Chapter 1

Introduction

Let f be a smooth, real-valued function that is compactly supported on $\mathbb{R}^n$ and consider the following inequalities:

$$\|\nabla f\|_{L^p(\mathbb{R}^n)} \le C_p \|\Delta^{1/2} f\|_{L^p(\mathbb{R}^n)} \tag{1.1}$$

$$\|\nabla^2 f\|_{L^p(\mathbb{R}^n)} \le C_p \|\Delta f\|_{L^p(\mathbb{R}^n)}. \tag{1.2}$$

The second is commonly known as the *Calderón–Zygmund inequality*. Here the constant C_p may depend on p and the dimension n, but not on f. Both inequalities are valid for all $1 < p < \infty$. Note here that $\Delta = \sum_j \partial_j^2$ is the Laplacian on $\mathbb{R}^n$, while ∇^2 is shorthand for $\partial_j \partial_k$.

Inequalities such as these, often referred to as 'L^p-estimates', along with their analogues (when the space $L^p(\mathbb{R}^n)$ is replaced by other function spaces) have been thoroughly studied in the harmonic analysis literature, motivated in part by their connections with partial differential equations. We shall restrict our attention to three particular classes of function spaces.

When one considers p below 1, inequalities (1.1) and (1.2) are valid for $p \le 1$ once we replace the $L^p(\mathbb{R}^n)$ spaces and their norms by the Hardy spaces $H^p(\mathbb{R}^n)$ and their respective norms. Another natural extension is to replace $L^p(\mathbb{R}^n)$ by weighted spaces $L^p(w)$, and it is well understood that the corresponding inequalities in this situation hold for the full range $1 < p < \infty$ precisely when the function w belongs to a family of so-called 'Muckenhoupt weights', denoted by $\mathcal{A}_p$. Another class of function spaces that bears some connection to the $\mathcal{A}_p$ classes is the class of Morrey spaces $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$, and the corresponding inequalities hold for these spaces for the full range of $1 < p < \infty$ and $0 < \lambda < n$. A standard reference for the results on Hardy and weighted spaces is the monograph [101]. For the results on Morrey spaces see [34, 86].

Our goal in this thesis is to replace the Laplacian $-\Delta$ in (1.1) and (1.2) by operators of the form

$$L = -\Delta + V \qquad \text{on} \quad \mathbb{R}^n, \quad n \geq 1,$$

where V is a non-negative and locally integrable function, and determine *to what extent are the inequalities still valid*. Historically, an operator of the form $-\Delta + V$ has been referred to as a Schrödinger operator, and V its *potential*. This operator plays a fundamental role in non-relativistic quantum mechanics. It is also studied in the field of partial differential equations, and has applications to spectral and scattering theory. See [20, 23, 56, 89, 90, 92, 96, 97].

By replacing f by $(-\Delta)^{-1/2}g$ in (1.1) and f by $(-\Delta)^{-1}g$ in (1.2), the inequalities (1.1) and (1.2) can be interpreted as the $L^p(\mathbb{R}^n)$ boundedness of the operators $\nabla(-\Delta)^{-1/2}$ and $\nabla^2(-\Delta)^{-1}$ respectively. These operators are also commonly referred to as the first- and second-order Riesz transforms respectively. These objects belong to a class of operators called *Calderón–Zygmund operators*.

A Calderón–Zygmund operator is an operator that is bounded on $L^2(\mathbb{R}^n)$ and whose kernel satisfies certain smoothness and decay properties. The Calderón–Zygmund theory of singular integrals was initiated in the 50s to systematically study such objects. Since then it has undergone a rich development and we refer the reader to [101] and [60] for complete details and historical references.

Broadly speaking, one shows that such operators are of weak type $(1, 1)$ – that is, they map $L^1(\mathbb{R}^n)$ into the larger space $L^{1,\infty}(\mathbb{R}^n)$ – through the Calderón–Zygmund decomposition, and then invoke interpolation to obtain the boundedness on $L^p(\mathbb{R}^n)$ for all $1 < p < 2$. For the range $2 < p < \infty$, a family of techniques referred to as 'good-λ' inequalities are often used.

One drawback of this approach is that the regularity of the kernel is required. Another limitation is that the boundedness given is the full range of $1 < p < \infty$. It is known however that there are operators that do not satisfy either of these restrictions. Some examples include elliptic operators in divergent form, operators on irregular domains, and operators on manifolds. Of relevance to us are the Riesz transforms associated to the Schrödinger operator $L = -\Delta + V$. Depending on the smoothness and size of V, the first-order Riesz transforms $\nabla L^{-1/2}$, $V^{1/2}L^{-1/2}$ and the second-order Riesz transforms $\nabla^2 L^{-1}$, VL^{-1} may not be Calderón–Zygmund operators. Therefore new techniques are needed. For the convenience of the reader we list the main operators studied in this thesis below. See also Chapter 2 for further details.

Underlying operator:	$L = -\Delta + V, \qquad V \geq 0$
First-order Riesz transforms:	$\nabla L^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \nabla e^{-tL} \frac{dt}{\sqrt{t}}$ $V^{1/2}L^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty V^{1/2} e^{-tL} \frac{dt}{\sqrt{t}}$
Second-order Riesz transforms:	$\nabla^2 L^{-1} = \int_0^\infty \nabla^2 e^{-tL}\, dt$ $VL^{-1} = \int_0^\infty V e^{-tL}\, dt$

Table 1: Main operators considered

Having dispensed with this brief overview we turn now to a more focussed description of the results in this thesis. The rest of this chapter will be devoted to explaining our results in their historical and mathematical context. We shall proceed as follows.

- In Section 1.1 we describe the known results in the literature and formulate our objectives as several key questions.
- The main results of this thesis are presented in Section 1.2 in the context of these objectives.
- We explain the key techniques and ideas behind the proofs in Section 1.3.
- We describe the organisation of the thesis in Section 1.4.

In this thesis we restrict our attention to non-negative potentials, and so we limit our survey accordingly. A body of literature also exists for other Schrödinger-type operators, including say, those with negative potentials and those that have magnetic components but we do not describe them here. We refer the reader to the survey [96], and also [10, 21, 22] for examples of more recent work and the references therein.

1.1 Known results

The study of $L^p(\mathbb{R}^n)$ estimates for the Schrödinger operator $L = -\Delta + V$ has attracted many authors. Under the assumption that V is a non-negative polynomial, J. Nourrigat [82] studied $L^2(\mathbb{R}^n)$ boundedness of $\nabla^2 L^{-1}$. This was extended to $L^p(\mathbb{R}^n)$, for $1 < p < \infty$, both by J. Zhong [110] and by D. Guibourg [62] independently. In particular Zhong showed that when V is a non-negative polynomial, the operators $\nabla L^{-1/2}$ and $\nabla^2 L^{-1}$ are Calderón–Zygmund operators. For the case $p = 1$, the operators ΔL^{-1} and VL^{-1} are known to be bounded on $L^1(\mathbb{R}^n)$ whenever $V \geq 0$ (see either [73] or [58]), and as a consequence it follows that the operator $\nabla^2 L^{-1}$ is weak $(1, 1)$.

It is natural to ask the following.

Question 1. *For which $V \geq 0$ and which $p \geq 1$ do the following inequalities hold?*

$$\|\nabla f\|_{L^p} + \|V^{1/2} f\|_{L^p} \leq C_p \|(-\Delta + V)^{1/2} f\|_{L^p} \qquad \forall f \in C_0^\infty, \tag{1.3}$$

$$\|\nabla^2 f\|_{L^p} + \|V f\|_{L^p} \leq C_p \|(-\Delta + V) f\|_{L^p} \qquad \forall f \in C_0^\infty. \tag{1.4}$$

Here C_0^∞ denotes the space of all functions on $\mathbb{R}^n$ that are smooth and compactly supported.

A consequence of (1.3) is the boundedness of $\nabla L^{-1/2}$ and $V^{1/2} L^{-1/2}$ on $L^p(\mathbb{R}^n)$, and similiarly (1.4) imply the boundedness of $\nabla^2 L^{-1}$ and VL^{-1}. Note also that if (1.4) holds for some p then (1.3) holds for $2p$ (see [12]).

For the range $1 < p \leq 2$, a complete answer for (1.3) is given in the following.

Theorem 1.1 ([46, 94])**.** *Assume that $n \geq 1$ and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then (1.3) holds for each $p \in (1, 2\,]$.*

The expression $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$ denotes that V is a non-negative and locally integrable function on $\mathbb{R}^n$. The validity of (1.3) at the endpoint $p = 2$ follows from the definition of L. For the other endpoint, A. Sikora [94] showed $\nabla L^{-1/2}$ is weak $(1, 1)$. Independently, the authors in [46] showed that the operators $\nabla L^{-1/2}$ and $V^{1/2}L^{-1/2}$ map the Hardy space $H^1_L(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ (see Section 1.1.3 for the definition of these Hardy spaces). In both cases, interpolation gives (1.3) for the range $1 < p < 2$. In light of this, it is therefore of interest to find conditions on V ensuring that (1.3) and (1.4) hold for $p > 2$.

A pivotal work in this area was done by Z. Shen [93] in 1995. In that article the author gives a systematic study of $L^p(\mathbb{R}^n)$ estimates for the operator $-\Delta + V$ in the situation where V satisfies a so-called *reverse Hölder inequality*:

$$\Big(\frac{1}{|B|}\int_B V^q\Big)^{1/q} \leq C\Big(\frac{1}{|B|}\int_B V\Big) \tag{1.5}$$

for every ball B. If V satisfies (1.5) for some $q > 1$, then we say that V belongs to the class of reverse Hölder weights of exponent q, and write $V \in \mathcal{B}_q$ (when $q = \infty$ we take the left-hand side of (1.5) to be the essential supremum of V on B). We remark that these classes form a decreasing scale in the sense that $\mathcal{B}_q \subset \mathcal{B}_p$ whenever $q > p$. One motivation for the introduction of these classes is that they give a generalisation for the polynomial potentials which have already been studied in the literature. In fact if V is a non-negative polynomial then V satisfies (1.5) with exponent $q = \infty$. Other examples include the functions $V(x) = |x|^{-\alpha}$, for which $V \in \mathcal{B}_q$ whenever $\alpha \in (-\infty, n/q)$.

The class of *reverse Hölder potentials* will be of central focus in this thesis. We summarise the results of relevance to (1.3) and (1.4) from [93] in the following statement. We adopt the notation

$$q^* := \begin{cases} \dfrac{nq}{n-q}, & q < n; \\ \infty, & q \geq n. \end{cases} \tag{1.6}$$

Theorem 1.2 ([93] Theorems 0.3, 5.10, 0.5, 0.8). *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 3$. Assume that $V \in \mathcal{B}_q$ for some $q > 1$.*

(a) *If* $q \geq n/2$ *then* (1.4) *holds for all* $p \in (1, q\,]$.

(b) *If* $q \geq n/2$ *then* (1.3) *holds for all* $p \in (1, 2q\,]$.

(c) *If* $q \in [\,n/2, n)$ *then* $\nabla L^{-1/2}$ *is bounded on* $L^p(\mathbb{R}^n)$ *for* $p \in (1, q^*]$.

(d) *If* $q \in [\,n, \infty\,]$ *then* $\nabla L^{-1/2}$ *is a Calderón–Zygmund operator, and hence bounded on* $L^p(\mathbb{R}^n)$ *for all* $p \in (1, \infty)$.

The author of [93] also shows that the ranges in (a) and (c) are sharp. Note the connection between higher regularity (in the sense that, say the operator $\nabla L^{-1/2}$ approaches a Calderón–Zygmund operator) with the increasing reverse Hölder exponent. We emphasise also the dependence of the intervals of boundedness on the reverse Hölder exponent q.

Shen's article was a source of influence for many subsequent authors in this area of research. A key idea was the introduction of the "critical radius" function (see also Definition 2.2)

$$\gamma(x) = \sup\Big\{ r > 0 : \ \frac{r^2}{|B(x,r)|} \int_{B(x,r)} V(y)\, dy \leq 1 \Big\}, \tag{1.7}$$

modelled on a similar tool used in [110] in the study of polynomial potentials. This tool has been a cornerstone in later investigations on Schrödinger operators with reverse Hölder potentials. Examples include [28, 29, 30, 49, 51, 53, 63, 76], some of which we touch upon in later sections. We direct also the reader to Section 1.3 for a discussion of (1.7) and its role in this thesis. Before closing this section we mention another important contribution in this direction of research. Throughout [93] there was a dimensional restriction of $n \geq 3$. This restriction was removed more recently by P. Auscher and B. Ben-Ali [12] using different methods.

Theorem 1.3 ([12])**.** *Let* $L = -\Delta + V$ *on* $\mathbb{R}^n$ *with* $n \geq 1$ *and* $V \in \mathcal{B}_q$ *for some* $q > 1$.

(a) *There exists* $\varepsilon > 0$ *such that* (1.4) *holds for every* $p \in (1, q + \varepsilon)$. *That is,*

$$\|\nabla^2 f\|_{L^p} + \|Vf\|_{L^p} \leq C_p \|Lf\|_{L^p}, \qquad \forall f \in C_0^\infty.$$

(b) *There exists* $\varepsilon > 0$ *such that* (1.3) *holds for every* $p \in (1, 2q + \varepsilon)$. *That is,*

$$\|\nabla f\|_{L^p} + \|V^{1/2} f\|_{L^p} \leq C_p \|L^{1/2} f\|_{L^p}, \qquad \forall f \in C_0^\infty.$$

(c) *If $q \geq n/2$ then there exists $\varepsilon > 0$ such that for every $p \in (1, q^* + \varepsilon)$,*

$$\|\nabla f\|_{L^p} \leq C_p \|L^{1/2} f\|_{L^p}, \qquad \forall f \in C_0^\infty.$$

Here q^* has been defined in (1.6).

Note that $q^* \geq 2q$ if and only if $q \geq n/2$ so item (c) improves over item (b) for the gradient part in this situation. In a sense the results of [12] gives a complete answer to Question 1 in the context of reverse Hölder weights. Hence the question of the validity of (1.3) and (1.4), in the range $p > 2$ for classes of potentials beyond the reverse Hölder classes, is open.

As in the classical situation of (1.1) and (1.2), once the L^p estimates have been resolved, it is natural to inquire about corresponding estimates in other function spaces.

1.1.1 Weighted spaces

Historically one motivation for the $\mathcal{A}_p$ classes is the characterisation of all the non-negative measures μ on $\mathbb{R}^n$ for which the Hardy–Littlewood maximal function M satisfies

$$\int_{\mathbb{R}^n} |Mf(x)|^p \, d\mu(x) \leq C \int_{\mathbb{R}^n} |f(x)|^p \, d\mu(x), \qquad \forall f \in C_0^\infty(\mathbb{R}^n), \tag{1.8}$$

for some $p \in (1, \infty)$. The operator M is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy, \qquad x \in \mathbb{R}^n, \tag{1.9}$$

where the supremum is taken over all balls containing x. A complete characterisation was given by Muckenhoupt [80]: estimate (1.8) holds if and only if $d\mu(x) = w(x)\, dx$ where w satisfies the so-called '$\mathcal{A}_p$-condition'

$$\Big(\frac{1}{|B|} \int_B w\Big)^{1/p} \Big(\frac{1}{|B|} \int_B w^{1-p'}\Big)^{1/p'} < \infty, \tag{1.10}$$

for all balls B (here p' is the conjugate exponent of p, defined by the relationship $1/p + 1/p' = 1$). In this case we say that w belongs to the class of Muckenhoupt weights $\mathcal{A}_p$ and write $w \in \mathcal{A}_p$. We note that the class $\mathcal{A}_1$ can be defined using (1.10) with $p = 1$, but we set $\mathcal{A}_\infty$ to be the union of all $\mathcal{A}_p$ with $1 \leq p < \infty$.

These weights have been extensively studied and many of their properties are now well known and considered an established part of the harmonic analysis canon. An important property that is of relevance to us is the connection with reverse Holder weights: if w is an $\mathcal{A}_\infty$ weight then it satisfies a reverse Hölder inequality (1.5) for some $q > 1$. A converse statement is also true, so that in this sense the class of all Muckenhoupt weights in fact coincides with the class of all reverse Hölder weights. We refer to [59] for a treatise on the subject.

It has been known since [35] that Calderón–Zygmund operators are bounded on weighted L^p spaces with $\mathcal{A}_p$ weights. The study of operators with $\mathcal{A}_p$ weights continues to be active area of research, motivated both by its traditional place within harmonic analysis, and also by its connection with boundary value problems.

We are interested in estimates related to the Schrödinger operator. *For which* $V \geq 0$, $p \geq 1$, *and* $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ *do the following hold?*

$$\|\nabla f\|_{L^p(w)} + \|V^{1/2} f\|_{L^p(w)} \leq C_p \|(-\Delta + V)^{1/2} f\|_{L^p(w)} \qquad \forall f \in C_0^\infty, \tag{1.11}$$

$$\|\nabla^2 f\|_{L^p(w)} + \|V f\|_{L^p(w)} \leq C_p \|(-\Delta + V) f\|_{L^p(w)} \qquad \forall f \in C_0^\infty. \tag{1.12}$$

Prior results in this direction can be found in [6, 30, 75, 98], which we now describe. We shall employ the following notation, first introduced in [16]. For $w \in \mathcal{A}_\infty$ and $1 \leq p_0 < q_0 \leq \infty$ we set

$$\mathcal{W}_w(p_0, q_0) := \left\{ p \in (p_0, q_0) : w \in \mathcal{A}_{p/p_0} \cap \mathcal{B}_{(q_0/p)'} \right\}.$$

(i) When V is a non-negative and locally integrable function, B.T. Anh [6] showed that (1.11) holds for each $w \in \mathcal{A}_\infty$ and $p \in \mathcal{W}_w(1, 2)$. This is the weighted counterpart to Theorem 1.1. The result for $\nabla L^{-1/2}$ was also obtained independently by L. Song and L. Yan [98]. It was also shown in [6] that the first-order Riesz transforms $\nabla L^{-1/2}$ and $V^{1/2} L^{-1/2}$ are weak $(1, 1)$ with respect to the measure $w\, dx$ with $w \in \mathcal{A}_1 \cap \mathcal{B}_2$.

(ii) Specializing to reverse Hölder potentials, when $V \in \mathcal{B}_q$ with $q \geq n$, recall from Theorem 1.2 above that $\nabla L^{-1/2}$ is a Calderón–Zygmund operator, and hence is bounded on $L^p(w)$ for each $p \in (1, \infty)$ and $w \in \mathcal{A}_p$.

In [75] the authors show that if $V \in \mathcal{B}_q$, with $q > \max\{n/2, p\}$ and $p \in (1,\infty)$, then VL^{-1} is bounded on $L^p(w)$ for $w^{1-p'} \in \mathcal{A}_{p'/q'}$. We observe that this is equivalent to $w \in \mathcal{A}_\infty$ and $p \in \mathcal{W}_w(1,q)$. It is a straightforward consequence that the estimate for $\nabla^2 L^{-1}$ follows from that of VL^{-1}. Indeed, from the boundedness of $\nabla^2(-\Delta)^{-1}$ on $L^p(w)$ for each $p \in (1,\infty)$ and $w \in \mathcal{A}_p$, we have

$$\left\|\nabla^2 L^{-1} f\right\|_{L^p(w)} \lesssim \left\|-\Delta L^{-1} f\right\|_{L^p(w)} = \left\|f - VL^{-1} f\right\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}. \tag{1.13}$$

In summary when $V \in \mathcal{B}_q$ for some $q > n/2$ with $n \geq 3$ then (1.12) holds for all $w \in \mathcal{A}_\infty$ and $p \in \mathcal{W}_w(1,q)$.

Note that the results here are subsumed by the results in the next item.

(iii) B. Bongioanni, E. Harboure, and O. Salinas [30] introduced a new class of weights $\mathcal{A}_\infty^L$, modelled on the classical $\mathcal{A}_\infty$ weights, but adapted to the Schrödinger operator in a certain sense. These weights are defined as those $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\Big(\frac{1}{|B|}\int_B w\Big)^{1/p}\Big(\frac{1}{|B|}\int_B w^{1-p'}\Big)^{1/p'} \leq C\Big(1 + \frac{r}{\gamma(x)}\Big)^{\theta} \tag{1.14}$$

for some $\theta \geq 0$ and every ball $B = B(x,r)$. Note that γ is the function defined in (1.7). In this case we say that $w \in \mathcal{A}_p^L$.

Observe that when $\theta = 0$ they coincide with the $\mathcal{A}_\infty$ classes, but in general they form a larger class of weights. To see this let $V \equiv 1$ and take $w(x) = (1+|x|)^{-(n+\varepsilon)}$ where $\varepsilon > 0$. Then $w \notin \mathcal{A}_\infty$ but satisfies (1.14) for any $\theta \geq \varepsilon$.

It was shown in [30] that, as in the classical situation, if w is a member of $\mathcal{A}_\infty^L$ then it satisfies a certain reverse Hölder inequality. This inequality is similar to (1.5) but with the extra growth term involving γ as in (1.14). Inspired by this result we introduce the reverse Hölder classes $\mathcal{B}_q^L$ for $q > 1$, adapted to L in Definition 5.3, which as far as we are aware has not appeared elsewhere in the literature. This allows us to introduce the following notation. Given $w \in \mathcal{A}_\infty^L$ and $1 \leq p_0 < q_0 \leq \infty$ we set

$$\mathcal{W}_w^L(p_0,q_0) := \Big\{p \in (p_0,q_0) : w \in \mathcal{A}_{p/p_0}^L \cap \mathcal{B}_{(q_0/p)'}^L\Big\}.$$

In their article [30] the authors proved that the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^p(w)$ for each $p \in (1, q^*)$, and weight w with $w^{1-p'} \in \mathcal{A}^L_{p'/(q^*)'}$ (equivalently $w \in \mathcal{A}^L_\infty$ and $p \in \mathcal{W}^L_w(1, q^*)$ in our notation), and satisfies a weighted weak $(1,1)$ estimate for weights w with $w^{(q^*)'} \in \mathcal{A}^L_1$.

A further study of these weights was undertaken by L. Tang in [104, 105, 106]. The author obtains, amongst other results, the boundedness of the operators $V^{1/2}L^{-1/2}$ and VL^{-1}, as well as another proof of the result for $\nabla L^{-1/2}$.

Theorem 1.4 ([106]). *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 3$. Assume that $V \in \mathcal{B}_q$ for some $q > n/2$. Then*

(a) *$V^{1/2}L^{-1/2}$ is bounded on $L^p(w)$ for each $p \in (1, 2q)$ and weight w with $w^{1-p'} \in \mathcal{A}^L_{p'/(2q)'}$ (equivalently $w \in \mathcal{A}^L_\infty$ and $p \in \mathcal{W}^L_w(1, 2q)$),*

(b) *VL^{-1} is bounded on $L^p(w)$ for each $p \in (1, q)$ and weight w with $w^{1-p'} \in \mathcal{A}^L_{p'/q'}$ (equivalently $w \in \mathcal{A}^L_\infty$ and $p \in \mathcal{W}^L_w(1, q)$).*

Let us summarise the situation for weighted estimates for the Schrödinger operator with a reverse Hölder potential (that is, items (ii) and (iii)). When $V \in \mathcal{B}_q$ for some $q > n/2$ with $n \geq 3$, results for the operators $\nabla L^{-1/2}$, $V^{1/2}L^{-1/2}$ and VL^{-1} are known for $\mathcal{A}^L_\infty$ (and therefore also $\mathcal{A}_\infty$) weights. We display this information in the table below.

Operator:	$V^{1/2}L^{-1/2}$	$\nabla L^{-1/2}$	VL^{-1}	$\nabla^2 L^{-1}$
$w \in \mathcal{A}_\infty$	$\mathcal{W}_w(1, 2q)$	$\mathcal{W}_w(1, q^*)$	$\mathcal{W}_w(1, q)$	$\mathcal{W}_w(1, q)$
$w \in \mathcal{A}^L_\infty$	$\mathcal{W}^L_w(1, 2q)$	$\mathcal{W}^L_w(1, q^*)$	$\mathcal{W}^L_w(1, q)$	?

Table 2: Known results for weighted spaces

We will show in this thesis that it is valid to place $\mathcal{W}^L_w(1, q)$ in the entry marked “ ? ”.

Unfortunately in contrast with the $\mathcal{A}_\infty$ situation, the calculation in (1.13) is of limited use in passing from estimates for VL^{-1} to estimates for $\nabla^2 L^{-1}$ for $\mathcal{A}^L_\infty$ weights because the mapping properties of $\nabla^2(-\Delta)^{-1}$ for these weights are not clear. In spite of this, since the $\mathcal{A}_\infty$ weights are a special case of the $\mathcal{A}_\infty^L$ weights, one might conjecture that a corresponding result holds for $\nabla^2 L^{-1}$ as in item (ii) above. This leads us to the main objective of this section.

Question 2. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 3$. Assume that $V \in \mathcal{B}_q$ for some $q > n/2$. For which $w \in \mathcal{A}_\infty^L$ and which $p > 1$ does the following inequality hold?*

$$\left\| \nabla^2 L^{-1} f \right\|_{L^p(w)} \leq C_p \left\| f \right\|_{L^p(w)}, \qquad \forall f \in C_0^\infty(\mathbb{R}^n).$$

We shall give an answer to this question in Theorem 1.7 (see also Theorem 5.1). Our techniques are different to those of [30] and [106]. The purpose of Chapter 5 is to develop these techniques and apply them to give the proof of this result.

Next we turn to a class of spaces that bears some connection with $\mathcal{A}_p$ weights.

1.1.2 Morrey spaces

Let $p \in [1, \infty)$ and $\lambda \in (0, n)$. A function f is said to belong to the Morrey space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{\mathcal{L}^{p,\lambda}} := \sup_B \left(\frac{r_B^\lambda}{|B|} \int_B |f - f_B|^p \right)^{1/p} < \infty.$$

These spaces were introduced by C.B. Morrey [79] to study the regularity of partial differential equations. Some of their properties were investigated in the 1960s by G. Stampacchia [99] and S. Campanato [33]. See [87] for a survey of some of these properties. Recently these spaces have garnered much attention in the study of non-linear equations. See for example [78] and the references therein. They are related to the Lebesgue spaces and the Sobolev spaces in their two parameters p and λ measuring size and smoothness respectively. For the limiting cases it is clear that when $\lambda = n$ the resulting Morrey space coincides with the space $L^p(\mathbb{R}^n)$, and for $\lambda = 0$ the resulting space is BMO, the space of bounded mean oscillation introduced in [71].

It is well known that classical singular integral operators such as the Hardy–Littlewood maximal function and Calderón–Zygmund operators are bounded on the Morrey spaces $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$

for all $p \in (1,\infty)$ and $\lambda \in (0,n)$. See [86] and [34]. However operators that fall outside the Calderón–Zygmund class have received less attention in the literature on Morrey spaces.

We are interested in estimates for the Schrödinger operator on these spaces. More precisely we seek to answer: *for which $V \geq 0$, $p \geq 1$ and $\lambda \in (0,n)$, do the following hold?*

$$\|\nabla f\|_{\mathcal{L}^{p,\lambda}} + \|V^{1/2} f\|_{\mathcal{L}^{p,\lambda}} \leq C_p \|(-\Delta + V)^{1/2} f\|_{\mathcal{L}^{p,\lambda}} \qquad \forall f \in C_0^\infty, \tag{1.15}$$

$$\|\nabla^2 f\|_{\mathcal{L}^{p,\lambda}} + \|V f\|_{\mathcal{L}^{p,\lambda}} \leq C_p \|(-\Delta + V) f\|_{\mathcal{L}^{p,\lambda}} \qquad \forall f \in C_0^\infty. \tag{1.16}$$

We observe again that when $V \in \mathcal{B}_n$, the Riesz transform $\nabla L^{-1/2}$ is a Calderón–Zygmund operator and therefore falls within the scope of the classical results obtained in [34]. That is, the gradient part of (1.15) holds for all $p \in (1,\infty)$ and $\lambda \in (0,n)$. When $V \in \mathcal{B}_\infty$ then the operators VL^{-1} and $\nabla^2 L^{-1}$ may not be of Calderón–Zygmund type, but the authors in [75] show nonetheless that (1.16) holds for all $p \in (1,\infty)$ and $\lambda \in (0,n)$.

Within this context we seek to establish the range of p and λ for which (1.15) and (1.16) holds when V is a reverse Hölder potential with $q < n$. We are motivated by the fact that taking $\lambda = n$ returns us to the situation of (1.3) and (1.4), where results are already known. Recall that in those L^p estimates there was an upper restriction on p that depended on the reverse Hölder exponent q, and so one expects a corresponding restriction to be transferred to the Morrey space scale in the p parameter. It would be of interest to uncover any lower restriction on the parameter λ. We ask the following question.

Question 3. *Let $n \geq 1$ with $V \in \mathcal{B}_q$ for some $q > 1$. For which $p > 1$ and $\lambda \in (0,n)$ do the inequalities* (1.15) *and* (1.16) *hold?*

We give a rather complete picture of this setting in Theorem 6.2.

1.1.3 Hardy spaces

For $0 < p < \infty$ the tempered distribution f is said to belong to the Hardy space $H^p(\mathbb{R}^n)$ if the so-called "square function"

$$Sf(x) = \Big(\int_0^\infty \int_{|x-y|<t} \big|t^2 \Delta e^{t^2\Delta} f(y)\big|^2 dy \frac{dt}{t^{n+1}}\Big)^{1/2}, \qquad x \in \mathbb{R}^n \tag{1.17}$$

satisfies $Sf \in L^p(\mathbb{R}^n)$. The study of these spaces began in [102] in the early 1960s. Real variable methods were introduced in [57], and since then, the theory of Hardy spaces has undergone a rich development. We refer the reader to the monograph [101] for an exposition on this subject.

For p below 1 these spaces are the natural continuation of the $L^p(\mathbb{R}^n)$ spaces because firstly it can be shown that L^p coincides with H^p for $p > 1$, and secondly, on replacing L^p by H^p then (1.1) and (1.2) holds for all $0 < p < \infty$.

In the context of Schrödinger operators, we are interested in answers to the following. *For which $V \geq 0$ and $p \leq 1$ do the following hold?*

$$\|\nabla f\|_{H^p} + \|V^{1/2} f\|_{H^p} \leq C_p \|(-\Delta + V)^{1/2} f\|_{H^p} \qquad \forall f \in C_0^\infty \tag{1.18}$$

$$\|\nabla^2 f\|_{H^p} + \|V f\|_{H^p} \leq C_p \|(-\Delta + V) f\|_{H^p} \qquad \forall f \in C_0^\infty \tag{1.19}$$

Part of the interest in the H^p spaces stems from their role in partial differential equations and in harmonic analysis. However it is known that there are many situations in which these classical spaces are not directly applicable. For instance the classical Riesz transforms $\nabla(-\Delta)^{-1/2}$ are bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ (and even $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$). In fact, $\nabla(-\Delta)^{-1/2} f \in L^p$ is one criterion for membership of f in H^p. See [101] and [60]. Unfortunately given an arbitrary differential operator L, its associated Riesz transform $\nabla L^{-1/2}$ may not necessarily be bounded from H^1 to L^1. This may happen, for example, when L is an elliptic operator in divergence form with complex coefficients (see the discussion in [67] and also [11, 27, 66] for results on the intervals of boundedness of $\nabla L^{-1/2}$ on $L^p(\mathbb{R}^n)$).

The notion of a *Hardy space adapted to an operator* was introduced to address some of these deficiencies. Given an operator L and in analogy with (1.17) we say that $f \in H^p_L(\mathbb{R}^n)$ provided the associated square function

$$S_L f(x) = \Big(\int_0^\infty \int_{|x-y|<t} \big| t^2 L e^{-t^2 L} f(y) \big|^2 dy \frac{dt}{t^{n+1}} \Big)^{1/2}, \qquad x \in \mathbb{R}^n$$

satisfies $S_L f \in L^p(\mathbb{R}^n)$. Depending on L, these spaces may or may not coincide with the classical Hardy spaces. Nevertheless under suitable conditions on L, the spaces $H^p_L(\mathbb{R}^n)$ may still interpolate with $L^p(\mathbb{R}^n)$. This is useful in applications. For instance the proof of Theorem 1.1

given in [46] takes advantage of this fact. These spaces were initially introduced (for operators whose heat kernels satisfy suitable pointwise bounds) in [14, 47, 48], and were further developed (for more general classes of operators) in [19, 65, 67]. We refer the reader to these articles for the details and relevant references as well as some historical notes on the evolution of these ideas. For some recent applications of these H^p_L spaces to partial differential equations we refer the reader to [43].

Our focus is on the case $L = -\Delta + V$, the Schrödinger operator with a non-negative potential V. The development of the Hardy spaces adapted to this operator has been taken up independently, on the one hand as a consequence of the theory mentioned above (see in particular [65] and [70]), and on the other hand by J. Dziubański and J. Zienkiewicz [51, 52, 53]. In the latter articles, the authors focus on situations with stronger conditions on the potential, namely where V is a reverse Hölder potential. We note that they give certain atomic decompositions for the spaces, and one advantage of these decompositions is they allow direct comparisons with the classical H^p. We note also that both spaces coincide for the range $n/(n+1) < p \leq 1$. In fact combining these results gives us the following: when $V \in \mathcal{B}_q$ for some $q > n/2$, then $H^p(\mathbb{R}^n) \subsetneq H^p_L(\mathbb{R}^n)$ for every $p \in (n/(n+p_L), 1\,]$ where $p_L = \min\{1, 2 - n/q\}$. See Section 7.1.1 for the details.

In [65] and [70] the authors show that under the condition that V is non-negative and locally integrable, the Riesz transform $\nabla L^{-1/2}$ is bounded from $H^p_L(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $p \in (0, 1\,]$, and bounded from $H^p_L(\mathbb{R}^n)$ into $H^p(\mathbb{R}^n)$ for $p \in (n/(n+1), 1\,]$. On restricting the class of potentials to the reverse Hölder potentials then the relationship between $H^p(\mathbb{R}^n)$ and $H^p_L(\mathbb{R}^n)$ mentioned in the previous paragraph gives us stronger conclusions. Indeed if $V \in \mathcal{B}_q$ for some $q > n/2$, then the gradient part of (1.18) holds for every $p \in (n/(n+p_L), 1\,]$.

Our aim is to give parallel results for the second-order Riesz transforms $\nabla^2 L^{-1}$ and VL^{-1} (when V is a reverse Hölder potential) on these spaces, which as far as we are aware, has not appeared in the literature. We wish to answer the following question.

Question 4. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 1$. Assume that $V \in \mathcal{B}_q$ for some $q > 1$. For which $p \leq 1$ is the following inequality valid?*

$$\left\|\nabla^2 L^{-1} f\right\|_{H^p} \leq C_p \|f\|_{H^p}, \qquad \forall f \in C_0^\infty.$$

Our main result in this direction is Theorem 1.9, which is proved in Chapter 7.

1.2 Main results

In this section we give the main results of this thesis, namely Theorems 1.5–1.9, framed as answers to the questions raised in the previous section. Before we address these questions we present a result in the general setting of non-negative potentials that demonstrates the estimates (1.3), (1.11), and (1.15) are intimately related. This is captured in the following result for the first-order Riesz transforms.

Theorem 1.5. *Fix $s > 2$. Let $n \geq 1$ and $L = -\Delta + V$ on $\mathbb{R}^n$ with $0 \leq V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then the following are equivalent.*

(a) *Estimate (1.3) holds for each $p \in (1, s)$. That is,*

$$\|\nabla f\|_{L^p} + \|V^{1/2} f\|_{L^p} \leq C_p \|L^{1/2} f\|_{L^p}, \qquad \forall f \in C_0^\infty.$$

(b) *Estimate (1.11) holds for each $w \in \mathcal{A}_\infty$ and each $p \in \mathcal{W}_w(1, s)$. That is,*

$$\|\nabla f\|_{L^p(w)} + \|V^{1/2} f\|_{L^p(w)} \leq C_p \|L^{1/2} f\|_{L^p(w)}, \qquad \forall f \in C_0^\infty.$$

(c) *Estimate (1.15) holds for each $p \in (1, s)$ and $\lambda \in \left(\frac{n}{s} p, n\right)$. That is,*

$$\|\nabla f\|_{\mathcal{L}^{p,\lambda}} + \|V^{1/2} f\|_{\mathcal{L}^{p,\lambda}} \leq C_p \|L^{1/2} f\|_{\mathcal{L}^{p,\lambda}}, \qquad \forall f \in C_0^\infty.$$

The proof of this is split over two theorems. The equivalence (a) $\iff$ (b) is contained in Theorem 4.1, while the equivalence (a) $\iff$ (c) is contained in Theorem 6.1. It is easy to see that on taking $w \equiv 1$ and $\lambda = n$, that we have (b) $\implies$ (a) and (c) $\implies$ (a) respectively. The hard work is in demonstrating (a) $\implies$ (b) and (b) $\implies$ (c), which are given in Chapters 4 and 6.

An extra statement may be added to this collection of equivalences. It involves a weighted weak type $(1,1)$ estimate. The reader is directed to Theorems 4.1 and 4.2.

Once the work of obtaining the L^p estimate is done (item (a)) then the Theorem grants us the estimates on weighted spaces and Morrey spaces immediately. Items (a) and (b) also generalise item (i) in Section 1.1.1.

The result also gives us a new counterpart to both Theorem 1.1 and Section 1.1.1 item (i), but for Morrey spaces. Indeed, if we let $s \to 2$, and taking into account Theorem 1.1, then we obtain a result as follows.

Theorem 1.6. *Let $n \geq 1$ and $0 \leq V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then (1.15) holds for each $p \in (1,2)$ and $\lambda \in \left(\frac{n}{2}p, n\right)$.*

Note the upper restriction on p and the lower restriction on λ, both governed by the auxiliary parameter s. On taking $s \to \infty$ we obtain boundedness for the full range of $p \in (1,\infty)$, $w \in \mathcal{A}_p$, and $\lambda \in (0,n)$. This happens, as we have seen for example when V is a non-negative polynomial, or when V is a reverse Hölder potential of order at least n. In fact in the latter case, $\nabla L^{-1/2}$ is a Calderón–Zygmund operator, which returns us to the classical situation.

We mention one other application to reverse Hölder potentials. If we take $V \in \mathcal{B}_q$ for some $q > 1$, and $n \geq 1$, then combining this with the result of [12] (specifically Theorem 1.3) we can recover the weighted results of Section 1.1.1 items (ii) and (iii) for the first-order Riesz transforms. In fact there is an improvement because the dimensional restriction of $n \geq 3$ is removed.

We now devote our attention to reverse Hölder potentials, and in particular address Questions 2, 3, and 4. With respect to the weighted Lebesgue spaces we give an answer to Question 2 in the following result.

Theorem 1.7. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 3$. Assume that $V \in \mathcal{B}_q$ for some $q > n/2$. Then the operator $\nabla^2 L^{-1}$ is bounded on $L^p(w)$ for each $w \in \mathcal{A}^L_\infty$ and $p \in \mathcal{W}^L_w(1,q)$.*

This result extends the one for $\nabla^2 L^{-1}$ and $\mathcal{A}_\infty$ weights and completes the picture for the first- and second-order Riesz transforms on weighted spaces with both $\mathcal{A}_\infty$ and $\mathcal{A}_\infty^L$ weights, at least for the range $q > n/2$. The proof of Theorem 1.7 is given in Theorem 5.1.

For Morrey spaces we answer Question 3 in the following result.

Theorem 1.8. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 1$ and assume $V \in \mathcal{B}_q$ for some $q > 1$. Then we have the following.*

(a) *Estimate (1.15) holds for each $p \in (1, 2q)$ and $\lambda \in \left(\frac{n}{2q}p, n\right)$.*

(b) *If $q \geq n/2$ then $\nabla L^{-1/2}$ is bounded on $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ for each $p \in (1, q^*)$ and $\lambda \in \left(\frac{n}{q^*}p, n\right)$.*

(c) *If $n \geq 2$ and $q \geq n/2$ then (1.16) holds for each $p \in (1, q)$ and $\lambda \in \left(\frac{n}{q}p, n\right)$.*

The proof of this is in Theorem 6.2. Items (a) and (b) are obtained using Theorem 1.5 and the results of [12] in Theorem 1.3. Note the lower restriction on λ and upper restriction on p, which as far as can tell appears to be the first result of its kind. If $q \to \infty$ then item (c) recovers the result from [75]. If $q \geq n$ then $q^* = \infty$ and so item (b) gives the result for $1 < p < \infty$ and $0 < \lambda < n$, which recovers the classical situation (that is, $\nabla L^{-1/2}$ is a Calderón–Zygmund operator) of [34]. We remark that our results improve over those in [75] in giving a restricted range on the parameters p and λ, which is as expected. Indeed boundedness cannot happen for λ going all the way to 0 because this implies boundedness on BMO.

For the Hardy spaces we give an answer to Question 4 in the following theorem.

Theorem 1.9. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 3$. Assume that $V \in \mathcal{B}_q$ with $q > \max\{2, n/2\}$. Then the following holds.*

(a) *The operators $\nabla^2 L^{-1}$ and VL^{-1} are bounded from $H^p_L(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for each $p \in (0, 1]$.*

(b) *The operator $\nabla^2 L^{-1}$ is bounded from $H^p_L(\mathbb{R}^n)$ into $H^p(\mathbb{R}^n)$ for each $p \in \left(\frac{n}{n+1}, 1\right]$.*

(c) *The operator $\nabla^2 L^{-1}$ is bounded from $H^p(\mathbb{R}^n)$ into $H^p(\mathbb{R}^n)$ for each $p \in \left(\frac{n}{n+p_L}, 1\right]$ where $p_L = \min\{1, 2 - n/q\}$.*

Parts (a) and (b) are proved in Theorem 7.1. Item (c) is a corollary of item (b) and is proved in Corollary 7.2. Theorem 1.9 extends the result mentioned in Section 1.1.3 for the first-order Riesz transforms $\nabla L^{-1/2}$ to the second-order Riesz transforms $\nabla^2 L^{-1}$ for reverse Hölder potentials. Item (c) gives an answer to Question 4.

We give some remarks about the condition $q > \max\{2, n/2\}$ in our results. The requirement $q > 2$ is a technical constraint used in two instances. The first is in the construction of the $H^p_L(\mathbb{R}^n)$ spaces, which in our work uses L^2-convergence of atomic sums (see Section 7.1.1). The other instance is the $L^2(\mathbb{R}^n)$ boundedness of the operators $\nabla^2 L^{-1}$ and VL^{-1}, which we recall is valid when $q > 2$. The careful reader will observe that our techniques and our heat kernel estimates will still follow through for the range $q < 2$, with suitable modifications, once an alternative construction of $H^p_L(\mathbb{R}^n)$ is available. For the time being however, the range $n = 3$ and $3/2 < q < 2$ remains open.

This result also admits extensions to weighted Hardy spaces for items (a) and (b), with $H^p_L(w)$ and $H^p(w)$ where $w \in \mathcal{A}_1 \cap \mathcal{B}_{(2/p)'}$. See Theorem 7.12. However item (c) remains open in this setting.

To conclude this section we also offer a result for $\nabla L^{-1/2}$ for a class of potentials V slightly larger than the reverse Hölder classes. This larger class, denoted $(DK_{\alpha,\theta,\sigma})$, is defined in Definition 8.1. We prove that the Riesz transform $\nabla L^{-1/2}$ in this setting is bounded on $L^p(\mathbb{R}^n)$ for some interval of p that is larger than $(1, 2]$. The result is Theorem 8.3 and its proof is given in Chapter 8. Theorem 1.5 then gives corresponding results on the weighted spaces and the Morrey spaces.

1.3 Key ideas and techniques behind the proofs

1.3.1 Overview

The basis of our techniques lies in two principles.

(i) The operator $L = -\Delta + V$ may be viewed as a 'local perturbation' of $-\Delta$.

(ii) Representation formulae for L through its heat semigroup:

$$L^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-tL} \frac{dt}{\sqrt{t}} \qquad \text{and} \qquad L^{-1} = \int_0^\infty e^{-tL} dt. \tag{1.20}$$

These are rooted in two major areas of influence for us: the work of [93] and a new framework for studying operators beyond the Calderón–Zygmund class, begun in [45].

As mentioned earlier Shen's work was the starting point of many subsequent lines of research by other authors. Two key ideas in Shen's work were perturbation and estimates on the fundamental solution for L. Recall that the auxiliary function γ was introduced (see (1.7)) and used to determine the 'local' and the 'global' regions. In the global regions (for scales larger than γ) estimates on the kernels related to L typically have stronger decay properties. Shen proved that whenever $V \in \mathcal{B}_{n/2}$ with $n \geq 3$, then the fundamental solution $\Gamma_L(x,y)$ of L satisfies

$$|\Gamma_L(x,y)| \leq \frac{C_N}{\left(1 + \frac{|x-y|}{\gamma(x)}\right)^N} \frac{1}{|x-y|^{n-2}} \tag{1.21}$$

for any $N > 0$, and C_N is a constant depending on the dimension n, on the potential V, and on N. Comparing this with the fundamental solution of $-\Delta$, given by

$$\Gamma_\Delta(x,y) = \frac{C_n}{|x-y|^{n-2}},$$

one sees that Γ_L has stronger decay whenever $|x-y| > \gamma(x)$.

On the other hand, in the local regions (for scales less than γ) the operator L behaves like $-\Delta$ in the following sense. When $|x-y| \leq \gamma(x)$ and $V \in \mathcal{B}_{n/2}$, then

$$|\Gamma_L(x,y) - \Gamma_\Delta(x,y)| \leq C \left(\frac{|x-y|}{\gamma(x)}\right)^{2-n/q} \frac{1}{|x-y|^{n-2}}\,. \tag{1.22}$$

These two ideas were utilised in later works including those mentioned in [63] and [30]. For instance the following estimates, proved in [93], on the kernel $K_L^*(x,y)$ of the adjoint of the Riesz transform $(\nabla L^{-1/2})^*$

$$|K_L^*(x,y)| \leq \frac{C_N}{\left(1 + \frac{|x-y|}{\gamma(x)}\right)^N} \frac{1}{|x-y|^{n-1}} \left\{ \int_{B\left(y, \frac{1}{4}|x-y|\right)} \frac{V(z)}{|z-y|^{n-1}} dz + \frac{1}{|x-y|} \right\}$$

for any $x \neq y$, and

$$|K_L^*(x,y) - K_\Delta^*(x,y)| \\ \leq \frac{C}{|x-y|^{n-1}} \left\{ \int_{B\left(y, \frac{1}{4}|x-y|\right)} \frac{V(z)}{|z-y|^{n-1}} dz + \frac{1}{|x-y|} \left(\frac{|x-y|}{\gamma(x)} \right)^{2-n/q} \right\} \quad (1.23)$$

whenever $|x - y| \leq \gamma(x)$ were crucial in [30] in obtaining the results for $\nabla L^{-1/2}$ on weighted spaces. Typical strategies taken in the study of such operators involve decomposing them into their local and global parts. For the local parts one further splits them into two operators, a local version of the classical operators whose boundedness are typically guaranteed, and a difference operator which is where estimates such as (1.22) and (1.23) play an important role. For the global parts one uses the stronger decay in the kernels. An explicit discussion of the idea of perturbation can be found in the recent articles [1, 25].

In our work we retain Shen's notion of perturbation, but replace estimates on the fundamental solutions by estimates on the *heat kernel* $p_t(x, y)$ of L. The kernel $p_t(x, y)$ is the integral kernel of the operator e^{-tL} (which forms a semigroup family of operators in the time variable t), and appears for us through the representation formulae in (1.20). For non-negative potentials it is well known [96] that the heat kernel satisfies

$$0 \leq p_t(x,y) \leq h_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t}, \quad (1.24)$$

for each $x, y \in \mathbb{R}^n$ and $t > 0$. Here $h_t(x, y)$ is the heat kernel of the Laplacian $-\Delta$ (see also (2.1)).

There are two advantages to working with heat kernels rather than fundamental solutions. Firstly they are stronger estimates in the sense that one can recover estimates on the fundamental solutions once estimates on the heat kernels are known. This may be done through the representation formula in (1.20) because Γ_L is the integral kernel of L^{-1}. The second advantage is they allow us to utilise the machinery begun in [45] in treating operators outside the Calderón–Zygmund class. We briefly survey these developments before explaining how they are used in our proofs.

A new paradigm was started in [45] and subsequently there were two strands of development. The first was in extending the Calderón–Zygmund theory of singular integrals to systematically handle operators with no kernel regularity (or even not possessing any kernels at all). These ideas were developed further in articles such as [11, 13, 16, 26, 27, 38] and also [12]. The other strand concerned function spaces built from the underlying operator. Already mentioned and of relevance to us are the Hardy spaces (see for example [19, 47, 65, 67] amongst others), but also other function spaces including the BMO and Besov spaces [48, 32]. Some cornerstones of this new framework include generating a theory that is intrinsically linked to the properties of the underlying operator and building averaging processes from this operator. This paradigm offers an elegant and unifying perspective on operator theory in harmonic analysis. Within this viewpoint, the classical Calderón–Zygmund theory is intimately connected with properties of the Laplacian and harmonic functions.

In this framework, the heat semigroup and the heat kernel plays an important role. For instance to study Riesz transforms associated to an operator L on $L^p(\mathbb{R}^n)$ (and also on the weighted L^p spaces) for $p < 2$, working through the ideas of [45] one is led, in practice, to studying estimates on the derivatives $\nabla_x p_t(x, y)$, where $p_t(x, y)$ is the heat kernel of L. While pointwise bounds may be much too strong a demand, it often suffices to work with weighted norm versions such as the following:

$$\int |\nabla_x p_t(x, y)|^2 \, e^{c|x-y|^2/t} \, dx \leq \frac{C}{t^{1+n/2}}. \tag{1.25}$$

We refer the reader to [38] where this inequality is applied (using the method of [45]) to obtain weak $(1, 1)$ estimates for the Riesz transform on a manifold. For the case $p > 2$, one applies (1.25) in the good-λ machinery developed in [11, 13, 16]. See Lemma 3.1 of [13], as well as Lemma 4.8 of this thesis and its proof. The derivation of (1.25) in [38] was modelled on a technique that originated in [61]. See also [39, 40, 46].

Estimate (1.25) is known to hold for Schrödinger operators with non-negative potentials (see Lemma 3.1) and therefore a similar approach can be applied to these operators. However for the second-order Riesz transform one needs a corresponding estimate for the second derivatives of the heat kernel. We adapt a technique from [43] to obtain, whenever $V \in \mathcal{B}_q$ with $q > 2$,

$$\int \left|\nabla_x^2 p_t(x,y)\right|^2 e^{c|x-y|^2/t}\, dx \le \frac{C}{t^{2+n/2}}\, e^{-c\left(1+t/\gamma(y)^2\right)^{\delta}} \tag{1.26}$$

for some constants $C, c, \delta > 0$. See Proposition 3.7 for the full statement of the result. The main idea is to use a weighted version of the Calderón–Zygmund inequality (Lemma 3.11) to transfer estimates involving mixed derivatives ∇^2 to estimates involving the Laplacian Δ and the potential V, and from there utilise the (reverse Hölder) properties of V.

Comparing (1.25) and (1.26), one observes the extra exponential decay in the latter in the time variable t, for the scale $t > \gamma(y)$. This extra decay in the kernel estimates is a feature of Schrödinger operators with reverse Hölder potentials, and has been manifested not only in estimates for the fundamental solutions as we saw earlier in (1.21), but also in estimates on the heat kernel. For instance in contrast with the case of non-negative potentials in (1.24), it was shown by K. Kurata [74] that when $V \in \mathcal{B}_q$ with $q \ge n/2$,

$$p_t(x,y) \le \frac{C}{t^{n/2}}\, e^{-c|x-y|^2/t}\, e^{-c\left(1+t/\gamma(x)^2\right)^{\delta}}. \tag{1.27}$$

A similar estimate was obtained independently in [49]. See Propositions 3.3 and 3.2 of this thesis. In Chapter 3 we show that it is possible to carry over this extra decay to estimates on the time derivatives (Proposition 3.4), an analogous version of (1.25) (Proposition 3.6), and finally the second derivative estimates (1.26). While this extra decay is not needed in the results of Chapters 4, 6, and 7, it is crucial for the results in Chapter 5.

As a sidenote, we mention that although pointwise bounds on spatial derivatives of the heat kernel $\nabla_x p_t(x,y)$ may be a highly non-trivial matter in general, we do show that for Schrödinger operators with reverse Hölder potentials it is possible to obtain such bounds provided the reverse Hölder exponent is sufficiently large (Proposition 3.5).

1.3.2 Techniques used in each main result

We move on to specifics of each main result. Let us first describe the case $p \le 1$. The proof of Theorem 1.9, and of the other results in Chapter 7, uses the same strategies as in the study of the first-order Riesz transform $\nabla L^{-1/2}$ in [65, 70], and the second-order results in [43]. These works show that elements of the spaces $H^p_L(\mathbb{R}^n)$ may be expressed as sums of localised representative functions called *atoms* or *molecules*. See Section 7.1.1, and in particular Definition 7.4. These atomic and molecular characterisations allow us to reduce the study of operators on $H^p_L(\mathbb{R}^n)$ to studying their behaviour on single atoms or molecules. The main technical tool is Lemma 7.5 and the estimate (1.26) (and suitable adaptations in Section 7.1.2) through the representation formulae (1.20) allow us to apply this Lemma to the operators $\nabla^2 L^{-1}$ and VL^{-1}. The last item of Theorem 1.9 follows as a consequence of the first two items, via the atomic characterisation of $H^p_L(\mathbb{R}^n)$ given in [52] (see Definition 7.6).

In Section 7.2 we give weighted extensions to items (a) and (b) of Theorem 1.9. That is, we show that similar estimates hold for the *weighted* Hardy spaces $H^p_L(w)$, where w is a Muckenhoupt weight. The techniques are similar to the unweighted case, and rely firstly on the structural properties of the spaces (already developed in [98, 108, 109]), and secondly on the heat kernel estimate (1.26). The results are summarised in Theorem 7.12.

The proofs of Theorems 1.5, 1.7, and 1.8 involve good-λ inequalities. A typical good-λ inequality for suitable non-negative functions F and G is the following: for each $0 < \varepsilon < 1$ there exists $C > 0$ and δ depending on ε such that for every $\lambda > 0$,

$$\big|\{x \in \mathbb{R}^n : F(x) > 2\lambda \ \text{ and } \ G(x) \leq \delta\lambda\}\big| \leq C\varepsilon\big|\{x \in \mathbb{R}^n : F(x) > \lambda\}\big|. \tag{1.28}$$

This estimate gives us a comparison of the (Lebesgue) measure of the level sets of F and G. They allow us to control various norms of F by that of G. This has direct applications for operators, where one tries in practice to control the operator under study by another (maximal) operator whose mapping properties are known. For instance if f_B is the average of f over the ball B and setting

$$M^{\#} f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f - f_B| \,, \tag{1.29}$$

then taking $F = Mf$ (where M is the Hardy–Littlewood operator defined in (1.9)) and $G = M^{\#}f$ in (1.28) gives us the following well known Fefferman–Stein sharp inequality [57]:

$$\|Mf\|_{L^p} \leq C_p \left\|M^{\#}f\right\|_{L^p}, \tag{1.30}$$

which is valid for all $0 < p < \infty$. Now if T is a Calderón–Zygmund operator then it can be shown that for almost every $x \in \mathbb{R}^n$,

$$M^{\#}(Tf)(x) \leq C\big(M\big(|f|^2\big)(x)\big)^{1/2}. \tag{1.31}$$

Since the operator $M\big(|\cdot|^2\big)^{1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p > 2$, then the conjunction of (1.30) and (1.31) leads to the conclusion that T is bounded on $L^p(\mathbb{R}^n)$ for all $p > 2$.

One can also obtain weighted versions of (1.28) and (1.30) by replacing the Lebesgue measure dx by wdx, where $w \in \mathcal{A}_\infty$. This allows us to obtain the boundedness of operators on $L^p(w)$. In particular combining these weighted versions of (1.28) and (1.30) with (1.31) gives the boundedness of Calderón–Zygmund operators on $L^p(w)$ for all $1 < p < \infty$ and $w \in \mathcal{A}_p$.

However, recall that singular integrals associated to Schrödinger operators fall outside the Calderón–Zygmund class, with one consequence being that boundedness on L^p for such operators may hold only for a strict subset of $(1, \infty)$. Therefore we need appropriate extensions of the good-λ techniques that can account for this. This was done in [13] and [11], inspired by the techniques in [77]. In these works the authors gave some good-λ inequalities for M and some *ad hoc* sharp functions,

$$M_A^{\#}f(x) = \sup_{B \ni x} \left(\frac{1}{|B|}\int_B |f - A_B f|^{p_0}\right)^{1/p_0}, \tag{1.32}$$

where the A_B is some suitably chosen averaging operator over the ball B that is related to the operator T under investigation. The first advantage is this allows one to obtain analogues of (1.31) with the exponent of 2 replaced by p_0, which can be any fixed number between 1 and ∞. The second advantage is that the good-λ inequalities involve *two* parameters, and therefore allow for an upper restriction on p in (1.30) that is strictly less than ∞. As a consequence the methods produce intervals of boundedness that can be a strict subset of $(1, \infty)$.

In [16] these inequalities were extended in several directions: from unweighted measures to weighted measures with $\mathcal{A}_\infty$ weights, and from operators to functions. They appear as follows: under suitable conditions on non-negative functions F and G, there exists $K_0 \geq 1$ and $C > 0$ such that for each $K \geq K_0$, $\delta \in (0,1)$, $w \in \mathcal{B}_{s'}$, and all $\lambda > 0$

$$w\big(\{x \in \mathbb{R}^n : MF(x) > K\lambda \text{ and } G(x) \leq \delta\lambda\}\big) \leq C\Big(\frac{1}{K^q} + \frac{\delta}{K}\Big)^{1/s} w\big(\{x \in \mathbb{R}^n : MF(x) > \lambda\}\big). \tag{1.33}$$

The two parameters are K and δ. See Theorem 4.4 for the full statement of this result, which is reproduced from [16]. This was used in [18] to study Riesz transforms on manifolds and the method was adapted in [6] to study the first-order Riesz transforms associated to magnetic Schrödinger operators.

Our observation in Chapter 4 is that the *a priori* $L^p(\mathbb{R}^n)$-boundedness of $\nabla L^{-1/2}$ and the pointwise bounds on the heat kernel of L are enough to allow the same approach to work, and allows us also to prove the implication (a) $\implies$ (b) in Theorem 1.5. We also use the same machinery combined with (1.26) to obtain results for the second-order Riesz transforms with reverse Hölder potentials (Theorem 4.3).

We now discuss the techniques of Chapter 5 and the proof of Theorem 1.7. As mentioned earlier, in [30] the authors introduced and studied some new weight classes $\mathcal{A}_\infty^L$ adapted to $L = -\Delta + V$. These weights are locally (within the region defined by γ) like the Muckenhoupt weights $\mathcal{A}_\infty$, but have larger growth outside γ. The techniques employed were in a similar spirit to that of [93] and depended heavily on kernel estimates, particularly the regularity as seen in (1.23), and on the comparison between operators associated to L with operators associated to $-\Delta$.

The article [106] brought methods that were closer in essence with classical harmonic analysis and Calderón–Zygmund theory in the sense of [100]. The author introduced two maximal operators adapted to L (with a parameter $\eta > 0$),

$$\mathcal{M}_\eta^L f(x) = \sup_{B \ni x} \frac{1}{\left(1 + \frac{r_B}{\gamma(x_B)}\right)^\eta} \frac{1}{|B|} \int_B |f(y)| \, dy \tag{1.34}$$

and

$$\mathcal{M}_\eta^{\#,L} f(x) = \sup_{\substack{B \ni x \\ r_B \leq \gamma(x_B)}} \frac{1}{|B|} \int_B |f - f_B| + \sup_{\substack{B \ni x \\ r_B > \gamma(x_B)}} \frac{1}{\left(1 + \frac{r_B}{\gamma(x_B)}\right)^\eta} \frac{1}{|B|} \int_B |f| \,, \tag{1.35}$$

generalising the Hardy–Littlewood maximal operator and the Fefferman–Stein sharp maximal operator (1.29) respectively. The author obtained the following key facts for these operators. The first is an analogue of (1.30), valid for all $0 < p < \infty$, $w \in \mathcal{A}_\infty^L$, and all $\eta > 0$:

$$\left\| \mathcal{M}_\eta^L f \right\|_{L^p(w)} \;\leq\; C_p \left\| \mathcal{M}_\eta^{\#,L} f \right\|_{L^p(w)} . \tag{1.36}$$

This was proved using good-λ inequalities similar to that in (1.28). The second is the following analogue of (1.31),

$$\mathcal{M}_\eta^{\#,L}(Tf)(x) \;\leq\; C \left(\mathcal{M}_\eta^L \left(|f|^s \right)(x) \right)^{1/s}. \tag{1.37}$$

Here T is an operator associated to L, and examples are (adjoints of) $\nabla L^{-1/2}$, $V^{1/2}L^{-1/2}$ and VL^{-1}. The exponent s depends on T. As before, (1.36) and (1.37) leads to the boundedness of T for $s < p < \infty$.

Since these estimates rely on classical techniques, and in particular some kind of kernel regularity, there is a restriction on the type of operators that can be handled. It is natural to wonder whether we can bring the flexibility of the machinery from [16] to the study of these weights. Three observations serve as motivation. The first is that with respect to $\mathcal{A}_\infty^L$ weights, the operator $\nabla^2 L^{-1}$ has remained untreated in the literature (Question 2). Secondly the techniques of Chapter 4 are able to handle the operator $\nabla^2 L^{-1}$ for $\mathcal{A}_\infty$ weights (notably Theorem 4.3). The third observation is that the $\mathcal{A}_\infty^L$ weights behave 'locally' like the $\mathcal{A}_\infty$ weights.

The task of adapting the machinery from [16] to $\mathcal{A}_\infty^L$ weights is carried out in Chapter 5. The first step is an adaptation of the good-λ result of [16] in Theorem 4.4, and this is done in

Theorem 5.10. There we extend (1.33) to $\mathcal{A}_\infty^L$ weights under suitable conditions on F and G. The reader may observe that besides replacing the maximal operator M by $\mathcal{M}_\eta^L$, the assumptions remain unchanged in the local scale. The key difference is that in the global scale (for balls $B(x,r)$ with radii that exceed a fixed multiple of γ) we impose the condition

$$\frac{1}{\big(1+\frac{r}{\gamma(x)}\big)^\eta}\frac{1}{|B(x,r)|}\int_{B(x,r)} F(y)\,dy \;\le\; G(z), \qquad \forall\, z\in B(x,r).$$

The second step is to use this to extend the maximal criterion in Theorem 4.6 to Theorem 5.16. Finally we prove Theorem 1.7 by applying this criterion with the kernel estimate (1.26). In contrast to the proof of Theorem 4.3, here the extra decay for the heat kernel estimates in the scale $t>\gamma(x)$ plays a decisive role.

We remark that our techniques allow us to take the study of operators with $\mathcal{A}_\infty^L$ weights in the direction of [13, 16, 45] discussed earlier, and as a consequence we can recover some of the results in [30] and [106]. For example estimate (1.36) can be obtained by choosing F and G appropriately in Theorem 5.10 (see Section 5.2.1). While our main application is to the operator $\nabla^2 L^{-1}$, we believe the same method can also be applied to the first-order Riesz transforms $\nabla L^{-1/2}$ and $V^{1/2}L^{-1/2}$, but we do not give these details in this thesis.

We now explain the techniques behind our Morrey space results. Recall that depending on V, the Riesz transforms in Theorem 1.8 may not be of Calderón–Zygmund type. One approach to studying singular integrals on Morrey spaces may be to follow the route of the L^p case. For instance one may attempt to obtain an analogue of (1.30):

$$\|Mf\|_{\mathcal{L}^{p,\lambda}} \;\le\; C_p\left\|M^\# f\right\|_{\mathcal{L}^{p,\lambda}}, \qquad 1<p<\infty, \quad 0<\lambda<n. \tag{1.38}$$

Then combining this with the pointwise bound of (1.31) already established allows us to obtain results for Calderón–Zygmund operators. One method of proving (1.38) is to first obtain a local version of (1.28) with $F=Mf$ and $G=M^\# f$ and then pass from this to the Morrey norm. This is what is done in [107] for some generalisations of the Morrey spaces.

However, since we are working with operators beyond the Calderón–Zygmund theory, the classical sharp operator $M^{\#}$ and (1.31) may not be sufficient. Motivated by the success of the new paradigm for $L^p(\mathbb{R}^n)$ and $L^p(w)$ spaces discussed earlier in this section, the question arises naturally of whether we can bring these techniques to the study of operators on Morrey spaces. The answer is yes, and follows directly from a principle that has been implicit in the literature on Morrey spaces since at least [34]. We formulate it here as follows.

Principle 1.10. *Results for weighted Lesbesgue spaces with $\mathcal{A}_\infty$ weights lead naturally to corresponding results for Morrey spaces.*

This idea was introduced first in [34] through the key observation that if $\mathbf{1}_B$ is the indicator function of a ball B, and M is the Hardy–Littlewood maximal operator, then the function $(M\mathbf{1}_B)^\delta$ is an $\mathcal{A}_1$ weight for any $\delta \in (0,1)$. This, combined with the decomposition

$$M\mathbf{1}_B \;\approx\; \mathbf{1}_B + \sum_{j=0}^{\infty} 2^{-jn}\mathbf{1}_{2^{j+1}B\setminus 2^jB},$$

allowed them to obtain firstly a simple proof of the boundedness of the maximal function:

$$\|Mf\|_{\mathcal{L}^{p,\lambda}} \;\leq\; C_p\,\|f\|_{\mathcal{L}^{p,\lambda}} \qquad\qquad 1<p<\infty, \quad 0<\lambda<n,$$

and secondly, new proofs for the boundedness of Riesz potentials (originally given in [2]) and of Calderón–Zygmund operators on Morrey spaces. In [42] these ideas were continued and used to give a simple proof of (1.38) which was then applied to give estimates for fractional maximal operators and for commutators. We give an explicit formulation of the calculation used in these results in Lemma 6.3.

