MACQUARIE UNIVERSITY

DOCTORAL DISSERTATION

January 31, 2019

Homotopy theory of Grothendieck ∞-groupoids and ∞-categories

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Summary

This work investigates the homotopy theory of globular models for higher categorical structures. In particular, we focus on weak ∞-groupoids, but most of the constructions can be performed also for weak ∞-categories, and we explicitly mention this when appropriate.

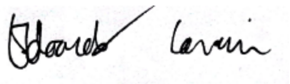
Motivated by Grothendieck's homotopy hypothesis, we study algebraic models of homotopy types in the form of ∞-groupoids, and we address the problem of constructing a path object for these structures, after having introduced their homotopy theory. In detail, we define (trivial) cofibrations and (trivial) fibrations of ∞-groupoids, and prove some basic facts about the induced factorization systems. The construction of a path-object endofunctor is a highly non-trivial task, and the first step we take is to characterize those globular theories whose category of models can be endowed with cofibrantly generated semi-model structure of a precise form, and we also give a sufficient condition for this to happen, based on the existence of a path object endofunctor.

We then construct the underlying globular set of this path object based on the notion of cylinders, and show how to endow it with systems of structures, involving compositions, identities and inverses. Using the combinatorics of finite planar rooted trees we construct an approximation of the algebraic structure needed for the construction of the path object and we introduce modifications to "correct" this approximation in low dimensions, and we thus interpret all operations of dimension less than or equal to 2.

Finally, we manage to complete this construction for the finite-dimensional case of Grothendieck 3-groupoids, thanks to the introduction of a bicategory of cylinders and modifications. We thus establish a semi-model structure on this category, which is conjectured to model homotopy 3-types.

Statement of Originality

This work has not previously been submitted for a degree or diploma in any university. To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.



EDOARDO LANARI   DATE: January 31, 2019

Acknowledgements

The following is a list of people I would like to thank, since they have been a part of this journey and have made it possible, even though some of them may not be aware of this. Also, the order of appearance is irrelevant.

I will start with my parents, for having always believed in me and having taught me to never give up pursuing my dreams.

I would then like to thank my supervisor Richard, for having always been present and for having greatly contributed to my growth as a mathematician. I am also very thankful to my co-supervisor Dom, for his always useful insights and fascinating perspectives, enjoyable conversations and wonderful cooking.

I want to thank all the past and current members of the Centre of Australian Category Theory for having been welcoming since day one, and for setting the bar high in category theory research.

I am very grateful to Dimitri Ara and Nicola Gambino for being very kind in giving me the chance to visit them, I have really enjoyed both these experiences.

A special thank goes to Sara, my partner in crime, for being patient in dealing with me, which is not an easy task.

I would also like to thank Rosa and Ramon, without whom there would have been less fun.

Finally, I want to thank my friends back in Italy, the ones all over the world, C&P and Eco Parcel, and all my instructors and training partners for having kept me mentally sane throughout these years.

A mamma

CHAPTER 1

Introduction

Ordinary category theory studies 1-categories, which can be informally described as collections of objects and arrows between them (i.e. 0-cells and 1-cells), that can be composed via an associative and unital binary composition operation. However, the intrinsic 1-dimensionality of these algebraic objects makes them not suitable to deal with certain kinds of structures or universal constructions, where the universal property encodes more structure than one detectable by 1-categories.

Strict n-categories, on the other hand, have k-cells for $0 \leq k \leq n$, that can be composed appropriately, and all the expected coherences hold strictly. For instance, in a 2-category $\mathcal{C}$, given a diagram of the form:

$$\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.2ex, "\Downarrow a"] \arrow[r, phantom, shift right=2.2ex, "\Downarrow b"] & \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.2ex, "\Downarrow c"] \arrow[r, phantom, shift right=2.2ex, "\Downarrow d"] & \bullet \end{tikzcd} \tag{1}$$

there exists a unique way of composing together the displayed 2-cells. Indeed, if one denotes by $\circ$ the vertical composition operation and by $*$ the horizontal one, we get:

$$(a \circ b) * (c \circ d) = (a * c) \circ (b * d).$$

These structures, despite allowing for higher dimensional data, are too strict to capture many phenomena that naturally arise in mathematics. An example of this deficiency is given by the fundamental ∞-groupoid ΠX of a space X: this is supposed to be an ∞-category where all the k-cells for $k > 0$ are invertible (hence the name groupoid) that captures all the homotopical data of a given space X. Its 1-cells are defined to be paths in X, i.e. maps $[0,1] \to X$, whose composition is defined by concatenation of paths. This operation is easily seen not to be associative, but it is so up to homotopy (relative to the boundary). What this means is that given paths in X of the form $a\colon w \rightsquigarrow x, b\colon x \rightsquigarrow y, c\colon y \rightsquigarrow z$ we can compose them in two different ways using concatenation of paths and then rescaling the domain, i.e. $c \circ (b \circ a)$ and $(c \circ b) \circ a$, and there is a map $H\colon [0,1]^2 \to X$ such that

$$H(s,0) = (c \circ (b \circ a))(s),\ H(s,1) = ((c \circ b) \circ a)(s) \text{ and } H(0,t) = *_w,\ H(1,t) = *_z \tag{2}$$

where $*_t$ denotes the constant path at $t \in X$. Such a map H is equivalently given by a map $D_2 \to X$, i.e. a 2-disk in X. We see with this example that associativity is not a property of the composition operation, but rather data encoded by the homotopy H. Therefore, we can think of H as being a 2-cell between the 1-cells $c \circ (b \circ a)$ and $(c \circ b) \circ a$.

Traditionally, the fundamental 1-groupoid $\Pi_1 X$ of a space X was defined in the following way: its objects are points $x \in X$, and maps $x \to y$ are homotopy classes of paths $a\colon x \rightsquigarrow y$ fixing the endpoints. This constitutes an ordinary category where all arrows are invertible, i.e. a 1-groupoid. However, it only retains 1-dimensional information about the space, i.e. one can only recover the homotopy groups $\pi_0(X)$ and $\pi_1(X, x)$ for every $x \in X$.

If we keep track of 2-cells such as H in (2), we can instead define a *weak 2-category* (i.e. a bicategory) $\Pi_2 X$ having the same objects as $\Pi_1 X$, but 1-cells $a\colon x \to y$ are now simply paths $a\colon x \rightsquigarrow y$ in X and 2-cells $H\colon a \Rightarrow b$ are homotopy classes (relative to their boundary) $[h]$ of homotopies of paths $h\colon a \simeq b$ (fixing the endpoints). This means that if $h, h'\colon D_2 \to X$ are two maps that agree on the boundary of the 2-disk, i.e. $h_{|S^1} = h'_{|S^1}$ then $[h] = [h']$ if and only if there exists a map $\chi\colon D_3 \to X$ whose restriction to $S^2 = D_2 \coprod_{S^1} D_2$ coincides with (h, h'). The adjective weak comes from the fact that constraints are not satisfied on the nose, i.e. by equality, but rather witnessed by higher cells. Specifically, associativity and unitality of composition of 1-cells only holds up to a 2-cell, whereas in top-dimension we still have strict constraints, as one should expect.

It turns out that every cell in $\Pi_2 X$ is invertible in the appropriate sense, thus giving rise to a *weak 2-groupoid*. Furthermore, this algebraic object captures the *homotopy 2-type* of the space X, i.e. one can recover $\pi_0(X)$ from it, as well as $\pi_n(X, x)$ for $n = 1, 2$ and all $x \in X$.

As the reader can easily imagine at this point, if one goes on in this way, eventually one can capture the full homotopy-type of X via an algebraic structure that deserves to be called *weak ∞-groupoid* (note that we will often avoid writing the adjective weak, and only specify when structures are strict). The only problem is, how can we make this into a formal definition? The most developed model of weak ∞-groupoid is given by Kan complexes, i.e. simplicial sets with the extension property with respect to horn inclusions. Diagrammatically, $X \in \mathbf{sSet}$ is a Kan complex if and only if we have solutions to all extension problems of the form:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{3}$$

where Λ^n_i denotes the simplicial set obtained from the representable n-simplex Δ^n by removing its unique non-degenerate n-simplex and its face opposite to the i-th vertex. Essentially all of the homotopy theory of topological spaces can be formulated in the language of Kan complexes, with the advantage of a fully combinatorial framework.

Equivalently, Kan complexes correspond to quasi-categories where all 1-simplices are invertible. Quasi-categories are, in turn, the most developed model for $(\infty, 1)$-categories ([**Lu**]), where (∞, n)-categories have cells in each dimension $k \geq 0$ which are invertible above dimension n. They can be characterized as those simplicial sets having the *inner* horn extension property, i.e. such that every extension problem as in (3) admits a solution if $0 < i < n$. This ensures the existence of (non-unique) composites, and every coherence is encoded by higher cells. Also recall that a 1-simplex f in a quasi-category X is said to be an equivalence (or *invertible*) if there are 2-simplices in X of the form:

$$\begin{tikzcd} & 1 \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow"] & \\ 0 \arrow[ur, "f"] \arrow[rr, equal] & {} & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dr, "f"] \arrow[d, phantom, "\Uparrow"] & \\ 0 \arrow[ur, "h"] \arrow[rr, equal] & {} & 2 \end{tikzcd} \tag{4}$$

The assignment:

$$[n] \mapsto |\Delta^n| \overset{def}{=} \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \colon \sum_{i=0}^{n} x_i = 1\}$$

defines, in a natural way, a functor $\Delta \to \mathbf{Top}$, i.e. a cosimplicial space. This determines, in turn, an adjunction of the form:

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, bend left, "|\cdot|"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, bend left, "\mathrm{Sing}"] \end{tikzcd} \tag{5}$$

with $\mathrm{Sing}(X)_n \overset{def}{=} \mathbf{Top}\left(|\Delta^n|, X\right)$. The fact that Kan complexes recover all homotopy types is encoded by the result that this adjunction induced an equivalence at the level of the associated *homotopy theories*, i.e. $(\infty, 1)$-categories. One way of obtaining this is to endow both categories with a model structure, the one on simplicial sets being the so-called Kan-Quillen model structure and the one on spaces being the Serre one, and then show that the adjunction in (5) is a Quillen equivalence (see, for instance, [**Ho**]).

In the discussion at the beginning of this chapter we sketched the idea behind the fundamental ∞-groupoid of a space, and a difference from the simplicial model outlined above is the following: in the former, the basic shapes are (glueing of) globes, whereas in the latter they are given by simplices. With regards to globes, this comes with advantages and disadvantages: on the upside, the structure one obtains inherits a well-defined notion of source and target of cells; on the downside, the combinatorics of simplicial sets is deeply understood and well documented in the mathematical literature, but not so much that of cellular structures.

Using a globular algebraic version of weak ∞-groupoids to model homotopy types was Grothendieck's idea, introduced in the famous letter to Quillen dated 1983. He wanted to have a completely algebraic model of these highly structured gadgets, encoding every possible composition operation, inverses in all codimensions and coherence constraints needed for a sensible definition of weak ∞-groupoid. The conjecture that these capture all homotopy types goes under the name of *homotopy hypothesis*. The basic idea is to start with a category of shapes Θ_0, that consists of globular pasting diagrams, an example of which is depicted in (1). Using this as a starting point, one freely adds operations via a universal construction, and if the resulting theory is *contractible* then it is a good theory for Grothendieck ∞-groupoids. The freeness provided by the universal construction ensures that all the relations between the operations only hold up to higher cells, rather than being properties given by equalities. In other words, models of these theories are *weak*. On the other hand, contractibility ensures the existence of all the structure that is needed, which is captured implicitly and never spelled out as a list of operations and coherences, which would be essentially impossible in infinite dimension and already quite challenging even in low dimensions.

Once having isolated the right notion of theory for these kind of algebraic structure, one then defines models (i.e. ∞-groupoids) of a given theory $\mathfrak{C}$ as presheaves $X \colon \mathfrak{C}^{\mathrm{op}} \to \mathbf{Set}$ that preserve the globular shapes. This is an analogy of the so-called *Segal condition* that characterizes Segal spaces, for instance, among all bisimplicial sets. This time, it is encoded as a strict condition in that it is a preservation of some limit cones, and that is because the required "weakness" is all in the theory. Moreover, by construction there is a map $\Theta_0^{\mathrm{op}} \to \mathfrak{C}^{\mathrm{op}}$ which induces, by precomposition, a forgetful functor of the form:

$$\mathbf{U} \colon \infty\text{-}\mathcal{G}pd \to [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$$

where the codomain category is that of *globular sets*, i.e. family of sets $(Y_k)_{k\in\mathbb{N}}$ equipped with source and target maps $s_i, t_i\colon Y_{i+1} \to Y_i$ satisfying sensible axioms. This forgetful functor is proven to be monadic in [**Ar1**], which formalizes the idea that these models of weak ∞-groupoids can be regarded as globular sets with algebraic structure, where we think of $(\mathbf{U}X)_k$ as the set of k-cells of the ∞-groupoid X, on which the various operations in $\mathfrak{C}$ act. For example, these operations take as input a diagram as in (1) labeled by cells of X, and return any of the 2-cell composites, or, say, a comparison 3-cell of the form:

$$\Phi\colon (a\circ b)*(c\circ d) \Rrightarrow (a*c)\circ(b*d). \tag{6}$$

In detail, one has (at least) two ways of composing the diagram (1) using the structure of X, once we have chosen a vertical and a horizontal composition of 2-cells. These are given by $(a\circ b)*(c\circ d)$ and $(a*c)\circ(b*d)$, which coincided in the strict case previously discussed. In contrast with that, in the context of weak structures, the two outcomes are not going to be equal, but instead there is a (structural) 3-cell depicted in (6). As we will see in the next chapter, this follows essentially from the fact that the free ∞-groupoid on the globular pasting diagram (1) is contractible, in a suitable sense.

All that has been said so far can be adapted to the context of (weak) ∞-categories, as done by Maltsiniotis in [**Ma**], and more generally to (∞,n)-categories for $0\le n\le\infty$, i.e. ∞-dimensional categorical structures where all cells above dimension n are invertible. Indeed, one keeps the cellularity or freeness aspect in the construction of the theories, but the notion of contractibility has to be suitably modified in order to capture the correct algebraic structure. For instance, (∞,∞)-categories should not have inverse operations as part of the theory, and since contractibility is responsible for their presence in the case of ∞-groupoids, it has to be modified appropriately. Informally speaking, a notion of "directed" contractibility is needed.

Turning to the homotopy hypothesis, firstly we need a formal way of expressing its statement. In order to achieve this, one has to define the homotopy theory of ∞-groupoids, and comparison functors between this homotopy theory and that of, say, topological spaces. Already in [**Gr**] there is a very natural and simple functorial definition of homotopy groups of a given ∞-groupoid, which induces, just like in the case of topological spaces or Kan complexes, a notion of weak equivalence. In details, a map $f\colon X\to Y$ in ∞-$\mathcal{G}pd$ is a weak equivalence if it induces a bijection on the set of path components $\pi_0(X)\cong\pi_0(Y)$, as well as isomorphisms of homotopy groups:

$$\pi_n(X,x)\xrightarrow{\pi_n(f)}\pi_n\left(Y,f(x)\right)$$

for every 0-cell x of X.

This choice of weak equivalences endows the category ∞-$\mathcal{G}pd$ with a relative category structure, and we can also consider **Top** as a relative category, with the usual notion of weak equivalences of topological spaces. In [**Ar2**], following Grothendiecks's original idea, Ara defines a fundamental ∞-groupoid functor $\Pi_\infty\colon \mathbf{Top}\to\infty\text{-}\mathcal{G}pd$ which fits into an adjunction of the form:

$$\begin{tikzcd} \infty\text{-}\mathcal{G}pd \arrow[r, bend left, "|\bullet|"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, bend left, "\Pi_\infty"] \end{tikzcd} \tag{7}$$

It turns out that this functor Π_∞ preserves weak equivalences, and thus a precise statement of the homotopy hypothesis is that Π_∞ is an equivalence of relative categories (e.g. it induces a Dwyer-Kan equivalence at the level of the simplicial localization of the relative categories involved). Note that its left adjoint may not be a morphism of relative categories and, therefore, the adjunction in (7) may not be one of relative categories. Recently, Simon Henry proved in [**Hen**] that it suffices to construct a semi-model structure on ∞-groupoids to validate the homotopy hypothesis. This semi-model structure should have, as generating cofibration (resp. trivial cofibrations), the "boundary inclusions" $\partial\colon S^{n-1} \to D_n$ (resp. the "source maps" $\sigma_n\colon D_n \to D_{n+1}$) for every $n \geq 0$, which clearly resemble the ones for the Serre model structure on topological spaces.

The only difficult part in proving the existence of this semi-model structure is showing that given a cocartesian square of the form:

$$\begin{tikzcd} D_n \arrow[r] \arrow[d, "\sigma_n"'] & X \arrow[d, "i"] \\ D_{n+1} \arrow[r] & X^+ \end{tikzcd} \tag{8}$$

where X is cofibrant, the map $i\colon X \to X^+$ is a weak equivalence of ∞-groupoids. In fact we will show that to prove this "pushout lemma" it is enough to construct a path object $\mathbb{P}X$ for every cofibrant object X (even finitely cellular would do), i.e. a functorial construction of a fibration $\mathbf{ev}\colon \mathbb{P}X \to X \times X$ which is a trivial fibration when composed with the product projections. This problem and, more generally, the homotopy hypothesis, are the main motivations for this work, along with the use of homotopical methods to understand higher structures.

Despite not having solved this problem in full generality, we have managed to make significant progress in this direction. We give a characterization of those globular theories whose models can be understood as ∞-groupoids in a broader sense and bear a cofibrantly generated semi-model structure which we describe in detail. Furthermore, we isolate a sufficient condition for the existence of said model structure on groupoids of the form "weak categories with weak inverses", i.e. $\mathfrak{C}^{\mathbf{W}}$-models as introduced in Definition 2.22. We construct the underlying globular set of $\mathbb{P}X$, and prove that the natural map $\mathbf{ev}\colon \mathbb{P}X \to \mathbf{U}X \times \mathbf{U}X$ is a fibration, which is trivial when composed with the product projections. This happens at the level of underlying globular sets, but is enough to conclude that if one endows $\mathbb{P}X$ with the structure of an ∞-groupoid and extends $\mathbf{ev}$ to a map of ∞-groupoids, then the same holds for that map, since (trivial) fibrations are detected at the level of the underlying globular sets. We endow $\mathbb{P}X$ with non-trivial algebraic structure and we also construct a non-functorial approximation to an interpretation of any operation of a theory for ∞-categories, dealing with inverses separately. Moreover, thanks to the use of modifications, we inductively adjust this interpretation in low dimensions, which allows us to construct the path object endofunctor on Grothendieck 3-groupoids and thus equip the category of these with a semi-model structure. We conclude this introduction with a more detailed summary of the contents of the thesis.

In Chapter 2 we introduce the language of globular theories and their corresponding models. The reader might want to have in mind the analogy with Lawvere theories: in that case the arities of the theories are given by natural numbers, in this case they are given by globular sums or finite planar trees, as we will explain. We also adapt the definition available in the literature to capture (weak) n-groupoids. Given a globular theory $\mathfrak{C}$ with enough structure, we define in Definition 2.22 an associated globular theory $\mathfrak{C}^{\mathbf{W}}$ with weak inverses, that will play a fundamental role in later chapters. Moreover, we introduce an orthogonal factorization system consisting of n-bijective and n-fully faithful maps, which is lifted from globular sets.

In Chapter 3 we develop basic aspects of the homotopy theory of Grothendieck ∞-groupoids. We begin by defining the relative category associated with them, i.e. we define weak equivalences of ∞-groupoids (as in [**Ar2**]), and we formulate the homotopy hypothesis in the form of Conjecture 3.4. We then define (trivial) cofibrations of ∞-groupoids by giving two generating sets I and J, and we prove the existence of a long exact sequence of homotopy groups associated with a fibration as well as the stability of weak equivalences under pullbacks along fibrations. After a brief detour into the theory of direct categories we prove some very useful results about extension problems in the realm of Grothendieck ∞-categories: these provide necessary and sufficient conditions for the existence of extensions conditional upon their existence in the case of strict ω-categories.

Chapter 4 is where semi-model structures are introduced. This weakening of the ordinary concept of model structure is needed since the structures involved are weak but the maps are strict, so there seems to be no obvious way to construct certain maps, as explained at the beginning of this chapter.

We prove in Theorem 4.2 a characterization of those globular theories whose category of models admits a semi-model structure of a precise form, which resemble ∞-groupoids: for instance, we prove that in such globular theories globular sums are contractible. Consequently, we have a sufficient condition for ∞-categories with weak inverses (i.e. $\mathfrak{C}^{\mathbf{W}}$-models) to be promoted to ∞-groupoids. This is formulated in Theorem 4.8, and depends essentially on the existence of a path object.

In Chapter 5 we first construct an adjunction representing the suspension-space of paths one, which in particular produces, given $1 \leq n \leq \infty$, an n-groupoid (resp. n-category) X and two of its 0-cells x, y, an $(n-1)$-groupoid (resp. $(n-1)$-category) of paths $X(x, y)$, which can be thought of as the $(n-1)$-groupoid (resp. $(n-1)$-category) of morphisms between x and y. Next, we construct a coglobular object $\mathbf{Cyl}(D_\bullet)$ of cylinders, which represents natural transformations between ∞-groupoids (resp. ∞-categories).

This coglobular object corepresents a functor that takes an ∞-groupoid X and associates with it its path object $\mathbb{P}X$, which is for the moment a bare globular set. In Chapter 6 we endow this globular set with some algebraic structure, consisting of a system of compositions, a system of identities and a system of inverses, as introduced in Definition 2.21.

In Chapter 7 we introduce some combinatorics of trees to construct an approximation $\hat{\varrho}$ (i.e. a non-functorial interpretation) of the map $\mathbf{Cyl}(\varrho)\colon \mathbf{Cyl}(D_n) \to \mathbf{Cyl}(A)$ for a given operation $\varrho\colon D_n \to A$. This would then corepresent the action of the operation ϱ on $\mathbb{P}X$, so it is a necessary part of endowing the path object with the structure of an ∞-groupoid. To achieve this, we construct a zig-zag of globular sums that only depends on A, whose colimit is $\mathbf{Cyl}(A)$, and then exploit the contractibility of globular sums (or the results in Section 3 of Chapter 2) to construct the map $\hat{\varrho}\colon \mathbf{Cyl}(D_n) \to \mathbf{Cyl}(A)$ both in the case of ∞-categories and ∞-groupoids. To avoid redundancy of data, a notion of degenerate cylinders is required.

In the eight and final chapter we define another coglobular object, which corepresents modifications (i.e. cells between natural transformations), and we use these to inductively correct the boundary of the elementary interpretation $\hat{\varrho}$ constructed in Chapter 7 for low dimensions. This allows to extend the construction of $\mathbb{P}X$ from a globular set to a 3-groupoid, so we get an endofunctor $\mathbb{P}$ on 3-$\mathcal{G}pd$. Together with the results of Chapter 4, this proves the existence of a semi-model structure on the category of Grothendieck 3-groupoids, as recorded in Theorem 8.30, which concludes this work.

It is worth mentioning that the strategy adopted in this work is substantially different from the one outlined by Maltsiniotis in [**Ma**], in which he makes use of the notion (introduced by A.Grothendieck) of test category. These are, in particular, small categories $\mathscr{C}$ for which the presheaf category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ admits a model structure which is Quillen equivalent to Quillen's one on simplicial sets, modeling homotopy types. The first step in Maltsiniotis' strategy is to show that any coherator for ∞-groupoids is a test category. One would then have to transfer the model structure on the presheaf category to that on the subcategory of models, and obtain a Quillen equivalent one. We preferred our approach given the unfamiliarity of the author with the techniques involved in proving that a certain category is test, but of course this remains a totally viable approach.

CHAPTER 2

Globular theories and models

1. Background

The preliminary concepts and definitions needed for understanding this work can be found in [**Ar1**] and [**Ma**]. We now present a summary of these, to ease the reading of this work.

We start by defining the category of globes, which will serve as the starting point for everything that follows.

Definition 2.1. Let $\mathbb{G}$ be the category obtained as the quotient of the free category on the graph

$$0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} 1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} \dots n \underset{\tau_n}{\overset{\sigma_n}{\rightrightarrows}} n+1 \underset{\tau_{n+1}}{\overset{\sigma_{n+1}}{\rightrightarrows}} \dots$$

by the set of relations $\sigma_k \circ \sigma_{k-1} = \tau_k \circ \sigma_{k-1}$, $\sigma_k \circ \tau_{k-1} = \tau_k \circ \tau_{k-1}$ for $k \geq 1$.

Given integers $j > i$, define $\sigma_i^j = \sigma_{j-1} \circ \sigma_i^{j-1}$, where $\sigma_i^{i+1} = \sigma_i$. The maps τ_i^j are defined similarly.

The category of globular sets is by definition the presheaf category $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$.

Definition 2.2. For $0 \leq n$, we denote with $\mathbb{G}_n$ the full subcategory of $\mathbb{G}$ generated by the set of objects $\{k \in \mathbb{G} \colon k \leq n\}$.

The category of n-globular sets is by definition the presheaf category $[\mathbb{G}_n^{op}, \mathbf{Set}]$.

Globes are not enough to capture a meaningful theory of n-groupoids, for which we need more complex shapes, called globular sums, which are a special kind of pasting of globes. Indeed, globes alone cannot encode, for instance, composition operations.

In what follows we let $0 \leq n \leq \infty$, where the case $n = \infty$ refers to globular sets.

Definition 2.3. A table of dimensions is a sequence of integers of the form

$$\begin{pmatrix} i_1 & & i_2 & \dots & i_{m-1} & & i_m \\ & i'_1 & & \dots & & i'_{m-1} & \end{pmatrix} \tag{9}$$

satisfying the following inequalities: $i'_k < i_k$ and $i'_k < i_{k+1}$ for every $1 \leq k \leq m-1$.

