Riesz transforms, function spaces, and weighted estimates for Schrödinger operators with non-negative potentials.

by

Fu Ken Ly

B.Sc. Macquarie University (2005)

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written by **Fu Ken Ly**

has been approved for the Department of Mathematics

Xuan Thinh Duong

Christopher Meaney

Date

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

Thesis directed by Prof. Xuan Thinh Duong and Dr. Christopher Meaney.

Summary

The main aim of this thesis is to obtain estimates for Riesz transforms associated to the Schrödinger operator with non-negative potentials on various function spaces over the Euclidean spaces. Our results concern the first- and second-order Riesz transforms on the weighted L^p spaces, the Hardy spaces, and the Morrey spaces. We describe our main results briefly in the following.

We show that the L^p boundedness of the first-order Riesz transforms within a range of exponents is *equivalent*, firstly to their boundedness on the weighted L^p spaces within a specific range of both p and (Muckenhoupt) weights, and secondly to their boundedness on the Morrey spaces $\mathcal{L}^{p,\lambda}$ within a specific range of parameters p and λ.

On specialising to the case where the potential satisfies a reverse Hölder inequality up to some exponent, we show that the second-order Riesz transforms are bounded on three classes of function spaces: the weighted Lebesgue spaces, the Hardy spaces associated to the Schrödinger operator, and the Morrey spaces, again within a specific range of parameters; this time depending on the reverse Hölder exponent. This is achieved through some new estimates on the heat kernel (associated to the Schrödinger operator) that we derive within this context. These estimates involve the time and the spatial derivatives and have extra global decay over the usual Gaussian.

In this setting we also study a class of weights that generalise the Muckenhoupt weights and are adapted to the Schrödinger operator in a certain sense. We develop some new good-λ estimates that provide a systematic framework for investigating operators lacking kernel regularity on L^p spaces with these weights.

Declaration

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree to any other university or institution other than Macquarie University.

I also certify that the thesis is an original piece of research and it has been written by me. Any help and assistance that I have received in my research work and the preparation of the thesis itself has been appropriately acknowledged.

In addition, I certify that all information sources and literature used are indicated in the thesis.

Fu Ken Ly

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