We observe that in the proofs of the above results, no other properties of the operators are used besides their boundedness on weighted spaces. In other words, if an inequality holds on the weighted spaces for a certain range of p and collection of weights then it should imply a corresponding inequality for the Morrey spaces for a certain range of p and λ. This is a quantitative version of Principle 1.10. An explicit statement of this was first given in [9] in the context of Morrey spaces on spaces of homogeneous type. We give a special case of their result for $\mathbb{R}^n$ in the following.

Theorem 1.11 ([9]). *Let F and G be non-negative Borel measurable functions on $\mathbb{R}^n$. Set*

$$\mathcal{A}_1^{(\alpha)} = \{w \in \mathcal{A}_1 : \|w\|_\infty \leq 1 \quad \textit{and} \quad \mathcal{A}_{1,w} \leq \alpha\}$$

where $\|w\|_\infty = \inf\{t > 0 : |\{x \in \mathbb{R}^n : w(x) > t\}| = 0\}$ and $\mathcal{A}_{1,w}$ is the infimum of all the constants $C > 0$ for which $Mw \leq Cw$ almost everywhere. Suppose that for every $\alpha \geq 1$, there exists $c(\alpha) > 0$ such that the following inequality holds.

$$\int_{\mathbb{R}^n} F(x)\, w(x)\, dx \;\leq\; c(\alpha) \int_{\mathbb{R}^n} G(x)\, w(x)\, dx, \qquad\qquad \forall\, w \in \mathcal{A}_1^{(\alpha)}. \tag{1.39}$$

Then there exists $C_1, \alpha_0 > 0$ both depending only on n such that

$$\|F\|_{\mathcal{L}^{1,\lambda}} \;\leq\; C_1 c(\alpha_0)\, \|G\|_{\mathcal{L}^{1,\lambda}}\,, \qquad\qquad 0 < \lambda < n. \tag{1.40}$$

We can apply this to the study of operators in the following fashion: if T is an operator and (1.39) holds with $F = |Tf|^p$, $G = |f|^p$, for some fixed $p \in (0, \infty)$ and any f from a suitable class of test functions, then through (1.40) T can be extended to a bounded operator on $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ for *all* $0 < \lambda < n$. This gives us a way to obtain results for operators on Morrey spaces that have no kernel regularity. In particular the machinery developed in the aforementioned works [13, 16, 17, 38, 45] can be made available through this principle. This forms our approach in this thesis.

Before proceeding we remark here that for completeness' sake, Section 6.1 provides proofs of some of these ideas. It contains the calculation mentioned earlier (Lemma 6.3) as well as a proof of a version of Theorem 1.11 with simpler hypotheses (Proposition 6.5). However we do give a new application of these ideas to the study of 'fractional type' operators, which are modelled on the classical Riesz potentials $(-\Delta)^{-\alpha/2}$. The work here is inspired by [17], where as an application of the ideas in [16], the authors give some criterion for the study of fractional type operators on the weighted Lebesgue spaces. These are applied to the fractional powers of divergence form elliptic operators with complex coefficients. Our result is an adaptation of theirs and is given in Theorem 6.11. In Section 6.3 we apply this to Schrödinger operators and divergence form operators.

It is natural to wonder if the class $\mathcal{A}_1$ can be relaxed in the hypotheses for (1.39). We do this in Theorem 6.15. Motivated by the results in Section 1.1.1, we show that if (1.39) holds (with $|F|^p$ and $|G|^p$ in place of F and G respectively) for some fixed numbers $1 \leq p_0 < p < q_0 \leq \infty$ and *all* weights $w \in \mathcal{A}_{p/p_0} \cap \mathcal{B}_{(q_0/p)'}$, then the Morrey inequality (1.40) holds for all $pn/q_0 < \lambda < n$. This is a refinement of Theorem 1.11 (and Proposition 6.5) in the sense that weakening the hypothesis to admit a larger collection of weights leads to a tightening of the range of Morrey spaces in the conclusion. The proof of this result utilises a new characterisation of the Morrey spaces given in the recent work of D.R. Adams and J. Xiao [4, 5]. There the authors characterise the Morrey spaces and their preduals in terms of Hausdorff capacity and $\mathcal{A}_1$ weights (see expression (6.6)).

With this result in hand, we can prove our main results for the Schrödinger operators. We combine Theorem 6.15 with the results from Chapter 4 to give the proof of Theorem 1.8 and the implication (b) $\Longrightarrow$ (c) in Theorem 1.5. The details can be found in Section 6.3.

We also give another quantitative version of Principle 1.10 in Theorem 6.16, an *extrapolation* result for Morrey spaces, which as far as we are aware is a first of its kind. The concept of extrapolation for $\mathcal{A}_\infty$ weights is well known (see [91]): if a weighted inequality holds on $L^{p_0}(w)$ for some $p_0 \in [1, \infty)$ and all $w \in \mathcal{A}_{p_0}$, then it holds for all $p \in (1, \infty)$ and $w \in \mathcal{A}_p$. In [16] it was shown that the range of exponents need not be all of $(1, \infty)$ (this is reproduced in Proposition 4.9 of this thesis). We use these ideas for extrapolation on $L^p(w)$ spaces with $\mathcal{A}_p$ weights to obtain a similar principle for Morrey spaces: we show that an inequality on the Morrey spaces for a fixed pair of parameters (p_0, λ_0) propagates to a certain range of (p, λ).

Finally, it is interesting to ask to what extent a converse to Principle 1.10 holds. That is, *whether results on Morrey spaces lead to corresponding results for weighted Lebesgue spaces with $\mathcal{A}_p$ weights.* However we have not obtained any results in this direction here.

We end this section with some comments on the proof of the main result in Chapter 8, which is Theorem 8.3. In this result we give $L^p(\mathbb{R}^n)$ estimates for the Riesz transform $\nabla L^{-1/2}$ associated to the Schrödinger operator $L = -\Delta + V$ with V belonging to a class that is slightly more general than the reverse Hölder class studied throughout the rest of the thesis. This class was introduced in [54] and [50] and is defined, roughly speaking, by three aspects: there is a collection of slowly varying cubes covering $\mathbb{R}^n$ that determines the 'local' regions; within these cubes the potential V satisfies a certain estimate involving the classical heat semigroup $e^{t\Delta}$; outside these cubes the heat kernel of L satisfies extra decay. We refer the reader to Definition 8.1 and the remarks following for more precise details. We mention also that we impose the condition

$$\left\|\sqrt{t}\nabla e^{-tL}\right\|_{L^p \to L^p} \leq C_p,$$

but since this is necessary for $\nabla L^{-1/2}$ to be L^p-bounded, this is relatively harmless.

Although the good-λ methods of [11, 13, 16] have been successfully applied to L^p estimates such as divergence form elliptic operators and to the Laplace–Beltrami operator on a manifold, it is not clear if the same approach can work for the Schrödinger operator. This is in spite of the fact that weighted (and even Morrey) estimates can follow through, as some of the earlier parts of this thesis shows. It appears the main obstacle here is that while these operators satisfy the so-called *conservation property* $e^{-tL}(\mathbf{1}) = 1$, this property is completely lacking for Schrödinger operators in general. However in [13] section 4, these methods are adapted to give L^p estimates for *local* Riesz transfoms $\nabla(-\Delta + \ a)^{-1/2}$, with $a > 0$. Since the operator $-\Delta + a$ is a special case of the Schrödinger operators studied thoughout this thesis, results in this direction may be indeed possible.

Instead we return in a sense to [93] and employ techniques in the spirit of that work. In [50] the authors give a Riesz transform characterisation of the Hardy space H^1_L associated to these operators. Our approach is to adapt their argument which proceeds as follows. The main point is to control the adjoint of $\nabla L^{-1/2}$ by the maximal operator $M\left(|\cdot|^s\right)^{1/s}$ for some suitable $s > 1$. We split our analysis into the local and global regions, where locality is defined by the cubes in the definition of V. In the global regions we use the extra decay in the heat kernel (again from the definition of V), while in the local regions we base our analysis on a comparison between the heat kernel $p_t(x,y)$ of L with the classical heat kernel $h_t(x,y)$ of $-\Delta$, through the well known perturbation formula

$$e^{t\Delta} - e^{-tL} = \int_0^t e^{(t-s)\Delta} V e^{-sL}\, ds.$$

We direct the reader to Chapter 8 for the details.

1.4 Organisation of the thesis

We describe the structure of the rest of this thesis. In Chapter 2 we give some basic definitions and preliminary facts concerning the Schrödinger operator, reverse Hölder classes, and Muckenhoupt weights. We also fix some notation that will be used throughout the thesis.

In Chapter 3 we collect together the various estimates on the heat kernel of the Schrödinger operator with a non-negative potential. The main result is Proposition 3.7 but we attempt to be exhaustive for the class of reverse Hölder potentials. Accordingly, we also give estimates for the first derivatives in Proposition 3.6, as well as pointwise bounds on the time derivatives (Proposition 3.4), and pointwise bounds on the gradient of the heat kernel when the potential is smooth enough (Proposition 3.5).

The results for weighted Lebesgue spaces, Morrey spaces, and Hardy spaces are given in Chapters 4, 5, 6 and 7 respectively.

Our study of weighted spaces is divided over two chapters. Chapter 4 is concerned with Muckenhoupt weights. We prove the equivalence of the first two items in Theorem 1.5 which is contained in Theorem 4.1 and 4.2. We also give a proof for weighted estimates for the second-order Riesz transforms in Theorem 4.3. The second objective of this chapter is to lay a foundation for the next chapter.

Chapter 5 continues the study of weights but with a class larger than that of the Muckenhoupt weights. We develop the machinery needed by adapting the techniques from the previous chapter, before applying this to prove Theorem 1.7 which is contained in Theorem 5.1.

In Chapter 6 we study Morrey spaces. The first goal of this chapter is to develop the theme of applying weighted estimates to obtain estimates for Morrey spaces (Sections 6.1 and 6.2) culminating in Theorem 6.15, which is the main result of these sections. The second goal is to apply this to Schrödinger operators: the proof of the equivalence between the first and last items in Theorem 1.5 (Theorem 6.1), and also results for the second-order Riesz transforms when we specialise to the reverse Hölder class (Theorem 6.2). Lastly we give some further applications of Theorem 6.15.

Chapter 7 is devoted to Hardy spaces. We give the proof of Theorem 1.9 in the first part of the chapter. In the second part we extend some of these results to weighted Hardy spaces.

We conclude this thesis with Chapter 8 where we give the result for the Riesz transform associated to a more general class of potentials.

Chapter 2

Some preliminaries

2.1 Schrödinger operators

In this section we give the definition of the Schrödinger operator via forms and introduce the semigroup associated to this operator. For more on forms, operators and semigroups we refer the reader to [41, 64, 84, 96].

Let $n \geq 1$ and V be a non-negative locally integrable function on $\mathbb{R}^n$. We define the form $\mathcal{Q}_V$ by

$$\mathcal{Q}_V(u,v) := \int_{\mathbb{R}^n} \nabla u \cdot \nabla v + \int_{\mathbb{R}^n} Vuv$$

with domain

$$\mathcal{D}(\mathcal{Q}_V) = \Big\{ u \in W^{1,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V\,|u|^2 < \infty \Big\}.$$

It is well known that this symmetric form is closed. It was also shown by Simon [95] that this form coincides with the minimal closure of the form given by the same expression but defined on $C_0^\infty(\mathbb{R}^n)$. In other words, $C_0^\infty(\mathbb{R}^n)$ is a *core* of the form $\mathcal{Q}_V$.

Let us denote by L the self-adjoint operator associated with $\mathcal{Q}_V$. Its domain is

$$\mathcal{D}(L) := \Big\{ u \in \mathcal{D}(\mathcal{Q}_V) : \exists\, v \in L^2(\mathbb{R}^n) \quad \text{with} \quad \mathcal{Q}_V(u,\varphi) = \int v\varphi, \quad \forall \varphi \in \mathcal{D}(\mathcal{Q}_V) \Big\}.$$

We write formally $L := -\Delta + V$.

We now introduce the heat kernel associated to L. Consider the following parabolic equation

$$\Big(\frac{\partial}{\partial t} + L\Big)u(x,t) = 0, \qquad\qquad (x,t) \in \mathbb{R}^n \times (0,\infty).$$

We are interested in the fundamental solution $\Gamma(x,y,t)$ of this equation. That is, Γ satisfies, for each $y \in \mathbb{R}^n$

$$\Big(\frac{\partial}{\partial t} + L\Big)\Gamma(x,y,t) = 0, \qquad\qquad \forall\, x \in \mathbb{R}^n,\ x \neq y,\ t > 0,$$
$$\lim_{t\to 0}\Gamma(x,y,t) = \delta(x-y).$$

This fundamental solution is called the *heat kernel* of L. We use the notation $p_t(x,y)$ in place of $\Gamma(x,y,t)$. The heat kernel generates a semigroup family of integral operators associated to L, which we shall denote by $\{e^{-tL}\}_{t>0}$ and refer to as the *heat semigroup associated to* L. That is, $p_t(x,y)$ is the integral kernel associated to e^{-tL} in the following sense.

$$e^{-tL}f(x) = \int_{\mathbb{R}^n} p_t(x,y)\, f(y)\, dy.$$

We denote by $h_t(x,y)$ the heat kernel of $-\Delta$ in $\mathbb{R}^n$. When $n \geq 3$ for each $x, y \in \mathbb{R}^n$ and $t > 0$ it is well known that

$$h_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t}. \tag{2.1}$$

This is the integral kernel of the semigroup generated by $-\Delta$. That is,

$$e^{t\Delta}f(x) = \int_{\mathbb{R}^n} h_t(x,y) f(x)\, dy.$$

We also record the following fact, which will be used in Section 3.2.2 :

$$\Big(\frac{\partial}{\partial t} + L\Big)^{-1} f(x,t) = \int_0^t e^{-(t-s)L} f(x,s)\, ds = \int_{\mathbb{R}^n}\int_0^t p_{t-s}(x,y) f(y,s)\, ds\, dy. \tag{2.2}$$

That is, the integral kernel of $\Big(\dfrac{\partial}{\partial t} + L\Big)^{-1}$ is $p_{t-s}(x,y)\mathbf{1}_{(0,t)}(s)$.

A useful formulation of the semigroup property is:

$$p_{2t}(x,y) = \int_{\mathbb{R}^n} p_t(x,u)\, p_t(u,y)\, du = e^{-tL} p_t(\cdot, y)(x). \tag{2.3}$$

for any $x, y \in \mathbb{R}^n$ and $t > 0$. Indeed,

$$\begin{aligned}
\int p_{2t}(x,y)f(y)\,dy &= e^{-2tL}f(x) = e^{-tL}e^{-tL}f(x) \\
&= \int p_t(x,u)e^{-tL}f(u)\,du \\
&= \int p_t(x,u)\Big(\int p_t(u,y)f(y)\,dy\Big)du \\
&= \int\Big(\int p_t(x,u)\,p_t(u,y)\,du\Big)f(y)\,dy \\
&= \int e^{-tL}p_t(x,y)f(y)\,dy.
\end{aligned}$$

The following *perturbation formula* holds as a consequence of perturbation for semigroups of operators (see for example [85]). It is used in the proof of Theorem 8.3.

$$e^{t\Delta} - e^{-tL} = \int_0^t e^{(t-s)\Delta}Ve^{-sL}ds = \int_0^t e^{s\Delta}Ve^{-(t-s)L}ds. \tag{2.4}$$

This gives

$$h_t(x,y) - p_t(x,y) = \int_0^t \int_{\mathbb{R}^n} h_{t-s}(x,z)V(z)p_s(z,y)\,dz\,ds = \int_0^t \int_{\mathbb{R}^n} h_s(x,z)V(z)p_{t-s}(z,y)\,dz\,ds.$$

We remark that we can interchange the role of $-\Delta$ and L in (2.4).

2.2 Notation

We collect here some standard notation we shall employ throughout this thesis.

If $k \in \mathbb{Z}_+$ we write ∂_k to mean the derivative in the k-th variable $\dfrac{\partial}{\partial x_k}$, and ∂_k^2 to mean the second derivative in the k-th variable $\dfrac{\partial^2}{\partial x_k^2}$. At times we will abuse notation and write ∇ and ∇^2 for ∂_j and $\partial_j\partial_k$ respectively. For $\alpha \in \mathbb{R}$ we use the notation $[\alpha]$ to mean the greatest integer not exceeding α.

On $\mathbb{R}^n$ the classical Riesz transforms $\partial_j(-\Delta)^{-1/2}$ for $j \in \{1, \dots, n\}$ are given (formally) by

$$\partial_j(-\Delta)^{-1/2} = \frac{1}{\sqrt{\pi}}\int_0^\infty \partial_j e^{t\Delta}\frac{dt}{\sqrt{t}}.$$

The second-order Riesz transforms for $j, k \in \{1, \dots, n\}$ are given by

$$\partial_j\partial_k(-\Delta)^{-1} = \int_0^\infty \partial_j\partial_k e^{t\Delta}\,dt.$$

We will often write $\nabla(-\Delta)^{-1/2}$ and $\nabla^2(-\Delta)^{-1}$ in place of $\partial_j(-\Delta)^{-1/2}$ and $\partial_j\partial_k(-\Delta)^{-1/2}$.

The first-order Riesz transforms associated to L are $\partial_j L^{-1/2}$ for $j \in \{1,\dots,n\}$ and $V^{1/2}L^{-1/2}$. The second-order Riesz transforms are $\partial_j\partial_k L^{-1}$ for $j,k \in \{1,\dots,n\}$ and VL^{-1}. We will often write $\nabla L^{-1/2}$ and $\nabla^2 L^{-1}$ as shorthand for $\partial_j L^{-1/2}$ and $\partial_j\partial_k L^{-1}$ respectively.

The following well known representation formulae will be used regularly:

$$\begin{aligned} L^{-\alpha/2} &= \frac{1}{\Gamma(\alpha/2)}\int_0^\infty e^{-tL}\frac{dt}{t^{1-\alpha/2}}, \qquad \alpha > 0, \\ \nabla L^{-1/2} &= \frac{1}{\sqrt{\pi}}\int_0^\infty \nabla e^{-tL}\frac{dt}{\sqrt{t}}, \\ \nabla^2 L^{-1} &= \int_0^\infty \nabla^2 e^{-tL}\,dt. \end{aligned}$$

Similar formulae hold for $V^{1/2}L^{-1/2}$ and VL^{-1}. One can arrive at these via functional calculus or spectral theory (see [64]).

Our underlying measure space, unless otherwise noted, will be $\mathbb{R}^n$ with the Lebesgue measure. Given a measurable set $E \subset \mathbb{R}^n$ we write $|E|$ to mean the Lebesgue measure of E. The notation $\int_E f(x)\,dx$ will mean the Lebesgue integral of f over E. At times we often drop the dx to simplify notation. We also use the notation

$$⨍_E f \;:=\; \frac{1}{|E|}\int_E f$$

to mean the average of f over the measurable set E. Given a measure space (X,μ) and $1 \le p < \infty$, we denote by $L^p(X,\mu)$ the Banach space of complex valued functions on X that are p-integrable. That is we say that $f \in L^p(X,\mu)$ if the $L^p(X,\mu)$-norm of f,

$$\|f\|_{L^p(X)} = \Big(\int_X |f|^p\,d\mu\Big)^{1/p}$$

is finite. When $X = \mathbb{R}^n$ and $d\mu = dx$ and we will often write L^p in place of $L^p(\mathbb{R}^n)$. If $d\mu = w dx$ for some locally integrable function w, then we write $L^p(w)$ instead. When we use the expressions *almost everywhere* or *almost every* x (abbreviated "a.e." or "a.e.x") we mean that the properties to which they refer hold except on a set of measure zero. Given normed spaces $\big(X, \|\cdot\|_X\big)$ and $\big(Y, \|\cdot\|_Y\big)$, the expression $T : X \to Y$ will mean that T is a bounded mapping or operator (or admits a bounded extension) from X into Y. In this case we write $\|T\|_{X\to Y}$ to mean the operator norm of T, defined as $\|T\|_{X\to Y} := \inf\{C > 0 : \|Tx\|_Y \le C\,\|x\|_X\}$. When $Y = X$ we will simply say that T is 'bounded on X'.

When we refer to a ball centred at $x \in \mathbb{R}^n$ with radius $r > 0$, we mean the open set $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$. When we mention 'a ball B' we mean that a ball with a designated centre x_B and radius r_B has been chosen and fixed. By a cube $Q = Q(x_Q, l_Q)$ in $\mathbb{R}^n$ we mean a cube centred at x_Q with sidelength l_Q, and with sides parallel to the coordinate axes. If $\lambda > 0$ then we write $\lambda B = B(x_B, \lambda r_B)$ (respectively $\lambda Q = Q(x_Q, \lambda l_Q)$) to mean the ball with the same centre as B but with radius dilated by a factor of λ (respectively a cube with the same centre as Q but with sidelength dilated by a factor of λ). We define the *distance* between two subsets $E, F \subset \mathbb{R}^n$ as $\operatorname{dist}(E,F) := \inf\{|x-y| : x \in E, y \in F\}$. The notation $\mathbf{1}_E$ will be used to denote the indicator or characteristic function of the set E: $\mathbf{1}_E(x) = 1$ if $x \in E$ and 0 if $x \notin E$.

Given a function $\gamma : \mathbb{R}^n \to (0,\infty)$, we define balls associated to γ by $B(x,\gamma(x))$. We shall use the notation $\mathfrak{B}^\gamma(x) := B(x,\gamma(x))$. When we mention a ball $\mathfrak{B}^\gamma$ we mean that a ball with a designated centre x_B and radius $\gamma(x_B)$ has been fixed. That is, $\mathfrak{B}^\gamma := B(x_B, \gamma(x_B))$.

We will often discretise the space $\mathbb{R}^n$ into concentric annuli centred at a fixed ball B as follows:

$$U_j(B) := \begin{cases} B, & j = 0; \\ 2^j B \backslash 2^{j-1} B, & j \ge 1. \end{cases} \tag{2.5}$$

We can replace B by the balls $\mathfrak{B}^\gamma$ or a cube Q, with the obvious modifications.

Given a number $p \in (0,\infty]$ we shall use the notation p' to denote the conjugate exponent of p. That is, p and p' satisfy the relationship

$$\frac{1}{p} + \frac{1}{p'} = 1. \tag{2.6}$$

More explicitly, $p' = p/(p-1)$ if $p \neq 1$. When $p = 1$ we set $p' = \infty$, and when $p = \infty$ we set $p' = 1$. We also write p^* to denote the Sobolev exponent of p. This is defined as

$$p^* := \begin{cases} \dfrac{np}{n-p}, & p < n; \\ \infty, & p \geq n. \end{cases}$$

Finally we follow the convention that the symbol C in a string of inequalities will mean a constant that may change over the course of the inequalities, but does not depend on the essential variables under focus. The symbol $\lesssim$ will mean the same thing but with the C suppressed.

2.3 The reverse Hölder class

In this section we define the class of potentials that is the focus of this thesis, and give a list of their known properties. These properties originated in [93].

Definition 2.1 (Reverse Hölder class)**.** *Let $1 < q < \infty$. We say that a non-negative and locally integrable function V belongs to the reverse Hölder class of order q if there exists $C > 0$ such that*

$$\Big(\fint_B V^q \Big)^{1/q} \leq C \fint_B V$$

for all balls B. In this case we write $V \in \mathcal{B}_q$. We say that $V \in \mathcal{B}_\infty$ if there exists $C > 0$ such that for all balls B

$$V(x) \leq C \fint_B V \qquad a.e.\ x \in B.$$

For all $1 < s < q$, it is easily seen that $\mathcal{B}_s \supset \mathcal{B}_q$. Furthermore, $V(x)dx$ is a doubling measure. That is, there is a constant $C_0 > 1$ such that

$$\int_{2B} V(x)\, dx \ \leq \ C_0 \int_B V(x)\, dx.$$

It follows that for each $\lambda \geq 1$ there exists $n_0 > 0$ and $C > 0$ such that

$$\int_{\lambda B} V(x)\, dx \ \leq \ C\lambda^{n_0} \int_B V(x)\, dx. \tag{2.7}$$

In fact we can take $n_0 = \log_2 C_0$.

Definition 2.2 (Critical radius)**.** *For $V \geq 0$ we define the* critical radius *associated to V at x by the following expression.*

$$\gamma(x) = \gamma(x, V) := \sup\Big\{r > 0 : \; r^2 \fint_{B(x,r)} V \leq 1\Big\}. \tag{2.8}$$

As an example if $V(x) = |x|^2$ then $\gamma(x) \sim \dfrac{1}{1 + |x|}$.

Lemma 2.3 ([93] Lemmas 1.2 and 1.8)**.** *If $n \geq 1$ and $V \in \mathcal{B}_q$ for some $q > 1$ then there exists $C > 0$ such that the following holds:*

(a) *for each $\lambda > 1$ and all balls B,*

$$r_B^2 \fint_B V \; \leq \; C\lambda^{n/q-2}(\lambda r_B)^2 \fint_{\lambda B} V \,,$$

(b) *for all balls B satisfying $r_B \geq \gamma(x_B)$,*

$$r_B^2 \fint_B V \; \leq \; C\Big(\frac{r_B}{\gamma(x_B)}\Big)^{\sigma}$$

where $\sigma = n_0 - n + 2$.

Lemma 2.4 ([93] estimates 1.6 and 1.7)**.** *Let $V \in \mathcal{B}_q$. Then the following holds.*

(a) *If $q > n/2$ then there exists $C = C(n, q, V)$ such that for any ball B,*

$$\int_B \frac{V(x)}{|x_B - x|^{n-2}}\, dx \leq \frac{C}{r_B^{n-2}} \int_B V(x)\, dx.$$

(b) *If $q \geq n$ then there exists $C > 0$ such that for any ball B,*

$$\int_B \frac{V(x)}{|x_B - x|^{n-1}}\, dx \leq \frac{C}{r_B^{n-1}} \int_B V(x)\, dx.$$

The next property states that the function γ is slowly varying.

Lemma 2.5 ([93] Lemma 1.4)**.** *Let $V \in \mathcal{B}_q$ with $q \geq n/2$. Then there exists $C_0 > 0$ and $\kappa_0 \geq 1$ with*

$$C_0^{-1}\, \gamma(x) \left(1 + \frac{|x-y|}{\gamma(x)}\right)^{-\kappa_0} \leq \gamma(y) \leq C_0\, \gamma(x) \left(1 + \frac{|x-y|}{\gamma(x)}\right)^{\frac{\kappa_0}{\kappa_0+1}}. \tag{2.9}$$

In particular if $x, y \in B(x_B, \lambda\gamma(x_B))$ for some $\lambda > 0$, then

$$\gamma(x) \leq C_\lambda \gamma(y) \tag{2.10}$$

where $C_\lambda = C_0^2(1+\lambda)^{\frac{2\kappa_0+1}{\kappa_0+1}}$.

A consequence of (2.10) is that $\mathbb{R}^n$ admits a covering with 'critical balls' that has bounded overlap.

Lemma 2.6 ([51])**.** *Let $V \in \mathcal{B}_q$ with $q \geq n/2$. Let $\gamma : \mathbb{R}^n \to (0, \infty)$ be as defined in (2.8). Then there exists a countable collection of critical balls $\{\mathfrak{B}_j^\gamma\}_j = \{B(x_{B_j}, \gamma(x_{B_j}))\}_j$ satisfying the following properties.*

(i) $\bigcup_j \mathfrak{B}_j^\gamma = \mathbb{R}^n$.

(ii) *For every $\sigma \geq 1$ there exists constants C and N such that $\sum_j \mathbf{1}_{\sigma\mathfrak{B}_j^\gamma} \leq C\sigma^N$.*

Remark 2.7. We only require the following dilation. Set $\sigma = C2^{\kappa/(\kappa+1)}$ where C and κ are from (2.9). Then there exists C and $\widetilde{N}$ such that $\sum_j \mathbf{1}_{\sigma\mathfrak{B}_j^\gamma} \leq C\sigma^{\widetilde{N}}$ and it follows from (2.9) that for each j,

$$\bigcup_{x \in \mathfrak{B}_j^\gamma} \mathfrak{B}^\gamma(x) \subseteq \widetilde{\mathfrak{B}_j^\gamma}$$

where $\widetilde{\mathfrak{B}_j^\gamma} = \sigma\mathfrak{B}_j^\gamma$.

2.4 Muckenhoupt weights

The class of Muckenhoupt weights will play an important role throughout this thesis. We introduce them here and give some of their well known properties. Some standard references for these weights include [59, 60, 101].

Definition 2.8 (Muckenhoupt weights)**.** *Let $p \in (1, \infty)$ and p' be its conjugate exponent as defined in (2.6). For a non-negative and locally integrable function w, we say that $w \in \mathcal{A}_p$ if there exists $C > 0$ such that for all balls B*

$$\left(\fint_B w\right)\left(\fint_B w^{1-p'}\right)^{p-1} \leq C.$$

We say that $w \in \mathcal{A}_1$ *if there exists* $C > 0$ *such that for all balls* B

$$\fint_B w \leq C w(x) \qquad a.e.\ x \in B.$$

We also define $\mathcal{A}_\infty := \bigcup_{1 \leq p < \infty} \mathcal{A}_p$.

Some well known properties concerning the class of Muckenhoupt weights and the class of reverse Hölder weights are summarised in the following. For their proofs see [16] Proposition 2.1 (or the standard references mentioned at the start of this section).

Proposition 2.9. *One has*

(a) $1 \leq p_1 \leq p_2 < \infty \implies \mathcal{A}_1 \subset \mathcal{A}_{p_1} \subset \mathcal{A}_{p_2}$.

(b) $1 < p_1 \leq p_2 \leq \infty \implies \mathcal{B}_{p_1} \supset \mathcal{B}_{p_2} \supset \mathcal{B}_\infty$.

(c) *Let* $p \in (1, \infty)$. *Then* $w \in \mathcal{A}_p \iff w^{1-p'} \in \mathcal{A}_{p'}$.

(d) $w \in \mathcal{A}_p$ *for some* $1 < p < \infty \implies w \in \mathcal{A}_{p_0}$ *for some* p_0 *such that* $1 < p_0 < p$.

(e) $w \in \mathcal{B}_q$ *for some* $1 < q < \infty \implies w \in \mathcal{B}_{q_0}$ *for some* q_0 *such that* $q < q_0 < \infty$.

(f) *Let* $p, q \in [1, \infty)$. *Then* $w \in \mathcal{A}_p \cap \mathcal{B}_q \iff w^q \in \mathcal{A}_{q(p-1)+1}$.

(g) $\mathcal{A}_\infty = \bigcup_{1 < q \leq \infty} \mathcal{B}_q$.

We also define the following sets of exponents associated to a fixed weight, first introduced in [16]. For $w \in \mathcal{A}_\infty$ and $1 \leq p_0 < q_0 \leq \infty$ we set

$$\mathcal{W}_w(p_0, q_0) := \left\{ p \in (p_0, q_0) : w \in \mathcal{A}_{p/p_0} \cap \mathcal{B}_{(q_0/p)'} \right\}.$$

If we define $r_w := \inf \{ r > 1 : w \in \mathcal{A}_r \}$ and $s_w := \sup \{ s > 1 : w \in \mathcal{B}_s \}$. then we have

$$\mathcal{W}_w(p_0, q_0) = \left(p_0 r_w \, , \, \frac{q_0}{s'_w} \right).$$

The case $w \equiv 1$ corresponds to the Lebesgue measure on $\mathbb{R}^n$ so that in this situation we have $\mathcal{W}_1(p_0, q_0) = (p_0, q_0)$. If $q_0 = \infty$ then $\mathcal{W}_w(p_0, \infty) = \left\{ p \in (p_0, q_0) : w \in \mathcal{A}_{p/p_0} \right\}$. Note also that these sets can be empty. See for instance [16] Remark 4.3. For more information on these sets of exponents we refer the reader to [16] Section 4.

The following describes the doubling property for Muckenhoupt weights.

Lemma 2.10 ([59]). *Let $w \in \mathcal{A}_p$ for some $p \geq 1$. Then for any ball B, there exists $C > 0$ such that*

$$w(2B) \leq Cw(B).$$

More generally for each $\lambda > 1$,

$$w(\lambda B) \leq C\lambda^{np}w(B).$$

where C is independent of λ and B.

Lemma 2.11 ([59]). *Let $w \in \mathcal{A}_p \cap \mathcal{B}_q$ for some $p \geq 1$ and $q > 1$. Then there exist $C_1, C_2 > 0$ such that*

$$C_1 \left(\frac{|E|}{|B|}\right)^p \leq \frac{w(E)}{w(B)} \leq C_2 \left(\frac{|E|}{|B|}\right)^{1-1/q}$$

for any ball B and measurable subset $E \subset B$.

Chapter 3

Heat kernel estimates

The heat kernel and the heat semigroup associated to $L = -\Delta + V$ play a crucial role in our techniques. In this chapter we present various estimates involving the heat kernels of Schrödinger operators and their derivatives. In the first section (Section 3.1) we summarise the known estimates for the heat kernel associated to L when V is non-negative and locally integrable. Then in Section 3.2 we specialise to reverse Hölder potentials and give our improvements on these estimates.

The main result of this chapter is Proposition 3.7 which is new. It plays an important role in the results of the subsequent chapters.

The following observation will be useful throughout the rest of the chapter: for any $c > 0$ there exists $C > 0$, depending only on c and n, such that for every $y \in \mathbb{R}^n$ and $t > 0$,

$$\int_{\mathbb{R}^n} e^{-c\frac{|x-y|^2}{t}}\, dx \leq C\, t^{n/2}.$$

3.1 Non-negative potentials

It is known that since V is non-negative and locally integrable, by the Feynman–Kac formula the heat kernel of L admits the following so-called *Gaussian upper bound* (see [96]):

$$0 \leq p_t(x,y) \leq (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}}. \tag{3.1}$$

However while pointwise bounds on the derivatives of the heat kernel are generally not available, we do have the following weighted integral estimates. These often suffice in the analysis of singular integrals associated to L.

Lemma 3.1. *Let $L = -\Delta + V$ on $\mathbb{R}^n$, $n \geq 1$ with $0 \leq V \in L^1_{\rm loc}(\mathbb{R}^n)$. Then the heat kernel $p_t(x,y)$ of L satisfies the following.*

For each $p \in [1,2]$ there exists positive constants α_p, C_p and c such that for all $y \in \mathbb{R}^n$, and $t > 0$,

$$\Big(\int |\nabla_x p_t(x,y)|^p e^{\alpha_p \frac{|x-y|^2}{t}}\, dx\Big)^{1/p} \leq \frac{C_p}{t^{1/2+n/(2p')}}\,, \tag{3.2}$$

$$\Big(\int \big|V^{1/2}(x)\, p_t(x,y)\big|^p e^{\alpha_p \frac{|x-y|^2}{t}}\, dx\Big)^{1/p} \leq \frac{C_p}{t^{1/2+n/(2p')}}\,. \tag{3.3}$$

For each $k \in \mathbb{N}$ there exists $C_k > 0, c > 0$ satisfying

$$\Big|\frac{\partial^k}{\partial t^k} p_t(x,y)\Big| \leq \frac{C_k}{t^{n/2+k}} e^{-c\frac{|x-y|^2}{t}} \tag{3.4}$$

for every $x, y \in \mathbb{R}^n$, and $t > 0$.

Proof. We show (3.2). The estimate for $p = 2$ is known. See Lemma 2.5 of [6] (and also [46]). We shall obtain the estimate below 2. Fix $p \in [1,2)$ and a constant $\alpha_p \in (0, \alpha_2/2)$. Applying Hölder's inequality with exponents $2/p$ and $(2/p)' = 2/(2-p)$ gives

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^p\, e^{\alpha_p \frac{|x-y|^2}{t}}\, dx &= \int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^p\, e^{p\alpha_p \frac{|x-y|^2}{t}}\, e^{-(p-1)\alpha_p \frac{|x-y|^2}{t}}\, dx \\ &\leq \Big(\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^2\, e^{2\alpha_p \frac{|x-y|^2}{t}} dx\Big)^{\frac{p}{2}} \Big(\int_{\mathbb{R}^n} e^{-2\alpha_p \frac{p-1}{2-p} \frac{|x-y|^2}{t}} dx\Big)^{\frac{2-p}{2}}.\end{aligned}$$

Now since $2\alpha_p < \alpha_2$ the first factor is bounded by a constant multiple of $(t^{-n/2-1})^{p/2}$. Also since $p \in (1,2)$ then $(p-1)/(2-p) > 0$ so that the second integral is bounded by a multiple of $(t^{n/2})^{1-p/2}$. Therefore we obtain

$$\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^p\, e^{\alpha_p \frac{|x-y|^2}{t}}\, dx \;\lesssim\; \Big(\frac{1}{t^{n/2+1}}\Big)^{p/2} (t^{n/2})^{1-p/2} \;=\; \frac{1}{t^{np/2+p/2-n/2}}$$

as required.

The estimate (3.3) can be obtained in a similar fashion and we omit the details. For the estimate on the time derivatives (3.4) we refer the reader to estimate (3.1) of [46] and the references therein. $\square$

One can improve the ranges for p above 2 in these estimates if V satisfies further conditions. We turn to this in Section 3.2.

3.2 Potentials from the reverse Hölder class

Under the extra condition that V belongs to a reverse Hölder class the pointwise bounds in (3.1) can be improved. These improvements were obtained independently by separate authors and we recall both of these results here. Throughout the rest of this chapter the function γ is the "critical radius" function defined in Definition 2.2.

Proposition 3.2 ([49] Proposition 2)**.** *Assume that $V \in \mathcal{B}_q$ for some $q \geq n/2$ and $n \geq 3$. Then for each $N > 0$, there exists $C_N > 0$ and $c > 0$ such that*

$$0 \leq p_t(x,y) \leq \frac{C_N}{t^{n/2}} e^{-\frac{|x-y|^2}{ct}} \Big(1 + \frac{\sqrt{t}}{\gamma(x)} + \frac{\sqrt{t}}{\gamma(y)}\Big)^{-N}. \tag{3.5}$$

Proposition 3.3 ([74] Theorem 1)**.** *Assume that $V \in \mathcal{B}_q$ with $q \geq n/2$ for $n \geq 3$, or $q > 1$ for $n = 2$. Then there exists $C_0, c_0, c > 0, 0 < \delta < 1$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,*

$$p_t(x,y) \leq \frac{C_0}{t^{n/2}} e^{-c_0 \frac{|x-y|^2}{t}} e^{-c\left(1 + \frac{t}{\gamma(x)^2}\right)^{\delta}}. \tag{3.6}$$

We remark that δ depends on the constant κ_0 in Lemma 2.5.

Our aim in this section is to show that this improvement (specifically the extra decay in (3.6)) can be carried over to estimates on various derivatives of the heat kernel. These estimates will be indispensable throughout the rest of this thesis. We first give a list of these estimates, before giving the proofs.

The first is an improvement over the time derivative estimates of (3.4).

Proposition 3.4 (Time derivatives)**.** *Assume that $V \in \mathcal{B}_q$ with $q \geq n/2$ for $n \geq 3$, or $q > 1$ for $n = 2$. Let δ be the constant from (3.6). Then there exists $c = c(\delta) > 0$ and $c_1 > 0$ such that for each $k \in \mathbb{N}$ there exists $C_k > 0$ satisfying*

$$\Big|\frac{\partial^k}{\partial t^k} p_t(x,y)\Big| \leq \frac{C_k}{t^{n/2+k}} e^{-c_1 \frac{|x-y|^2}{t}} e^{-c\left(1 + \frac{t}{\gamma(x)^2}\right)^{\delta}} \tag{3.7}$$

for every $x, y \in \mathbb{R}^n$, and $t > 0$.

This will be proved in Section 3.2.1. Similar estimates can be found in [49] where the improvement factor is similar to (3.5).

Next we show that for potentials with enough regularity, one can obtain pointwise bounds on the first derivatives of the heat kernel.

Proposition 3.5 (Gradient bounds). *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 3$. Assume that $V \in \mathcal{B}_q$ with $q \geq n$. Then the heat kernel $p_t(x, y)$ of L satisfies*

$$\left|\nabla_x p_t(x,y)\right| \leq \frac{C}{t^{n/2+1/2}} e^{-c\frac{|x-y|^2}{t}} e^{-c\left(1+\frac{\sqrt{t}}{\gamma(x)}\right)^\delta}. \tag{3.8}$$

This is proved in Section 3.2.2.

For $q < n$, pointwise bounds are not available. However we do have the following weighted estimate. It is an improvement over estimates (3.2) and (3.3). We remind the reader that the Sobolev exponent q_+^* has been defined in Section 2.2.

Proposition 3.6 (First derivatives). *Assume that $V \in \mathcal{B}_q$ with $q \geq n/2$ for $n \geq 3$, or $q > 1$ for $n = 2$. Let δ be the constant from (3.6). Set $q_+ := \sup\{q > n/2 : V \in \mathcal{B}_q\}$. Then for each $p \in [1, q_+^*)$ there exists positive constants α_p, C_p, c such that for all $y \in \mathbb{R}^n$, and $t > 0$,*

$$\left(\int |\nabla_x p_t(x,y)|^p \, e^{\alpha_p\frac{|x-y|^2}{t}}\, dx\right)^{1/p} \leq \frac{C_p}{t^{1/2+n/(2p')}} e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^\delta}. \tag{3.9}$$

Also for each $p \in [1, 2q_+)$ there exists positive constants α_p, C_p, c such that for all $y \in \mathbb{R}^n$, and $t > 0$,

$$\left(\int \left|V^{1/2}(x) p_t(x,y)\right|^p e^{\alpha_p\frac{|x-y|^2}{t}}\, dx\right)^{1/p} \leq \frac{C_p}{t^{1/2+n/(2p')}} e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^\delta}. \tag{3.10}$$

Note that α_p also depends on q_+. The proof of this result can be found in Section 3.2.3.

The following is the main result of this chapter. It gives the technical estimates behind the results for the second-order Riesz transforms $\nabla^2 L^{-1}$ and VL^{-1} in Chapters 4, 7, and 6. It will be proved in Section 3.2.4.

Proposition 3.7 (Second derivatives). *Assume that $V \in \mathcal{B}_q$ with $q \geq n/2$ for $n \geq 3$, or $q > 1$ for $n = 2$. Let δ be the constant from (3.6). Set $q_+ := \sup\{q > n/2 : V \in \mathcal{B}_q\}$. Then for each*

$p \in [1, q_+)$ *there exists* $\beta_p, C_p, c > 0$ *such that for all* $y \in \mathbb{R}^n$*, and* $t > 0$*,*

$$\left(\int \left| \nabla_x^2 p_t(x,y) \right|^p e^{\beta_p \frac{|x-y|^2}{t}} \, dx \right)^{1/p} \leq \frac{C_p}{t^{1+n/(2p')}} e^{-c\left(1 + \frac{t}{\gamma(y)^2}\right)^\delta}, \tag{3.11}$$

$$\left(\int |V(x) p_t(x,y)|^p e^{\beta_p \frac{|x-y|^2}{t}} \, dx \right)^{1/p} \leq \frac{C_p}{t^{1+n/(2p')}} e^{-c\left(1 + \frac{t}{\gamma(y)^2}\right)^\delta}. \tag{3.12}$$

3.2.1 Time derivative bounds

In this section we obtain the proof of Proposition 3.4. Our approach is to work with a holomorphic extension of the heat semigroup to an appropriate sector in the complex plane, and then invoke Cauchy's integral formula. This holomorphic extension is contained in

Lemma 3.8 ([53] Corollary 6.2)**.** *The semigroup* $\left\{ e^{-tL} \right\}$ *has a unique holomorphic extension on* $L^2(e^{\eta|x-y|}dx)$ *for every* $\eta > 0$ *and* $y \in \mathbb{R}^n$ *in the sector* $\Sigma_{\pi/4} := \{\xi \in \mathbb{C} : |\arg \xi| < \pi/4\}$*. Moreover there exists constants* $C, c > 0$ *such that*

$$\|e^{-zL}\|_{L^2(e^{\eta|x-y|}dx) \to L^2(e^{\eta|x-y|}dx)} \leq C e^{c\eta^2 \Re z}$$

for every $y \in \mathbb{R}^n$*,* $z \in \Sigma_{\pi/4}$*, and* $\eta > 0$*.*

Proof of Proposition 3.4. In the following we shall write $p_z(x,y)$ to mean the integral kernel of the operator e^{-zL}. Our aim is to obtain the following pointwise bounds on this integral kernel, which is an extension of (3.6) to complex times.

Lemma 3.9. *Assume that the conditions in Proposition 3.4 hold. Then there exists* $C, c > 0$ *such that for all* $x, y \in \mathbb{R}^n$ *and* $z \in \Sigma_{\pi/5}$*, one has*

$$|p_z(x,y)| \leq \frac{C}{(\Re z)^{n/2}} e^{-c\left(1 + \frac{\Re z}{\gamma(x)^2}\right)^\delta} e^{-c\frac{|x-y|^2}{\Re z}}. \tag{3.13}$$

Let us demonstrate how (3.13) readily leads to (3.7). Fix $x, y \in \mathbb{R}^n$ and $t > 0$. We shall apply Cauchy's integral formula to $p_z(x,y)$ in the disk

$$\Gamma(t) := \{\xi \in \mathbb{C} : |\xi - t| \leq t/2\}.$$

Observe that $\Gamma(t) \subset \Sigma_{\pi/5}$. Hence $p_z(x,y)$ is holomorphic over $\Gamma(t)$, and so for each $k \in \mathbb{N}$, Cauchy's integral formula gives

$$\frac{\partial^k}{\partial t^k} p_t(x,y) = \frac{k!}{2\pi i} \int_{\Gamma(t)} \frac{p_z(x,y)}{(z-t)^{k+1}}\, dz.$$

Using (3.13) and noting that when $z \in \Gamma(t)$ one has $t/2 \le \Re z \le 3t/2$ and $|z-t| = t/2$, we get

$$\begin{aligned}
\Big|\frac{\partial^k}{\partial t^k} p_t(x,y)\Big| &\le C_k \int_{\Gamma(t)} e^{-c\frac{|x-y|^2}{\Re z}}\, e^{-c\left(1+\frac{\Re z}{\gamma(x)^2}\right)^\delta} \frac{|dz|}{(\Re z)^{n/2}(t/2)^{k+1}} \\
&\le \frac{C_k}{t^{n/2+k+1}} e^{-c_1\frac{|x-y|^2}{t}}\, e^{-c\left(1+\frac{t}{2\gamma(x)^2}\right)^\delta} \int_{\Gamma(t)} |dz| \\
&\le \frac{C_k}{t^{n/2+k}} e^{-c_1\frac{|x-y|^2}{t}}\, e^{-c2^{-\delta}\left(1+\frac{t}{\gamma(x)^2}\right)^\delta}
\end{aligned}$$

which is (3.7). □

We turn to the

Proof of Lemma 3.9. We claim that (3.13) follows from the following weighted estimate: there exists $C, c, \epsilon > 0$ such that for every $y \in \mathbb{R}^n$, $\eta > 0$, and $z \in \Sigma_{\pi/5}$,

$$\int_{\mathbb{R}^n} |p_z(x,y)|^2\, e^{\eta|x-y|}\, dx \le \frac{Ce^{\epsilon\eta^2\Re z}}{(\Re z)^{n/2}} e^{-c\left(1+\frac{\Re z}{\gamma(y)^2}\right)^\delta}. \tag{3.14}$$

Assume this estimate for the moment. Then the semigroup property, the Cauchy-Schwarz inequality, and estimate (3.14) give

$$\begin{aligned}
|p_z(x,y)|\, e^{\eta|x-y|} &= \Big|\int_{\mathbb{R}^n} p_{z/2}(x,u)\, p_{z/2}(u,y)\, du\Big|\, e^{\eta|x-y|} \\
&\le \int_{\mathbb{R}^n} |p_{z/2}(x,u)||p_{z/2}(u,y)|\, e^{\eta|x-u|} e^{\eta|u-y|}\, du \\
&\le \big\|p_{z/2}(x,\cdot)\, e^{\eta|x-\cdot|}\big\|_{L^2} \big\|p_{z/2}(\cdot,y)\, e^{\eta|\cdot-y|}\big\|_{L^2} \\
&\le \frac{Ce^{4\epsilon\eta^2\Re z}}{(\Re z)^{n/2}} e^{-c\left(1+\frac{\Re z}{\gamma(x)^2}\right)^\delta}.
\end{aligned}$$

Now fix $\epsilon_0 \in (0, 1/4\epsilon)$ and choose $\eta = \epsilon_0\,|x-y|\,/\Re z$. Then our estimate becomes

$$|p_z(x,y)| \le \frac{C}{(\Re z)^{n/2}}\, e^{(4\epsilon\epsilon_0^2-\epsilon_0)\frac{|x-y|^2}{\Re z}}\, e^{-c\left(1+\frac{\Re z}{\gamma(x)^2}\right)^\delta}.$$

Since $4\epsilon\epsilon_0^2 - \epsilon_0 < 0$, this establishes (3.13).

Hence our proof of Lemma 3.9 will be complete provided we show (3.14). Accordingly fix $x, y \in \mathbb{R}^n$, $\eta > 0$, $z \in \Sigma_{\pi/5}$ and set $t := \Re z$. Then the semigroup property implies that

$$p_z(x,y) = \big(e^{-(z-\frac{t}{10})L} p_{\frac{t}{10}}(\cdot,y)\big)(x).$$

Since $z \in \Sigma_{\pi/5}$ then $z - \frac{t}{10} \in \Sigma_{\pi/4}$, and hence by Lemma 3.8

$$\begin{aligned}\big\|p_z(\cdot,y)\, e^{\eta|\cdot - y|}\big\|_{L^2} &= \Big(\int_{\mathbb{R}^n} \big|e^{-(z-\frac{t}{10})L} p_{\frac{t}{10}}(\cdot,y)(x)\big|^2 e^{\eta|x-y|}\, dx\Big)^{1/2}\\ &\le Ce^{c\eta^2 t}\big\|p_{\frac{t}{10}}(\cdot,y)\, e^{\eta|\cdot - y|}\big\|_{L^2}.\end{aligned}$$

The bounds for the heat kernel from (3.6) give

$$\big\|p_{\frac{t}{10}}(\cdot,y)\, e^{\eta|\cdot - y|}\big\|_{L^2} \le \frac{C}{t^{n/2}} e^{-c10^{-\delta}\left(1+\frac{t}{\gamma(y)^2}\right)^{\delta}} \Big(\int_{\mathbb{R}^n} e^{-20c_0\frac{|x-y|^2}{t}} e^{\eta|x-y|}\, dx\Big)^{1/2}.$$

We shall prove that for any $\theta > 0$ there exists $C_\theta > 0$ and $c_\theta > 0$ such that for all $\eta > 0$ and $t > 0$,

$$\int_{\mathbb{R}^n} e^{-\theta\frac{|x-y|^2}{t}} e^{\eta|x-y|}\, dx \le C_\theta\, t^{n/2}\, e^{c_\theta \eta^2 t}. \tag{3.15}$$

Combining (3.15) with the previous two estimates will give (3.14).

We shall obtain (3.15) by considering two cases: (i) $\eta\sqrt{t} \ge 1$, and (ii) $\eta\sqrt{t} < 1$. Fix a constant $c \ge 8/\theta$. In the first case we write

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\theta\frac{|x-y|^2}{t}} e^{-\eta|x-y|}\, dx &\le\ 2\int_{B(y,2c\eta t)} e^{\eta|x-y|}\, dx + \sum_{j=2}^{\infty}\int_{U_j(B(y,c\eta t))} e^{-\theta\frac{|x-y|^2}{t}} e^{-\eta|x-y|}\, dx\\ &\le\ 2e^{2c\eta^2 t}\,|B(y,2c\eta t)| + \sum_{j=2}^{\infty} e^{-\theta\frac{c^2}{4}4^j\eta^2 t} e^{2^j c\eta^2 t}\,\big|B(y,2^j c\eta t)\big|\ .\end{aligned}$$

Now using that $\theta c \ge 8$ we have that $e^{\eta^2 t(c2^j - \theta 4^j c^2/8)} \le 1$, and hence

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\theta\frac{|x-y|^2}{t}} e^{-\eta|x-y|}\, dx &\le Ct^{n/2}e^{3c\eta^2 t} + C\sum_{j=0}^{\infty} e^{-\theta\frac{c^2}{8}4^j\eta^2 t}(2^j c\eta t)^n\\ &\le Ct^{n/2}e^{3c\eta^2 t} + C\frac{t^{n/2}}{(\eta^2 t)^n}\sum_{j=2}^{\infty} 2^{-nj}\\ &\le Ct^{n/2}e^{c_\theta\eta^2 t}\end{aligned}$$

where in the next to last line we have used the fact that $\eta^2 t \ge 1$.

For the second case, with the same $c \geq 8\theta$, we write

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\theta\frac{|x-y|^2}{t}} e^{-\eta|x-y|}\,dx &\leq 2e^{2c\eta\sqrt{t}}\big|B(y,2c\sqrt{t})\big| + \sum_{j=2}^{\infty}\int_{U_j(B(y,c\sqrt{t}))} e^{-\theta\frac{|x-y|^2}{t}} e^{-\eta|x-y|}\,dx \\ &\leq Ct^{n/2} + \sum_{j=2}^{\infty} e^{-\theta\frac{c^2}{4}4^j} e^{2^j c\eta\sqrt{t}}\big|B(y,2^j c\sqrt{t})\big| \\ &\leq Ct^{n/2} + Ct^{n/2}\sum_{j=2}^{\infty} e^{-\theta\frac{c^2}{8}4^j}(2^j\sqrt{t})^n \\ &\leq Ct^{n/2} \leq Ct^{n/2}e^{c_\theta\eta^2 t}.\end{aligned}$$

In the second line we have used that $\eta\sqrt{t} < 1$.

This completes the proof of (3.15), and hence also of Lemma 3.9. $\square$

3.2.2 Pointwise gradient bounds

In this section we obtain the proof of Proposition 3.5. Before turning to the details, we address some notational matters. In this section we will be working with “parabolic cylinders”. We define the open parabolic cylinder $Q(x,t,r)$ by

$$Q(x,t,r) = \big\{(y,s) \in \mathbb{R}^n \times (0,\infty) : \ |x-y| < r \quad \text{and} \quad t - r^2 < s < t\big\}$$

which in simple terms, describes the open cylinder in the half space $\mathbb{R}^n \times (0,\infty)$, with centre (x,t) at the top, radius r, and height r^2. It may be helpful to note that $Q(x,t,r) = B(x,r)\times(t-r^2,t)$. When we speak of the ‘cylinder Q’ we shall mean a cylinder $Q(x_Q,t_Q,r_Q)$ with fixed centre (x_Q,t_Q) and radius r_Q. Given $\lambda > 0$ and a cylinder $Q(x,t,r)$ we define the dilated cylinder by $\lambda Q = Q(x,t,\lambda r)$. We also write $\overline{Q}$ to mean the closure of Q.

Proof of Proposition 3.5. The main idea is to exploit the local gradient estimates of solutions to the operator $\dfrac{\partial}{\partial t} - \Delta$, which are themselves well known. To do so we study the construction

$$\tilde{u} := u + \Big(\frac{\partial}{\partial t} - \Delta\Big)^{-1}(u\mathbf{1}_Q V)$$

where u is a solution to $\Big(\dfrac{\partial}{\partial t} + L\Big)u = 0$ in the parabolic cylinder Q. It follows then, that $\tilde{u}$ is a solution to $\Big(\dfrac{\partial}{\partial t} - \Delta\Big)\tilde{u} = 0$ in Q. Hence we may pass from bounds on $\nabla\tilde{u}$ to bounds on $\tilde{u}$. We also need bounds on the gradients of the kernel of $\left(\frac{\partial}{\partial t}-\Delta\right)^{-1}$ but since this involves the usual heat kernel of $-\Delta$ it is readily calculated. Lastly we need the reverse Hölder properties of V to finish the estimates.

We first obtain local estimates for the parabolic Schrödinger operator. These local estimates will be used to obtain full gradient bounds. More precisely we shall first prove:

Lemma 3.10. *Let $L=-\Delta+V$ with $n\geq 3$. Assume that $V\in\mathcal{B}_n$. Let $Q=Q(x_Q,t_Q,r_Q)$ be the open parabolic cylinder centred (x_Q,t_Q), with radius r_Q, height r_Q^2, and $\overline{Q}$ is its closure. Suppose that u satisfies*

$$\left(\frac{\partial}{\partial t}+L\right)u(x,t)=0, \qquad\qquad \forall\,(x,t)\in\overline{Q}.$$

Then there exists $C>0$ and $k>0$ independent of Q such that

$$\sup_{(x,t)\in\frac{1}{2}\overline{Q}}|\nabla_x u(x,t)|\leq \frac{C}{r_Q}\left(1+\tfrac{r_Q}{\gamma(x_Q)}\right)^k\sup_{(x,t)\in\overline{Q}}|u(x,t)|\,. \tag{3.16}$$

Proof. Construct the following

$$\begin{aligned}\tilde{u}(x,t)&:=u(x,t)+\left(\frac{\partial}{\partial t}-\Delta_x\right)^{-1}(u\mathbf{1}_QV)(x,t)\\&=u(x,t)+\int_{\mathbb{R}^n}\int_0^t h_{t-s}(x,y)u(y,s)V(y)\mathbf{1}_Q(y,s)\,ds\,dy.\end{aligned}$$

The second line follows from (2.2) applied to the case $L=-\Delta$. Let us see that $\left(\frac{\partial}{\partial t}-\Delta\right)\tilde{u}=0$ in $\overline{Q}$. Given $(x,t)\in\overline{Q}$ we have

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta_x\right)\tilde{u}(x,t)&=\left(\frac{\partial}{\partial t}-\Delta_x\right)u(x,t)+\left(\frac{\partial}{\partial t}-\Delta_x\right)\left(\frac{\partial}{\partial t}-\Delta_x\right)^{-1}(u\mathbf{1}_QV)(x,t)\\&=\frac{\partial}{\partial t}u(x,t)-\Delta_xu(x,t)+u(x,t)V(x)\mathbf{1}_Q(x,t)\\&=\frac{\partial}{\partial t}u(x,t)-\Delta_xu(x,t)+u(x,t)V(x)\\&=\left(\frac{\partial}{\partial t}+L\right)u(x,t)=0\end{aligned}$$

because by assumption u satisfies the parabolic Schrödinger equation in $\overline{Q}$. Hence according to [55] Chapter 2.3, Theorem 9 on page 61, $\tilde{u}$ satisfies the following local gradient estimate in $\overline{Q}$:

$$\max_{(x,t)\in\frac{1}{2}\overline{Q}}|\nabla_x\tilde{u}(x,t)|\ \leq\ \frac{C}{r_Q^{n+3}}\,\|\tilde{u}\|_{L^1(\overline{Q})} \tag{3.17}$$

After rearranging $u = \tilde{u} - \Big(\frac{\partial}{\partial t} - \Delta\Big)^{-1}(u\mathbf{1}_Q V)$, we have

$$\sup_{\frac{1}{2}\overline{Q}} |\nabla u| \;\le\; \sup_{\frac{1}{2}\overline{Q}} |\nabla \tilde{u}| \;+\; \sup_{\frac{1}{2}\overline{Q}} \Big|\nabla\Big(\frac{\partial}{\partial t} - \Delta\Big)^{-1}(u\mathbf{1}_Q V)\Big| \;=:\; I + II.$$

Now since

$$\|\tilde{u}\|_{L^1(\overline{Q})} \;\lesssim\; |Q| \sup_{\overline{Q}} |\tilde{u}| \;\lesssim\; r_Q^2\, |B(x_Q, r_Q)| \sup_{\overline{Q}} |\tilde{u}| \,,$$

then by (3.17) we have

$$\begin{aligned}
I &\lesssim \frac{1}{r_Q} \sup_{\overline{Q}} |\tilde{u}| \\
&\lesssim \frac{1}{r_Q} \sup_{\overline{Q}} \Big\{ |u| + \Big|\Big(\frac{\partial}{\partial t} - \Delta\Big)^{-1}(u\mathbf{1}_Q V)\Big| \Big\} \\
&\lesssim \frac{1}{r_Q} \sup_{\overline{Q}} |u| + \frac{1}{r_Q} \sup_{(x,t)\in\overline{Q}} \int_{\mathbb{R}^n} \int_0^t h_{t-s}(x,y)\, |u(y,s)|\, V(y) \mathbf{1}_Q(y,s)\, ds\, dy \\
&=: I_1 + I_2.
\end{aligned}$$

Now for each $(x,t) \in Q$

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_0^t h_{t-s}(x,y)\, |u(y,s)|\, V(y) \mathbf{1}_Q(y,s)\, ds\, dy \\
&\qquad = \int_{B(x_Q, r_Q)} \int_{t_Q - r_Q^2}^t h_{t-s}(x,y)\, |u(y,s)|\, V(y)\, ds\, dy \\
&\qquad = \frac{1}{(4\pi)^{n/2}} \int_{B(x_Q, r_Q)} \int_{t_Q - r_Q^2}^t \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{n/2}}\, |u(y,s)|\, V(y)\, ds\, dy \\
&\qquad \lesssim \sup_{\overline{Q}} |u| \int_{B(x_Q, r_Q)} V(y) \int_{t_Q - r_Q^2}^t \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{n/2}}\, ds\, dy.
\end{aligned}$$

Since

$$\int_{t_Q - r_Q^2}^t \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{n/2}}\, ds \;\lesssim\; \int_0^\infty \frac{e^{-\frac{|x-y|^2}{4t}}}{t^{n/2}}\, ds \;\lesssim\; \frac{1}{|x-y|^{n-2}}$$

we obtain

$$I_2 \;\lesssim\; \frac{1}{r_Q} \sup_{\overline{Q}} |u| \sup_{(x,t)\in\overline{Q}} \int_{B(x_Q, r_Q)} \frac{V(y)}{|x-y|^{n-2}} dy.$$

We can estimate this using the fact that $V \in \mathcal{B}_{n/2}$. Indeed by Lemma 2.4 part (a),

$$\sup_{(x,t)\in\overline{Q}} \int_{B(x_Q, r_Q)} \frac{V(y)}{|x-y|^{n-2}} dy \;\;\lesssim\;\; \frac{1}{r_Q^{n-2}} \int_{B(x_Q, r_Q)} V(y)\, dy.$$

This then gives

$$I \lesssim I_1 + I_2 \lesssim \frac{1}{r_Q} \sup_{\overline{Q}} |u| \Big(1 + \frac{1}{r_Q^{n-2}} \int_{B(x_Q, r_Q)} V\Big).$$

Let us turn to the second term II.

$$\begin{aligned}
II &= \sup_{(x,t)\in \frac{1}{2}\overline{Q}} \Big| \int_{B(x_Q,r_Q)} \int_{t_Q - r_Q^2}^{t} \nabla_x h_{t-s}(x,y) u(y,s) V(y)\, ds\, dy \Big| \\
&\le \sup_{(x,t)\in \frac{1}{2}\overline{Q}} \int_{B(x_Q,r_Q)} \int_{t_Q - r_Q^2}^{t} |\nabla_x h_{t-s}(x,y)|\, |u(y,s)|\, V(y)\, ds\, dy \\
&\lesssim \sup_{\overline{Q}} |u| \sup_{(x,t)\in \frac{1}{2}\overline{Q}} \int_{B(x_Q,r_Q)} \int_{t_Q - r_Q^2}^{t} \frac{e^{-\frac{|x-y|^2}{8(t-s)}}}{(t-s)^{\frac{n+1}{2}}} V(y)\, ds\, dy.
\end{aligned}$$

In the third line we have used that since $z^2/8 \le e^{z^2/8}$ for all $z \in \mathbb{R}$, then

$$|\nabla_x h_t(x,y)| = \frac{|x-y|}{2t} h_t(x,y) \le \frac{4}{\sqrt{t}}\, e^{\frac{|x-y|^2}{8t}}\, h_t(x,y) = \frac{4}{(4\pi)^{n/2}} \frac{e^{-\frac{|x-y|^2}{8t}}}{(t-s)^{\frac{n+1}{2}}}.$$

Next, a change of variable $r = t - s$ gives

$$\begin{aligned}
\int_{t_Q - r_Q^2}^{t} \frac{e^{-\frac{|x-y|^2}{8(t-s)}}}{(t-s)^{\frac{n+1}{2}}} ds &= \int_0^{t - t_Q + r_Q^2} e^{-\frac{|x-y|^2}{8s}} \frac{ds}{s^{\frac{n+1}{2}}} \\
&\le \int_0^{\infty} e^{-\frac{|x-y|^2}{8s}} \frac{ds}{s^{\frac{n+1}{2}}} \\
&= \Big(\int_0^{|x-y|^2} + \int_{|x-y|^2}^{\infty} \Big) e^{-\frac{|x-y|^2}{8s}} \frac{ds}{s^{\frac{n+1}{2}}} \\
&= J_1 + J_2.
\end{aligned}$$

We have firstly that

$$J_1 \lesssim \int_0^{|x-y|^2} \Big(\frac{s}{|x-y|^2} \Big)^{n/2} \frac{ds}{s^{\frac{n+1}{2}}} = \frac{1}{|x-y|^n} \int_0^{|x-y|^2} \frac{ds}{s^{1/2}} \lesssim \frac{1}{|x-y|^{n-1}}$$

and secondly that

$$J_2 \le \int_{|x-y|^2}^{\infty} \frac{ds}{s^{\frac{n+1}{2}}} \lesssim \frac{1}{|x-y|^{n-1}}.$$

Now, using Lemma 2.4 part (b), we have

$$\sup_{(x,y)\in \frac{1}{2}\overline{Q}} \int_{B(x_Q,r_Q)} \frac{V(y)}{|x-y|^{n-1}} dy \lesssim \frac{1}{r_Q^{n-1}} \int_{B(x_Q,r_Q)} V(y)\, dy.$$

Inserting these into II gives

$$II \;\lesssim\; \Big(\frac{1}{r_Q^{n-1}}\int_{B(x_Q,r_Q)} V\Big)\sup_{\overline{Q}}|u|\,.$$

Collecting these estimates we have

$$\begin{aligned}
I + II \;&\lesssim\; \frac{1}{r_Q}\sup_{\overline{Q}}|u|\,\Big(1+\frac{1}{r_Q^{n-2}}\int_{B(x_Q,r_Q)} V\Big) + \sup_{\overline{Q}}|u|\,\frac{1}{r_Q^{n-1}}\int_{B(x_Q,r_Q)} V \\
&=\; \frac{1}{r_Q}\sup_{\overline{Q}}|u|\,\Big(1+\frac{2}{r_Q^{n-2}}\int_{B(x_Q,r_Q)} V\Big),
\end{aligned}$$

and hence

$$\sup_{\frac{1}{2}\overline{Q}}|\nabla u| \;\lesssim\; \frac{1}{r_Q}\Big(1+\frac{1}{r_Q^{n-2}}\int_{B(x_Q,r_Q)} V\Big)\sup_{\overline{Q}}|u|\,.$$

Finally using items (a) and (b) of Lemma 2.3 we can show that

$$\Big(1+\frac{1}{r_Q^{n-2}}\int_{B(x_Q,r_Q)} V\Big) \;\lesssim\; \Big(1+\frac{r_Q}{\gamma(x_Q)}\Big)^k$$

where $k = \max\{\sigma, 1\}$ and σ is the constant from Lemma 2.3 (b). From this one can obtain the desired result. This completes the proof of Lemma 3.10. $\square$

We now turn to the full gradient bounds. Fix $x, y \in \mathbb{R}^n$ and $t > 0$ with $x \neq y$. We shall show (3.8) for $p_t(x,y)$. Set $u(z,s) := p_s(z,y)$ for each $s > 0$ and $z \neq y$. We also define the cylinder Q by setting $x_Q = x$, $t_Q = t$, and r_Q is a number satisfying $0 < r_Q^2 < t$. Then clearly $(x,t) \in \frac{1}{2}\overline{Q}$ and u is a weak solution of $\dfrac{\partial}{\partial t} + L$ in $\overline{Q}$. Therefore by the local estimates of Lemma 3.10 and the bounds of the heat kernel in (3.6), we have

$$\begin{aligned}
\big|\nabla_x p_t(x,y)\big| &\le \sup_{(z,s)\in\frac{1}{2}\overline{Q}} |\nabla_z u(z,s)| \\
&\le \frac{1}{r_Q}\Big(1+\frac{r_Q}{\gamma(x_Q)}\Big)^k \sup_{(z,s)\in\overline{Q}} |u(z,s)| \\
&= \frac{1}{r_Q}\Big(1+\frac{r_Q}{\gamma(x)}\Big)^k \sup_{(z,s)\in\overline{Q}} |p_s(z,y)| \\
&\lesssim \frac{1}{r_Q}\Big(1+\frac{r_Q}{\gamma(x)}\Big)^k \sup_{(z,s)\in\overline{Q}} \frac{1}{s^{n/2}} e^{-c_0\frac{|z-y|^2}{s}} e^{-c\left(1+\frac{s}{\gamma(z)^2}\right)^{\delta}}.
\end{aligned} \tag{3.18}$$

We shall estimate (3.18) over two cases, depending on the size of t in comparison to the size of $\gamma(x)^2$. Suppose firstly that $t \le \gamma(x)^2$. Then we set $r_Q := t/2$ and observe that

$$\Big(1 + \frac{r_Q}{\gamma(x)}\Big)^k \le \Big(1 + \frac{1}{\sqrt{2}}\Big)^k \le C$$

so that (3.18) becomes

$$\big|\nabla_x p_t(x,y)\big| \lesssim \frac{1}{t^{1/2}} \sup_{(z,s)\in\overline{Q}} \frac{1}{s^{n/2}} e^{-c_0\frac{|z-y|^2}{s}}. \tag{3.19}$$

For the time variable, we mention that when $(z,s) \in \overline{Q}$ then $s \approx t$. Indeed,

$$t \;\ge\; s \;\ge\; t - r_Q^2 \;=\; t - t/2 \;=\; t/2.$$

Now if $|x-y|^2 > 2t$ then for each $z \in B(x, r_Q)$,

$$|z-x| \;\ge\; |x-y| - r_Q \;=\; |x-y| - \sqrt{\tfrac{t}{2}} \;\ge\; |x-y| - \tfrac{|x-y|}{2} \;=\; \tfrac{|x-y|}{2}$$

so that

$$\frac{1}{s^{n/2}} e^{-c_0\frac{|z-y|^2}{s}} \lesssim \frac{1}{t^{n/2}} e^{-c\frac{|x-y|^2}{t}}.$$

On the other hand if $|x-y|^2 \le 2t$ then $e^{-2} \le e^{-|x-y|^2/t}$. In either, case we further reduce estimate (3.19) to

$$\big|\nabla_x p_t(x,y)\big| \lesssim \frac{1}{t^{n/2+1/2}} e^{-c\frac{|x-y|^2}{t}}. \tag{3.20}$$

Finally, we may introduce the extra decay term by observing the inequality $t \le \gamma(x)^2$ implies

$$e^{-2^\delta} \le e^{-\left(1+\frac{t}{\gamma(x)^2}\right)^\delta},$$

so that (3.20) can be further improved to

$$\big|\nabla_x p_t(x,y)\big| \lesssim \frac{1}{t^{n/2+1/2}} e^{-c\frac{|x-y|^2}{t}} e^{-c\left(1+\frac{t}{\gamma(x)^2}\right)^\delta}.$$

This gives (3.8) for the case $t \le \gamma(x)^2$.