Given a category $\mathcal{C}$ and a functor $F \colon \mathbb{G}_n \to \mathcal{C}$, a table of dimensions as above, with $i_k \leq n$ for all $1 \leq k \leq m$, induces a diagram of the form

$$\begin{tikzcd}[column sep=small]
F(i_1) & & F(i_2) & & F(i_3) & F(i_{m-1}) & & F(i_m) \\
& F(i'_1) \arrow[ul, "F(\sigma^{i_1}_{i'_1})"] \arrow[ur, "F(\tau^{i_2}_{i'_1})"'] & & F(i'_2) \arrow[ul, "F(\sigma^{i_2}_{i'_2})"] \arrow[ur, "F(\tau^{i_3}_{i'_2})"'] & \cdots & & F(i'_{m-1}) \arrow[ul, "F(\sigma^{i_{m-1}}_{i'_{m-1}})"] \arrow[ur, "F(\tau^{i_m}_{i'_{m-1}})"'] &
\end{tikzcd}$$

The n-globular sum (of type F) associated with (9) is the colimit in $\mathcal{C}$ (if it exists) of the diagram above.

We also define the height of this n-globular sum to be $\mathrm{ht}(A) = \max\{i_k\}_{k\in\{1,\dots,\ m\}}$. Given an n-globular sum A, we denote with ι_k^A the colimit inclusion $F(i_k) \to A$, dropping subscripts when there is no risk of confusion.

Definition 2.4. We denote by Θ_0 the full subcategory of globular sets spanned by the globular sums of type $y\colon \mathbb{G} \to [\mathbb{G}^{op}, \mathbf{Set}]$, where y is the Yoneda embedding. Moreover, we denote $y(i)$ by D_i and the globular sum corresponding to the table of dimensions:

$$\begin{pmatrix} 1 & & 1 & \dots & 1 & & 1 \\ & 0 & & \dots & & 0 & \end{pmatrix}$$

by $D_1^{\otimes k}$, where the integer 1 appears exactly k times.

Also define the subcategory $\Theta_0^{\leq n} \subset \Theta_0$ to be that spanned by globular sums of height less or equal than n.

It is not hard to see that there is a fully faithful embedding functor $\Theta_0^{\leq n} \to [\mathbb{G}_n^{op}, \mathbf{Set}]$. The category $\Theta_0^{\leq n}$ plays a similar role for n-groupoids as Θ_0 does for ∞-groupoids.

Definition 2.5. An n-truncated globular theory is a pair $(\mathfrak{C}, \mathbf{F})$, where $\mathfrak{C}$ is a category and $\mathbf{F}\colon \Theta_0^{\leq n} \to \mathfrak{C}$ is a bijective on objects functor that preserves globular sums of height less than or equal to n.

We denote by $\mathbf{GlTh_n}$ the category of n-globular theories and n-globular sums preserving functors. More precisely, a morphism $H\colon (\mathfrak{C}, \mathbf{F}) \to (\mathfrak{C}, \mathbf{G})$ is a functor $H\colon \mathfrak{C} \to \mathfrak{C}$ such that $\mathbf{G} = H \circ \mathbf{F}$.

If there is no risk of confusion we will omit the structural map $\mathbf{F}\colon \Theta_0^{\leq n} \to \mathfrak{C}$ and simply denote the globular theory $(\mathfrak{C}, \mathbf{F})$ by $\mathfrak{C}$.

Definition 2.6. Given an n-globular theory $\mathfrak{C}$, we define the category of its models, denoted $\mathbf{Mod}(\mathfrak{C})$, to be the category of n-globular product preserving functors $G\colon \mathfrak{C}^{op} \to \mathbf{Set}$. Clearly, the Yoneda embedding $y\colon \mathfrak{C} \to [\mathfrak{C}^{op}, \mathbf{Set}]$ factors through $\mathbf{Mod}(\mathfrak{C})$, and it will still be denoted by $y\colon \mathfrak{C} \to \mathbf{Mod}(\mathfrak{C})$. Also, notice that $\mathbf{Mod}(\Theta_0^{\leq n}) \cong [\mathbb{G}_n^{op}, \mathbf{Set}]$. Again, we denote the image of i under y by D_i.

We now record the universal property of the category of models of an n-globular theory.

Proposition 2.7. *Given an n-globular theory $\mathfrak{C}$, its category of models $\mathbf{Mod}(\mathfrak{C})$ enjoys a universal property: given any cocomplete category $\mathcal{D}$, a cocontinuous functor $F\colon \mathbf{Mod}(\mathfrak{C}) \to \mathcal{D}$ is determined up to a unique isomorphism by an n-globular sums-preserving functor $\overline{F}\colon \mathfrak{C} \to \mathcal{D}$, corresponding to its restriction along the Yoneda embedding. Conversely, any such functor $\overline{F}\colon \mathfrak{C} \to \mathcal{D}$ extends essentially-uniquely to a cocontinuous one on $\mathbf{Mod}(\mathfrak{C})$.*

Proof. The presheaf category $[\mathfrak{C}^{op}, \mathbf{Set}]$ is the free cocompletion of $\mathfrak{C}$, therefore we get a natural equivalence induced by the Yoneda embedding of the form:

$$[\mathfrak{C}, \mathcal{D}] \cong [[\mathfrak{C}^{op}, \mathbf{Set}], \mathcal{D}]_c$$

where $[\cdot,\cdot]_c$ denotes the class of cocontinuous functors. It is easy to check that this restricts to an equivalence of the form:

$$[\mathfrak{C}, \mathcal{D}]_{gl} \cong [\mathbf{Mod}(\mathfrak{C}), \mathcal{D}]_c$$

where $[\cdot,\cdot]_{gl}$ denotes the set of globular sum-preserving maps. $\square$

Grothendieck groupoids are presented as models of a certain class of globular theories, namely the cellular and contractible ones.

Definition 2.8. Given $k \leq n$, two maps $f, g \colon D_k \to A$ in an n-globular theory are said to be parallel if either $k = 0$ or $f \circ \varepsilon = g \circ \varepsilon$ for $\varepsilon = \sigma, \tau$. A pair of parallel maps (f, g) is said to be admissible if $\mathrm{ht}(A) \leq k + 1$. A globular theory $(\mathfrak{C}, F)$ is called contractible if for every admissible pair of maps $f, g \colon D_k \to A$ either $k = n$ and $f = g$, or $k < n$ and there exists an extension $h \colon D_{k+1} \to A$ rendering the following diagram serially commutative

$$\begin{tikzcd} D_k \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift right, "\sigma_k"'] \arrow[d, shift left, "\tau_k"] & A \\ D_{k+1} \arrow[ur, "h"'] & \end{tikzcd}$$

Contractibility ensures the existence of all the operations that ought to be part of the structure of an n-groupoid. However, it does not guarantee weakness of the models, and indeed there exists a contractible globular theory (which we denote by $\tilde{\Theta}^{\leq n}$) whose models are strict n-groupoids.

To remedy this, we need the concept of cellularity, or freeness, to restrict the class of globular theories we consider. This notion is based on a slight variation of a construction explained in paragraph 4.1.3 of [**Ar1**], which we record in the following proposition.

Proposition 2.9. *Given an n-globular theory $\mathfrak{C}$ and set X of admissible pairs in it, there exists another n-globular theory $\mathfrak{C}[X]$ equipped with a morphism $\varphi \colon \mathfrak{C} \to \mathfrak{C}[X]$ in $\mathbf{GlTh_n}$ with the following universal property: given an n-globular theory $\mathfrak{C}$, a morphism $H \colon \mathfrak{C}[X] \to \mathfrak{C}$ is determined up to a unique isomorphism by $F \overset{def}{=} H \circ \varphi$, together with a choice of an extension to D_{k+1} of the image under F of each admissible pair $f, g \colon D_k \to A$ in X with $k < n$, or the requirement that $F(f) = F(g)$ if $k = n$.*

In words, $\mathfrak{C}[X]$ is obtained from $\mathfrak{C}$ by universally adding a lift for each pair in X of non-maximal dimension and by equalizing parallel n-dimensional operations in X.

Definition 2.10. An n-globular theory $\mathfrak{C}$ is said to be cellular if there exists a functor $\mathfrak{C}_\bullet \colon \omega \to \mathbf{GlTh_n}$, where ω is the first infinite ordinal, such that:

(1) $\mathfrak{C}_0 \cong \Theta_0^{\leq n}$;
(2) for every $m \geq 0$, there exists a family X_m of admissible pairs of arrows in $\mathfrak{C}_m$ (as in Definition 2.8) such that $\mathfrak{C}_{m+1} \cong \mathfrak{C}_m[X_m]$;
(3) $\mathrm{colim}_{m \in \omega} \mathfrak{C}_m \cong \mathfrak{C}$.

Equivalently, one can consider arbitrary ordinals γ and assume X_α to be a singleton for each $\alpha < \gamma$.

As anticipated earlier, we now define the class of n-globular theories which are appropriate to develop a theory of n-groupoids.

Definition 2.11. An n-truncated (groupoidal) coherator, or, briefly, an n-coherator, is a cellular and contractible n-globular theory. Given an n-coherator $\mathfrak{G}$, the category of n-groupoids of type $\mathfrak{G}$ is the category $\mathbf{Mod}(\mathfrak{G})$ of models of $\mathfrak{G}$. In what follows, $\mathfrak{G}$ will always denote a coherator for n-groupoids, with $0 \leq n \leq \infty$, and sometimes we will denote the category of its models by n-$\mathcal{G}pd$, with no reference to $\mathfrak{G}$.

The restriction of an n-groupoid $X\colon \mathfrak{G}^{op} \to \mathbf{Set}$ to $\Theta_0^{\leq n\, op}$ gives an object of $\mathbf{Mod}(\Theta_0^{\leq n}) \simeq [\mathbb{G}_n^{op}, \mathbf{Set}]$, which we call the underlying n-globular set of X. The set X_i represents the set of i-cells of X for each $i \leq n$.

Let us now consider the algebraic structure acting on these sets of cells. Section 3 of [**Ar2**] shows how to endow the underlying globular set of an ∞-groupoid with all the sensible operations one would expect it to have. A completely analogous argument applies to the case of n-groupoids.

For example, we can build operations that represent binary composition of a pair of 1-cells, codimension-1 inverses for 2-cells and an associativity constraint for composition of 1-cells by solving, respectively, the following extension problems:

$$\begin{tikzcd}
D_0 \arrow[r, shift left, "i_0 \circ \sigma_0"] \arrow[r, shift right, "i_1 \circ \tau_0"'] \arrow[d, shift right, "\sigma_0"'] \arrow[d, shift left, "\tau_0"] & D_1 \underset{D_0}{\amalg} D_1 \\
D_1 \arrow[ur, "\nabla_0^1"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
D_1 \arrow[r, shift left, "\tau_1"] \arrow[r, shift right, "\sigma_1"'] \arrow[d, shift right, "\sigma_1"'] \arrow[d, shift left, "\tau_1"] & D_2 \\
D_2 \arrow[ur, "\omega_1^2"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
D_1 \arrow[r, shift left, "(\nabla_0^1 \underset{D_0}{\amalg} 1_{D_1}) \circ \nabla_0^1"] \arrow[r, shift right, "(1_{D_1} \underset{D_0}{\amalg} \nabla_0^1) \circ \nabla_0^1"'] \arrow[d, shift right, "\sigma_1"'] \arrow[d, shift left, "\tau_1"] & D_1 \underset{D_0}{\amalg} D_1 \underset{D_0}{\amalg} D_1 \\
D_2 \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

We will need to choose some operations once and for all, so we record here their definition. Choose an operation $\nabla_0^1\colon D_1 \to D_1 \amalg_{D_0} D_1$ as above, and define $w = \nabla_0^1$. Next, pick operations ${}_2w\colon D_2 \to D_2 \amalg_{D_0} D_1$ and $w_2\colon D_2 \to D_1 \amalg_{D_0} D_2$ whose source and target are given, respectively by $((\sigma \amalg_{D_0} 1) \circ w, (\tau \amalg_{D_0} 1) \circ w)$ and $((1 \amalg_{D_0} \sigma) \circ w, (1 \amalg_{D_0} \tau) \circ w)$. Proceeding in this way we get specified whiskering maps for every $k \leq n$ of the form:

$$\begin{aligned} {}_kw\colon D_k &\to D_k \underset{D_0}{\amalg} D_1 \\ w_k\colon D_k &\to D_1 \underset{D_0}{\amalg} D_k \end{aligned} \tag{10}$$

We will often avoid writing down all the subscripts, when they are clear from the context.

Definition 2.12. Given a globular sum A, whose table of dimensions is

$$\begin{pmatrix} i_1 & & i_2 & \dots & i_{m-1} & & i_m \\ & i'_1 & & \dots & & i'_{m-1} & \end{pmatrix}$$

with $i'_k > 0$ for every $1 \leq k \leq m-1$, we define a map ${}_Aw\colon A \to A \amalg_{D_0} D_1$ by

$$w_{i_1} \underset{w_{i'_1}}{\amalg} \dots \underset{w_{i'_{m-1}}}{\amalg} w_{i_m} \colon D_{i_1} \underset{D_{i'_1}}{\amalg} \dots \underset{D_{i'_{m-1}}}{\amalg} D_{i_m} \to (D_{i_1} \underset{D_{i'_1}}{\amalg} \dots \underset{D_{i'_{m-1}}}{\amalg} D_{i_m+1}) \underset{D_0}{\amalg} D_1$$

which makes sense since the target is isomorphic to

$$(D_{i_1} \underset{D_0}{\amalg} D_1) \underset{D_{i'_1} \underset{D_0}{\amalg} D_1}{\amalg} \dots \underset{D_{i'_{m-1}} \underset{D_0}{\amalg} D_1}{\amalg} (D_{i_m} \underset{D_0}{\amalg} D_1)$$

In a completely analogous manner we define a map $w_A\colon A \to D_1 \amalg_{D_0} A$.

Let us now see how to adapt the main definitions to the case of n-categories, following [**Ar1**]. The definition is essentially the same as that of n-groupoids, except we have to restrict the class of admissible maps.

Definition 2.13. We define a globular theory Θ by considering the full subcategory of ω-$\mathcal{C}at_s$, i.e. strict ∞-categories, spanned by the free ∞-categories on globular pasting diagrams. More precisely, consider the monadic forgetful functor $\mathbf{U}\colon \omega\text{-}\mathcal{C}at_s \to [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$, whose left adjoint we denote by $\mathbf{F}$. Then Θ is the full subcategory of ω-$\mathcal{C}at_s$ spanned by object of the form $\mathbf{F}A$ for all globular sum A in $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ of type $y\colon \mathbb{G} \to [\mathbb{G}^{op}, \mathbf{Set}]$, where y denotes the Yoneda embedding.

Definition 2.14. Given an n-globular theory $(\mathfrak{C}, F)$, we say that a map f in $\mathfrak{C}$ is globular if it is in the image of $\Theta_0^{\leq n}$ under F.

On the other hand, f is called homogeneous if for every factorization $f = g \circ f'$ where g is a globular map, g must be the identity.

$\mathfrak{C}$ is said to be homogeneous if it comes endowed with an n-globular sum preserving functor $H\colon \mathfrak{C} \to \Theta^{\leq n}$ that detects homogeneous maps, in the sense that a map f in $\mathfrak{C}$ is homogeneous if and only if $H(f)$ is such, where Θ is the globular theory for strict ∞-categories, as defined in [**Ar1**], and $\Theta^{\leq n}$ is its subcategory spanned by all globular sums of height less or equal to n. We observe that, given a homogeneous map $\varrho\colon D_m \to A$ in $\mathfrak{C}$, we have $m \geq \mathrm{ht}(A)$, since this is holds true in Θ, see [**Ar1**].

If globular maps are monomorphisms in $\mathfrak{C}$ (for instance if $\mathfrak{C}$ is a coherator for n-categories), then every map f admits a unique factorization as a homogeneous map followed by a globular one. Indeed, either f is homogeneous or $f = i \circ f'$, with i globular. Iterating this process for f' and so on, we eventually have to stop, since the sum of the entries in the first row of the table of dimensions of the domain of a globular map is less than the one of the codomain. Therefore, for every f, we obtain a factorization of the form $f = j \circ h$, with j globular and h homogeneous. To prove its unicity, we observe that its image in $\Theta^{\leq n}$ is again a factorization into a globular map followed by a homogeneous one, so every two factorizations of f have the same globular part. These maps being monomorphisms, we obtain that also the homogeneous parts coincide.

Remark 2.15. A map $f\colon A \to B$ in a homogeneous globular theory $\mathfrak{C}$ is homogeneous if and only if, for every D_{i_k} appearing in the globular decomposition of A, the homogeneous-globular factorizations of $D_{i_k} \to A \to B$ given by $D_{i_k} \to B_k \to B$ induce an isomorphism of the form:

$$\mathrm{colim}_k\, B_k \cong B$$

This holds true since it does in Θ.

Definition 2.16. Given a globular sum A such that $\mathrm{ht}(A) = m > 0$, whose table of dimensions is

$$\begin{pmatrix} i_1 & & i_2 & \dots & i_{q-1} & & i_q \\ & i'_1 & & \dots & & i'_{q-1} & \end{pmatrix}$$

we define its boundary to be the globular sum whose table of dimensions is

$$\begin{pmatrix} \bar{\imath}_1 & & \bar{\imath}_2 & \dots & \bar{\imath}_{q-1} & & \bar{\imath}_q \\ & i'_1 & & \dots & & i'_{q-1} & \end{pmatrix}$$

where we set

$$\bar{\imath}_k = \begin{cases} i_k - 1 & \text{if } i_k = m \\ i_k & \text{otherwise} \end{cases}$$

and we replace each occurrence of $\bar{\imath}_{k-1} = i'_{k-1} = \bar{\imath}_k$ with a single $\bar{\imath}_{q-1}$. The maps $\sigma, \tau\colon D_{m-1} \to D_m$ induce maps:

$$\partial_\sigma, \partial_\tau \colon \partial A \to A \tag{11}$$

Definition 2.17. Let $(\mathfrak{C}, F)$ be an n-globular theory. A pair of maps (f, g) with $f, g\colon D_k \to A$ is said to be admissible for a theory of n-categories (or just admissible, in case there is no risk of confusion with the groupoidal case) if either $k = 0$, or both of them are homogeneous maps or else if there exists homogeneous maps $f', g'\colon D_k \to \partial A$ such that the following diagrams commute

$$\begin{tikzcd} D_k \arrow[r, "f"] \arrow[d, "f'"'] & A \\ \partial A \arrow[ur, "\partial_\sigma"'] & \end{tikzcd} \qquad \begin{tikzcd} D_k \arrow[r, "g"] \arrow[d, "g'"'] & A \\ \partial A \arrow[ur, "\partial_\tau"'] & \end{tikzcd}$$

The definition of a coherator for n-categories is totally analogous to that for n-groupoids.

Definition 2.18. An n-truncated coherator for categories, or, briefly, an n-coherator for categories, is a cellular and contractible n-globular theory with respect to the class of admissible pairs defined above. More precisely, the pairs appearing in Definition 2.8 and in point 2 of Definition 2.10 must be pairs of admissible maps for n-categories.

The following definition is analogous to the one given for the groupoidal case.

Definition 2.19. A (Grothendieck) n-category is a model of a coherator for n-categories.

Unless specified otherwise, n-category and n-groupoid will always mean weak ones, i.e. Grothendieck n-categories and Grothendieck n-groupoids. Notice that the maps introduced in Definition 2.12 exist also in coherators for n-categories.

Remark 2.20. We observe here that one can give a definition of (∞, k)-categories for $k > 0$ in the same spirit as Definition 2.11 and 2.19. It is enough to define them as models of an appropriate modified notion of coherator. In detail, we only alter the class of admissible pairs, imposing it consists of the union of the admissible pairs for a theory for ∞-categories and all parallel pairs $(f, g)\colon D_n \to A$ with $n \geq k$.

We are now going to introduce globular theories whose models can be thought of as weak n-categories with weak inverses. These will appear in Theorem 4.8, where we prove that under the existence of a path object fibration these globular theories are coherators for n-groupoids. We consider left and right inverses instead of two-sided inverses, and this is done in order to produce the correct homotopy type of globular sums in the corresponding category of models.

Definition 2.21. A system of compositions in an n-globular theory $\mathfrak{C}$ consists of a family of maps $\{\mathbf{c}_k\colon D_k \to D_k \amalg_{D_{k-1}} D_k\}_{1 \leq k \leq n}$ such that $\mathbf{c}_k \circ \sigma = i_1 \circ \sigma$ and $\mathbf{c}_k \circ \tau = i_2 \circ \tau$, where i_1 (resp. i_2) denotes the colimit inclusion onto the first (resp. second) factor.

A system of identities (with respect to a chosen system of compositions) consists of a family of maps $\{\mathbf{id}_k\colon D_{k+1} \to D_k\}_{0 \leq k \leq n-1} \cup \{\mathbf{l}_k, \mathbf{r}_k\colon D_k \to D_{k-1}\}_{2 \leq k \leq n+1}$ such that $\mathbf{id}_k \circ \varepsilon = 1_{D_k}$, for every $k \geq 0$ and $\varepsilon = \sigma, \tau$, $\mathbf{l}_k \circ \sigma = 1_{D_{k-1}}$, $\mathbf{l}_k \circ \tau = (1_{D_{k-1}}, \tau \circ \mathbf{id}_{k-2}) \circ \mathbf{c}_{k-1}$ and $\mathbf{r}_k \circ \sigma = 1_{D_{k-1}}$, $\mathbf{r}_k \circ \tau = (\sigma \circ \mathbf{id}_{k-2}, 1_{D_{k-1}}) \circ \mathbf{c}_{k-1}$.

A system of (left and right) inverses (with respect to chosen systems of compositions and identities) consists of a family of maps $\{\mathbf{i}^l_k, \mathbf{i}^r_k\colon D_k \to D_k\}_{1 \leq k \leq n} \cup \{\mathbf{k}^l_k, \mathbf{k}^r_k\colon D_k \to D_{k-1}\}_{2 \leq k \leq n+1}$ such that $\mathbf{i}^\varepsilon_k \circ \sigma = \tau$, $\mathbf{i}_k \circ \tau = \sigma$ for $\varepsilon = l, r$, $\mathbf{k}^l_k \circ \sigma = \sigma \circ \mathbf{id}_{k-2}$, $\mathbf{k}^l_k \circ \tau = (\mathbf{i}^l_{k-1}, 1_{D_{k-1}}) \circ \mathbf{c}_{k-1}$, $\mathbf{k}^r_k \circ \sigma = \tau \circ \mathbf{id}_{k-2}$ and $\mathbf{k}^r_k \circ \tau = (\mathbf{i}^r_{k-1}, 1_{D_{k-1}}) \circ \mathbf{c}_{k-1}$.

If $\mathfrak{C}$ admits a choice of such three systems, given a globular theory $\mathfrak{G}$ and a globular functor $\mathbf{F}\colon \mathfrak{C} \to \mathbf{Mod}(\mathfrak{G})$ we say that for every $\mathfrak{G}$-model X, the $\mathfrak{C}$-model $\mathfrak{G}(\mathbf{F}, X)$ can be endowed with such systems.

Definition 2.22. Given an n-coherator for categories $\mathfrak{C}$, we define a new globular theory $\mathfrak{C}^{\mathbf{W}}$ by means of the following pushout of globular theories:

$$\begin{tikzcd}
& \Theta_0^{\leq n}[\mathbf{comp}, \mathbf{id}] \arrow[dl] \arrow[dr, "\mathbf{i}"] & \\
\Theta_0^{\leq n}[\mathbf{comp}, \mathbf{id}, \mathbf{inv}] \arrow[dr] & & \mathfrak{C} \arrow[dl] \\
& \mathfrak{C}^{\mathbf{W}} &
\end{tikzcd} \tag{12}$$

Here, we denote with $\Theta_0^{\leq n}[\mathbf{comp}, \mathbf{id}]$ the free globular theory on a system of compositions and identities (i.e. morphisms of n-globular theories $\Theta_0^{\leq n}[\mathbf{comp}, \mathbf{id}] \to \mathfrak{D}$ corresponds to choices of a system of compositions and identities in $\mathfrak{D}$), and with $\Theta_0^{\leq n}[\mathbf{comp}, \mathbf{id}, \mathbf{inv}]$ the free globular theory on a system of compositions, identities and inverses. There is a canonical map as depicted in the upper left of the square and the map denoted by $\mathbf{i}$ is defined (non-canonically) by choosing a system of compositions and a system of identities in $\mathfrak{C}$, using the fact that it is a coherator for n-categories.

It follows from the definition that $\mathfrak{C}^{\mathbf{W}}$-models are Grothendieck n-categories in which each k-cell (for $k > 0$) admits a left and a right inverse up to homotopy.

Remark 2.23. In the presence of both left and right inverses for every cell, together with chosen associativity constraints as below, any of the two inverses can be promoted to a two-sided one. For instance, assume f is an m-cell with both a left inverse k and a right inverse g, and let us show that k is also a right inverse for f, the other case being similar. It is enough to provide a cell from k to g as follows, where the arrows are obtained by whiskering an m-cell with an $(m+1)$-cell, i.e. by composing the $(m+1)$-cell with the identity on the m-cell involved:

$$k \xrightarrow{k\mathbf{k}^r_m(f)} k(fg) \xrightarrow{\simeq} (kf)g \xrightarrow{\mathbf{i}^r_{m+1}(\mathbf{k}^l_m(f))g} g$$

We now introduce two classes of maps of globular sets, that constitute an orthogonal factorization system in that category. This will be lifted to the category of models of coherators for n-categories and n-groupoids, and will be used to prove Proposition 3.9, i.e. the contractibility of globular sums in the category of n-groupoids.