We turn to the case $t > \gamma(x)^2$. Set $r_Q^2 := \gamma(x)^2/2$. Then $(z,s) \in \overline{Q}$ implies that $s \approx t$ and $\gamma(z) \le C_1\gamma(x)$, with $C_1 > 1$. Indeed, the inequality

$$t \;\ge\; s \;\ge\; t - r_Q^2 \;=\; t - \gamma(x)^2/2 \;>\; t - t/2 \;=\; t/2,$$

and the definition of r_Q implies that $B(x, r_Q) \subseteq \mathfrak{B}^\gamma(x)$, so that by (2.10), one has $\gamma(z) \le C_1\gamma(x)$ for every $z \in B(x, r_Q)$, where $C_1 = 4C_0$. Combining these facts, one has for each $(z, s) \in \overline{Q}$,

$$e^{-c\left(1+\frac{s}{\gamma(z)^2}\right)^\delta} \le e^{-c'\left(1+\frac{t}{\gamma(x)^2}\right)^\delta}.$$

Then we further estimate (3.18) by

$$\begin{aligned}
\left|\nabla_x p_t(x,y)\right| &\lesssim \frac{1}{\gamma(x)}\left(1+\tfrac{1}{\sqrt{2}}\right)^k \frac{1}{t^{n/2}} e^{-c'\left(1+\frac{t}{\gamma(x)^2}\right)^\delta} \sup_{(z,s)\in\overline{Q}} e^{-c_0\frac{|z-y|^2}{s}} \\
&= \frac{1}{\sqrt{t}}\left(\frac{\sqrt{t}}{\gamma(x)}\right)\frac{1}{t^{n/2}} e^{-c'\left(1+\frac{t}{\gamma(x)^2}\right)^\delta} \sup_{(z,s)\in\overline{Q}} e^{-c_0\frac{|z-y|^2}{s}} \\
&\lesssim \frac{1}{t^{n/2+1/2}} e^{-c''\left(1+\frac{t}{\gamma(x)^2}\right)^\delta} \sup_{(z,s)\in\overline{Q}} e^{-c_0\frac{|z-y|^2}{s}}.
\end{aligned} \tag{3.21}$$

Finally we may estimate the Gaussian term in a similar fashion to the previous case. Namely, if $|x-y|^2 > 2t$ then whenever $z \in B(x, r_Q)$, we have

$$\begin{aligned}
|z-x| \ge\ |x-y| - r_Q\ &=\ |x-y| - \tfrac{\gamma(x)}{\sqrt{2}} \\
&\ge\ |x-y| - \sqrt{\tfrac{t}{2}}\ \ge\ |x-y| - \tfrac{|x-y|}{2}\ =\ \tfrac{|x-y|}{2},
\end{aligned}$$

so that

$$\frac{1}{s^{n/2}} e^{-c_0\frac{|z-y|^2}{s}} \lesssim \frac{1}{t^{n/2}} e^{-c\frac{|x-y|^2}{t}}.$$

On the other hand if $|x-y|^2 \le 2t$ then $e^{-2} \le e^{-|x-y|^2/t}$. In either case we further reduce estimate (3.21) to

$$\left|\nabla_x p_t(x,y)\right| \lesssim \frac{1}{t^{n/2+1/2}} e^{-c\frac{|x-y|^2}{t}} e^{-c''\left(1+\frac{t}{\gamma(x)^2}\right)^\delta} \tag{3.22}$$

which gives (3.8) for the case $t > \gamma(x)^2$.

This completes the proof of Proposition 3.5. □

3.2.3 Weighted Sobolev bounds: first derivatives

In this section we give the proof of Proposition 3.6. We will consider three separate cases: $p = 2$, $p < 2$, and $p > 2$.

We first obtain the case $p = 2$. Let c_0 be the constant in (3.6), and choose $\alpha_2 \in (0, \frac{2}{3}c_0)$. We shall proceed as in [46] with some slight modifications. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $0 \le \varphi \le 1$, support in $B(0,2)$, $|\nabla\varphi| \le 1$, and $\varphi \equiv 1$ on $B(0,1)$. Define for each $R \ge 1$,

$$\varphi_R(\cdot) := \varphi\left(\frac{\cdot}{R}\right).$$

Then it follows that $|\nabla\varphi_R| \lesssim 1/R$.

Fix $y \in \mathbb{R}^n$, $t > 0$, $R \ge 1$, and set

$$I_R(t,y) := \sum_{k=1}^n \int_{\mathbb{R}^n} |\partial_k p_t(x,y)|^2 \, e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x) \, dx.$$

Then one has

$$I_R(t,y) = I_R^1(t,y) - I_R^2(t,y)$$

where

$$I_R^1(t,y) := \sum_{k=1}^n \int_{\mathbb{R}^n} \partial_k p_t(x,y) \, \partial_k \Big[p_t(x,y) \, e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x) \Big] \, dx$$
$$I_R^2(t,y) := \sum_{k=1}^n \int_{\mathbb{R}^n} \partial_k p_t(x,y) \, p_t(x,y) \, \partial_k \Big[e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x) \Big] \, dx.$$

Let us study the first term. Since φ_R has compact support then

$$p_t(\cdot,y) \, e^{\alpha_2 \frac{|\cdot-y|^2}{t}} \varphi_R(\cdot) \ \in \ \mathcal{D}(\mathcal{Q}_V).$$

Therefore since both V and φ_R are non-negative,

$$\begin{aligned} I_R^1(t,y) &\le I_R^1(t,y) + \sum_{k=1}^n \int_{\mathbb{R}^n} V(x) \, p_t(x,y)^2 e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x) \, dx \\ &= \mathcal{Q}_V \Big(p_t(\cdot,y) \, , \, p_t(\cdot,y) \, e^{\alpha_2 \frac{|\cdot-y|^2}{t}} \varphi_R(\cdot) \Big) \\ &= \int_{\mathbb{R}^n} L p_t(x,y) \, p_t(x,y) \, e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x) \, dx \\ &= \int_{\mathbb{R}^n} \frac{\partial}{\partial t} p_t(x,y) \, p_t(x,y) \, e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x) \, dx. \end{aligned}$$

Now using the bounds on the heat kernel (3.6) and on its time derivative (3.7) we have

$$I_R^1(t,y) \ \le \ \frac{C}{t^{n+1}} e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^\delta} \int_{\mathbb{R}^n} e^{-(c_0-\alpha_2)\frac{|x-y|^2}{t}} \varphi_R(x) \, dx.$$

Since $\alpha_2 < c_0$ and $\varphi_R \le 1$ we can control the integral by a multiple of $t^{n/2}$ and obtain

$$I_R^1(t,y) \;\le\; \frac{C}{t^{n/2+1}} e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^\delta}. \tag{3.23}$$

For the second term we have

$$\begin{aligned}
I_R^2(t,y) = \;& \sum_{k=1}^n \int_{\mathbb{R}^n} \partial_k p_t(x,y)\, p_t(x,y)\, e^{\alpha_2 \frac{|x-y|^2}{t}} \Big[\partial_k \varphi_R(x) + \frac{2\alpha_2}{t}(x_k - y_k)\, \varphi_R(x)\Big] dx \\
\le \;& \sum_{k=1}^n \frac{C}{\sqrt{t}} \int_{\mathbb{R}^n} |\partial_k p_t(x,y)||p_t(x,y)|\, e^{2\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x)\, dx \\
& \qquad + \sum_{k=1}^n \int_{\mathbb{R}^n} |\partial_k p_t(x,y)||p_t(x,y)|\, e^{\alpha_2 \frac{|x-y|^2}{t}}\, |\partial_k \varphi_R(x)|\; dx \\
=: \;& I_R^{2.1}(t,y) + I_R^{2.2}(t,y).
\end{aligned} \tag{3.24}$$

To estimate the first term we use the Cauchy-Schwarz inequality, the heat kernel bounds (3.6), and that $2c_0 > 3\alpha_2$ to obtain

$$\begin{aligned}
I_R^{2.1}(t,y) &\le \sum_{k=1}^n \frac{C}{\sqrt{t}} \big\| p_t(\cdot,y) e^{\frac{3\alpha_2}{2} \frac{|\cdot-y|^2}{t}} \varphi_R \big\|_{L^2} \big\| |\partial_k p_t(\cdot,y)|\, e^{\frac{\alpha_2}{2} \frac{|\cdot-y|^2}{t}} \varphi_R \big\|_{L^2} \\
&\le \frac{C e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^\delta}}{t^{n/2+1/2}} \sum_{k=1}^n \big\| e^{-\frac{(2c_0-3\alpha_2)}{2} \frac{|\cdot-y|^2}{t}} \big\|_{L^2} \big\| |\partial_k p_t(\cdot,y)|\, e^{\frac{\alpha_2}{2} \frac{|\cdot-y|^2}{t}} \varphi_R \big\|_{L^2} \\
&\le \frac{C}{\sqrt{t^{n/2+1}}} \sqrt{I_R(t,y)}\, e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^\delta}.
\end{aligned} \tag{3.25}$$

Combining (3.23), (3.24), and (3.25), with the inequality $\sqrt{AB} \le \dfrac{\varepsilon}{2} A + \dfrac{1}{2\varepsilon} B$, valid for all $\varepsilon, A, B > 0$, we obtain

$$\begin{aligned}
I_R(t,y) &\le C e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^\delta} \left(\frac{1}{t^{n/2+1}} + \frac{1}{\sqrt{t^{n/2+1}}} \sqrt{I_R(t,y)} \right) + I_R^{2.2}(t,y) \\
&\le C e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^\delta} \left(\frac{1+2\varepsilon}{t^{n/2+1}} + \frac{1}{2\varepsilon} I_R(t,y) \right) + I_R^{2.2}(t,y).
\end{aligned}$$

Choosing ε large enough therefore gives

$$I_R(t,y) \;\le\; \frac{C}{t^{n/2+1}} e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^\delta} + C\, I_R^{2.2}(t,y).$$

Now using that $|\nabla \varphi_R| \lesssim 1/R$ we see that

$$I_R^{2.2}(t,y) \;\le\; \frac{C}{R} \left\{ \sum_{k=1}^n \int_{\mathbb{R}^n} |\partial_k p_t(x,y)|\, |p_t(x,y)|\; e^{\alpha_2 \frac{|x-y|^2}{t}}\; dx \right\}^{1/2} \;\longrightarrow\; 0$$

as $R \to \infty$. Hence by Fatou's Lemma,

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^2\, e^{\alpha_2|x-y|^2/t}\, dx &\le \int_{\mathbb{R}^n} \liminf_{R\to\infty} \Big\{ |\nabla_x p_t(x,y)|^2\, e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x) \Big\} dx \\ &\le \liminf_{R\to\infty} \int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^2\, e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x)\, dx \\ &= \liminf_{R\to\infty} I_R(t,y) \\ &\le \liminf_{R\to\infty} \Big\{ \frac{C}{t^{n/2+1}} e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^\delta} + I_R^{2.2}(t,y) \Big\} \\ &\le \frac{C}{t^{n/2+1}} e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^\delta}.\end{aligned}$$

This proves (3.9) for $p=2$.

To obtain (3.10) for $p=2$, we observe that

$$\int_{\mathbb{R}^n} V(x)\, p_t(x,y)^2 e^{\alpha_2 \frac{|x-y|^2}{t}} \varphi_R(x)\, dx \;=\; \mathcal{Q}_V\Big(p_t(\cdot,y)\,,\, p_t(\cdot,y)\, e^{\alpha_2 \frac{|\cdot-y|^2}{t}} \varphi_R\Big) - I_R^1(t,y).$$

Since both terms have been estimated we can apply the same computations as in (3.9) and yield (3.10). This completes the proof of Proposition 3.6 for the case $p=2$.

Next we turn to the case $p<2$. Let $p \in [1,2)$ and fix $\alpha_p \in (0, \alpha_2/4)$. Applying Hölder's inequality with exponents $2/p$ and $(2/p)' = 2/(2-p)$ gives

$$\begin{aligned}&\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^p\, e^{\alpha_p \frac{|x-y|^2}{t}}\, dx \\ &\qquad = \int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^p\, e^{2p\alpha_p \frac{|x-y|^2}{t}}\, e^{-(2p-1)\alpha_p \frac{|x-y|^2}{t}}\, dx \\ &\qquad \le \Big(\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^2\, e^{2p\alpha_p \frac{|x-y|^2}{t}}\, dx\Big)^{\frac{p}{2}} \Big(\int_{\mathbb{R}^n} e^{-\frac{2(2p-1)}{2-p}\alpha_p \frac{|x-y|^2}{t}}\, dx\Big)^{1-\frac{p}{2}}\end{aligned}$$

Since $4\alpha_p < \alpha_2$ we can control the first term by a constant multiple of

$$\left[\frac{1}{t^{n/2+1}}\, e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^\delta}\right]^{p/2},$$

and since $(2p-1)/(2-p) > 0$ we can bound the second integral by a multiple of $(t^{n/2})^{1-p/2}$. Therefore

$$\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^p\, e^{\alpha_p \frac{|x-y|^2}{t}}\, dx \le \frac{C}{t^{p/2+(p-1)n/2}}\, e^{-\frac{cp}{2}\left(1+\frac{t}{\gamma(y)^2}\right)^\delta}.$$

which gives (3.9) for $p \in [1,2)$. Similar calculations gives (3.10) for the same range of p.

We now consider the case $2 < p < q_+^*$. We shall make use of the following estimate, valid for each $q \in (2, q_+^*)$,

$$\|\nabla p_t(\cdot, y)\|_q \leq \frac{C_q}{t^{1/2+n/2q'}} e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^\delta} \qquad \forall\, y \in \mathbb{R}^n,\ t > 0. \tag{3.26}$$

Assume this estimate for the moment. We shall show how an interpolation between (3.26) and the estimate (3.9) for $p = 2$ yields (3.9) for all $p \in (2, q_+^*)$. Indeed for each $p \in (2, q_+^*)$ set (recall that $q_+^* = \infty$ if and only if $q_+ \geq n$)

$$q := \begin{cases} \dfrac{p+q_+^*}{2} & \text{if } q_+^* < \infty, \\ 2p & \text{if } q_+^* = \infty \end{cases}$$

and $\alpha_p := \alpha_2(q-p)/(q-2)$. Note that p and q satisfy

$$p = 2\Big(\frac{q-p}{q-2}\Big) + q\Big(\frac{p-2}{q-2}\Big), \qquad 0 < \frac{q-p}{q-2} < 1, \qquad 1 < \frac{q-2}{q-p} < \infty.$$

Applying Hölder's inequality with exponents

$$\frac{q-2}{q-p} \qquad \text{and} \qquad \Big(\frac{q-2}{q-p}\Big)' = \frac{q-2}{p-2}$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^p\, e^{\alpha_p \frac{|x-y|^2}{t}}\, dx \\
&\qquad = \int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^{2\frac{q-p}{q-2}}\, e^{\alpha_p \frac{|x-y|^2}{t}}\, |\nabla_x p_t(x,y)|^{q\frac{p-2}{q-2}}\, dx \\
&\qquad \leq \Big(\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^2\, e^{\alpha_2 \frac{|x-y|^2}{t}}\, dx\Big)^{\frac{q-p}{q-2}} \Big(\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^q\, dx\Big)^{\frac{p-2}{q-2}}.
\end{aligned}$$

Estimate (3.9) for the case $p = 2$ allows us to control the first term by a multiple of

$$\big[t^{\frac{n}{2}+1}\big]^{-\frac{q-p}{q-2}} e^{-2c\frac{q-p}{q-2}\left(1+\frac{t}{\gamma(y)^2}\right)^\delta},$$

while estimate (3.26) allows us to control the second by a multiple of

$$\Big[t^{\frac{q}{2}+\frac{n(q-1)}{2}}\Big]^{-\frac{p-2}{q-2}} e^{-cq\frac{p-2}{q-2}\left(1+\frac{t}{\gamma(y)^2}\right)^\delta}.$$

Combining these estimates we obtain

$$\int_{\mathbb{R}^n} |\nabla_x p_t(x,y)|^p\, e^{\alpha_p \frac{|x-y|^2}{t}}\, dx \ \leq\ \frac{C}{t^{p/2+(p-1)n/2}} e^{-pc\left(1+\frac{t}{\gamma(y)^2}\right)^\delta}$$

which is (3.9).

It remains to obtain (3.26). Firstly observe that the semigroup property (2.3) implies

$$\nabla_x p_{2t}(x,y) = \nabla_x e^{-tL} p_t(\cdot,y)(x). \tag{3.27}$$

Now recall from Theorem 1.3 that under our assumptions on $L = -\Delta + V$ the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^q(\mathbb{R}^n)$ for every $q \in (1, q_+^*)$. This implies that for each $q \in (1, q_+^*)$

$$\left\| \nabla e^{-tL} \right\|_{q \to q} \le \frac{C_q}{\sqrt{t}}.$$

Indeed by the analyticity of the semigroup $\left\{e^{-tL}\right\}_{t>0}$ (see [85] p74, Theorem 6.13)

$$\left\| \sqrt{t} \nabla e^{-tL} f \right\|_q = \left\| \sqrt{t} \nabla L^{-1/2} L^{1/2} e^{-tL} f \right\|_q \lesssim \left\| \sqrt{t} L^{1/2} e^{-tL} \right\|_q \lesssim \|f\|_q.$$

Hence from (3.27)

$$\|\nabla p_2 t(\cdot,y)\|_q = \left\| \nabla e^{-tL} p_t(\cdot,y) \right\|_q \lesssim \frac{1}{\sqrt{t}} \|p_t(\cdot,y)\|_q. \tag{3.28}$$

Now using the bounds (3.6), we have

$$\|p_t(\cdot,y)\|_q^q \;\le\; \frac{C}{t^{qn/2}} e^{-qc\left(1+\frac{t}{\gamma(y)^2}\right)^\delta} \int_{\mathbb{R}^n} e^{-qc\frac{|x-y|^2}{t}}\, dx \;\le\; \frac{C}{t^{(q-1)n/2}} e^{-qc\left(1+\frac{t}{\gamma(y)^2}\right)^\delta}.$$

Combining this with (3.28) gives[1] (3.26).

Finally to obtain (3.10) for $p \in (2, 2q_+)$ we may argue in a similar fashion as above, except in place of (3.26) we use

$$\left\| V(\cdot)^{1/2} p_t(\cdot,y) \right\|_q \le \frac{C_q}{t^{1/2+n/2q'}} e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^\delta} \qquad\qquad \forall\, y \in \mathbb{R}^n, t > 0$$

which follows similarly from the heat kernel bounds (3.6), and the boundedness of $V^{1/2} L^{-1/2}$ on $L^q(\mathbb{R}^n)$ for all $q \in (1, 2q_+)$ (see Theorem 1.3).

This concludes the proof of Proposition 3.6.

3.2.4 Weighted Sobolev bounds: second derivatives

In this section we give the proof of Proposition 3.7. We shall first obtain the Proposition for $p \in (1, q_+)$. The case $p = 1$ can then be obtained by Hölder's inequality (we omit the details

[1] We remark that for $n \ge 3$ we can also use Proposition 3.5 to obtain (3.26) when $q_+ \ge n$.

for this case). Fix $p \in (1, q_+)$. Let α_p be the constant in Proposition 3.6, c_1 be the constant in Proposition 3.4, and c_0 the constant in (3.6). Pick $\beta \in (0, \min\{\alpha_p, pc_1, pc_0\})$ and set $\beta_p = \beta/2$.

We shall require the following by-parts inequality that is in some sense based on the Calderón–Zygmund inequality. It is inspired by a similar inequality in [43] but valid only on certain domains of $\mathbb{R}^n$. The following applies to $\mathbb{R}^n$ and we defer its proof to the end of this section.

Lemma 3.11. *Let $p \in (1, \infty)$ and $f \in W^{2,p}(\mathbb{R}^n)$. Then there exists $C = C(p, n)$ such that for each $1 \leq j, k \leq n$ one has*

$$\|\phi\, \partial_j \partial_k f\|_{L^p} \leq C\Big(\big\|f|\nabla^2 \phi|\big\|_{L^p} + \big\||\nabla f||\nabla \phi|\big\|_{L^p} + \big\|\phi \Delta f\big\|_{L^p}\Big)$$

for every $\phi \in C_0^\infty(\mathbb{R}^n)$.

We will prove (3.11) by using a family of weight functions $\{w_{t,R}(\cdot, y)\}_R \subset C_0^\infty(\mathbb{R}^n)$ that forms a smooth cutoff of $e^{\beta|x-y|^2/t}$, and then applying an approximation argument. Accordingly fix $t > 0$ and let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a function satisfying the following (for some fixed constant C):

$$\operatorname{supp} \varphi \subset B(0, 2\sqrt{t}), \quad \varphi \equiv 1 \text{ on } B(0, \sqrt{t}), \quad |\varphi| \leq 1, \quad |\nabla \varphi| \leq C/\sqrt{t}, \quad \big|\nabla^2 \varphi\big| \leq C/t.$$

Now for each $R \geq 1$ set $\varphi_R := \varphi(\frac{\cdot}{R})$. Then φ_R satisfies:

$$\varphi_R \equiv 1 \text{ on } B(0, R\sqrt{t}), \qquad |\varphi_R| \leq 1, \qquad |\nabla \varphi_R| \leq \frac{C}{\sqrt{t}}, \qquad \big|\nabla^2 \varphi\big| \leq \frac{C}{\sqrt{t}}.$$

Now define

$$w_{t,R}(x, y) \;:=\; \varphi_R\big(|x - y|\big) e^{\beta_p \frac{|x-y|^2}{pt}}.$$

Then supp $w_{t,R}(x, y) \subset B(y, 2R\sqrt{t})$ and one can show easily that

$$|\nabla_x w_{t,R}(x, y)| \leq \frac{C}{\sqrt{t}} e^{\beta \frac{|x-y|^2}{pt}} \qquad \text{and} \qquad \big|\nabla_x^2 w_{t,R}(x, y)\big| \leq \frac{C}{t} e^{\beta \frac{|x-y|^2}{pt}}. \tag{3.29}$$

Next define for each $t > 0$, $y \in \mathbb{R}^n$ and $R \geq 1$,

$$J_R(t, y) \;:=\; \big\|w_{t,R}(\cdot, y)\, \big|\nabla^2 p_t(\cdot, y)\big|\big\|_p.$$

We apply Lemma 3.11 with $f := p_t(\cdot, y)$ and $\phi := w_{t,R}(\cdot, y)$. Note that $p_t(\cdot, y) \in W^{2,p}(\mathbb{R}^n)$. To see this recall firstly that $\nabla^2 L^{-1}$ is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1, q_+)$ (from Theorem 1.3), and secondly that $\frac{\partial}{\partial t} p_t(\cdot, y) \in L^p(\mathbb{R}^n)$ (due to the pointwise bounds (3.7) on the time derivative of the heat kernel of L). Therefore one has

$$\left\|\nabla^2 p_t(\cdot, y)\right\|_p \;=\; \left\|-\nabla^2 L^{-1} \frac{\partial}{\partial t} p_t(\cdot, y)\right\|_p \;\lesssim\; \left\|\frac{\partial}{\partial t} p_t(\cdot, y)\right\|_p \;<\; \infty$$

so that $\nabla^2 p_t(\cdot, y) \in L^p(\mathbb{R}^n)$. Hence by Lemma 3.11, for each $t > 0$, $y \in \mathbb{R}^n$, and $R \geq 1$, we obtain

$$\begin{aligned} J_R(t, y) &\lesssim \; \left\||\nabla^2 w_{t,R}(\cdot, y)| p_t(\cdot, y)\right\|_p + \left\||\nabla w_{t,R}(\cdot, y)||\nabla p_t(\cdot, y)|\right\|_p + \left\|w_{t,R}(\cdot, y)\Delta p_t(\cdot, y)\right\|_p \\ &=: \; J_R^1(t, y) + J_R^2(t, y) + J_R^3(t, y). \end{aligned}$$

To estimate the first term we use the bounds of our constructed weight functions (3.29), the bounds on the heat kernel in (3.6), and that $\beta - pc_0 < 0$:

$$\begin{aligned} J_R^1(t, y)^p &= \int_{\mathbb{R}^n} \left|\nabla_x^2 w_{t,R}(x, y)\right|^p p_t(x, y)^p \, dx \\ &\leq \frac{C}{t^{p+pn/2}} e^{-pc\left(1+\frac{t}{\gamma(y)^2}\right)^\delta} \int_{\mathbb{R}^n} e^{(\beta - pc_0)\frac{|x-y|^2}{t}} \, dx \\ &\leq \frac{C}{t^{p+(p-1)n/2}} e^{-pc\left(1+\frac{t}{\gamma(y)^2}\right)^\delta}. \end{aligned}$$

For the second term J_R^2 we observe that since $q_+^* \geq q_+$ then Proposition 3.6 applies. Therefore because $\beta \leq \alpha_p$ we may combine (3.9) with (3.29) to obtain

$$\begin{aligned} J_R^2(t, y)^p &= \int_{\mathbb{R}^n} |\nabla_x w_{t,R}(x, y)|^p \, |\nabla_x p_t(x, y)|^p \, dx \\ &\leq \frac{C}{t^{p/2}} \int_{\mathbb{R}^n} |\nabla_x p_t(x, y)|^p \, e^{\beta\frac{|x-y|^2}{t}} \, dx \\ &\leq \frac{C}{t^{p+(p-1)n/2}} e^{-pc\left(1+\frac{t}{\gamma(y)^2}\right)^\delta}. \end{aligned}$$

Now for the third term

$$\begin{aligned} J_R^3(t, y) &= \|w_{t,R}(\cdot, y)(L - V)p_t(\cdot, y)\|_p \\ &\leq \|w_{t,R}(\cdot, y)Lp_t(\cdot, y)\|_p + \|w_{t,R}(\cdot, y)Vp_t(\cdot, y)\|_p \end{aligned}$$

$$=: J_R^{3.1}(t,y) + J_R^{3.2}(t,y).$$

Using the pointwise bounds on the time derivative of the heat kernel (3.7) and that $|w_{t,R}(x,y)| \leq e^{\beta_p |x-y|^2/t}$ we have

$$\begin{aligned} J_R^{3.1}(t,y)^p &= \int_{\mathbb{R}^n} \Big| \frac{\partial}{\partial t} p_t(x,y) \Big|^p w_{t,R}(x,y)^p \, dx \\ &\leq \frac{C}{t^{p+pn/2}} e^{-pc\left(1+\frac{t}{\gamma(y)^2}\right)^\delta} \int_{\mathbb{R}^n} e^{(\beta_p - pc_1)\frac{|x-y|^2}{t}} \, dx \\ &\leq \frac{C}{t^{p+(p-1)n/2}} e^{-pc\left(1+\frac{t}{\gamma(y)^2}\right)^\delta}, \end{aligned}$$

where in the last line we have used that $\beta_p - pc_1 < 0$. For the final term $J_R^{3.2}(t,y)$ we employ the reverse Hölder properties of V, and the improved decay inherent in the heat kernel of L, namely (3.6). Indeed one has

$$\begin{aligned} J_R^{3.2}(t,y)^p &= \int_{\mathbb{R}^n} V(x)^p p_t(x,y)^p w_{t,R}(x,y)^p \, dx \\ &\leq \frac{C}{t^{pn/2}} e^{-pc\left(1+\frac{t}{\gamma(y)^2}\right)^\delta} \int_{\mathbb{R}^n} V(x)^p e^{(\beta_p - pc_0)\frac{|x-y|^2}{t}} \, dx \\ &= \frac{C}{t^{pn/2}} e^{-pc\left(1+\frac{t}{\gamma(y)^2}\right)^\delta} \sum_{j=0}^{\infty} \int_{U_j(B(y,\sqrt{t}))} V(x)^p e^{-\beta_0 \frac{|x-y|^2}{t}} \, dx \end{aligned}$$

where $\beta_0 := pc_0 - \beta_p > 0$. Now for each $j \geq 1$,

$$\begin{aligned} \int_{U_j(B(y,\sqrt{t}))} V(x)^p e^{-\beta_0 \frac{|x-y|^2}{t}} \, dx &\leq e^{-\beta_0 2^{2j}} \int_{B(y,2^j\sqrt{t})} V(x)^p \, dx \\ &\leq C e^{-\beta_0 4^j} \big|B(y,2^j\sqrt{t})\big| \Big(\fint_{B(y,2^j\sqrt{t})} V(x)\, dx \Big)^p \\ &\leq C e^{-\beta_0 4^j} 2^{jn} t^{n/2} \Big(2^{j(n_0-n)} \fint_{B(y,\sqrt{t})} V(x)\, dx \Big)^p \\ &= \frac{C e^{-\beta_0 4^j} 2^{j(n+n_0p-np)}}{t^{p-n/2}} \Big(t \fint_{B(y,\sqrt{t})} V(x)\, dx \Big)^p. \end{aligned}$$

In the second inequality we have used that $V \in \mathcal{B}_p$ because $p < q_+$ and hence $\mathcal{B}_p \supset \mathcal{B}_q$. In the next to last line we have used that $V\,dx$ is a doubling measure (see (2.7)). Next we remark that if $\sqrt{t} \leq \gamma(y)$, then by Lemma 2.3 (a) and the definition of γ in (2.8), one has

$$t \fint_{B(y,\sqrt{t})} V(x)\, dx \;\leq\; C \Big(\frac{\sqrt{t}}{\gamma(y)} \Big)^{2-n/q} \;\leq\; C$$

since $q > n/2$. On the other hand if $\sqrt{t} > \gamma(y)$, then Lemma 2.3 (b) implies that

$$t \fint_{B(y,\sqrt{t})} V(x)\,dx \;\leq\; C\Big(\frac{\sqrt{t}}{\gamma(y)}\Big)^{\sigma} \;\leq\; C\Big(\frac{\sqrt{t}}{\gamma(y)}\Big)^{|\sigma|}.$$

In either case we can bound

$$e^{-\frac{pc}{2}\left(1+\frac{t}{\gamma(y)^2}\right)^{\delta}}\Big(t \fint_{B(y,\sqrt{t})} V(x)\,dx\Big)^{p}$$

by a fixed constant independent of t and y. Therefore it follows that

$$\begin{aligned} J_R^{3.2}(t,y)^p &\leq \frac{C}{t^{p+(p-1)n/2}} e^{-pc\left(1+\frac{t}{\gamma(y)^2}\right)^{\delta}} \Big(t \fint_{B(y,\sqrt{t})} V(x)\,dx\Big)^{p} \Big\{1 + \sum_{j=1}^{\infty} e^{-\beta_0 4^j} 2^{j(n+n_0 p - np)}\Big\} \\ &\leq \frac{C}{t^{p+(p-1)n/2}} e^{-\frac{pc}{2}\left(1+\frac{t}{\gamma(y)^2}\right)^{\delta}}. \end{aligned}$$

Collecting the estimates for J_R^1, J_R^2 and J_R^3 we obtain

$$J_R(t,y) \;\leq\; \frac{C}{t^{1+n/(2p')}} e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^{\delta}}$$

with C, c independent of R. Therefore

$$\Big(\int_{\mathbb{R}^n} \big|\nabla_x^2 p_t(x,y)\big|^p e^{\beta\frac{|x-y|^2}{t}}\,dx\Big)^{1/p} \;=\; \sup_{R\geq 1} J_R(t,y) \;\leq\; \frac{C}{t^{1+n/(2p')}} e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^{\delta}}.$$

This establishes (3.11).

To prove (3.12) we simply note that

$$\Big(\int |V(x)p_t(x,y)|^p e^{\beta_p\frac{|x-y|^2}{t}}\,dx\Big)^{1/p} \;=\; \sup_{R\geq 1} J_R^{3.2}(t,y) \;\leq\; \frac{C}{t^{1+n/(2p')}} e^{-c\left(1+\frac{t}{\gamma(y)^2}\right)^{\delta}}$$

which follows from our previous estimates.

This concludes the proof of Proposition 3.7, save for the proof of Lemma 3.11 which was deferred. We turn to this now.

Proof of Lemma 3.11. Fix $p \in (1,\infty)$, $f \in W^{2,p}(\mathbb{R}^n)$ and $j,k \in \{1,2,...,n\}$. Let $\phi \in C_0^{\infty}(\mathbb{R}^n)$. Then the product rule gives the following

$$\begin{aligned} \phi\,\partial_j\partial_k f &= \partial_j(\phi\,\partial_k f) - \partial_j\phi\,\partial_k f \\ &= \partial_j(\partial_k(\phi f) - f\partial_k\phi) - \partial_j\phi\,\partial_k f \end{aligned}$$

$$= \partial_j \partial_k(\phi f) - \partial_j(f \partial_k \phi) - \partial_j \phi \, \partial_k f$$

$$= \partial_j \partial_k(\phi f) - f \partial_j \partial_k \phi - \partial_j f \partial_k \phi - \partial_j \phi \, \partial_k f.$$

Taking L^p norms gives

$$\|\phi \, \partial_j \partial_k f\|_p \ \leq \ \|\partial_j \partial_k(\phi f)\|_p + \|f \partial_j \partial_k \phi\|_p + \|\partial_j f \partial_k \phi\|_p + \|\partial_j \phi \, \partial_k f\|_p \, . \tag{3.30}$$

Note that the left hand side is finite because $f \in W^{2,p}(\mathbb{R}^n)$ and $\phi \in C_0^\infty(\mathbb{R}^n)$. Let us consider each term on the right hand side in turn.

Firstly by noting that $|\partial_j \phi| \leq \left(\sum_k |\partial_k \phi|^2\right)^{1/2} \leq |\nabla f|$ for every $j \in \{1, ..., n\}$, we have

$$\|\partial_j f \partial_k \phi\|_p + \|\partial_j \phi \partial_k f\|_p \ \leq \ 2\big\||\nabla f| \, |\nabla \phi|\big\|_p. \tag{3.31}$$

Similarly $|\partial_j \partial_k \phi| \leq \left(\sum_j \sum_k |\partial_j \partial_k \phi|^2\right)^{1/2} = |\nabla^2 \phi|$ for every $j, k \in \{1, ..., n\}$, so that

$$\|f \partial_j \partial_k \phi\|_p \ \leq \ \big\|f |\nabla^2 \phi|\big\|_p. \tag{3.32}$$

Next since $\phi f \in W^{2,p}(\mathbb{R}^n)$ then by the Calderón–Zygmund inequality (1.2) (see also [100] Chapter 3, Proposition 3) on $\mathbb{R}^n$,

$$\|\partial_j \partial_k(\phi \, f)\|_p \ \leq \ \big\||\nabla^2(\phi \, f)|\big\|_p \ \leq \ C_p \, \|\Delta(\phi \, f)\|_p \, .$$

Now direct computations give

$$\begin{aligned}
\Delta(\phi \, f) &= \sum_{j=1}^n \partial_j^2(\phi \, f) \ = \ \sum_{j=1}^n \partial_j \big(\phi \, \partial_j f + f \partial_j \phi\big) \\
&= \sum_{j=1}^n \big(\partial_j \phi \, \partial_j f + \phi \, \partial_j^2 f + \partial_j f \partial_j \phi + f \partial_j^2 \phi\big) \\
&= \phi \sum_{j=1}^n \partial_j^2 f + f \sum_{j=1}^n \partial_j^2 \phi + 2 \sum_{j=1}^n \partial_j \phi \, \partial_j f \\
&= \phi \Delta f + f \Delta \phi + 2 \nabla \phi \cdot \nabla f \, .
\end{aligned}$$

By Cauchy-Schwarz,

$$|\Delta(\phi \, f)| \ \ \leq \ \ |\phi \Delta f| + |f \Delta \phi| + 2 \, |\nabla \phi||\nabla f| \ \ \leq \ \ |\phi \Delta f| + |f||\nabla^2 \phi| + 2|\nabla \phi||\nabla f| \, .$$

Hence

$$\|\partial_j\partial_k(\phi\, f)\|_p \;\le\; C_p\,\|\phi\Delta f\|_p + C_p\big\|f|\nabla^2\phi|\big\|_p + 2C_p\big\||\nabla\phi|\,|\nabla f|\big\|_p. \tag{3.33}$$

Inserting (3.31), (3.32), and (3.33) into (3.30) we obtain

$$\|\phi\,\partial_j\partial_k f\|_p \;\le\; C_p\,\|\phi\Delta f\|_p + C_p\big\|f|\nabla^2\phi|\big\|_p + C_p\big\||\nabla\phi|\,|\nabla f|\big\|_p,$$

and in fact

$$\big\|\phi|\nabla^2 f|\big\|_p \;\le\; \sum_{j,k=1}^{n}\|\phi\,\partial_j\partial_k f\|_p \;\le\; C\Big(\|\phi\Delta f\|_p + \big\|f|\nabla^2\phi|\big\|_p + \big\||\nabla\phi||\nabla f|\big\|_p\Big),$$

where C depends on p and the dimension n. This ends the proof of Lemma 3.11. $\square$

Chapter 4

Weighted Lebesgue spaces I: Muckenhoupt weights

This chapter studies the first- and second-order Riesz transforms associated to the Schrödinger operator $L = -\Delta + V$ on weighted Lebesgue spaces with weights belonging to the Muckenhoupt class $\mathcal{A}_\infty$. We are interested in the conditions on p and $w \in \mathcal{A}_\infty$ for which the following

$$\|Tf\|_{L^p(w)} \leq C \|f\|_{L^p(w)}, \qquad \forall f \in L_c^\infty(\mathbb{R}^n)$$

holds. Here T is one of the operators $\nabla L^{-1/2}$, $V^{1/2}L^{-1/2}$, $\nabla^2 L^{-1}$ or VL^{-1}.

In this chapter we combine the techniques in [13] and [18] to show that for non-negative potentials, boundedness of the first-order Riesz transforms on $L^p(\mathbb{R}^n)$ for p above 2 is *equivalent* to their boundedness on the weighted spaces, for a certain range of p and w. This is encapsulated in the following two theorems, which are the main results of this chapter. Recall that the notation for the sets $\mathcal{W}_w(p_0, q_0)$ was defined in Section 2.4.

Theorem 4.1. *Let $n \geq 1$ and $L = -\Delta + V$ on $\mathbb{R}^n$ with $0 \leq V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Fix $s > 2$. Then the following are equivalent.*

(a) $\nabla L^{-1/2}$ *is bounded on $L^p(\mathbb{R}^n)$ for each $p \in (1, s)$*

(b) $\nabla L^{-1/2}$ *is bounded on $L^p(w)$ for each $w \in \mathcal{A}_\infty$ and each $p \in \mathcal{W}_w(1, s)$.*

(c) $\nabla L^{-1/2}$ *is bounded from $L^1(w)$ to $L^{1,\infty}(w)$ for each $w \in \mathcal{A}_1 \cap \mathcal{B}_{s'}$.*

Theorem 4.2. *Let $n \geq 1$ and $L = -\Delta + V$ on $\mathbb{R}^n$ with $0 \leq V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Fix $s > 2$. Then the following are equivalent.*

(a) $V^{1/2}L^{-1/2}$ *is bounded on* $L^p(\mathbb{R}^n)$ *for each* $p \in (1, s)$

(b) $V^{1/2}L^{-1/2}$ *is bounded on* $L^p(w)$ *for each* $w \in \mathcal{A}_\infty$ *and each* $p \in \mathcal{W}_w(1, s)$.

(c) $V^{1/2}L^{-1/2}$ *is bounded from* $L^1(w)$ *to* $L^{1,\infty}(w)$ *for each* $w \in \mathcal{A}_1 \cap \mathcal{B}_{s'}$.

On specialising to reverse Hölder potentials, these same techniques in conjunction with the heat kernel estimates we obtained in Chapter 3 (particularly Proposition 3.7) allow us to also obtain boundedness for the second-order Riesz transforms $\nabla^2 L^{-1}$ and VL^{-1}.

Theorem 4.3. *Let* $L = -\Delta + V$ *on* $\mathbb{R}^n$ *with* $n \geq 2$, *and suppose that* $V \in \mathcal{B}_q$ *for some* $q > n/2$. *Set* $q_+ := \sup\left\{q > \frac{n}{2} : V \in \mathcal{B}_q\right\}$. *Then the following holds.*

(a) *For each* $w \in \mathcal{A}_\infty$, *the operators* VL^{-1} *and* $\nabla^2 L^{-1}$ *are both bounded on* $L^p(w)$ *for all* $p \in \mathcal{W}_w(1, q_+)$.

(b) *For each* $w \in \mathcal{A}_1 \cap \mathcal{B}_{q'_+}$ *the operators* VL^{-1} *and* $\nabla^2 L^{-1}$ *map* $L^1(w)$ *into* $L^{1,\infty}(w)$.

We note here that the first conclusion recovers the results in [75] (see also Section 1.1.1 item (ii)). The second conclusion is new.

This chapter is organised as follows. Section 4.1 gives the main technical tools required to prove our results, with the key result here being Theorem 4.6. We apply this to give the proofs of Theorems 4.1 and 4.2 in Section 4.2, and the proof of Theorem 4.3 in Section 4.3.

4.1 Main tools

We give here the main tools we use to prove boundedness on weighted spaces, which are taken from [16]. The first is a maximal type theorem, which will also play a role in Section 6.1.1. In the following M is the Hardy–Littlewood maximal function defined in (1.9).

Theorem 4.4 ([16] Theorem 3.1)**.** *Fix* $q \in (1, \infty)$, $\xi \geq 1, s \in (1, q), v \in \mathcal{B}_{s'}$. *Assume that* F, G, H_1, H_2 *are non-negative measurable functions on* $\mathbb{R}^n$ *such that for each ball* B *there exist non-negative functions* G_B *and* H_B *such that*

$$F(x) \leq G_B(x) + H_B(x), \qquad a.e.\ x \in B, \tag{4.1}$$

$$\Big(\fint_B H_B^q\Big)^{1/q} \le \xi(MF(x) + MH_1(x) + H_2(y)), \qquad \forall\, x, y \in B, \tag{4.2}$$

$$\fint_B G_B \le G(x), \qquad \forall\, x \in B. \tag{4.3}$$

Then there exists $K_0 = K_0(n, \xi) \ge 1$ and $C = C(q, n, \xi, v, s)$ such that the following holds: for each $\lambda > 0$, $K \ge K_0$, and $\delta \in (0, 1)$,

$$v\big(\{x \in \mathbb{R}^n : MF(x) > K\lambda \text{ and } G(x) \le \delta\lambda\}\big) \le C\Big(\frac{\xi^q}{K^q} + \frac{\delta}{K}\Big)^{1/s} v\big(\{x \in \mathbb{R}^n : MF(x) > \lambda\}\big).$$

As a consequence, if $r \in (1, q/s\,]$ and $F \in L^1(\mathbb{R}^n)$, then

$$\|MF\|_{L^r(v)} \le C\Big(\|G\|_{L^r(v)} + \|MH_1\|_{L^r(v)} + \|H_2\|_{L^r(v)}\Big). \tag{4.4}$$

The next tool is a weak type criterion.

Theorem 4.5 ([18] Theorem 3.3)**.** *Fix $1 \le p_0 < q_0 \le \infty$ and $w \in \mathcal{A}_\infty$. Let T be a sublinear operator defined on $\mathcal{D}$, a subspace of $L^{q_0}(\mathbb{R}^n)$, and $\{A_B\}_B$ be a family of operators indexed by balls acting from $L^\infty_c(\mathbb{R}^n)$ into $\mathcal{D}$. Assume that T and $\{A_B\}_B$ satisfy the following (recall that the sets $U_j(B)$ have been defined in (2.5)).*

(i) *There exists $q \in \mathcal{W}_w(p_0, q_0)$ such that T maps $L^q(w)$ continuously into $L^{q,\infty}(w)$.*

(ii) *For each $j \ge 0$ there exists a constant α_j such that for any ball B and $f \in L^\infty_c(\mathbb{R}^n)$ supported in B,*

$$\left(\fint_{U_j(B)} |A_B f|^{q_0}\right)^{1/q_0} \le \alpha_j \left(\fint_B |f|^{p_0}\right)^{1/p_0}. \tag{4.5}$$

(iii) *There exists p such that $w \in \mathcal{B}_{p'}$ with the following property: for each $j \ge 2$, there is a constant α_j such that for any ball B and $f \in L^\infty_c(\mathbb{R}^n)$ supported in B,*

$$\Big(\fint_{U_j(B)} |T(I - A_B)f|^p\Big)^{1/p} \le \alpha_j\Big(\fint_B |f|^{p_0}\Big)^{1/p_0}. \tag{4.6}$$

(iv) *The constants $\{\alpha_j\}_j$ from* (ii) *and* (iii) *satisfy $\sum_j \alpha_j 2^{jD_w} < \infty$, where D_w is the doubling constant of $w\,dx$.*

Under these hypotheses, if $w \in \mathcal{A}_1 \cap \mathcal{B}_{(q_0/p_0)'}$ *then* T *is weak* (p_0, p_0) *with respect to* wdx*. That is, for each* $f \in L^\infty_c(\mathbb{R}^n)$*,*

$$\|Tf\|_{L^{p_0,\infty}(w)} \leq C \|f\|_{L^{p_0}(w)} .$$

We next give a particular case of Theorem 4.4 and Theorem 4.5. Our aim is to apply this to the operators $\nabla L^{-1/2}$, $V^{1/2}L^{-1/2}$, VL^{-1}.

Theorem 4.6. *Let* $1 \leq p_0 < q_0 \leq \infty$ *and* T *be a linear operator. Suppose that for each* $\tilde{q} \in (p_0, q_0)$ *there exists a family of operators* $\{A_B\}_B$ *indexed by balls, and a collection of scalars* $\{\alpha_j\}_{j=0}^\infty$ *such that the following holds.*

(i) T *is bounded on* $L^{\tilde{q}}(\mathbb{R}^n)$*.*

(ii) *For every ball* B *and* $f \in L^\infty_c(\mathbb{R}^n)$ *supported in* B*,*

$$\Big(\fint_{U_j(B)} |A_B f|^{\tilde{q}} \Big)^{1/\tilde{q}} \leq \alpha_j \Big(\fint_B |f|^{p_0} \Big)^{1/p_0}, \qquad \forall j \geq 0 \tag{4.7}$$

$$\Big(\fint_{U_j(B)} |T(I - A_B)f|^{\tilde{q}} \Big)^{1/\tilde{q}} \leq \alpha_j \Big(\fint_B |f|^{p_0} \Big)^{1/p_0}, \qquad \forall j \geq 2. \tag{4.8}$$

Here the sets $U_j(B)$ *have been defined in* (2.5)*.*

(iii) *The constants* $\{\alpha_j\}_j$ *satisfy* $\sum_j \alpha_j 2^{jn} < \infty$*.*

Then we have the following.

(a) *If* $w \in \mathcal{A}_\infty$ *then* T *extends to a bounded operator on* $L^p(w)$ *for all* $p \in \mathcal{W}_w(p_0, q_0)$*.*

(b) *If* $w \in \mathcal{A}_1 \cap \mathcal{B}_{(q_0/p_0)'}$ *and in addition* $\sum_j \alpha_j 2^{jD_w} < \infty$ *(where* D_w *is the doubling order of* w*), then* T *maps* $L^{p_0}(w)$ *into* $L^{p_0,\infty}(w)$*.*

Proof. The ideas in the proof originate from [24] and were applied in [18] to study weighted norm inequalities of Riesz transforms associated to the Laplace Beltrami operator on doubling manifolds. The proof we describe here follows closely that of [18] Theorem 1.2 (i). We remark also that the spirit of the proof is akin to that of [13] section 3.1.

We first prove (a). Fix $w \in \mathcal{A}_\infty$ and $p \in \mathcal{W}_w(p_0, q_0)$. Denote by T^* the dual operator to T. We first observe that the $L^p(w)$ boundedness of T is equivalent to the $L^{p'}(w^{1-p'})$ boundedness of T^* (see Remark 4.7 (a)). We shall apply Theorem 4.4 to obtain the latter.

Firstly there exists numbers p_1 and q_1 such that

$$p_0 < p_1 < p < q_1 < q_0 \qquad \text{and} \qquad w \in \mathcal{A}_{p/p_1} \cap \mathcal{B}_{(q_1/p)'}$$

holds. See Remark 4.7 (b). It follows from Remark 4.7 (c) that

$$w^{1-p'} \in \mathcal{A}_{p'/q_1'} \cap \mathcal{B}_{(p_1'/p')'}\,.$$

Now we apply Theorem 4.4 to the following data. For each $f \in L_c^\infty(\mathbb{R}^n)$ set

$$F := |T^* f|^{q_1'}\,, \qquad H_1 = H_2 := 0, \qquad v := w^{1-p'},$$
$$s := \frac{p_1'}{p'}, \qquad r := \frac{p'}{q_1'}, \qquad q := \frac{p_1'}{q_1'}.$$

Let $\tilde{q} = q_1$ and $\{A_B\}_B$ and $\{\alpha_j\}_j$ be as in the hypotheses. We will check that Theorem 4.4 conditions (4.1), (4.2) and (4.3) hold with

$$G_B := 2^{q_1'-1}\,|(I-A_B)^* T^* f|^{q_1'} \qquad \text{and} \qquad H_B := 2^{q_1'-1}\,|A_B^* T^* f|^{q_1'}$$

and G is a fixed constant multiple of $M\big(|f|^{q_1'}\big)$, with M the Hardy–Littlewood maximal function. We first check condition (4.1). By noting that $(I - A_B^*) = (I - A_B)^*$ one has

$$\begin{aligned} F(x) = |T^* f(x)|^{q_1'} &= |(I-A_B)^* T^* f(x) + A_B^* T^* f(x)|^{q_1'} \\ &\le 2^{q_1'-1}\,|(I-A_B^*) T^* f(x)|^{q_1'} + 2^{q_1'-1}\,|A_B^* T^* f(x)|^{q_1'} \\ &= G_B(x) + H_B(x). \end{aligned}$$

We have used that $|a+b|^r \le 2^{r-1}|a|^r + 2^{r-1}|b|^r$, which is valid for all $r \ge 1$ and $a, b \in \mathbb{R}$. We now check condition (4.2). We first write

$$\Big(\fint_B H_B^q\Big)^{1/q} = \Big(\fint_B 2^{p_1'-p_1'/q_1'}\,|A_B^* T^* f|^{p_1'}\Big)^{q_1'/p_1'} \lesssim \Big(\fint_B |A_B^* T^* f|^{p_1'}\Big)^{q_1'/p_1'}.$$

To estimate the integral we apply duality to $R := T^*$, $S := A_B^*$ with some $g \in L^{p_1}(B, dx/|B|)$ with norm 1 (Remark 4.7 (d)) to obtain for each $x \in B$,

$$\begin{aligned}\Big(\fint_B H_B^q\Big)^{1/qq_1'} &\lesssim \Big(\fint_B |A_B^* T^* f|^{p_1'}\Big)^{1/p_1'}\\ &\le \fint_B |T^* f|\,|A_B g|\\ &\le \sum_{j=0}^{\infty} 2^{jn} \fint_{U_j(B)} |T^* f|\,|A_B g|\\ &\le \sum_{j=0}^{\infty} 2^{jn} \Big(\fint_{2^j B} |T^* f|^{q_1'}\Big)^{1/q_1'} \Big(\fint_{U_j(B)} |A_B g|^{q_1}\Big)^{1/q_1}\\ &\le M\big(|T^* f|^{q_1'}\big)(x)^{1/q_1'} \sum_{j=0}^{\infty} 2^{jn} \Big(\fint_{U_j(B)} |A_B g|^{q_1}\Big)^{1/q_1}.\end{aligned}$$

Now from condition (ii) estimate (4.7) with exponent $\tilde{q} = q_1$, we have for each $j \ge 0$

$$\Big(\fint_{U_j(B)} |A_B g|^{q_1}\Big)^{1/q_1} \le \alpha_j \Big(\fint_B |g|^{p_0}\Big)^{1/p_0} \le \alpha_j \Big(\fint_B |g|^{p_1}\Big)^{1/p_1} = \alpha_j.$$

Where we have used Hölder's inequality (since $p_1 > p_0$) and the normalisation of g. It follows then that for each $x \in B$,

$$\Big(\fint_B H_B^q\Big)^{1/qq_1'} \lesssim M\big(|T^* f|^{q_1'}\big)(x)^{1/q_1'} \sum_{j=0}^{\infty} \alpha_j 2^{jn} \lesssim M\big(|T^* f|^{q_1'}\big)(x)^{1/q_1'}$$

so that (4.2) holds with $H_1 = H_2 = 0$. We check condition (4.3). We first write

$$\Big(\fint_B G_B\Big)^{1/q_1'} = \Big(\fint_B 2^{q_1'-1} \,|(I - A_B)^* T^* f|^{q_1'}\,dx\Big)^{1/q_1'} \lesssim \Big(\fint_B |(I - A_B)^* T^* f|^{q_1'}\,dx\Big)^{1/q_1'}.$$

We apply duality again now with $R := I$, $S := (I - A_B)^* T^*$ and $g \in L^{q_1}(B, dx/|B|)$ with norm 1. Then for each $x \in B$,

$$\begin{aligned}\Big(\fint_B G_B\Big)^{1/q_1'} &\lesssim \fint_B |f|\,|T(I - A_B) g|\\ &\le \sum_{j=0}^{\infty} 2^{jn} \fint_{U_j(B)} |f|\,|T(I - A_B) g|\\ &\le \sum_{j=0}^{\infty} 2^{jn} \Big(\fint_{2^j B} |f|^{q_1'}\Big)^{1/q_1'} \Big(\fint_{U_j(B)} |T(I - A_B) g|^{q_1}\Big)^{1/q_1}\\ &\le M\big(|f|^{q_1'}\big)(x)^{1/q_1'} \sum_{j=0}^{\infty} 2^{jn} \Big(\fint_{U_j(B)} |T(I - A_B) g|^{q_1}\Big)^{1/q_1}.\end{aligned}$$

To estimate the summands we observe that since $q_1 \in (p_0, q_0)$, we may apply (4.8) with exponent $\tilde{q} = q_1$ to obtain for $j \geq 2$,

$$\Big(\fint_{U_j(B)} |T(I - A_B)g|^{q_1} \Big)^{1/q_1} \leq \alpha_j \Big(\fint_B |g|^{p_0} \Big)^{1/p_0} \leq \alpha_j \Big(\fint_B |g|^{q_1} \Big)^{1/q_1} = \alpha_j.$$

We have used Hölder's inequality (because $q_1 > p_0$) and the normalisation of g. For $j = 0, 1$ we use hypothesis (i) with exponent $\tilde{q} = q_1$ to give

$$\fint_{U_j(B)} |T(I - A_B)g|^{q_1} \lesssim \frac{1}{|B|} \int_{\mathbb{R}^n} |(I - A_B)g|^{q_1} \lesssim \frac{1}{|B|} \Big\{ \int_B |g|^{q_1} + \sum_{k=0}^{\infty} \int_{U_k(B)} |A_B g|^{q_1} \Big\}.$$

For the summands we use the approach as before, namely applying (4.7) for $k \geq 0$ and Hölder's inequality to get

$$\Big(\fint_{U_k(B)} |A_B g|^{q_1} \Big)^{1/q_1} \leq \alpha_k \Big(\fint_B |g|^{p_0} \Big)^{1/p_0} \leq \alpha_k \Big(\fint_B |g|^{q_1} \Big)^{1/q_1} = \alpha_k.$$

Collecting these estimates we have for $j = 0, 1$,

$$\fint_{U_j(B)} |T(I - A_B)g|^{q_1} \;\lesssim\; \fint_B |g|^{q_1} + \sum_{k=0}^{\infty} 2^{kn} \fint_{U_k(B)} |A_B g|^{q_1} \;\lesssim\; \fint_B |g|^{q_1} + \sum_{k=0}^{\infty} \alpha_k^{q_1} 2^{kn}$$

which is finite because $\sum_k \alpha_k 2^{kn}$ is. Finally we can estimate G_B:

$$\Big(\fint_B G_B \Big)^{1/q_1'} \;\lesssim\; M\big(|f|^{q_1'}\big)(x)^{1/q_1'} \Big\{ \sum_{j=2}^{\infty} \alpha_j 2^{jn} + C \Big\} \;\lesssim\; M\big(|f|^{q_1'}\big)(x)^{1/q_1'} \;=:\; G(x)^{1/q_1'}.$$

This finishes the proof of (4.3).

Theorem 4.4 allows us to conclude that

$$\big\| M\, |T^* f|^{q_1'} \big\|_{L^r(v)} \leq C \big\| M\, |f|^{q_1'} \big\|_{L^r(v)} \tag{4.9}$$

where we recall that $r = p'/q_1'$ and $v = w^{1-p'}$. The $L^{p'}(v)$ boundedness of T^* then follows because

$$\|T^* f\|_{L^{p'}(v)}^{q_1'} \leq \big\| M\, |T^* f|^{q_1'} \big\|_{L^r(v)} \leq C \big\| M\, |f|^{q_1'} \big\|_{L^r(v)} \leq C\, \|f\|_{L^{p'}(v)}^{q_1'}.$$

The first inequality holds by domination of the maximal function. Indeed for almost every $x \in \mathbb{R}^n$ and any $\delta \geq 1$,

$$|g(x)| \;\leq\; \sup_{B \ni x} \fint_B |g| \;\leq\; \sup_{B \ni x} \Big(\fint_B |g|^{\delta} \Big)^{1/\delta} = M(|g|^{\delta})(x)^{1/\delta}.$$

Therefore

$$\|T^*f\|_{L^{p'}(v)}^{q_1'} = \Big(\int |T^*f|^{p'}\, v\Big)^{q_1'/p'} \le \Big(\int M\big(|T^*f|^{q_1'}\big)^{p'/q_1'} v\Big)^{q_1'/p'} = \big\|M\,|T^*f|^{q_1'}\big\|_{L^r(v)}.$$

The second inequality is the conclusion of the maximal theorem (4.9). The final inequality follows from the boundedness of the maximal function $(M\,|\cdot|^{\delta})^{1/\delta}$ on weighted spaces $L^p(w)$ for any $p > \delta$:

$$\big\|M\,|f|^{q_1'}\big\|_{L^r(v)} = \Big(\int M\big(|f|^{q_1'}\big)^{p'/q_1'} v\Big)^{q_1'/p'} \le C\Big(\int |f|^{p'}\, v\Big)^{q_1'/p'} = C\,\|f\|_{L^{p'}(v)}^{q_1'}$$

because $p' > q_1'$. By duality we obtain therefore that T is bounded on $L^p(w)$.

We now prove (b). Fix a weight $w \in \mathcal{A}_1 \cap \mathcal{B}_{(q_0/p_0)'}$. We shall apply Theorem 4.5 to T and A_B as given in Theorem 4.6.

Let us check Theorem 4.5 (i). We first explain why, for our weight w, the set $\mathcal{W}_w(p_0, q_0)$ is non-empty. Since $w \in \mathcal{B}_{(q_0/p_0)'}$ by Proposition 2.9 (e) there exists q such that $(q_0/p_0)' < (q_0/q)' < \infty$ with $w \in \mathcal{B}_{(q_0/q)'}$. This means that $1 < q_0/q < q_0/p_0$ and hence $p_0 < q < q_0$. In particular $q/p_0 > 1$ and so by Proposition 2.9 (a) we have the containment $\mathcal{A}_1 \subset \mathcal{A}_{q/p_0}$. We have shown that $p_0 < q < q_0$ and $w \in \mathcal{A}_{q/p_0} \cap \mathcal{B}_{(q_0/q)'}$ so that $q \in \mathcal{W}_w(p_0, q_0)$. It now follows from conclusion (a) that T is bounded on $L^q(w)$ and hence maps $L^q(w)$ into $L^{q,\infty}(w)$. Next we observe that Theorem 4.5 (ii) is contained in hypothesis (ii) of Theorem 4.6. Let us turn to Theorem 4.5 (iii). By Proposition 2.9 (e) there exists $p' \in (q_0', p_0')$ such that $w \in \mathcal{B}_{p'}$. Hence $p \in (p_0, q_0)$ and Theorem 4.5 (iii) holds by hypothesis (iii) of Theorem 4.6. Since we chose α_j such that $\sum_j \alpha_j 2^{jD_w} < \infty$ we see that condition (iv) of Theorem 4.5 is also satisfied.

The Theorem now lets us conclude that

$$\|Tf\|_{L^{p_0,\infty}(w)} \le C\,\|f\|_{L^{p_0}(w)}$$

which was to be proved. $\square$

Remark 4.7. The following facts are well known but we give the details here for the reader's convenience.

(a) Fix $p \in (1,\infty)$ and let w be a weight and that $w^{1-p'} \in L^1_{\text{loc}}(\mathbb{R}^n)$. Let T be a linear operator and T^* be its adjoint with respect to dx. We explain why the boundedness of T on $L^p(w)$ is equivalent to the boundedness of T^* on $L^{p'}(w^{1-p'})$ (see also [16] remark 4.5).

First assume that T is bounded on $L^p(w)$. One has

$$\|T^* f\|_{L^{p'}(w^{1-p'})} = \int |T^* f|^{p'} \, w^{1-p'} dx = \int |w^{-1} T^* f|^{p'} w \, dx \, .$$

By duality there exists $g \in L^p(w)$ with norm 1 such that

$$\int |w^{-1} T^* f|^{p'} w \, dx \leq \left| \int w^{-1} T^* f \, g \, w \, dx \right| = \left| \int T^* f \, g \, dx \right| \leq \int |f| \, |Tg| \; dx \, .$$

Hölder's inequality with respect to wdx gives

$$\begin{aligned} \int |f| \, |Tg| \; dx &= \int \left| f w^{-1} \right| |Tg| \; w \, dx \\ &\leq \Big(\int \left| f w^{-1} \right|^{p'} w \, dx \Big)^{1/p'} \Big(\int |Tg|^p \, w \, dx \Big)^{1/p} \\ &= \|f\|_{L^{p'}(w^{1-p'})} \, \|Tg\|_{L^p(w)} \; . \end{aligned}$$

Using that T is bounded on $L^p(w)$ and that $\|g\|_{L^p(w)} = 1$ we obtain

$$\|T^* f\|_{L^{p'}(w^{1-p'})} \leq C \, \|f\|_{L^{p'}(w^{1-p'})} \; .$$

To prove the other direction, we remark that if T^* is bounded on $L^{p'}(w^{1-p'})$, then the previous proof implies T is bounded on $L^p(w^{(1-p')(1-p)}) \equiv L^p(w)$.

(b) Fix $w \in \mathcal{A}_\infty$ and $p \in \mathcal{W}_w(p_0, q_0)$. The latter condition implies that $w \in \mathcal{A}_{p/p_0} \cap \mathcal{B}_{(q_0/p)'}$. By Proposition 2.9 (d) there exists p_1 such that $1 < p/p_1 < p/p_0$ and $w \in \mathcal{A}_{p/p_1}$. This implies $p_0 < p_1 < p$. By Proposition 2.9 (e) there exists q_1 such that $(q_0/p)' < (q_1/p)' < \infty$ and $w \in \mathcal{B}_{(q_1/p)'}$. This implies that $p < q_1 < q_0$. Hence $w \in \mathcal{A}_{p/p_1} \cap \mathcal{B}_{(q_1/p)'}$.

(c) Given $p_1 < p < q_1$ the equivalence

$$w \in \mathcal{A}_{p/p_1} \cap \mathcal{B}_{(q_1/p)'} \qquad \Longleftrightarrow \qquad w^{1-p'} \in \mathcal{A}_{p'/q_1'} \cap \mathcal{B}_{(p_1'/p')'}$$

follows from [16] Lemma 4.4.

(d) We recall we may estimate the L^p norms by using duality with $L^{p'}$:

$$\|f\|_p = \sup_{\|g\|_{p'}=1} |\langle f, g\rangle| = \sup_{\|g\|_{p'}=1} \Big| \int f\, g \Big| .$$

Hence if T is a sublinear operator, by the same token

$$\|Tf\|_p = \sup_{\|g\|_{p'}=1} |\langle Tf, g\rangle| = \sup_{\|g\|_{p'}=1} |\langle f, T^* g\rangle| \ .$$

If S and R are sublinear operators and B is a ball, then writing $L^p(X) := L^p\left(B, dx/|B|\right)$ we have

$$\begin{aligned}\Big(\fint |SRf|^p\Big)^{1/p} &= \|SRf\|_{L^p(X)} = \sup_{\|g\|_{L^{p'}(X)}=1} |\langle SRf, g\rangle_X| \\ &= \sup_{\|g\|_{L^{p'}(X)}=1} |\langle Rf, S^* g\rangle_X| \le \fint_B |Rf|\,|S^*g|\ dx,\end{aligned}$$

where the inner product $\langle , \rangle_X$ is the $L^2(X)$ inner product. That is, $\langle u, v\rangle_X = \fint_B uv\, dx$.

4.2 First Order Riesz Transforms

In this section we give the proofs of Theorems 4.1 and 4.2.

Proof of Theorem 4.1. Proof of (a) $\Rightarrow$ (b). We invoke Theorem 4.6 with $T = \nabla L^{-1/2}$, $p_0 = 1$, $q_0 = s$ and

$$A_B = I - (I - e^{-r_B^2 L})^m$$

where $m > n/2$ is an integer. Then condition (i) of Theorem 4.6 holds from our hypothesis (a). We shall show that condition (ii) holds for any $q, p_0 \ge 1$ (and any $m \ge 1$), with $\alpha_j = C4^{-jm}$ for $j \ge 0$. Here C is a constant independent of j and B. To see this we expand for each $j \ge 0$,

$$A_B = \sum_{k=1}^{m} \tbinom{m}{k} (-1)^k e^{-kr_B^2 L}.$$

Therefore for each $x \in \mathbb{R}^n$ one has

$$|A_B f(x)| \le \sum_{k=1}^{m} \tbinom{m}{k} \big|e^{-kr_B^2 L} f(x)\big|.$$

Now for each $j \geq 2$, $x \in U_j(B)$, $y \in B$, we observe that $|x-y| \geq 2^j r_B/4$. Hence for each $k \geq 1$, the Gaussian bounds (3.1) on the heat kernel of L imply that

$$\begin{aligned}
\sup_{x\in U_j(B)} \left|e^{-kr_B^2 L} f(x)\right| &\leq \sup_{x\in U_j(B)} \int_B \left|p_{kr_B^2}(x,y)\right| |f(y)|\, dy \\
&\leq \sup_{x\in U_j(B)} (kr_B^2)^{-n/2} e^{-c4^j} \int_B |f| \\
&\lesssim e^{-c4^j} \fint_B |f|\,.
\end{aligned}$$

These bounds give for each $j \geq 2$, $q \geq 1$, and $p_0 \geq 1$,

$$\begin{aligned}
\Big(\fint_{U_j(B)} |A_B f|^q \, dx\Big)^{1/q} &\lesssim \Big(\fint_{U_j(B)} e^{-c4^j} \Big(\fint_B |f|\Big)^q dx\Big)^{1/q} \\
&\leq e^{-c4^j} \fint_B |f| \ \leq \ e^{-c4^j} \Big(\fint_B |f|^{p_0}\Big)^{1/p_0}
\end{aligned} \tag{4.10}$$

by Hölder's inequality. The same approach gives for $j = 0, 1$

$$\Big(\fint_{U_j(B)} |A_B f|^q\Big)^{1/q} \lesssim \Big(\fint_B |f|^{p_0}\Big)^{1/p_0}.$$

Next we show that hypothesis (a) leads to condition (iii) of Theorem 4.6. Condition (iii) is contained in the conclusion of the following lemma, whose proof we postpone to the end of the section.

Lemma 4.8. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 1$ and $0 \leq V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Assume that for some $q > 2$ there exists $C_q > 0$ such that*

$$\left\|\sqrt{t}\nabla e^{-tL}\right\|_{q\to q} \leq C_q. \tag{4.11}$$

Then for each $p \in [1, q)$ there exists $C_p > 0$ such that for each $m \geq 1$, $j \geq 2$ and any $p_0 \geq 1$,

$$\Big(\fint_{U_j(B)} \left|\nabla L^{-1/2}(I - e^{-r_B^2 L})^m f\right|^p\Big)^{1/p} \leq C_p 4^{-jm} \Big(\fint_B |f|^{p_0}\Big)^{1/p_0}. \tag{4.12}$$

for all balls B and $f \in L^1(B)$.

Hence our proof of (iii) will be complete provided we show how (a) leads to estimate (4.11). Indeed for each $t > 0$ and $q \in (1, s)$, by the $L^q(\mathbb{R}^n)$ boundedness of $\nabla L^{-1/2}$,

$$\left\|\sqrt{t}\nabla e^{-tL} f\right\|_q = \left\|\sqrt{t}\nabla L^{-1/2} L^{1/2} e^{-tL} f\right\|_q \lesssim \left\|\sqrt{t} L^{1/2} e^{-tL} f\right\|_q.$$

Now by the analyticity of $\left\{e^{-tL}\right\}_{t>0}$ ([85] p74, Theorem 6.13), one therefore obtains

$$\left\|\sqrt{t}L^{1/2}e^{-tL}f\right\|_q \lesssim \|f\|_q$$

as required. To complete the proof of (a) $\Rightarrow$ (b), we remark that since $m \geq n/2$ our constants α_j satisfy $\sum_j \alpha_j 2^{jn} < \infty$.

Proof of (a) $\Rightarrow$ (c). To prove this implication we invoke Theorem 4.6 again, and appeal this time to the second conclusion with almost the same datum as the previous case. The exception is that we take $m > D_w/2$, where D_w is the doubling order of w instead.

Proof of (b) $\Rightarrow$ (a). Simply take $w \equiv 1$ and observe that $\mathcal{W}_1(1,s) = (1,s)$.

Proof of (c) $\Rightarrow$ (a). For this implication we apply the following extrapolation result due to Auscher and Martell.

Proposition 4.9 ([16] Corollary 4.10)**.** *Let* $0 < p_0 < q_0 \leq \infty$*. Suppose that there exists* $q \in [\,p_0\,,\,q_0\,]$ *(with* $q < \infty$ *if* $q_0 = \infty$*) such that*

$$T : L^q(w) \to L^{q,\infty}(w), \qquad \forall\, w \in \mathcal{A}_{q/p_0} \cap \mathcal{B}_{(q_0/q)'}\,. \tag{4.13}$$

Then for all $p \in (p_0, q_0)$ *we have*

$$T : L^p(w) \to L^{p,\infty}(w), \qquad \forall\, w \in \mathcal{A}_{p/p_0} \cap \mathcal{B}_{(q_0/p)'}\,. \tag{4.14}$$

We apply this Proposition to $T = \nabla L^{-1/2}$ with $p_0 = 1$ and $q_0 = s$. Hence from hypothesis (c), condition (4.13) holds for $q = 1$. We conclude therefore that

$$\nabla L^{-1/2} : L^p(w) \to L^{p,\infty}(w), \qquad \forall\, w \in \mathcal{A}_p \cap \mathcal{B}_{(s/p)'} \tag{4.15}$$

for every $p \in (1,s)$. This implies, by setting $w \equiv 1$, that $\nabla L^{-1/2}$ is weak(p,p) for each $p \in (1,s)$ and hence by interpolation is bounded on $L^p(\mathbb{R}^n)$ for every $p \in (1,s)$, which is (a). $\square$

Proof of Theorem 4.2. The proof of Theorem 4.2 follows that of Theorem 4.1 with $V^{1/2}L^{-1/2}$ in place of $\nabla L^{-1/2}$. The main modification is that in place of Lemma 4.8 we use the following.

Lemma 4.10. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 1$ and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. Assume that for some $q > 2$ there exists $C_q > 0$ such that*

$$\left\| \sqrt{t} V^{1/2} e^{-tL} \right\|_{L^q \to L^q} \leq C_q \,. \tag{4.16}$$

Then for each $p \in [1, q)$ there exists $C_p > 0$ such that for each $m \geq 1$, $j \geq 2$

$$\left(\fint_{U_j(B)} \left| V^{1/2} L^{-1/2} (I - e^{-r_B^2 L})^m f \right|^p \right)^{1/p} \leq C_p 4^{-jm} \fint_B |f| \tag{4.17}$$

for all balls B and $f \in L^1(B)$.

The proof of this lemma is almost the same as the proof for Lemma 4.8 and we omit the details. $\square$

Proof of Lemma 4.8. It is known that (4.12) holds for $p = 2$ ([6] Proposition 2.4). Hence (4.12) holds for all $p \in [1, 2)$. To see this we simply apply Hölder's inequality (with exponents $2/p$ and its conjugate $2/(2-p)$) to the left hand side of (4.12), and then invoke the estimate for $p = 2$. It remains to prove (4.12) for $p \in (2, q)$. The argument given here follows that of [13] p944.

The first step is to show that (4.11) leads to the following: there exists $C > 0$ such that for all $y \in \mathbb{R}^n$ and $t > 0$,

$$\|\nabla p_t(\cdot, y)\|_q \leq \frac{C}{t^{1/2+n/2-n/2q}} \,. \tag{4.18}$$

Firstly, the semigroup property (2.3) implies that $\nabla_x p_{2t}(x, y) = \nabla_x e^{-tL} p_t(x, y)$. Therefore by (4.11),

$$\|\nabla_x p_{2t}(\cdot, y)\|_q \;=\; \left\|\nabla_x e^{-tL} p_t(\cdot, y)\right\|_q \;\lesssim\; \frac{1}{\sqrt{t}} \|p_t(\cdot, y)\|_q \,.$$

Now using the Gaussian upper bounds (3.1) for $p_t(x, y)$, we obtain

$$\|p_t(\cdot, y)\|_q^q \;\leq\; \frac{C}{t^{qn/2}} \int e^{-q|x-y|^2/ct} dx \;\leq\; \frac{C}{t^{nq/2-n/2}},$$

and combining this with the previous estimate gives (4.18).

The second step is to obtain the following weighted estimate for $p \in (2, q)$: there exists $\gamma_p > 0$ and $C_p > 0$ such that for all $y \in \mathbb{R}^n$ and $t > 0$,

$$\left\|\nabla p_t(\cdot, y) e^{\gamma_p |\cdot - y|^2/t}\right\|_p \leq \frac{C_p}{t^{1/2+n/2-n/2p}} \,. \tag{4.19}$$

This is known to hold for $p = 2$ (see Lemma 3.1). We shall obtain (4.19) by interpolating between the case $p = 2$ and estimate (4.18). Fix $p \in (2, q)$ and let γ_2 be the constant from the case $p = 2$. Define $\gamma_p := \gamma_2(q-p)/(q-2)$. Note that

$$p \;=\; 2\,\frac{q-p}{q-2} + q\frac{p-2}{q-2}, \qquad\qquad 0 < \frac{q-p}{q-2} < 1, \qquad\qquad 1 < \frac{q-2}{q-p} < \infty\,.$$

We apply Hölder's inequality with exponents $(q-2)/(q-p)$ and $((q-2)/(q-p))' = (q-2)/(p-2)$ to obtain

$$\begin{aligned}
&\int |\nabla_x p_t(x,y)|^p\, e^{\gamma_p|x-y|^2/t} dx \\
&\qquad = \int |\nabla_x p_t(x,y)|^{2\frac{q-p}{q-2}}\, e^{\gamma_p|x-y|^2/t}\, |\nabla_x p_t(x,y)|^{q\frac{p-2}{q-2}}\, dx \\
&\qquad \le \Big(\int |\nabla_x p_t(x,y)|^2\, e^{\gamma_2|x-y|^2/t} dx\Big)^{\frac{q-p}{q-2}} \Big(\int |\nabla_x p_t(x,y)|^q\, dx\Big)^{\frac{p-2}{q-2}} \\
&\qquad \le C\big(t^{-1-n/2}\big)^{\frac{q-p}{q-2}} \big(t^{-q/2-nq/2+n/2}\big)^{\frac{p-2}{q-2}} \\
&\qquad = Ct^{-p/2-np/2+n/2}
\end{aligned}$$

and the required estimate follows.

In the third and final step we prove that the weighted estimate (4.19) for some $p \in (2, q)$ leads to (4.12) for the same p. Fix a ball B, $f \in L^1(B)$, $m \ge 1$, and $p \in (2, q)$. Then for each $j \ge 2$, $y \in B$ and $x \in U_j(B)$ one has $|x - y| \ge 2^j r_B/4$. This combined with (4.19) gives

$$\begin{aligned}
\int_{U_j(B)} |\nabla_x p_t(x,y)|^p\, dx &= \int_{U_j(B)} |\nabla_x p_t(x,y)|^p\, e^{\gamma_p|x-y|^2/t} e^{-\gamma_p|x-y|^2/t} dx \\
&\le e^{-c4^j r_B^2/t} \big\|\nabla_x p_t(\cdot,y) e^{\gamma_p|\cdot-y|^2/t}\big\|_p^p \\
&\le \frac{C}{t^{p/2+np/2-n/2}} e^{-c4^j r_B^2/t}\,.
\end{aligned}$$

Next we write

$$\nabla L^{-1/2}(I - e^{r_B^2 L})^m f = \int_0^\infty g_{r_B}(t) \nabla e^{-tL} f\, dt$$

where $g_r : \mathbb{R}^+ \to \mathbb{R}$ is a function such that (see [13] p931)

$$\int_0^\infty |g_r(t)|\, e^{-c4^j r^2/t} \frac{dt}{\sqrt{t}} \le C_m 4^{-jm}.$$

By Minkowski's inequality and the previous estimate,

$$
\begin{aligned}
&\Big(\fint_{U_j(B)} \big|\nabla L^{-1/2}(I-e^{-r_B^2 L})^m f\big|^p dx\Big)^{1/p} \\
&\qquad = \frac{1}{|2^j B|^{1/p}} \Big\| \int_0^\infty g_{r_B}(t) \nabla e^{-tL} f\, dt \Big\|_{L^p(U_j(B))} \\
&\qquad \le \frac{1}{|2^j B|^{1/p}} \int_0^\infty |g_{r_B}(t)| \int_B |f(y)| \Big(\int_{U_j(B)} |\nabla_x p_t(x,y)|^p\, dx\Big)^{1/p} dy\, dt \\
&\qquad \le \frac{C}{|2^j B|^{1/p}} \int_0^\infty |g_{r_B}(t)| \int_B |f(y)|\ dy \frac{e^{-c4^j r_B^2/t}}{t^{1/2+n/2-n/2p}} dt \\
&\qquad = C\Big(\int_0^\infty |g_{r_B}(t)|\, \frac{|B|}{|2^j B|^{1/p}} \frac{1}{t^{n/2(1-1/p)}} e^{-c4^j r_B^2/t} \frac{dt}{\sqrt{t}}\Big) \Big(\fint_B |f| \Big) \\
&\qquad \le C 2^{-jn/p} \Big(\int_0^\infty |g_{r_B}(t)| \Big(\frac{r_B}{\sqrt{t}}\Big)^{n(1-1/p)} e^{-c4^j r_B^2/t} \frac{dt}{\sqrt{t}}\Big)\Big(\fint_B |f| \Big).
\end{aligned}
$$

By absorbing $\big(2^j r_B/\sqrt{t}\big)^{n(1-1/p)}$ into another exponential with some constant $c' < c$, and applying Hölder's inequality with any $p_0 \ge 1$, we obtain

$$
\begin{aligned}
\Big(\fint_{U_j(B)} \big|\nabla L^{-1/2}(I-e^{-r_B^2 L})^m f\big|^p dx\Big)^{1/p} &\le C \int_0^\infty |g_{r_B}(t)|\, e^{-c'4^j r_B^2/t} \frac{dt}{\sqrt{t}} \fint_B |f| \\
&\le C 4^{-jm} \fint_B |f| \ \ \le\ \ C 4^{-jm} \Big(\fint_B |f|^{p_0} \Big)^{1/p_0}.
\end{aligned}
$$

This gives (4.12) and our proof of Lemma 4.8 is complete. □

4.3 Second order Riesz Transforms

In this section we study the second-order Riesz transforms $\nabla^2 L^{-1}$ and VL^{-1} associated to $L = -\Delta + V$ under the additional condition that V belongs to some reverse Hölder class. Our main task is to give the proof of Theorem 4.3.

Firstly the weighted estimates in Proposition 3.7 allow us to obtain an analogue of Lemma 4.8.

Lemma 4.11. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \ge 2$. Set $q_+ := \sup\big\{q > \frac{n}{2} : V \in \mathcal{B}_q\big\}$. Then for each $j \ge 2$, $m \ge 1$, $p \in (1, q_+)$, ball B, and $f \in L^1(B)$ we have*

$$
\Big(\fint_{U_j(B)} |\nabla^2 L^{-1}(I-e^{-r_B^2 L})^m f|^p \Big)^{1/p} \le C e^{-c4^j} \fint_B |f| \ , \tag{4.20}
$$

$$\Big(\fint_{U_j(B)} |VL^{-1}(I - e^{-r_B^2 L})^m f|^p \Big)^{1/p} \le C e^{-c4^j} \fint_B |f| \,. \tag{4.21}$$

Proof. We first prove (4.21). The first step is to write, using the binomial theorem,

$$\begin{aligned} VL^{-1}(I - e^{-r_B^2 L})^m &= \sum_{k=0}^m \binom{m}{k} (-1)^k \int_0^\infty V e^{-(kr_B^2 + t)L} \, dt \\ &= \sum_{k=0}^m \binom{m}{k} (-1)^k \int_0^\infty V e^{-tL} \mathbf{1}_{(kr_B^2, \infty)}(t) \, dt \\ &= \int_0^\infty h_{r_B}(t) V e^{-tL} \, dt \end{aligned}$$

where

$$h_r(t) := \sum_{k=0}^m (-1)^k \binom{m}{k} \mathbf{1}_{(kr_B^2, \infty)}(t).$$

Now observing that $\sum_{k=0}^m (-1)^k \binom{m}{k} = 0$ we can write

$$\begin{aligned} h_r(t) = &\sum_{k=0}^m (-1)^k \binom{m}{k} \mathbf{1}_{(mr^2, \infty)}(t) + \sum_{k=0}^m (-1)^k \binom{m}{k} \mathbf{1}_{(kr^2, mr^2]}(t) \\ = &\sum_{k=0}^m (-1)^k \binom{m}{k} \mathbf{1}_{(kr^2, mr^2]}(t) \,. \end{aligned}$$

Therefore

$$|h_r(t)| \;\le\; \sum_{k=0}^m \binom{m}{k} \mathbf{1}_{(0, mr^2]}(t) \le 2^m \mathbf{1}_{(0, mr^2]}(t) \,.$$

Now by Minkowski's inequality,

$$\begin{aligned} \big\| VL^{-1}(I - e^{-r_B^2 L})^m f \big\|_{L^p(U_j(B))} &= \Big\| \int_0^\infty h_{r_B}(t) V e^{-tL} f \, dt \Big\|_{L^p(U_j(B))} \\ &\le \int_0^\infty |h_{r_B}(t)| \, \big\| V e^{-tL} f \big\|_{L^p(U_j(B))} \, dt \\ &\le \int_0^\infty |h_{r_B}(t)| \int_B |f(y)| \, \| V(\cdot) p_t(\cdot, y) \|_{L^p(U_j(B))} \, dy \, dt \,. \end{aligned}$$

Next for each $y \in B$ and $t > 0$, by estimate (3.12),

$$\begin{aligned} \| V(\cdot) p_t(\cdot, y) \|_{L^p(U_j(B))} &\le \Big(\int_{U_j(B)} |V(x) p_t(x,y)|^p \, e^{\beta_p \frac{|x-y|^2}{t}} e^{-\beta_p \frac{|x-y|^2}{t}} \, dx \Big)^{1/p} \\ &\le \sup_{x \in U_j(B)} e^{-\beta_p \frac{|x-y|^2}{t}} \big\| V(\cdot) p_t(\cdot, y) e^{\beta_p \frac{|\cdot - y|^2}{t}} \big\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \frac{1}{t^{1+n/2-n/2p}} e^{-c4^j r_B^2 / t} \,. \end{aligned}$$

Therefore one has

$$\begin{aligned}\Big(\fint_{U_j(B)}\big|VL^{-1}(I-e^{-r_B^2L})^m f\big|^p\Big)^{1/p} &= \frac{1}{|2^jB|^{1/p}}\|VL^{-1}(I-e^{-r_B^2L})^m f\|_{L^p(U_j(B))}\\ &\lesssim \Big(\int_0^\infty |h_{r_B}(t)|\,\frac{|B|}{|2^jB|^{1/p}}\frac{e^{-c4^jr_B^2/t}}{t^{1+n/2-n/2p}}\,dt\Big)\Big(\fint_B |f|\Big)\\ &\lesssim \Big(\int_0^{mr_B^2} \frac{|B|}{|2^jB|^{1/p}}\frac{e^{-c4^jr_B^2/t}}{t^{1+n/2-n/2p}}\,dt\Big)\Big(\fint_B |f|\Big).\end{aligned}$$

Since

$$\frac{|B|}{|2^jB|^{1/p}} \approx \frac{r_B^n}{(2^{jn}r_B^n)^{1/p}} \approx \frac{r_B^{n(1-1/p)}}{2^{jn/p}} = 2^{-jn}(2^jr_B)^{n(1-1/p)},$$

then it follows that for some $\epsilon > 0$

$$\frac{|B|}{|2^jB|^{1/p}}\frac{1}{t^{n/2(1-1/p)}} \lesssim 2^{-jn}\Big(\frac{2^jr_B}{\sqrt{t}}\Big)^{n(1-1/p)} \leq \Big(\frac{2^jr_B}{\sqrt{t}}\Big)^{n(1-1/p)} \lesssim e^{\epsilon 4^jr_B^2/t}.$$

Collecting these estimates we obtain

$$\begin{aligned}\Big(\fint_{U_j(B)}\big|VL^{-1}(I-e^{-r_B^2L})^m f\big|^p\Big)^{1/p} &\lesssim \Big(\int_0^{mr_B^2} e^{-c'4^jr_B^2/t}\frac{dt}{t}\Big)\Big(\fint_B |f|\Big)\\ &\lesssim e^{-c4^j/m}\Big(\int_0^{mr_B^2}\frac{t}{4^jr_B^2}\frac{dt}{t}\Big)\Big(\fint_B |f|\Big)\\ &\lesssim e^{-c4^j/m}\fint_B |f|\end{aligned}$$

provided $m > 0$.

The proof of (4.20) is similar but uses (3.11) in place of (3.12) and we omit the details. □

We are now ready to give the

Proof of Theorem 4.3. The proofs of (a) and (b) are contained in the conclusions of Theorem 4.6 provided we can show conditions Theorem 4.6 (i)-(iii) hold for the following: $p_0 = 1$, $q_0 = q_+$, T one of ∇^2L^{-1} or VL^{-1}, and $A_B = I-(I-e^{-r_B^2L})^m$ for m large enough. We need to take $m > n/2$ for conclusion (a), and $m > D_w/2$ for conclusion (b), where D_w is the doubling order of the chosen weight w.

From the work of [93] and also [12] (see Theorems 1.2 and 1.3) we know that the operators ∇^2L^{-1} and VL^{-1} are bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, q_+)$, and hence Theorem 4.6 (i) holds.

The proof of condition (ii) can be found in the proof of Theorem 4.1, and condition (iii) is contained in Lemma 4.11. $\square$

Chapter 5

Weighted Lebesgue spaces II: weights adapted to the Schrödinger operator

In this chapter we extend the results from Chapter 4 to weighted spaces with weights adapted to the Schrödinger operator. These weights were introduced in [30] and further investigated in [104, 105, 106]. These weights form a larger class of weights than the $\mathcal{A}_\infty$ class. See Definition 5.2 below.

The main result of this chapter is

Theorem 5.1. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 3$. Assume that $V \in \mathcal{B}_s$ for some $s > n/2$. Then the operators $\nabla^2 L^{-1}$ and VL^{-1} are bounded on $L^p(w)$ for each $p \in (1, s)$ and $w^{1-p'} \in \mathcal{A}^L_{p'/s'}$.*

We give some brief remarks on this result. Firstly, the condition $w^{1-p'} \in \mathcal{A}^L_{p'/s'}$ can be equivalently expressed as $w \in \mathcal{A}^L_p \cap \mathcal{B}^L_{(s/p)'}$. Another formulation (see Section 1.1.1 (iii) and also Theorem 1.7) of the hypothesis $p \in (1, s)$ and $w^{1-p'} \in \mathcal{A}^L_{p'/s'}$ is the statement: $w \in \mathcal{A}^L_\infty$ and $p \in \mathcal{W}^L_w(1, s)$. Note also that the result for VL^{-1} is known [106] but the result for $\nabla^2 L^{-1}$ is new. Unlike the situation for $\mathcal{A}_\infty$ weights, we are not able to pass from the result for VL^{-1} to $\nabla^2 L^{-1}$ easily. See the calculation in (1.13) and item (iii) in Section 1.1.1 of this thesis. Finally the techniques developed in this chapter also allow us to give new proofs of boundedness of the first-order Riesz transforms $\nabla L^{-1/2}$ and $V^{1/2} L^{-1/2}$ (which are known to hold in [30, 106]), but we do not give the details here.

This chapter is organised as follows. In Section 5.1 we give the appropriate definitions and collect some useful estimates related to these weights. In Section 5.2 we develop the good-λ

techniques needed to prove the main result, and also give some applications of these techniques. The proof of Theorem 5.1 is given in Section 5.3. In the final section we give an alternate proof of Theorem 5.1 for the operator VL^{-1}, using the approach in [30].

5.1 Weights adapted to Schrödinger operators

In this section we define weights adapted to Schrödinger operators and give some of their properties. Throughout this chapter we use the following notation. For a given ball B and a number $\theta \geq 0$, we set

$$\psi_\theta(B) := \Big(1 + \frac{r_B}{\gamma(x_B)}\Big)^\theta.$$

Here $\gamma : \mathbb{R}^n \to (0, \infty)$ is the auxiliary weight function defined in Definition 2.2. Observe that for any $\lambda \geq 1$, we have $\psi_\theta(B) \leq \psi_\theta(\lambda B) \leq \lambda^\theta \psi_\theta(B)$. We will also often interchange balls with cubes. In this case if Q is a cube, the expression for $\psi_\theta(Q)$ is the same as above but with r_B replaced by l_Q (the sidelength of Q), and x_B replaced by x_Q (the centre of Q).

The following maximal operator was first defined in [31, 106] and will be an essential tool throughout this chapter. For each $\theta \geq 0$, we set

$$\mathcal{M}_\theta^L f(x) := \sup_{B \ni x} \frac{1}{\psi_\theta(B)} \fint_B |f(y)| \, dy.$$

We mention here that f is pointwise controlled by $\mathcal{M}_\theta^L f$. Indeed, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\theta \geq 0$, we have for almost every x,

$$|f(x)| \;\; \leq \;\; 2^\theta \mathcal{M}_\theta^L f(x). \tag{5.1}$$

To see this, we let $r \leq \gamma(x)$ and observe that

$$\fint_{B(x,r)} |f| \;\; \leq \;\; \psi_\theta(B(x,r))\, \mathcal{M}_\theta^L f(x) \;\; \leq \;\; 2^\theta \mathcal{M}_\theta^L f(x).$$

Now let $r \to 0$ and apply Lebesgue's differentiation theorem (see [100]) to obtain (5.1).

Definition 5.2 (Weights adapted to the Schrödinger operator)**.** *Let w be a non-negative locally integrable function. For $p \in (1, \infty)$ and $\theta \geq 0$, we say that $w \in \mathcal{A}_p^{L,\theta}$ if there exists $C =$*

$C(w, \theta, p) > 0$ *such that for all balls* B,

$$\Big(\fint_B w \Big)^{1/p} \Big(\fint_B w^{1-p'} \Big)^{1/p'} \leq C \psi_\theta(B).$$

We say that $w \in \mathcal{A}_1^{L,\theta}$ *if there exists* $C = C(w, \theta) > 0$ *such that for all balls* B

$$\fint_B w \leq C\, \psi_\theta(B)\, w(x) \qquad\qquad a.e.\ x \in B.$$

For $p \in [1, \infty)$ *we set*

$$\mathcal{A}_p^L := \bigcup_{\theta \geq 0} \mathcal{A}_p^{L,\theta}.$$

We also define $\mathcal{A}_\infty^L := \bigcup\limits_{1 \leq p < \infty} \mathcal{A}_p^L$.

By taking $\theta = 0$ we see that these weights contain the $\mathcal{A}_\infty$ weights. That is, $\mathcal{A}_p \subset \mathcal{A}_p^{L,\theta}$ for every $p \in [1, \infty)$ and every $\theta > 0$. However the inclusion is proper. For example let $V \equiv 1$ and take $w(x) = 1 + |x|^\epsilon$ with $\epsilon > n(p-1)$. Then w is a member of $\mathcal{A}_p^L$ but w does not belong to $\mathcal{A}_p$.

We also introduce a class of reverse Hölder weights adapted to the Schrödinger operator. As far as we are aware, these classes have not been explicitly defined elsewhere in the literature.

Definition 5.3 (Reverse Hölder weights adapted to the Schrödinger operator)**.** *Let* w *be a non-negative locally integrable function. For* $q \in (1, \infty)$ *and* $\theta \geq 0$, *we say that* $w \in \mathcal{B}_q^{L,\theta}$ *if there exists* $C = C(w, q, \theta) > 0$ *such that for all balls* B,

$$\Big(\fint_B w^q \Big)^{1/q} \leq C\, \psi_\theta(B) \Big(\fint_B w \Big).$$

We say that $w \in \mathcal{B}_\infty^{L,\theta}$ *if there exists* $C = C(w, \theta) > 0$ *such that for all balls* B,

$$w(x) \leq C \psi_\theta(B) \Big(\fint_B w \Big), \qquad\qquad a.e.\ x \in B.$$

For $q \in (1, \infty]$ *we set*

$$\mathcal{B}_q^L := \bigcup_{\theta \geq 0} \mathcal{B}_q^{L,\theta}.$$

We remark that in the definitions one can interchange balls by cubes and obtain the same classes of weights.

The next property of the $\mathcal{B}_q^L$ classes is an analogue of Lemma 2.11.

Lemma 5.4. *Let $w \in \mathcal{B}_{s'}^{L,\theta}$ for some $\theta \geq 0$ and $1 \leq s \leq \infty$. Then there exists $C_w > 0$ such that for any cube Q and measurable $E \subset Q$,*

$$\frac{w(E)}{w(Q)} \leq C_w\, \psi_\theta(Q) \left(\frac{|E|}{|Q|}\right)^{1/s}.$$

Proof. If $s' < \infty$ then by Hölder's inequality with exponents s' and s,

$$\begin{aligned}\frac{w(E)}{w(Q)} = \frac{|Q|}{w(Q)} \frac{1}{|Q|} \int_E w &\leq \frac{|Q|}{w(Q)} \Big(\fint_Q w^{s'}\Big)^{1/s'} \left(\frac{|E|}{|Q|}\right)^{1/s} \\ &\leq C_w \frac{|Q|}{w(Q)} \psi_\theta(Q) \Big(\fint_Q w\Big) \left(\frac{|E|}{|Q|}\right)^{1/s} \\ &= C_w \psi_\theta(Q) \left(\frac{|E|}{|Q|}\right)^{1/s}.\end{aligned}$$

If $s' = \infty$ then the same conclusion holds. $\square$

As in the classical situation these two weight classes are intimately connected. It was shown in [30] that if $w \in \mathcal{A}_p^L$ for some $p \in [1, \infty)$, then $w \in \mathcal{B}_q^{L,\theta}$ for some $q > 1$ and $\theta \geq 0$ (see [30] Lemma 5). We give a more explicit statement of this connection in the next result, itself modelled on [12] Proposition 11.1.

Lemma 5.5. *Let $w \geq 0$ be a measurable function. Then the following are equivalent.*

(a) $w \in \mathcal{A}_\infty^L$.

(b) *For all $\sigma \in (0, 1)$, $w^\sigma \in \mathcal{B}_{1/\sigma}^L$.*

(c) *There exists $\sigma \in (0, 1)$ such that $w^\sigma \in \mathcal{B}_{1/\sigma}^L$.*

Proof. If $w^\sigma \in \mathcal{B}_{1/\sigma}^L$ for some $\sigma \in (0, 1)$, then the self improvement property of these classes (see Lemma 5.6 (v) below) implies that $w^\sigma \in \mathcal{B}_{1/\sigma+\varepsilon}^L$ for some $\varepsilon > 0$. Therefore $w \in \mathcal{B}_{1+\sigma\varepsilon}^L$, which implies that $w \in \mathcal{A}_\infty^L$. Hence we have (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a).

We now show (a) $\Longrightarrow$ (b). Let $w \in \mathcal{A}_\infty^L$ and $\sigma \in (0, 1)$. Then $w \in \mathcal{B}_r^{L,\theta}$ for some $r > 1$ and $\theta \geq 0$ (by [30] Lemma 5). Therefore for any $\alpha > 1$ and cube Q, the set

$$E_Q := \left\{x \in Q : \ w^\sigma(x) > \alpha \fint_Q w^\sigma\right\}$$

satisfies, by Lemma 5.4,

$$\frac{w(E_Q)}{w(Q)} \;\leq\; C\psi_\theta(Q)\Big(\frac{|E_Q|}{|Q|}\Big)^{1/r'}.$$

Then it follows that

$$|E_Q| \;=\; \frac{1}{\alpha}\int_{E_Q} \alpha\, dx \;<\; \frac{1}{\alpha}\int_{E_Q} \frac{w^\sigma}{\fint_Q w^\sigma} dx \;\leq\; \frac{|Q|}{\alpha}.$$

Hence we obtain that

$$w(E_Q) \;\leq\; C\,\alpha^{-1/r'}\psi_\theta(Q)\,w(Q).$$

We choose α such that $C\alpha^{-1/r'}\psi_\theta(Q) = 1/2$ (note that $\alpha > 1$). Next, observe that for each $x \in Q\backslash E_Q$ we have $w(x) \leq \Big(\alpha \fint_Q w^\sigma\Big)^{1/\sigma}$. Therefore

$$\begin{aligned}
\int_Q w\,dx &= \int_{E_Q} w\,dx + \int_{Q\backslash E_Q} w\,dx \\
&\leq \frac{1}{2}\int_Q w\,dx + \Big(\alpha \fint_Q w^\sigma\Big)^{1/\sigma} \int_{Q\backslash E_Q} dx \\
&\leq \frac{1}{2}\int_Q w\,dx + \alpha^{1/\sigma}\,|Q|\,\Big(\fint_Q w^\sigma\Big)^{1/\sigma}.
\end{aligned}$$

Rearranging this statement gives us

$$\fint_Q w\,dx \;\leq\; 2\alpha^{1/\sigma}\Big(\fint_Q w^\sigma\Big)^{1/\sigma} \;=\; 2^{r'/\sigma+1}\,C^{r'/\sigma}\,\psi_{\theta r'/\sigma}(Q)\Big(\fint_Q w^\sigma\Big)^{1/\sigma}.$$

That is, $w^\sigma \in \mathcal{B}^{L,\theta r'}_{1/\sigma} \subset \mathcal{B}^{L}_{1/\sigma}$. □

We now describe some further properties of these weights. The reader may find it useful to compare these with those in Proposition 2.9 (and also Remark 4.7 (b) and (c)).

Lemma 5.6. *One has*

(i) *For each* $\theta \geq 0$, *if* $1 \leq p_1 \leq p_2 < \infty$ *then* $\mathcal{A}^{L,\theta}_1 \subset \mathcal{A}^{L,\theta}_{p_1} \subset \mathcal{A}^{L,\theta}_{p_2}$.

(ii) *For each* $\theta \geq 0$, *if* $1 < p_1 \leq p_2 \leq \infty$ *then* $\mathcal{B}^{L,\theta}_{p_1} \supset \mathcal{B}^{L,\theta}_{p_2} \supset \mathcal{B}^{L,\theta}_{\infty}$.

(iii) *For each* $1 \leq p \leq \infty$ *and* $\theta \geq 0$, $w \in \mathcal{A}^{L,\theta}_p$ *if and only if* $w^{1-p'} \in \mathcal{A}^{L,\theta}_{p'}$.

(iv) *If* $w \in \mathcal{A}^L_p$ *for some* $p \in (1,\infty)$ *then there exists* $p_0 \in (1,p)$ *with* $w \in \mathcal{A}^L_{p_0}$.

(v) *If* $w \in \mathcal{B}_q^L$ *for some* $q \in (1,\infty)$ *then there exists* $q_0 \in (q,\infty)$ *with* $w \in \mathcal{B}_{q_0}^L$.

(vi) *For each* $r \in (1,\infty)$, $w^r \in \mathcal{A}_\infty^L \iff w \in \mathcal{B}_r^L$.

(vii) *Suppose* $w^\sigma \in \mathcal{A}^L_{\sigma(s-1)+1}$ *for some* $\sigma \in (0,\infty)$ *and* $s \in [1,\infty)$. *Then* $w \in \mathcal{A}_s^L$ *if and only if* $w \in \mathcal{A}_\infty^L$.

(viii) *For each* $1 \le p \le \infty$ *and* $1 \le q < \infty$, *we have*

$$w^q \in \mathcal{A}^L_{q(p-1)+1} \qquad \iff \qquad w \in \mathcal{A}_p^L \cap \mathcal{B}_q^L\,.$$

(ix) *Suppose* $p_0 < p < q_0$ *and* $w \in \mathcal{A}^L_{p/p_0} \cap \mathcal{B}^L_{(q_0/p)'}$. *Then there exists* p_1 *and* q_1 *such that*

$$p_0 < p_1 < p < q_1 < q_0 \qquad \text{and} \qquad w \in \mathcal{A}^L_{\frac{p}{p_1}} \cap \mathcal{B}^L_{\left(\frac{q_1}{p}\right)'}\,.$$

(x) *Given* $p_0 < p < q_0$, *we have*

$$w \in \mathcal{A}^L_{\frac{p}{p_0}} \cap \mathcal{B}^L_{\left(\frac{q_0}{p}\right)'} \qquad \iff \qquad w^{1-p'} \in \mathcal{A}^L_{\frac{p'}{q_0'}} \cap \mathcal{B}^L_{\left(\frac{p_0'}{p'}\right)'}\,.$$

Proof. The proofs of (i), (ii) and (iii) follow easily from the definition of the $\mathcal{A}_p^L$ and $\mathcal{B}_q^L$ classes. For the proof of (iv) see [30] and also [104] Proposition 2.1 (iii). Property (v) is the self-improvement property of the $\mathcal{B}_q^L$ classes mentioned in [30]. Property (vi) is a restatement of Lemma 5.5. Indeed by replacing $1/\sigma$ by r and w^σ by w in Lemma 5.5 we obtain (vi).

The proofs of the next two properties are adapted from [59] and [72].

Proof of (vii). We note that $\mathcal{A}_s^L \subset \mathcal{A}_\infty^L$ for every $s \ge 1$, and so necessity is clear. It suffices to consider the converse. Let $w \in \mathcal{A}_\infty^L$. Suppose firstly that $0 < \sigma < 1$. Since $w \in \mathcal{A}_\infty^L$ then by Lemma 5.5 (or property (vi) above) we have $w^\sigma \in \mathcal{B}^{L,\theta}_{1/\sigma}$ for some $\theta \ge 0$. This means that for any ball B,

$$\Big(\fint_B w\Big)^\sigma = \Big(\fint_B (w^\sigma)^{1/\sigma}\Big)^\sigma \le C\,\psi_\theta(B)\fint_B w^\sigma\,.$$

Let $r := \sigma(s-1)+1$. Then since $w^\sigma \in \mathcal{A}_r^L$,

$$\Big(\fint_B w\Big)\Big(\fint_B w^{-1/(s-1)}\Big)^{s-1} \le C\,\psi_{\theta/\sigma}(B)\Big(\fint_B w^\sigma\Big)^{1/\sigma}\Big(\fint_B w^{-1/(s-1)}\Big)^{s-1}$$

$$= C\,\psi_{\theta/\sigma}(B)\Big(\fint_B w^\sigma\Big)^{1/\sigma}\Big(\fint_B (w^\sigma)^{-1/(r-1)}\Big)^{(r-1)/\sigma}$$
$$\le C\,\psi_{(r+1)\theta/\sigma}(B)\,.$$

That is, $w \in \mathcal{A}_s^{L,(r+1)\theta/(s\sigma)} \subset \mathcal{A}_s^L$. Suppose now that $1 \le \sigma < \infty$. Let B be a ball and $r := \sigma(s-1)+1$. Note $w^\sigma \in \mathcal{A}_r^L$ implies that $w^\sigma \in \mathcal{A}_r^{L,\theta}$ for some $\theta \ge 0$. Since $\sigma \ge 1$ we may apply Hölder's inequality with exponents σ and σ' to get

$$\begin{aligned}\Big(\fint_B w\Big)\Big(\fint_B w^{-1/(s-1)}\Big)^{s-1} &\le \Big(\fint_B w^\sigma\Big)^{1/\sigma}\Big(\fint_B w^{-1/(s-1)}\Big)^{s-1}\\ &= \Big(\fint_B w^\sigma\Big)^{1/\sigma}\Big(\fint_B (w^\sigma)^{-1/(r-1)}\Big)^{(r-1)/\sigma}\\ &\le C\,\psi_{r\theta/\sigma}(B)\,.\end{aligned}$$

We have shown that $w \in \mathcal{A}_s^{L,\theta r/(\sigma s)} \subset \mathcal{A}_s^L$. This concludes the proof of (vii).

Proof of (viii). We first show the $\Longrightarrow$ direction . Assume that $w^q \in \mathcal{A}^L_{q(p-1)+1}$. Then $w^q \in \mathcal{A}^L_\infty$, and by property (vi) above $w \in \mathcal{B}^L_q$. If in addition $w \in \mathcal{A}^L_\infty$, then applying property (vii) with $\sigma = q$ and $s = p$ we obtain $w \in \mathcal{A}^L_p$. We now prove the converse $\Longleftarrow$ direction. Assume that $w \in \mathcal{A}^L_p \cap \mathcal{B}^L_q$. Then $w \in \mathcal{B}^L_q$ and this implies, by property (vi), that $w^q \in \mathcal{A}^L_\infty$. Hence $(w^q)^{1/q} = w \in \mathcal{A}^L_p$, and property (vii) with $\sigma = 1/q$ and $p = \sigma(s-1)+1$ gives $w^q \in \mathcal{A}^L_s \equiv \mathcal{A}^L_{q(p-1)+1}$.

Proof of (ix). Firstly, property (iv) implies there exists p_1 such that

$$1 < \frac{p}{p_1} < \frac{p}{p_0} \qquad \text{and} \qquad w \in \mathcal{A}^L_{\frac{p}{p_1}}\,.$$

This implies $p_0 < p_1 < p$. Secondly, property (v) implies there exists q_1 such that

$$\Big(\frac{q_0}{p}\Big)' < \Big(\frac{q_1}{p}\Big)' < \infty \qquad \text{and} \qquad w \in \mathcal{B}^L_{\left(\frac{q_1}{p}\right)'}\,.$$

This implies $p < q_1 < q_0$.

Proof of (x). The proof is almost the same as that of Lemma 4.4 from [16]. We give the details here for convenience. Set $q = \big(\frac{q_0}{p}\big)'\big(\frac{p}{p_0}-1\big)+1$. Using properties (iii) and (viii), we have

$$w \in \mathcal{A}^L_{\frac{p}{p_0}} \cap \mathcal{B}^L_{\left(\frac{q_0}{p}\right)'} \iff w^{\left(\frac{q_0}{p}\right)'} \in \mathcal{A}^L_{\left(\frac{q_0}{p}\right)'\left(\frac{p}{p_0}-1\right)+1} \equiv \mathcal{A}^L_q \iff w^{\left(\frac{q_0}{p}\right)'(1-q')} \in \mathcal{A}^L_{q'}$$

and

$$w^{1-p'} \in \mathcal{A}^{L}_{\frac{p'}{q_0'}} \cap \mathcal{B}^{L}_{\left(\frac{p_0'}{p'}\right)'} \qquad \Longleftrightarrow \qquad w^{(1-p')\left(\frac{p_0'}{p'}\right)'} \in \mathcal{A}^{L}_{\left(\frac{p_0'}{p'}\right)'\left(\frac{p'}{q_0'}-1\right)+1}.$$

Direct computations show

$$\left(\frac{q_0}{p}\right)'(1-q') = (1-p')\left(\frac{p_0'}{p'}\right)' \qquad \text{and} \qquad q' = \left(\frac{p_0'}{p'}\right)'\left(\frac{p'}{q_0'}-1\right)+1.$$

□

The following weak type property of the operator $\mathcal{M}^L$ is implicit throughout [106], but we supply a proof here.

Lemma 5.7. *For each $\eta \geq 0$, the operator $\mathcal{M}^L_\eta$ is weak type (p,p) for every $p \in [1,\infty)$.*

Proof. We observe that $\mathcal{M}^L_\eta$ is controlled pointwise by M, the Hardy–Littlewood maximal function. Indeed, for each x and ball B containing x, we have

$$\frac{1}{\psi_\eta(B)} \fint_B |f| \leq \fint_B |f| \leq Mf(x).$$

Hence the weak type properties of M carry over to $\mathcal{M}^L_\eta$, since

$$\left\{x \in \mathbb{R}^n : \mathcal{M}^L_\eta f(x) > \lambda\right\} \subseteq \left\{x \in \mathbb{R}^n : Mf(x) > \lambda\right\}.$$

Therefore

$$\left|\left\{x \in \mathbb{R}^n : \mathcal{M}^L_\eta f(x) > \lambda\right\}\right| \;\leq\; \left|\left\{x \in \mathbb{R}^n : Mf(x) > \lambda\right\}\right| \;\leq\; \frac{C_p}{\lambda^p}\left\|f\right\|^p_{L^p}.$$

In fact the weak (p,p) bound of $\mathcal{M}^L_\eta$ is controlled by that of M. □

The main mapping property of the operator $\mathcal{M}^L$ we will require is the following.

Lemma 5.8 ([106] Theorem 2.2)**.** *Let $p \in (1,\infty)$ and $\theta \geq 0$. Then for each $w \in \mathcal{A}^{L,\theta}_p$ there exists $C > 0$ such that*

$$\left\|\mathcal{M}^L_{p'\theta} f\right\|_{L^p(w)} \leq C\left\|f\right\|_{L^p(w)}.$$

Proof. A proof of this can be found in [106] Theorem 2.2. See also [104] Lemma 2.2. □

Remark 5.9. As a consequence of Lemma 5.8, if $p > s$, $w \in \mathcal{A}_{p/s}^{L,\theta}$ and $\eta = (p/s)'\theta$ then the operator $\mathcal{M}_\eta^L(|\cdot|^s)^{1/s}$ is bounded on $L^p(w)$. In fact, since $\mathcal{M}_\eta^L$ is bounded on $L^{p/s}(w)$ for each $w \in \mathcal{A}_{p/s}^L\theta$, then we have

$$\left\|\mathcal{M}_\eta^L(|f|^s)^{1/s}\right\|_{L^p(w)}^p \;=\; \int \left(\mathcal{M}_\eta^L |f|^s\right)^{p/s} w \;\lesssim\; \int |f|^p\, w\,.$$

5.2 A new good-λ inequality

The main result of this section is the following extension of Theorem 4.4 to $\mathcal{A}_\infty^L$ weights. It is the key technical tool of this chapter and is also of independent interest.

Theorem 5.10. *Fix $\eta > 0$, $q \in (1, \infty]$, $\xi \geq 1$, $s \in [1, \infty)$, and $\nu \in \mathcal{B}_{s'}^L$. Assume that F,G, and H are non-negative functions on $\mathbb{R}^n$ such that for each ball B with $r_B \leq 12\sqrt{n}\,\gamma(x_B)$, there exist non-negative functions H_B and G_B with*

$$F(x) \;\leq\; H_B(x) + G_B(x) \qquad a.e.\ x \in B, \tag{5.2}$$

$$\fint_B G_B \;\leq\; G(x), \qquad \forall x \in B, \tag{5.3}$$

$$\Big(\fint_B H_B^q\Big)^{1/q} \;\leq\; \xi\left(\mathcal{M}_\eta^L F(x) + H(y)\right), \qquad \forall x, y \in B \tag{5.4}$$

and for each ball B with $r_B > 12\sqrt{n}\,\gamma(x_B)$,

$$\frac{1}{\psi_\eta(B)} \fint_B F \;\leq\; G(x), \qquad \forall x \in B\,. \tag{5.5}$$

Then there exists $C = C(n, q, \nu, \xi, s, \eta, \gamma) > 0$ and $K_0 = K_0(m, \eta, \xi, \gamma) \geq 1$ with the following property: for each $\lambda > 0$, $K \geq K_0$, and $\delta \in (0, 1)$,

$$\begin{aligned}\nu\big(\{x \in \mathbb{R}^n : \mathcal{M}_\eta^L F(x) > K\lambda,\, G(x) \leq \delta\lambda\}\big)& \\ \leq C\Big(\frac{\xi^q}{K^q} + \frac{\delta}{K}\Big)^{1/s} \nu\big(\{x \in \mathbb{R}^n : \mathcal{M}_\eta^L F(x) > \lambda\}\big).&\end{aligned} \tag{5.6}$$

As a consequence, for all $r \in \big(0, q/s\big)$, we have

$$\left\|\mathcal{M}_\eta^L F\right\|_{L^r(\nu)} \;\leq\; C\big(\|G\|_{L^r(\nu)} + \|H\|_{L^r(\nu)}\big) \tag{5.7}$$

provided $\left\|\mathcal{M}_\eta^L F\right\|_{L^r(\nu)} < \infty$. If $r \geq 1$ then (5.7) holds provided $F \in L^1(\mathbb{R}^n)$.

Remark 5.11. We mention that the term H is an error term, which is useful in applications. For instance it allows us to consider commutators (see Theorem 3.16 in [16] for the case of $\mathcal{A}_\infty$ weights). However we do not give any results in this direction in this thesis.

Proof of Theorem 5.10. The proof is an adaptation of the proof of Theorem 4.4 (found in [16] Theorem 3.1). We begin by mentioning that it will suffice to consider the case $G = H$. Indeed if we set $\widetilde{G} := G + H$, then (5.3) holds with $\widetilde{G}$ in place of G and (5.4) holds with $\widetilde{G}$ in place of H. Henceforth we shall assume that $H = G$.

We shall first demonstrate (5.6). Fix $\lambda > 0$ and set

$$\Omega_\lambda := \left\{x \in \mathbb{R}^n : \mathcal{M}_\eta^L F(x) > \lambda\right\}$$

$$E_\lambda := \left\{x \in \mathbb{R}^n : \mathcal{M}_\eta^L F(x) > K\lambda,\ 2G(x) \le \delta\lambda\right\}.$$

Note that Ω_λ is an open set, and hence the Whitney decomposition lemma (see [60]) allows us to decompose it into a family of pairwise disjoint cubes $\mathcal{Q} = \{Q_j\}_j$, with $\Omega_\lambda = \cup_j Q_j$, and such that $4Q_j$ meets Ω_λ^c for every j. Our aim is to show the following estimate: there exists $C > 0$ such that for every j for which $E_\lambda \cap Q_j$ is not empty,

$$\nu(E_\lambda \cap Q_j) \le C\Big(\frac{\xi^q}{K^q} + \frac{\delta}{K}\Big)^{1/s} \nu(Q_j)\,. \tag{5.8}$$

Then since $E_\lambda \subset \bigcup_j E_\lambda \cap Q_j$, we may sum over all the disjoint cubes in $\mathcal{Q}$ to obtain

$$\nu(E_\lambda) \;\le\; \sum_j \nu(E_\lambda \cap Q_j) \;\le\; C\Big(\frac{\xi^q}{K^q} + \frac{\delta}{K}\Big)^{1/s} \sum_j \nu(Q_j) \;=\; C\Big(\frac{\xi^q}{K^q} + \frac{\delta}{K}\Big)^{1/s} \nu(\Omega_\lambda)\,.$$

which is (5.6).

We proceed with the proof of (5.8). We shall consider two regimes.

$$\mathcal{J}_0 := \left\{j : Q_j \in \mathcal{Q} \quad \text{and} \quad l_{Q_j} \le 2\gamma(x_{Q_j})\right\}$$

$$\mathcal{J}_\infty := \left\{j : Q_j \in \mathcal{Q} \quad \text{and} \quad l_{Q_j} > 2\gamma(x_{Q_j})\right\}.$$

We first study the case $j \in \mathcal{J}_0$. For each such j we define B_j to be the ball with the same centre as Q_j but with radius $r_{B_j} = \frac{\sqrt{n}}{2} l_{Q_j}$. (That is, B_j is the 'smallest' ball concentric with and containing Q_j). Our task will be to show that for each $j \in \mathcal{J}_0$ with $E_\lambda \cap Q_j$ non-empty, the following estimate holds:

$$|E_\lambda \cap Q_j| \le C\Big(\frac{\xi^q}{K^q} + \frac{\delta}{K}\Big)\,|Q_j| \tag{5.9}$$

with C depending only on q, n , η, γ, and the weak type bounds of $\mathcal{M}^L_\eta$. (We remark here that if $q = \infty$ then the first term ξ^q/K^q is taken to be zero in inequality (5.9)). Once (5.9) is proven we may obtain (5.8) as follows. Recall that since $\nu \in \mathcal{B}^L_{s'}$, then there exists $\theta \ge 0$ for which $\nu \in \mathcal{B}^{L,\theta}_{s'}$. We then apply Lemma 5.4 to ν, and to the sets $E_\lambda \cap Q_J \subset Q_j$, to obtain

$$\nu(E_\lambda \cap Q_j) \;\le\; C_\nu\,\psi_\theta(Q_j)\Big(\frac{|E_\lambda \cap Q_j|}{|Q_j|}\Big)^{1/s}\nu(Q_j) \;\le\; C\,\Big(\frac{\xi^q}{K^q} + \frac{\delta}{K}\Big)^{1/s}\nu(Q_j)\,. \tag{5.10}$$

Note we have used that $\psi_\theta(Q_j) \le 3^\theta$ since $j \in \mathcal{J}_0$. This gives estimate (5.8).

We proceed with obtaining (5.9). We shall need a localisation lemma whose proof we postpone to the end of the section.

Lemma 5.12. *Fix $\eta > 0$. Then there exists $\widetilde{K_0} > 1$ depending only on n, η, and the growth function γ, with the following property: for each $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, each $\lambda > 0$, each $K \ge \widetilde{K_0}$, and each ball $\widetilde{B}$ for which there exists $\widetilde{x} \in \widetilde{B}$ with $\mathcal{M}^L_\eta f(\widetilde{x}) \le \lambda$,*

$$\big\{x \in \widetilde{B} : \mathcal{M}^L_\eta f(x) > K\lambda\big\} \subset \big\{x \in \mathbb{R}^n : \mathcal{M}^L_\eta\big(f\mathbf{1}_{3\widetilde{B}}\big) > (K/\widetilde{K_0})\lambda\big\}.$$

Now recall that $4Q_j$ meets Ω^c_λ. This means that there exists $x_j \in 4Q_j \subset 4B_j$ with

$$\mathcal{M}^L_\eta F(x_j) \le \lambda. \tag{5.11}$$

Hence applying Lemma 5.12 to the ball $4B_j$ and F implies that there exists $\widetilde{K_0} \ge 1$ so that, for all $K \ge \widetilde{K_0}$,

$$\big\{x \in 4B_j : \mathcal{M}^L_\eta F(x) > K\lambda\big\} \subset \big\{x \in \mathbb{R}^n : \mathcal{M}^L_\eta\big(F\mathbf{1}_{12B_j}\big)(x) > (K/\widetilde{K_0})\lambda\big\} \tag{5.12}$$

Now we observe that the hypotheses (5.2), (5.3), and (5.4) may be applied to the ball $12B_j$ (since $j \in \mathcal{J}_0$) and hence $12B_j$ satisfies

$$r_{12B_j} \;=\; 12r_{B_j} \;=\; 6\sqrt{n}\,l_{Q_j} \;\le\; 12\sqrt{n}\,\gamma(x_{Q_j}) \;=\; 12\sqrt{n}\,\gamma\big(x_{12B_j}\big). \tag{5.13}$$

Combining (5.2) with (5.12), and the fact that $\mathcal{M}^L_\eta$ is sublinear,

$$
\begin{aligned}
|E_\lambda \cap B_j| &\le \left|\{x \in \mathbb{R}^n : \mathcal{M}^L_\eta F(x) > K\lambda\} \cap B_j\right| \\
&\le \left|\{x \in 4B_j : \mathcal{M}^L_\eta F(x) > K\lambda\}\right| \\
&\le \Big|\big\{x \in \mathbb{R}^n : \mathcal{M}^L_\eta\big(F\mathbf{1}_{12B_j}\big) > (K/\widetilde{K_0})\lambda\big\}\Big| \\
&\le \Big|\big\{x \in \mathbb{R}^n : \mathcal{M}^L_\eta\big(G_{12B_j}\mathbf{1}_{12B_j}\big)(x) > (K/2\widetilde{K_0})\lambda\big\}\Big| \\
&\qquad + \Big|\big\{x \in \mathbb{R}^n : \mathcal{M}^L_\eta\big(H_{12B_j}\mathbf{1}_{12B_j}\big)(x) > (K/2\widetilde{K_0})\lambda\big\}\Big|.
\end{aligned} \tag{5.14}
$$

Now recall that $E_\lambda \cap Q_j$ is assumed to be not empty. Hence there exists $\widetilde{x_j} \in Q_j \subset B_j$ with

$$
G(\widetilde{x_j}) \le \frac{\delta}{2}\lambda. \tag{5.15}
$$

Let c_p be the weak (p,p) bound of $\mathcal{M}^L_\eta$ (from Lemma 5.7). Applying assumption (5.3), valid because of (5.13), we obtain

$$
\begin{aligned}
\Big|\big\{x \in \mathbb{R}^n : \mathcal{M}^L_\eta\big(G_{12B_j}\mathbf{1}_{12B_j}\big)(x) > (K/2\widetilde{K_0})\lambda\big\}\Big| &\le \frac{c_1 2\widetilde{K_0}}{K\lambda}\int_{12B_j} G_{12B_j} \\
&\le \frac{c_1 2\widetilde{K_0}}{K\lambda}\,|12B_j|\,G(\widetilde{x_j}) \\
&\le \frac{12^n c_1 \widetilde{K_0}}{K}\,|B_j|\,\delta\,.
\end{aligned} \tag{5.16}
$$

Next suppose that $q < \infty$. We apply (5.4) — again since (5.13) holds — to get

$$
\begin{aligned}
\big|\big\{x \in \mathbb{R}^n : \mathcal{M}^L_\eta\big(H_{12B_j}\mathbf{1}_{12B_j}\big)(x) > (K/2\widetilde{K_0})\lambda\big\}\big| &\le \Big(\frac{2\widetilde{K_0}c_q}{K\lambda}\Big)^q \int_{12B_j} H^q_{12B_j} \\
&\le \Big(\frac{2\widetilde{K_0}c_q}{K\lambda}\Big)^q \xi^q\left(\mathcal{M}^L_\eta F(x_j) + G(\widetilde{x_j})\right)^q\,|12B_j| \\
&\le (4\widetilde{K_0}c_q)^q\,12^n\,\frac{\xi^q}{K^q}\,|B_j|\,,
\end{aligned} \tag{5.17}
$$

where the points x_j and $\widetilde{x_j}$ satisfy (5.12) and (5.15) respectively. We insert now estimates (5.16) and (5.17) into (5.14) to arrive at

$$
|E_\lambda \cap Q_j| \;\le\; |E_\lambda \cap B_j| \;\le\; C\Big(\frac{\xi^q}{K^q} + \frac{\delta}{K}\Big)\,|Q_j|
$$

where C depends on q, n, $\widetilde{K_0}$ and the weak type bounds of $\mathcal{M}^L_\eta$. This gives (5.9) for the case $q < \infty$, and hence from (5.10) we get (5.8) for those cubes Q_j with $j \in \mathcal{J}_0$.

If $q = \infty$, then firstly notice that

$$\left\| \mathcal{M}_\eta^L \big(H_{12B_j} \mathbf{1}_{12B_j} \big) \right\|_{L^\infty} \;\le\; \left\| H_{12B_j} \mathbf{1}_{12B_j} \right\|_{L^\infty} \;\le\; \xi \left(\mathcal{M}_\eta^L F(x_j) + G(\widetilde{x_j}) \right) \;\le\; 2\xi\lambda.$$

Therefore it follows that whenever $K \ge 4\xi \widetilde{K_0}$, then

$$\left\{ x \in \mathbb{R}^n : \mathcal{M}_\eta^L \big(H_{12B_j} \mathbf{1}_{12B_j} \big)(x) > (K/2\widetilde{K_0})\lambda \right\} = \emptyset.$$

So we set $K_0 = 4\xi\widetilde{K_0} \ge 1$, and for each $K \ge K_0$ we may proceed as before with estimates (5.16) and (5.17) to obtain the following variant of (5.9):

$$|E_\lambda \cap Q_j| \;\le\; C \left(\frac{\delta}{K} \right)^{1/s} |Q_j| \,.$$

Before concluding the proof of the case $j \in \mathcal{J}_0$, we remark that taking the choice $K_0 = 4\xi\widetilde{K_0}$ will allow us to cover both of the situations $q < \infty$, and $q = \infty$.