Definition 2.24. Given $m \leq n$, a map $f\colon X \to Y$ of n-globular sets is said to be m-bijective if $f_k\colon X_k \to Y_k$ is a bijection of sets for every $k \leq m$, and m-fully faithful if the following square is cartesian for all $m \leq i \leq n$:

$$\begin{tikzcd}
X_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "{(s,t)}"'] & Y_{i+1} \arrow[d, "{(s,t)}"] \\
X_i \times X_i \arrow[r, "f_i \times f_i"'] & Y_i \times Y_i
\end{tikzcd}$$

Here, if $i = n$, we set $X_{n+1} = Y_{n+1} \overset{def}{=} \emptyset$. We denote the class of m-bijective morphisms by $\mathbf{bij_m}$, and that of m-fully faithful ones by $\mathbf{ff_m}$.

The following result holds true, and its proof is left as a simple exercise

Proposition 2.25. *The pair* $(\mathbf{bij_m}, \mathbf{ff_m})$ *is an orthogonal factorization system on the category of n-globular sets* $[\mathbb{G}^{op}, \mathbf{Set}]$ *for every* $m \leq n$.

We now want to lift the factorization system of Proposition 2.25 to models of an arbitrary coherator for n-categories or n-groupoids $\mathfrak{A}$. To do so, consider the forgetful functor

$$\mathbf{U_n} \colon \mathbf{Mod}(\mathfrak{A}) \to \mathbf{Mod}(\Theta_0^{\leq n}) \simeq [\mathbb{G}_n^{op}, \mathbf{Set}]$$

induced by the structural map $\Theta_0^{\leq n} \to \mathfrak{A}$. Given a map of $\mathfrak{A}$-models $f \colon X \to Y$ and a natural number $m \leq n$, we can factor the map $\mathbf{U_n}(f)$ as $\mathbf{U_n}(f) = g \circ h$, where h is m-bijective and g is m-fully faithful thanks to Proposition 2.25. It is not hard to see that the target of h can be endowed with the structure of an $\mathfrak{A}$-model, in such a way that g and h are maps of such. This follows from the dimensional constraint in the definition of admissible pairs, both in the case of n-categories and n-groupoids. Thanks to Proposition 2 of [**BG**], we obtain the following result:

Proposition 2.26. *Given* $m \leq n$, *the orthogonal factorization system* $(\mathbf{bij_m}, \mathbf{ff_m})$ *on n-globular sets lifts to one on* $\mathbf{Mod}(\mathfrak{A})$, *where* $\mathfrak{A}$ *is any given coherator for n-categories or n-groupoids, via the forgetful functor* $\mathbf{U_n} \colon \mathbf{Mod}(\mathfrak{A}) \to [\mathbb{G}_n^{op}, \mathbf{Set}]$.

This means, in particular, that every map in $\mathbf{Mod}(\mathfrak{A})$ admits a unique factorization $f = g \circ h$ where $\mathbf{U_n}(h)$ is m-bijective and $\mathbf{U_n}(g)$ is m-fully faithful, and that m-bijective maps are closed under colimits in $\mathbf{Mod}(\mathfrak{A})$.

Example 2.27. The maps $\sigma_k, \tau_k \colon D_k \to D_{k+1}$ are $(k-1)$-bijective. Indeed, since the forgetful functor $\mathbf{U_n}$ preserves the right class of the factorization system $(\mathbf{bij_k}, \mathbf{ff_k})$ on $\mathbf{Mod}(\mathfrak{A})$ for every $k \leq n$, its left adjoint $\mathbf{F_n} \colon [\mathbb{G}_n^{op}, \mathbf{Set}] \to \mathbf{Mod}(\mathfrak{A})$ preserves the left class. Now it is enough to observe that $\mathbf{F_n}$ sends σ_k and τ_k in globular sets to σ_k and τ_k in $\mathfrak{A}$-models, and for the former it is easy to check the statement on $(k-1)$-bijectivity.

Thanks to what we observed in Example 2.27, we have the following result.

Proposition 2.28. *Given a globular sum A, with* $0 < m = \mathrm{ht}(A)$, *the maps* $\partial_\sigma, \partial_\tau \colon \partial A \to A$ *are* $(m-1)$*-bijective.*

CHAPTER 3

Basic homotopy theory of ∞-groupoids

In this chapter, we are going to prove some basic facts about the homotopy theory of ∞-groupoids, that better illustrate the similarities between these sophisticated algebraic gadgets and, say, ordinary topological spaces. The definition of (co)fibrations and trivial (co)fibrations mimicks the one given in the context of the usual model structure on topological spaces or the Kan-Quillen one on simplicial sets, and the aim is to get one on weak ∞-groupoids as it has been done in the strict case in [**ArMe**] (which, in turn, is based on the categorical case dealt with in [**LMW**]). We will see in later chapters that this may be a too high expectation, and one may have to settle for a semi-model structure instead, as we justify at the beginning of the next chapter.

1. Weak equivalences and the fundamental ∞-groupoid of a space

According to Grothendieck's homotopy hypothesis, (weak) ∞-groupoids should constitute an algebraic model of homotopy types. Let us now outline a possible way of formalizing this. First of all, this is a statement about the homotopy theory (i.e. $(\infty, 1)$-category) of homotopy types, therefore we should have a model for this and a model for the homotopy theory of Grothendieck ∞-groupoids and the goal would then be to compare them.

The simplest way of describing an $(\infty, 1)$-category is that of a relative category, a concept introduced in [**BK**], which we now recall.

Definition 3.1. A relative category is a pair $(\mathcal{C}, \mathbf{W})$ where $\mathcal{C}$ is a category and $\mathbf{W}$ is a subcategory of it that contains all the objects.

Given a relative category $(\mathcal{C}, \mathbf{W})$ we can form its simplicial localization $\mathbf{L}(\mathcal{C}, \mathbf{W})$ (see [**DK**]) which, in particular, produces a simplicial set of morphisms $\mathbf{L}(\mathcal{C}, \mathbf{W})(x, y)$ between any pair of objects x, y in $\mathcal{C}$. An equivalence of relative categories $F\colon (\mathcal{C}, \mathbf{W}) \to (\mathcal{D}, \mathbf{V})$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that $F(\mathbf{W}) \subset \mathbf{V}$ and F induces an equivalence of simplicial categories

$$\mathbf{L}(\mathcal{C}, \mathbf{W}) \simeq \mathbf{L}(\mathcal{D}, \mathbf{V})$$

the latter meaning that the induced map on ordinary localizations $\mathcal{C}[\mathbf{W}^{-1}] \to \mathcal{D}[\mathbf{V}^{-1}]$ is essentially surjective on objects and F induces weak equivalences of the form:

$$\mathbf{L}(\mathcal{C}, \mathbf{W})(x, y) \simeq \mathbf{L}(\mathcal{D}, \mathbf{V})(Fx, Fy)$$

Modeling the homotopy theory of homotopy types by means of a relative category is a pretty straightforward task, for instance we can consider a (well behaved) category of spaces and weak equivalences between them to be given by maps which induce isomorphisms on all homotopy groups.

Turning to Grothendieck ∞-groupoids, we can define homotopy groups in this context too, as follows. Given an ∞-groupoid X, an integer $n \geq 0$ and a pair of parallel $(n-1)$-cells $a, b \in X_{n-1}$ (ignore this part if $n = 0$), we define the set $\pi_n(X, a, b)$ to be the set of equivalences classes of n-cells $H\colon a \to b$ in X, where the relation is given by $H \approx H'$ if and only if there exists an $(n+1)$-cell $K\colon H \to H'$. It is easy to show that this set can be naturally endowed with a group structure provided $a = b$, by choosing appropriate operations from the groupoid structure on X (and the structure is independent of these choices, see [**Ar2**] for a reference). Finally, if x is a 0-cell of X, we denote by $\pi_n(X, x)$ the group $\pi_n(X, \mathbf{id}(x), \mathbf{id}(x))$, where $\mathbf{id}(x)$ is a choice (in fact, any) of an $(n-1)$-dimensional identity cell on x.

Definition 3.2. A map $f\colon X \to Y$ of ∞-groupoids is a weak equivalence if it induces isomorphisms on all homotopy sets, i.e.:

$$\pi_0(f)\colon \pi_0(X) \overset{\cong}{\to} \pi_0(Y) \text{ and } \pi_n(f)\colon \pi_n(X, a, b) \overset{\cong}{\to} \pi_n(Y, f(a), f(b))$$

The following result is an important characterization of the class of weak equivalences of ∞-groupoids, see Theorem 4.18 of [**Ar2**].

Proposition 3.3. *Let $f\colon X \to Y$ be a map of ∞-groupoids. Then the following are equivalent:*

(1) *f is a weak equivalence;*
(2) *for every $n \geq 0$ and $(a, b)\colon S^{n-1} \to X$ the map $\pi_n(f)\colon \pi_n(X, a, b) \to \pi_n(Y, f(a), f(b))$ is a surjection (when $n = 0$ this map is simply $\pi_0(f)$);*
(3) *for every $x \in X_0$ the map $\pi_n(f)\colon \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism and $\pi_0(f)\colon \pi_0(X) \overset{\cong}{\to} \pi_0(Y)$.*

Diagrammatically, f is a weak equivalence if and only if given a solid commutative square of the form:

$$\begin{tikzcd} S^{k-1} \arrow[r] \arrow[d, "j_k"'] & X \arrow[d, "f"] \\ D_k \arrow[r, "\gamma"'] \arrow[ur, dotted, "\Gamma"] \arrow[ur, phantom, "\simeq", shift right=3] & Y \end{tikzcd} \tag{13}$$

there exists a map Γ (as displayed by the dotted arrow) that renders the top triangle strictly commutative and the bottom one commutative up to an $(n+1)$-cell (i.e. such that there exists a $(k+1)$-cell $H\colon f \circ \Gamma \to \gamma$ in Y. Here, when $k = 0$, we let $S^{-1} \overset{def}{=} \emptyset$ and, for $k > 0$, S^k is defined to be the ∞-groupoid that corepresents pairs of parallel k-cells. Note that this notion makes sense in any globular theory, and indeed it can be defined at that level of generality.

Let $\mathbf{W}_g$ be the class of weak equivalences of ∞-groupoids, and $\mathbf{W}_s$ that of weak equivalences of spaces. In [**Ar2**], Ara constructs a map of relative categories

$$\Pi_\infty\colon (\mathbf{Top}, \mathbf{W}_s) \to (\infty\text{-}\mathcal{G}pd, \mathbf{W}_g)$$

with $\Pi_\infty(X)_n = \mathbf{Top}(D_n, X)$. In fact, it is proven there that given any model category $\mathcal{M}$ where every object is fibrant, the choice of an object $X_0 \in \mathcal{M}$ induces an adjunction

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "|\cdot|"] \arrow[r, phantom, "\perp"] & \infty\text{-}\mathcal{G}pd \arrow[l, bend left, "\Pi_\infty^{X_0}"] \end{tikzcd}$$

The abovementioned functor is associated with the choice of $\mathcal{M} = \mathbf{Top}$ and $X_0 = *$.

The homotopy hypothesis can now be formulated precisely as follows:

Conjecture 3.4. *The map $\Pi_\infty\colon (\mathbf{Top}, \mathbf{W}_s) \to (\infty\text{-}\mathcal{G}pd, \mathbf{W}_g)$ is an equivalence of relative categories.*

In [**Hen**], Henry proves that in order for this to hold, it is enough to show that, given a pushout square of ∞-groupoids of the form:

$$\begin{tikzcd} D_k \arrow[r, "x"] \arrow[d, "\sigma"'] & X \arrow[d, "j"] \\ D_{k+1} \arrow[r, "h"'] & X^+ \end{tikzcd}$$

then j is a weak equivalence. To address this problem, we will embark on the process of defining a path object on the category of ∞-groupoids, and although we do not completely succeed in doing so, we lay the foundations of a possible approach and we fully construct it in the case of 3-dimensional Grothendieck groupoids. The reason why such a path object would be enough to prove the "pushout lemma" is part of the content of Theorem 4.2.

2. (Co)fibrations of ∞-groupoids

In this section, we introduce two important classes of maps, namely cofibrations and trivial cofibrations, and their corresponding weak factorization systems, which will play a fundamental role in what follows. As before, we consider the case of n-groupoids for $0 \leq n \leq \infty$. Also, we denote by $\pitchfork$ the relation of weak orthogonality between arrows, and given a set of morphisms C in a category $\mathscr{C}$, we let $C^{\pitchfork}$ be the class of maps f in $\mathscr{C}$ such that $c \pitchfork f$ for every c in C, and we define ${}^{\pitchfork}C$ in a similar way.

Definition 3.5. Let I_n be the set $\{S^{k-1} \to D_k\}_{0 \leq k \leq n} \cup \{(1,1)\colon S^n \to D_n\}$ of boundary inclusions in $\mathbf{Mod}(\mathfrak{G})$, where S^k is the free model on a pair of parallel k-cells, together with the map collapsing a pair of parallel n-cells to a single n-cell (disregard this last element in the case $n = \infty$). Also, let J_n be the set of source maps $\{\sigma_k\colon D_k \to D_{k+1}\}_{0 \leq k \leq n-1}$, and $\mathbb{I}_n$ (resp. $\mathbb{J}_n$) be the saturation of I_n (resp. J_n), i.e. the set ${}^{\pitchfork}(I_n^{\pitchfork})$ (resp. $\mathbb{J}_n = {}^{\pitchfork}(J_n^{\pitchfork})$).

We say that a map of n-groupoids $f\colon X \to Y$ is a cofibration (resp. trivial cofibration) of n-groupoids if it belongs to $\mathbb{I}_n$ (resp. $\mathbb{J}_n$).

The maps in the class $J_n^{\pitchfork}$ (resp. $I_n^{\pitchfork}$) are called fibrations (resp. trivial fibrations).

Remark 3.6. We observe that the previous definition makes sense in the category of models of any globular theory, so that we get a notion of (trivial) fibrations and (trivial) cofibrations of such models, although these may not be sensible notions, depending on the context.

Let $*$ denote the terminal object in the category of n-groupoids. Since every map in J admits a retraction, the following result is straightforward.

Proposition 3.7. *Every n-groupoid is fibrant, i.e. the unique map $X \to *$ is a fibration for every $X \in \mathbf{Mod}(\mathfrak{G})$.*

Definition 3.8. An n-groupoid X is said to be contractible if the unique map $X \to *$ is a trivial fibration.

Proposition 3.9. *Globular sums, seen as objects in the image of the Yoneda embedding functor $y\colon \mathfrak{G} \to \mathbf{Mod}(\mathfrak{G})$, are contractible n-groupoids.*

Proof. We proceed by induction on $m = \mathrm{ht}(A)$. If $m = 0$ then $A = D_0$, in which case the statement is obvious.

Let $m > 0$ and let us prove that any map $\alpha\colon S^{k-1} \to A$ extends to D_k. By contractibility of $\mathfrak{G}$ we already know this is possible whenever $\mathrm{ht}(A) = m \leq k$, so we assume $k < m$. Consider ∂A, whose height is $m-1$ by construction, and is therefore contractible by inductive assumption. The map $\partial_\sigma\colon \partial A \to A$ is $(m-2)$-bijective, thanks to Proposition 2.28, thus α must factor through it since it consists of a pair of parallel $(k-1)$-cells, and contractibility of ∂A allows us to find the desired extension.

By definition of a coherator for n-groupoids, it follows that every globular sum in n-groupoids lifts against the map $(1,1)\colon S^n \to D_n$, which concludes the proof. $\square$

We now construct the long exact sequence of homotopy groups associated with a fibration of ∞-groupoids. Assume given a fibration $p\colon E \to B$ in ∞-$\mathcal{G}pd$ and a 0-cell $e \in E$, define an ∞-groupoid F by means of the following pullback square:

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d] & E \arrow[d, two heads, "p"] \\ D_0 \arrow[r, "p(e)"] & B \end{tikzcd}$$

To begin with, we want to define morphisms of sets $\partial_n\colon \pi_n(B, p(e)) \to \pi_{n-1}(E, e)$. Given $[\beta] \in \pi_n(B, p(e))$, where $\beta\colon \mathbf{id}(p(e)) = p(\mathbf{id}(e)) \to \mathbf{id}(p(e))$ is an $(n+1)$-cell in B, we can lift it to an $(n+1)$-cell $\alpha\colon \mathbf{id}(e) \to e'$ which satisfies $p(\alpha) = \beta$, since p is a fibration. We now set $\partial_n[\beta] = [e']$. It is a routine calculation to show this does not depend on the choices we made, and that ∂_n is a group morphism.

Proposition 3.10. *Given a fibration of ∞-groupoids $p\colon E \to B$ and a 0-cell $e \in E$, we get a long exact sequence of homotopy groups of the form:*

$$\cdots\ \pi_{n+1}(B, p(e)) \xrightarrow{\partial_{n+1}} \pi_n(F, e) \xrightarrow{\pi_n(i)} \pi_n(E, e) \xrightarrow{\pi_n(p)} \pi_n(B, p(e)) \xrightarrow{\partial_n} \pi_{n-1}(F, e)\ \cdots$$

Proof. Let us begin by proving $\mathrm{Ker}\,(\pi_n(p)) = \mathrm{Im}\,(\pi_n(i))$. Clearly, $\pi_n(p) \circ \pi_n(i) = 0$. Also, if $\pi_n(p)([x]) = 0$, i.e. there exists an $(n+1)$-cell $H\colon p(x) \to \mathbf{id}(p(e))$ in B, then we can use the lifting property of p to get an $(n+1)$-cell in E of the form $\overline{H}\colon x \to x'$ with $p(x') = \mathbf{id}(p(e))$. Therefore, $x \approx i(x')$, so that $\mathrm{Ker}\,(\pi_n(p)) = \mathrm{Im}\,(\pi_n(i))$.

Turning to the equality $\mathrm{Ker}\,(\partial_n) = \mathrm{Im}\,(\pi_n(p))$, assume we have $\partial_n[\beta] = 0$, i.e there exists a lift $\alpha\colon \mathbf{id}(e) \to e'$ of β and an n-cell $H\colon e' \to \mathbf{id}(e)$ in F. The cell $p(H\alpha)$ is then homotopic to β by construction, thus showing that $\mathrm{Ker}\,(\partial_n) \subset \mathrm{Im}\,(\pi_n(p))$. Conversely, if $\beta \approx p(\alpha)$ for an $[\alpha] \in \pi_n(B, p(e))$, then we have $\partial_n([\beta]) = \partial_n([p(\alpha)]) = [t(\alpha)] = [\mathbf{id}(e)] = 0$.

The verification of the identity $\mathrm{Ker}\,(\pi_n(i)) = \mathrm{Im}\,(\partial_{n+1})$ is left as an exercise for the interested reader. $\square$

Since every ∞-groupoid is a fibrant object, the following fact should not come as a surprise, although it is not straightforward since we do not have a model structure on ∞-$\mathcal{G}pd$.

Proposition 3.11. *Given a pullback square of the form:*

$$\begin{tikzcd} G \arrow[r, "g"] \arrow[d, two heads, "q"'] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

where p is a fibration and f is a weak equivalence, then g is also a weak equivalence.

Proof. We need to show that for every 0-cell $x \in G_0$ and every $n \geq 0$ the induced map $\pi_n(G,x) \to \pi_n(E,g(x))$ is an isomorphism. Since q is also a fibration, we get a morphism of fiber sequences of the form:

$$\begin{tikzcd} F'_{q(x)} \arrow[r, "\mathrm{i}'"] \arrow[d, "\cong"'] & G \arrow[r, "q"] \arrow[d, "g"'] & A \arrow[d, "f"'] \\ F_{f\circ q(x)} \arrow[r, "\mathrm{i}"] & E \arrow[r, "p"] & B \end{tikzcd}$$

where $F'_{q(x)}$ denotes the fiber of q over $q(x)$ and $F_{f\circ q(x)}$ that of p over $f \circ q(x)$ (which are isomorphic). This induce a morphism between corresponding long exact sequences of homotopy groups, part of which is displayed below:

$$\begin{tikzcd} \pi_{n+1}(A,q(x)) \arrow[r, "\partial_{n+1}"] \arrow[d, "\pi_{n+1}(f)"'] & \pi_n\left(F'_{q(x)},x\right) \arrow[r, "\pi_n(\mathrm{i}')"] \arrow[d, "\cong"'] & \pi_n(G,x) \arrow[r, "\pi_n(q)"] \arrow[d, "\pi_n(g)"'] & \pi_n(A,q(x)) \arrow[r, "\partial_n"] \arrow[d, "\pi_n(f)"'] & \pi_{n-1}\left(F'_{q(x)},x\right) \arrow[d, "\cong"'] \\ \pi_{n+1}(B,fq(x)) \arrow[r, "\partial_{n+1}"] & \pi_n\left(F_{fq(x)},g(x)\right) \arrow[r, "\pi_n(\mathrm{i})"] & \pi_n(E,g(x)) \arrow[r, "\pi_n(q)"] & \pi_n(B,fq(x)) \arrow[r, "\partial_n"] & \pi_{n-1}\left(F_{fq(x)},x\right) \end{tikzcd}$$

Thanks to the five lemma, $\pi_n(g)$ must be an isomorphism since f is a weak equivalence, and this concludes the proof. □

2.1. Direct categories. The small object argument provides a factorization system on n-groupoids given by cofibrations and trivial fibrations. Lemma 3.13 will be applied to this factorization system and to the direct category structure on $\mathbb{G}_n$ as defined in Example 3.15, to provide a way of inductively extending certain maps in $\mathbf{Mod}(\mathfrak{G})^{\mathbb{G}}$.

Definition 3.12 (see also [**Ho**], Chapter 5). A direct category is a pair $(\mathscr{C}, d)$, where $\mathscr{C}$ is a small category and $d : \mathrm{Ob}(\mathscr{C}) \to \lambda$ is a function into an ordinal λ, such that if there is a non-identity morphism $f : a \to b$ in $\mathscr{C}$, then $d(a) < d(b)$.

Given a cocomplete category $\mathcal{D}$ and a functor $X \colon \mathscr{C} \to \mathcal{D}$, we define the latching object of X at an object $c \in \mathscr{C}$ to be the object of $\mathcal{D}$ given by

$$L_c(X) = \mathrm{colim}_{c' \in \mathscr{C}_{<d(c)} \downarrow c}\, X(c')$$

This defines a functor L_c from the functor category $[\mathscr{C}, \mathcal{D}]$ to the category $\mathcal{D}$, together with a natural transformation $\varepsilon_c \colon L_c \Rightarrow ev_c$, with codomain the functor given by evaluation at c. We also define the latching map of a natural transformation $\alpha \colon X \to Y$ in $\mathcal{D}^{\mathscr{C}}$ at an object $c \in \mathscr{C}$ to be the map of $\mathcal{D}$

$$\hat{L}_c(\alpha) \colon X(c) \coprod_{L_c(X)} L_c(Y) \to Y(c)$$

induced by $L_c(f)$ and ε_c.

The following results on direct categories are well known, therefore we omit their proofs.

Lemma 3.13. *Let $\mathcal{D}$ be a direct category and $\mathscr{C}$ a category equipped with two classes of arrows $(\mathscr{L}, \mathcal{R})$ such that $\mathscr{L} \pitchfork \mathcal{R}$. If we define*

$$\mathscr{L}^{\mathcal{D}} = \{\alpha \colon X \to Y \text{ in } \mathscr{C}^{\mathcal{D}} \mid \hat{L}_d(\alpha) \in \mathscr{L}\ \forall d \in \mathcal{D}\}$$

and

$$\mathcal{R}^{\mathcal{D}} = \{\alpha \colon X \to Y \text{ in } \mathscr{C}^{\mathcal{D}} \mid \alpha_d \colon X(d) \to Y(d) \in \mathcal{R}\ \forall d \in \mathcal{D}\}$$

we have $\mathscr{L}^{\mathcal{D}} \pitchfork \mathcal{R}^{\mathcal{D}}$.

Lemma 3.14. *Let* A, B *be two cocomplete categories equipped, respectively, with two classes of arrows* $(\mathscr{L}_A, \mathcal{R}_A)$ *and* $(\mathscr{L}_B, \mathcal{R}_B)$ *such that* $\mathscr{L}_A \pitchfork \mathcal{R}_A$ *and* $\mathscr{L}_B \pitchfork \mathcal{R}_B$. *Given a cocontinuous functor* $F\colon A \to B$ *such that* $F(\mathscr{L}_A) \subset \mathscr{L}_B$ *and a direct category* $\mathcal{D}$, *the induced map* $F^{\mathcal{D}}\colon A^{\mathcal{D}} \to B^{\mathcal{D}}$ *preserves the direct cofibrations, i.e.*

$$F(\mathscr{L}_A^{\mathcal{D}}) \subset \mathscr{L}_B^{\mathcal{D}}$$

Example 3.15. The category $\mathbb{G}_n$ has a natural structure of direct category, with degree function defined by

$$\begin{aligned} \deg\colon \mathbb{G}_n &\to \mathbb{N} \\ m &\mapsto m \end{aligned}$$

Every time we have an n-coglobular object $\mathbf{D}_\bullet\colon \mathbb{G}_n \to \mathscr{C}$ in a finitely cocomplete category, we can consider the latching map of $!\colon \emptyset \to \mathbf{D}_\bullet$ at m, i.e. the map

$$\hat{L}_m(!)\colon L_m(\mathbf{D}_\bullet) \to \mathbf{D}_m$$

Notice that

$$\hat{L}_1(!) = (\mathbf{D}(\sigma_0), \mathbf{D}(\tau_0))\colon \mathbf{D}_0 \amalg \mathbf{D}_0 \to \mathbf{D}_1$$

and the other latching maps are obtained inductively from the following cocartesian square

$$\begin{tikzcd}
L_m(\mathbf{D}_\bullet) \arrow[r, "\hat{L}_m(!)"] \arrow[d, "\hat{L}_m(!)"'] & \mathbf{D}_m \arrow[d] \arrow[ddr, bend left, "\mathbf{D}(\sigma_m)"] & \\
\mathbf{D}_m \arrow[r] \arrow[drr, bend right, "\mathbf{D}(\tau_m)"'] & L_{m+1}(\mathbf{D}_\bullet) \arrow[dr, dashed, "\exists! \hat{L}_{m+1}(!)"'] & \\
 & & \mathbf{D}_{m+1}
\end{tikzcd}$$

When $\mathbf{D}_\bullet\colon \mathbb{G}_n \to \mathfrak{G} \to \mathbf{Mod}(\mathfrak{G})$ is the canonical coglobular n-groupoid, we observe that $L_m(\mathbf{D}_\bullet) \cong S^{m-1}$, i.e. the free model on a pair of parallel $(m-1)$-cells.