We turn to the proof of (5.8) for the case $j \in \mathcal{J}_\infty$. We shall require the following decomposition lemma.

Lemma 5.13 ([106] Lemma 3.1)**.** *For any cube Q with $l_Q > 2\gamma(x_Q)$ there exists a finite collection of disjoint subcubes $\{Q_k\}_{k=1}^N$ such that $Q = \cup_{k=1}^N Q_k$ with the following property: for every $k \in \{1, ..., N\}$, there exists $x_k \in Q_k$ with*

$$\tfrac{1}{2}\, l_{Q_k} \;\le\; \gamma(x_k) \;\le\; 2\sqrt{n}\, C_0\, l_{Q_k},$$

where C_0 is the constant from Lemma 2.5.

Recall that when $j \in \mathcal{J}_\infty$ the cube Q_j satisfies $l_{Q_j} > 2\gamma(x_{Q_j})$. Hence we may apply Lemma 5.13 to Q_j and obtain a finite collection of disjoint subcubes $\{Q_{j,k}\}_{k=1}^{N_j}$, with $Q_j = \cup_{k=1}^{N_j} Q_{j,k}$, such that for each $k \in \{1, ..., N_j\}$ there exists $x_{j,k} \in Q_{j,k}$ with

$$\tfrac{1}{2}\, l_{Q_{j,k}} \;\le\; \gamma(x_{j,k}) \;\le\; 2\sqrt{n}\, C_0\, l_{Q_{j,k}}. \tag{5.18}$$

We observe that this implies $\gamma(x_{j,k}) \approx \gamma(x_{Q_{j,k}})$ with constants depending only on n and C_0, where $x_{Q_{j,k}}$ is the centre of the cube $Q_{j,k}$. Indeed, since $x_{j,k}, x_{Q_{j,k}} \in Q_{j,k}$ then $x_{Q_{j,k}} \in$

$B(x_{j,k}, \frac{\sqrt{n}}{2} l_{Q_{j,k}}) \subseteq \sqrt{n}\,\mathfrak{B}^{\gamma}(x_{j,k})$ and hence by (2.10) we have $\gamma(x_{Q_{j,k}}) \leq C_0^2(1+\sqrt{n})^2\gamma(x_{j,k})$. The other inequality can be obtained similarly.

Now for each j and k we set $B_{j,k}$ to be the ball concentric with $Q_{j,k}$ but with radius $\frac{\sqrt{n}}{2} l_{Q_{j,k}}$. That is, $B_{j,k}$ is the smallest ball concentric with, and containing $Q_{j,k}$. We claim the following property holds, whose proof we defer to the end of this section.

Lemma 5.14. *There exists $\alpha \geq 1$, depending only on n, η and C_0, with the following property: for every cube $Q_{j,k}$ for which $E_\lambda \cap Q_{j,k}$ is non-empty, one has*

$$E_\lambda \cap Q_{j,k} \;\subset\; \left\{x \in Q_{j,k} : \mathcal{M}_\eta^L\big(F\mathbf{1}_{\alpha B_{j,k}}\big)(x) > K\lambda\right\} \tag{5.19}$$

$$r_{\alpha B_{j,k}} \;>\; 12\sqrt{n}\,\gamma\big(x_{\alpha B_{j,k}}\big). \tag{5.20}$$

Let us fix k and assume that $E_\lambda \cap Q_{j,k}$ is not empty, since otherwise there is nothing to prove for the cube $Q_{j,k}$. This implies that there exists a point $\widetilde{x}_{j,k} \in Q_{j,k} \subset \alpha B_{j,k}$ with

$$G\big(\widetilde{x}_{j,k}\big) \leq \tfrac{\delta}{2}\lambda, \tag{5.21}$$

Let c_1 be the weak $(1,1)$ bound of $\mathcal{M}_\eta^L$. Then (5.19) gives

$$\begin{aligned} |E_\lambda \cap Q_{j,k}| \leq\; & \left|\left\{x \in Q_{j,k} : \mathcal{M}_\eta^L\big(F\mathbf{1}_{\alpha B_{j,k}}\big)(x) > K\lambda\right\}\right| \;\leq\; \frac{c_1}{K\lambda}\int_{\alpha B_{j,k}} F \\ \leq\; & \frac{c_1}{K\lambda}\,|\alpha B_{j,k}|\;\psi_\eta(\alpha B_{j,k})\,G(\widetilde{x}_{j,k}) \;\leq\; C\,\frac{\delta}{K}\,|Q_{j,k}|\,. \end{aligned} \tag{5.22}$$

In the third inequality we have applied hypothesis (5.5) – since the ball $\alpha B_{j,k}$ satisfies (5.20) – and in the final inequality we used (5.21), the doubling property for the Lebesgue measure, and that

$$\psi_\eta(\alpha B_{j,k}) \;\leq\; \alpha^\eta\,\psi_\eta(B_{j,k}) \;\leq\; C,$$

which follows from (5.18). We remark that the constant C in (5.22) depends only on n, η, C_0 and is independent of j and k.

In a similar fashion to estimate (5.10), we apply Lemma 5.4 to $\nu \in \mathcal{B}_{s'}^{L,\theta}$ and the sets $E_\lambda \cap Q_{j,k} \subset Q_{j,k}$ and evoke (5.22) to obtain

$$\nu(E_\lambda \cap Q_{j,k}) \;\leq\; C_\nu\,\psi_\theta(Q_{j,k})\left(\frac{|E_\lambda \cap Q_{j,k}|}{|Q_{j,k}|}\right)^{1/s}\nu(Q_{j,k}) \;\leq\; C\left(\frac{\delta}{K}\right)^{1/s}\nu(Q_{j,k})$$

where C depends on n, C_0, η and ν. Summing this over k gives

$$\nu(E_\lambda \cap Q_j) \;\le\; \sum_{k=1}^{N_j} \nu(E_\lambda \cap Q_{j,k}) \;\le\; C\Big(\frac{\delta}{K}\Big)^{1/s} \sum_{k=1}^{N_j} \nu(Q_{j,k}) \;\le\; C\Big(\frac{\xi^q}{K^q} + \frac{\delta}{K}\Big)^{1/s} \nu(Q_j)$$

which gives (5.8) for $j \in \mathcal{J}_\infty$. Note that when $q = \infty$ we end the estimate at the second inequality. This concludes the proof of (5.8), and hence of (5.6).

Since (5.6) holds we may prove (5.7) using the same approach as the final part of the proof of Theorem 3.1 from [16], pp. 20-21. In fact the proof is identical but with $\mathcal{M}^L_\eta$ in place of the Hardy–Littlewood maximal operator M, and $\mathcal{B}^{L,\theta}_{s'}$ in place of $\mathcal{B}_{s'}$. We omit the details. $\square$

We end this section with the proofs of the lemmata that were deferred during the proof of Theorem 5.10.

Proof of Lemma 5.12. This proof is an adaptation of the localisation lemma from [11]. Let $x \in \widetilde{B}$ with $\mathcal{M}^L_\eta f(x) > K\lambda$. Then there exists a ball B containing x with

$$\frac{1}{\psi_\eta(B)} \fint_B |f| > K\lambda.$$

Note that $B \subset B(x, 2r_B) \subset 3B$ so that $|B| \ge 3^{-n}\,|B(x, 2r_B)|$.

From Lemma 2.5, since $x \in B$ then $\gamma(x_B) \le C_1\gamma(x)$, where $C_1 = 4C_0^2$. This gives (because $C_0 \ge 1$)

$$\psi_\eta(B) \;\ge\; \Big(1 + \frac{r_B}{4C_0^2\gamma(x)}\Big)^\eta \;\ge\; (8C_0^2)^{-\eta}\Big(1 + \frac{2r_B}{\gamma(x)}\Big)^\eta \;=\; (8C_0^2)^{-\eta}\psi_\eta\big(B(x, 2r_B)\big).$$

Therefore

$$\int_{B(x,2r_B)} |f| \;>\; \int_B |f| \;>\; K\lambda|B|\psi_\eta(B) \;\ge\; \frac{K\lambda\,|B(x, 2r_B)|}{3^n}\,\frac{\psi_\eta\big(B(x, 2r_B)\big)}{(8C_0^2)^\eta}.$$

This implies

$$\frac{1}{\psi_\eta(B(x, 2r_B))} \fint_{B(x,2r_B)} |f| \;>\; \frac{K}{\widetilde{K_0}}\lambda \tag{5.23}$$

where $\widetilde{K_0} = 3^n 8^\eta C_0^{2\eta}$. Now since $K \ge \widetilde{K_0}$, then in fact

$$\frac{1}{\psi_\eta(B(x, 2r_B))} \fint_{B(x,2r_B)} |f| \;>\; \lambda$$

and this combined with the point $\widetilde{x}$ from the hypothesis implies that $\widetilde{x} \notin B(x, 2r_B)$, for otherwise this contradicts $\mathcal{M}_\eta^L f(\widetilde{x}) \leq \lambda$. This final fact implies that $B(x, 2r_B) \subset 3\widetilde{B}$, and combining this with (5.23) gives

$$\frac{1}{\psi_\eta(B(x,2r_B))} \fint_{B(x,2r_B)} |f| \, \mathbf{1}_{3\widetilde{B}} = \frac{1}{\psi_\eta(B(x,2r_B))} \fint_{B(x,2r_B)} |f| > \frac{K}{\widetilde{K_0}} \lambda.$$

This last step ensures $\mathcal{M}_\eta^L \big(f \mathbf{1}_{3\widetilde{B}}\big)(x) > (K/\widetilde{K_0})\lambda$. □

Proof of Lemma 5.14. Let $x \in E_\lambda \cap Q_{j,k}$. Then it follows that

$$G(x) \leq \tfrac{\delta}{2} \lambda, \tag{5.24}$$

$$\mathcal{M}_\eta^L F(x) > K\lambda. \tag{5.25}$$

The latter property ensures that there exists a ball B containing x such that

$$\frac{1}{\psi_\eta(B)} \fint_B F > K\lambda. \tag{5.26}$$

Then we necessarily have

$$r_B \leq 12\sqrt{n}\, \gamma(x_B). \tag{5.27}$$

Suppose otherwise. Then hypothesis (5.5) applies to B. Combining this with (5.24) and (5.26), we arrive at the statement

$$K\lambda < \frac{1}{\psi_\eta(B)} \fint_B F \leq G(x) \leq \frac{\delta}{2} \lambda,$$

which is impossible, since $K \geq 1$ and $\delta \in (0,1)$. Therefore the ball B necessarily satisfies (5.27).

Next we claim that there exists $\alpha \geq 1$, depending only on C_0, n and η, such that (5.20) holds and

$$B \subset \alpha B_{j,k}. \tag{5.28}$$

Let us demonstrate this claim. This will involve repeated application of (2.10). Firstly (5.27) implies $B \subset 12\sqrt{n}\, \mathfrak{B}^\gamma$, so that

$$\gamma(x_B) \leq C_1 \gamma(x), \tag{5.29}$$

where $C_1 = C_0^2(1+12\sqrt{n})^2$. Secondly since both $x, x_{j,k} \in Q_{j,k}$, then the distance between x and $x_{j,k}$ is at most the diameter of $Q_{j,k}$. That is,

$$|x - x_{j,k}| \;\le\; \operatorname{diam}(Q_{j,k}) \;=\; \sqrt{n}\, l_{Q_{j,k}}.$$

It follows that $x \in B\big(x_{j,k}, \sqrt{n}\, l_{Q_{j,k}}\big) \subset 2\sqrt{n}\,\mathfrak{B}^\gamma(x_{j,k})$, and hence

$$\gamma(x) \;\le\; C_2\, \gamma(x_{j,k}) \tag{5.30}$$

where $C_2 = C_0^2(1+2\sqrt{n})^2$. We now combine (5.29) and (5.30) with (5.18) and (5.27) to obtain

$$r_B \;\le\; 12\sqrt{n}\,\gamma(x_B) \;\le\; 12\sqrt{n}\, C_1\, \gamma(x) \;\le\; 12\sqrt{n}\, C_1\, C_2\, \gamma(x_{j,k}) \;\le\; \alpha_0 \tfrac{\sqrt{n}}{2}\, l_{Q_{j,k}} \;=\; \alpha_0\, r_{B_{j,k}},$$

where $\alpha_0 = 48C_0C_1C_2\sqrt{n}$. Therefore it follows that $B \subset (1+2\alpha_0)B_{j,k}$. Next we set $\widetilde{\alpha}$ to be a number such that

$$r_{\widetilde{\alpha}B_{j,k}} > 12\sqrt{n}\,\gamma\big(x_{Q_{j,k}}\big).$$

Note that this number exists because we recall that $\gamma(x_{Q_{j,k}}) \approx \gamma(x_{j,k}) \approx l_{Q_{j,k}} \approx r_{B_{j,k}}$ with constants depending on C_0 and n. In fact, $\gamma(x_{Q_{j,k}}) \ge C_3\gamma(x_{j,k})$ where $C_3 = C_0^2(1+\sqrt{n})^2$, so that $\widetilde{\alpha}\frac{\sqrt{n}}{2} l_{Q_{j,k}} > 12\sqrt{n}C_3\,\gamma(x_{j,k})$, which holds provided $\widetilde{\alpha} \ge 43\sqrt{n}C_0C_3$ by (5.18). On choosing $\alpha = \max\{1+2\alpha_0, \widetilde{\alpha}\}$, the estimate (5.20) and the claim (5.28) both hold.

Finally to obtain the inclusion (5.19), we see that (5.28) with (5.26) implies

$$\frac{1}{\psi_\eta(B)} \fint_B F\mathbf{1}_{\alpha B_{j,k}} \;=\; \frac{1}{\psi_\eta(B)} \fint_B F \;>\; K\lambda.$$

It necessarily follows that

$$\mathcal{M}^L_\eta(F\mathbf{1}_{\alpha B_{j,k}})(x) > K\lambda,$$

and as a consequence (5.19) holds. $\square$

5.2.1 Applications

In this section we give some applications of Theorem 5.10.

We give here a proof of a Fefferman–Stein type inequality (1.36), which was first given and proved in [106], Theorem 2.1. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, set

$$\mathcal{M}_\eta^{\#,L} f(x) := \sup_{\substack{B \ni x \\ r_B \le \gamma(x_B)}} \fint_B |f - f_B| + \sup_{\substack{B \ni x \\ r_B > \gamma(x_B)}} \frac{1}{\psi_\eta(B)} \fint_B |f| \,.$$

Then the following holds for each $\eta > 0$, $p \in (0, \infty)$, and $w \in \mathcal{A}_\infty^L$.

$$\left\| \mathcal{M}_\eta^L f \right\|_{L^p(w)} \le C_p \left\| \mathcal{M}_\eta^{\#,L} f \right\|_{L^p(w)} .$$

To see this we apply Theorem 5.10 to any $s \in [1, \infty)$, and

$$F = |f| \,, \qquad H = 0, \qquad G = 2(1 + 12\sqrt{n})^\eta \, \mathcal{M}_\eta^{\#,L} f,$$

$$q = \infty, \qquad \nu = w, \qquad r = p.$$

Now let B be a ball with $r_B \le 12\sqrt{n}\,\gamma(x_B)$. We have

$$F = |f| \le |f - f_B| + |f_B| =: G_B + H_B.$$

Then (5.4) holds because

$$\Big(\fint_B H_B^q \Big)^{1/q} \;=\; |f_B| \;\le\; \psi_\eta(B) \, \mathcal{M}_\eta^L f \;\le\; (1 + 12\sqrt{n})^\eta \, \mathcal{M}_\eta^L f \,.$$

Next we check (5.3). If $r_B \le \gamma(x_B)$ then

$$\fint_B G_B \;=\; \fint_B |f - f_B| \;\le\; \mathcal{M}_\eta^{\#,L} f \;\le\; G.$$

If $\gamma(x_B) < r_B \le 12\sqrt{n}\,\gamma(x_B)$, then

$$\fint_B G_B \;\le\; 2\,|f|_B \;\le\; 2\,\psi_\eta(B) \, \mathcal{M}_\eta^{\#,L} f \;\le\; 2(1 + 12\sqrt{n})^\eta \, \mathcal{M}_\eta^{\#,L} f \;=\; G.$$

Finally we check (5.5). If $r_B > 12\sqrt{n}\,\gamma(x_B)$, we have

$$\frac{1}{\psi_\eta(B)} \fint_B F \;=\; \frac{1}{\psi_\eta(B)} \fint_B |f| \;\le\; \mathcal{M}_\eta^{\#,L} f \;\le\; G$$

and we are done.

The following application is an adaptation of [16] Theorem 3.1 for $\mathcal{A}_\infty^L$ weights.

Theorem 5.15. *Let $1 \le p_0 < q_0 \le \infty$ and $\mathcal{E}, \mathcal{D}$ be vector spaces such that $\mathcal{D} \subset \mathcal{E}$. Let T, S be operators such that S acts from $\mathcal{D}$ into the set of measurable functions, and T is sublinear acting from $\mathcal{E}$ into $L^{p_0}(\mathbb{R}^n)$. Let $\{A_B\}_B$ be a family of operators indexed by balls on $\mathbb{R}^n$, acting from $\mathcal{D}$ into $\mathcal{E}$. Assume the following: for any $\eta > 0$ there exists $C_1, C_2 > 0$ such that for each ball B with $r_B \le 12\sqrt{n}\,\gamma(x_B)$, $f \in \mathcal{D}$, and $x \in B$,*

$$\Big(\fint_B |T(I - A_B)f|^{p_0} \Big)^{1/p_0} \le C_1\, \mathcal{M}_\eta^L \big(|Sf|^{p_0}\big)(x)^{1/p_0}, \tag{5.31}$$

$$\Big(\fint_B |TA_B f|^{q_0} \Big)^{1/q_0} \le C_1\, \mathcal{M}_\eta^L \big(|Tf|^{p_0}\big)(x)^{1/p_0}, \tag{5.32}$$

and for all balls B with $r_B > 12\sqrt{n}\gamma(x_B)$, $f \in \mathcal{D}$, and $x \in B$,

$$\frac{1}{\psi_\eta(B)} \fint_B |Tf|^{p_0} \le C_2 \mathcal{M}_\eta^L \big(|Sf|^{p_0}\big)(x). \tag{5.33}$$

Let $p_0 < p < q_0$ (with $p = q_0$ if $q_0 < \infty$) and $w \in \mathcal{A}^L_{p/p_0} \cap \mathcal{B}^L_{(q_0/p)'}$. Then there exists $C > 0$ such that

$$\|Tf\|_{L^p(w)} \le C\, \|Sf\|_{L^p(w)}, \qquad \forall f \in \mathcal{D}. \tag{5.34}$$

We can take $\mathcal{E} = L^{p_0}$ and $\mathcal{D}$ to be a class of 'nice' functions such as $L^\infty_c, L^{p_0} \cap L^2, C^\infty_0$ etc.

Proof of Theorem 5.15. We first consider the case $q_0 < \infty$. Let $p_0 < p \le q_0$, $f \in \mathcal{D}$ and $w \in \mathcal{A}^L_{p/p_0} \cap \mathcal{B}^L_{(q_0/p)'}$. Then there exists $\theta \ge 0$ such that $w \in \mathcal{A}^{L,\theta}_{p/p_0}$. We shall apply Theorem 5.10 with

$$s = \frac{q_0}{p}, \qquad q = \frac{q_0}{p_0}, \qquad r = \frac{p}{p_0} > 1,$$

$$\xi = C_1^{p_0}\, 2^{p_0 - 1}, \qquad \nu = w, \qquad \eta = \big(\tfrac{p}{p_0}\big)'\theta = \tfrac{p\theta}{p-\theta},$$

$$F = |Tf|^{p_0}, \qquad H = 0, \qquad G = C_3\, \mathcal{M}_\eta^L \big(|Sf|^{p_0}\big),$$

where $C_3 = \max\big\{C_1^{p_0} 2^{p_0-1}, C_2\big\}$.

Let B be a ball with $r_B \le 12\sqrt{n}\,\gamma(x_B)$. We will check that conditions (5.2), (5.3), and (5.4) hold for this ball. Firstly (5.2) follows easily because by sublinearity of T,

$$|Tf|^{p_0} \;\; \le \;\; 2^{p_0-1}\, |T(I - A_B)f|^{p_0} + 2^{p_0-1}\, |TA_B f|^{p_0} \;\; =: \;\; G_B + H_B.$$

Next we check (5.3). For each $x \in B$, by hypothesis (5.31),

$$\begin{aligned}\fint_B G_B &= 2^{p_0-1} \fint_B |T(I-A_B)f|^{p_0} \\ &\le 2^{p_0-1} C_1^{p_0}\, \mathcal{M}_\eta^L\big(|Sf|^{p_0}\big)(x) \\ &\le C_3\, \mathcal{M}_\eta^L\big(|Sf|^{p_0}\big)(x) = G(x).\end{aligned}$$

Thirdly we check (5.4). For each $x \in B$, by hypothesis (5.32),

$$\Big(\fint_B H_B^q\Big)^{1/q'} \;=\; 2^{p_0-1}\Big(\fint_B |TA_Bf|^{q_0}\Big)^{p_0/q_0} \;\le\; 2^{p_0-1}C_1^{p_0}\, \mathcal{M}_\eta^L\big(|Tf|^{p_0}\big)(x) \;=\; \xi\, \mathcal{M}_\eta^L\big(|Tf|^{p_0}\big)(x).$$

Finally we check (5.5). For any ball B with $r_B > 12\sqrt{n}\,\gamma(x_B)$, by (5.33), then

$$\frac{1}{\psi_\eta(B)}\fint_B F \;=\; \frac{1}{\psi_\eta(B)}\fint_B |Tf|^{p_0} \;\le\; C_2\, \mathcal{M}_\eta^L\big(|Sf|^{p_0}\big)(x) \;\le\; C_3\, \mathcal{M}_\eta^L\big(|Sf|^{p_0}\big)(x) \;=\; G(x).$$

Since $w \in \mathcal{B}^L_{(q_0/p)'} = \mathcal{B}^L_{s'}$ then the conclusion of Theorem 5.10 gives

$$\|Tf\|_{L^p(w)}^{p_0} \le 2^\eta \left\|\mathcal{M}_\eta^L F\right\|_{L^{p/p_0}(w)} \le C\,\|G\|_{L^{p/p_0}(w)} = C\left\|\mathcal{M}_\eta^L(|Sf|^{p_0})\right\|_{L^{p/p_0}(w)} \le C\,\|Sf\|_{L^p(w)}^{p_0},$$

which is (5.34). The first inequality holds from the pointwise control of the maximal operator $\mathcal{M}_\eta^L$ (see (5.1)). The last inequality follows from the boundedness of the operator $\mathcal{M}_\eta^L$ on $L^{p/p_0}(w)$. Indeed Lemma 5.8 applies in this situation since $p > p_0$, $w \in \mathcal{A}^{L,\theta}_{p/p_0}$, and $\eta = (p/p_0)'\theta$.

If $q_0 = \infty$ then we can apply Theorem 5.10 as before with $p_0 < p < \infty$ and $w \in \mathcal{A}^L_{p/p_0}$ to conclude the proof of the theorem. □

The next application is an extension of Theorem 4.6 to $\mathcal{A}^L_\infty$ weights, and is the main tool used in the proof of Theorem 5.1.

Theorem 5.16. *Let $1 \le p_0 < q_0 \le \infty$ and T be a linear operator. Assume that for each $\tilde{q} \in (p_0, q_0)$ and $\eta > 0$ there exists a family of operators $\{A_B\}_B$ indexed by balls and a collection of scalars $\{\alpha_j\}_{j=0}^\infty$ such that the following holds.*

(i) *T is bounded on $L^{\tilde{q}}(\mathbb{R}^n)$.*

(ii) *For every ball B with $r_B \le 12\sqrt{n}\,\gamma(x_B)$, and every $f \in L^\infty_c(\mathbb{R}^n)$ supported in B,*

$$\Big(\fint_{U_j(B)} |A_Bf|^{\tilde{q}}\Big)^{1/\tilde{q}} \le \alpha_j\Big(\fint_B |f|^{p_0}\Big)^{1/p_0}, \qquad \forall j \ge 0 \tag{5.35}$$

$$\Big(\fint_{U_j(B)} |T(I-A_B)f|^{\tilde{q}}\Big)^{1/\tilde{q}} \le \alpha_j \Big(\fint_B |f|^{p_0}\Big)^{1/p_0}, \qquad \forall j \ge 2. \tag{5.36}$$

(iii) *There exists* $\widetilde{C} > 0$ *such that for every ball* B *with* $r_B > 12\sqrt{n}\,\gamma(x_B)$ *and* $f \in L_c^\infty(\mathbb{R}^n)$,

$$\Big(\frac{1}{\psi_\eta(B)} \fint_B |T^* f|^{\tilde{q}'}\Big)^{1/\tilde{q}'} \le \widetilde{C}\,\mathcal{M}_\eta^L\big(|f|^{\tilde{q}'}\big)(x)^{1/\tilde{q}'}, \qquad \forall x \in B. \tag{5.37}$$

(iv) $\sum_j \alpha_j 2^{j(n+\eta)} < \infty$.

Let $p \in (p_0, q_0)$ *and* $w \in \mathcal{A}^L_{p/p_0} \cap \mathcal{B}^L_{(q_0/p)'}$. *Then* T *extends to a bounded operator on* $L^p(w)$.

Proof of Theorem 5.16. The proof is an adaptation of the proof of Theorem 4.6 (a). We fix $p \in (p_0, q_0)$ and $w \in \mathcal{A}^L_{p/p_0} \cap \mathcal{B}^L_{(q_0/p)'}$. Denote by T^* the adjoint of T. Then it will suffice to prove that T^* is bounded on $L^{p'}(w^{1-p'})$, because this is equivalent to the $L^p(w)$ boundedness of T (see Remark 4.7 (a)). We shall apply Theorem 5.10 to T^*.

Firstly, by Lemma 5.6 property (ix), there exists numbers p_1 and q_1 such that

$$p_0 < p_1 < p < q_1 < q_0 \qquad \text{and} \qquad w \in \mathcal{A}^L_{\frac{p}{p_1}} \cap \mathcal{B}^L_{\left(\frac{q_1}{p}\right)'}.$$

Then it follows from property (x) of Lemma 5.6 that

$$w^{1-p'} \in \mathcal{A}^L_{\frac{p'}{q_1'}} \cap \mathcal{B}^L_{\left(\frac{p_1'}{p'}\right)'}.$$

Next there exists $\theta \ge 0$ such that

$$w^{1-p'} \in \mathcal{A}^{L,\theta}_{\frac{p'}{q_1'}}.$$

We now apply Theorem 5.10 to the following datum. For each $f \in L_c^\infty(\mathbb{R}^n)$ we set

$$s := \frac{p_1'}{p'}, \qquad q := \frac{p_1'}{q_1'}, \qquad r := \frac{p'}{q_1'}, \qquad \eta := r'\theta,$$

$$F := |T^* f|^{q_1'}, \qquad H := 0, \qquad \nu := w^{1-p'}.$$

Let $\tilde{q} = q_1$. Take $\{A_B\}_B$ and $\{\alpha_j\}_j$ to be as in the hypotheses. We shall show that conditions (5.2)–(5.5) hold with

$$G_B := 2^{q_1'-1}\,|(I-A_B)^* T^* f|^{q_1'} \qquad \text{and} \qquad H_B := 2^{q_1'-1}\,|A_B^* T^* f|^{q_1'},$$

and G is a fixed constant multiple of $\mathcal{M}_\eta^L\big(|f|^{q_1'}\big)$ (with the constant to be specified later).

We first check condition (5.2). By noting that $(I - A_B^*) = (I - A_B)^*$, one has

$$\begin{aligned} F(x) = |T^* f(x)|^{q_1'} &= |(I - A_B)^* T^* f(x) + A_B^* T^* f(x)|^{q_1'} \\ &\le 2^{q_1'-1} \, |(I - A_B^*) T^* f(x)|^{q_1'} + 2^{q_1'-1} \, |A_B^* T^* f(x)|^{q_1'} \\ &= G_B(x) + H_B(x) \end{aligned}$$

where we have used that $|a+b|^r \le 2^{r-1}\,|a| + 2^{r-1}\,|b|^r$ valid for all $r \ge 1$ and $a, b \in \mathbb{R}$.

We now check condition (5.4). Let B be a ball with $r_B \le 12\sqrt{n}\,\gamma(x_B)$. We first write

$$\Big(\fint_B H_B^q\Big)^{1/q} = \Big(\fint_B 2^{p_1' - p_1'/q_1'}\,|A_B^* T^* f|^{p_1'}\Big)^{q_1'/p_1'} \lesssim \Big(\fint_B |A_B^* T^* f|^{p_1'}\Big)^{q_1'/p_1'}.$$

To estimate the integral we apply duality to $R := T^*$, $S := A_B^*$ with some $g \in L^{p_1}(B, dx/|B|)$ with norm 1 (Remark 4.7(d)), to obtain for each $x \in B$,

$$\begin{aligned} \Big(\fint_B H_B^q\Big)^{1/qq_1'} &\lesssim \Big(\fint_B |A_B^* T^* f|^{p_1'}\Big)^{1/p_1'} \;\le\; \fint_B |T^* f|\,|A_B g| \;\le\; \sum_{j=0}^{\infty} 2^{jn} \fint_{U_j(B)} |T^* f|\,|A_B g| \\ &\le \sum_{j=0}^{\infty} 2^{jn} \psi_\eta(2^j B)\Big(\frac{1}{\psi_\eta(2^j B)} \fint_{2^j B} |T^* f|^{q_1'}\Big)^{1/q_1'} \Big(\frac{1}{\psi_\eta(2^j B)} \fint_{U_j(B)} |A_B g|^{q_1}\Big)^{1/q_1} \\ &\lesssim \mathcal{M}_\eta^L\big(|T^* f|^{q_1'}\big)(x)^{1/q_1'} \sum_{j=0}^{\infty} 2^{j(n+\eta)} \Big(\fint_{U_j(B)} |A_B g|^{q_1}\Big)^{1/q_1}. \end{aligned} \tag{5.38}$$

In the last line we have used that since $r_B \le 12\sqrt{n}\,\gamma(x_B)$, then

$$\psi_\eta(2^j B) \;\le\; 2^{j\eta}\,\psi_\eta(B) \;\le\; 2^{j\eta}\,(1 + 12\sqrt{n})^\eta \tag{5.39}$$

valid for every $j \ge 0$. Now from (5.35) with exponent $\tilde{q} = q_1$, we have for each $j \ge 0$,

$$\Big(\fint_{U_j(B)} |A_B g|^{q_1}\Big)^{1/q_1} \;\le\; \alpha_j \Big(\fint_B |g|^{p_0}\Big)^{1/p_0} \;\le\; \alpha_j \Big(\fint_B |g|^{p_1}\Big)^{1/p_1} \;=\; \alpha_j,$$

where we have used Hölder's inequality (with exponents p_1/p_0 and $(p_1/p_0)'$) and the the normalisation of g. Inserting this estimate into (5.38) gives, for each $x \in B$,

$$\Big(\fint_B H_B^q\Big)^{1/qq_1'} \;\lesssim\; \mathcal{M}_\eta^L\big(|T^* f|^{q_1'}\big)(x)^{1/q_1'} \sum_{j=0}^{\infty} \alpha_j 2^{j(n+\eta)} \;\le\; C_1 \mathcal{M}_\eta^L\big(|T^* f|^{q_1'}\big)(x)^{1/q_1'}$$

by hypothesis (iv). Hence (5.4) holds with $H = 0$ and $\xi = C_1$.

Next we check condition (5.3). Let B be a ball with $r_B \le 12\sqrt{n}\,\gamma(x_B)$. We first write

$$\Big(\fint_B G_B\Big)^{1/q_1'} = \Big(\fint_B 2^{q_1'-1}\,|(I-A_B)^*T^*f|^{q_1'}\,dx\Big)^{1/q_1'} \lesssim \Big(\fint_B |(I-A_B)^*T^*f|^{q_1'}\,dx\Big)^{1/q_1'}.$$

We apply duality again now with $R := I$, with $S := (I-A_B)^*T^*$, and with $g \in L^{q_1}(B, dx/\,|B|)$ of norm 1. Then for each $x \in B$,

$$\begin{aligned}\Big(\fint_B G_B\Big)^{1/q_1'} &\lesssim \fint_B |f|\,|T(I-A_B)g| \;\le\; \sum_{j=0}^{\infty} 2^{jn} \fint_{U_j(B)} |f|\,|T(I-A_B)g| \\ &\le \sum_{j=0}^{\infty} 2^{jn}\psi_\eta(2^jB)\Big(\frac{1}{\psi_\eta(2^jB)}\fint_{2^jB}|f|^{q_1'}\Big)^{1/q_1'}\Big(\frac{1}{\psi_\eta(2^jB)}\fint_{U_j(B)}|T(I-A_B)g|^{q_1}\Big)^{1/q_1} \\ &\lesssim \mathcal{M}_\eta^L\big(|f|^{q_1'}\big)(x)^{1/q_1'}\sum_{j=0}^{\infty}2^{j(n+\eta)}\Big(\fint_{U_j(B)}|T(I-A_B)g|^{q_1}\Big)^{1/q_1}, \end{aligned} \tag{5.40}$$

where in the last line we have used (5.39) again. Now for each $j \ge 2$, estimate (5.36) with exponent $\tilde q = q_1$ gives

$$\Big(\fint_{U_j(B)}|T(I-A_B)g|^{q_1}\Big)^{1/q_1} \;\le\; \alpha_j\Big(\fint_B|g|^{p_0}\Big)^{1/p_0} \;\le\; \alpha_j\Big(\fint_B|g|^{q_1}\Big)^{1/q_1} \;=\; \alpha_j, \tag{5.41}$$

where we have used Hölder's inequality (with exponents q_1/p_0 and $(q_1/p_0)'$) and the normalisation of g. For $j = 0, 1$ we use hypothesis (i) with $\tilde q = q_1$ to give

$$\fint_{U_j(B)}|T(I-A_B)g|^{q_1} \;\lesssim\; \frac{1}{|B|}\int_{\mathbb{R}^n}|(I-A_B)g|^{q_1} \;\lesssim\; \frac{1}{|B|}\Big\{\int_B|g|^{q_1} + \sum_{k=0}^{\infty}\int_{U_k(B)}|A_Bg|^{q_1}\Big\}.$$

For the summands we use the approach as before, namely applying (5.35) for $k \ge 0$, and Hölder's inequality to get

$$\Big(\fint_{U_k(B)}|A_Bg|^{q_1}\Big)^{1/q_1} \;\le\; \alpha_k\Big(\fint_B|g|^{p_0}\Big)^{1/p_0} \;\le\; \alpha_k\Big(\fint_B|g|^{q_1}\Big)^{1/q_1} \;=\; \alpha_k.$$

Collecting these estimates we have for $j = 0, 1$,

$$\fint_{U_j(B)}|T(I-A_B)g|^{q_1} \;\lesssim\; \fint_B|g|^{q_1} + \sum_{k=0}^{\infty}2^{kn}\fint_{U_k(B)}|A_Bg|^{q_1} \;\lesssim\; \fint_B|g|^{q_1} + \sum_{k=0}^{\infty}\alpha_k^{q_1}2^{kn}, \tag{5.42}$$

which is finite because the expression $\sum_k \alpha_k 2^{k(n+\eta)}$ is finite. Inserting (5.41) and (5.42) into (5.40) gives

$$\Big(\fint_B G_B\Big)^{1/q_1'} \;\lesssim\; \mathcal{M}_\eta^L\big(|f|^{q_1'}\big)(x)^{1/q_1'}\Big\{\sum_{j=2}^{\infty}\alpha_j 2^{j(n+\eta)} + C\Big\} \;\le\; C_2\mathcal{M}_\eta^L\big(|f|^{q_1'}\big)(x)^{1/q_1'} \tag{5.43}$$

for each $x \in B$.

Now let $G(x) := C_3\, \mathcal{M}_\eta^L\big(|f|^{q_1'}\big)(x)^{1/q_1'}$, where $C_3 = \max\{\widetilde{C}, C_2\}$. Here $\widetilde{C}$ is the constant from hypothesis (iii), and C_2 is the constant from (5.43). With this choice of G, firstly estimate (5.43) implies that (5.3) holds, and secondly estimate (5.37) implies that (5.5) holds.

We have shown that (5.2)–(5.5) holds. Therefore, since $\nu \in \mathcal{B}^L_{(p_1'/p')'} \equiv \mathcal{B}^L_{s'}$, then Theorem 5.10 allows us to conclude that

$$\big\|\mathcal{M}_\eta^L\big(|T^* f|^{q_1'}\big)\big\|_{L^r(\nu)} \;\le\; C\big\|\mathcal{M}_\eta^L\big(|f|^{q_1'}\big)\big\|_{L^r(\nu)} \tag{5.44}$$

for some $C > 0$, depending only on ν, q, n, ξ, s, η, γ, C_3, and hence only on w, p, p_1, q_1, C_1, C_2, $\widetilde{C}$. Recalling that $r = p'/q_1'$ and $\nu = w^{1-p'}$, we observe that the $L^{p'}(w^{1-p'})$ boundedness of T^* now follows, because

$$\|T^* f\|^{q_1'}_{L^{p'}(\nu)} \;\le\; 2^\eta\big\|\mathcal{M}_\eta^L\big(|T^* f|^{q_1'}\big)\big\|_{L^r(\nu)} \;\le\; C\big\|\mathcal{M}_\eta^L\big(|f|^{q_1'}\big)\big\|_{L^r(\nu)} \;\le\; C\,\|f\|^{q_1'}_{L^{p'}(\nu)}. \tag{5.45}$$

The first inequality in (5.45) holds by the pointwise control of the operator $\mathcal{M}_\eta^L$ (see (5.1)). The second inequality in (5.45) follows from the conclusion (5.44) above. The final inequality in (5.45) follows from the boundedness of the maximal operator $\mathcal{M}_\eta^L\big(|\cdot|^{q_1'}\big)^{1/q_1'}$ on $L^{p'}(\nu)$. Indeed, Remark 5.9 applies in this situation because firstly $p' > q_1'$, secondly $\nu = w^{1-p'} \in \mathcal{A}^{L,\theta}_{p'/q_1'}$, and lastly $\eta = r'\theta = (p'/q_1')'\theta$.

By duality, (5.45) implies the boundedness of T on $L^p(w)$. $\square$

5.3 Proof of the main result

In this section we give the proof of the main result of this chapter, namely Theorem 5.1.

Proof of Theorem 5.1. The proof is similar to the proofs of Theorems 4.1 and 4.3, but we apply Theorem 5.16 in place of Theorem 4.6.

We first consider the operator $\nabla^2 L^{-1}$. We apply Theorem 5.16 to $T = \nabla^2 L^{-1}$, $p_0 = 1$, $q_0 = s$, and $A_B = e^{-r_B^2 L}$. Fix $\widetilde{q} \in (1, s)$ and $\eta > 0$. We shall show that conditions (i)-(iv) of Theorem 5.16 hold. For simplicity we shall write q to denote $\widetilde{q}$ throughout the rest of this proof.

Firstly the $L^q(\mathbb{R}^n)$ boundedness of T holds from Theorem 1.3 (a), and so Theorem 5.16 (i) holds easily. Next we check conditions Theorem 5.16 (ii) and (iv). Fix a ball B and a function $f \in L^\infty_c(\mathbb{R}^n)$ supported in B. Recall from the proof of Theorem 4.1, estimate (4.10), that we have (via the Gaussian upper bounds (3.1) on the heat kernel of L)

$$\Big(\fint_{U_j(B)} |A_B f|^q \Big)^{1/q} \leq \beta_j \fint_B |f| \,, \tag{5.46}$$

with $\beta_j = C_1 e^{-c_1 4^j}$ if $j \geq 0$. Note that the constants C_1, c_1 depend on q and n only. Next we recall from Lemma 4.11, and in particular estimate (4.20), that

$$\Big(\fint_{U_j(B)} \big|\nabla^2 L^{-1}(I - e^{-r_B^2 L}) f\big|^q \Big)^{1/q} \leq C_2 \, e^{-c_2 4^j} \fint_B |f| \,, \qquad \forall j \geq 2. \tag{5.47}$$

Let us take $\alpha_j = Ce^{-c4^j}$ for $j \geq 0$, where $C = \max\{C_1, C_2\}$ and $c = \min\{c_2, c_3\}$. Then Theorem 5.16 (iv) is satisfied, and by (5.46) and (5.47), conditions (5.35) and (5.36) are also satisfied. This proves (ii) and (iv).

Finally we turn to condition (iii) of Theorem 5.16. Let $f \in L^\infty_c(\mathbb{R}^n)$ and fix a ball B with $r_B > 12\sqrt{n}\, \gamma(x_B)$. We write

$$f = \sum_{j=0}^{\infty} f \mathbf{1}_{U_j(B)} =: \sum_{j=0}^{\infty} f_j \,.$$

Then

$$\Big(\frac{1}{\psi_\eta(B)} \fint_B |T^* f|^{q'} \Big)^{1/q'} \;\leq\; \sum_{j=0}^{\infty} \Big(\frac{1}{\psi_\eta(B)} \fint_B |T^* f_j|^{q'} \Big)^{1/q'} . \tag{5.48}$$

To estimate the terms for $j = 0, 1$, we use that T^* is bounded on $L^{q'}(\mathbb{R}^n)$ by Theorem 1.3, and that $\psi_\eta(2B) \leq 2^\eta \, \psi_\eta(B)$ to obtain, for any $x \in B$,

$$\begin{aligned}
\Big(\frac{1}{\psi_\eta(B)} \fint_B |T^* f_j|^{q'} \Big)^{1/q'} &\leq C \Big(\frac{1}{\psi_\eta(B)\, |B|} \int_B |f_j|^{q'} \Big)^{1/q'} \\
&= C \Big(\frac{\psi_\eta(2B)\, |2B|}{\psi_\eta(B)\, |B|} \Big)^{1/q'} \Big(\frac{1}{\psi_\eta(2B)} \fint_{2B} |f|^{q'} \Big)^{1/q'} \\
&\leq C \mathcal{M}^L_\eta \big(|f|^{q'}\big)(x)^{1/q'}
\end{aligned} \tag{5.49}$$

Note that C depends on n, q and η. To estimate the terms for $j \geq 2$, we first write

$$|T^* f_j(y)| = \Big| \int_0^\infty \int_{U_j(B)} \nabla_z^2 p_t(z, y) \, f(z) \, dz \, dt \Big|$$

$$\le \Big(\int_{U_j(B)} |f|^{q'}\Big)^{1/q'} \int_0^\infty \Big(\int_{U_j(B)} \big|\nabla_z^2 p_t(z,y)\big|^q \, dz\Big)^{1/q} dt \tag{5.50}$$

by Hölder's inequality. Next, using that $\psi_\eta(2^jB) \le 2^{j\eta}\,\psi_\eta(B)$ we have for any $x \in B$,

$$\begin{aligned}\Big(\int_{U_j(B)} |f|^{q'}\Big)^{1/q'} &= \big(\psi_\eta(2^jB)\,|2^jB|\big)^{1/q'}\Big(\frac{1}{\psi_\eta(2^jB)}\int_{U_j(B)} |f|^{q'}\Big)^{1/q'} \\ &\le \big(\psi_\eta(B)\,|B|\big)^{1/q'}\, 2^{j(n+\eta)/q'}\, \mathcal{M}_\eta^L\big(|f|^{q'}\big)(x)^{1/q'}. \end{aligned} \tag{5.51}$$

Therefore using (5.50) and (5.51) we obtain, for each $j \ge 2$,

$$\begin{aligned}&\Big(\frac{1}{\psi_\eta(B)} \fint_B |T^* f_j|^{q'}\Big)^{1/q'} \\ &\quad\le \frac{1}{\psi_\eta(B)^{1/q'}}\Big(\int_{U_j(B)} |f|^{q'}\Big)^{1/q'}\Big(\fint_B\Big(\int_0^\infty \big\|\nabla^2 p_t(\cdot,y)\big\|_{L^q(U_j(B))}\,dt\Big)^{q'} dy\Big)^{1/q'} \\ &\quad\le 2^{j(n+\eta)/q'}\,|B|^{1/q'}\,\mathcal{M}_\eta^L\big(|f|^{q'}\big)(x)^{1/q'}\,\mathcal{I}(j,q,B) \end{aligned} \tag{5.52}$$

where

$$\mathcal{I}(j,q,B) := \Big(\fint_B\Big(\int_0^\infty \big\|\nabla^2 p_t(\cdot,y)\big\|_{L^q(U_j(B))}\,dt\Big)^{q'} dy\Big)^{1/q'}.$$

Now we estimate the final term in (5.51) by using the heat kernel bounds in Proposition 3.7. For each $j \ge 2$ and $y \in B$, we have $|z-y| \ge 2^{j-2} r_B$. Hence for all $t > 0$ estimate (3.11) gives

$$\begin{aligned}\big\|\nabla^2 p_t(\cdot,y)\big\|_{L^q(U_j(B))} &= \big\|\nabla^2 p_t(\cdot,y)\, e^{\beta_q|\cdot-y|^2/t}\, e^{-\beta_q|\cdot-y|^2/t}\big\|_{L^q(U_j(B))} \\ &\le e^{-c4^j r_B^2/t}\big\|\nabla^2 p_t(\cdot,y)\, e^{\beta_q|\cdot-y|^2/t}\big\|_{L^q(U_j(B))} \\ &\le \frac{C}{t^{1+n/2q'}}\, e^{-c4^j r_B^2/t}\, e^{-c\left(1+t/\gamma(y)^2\right)^\delta} \\ &\le \frac{C}{t^{1+n/2q'}}\, e^{-c4^j r_B^2/t}\, e^{-c\left(1+t/r_B^2\right)^\delta}. \end{aligned} \tag{5.53}$$

In the last step we used that since $r_B > 12\sqrt{n}\,\gamma(x_B)$, then for each $y \in B$, by Lemma 2.5,

$$\gamma(y) \;\le\; C_0\gamma(x_B)\Big(1 + \frac{r_B}{\gamma(x_B)}\Big) \;<\; C_0\big(\tfrac{1}{12\sqrt{n}} + 1\big)\, r_B \;=\; C' r_B.$$

Estimate (5.53) gives us

$$\mathcal{I}(j,q,B) \;\le\; C\int_0^\infty e^{-c4^j r_B^2/t} e^{-c\left(1+t/r_B^2\right)^\delta}\,\frac{dt}{t^{1+n/2q'}} \;=\; C\left\{\mathcal{I}_j + \mathcal{II}_j\right\} \tag{5.54}$$

where

$$\mathcal{I}_j := \int_0^{2^j r_B^2} e^{-c4^j r_B^2/t} e^{-c\left(1+t/r_B^2\right)^\delta} \frac{dt}{t^{1+n/2q'}},$$

$$\mathcal{II}_j := \int_{2^j r_B^2}^{\infty} e^{-c4^j r_B^2/t} e^{-c\left(1+t/r_B^2\right)^\delta} \frac{dt}{t^{1+n/2q'}}.$$

To estimate the first term we observe that since $t \leq 2^j r_B^2$ then $e^{-c4^j r_B^2/t} \leq e^{-c2^j}$, so that

$$\begin{aligned} \mathcal{I}_j &\leq Ce^{-c2^j} \int_0^{2^j r_B^2} \Big(\frac{t}{4^j r_B^2}\Big)^{1+n/2q'} \frac{dt}{t^{1+n/2q'}} \\ &\leq \frac{Ce^{-c2^j}}{4^{j(1+n/2q')} r_B^{2+n/q'}} \int_0^{2^j r_B^2} dt \\ &= \frac{Ce^{-c2^j}}{2^{j(1+n/q')} r_B^{n/q'}} \leq \frac{Ce^{-c2^{j\delta}}}{r_B^{n/q'}} \end{aligned} \tag{5.55}$$

since $0 < \delta < 1$. To estimate the second term we observe now that $t \geq 2^j r_B^2$ implies that $e^{-c\left(1+t/r_B^2\right)^\delta} \leq e^{-c2^{j\delta}}$, and hence

$$\mathcal{II}_j \leq \int_{2^j r_B^2}^{\infty} e^{-c\left(1+t/r_B^2\right)^\delta} \frac{dt}{t^{1+n/2q'}} \leq Ce^{-c2^{j\delta}} \int_{2^j r_B^2}^{\infty} \frac{dt}{t^{1+n/2q'}} \leq \frac{Ce^{-c2^{j\delta}}}{2^{jn/2q'} r_B^{n/q'}} \leq \frac{Ce^{-c2^{j\delta}}}{r_B^{n/q'}}. \tag{5.56}$$

By collecting (5.55) and (5.56) into (5.54), and then inserting the result into (5.52), gives for each $j \geq 2$,

$$\Big(\frac{1}{\psi_\eta(B)} \fint_B |T^* f_j|^{q'}\Big)^{1/q'} \leq C\, 2^{j(n+\eta)/q'}\, e^{-c2^{j\delta}}\, \mathcal{M}_\eta^L\big(|f|^{q'}\big)(x)^{1/q'} \tag{5.57}$$

for any $x \in B$. Finally on combining (5.57) with (5.49) into (5.48) we have, for every $x \in B$,

$$\Big(\frac{1}{\psi_\eta(B)} \fint_B |T^* f|^{q'}\Big)^{1/q'} \leq C\mathcal{M}_\eta^L\big(|f|^{q'}\big)(x)^{1/q'} \Big\{1 + \sum_{j=2}^{\infty} 2^{j(n+\eta)/q'} e^{-c2^{j\delta}}\Big\} \leq C_4 \mathcal{M}_\eta^L\big(|f|^{q'}\big)(x)^{1/q'}$$

which gives us (5.37) with $\widetilde{C} = C_4$.

Therefore Theorem 5.16 applies and we obtain the required result for $T = \nabla^2 L^{-1}$. We can obtain the result for the operator VL^{-1} in a similar manner but with estimate (3.12) in place of (3.12), and (4.21) in place of (4.20). We omit these details, but we refer the reader to the next section for a different proof. $\square$

5.3.1 An alternate proof of the estimate for VL^{-1}

In this section we give an alternate proof of Theorem 5.1 for the operator VL^{-1} that utilises the techniques of [30]. This proof does not seem to work for the operator $\nabla^2 L^{-1}$.

We first require a definition of the localised $\mathcal{A}_\infty^{L,loc}$ weights, first defined in [30]. It is clear that these include the $\mathcal{A}_\infty^{L}$ weights.

Definition 5.17. *Let w be a non-negative locally integrable function. For $p \in (1,\infty)$ we say that $w \in \mathcal{A}_p^{L,loc}$ if there exists $C = C(w,p)$ such that for all balls B with $r_B \le \gamma(x_B)$,*

$$\Big(\fint_B w\Big)^{1/p}\Big(\fint_B w^{1-p'}\Big)^{1/p'} \le C.$$

We say that $w \in \mathcal{A}_1^{L,loc}$ if there exists $C = C(w)$ such that for all balls B with $r_B \le \gamma(x_B)$,

$$\fint_B w \le Cw(x), \qquad\qquad a.e.\ x \in B.$$

We also set $\mathcal{A}_\infty^{L,loc} := \bigcup_{1\le p<\infty} \mathcal{A}_p^{L,loc}$.

We next define certain maximal operators related to these weights. Special cases of these operators were previously introduced in [30]. For each $s \ge 1$ we define a localised maximal operator associated to γ as follows.

$$\mathcal{M}_s^{loc} f(x) := \sup_{B\ni x}\Big(\frac{1}{|B|}\int_{B\cap\mathfrak{B}^\gamma(x)} |f(y)|^s\ dy\Big)^{1/s}.$$

Given a sequence $\alpha := \{\alpha(k)\}_{k=0}^\infty \in l^1(\mathbb{R}^n)$ we set

$$\mathcal{G}_s^\alpha f(x) := \sum_{k=0}^\infty \alpha(k)\Big(\fint_{2^k\mathfrak{B}^\gamma(x)} |f(y)|^s\ dy\Big)^{1/s}.$$

Then the following result holds.

Lemma 5.18. *Fix $s \ge 1$ and $\theta \ge 0$. Let $\widetilde{N}$ be the constant in Remark 2.7. Then for each $p > s$,*

(a) *the operator $\mathcal{M}_s^{loc}$ is bounded on $L^p(w)$ if and only if $w \in \mathcal{A}_{p/s}^{L,loc}$,*

(b) *if α satisfies $\sum_k \alpha(k)2^{k(\theta/s+\widetilde{N}/p)} < \infty$ for every $\epsilon > 0$, then the operator $\mathcal{G}_s^\alpha$ is bounded on $L^p(w)$ for each $w \in \mathcal{A}_{p/s}^{L,\theta}$.*

Proof. For the proof of part (a) we note that the authors in [30] show that for each $p \in (1,\infty)$ the operator $\mathcal{M}_1^{loc}$ is bounded on $L^p(w)$ if and only if $w \in \mathcal{A}_p^{L,loc}$ (see Theorem 1 and Remark 1 in [30]). It follows easily then that $\mathcal{M}_1^{loc}$ is bounded on $L^{p/s}(w)$ if and only if $w \in \mathcal{A}_{p/s}^{L,loc}$ for each $p > s$. Therefore

$$\left\|\mathcal{M}_s^{loc}(f)\right\|_{L^p(w)}^p = \left\|\mathcal{M}_1^{loc}\left(|f|^s\right)\right\|_{L^{p/s}(w)}^p \lesssim \left\||f|^s\right\|_{L^{p/s}(w)}^p = \|f\|_{L^p(w)}^p .$$

We turn to the proof of (b). The argument given here essentially follows that given in estimate (20) of [30].

Let $\left\{\mathfrak{B}_j^\gamma\right\}_j$ be the covering of $\mathbb{R}^n$ given in Lemma 2.6, and let $\left\{\widetilde{\mathfrak{B}_j^\gamma}\right\}_j$ be the dilation of this covering specified in Remark 2.7. Then one may use this covering to write

$$\begin{aligned}
\|\mathcal{G}_s^\alpha f\|_{L^p(w)} &= \Big(\int_{\mathbb{R}^n}\Big|\sum_{k=0}^\infty \alpha(k)\Big(⨍_{2^k\mathfrak{B}^\gamma(x)} |f(y)|^s\,dy\Big)^{1/s}\Big|^p w(x)\,dx\Big)^{1/p} \\
&\le \sum_{k=0}^\infty \alpha(k)\Big(\int_{\mathbb{R}^n}\Big(⨍_{2^k\mathfrak{B}^\gamma(x)} |f(y)|^s\,dy\Big)^{p/s} w(x)\,dx\Big)^{1/p} \\
&\le \sum_{k=0}^\infty \alpha(k)\Big\{\sum_j \int_{\mathfrak{B}_j^\gamma}\Big(⨍_{2^k\mathfrak{B}^\gamma(x)} |f(y)|^s\,dy\Big)^{p/s} w(x)\,dx\Big\}^{1/p} \\
&=: \sum_{k=0}^\infty \alpha(k)\Big\{\sum_j I(j,k)\Big\}^{1/p}.
\end{aligned}$$

Now for each j and $x \in \mathfrak{B}_j^\gamma$ we have $2^k\mathfrak{B}^\gamma(x) \subset 2^k\widetilde{\mathfrak{B}_j^\gamma}$ for all $k \ge 0$. Also by (2.10), $x \in \mathfrak{B}_j^\gamma$ implies that $\gamma(x) \approx \gamma(x_{\mathfrak{B}_j^\gamma})$, and hence $|\mathfrak{B}^\gamma(x)| \approx \left|\mathfrak{B}_j^\gamma\right|$. Since $\left|\widetilde{\mathfrak{B}_j^\gamma}\right| = \sigma^n\left|\mathfrak{B}_j^\gamma\right|$, then $\left|2^k\widetilde{\mathfrak{B}_j^\gamma}\right| \approx \left|2^k\mathfrak{B}^\gamma(x)\right|$. This combined with Hölder's inequality with exponents p/s and $p/(p-s)$ gives for each j, k, and $x \in \mathfrak{B}_j^\gamma$,

$$\begin{aligned}
\Big(⨍_{2^k\mathfrak{B}^\gamma(x)} |f(y)|^s\,dy\Big)^{p/s} &\lesssim \Big(⨍_{2^k\widetilde{\mathfrak{B}_j^\gamma}} |f(y)|^s\,dy\Big)^{p/s} \\
&= \Big(⨍_{2^k\widetilde{\mathfrak{B}_j^\gamma}} |f(y)|^s\,w(y)^{s/p}w(y)^{-s/p}dy\Big)^{p/s} \\
&\le \Big(⨍_{2^k\widetilde{\mathfrak{B}_j^\gamma}} |f(y)|^p\,w(y)\,dy\Big)\Big(⨍_{2^k\widetilde{\mathfrak{B}_j^\gamma}} w(y)^{-s/(p-s)}\Big)^{(p-s)/s}.
\end{aligned}$$

Since $w \in \mathcal{A}_{p/s}^{L,\theta}$ then

$$\begin{aligned}
I(j,k) &\le \frac{w(\mathfrak{B}_j^\gamma)}{\left|2^k\widetilde{\mathfrak{B}_j^\gamma}\right|}\Big(⨍_{2^k\widetilde{\mathfrak{B}_j^\gamma}} w(y)^{-s/(p-s)}dy\Big)^{(p-s)/s}\Big(\int_{2^k\widetilde{\mathfrak{B}_j^\gamma}} |f(y)|^p\,w(y)\,dy\Big) \\
&\le \Big(⨍_{2^k\widetilde{\mathfrak{B}_j^\gamma}} w(y)\,dy\Big)\Big(⨍_{2^k\widetilde{\mathfrak{B}_j^\gamma}} w(y)^{-s/(p-s)}dy\Big)^{(p-s)/s}\int_{2^k\widetilde{\mathfrak{B}_j^\gamma}} |f(y)|^p\,w(y)\,dy
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(1+\frac{2^k\,\sigma\, r_{\mathfrak{B}_j^\gamma}}{\gamma(x_{\mathfrak{B}_j^\gamma})}\right)^{\theta p/s} \int_{2^k\widetilde{\mathfrak{B}_j^\gamma}} |f(y)|^p\, w(y)\, dy \\
&= (1+2^k\sigma)^{\theta p/s} \int_{2^k\widetilde{\mathfrak{B}_j^\gamma}} |f(y)|^p\, w(y)\, dy\,.
\end{aligned}
$$

Collecting these estimates for j and k and using the bounded overlap of the family $2^k\widetilde{\mathfrak{B}_j^\gamma}$ we obtain

$$
\begin{aligned}
\|\mathcal{G}_s^\alpha f\|_{L^p(w)} &\lesssim \sum_{k=0}^\infty \alpha(k) \left\{ \sum_j (1+2^k\sigma)^{\theta p/s} \int_{2^k\widetilde{\mathfrak{B}_j^\gamma}} |f(y)|^p\, w(y)\, dy \right\}^{1/p} \\
&= \sum_{k=0}^\infty \alpha(k)(1+2^k\sigma)^{\theta/s} \left\{ \sum_j \int_{2^k\widetilde{\mathfrak{B}_j^\gamma}} |f(y)|^p\, w(y)\, dy \right\}^{1/p} \\
&\lesssim \|f\|_{L^p(w)} \sum_{k=0}^\infty \alpha(k)(1+2^k\sigma)^{\theta/s} 2^{k\widetilde{N}/p} \\
&\lesssim \|f\|_{L^p(w)} \sum_{k=0}^\infty 2^{k(\theta/s+\widetilde{N}/p)},
\end{aligned}
$$

and using the hypothesis that $\sum_k 2^{k(\theta/s+\widetilde{N}/p)} < \infty$ we are done. □

We now give the main result of this section. It coincides with Theorem 5.1 for the VL^{-1} operator.

Theorem 5.19. *Let $n \geq 3$ and $L = -\Delta + V$ on $\mathbb{R}^n$, and assume that $V \in \mathcal{B}_q$ for some $q > n/2$. Then the operator VL^{-1} is bounded on $L^p(w)$ for each $p \in (1,q)$, $\theta \geq 0$ and $w^{1-p'} \in \mathcal{A}^{L,\theta}_{p'/q'}$.*

Proof of Theorem 5.19. Set $T := VL^{-1}$. We shall obtain the theorem by studying the dual operator $T^* = L^{-1}V$. Our strategy is to split T^* into its 'local' and 'global' parts, and show that they can be controlled by the two maximal operators $\mathcal{M}^{loc}$ and $\mathcal{G}^\alpha$ for a suitable $\alpha = \{\alpha(k)\}$. More precisely we split $T^*f(x) = T_0^*f(x) + T_\infty^*f(x)$ where

$$
\begin{aligned}
T_0^*f(x) &:= \int_{\mathfrak{B}^\gamma(x)} \int_0^\infty V(y)\, p_t(y,x)\, dt\, f(y)\, dy, \\
T_\infty^*f(x) &:= \int_{\mathbb{R}^n\setminus\mathfrak{B}^\gamma(x)} \int_0^\infty V(y)\, p_t(y,x)\, dt\, f(y)\, dy.
\end{aligned}
$$

We shall prove that there exists $C > 0$ such that for almost every x the following hold

$$
|T_0^*f(x)| \leq C\, \mathcal{M}_{q'}^{loc} f(x), \tag{5.58}
$$

$$
|T_\infty^*f(x)| \leq C\, \mathcal{G}_{q'}^{\alpha} f(x). \tag{5.59}
$$

Here $\alpha(k) = e^{-c2^{k\delta}}$, for some $c > 0$ fixed, and δ is the constant in Proposition 3.7. By Lemma 5.18 this shows that T^* is bounded on $L^p(w)$ for all $p > q'$ and all $w \in \mathcal{A}^{L,\theta}_{p/q'}$ for any $\theta \geq 0$. By duality (combining Remark 4.7 (a) and Lemma 5.6 (iii)) this gives the $L^p(w)$-boundedness of T for each $p \in (1, q)$, $\theta \geq 0$, and $w^{1-p'} \in \mathcal{A}^{L,\theta}_{p'/q'}$.

We first show (5.59). We write

$$
\begin{aligned}
T^*_\infty f(x) &= \sum_{j=1}^{\infty} \int_{U_j(\mathfrak{B}^\gamma(x))} \int_0^\infty V(y)\, p_t(y,x)\, dt\, f(y)\, dy \\
&\leq \sum_{j=1}^{\infty} \left|2^j \mathfrak{B}^\gamma(x)\right|^{1/q'} \Big(⨍_{U_j(\mathfrak{B}^\gamma(x))} |f(y)|^{q'}\, dy \Big)^{1/q'} \times \\
&\qquad\qquad \Big(\int_{U_j(\mathfrak{B}^\gamma(x))} \Big| \int_0^\infty V(y)\, p_t(y,x)\, dt \Big|^q dy \Big)^{1/q}.
\end{aligned}
$$

Now for each $j \geq 1$, and $y \in U_j(\mathfrak{B}^\gamma(x))$ we have that $|x-y| > 2^{j-1}\gamma(x)$. Let β_q be the constant from Proposition 3.7. Then by (3.12),

$$
\begin{aligned}
\Big(\int_{U_j(\mathfrak{B}^\gamma(x))} V(y)\, p_t(y,x)^q\, dy \Big)^{1/q} &= \Big(\int_{U_j(\mathfrak{B}^\gamma(x))} V(y)\, p_t(y,x)^q e^{\beta_q |x-y|^2/t} e^{-\beta_q |x-y|^2/t}\, dy \Big)^{1/q} \\
&\leq e^{-c4^j \gamma(x)^2/t} \Big(\int_{\mathbb{R}^n} V(y)\, p_t(y,x)^q e^{\beta_q |x-y|^2/t}\, dy \Big)^{1/q} \\
&\leq \frac{C}{t^{1+n/2q'}} e^{-c4^j\gamma(x)^2/t} e^{-c\left(1+t/\gamma(x)^2\right)^\delta}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\Big(\int_{U_j(\mathfrak{B}^\gamma(x))} \Big| \int_0^\infty V(y)\, p_t(y,x)\, dt \Big|^q dy \Big)^{1/q} &\leq \int_0^\infty \Big(\int_{U_j(\mathfrak{B}^\gamma(x))} V(y)\, p_t(y,x)^q dy \Big)^{1/q} dt \\
&\leq C \int_0^\infty e^{-c4^j\gamma(x)^2/t} e^{-c\left(1+t/\gamma(x)^2\right)^\delta} \frac{dt}{t^{1+n/2q'}} \\
&=: C \left\{ I_j + II_j \right\},
\end{aligned}
$$

where

$$
\begin{aligned}
I_j &:= \int_0^{2^j\gamma(x)^2} e^{-c4^j\gamma(x)^2/t} e^{-c\left(1+t/\gamma(x)^2\right)^\delta} \frac{dt}{t^{1+n/2q'}}\,, \\
II_j &:= \int_{2^j\gamma(x)^2}^\infty e^{-c4^j\gamma(x)^2/t} e^{-c\left(1+t/\gamma(x)^2\right)^\delta} \frac{dt}{t^{1+n/2q'}}\,.
\end{aligned}
$$

To estimate the first term we observe that since $t \leq 2^j\gamma(x)^2$ then $e^{-c4^j\gamma(x)^2/t} \leq e^{-c2^j}$. Hence

$$
I_j \leq Ce^{-c2^j} \int_0^{2^j\gamma(x)^2} \Big(\frac{t}{4^j\gamma(x)^2} \Big)^{1+n/2q'} \frac{dt}{t^{1+n/2q'}}
$$

$$
\begin{aligned}
&\le \frac{Ce^{-c2^j}}{4^{j(1+n/2q')}\gamma(x)^{2+n/q'}} \int_0^{2^j\gamma(x)^2} dt \\
&= \frac{Ce^{-c2^j}}{2^{j(1+n/2q')}\gamma(x)^{n/q'}} \;\le\; \frac{Ce^{-c2^{j\delta}}}{\gamma(x)^{n/q'}}
\end{aligned}
$$

since $0 < \delta < 1$. To estimate the second term we observe now that $t \ge 2^j\gamma(x)^2$ implies that $e^{-c\left(1+t/\gamma(x)^2\right)^\delta} \le e^{-c2^{j\delta}}$. Hence

$$
\begin{aligned}
II_j &\le \int_{2^j\gamma(x)^2}^\infty e^{-c\left(1+t/\gamma(x)^2\right)^\delta} \frac{dt}{t^{1+n/2q'}} \\
&\le Ce^{-c2^{j\delta}} \int_{2^j\gamma(x)^2}^\infty \frac{dt}{t^{1+n/2q'}} \\
&\le \frac{Ce^{-c2^{j\delta}}}{2^{jn/2q'}\gamma(x)^{n/q'}} \;\le\; \frac{Ce^{-c2^{j\delta}}}{\gamma(x)^{n/q'}}\,.
\end{aligned}
$$

Collecting these estimates for $j \ge 1$ gives

$$
\begin{aligned}
|T_\infty^* f(x)| &\le C\sum_{j=1}^\infty \left|2^j\mathfrak{B}^\gamma(x)\right|^{1/q'} \frac{e^{-c2^{j\delta}}}{\gamma(x)^{n/q'}} \Big(⨍_{U_j(\mathfrak{B}^\gamma(x))} |f(y)|^{q'}\,dy\Big)^{1/q'} \\
&\le C\sum_{j=1}^\infty 2^{jn/q'} e^{-c2^{j\delta}} \Big(⨍_{2^j\mathfrak{B}^\gamma(x)} |f(y)|^{q'}\,dy\Big)^{1/q'} \\
&\le C\sum_{j=1}^\infty e^{-c'e^{j\delta}} \Big(⨍_{2^j\mathfrak{B}^\gamma(x)} |f(y)|^{q'}\,dy\Big)^{1/q'} \\
&= C\,\mathcal{G}_{q'}^\alpha f(x)
\end{aligned}
$$

with $\alpha(j) = e^{-c'2^{j\delta}}$. This gives (5.59).

We now consider estimate (5.58). Write

$$
\begin{aligned}
T_0^* f(x) &= \sum_{j\le 0} \int_{U_j(\mathfrak{B}^\gamma(x))} \int_0^\infty p_t(y,x)\,V(y)\,dt\,f(y)\,dy \\
&\le \sum_{j\le 0} \left|2^j\mathfrak{B}^\gamma(x)\right| \Big(⨍_{U_j(\mathfrak{B}^\gamma(x))} |f(y)|^{q'}\,dy\Big)^{1/q'} \times \\
&\qquad\qquad \Big(⨍_{U_j(\mathfrak{B}^\gamma(x))} \Big|\int_0^\infty V(y)\,p_t(y,x)\,dt\Big|^q dy\Big)^{1/q}.
\end{aligned}
$$

Now for each $j \le 0$ we have $2^j\mathfrak{B}^\gamma(x) \subseteq \mathfrak{B}^\gamma(x)$ and hence

$$
\Big(⨍_{2^j\mathfrak{B}^\gamma(x)} |f(y)|^{q'}\,dy\Big)^{1/q'} \;\le\; \mathcal{M}_{q'}^{loc} f(x).
$$

Also if $y \in 2^j\mathfrak{B}^\gamma(x)\backslash 2^{j-1}\mathfrak{B}^\gamma(x)$ then by the heat kernel bounds (3.6),

$$
\Big(⨍_{U_j(\mathfrak{B}^\gamma(x))} \Big|\int_0^\infty V(y)\,p_t(y,x)\,dt\Big|^q dy\Big)^{1/q}
$$

$$\leq \int_0^\infty \Big(\fint_{U_j(\mathfrak{B}^\gamma(x))} V(y)^q p_t(y,x)^q dy \Big)^{1/q} dt$$
$$\leq C \Big(\fint_{2^j \mathfrak{B}^\gamma(x)} V(y)^q dy \Big)^{1/q} \int_0^\infty e^{-c4^j\gamma(x)^2/t} e^{-c\left(1+t/\gamma(x)^2\right)^\delta} \frac{dt}{t^{n/2}} \,.$$

Since $V \in \mathcal{B}_q$ then Lemma 2.3 (a) applied to the ball $\mathfrak{B}^\gamma(x)$ with $\lambda = 2^{-j} \geq 1$ gives

$$\begin{aligned}
\Big(\fint_{2^j \mathfrak{B}^\gamma(x)} V(y)^q dy \Big)^{1/q} &\leq C \fint_{2^j \mathfrak{B}^\gamma(x)} V(y)\, dy \\
&= C \big(2^j \gamma(x)\big)^{-2} \big(2^j \gamma(x)\big)^2 \fint_{2^j \mathfrak{B}^\gamma(x)} V(y)\, dy \\
&\leq C \big(2^j \gamma(x)\big)^{-2} \Big(\frac{2^j \gamma(x)}{\gamma(x)} \Big)^{2-n/q} \gamma(x)^2 \fint_{\mathfrak{B}^\gamma(x)} V(y)\, dy \\
&\leq C 2^{-jn/q} \gamma(x)^{-2}.
\end{aligned}$$

Next we write for each $j \leq 0$,

$$\int_0^\infty e^{-c4^j\gamma(x)^2/t} e^{-c\left(1+t/\gamma(x)^2\right)^\delta} \frac{dt}{t^{n/2}} \;\leq\; \int_0^{4^j\gamma(x)^2} e^{-c4^j\gamma(x)^2/t} \frac{dt}{t^{n/2}} + \int_{4^j\gamma(x)^2}^\infty \frac{dt}{t^{n/2}} \;=:\; I_j + II_j \,.$$

Then

$$I_j \;\leq\; C \int_0^{4^j\gamma(x)^2} \Big(\frac{t}{4^j \gamma(x)^2} \Big)^{n/2} \frac{dt}{t^{n/2}} \;=\; C\, 2^{j(2-n)}\, \gamma(x)^{2-n}$$

and since $n \geq 3$,

$$II_j \;\leq\; \frac{C}{\big(4^j \gamma(x)^2\big)^{n/2-1}} \;=\; C\, 2^{j(2-n)}\, \gamma(x)^{2-n}.$$

Collecting these estimates we obtain

$$\begin{aligned}
|T_0^* f(x)| &\leq C\, \mathcal{M}_{q'}^{loc} f(x) \sum_{j \leq 0} 2^{j(2-n/q-n)} \gamma(x)^{-n} \left| 2^j \mathfrak{B}^\gamma(x) \right| \\
&\leq C\, \mathcal{M}_{q'}^{loc} f(x) \sum_{j \leq 0} 2^{j(2-n/q)} \\
&\leq C\, \mathcal{M}_{q'}^{loc} f(x)
\end{aligned}$$

with the sum being convergent because $q > n/2$.

This concludes the proof of estimate (5.58) and the theorem. $\square$

Chapter 6

Morrey spaces and Muckenhoupt weights

Let $p \in [1, \infty)$ and $\lambda \in (0, n)$. A function f is said to belong to the Morrey space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{\mathcal{L}^{p,\lambda}} := \sup_B \left(r_B^\lambda \fint_B |f - f_B|^p \right)^{1/p} < \infty.$$

It is possible to take $\lambda \in (-\infty, n\,]$ in the definition, but outside the range of $(0, n)$ the spaces coincide with other well known spaces. When $\lambda \in (-n, 0)$ the Morrey spaces coincide with the Lipschitz spaces of order λ. For $\lambda < 0$ the spaces are also commonly known as the Morrey-Campanato spaces. We also have $\mathcal{L}^{p,0} = \mathrm{BMO}$ and $\mathcal{L}^{p,n} = L^p$. Some standard references for these spaces include [33, 86, 87, 99].

In this chapter we study the Riesz transforms associated to Schrödinger operators on the Morrey spaces. The main results of this chapter are the following two theorems.

Theorem 6.1. *Fix $s > 2$. Let $n \geq 1$ and $L = -\Delta + V$ on $\mathbb{R}^n$ with $0 \leq V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then the following are equivalent.*

(a) *The operator $\nabla L^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for each $p \in (1, s)$.*

(b) *The operator $\nabla L^{-1/2}$ is bounded on $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ or each $p \in (1, s)$ and each $\lambda \in \left(\frac{n}{s}p, n\right)$.*

We mention that a corresponding result also holds for the operator $V^{1/2}L^{-1/2}$ but we do not give it here.

Next we specialise to the case that V is a reverse Hölder potential. Note that q^* has been defined in Section 2.2 (see also (1.6)).

Theorem 6.2. *Let $n \geq 1$ and suppose $V \in \mathcal{B}_q$ for some $q \geq n/2$. Then the following holds.*

(a) *The operator $\nabla L^{-1/2}$ is bounded on $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ for all $p \in (1, q^*)$ and $\lambda \in \Big(\dfrac{n}{q^*}p, n\Big)$.*

(b) *The operator $V^{1/2}L^{-1/2}$ is bounded on $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ for all $p \in (1, 2q)$ and $\lambda \in \Big(\dfrac{n}{2q}p, n\Big)$.*

(c) *If $n \geq 2$ then the operators $\nabla^2 L^{-1}$ and VL^{-1} are bounded on $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ for each $p \in (1, q)$ and $\lambda \in \Big(\dfrac{n}{q}p, n\Big)$.*

We direct the reader to the discussion in Sections 1.1.2 and 1.2 for a comparison of the above two results with the known results from the literature.

The first objective of this chapter is to develop the techniques needed to prove these results. This involves a principle that allows us to obtain Morrey space estimates from weighted L^p estimates, with weights from the Muckenhoupt class (Theorem 6.15). The second objective is to apply this to prove Theorem 6.2 and Theorem 6.1.

This chapter is organised as follows. The first section gives an exposition of the main technique in [9, 34, 42]. This is encapsulated in Lemma 6.3. We then show how this can be used to obtain Morrey estimates from weighted estimates. We also apply these to give a new maximal theorem (Theorem 6.11) for fractional type operators on Morrey spaces in the spirit of [17]. The next two sections form the main parts of this chapter, and may be read independently of the first. Section 6.2 applies the ideas in [4, 5] to improve upon some of the results from the first section, while Section 6.3 applies the results from Sections 6.1 and 6.2 to prove Theorems 6.2 and 6.1.

6.1 From Muckenhoupt weights to Morrey spaces I

The following appears in [34, 42] in the proof of their main results but is not explicitly stated.

Lemma 6.3. *Let $\delta \in (0, 1)$, $\lambda \in \big(n(1-\delta), n\big)$, and B be a ball in $\mathbb{R}^n$. Assume that h is a non-negative function for which $h(M\mathbf{1}_B)^\delta \in L^1(\mathbb{R}^n)$. Then there exists $C = C(n, \lambda, \delta) > 0$*

such that

$$\int_{\mathbb{R}^n} h(x)\big(M\mathbf{1}_B(x)\big)^\delta\,dx \;\le\; C\,|B|\,r_B^{-\lambda}\,\|h\|_{\mathcal{L}^{1,\lambda}}\,.$$

Proof. We shall need the following estimate: there exists $C = C(n) > 0$ such that for every ball B,

$$M\mathbf{1}_B(x) \;\le\; C\frac{r_B^n}{(|x-x_B|-r_B)^n}\,. \tag{6.1}$$

Indeed,

$$M\mathbf{1}_B(x) \;=\; \sup_{\widetilde{B}\ni x} \fint_{\widetilde{B}} \mathbf{1}_B(y)\,dy \;=\; \sup_{\widetilde{B}\ni x} \frac{|B\cap\widetilde{B}|}{|\widetilde{B}|}\,.$$

Now the quantity $|B\cap\widetilde{B}|$ is maximised when $\widetilde{B}$ covers B, so that $|B\cap\widetilde{B}| = |B| \approx r_B^n$. The quantity $|\widetilde{B}|$ is minimised when $\widetilde{B}$ just touches B (recall that $\widetilde{B}$ must also contain x). In this case we have $|\widetilde{B}| \approx (|x-x_B|-r_B)^n$. Hence for each ball $\widetilde{B}$ containing x, we have

$$\frac{|B\cap\widetilde{B}|}{|\widetilde{B}|} \;\le\; C\frac{r_B^n}{(|x-x_B|-r_B)^n}\,.$$

which gives (6.1).

With estimate (6.1) in hand, then for each $j\ge 2$ and $x\in U_j(B)$, one has $|x-x_B|-r_B \ge 2^{j-2}r_B$, so that

$$\big(M\mathbf{1}_B(x)\big)^\delta \;\le\; C\frac{r_B^{n\delta}}{(|x-x_B|-r_B)^{n\delta}} \;\le\; C2^{-jn\delta}\,.$$

This combined with the fact that $\big(M\mathbf{1}_B\big)^\delta \le 1$, gives

$$\begin{aligned}
\int_{\mathbb{R}^n} h(x)\big(M\mathbf{1}_B(x)\big)^\delta\,dx &= \int_{2B} h(x)\big(M\mathbf{1}_B(x)\big)^\delta\,dx + \sum_{j=2}^{\infty}\int_{U_j(B)} h(x)(M\mathbf{1}_B(x))^\delta\,dx \\
&\lesssim \int_{2B} h(x)\,dx + \sum_{j=2}^{\infty} 2^{-jn\delta}\int_{U_j(B)} h(x)\,dx \\
&\lesssim |B|\,r_B^{-\lambda}\,\|h\|_{\mathcal{L}^{1,\lambda}}\left\{1+\sum_{j=2}^{\infty}2^{-j(n\delta+\lambda-n)}\right\} \\
&\lesssim |B|\,r_B^{-\lambda}\,\|h\|_{\mathcal{L}^{1,\lambda}}\,.
\end{aligned}$$

Note that the sum is convergent in the last line because the hypotheses on λ and δ ensure $n\delta+\lambda-n>0$. $\square$

We illustrate the usefulness of this Lemma by giving some applications. These applications are based on the well known fact (due to Coifman and Rochberg [36]) that $(M\mathbf{1}_B)^\delta$ is an $\mathcal{A}_1$ weight whenever $\delta \in (0,1)$. We record this here.

Lemma 6.4 ([59], Theorem 3.4 p158)**.** *Let μ be any Borel measure such that $M\mu < \infty$ almost everywhere. Let $\delta \in (0,1)$. Then $(M\mu)^\delta \in \mathcal{A}_1$, with constant depending on n and δ.*

In particular $(M\mathbf{1}_B)^\delta$ is an $\mathcal{A}_1$ weight for each ball B and $\delta \in (0,1)$. Note also that we have $\mu \leq (M\mu)^\delta \leq \|\mu\|_\infty$.

Our first application is to show that estimates in weighted spaces with Muckenhoupt weights imply estimates in Morrey spaces. A more general version of this appears in Theorem 3.1 of [9] (see Theorem 1.11 in Chapter 1).

Proposition 6.5. *Let $p \in (0,\infty)$ and F and G be a pair of functions satisfying the following property: for each $w \in \mathcal{A}_1$ there exists $C_0 = C_0(w,p) > 0$ such that*

$$\|F\|_{L^p(w)} \leq C_0 \|G\|_{L^p(w)}. \tag{6.2}$$

Then it follows that for each $\lambda \in (0,n)$ there exists $C_1 = C_1(p,\lambda,n) > 0$ such that

$$\|F\|_{\mathcal{L}^{p,\lambda}} \leq C_1 \|G\|_{\mathcal{L}^{p,\lambda}}.$$

A straightforward consequence of this result is that if T is an operator bounded on $L^p(w)$ for every $p \in (1,\infty)$ and $w \in \mathcal{A}_p$, then it is bounded on $\mathcal{L}^{p,\lambda}$ for each $p \in (1,\infty)$ and $\lambda \in (0,n)$. In particular this applies to the Hardy–Littlewood maximal function and to Calderón–Zygmund operators, which recovers the results in [34, 86].

Proof of Proposition 6.5. Fix a ball B and $\delta \in \big(1-\lambda/n, 1\big)$. Then by Lemma 6.4 we have that $(M\mathbf{1}_B)^\delta \in \mathcal{A}_1$. The hypothesis (6.2) gives

$$\int_B |F|^p \;\leq\; \int_B |F|^p\,(M\mathbf{1}_B)^\delta \;\leq\; C\int |G|^p\,(M\mathbf{1}_B)^\delta.$$

Now we apply Lemma 6.3 with $h := |G|^p$ to obtain

$$\int_B |F|^p \;\leq\; C\,|B|\,r_B^{-\lambda}\big\||G|^p\big\|_{\mathcal{L}^{1,\lambda}} \;=\; C\,|B|\,r_B^{-\lambda}\,\|G\|^p_{\mathcal{L}^{p,\lambda}}.$$

Since the constant C depends on n, λ, δ, C_0, and is independent of B we obtain

$$r_B^{\lambda} \fint_B |F|^p \;\leq\; C \, \|G\|^p_{\mathcal{L}^{p,\lambda}} \,.$$

Taking supremum over all balls gives the required result. $\square$

A corresponding weak type result is also possible. For $p \geq 1$ and $\lambda \in (0, n)$ we define the weak Morrey space $\mathcal{L}^{p,\lambda}_{\infty}(\mathbb{R}^n)$ as

$$\mathcal{L}^{p,\lambda}_{\infty}(\mathbb{R}^n) := \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{L}^{p,\lambda}_{\infty}} < \infty\},$$

where

$$\|f\|_{\mathcal{L}^{p,\lambda}_{\infty}} := \inf\Big\{C > 0 : \; \alpha^p \sup_B \Big(\frac{r_B^{\lambda}}{|B|}\big|\{x \in B : |f(x)| > \alpha\}\big|\Big) \leq C\Big\}.$$

Proposition 6.6. *Suppose F and G are a pair of measurable functions such that for some $p \in (0, \infty)$ and any $w \in \mathcal{A}_1$, there exists $C_0 = C_0(w, p) > 0$ such that*

$$\|F\|_{L^{p,\infty}(w)} \leq C_0 \, \|G\|_{L^p(w)} \,.$$

Then it follows that for each $\lambda \in (0, n)$ there exists $C_1 = C_1(p, \lambda, n) > 0$ such that

$$\|F\|_{\mathcal{L}^{p,\lambda}_{\infty}} \leq C_1 \, \|G\|_{\mathcal{L}^{p,\lambda}} \,.$$

Proof. Fix a ball B and $\delta \in \big(1 - \lambda/n, 1\big)$. Then $(M\mathbf{1}_B)^{\delta} \in \mathcal{A}_1$ and we can apply the inequality in the hypothesis to obtain

$$\begin{aligned}
\big|\{x \in B : |F(x)| > \alpha\}\big| &= \int_{\{x : |F(x)| > \alpha\}} \mathbf{1}_B(x) \, dx \\
&\leq \int_{\{x : |F(x)| > \alpha\}} (M\mathbf{1}_B)^{\delta} \, dx \\
&\leq \frac{C^p}{\alpha^p} \int_{\mathbb{R}^n} |G|^p \, (M\mathbf{1}_B)^{\delta} \, dx \,.
\end{aligned}$$

Now since $\delta > 1 - \lambda/n$, applying Lemma 6.3 to $h = |G|^p$, we have

$$\int_{\mathbb{R}^n} |G|^p \, (M\mathbf{1}_B)^{\delta} \, dx \;\leq\; C r_B^{-\lambda} \, |B| \, \|G\|^p_{\mathcal{L}^{p,\lambda}} \,.$$

Therefore

$$\alpha^p \frac{r_B^{\lambda}}{|B|}\big|\{x \in B : |F(x)| > \alpha\}\big| \;\leq\; C \, \|G\|^p_{\mathcal{L}^{p,\lambda}} \,,$$

and taking supremum over all balls gives the result as required. $\square$

6.1.1 An application to fractional powers

Our next application of Lemma 6.3 concerns fractional type operators on Morrey spaces, which are modelled on the classical Riesz potentials. These potentials are the collection of operators $(-\Delta)^{-\alpha/2}$, for $\alpha \in (0, n)$, defined as

$$(-\Delta)^{-\alpha/2} f(x) := \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{t\Delta} f(x) \frac{dt}{t^{1-\alpha/2}}$$

which, up to a constant multiple, is equivalent to the operator

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy\,.$$

A related operator is the fractional maximal operator

$$M_\alpha f(x) := \sup_{B \ni x} r_B^\alpha \fint_B |f|\,.$$

The two operators I_α and M_α are intimately related. On the one hand we have pointwise control $M_\alpha f \lesssim I_\alpha(|f|)$, and on the other hand, while the converse does not hold pointwise, one has norm equivalence. This is contained in the following result, first obtained by B. Muckenhoupt and R. Wheeden.

Theorem 6.7 ([81] Theorem 1)**.** *Let $\alpha \in (0, n)$, $p \in (0, \infty)$ and $w \in \mathcal{A}_\infty$. Then*

$$\|I_\alpha f\|_{L^p(w)} \approx \|M_\alpha f\|_{L^p(w)}\,.$$

Hence the boundedness of I_α on weighted spaces follows immediately from that of M_α. This is given in

Theorem 6.8 ([81] Theorem 3)**.** *Let $\alpha \in (0, n)$, $p \in (1, \frac{n}{\alpha})$, $1/p - 1/q = \alpha/n$. Then*

$$M_\alpha : L^p(w^p) \longrightarrow L^q(w^q) \qquad \Longleftrightarrow \qquad w \in \mathcal{A}_{1+\frac{1}{p'}} \cap \mathcal{B}_q.$$

Estimates for these operators on Morrey spaces have been obtained by several authors. Soon after the result by Muckenhoupt and Wheeden, D.R. Adams obtained the following estimates for the operators $(-\Delta)^{-\alpha/2}$.

Theorem 6.9 ([2] Theorem 3.1)**.** *Let $\alpha \in (0,n)$ and $\lambda \in (0,n)$. Suppose that p and q satisfy $p \in \big(1, \lambda/\alpha\big)$ and $1/p - 1/q = \alpha/\lambda$. Then $(-\Delta)^{-\alpha/2}$ is bounded from $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ into $\mathcal{L}^{q,\lambda}(\mathbb{R}^n)$.*

We remark that if we take $\lambda = n$, then we recover the Sobolev embedding theorem on $L^p(\mathbb{R}^n)$. Independently the following was attributed to S. Spanne, first stated by J. Peetre in [87].

Theorem 6.10 (S. Spanne, unpublished)**.** *Let $\alpha \in (0,n)$, $p \in \big(1, n/\alpha\big)$ and $\lambda \in (\alpha p, n)$. Suppose that q and μ satisfy $1/p - 1/q = \alpha/n$ and $(n-\lambda)/p = (n-\mu)/q$. Then $(-\Delta)^{-\alpha/2}$ is bounded from $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ into $\mathcal{L}^{q,\mu}(\mathbb{R}^n)$.*

In the late 1980s Chiarenza and Frasca gave simpler proofs of both results. See [34] Theorem 2 and its Corollary. They state Adams' result with an equivalent formulation:

$$\alpha \in (0,n), \qquad p \in \Big(1, \frac{n}{\alpha}\Big), \qquad \lambda \in (\alpha p, n), \qquad \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\lambda}\,.$$

They also show Spanne's result actually follows from Adams' result. See also Remark 6.12 (vi) below.

The main goal of this section is to generalise Adams' result, Theorem 6.9, to 'non-integral' operators that are of 'fractional type'. This is motivated by the study in [17]. Typical examples of such operators are $L^{-\alpha/2}$ with $\alpha \in (0,n)$, where L is a Schrödinger operator or an elliptic operator in divergence form. This is contained in the following, which is a variant of Theorem 2.2 from [17]. We will give applications of this in Section 6.3.

Theorem 6.11. *Let $\alpha \in (0,n)$ and $1 \le p_0 < s_0 < q_0 \le \infty$ be numbers such that $1/p_0 - 1/s_0 = \alpha/n$. Suppose that T is a bounded sublinear operator from $L^{p_0}(\mathbb{R}^n)$ to $L^{s_0}(\mathbb{R}^n)$, and that $\{A_B\}_B$ is a family of operators indexed by balls acting from $L^\infty_c(\mathbb{R}^n)$ into $L^{p_0}(\mathbb{R}^n)$. Assume that there exists $C_0 > 0$ such that*

$$\Big(\fint_B |T(I - A_B)f|^{s_0}\Big)^{1/s_0} \le C_0\, M_{\alpha p_0}\big(|f|^{p_0}\big)(x)^{1/p_0}, \tag{6.3}$$

$$\Big(\fint_B |TA_B f|^{q_0}\Big)^{1/q_0} \le C_0\,\big\{M\big(|Tf|^{s_0}\big)(x)^{1/s_0} + M_{\alpha p_0}\big(|f|^{p_0}\big)(y)^{1/p_0}\big\}, \tag{6.4}$$

for each $f \in L^\infty_c(\mathbb{R}^n)$, ball $B \subset \mathbb{R}^n$, and every $x, y \in B$.

Let $p_0 < p < q < q_0$ and $\lambda \in \big(p(\frac{n}{q_0} + \alpha), n\big)$ be such that $1/p - 1/q = \alpha/\lambda$. Then T is bounded from $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{L}^{q,\lambda}(\mathbb{R}^n)$.

Remark 6.12. (i) In the case $q_0 = \infty$, the left hand side of (6.4) is the essential supremum of $|TA_B|$ over B.

(ii) If L is an operator with Gaussian upper bounds on its heat kernel then Theorem 6.11 is satisfied with $T = L^{-\alpha/2}$, $A_B = I - (I - e^{-r_B^2 L})^m$, $p_0 = 1$, $q_0 = \infty$, and $m > 0$ large enough. We shall give the details for this fact in the context of the Schrödinger operator with non-negative potentials within the proof of Theorem 6.17.

(iii) Theorem 6.11 generalises Adams' result because by Remarks (i) and (ii), with $p_0 = 1$, $q_0 = \infty$, and $T = (-\Delta)^{-\alpha/2}$, we can obtain that T is bounded from $\mathcal{L}^{p,\lambda}$ to $\mathcal{L}^{q,\lambda}$, where $p \in (1, n/\alpha)$, $\lambda \in (\alpha p, n)$, and $1/p - 1/q = \alpha/\lambda$.

(iv) The fact that the set $\big(1 - \lambda/n, 1 - q/q_0\big)$ is not empty and contained in $(0,1)$ is crucial to the proof, because for each ball B and $\delta \in \big(1 - \lambda/n, 1 - q/q_0\big)$ we obtain the following two desirable properties:

(1) $\delta > 1 - \lambda/n \Longrightarrow \sum_j 2^{-j(\delta n + \lambda - n)} < \infty$, and

(2) $\delta < 1 - q/q_0 \Longrightarrow (M\mathbf{1}_B)^{\delta(q_0/q)'} \in \mathcal{A}_1$ because $\delta(q_0/q)' < 1$. This is equivalent to $(M\mathbf{1}_B)^{\delta} \in \mathcal{A}_1 \cap \mathcal{B}_{(q_0/q)'}$.

(v) We mention that the conditions of Theorem 6.11 are the same as that in [17] Theorem 2.2, but the conclusion is different in sense that direct extension of the result from [17] leads to a conclusion that generalises Spanne's result. Instead we modify the proof to obtain a generalisation of Adams' result. In fact Spanne's result is a consequence of Adams':

Corollary 6.13. *Under the same conditions as Theorem 6.11 and assuming that $p_0 < p < q < q_0$, $\lambda \in \left(p\left(n/q_0 + \alpha\right), n\right)$, $1/p - 1/q = \alpha/n$ and μ satisfying $(n - \lambda)/p = (n - \mu)/q$, then T is bounded from $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{L}^{q,\mu}(\mathbb{R}^n)$.*

To see this, it is sufficient to show that if $p_0 < p < q < q_0$, $\lambda \in (p(n/q_0 + \alpha), n)$, and $1/p - 1/q_1 = \alpha/\lambda$, then whenever q_2 and μ satisfies $(n-\lambda)/p = (n-\mu)/q_2$ and $1/p - 1/q_2 = \alpha/n$, then one has $\mathcal{L}^{q_1,\lambda} \subseteq \mathcal{L}^{q_2,\mu}$. This holds by Hölder's inequality and the following two facts: (a) $q_1 > q_2$, and (b) $\mu = \lambda q_2/q_1$. Property (a) follows from $1/p - 1/q_1 = \alpha/\lambda$ and $1/p - 1/q_2 = \alpha/n$ in tandem with $n > \lambda$. To see (b) we observe that the conditions on q_2 gives $\mu = n(\lambda - \alpha p)/(n - \alpha p)$. Note also that $q_1 = p\lambda/(\lambda - \alpha p)$ and $q_2 = pn/(n - \alpha p)$. Combining these three equalities gives (b).

Proof of Theorem 6.11. Our aim is to show that, under the conditions of Theorem 6.11, one has the following norm control on the operator T.

$$\|Tf\|_{\mathcal{L}^{q,\lambda}} \lesssim \left\| M_{\alpha p_0}\big(|f|^{p_0}\big)^{1/p_0} \right\|_{\mathcal{L}^{q,\lambda}}. \tag{6.5}$$

This suffices to give the conclusion of the theorem by a classical result for fractional maximal operators on Morrey spaces due to G. Di Fazio and M.A. Ragusa.

Lemma 6.14 ([42] Lemma 4)**.** *Let $\alpha \in (0, n)$ and $p \in \big(1, n/\alpha\big)$. Suppose that λ and q satisfy $\lambda \in (\alpha p, n)$ and $1/p - 1/q = \alpha/\lambda$. Then M_α is bounded from $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ into $\mathcal{L}^{q,\lambda}(\mathbb{R}^n)$.*

Let us explain how (6.5) leads to our result. Our conditions on the parameters implies the following

$$\alpha p_0 \in (0, n), \qquad p/p_0 \in \Big(1, \frac{n}{\alpha p_0}\Big), \qquad \lambda \in (\alpha p, n), \qquad \frac{1}{p/p_0} - \frac{1}{q/p_0} = \frac{\alpha p_0}{\lambda}.$$

Hence Lemma 6.14 implies that $M_{\alpha p_0}$ maps $\mathcal{L}^{p/p_0,\lambda}$ to $\mathcal{L}^{q/p_0,\lambda}$, and we obtain

$$\begin{aligned}
\left\| M_{\alpha p_0}\big(|f|^{p_0}\big)^{1/p_0} \right\|_{\mathcal{L}^{q,\lambda}} &= \sup_B \Big(r_B^{\lambda} \fint_B \big(M_{\alpha p_0} |f|^{p_0}\big)^{q/p_0} \Big)^{1/q} \\
&= \left\| M_{\alpha p_0}\big(|f|^{p_0}\big) \right\|_{\mathcal{L}^{q/p_0,\lambda}}^{1/p_0} \\
&\lesssim \|f^{p_0}\|_{\mathcal{L}^{p/p_0,\lambda}}^{1/p_0} \; = \; \|f\|_{\mathcal{L}^{p,\lambda}}.
\end{aligned}$$

The rest of the proof is devoted to obtaining estimate (6.5). We shall follow the strategy of Theorem 2.2 in [17] and apply the good-λ result of Theorem 4.4 with modifications to suit our purposes.

We first consider the case $q_0 < \infty$. Fix $f \in L^\infty_c(\mathbb{R}^n)$ and set $F := |Tf|^{s_0} \in L^1(\mathbb{R}^n)$. Sublinearity of T then gives for each ball B,

$$F \;\le\; 2^{s_0-1}\,|T(I-A_B)f|^{s_0} + 2^{s_0-1}\,|TA_Bf|^{s_0} \;=:\; G_B + H_B.$$

We apply Theorem 4.4 with

$$G := 2^{s_0-1}C^{s_0}\big(M_{\alpha p_0}\,|f|^{p_0}\big)^{s_0/p_0}, \qquad H_1 := 0, \qquad H_2 := 2^{s_0-1}G,$$

$$\xi := 2^{2(s_0-1)}C^{s_0}, \qquad r := \frac{q_0}{s_0}\,.$$

We check condition (4.2). By (6.4) we obtain, for each x and y in B,

$$\begin{aligned}
\Big(\fint_B H_B^r\Big)^{1/r} &= 2^{s_0-1}\Big(\fint_B |TA_Bf|^{q_0}\Big)^{s_0/q_0}\\
&\le 2^{s_0-1}\Big\{C_0M\big(|Tf|^{s_0}\big)(x)^{1/s_0} + C_0M_{\alpha p_0}\big(|f|^{p_0}\big)(y)^{1/p_0}\Big\}^{s_0}\\
&\le 2^{2(s_0-1)}C_0^{s_0}\Big\{M\big(|Tf|^{s_0}\big)(x) + M_{\alpha p_0}\big(|f|^{p_0}\big)(y)^{s_0/p_0}\Big\}\\
&= \xi\,\big\{MF(x) + H_2(y)\big\}.
\end{aligned}$$

Next we check condition (4.3). By (6.3) we obtain, for each $x \in B$,

$$\fint_B G_B \;=\; 2^{s_0-1}\fint_B |T(I-A_B)f|^{s_0} \;\le\; 2^{s_0-1}C_0^{s_0}M_{\alpha p_0}\big(|f|^{p_0}\big)(x)^{s_0/p_0} \;=\; G(x).$$

Now from our hypotheses on n, λ, q, q_0 the set $\big(1-\lambda/n, 1-q/q_0\big)$ is non-empty and is contained in $(0,1)$. Fix $\delta \in \big(1-\lambda/n, 1-q/q_0\big)$. Then for any ball B, we have $(M\mathbf{1}_B)^{\delta(q_0/q)'} \in \mathcal{A}_1$ (because $\delta < 1-q/q_0$ is equivalent to $\delta(q_0/q)' < 1$). Now recall that by [16] Proposition 2.1 (vii),

$$(M\mathbf{1}_B)^{\delta(q_0/q)'} \in \mathcal{A}_1 \qquad \Longleftrightarrow \qquad (M\mathbf{1}_B)^{\delta} \in \mathcal{A}_1 \cap \mathcal{B}_{\left(\frac{q_0}{q}\right)'}$$

and hence by [16] Proposition 2.1 (iv), there exists $s \in (1, q_0/q)$ such that $(M\mathbf{1}_B)^\delta \in \mathcal{B}_{s'}$ (because $(q_0/q)' < s'$). Now let us take q/s_0 in place of p in Theorem 4.4. From our hypotheses on q, s_0, q_0, s we see that $1 < q/s_0 < r/s$. Indeed, $s < q_0/q$ implies $r/s > (q_0/s_0)/(q_0/q) = q/s_0$, and the three conditions $1/p - 1/q = \alpha/\lambda$, $1/p_0 - 1/s_0 = \alpha/n$, and $\lambda < n$ implies $1/p_0 - 1/s_0 < 1/p - 1/q$. Rearranging this gives

$$\frac{1}{s_0} - \frac{1}{q} > \frac{p-p_0}{pp_0} > 0,$$

so that $q > s_0$. Therefore, with q/s_0 in place of p, and with $w := (M\mathbf{1}_B)^\delta$, Theorem 4.4 gives

$$\begin{aligned}\|Tf\|_{L^q(w)}^{s_0} &\le \Big(\int_{\mathbb{R}^n} M\big(|Tf|^{s_0}\big)^{q/s_0} w\Big)^{s_0/q} \;=\; \|MF\|_{L^{q/s_0}(w)} \\ &\le C\,\|G\|_{L^{q/s_0}(w)} \;=\; C\big\|M_{\alpha p_0}\big(|f|^{p_0}\big)^{s_0/p_0}\big\|_{L^{q/s_0}(w)} \\ &= C\big\|M_{\alpha p_0}\big(|f|^{p_0}\big)^{1/p_0}\big\|_{L^q(w)}^{s_0}.\end{aligned}$$

For each ball B, one therefore has

$$\Big(\int_B |Tf|^q\Big)^{1/q} \le \Big(\int_{\mathbb{R}^n} |Tf|^q\,(M\mathbf{1}_B)^\delta\Big)^{1/q} \lesssim \Big(\int_{\mathbb{R}^n} M_{\alpha p_0}\big(|f|^{p_0}\big)^{q/p_0}(M\mathbf{1}_B)^\delta\Big)^{1/q}.$$

Since $\delta > 1-\lambda/n$, then Lemma 6.3 applied to $h = M_{\alpha p_0}\big(|f|^{p_0}\big)^{q/p_0}$ and then taking supremum over all balls gives

$$\|Tf\|_{\mathcal{L}^{q,\lambda}} \lesssim \big\|M_{\alpha p_0}\big(|f|^{p_0}\big)^{1/p_0}\big\|_{\mathcal{L}^{q,\lambda}},$$

which is estimate (6.5).

Let us turn to the case $q_0 = \infty$. We fix $\delta \in \big(1-\lambda/n, 1\big)$. Then $(M\mathbf{1}_B)^\delta \in \mathcal{A}_1$ for any ball B. Hence $(M\mathbf{1}_B)^\delta \in \mathcal{B}_{s'}$ for some $1 < s < \infty$. We apply Theorem 4.4 again and see that the proof follows the same argument with $r = \infty$. Condition (4.1) can be checked as follows:

$$\begin{aligned}\Big(\fint_B H_B^r\Big)^{1/r} &= \operatorname{ess\,sup}_B 2^{s_0-1}\,|TA_Bf|^{s_0} \\ &\le 2^{2(s_0-1)}C_0^{s_0}\Big\{M\big(|Tf|^{s_0}\big)(x) + M_{\alpha p_0}\big(|f|^{p_0}\big)(y)^{s_0/p_0}\Big\} \\ &= \xi\,\big\{MF(x) + H_2(y)\big\}\end{aligned}$$

for each $x, y \in B$. We also take, as before, q/s_0 in place of p. It is trivial that $1 < q/s_0 < r/s$. The rest of the proof is the same as before and we obtain estimate (6.5) as required. $\square$

6.2 From Muckenhoupt weights to Morrey spaces II

D.R. Adams and J. Xiao [4, 5] give a new characterisation of Morrey spaces and their preduals in terms of Hausdorff capacity and $\mathcal{A}_1$ weights. For $\alpha \in (0, n\,]$ the *α-Hausdorff capacity* of $\Omega \subset \mathbb{R}^n$ is defined to be

$$\Lambda_\alpha^{(\infty)}(\Omega) := \inf\Big\{\sum_j r_{B_j}^\alpha \,:\, \Omega \subseteq \bigcup_j B_j\Big\}.$$

From this capacity one can define, for $p \in [1,\infty)$, the Choquet-p integral of $f \in C_0^\infty(\mathbb{R}^n)$ as

$$\int_{\mathbb{R}^n} |f|^p \, d\Lambda_\alpha^{(\infty)} := \int_0^\infty \Lambda_\alpha^{(\infty)}(\{x \in \mathbb{R}^n : |f(x)| > t\})\, dt.$$

We refer the reader to [3] for more on the Choquet integral. Next we set

$$\mathcal{A}_1^{(n-\lambda)} := \Big\{ w \in \mathcal{A}_1 : \int_{\mathbb{R}^n} w \, d\Lambda_{n-\lambda}^{(\infty)} \leq 1 \Big\}.$$

We can now give the characterisation of the Morrey spaces from [4, 5]: for each $p \in (1,\infty)$ and $\lambda \in (0,n)$ we have

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^n) = \Big\{ f \in L^p_{\mathrm{loc}} : \sup_{w \in \mathcal{A}_1^{(n-\lambda)}} \Big(\int_{\mathbb{R}^n} |f|^p \, w \Big)^{1/p} < \infty \Big\}. \tag{6.6}$$

These new characterisations allow us to transfer information from $\mathcal{A}_p$ weights to Morrey spaces. The next result is a more refined version of Proposition 6.5, in the sense that the hypotheses are weakened to allow a larger class of weights.

Theorem 6.15 ($\mathcal{A}_p$ boundedness gives Morrey space boundedness)**.** *Let $1 \leq p_0 < q_0 \leq \infty$ and assume that F and G are measurable functions for which the following holds: for some $p \in (p_0, q_0)$ and all $w \in \mathcal{A}_{p/p_0} \cap \mathcal{B}_{(q_0/p)'}$, there exists $C = C(p,w) > 0$ such that,*

$$\|F\|_{L^p(w)} \leq C_p \, \|G\|_{L^p(w)} .$$

Then there exists $C = C(p,n,\lambda)$ such that

$$\|F\|_{\mathcal{L}^{p,\lambda}} \leq C \, \|G\|_{\mathcal{L}^{p,\lambda}} , \qquad \forall\, \lambda \in \big(\tfrac{n}{q_0} p, n\big).$$

In the next result we use ideas from the extrapolation of $L^p(w)$ spaces with $\mathcal{A}_p$ weights to obtain an extrapolation theorem for Morrey spaces. We show that an inequality for a fixed pair of parameters (p_0, λ_0) automatically propagates to a range of (p, λ).

Theorem 6.16 (Extrapolation for Morrey spaces)**.** *Assume that for some $p_0 \in [1,\infty)$ and some $\lambda_0 \in (0,n)$ the following holds.*

$$\|f\|_{\mathcal{L}^{p_0,\lambda_0}} \leq C_0 \, \|g\|_{\mathcal{L}^{p_0,\lambda_0}} .$$

Then for each $p \in \big(p_0, np_0/\lambda_0\big)$ *and* $\lambda = p\lambda_0/p_0$ *there exists* $C = C(p, p_0, \lambda_0) > 0$ *such that*

$$\|f\|_{\mathcal{L}^{p,\lambda}} \leq C\, \|g\|_{\mathcal{L}^{p,\lambda}}\,.$$

Before presenting the proofs of the above two results, we collect together some useful facts. The first is the main result of [83]. Let $\alpha \in (0, n)$. Then there exists $C = C(\alpha, p, n)$, such that for each $p > \alpha/n$,

$$\int (Mf)^p \, d\Lambda_\alpha^{(\infty)} \leq C \int |f|^p \; d\Lambda_\alpha^{(\infty)}. \tag{6.7}$$

The next so-called 'quasi-Hölder' inequality can be found in [5] in the proof of Theorem 18.

$$\int w_0^{1-\theta} w_1^\theta \, d\Lambda_\alpha^{(\infty)} \leq C \left(\int w_0 \, d\Lambda_{n-\alpha_0}^{(\infty)}\right)^{1-\theta} \left(\int w_1 d\Lambda_{n-\alpha_1}^{(\infty)}\right)^\theta \tag{6.8}$$

where $\alpha = (n - \alpha_0)(1-\theta) + (n-\alpha_1)\theta$.

Proof of Theorem 6.15. For each $w \in \mathcal{A}_1^{(n-\lambda)}$ and $\theta \in \big(1 - \lambda/n, (q_0 - p)/p_0\big)$ (this set is not empty due to our hypotheses on p_0, q_0, p, λ), we construct the weight $w_\theta := \big(Mw^{1/\theta}\big)^\theta$. Then combining the fact that $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and that $0 < \theta < 1$ with Lebesgue's Differentiation Theorem gives

$$w(x) \leq w_\theta(x), \qquad\qquad \text{a.e. } x \in \mathbb{R}^n. \tag{6.9}$$

One also has the following:

$$w_\theta \in \mathcal{A}_{\frac{p}{p_0}} \cap \mathcal{B}_{\left(\frac{q_0}{p}\right)'}\,. \tag{6.10}$$

This holds because our hypothesis on θ imply that $\theta(q_0/p)' < 1$, and so $w_\theta^{(q_0/p)'} \in \mathcal{A}_1$. Hence $w_\theta^{(q_0/p)'} \in \mathcal{A}_{(q_0/p)'(p/p_0-1)+1}$ which is equivalent to (6.10), by Proposition 2.9 (f). Next, there also exists a constant $c_\theta > 0$ such that

$$\frac{w_\theta}{c_\theta} \in \mathcal{A}_1^{(n-\lambda)}. \tag{6.11}$$

Indeed since $w_\theta \in \mathcal{A}_1$, and since (6.7) applies (because $1 - \lambda/n < \theta$), then there exists $C_1 = C_1(\theta, n, \lambda)$ such that

$$\int w_\theta \, d\Lambda_{n-\lambda}^{(\infty)} \;=\; \int \big(Mw^{1/\theta}\big)^\theta \, d\Lambda_{n-\lambda}^{(\infty)} \;\leq\; C_1 \int w \, d\Lambda_{n-\lambda}^{(\infty)} \;\leq\; C_1.$$

The last inequality holds because $w \in \mathcal{A}_1^{(n-\lambda)}$. Taking $c_\theta = C_1$ gives us (6.11).

Now combining the facts (6.9), (6.10), and (6.11) one has for such w and θ, and each $f \in C_0(\mathbb{R}^n)$,

$$\frac{1}{c_\theta}\int |F|^p\, w \;\le\; \frac{1}{c_\theta}\int |F|^p\, w_\theta \;\le\; C\int |G|^p \left(\frac{w_\theta}{c_\theta}\right) \;\le\; C \sup_{\nu \in \mathcal{A}_1^{(n-\lambda)}} \int |G|^p\, \nu \;=\; C\, \|G\|^p_{\mathcal{L}^{p,\lambda}}\,.$$

The first inequality follows from (6.9), and the second from the hypothesis on F and G. Taking supremum over all $w \in \mathcal{A}_1^{(n-\lambda)}$ gives

$$\|F\|^p_{\mathcal{L}^{p,\lambda}} \;\le\; c_\theta\, C\, \|G\|^p_{\mathcal{L}^{p,\lambda}}$$

as required. □

Proof of Theorem 6.16. Fix $p \in \big(p_0, np_0/\lambda_0\big)$ and let $\lambda = p\lambda_0/p_0$. We aim to show that there exists some $C > 0$ such that for any $w \in \mathcal{A}_1^{(n-\lambda)}$,

$$\|f\|_{L^p(w)} \le C\, \|g\|_{\mathcal{L}^{p,\lambda}}\,.$$

Taking supremum over all such w will give the desired result.

Now since $w \in \mathcal{A}_1^{(n-\lambda)}$, then in particular $w \in \mathcal{A}_{p/p_0}$ and so by duality there exists $h \in L^{(p/p_0)'}(w)$ with norm 1 such that $\|f\|^{p_0}_{L^p(w)} = \|f\|^{p_0}_{L^{p_0}(hw)}$. Next for each $0 < s < 1$ one has $hw \le \big(M(hw)^{1/s}\big)^s \in \mathcal{A}_1$. Now provided we show

$$\big(M(hw)^{1/s}\big)^s \in \mathcal{A}_1^{(n-\lambda_0)} \tag{6.12}$$

then the result follows. Indeed (6.12) and our hypothesis on f and g imply that

$$\int |f|^p \big(M(gw)^{1/s}\big)^s \;\le\; C\, \|g\|^{p_0}_{\mathcal{L}^{p_0,\lambda_0}}\,,$$

and since $\lambda_0 = \lambda p_0/p$ and $p > p_0$, then by Hölder's inequality,

$$\|g\|_{\mathcal{L}^{p_0,\lambda_0}} \le \|g\|_{\mathcal{L}^{p,\lambda}}\,.$$

Putting these estimates together yields

$$\|f\|^{p_0}_{L^p(w)} \;=\; \|f\|^{p_0}_{L^{p_0}(gw)} \;\le\; C\, \|g\|^{p_0}_{\mathcal{L}^{p_0,\lambda_0}} \;\le\; C\, \|g\|^{p_0}_{\mathcal{L}^{p,\lambda}}\,.$$

It remains to check (6.12). Let $s > 1 - \lambda_0/n$. By (6.7) we have

$$\int \left(M(hw)^{1/s}\right)^s d\Lambda^{(\infty)}_{n-\lambda_0} \lesssim \int hw \, d\Lambda^{(\infty)}_{n-\lambda_0}.$$

We apply (6.8) with

$$\alpha_0 = 0, \qquad \alpha_1 = \lambda_0, \qquad \theta = \frac{p_0}{p}, \qquad w_0 = h^{p/(p-p_0)} w, \qquad w_1 = w.$$

Then it follows that

$$\begin{aligned}
\int h\, w \, d\Lambda^{(\infty)}_{n-\lambda_0} &= \int \left(h^{p/(p-p_0)} w\right)^{1-p_0/p} w^{p_0/p} \, d\Lambda^{(\infty)}_{n-\lambda_0} \\
&\lesssim \left(\int h^{p/(p-p_0)} w \, d\Lambda^{(\infty)}_{n}\right)^{1-p_0/p} \left(\int w \, d\Lambda^{(\infty)}_{n-\lambda}\right)^{p_0/p} \\
&= \left(\int h^{(p/p_0)'} w \, d\Lambda^{(\infty)}_{n}\right)^{1/(p/p_0)'} \left(\int w \, d\Lambda^{(\infty)}_{n-\lambda}\right)^{p_0/p}.
\end{aligned}$$

This can be controlled by the constant 1 because, firstly $w \in \mathcal{A}_1^{(n-\lambda)}$, and secondly

$$\left(\int h^{(p/p_0)'} w \, d\Lambda^{(\infty)}_{n}\right)^{1/(p/p_0)'} = \left(\int h^{(p/p_0)'} w \, dx\right)^{1/(p/p_0)'} = 1$$

by our choice of h. This concludes the proof of (6.12). □

6.3 Applications

In this section we give applications of the results from the previous two sections to some differential operators.

6.3.1 Schrödinger operators

Here we present the proofs of the results in mentioned in the introduction to this chapter.

Proof of Theorem 6.1. We show that (a) implies (b). From Theorem 4.1 we see that (a) implies that $\nabla L^{-1/2}$ is bounded on $L^p(w)$ for all $p \in (1, s)$ and $w \in \mathcal{A}_p \cap \mathcal{B}_{(s/p)'}$. We then apply Theorem 6.15 with $F = \left|\nabla L^{-1/2} f\right|$, $G = f$, $p_0 = 1$, and $q_0 = s$ to obtain (b).

For the converse if (b) holds, then (a) follows simply by taking $\lambda = n$ and recalling that $\mathcal{L}^{p,n}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. □

Proof of Theorem 6.2. We prove (a) and (b). We recall that in [93, 12] (see also Theorems 1.2 and 1.3 in Chapter 1 of this thesis) the operators $\nabla L^{-1/2}$ and $V^{1/2}L^{-1/2}$ are bounded on $L^p(\mathbb{R}^n)$ for $p \in (1, q^*)$ and $p \in (1, 2q)$ respectively. Applying Theorem 4.1 and Theorem 4.2 to these operators, with $s = q^*$ and $s = 2q$ respectively gives firstly the boundedness of $\nabla L^{-1/2}$ on $L^p(w)$ for $p \in (1, q^*)$, $w \in \mathcal{A}_p \cap \mathcal{B}_{(q^*/p)'}$, and secondly the boundedness of $V^{1/2}L^{-1/2}$ on $L^p(w)$ for $p \in (1, 2q)$, $w \in \mathcal{A}_p \cap \mathcal{B}_{(2q/p)'}$. Next we may apply Theorem 6.15, firstly to $F = \left|\nabla L^{-1,2}\right|$ and $G = f$ with $p_0 = 1$ and $q_0 = q^*$, and secondly to $F = V^{1/2}L^{-1/2}$ and $G = f$ with $p_0 = 1$ and $q_0 = 2q$.

We prove (c). To do this we combine Theorem 6.15 with Theorem 4.3. Indeed from Theorem 4.3 we know that VL^{-1} and $\nabla^2 L^{-1}$ are both bounded on $L^p(w)$ for each $p \in (1, q)$ and $w \in \mathcal{A}_p \cap \mathcal{B}_{(q/p)'}$. Hence for each such p and w one has, for each $f \in C_0^\infty(\mathbb{R}^n)$,

$$\left\|\nabla^2 L^{-1} f\right\|_{L^p(w)} \leq C \left\|f\right\|_{L^p(w)}.$$

Theorem 6.15 applied to $F := \left|\nabla^2 L^{-1} f\right|$, $G := f$, $p_0 = 1$, and $q_0 = q$ gives

$$\left\|\nabla^2 L^{-1} f\right\|_{\mathcal{L}^{p,\lambda}} \leq C \left\|f\right\|_{\mathcal{L}^{p,\lambda}}$$

for each $\lambda \in \big(np/q, n\big)$. □

Recall earlier in Remark 6.12 (iii) that conditions (6.3) and (6.4) are satisfied with $p_0 = 1$, $q_0 = \infty$, $T = L^{-\alpha/2}$, and $A_B = e^{-r_B^2 L}$ whenever L admits Gaussian upper bounds on its heat kernel. We shall now give the details of this fact, which will be contained in the proof of the following result.

Theorem 6.17. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 1$ and $0 \leq V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let $\alpha \in (0, n)$, $p \in \big(1, n/\alpha\big)$, and $\lambda \in (\alpha p, n)$ with $1/p - 1/q = \alpha/\lambda$. Then $L^{-\alpha/2}$ is bounded from $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{L}^{q,\lambda}(\mathbb{R}^n)$.*

Proof. Recall that under our assumptions on V the heat kernel of L satisfies the Gaussian upper bound (3.1). This implies the pointwise control $|L^{-\alpha/2} f| \lesssim I_\alpha |f|$ by the Riesz potential I_α and hence Theorem 6.17 can be obtained as a consequence of Adams' result for the Riesz potential I_α

(Theorem 6.9). However we shall prove this result by utilising the machinery we have developed in this chapter, namely Theorem 6.11.

We first state a boundedness result for $L^{-\alpha/2}$ on the $L^p(\mathbb{R}^n)$ spaces, which follows from the boundedness of the classical Riesz potentials.

Lemma 6.18. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 1$ and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. Let $\alpha \in (0, n)$ and $p \in \big(1, n/\alpha\big)$, with $1/p - 1/q = \alpha/n$. Then $L^{-\alpha/2}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Proof. From the Gaussian upper bounds for the heat kernel of L, it is easy to show the following pointwise bounds.

$$|L^{-\alpha/2} f(x)| \;\leq\; C\, I_\alpha\big(|f|\big)(x), \qquad \text{a.e. } x \in \mathbb{R}^n. \tag{6.13}$$

The conclusion of the lemma for $L^{-\alpha/2}$ then follows from the corresponding result for I_α, which can be found in Chapter 5 of [100].

Let us show (6.13). Firstly,

$$\begin{aligned}
\big|L^{-\alpha/2} f(x)\big| &= \left|\frac{1}{\Gamma(\alpha/2)} \int_0^\infty \int_{\mathbb{R}^n} p_t(x,y) f(y)\, dy \frac{dt}{t^{1-\alpha/2}}\right| \\
&\leq \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}^n} |f(y)| \int_0^\infty |p_t(x,y)| \frac{dt}{t^{1-\alpha/2}} dy \\
&= \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}^n} k(x,y)\, |f(y)|\, dy
\end{aligned}$$

where

$$k(x,y) = \int_0^\infty |p_t(x,y)|\, t^{\alpha/2-1}\, dt\,.$$

Our task is to show that $k(x,y) \leq C\,|x-y|^{\alpha-n}$. Write $k(x,y) = k_0(x,y) + k_\infty(x,y)$ where

$$k_0(x,y) \;=\; \int_0^{|x-y|^2} |p_t(x,y)| \frac{dt}{t^{1-\alpha/2}} \;\leq\; \int_0^{|x-y|^2} e^{-|x-y|^2/t} \frac{dt}{t^{1+(n-\alpha)/2}}\,.$$

If we assume that $x \neq y$ and let $s = t/|x-y|^2$, we have

$$\begin{aligned}
k_0(x,y) &\leq \int_0^1 \frac{|x-y|^2 e^{-1/s}\, ds}{|x-y|^{2+n-\alpha} s^{1+(n-\alpha)/2}} \;=\; \frac{1}{|x-y|^{n-\alpha}} \int_0^1 \frac{e^{-1/s}\, ds}{s^{1+(n-\alpha)/2}} \\
&\leq \frac{C_{n,\alpha}}{|x-y|^{n-\alpha}} \int_0^1 ds \;=\; \frac{C_{n,\alpha}}{|x-y|^{n-\alpha}}\,.
\end{aligned}$$

We also have

$$\begin{aligned} k_\infty(x,y) &\le \int_{|x-y|^2}^\infty \frac{e^{-|x-y|^2/t}\,dt}{t^{1+(n-\alpha)/2}} \;\le\; \int_{|x-y|^2}^\infty \frac{dt}{t^{1+(n-\alpha)/2}} \\ &= C_{n,\alpha}\Big[\frac{1}{t^{(n-\alpha)/2}}\Big]_\infty^{|x-y|^2} \;=\; \frac{C_{n,\alpha}}{|x-y|^{n-\alpha}}\,. \end{aligned}$$

Finally we note that if $x = y$ the estimate holds trivially. □

Now set $T = L^{-\alpha/2}$, $A_B = e^{-r_B^2 L}$, $p_0 = 1$ and $q_0 = \infty$ and $s_0 = s = n/(n-\alpha)$. Then Lemma 6.18 implies that $L^{-\alpha/2}$ is bounded from $L^1(\mathbb{R}^n)$ into $L^s(\mathbb{R}^n)$. Fix $f \in L_c^\infty(\mathbb{R}^n)$ and a ball B. We first show (6.4). From the bounds (3.1) we have, for any $x \in B$,

$$\begin{aligned} |e^{-r_B^2 L}L^{-\alpha/2}f(x)| &\le \sum_{j=0}^\infty \int_{U_j(B)} |p_{r_B^2}(x,y)|\,|L^{-\alpha/2}f(y)|\,dy \\ &\le \sum_{j=0}^\infty \big|2^jB\big|\Big(\fint_{U_j(B)} |p_{r_B^2}(x,y)|^{s'}dy\Big)^{1/s'}\Big(\fint_{2^jB}|L^{-\alpha/2}f(y)|^s dy\Big)^{1/s} \\ &\lesssim \sum_{j=0}^\infty e^{-4^j}2^{jn}\Big(\fint_{2^jB}|L^{-\alpha/2}f(y)|^s dy\Big)^{1/s} \\ &\le M\big(|L^{-\alpha/2}f|^s\big)(x)^{1/s}. \end{aligned}$$

Next we show (6.3). For each $j \ge 0$ set $f_j := f\mathbf{1}_{U_j(B)}$. Then we write

$$\Big(\fint_B |L^{-\alpha/2}(I-e^{-r_B^2L})f|^s\Big)^{1/s} \le \sum_{j=0}^\infty\Big(\fint_B |L^{-\alpha/2}(I-e^{-r_B^2L})f_j|^s\Big)^{1/s}. \tag{6.14}$$

For the terms $j = 0, 1$ we use the $L^1(\mathbb{R}^n) \to L^s(\mathbb{R}^n)$ boundedness of $L^{-\alpha/2}$ (from Lemma 6.18) and the Gaussian bounds (3.1) to obtain

$$\begin{aligned} \Big(\fint_B |L^{-\alpha/2}(I-e^{-r_B^2L})f_j(x)|^s dx\Big)^{1/s} &\lesssim \frac{1}{|B|^{1/s}}\int_{\mathbb{R}^n}|(I-e^{-r_B^2L})f_j(x)|\,dx \\ &\lesssim \frac{1}{|B|^{1/s}}\int_{\mathbb{R}^n}|f_j(x)|\,dx \;=\; \frac{1}{|B|^{1/s}}\int_{2B}|f(x)|\,dx \\ &\lesssim r_B^{n-n/s}\fint_{2B}|f(x)|\,dx \;=\; r_B^\alpha\fint_{2B}|f(x)|\,dx\,. \end{aligned}$$

For $j \ge 2$ we use the following identity (from integration by parts),

$$\begin{aligned} L^{-\alpha/2}(I-e^{-r_B^2L}) &= \frac{1}{\Gamma(\alpha/2)}\int_0^\infty e^{-tL}(I-e^{-r_B^2L})\frac{dt}{t^{1-\alpha/2}} \\ &= \frac{1}{\Gamma(\alpha/2)}\int_0^\infty\int_t^{t+r_B^2} -\frac{\partial}{\partial s}e^{-sL}\,ds\frac{dt}{t^{1-\alpha/2}} \end{aligned}$$

$$= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \int_t^{t+r_B^2} Le^{-sL}\, ds \frac{dt}{t^{1-\alpha/2}}\,,$$

and apply Minkowski's inequality and the bounds on the time derivative of the heat kernel (3.4) to obtain

$$\begin{aligned}
&\Big(\fint_B |L^{-\alpha/2}(I - e^{-r_B^2 L}) f_j(x)|^s dx \Big)^{1/s} \\
&\qquad \lesssim \int_0^\infty \int_t^{t+r_B^2} \Big(\fint_B |Le^{-sL} f_j(x)|^s dx \Big)^{1/s} ds \frac{dt}{t^{1-\alpha/2}} \\
&\qquad \le \int_0^\infty \int_t^{t+r_B^2} \int_{U_j(B)} \Big(\fint_B \Big| \frac{\partial}{\partial s} p_s(x,y) \Big|^s dx \Big)^{1/s} |f(y)|\; dy\, ds \frac{dt}{t^{1-\alpha/2}} \\
&\qquad \lesssim \int_0^\infty \int_t^{t+r_B^2} e^{-4^j r_B^2/s} \frac{ds}{s^{n/2+1}} \frac{dt}{t^{1-\alpha/2}} \int_{U_j(B)} |f(y)|\; dy\,.
\end{aligned}$$

To complete the estimates we split the integral

$$\int_0^\infty \int_t^{t+r_B^2} e^{-4^j r_B^2/s} \frac{ds}{s^{n/2+1}} \frac{dt}{t^{1-\alpha/2}} \;=\; I_j + II_j,$$

where

$$\begin{aligned}
I_j &= \int_0^{4^j r_B^2} \int_t^{t+r_B^2} e^{-4^j r_B^2/s} \frac{ds}{s^{n/2+1}} \frac{dt}{t^{1-\alpha/2}} \\
&\lesssim \int_0^\infty \int_t^{t+r_B^2} \Big(\frac{s}{4^j r_B^2} \Big)^{n/2+1} \frac{ds}{s^{n/2+1}} \frac{dt}{t^{1-\alpha/2}} \\
&\lesssim (2^j r_B)^{\alpha-n} 4^{-j},
\end{aligned}$$

and

$$II_j \;=\; \int_{4^j r_B^2}^\infty \int_t^{t+r_B^2} e^{-4^j r_B^2/s} \frac{ds}{s^{n/2+1}} \frac{dt}{t^{1-\alpha/2}} \;\le\; r_B^2 \int_{4^j r_B^2}^\infty \frac{dt}{t^{n/2+2-\alpha/2}} \;\lesssim\; (2^j r_B)^{\alpha-n} 4^{-j}.$$

We remark that both estimates follow because $\alpha \in (0, n)$. Collecting these estimates into (6.14) we obtain for any $x \in B$,

$$\Big(\fint_B |L^{-\alpha/2}(I - e^{-r_B^2 L}) f|^s \Big)^{1/s} \;\le\; \sum_{j=0}^\infty 4^{-j} (2^j r_B)^\alpha \fint_{2^j B} |f| \;\lesssim\; M_\alpha\big(|f|\big)(x)$$

which is (6.3). □

6.3.2 Elliptic operators in divergence form

The following section is mostly taken from [15]. We refer the reader to that article for a more complete treatment.

Let $A = (a_{j,k})_{j,k}$ be an $n \times n$ matrix of complex and L^∞ valued coefficients defined on $\mathbb{R}^n$. We assume that this matrix satisfies the following ellipticity (or 'accretivity') condition: there exists $0 < \pi \leq \Pi < \infty$ such that

$$\pi\,|\xi|^2 \leq \Re\, A(x)\,\xi \cdot \overline{\xi} \qquad \text{and} \qquad \left|A(x)\,\xi \cdot \overline{\zeta}\right| \leq \Pi\,|\xi|\,|\zeta|\,,$$

for all $\xi, \zeta \in \mathbb{C}^n$ and almost every $x \in \mathbb{R}^n$. Note that $\xi \cdot \zeta = \xi_1\zeta_1 + \cdots + \xi_n\zeta_n$ is the usual inner product on $\mathbb{C}^n$, and hence $A(x)\xi \cdot \overline{\zeta} = \sum_{j,k} a_{j,k}(x)\xi_k\overline{\zeta}_j$.

Associated with this matrix we define the second-order divergence form operator

$$Lf = -\operatorname{div}(A\nabla f),$$

which is understood in the standard weak sense as a maximal accretive operator on $L^2(\mathbb{R}^n)$ with domain $\mathcal{D}(L)$ by means of a sesquilinear form. The operator $-L$ generates a C^0-semigroup $\left\{e^{-tL}\right\}_{t>0}$ of contractions on $L^2(\mathbb{R}^n)$. We set

$$p_- = p_-(L) = \inf\{p \geq 1 :\ e^{-tL} \ \text{ is bounded uniformly on } \ L^p(\mathbb{R}^n) \ \text{ for all } \ t > 0\},$$

$$p_+ = p_+(L) = \sup\{p < \infty :\ e^{-tL} \ \text{ is bounded uniformly on } \ L^p(\mathbb{R}^n) \ \text{ for all } \ t > 0\},$$

and also

$$q_- = q_-(L) = \inf\bigl\{q \geq 1 :\ \sqrt{t}\nabla e^{-tL} \ \text{ is bounded uniformly on } \ L^p(\mathbb{R}^n) \ \text{ for all } \ t > 0\bigr\},$$

$$q_+ = q_+(L) = \sup\bigl\{q < \infty :\ \sqrt{t}\nabla e^{-tL} \ \text{ is bounded uniformly on } \ L^p(\mathbb{R}^n) \ \text{ for all } \ t > 0\bigr\}.$$

The following are known results concerning the Riesz transform and fractional powers associated to $L = -\operatorname{div}(A\nabla)$.