3. Relative lifting properties of $\mathbf{Mod}(\mathfrak{C})$

In this section, $\mathfrak{C}$ will denote a fixed coherator for n-categories. Let us assume, for sake of simplicity, that $n = \infty$, leaving to the interested reader the task of adapting all what follows to the finite dimensional case.

We are going to prove some useful lemmas on relative liting properties of $\mathfrak{C}$-models with respect to Θ-models, i.e. strict ∞-categories. These are needed to produce fillers of diagrams which are essentially provided by the algebraic structure globular sums are endowed with. In the groupoidal case, we show that globular sums are contractible in Proposition 3.9, but this is no longer true in this context. However, we establish some criteria that allow us to produce such fillers without having to explicitly spell them out, provided they exist in their strict counterpart.

Recall that the structural functor $F\colon \mathfrak{C} \to \Theta$ of the homogeneous coherator $\mathfrak{C}$ gives rise to a cocontinuous functor $F\colon \mathbf{Mod}(\mathfrak{C}) \to \mathbf{Mod}(\Theta) \cong \omega\text{-}\mathcal{C}at_s$, where $\omega\text{-}\mathcal{C}at_s$ denotes the category of strict ∞-categories, thanks to Proposition 2.7, by considering the following Kan extension:

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "y \circ F"] \arrow[d, "y"'] & \omega\text{-}\mathcal{C}at_s \\ \mathbf{Mod}(\mathfrak{C}) \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

where the y's denote two (different) instances of the Yoneda embedding.

Lemma 3.16. *An extension problem in $\mathfrak{C}$ of the form:*

$$\begin{tikzcd} D_{n-1} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & A \\ D_n \arrow[ur, dashed] & \end{tikzcd}$$

admits a solution if and only its image under $F\colon \mathfrak{C} \to \Theta$ does so, and moreover such an extension can be chosen so as to live over the one in Θ.

Proof. Let's prove the non-trivial implication. Suppose we have a map $H\colon D_n \to A$ in Θ, with boundary $(F(f), F(g))$. By factoring H into a homogeneous map $p\colon D_n \to A'$ followed by a globular map $i\colon A' \to A$, we see by inspection that the pair $(f', g') \overset{def}{=} (p \circ \sigma, p \circ \tau)\colon D_{n-1} \to A'$ is admissible: indeed, such are the boundaries of homogeneous maps in Θ. By uniqueness of homogeneous-globular factorizations in $\mathfrak{C}$, we see that f and g have to factor through A' via an admissible pair $(\overline{f}, \overline{g})\colon S^{n-1} \to A'$ that lives over (f', g'). More precisely, we can factor $f = j \circ h$ into a homogeneous map followed by a globular one, so that $F(j) = i$ by uniqueness of the factorization in Θ. Similarly, the globular part of g also must be j, and the rest of the claim follows from the simple fact that F detects admissible pairs. It follows that there exists an extension of $(\overline{f}, \overline{g})$ to a map $\overline{p}\colon D_n \to A'$, and therefore the composite $i \circ \overline{p}$ is the extension we are looking for, and lives over H by construction. $\square$

Lemma 3.17. *Let $i\colon X \to Y$ be an $\mathbf{I}$-cellular map in $\mathbf{Mod}(\mathfrak{C})$ (i.e. a transfinite composite of pushouts of maps in $\mathbf{I}$), and consider the following extension problem, where A is a globular sum:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & A \\ Y \arrow[ur, dashed] & \end{tikzcd}$$

Then such an extension exists if and only if $F(f)$ admits an extension along $F(i)$. Moreover, if we fix an extension in $\mathbf{Mod}(\Theta)$ then the one in $\mathbf{Mod}(\mathfrak{C})$ can be chosen to live over that in $\mathbf{Mod}(\Theta)$.

Proof. There is only one non-trivial implication, which follows from Lemma 3.16 and cocontinuity of F by constructing the extension cell by cell. $\square$

LEMMA 3.18. *Let $i\colon X \to Y$ be a map in $\mathbf{Mod}(\mathfrak{C})^{\mathbb{G}}$, such that its latching maps $\hat{L}_n(i)$ are $\mathbf{I}$-cellular maps for every $n \geq 0$. Then an extension problem of the form:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & A \\ Y \arrow[ur, dashed] & \end{tikzcd}$$

where A factors through $\mathfrak{C}^{\mathbb{G}}$ (i.e. is pointwise a globular sum), admits a solution if and only if its image under $F^{\mathbb{G}}$ does so in $\mathbf{Mod}(\Theta)^{\mathbb{G}}$.

PROOF. The non-trivial implication follows from the observation that $F(\hat{L}_n(i)) \cong \hat{L}_n(F(i))$ by cocontinuity of F, so that one can construct an extension using the usual inductive argument for direct categories and the previous lemmas. □

Let us conclude this section with a very useful lemma on fillers of spheres in globular sums.

LEMMA 3.19. *Let A be a globular sum in $\mathfrak{C}$ with $n = \mathrm{ht}(A)$. Then every k-sphere in A with $k \geq n$ admits a filler. In particular, D_0 is contractible, i.e. the unique map $D_0 \to *$ has the right lifting property with respect to all boundary inclusions $S^{k-1} \to D_k$.*

PROOF. Thanks to Lemma 3.16, it is enough to prove the statement in $\mathbf{Mod}(\Theta)$. If $k > n$ then the only sphere $S^k \to A$ is given by a pair of identities on the same cell, and therefore it surely admits a filler. If $k = n$ and the restriction along one of the inclusions $D_k \to S^k$ is an identity cell, then the other must be as well, since globular sums in Θ admits no non-trivial endomorphisms of cells. In this case too, a filler exists. Finally, if we have an n-sphere in A consisting of a pair of parallel n-cells none of which is an identity, then the claim follows from the fact that an n-cell in a globular sum in Θ of height n is uniquely determined by its boundary, as can easily be proven using the combinatorial description of Θ in terms of trees given in Section 3.3 of [**Ar1**]. □

CHAPTER 4

Semi-model structures on categories of models of globular theories

1. Recognition principle

In this section we are going to characterize those globular theories $\mathfrak{C}$ for which the category of models $\mathbf{Mod}(\mathfrak{C})$ bears a cofibrantly generated semi-model structure that satisfies some natural conditions for objects of $\mathbf{Mod}(\mathfrak{C})$ to look like ∞-groupoids. Cofibrantly generated semi-model structures are defined in [**FR**], Definition 12.1.3: essentially we have sets of maps I and J such that the class of fibrations (resp. trivial fibrations) is precisely the class of maps that has the right lifting property with respect to maps in J (resp. I). The need of a semi-model structure instead of a full one is that it is not clear how to construct a factorization of the diagonal $X \to X \times X$ through some object $\mathbb{P}X$ via a strict map. Nevertheless, this ought to be doable for cofibrant objects, and this is where the weaker axioms of a semi-model structure come into play. Indeed, the recognition principle for cofibrantly generated model categories (i.e. Theorem 2.1.19 in [**Ho**]) can be adapted to this context as follows

THEOREM 4.1. *Suppose $\mathscr{C}$ is a category with all small limits and colimits, let $\mathbf{W}$ be a subcategory of $\mathscr{C}$ which contains all the identities and let I, J be sets of maps in $\mathscr{C}$. Then there exists a cofibrantly generated semi-model structure on $\mathscr{C}$ with I as the set of generating cofibrations, J as the set of generating trivial cofibrations and $\mathbf{W}$ as the class of weak equivalences if and only if the following conditions are satisfied:*

- *$\mathbf{W}$ has the two out of three property and is closed under retracts;*
- *The domains of I are small relative to I-cell complexes with cofibrant domain;*
- *The domains of J are small relative to J-cell complexes with cofibrant domain;*
- *J-cell complexes with cofibrant domain belong to the intersection $\mathbf{W} \cap I$-cof;*
- *I-inj $\subset \mathbf{W} \cap J$-inj;*
- *Either $\mathbf{W} \cap I$-cof $\subset J$-cof or $\mathbf{W} \cap J$-inj $\subset I$-inj.*

It is clear that everything that follows still holds true, mutatis mutandis, to the case of n-globular theories for $n < \infty$.

To begin with, we define a class of maps $\mathbf{W}$ in $\mathbf{Mod}(\mathfrak{C})$ that consists of the maps $f \colon X \to Y$ satisfying the property described in (13).

THEOREM 4.2. *Given a globular theory $\mathfrak{C}$, there exists a cofibrantly generated semi-model structure on the category of models $\mathbf{Mod}(\mathfrak{C})$ with weak equivalences given by the class $\mathbf{W}$, where every object is fibrant, globular sums are contractible and the set of generating cofibrations (resp. trivial cofibrations) consists of the boundary inclusions $\mathbf{I} \overset{def}{=} \{j_k \colon S^{k-1} \to D_k\}_{k \geq 0}$ (resp. source maps $\mathbf{J} \overset{def}{=} \{\sigma_k \colon D_k \to D_{k+1}\}_{k\geq 0}$) if and only if:*

- *D_0 is contractible (i.e. the unique map $D_0 \to *$ is a trivial fibration);*
- *$\mathfrak{C}$ admits a system of composition and identities, as defined in Definition 2.21;*

- *for every cofibrant object X in $\mathbf{Mod}(\mathfrak{C})$ there exists a fibration $\mathbf{ev}\colon \mathbb{P}X \to X \times X$ such that $\mathbf{ev}_i = \pi_i \circ \mathbf{ev}$ is a trivial fibration for $i = 0, 1$, where $\pi_i\colon X \times X \to X$ denote the product projections.*

In particular, under such assumptions, if $\mathfrak{C}$ is cellular then it is a coherator for ∞-groupoids.

If such a semi-model structure exists on $\mathbf{Mod}(\mathfrak{C})$, then clearly the four conditions are satisfied. Let us check that the converse also holds true.

The proof is a matter of checking that the recognition principle given in Theorem 4.1 applies to this situation. The third condition implies that all maps in $\mathbf{J}$ admit a retraction, therefore all objects are fibrant. Moreover, $\mathbf{Mod}(\mathfrak{C})$ is complete and cocomplete, and both the domains of $\mathbf{I}$ and $\mathbf{J}$ permit the small object argument.

Lemma 4.3. $\mathbf{W}$ *is closed under retracts.*

Proof. Assume f is a retract of $g \in \mathbf{W}$, so that we have a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & W \arrow[r, "b"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "c"] & Z \arrow[r, "d"] & Y \end{tikzcd}$$

with $b \circ a = 1_X$ and $d \circ c = 1_Y$. Given a $(k-1)$-sphere (α, β) in X and a k-cell $H\colon f(\alpha) \to f(\beta)$ in Y, we get a k-cell $\varphi\colon a(\alpha) \to a(\beta)$ together with a $(k+1)$-cell $\Gamma\colon g(\varphi) \to c(H)$ in Z, since g is a weak equivalence. If we consider the k-cell $\overline{H} \overset{def}{=} b(\varphi)$ in X, we see that $\overline{H}\colon \alpha \to \beta$ and $d(\Gamma)\colon f(\overline{H}) = d(g(\varphi)) \to d(c(H)) = H$. $\square$

Lemma 4.4. *Let $f\colon X \to Y, g\colon Y \to Z$ be maps in $\mathbf{Mod}(\mathfrak{C})$. Then:*

(1) *If f and g belong to $\mathbf{W}$ then so does $g \circ f$;*
(2) *If g and $g \circ f$ belong to $\mathbf{W}$, then so does f;*
(3) *If $g \circ f = 1_X$ and $f \circ g$ belongs to $\mathbf{W}$, then both f and g belong to $\mathbf{W}$.*

Proof. Firstly, assume f and g belong to $\mathbf{W}$, and assume given a $(k-1)$-sphere (a, b) in X, together with a k-cell $\gamma\colon g \circ f(a) \to g \circ f(b)$ in Z. By assumption we get a k-cell $\beta\colon f(a) \to f(b)$ in Y and a $(k+1)$-cell $H\colon g(\beta) \to \gamma$. Again by assumption we get a k-cell $\alpha\colon a \to b$ in X, together with a $(k+1)$-cell $H'\colon f(\alpha) \to \beta$. The composite $H \circ g(H')\colon g \circ f(\alpha) \to \gamma$ (obtained using the system of composition on $\mathfrak{C}$) is the data we need to conclude the proof of the first statement.

Turning to the second statement, assume g and $g \circ f$ belong to $\mathbf{W}$ and consider a $(k-1)$-sphere (a, b) in X, together with a k-cell $\alpha\colon f(a) \to f(b)$ in Y. We can lift the $(k-1)$-sphere $(g \circ f(a), g \circ f(b))$ in Z along $g \circ f$ to get a k-cell in X of the form $H\colon a \to b$, together with a $(k+1)$-cell $\Gamma\colon g \circ f(H) \to g(\alpha)$. We now have a k-sphere in Y given by $(f(H), \alpha)$, and an extension to a $(k+1)$-cell in Z between its image under g. By assumption, we get a lift to a $(k+1)$-cell $\overline{H}\colon f(H) \to \alpha$, which concludes the proof of the second statement.

Finally, if $g \circ f = 1_X$ then g is a retract of $f \circ g$, and is thus a weak equivalence thanks to Lemma 4.3. Therefore, $f \in \mathbf{W}$ thanks to the second point of this lemma, since $1_X\colon X \to X$ is a weak equivalence thanks to the existence of a system of identities in $\mathfrak{C}$. $\square$

Lemma 4.5. $cof(\mathbf{J}) \subset cof(\mathbf{I})$.

PROOF. Of course it is enough to check that $\mathbf{J} \subset cof(\mathbf{I})$. We thus have to prove that, for every $k \geq 0$, we have that $\sigma_k \colon D_k \to D_{k+1}$ belongs to $cof(\mathbf{I})$. We know by assumption that $S^{k-1} \to D_k$ is a cofibration, so that the colimit injection $i_0 \colon D_k \to S^k \overset{def}{=} D_k \coprod_{S^{k-1}} D_k$ is also such, being a pushout of it. We can now compose that with the boundary inclusion $S^k \to D_{k+1}$ to conclude the proof. $\square$

LEMMA 4.6. *$inj(\mathbf{I}) = inj(\mathbf{J}) \cap \mathbf{W}$.*

PROOF. We start by proving $inj(\mathbf{I}) \subset inj(\mathbf{J}) \cap \mathbf{W}$. Thanks to Lemma 4.5 we only have to prove that $inj(\mathbf{I}) \subset \mathbf{W}$, which is obvious, since a cell $f \to f$ exists for every cell in Y thanks to the system of identities in $\mathfrak{C}$. Conversely, assume f is both a fibration and a weak equivalence, and consider a $(k-1)$-sphere (a, b) in X together with a k-cell $H \colon f(a) \to f(b)$ in Y. Since f belongs to $\mathbf{W}$, we find a k-cell $\overline{H} \colon a \to b$ in X, together with a $(k+1)$-cell $\Gamma \colon f(\overline{H}) \to H$ in Y. Because f is a fibration, we can lift Γ to a cell $\gamma \colon \overline{H} \to \beta$, so that $f(\beta) = H$ and $\beta \colon a \to b$, since it is parallel to $\overline{H}$, and this concludes the proof. $\square$

Since relative $\mathbf{J}$-cell complexes relative to D_0 include all globular sums, if we prove that such maps are weak equivalences we then obtain for free the contractibility of globular sums, since D_0 is contractible by hypothesis. We actually prove a little bit more, namely the following result.

LEMMA 4.7. *Let $f \in cof(\mathbf{J})$ have a cofibrant domain. Then $f \in \mathbf{W}$.*

PROOF. Let $f \colon X \to Y$ be as in the statement. Pick a section i_Y of the trivial fibration $\mathbf{ev}_0 \colon \mathbb{P}Y \to Y$, which exists since Y is cofibrant and is a weak equivalence thanks to Lemma 4.4, and denote by α the endomorphism $\mathbf{ev}_1 \circ i_Y$, which is a weak equivalence thanks to Lemma 4.4 and Lemma 4.6. Consider the following commutative square:

$$\begin{tikzcd} X \arrow[r, "i_Y \circ f"] \arrow[d, "f"'] & \mathbb{P}Y \arrow[d, "\mathbf{ev}"] \\ Y \arrow[r, "(1, \alpha \circ f \circ r)"'] \arrow[ur, dotted, "\Gamma"] & Y \times Y \end{tikzcd}$$

where r denotes the choice of a retraction of f, which exists since X is fibrant, and the lift Γ exists by assumption, since $f \in cof(\mathbf{J})$, X is cofibrant and $\mathbf{ev}$ is a fibration. We have $\mathbf{ev}_0 \circ \Gamma = 1_Y$ which implies that Γ is a weak equivalence, thanks to Lemma 4.4. Therefore, thanks to the same lemma and Lemma 4.6, we see that $\alpha \circ f \circ r = \mathbf{ev}_1 \circ \Gamma$ is also a weak equivalence. A further application of Lemma 4.4 yields that $f \circ r$ belongs to $\mathbf{W}$, which in turn implies that f is a weak equivalence thanks to Lemma 4.4 again, since $r \circ f = 1_X$. $\square$

Since we have proven that globular sums are contractible in $\mathbf{Mod}(\mathfrak{C})$, we can endow models of $\mathfrak{C}$ with the structure of ∞-groupoids, and use the results in Section 4 of [**Ar2**] to obtain the missing piece: namely, the 2-out-of-3 property for $\mathbf{W}$. Indeed, the maps in $\mathbf{W}$ can be characterized as in Theorem 4.18 (ibid.) and we can use the invariance of basepoints (i.e. Corollary 4.14) to conclude that if f and $g \circ f$ are weak equivalences then g is also such. Indeed, it is clear that $\pi_0(g)$ is a bijection. Now, let y be a 0-cell of Y, we want to prove that $\pi_n(g) \colon \pi_n(Y, y) \to \pi_n(Z, g(y))$ is an isomorphism. Choose a 1-cell $f(x) \to y$, whose existence is ensured by the fact that $\pi_0(f)$ is bijective, so that we have an isomorphism $\pi_n(Y,y) \cong \pi_n(Y,f(x))$ as well as $\pi_n(Z,g(y)) \cong \pi_n(Y,g(f(x)))$. Consider the following commutative diagram:

$$\begin{tikzcd}
\pi_n(X,x) \arrow[r, "\pi_n(f)"] & \pi_n\left(Y,f(x)\right) \arrow[r, "\pi_n(g)"] \arrow[d, "\cong"'] & \pi_n\left(Z,g(f(x))\right) \arrow[d, "\cong"] \\
& \pi_n\left(Y,y\right) \arrow[r, "\pi_n(g)"] & \pi_n\left(Z,g(y)\right)
\end{tikzcd}$$

By assumption, the upper horizontal arrow of the square is bijective, which implies that the bottom one is also such and this concludes the proof of Theorem 4.2.

2. Semi-model structure on $\mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$

The part of Theorem 4.2 that is hard to check in practice is the functorial construction of a path object, i.e. fibration $\mathbf{ev}\colon \mathbb{P}X \to X \times X$ such that the composition with both projections is a trivial fibration. Usually, a path object is defined to be a little bit more than that, in that it requires the existence of a functorial factorization of the diagonal map $X \to X \times X$ through it, but it turns out that our weakened version is enough to get a semi-model structure, whereas a full path object would produce a model structure in the ordinary sense. As we now prove, it is enough to construct our weakened path object for a globular theory obtained from a coherator for n-categories (with $0 \leq n \leq \infty$) by freely adjoining a left and a right inverse for each map. This appears to be easier than building a path object for a coherator for n-groupoids, since we can use the homogeneity property.

Theorem 4.8. *Let $\mathfrak{C}$ be a coherator for n-categories (with $0 \leq n \leq \infty$), and suppose there is a functor $\mathbb{P}\colon \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}) \to \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$ endowed with a natural transformation $\mathbf{ev}\colon \mathbb{P} \Rightarrow \mathbf{Id} \times \mathbf{Id}$ which is a pointwise fibration with the property that $\mathbf{ev}_i \overset{def}{=} \pi_i \circ \mathbf{ev}$ is a pointwise trivial fibration. Then $\mathfrak{C}^{\mathbf{W}}$ satisfies the hypotheses of Theorem 4.2, and therefore is a coherator for n-groupoids. Moreover, $\mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$ admits a semi-model structure as described in 4.2.*

Proof. We denote by ${D_0}^{\mathfrak{C}^{\mathbf{W}}}$ the representable $\mathfrak{C}^{\mathbf{W}}$-model on D_0, and we adopt a similar convention for ${D_0}^{\mathfrak{C}}$. All the hypotheses of the theorem are trivially satisfied, except for the contractibility of ${D_0}^{\mathfrak{C}^{\mathbf{W}}}$. We know from Lemma 3.19 that ${D_0}^{\mathfrak{C}}$ is contractible, so it can be endowed with the structure of $\mathfrak{C}^{\mathbf{W}}$-model, that we still denote by ${D_0}^{\mathfrak{C}}$. The claim would then follow if we can prove that the counit of the adjunction

$$\begin{tikzcd}
\mathbf{Mod}(\mathfrak{C}) \arrow[r, bend left, "\mathbf{F}"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}) \arrow[l, bend left, "\mathbf{U}"]
\end{tikzcd}$$

is a weak equivalence at ${D_0}^{\mathfrak{C}}$, since $\mathbf{FU}{D_0}^{\mathfrak{C}} = {D_0}^{\mathfrak{C}^{\mathbf{W}}}$. This is a consequence of a more general result, proven in Proposition 4.9. $\square$

Proposition 4.9. *Let X be a $\mathfrak{C}^{\mathbf{W}}$-model such that $\mathbf{FU}X$ is cofibrant. Then the counit ε of the adjunction*

$$\begin{tikzcd}
\mathbf{Mod}(\mathfrak{C}) \arrow[r, bend left, "\mathbf{F}"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}) \arrow[l, bend left, "\mathbf{U}"]
\end{tikzcd}$$

is a weak equivalence at X.

Proof. It is enough to show that $\mathbf{U}(\varepsilon_X)$ is a weak equivalence of $\mathfrak{C}$-models. Let us consider the following commutative square in $\mathbf{Mod}(\mathfrak{C})$:

$$\begin{tikzcd} \mathbf{U}X \arrow[rr, "\mathbf{U}(i)\circ\eta_{\mathbf{U}X}"] \arrow[d, "\eta_{\mathbf{U}X}"'] & & \mathbf{U}\mathbb{P}\mathbf{FU}X \arrow[d, two heads, "\mathbf{U}(\mathbf{ev})"] \\ \mathbf{UFU}X \arrow[rr, "(\eta_{\mathbf{U}X}\circ\mathbf{U}(\varepsilon_X),\mathbf{U}(\mathbf{ev}_1\circ i))"] & & \mathbf{UFU}X\times\mathbf{UFU}X \end{tikzcd} \tag{14}$$

Here, i denotes a choice of a section of the trivial fibration $\mathbf{ev}_0\colon \mathbb{P}\mathbf{FU}X \to \mathbf{FU}X$, whose existence of i is ensured by the cofibrancy assumption on $\mathbf{FU}X$. Suppose we manage to find a diagonal filler $\Gamma\colon \mathbf{UFU}X \to \mathbf{U}\mathbb{P}\mathbf{FU}X$ for such square, we would then have that Γ is a weak equivalence by Lemma 4.4, since $\mathbf{ev}_1$ and $\mathbf{U}(\mathbf{ev}_1\circ i)$ both are, and by construction $\mathbf{ev}_1\circ\Gamma = \mathbf{U}(\mathbf{ev}_1\circ i)$. This, in turn, implies that $\eta_{\mathbf{U}X}\circ\mathbf{U}(\varepsilon_X)$ is also a weak equivalence, and moreover, thanks to the triangle identities, we have $\mathbf{U}(\varepsilon_X)\circ\eta_{\mathbf{U}X} = 1_{\mathbf{U}X}$. This implies, again thanks to Lemma 4.4, that $\mathbf{U}(\varepsilon_X)$ is a weak equivalence. To conclude the proof, all is left to do is to find the filler Γ, and this is accomplished separately in the next lemmas. □

We define a set of maps $\alpha_k\colon D_k \to \mathbb{I}_k$, where the codomain is obtained by freely adding a pair of k-cell going in the opposite direction as well as a pair of $(k+1)$-cells connecting the two possible composites with identities (with respect to the system of composition chosen to define (12)). For example, if $k=1$, then $\mathbb{I}_k$ is the free $\mathfrak{C}$-model on the following pasting diagram:

$$\begin{tikzcd} & 1 \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Uparrow"] & & 1 \\ 0 \arrow[ur, "g"] \arrow[rr, equal] & {} & 0 \arrow[ur, "k"'] \arrow[u, phantom, "\Downarrow"] & \end{tikzcd} \tag{15}$$

and α_1 picks out f.