Theorem 6.19 ([15])**.** *The operator $\nabla L^{-1/2}$ is bounded on $L^p(w)$ for each $p \in (q_-, q_+)$ and $w \in \mathcal{W}_w(q_-, q_+)$.*

Theorem 6.20 ([17])**.** *Let p, q, α satisfy $p_- < p < q < p_+$ and $\alpha/n = 1/p - 1/q$. Then the operator $L^{-\alpha/2}$ is bounded from $L^p(w^p)$ to $L^q(w^q)$ for each $w \in \mathcal{A}_{1+\frac{1}{p_-}-\frac{1}{p}} \cap \mathcal{B}_{q\left(\frac{p_+}{q}\right)'}$.*

Sections 6.1 and 6.2 allow us to extend these results to Morrey spaces readily.

Theorem 6.21. *The operator $\nabla L^{-1/2}$ is bounded on $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ for each $p \in (q_-, q_+)$ and $\lambda \in \big(np/q_+, n\big)$.*

Proof. We combine Theorem 6.19 with Theorem 6.15 with $p_0 = q_-$ and $q_0 = q_+$. $\square$

Theorem 6.22. *Let p, q, α, and λ satisfy $p_- < p < q < p_+$, $\lambda \in \big(p\big(\frac{n}{p_+} + \alpha\big), n\big)$ and $\alpha/\lambda = 1/p - 1/q$. Then the operator $L^{-\alpha/2}$ is bounded from $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{L}^{q,\lambda}(\mathbb{R}^n)$.*

Proof. We remark that $L^{-\alpha/2}$ satisfies conditions (6.3) and (6.4) by Lemma 3.2 in [17], with $\alpha \in (0, n)$, $p_0 = p_-$, $q_0 = p_+$, and s_0 satisfying $1/p_0 - 1/s_0 = \alpha/n$. Then we may apply Theorem 6.11 to obtain the desired conclusion. $\square$

Chapter 7

Hardy spaces and Schrödinger operators

In this chapter we are interested in studying the second-order Riesz transforms $\nabla^2 L^{-1}$ and VL^{-1} (and their commutators with BMO functions) associated to the Schrödinger operator in the range $p \leq 1$. This will involve both the classical Hardy spaces $H^p(\mathbb{R}^n)$ and the Hardy spaces $H^p_L(\mathbb{R}^n)$ associated to $L = -\Delta + V$ (see Section 7.1.1 for a definition). We mention that the results for the first order Riesz transform $\nabla L^{-1/2}$ under the condition that V is non-negative and locally integrable are known [65, 70]. See the discussion in Section 1.1.3.

The main results of this chapter are the following theorem and its corollary.

Theorem 7.1. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 3$. Assume that $V \in \mathcal{B}_q$ with $q >\ \max\{2, n/2\}$. Then the following holds.*

(a) *The operators $\nabla^2 L^{-1}$ and VL^{-1} are bounded from $H^p_L(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for each $p \in (0, 1]$.*

(b) *The operator $\nabla^2 L^{-1}$ is bounded from $H^p_L(\mathbb{R}^n)$ into $H^p(\mathbb{R}^n)$ for each $p \in \big(n/(n+1), 1\big]$.*

Under reverse Hölder conditions on V, our result admits a straightforward consequence. The atomic characterisation given in [52] (see Definition 7.6 below) allow us to state the range of boundedness on the *classical* Hardy spaces.

Corollary 7.2. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 3$. Assume that $V \in \mathcal{B}_q$ with $q >\ \max\{2, n/2\}$. Then the operator $\nabla^2 L^{-1}$ is bounded from $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ for each $p \in \big(n/(n+p_L), 1\big]$, where $p_L = \min\{1, 2 - n/q\}$.*

The proof of this Corollary is given at the end of Section 7.1.1.

We also study the *commutator* between the operator $\nabla^2 L^{-1}$ and a BMO function b, which is defined as

$$[b, \nabla^2 L^{-1}]f \;:=\; \nabla^2 L^{-1}(bf) - b\nabla^2 L^{-1} f.$$

The commutator $[b, VL^{-1}]$ of VL^{-1} and b is defined similarly. Commutators of a singular integral operator with BMO functions are also objects that arise naturally in harmonic analysis and partial differential equations. They were introduced in [37] and were further studied in [69] and [88].

In [63] the authors show that when $V \in \mathcal{B}_q$ with $q \geq n/2$ and $n \geq 3$, the commutators $[b, \nabla^2 L^{-1}]$ and $[b, VL^{-1}]$ as defined above are bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, q]$. Here we give an estimate for the endpoint $p = 1$.

Theorem 7.3. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \geq 3$. Assume that $V \in \mathcal{B}_q$ with $q > \max\{2, n/2\}$. Let $b \in \mathrm{BMO}$. Then the commutators $[b, \nabla^2 L^{-1}]$ and $[b, VL^{-1}]$ map $H^1_L(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.*

We give some remarks on results for the commutators of the first-order Riesz transforms $\nabla L^{-1/2}$ and $V^{1/2}L^{-1/2}$ with a BMO function b. When V is a non-negative and locally integrable function the boundedness of $[b, \nabla L^{-1/2}]$ and $[b, V^{1/2}L^{-1/2}]$ on $L^p(\mathbb{R}^n)$ for $p \in (1, 2]$ can be obtained as a consequence of the results in [6]. When $V \in \mathcal{B}_q$ with $q \geq n/2$, it is shown in [63] that the range of boundedness of $[b, V^{1/2}L^{-1/2}]$ can be improved to $(1, 2q]$, while the range for $[b, \nabla L^{-1/2}]$ can be improved to $(1, q^*]$. Note that q^* is defined in Section 2.2. Similar endpoint estimates to Theorem 7.3 for the first-order Riesz transforms are proved in [7].

In [98] the authors introduce the notion of a weighted Hardy space $H^p_L(w)$ associated to an operator L. The standard reference for the classical counterparts of these spaces, the weighted Hardy spaces $H^p(w)$, is the monograph [103]. We give an extension of Theorem 7.1 to these weighted spaces $H^p_L(w)$ in section 7.2.

This chapter is organised as follows. Section 7.1 presents the proofs of the unweighted results. Section 7.1.1 collects together the required definitions and properties of the Hardy spaces $H^p_L(\mathbb{R}^n)$ and gives the proof of Corollary 7.2. We give some kernel estimates that will be needed throughout the rest of the chapter in Section 7.1.2, before moving on to the proofs of Theorems 7.1 and 7.3. Section 7.2 gives the extensions to weighted Hardy spaces.

7.1 Unweighted Hardy spaces

In this section we give the proofs of Theorem 7.1, Theorem 7.3 and Corollary 7.2. We begin with describing the constructions of the Hardy spaces under consideration before turning to the proofs of the main results.

7.1.1 Hardy spaces associated to Schödinger operators

We give a brief survey on the Hardy spaces adapted to the Schrödinger operator $L = -\Delta + V$. Unless otherwise noted, we will assume the potential V is a non-negative and locally integrable function. The material in this section can be found in more complete form in [65, 44, 70], where more general classes of operators are treated. See also [43]. For a description of the classical Hardy spaces and their properties see [101] (we can also take $L = -\Delta$ throughout this section).

Firstly we set

$$\mathbb{H}^2(\mathbb{R}^n) := \overline{\{Lu \in L^2(\mathbb{R}^n) : u \in L^2(\mathbb{R}^n)\}}. \tag{7.1}$$

For each $f \in L^2(\mathbb{R}^n)$, we define the area integral function of f associated to L as

$$S_L(f)(x) := \Big(\int_0^\infty \int_{|x-y|<t} \big|t^2 L e^{-t^2 L} f(y)\big|^2 dy \frac{dt}{t^{n+1}}\Big)^{1/2}, \qquad x \in \mathbb{R}^n. \tag{7.2}$$

For each $p \in (0, 1]$ we define the *Hardy space* $H^p_L(\mathbb{R}^n)$ *associated to* L as the completion of

$$\big\{f \in \mathbb{H}^2(\mathbb{R}^n) : \|S_L(f)\|_{L^p(\mathbb{R}^n)} < \infty\big\}$$

in the metric $\|f\|_{H^p_L} := \|S_L(f)\|_{L^p}$.

Next we introduce the notion of $(p, 2, M)$-atoms for L.

Definition 7.4 (Atoms for H^p_L). *Let $0 < p \leq 1$ and $M \in \mathbb{N}$. A function $a \in L^2(\mathbb{R}^n)$ is called a $(p, 2, M)$-atom for L associated to the ball B if for some $b \in \mathcal{D}(L^M)$ we have*

(i) $a = L^M b$,

(ii) $\operatorname{supp} L^k b \subseteq B$ *for each* $k = 0, 1, \ldots, M$,

(iii) $\left\|(r_B^2 L)^k b\right\|_2 \leq r_B^{2M} \left|B\right|^{1/2-1/p}$ *for each* $k = 0, 1, \ldots, M$.

Let $M > \frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$. Then it follows that for each $f \in H_L^p(\mathbb{R}^n)$ there exists a sequence $\{a_B\}_B$ of $(p, 2, M)$-atoms for L, and a sequence of scalars $\{\lambda_B\}_B \subset \mathbb{C}$, such that

$$f = \sum_B \lambda_B a_B \qquad \text{and} \qquad \sum_B |\lambda|^p \leq \|f\|_{H_L^p}^p .$$

The convergence is in both $H_L^p(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$.

These atoms allow us to reduce the study of operators on $H_L^p(\mathbb{R}^n)$ to studying their behaviour on single atoms. This is recorded in the following fact, and will be crucial in the proof of Theorem 7.1 (a).

Lemma 7.5. *Let* $0 < p \leq 1$ *and fix an integer* $M > \frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$. *Assume that* T *is a linear operator* (*resp. non-negative sublinear*) *operator that maps* $L^2(\mathbb{R}^n)$ *continuously into* $L^{2,\infty}(\mathbb{R}^n)$ *satisfying the following property: there exists* $C > 0$ *such that for each* $(p, 2, M)$*-atom* a,

$$\|Ta\|_{L^p(\mathbb{R}^n)} \leq C.$$

Then T *extends to a bounded linear* (*resp. sublinear*) *operator from* $H_L^p(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$. *Furthermore, there exists* $C' > 0$ *such that*

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leq C' \|f\|_{H_L^p(\mathbb{R}^n)}$$

for every $f \in H_L^p(\mathbb{R}^n)$.

For a proof of this Lemma we refer the reader to [65] Lemma 4.3 or [44] Lemma 3.15.

Next we describe the Hardy spaces adapted to the Schrödinger operator defined and studied by Dziubański and Zienkiewicz in the series of papers [51, 52, 53]. These will turn out to be equivalent to the spaces H_L^p defined earlier for a certain range of p. We will use this fact to give the proof of Corollary 7.2 at the end of this section.

For each $p \in (0,1]$ we define (note the calligraphic $\mathcal{H}$) the space $\mathcal{H}^p_L(\mathbb{R}^n)$ as the completion of

$$\left\{ f \in L^1_c(\mathbb{R}^n) : \|\mathcal{M}_L f\|_{L^p} < \infty \right\}$$

in the metric $\|f\|_{\mathcal{H}^p_L} = \|\mathcal{M}_L f\|_{L^p}$. Here $L^1_c(\mathbb{R}^n)$ is the space of compactly supported functions on $\mathbb{R}^n$, and the operator $\mathcal{M}_L$ is defined as

$$\mathcal{M}_L f(x) := \sup_{t>0} \left| e^{-tL} f(x) \right| .$$

When $V \in \mathcal{B}_q$ with $q \geq n/2$ and $n \geq 3$, the authors in [52] give a special atomic characterisation of $\mathcal{H}^p_L(\mathbb{R}^n)$. In the following γ is the function defined in Definition 2.2.

Definition 7.6 (Special L atoms)**.** *A function a is called a special L-atom associated to the ball $B = B(x_B, r_B)$ if $r_B \leq \gamma(x_B)$ and*

(i) supp $a \subseteq B$*,*

(ii) $\|a\|_{L^\infty} \leq |B|^{-1/p}$*,*

(iii) $\int a(x)\, dx = 0$ *whenever* $r_B \leq \frac{1}{4}\gamma(x_B)$*.*

Let $p_L := \min\{1, 2 - n/q\}$. Then the authors show that when $p \in (n/(n+p_L), 1]$, each $f \in \mathcal{H}^p_L(\mathbb{R}^n)$ has a special atomic decomposition $f = \sum_B \lambda_B a_B$ where the a_B are special L-atoms.

Recall that in the atomic characterisation for the classical $H^p(\mathbb{R}^n)$ spaces, the cancellation condition is required for all balls [101]. Comparing this with Definition 7.6 (iii) above, we therefore have the following strict inclusion,

$$H^p(\mathbb{R}^n) \subsetneq \mathcal{H}^p_L(\mathbb{R}^n), \qquad p \in \left(\tfrac{n}{n+p_L}, 1\right]. \tag{7.3}$$

It is also known (see [70], Section 6) that

$$H^p_L(\mathbb{R}^n) = \mathcal{H}^p_L(\mathbb{R}^n), \qquad p \in \left(\tfrac{n}{n+1}, 1\right]. \tag{7.4}$$

We end this section with the proof of Corollary 7.2.

Proof of Corollary 7.2. We simply observe that $p_L \le 1$ and hence $n/(n+p_L) \ge n/(n+1)$. Therefore (7.3) and (7.4) gives

$$H^p(\mathbb{R}^n) \subsetneq \mathcal{H}^p_L(\mathbb{R}^n) = H^p_L(\mathbb{R}^n), \qquad p \in \big(\tfrac{n}{n+p_L}, 1\big].$$

Combining this with Theorem 7.1 (b) we obtain the corollary. □

7.1.2 Some kernel estimates

We collect here the heat kernel estimates that we will use throughout the rest of this chapter. The following is an extension of Proposition 3.7 to time derivatives on the heat kernel.

Proposition 7.7. *Assume $V \in \mathcal{B}_q$ with $q \ge n/2$ for $n \ge 3$ or $q > 1$ for $n = 2$. Let δ be the constant from (3.6). Set $q_+ = \sup\{q > n/2 : V \in \mathcal{B}_q\}$. Then for each $p \in [1, q_+)$ and $k \in \mathbb{Z}_+$ there exists $\xi = \xi(k,p) > 0$ and $C_{k,p} > 0$ such that*

$$\Big(\int_{\mathbb{R}^n} \Big|\nabla_x^2 \frac{\partial^k}{\partial t^k} p_t(x,y)\Big|^p e^{\xi\frac{|x-y|^2}{t}}\,dx\Big)^{1/p} \le \frac{C_{k,p}}{t^{1+n/(2p')+k}} e^{-c\left(1+\frac{t}{\gamma(x)^2}\right)^{\delta}}, \tag{7.5}$$

$$\Big(\int_{\mathbb{R}^n} \Big|V(x) \frac{\partial^k}{\partial t^k} p_t(x,y)\Big|^p e^{\xi\frac{|x-y|^2}{t}}\,dx\Big)^{1/p} \le \frac{C_{k,p}}{t^{1+n/(2p')+k}}\, e^{-c\left(1+\frac{t}{\gamma(x)^2}\right)^{\delta}}. \tag{7.6}$$

for every $y \in \mathbb{R}^n$ and $t > 0$.

Proof. We shall make use of the commutativity property of the semigroup e^{-tL} to see that for each $k \ge 1$,

$$\frac{\partial^k}{\partial t^k} e^{-2tL} = (-2L)^k e^{-2tL} = 2^k e^{-tL} \frac{\partial^k}{\partial t^k} e^{-tL}.$$

In particular this implies

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\partial^k}{\partial t^k} p_{2t}(x,y) f(y)\,dy &= \frac{\partial^k}{\partial t^k} e^{-2tL} f(x) = 2^k e^{-tL} \frac{\partial^k}{\partial t^k} e^{-tL} f(x) \\
&= 2^k \int_{\mathbb{R}^n} p_t(x,w) \frac{\partial^k}{\partial t^k} e^{-tL} f(w)\,dw \\
&= 2^k \int_{\mathbb{R}^n} p_t(x,w) \int_{\mathbb{R}^n} \frac{\partial^k}{\partial t^k} p_t(w,y) f(y)\,dy\,dw \\
&= 2^k \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} p_t(x,w) \frac{\partial^k}{\partial t^k} p_t(w,y)\,dw\Big) f(y)\,dy,
\end{aligned}$$

giving the identity

$$\frac{\partial^k}{\partial t^k} p_{2t}(x,y) = \int_{\mathbb{R}^n} p_t(x,w) \frac{\partial^k}{\partial t^k} p_t(w,y)\,dw \tag{7.7}$$

for each $x, y \in \mathbb{R}^n$.

Now fix $k \geq 1$ and $p \in [1, q_+)$. We first estimate (7.5). Let ξ be a constant such that $0 < \xi < \min\{\beta_p/2,\, pc_1/4\}$ where c_1 is the constant in the time derivative bounds of Proposition 3.4 and β_p is the constant in Proposition 3.7. Then using (7.7) we have for each $y \in \mathbb{R}^n$ and $t > 0$,

$$\int_{\mathbb{R}^n} \Big|\nabla_x^2 \frac{\partial^k}{\partial t^k} p_{2t}(x,y)\Big|^p e^{\xi \frac{|x-y|^2}{t}}\, dx \;=\; 2^k \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \nabla_x^2 p_t(x,w)\, \frac{\partial^k}{\partial t^k} p_t(w,y)\, e^{\xi \frac{|x-y|^2}{pt}}\, dw \Big|^p dx.$$

Now for each $w \in \mathbb{R}^n$ the triangle inequality gives

$$e^{\xi \frac{|x-y|^2}{pt}} \;\leq\; e^{2\xi \frac{|x-w|^2}{pt}}\, e^{2\xi \frac{|w-y|^2}{pt}} \;=\; e^{2\xi \frac{|x-w|^2}{pt}}\, e^{-2\xi \frac{|w-y|^2}{pt}}\, e^{4\xi \frac{|w-y|^2}{pt}}.$$

Therefore for each $x, y \in \mathbb{R}^n$, by Hölder's inequality with exponent p and p',

$$\begin{aligned} &\Big| \int_{\mathbb{R}^n} \nabla_x^2 p_t(x,w)\, \frac{\partial^k}{\partial t^k} p_t(w,y)\, e^{\xi \frac{|x-y|^2}{pt}}\, dw \Big|^p \\ &\quad \leq \Big(\int_{\mathbb{R}^n} \big|\nabla_x^2 p_t(x,w)\big|^p e^{2\xi \frac{|x-w|^2}{t}} e^{-2\xi \frac{|w-y|^2}{t}}\, dw \Big) \Big(\int_{\mathbb{R}^n} \Big| \frac{\partial^k}{\partial t^k} p_t(w,y)\Big|^{p'} e^{4p'\xi \frac{|w-y|^2}{pt}}\, dw \Big)^{p/p'}. \end{aligned}$$

Using that $\xi < pc_1/4$ the time derivative bounds of Proposition 3.4 give

$$\int_{\mathbb{R}^n} \Big| \frac{\partial^k}{\partial t^k} p_t(w,y)\Big|^{p'} e^{4p'\xi \frac{|w-y|^2}{pt}}\, dw \;\leq\; \frac{C_{k,p}}{t^{np'/2+kp'}} \int_{\mathbb{R}^n} e^{-p'(c_1 - 4\xi/p)\frac{|w-y|^2}{t}}\, dw \;\leq\; \frac{C_{k,p}}{t^{np'/2p+kp'}}$$

since $p' - 1 = p'/p$. Note that the constant $C_{k,p}$ is independent of y. We therefore obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \Big|\nabla_x^2 \frac{\partial^k}{\partial t^k} p_{2t}(x,y)\Big|^p e^{\xi \frac{|x-y|^2}{t}}\, dx &\leq \frac{C_{k,p}}{t^{n/2+kp}} \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \big|\nabla_x^2 p_t(x,w)\big|^p e^{2\xi \frac{|x-w|^2}{t}}\, dx \Big) e^{-2\xi \frac{|w-y|^2}{t}}\, dw \\ &\leq C_{k,p} \frac{e^{-cp\left(1+\frac{t}{\gamma(x)^2}\right)^{\delta}}}{t^{n/2+kp+np/2p'}} \int_{\mathbb{R}^n} e^{-2\xi \frac{|w-y|^2}{t}}\, dw \\ &\leq C_{k,p} \frac{e^{-cp\left(1+\frac{t}{\gamma(x)^2}\right)^{\delta}}}{t^{p+kp+np/2p'}}, \end{aligned}$$

where we have applied (3.11) in the second inequality because $2\xi < \beta_p$. This concludes the proof of estimate (7.5).

We can obtain (7.6) in the same way, but we use (3.12) in place of (3.11). □

These estimates allow us to obtain the following decay estimates, which will be crucial in the subsequent sections.

Lemma 7.8. *Assume $V \in \mathcal{B}_q$ with $q > \max\{2, n/2\}$ and $n \geq 3$. Then for each $k \in \mathbb{N} \cup \{0\}$, there exists $C_k, c > 0$ such that*

$$\Big(\int_{|x-y|\geq\sqrt{s}} \Big|\nabla_x^2 \frac{\partial^k}{\partial t^k} p_t(x,y)\Big|^2 dx\Big)^{1/2} \leq \frac{C_k}{t^{1+n/4+k}}\, e^{-cs/t}, \tag{7.8}$$

$$\Big(\int_{|x-y|\geq\sqrt{s}} \Big|V(x) \frac{\partial^k}{\partial t^k} p_t(x,y)\Big|^2 dx\Big)^{1/2} \leq \frac{C_k}{t^{1+n/4+k}}\, e^{-cs/t}, \tag{7.9}$$

for each $y \in \mathbb{R}^n$ and $s, t > 0$.

Proof. Since $q > 2$ we may apply Proposition 7.7 with $p = 2$. Let ξ be the constant in Proposition 7.7. Then by (7.5),

$$\begin{aligned}\Big(\int_{|x-y|\geq\sqrt{s}} \Big|\nabla_x^2 \frac{\partial^k}{\partial t^k} p_t(x,y)\Big|^2 dx\Big)^{1/2} &= \Big(\int_{|x-y|\geq\sqrt{s}} \Big|\nabla_x^2 \frac{\partial^k}{\partial t^k} p_t(x,y)\Big|^2 e^{\xi\frac{|x-y|^2}{t}}\, e^{-\xi\frac{|x-y|^2}{t}} dx\Big)^{1/2} \\ &\leq \sup_{|x-y|\geq\sqrt{s}} e^{-\xi\frac{|x-y|^2}{t}} \Big(\int_{\mathbb{R}^n} \Big|\nabla_x^2 \frac{\partial^k}{\partial t^k} p_t(x,y)\Big|^2 e^{\xi\frac{|x-y|^2}{t}} dx\Big)^{1/2} \\ &\leq \frac{C}{t^{1+n/4+k}}\, e^{-\xi s/t}.\end{aligned}$$

Estimate (7.9) can be obtained similarly but with (7.6) in place of (7.5). □

We also record corresponding estimates for the first spatial derivative. These are needed in the proofs of Theorem 7.1 (b) and Theorem 7.12 (b).

Lemma 7.9. *Assume $n \geq 1$ and $0 \leq V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then for each $k \in \mathbb{N} \cup \{0\}$, there exists $C_k, c > 0$ such that*

$$\int_{|x-y|\geq\sqrt{s}} \Big|\nabla_x \frac{\partial^k}{\partial t^k} p_t(x,y)\Big|\, dx \;\leq\; \frac{C_k}{t^{1/2+k}}\, e^{-cs/t}, \tag{7.10}$$

for each $y \in \mathbb{R}^n$ and $s, t > 0$.

Proof. We first observe that a similar argument to the proof of Proposition 7.7, but with (3.2) in place of (3.11), and with the time derivative bounds in (3.4) in place of Proposition 3.4, give the following estimates: for each $p \in [1, 2]$ and $k \in \mathbb{N} \cup \{0\}$, there exists $\xi = \xi(k, p) > 0$ and $C_{k,p} > 0$ such that

$$\Big(\int_{\mathbb{R}^n} \Big|\nabla_x \frac{\partial^k}{\partial t^k} p_t(x,y)\Big|^p e^{\xi\frac{|x-y|^2}{t}} dx\Big)^{1/p} \leq \frac{C_{k,p}}{t^{1/2+n/(2p')+k}}\,. \tag{7.11}$$

Note that the case $k = 0$ is simply the estimate in (3.2).

Now we combine (7.11) for $p = 2$ with the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
&\int_{|x-y|\geq\sqrt{s}} \left|\nabla_x \frac{\partial^k}{\partial t^k} p_t(x,y)\right| dx \\
&\qquad \leq \Big(\int_{\mathbb{R}^n} \left|\nabla_x \frac{\partial^k}{\partial t^k} p_t(x,y)\right|^2 e^{\xi\frac{|x-y|^2}{t}} dx\Big)^{1/2} \Big(\int_{|x-y|\geq\sqrt{s}} e^{-\xi\frac{|x-y|^2}{t}} dx\Big)^{1/2} \\
&\qquad \leq \frac{C_k}{t^{1/2+k}}\, e^{-cs/t}
\end{aligned}$$

as desired. □

7.1.3 Proof of the main result

In this section we prove Theorem 7.1.

Proof of Theorem 7.1 (a). We show that Lemma 7.5 holds for each of the operators $\nabla^2 L^{-1}$ and VL^{-1}, for all $0 < p \leq 1$. More precisely let $M > \frac{n}{2}\big(\frac{1}{p} - \frac{1}{2}\big)$ be an integer and a_B be a $(p, 2, M)$-atom for L associated to the ball $B = B(x_B, r_B)$.

We first consider the operator $\nabla^2 L^{-1}$. By Lemma 7.5 it suffices to show that

$$\left\|\nabla^2 L^{-1} a_B\right\|_{L^p} \leq C \tag{7.12}$$

with C independent of a_B.

Since $0 < p \leq 1$ we may apply Hölder's inequality with exponents $2/p$ and $2/(2-p)$ to obtain

$$\begin{aligned}
\left\|\nabla^2 L^{-1} a_B\right\|_{L^p}^p &= \sum_{j=0}^{\infty} \left\|\left|\nabla^2 L^{-1} a_B\right|^p\right\|_{L^1(U_j(B))} \\
&\leq \sum_{j=0}^{\infty} \left|2^j B\right|^{1-p/2} \left\|\nabla^2 L^{-1} a_B\right\|_{L^2(U_j(B))}^p \\
&\leq |B|^{1-p/2} \sum_{j=0}^{\infty} 2^{jn(1-p/2)} \left\|\nabla^2 L^{-1} a_B\right\|_{L^2(U_j(B))}^p .
\end{aligned} \tag{7.13}$$

Since $q > 2$ the operator $\nabla^2 L^{-1}$ is bounded on $L^2(\mathbb{R}^n)$, and hence for $j = 0, 1, 2,$

$$\left\|\nabla^2 L^{-1} a_B\right\|_{L^2(U_j(B))} \;\leq\; C \left\|a_B\right\|_{L^2} \;\leq\; C \left|B\right|^{1/2-1/p} . \tag{7.14}$$

Now for each $j \geq 3$ we note that

$$\operatorname{dist}\big(U_j(B), B\big) \;\geq\; 2^{j-1} r_B - r_B \;\geq\; 2^{j-2} r_B.$$

Then using the identity

$$L^{-1} = \int_0^\infty e^{-tL} dt,$$

we obtain

$$\begin{aligned}
\big\|\nabla^2 L^{-1} a_B\big\|_{L^2(U_j(B))} &\leq \Big\|\int_0^{r_B^2} \nabla^2 e^{-tL} a_B\, dt\Big\|_{L^2(U_j(B))} + \Big\|\int_{r_B^2}^\infty \nabla^2 e^{-tL} a_B\, dt\Big\|_{L^2(U_j(B))} \\
&=: I_j + II_j\,.
\end{aligned}$$

We first estimate term I_j. Using estimate (7.8) with $k = 0$ we have

$$\begin{aligned}
\big\|\nabla^2 e^{-tL} a_B\big\|_{L^2(U_j(B))} &= \Big(\int_{U_j(B)} \Big|\int_B \nabla_x^2 p_t(x,y)\, a_B(y)\, dy\Big|^2 dx\Big)^{1/2} \\
&\leq \int_B |a_B(y)| \Big(\int_{|x-y| \geq 2^{j-2} r_B} \big|\nabla_x^2 p_t(x,y)\big|^2\, dx\Big)^{1/2} dy \\
&\leq C\, \|a_B\|_{L^1}\, \frac{e^{-c4^j r_B^2/t}}{t^{1+n/4}}\,. \qquad (7.15)
\end{aligned}$$

In the following let α be a number satisfying $\frac{n}{2}\big(\frac{1}{p} - \frac{1}{2}\big) < \alpha < M$. Then (7.15) gives

$$\begin{aligned}
I_j &\leq \int_0^{r_B^2} \big\|\nabla^2 e^{-tL} a_B\big\|_{L^2(U_j(B))}\, dt \\
&\leq C\, \|a_B\|_{L^1} \int_0^{r_B^2} e^{-c4^j r_B^2/t} \frac{dt}{t^{n/4+1}} \\
&\leq C\, |B|^{1-1/p} \int_0^{r_B^2} \Big(\frac{t}{4^j r_B^2}\Big)^\alpha \frac{dt}{t^{n/4+1}} \\
&\leq C 2^{-2j\alpha}\, |B|^{1/2-1/p}\,. \qquad (7.16)
\end{aligned}$$

In the last line we used that $\alpha > n/4$, which is valid because $p \leq 1$ implies that $\frac{n}{2}\big(\frac{1}{p} - \frac{1}{2}\big) \geq \frac{n}{4}$.

We turn to the term II_j. For this estimate we apply L-cancellation to transfer powers of L to powers of t^{-1} increasing the decay as $t \to \infty$. More precisely we write $a_B = L^M b_B$ for some $b_B \in \mathcal{D}(L^M)$, and obtain

$$e^{-tL} a_B \;=\; e^{-tL} L^M b_B \;=\; L^M e^{-tL} b_B \;=\; (-1)^M \frac{\partial^M}{\partial t^M} e^{-tL} b_B.$$

Now we apply (7.8) with $k = M$ to obtain the extra powers of t^{-1}. This gives

$$\begin{aligned}
\left\| \nabla^2 \frac{\partial^M}{\partial t^M} e^{-tL} b_B \right\|_{L^2(U_j(B))} &= \Big(\int_{U_j(B)} \Big| \int_B \nabla_x^2 \frac{\partial^M}{\partial t^M} p_t(x,y)\, b_B(y)\, dy \Big|^2 dx \Big)^{1/2} \\
&\leq \int_B |b_B(y)| \Big(\int_{|x-y| \geq 2^{j-2} r_B} \Big| \nabla_x^2 \frac{\partial^M}{\partial t^M} p_t(x,y) \Big|^2 dx \Big)^{1/2} dy \\
&\leq C \, \|b_B\|_{L^1} \frac{e^{-c4^j r_B^2/t}}{t^{M+n/4+1}} . \qquad (7.17)
\end{aligned}$$

Then, with α as before, we use 7.17 to get

$$\begin{aligned}
II_j &\leq \int_{r_B^2}^{\infty} \left\| \nabla^2 \frac{\partial^M}{\partial t^M} e^{-tL} b_B \right\|_{L^2(U_j(B))} dt \\
&\leq C \, \|b_B\|_{L^1} \int_{r_B^2}^{\infty} e^{-c4^j r_B^2/t} \frac{dt}{t^{M+n/4+1}} \\
&\leq C r_B^{2M} \, |B|^{1-1/p} \int_{r_B^2}^{\infty} \Big(\frac{t}{4^j r_B^2} \Big)^{\alpha} \frac{dt}{t^{M+n/4+1}} \\
&\leq C 2^{-2j\alpha} \, |B|^{1/2-1/p} . \qquad (7.18)
\end{aligned}$$

In the last line we used that $\alpha < M + n/4$.

Collecting estimates (7.14), (7.16) and (7.18) into (7.13) we obtain

$$\left\| \nabla^2 L^{-1} a_B \right\|_{L^p}^p \;\leq\; C + |B|^{1-p/2} \sum_{j=3}^{\infty} 2^{jn(1-p/2)} \big\{ I_j + II_j \big\}^p \;\leq\; C + C \sum_{j=3}^{\infty} 2^{-j(2\alpha p - n(1-p/2))} \;\leq\; C,$$

with the sum converging because $\alpha > \frac{n}{2}\big(\frac{1}{p} - \frac{1}{2}\big)$. Therefore (7.12) holds.

Turning to the operator VL^{-1} we observe that we can repeat the proof to obtain

$$\left\| V L^{-1} a_B \right\|_{L^p} \leq C$$

using (7.9) in place of (7.8). □

Proof of Theorem 7.1 (b). The proof we give here follows the same strategy as in [68] Proposition 5.6. We utilise a certain characterisation of $H^p(\mathbb{R}^n)$ for $p \leq 1$ given there on p38: for each $p \in (0,1]$, $\varepsilon > 0$, and $N \in \mathbb{N} \cup \{0\}$ with $N \geq \big[n\big(\frac{1}{p} - 1\big) \big]$, we call $m \in L^2(\mathbb{R}^n)$ a $(p, 2, N, \varepsilon)$-molecule for $H^p(\mathbb{R}^n)$ associated to a ball B if

(a) $\int_{\mathbb{R}^n} x^{\alpha} m(x)\, dx = 0$ for all multi-indices $0 \leq |\alpha| \leq N$,

(b) $\|m\|_{L^2(U_j(B))} \leq 2^{-j\varepsilon} \left|2^j B\right|^{1/2-1/p}$ for all $j = 0, 1, \ldots.$

Then one may characterise the classical $H^p(\mathbb{R}^n)$ as follows

$$H^p(\mathbb{R}^n) = \Big\{\sum_j \lambda_j m_j : \{\lambda_j\} \in l^p,\, m_j \text{ are } (p, 2, N, \varepsilon) - \text{molecules}\Big\}$$

with

$$\|f\|_{H^p} \approx \inf\Big\{\Big(\sum_j |\lambda_j|^p\Big)^{1/p}\Big\}, \tag{7.19}$$

where the infimum being taken over all decompositions $f = \sum_j \lambda_j m_j$ and the sum converging the space of tempered distributions $\mathcal{S}'$.

We shall show that for each $p \in (n/(n+1), 1]$ and $M > \frac{n}{2}\big(\frac{1}{p} - \frac{1}{2}\big)$, the operator $\nabla^2 L^{-1}$ maps $(p, 2, M)$-atoms for H^p_L to multiples of $(p, 2, 0, \varepsilon)$-molecules for H^p with some $\varepsilon > 0$. Fix a $(p, 2, M)$-atom a_B for L associated to a ball $B = B(x_B, r_B)$. Set $m_B := \nabla^2 L^{-1}$. Since $p > n/(n+1)$ then we may take $N = 0$ in the above cancellation condition (a). Then we aim to show that there exists $C > 0$ and $\varepsilon > 0$ such that

$$\|m_B\|_{L^2(U_j(B))} \leq C\, 2^{-j\varepsilon} \left|2^j B\right|^{1/2-1/p}, \tag{7.20}$$

$$\int_{\mathbb{R}^n} m_B(x)\, dx = 0, \tag{7.21}$$

for all $j \geq 0$.

Before we prove (7.20) and (7.21) we explain how these imply the estimate

$$\left\|\nabla^2 L^{-1} f\right\|_{H^p} \leq C \|f\|_{H^p_L}.$$

Since $f \in H^p_L(\mathbb{R}^n)$ there is a sequence of $(p, 2, M)$-atoms $\{a_B\}_B$ for L and constants $\{\lambda_B\}_B$ such that $f = \sum_B \lambda_B a_B$ in $L^2(\mathbb{R}^n)$ and

$$\|f\|_{H^p_L} \approx \Big(\sum_B |\lambda_B|^p\Big)^{1/p}. \tag{7.22}$$

Now since the sum converges in $L^2(\mathbb{R}^n)$ we have

$$\nabla^2 L^{-1} f \;=\; \sum_B \lambda_B \left(\nabla^2 L^{-1} a_B\right) \;=:\; \sum_B \lambda_B m_B\,.$$

By (7.20) and (7.21) each m_B is a $(p, 2, 0, \varepsilon)$-molecule and hence this last sum converges in $L^2(\mathbb{R}^n)$, and hence also in $\mathcal{S}'$. Therefore $\sum_B \lambda_B m_B \in H^p(\mathbb{R}^n)$ and furthermore

$$\left\|\nabla^2 L^{-1} f\right\|_{H^p} = \Big\|\sum_B \lambda_B m_B\Big\|_{H^p} \le \Big(\sum_B |\lambda_B|^p\Big)^{1/p} \approx \|f\|_{H^p_L}$$

from (7.19) and (7.22).

Having these facts in hand we now proceed to estimate (7.20). We recall from the proof of Theorem 7.1 (a) that for any α with $\frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right) < \alpha < M$ we have from estimates (7.14), (7.16), and (7.18) that there exists $C > 0$ with

$$\begin{aligned} \|m_B\|_{L^2(U_j(B))} = \left\|\nabla^2 L^{-1} a_B\right\|_{L^2(U_j(B))} &\le C\, 2^{-2j\alpha}\, |B|^{1/2-1/p} \\ &= C\, 2^{-j(2\alpha+n/2-n/p)}\, \left|2^j B\right|^{1/2-1/p}. \end{aligned} \tag{7.23}$$

Since $\alpha > \frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$ then $2\alpha + n/2 - n/p > 0$ and we obtain (7.20) with $\varepsilon = 2\alpha + n/2 - n/p$.

We now prove the moment condition (7.21). To do so we shall need the following result. It is implicit in [70] but we give a proof here for completeness.

Lemma 7.10. *Assume that $f \in L^1(\mathbb{R}^n)$ and $\partial_k f \in L^1(\mathbb{R}^n)$ for some $1 \le k \le n$. Then*

$$\int_{\mathbb{R}^n} \partial_k f(x)\, dx = 0. \tag{7.24}$$

Proof of Lemma 7.10. It is clear that (7.24) holds for any function in $C_0^\infty(\mathbb{R}^n)$. It turns out that integrability of f and of its derivative are also enough for (7.24) to hold. The idea of the proof is to apply a smooth partition of unity to split f into smooth and compactly supported pieces.

Let $\{\phi_j\}_{j=0}^\infty \subset C_0^\infty(\mathbb{R}^n)$ be a partition of unity subordinate to the cover $\{B_j\}_j$ such that each $B_j = B(x_j, r_j)$ is a ball and

$$\bigcup_j B_j = \mathbb{R}^n, \qquad \sum_j \mathbf{1}_{2B_j} \le N, \qquad \sum_j \phi_j(x) = 1, \qquad 0 \le \phi_j \le 1,$$

$$\operatorname{supp} \phi_j \subset 2B_j, \qquad \phi_j = 1 \text{ on } B_j, \qquad |\phi_j(x)| + |\nabla \phi_j(x)| \le C.$$

We also use $\{\eta_j\}_j \subset C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \eta_j \subset 4B_j$ and $\eta_j = 1$ on $2B_j$. By the dominated convergence theorem this lets us write

$$\int_{\mathbb{R}^n} \partial_k f = \sum_j \int_{\mathbb{R}^n} \partial_k(\phi_j f).$$

For each $j \geq 0$ we have

$$\int_{\mathbb{R}^n} \partial_k(\phi_j f) = \int_{\mathbb{R}^n} \eta_j \partial_k(\phi_j f).$$

We then apply the divergence theorem on a ball containing $4B_j$ to the vector field

$$(0, \ldots, 0,\, \eta_j(\phi_j f)\,, 0 \ldots, 0)\,,$$

with the non-zero entry occurring in the kth component. This gives

$$\int_{\mathbb{R}^n} \eta_j \partial_k(\phi_j f) \;=\; -\int_{\mathbb{R}^n} (\phi_j f)\partial_k \eta_j \;=\; 0$$

because $\operatorname{supp} \phi_j f \subset 2B_j$ and $\partial_k \eta_j = 0$ on $2B_j$. □

By Lemma 7.10, to show that

$$\int_{\mathbb{R}^n} \partial_k \partial_l L^{-1} a_B(x)\, dx = 0$$

for each $1 \leq k, l \leq n$, it suffices to show that the functions $\partial_k L^{-1} a_B$ and $\partial_k \partial_l L^{-1} a_B$ are integrable. We note that $\partial_k \partial_l L^{-1} a_B \in L^1(\mathbb{R}^n)$ follows from (7.23). Indeed,

$$\begin{aligned}
\left\|\partial_k \partial_l L^{-1} a_B\right\|_{L^1} \leq \left\|\nabla^2 L^{-1} a_B\right\|_{L^1} &= \sum_{j=0}^{\infty} \left\|\nabla^2 L^{-1} a_B\right\|_{L^1(U_j(B))} \\
&\leq \sum_{j=0}^{\infty} |B|^{1/2} \left\|\nabla^2 L^{-1} a_B\right\|_{L^2(U_j(B))} \\
&\leq C\, |B|^{1/2} \sum_{j=0}^{\infty} 2^{-j\varepsilon} \left|2^j B\right|^{1/2-1/p} \\
&= C\, |B|^{1-1/p} \sum_{j=0}^{\infty} 2^{-j(\varepsilon + n/p - n/2)} \\
&\leq C\, |B|^{1-1/p}\,,
\end{aligned}$$

with the sum being convergent since $\varepsilon + n/p - n/2 = 2\alpha > 0$. To check $\partial_k L^{-1} a_B \in L^1(\mathbb{R}^n)$ we write

$$\left\|\partial_k L^{-1} a_B\right\|_{L^1} \;\leq\; \left\|\nabla L^{-1} a_B\right\|_{L^1} \;=\; \sum_{j=0}^{\infty} \left\|\nabla L^{-1} a_B\right\|_{L^1(U_j(B))}.$$

For $j \geq 3$,

$$
\begin{aligned}
\left\|\nabla L^{-1} a_B\right\|_{L^1(U_j(B))} &\leq \left\|\int_0^{r_B^2} \nabla e^{-tL} a_B \, dt\right\|_{L^1(U_j(B))} + \left\|\int_{r_B^2}^{\infty} \nabla e^{-tL} a_B \, dt\right\|_{L^1(U_j(B))} \\
&=: I_j + II_j \, .
\end{aligned}
$$

Let β be a number satisfying $0 < \beta < M - \frac{1}{2}$. Then using (7.10) with $k = 0$, we have

$$
\begin{aligned}
I_j &\leq \int_0^{r_B^2} \left\|\nabla e^{-tL} a_B\right\|_{L^1(U_j(B))} dt \\
&= \int_0^{r_B^2} \int_{U_j(B)} \left| \int_B \nabla_x p_t(x,y) a_B(y) \, dy \right| dx \, dt \\
&\leq \int_0^{r_B^2} \int_B |a_B(y)| \int_{|x-y| \geq 2^{j-2} r_B} |\nabla_x p_t(x,y)| \; dx \, dy \, dt \\
&\leq C \left\|a_B\right\|_{L^1} \int_0^{r_B^2} e^{-c4^j r_B^2/t} \frac{dt}{\sqrt{t}} \\
&\leq C \, |B|^{1-1/p} \int_0^{r_B^2} \Big(\frac{t}{4^j r_B^2}\Big)^{\beta} \frac{dt}{\sqrt{t}} \\
&\leq 4^{-j\beta} \, |B|^{1-1/p+1/n} \, .
\end{aligned} \tag{7.25}
$$

For the second term we use L-cancellation and estimate (7.10) with $k = M$ to obtain

$$
\begin{aligned}
II_j &\leq \int_{r_B^2}^{\infty} \left\| \nabla \frac{\partial^M}{\partial t^M} e^{-tL} b_B \right\|_{L^1(U_j(B))} dt \\
&\leq \int_{r_B^2}^{\infty} \int_B |b_B(y)| \int_{|x-y| \geq 2^{j-2} r_B} \left| \nabla_x \frac{\partial^M}{\partial t^M} p_t(x,y) \right| dx \, dy \, dt \\
&\leq C \left\|b_B\right\|_{L^1} \int_{r_B^2}^{\infty} e^{-c4^j r_B^2/t} \frac{dt}{t^{M+1/2}} \\
&\leq C r_B^{2M} \, |B|^{1-1/p} \int_{r_B^2}^{\infty} \Big(\frac{t}{4^j r_B^2}\Big)^{\beta} \frac{dt}{t^{M+1/2}} \\
&\leq C 4^{-j\beta} \, |B|^{1-1/p+1/n} \, .
\end{aligned} \tag{7.26}
$$

The last line holds because $0 < \beta < M - \frac{1}{2}$. For $j = 0, 1, 2$ we use that the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^2(\mathbb{R}^n)$, and that the fractional power $L^{-1/2}$ maps $L^{2n/(n+2)}(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. The latter holds because the heat kernel of L has Gaussian upper bounds, and hence fractional powers of L satisfies the same mapping properties of the classical Riesz potentials $(-\Delta)^{-\alpha/2}$. We refer the reader to Lemma 6.18 for a precise statement of this fact. More precisely we have

$$\left\|\nabla L^{-1}a_B\right\|_{L^2(8B)} \;=\; \left\|\nabla L^{-1/2}L^{-1/2}a_B\right\|_{L^2(8B)} \;\leq\; C\left\|L^{-1/2}a_B\right\|_{L^2} \;\leq\; C\left\|a_B\right\|_{L^{2n/(n+2)}}.$$

Now we apply Hölder's inequality with exponents $s := (n+2)/n$ and $s' := (n+2)/2$ to obtain

$$\|a_B\|_{L^{2/s}}^{2/s} \;\leq\; \|a_B\|_{L^2}^{2/s}\,|B|^{1/s'} \;\leq\; |B|^{1-2/ps},$$

and therefore

$$\left\|\nabla L^{-1}a_B\right\|_{L^1(8B)} \;\leq\; C\,|B|^{1/2}\left\|\nabla L^{-1}a_B\right\|_{L^2(8B)} \;\leq\; C\,|B|^{1-1/p+1/n}.$$

Collecting these estimates for $j \geq 0$ we obtain for some $\;0 < \beta < M - \frac{1}{2}$,

$$\left\|\nabla L^{-1}a_B\right\|_{L^1} \;\leq\; C + C\,|B|^{1-1/p+1/n}\sum_{j=3}^{\infty}4^{-j\beta} \;\leq\; C\,|B|^{1-1/p+1/n}.$$

We have shown that $\partial_k L^{-1}a_B \in L^1(\mathbb{R}^n)$ for each $1 \leq k \leq n$, and hence by Lemma 7.10, estimate (7.21) holds.

The proof of Theorem 7.1 (b) is therefore complete. □

7.1.4 Commutators

In this section we give the proof of Theorem 7.3. We shall employ the following result which is a slight variation of Theorem 1.2 from [7] taking into account Remark 3.2 of the same paper.

Proposition 7.11 ([7]). *Let L be a non-negative self adjoint operator satisfying the Davies–Gaffney condition. That is, L generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ and there exists $C_1, C_2 > 0$ such that for all bounded open subsets $U_1, U_2 \subset \mathbb{R}^n$,*

$$\left|\langle e^{-tL}f_1, f_2\rangle\right| \leq C_1 exp\Big(-C_2\frac{\mathrm{dist}(U_1,U_2)^2}{t}\Big)\,\|f_1\|_{L^2}\,\|f_2\|_{L^2} \tag{7.27}$$

for all $f_i \in L^2(U_i)$, $i = 1, 2$ and all $t > 0$.

Let $p \in (0,1]$ and $M > \left[\frac{n}{2}\left(\frac{1}{p} - 1\right)\right]$. Assume that T is a bounded operator on $L^2(\mathbb{R}^n)$ such that for some $M_0 > \frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$ and $C > 0$,

$$\|Ta_B\|_{L^2(U_j(B))} \leq C4^{-jM_0}\,|B|^{1/2-1/p} \tag{7.28}$$

for each $(p,2,M)$-atom a_B for L associated to a ball B and all $j \geq 0$. Then T is bounded from $H^p_L(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Furthermore if T satisfies (7.28) for $p=1$ and is of weak type $(1,1)$ then for all $b \in \mathrm{BMO}$, the commutator $[b,T]$ is bounded from $H^1_L(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

We shall apply this Proposition to T being either $\nabla^2 L^{-1}$ or VL^{-1}, by obtaining estimate (7.28) for $p=1$ and $M_0 = M-1$, where M is any integer satisfying $M > \frac{n}{4}+1$. It is clear that (7.27) is satisfied by $L = -\Delta + V$ with $0 \leq V \in L^1_{\rm loc}(\mathbb{R}^n)$ (see Section 5 of [7]).

Proof of Theorem 7.3. To begin, fix an integer $M > \frac{n}{4}+1$ and let a_B be a $(1,2,M)$-atom for L associated to a ball B. We first show that (7.28) holds for the operator $\nabla^2 L^{-1}$ for every $j \geq 0$.

We begin with $j \geq 3$. Perform the following decomposition

$$\nabla^2 L^{-1} a_B \;=\; \nabla^2 L^{-1}(I - e^{-r_B^2 L})^M a_B + \nabla^2 L^{-1}\big(I - (I - e^{-r_B^2 L})^M\big) a_B\,.$$

Hence to show (7.28) for $j \geq 3$ it suffices to show the following two estimates

$$\big\|\nabla^2 L^{-1}(I - e^{-r_B^2 L})^M a_B\big\|_{L^2(U_j(B))} \;\leq\; C\, 4^{-jM_0}\, |B|^{-1/2}\,, \tag{7.29}$$

$$\big\|\nabla^2 L^{-1}\big(I - (I - e^{-r_B^2 L})^M\big) a_B\big\|_{L^2(U_j(B))} \;\leq\; C\, 4^{-jM_0}\, |B|^{-1/2}\,, \tag{7.30}$$

with $M_0 > n/4$.

Let us first check (7.29). The binomial theorem gives

$$\nabla^2 e^{-tL}(I - e^{-r_B^2 L})^M \;=\; \sum_{k=0}^{M} (-1)^k \binom{M}{k}\, \nabla^2 e^{-(t+kr_B^2)L}.$$

Now for each $k = 0, 1, \ldots, M$, on making a change of variable we obtain

$$\int_0^\infty \nabla^2 e^{-(t+kr_B^2)L} dt \;=\; \int_0^\infty \mathbf{1}_{(kr_B^2,\infty)}(t)\, \nabla^2 e^{-tL} dt\,.$$

Therefore

$$\nabla^2 L^{-1}(I - e^{-r_B^2 L})^M a_B \;=\; \int_0^\infty g_{r_B}(t)\, \nabla^2 e^{-tL} a_B\, dt,$$

where

$$g_r(t) \;=\; \sum_{k=0}^{M} (-1)^k \binom{M}{k}\, \mathbf{1}_{(kr^2,\infty)}(t)\,.$$

Now noting that $\sum_{k=0}^{M}(-1)^k\binom{M}{k}=0$ we have

$$g_r(t) \;=\; \sum_{k=0}^{M}(-1)^k\binom{M}{k}\mathbf{1}_{(Mr^2,\infty)}(t)+\sum_{k=0}^{M}(-1)^k\binom{M}{k}\mathbf{1}_{(kr^2,Mr^2]}(t) \;=\; \sum_{k=0}^{M}(-1)^k\binom{M}{k}\mathbf{1}_{(kr^2,Mr^2]}(t)\,.$$

It follows that

$$|g_r(t)| \;\le\; \Big|\sum_{k=0}^{M}(-1)^k\binom{M}{k}\mathbf{1}_{(0,Mr^2]}(t)\Big| \;\le\; 2^M\mathbf{1}_{(0,Mr^2)}(t)\,.$$

We proceed with estimating (7.29). For $j\ge 3$ by Minkowski's inequality,

$$\begin{aligned}\big\|\nabla^2L^{-1}(I-e^{-r_B^2L})^Ma_B\big\|_{L^2(U_j(B))} &= \Big\|\int_0^\infty g_{r_B}(t)\nabla^2e^{-tL}a_B\,dt\Big\|_{L^2(U_j(B))}\\ &\le \int_0^\infty |g_{r_B}(t)|\big\|\nabla^2e^{-tL}a_B\big\|_{L^2(U_j(B))}dt\,.\end{aligned}$$

For each $t>0$ by Minkowski's inequality again, and estimate (7.8) with $k=0$,

$$\big\|\nabla^2e^{-tL}a_B\big\|_{L^2(U_j(B))} \;\le\; \int_B|a_B(x)|\Big(\int_{|x-y|\ge 2^{j-2}r_B}\big|\nabla_x^2p_t(x,y)\big|^2\,dx\Big)^{1/2}dy \;\le\; \frac{C}{t^{1+n/4}}e^{-c4^jr_B^2/t}$$

since $\|a_B\|_{L^1}\le 1$. Therefore

$$\begin{aligned}\big\|\nabla^2L^{-1}(I-e^{-r_B^2L})^Ma_B\big\|_{L^2(U_j(B))} &\le C\int_0^\infty|g_{r_B}(t)|\,e^{-c4^jr_B^2/t}\frac{dt}{t^{n/4+1}}\\ &\le C\int_0^{Mr_B^2}e^{-c4^jr_B^2/t}\frac{dt}{t^{n/4+1}}\\ &\le C4^{-jM}r_B^{-2M}\int_0^{Mr_B^2}t^{M-n/4-1}dt\,.\end{aligned}$$

Recalling that $M>\frac{n}{4}+1$ we see that the integral is convergent and dominated by $Cr_B^{2M-n/2}$.

Finally

$$\big\|\nabla^2L^{-1}(I-e^{-r_B^2L})^Ma_B\big\|_{L^2(U_j(B))} \;\le\; C4^{-jM}r_B^{-n/2} \;\le\; C4^{-j(M-1)}\,|B|^{-1/2}$$

and the proof of (7.29) is complete.

To study (7.30) we observe that

$$I-(I-e^{-r_B^2L})^M=\sum_{k=1}^{M}\beta_ke^{-kr_B^2L}$$

where $\beta_k=(-1)^{k+1}\binom{M}{k}$. Next by using the L-cancellation of $a_B=L^Mb_B$ for some $b_B\in\mathcal{D}(L^M)$, we obtain

$$\nabla^2L^{-1}\big(I-(I-e^{-r_B^2L})^M\big)a_B \;=\; \sum_{k=1}^{M}\beta_k\,\nabla^2L^{-1}(L^Me^{-kr_B^2L})b_B\,.$$

Therefore estimate (7.30) will follow once we show

$$\left\|\nabla^2 L^{-1}(L^M e^{-kr_B^2 L})b_B\right\|_{L^2(U_j(B))} \leq C4^{-jM_0}\,|B|^{-1/2} \tag{7.31}$$

for each $k = 1, 2, \ldots, M$ with C independent of k. Fix $1 \leq k \leq M$ and write via a change of variable

$$\nabla^2 L^{-1}(L^M e^{-kr_B^2 L})b_B \;=\; \int_0^\infty \nabla^2 L^M e^{-(t+kr_B^2)L} b_B\,dt \;=\; (-1)^M \int_{kr_B^2}^\infty \nabla^2 \frac{\partial^M}{\partial t^M} e^{-tL} b_B\,dt\,.$$

Applying this identity and Minkowski's inequality gives

$$\left\|\nabla^2 L^{-1}(L^M e^{-kr_B^2 L})b_B\right\|_{L^2(U_j(B))} \;\leq\; \int_{kr_B^2}^\infty \left\|\nabla^2 \frac{\partial^M}{\partial t^M} e^{-tL} b_B\right\|_{L^2(U_j(B))} dt\,.$$

By estimate (7.8) with $k = M$ one has

$$\begin{aligned}\left\|\nabla^2 \frac{\partial^M}{\partial t^M} e^{-tL} b_B\right\|_{L^2(U_j(B))} &\leq \int_B |b_B(y)| \left(\int_{|x-y|\geq 2^{j-2}r_B} \left|\nabla^2 \frac{\partial^M}{\partial t^M} p_t(x,y)\right|^2 dx\right)^{1/2} dy \\ &\leq C\frac{r_B^{2M}}{t^{M+n/4+1}} e^{-c4^j r_B^2/t}\end{aligned}$$

because $\|b_B\|_{L^1} \leq r_B^{2M}$. Therefore for each $j \geq 3$, noting that $r_B^2 \leq kr_B^2$ because $k \geq 1$, and applying the previous calculation we have

$$\left\|\nabla^2 L^{-1}(L^M e^{-kr_B^2 L})b_B\right\|_{L^2(U_j(B))} \leq C\,r_B^{2M} \int_{r_B^2}^\infty e^{-c4^j r_B^2/t} \frac{dt}{t^{M+n/4+1}}\,.$$

Finally by noting that $t > r_B^2$, we have

$$\begin{aligned}\left\|\nabla^2 L^{-1}(L^M e^{-kr_B^2 L})b_B\right\|_{L^2(U_j(B))} &\leq Cr_B^{2M-n/2} \int_{r_B^2}^\infty e^{-c4^j r_B^2/t} \frac{dt}{t^{1+M}} \\ &\leq Cr_B^{2M-n/2} \int_{r_B^2}^\infty \left(\frac{t}{4^j r_B^2}\right)^{M-1} \frac{dt}{t^{M+1}} \\ &= Cr_B^{2-n/2} 4^{-j(M-1)} \int_{r_B^2}^\infty \frac{dt}{t^2} \\ &\leq C4^{-j(M-1)}\,|B|^{-1/2}\end{aligned}$$

and the proof of estimate (7.31) for each $k \geq 1$ is complete, which as mentioned earlier implies estimate (7.30). This together with (7.29) shows that $\nabla^2 L^{-1}$ satisfies (7.28) for each $j \geq 3$ with $M_0 = M - 1$.

We are left to check (7.28) for $j = 0, 1, 2$. However this follows from the $L^2(\mathbb{R}^n)$ boundedness of $\nabla^2 L^{-1}$.

$$\left\| \nabla^2 L^{-1} a_B \right\|_{L^1(U_j(B))} \le C \left\| a_B \right\|_{L^2} \le C \left| B \right|^{1/2 - 1/p}$$

and (7.28) follows readily.

Finally we mention that one can show (7.28) for the operator VL^{-1} in a similar fashion but applying (7.9) in place of (7.8). □

7.2 Weighted Hardy spaces

In this section we give the extensions of Theorems 7.1 (a) and 7.1 (b) to weighted Hardy spaces adapted to L. The study of such spaces originated in [98], and was further developed in the work of [8, 109, 108]. In these papers the authors study the Schrödinger operator with an arbitrary non-negative potential and obtained results for the first-order Riesz transform associated to such operators. Here we obtain results for the second-order Riesz transforms under the extra condition that the potential belongs to a reverse Hölder class. In this section we give a proof of the following result.

Theorem 7.12. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with $n \ge 3$. Assume that $V \in \mathcal{B}_q$ with $q > \max\{2, n/2\}$. Then the following holds.*

(a) *The operators $\nabla^2 L^{-1}$ and VL^{-1} are bounded from $H^p_L(w)$ into $L^p(w)$ for each $p \in (0, 1]$ and each $w \in \mathcal{A}_1 \cap \mathcal{B}_{(2/p)'}$.*

(b) *The operator $\nabla^2 L^{-1}$ is bounded from $H^p_L(w)$ into $H^p(w)$ for each $p \in \big(n/(n+1), 1\big]$ and each $w \in \mathcal{A}_1 \cap \mathcal{B}_{(2/p)'}$.*

7.2.1 Weighted Hardy spaces associated to Schrödinger operators

We first define the weighted Hardy spaces $H^p_L(w)$ associated to the Schrödinger operator where w is an $\mathcal{A}_\infty$ weight. The constructions given here are similar their unweighted counterparts. Further details can be found in [98, 8, 109].

Recall the definitions of $\mathbb{H}^2(\mathbb{R}^n)$ from (7.1), and of the area function S_L associated to L in (7.2). Given $w \in \mathcal{A}_\infty$ and $p \in (0,1]$ we define the *weighted Hardy space* $H^p_L(\mathbb{R}^n)$ *associated to* L as the completion of

$$\left\{f \in \mathbb{H}^2(\mathbb{R}^n) : \|S_L(f)\|_{L^p(w)} < \infty\right\}$$

in the metric $\|f\|_{H^p_L(w)} := \|S_L(f)\|_{L^p(w)}$.

As in Definition 7.4 we define the notion of atoms for $H^p_L(w)$.

Definition 7.13 (Atoms for $H^p_L(w)$)**.** *Let $0 < p \le 1$ and $M \in \mathbb{N}$. A function $a \in L^2(\mathbb{R}^n)$ is called a $(w, p, 2, M)$-atom for L associated to the ball B if for some $b \in \mathcal{D}(L^M)$ we have*

(i) $a = L^M b$,

(ii) $\operatorname{supp} L^k b \subseteq B$ *for each* $k = 0, 1, \dots, M$,

(iii) $\left\|(r_B^2 L)^k b\right\|_2 \le r_B^{2M} |B|^{1/2} w(B)^{-1/p}$ *for each* $k = 0, 1, \dots, M$.

Then the following decomposition of the weighted Hardy spaces hold (see [109]). Let $M > \frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$ and $w \in \mathcal{A}_1$. Then it follows that for each $f \in H^p_L(\mathbb{R}^n)$ there exists a sequence $\{a_B\}_B$ of $(w, p, 2, M)$-atoms for L, and a sequence of scalars $\{\lambda_B\}_B \subset \mathbb{C}$, such that

$$f = \sum_B \lambda_B a_B \qquad \text{and} \qquad \sum_B |\lambda|^p \le \|f\|^p_{H^p_L(w)}.$$

The convergence is in both $H^p_L(w)$ and $L^2(\mathbb{R}^n)$.

We also have an analogous version of Lemma 7.5.

Lemma 7.14. *Let $0 < p \le 1$, $w \in \mathcal{A}_\infty$, and fix an integer $M > \frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$. Assume that T is a linear operator (resp. non-negative sublinear) operator that maps $L^2(\mathbb{R}^n)$ continuously into $L^{2,\infty}(\mathbb{R}^n)$ satisfying the following property: there exists $C > 0$ such that for each $(w, p, 2, M)$-atom a*

$$\|Ta\|_{L^p(w)} \le C.$$

Then T extends to a bounded linear (resp. sublinear) operator from $H^p_L(w)$ to $L^p(w)$. Furthermore, there exists $C' > 0$ such that

$$\|Tf\|_{L^p(w)} \le C' \|f\|_{H^p_L(w)}$$

for ever $f \in H^p_L(\mathbb{R}^n)$.

We refer the reader to [8] and [109] for further details.

7.2.2 Proof of the weighted result

In this section we give the proof of Theorem 7.12

Proof of Theorem 7.12 (a). We remark that the argument for this result is similar to the argument given in the proof of the unweighted version of Theorem 7.1 (a) with some modifications.

We shall show that Lemma 7.14 holds for each of the operators $\nabla^2 L^{-1}$ and VL^{-1} for $p \in (0,1]$. More precisely let $M > \frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$ be an integer and a_B be a $(w, p, 2, M)$-atom for $H^p_L(w)$ associated to a ball B.

We first consider the operator $\nabla^2 L^{-1}$. It will suffice to show that

$$\left\|\nabla^2 L^{-1} a_B\right\|_{L^p(w)} \le C \tag{7.32}$$

with C independent of a_B.