Lemma 4.10. *The map $\eta_{\mathbf{U}X}$ is obtained as a transfinite composite of pushouts of maps of the form $\alpha_k\colon D_k \to \mathbb{I}_k$ for $k \geq 1$.*

Proof. The result follows from the same argument given in Proposition 2.2 in [**Nik**], where it is proven that the unit of the adjunction between a model category and the (model) category of its algebraically fibrant objects is an FJ-cell complex (F being the free functor from the original model category and J being a set of trivial cofibrations that detects the fibrant objects) provided the J's are monomorphisms. In this case, $J = \{\alpha_k\colon D_k \to \mathbb{I}_k\}_{k\geq 1}$ so these maps are cofibrations, hence it suffices to show that cofibrations are monomorphisms. In the language of [**JB2**], we can view $\mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$ as the cofiltered limit of a tower of iterated injectives, starting from $\Big([\mathbb{G}^{op}, \mathbf{Set}], \mathbb{I}_0 = \{S^{k-1} \to D_k\}_{k\geq 0}\Big)$. Since maps in $\mathbb{I}_0$ are monomorphisms, we see that $F\mathbb{I}_0$-cell complexes in $\mathbf{Inj}(\mathbb{I}_1)$, where F_0 is the left adjoint to the forgetful functor into globular sets, are again monomorphisms since, by Proposition 2.18 (ibid.), these are $\mathbb{I}_0$-cell complexes. Therefore we can iterate this construction and get $\mathbb{I} \subset \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$ as a filtered colimit of $F_i(\mathbb{I}_0)$, with F_i being the left adjoint to the forgetful functor down to globular sets, where each set $F_i(\mathbb{I}_0)$ consists of monomorphisms by induction. It follows that $\mathbb{I}$ consists of monomorphisms. □

Lemma 4.11. *If $p\colon E \to B$ is a fibration in $\mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$, then p has the right lifting property with respect to the set of maps $\{\tau_k\colon D_k \to D_{k+1}\}_{k\geq 0}$.*

PROOF. Suppose given a $(k+1)$-cell $H \in B_{k+1}$ with $H\colon g \to p(f)$. By assumption there exists a $(k+1)$-cell $h_0 \in E_{k+1}$ with $h_0\colon f \to \bar{g}$ and $p(h_0) = H^{-1}$. We have $p({h_0}^{-1}) = (H^{-1})^{-1}$, so that there exists a $(k+2)$-cell $\gamma\colon p({h_0}^{-1}) \to H$ which we can lift to get a $(k+2)$-cell $\overline{\gamma}\colon {h_0}^{-1} \to \overline{H}$. By construction, $p(\overline{H}) = H$. $\square$

LEMMA 4.12. *The commutative square* (14) *admits a diagonal filler* $\Gamma\colon \mathbf{UFU}X \to \mathbf{U}\mathbb{P}\mathbf{FU}X$.

PROOF. Thanks to Lemma 4.10 and to the fact that $F(\alpha_k) = \alpha_k$ (where, with a minor abuse of language, we have denoted with the same expression the interpretation of α_k in $\mathbf{Mod}(\mathfrak{C})$ on the left and in $\mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$ on the right) with F being the left adjoint to the forgetful functor $\mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}) \to \mathbf{Mod}(\mathfrak{C})$, it is enought to show that α_k is a trivial cofibration in $\mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$, i.e. it has the left lifting properties with respect to fibrations. Suppose given a fibration $p\colon E \to B$ and a diagram of k-cells and $(k+1)$-cells in B of the form:

$$\begin{tikzcd}
 & p(y) \arrow[rr, equal] \arrow[dr, "p(f)"] \arrow[d, phantom, "\Uparrow\gamma"] & \arrow[d, phantom, "\Downarrow\beta"] & p(y) \\
p(x) \arrow[ur, "g"] \arrow[rr, equal] & & p(x) \arrow[ur, "k"'] &
\end{tikzcd}$$

Since identities are preserved by any map, the domain of γ is of the form $p(\mathbf{id}_{k-1}(x))$, therefore we can lift γ to a $(k+1)$-cell $\gamma'\colon \mathbf{id}_{k-1}(x) \to g_0$ in E. There exists a $(k+1)$-cell $\delta\colon p(f_l^{-1}g_0) \to g$ in B obtained as the following composite (where $(\cdot)_l^{-1}$ denotes the left inverse operation):

$$p(f_l^{-1}g_0) = p(f_l^{-1})(p(f)g) \xrightarrow{\simeq} (p(f_l^{-1})p(f))g \xrightarrow{(\mathbf{k}_k^l)^{-1}g} 1g \xrightarrow{\simeq} g$$

which we can lift it to get a $(k+1)$-cell $\overline{\delta}\colon f_l^{-1}g_0 \to \overline{g}$ in E, with in particular $p(\overline{g}) = g$. Thanks to Remark 2.23, there exists a cell $\chi\colon g_0 \to f(f_l^{-1}g_0)$, and the composite $f\overline{\delta}\circ\chi\circ\gamma'\colon \mathbf{id}_{k-1}(x) \to f\overline{g}$ lives over a cell of the form $\mathbf{id}_{k-1}(p(x)) \to p(f)g$ which is homotopic to γ. In fact, its image is a composite of γ, associativity constraints, left and right inverse constraints and their respective inverses, so a simple diagram chasing achieves the desired result. Therefore, by lifting this homotopy and taking its target, we get a cell $\overline{\gamma}\colon \mathbf{id}_{k-1}(x) \to f\overline{g}$ which is the lift we were looking for. The case of β is similar to the one we have just considered thanks to Lemma 4.11. $\square$

This concludes the proof of Proposition 4.9 and this chapter as well.

CHAPTER 5

Main constructions

In this chapter we define a functor

$$\mathbf{Cyl}\colon \mathbb{G} \to \mathbf{Mod}(\mathfrak{A})$$

where $\mathfrak{A}$ is a coherator for ∞-categories, ∞-groupoids or one of the form $\mathfrak{C}^{\mathbf{W}}$. It follows from the results of this section and of Chapter 6 that if one proves the division lemma (see Lemma 6.2 below) holds for $\mathfrak{C}^{\mathbf{W}}$-models, then it is enough to extend the functor above (in the case of $\mathfrak{C}^{\mathbf{W}}$-models) to one of the form:

$$\mathbf{Cyl}\colon \mathfrak{C} \to \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$$

and set

$$\mathbb{P}X \overset{def}{=} \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})\,(\mathbf{Cyl}(\bullet), X) : \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}) \to \mathbf{Mod}(\mathfrak{C})$$

to prove $\mathfrak{C}^{\mathbf{W}}$ is a coherator for ∞-groupoids and get a semi-model structure on Grothendieck ∞-groupoids of type $\mathfrak{C}^{\mathbf{W}}$, thanks to Theorem 4.8. Indeed, at that point the remaining work to do in order to get an endofunctor on $\mathfrak{C}^{\mathbf{W}}$-models would be to define inverses, which we address in Chapter 6 of this work. This would solve the open problem of making an ∞-groupoid à la Batanin, i.e. a $\mathfrak{C}^{\mathbf{W}}$-model (see [**Bat**]), into a Grothendieck one and it would also prove the homotopy hypothesis thanks to the main results in [**Hen**].

1. Suspension-space of paths adjunction

We construct here the adjunction given by suspension-space of paths that generalizes the loop space one for, say, topological spaces. It is given in a slightly more general fashion, since the space of paths that we get takes as input an n-groupoid and two points, and its 0-cells are not necessarily loops but rather paths. We define it both for $\mathfrak{C}$-models and $\mathfrak{C}^{\mathbf{W}}$-models. In a similar (and simpler) way, one can get the analogous construction in the case of a coherator for groupoids $\mathfrak{G}$, simply using the contractibility of globular sums in there.

In detail, given $X \in \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}_n)$ (resp. $X \in \mathbf{Mod}(\mathfrak{C}_n)$ and two 0-cells $a, b \in X_0$ we produce the $\mathfrak{C}^{\mathbf{W}}_{n-1}$-model (resp.$\mathfrak{C}_{n-1}$-model) of morphisms from a to b, denoted by $\Omega(X, a, b)$.

This construction is obtained by firstly defining a suspension functor $\Sigma\colon \mathfrak{C}^{\mathbf{W}}_{n-1} \to S^0 \downarrow \mathfrak{C}^{\mathbf{W}}_n$, where the codomain is defined to be the subcategory of the slice category $S^0 \downarrow \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}_n)$ spanned by the representable functors. We then extend this functor (thanks to Proposition 2.7) to a cocontinuous functor having $\mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}_{n-1})$ as domain and $S^0 \downarrow \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}_n)$ as codomain. This is defined as a left Kan extension of the following form:

$$\begin{tikzcd}
\mathfrak{C}^{\mathbf{W}}_{n-1} \arrow[r, "\Sigma"] \arrow[d] & S^0 \downarrow \mathfrak{C}^{\mathbf{W}}_n \arrow[r] & S^0 \downarrow \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}_n) \\
\mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}_{n-1}) \arrow[urr, dashed, "\Sigma"'] & &
\end{tikzcd}$$

where the unlabelled maps are induced from the appropriate Yoneda embeddings. This suspension functor must have a right adjoint, which we denote by Ω, since both its domain and codomain are presentable categories, so that we end up with the following adjunction:

$$\begin{tikzcd} \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}_{n-1}) \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\perp"] & S^0 \downarrow \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}_n) \arrow[l, bend left, "\Omega"] \end{tikzcd} \tag{16}$$

or with a similar adjunction involving $\mathfrak{C}$ in place of $\mathfrak{C}^{\mathbf{W}}$.

Using the language of trees it is straightforward to construct a functor

$$\Sigma\colon (n-1)\text{-}\mathcal{C}at_s \to S^0 \downarrow n\text{-}\mathcal{C}at_s$$

that models suspension, sending D_k to D_{k+1}, where m-$\mathcal{C}at_s$ denotes the category $\mathbf{Mod}(\Theta^{\leq m})$ of strict m-categories. As outlined above, we will firstly construct $\Sigma\colon \mathbf{Mod}(\mathfrak{C}_{n-1}) \to S^0 \downarrow \mathbf{Mod}(\mathfrak{C}_n)$ as the left Kan extension of a functor $\Sigma\colon \mathfrak{C}_{n-1} \to S^0 \downarrow \mathbf{Mod}(\mathfrak{C}_n)$, defined by induction on the defining tower $\mathfrak{C}^{\bullet}_{n-1}$ of $\mathfrak{C}_{n-1}$, and then we will extend that construction to the case where we have inverses too. We assume by inductive hypothesis that at each ordinal α for which we assume we already have this construction, the following square commutes:

$$\begin{tikzcd} \mathfrak{C}^{\alpha}_{n-1} \arrow[r, "\Sigma"] \arrow[d, "F"'] & S^0 \downarrow \mathfrak{C}_n \arrow[d] \\ \Theta^{\leq n-1} \arrow[r, "\Sigma"] & \Theta^{\leq n} \end{tikzcd}$$

The case $\Theta_0 = \mathfrak{C}^0_{n-1}$ has already been discussed, and the limit ordinal case is trivial. Let us then suppose we have the construction on $\mathfrak{C}^{\alpha}_{n-1}$, and that $\mathfrak{C}^{\alpha+1}_{n-1}$ is obtained by adding an operation $\varrho\colon D_n \to A$ with boundary an admissible pair (f, g). It is easy to see that $\Sigma\colon \Theta^{\leq n-1} \to \Theta^{\leq n}$ preserves admissible pairs, so that we can define $\Sigma(\varrho)$ as the choice of an extension to the following diagram in $\mathfrak{C}_n$, which is automatically under S^0 and again satisfies the inductive hypothesis:

$$\begin{tikzcd} D_n \arrow[r, shift left, "\Sigma(f)"] \arrow[r, shift right, "\Sigma(g)"'] \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & \Sigma A \\ D_{n+1} \arrow[ur, dashed, "\Sigma(\varrho)"'] & \end{tikzcd}$$

This argument can be adapted in a straightforward manner to the case in which the dimension of (f, g) is maximal.

If we want to extend this construction to the case of $\mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$, we simply consider the pushout (12) that defines this globular theory. Thanks to that definition and the work done so far, constructing a functor $\Sigma\colon \mathfrak{C}^{\mathbf{W}}_{n-1} \to \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}_n)$ amounts to define an action of Σ on inverses and inverses constraints in a compatible way. More precisely, we define a map $\Sigma\colon \mathfrak{C}_{n-1} \to S^0 \downarrow \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}_n)$ by composing the functor $\Sigma\colon \mathfrak{C}_{n-1} \to S^0 \downarrow \mathbf{Mod}(\mathfrak{C}_n)$ defined above with the natural map $S^0 \downarrow \mathbf{Mod}(\mathfrak{C}_n) \to S^0 \downarrow \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}_n)$. In addition, we define a map $\Sigma\colon \Theta_0[\mathbf{comp}, \mathbf{id}, \mathbf{inv}] \to S^0 \downarrow \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}_n)$ by setting $\Sigma(\mathbf{i}^{\varepsilon}_k) = \Sigma^k(\mathbf{i}_1)$ for $\varepsilon = l, r$ and similarly for the maps $\mathbf{k}_m{}^{\varepsilon}$. The universal property of pushouts yields the desired map $\Sigma\colon \mathfrak{C}^{\mathbf{W}}_{n-1} \to S^0 \downarrow \mathbf{Mod}\left(\mathfrak{C}^{\mathbf{W}}_n\right)$, and consequently a functor

$$\Sigma\colon \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}_{n-1}) \to S^0 \downarrow \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}_n)$$

by left Kan-extension. This, in turn, can be promoted to the adjunction (16) as observed earlier.

By adjunction, the underlying globular set of $\Omega(X,a,b)$ is given by

$$\Omega(X,a,b)_k\colon = \{x \in X_{k+1} |\ s_0^{k+1}(x) = a,\ t_0^{k+1}(x) = b\}$$

We will often denote $\Omega(X,a,b)$ simply by $X(a,b)$.

REMARK 5.1. If we compose Σ with the forgetful functor $U\colon S^0 \downarrow \mathbf{Mod}(\mathfrak{C}_n^{\mathbf{W}}) \to \mathbf{Mod}(\mathfrak{C}_n^{\mathbf{W}})$, we get a functor which is no longer cocontinuous. Nevertheless, it is well known that U creates connected colimits, therefore $U \circ \Sigma$ preserves all such. Because $\Sigma(I_{n-1}) \subset I_n$, where I_k is the set of maps defined in Definition 3.5, we therefore have that $U \circ \Sigma$ preserves cofibrations (i.e. it sends maps in $\mathbb{I}_{n-1}$ to maps in $\mathbb{I}_n$, the respective saturations of I_{n-1} and I_n). A similar situation is treated in Lemma 1.3.52 of [**Cis**].

As a justification for the notation we have chose, we observe that given a map of n-groupoids $(\alpha,\beta)\colon S^k \to X$, seen as a map $(\hat{\alpha},\hat{\beta})\colon S^{k-1} \to X(a,b)$, where $a = s_0^k(\alpha)$ and $b = t_0^k(\beta)$, then it holds true that

$$\pi_{k-1}(X(a,b),\hat{\alpha},\hat{\beta}) \cong \pi_k(X,\alpha,\beta)$$

(see also [**Ar2**], Definition 4.11).

PROPOSITION 5.2. *Let $(X,(a,b))$ be an object in $S^0 \downarrow \infty$-$\mathcal{G}pd$. Assume that X is a contractible ∞-groupoid. Then $X(a,b)$ is again contractible.*

PROOF. Diagrams of the form:

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & X(a,b) \\ D_n \arrow[ur, dashed] & \end{tikzcd}$$

correspond, under the adjunction $\Sigma \dashv \Omega$, to diagrams under S^0 of the form

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & X \\ D_{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

By assumption, all such diagrams admit an extension, which concludes the proof. $\square$

The following lemma holds for the class of all globular theories considered in this section, and will be used quite frequently in the forthcoming sections. Its proof is straightforward and it is thus left to the reader.

LEMMA 5.3. *Let $\mathfrak{A}_n$ be a coherator for n-categories, a coherator for n-groupoids or a globular theory of the form $\mathfrak{A}_n \cong \mathfrak{C}_n^{\mathbf{W}}$. For every globular sum A in $\mathfrak{A}$ there exist unique globular sums $\alpha_1,\ldots,\alpha_q$ such that:*

$$A \cong \Sigma\alpha_1 \coprod_{D_0} \Sigma\alpha_2 \coprod_{D_0} \ldots \coprod_{D_0} \Sigma\alpha_q \tag{17}$$

the colimit being taken over the maps

$$\begin{tikzcd} & D_0 \arrow[dl, "\top"'] \arrow[dr, "\perp"] & \\ \Sigma\alpha_i & & \Sigma\alpha_{i+1} \end{tikzcd}$$

where we denote by $(\Sigma B, \perp, \top)$ *the image under the functor* $\Sigma \colon \mathbf{Mod}(\mathfrak{A}_{n-1}) \to S^0 \downarrow \mathbf{Mod}(\mathfrak{A}_n)$ *of any globular sum* B *.*

PROOF. The proof consists of a simple induction on the number of 0-cells of A. □

We conclude this section with the construction of iterated suspension functors, which we denote by Σ^n. We assume that for every n, a sequence of coherators for n-categories (resp. n-groupoids or of the form $\mathfrak{C}_n^{\mathbf{W}}$) $\mathfrak{A}_n$ is given.

LEMMA 5.4. *For every* $k > 0, n \geq 0$ *we have adjunctions of the form:*

$$\begin{tikzcd} \mathbf{Mod}(\mathfrak{A}_n) \arrow[r, bend left, "\Sigma^k"] \arrow[r, phantom, "\perp"] & S^{k-1} \downarrow \mathbf{Mod}(\mathfrak{A}_{n+k}) \arrow[l, bend left, "\Omega^k"] \end{tikzcd}$$

PROOF. The proof proceeds by induction, and we have already proven the case $k = 1$. Assume $k > 1$, and define Σ^k by setting:

$$\Sigma^k X \overset{def}{=} \Sigma(\Sigma^{k-1} X)$$

where $\Sigma^{k-1}X$ denotes, with a small abuse of language, the map $S^{k-2} \to \Sigma^{k-1}X$ defined by inductive assumption. On the other hand, we define Ω^k by setting:

$$\Omega^k(f \colon S^{k-1} \to Y) \overset{def}{=} \Omega^{k-1}(\hat{f} \colon S^{k-2} \to \Omega Y)$$

where Ω^{k-1} is defined by inductive assumption and $\hat{f}$ is the transpose of f under the adjuntion $\Sigma \dashv \Omega$. At this point we can conclude the proof by observing that there is a bijection between commutative triangles of the form:

$$\begin{tikzcd} & S^{k-1} \arrow[dl] \arrow[dr, "f"] & \\ \Sigma^k X \arrow[rr] & & Y \end{tikzcd}$$

and maps $X \to \Omega^k Y$. □

2. Cylinders

Cylinders should be thought as homotopies between cells that are not parallel, so that one needs to provide first homotopies between the 0-dimensional boundary, then between the 1-dimensional boundary adjusted using those homotopies, and so on. This is the right notion of natural transformation in this context.

Cylinders first appear in higher category theory in Bénabou's work on bicategories (see [**Ben**]). Later on, Lack makes use of them to construct a model structure on bicategories and on Gray-categories, see [**La**] and [**La2**]. These have later been generalized by Lafont, Métayer and Worytkiewicz in [**LMW**] to build a model structure on the category of strict ω-categories.

We give all the definitions in the ∞-dimensional case, leaving the appropriate modifications for the finite case to the interested reader. Also, in what follows $\mathfrak{A}$ is either a coherator for ∞-categories, a coherator for ∞-groupoids or a globular theory of the form $\mathfrak{A} \cong \mathfrak{C}^{\mathbf{W}}$.

Definition 5.5. We define, by induction on $n \in \mathbb{N}$, a coglobular object $\mathbf{Cyl}(D_\bullet) \in \mathbf{Mod}(\mathfrak{A})^{\mathbb{G}}$, together with a map

$$\iota = (\iota_0, \iota_1) \colon D_\bullet \amalg D_\bullet \to \mathbf{Cyl}(D_\bullet)$$

We begin by setting

$$\mathbf{Cyl}(D_0) = D_1,\ (\iota)_0 = (\sigma, \tau) \colon D_0 \amalg D_0 \to D_1$$

Now, let $n > 0$ and assume we have constructed

$$\mathbf{Cyl}(D_\bullet) \in \mathbf{Mod}(\mathfrak{A})^{\mathbb{G}_{\leq n-1}} \text{ and } \iota+\colon D_\bullet \amalg D_\bullet \to \mathbf{Cyl}(D_\bullet)$$

We then define $\mathbf{Cyl}(D_n)$ as the colimit in $\mathbf{Mod}(\mathfrak{A})$ of the following diagram:

$$\begin{tikzcd}
 & D_n \underset{D_0}{\amalg} D_1 \\
D_n \arrow[ur, "w"] \arrow[dr, "\Sigma(\iota_0)"'] & \\
 & \Sigma\mathbf{Cyl}(D_{n-1}) \\
D_n \arrow[ur, "\Sigma(\iota_1)"'] \arrow[dr, "w"'] & \\
 & D_1 \underset{D_0}{\amalg} D_n
\end{tikzcd} \tag{18}$$

Next, we define $\iota_0, \iota_1 \colon D_n \to \mathbf{Cyl}(D_n)$ respectively as the composites

$$D_n \xrightarrow{\iota} D_n \underset{D_0}{\amalg} D_1 \longrightarrow \mathbf{Cyl}(D_n)$$

$$D_n \xrightarrow{\iota} D_1 \underset{D_0}{\amalg} D_n \longrightarrow \mathbf{Cyl}(D_n)$$

where the unlabelled maps are given by the colimit inclusions.

Finally, for $\varepsilon = \sigma, \tau$, we construct the induced map $\mathbf{Cyl}(\varepsilon) \colon \mathbf{Cyl}(D_{n-1}) \to \mathbf{Cyl}(D_n)$ by induction. We define $\mathbf{Cyl}(\sigma), \mathbf{Cyl}(\tau) \colon \mathbf{Cyl}(D_0) \to \mathbf{Cyl}(D_1)$ respectively as the lower and upper composite maps

$$\begin{tikzcd}
 & & D_1 \underset{D_0}{\amalg} D_1 \arrow[ddrr, bend left] & & \\
 & D_1 \arrow[ur, "w"] \arrow[dr, "\Sigma(\iota_0)"'] & & & \\
\mathbf{Cyl}(D_0) \cong D_1 \arrow[uurr, bend left, "i_1"] \arrow[ddrr, bend right, "i_0"'] & & \Sigma\mathbf{Cyl}(D_0) & & \mathbf{Cyl}(D_1) \\
 & D_1 \arrow[ur, "\Sigma(\iota_1)"'] \arrow[dr, "w"'] & & & \\
 & & D_1 \underset{D_0}{\amalg} D_1 \arrow[uurr, bend right] & &
\end{tikzcd} \tag{19}$$

We then inductively define for $\varepsilon = \sigma, \tau$ the structural map $\mathbf{Cyl}(\varepsilon) \colon \mathbf{Cyl}(D_{n-1}) \to \mathbf{Cyl}(D_n)$ as the map induced on colimits by the following natural transformation

$$
\begin{tikzcd}
 & D_{n-1} \amalg_{D_0} D_1 \arrow[rr, "\varepsilon \amalg_{D_0} 1"] & & D_n \amalg_{D_0} D_1 \\
D_{n-1} \arrow[ur, "w"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Sigma(\iota_0)"'] & & D_n \arrow[ur, "w"] \arrow[dr, "\Sigma(\iota_0)"] & \\
 & \Sigma\mathbf{Cyl}(D_n - 2) \arrow[rr, "\Sigma\mathbf{Cyl}(\varepsilon)"] & & \Sigma\mathbf{Cyl}(D_n - 1) \\
D_{n-1} \arrow[ur, "\Sigma(\iota_1)"] \arrow[rr, "\varepsilon"] \arrow[dr, "w"'] & & D_n \arrow[ur, "\Sigma(\iota_1)"] \arrow[dr, "w"'] & \\
 & D_1 \amalg_{D_0} D_{n-1} \arrow[rr, "1 \amalg_{D_0} \varepsilon"] & & D_1 \amalg_{D_0} D_n
\end{tikzcd}
$$

Definition 5.6. Given an $\mathfrak{A}$-model X, an n-cylinder in X is a map $C : \mathbf{Cyl}(D_n) \to X$. We denote the source and target cylinders of C by, respectively

$$s(C) = C \circ \mathbf{Cyl}(\sigma), \;\; t(C) = C \circ \mathbf{Cyl}(\tau)$$

If $C \circ \iota_0 = A, C \circ \iota_1 = B$, then we write $C \colon A \curvearrowright B$.

By (18), an n-cylinder $C \colon A \curvearrowright B$ in an $\mathfrak{A}$-model X is given by a pair of 1-cells C_s, C_t in X and an $(n-1)$-cylinder $\bar{C} \colon C_t A \curvearrowright B C_s$ in $\Omega\,(X, s(C_s), t(C_t))$. C and $\bar{C}$ will often be referred to as mutually transpose. We will sometimes refer to the cell $C \circ \iota_0$ (resp. $C \circ \iota_1$) as the top (resp. bottom) cell of C, and denote it with C_0 (resp. C_1).

Example 5.7. A 2-cylinder $C \colon A \curvearrowright B$ in X consists of a pair of 1-cells $f = C_s, g = C_t$ and a 1-cylinder $\bar{C} \colon gA \curvearrowright Bf$ in $X\,(s(f), t(g))$. It can also be represented as the following data in X

$$
\begin{tikzcd}
s(f) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & s(g) \arrow[d, "g"] \arrow[r, phantom, "\Rrightarrow"] & s(f) \arrow[r, "s(A)"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & s(g) \arrow[d, "g"] \\
t(f) \arrow[r, "t(B)"] & t(g) & t(f) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow B"] & t(g)
\end{tikzcd}
$$

Or, in a way that better justifies its name, as

$$
\begin{tikzcd}
A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \;\; \Nwarrow \;\; \Downarrow"] & B \arrow[d, "g"] \\
A' \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow B"] & B'
\end{tikzcd}
\tag{20}
$$

where the front face is the square (i.e. 1-cylinder) given by $t(C)$, and the back one is $s(C)$.