Since $p \le 1$ we may apply Hölder's inequality with exponents $2/p$ and $2/(2-p)$ to obtain

$$\begin{aligned}
\left\|\nabla^2 L^{-1} a_B\right\|^p_{L^p(w)} &= \sum_{j=0}^{\infty} \int_{U_j(B)} \left|\nabla^2 L^{-1} a_B\right|^p w(x)\, dx \\
&\le \sum_{j=0}^{\infty} \Big(\int_{2^j B} w^{2/(2-p)}\Big)^{1-p/2} \left\|\nabla^2 L^{-1} a_B\right\|^p_{L^2(U_j(B))} \\
&= \sum_{j=0}^{\infty} \left|2^j B\right|^{1-p/2} \left(⨍_{2^j B} w^{2/(2-p)}\right)^{1-p/2} \left\|\nabla^2 L^{-1} a_B\right\|^p_{L^2(U_j(B))} \\
&\le C \sum_{j=0}^{\infty} \frac{w(2^j B)}{|2^j B|^{p/2}} \left\|\nabla^2 L^{-1} a_B\right\|^p_{L^2(U_j(B))} \\
&\le C \sum_{j=0}^{\infty} 2^{jn(1-p/2)} \frac{w(B)}{|B|^{p/2}} \left\|\nabla^2 L^{-1} a_B\right\|^p_{L^2(U_j(B))}.
\end{aligned} \tag{7.33}$$

The second inequality follows because $w \in \mathcal{B}_{(2/p)'}$. The last inequality follows from the doubling property of w (Lemma 2.10), since $w \in \mathcal{A}_1$. Since $q > 2$ the operator $\nabla^2 L^{-1}$ is bounded on $L^2(\mathbb{R}^n)$, and hence for $j = 0, 1, 2$,

$$\left\|\nabla^2 L^{-1} a_B\right\|_{L^2(U_j(B))} \;\le\; C\,\|a_B\|_{L^2} \;\le\; C\,|B|^{1/2}\,w(B)^{-1/p}. \tag{7.34}$$

We have used the doubling property of w in the last inequality. Now for each $j \ge 3$ we note that

$$\operatorname{dist}(U_j(B), B) \;\ge\; 2^{j-1} r_B - r_B \;\ge\; 2^{j-2} r_B\,.$$

Then using the identity

$$L^{-1} = \int_0^\infty e^{-tL}\,dt$$

we obtain

$$\begin{aligned}\left\|\nabla^2 L^{-1} a_B\right\|_{L^2(U_j(B))} &\le \left\|\int_0^{r_B^2} \nabla^2 e^{-tL} a_B\,dt\right\|_{L^2(U_j(B))} + \left\|\int_{r_B^2}^\infty \nabla^2 e^{-tL} a_B\,dt\right\|_{L^2(U_j(B))}\\ &=: I_j + II_j\,.\end{aligned}$$

Now let α be a number satisfying $\frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right) < \alpha < M$. Using (7.15) we obtain

$$\begin{aligned} I_j &\le \int_0^{r_B^2} \left\|\nabla^2 e^{-tL} a_B\right\|_{L^2(U_j(B))} dt\\ &\le C\,\|a_B\|_{L^1} \int_0^{r_B^2} e^{-c4^j r_B^2/t} \frac{dt}{t^{n/4+1}}\\ &\le C\,|B|^{1/2}\,\|a_B\|_{L^2} \int_0^{r_B^2} \left(\frac{t}{4^j r_B^2}\right)^\alpha \frac{dt}{t^{n/4+1}}\\ &\le C 2^{-2j\alpha}\,|B|^{1/2}\,w(B)^{-1/p}. \end{aligned} \tag{7.35}$$

In the last line we used that $\alpha > n/4$, which is valid because $p \le 1$ implies that $\frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right) \ge \frac{n}{4}$.

We turn to the term II_j. For this estimate we apply L-cancellation to transfer powers of L to powers of t^{-1} increasing the decay as $t \to \infty$. More precisely we write $a_B = L^M b_B$ for some $b_B \in \mathcal{D}(L^M)$ and obtain

$$e^{-tL} a_B \;=\; e^{-tL} L^M b_B \;=\; L^M e^{-tL} b_B \;=\; (-1)^M \frac{\partial^M}{\partial t^M} e^{-tL} b_B\,.$$

Then, with α as before, we use (7.17) to get

$$\begin{aligned}
II_j &\leq \int_{r_B^2}^{\infty} \left\| \nabla^2 \frac{\partial^M}{\partial t^M} e^{-tL} b_B \right\|_{L^2(U_j(B))} dt \\
&\leq C \left\| b_B \right\|_{L^1} \int_{r_B^2}^{\infty} e^{-c4^j r_B^2/t} \frac{dt}{t^{M+n/4+1}} \\
&\leq C \left| B \right|^{1/2} \left\| b_B \right\|_{L^2} \int_{r_B^2}^{\infty} \left(\frac{t}{4^j r_B^2} \right)^{\alpha} \frac{dt}{t^{M+n/4+1}} \\
&\leq C 2^{-2j\alpha} r_B^{2M-2\alpha} \left| B \right| w(B)^{-1/p} \int_{r_B^2}^{\infty} \frac{dt}{t^{M+n/4+1-\alpha}} \\
&\leq C 2^{-2j\alpha} \left| B \right|^{1/2} w(B)^{-1/p}. \qquad (7.36)
\end{aligned}$$

In the last line we used that $\alpha < M + n/4$.

Collecting estimates (7.34), (7.35) and (7.36) into (7.33) we obtain

$$\begin{aligned}
\left\| \nabla^2 L^{-1} a_B \right\|_{L^p(w)}^p &\leq \; C + C \frac{w(B)}{|B|^{p/2}} \sum_{j=3}^{\infty} 2^{jn(1-p/2)} \left\{ I_j + II_j \right\}^p \\
&\leq \; C + C \sum_{j=3}^{\infty} 2^{-j(2\alpha p - n(1-p/2))} \; \leq \; C,
\end{aligned}$$

with the sum converging because $\alpha > \frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$. Therefore (7.32) holds.

The corresponding estimate for the operator VL^{-1} may be proved similarly with (7.9) in place of (7.8). This concludes the proof of Theorem 7.12 (a). □

Proof of Theorem 7.12 (b). The strategy for the proof follows that of Theorem 7.1 (b). The key step is to obtain a suitable molecular characterisation of $H^p(w)$ and then show that $\nabla^2 L^{-1}$ maps appropriate $(w, p, 2, M)$-atoms for $H_L^p(w)$ to such molecules. One such characterisation is given in [109], which is an extension of the case $p = 1$ given in [98].

For each $p \in \big(n/(n+1), 1\big]$, $w \in \mathcal{A}_1$, and $\varepsilon > 0$, we say that $m \in L^2(\mathbb{R}^n)$ is a $(w, p, 2, 0, \varepsilon)$-molecule for $H^p(w)$ associated to the ball B if

(a) $\int_{\mathbb{R}^n} m(x)\, dx = 0$,

(b) $\|m\|_{L^2(U_j(B))} \leq 2^{-j\varepsilon} \left| 2^j B \right|^{1/2} w(2^j B)^{-1/p}$ for all $j = 0, 1, 2, \ldots$.

Then the following holds.

Theorem 7.15 ([109] Theorem 4.4). *Let $p \in \big(n/(n+1),1\big)$, $w \in \mathcal{A}_1 \cap \mathcal{B}_{(2/p)'}$ and $\varepsilon > n/2$. Then*

$$H^p(w) = \Big\{ \sum_j \lambda_j m_j : \; \{\lambda_j\} \in l^p, \, m_j \;\text{ are }\; (w,p,2,0,\varepsilon) - \text{molecules} \Big\}$$

and

$$\|f\|_{H^p(w)} \approx \inf \Big\{ \Big(\sum_j |\lambda_j|^p \Big)^{1/p} \Big\},$$

with the infimum being taken over all decompositions $f = \sum_j \lambda_j m_j$.

With this characterisation of $H^p(w)$ in hand, it will suffice to show that for each $p \in \big(n/(n+1), 1\big]$ and $M > n\big(\frac{1}{p} - \frac{1}{2}\big)$ the operator $\nabla^2 L^{-1}$ maps $(w,p,2,M)$-atoms for $H^p_L(w)$ to $(w,p,2,0,\varepsilon)$-molecules for $H^p(w)$ for some $\varepsilon > n/2$. Accordingly, fix a $(w,p,2,M)$-atom a_B associated to a ball B and set $m_B = \nabla^2 L^{-1}$. We aim to show

$$\|m_B\|_{L^2(U_j(B))} \;\le\; C\, 2^{-j\varepsilon} \left|2^j B\right|^{1/2} w(2^j B)^{-1/p}, \qquad j \ge 0, \tag{7.37}$$

$$\int_{\mathbb{R}^n} m_B(x)\, dx = 0, \tag{7.38}$$

for some $\varepsilon > n/2$.

We first obtain (7.37). Recall from the proof of Theorem 7.12 (a), that for any $n/4 < \alpha < M + n/4$ the estimates (7.34), (7.35), and 7.36 give

$$\|m_B\|_{L^2(U_j(B))} \;=\; \left\|\nabla^2 L^{-1} a_B\right\|_{L^2(U_j(B))} \;\le\; C 2^{-2j\alpha} \, |B|^{1/2} \, w(B)^{-1/p}, \tag{7.39}$$

for each $j \ge 0$. In particular we can pick α satisfying $n\big(\frac{1}{p} - \frac{1}{2}\big) < \alpha < M$ since $p \le 1$ implies $n\big(\frac{1}{p} - \frac{1}{2}\big) > n/4$. Next we note that since $w \in \mathcal{A}_1$ then Lemma 2.11 applied to $B \subset 2^j B$ gives

$$\frac{|B|}{|2^j B|} \le C \frac{w(B)}{w(2^j B)}.$$

Therefore

$$\begin{aligned} |B|^{1/2} \, w(B)^{-1/p} &\le\; C \left|2^j B\right|^{1/2} w(B)^{1/2-1/p} \, w(2^j B)^{-1/2} \\ &=\; C \left|2^j B\right|^{1/2} w(2^j B)^{-1/p} \, w(B)^{1/2-1/p} \, w(2^j B)^{1/p-1/2} \\ &\le\; C\, 2^{jn(1/2-1/p)} \left|2^j B\right|^{1/2} w(2^j B)^{-1/p}, \end{aligned}$$

where in the last step we have used the doubling property of w (Lemma 2.10). Therefore (7.39) becomes

$$\|m_B\|_{L^2(U_j(B))} \leq C\,2^{-j(2\alpha+n/2-n/p)}\left|2^j B\right|^{1/2} w(2^j B)^{-1/p},$$

and hence (7.37) holds with $\varepsilon = 2\alpha + n/2 - n/p$. We observe that $\varepsilon > n/2$, because $p \leq 1$ implies $2\alpha + n/2 - n/p > n\left(\frac{1}{p} - \frac{1}{2}\right) \geq n/2$.

We now turn to the moment condition (7.38). By Lemma 7.10 to show that

$$\int_{\mathbb{R}^n} \partial_k \partial_l L^{-1} a_B(x)\,dx = 0$$

for each $1 \leq k, l \leq n$, it suffices to show that the functions $\partial_k L^{-1} a_B$ and $\partial_k \partial_l L^{-1} a_B$ are integrable. We note that $\partial_k \partial_l L^{-1} a_B \in L^1(\mathbb{R}^n)$ follows from (7.39). Indeed,

$$\begin{aligned}
\left\|\partial_k \partial_l L^{-1} a_B\right\|_{L^1} \leq \left\|\nabla^2 L^{-1} a_B\right\|_{L^1} &= \sum_{j=0}^{\infty} \left\|\nabla^2 L^{-1} a_B\right\|_{L^1(U_j(B))} \\
&\leq \sum_{j=0}^{\infty} |B|^{1/2} \left\|\nabla^2 L^{-1} a_B\right\|_{L^2(U_j(B))} \\
&\leq C\,|B|^{1/2} \sum_{j=0}^{\infty} 2^{-2j\alpha}\,|B|^{1/2}\,w(B)^{-1/p} \\
&= C\,|B|\,w(B)^{-1/p} \sum_{j=0}^{\infty} 2^{-2j\alpha} \\
&\leq C\,|B|\,w(B)^{-1/p},
\end{aligned}$$

the sum being convergent since $2\alpha > 0$. To check $\partial_k L^{-1} a_B \in L^1(\mathbb{R}^n)$ we write

$$\left\|\partial_k L^{-1} a_B\right\|_{L^1} \leq \left\|\nabla L^{-1} a_B\right\|_{L^1} = \sum_{j=0}^{\infty} \left\|\nabla L^{-1} a_B\right\|_{L^1(U_j(B))}.$$

A similar calculation to that in (7.25) and (7.26) gives, for each $j \geq 3$,

$$\left\|\nabla L^{-1} a_B\right\|_{L^1(U_j(B))} \leq C 4^{-j\beta}\,|B|^{1+1/n}\,w(B)^{-1/p},$$

for any β satisfying $0 < \beta < M - 1/2$. For $j = 0, 1, 2$, as in Theorem 7.1 (b), we use that $L^{-1/2}$ maps $L^{2n/(n+2)}(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ (see Lemma 6.18). Then

$$\left\|\nabla L^{-1} a_B\right\|_{L^2(8B)} = \left\|\nabla L^{-1/2} L^{-1/2} a_B\right\|_{L^2(8B)} \leq C\left\|L^{-1/2} a_B\right\|_{L^2} \leq C\,\|a_B\|_{L^{2n/(n+2)}}.$$

Now we apply Hölder's inequality with exponents $s := (n+2)/n$ and $s' := (n+2)/2$ to obtain

$$\|a_B\|_{L^{2/s}}^{2/s} \;\le\; \|a_B\|_{L^2}^{2/s} \, |B|^{1/s'} \;\le\; |B| \, w(B)^{-2/ps},$$

and therefore

$$\left\|\nabla L^{-1} a_B\right\|_{L^1(8B)} \;\le\; C \, |B|^{1/2} \left\|\nabla L^{-1} a_B\right\|_{L^2(8B)} \;\le\; C \, |B|^{1+1/n} \, w(B)^{-1/p}.$$

Collecting our terms for each $j \ge 0$ we obtain

$$\left\|\nabla L^{-1} a_B\right\|_{L^1} \;\le\; |B|^{1+1/n} \, w(B)^{-1/p} \Big\{ C + C \sum_{j=3}^{\infty} 4^{-j\beta} \Big\} \;\le\; C \, |B|^{1+1/n} \, w(B)^{-1/p}.$$

This concludes the proof of (7.38), and hence Theorem 7.12 (b). $\square$

Chapter 8

A class of potentials beyond the reverse Hölder class

We end this thesis with an application of Theorem 1.5. We obtain boundedness of the Riesz transform $\nabla L^{-1/2}$ on $L^p(\mathbb{R}^n)$ for suitable p for a class of potentials slightly more general than the reverse Hölder class studied throughout the rest of the thesis. We direct the reader to Theorem 8.3 below for a precise statement of the result. Theorem 1.5 then allows us to obtain boundedness on the weighted Lebesgue spaces and Morrey spaces with no extra effort.

These potentials, defined in Definition 8.1 below, were introduced in [50] and [54]. There the authors give atomic and Riesz transform characterisations of the Hardy space $H^1_L(\mathbb{R}^n)$ associated to $L = -\Delta + V$, with V a potential from this new class.

We are interested in studying these Riesz transforms on the $L^p(\mathbb{R}^n)$ spaces. Let us remind the reader that the $L^p(\mathbb{R}^n)$-boundedness of the Riesz transform $\nabla L^{-1/2}$ is known to hold for all $1 < p \leq 2$, assuming only that V is non-negative and locally integrable. We mention also that the following condition is a necessary condition for $L^p(\mathbb{R}^n)$-boundedness of the Riesz transform $\nabla L^{-1/2}$:

$$\left\| \nabla e^{-tL} \right\|_{L^p \to L^p} \leq \frac{C_p}{\sqrt{t}} \,. \tag{G_p}$$

It is of interest to find sufficient conditions on the potential V ensuring boundedness for $p > 2$. The reverse Hölder class studied throughout the rest of this thesis is one such class for which this boundedness is known to be valid. Our aim is to show that for suitable potentials in the new class, the $L^p(\mathbb{R}^n)$ boundedness for p above 2 also holds.

Before stating our main result we first describe some notation we will use throughout the rest of this chapter. For a given cube Q we write $d(Q) := \sup\{|x-y| : x, y \in Q\}$ to mean the diameter of Q, and $l(Q)$ to mean the sidelength of Q. For $\beta \geq 1$ we use βQ to mean the cube concentric with Q but with β times the sidelength. Given such a β we also use the notation $U_{j,\beta}(Q) := \beta^j Q \backslash \beta^{j-1} Q$ for $j \geq 1$ and $U_{0,\beta}(Q) := Q$. When $\beta = 2$ we drop the subscript for β and write $U_j(Q)$ in place of $U_{j,2}(Q)$.

The following potentials were introduced in [50, 54]. The letters (D) and (K) were also used in those papers.

Definition 8.1. *Let $L = -\Delta + V$ on $\mathbb{R}^n$ with V non-negative and locally integrable. We say that V belongs to the class (DK) of order (α, θ, σ) for some $\alpha > 1$, $\theta > 0$, and $\sigma > 0$, if there exists constants $C_0, C_1, C_2 > 0$ and a countable collection of cubes $\mathcal{Q} = \{Q_j\}_j$ with parallel sides and disjoint interiors satisfying $\left|\mathbb{R}^n \backslash \bigcup_j Q_j\right| = 0$, and the following properties:*

$$\alpha^4 Q_i \cap \alpha^4 Q_j \neq \emptyset \quad \Longrightarrow \quad d(Q_i) \leq C_0\, d(Q_j), \qquad \forall\, Q_i, Q_j \in \mathcal{Q}, \tag{O_α}$$

and for each cube $Q \in \mathcal{Q}$ and $x \in \mathbb{R}^n$,

$$\sup_{y \in \alpha Q} e^{-2^k d(Q)^2 L}(\mathbf{1})(x) \;\leq\; \frac{C_1}{k^{1+\theta}}, \qquad \forall\, k \in \mathbb{N}, \tag{D_θ}$$

$$\int_0^{2t} e^{s\Delta}(\mathbf{1}_{\alpha^3 Q} V)(x)\, ds \;\leq\; C_2 \Big(\frac{t}{d(Q)^2}\Big)^{\sigma}, \qquad \forall\, 0 < t \leq d(Q)^2. \tag{K_σ}$$

In this case we shall write $V \in (DK_{\alpha,\theta,\sigma})$.

Remark 8.2. (i) Condition (O_α) implies that the collection of cubes $\mathcal{Q}$ has slowly varying diameters. In particular, the collection of dilates $\alpha^4 \mathcal{Q} = \left\{\alpha^4 Q_j\right\}_j$ has bounded overlap.

(ii) Condition (D_θ) is a decay condition on the heat kernel of the Schrödinger operator. This extra decay allows us to handle the global singularities of singular integrals associated to L.

(iii) Condition (K_σ) captures the extent to which L is a local perturbation of $-\Delta$. This allows us to handle the local singularities of singular integrals associated to L.

(iv) This class of potentials generalise the reverse Hölder class in the following sense. Suppose that $V \in \mathcal{B}_q$ for some $q \geq n/2$ and $n \geq 3$. If we define $\{Q_j\}_j$ to be the maximal cubes on $\mathbb{R}^n$ for which $d(Q)^2 \fint_Q V \leq 1$, then (D_θ), (K_σ), and (O_α) holds for some θ, σ, α. We refer the reader to Section 8 of [54] for the details.

(v) It is not known whether conditions (O_α), (D_θ), and (K_σ) imply (G_p).

We can now state the main result of this chapter.

Theorem 8.3. *Let $L = -\Delta + V$ on $\mathbb{R}^n$, $n \geq 1$ with V non-negative and locally integrable. Assume further that $V \in (DK_{\alpha,\theta,\sigma})$ for some $\sigma \in \left(\frac{n+2}{4}, \frac{n+1}{2}\right)$, $\theta > 1$, and $\alpha > 1$. Assume also that (G_p) holds for each $p \in \left(1, \frac{n}{n+1-2\sigma}\right)$. Then $\nabla L^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for each $p \in \left(1, \frac{n}{n+1-2\sigma}\right)$.*

Before we begin the proof we mention that by combining this with our results from Chapters 4 and 6 (specifically Theorems 4.1 and 6.1) we can obtain the following consequence.

Corollary 8.4. *Under the assumptions of Theorem 8.3 the following holds.*

(a) *$\nabla L^{-1/2}$ is bounded on $L^p(w)$ for each $w \in \mathcal{A}_\infty$ and each $p \in \mathcal{W}_w\left(1, \frac{n}{n+1-2\sigma}\right)$.*

(b) *$\nabla L^{-1/2}$ is bounded on $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ for each $p \in \left(1, \frac{n}{n+1-2\sigma}\right)$ and each $\lambda \in \left((n+1-2\sigma)p,\, n\right)$.*

Proof of Theorem 8.3. The spirit of the following argument is adapted from [50]. Our strategy is to show that the adjoint of $\nabla L^{-1/2}$ is controlled pointwise by the maximal operator $M\left(|\cdot|^{p'}\right)^{1/p'}$ for the appropriate range of p. To do so we exploit the principle that L is a local perturbation of $-\Delta$, with the region of locality determined by the cubes in $\mathcal{Q}$. We shall split our analysis into 'local' and 'global' regions. In the global region we use the extra decay given by condition (D_θ), while in the local region we base our analysis on a comparison between the heat kernel $p_t(x, y)$ of e^{-tL} with the classical heat kernel $h_t(x, y)$ of $e^{t\Delta}$ (as defined in (2.1)), via the perturbation formula (2.4). The theorem then follows by duality.

We turn to the details. Set $R_L := \nabla L^{-1/2}$ and $R_\Delta := \nabla(-\Delta)^{-1/2}$. We aim to show that for almost every $x \in \mathbb{R}^n$ and each $p \in \left(1, \frac{n}{n+1-2\sigma}\right)$ we have

$$|R_L^* f(x)| \lesssim M\big(|f|^{p'}\big)(x)^{1/p'}. \tag{8.1}$$

Now for almost every x there exists a unique cube $Q \in \mathcal{Q}$ with $x \in Q$. Hence the following decomposition of R_L^* is well defined:

$$R_L^* f = R_{L,glob}^* f + (R_{L,loc}^* - R_{\Delta,loc}^*) f + R_{\Delta,loc}^* f,$$

where

$$\begin{aligned} R_{L,glob} f(x) &:= \int_{d(Q)^2}^{\infty} \nabla e^{-tL} f(x) \frac{dt}{\sqrt{t}}, \\ R_{L,loc} f(x) &:= \int_0^{d(Q)^2} \nabla e^{-tL} f(x) \frac{dt}{\sqrt{t}}, \\ R_{\Delta,loc} f(x) &:= \int_0^{d(Q)^2} \nabla e^{t\Delta} f(x) \frac{dt}{\sqrt{t}}, \end{aligned}$$

with Q the unique cube from $\mathcal{Q}$ containing x. Note that we have ommitted the constant $1/\sqrt{\pi}$ which should appear on the right hand side of the above formulae – see Section 2.2.

We first study the global operator $R_{L,glob}^*$. Write

$$R_{L,glob}^* f(x) = \int_{\mathbb{R}^n} k_{glob}(y,x) f(y)\, dy, \qquad \text{where} \qquad k_{glob}(y,x) := \int_{d(Q)^2}^{\infty} \nabla_y p_t(y,x) \frac{dt}{\sqrt{t}}.$$

Then by Hölder's inequality

$$\begin{aligned} \left|R_{L,glob}^* f(x)\right| &\leq \sum_{j=0}^{\infty} \left|2^j Q\right| \Big(\fint_{U_j(Q)} |f(y)|^{p'}\ dy \Big)^{1/p'} \Big(\fint_{U_j(Q)} |k_{glob}(y,x)|^{p}\ dy \Big)^{1/p} \\ &\lesssim M\big(|f|^{p'}\big)(x)^{1/p'} \sum_{j=0}^{\infty} 2^{jn/p'} d(Q)^{n/p'} \left\|k_{glob}(\cdot,x)\right\|_{L^p(U_j(Q))}. \end{aligned}$$

We shall prove that the series is uniformly bounded with respect to x and Q. That is, for some $C_p > 0$ independent of x and Q,

$$\Sigma(x,Q,p) := \sum_{j=0}^{\infty} 2^{jn/p'} d(Q)^{n/p'} \left\|k_{glob}(\cdot,x)\right\|_{L^p(U_j(Q))} \leq C_p. \tag{8.2}$$

We first consider $j = 0,1,2$. Let us point out that condition (G_p) for $p \in \left(1, \frac{n}{n+1-2\sigma}\right)$ implies that

$$\|\nabla p_t(\cdot,x)\|_{L^p} \lesssim \frac{1}{t^{1/2+n/(2p')}} \tag{8.3}$$

for the same range of p. This can be seen from the argument in Lemma 4.8. See in particular inequality (4.18). Estimate (8.3) combined with Minkowski's inequality implies

$$\begin{aligned}\|k_{glob}(\cdot,x)\|_{L^p(U_j(Q))} \le \|k_{glob}(\cdot,x)\|_{L^p(4Q)} &\le \int_{d(Q)^2}^{\infty} \|\nabla p_t(\cdot,x)\|_{L^p(4Q)} \frac{dt}{\sqrt{t}} \\ &\lesssim \int_{d(Q)^2}^{\infty} \frac{dt}{t^{1+n/(2p')}} \lesssim d(Q)^{-n/p'}.\end{aligned}$$

Next for $j \ge 3$ we decompose

$$\begin{aligned}\|k_{glob}(\cdot,x)\|_{L^p(U_j(Q))} &\le \int_{d(Q)^2}^{\infty} \|\nabla p_t(\cdot,x)\|_{L^p(U_j(Q))} \frac{dt}{\sqrt{t}} \\ &= \sum_{k=0}^{\infty} \int_{2^k d(Q)^2}^{2^{k+1}d(Q)^2} \|\nabla p_t(\cdot,x)\|_{L^p(U_j(Q))} \frac{dt}{\sqrt{t}}.\end{aligned}$$

Using the semigroup property (2.3), we have for each $k \ge 0$ and $2^k d(Q)^2 \le t \le 2^{k+1}d(Q)^2$,

$$\nabla_y p_t(y,x) = \int_{\mathbb{R}^n} \nabla_y p_{t-2^{k-1}d(Q)^2}(y,z)\, p_{2^{k-1}d(Q)^2}(z,x)\, dz.$$

Applying (G_p) with $p \in \big(1, \frac{n}{n+1-2\sigma}\big)$ we obtain (see (8.3) and also the proof of Lemma 4.8)

$$\begin{aligned}\|\nabla p_t(\cdot,x)\|_{L^p(U_j(Q))} &\le \int_{\mathbb{R}^n} \Big(\int_{U_j(Q)} \big|\nabla_y p_{t-2^{k-1}d(Q)^2}(y,z)\big|^p\, dy\Big)^{1/p} p_{2^{k-1}d(Q)^2}(z,x)\, dz \\ &\lesssim \frac{\exp\big(-c\frac{4^j d(Q)^2}{t-2^{k-1}d(Q)^2}\big)}{\big(t-2^{k-1}d(Q)^2\big)^{1/2+n/(2p')}} \int_{\mathbb{R}^n} p_{2^{k-1}d(Q)^2}(z,x)\, dz.\end{aligned}$$

Since $2^k d(Q)^2 \le t \le 2^{k+1}d(Q)^2$ implies that $2^{k-1}d(Q)^2 \le t - 2^{k-1}d(Q)^2 \le 2^{k+1}d(Q)^2$ we obtain

$$\|\nabla p_t(\cdot,x)\|_{L^p(U_j(Q))} \lesssim \frac{e^{-c' 2^{2j-k}}}{\big(2^k d(Q)^2\big)^{1/2+n/(2p')}} \int_{\mathbb{R}^n} p_{2^{k-1}d(Q)^2}(z,x)\, dz.$$

Noting that for each $k \ge 0$,

$$\int_{2^k d(Q)^2}^{2^{k+1}d(Q)^2} \frac{dt}{\sqrt{t}} = 2^{k/2}\, d(Q)^2 \big(2\sqrt{2}-2\big),$$

we have that

$$\|k_{glob}(\cdot,x)\|_{L^p(U_j(Q))} \lesssim \sum_{k=0}^{\infty} \frac{e^{-c' 2^{2j-k}}}{2^{kn/(2p')} d(Q)^{n/p'}} \int_{\mathbb{R}^n} p_{2^{k-1}d(Q)^2}(z,x)\, dz.$$

For $k = 0,1,2$ we use the Gaussian bounds on the heat kernel of L. For $k \ge 3$ we use that $k/2 < k-1 < k$ and combine this with (D_θ). Together these allow us to obtain (for $j \ge 3$)

$$\|k_{glob}(\cdot,x)\|_{L^p(U_j(Q))} \lesssim d(Q)^{-n/p'} \Big\{ e^{-c' 4^j} + \sum_{k=3}^{\infty} \frac{e^{-c' 2^{2j-k}}}{2^{kn/(2p')}} \frac{1}{k^{1+\theta}} \Big\}.$$

Therefore returning to (8.2) we have

$$\Sigma(x,Q,p) \;\lesssim\; 1+\sum_{j=3}^{\infty}\Big\{2^{jn/p'}e^{-c'4^j}+\sum_{k=3}^{\infty}2^{n/p'(j-k/2)e^{-c'2^{2j-k}}}\frac{1}{k^{1+\theta}}\Big\}.$$

Thus estimate (8.2) follows once we show that

$$\Sigma_2 \;:=\; \sum_{j=3}^{\infty}\sum_{k=3}^{\infty}2^{n/p'(j-k/2)}e^{-c'2^{2j-k}}\frac{1}{k^{1+\theta}} \;<\; C_p. \tag{8.4}$$

We split the sum into

$$\Sigma_2 \;=\; \sum_{j=3}^{\infty}\left(\sum_{k=3}^{j}+\sum_{k=j+1}^{\infty}\right)2^{n/p'(j-k/2)}e^{-c'2^{2j-k}}\frac{1}{k^{1+\theta}} \;=:\; \Sigma_{2.1}+\Sigma_{2.2}\,.$$

For the first term, for each $k\le j$ we have that $e^{-c'2^{2j-k}}\le e^{-c'2}$. Then

$$\Sigma_{2.1} \;\lesssim\; \sum_{j=3}^{\infty}\sum_{k=3}^{j}e^{-c''2^j}\frac{1}{k^{1+\theta}} \;\lesssim\; \sum_{j=3}^{\infty}je^{-c''2^j} \;<\; C.$$

For the second term, since $\theta>1$,

$$\Sigma_{2.2} \;\lesssim\; \sum_{j=3}^{\infty}\sum_{k=j+1}^{\infty}\frac{1}{k^{1+\theta}} \;\le\; \sum_{j=3}^{\infty}j^{-(1+\theta)/2}\sum_{k=j+1}^{\infty}k^{-(1+\theta)/2} \;<\; C.$$

Combining these two estimates gives (8.4)

Next turn to the localised operator $R^*_{L,loc}-R^*_{\Delta,loc}$. We write

$$(R^*_{L,loc}-R^*_{\Delta,loc})f(x)=\int_{\mathbb{R}^n}k_{loc}(y,x)f(y)\,dy$$

where

$$k_{loc}(y,x):=\int_0^{d(Q)^2}\big(\nabla_y p_t(y,x)-\nabla_y h_t(y,x)\big)\,\frac{dt}{\sqrt{t}}\,.$$

We split the domain of integration over α-dilates of Q and apply Hölder's inequality to obtain

$$\begin{aligned}\big|(R^*_{L,loc}-R^*_{\Delta,loc})f(x)\big| &\le \sum_{j=0}^{\infty}\big|\alpha^jQ\big|\Big(\fint_{U_{j,\alpha}(Q)}|f(y)|^{p'}\;dy\Big)^{1/p'}\Big(\fint_{U_{j,\alpha}(Q)}|k_{loc}(y,x)|^{p}\;dy\Big)^{1/p}\\ &\lesssim M\big(|f|^{p'}\big)(x)^{1/p'}\sum_{j=0}^{\infty}d(Q)^{n/p'}\alpha^{jn/p'}\,\|k_{loc}(\cdot,x)\|_{L^p(U_{j,\alpha}(Q))}\,.\end{aligned}$$

We aim to show that the series is uniformly bounded in x and Q. That is, for some $C_p>0$ independent of x and Q,

$$\sum_{j=0}^{\infty}d(Q)^{n/p'}\alpha^{jn/p'}\,\|k_{loc}(\cdot,x)\|_{L^p(U_{j,\alpha}(Q))} \;\le\; C_p. \tag{8.5}$$

We first consider the terms $j \geq 3$. Now

$$\begin{aligned}
\|k_{loc}(\cdot,x)\|_{L^p(U_{j,\alpha}(Q))} &\leq \Big\|\int_0^{d(Q)^2} |\nabla p_t(\cdot,x)| \,\frac{dt}{\sqrt{t}}\Big\|_{L^p(U_{j,\alpha}(Q))} + \Big\|\int_0^{d(Q)^2} |\nabla h_t(\cdot,x)| \,\frac{dt}{\sqrt{t}}\Big\|_{L^p(U_{j,\alpha}(Q))} \\
&\leq \int_0^{d(Q)^2} \|\nabla p_t(\cdot,x)\|_{L^p(U_{j,\alpha}(Q))} \frac{dt}{\sqrt{t}} + \int_0^{d(Q)^2} \|\nabla h_t(\cdot,x)\|_{L^p(U_{j,\alpha}(Q))} \frac{dt}{\sqrt{t}} \\
&=: \mathcal{J}_1 + \mathcal{J}_2\,.
\end{aligned}$$

To handle these terms we use (G_p) and the following fact: there exists $C = C(\alpha, n) > 0$ such that for every $x \in Q$, $y \in U_{j,\alpha}(Q)$ and $j \geq 3$, one has $|x-y| \geq C\alpha^j d(Q)$. To see note that the distance between the cubes Q and $\alpha^{j-1}Q$ is at least $\alpha^{j-1}l(Q)/2 - l(Q)/2$. Since $l(Q) = d(Q)/\sqrt{n}$, with $\alpha > 1$, and $j \geq 3$, then for all such x and y,

$$|x-y| \;\geq\; \tfrac{d(Q)}{2\sqrt{n}}(\alpha^{j-1}-1) \;\geq\; \big(\tfrac{\alpha^3-1}{2\alpha^3\sqrt{n}}\big)\,\alpha^j\, d(Q) \;=\; C\alpha^j d(Q),$$

with $C = (\alpha^2-1)/(2\alpha^3\sqrt{n})$. Fix $\varepsilon > n/(2p')$. Then for each $j \geq 3$, it follows that

$$\begin{aligned}
\mathcal{J}_1 &\lesssim \int_0^{d(Q)^2} \frac{e^{-c\alpha^{2j}d(Q)^2/t}}{t^{1+n/(2p')}}\,dt \\
&\lesssim \int_0^{d(Q)^2} \Big(\frac{t}{\alpha^{2j}d(Q)^2}\Big)^{\varepsilon} \frac{dt}{t^{1+n/(2p')}} \\
&= \frac{1}{\alpha^{2j\varepsilon}d(Q)^{2\varepsilon}} \int_0^{d(Q)^2} \frac{dt}{t^{1+n/(2p')-\varepsilon}} \\
&\lesssim d(Q)^{-n/p'}\alpha^{-2j\varepsilon}.
\end{aligned}$$

For $\mathcal{J}_2$ we use the well known bounds on $|\nabla h_t|$ to obtain, in a similar fashion

$$\mathcal{J}_2 \;\lesssim\; d(Q)^{-n/p'}\alpha^{-2j\varepsilon}$$

with the same ε. Then the estimates for $\mathcal{J}_1$ and $\mathcal{J}_2$ allow us to conclude that

$$\|k_{loc}(\cdot,x)\|_{L^p(U_{j,\alpha}(Q))} \;\lesssim\; d(Q)^{-n/p'}\alpha^{-2j\varepsilon}, \qquad \forall\, j \geq 3. \tag{8.6}$$

We turn to the terms corresponding to $j = 0, 1, 2$ in (8.5). We shall show that

$$\|k_{loc}(\cdot,x)\|_{L^p(U_{j,\alpha}(Q))} \;\lesssim\; d(Q)^{-n/p'} \qquad j = 0, 1, 2. \tag{8.7}$$

From the perturbation formula (2.4) we have for each $x \in Q$, $y \in \mathbb{R}^n$ and $t > 0$

$$k_{loc}(y,x) \;=\; -\int_0^{d(Q)^2}\int_0^t\int_{\mathbb{R}^n} \nabla_y h_{t-s}(y,z)\,V(z)\,p_s(z,x)\,dz\,ds\,\frac{dt}{\sqrt{t}}\,.$$

Then

$$
\begin{aligned}
|k_{loc}(y,x)| &\leq \int_0^{d(Q)^2} \int_0^t \int_{\mathbb{R}^n} |\nabla_y h_{t-s}(y,z)|\; V(z)\, p_s(z,x)\, dz\, ds\, \frac{dt}{\sqrt{t}} \\
&= k^1_{\rm loc}(y,x) + k^2_{\rm loc}(y,x) + k^3_{\rm loc}(y,x),
\end{aligned}
$$

where

$$
\begin{aligned}
k^1_{\rm loc}(y,x) &:= \int_0^{d(Q)^2} \int_0^{t/2} \int_{\alpha^3 Q} |\nabla_y h_{t-s}(y,z)|\, V(z) p_s(z,x)\, dz\, ds\, \frac{dt}{\sqrt{t}}, \\
k^2_{\rm loc}(y,x) &:= \int_0^{d(Q)^2} \int_{t/2}^{t} \int_{\alpha^3 Q} |\nabla_y h_{t-s}(y,z)|\, V(z) p_s(z,x)\, dz\, ds\, \frac{dt}{\sqrt{t}}, \\
k^3_{\rm loc}(y,x) &:= \int_0^{d(Q)^2} \int_0^{t} \int_{\mathbb{R}^n \setminus \alpha^3 Q} |\nabla_y h_{t-s}(y,z)|\, V(z) p_s(z,x)\, dz\, ds\, \frac{dt}{\sqrt{t}}.
\end{aligned}
$$

Then it follows that

$$
\|k_{loc}(\cdot,x)\|_{L^p(\alpha^2 Q)} \;\leq\; \left\|k^1_{\rm loc}(\cdot,x)\right\|_{L^p(\alpha^2 Q)} + \left\|k^2_{\rm loc}(\cdot,x)\right\|_{L^p(\alpha^2 Q)} + \left\|k^3_{\rm loc}(\cdot,x)\right\|_{L^p(\alpha^2 Q)}. \tag{8.8}
$$

To study the first term in (8.8) we observe that $s \in (0, t/2)$ implies that $t - s \in (t/2, t)$, and hence the well known bounds on $|\nabla h_{t-s}|$ give

$$
|\nabla_y h_{t-s}(y,z)| \;\lesssim\; \frac{1}{t^{n/2+1/2}} e^{-c\frac{|y-z|^2}{t}}
$$

for any $y, z \in \mathbb{R}^n$. Hence for any $z \in \mathbb{R}^n$,

$$
\|\nabla h_{t-s}(\cdot,z)\|_{L^p(\alpha^2 Q)} \;\lesssim\; t^{-1/2-n/(2p')}.
$$

This estimate gives, for each $x \in Q$,

$$
\begin{aligned}
\left\|k^1_{\rm loc}(\cdot,x)\right\|_{L^p(\alpha^2 Q)} &\leq \int_0^{d(Q)^2} \int_0^{t/2} \int_{\alpha^3 Q} \|\nabla h_{t-s}(\cdot,z)\|_{L^p(\alpha^2 Q)}\, V(z) p_s(z,x)\, dz\, ds\, \frac{dt}{\sqrt{t}} \\
&\lesssim \int_0^{d(Q)^2} \int_0^{t/2} \int_{\alpha^3 Q} V(z) p_s(z,x)\, dz\, ds\, \frac{dt}{t^{1+n/(2p')}} \\
&\leq \int_0^{d(Q)^2} \int_0^{t/2} e^{s\Delta}(\mathbf{1}_{\alpha^3 Q} V)(x)\, ds\, \frac{dt}{t^{1+n/(2p')}}.
\end{aligned}
$$

Applying (K_σ) and that $\sigma > n/(2p')$ we obtain

$$
\left\|k^1_{\rm loc}(\cdot,x)\right\|_{L^p(\alpha^2 Q)} \;\lesssim\; \int_0^{d(Q)^2} \Big(\frac{t}{d(Q)^2}\Big)^{\sigma} \frac{dt}{t^{1+n/(2p')}} \;\lesssim\; d(Q)^{-n/p'}.
$$

We turn to the second term in (8.8). From the well known bounds on $|\nabla h_t|$ we obtain,

$$\|\nabla h_{t-s}(\cdot,z)\|_{L^p(\alpha^2 Q)} \lesssim (t-s)^{-1/2-n/(2p')}, \qquad \forall\, z\in\mathbb{R}^n,\ t>s,$$

so that for each $x\in Q$,

$$\begin{aligned}
\left\|k^2_{\mathrm{loc}}(\cdot,x)\right\|_{L^p(\alpha^2 Q)} &\le \int_0^{d(Q)^2}\int_{t/2}^t\int_{\alpha^3 Q}\|\nabla h_{t-s}(\cdot,z)\|_{L^p(\alpha^2 Q)}\,V(z)p_s(z,x)\,dz\,ds\,\frac{dt}{\sqrt{t}}\\
&\lesssim \int_0^{d(Q)^2}\int_{t/2}^t\int_{\alpha^3 Q}V(z)p_s(z,x)\,dz\,\frac{ds}{(t-s)^{1/2+n/(2p')}}\,\frac{dt}{\sqrt{t}}\\
&\le \int_0^{d(Q)^2}\int_{t/2}^t\int_{\alpha^3 Q}V(z)h_s(z,x)\,dz\,\frac{ds}{(t-s)^{1/2+n/(2p')}}\,\frac{dt}{\sqrt{t}}\\
&= \int_0^{d(Q)^2}\int_{t/2}^t e^{s\Delta}(\mathbf{1}_{\alpha^3 Q}V)(x)\,\frac{ds}{(t-s)^{1/2+n/(2p')}}\,\frac{dt}{\sqrt{t}}\\
&= \int_0^{d(Q)^2}\int_0^{t/2} e^{(t-r)\Delta}(\mathbf{1}_{\alpha^3 Q}V)(x)\,\frac{dr}{r^{1/2+n/(2p')}}\,\frac{dt}{\sqrt{t}},
\end{aligned}$$

where in the last line we made a change of variable $r:=t-s$. Since $r<t/2$ and by the semigroup property (2.3),

$$\begin{aligned}
e^{(t-r)\Delta}(\mathbf{1}_{\alpha^3 Q}V)(x) &= e^{(t-2r)\Delta}e^{r\Delta}(\mathbf{1}_{\alpha^3 Q}V)(x)\\
&= \int_{\mathbb{R}^n} h_{t-2r}(x,y)e^{r\Delta}(\mathbf{1}_{\alpha^3 Q}V)(y)\,dy\\
&\le \operatorname{ess\,sup}_{y\in\mathbb{R}^n} e^{r\Delta}(\mathbf{1}_{\alpha^3 Q}V)(y).
\end{aligned}$$

To continue we require the following technical estimate.

Lemma 8.5. *Suppose $f\ge 0$ and for some $\delta>0$, the following holds*

$$\int_0^t f(s)\,ds \le \left(\frac{t}{R}\right)^{\delta}, \qquad 0<t\le R.$$

Then for each $0<\epsilon<\delta$, there exists $C=C(\epsilon,\delta)>0$ such that

$$\int_0^t f(s)\,\frac{ds}{s^{\epsilon}} \le C\frac{t^{\delta-\epsilon}}{R^{\delta}}\,.$$

Proof. We compute

$$\int_0^t f(s)\,\frac{ds}{s^{\epsilon}} = \sum_{k=0}^{\infty}\int_{2^{-k-1}t}^{2^{-k}t} f(s)\,\frac{ds}{s^{\epsilon}} \ \le\ \sum_{k=0}^{\infty}\frac{2^{(k+1)\epsilon}}{t^{\epsilon}}\int_0^{2^{-k}t} f(s)\,ds$$

$$\leq \left(\frac{2}{t}\right)^{\epsilon} \sum_{k=0}^{\infty} 2^{k\epsilon} \left(\frac{2^{-k} t}{R}\right)^{\delta} = 2^{\epsilon} \frac{t^{\delta-\epsilon}}{R^{\delta}} \sum_{k=0}^{\infty} 2^{k(\epsilon-\delta)} = C \frac{t^{\delta-\epsilon}}{R^{\delta}} .$$

where $C = 2^{\delta}/(2^{\delta-\epsilon} - 1)$. □

Since $\sigma > 1/2 + n/(2p')$, we can apply Lemma 8.5 with

$$f(r) := e^{r\Delta}(\mathbf{1}_{\alpha^3 Q} V)(y), \qquad R := d(Q)^2, \qquad \epsilon := \frac{1}{2} + \frac{n}{2p'} ,$$

to obtain

$$\begin{aligned} \left\|k_{\mathrm{loc}}^2(\cdot, x)\right\|_{L^p(\alpha^2 Q)} &\lesssim \operatorname{ess\,sup}_{y \in \mathbb{R}^n} \int_0^{d(Q)^2} \int_0^{t/2} e^{r\Delta}(\mathbf{1}_{\alpha^3 Q} V)(y) \frac{dr}{r^{1/2+n/(2p')}} \frac{dt}{t^{1/2}} \\ &\lesssim d(Q)^{-2\sigma} \int_0^{d(Q)^2} \frac{dt}{t^{1+n/(2p')-\sigma}} \\ &\lesssim d(Q)^{-n/p'} . \end{aligned}$$

We turn to the last term in (8.8). Firstly note that for $0 < s < t < d(Q)^2$, $z \notin \alpha^3 Q$, and $y \in \alpha^2 Q$ we have

$$t - s \leq d(Q)^2 - s \leq d(Q)^2 \qquad \text{and} \qquad |y - z| \geq \left(\tfrac{\alpha^3 - \alpha^2}{2\sqrt{n}}\right) d(Q) = C d(Q),$$

so that for all such z, y, t, s,

$$|\nabla_y h_{t-s}(y,z)| \lesssim \frac{\exp\left(-c \frac{|y-z|^2}{t-s}\right)}{(t-s)^{n/2+1/2}} \lesssim \frac{\exp\left(-c \frac{|y-z|^2}{t-s}\right)}{|y-z|^{n+1}} \lesssim \frac{\exp\left(-c \frac{|y-z|^2}{d(Q)^2}\right)}{d(Q)^{n+1}} .$$

Therefore

$$\begin{aligned} \left\|k_{\mathrm{loc}}^3(\cdot, x)\right\|_{L^p(\alpha^2 Q)} &\lesssim \tfrac{1}{d(Q)^{n+1}} \int_0^{d(Q)^2} \int_0^t \int_{\mathbb{R}^n \setminus \alpha^3 Q} \left\|e^{-c|\cdot - z|^2/d(Q)^2}\right\|_{L^p(\alpha^2 Q)} V(z) p_s(z,x)\, dz\, ds \frac{dt}{\sqrt{t}} \\ &\lesssim \tfrac{1}{d(Q)^{1+n/p'}} \int_0^{d(Q)^2} \int_0^t \int_{\mathbb{R}^n \setminus \alpha^3 Q} V(z) p_s(z,x)\, dz\, ds \frac{dt}{\sqrt{t}} \\ &\leq \tfrac{1}{d(Q)^{1+n/p'}} \int_0^{d(Q)^2} \int_0^{\infty} \int_{\mathbb{R}^n} V(z) p_s(z,x)\, dz\, ds \frac{dt}{\sqrt{t}} . \end{aligned}$$

To proceed we recall the following fact, which appears to be well known. We refer the reader to [54] and the references there for a proof.

Lemma 8.6 ([54] Lemma 3.10)**.** *Assume that $0 \leq V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then for each $f \in L^1(\mathbb{R}^n)$, one has*

$$\int_{\mathbb{R}^n} \int_0^{\infty} V(x)\, e^{-tL}\big(|f|\big)(x)\, dt\, dx \leq \|f\|_{L^1} .$$

Now by the semigroup property (2.3) and Lemma 8.6,

$$\int_0^\infty \int_{\mathbb{R}^n} V(z) p_s(z,x)\, dz\, ds = \int_0^\infty \int_{\mathbb{R}^n} V(z) e^{-(s/2)L}\big(p_{s/2}(\cdot,x)\big)(z)\, dz\, ds$$
$$\leq \left\| p_{s/2}(\cdot,x) \right\|_{L^1(\mathbb{R}^n)} \;\leq\; 1\,.$$

This immediately gives

$$\left\| k^3_{\mathrm{loc}}(\cdot,x) \right\|_{L^p(\alpha^2 Q)} \;\lesssim\; d(Q)^{-n/p'}.$$

Inserting these estimates in (8.8) yields (8.7).

Gathering (8.7) and (8.6) we then obtain

$$\sum_{j=0}^{\infty} d(Q)^{n/p'} \alpha^{jn/p'} \left\| k_{loc}(\cdot,x) \right\|_{L^p(U_{j,\alpha}(Q))} \;\;\lesssim\;\; 1 + \sum_{j=3}^{\infty} \alpha^{j(n/p'-2\varepsilon)}.$$

Recalling that $\varepsilon > n/(2p')$, we see the series is convergent and hence (8.5) follows.

Finally, it is well known that the classical Riesz transform satisfies for every $p > 1$,

$$|R^*_{\Delta,loc} f(x)| \lesssim M\big(|f|^{p'}\big)(x)^{1/p'}, \qquad\qquad \text{a.e. } x.$$

Combining this with our previous estimates we obtain, for each $p \in \big(1, \frac{n}{n+1-2\sigma}\big)$ and for almost every $x \in \mathbb{R}^n$,

$$|R^*_L f(x)| \;\leq\; |R^*_{L,glob} f(x)| + |(R^*_{L,loc} - R^*_{\Delta,loc} f)(x)| + |R^*_{\Delta,loc} f(x)| \;\lesssim\; M\big(|f|^{p'}\big)(x)^{1/p'},$$

which is (8.1). This completes the proof of Theorem 8.3. $\square$

Bibliography

[1] I. Abu-Falahah, P.R. Stinga, and J.L. Torrea, *Square functions associated to Schrödinger operators*, Studia Math. **203** (2011), no. 2, 171–194.

[2] D.R. Adams, *A note on Riesz potentials*, Duke Math. J. **42** (1975), no. 4, 765–778.

[3] ______, *Choquet integrals in potential theory*, Publ. Mat. **42** (1998), no. 1, 3–66.

[4] D.R. Adams and J. Xiao, *Nonlinear potential analysis on Morrey spaces and their capacities*, Indiana Univ. Math. J. **53** (2004), no. 6, 1629–1663.

[5] ______, *Morrey spaces in harmonic analysis*, Ark. Mat. **50** (2012), no. 2, 201–230.

[6] B.T. Anh, *Weighted norm inequalities for Riesz transforms of magnetic Schrödinger operators*, Differential Integral Equations **23** (2010), no. 9-10, 811–826.

[7] B.T. Anh and X.T. Duong, *Boundedness of singular integrals and their commutators with BMO functions on Hardy spaces*, preprint (2011), arXiv:1110.1770v1.

[8] ______, *Weighted Hardy spaces associated to operators and boundedness of singular integrals*, preprint (2012), arXiv:1202.2063v1.

[9] H. Arai and T. Mizuhara, *Morrey spaces on spaces of homogeneous type and estimates for $\Box_b$ and the Cauchy-Szegő projection*, Math. Nachr. **185** (1997), 5–20.

[10] J. Assaad, *Riesz transforms associated to Schrödinger operators with negative potentials*, Publ. Mat. **55** (2011), no. 1, 123–150.

[11] P. Auscher, *On necessary and sufficient conditions for L^p-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates*, Mem. Amer. Math. Soc. **186** (2007), no. 871, xviii+75.

[12] P. Auscher and B. Ben Ali, *Maximal inequalities and Riesz transform estimates on L^p spaces for Schrödinger operators with nonnegative potentials*, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 6, 1975–2013.

[13] P. Auscher, T. Coulhon, X.T. Duong, and S. Hofmann, *Riesz transform on manifolds and heat kernel regularity*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 6, 911–957.

[14] P. Auscher, X.T. Duong, and A. McIntosh, *Boundedness of Banach space valued singular integral operators and Hardy spaces*, unpublished preprint (2005).

[15] P. Auscher and J.M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. III. Harmonic analysis of elliptic operators*, J. Funct. Anal. **241** (2006), no. 2, 703–746.

[16] ______, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. I. General operator theory and weights*, Adv. Math. **212** (2007), no. 1, 225–276.

[17] ______, *Weighted norm inequalities for fractional operators*, Indiana Univ. Math. J. **57** (2008), no. 4, 1845–1869.

[18] ______, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. IV. Riesz transforms on manifolds and weights*, Math. Z. **260** (2008), no. 3, 527–539.

[19] P. Auscher, A. McIntosh, and E. Russ, *Hardy spaces of differential forms on Riemannian manifolds*, J. Geom. Anal. **18** (2008), no. 1, 192–248.

[20] J. Avron, I. Herbst, and B. Simon, *Schrödinger operators with magnetic fields. I. General interactions*, Duke Math. J. **45** (1978), no. 4, 847–883.

[21] B. Ben Ali, *Maximal inequalities and Riesz transform estimates on L^p spaces for magnetic Schrödinger operators II*, preprint (2009), arXiv:0905.0265.

[22] ______, *Maximal inequalities and Riesz transform estimates on L^p spaces for magnetic Schrödinger operators I*, J. Funct. Anal. **259** (2010), no. 7, 1631–1672.

[23] F.A. Berezin and M.A. Shubin, *The Schrödinger equation*, Mathematics and its Applications (Soviet Series), vol. 66, Kluwer Academic Publishers Group, Dordrecht, 1991.

[24] F. Bernicot and J. Zhao, *New abstract Hardy spaces*, J. Funct. Anal. **255** (2008), no. 7, 1761–1796.

[25] J.J. Betancor, J.C. Fariña, E. Harboure, and L. Rodríguez-Mesa, *L^p-boundedness properties of variation operators in the Schrödinger setting*, preprint (2011), arXiv:1010.3117v1.

[26] S. Blunck and P.C. Kunstmann, *Calderón-Zygmund theory for non-integral operators and the H^∞ functional calculus*, Rev. Mat. Iberoamericana **19** (2003), no. 3, 919–942.

[27] ______, *Weak type (p, p) estimates for Riesz transforms*, Math. Z. **247** (2004), no. 1, 137–148.

[28] B. Bongioanni, E. Harboure, and O. Salinas, *Weighted inequalities for negative powers of Schrödinger operators*, J. Math. Anal. Appl. **348** (2008), no. 1, 12–27.

[29] ______, *Riesz transforms related to Schrödinger operators acting on BMO type spaces*, J. Math. Anal. Appl. **357** (2009), no. 1, 115–131.

[30] ______, *Classes of weights related to Schrödinger operators*, J. Math. Anal. Appl. **373** (2011), no. 2, 563–579.

[31] ______, *Weighted inequalities for commutators of Schrödinger-Riesz transforms*, J. Math. Anal. Appl. **392** (2012), no. 1, 6–22.

[32] H.-Q. Bui, X.T. Duong, and L. Yan, *Calderón reproducing formulas and new Besov spaces associated with operators*, Adv. Math. **229** (2012), no. 4, 2449–2502.

[33] S. Campanato, *Proprietà di inclusione per spazi di Morrey*, Ricerche Mat. **12** (1963), 67–86.

[34] F. Chiarenza and M. Frasca, *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat. Appl. (7) **7** (1987), no. 3-4, 273–279 (1988).

[35] R.R. Coifman, *Distribution function inequalities for singular integrals*, Proc. Nat. Acad. Sci. U.S.A. **69** (1972), 2838–2839.

[36] R.R. Coifman and R. Rochberg, *Another characterization of BMO*, Proc. Amer. Math. Soc. **79** (1980), no. 2, 249–254.

[37] R.R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) **103** (1976), no. 3, 611–635.

[38] T. Coulhon and X.T. Duong, *Riesz transforms for* $1 \leq p \leq 2$, Trans. Amer. Math. Soc. **351** (1999), no. 3, 1151–1169.

[39] ______, *Maximal regularity and kernel bounds: observations on a theorem by Hieber and Prüss*, Adv. Differential Equations **5** (2000), no. 1-3, 343–368.

[40] T. Coulhon, X.T. Duong, and X.D. Li, *Littlewood-Paley-Stein functions on complete Riemannian manifolds for* $1 \leq p \leq 2$, Studia Math. **154** (2003), no. 1, 37–57.

[41] E.B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989.

[42] G. Di Fazio and M.A. Ragusa, *Commutators and Morrey spaces*, Boll. Un. Mat. Ital. A (7) **5** (1991), no. 3, 323–332.

[43] X.T. Duong, S. Hofmann, D. Mitrea, M. Mitrea, and L. Yan, *Hardy spaces and regularity for the inhomogeneous Dirichlet and Neummann problems*, Rev. Mat. Iberoamericana (to appear).

[44] X.T. Duong and J. Li, *Hardy spaces associated to operators satisfying Davies–Gaffney estimates and bounded holomorphic functional calculus*, submitted (2010).

[45] X.T. Duong and A. McIntosh, *Singular integral operators with non-smooth kernels on irregular domains*, Rev. Mat. Iberoamericana **15** (1999), no. 2, 233–265.

[46] X.T. Duong, E.M. Ouhabaz, and L. Yan, *Endpoint estimates for Riesz transforms of magnetic Schrödinger operators*, Ark. Mat. **44** (2006), no. 2, 261–275.

[47] X.T. Duong and L. Yan, *Duality of Hardy and BMO spaces associated with operators with heat kernel bounds*, J. Amer. Math. Soc. **18** (2005), no. 4, 943–973 (electronic).

[48] ______, *New function spaces of BMO type, the John-Nirenberg inequality, interpolation, and applications*, Comm. Pure Appl. Math. **58** (2005), no. 10, 1375–1420.

[49] J. Dziubański, G. Garrigós, T. Martínez, J.L. Torrea, and J. Zienkiewicz, *BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality*, Math. Z. **249** (2005), no. 2, 329–356.

[50] J. Dziubański and M. Preisner, *On Riesz transforms characterization of* H^1 *spaces associated with some Schrödinger operators*, Potential Anal. **35** (2011), no. 1, 39–50.

[51] J. Dziubański and J. Zienkiewicz, *Hardy space* H^1 *associated to Schrödinger operator with potential satisfying reverse Hölder inequality*, Rev. Mat. Iberoamericana **15** (1999), no. 2, 279–296.

[52] ______, H^p *spaces for Schrödinger operators*, Fourier analysis and related topics (Będlewo, 2000), Banach Center Publ., vol. 56, Polish Acad. Sci., Warsaw, 2002, pp. 45–53.

[53] ______, H^p *spaces associated with Schrödinger operators with potentials from reverse Hölder classes*, Colloq. Math. **98** (2003), no. 1, 5–38.

[54] ______, *Hardy spaces* H^1 *for Schrödinger operators with certain potentials*, Studia Math. **164** (2004), no. 1, 39–53.

[55] L.C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.

[56] C. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. (N.S.) **9** (1983), no. 2, 129–206.

[57] C. Fefferman and E.M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3-4, 137–193.

[58] T. Gallouët and J.-M. Morel, *Resolution of a semilinear equation in L^1*, Proc. Roy. Soc. Edinburgh Sect. A **96** (1984), no. 3-4, 275–288.

[59] J. García-Cuerva and J.L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985, Notas de Matemática [Mathematical Notes], 104.

[60] L. Grafakos, *Classical and modern Fourier analysis*, Pearson Education, Inc., Upper Saddle River, NJ, 2004.

[61] A. Grigor′yan, *Upper bounds of derivatives of the heat kernel on an arbitrary complete manifold*, J. Funct. Anal. **127** (1995), no. 2, 363–389.

[62] D. Guibourg, *Inégalités maximales pour l'opérateur de Schrödinger*, C. R. Acad. Sci. Paris Sér. I Math. **316** (1993), no. 3, 249–252.

[63] Z. Guo, P. Li, and L. Peng, *L^p boundedness of commutators of Riesz transforms associated to Schrödinger operator*, J. Math. Anal. Appl. **341** (2008), no. 1, 421–432.

[64] M. Haase, *The functional calculus for sectorial operators*, Operator Theory: Advances and Applications, vol. 169, Birkhäuser Verlag, Basel, 2006.

[65] S. Hofmann, G. Lu, D. Mitrea, M. Mitrea, and L. Yan, *Hardy spaces associated to non-negative self adjoint operators satisfying Davies–Gaffney estimates*, Mem. Amer. Math. Soc. **214** (2011), no. 1007, vi+78.

[66] S. Hofmann and J.M. Martell, *L^p bounds for Riesz transforms and square roots associated to second order elliptic operators*, Publ. Mat. **47** (2003), no. 2, 497–515.

[67] S. Hofmann and S. Mayboroda, *Hardy and BMO spaces associated to divergence form elliptic operators*, Math. Ann. **344** (2009), no. 1, 37–116.

[68] S. Hofmann, S. Mayboroda, and A. McIntosh, *Second order elliptic operators with complex bounded measurable coefficients in L^p, Sobolev and Hardy spaces*, Ann. Sci. Éc. Norm. Supér. (4) **44** (2011), no. 5, 723–800.

[69] S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Mat. **16** (1978), no. 2, 263–270.

[70] R. Jiang and D. Yang, *Orlicz–Hardy spaces associated with operators satisfying Davies–Gaffney estimates*, Commun. Contemp. Math. **13** (2011), no. 2, 331–373.

[71] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

[72] R. Johnson and C.J. Neugebauer, *Change of variable results for A_p- and reverse Hölder RH_r-classes*, Trans. Amer. Math. Soc. **328** (1991), no. 2, 639–666.

[73] T. Kato, *L^p-theory of Schrödinger operators with a singular potential*, Aspects of positivity in functional analysis (Tübingen, 1985), North-Holland Math. Stud., vol. 122, North-Holland, Amsterdam, 1986, pp. 63–78.

[74] K. Kurata, *An estimate on the heat kernel of magnetic Schrödinger operators and uniformly elliptic operators with non-negative potentials*, J. London Math. Soc. (2) **62** (2000), no. 3, 885–903.

[75] K. Kurata and S. Sugano, *A remark on estimates for uniformly elliptic operators on weighted L^p spaces and Morrey spaces*, Math. Nachr. **209** (2000), 137–150.

[76] Y. Liu and J. Dong, *Some estimates of higher order Riesz transform related to Schrödinger type operators*, Potential Anal. **32** (2010), no. 1, 41–55.

[77] J.M. Martell, *Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications*, Studia Math. **161** (2004), no. 2, 113–145.

[78] A.L. Mazzucato, *Besov-Morrey spaces: function space theory and applications to nonlinear PDE*, Trans. Amer. Math. Soc. **355** (2003), no. 4, 1297–1364.

[79] C.B. Morrey, Jr., *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), no. 1, 126–166.

[80] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

[81] B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **192** (1974), 261–274.

[82] J. Nourrigat, *Une inégalité L^2*, unpublished manuscript.

[83] J. Orobitg and J. Verdera, *Choquet integrals, Hausdorff content and the Hardy-Littlewood maximal operator*, Bull. London Math. Soc. **30** (1998), no. 2, 145–150.

[84] E.M. Ouhabaz, *Analysis of heat equations on domains*, London Mathematical Society Monographs Series, vol. 31, Princeton University Press, Princeton, NJ, 2005.

[85] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.

[86] J. Peetre, *On convolution operators leaving $L^{p,\,\lambda}$ spaces invariant*, Ann. Mat. Pura Appl. (4) **72** (1966), 295–304.

[87] ——, *On the theory of $\mathcal{L}_{p,\,\lambda}$ spaces*, J. Functional Analysis **4** (1969), 71–87.

[88] C. Pérez, *Endpoint estimates for commutators of singular integral operators*, J. Funct. Anal. **128** (1995), no. 1, 163–185.

[89] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.

[90] ——, *Methods of modern mathematical physics. III. Scattering theory*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1979.

[91] J.L. Rubio de Francia, *Factorization theory and A_p weights*, Amer. J. Math. **106** (1984), no. 3, 533–547.

[92] M. Schechter, *Operator methods in quantum mechanics*, North-Holland Publishing Co., New York, 1981.

[93] Z. Shen, *L^p estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble) **45** (1995), no. 2, 513–546.

[94] A. Sikora, *Riesz transform, Gaussian bounds and the method of wave equation*, Math. Z. **247** (2004), no. 3, 643–662.

[95] B. Simon, *Maximal and minimal Schrödinger forms*, J. Operator Theory **1** (1979), no. 1, 37–47.

[96] ———, *Schrödinger semigroups*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 3, 447–526.

[97] ———, *Schrödinger operators in the twentieth century*, J. Math. Phys. **41** (2000), no. 6, 3523–3555.

[98] L. Song and L. Yan, *Riesz transforms associated to Schrödinger operators on weighted Hardy spaces*, J. Funct. Anal. **259** (2010), no. 6, 1466–1490.

[99] G. Stampacchia, *The spaces $\mathcal{L}^{(p,\lambda)}$, $N^{(p,\lambda)}$ and interpolation*, Ann. Scuola Norm. Sup. Pisa (3) **19** (1965), 443–462.

[100] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

[101] ———, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[102] E.M. Stein and G. Weiss, *On the theory of harmonic functions of several variables. I. The theory of H^p-spaces*, Acta Math. **103** (1960), 25–62.

[103] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy spaces*, Lecture Notes in Mathematics, vol. 1381, Springer-Verlag, Berlin, 1989.

[104] L. Tang, *Extrapolation from $A_\infty^{\rho,\infty}$, vector-valued inequalities and applications in the Schrödinger settings*, preprint (2011), arXiv:1109.0101v1.

[105] ———, *Weighted norm inequalities for commutators of Littlewood–Paley functions related to Schrödinger operators*, preprint (2011), arXiv:1109.0100v1.

[106] ———, *Weighted norm inequalities for Schrödinger operators*, preprint (2011), arXiv:1109.0099v1.

[107] Y. Tsutsui, *Sharp maximal inequalities and its application to some bilinear estimates*, J. Fourier Anal. Appl. **17** (2011), no. 2, 265–289.

[108] H. Wang, *The boundedness of some singular integral operators acting on weighted Hardy spaces associated to Schrödinger operators*, preprint (2011), arXiv:1103.4752v1.

[109] ———, *Riesz transforms associated with Schrödinger operators acting on weighted Hardy spaces*, preprint (2011), arXiv:1102.5467v1.

[110] J. Zhong, *The Sobolev estimates for some Schrödinger type operators*, Math. Sci. Res. Hot-Line **3** (1999), no. 8, 1–48.