We will often denote the source of the n-th latching map $\hat{L}_n(\iota)$ as $\partial\mathbf{Cyl}(D_n)$. This can be constructed as the following pushout:

$$\begin{tikzcd} S^{n-1} \amalg S^{n-1} \arrow[r, "(\iota_0,\iota_1)"] \arrow[d] & \mathbf{Cyl}(S^{n-1}) \arrow[d] \\ D_n \amalg D_n \arrow[r] & \partial\mathbf{Cyl}(D_n) \end{tikzcd} \tag{21}$$

Definition 5.8. We call $\partial\mathbf{Cyl}(D_n)$ the boundary of the n-cylinder. Given an n-cylinder in $C\colon \mathbf{Cyl}(D_n) \to X$, we call the boundary of C, denoted by ∂C, the following composite

$$\partial\mathbf{Cyl}(D_n) \rightarrowtail \mathbf{Cyl}(D_n) \xrightarrow{C} X$$

Thanks to (21), we know that specifying the boundary of an n-cylinder in an $\mathfrak{A}$-model X is equivalent to providing the following data:

- a pair of parallel $(n-1)$-cylinders $C\colon A \curvearrowright B, D\colon A' \curvearrowright B'$ in X;
- a pair of n-cells $\alpha\colon A \to A', \beta\colon B \to B'$ in X.

Let us now prove the following result:

Proposition 5.9. *The natural map*

$$\iota\colon D_\bullet \amalg D_\bullet \to \mathbf{Cyl}(D_\bullet)$$

is a direct cofibration in $\mathbf{Mod}(\mathfrak{A})^{\mathbb{G}}$ *(i.e. it belongs to the class* $\mathbb{I}^{\mathbb{G}}$ *according to the notation established in Lemma 3.13).*

Proof. We prove by induction on n that the latching map at n

$$\hat{L}_n(\iota)\colon \partial\mathbf{Cyl}(D_n) \to \mathbf{Cyl}(D_n)$$

fits into a cocartesian square of the form

$$\begin{tikzcd} S^n \arrow[r, "\partial"] \arrow[d] & D_{n+1} \arrow[d] \\ \partial\mathbf{Cyl}(D_n) \arrow[r, "\hat{L}_n(\iota_0,\iota_1)"] & \mathbf{Cyl}(D_n) \end{tikzcd} \tag{22}$$

and is therefore in $\mathbb{I}$. Observe that the statement is trivially true by definition if $n = 0$, so we assume $n > 0$ and its validity for every $k < n$. In fact, we are going to prove that there is a pushout square of the form

$$\begin{tikzcd} \Sigma\partial\mathbf{Cyl}(D_{n-1}) \arrow[r, "\Sigma(\hat{L}_{n-1}(\iota))"] \arrow[d] & \Sigma\mathbf{Cyl}(D_{n-1}) \arrow[d] \\ \partial\mathbf{Cyl}(D_n) \arrow[r, "\hat{L}_n(\iota)"] & \mathbf{Cyl}(D_n) \end{tikzcd} \tag{23}$$

and then conclude by applying the inductive hypothesis.

We prove this representably, i.e. we have to prove that, given n-cells A, B in X and a pair of parallel $(n-1)$-cylinders in X of the form $C^1\colon s(A) \curvearrowright s(B), C^2\colon t(A) \curvearrowright t(B)$ (i.e. a map $\partial\mathbf{Cyl}(D_n) \to X$, as observed in Definition 5.8), together with an $(n-1)$-cylinder

$C'\colon C^2_t A \curvearrowright BC^1_s$ in $X\,(s^n(A), t^n(B))$, there exists a unique n-cylinder $C\colon A \curvearrowright B$ in X with $\overline{C} = C', s(C) = C^1, t(C) = C^2$ provided $s(C') = \overline{C^1}, t(C') = \overline{C^2}$. This fact is an easy consequence of Definition 5.5. $\square$

We can define a map of coglobular $\mathfrak{A}$-models $\mathbf{C}_\bullet\colon \mathbf{Cyl}(D_\bullet) \to D_\bullet$ that fits into the following factorization of the codiagonal map

$$D_\bullet \amalg D_\bullet \xrightarrow{(\iota_0,\iota_1)} \mathbf{Cyl}(D_\bullet) \xrightarrow{\mathbf{C}_\bullet} D_\bullet$$

by solving the extension problem

$$\begin{tikzcd} D_\bullet \amalg D_\bullet \arrow[r, "\nabla"] \arrow[d, tail, "(\iota_0,\iota_1)"'] & D_\bullet \\ \mathbf{Cyl}(D_\bullet) \arrow[ur, dashed, "\mathbf{C}_\bullet"'] & \end{tikzcd} \tag{24}$$

This is achieved, in the case of ∞-groupoids, simply by contractibility of the coglobular object $D_\bullet$. In the case of ∞-categories, it is enough to apply Lemma 3.18, knowing that such an extension exists in the case of ω-$\mathcal{C}at$ since the n-th latching map of the cofibration on the left is a pushout of a boundary inclusion $S^n \to D_{n+1}$ and $\mathrm{ht}(D_n) = n < n+1$, so that we conclude thanks to Lemma 3.19.

In the remaining case $\mathfrak{A} \cong \mathfrak{C}^{\mathbf{W}}$, if we let $F\colon \mathbf{Mod}(\mathfrak{C}) \to \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$ be the left adjoint to the obvious restriction functor, then we can set

$$\mathbf{C}_\bullet \overset{def}{=} F\,(\mathbf{C}_\bullet)$$

since F preserves globes and cylinders on globes.

CHAPTER 6

Systems of structure on the path object

In this chapter we define the underlying globular set of our candidate for the path-object associated with an $\mathfrak{A}$-model X, and we endow it with a non-trivial algebraic structure that depends on the choice of $\mathfrak{A}$. This can be either a coherator for n-categories, a coherator for n-groupoids or a coherator of the form $\mathfrak{C}^{\mathbf{W}}$, unless specified otherwise. We will indicate the last two cases by saying that $\mathfrak{A}$ "admits inverses". Also, n can be any natural number or $n = \infty$: we only provide proofs for the latter case, the other being simple adaptations of it.

Definition 6.1. We define a functor $\mathbb{P}\colon \mathbf{Mod}(\mathfrak{A}) \to [\mathbb{G}^{op}, \mathbf{Set}]$ by setting:

$$\mathbb{P}X = \mathbf{Mod}(\mathfrak{A})(\mathbf{Cyl}(D_\bullet), X)$$

where the globular structure is induced by the coglobular object $\mathbf{Cyl}(D_\bullet)\colon \mathbb{G} \to \mathbf{Mod}(\mathfrak{A})$.

Precomposition with $\iota\colon D_\bullet \amalg D_\bullet \to \mathbf{Cyl}(D_\bullet)$ yields a natural map

$$p_X = (p_0, p_1)\colon \mathbb{P}X \to \mathbf{U}X \times \mathbf{U}X$$

where $\mathbf{U}\colon \mathbf{Mod}(\mathfrak{A}) \to [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ is the forgetful functor induced by precomposition with the structural map $\Theta_0 \to \mathfrak{A}$.

In what follows, we will need a construction representing the composition of an n-cylinder with a pair of $(n+1)$-cells attached at the top and bottom, respectively. This is an instance of a more general operation of vertical composition of degenerate cylinders, that will be defined in Section 5.2 of Chapter 7. Unless specified otherwise, $\mathfrak{A}$ is a coherator of any of the abovementioned kinds.

In the proof of Lemma 6.5 and of Proposition 6.6 we will make use of Lemma 4.12 in [**Ar2**], the so-called "division lemma", which we now state for sake of clarity. We record here the original statement in Ara's paper, although we observe that it can easily be adapted to the finite-dimensional case.

Lemma 6.2 (Division lemma). *The operation of whiskering n-cells with a given 1-cell in a coherator for ∞-groupoids is bijective up to homotopy. More precisely, suppose given a pair of n-cells A, B and a 1-cell γ in an ∞-groupoid X, satisfying $t^n(A) = t^n(B) = s(\gamma)$. Then, given an $(n+1)$-cell of the form $H\colon \gamma A \to \gamma B$ there exists an $(n+1)$-cell of the form $\overline{H}\colon A \to B$ such that $\gamma\overline{H} \simeq H$, where juxtaposition is the result of a whiskering operation in $\mathfrak{G}$ (for example, $w_n\colon D_n \to D_n \coprod_{D_0} D_1$).*

Ara's proof requires contractibility, and we were not able to generalize it to $\mathfrak{C}^{\mathbf{W}}$ (as defined in Definition 12) in the case where $\mathfrak{C}$ is a coherator for ∞-categories. The three dimensional case can still be proven by hands, as follows. Note that, in the presence of both a left and a right inverse for every cell, any of them can be promoted to a two-sided inverse, therefore we will use the notation f^{-1} with no reference to left or right.

Lemma 6.3. *The division lemma holds true in the context of $\mathfrak{C}^{\mathbf{W}}$-models, where $\mathfrak{C}$ denotes a coherator for 3-categories.*

Proof. If $n = 1$ and we have a 2-cell in X of the form:

$$\begin{tikzcd} a \arrow[r, bend left, "fA"] \arrow[r, bend right, "fB"'] \arrow[r, phantom, "H\Downarrow"] & c \end{tikzcd}$$

for $A, B\colon a \to b$ and $f\colon b \to c$, then we can define $\overline{H}$ as the following composite:

$$A \xrightarrow{\simeq} f^{-1}fA \xrightarrow{f^{-1}H} f^{-1}fB \xrightarrow{\simeq} B$$

where "$\simeq$" denotes coherence constraints that exist in $\mathfrak{C}^{\mathbf{W}}$. It is a routine exercise to check that $f\overline{H}$ is homotopic to H.

Turning to $n = 2$, we assume we have a 3-cell $H\colon fA \to fB$. We define $\overline{H}\colon A \to B$ as the following composite of 3-cells:

$$\begin{tikzcd} a \arrow[r, bend left, "s(A)"] \arrow[r, bend right, "t(A)"'] \arrow[r, phantom, "A\Downarrow"] & b \end{tikzcd}
\Rrightarrow
\begin{tikzcd} a \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "A\Downarrow"] \arrow[rrr, bend left=70, "s(A)"] \arrow[rrr, bend right=70, "t(A)"'] \arrow[rr, phantom, bend left=45, "\simeq\Downarrow"] \arrow[rr, phantom, bend right=45, "\simeq\Downarrow"] & b \arrow[r, "f"] & c \arrow[r, "f^{-1}"] & b \end{tikzcd}
\Rrightarrow
\begin{tikzcd} a \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "B\Downarrow"] \arrow[rrr, bend left=70, "s(B)"] \arrow[rrr, bend right=70, "t(B)"'] \arrow[rr, phantom, bend left=45, "\simeq\Downarrow"] \arrow[rr, phantom, bend right=45, "\simeq\Downarrow"] & b \arrow[r, "f"] & c \arrow[r, "f^{-1}"] & b \end{tikzcd}
\Rrightarrow
\begin{tikzcd} a \arrow[r, bend left, "s(B)"] \arrow[r, bend right, "t(B)"'] \arrow[r, phantom, "B\Downarrow"] & b \end{tikzcd}$$

Here, the 2-cells denoted with "$\simeq$" denote coherence constraints that exist in $\mathfrak{C}^{\mathbf{W}}$, and the first and last 3-cells are also composite of constraints, whereas the one in the middle is a whiskering of H with the other cells depicted there. Again, it is a tedious but straightforward exercise to check that $f\overline{H}$ is homotopic (i.e. equal,for dimensionality reasons) to $\overline{H}$.

Finally, if $n = 3$ we have to prove that $fA = fB$ implies $A = B$, which is entirely analogous to the arguments given so far. □

Lemma 6.4. *Given an $\mathfrak{A}$-model X, an n-cylinder $C\colon A \curvearrowright B$ in X and $(n+1)$-cells $\alpha\colon A' \to A$ and $\beta\colon B \to B'$ we can compose these data to get an n-cylinder $\beta C\alpha\colon A' \curvearrowright B'$. Moreover, $\varepsilon(\beta C\alpha) = \varepsilon(C)$ for $\varepsilon = s, t$.*

Proof. We prove this by induction on n, the case $n = 0$ being straightforward. Let's assume $n > 0$ and that we have already defined this operation for every $k < n$. We can transpose the data at hand to get an $(n-1)$-cylinder $\overline{C}\colon C_tA \curvearrowright BC_s$ in $X\,(s^n(A), t^n(B))$ and n-cells $C_t\alpha\colon C_tA' \to C_tA$, $\beta C_s\colon BC_s \to B'C_s$, where juxtaposition denotes the result of composing using the whiskering w's defined in 2.12. By inductive hypothesis we can compose these data to get an $(n-1)$-cylinder $(\beta C_s)\overline{C}(\alpha C_t)\colon C_tA' \curvearrowright B'C_s$. Finally, we define $(\beta C\alpha)_\varepsilon = C_\varepsilon$ for $\varepsilon = s, t$, and $\overline{\beta C\alpha} = (\beta C_s)\overline{C}(\alpha C_t)$.

The statement on source and target cylinders follows easily from the inductive argument we have just outlined. □

This operation also comes endowed with a "comparison cylinder", as explained in the following result.

Lemma 6.5. *In the situation of the previous lemma, if $\mathfrak{A}$ admits inverses and the division lemma holds for $\mathfrak{A}$-models (see Lemma 6.2) then there exists an $(n+1)$-cylinder $\Gamma_{\beta,C,\alpha}$ in X such that $s(\Gamma_{\beta,C,\alpha}) = C$ and $t(\Gamma_{\beta,C,\alpha}) = \beta C\alpha$.*

Proof. For sake of simplicity we drop the subscripts of Γ in what follows. We prove this result by induction on n. The base case $n = 0$ is straightforward, once we set $\Gamma_0 = \alpha^{-1}$ and $\Gamma_1 = \beta$, where $(\cdot)^{-1}$ is a (left or right) inverse promoted to a two-sided one. Let $n > 0$, and assume the result holds for each $k < n$. By inductive hypothesis we get an n-cylinder $\gamma\colon \overline{C} \curvearrowright (C_t\alpha)\overline{C}(\beta C_s)$. If we analyze the source and target of γ_0 and γ_1, we see that, thanks to Lemma 6.2, there exist a pair of n-cells E, F and $(n+1)$-cells $\vartheta\colon C_tE \to \gamma_0, \varphi\colon \gamma_1 \to FC_s$. We now define $\Gamma_\varepsilon = C_\varepsilon$ for $\varepsilon = s, t$, and $\overline{\Gamma} = \varphi\gamma\vartheta$, which concludes the proof. □

Recall that a map of globular sets $f\colon X \to Y$ is a fibration (resp. trivial fibration) if it has the right lifting property with respect to the set of maps $\{\sigma_n\colon D_n \to D_{n+1}\}_{n\geq 0}$ (resp. $\{S^{n-1} \to D_n\}_{n\geq 0}$).

Proposition 6.6. *Let $\mathfrak{A} = \mathfrak{G}$ be a coherator for ∞-groupoids, and X a $\mathfrak{G}$-model. The map $p_X = (p_0, p_1)\colon \mathbb{P}X \to \mathbf{U}X \times \mathbf{U}X$ (resp. $p_i\colon \mathbb{P}X \to X$ for $i = 0, 1$) is a fibration (resp. trivial fibration) of globular sets.*

Proof. Let us first prove the claim about p_X. We have to prove it lifts against maps of the form $\sigma_n\colon D_n \to D_{n+1}$ for $n \geq 0$. This is equivalent to saying that given $(n+1)$-cells A, B in X and an n-cylinder in X of the form $C\colon s(A) \curvearrowright s(B)$ we can always extend C to $C'\colon A \curvearrowright B$, so that $s(C') = C$.

We start by composing C with A^{-1} on the top and B at the bottom, using Lemma 6.4, to get a cylinder $D \overset{def}{=} BCA^{-1}$. Observe that, thanks to Lemma 6.5, there is an $(n+1)$-cylinder Γ whose source is C and whose target is D. Moreover, thanks to the proof of the same lemma, we see that $\Gamma_1 = B$ and Γ_0 is homotopic to $(A^{-1})^{-1}$. Therefore we can compose Γ with an $(n+1)$-cell χ in X witnessing the coherence constraint $A \simeq (A^{-1})^{-1}$, to get an $(n+1)$-cylinder $\Delta \overset{def}{=} \Gamma\chi$ that satisfies the desired properties.

We now prove that the map $p_0\colon \mathbb{P}X \to \mathbf{U}X$ is a trivial fibration, the other case being entirely similar. This amounts to prove it lifts against all the maps of the form $S^{n-1} \to D_n$. The case $n = 0$ is equivalent to proving that, for every given 0-cell $x \in X_0$ we can find a 0-cylinder $C\colon \mathbf{Cyl}(D_0) \to X$, i.e. a 1-cell in X, such that its source is precisely x. A possible solution is to take the trivial cylinder on x, i.e. $x \circ \mathbf{C}_0$.

If $n = 1$, we are given 1-cells C, D and γ, and we have to extend this to a 1-cylinder $\Gamma\colon \gamma \curvearrowright \delta$, whose source and target are, respectively, C and D. If we set $\delta = D\gamma C^{-1}$ (the meaning of $(\cdot)^{-1}$ being the same as above) then we are left with providing a 2-cell $\overline{\Gamma}\colon (D\gamma C^{-1})C \to D\gamma$, which surely exists thanks to the structure of $\mathfrak{G}$.

Now let $n > 1$, and assume we have a pair of parallel $(n-1)$-cylinders (C, D) in X, i.e. $\varepsilon(C) = \varepsilon(D)$ for $\varepsilon = s, t$, together with an n-cell $\Gamma\colon C_0 \to D_0$. Notice that, in particular, we have that C_ε and D_ε are parallel for $\varepsilon = 0, 1$. These data transpose to give a pair of parallel $(n-2)$-cylinders $(\bar{C}, \bar{D})$ in $X\,(s(C_s), t(C_t))$. Moreover, we also get an $(n-1)$-cell $C_t\Gamma\colon \bar{C}_0 = C_t(C_0) \to \bar{D}_0 = C_t(D_0)$ in $X\,(s(C_s), t(C_t))$.

By inductive hypothesis we thus get an $(n-1)$-cylinder $\chi\colon C_t\Gamma \curvearrowright \varepsilon$. By construction, the source (resp. target) of ε are of the form $(\bar{C}_1)C_s$ (resp. $(\bar{D}_1)C_s$). Thanks to Lemma 6.2, we see that there exists an $(n-1)$-cell $\Delta\colon \bar{C}_1 \to \bar{D}_1$ and an n-cell $\varepsilon \to \Delta C_s$ in $X\,(s(C_s), t(C_t))$. We can compose these data with χ using Lemma 6.4, getting an $(n-1)$-cylinder $\bar{C}'\colon C_t\Gamma \curvearrowright \Delta C_s$ in $X\,(s(C_s), t(C_s))$, which transposes to give the desired cylinder $C'\colon \Gamma \curvearrowright \Delta$ in X, having C as source and D as target. $\square$

Since the division lemma is the only non-trivial obstruction to the generalization of the previous result to the case of $\mathfrak{C}^{\mathbf{W}}$-models, we have the following corollary.

Corollary 6.7. *Let $\mathfrak{C}$ be a coherator for n-categories (with $0 \leq n \leq \infty$) such that the division lemma holds for $\mathfrak{C}^{\mathbf{W}}$-models. Then, for every $\mathfrak{C}^{\mathbf{W}}$-model X, we have that the map $p_X = (p_0, p_1)\colon \mathbb{P}X \to X \times X$ (resp. $p_i\colon \mathbb{P}X \to X$ for $i = 0, 1$) is a fibration (resp. trivial fibration) of globular sets. In particular, thanks to Lemma 6.3, this holds for $n \leq 3$.*

The factorization of the codiagonal map:

$$D_\bullet \amalg D_\bullet \overset{(\iota_0,\iota_1)}{\rightarrowtail} \mathbf{Cyl}(D_\bullet) \xrightarrow{\mathbf{C}_\bullet} D_\bullet$$

induces, by applying the functor $\mathbf{Mod}(\mathfrak{A})(\bullet, X)$, a factorization of the diagonal map in $[\mathbb{G}^{op}, \mathbf{Set}]$ of the form

$$\mathbf{U}X \overset{c}{\rightarrowtail} \mathbb{P}X \overset{p}{\twoheadrightarrow} \mathbf{U}(X \times X) \cong \mathbf{U}X \times \mathbf{U}X$$

It seems unlikely that any sensible structure of $\mathfrak{A}$-model on the globular set $\mathbb{P}X$ will make c into a (strict) map of $\mathfrak{A}$-models, and this is the reason for the choice of working with semi-model structures.

Let us now endow $\mathbb{P}X$ with some algebraic structure. More precisely, we will show how to endow it with a system of compositions if X is an n-category (i.e. if $\mathfrak{A}$ is a coherator for n-categories), and we will endow it with a system of identities and a system of inverses too if $\mathfrak{A}$ is a coherator for n-groupoids or is of the form $\mathfrak{C}^{\mathbf{W}}$. We refer the reader to Definition 2.21 for the concepts here involved.

To make this more precise, we consider $\Theta_0[\mathbf{comp}]$ in Definition 2.22, i.e. the globular theory freely generated by a system of composition and a system of identities, and, similarly, we consider $\Theta_0[\mathbf{comp}, \mathbf{id}, \mathbf{inv}]$, i.e. the free globular theory on a system of compositions, identities and inverses. We want to construct an extension of the form:

$$\begin{tikzcd}
\Theta_0 \arrow[r, "\mathbf{Cyl}"] \arrow[d] & \mathbf{Mod}(\mathfrak{A}) \\
\Theta_0[\mathbf{comp}, \mathbf{id}, \mathbf{inv}] \arrow[ur, dashed, "\mathbf{Cyl}"'] &
\end{tikzcd}$$

if $\mathfrak{A}$ admits inverses, and a similar extension with $\Theta_0[\mathbf{comp}]$ in place of $\Theta_0[\mathbf{comp}, \mathbf{id}, \mathbf{inv}]$ if $\mathfrak{A}$ is a coherator for n-categories. In turn, this is equivalent to defining interpretations $\mathbf{Cyl}(f)$ for each of the generators f of $\Theta_0[\mathbf{comp}, \mathbf{id}, \mathbf{inv}]$ (or $\Theta_0[\mathbf{comp}]$ in the categorical case), satisfying the appropriate equations.

Theorem 6.8. *Let $\mathfrak{A}$ be a coherator for n-categories, then the functor $\mathbf{Cyl}\colon \Theta_0 \to \mathbf{Mod}(\mathfrak{A})$ admits an extension to $\Theta_0[\mathbf{comp}]$. If $\mathfrak{A}$ admits inverses and the division lemma (see Remark 6.2) holds for $\mathfrak{A}$-models, then it can be extended to $\Theta_0[\mathbf{comp}, \mathbf{id}, \mathbf{inv}]$.*

Equivalently, given any $\mathfrak{A}$-model X, the globular set $\mathbb{P}X$ can be endowed with said systems of structures.

We observe here that the techniques employed in the proof of this theorem do not seem to provide (or at least not as easily as in this context) a system of identities in the case of Grothendieck categories. One would need, in particular, a version of the division lemma for invertible cells in Grothendieck categories.

The proof of Theorem 6.8 will be subdivided into several lemmas, the first one addressing the system of compositions. In what follows, we will use the expression "the structure of the globular sum A" to mean the relative lifting properties of A in $\mathfrak{A}$ against boundary inclusions, where $\mathfrak{A}$ is a coherator for categories, as described in Section 3, or its contractibility in the groupoidal case. Moreover, in all the proofs we argue representably, i.e. we assume given, say, a pair of composable n-cylinders $C, D\colon \mathbf{Cyl}(D_n) \to X$ and we construct their composite n-cylinder $D \circ C$. These constructions are easily seen to be natural in maps of $\mathfrak{A}$-models $f\colon X \to Y$, so that we get the desired extension thanks to the Yoneda lemma.

Lemma 6.9. *Let $\Theta_0[\mathbf{comp}]$ be the globular theory freely generated by a system of compositions. Then there exists an extension of the form:*

$$\begin{tikzcd} \Theta_0 \arrow[r, "\mathbf{Cyl}"] \arrow[d] & \mathbf{Mod}(\mathfrak{A}) \\ \Theta_0[\mathbf{comp}] \arrow[ur, dashed, "\mathbf{Cyl}"'] & \end{tikzcd}$$

Proof. We define this extension by induction. Firstly, we need to define $\mathbf{Cyl(c}_1)$. Given a pair of composable 1-cylinders C, C' in X, we denote $\mathbb{P}X(\mathbf{c}_1)(C, C')$ with $C' \circ_{\mathbf{c}_1} C$, and define the top cell (resp. bottom cell) of it to be $C'_0 \circ C_0$ (resp. $C'_1 \circ C_1$), composed using c_1. We then set $(C' \circ_{\mathbf{c}_1} C)_s = C_s$ and $(C' \circ_{\mathbf{c}_1} C)_t = C'_t$ and declare the 2-cell $\overline{C' \circ_{\mathbf{c}_1} C}$ (i.e. a 0-cylinder in $X\,(s(C_0), t(C'_1))$) to be the composite of:

$$C'_t(C'_0C_0) \xrightarrow{\simeq} (C'_tC'_0)C_0 \xrightarrow{\overline{C'}C_0} (C'_1C'_s)C_0 \xrightarrow{\simeq} C'_1(C'_sC_0) \xrightarrow{C'_1\overline{C}} C'_1(C_1C_s) \xrightarrow{\simeq} (C'_1C_1)C_s$$

where we have used the fact that $C'_s = C_t$, and we have denoted instances of associativity of composition of 1-cells with "$\simeq$" and the effect of composing using c_1 with juxtaposition.

Given $n > 1$, suppose we have defined $\mathbf{Cyl(c}_k)$ for each $k < n$, and denote $(F, G) \circ \mathbf{Cyl(c}_k)$ by $G \circ_{\mathbf{c}_k} F$ for each composable pair of k-cylinders in an $\mathfrak{A}$-model X. For every $\mathfrak{A}$-model X and every pair of n-cylinders $F, G\colon \mathbf{Cyl}(D_n) \to X$ such that $t(f) = s(G)$, we define $\overline{G \circ_{c_n} F}$ to be the following composite, obtained applying Lemma 6.4 to the following data in $X\,(s^n(F_0), t^n(F_1))$:

$$\begin{tikzcd} F_t(G_0F_0) \arrow[d, "D_1"'] \\ (F_tG_0)(F_tF_0) \arrow[d, bend left, "\overline{G}\circ_{\mathbf{c}_{n-1}}\overline{F}"'] \\ (G_1F_s)(F_1F_s) \arrow[d, "D_2"'] \\ (G_1F_1)F_s \end{tikzcd}$$

Here, D_1 is an n-cell obtained by using the structure of the globular sum $D_n \underset{D_{n-1}}{\amalg} D_n \underset{D_0}{\amalg} D_1$, and D_2 is defined similarly. This assignment defines, by the Yoneda lemma, a map

$$\mathbf{Cyl}(D_n) \to \mathbf{Cyl}(D_n) \underset{\mathbf{Cyl}(D_{n-1})}{\amalg} \mathbf{Cyl}(D_n)$$

which we take as the definition of $\mathbf{Cyl}(c_n)$. We also have the following chain of equalities, provided by the inductive hypothesis together with Lemma 6.4

$$\overline{s(G \circ_{\mathbf{c_n}} F)} = s(\overline{G \circ_{\mathbf{c_n}} F}) = s(\overline{G} \circ_{\mathbf{c_{n-1}}} \overline{F}) = s(\overline{F}) = \overline{s(F)}$$

and

$$(s(G \circ_{\mathbf{c_n}} F))_\varepsilon = s(F_\varepsilon)$$

for $\varepsilon = s, t$, which imply that $s(G \circ_{\mathbf{c_n}} F) = s(F)$, and a similar argument can be provided for the target. $\square$

Remark 6.10. It follows from Lemma 6.5 that if $\mathfrak{A}$ admits inverses, then there exists an n-cylinder $T_{F,G}$ in $X\,(s(F_s), t(G_t))$ such that $s(T_{F,G}) = \overline{G \circ_{\mathbf{c}_n} F}$ and $t(T_{F,G}) = \overline{G} \circ_{\mathbf{c}_{n-1}} \overline{F}$.

We now address the problem of definining a system of identities.

Lemma 6.11. *Let $\Theta_0[\mathbf{comp}, \mathbf{id}]$ be the globular theory freely generated by a system of compositions and identities. If $\mathfrak{A}$ admits inverses and the division lemma holds for $\mathfrak{A}$-models there exists an extension of the form:*

$$\begin{tikzcd} \Theta_0 \arrow[r, "\mathbf{Cyl}"] \arrow[d] & \mathbf{Mod}(\mathfrak{A}) \\ \Theta_0[\mathbf{comp}, \mathbf{id}] \arrow[ur, dashed, "\mathbf{Cyl}"'] & \end{tikzcd}$$

Proof. We only need to define a system of identities. Firstly, set

$$\mathbf{Cyl}(\mathbf{id}_0) = \mathbf{C}_1 \colon \mathbf{Cyl}(D_1) \to D_1 = \mathbf{Cyl}(D_0)$$

as defined in (24). Let $n > 1$, and assume we have already defined $\mathbf{Cyl}(\mathbf{id}_k)$ for each $k < n$. Given $F \colon \mathbf{Cyl}(D_k) \to X$, denote the $(k+1)$-cylinder $F \circ \mathbf{Cyl}(\mathbf{id}_k)$ by $\mathbf{id}_k(F)$. We have to define, for every $\mathfrak{A}$-model X and every n-cylinder $F \colon A \curvearrowright B$ in X, an $(n+1)$-cylinder $\mathbf{id}_n(F) \colon \mathbf{Cyl}(D_{n+1}) \to X$. Define its transpose $\overline{\mathbf{id}_n(F)}$ as the vertical composite of the following diagram:

$$\begin{tikzcd} F_t 1_A \arrow[d, "C_1"'] \\ 1_{F_t A} \arrow[d, bend left, "\mathbf{id}_{n-1}(\overline{F})"'] \\ 1_{B F_s} \arrow[d, "C_2"'] \\ 1_B F_s \end{tikzcd}$$

Here, juxtaposition of cells indicates, as usual, the whiskering operations w introduced in Definition 2.12, C_1 and C_2 are n-cells provided by the structure of the globular sums $D_n \amalg_{D_0}$

D_1 and $D_1 \amalg_{D_0} D_n$ respectively, and the composition operation is the one defined in Lemma 6.4.

Having defined identities and binary compositions, we can construct whiskering maps

$$*_k\colon \mathbf{Cyl}(D_k) \to \mathbf{Cyl}(D_k) \coprod_{\mathbf{Cyl}(D_{k-2})} \mathbf{Cyl}(D_{k-1})$$

$$_k*\colon \mathbf{Cyl}(D_k) \to \mathbf{Cyl}(D_{k-1}) \coprod_{\mathbf{Cyl}(D_{k-2})} \mathbf{Cyl}(D_k)$$

by setting

$$*_k = (1 \coprod_{\mathbf{Cyl}(D_{k-2})} \mathbf{Cyl}(\mathbf{id}_{k-1})) \circ \mathbf{Cyl}(\mathbf{c}_k) \tag{25}$$

and

$$_k* = (\mathbf{Cyl}(\mathbf{id}_{k-1}) \coprod_{\mathbf{Cyl}(D_{k-2})} 1) \circ \mathbf{Cyl}(\mathbf{c}_k) \tag{26}$$

When no confusion arises, subscripts will be dropped.

Remark 6.12. Thanks to Lemma 6.5, if $\mathfrak{A}$ admits inverses then for every k-cylinder F in X, there exists a $(k+1)$-cylinder λ_F such that:

$$s(\lambda_F) = \overline{\mathbf{id}_k(F)} \text{ and } t(\lambda_F) = \mathbf{id}_{k-1}(\overline{F})$$

These identity cylinders satisfy the required properties thanks to Lemma 6.4, so we are left with defining the action of $\mathbf{Cyl}$ on the maps $\mathbf{l}_n, \mathbf{r}_n$. We will only construct the $\mathbf{l}_n$'s, the other case being similar. The construction of $\mathbf{l}_2$ reduces to definining a 2-cylinder Γ whose source is $C \circ_{\mathbf{c}_1} \mathbf{id}_0(s(C))$ and whose target is C. We use the structure of $\mathfrak{A}$ once to find a pair of 2-cells $\Gamma_0\colon C_0 1_{s(C_0)} \to C_0$, $\Gamma_1\colon C_1 1_{s(C_1)} \to C_1$, and then again to choose a 3-cell between the following composites:

$$C_t(C_0 1_{s(C_0)}) \xrightarrow{C_t\Gamma_0} C_tC_0 \xrightarrow{\quad\overline{C}\quad} C_1C_s$$

$$C_t(C_0 1_{s(C_0)}) \xrightarrow{\overline{C\circ_{\mathbf{c}_1}\mathbf{id}_0(s(C))}} (C_1 1_{s(C_1)})C_s \xrightarrow{\Gamma_1 C_s} C_1C_s$$

In detail, this is obtained by finding a filler for the following pasting diagram (using the structure of $\mathfrak{A}$ and its relative lifting properties):

$$\begin{tikzcd}
C_t(C_0 1_{s(C_0)}) \arrow[r, "C_t\Gamma_0"] \arrow[d, "\simeq"'] & C_tC_0 \arrow[r, "\overline{C}"] & C_1C_s & C_1(1_{t(C_s)}C_s) \arrow[l, "\simeq"'] \\
(C_tC_0)1_{s(C_0)} \arrow[r, "\overline{C}1_{s(C_0)}"'] \arrow[ur, "\simeq"] & (C_1C_s)1_{s(C_0)} \arrow[r, "\simeq"'] \arrow[ur, "\simeq"] & C_1(C_s 1_{s(C_0)}) \arrow[ur, "C_1\overline{\mathbf{id}_0(s(C))}"']
\end{tikzcd}$$

Given an $(n-1)$-cylinder $F\colon A \curvearrowright B$ in X, we let $\alpha\colon A \circ 1_{s(A)} \to A$ be an instance of unitality of composition in $\mathfrak{A}$, and thanks to the structure of $D_{n-1} \amalg_{D_0} D_1$ we get an n-cell E_1 whose source is $F_t\alpha$ and whose target is:

$$(\mathbf{l}_{n-2}(\overline{F}) \circ_{\mathbf{c}_{n-1}} ((\overline{F} * \lambda_{s(F)}) \circ_{\mathbf{c}_{n-1}} T_{1_{s(F)},F}))_0$$

where the notation we use was introduced in Remark 6.12 and 6.10, and in (25) and (26). We observe that, by definition, this coincides with the composite:

$$F_t\left(A1_{s(A)}\right) \overset{\simeq}{\to} (F_tA)\left(F_t1_{s(A)}\right) \overset{\simeq}{\to} (F_tA)\left(1_{F_tA}\right) \overset{\simeq}{\to} F_tA$$

where each one of the displayed cells is obtained by making use of the structure of $\mathfrak{A}$. Now, we define $\overline{\mathbf{l}_{n-1}(F)}$ to be the following composite:

$$\begin{tikzcd}
F_t\alpha \arrow[d, "E_1"'] \\
(\mathbf{l}_{n-2}(\overline{F}) \circ_{\mathbf{c}_{n-1}} (\overline{F} * \lambda_{s(F)}) \circ_{\mathbf{c}_{n-1}} T_{1_{s(F)},F})_0 \arrow[d, bend left, "(\mathbf{l}_{n-2}(\overline{F})\circ_{\mathbf{c}_{n-1}}(\overline{F}*\lambda_{s(F)})\circ_{\mathbf{c}_{n-1}}T_{1_{s(F)},F})"'] \\
(\mathbf{l}_{n-2}(\overline{F}) \circ_{\mathbf{c}_{n-1}} (\overline{F} * \lambda_{s(F)}) \circ_{\mathbf{c}_{n-1}} T_{1_{s(F)},F})_1 \arrow[d, "E_2"'] \\
\alpha' F_s
\end{tikzcd}$$

Here, $\alpha' \colon B \circ 1_{s(B)} \to B$ is another instance of unitality of composition in $\mathfrak{A}$. E_2 is obtained similarly to E_1, and we compose the diagram using Lemma 6.4. $\square$

We are now ready to conclude the proof of Theorem 6.8:

Proof. The only thing left to define is a system of inverses on $\mathbb{P}X$ under the appropriate hypotheses, and again we do so by induction. Also, we interpret the left and the right inverse in the same manner, as follows. Given an $\mathfrak{A}$-model X and a 1-cylinder $F \colon A \curvearrowright B$ in X, we define $\overline{\mathbf{i}_1(F)}$ as the composite:

$$F_sA^{-1} \longrightarrow B^{-1}BF_sA^{-1} \xrightarrow{B^{-1}(\overline{F})^{-1}A^{-1}} B^{-1}F_tAA^{-1} \longrightarrow B^{-1}F_t$$

where the unlabelled cells are obtained by making use of the structure of $\mathfrak{A}$ and $()^{-1}$ is the action of taking the inverse of a given cell, encoded by promoting the (left or right) inverse operation of the globular theory $\mathfrak{A}$ to a two-sided one.

As before, we can construct $\mathbf{k}_2^s, \mathbf{k}_2^t$ by hands, using the structure of $\mathfrak{A}$, which concludes the proof of the base case.

To address the inductive step, assume given an n-cylinder $F \colon A \curvearrowright B$ in X with $n > 1$. We define $\overline{\mathbf{i}_n(F)}$ as the composite of the following diagram, obtained using Lemma 6.4:

$$\begin{tikzcd}
F_tA^{-1} \arrow[d, "M_1"'] \\
(F_tA)^{-1} \arrow[d, bend left, "\mathbf{i}_{n-1}(\overline{F})"'] \\
(BF_s)^{-1} \arrow[d, "M_2"'] \\
B^{-1}F_s
\end{tikzcd}$$

Here, M_1 and M_2 are n-cells obtained thanks to the structure of $\mathfrak{A}$. Again, observe that it follows from Lemma 6.5 that there exists a cylinder μ_F whose source is $\overline{\mathbf{i}_n(F)}$ and whose target is $\mathbf{i}_{n-1}(\overline{F})$.

We are now left with constructing $\mathbf{k}^{\varepsilon}_{n+1}$ for $\varepsilon = s, t$. The two cases being similar, we only construct $\mathbf{k}^{s}_{n+1}$. Let $\beta\colon AA^{-1} \to 1_{t(A)}$ be an instance of a coherence constraint for inverses, provided by the structure of $\mathfrak{A}$. The latter also provides a cell H_1, whose source is $F_t\beta$ and whose target is given by:

$$(((\mathbf{i}_n(\lambda_F)) \circ_{\mathbf{c}_n} (\mathbf{k}^s_{n-1}(\overline{F})) \circ_{\mathbf{c}_n} (\mu_F * \overline{F})) \circ_{\mathbf{c}_n} (T_{F^{-1},F}))_0$$

which by definition corresponds the following composite:

$$F_t(AA^{-1}) \xrightarrow{\cong} (F_tA)(F_tA^{-1}) \overset{\cong}{\rightarrowtail} (F_tA)(F_tA)^{-1} \xrightarrow{\cong} 1_{s(F_tA)} \xrightarrow{\cong} F_t1_{s(A)}$$

Finally, we define $\mathbf{l}^s_{n+1}(F)$ as the composite of the following diagram, using Lemma 6.4:

$$\begin{tikzcd}
F_t\beta \arrow[d, "H_1"'] \\
((\mathbf{i}_n(\lambda_F)) \circ_{\mathbf{c}_n} (\mathbf{k}^s_{n-1}(\overline{F})) \circ_{\mathbf{c}_n} (\mu_F * \overline{F}) \circ_{\mathbf{c}_n} (T_{F^{-1},F}))_0 \arrow[d, "((\mathbf{i}_n(\lambda_F))\circ_{\mathbf{c}_n}(\mathbf{k}^s_{n-1}(\overline{F}))\circ_{\mathbf{c}_n}(\mu_F*\overline{F}))\circ_{\mathbf{c}_n}(T_{F^{-1},F})"'] \\
((\mathbf{i}_n(\lambda_F)) \circ_{\mathbf{c}_n} (\mathbf{k}^s_{n-1}(\overline{F})) \circ_{\mathbf{c}_n} (\mu_F * \overline{F}) \circ_{\mathbf{c}_n} (T_{F^{-1},F}))_1 \arrow[d, "H_2"'] \\
\gamma F_s
\end{tikzcd}$$

Here, $\gamma\colon BB^{-1} \to 1_{t(B)}$ is an instance of a coherence constraint for inverses, and H_2 is obtained analogously to H_1, both being provided by the structure of $\mathfrak{A}$. $\square$

CHAPTER 7

Elementary interpretation of operations on cylinders

In this chapter we make use of the combinatorics of finite planar rooted trees to describe, given a globular sum A, a zig-zag diagram of globular sums whose colimit is isomorphic to $\mathbf{Cyl}(A)$. We then use this decomposition to construct, given a map $\varrho\colon D_n \to A$ in a coherator for categories $\mathfrak{C}$, a map $\hat{\varrho}\colon \mathbf{Cyl}(D_n) \to \mathbf{Cyl}(A)$. We call this map the elementary interpretation of ϱ (under the functor $\mathbf{Cyl}$), and we consider it an approximation, due to its non-functoriality, of the map $\mathbf{Cyl}(\varrho)$ that one needs to define in order to endow $\mathbb{P}X$ with the structure of a Grothendieck ∞-category. More precisely, given composable maps $\varrho\colon A \to B, \varphi\colon B \to C$ in $\mathfrak{C}$, one has $\widehat{\varphi \circ \varrho} \neq \hat{\varphi} \circ \hat{\varrho}$ in general.

1. Globular decomposition of Cylinders on globular sums

The goal of this section is, given a globular sum $A \in \Theta_0$, to express $\mathbf{Cyl}(A)$ as the colimit in $\mathbf{Mod}(\mathfrak{A})$ of a zig-zag diagram in $\mathfrak{A}$, which will be explicitly described. Here, $\mathfrak{A}$ is a coherator of the kind considered so far.

1.1. Zig-zag diagrams. To begin with, we define zig-zags and record their basic properties.

Definition 7.1. Given a natural number n, define a category $\mathbf{I}_n$ as the one associated to the poset $(\{(0,k)\colon 0 \leq k \leq n\} \cup \{(1,m)\colon 0 \leq m \leq n-1\}, \prec)$, where the relation is completely described by

$$\begin{cases} (0,k) \prec (1,k) \ \forall k \in \{0,\ldots,n-1\} \\ (0,m) \prec (1,m-1) \ \forall m \in \{1,\ldots,n\} \end{cases}$$

Notice that, if $k < n$, there is a natural inclusion $\mathbf{I}_k \to \mathbf{I}_n$.

Pictorially, $\mathbf{I}_n$ looks like

$$\begin{array}{ccccccccc} (0,0) & & (0,1) & & \ldots & & (0,n-1) & & (0,n) \\ & \searrow \ \swarrow & & \searrow & & \swarrow & & \searrow \ \swarrow & \\ & (1,0) & & & \ldots & & & (1,n-1) & \end{array} \tag{27}$$

We have two natural inclusions $* \overset{(0,0)=\iota_0}{\underset{(0,n)=\iota_n}{\rightrightarrows}} \mathbf{I}_n$ for any positive natural number n, where $*$ denotes the terminal category.

$\mathbf{I}_n$ is the free-living zig-zag of length n, in a sense made precise by the following

Definition 7.2. A zig-zag of length n in a category $\mathscr{C}$ is a functor $F\colon \mathbf{I}_n \to \mathscr{C}$. If $F(0,0) = a$ and $F(0,n) = b$ we write $F\colon a \rightsquigarrow b$.

We can also define a partial binary operation on zig-zags, which satisfies an associativity property and will be used in the next section.

Definition 7.3. Define the category $\mathbf{I}_m \bullet \mathbf{I}_n$ as the pushout

$$\begin{tikzcd} * \arrow[r, "\iota_n"] \arrow[d, "\iota_0"'] & \mathbf{I}_n \arrow[d] \\ \mathbf{I}_m \arrow[r] & \mathbf{I}_m \bullet \mathbf{I}_n \end{tikzcd}$$

Note that $\mathbf{I}_m \bullet \mathbf{I}_n \cong \mathbf{I}_{m+n}$.

Given a pair of zig-zags $F\colon \mathbf{I}_n \to \mathcal{C}$, $G\colon \mathbf{I}_m \to \mathcal{C}$ such that $F\colon a \rightsquigarrow b$ and $G\colon b \rightsquigarrow c$ we define

$$G \bullet F\colon \mathbf{I}_m \bullet \mathbf{I}_n \to \mathcal{C}$$

as the unique functor making the following diagram commute

$$\begin{tikzcd} * \arrow[r, "\iota_n"] \arrow[d, "\iota_0"'] & \mathbf{I}_n \arrow[d] \arrow[ddr, bend left, "F"] & \\ \mathbf{I}_m \arrow[r] \arrow[drr, bend right, "G"'] & \mathbf{I}_m \bullet \mathbf{I}_n \arrow[dr, dashed, "\exists ! G\bullet F"'] & \\ & & \mathcal{C} \end{tikzcd} \tag{28}$$

Note that $G \bullet F\colon a \rightsquigarrow c$.

Obviously, this can be iterated to express $\mathbf{I}_{n_1+n_2+\dots n_k}$ as an iterated pushout $\mathbf{I}_{n_k} \bullet \dots \bullet \mathbf{I}_{n_1}$.

Lemma 7.4. *Concatenation of zig-zags is associative. More precisely, if we are given $F\colon a \rightsquigarrow b$, $G\colon b \rightsquigarrow c$ and $H\colon c \rightsquigarrow d$ then it holds true that*

$$H \bullet (G \bullet F) = (H \bullet G) \bullet F$$

Definition 7.5. Let $\mathcal{C}$ be a cocomplete category. Given a zig-zag $F\colon \mathbf{I}_n \to \mathcal{C}$, we define a zig-zag $\tilde{F}\colon \mathbf{I}_1 \to \mathcal{C}$ of length 1 by setting $\tilde{F}(0,0) = F(0,0)$, $\tilde{F}(0,1) = F(0,n)$ and $\tilde{F}(1,0) = \operatorname{colim}_{\mathbf{I}_n} F$, where the structural maps are given by the colimit inclusions.

Given integers n_i for $1 \le i \le k$, we let $n = \sum_1^k n_i$. Given a zig-zag $F\colon \mathbf{I}_n \cong \mathbf{I}_{n_k} \bullet \dots \bullet \mathbf{I}_{n_1} \to \mathcal{C}$, where the target is a cocomplete category, we can consider its restrictions $F_i\colon \mathbf{I}_{n_i} \to \mathcal{C}$, obtained as in Definition 7.3. The next result then holds true, thanks to the universal property of colimits.

Lemma 7.6. *In the situation just described, we have the following isomorphism in $\mathcal{C}$:*

$$\operatorname{colim}_{\mathbf{I}_n} F \cong \operatorname{colim}_{\mathbf{I}_k} \left(\tilde{F}_k \bullet \dots \bullet \tilde{F}_1\right)$$

1.2. Trees and globular sums. We now need an alternative way of representing globular sums. In [**Ar1**] this is done (following [**Bat**]) by associating to any given globular sum A a finite planar rooted tree that uniquely represents it.

Definition 7.7. Consider the functor category $\mathrm{Ord}_{fin}^{\omega}$, where ω is viewed as a poset with respect to inclusion, and Ord_{fin} is the category of finite linearly ordered sets.

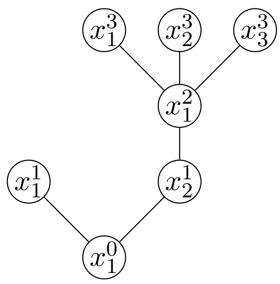
The category $\mathcal{T}$ of finite planar rooted trees is the full subcategory of $\mathrm{Ord}_{fin}^{\omega}$ spanned by the objects T such that T_0 is the terminal object of Ord_{fin} (i.e. the singleton with its unique ordering) and there exists an $n \in \mathbb{N}$ such that $T_i = \emptyset$ for each $i \ge n$.

We call the elements of $v(T) = \bigcup_{k\in\omega} T_k$ vertices of T, and we say that a vertex x has height m, denoted by $\mathrm{ht}(x) = m$, if $x \in T_m$. Finally, we set $\mathrm{ht}(T) = \max_{x\in v(T)} \mathrm{ht}(x)$.

Explicitly, a finite planar roote tree T consists of a family of finite linearly ordered sets $(T_i, \leq_i)_{0\leq i\leq n}$ for some $n \in \mathbb{N}$, with $T_0 = \{*\}$, together with order-preserving maps $\iota_i^T : T_{i+1} \to T_i$.

Definition 7.8. Given $n \geq 0$, a tree T and an element $x \in T_{n+1}$, we define the fiber of x to be the set $(\iota_n^T)^{-1}(\iota_n^T(x))$. Clearly, x belongs to such fiber.

Example 7.9. Consider the finite planar tree T given by $T_1 = \{x_1^1 < x_2^1\}$, $T_2 = \{x_1^2\}$ and $T_3 = \{x_1^3 < x_2^3 < x_3^3\}$, whose only non trivial structural map $T_2 \to T_1$ is given by $x_1^2 \mapsto x_2^1$.
Such a tree T can be depicted as



Definition 7.10. Given a tree T, we can define a relation on the set of vertices $v(T)$ as follows. Consider $x \neq y \in v(T)$, and set

$$x \prec y \iff \begin{cases} \mathrm{ht}(x) = \mathrm{ht}(y) \text{ and } y < x \text{ in } T_{\mathrm{ht}(x)} \\ \mathrm{ht}(x) < \mathrm{ht}(y) \text{ and } \iota_T^k(y) \leq x \text{ in } T_{\mathrm{ht}(x)} \\ \mathrm{ht}(x) > \mathrm{ht}(y) \text{ and } y < \iota_T^{k'} x \text{ in } T_{\mathrm{ht}(y)} \end{cases} \tag{29}$$

where $k = \mathrm{ht}(y) - \mathrm{ht}(x)$ and $k' = \mathrm{ht}(x) - \mathrm{ht}(y)$, and $\iota_T^k : T_{\mathrm{ht}(y)} \to T_{\mathrm{ht}(x)}, \iota_T^{k'} : T_{\mathrm{ht}(x)} \to T_{\mathrm{ht}(y)}$ are composite of the structural maps of T. Clearly, this defines a linear order on $v(T)$, which can be described as depth-first, right-to-left ordering.

For instance, given the tree T of the previous example, the totally ordered set of its vertices is given by

$$\{x_1^0 \prec x_2^1 \prec x_1^2 \prec x_3^3 \prec x_2^3 \prec x_1^3 \prec x_1^1\}$$

The proof of the following lemma is straightforward.

Lemma 7.11. *Maps of trees $f : T \to S$ preserve the order $\prec$.*

We can associate a tree with every given globular sum A. To do so we need the following definition.

Definition 7.12. Given a tree T and a vertex $x \in T$, we say that x is maximal (also called a leaf) if

$$\left(\iota_{\mathrm{ht}(x)}^T\right)^{-1}(x) = \emptyset$$

Let $\{x_1 \prec x_2 \prec \ldots \prec x_k\}$ be the ordered set of maximal vertices of T. Let h_i be the height of the highest vertex y such that both x_i and x_{i+1} belong to the fiber of (an iteration of) ι^T over y. h_i is called the height of the region between x_i and x_{i+1}.

It is an easy exercise to prove the following result.

Lemma 7.13. *Finite planar rooted trees are in bijection with globular sums. More precisely, the bijection is given by the function that associates the table of dimensions*

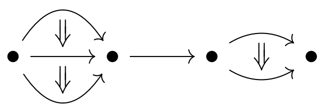
$$\begin{pmatrix} i_1 & & i_2 & \dots & i_{m-1} & & i_m \\ & i'_1 & & \dots & & i'_{m-1} & \end{pmatrix}$$

with a given tree T, where i_k is defined to be the height of the k-th maximal vertex x_k of T, and i'_k is the height of the region between x_k and x_{k+1}.

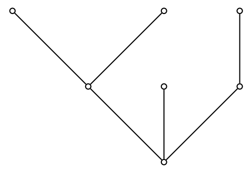
Having this in mind, we will often blur the distinction between trees and globular sums. For example, the globular sum

$$\begin{pmatrix} 2 & & 2 & & 1 & & 2 \\ & 1 & & 0 & & 0 & \end{pmatrix}$$

can be represented, equivalently, as



or



Example 7.14. Given a globular sum $A \in \Theta_0$, we have defined its suspension ΣA in Chapter 5. It is very easy to define the tree $T^{\Sigma A}$ in terms of T^A, where we denote by T^B the tree associated with the globular sum B. In fact, suppose T^A consists of the family of finite linearly ordered sets $(T_i, \leq_i)_{0 \leq i \leq n}$ for some natural number n. Then we define $T_1^{\Sigma A} = \{*\}$ and for every $k > 1$:

$$T_k^{\Sigma A} \overset{def}{=} T_{k-1}^A$$

Moreover, $\iota_k^{T^{\Sigma A}} \overset{def}{=} \iota_{k-1}^{T^A}$.

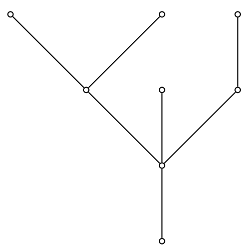
For instance, if we let A be the globular sum whose table of dimensions is

$$\begin{pmatrix} 2 & & 2 & & 1 & & 2 \\ & 1 & & 0 & & 0 & \end{pmatrix}$$

then ΣA has table of dimensions given by

$$\begin{pmatrix} 3 & & 3 & & 2 & & 3 \\ & 2 & & 1 & & 1 & \end{pmatrix}$$

Moreover, the tree $T^{\Sigma A}$ can be depicted as

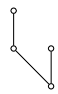


Remark 7.15. The decomposition given in Lemma 5.3 has a more geometric interpretation in the language of trees. It corresponds to the elementary fact that any tree can be realized as the glueing at the root of a family of trees all having a single edge at the bottom.

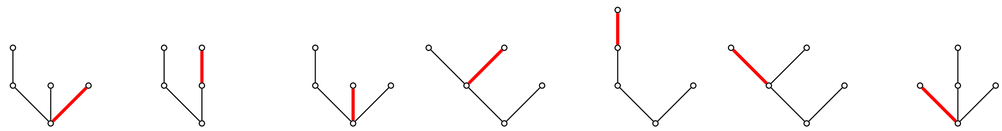
It turns out that the cylinder on a given globular sum A has a quite simple description in terms of trees. In fact, it is the colimit of a suitable zig-zag diagram of globes $\underline{\mathbf{Cyl}}(A)\colon \mathbf{I}_{n_A} \to \mathbf{Mod}(\mathfrak{A})$, for an integer n_A that will be defined in what follows. More precisely, this diagram is the composite of a diagram $\mathbf{I}_{n_A} \to \mathfrak{A}$ followed by the Yoneda embedding.

To begin with, we want to list the globular sums $\{\underline{\mathbf{Cyl}}(A)(1,k)\}_{0\leq k\leq n_A-1}$, i.e. those appearing on the bottom row (see (27)) of the zig-zag diagram associated with $\mathbf{Cyl}(A)$. These are obtained by considering $n_A - 1$ copies of the tree associated with A, and adding to each of these a single new branch, following the procedure we now describe. We start by sticking it at the bottom right and then we traverse the tree going upward and to the left, counterclockwise.

For example, for $A = D_2 \amalg_{D_0} D_1$, whose associated tree is



one gets the trees:



(30)

On the other hand, the upper row is constant on A, i.e.

$$\underline{\mathbf{Cyl}}(A)(0,k) = A \ \forall \ 0 \leq k \leq n_A$$

Let us now formalize what we have said so far.

Definition 7.16. Given a tree T of height n, seen as a family of linearly ordered sets $(T_i)_{0\leq i\leq n}$ together with compatible maps $\iota_k : T_{k+1} \to T_k$, we define a set of trees $\mathscr{L}(A)$ by considering all the trees obtained from A by adjoining a single edge.

Formally, this means that we consider all possible trees B such that there exists a unique $1 \leq k \leq n+1$ such that $B_k = A_k \cup \{*_B\}$ and $B_i = A_i$ for each $i \neq k$, in such a way that the obvious map $A \to B$ is a map of trees.

Note that, by construction, for every B in $\mathscr{L}(A)$ there is a natural inclusion of trees $\chi^A_B\colon A \to B$. We now define a relation on the set $\mathscr{L}(A)$ introduced in Definition 7.16.

Definition 7.17. Given $B \neq C$ in $\mathscr{L}(A)$, set $B \lessdot C$ if and only if there exists an $x \in A$ such that $*_B \prec_B x \prec_C *_C$, where the subscripts denote in which tree we are considering the ordering $\prec$.

Lemma 7.18. *Given a globular sum A, the relation on $\mathscr{L}(A)$ just defined is a linear order.*

Proof. The only non-trivial thing to check is transitivity. If $B \lessdot C \lessdot D$ then there exist x, y in A with $*_B \prec_B x \prec_C *_C$ and $*_C \prec_C y \prec_D *_D$. Then x (or y) is a witness for the relation $B \lessdot D$. $\square$

Lemma 7.19. *Given a tree T and an n-tuple of vertices $(v_1, \dots, v_n)$ of T (ordered from left to right) such that every leaf of T belongs to the tree C_i above v_i for some i, and the C_i's are disjoint, we have the following isomorphism*

$$T \cong \Sigma^{m_1} C_1 \coprod_{D_{h_1}} \Sigma^{m_2} C_2 \coprod_{D_{h_2}} \dots \coprod_{D_{h_{n-1}}} \Sigma^{m_n} C_n$$

where $m_i = \mathrm{ht}(v_i)$, h_i is the height of the region between v_i and v_{i+1} and the maps are the obvious maps in Θ_0 that there are between those objects.

Proof. We argue by induction on the total number m of vertices of the tree T . Assume $v_i \neq T(0)$, i.e. the root, in which case there is nothing to prove. Decompose T as $\Sigma T_1 \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma T_k$, as in Remark 7.15. If $k > 1$ then we can divide and reorder (if needed) the set of vertices $\{v_i\}_{1 \le i \le n}$ as $\{v_j\}_{1 \le j \le r} \cup \{v_q\}_{r+1 \le q \le n}$, so that the elements of $\{v_q\}_{r+1 \le q \le n}$ are precisely those v_i's that belong to ΣT_k. Therefore, because $h_r = 0$ by construction, the statement about the decomposition of the tree T holds true since we can apply the inductive hypothesis to the trees $\Sigma T_1 \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma T_{k-1}$ and ΣT_k, which have strictly fewer than m vertices.

If instead $k = 1$, then $T = \Sigma T'$ and all the v_i's belong to T'. Now, the result follows by induction, since T' has $m - 1$ vertices. $\square$

In what follows we assume A is a globular sum, decomposed as $A \cong \Sigma\alpha_1 \underset{D_0}{\amalg} \Sigma\alpha_2 \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma\alpha_q$, and we define the maps in the zig-zag diagram associated with $\mathbf{Cyl}(A)$.

Definition 7.20. Consider a globular sum $B \in \mathscr{L}(A)$. We define a map $z^A_B \colon A \to B$ as follows. Suppose B is obtained by adjoining a new vertex $*_B$ to A, and let $m = \mathrm{ht}(*_B)$. Observe that if $m > 1$ then $*_B$ is necessarily adjoined to a unique $\Sigma\alpha_i$. Denote the fiber of $*_B$, defined in Definition 7.8, with F. We now have different possible cases, and in each of these the map z^A_B will be defined by making use of Lemma 7.19 and the universal property of pushouts, i.e. we will specify the tuple of vertices v_i's and the maps on each factor $\Sigma^{m_i} C_i$.

(1) If $*_B = \min F$ and $F \neq \{*_B\}$, then we choose as v_i's the vertex x together with all the maximal vertices of A except for those over x, and we order them from left to right. $z^A_B \in \Theta_0$ is then defined as the unique map obtained by considering the identity on all the factors not associated with x and the unique map $\Sigma^m C \to D_{m+1} \underset{D_m}{\amalg} \Sigma^m C$ in Θ_0 on the x-factor.
(2) If $*_B = \min F$ and $F = \{*_B\}$ then z^A_B is defined analogously to the previous case except on the factor associated with the vertex x it is given by $\tau \colon D_m \to D_{m+1}$, where $m = \mathrm{ht}(x)$.
(3) If $*_B \neq \min F$, let $y \in F$ be the predecessor of $*_B$ in F, and let C be the subtree of B over y. Then we apply Lemma 7.19 to the tree associated with A and the set of vertices $\{v_1, \ldots, v_n, y\}$ where the v_i's are all the leaves which do not lie above y. This allows us to define the map z^A_B by imposing it to be $\Sigma^m({}_C w) \colon \Sigma^m C \to \Sigma^m C \underset{D_m}{\amalg} D_{m+1}$ (as in Definition 2.12) on $\Sigma^m C$ and the identity on all the other factors.

Dually, we define a map $v^A_B \colon A \to B$ by cases:

(1) If $*_B = \max F$ and $F \neq \{*_B\}$, then we choose as v_i's the vertex x together with all the maximal vertices of A except for those over x, and we order them from left to right. $z^A_B \in \Theta_0$ is then defined as the unique map obtained by considering the identity on all the factors not associated with x and the unique map $\Sigma^m C \to \Sigma^m C \underset{D_m}{\amalg} D_{m+1}$ in Θ_0 on the x-factor.
(2) If $*_B = \max F$ and $F = \{*_B\}$ then z^A_B is defined analogously except on the factor associated with the vertex x it is given by $\sigma \colon D_m \to D_{m+1}$, where $m = \mathrm{ht}(x)$.

(3) If $*_B \neq \max F$, let $y \in F$ be the successor of $*_B$ in F, and let C be the subtree of B over y. Then we apply Lemma 7.19 to the tree associated with A and the set of vertices $\{v_1, \ldots, v_n, y\}$ where the v_i's are all the leaves which do not lie above y. This allows us to define the map z^A_B by imposing it to be $\Sigma^m(w_C) \colon \Sigma^m C \to D_{m+1} \underset{D_m}{\amalg} \Sigma^m C$ (as in Definition 2.12) on $\Sigma^m C$ and the identity on all the other factors.

We are now ready to give the following definition

Definition 7.21. Given a globular sum A, let $\mathscr{L}(A) = \{A_1 \lessdot A_2 \lessdot \ldots \lessdot A_m\}$, so that $m = |\mathscr{L}(A)|$. We define a functor $\underline{\mathbf{Cyl}}(A) \colon \mathbf{I}_m \to \infty\text{-}\mathcal{G}pd$ by setting:

- $\underline{\mathbf{Cyl}}(A)(0,k) = A$ for every $0 \leq k \leq n$.
- $\underline{\mathbf{Cyl}}(A)(1,q) = A_{q+1}$ for every $0 \leq q \leq |\mathscr{L}(A)| - 1$.
- $\underline{\mathbf{Cyl}}(A)\left((0,r) \to (1,r-1)\right) = z^A_{A_{r-1}}$.
- $\underline{\mathbf{Cyl}}(A)\left((0,r) \to (1,r)\right) = v^A_{A_r}$.

$$\begin{tikzcd}
A \arrow[dr, "v^A_{A_1}"'] & & A \arrow[dl, "z^A_{A_1}"] \arrow[dr] & & A \arrow[dl] \arrow[dr, "v^A_{A_m}"'] & & A \arrow[dl, "z^A_{A_m}"] \\
& A_1 & & \ldots & & A_m &
\end{tikzcd}$$

Using the trees listed in (30), we can write down the full zig-zag diagram corresponding to $\mathbf{Cyl}(D_2 \amalg_{D_0} D_1)$, see Figure 1.

We will now prove that the colimit of the zig-zag diagram associated with a globular sum A we have just defined is precisely $\mathbf{Cyl}(A)$. To do so we need two preliminary lemmas, which we now present.

Lemma 7.22. *$\mathbf{Cyl}(\Sigma B)$ can be expressed as the colimit of the following diagram*

$$\begin{tikzcd}
\Sigma B \arrow[dr, "\iota"] & & \Sigma B \arrow[dl, "\Sigma B w"'] \arrow[dr, "\Sigma(\iota_0)"] & & \Sigma B \arrow[dl, "\Sigma(\iota_1)"'] \arrow[dr, "w \Sigma B"] & & \Sigma B \arrow[dl, "\iota"'] \\
& \Sigma B \underset{D_0}{\amalg} D_1 & & \Sigma \mathbf{Cyl}(B) & & D_1 \underset{D_0}{\amalg} \Sigma B &
\end{tikzcd} \tag{32}$$

Proof. As we let B vary in Θ_0, we see that the colimit of the zig-zag in the statement defines a globular functor $\Theta_0 \to \mathbf{Mod}(\mathfrak{A})$, which coincides with $\mathbf{Cyl}(\Sigma D_\bullet)$ on $\mathbb{G}$. Therefore, there exists a natural isomorphism as stated thanks to the universal property of Θ_0. $\square$

Lemma 7.23. *Given a category with pushouts $\mathcal{C}$ and a diagram in it of the form*

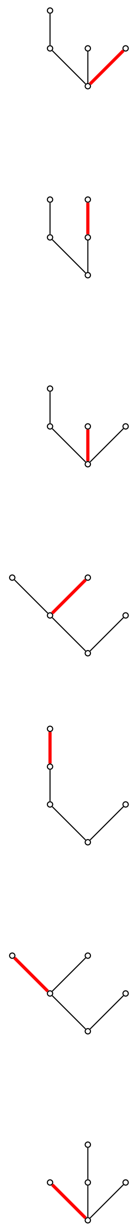
$$\begin{tikzcd}
& & C \arrow[dl] \arrow[dr] & & \\
& A \arrow[dl] & & B \arrow[dr] & \\
A' & & & & B'
\end{tikzcd}$$

we get a pushout square

$$\begin{tikzcd}
A \underset{C}{\amalg} B \arrow[r] \arrow[d] & A \underset{C}{\amalg} B' \arrow[d] \\
A' \underset{C}{\amalg} B \arrow[r] & A' \underset{C}{\amalg} B'
\end{tikzcd}$$

FIGURE 1. Zig-zag diagram corresponding to $\mathbf{Cyl}(D_2 \underset{D_0}{\amalg} D_1)$

$$\begin{tikzcd}
 & D_2 \underset{D_0}{\amalg} D_1 \underset{D_0}{\amalg} D_1 \\
D_2 \underset{D_0}{\amalg} D_1 \arrow[ur, "1 \amalg w"] \arrow[dr, "1 \amalg \sigma"] & \\
 & D_2 \underset{D_0}{\amalg} D_2 \\
D_2 \underset{D_0}{\amalg} D_1 \arrow[ur, "1 \amalg \tau"] \arrow[dr, "1 \amalg w"] & \\
 & D_2 \underset{D_0}{\amalg} D_1 \underset{D_0}{\amalg} D_1 \\
D_2 \underset{D_0}{\amalg} D_1 \arrow[ur, "w \amalg 1"] \arrow[dr, "{(i_0, i_2)}"] & \\
 & D_2 \underset{D_1}{\amalg} D_2 \underset{D_0}{\amalg} D_1 \\
D_2 \underset{D_0}{\amalg} D_1 \arrow[ur, "w \amalg 1"] \arrow[dr, "\sigma \amalg 1"] & \\
 & D_3 \underset{D_0}{\amalg} D_1 \\
D_2 \underset{D_0}{\amalg} D_1 \arrow[ur, "\tau \amalg 1"] \arrow[dr, "w \amalg 1"] & \\
 & D_2 \underset{D_1}{\amalg} D_2 \underset{D_0}{\amalg} D_1 \\
D_2 \underset{D_0}{\amalg} D_1 \arrow[ur, "{(i_1, i_2)}"] \arrow[dr, "w \amalg 1"] & \\
 & D_1 \underset{D_0}{\amalg} D_2 \underset{D_0}{\amalg} D_1
\end{tikzcd} \tag{31}$$



PROPOSITION 7.24. *Given a globular sum A, there exists a natural isomorphism in* $\mathbf{Mod}(\mathfrak{A})$*:*

$$\mathrm{colim}_{\mathbf{I}_{|\mathscr{L}(A)|}} \underline{\mathbf{Cyl}}(A) \cong \mathbf{Cyl}(A)$$

PROOF. In what follows, ι will denote a colimit inclusion, unless otherwise stated.

We make use of the (unique) decomposition of globular sums described in Lemma 5.3, which gives:

$$A \cong \Sigma\alpha_1 \underset{D_0}{\amalg} \Sigma\alpha_2 \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma\alpha_q$$

and we inductively assume the result holds for the α_i's.

By globularity of the cylinder functor, we have the isomorphism:

$$\mathbf{Cyl}(A) \cong \mathbf{Cyl}(\Sigma\alpha_1) \underset{\mathbf{Cyl}(D_0)}{\amalg} \mathbf{Cyl}(\Sigma\alpha_2) \underset{\mathbf{Cyl}(D_0)}{\amalg} \cdots \underset{\mathbf{Cyl}(D_0)}{\amalg} \mathbf{Cyl}(\Sigma\alpha_q)$$

We can break the ordered set $\mathscr{L}(A)$ into subintervals by taking into consideration the globular sums A_i for which the new edge is joined at the root. More precisely, let $1, m_1, \ldots, m_k, m_{k+1} = |\mathscr{L}(A)|$ be the ordered sequence of integers such that A_{m_i} is obtained from A by adding a new vertex at height 1. We then have:

$$\mathscr{L}(A) = \{A_1\} \cup \{A_2, \ldots, A_{m_1-1}\} \cup \{A_{m_1}\} \cup \ldots \cup \{A_{m_k}\} \cup \{A_{m_k+1}, \ldots A_{m_{k+1}-1}\} \cup \{A_{m_{k+1}}\}$$

and a corresponding isomorphism:

$$\mathbf{I}_{|\mathscr{L}(A)|} \cong \mathbf{I_1} \bullet \mathbf{I}_{m_{k+1}-m_k-1} \bullet \mathbf{I_1} \bullet \ldots \bullet \mathbf{I}_{m_1-2} \bullet \mathbf{I_1}$$

This, in turn, induces an isomorphism of diagrams:

$$\underline{\mathbf{Cyl}}(A) \cong \mathbf{A_{m_{k+1}}} \bullet \underline{\mathbf{Cyl}}(A)'_{k+1} \bullet \mathbf{A_{m_{k-1}}} \bullet \underline{\mathbf{Cyl}}(A)'_{k-1} \bullet \ldots \bullet \underline{\mathbf{Cyl}}(A)'_1 \bullet \mathbf{A_1}$$

where we define $\underline{\mathbf{Cyl}}(A)'_i = \underline{\mathbf{Cyl}}(A)_{|\mathbf{I}_{(m_i-1)-(m_{i-1}+1)+1}}$, and $\mathbf{A_1}, \mathbf{A_{m_i}}$ are zig-zags of length 1 which we now describe. By definition, $\mathbf{A_{m_i}}(1,0) = A_{m_i}$, and one has that $\mathbf{A_{m_i}}\,((0,0) \to (1,0))$ is given by:

$$1 \underset{D_0}{\amalg} w_{\Sigma(\alpha_i)} \underset{D_0}{\amalg} 1 \colon \Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k) \to \Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \left(D_1 \underset{D_0}{\amalg} \Sigma(\alpha_i)\right) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k) \underset{D_0}{\amalg} D_1$$

if $i \neq k+1$.

On the other hand, we have that $\mathbf{A_{m_i}}\,((0,1) \to (1,0))$ coincides with

$$1 \underset{D_0}{\amalg} {}_{\Sigma(\alpha_{k-i+1})}w \underset{D_0}{\amalg} 1 \colon \Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k) \to \Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \left(\Sigma(\alpha_{k-i+1}) \underset{D_0}{\amalg} D_1\right) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k)$$

$\mathbf{A_1}\,((0,0) \to (1,0))$ is the map

$$\iota \colon \Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k) \to \Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k) \underset{D_0}{\amalg} D_1$$

and $\mathbf{A_1}\,((0,1) \to (1,0))$ is given by

$$1 \underset{D_0}{\amalg} {}_{\Sigma(\alpha_k)}w \colon \Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k) \to \Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \left(\Sigma(\alpha_k) \underset{D_0}{\amalg} D_1\right)$$

Finally, $\mathbf{A_{m_{k+1}}}\,((0,0) \to (1,0))$ coincides with

$$w_{\Sigma(\alpha_1)} \underset{D_0}{\amalg} 1 \colon \Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k) \to \left(D_1 \underset{D_0}{\amalg} \Sigma(\alpha_1)\right) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k)$$

and $\mathbf{A_{m_{k+1}}}\,((0,1) \to (1,0))$ is the map

$$\iota \colon \Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k) \to D_1 \underset{D_0}{\amalg} \Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k)$$

We now want to show that

$$\operatorname{colim}_{\mathbf{I}_{\mathbf{I}_{(m_i-1)-(m_{i-1}+1)+1}}} \underline{\mathbf{Cyl}}(A)'_i \cong \Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_{i-1}) \underset{D_0}{\amalg} \Sigma\mathbf{Cyl}(\alpha_i) \underset{D_0}{\amalg} \Sigma(\alpha_{i+1}) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k) \tag{33}$$

Firstly, notice that the interval $\{A_{m_{i-1}+1}, \ldots, A_{m_i-1}\}$ is linearly isomorphic to $\Sigma(\mathscr{L}(\alpha_i))$, i.e. the image of the set $\mathscr{L}(\alpha_i)$ under the object-part function of the functor Σ.

By inspection of the maps z_B^A and v_B^A of Definition 7.20, we see that the diagram $\underline{\mathbf{Cyl}}(A)'_i$ coincides with

$$\Sigma(\alpha_1) \coprod_{D_0} \dots \coprod_{D_0} \Sigma(\alpha_{i-1}) \coprod_{D_0} \Sigma \circ \underline{\mathbf{Cyl}}(\alpha_i) \coprod_{D_0} \Sigma(\alpha_{i+1}) \coprod_{D_0} \dots \coprod_{D_0} \Sigma(\alpha_k)$$

Thus, using Remark 5.1, we see that (33) holds by inductive hypothesis. Thus, thanks to Lemma 7.6, the colimit of the diagram $\underline{\mathbf{Cyl}}(A)$ coincides with the colimit of the zig-zag on the left-hand side below, where $\amalg$ denotes the operation $\coprod\limits_{D_0}$. A further application of Lemma 7.6 and Lemma 7.22 proves that this last colimit is in turn isomorphic to the colimit of the right-hand side zig-zag below.

$$\begin{tikzcd}
A \arrow[dr, "\iota"] & \\
& \Sigma(\alpha_1) \coprod\limits_{D_0} \dots \coprod\limits_{D_0} \Sigma(\alpha_k) \coprod\limits_{D_0} D_1 \\
A \arrow[ur, "1 \amalg_{\Sigma(\alpha_k)} w"] \arrow[dr, "1 \amalg \Sigma(\iota_0)"] & \\
& \Sigma(\alpha_1) \coprod\limits_{D_0} \dots \coprod\limits_{D_0} \Sigma\left(\mathbf{Cyl}(\alpha_k)\right) \\
A \arrow[ur, "1 \amalg \Sigma(\iota_1)"] \arrow[dr] & \\
& \dots \\
A \arrow[ur] \arrow[dr, "\Sigma(\iota_0) \amalg 1"] & \\
& \Sigma\left(\mathbf{Cyl}(\alpha_1)\right) \coprod\limits_{D_0} \dots \coprod\limits_{D_0} \Sigma(\alpha_k) \\
A \arrow[ur, "\Sigma(\iota_1) \amalg 1"] \arrow[dr, "w_{\Sigma(\alpha_1)} \amalg 1"] & \\
& D_1 \coprod\limits_{D_0} \Sigma(\alpha_1) \coprod\limits_{D_0} \dots \coprod\limits_{D_0} \Sigma(\alpha_k) \\
A \arrow[ur, "\iota"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dr, "1 \amalg \iota_0"] & \\
& \Sigma(\alpha_1) \coprod\limits_{D_0} \dots \coprod\limits_{D_0} \mathbf{Cyl}\left(\Sigma(\alpha_k)\right) \\
A \arrow[ur, "1 \amalg (i \circ w_{\Sigma(\alpha_k)})"] \arrow[dr] & \\
& \dots \\
A \arrow[ur] \arrow[dr, "(i \circ_{\Sigma(\alpha_1)} w) \amalg 1"] & \\
& \mathbf{Cyl}\left(\Sigma(\alpha_1)\right) \coprod\limits_{D_0} \dots \coprod\limits_{D_0} \Sigma(\alpha_k) \\
A \arrow[ur, "\iota_1 \amalg 1"] &
\end{tikzcd}$$

To finish the proof we now apply Lemma 7.23 $(k-1)$ times to diagrams of the form:

$$\begin{tikzcd}
& D_1 \arrow[dl] \arrow[dr] & \\
\Sigma(\alpha_1) \coprod\limits_{D_0} \dots \coprod\limits_{D_0} \left(\Sigma(\alpha_i) \coprod\limits_{D_0} D_1\right) \coprod\limits_{D_0} \dots \coprod\limits_{D_0} \Sigma(\alpha_k) \arrow[d] & & \Sigma(\alpha_1) \coprod\limits_{D_0} \dots \coprod\limits_{D_0} \left(D_1 \coprod\limits_{D_0} \Sigma(\alpha_{i+1})\right) \coprod\limits_{D_0} \dots \coprod\limits_{D_0} \Sigma(\alpha_k) \arrow[d] \\
\Sigma(\alpha_1) \coprod\limits_{D_0} \dots \coprod\limits_{D_0} \mathbf{Cyl}\left(\Sigma(\alpha_i)\right) \coprod\limits_{D_0} \dots \coprod\limits_{D_0} \Sigma(\alpha_k) & & \Sigma(\alpha_1) \coprod\limits_{D_0} \dots \coprod\limits_{D_0} \mathbf{Cyl}\left(\Sigma(\alpha_{i+1})\right) \coprod\limits_{D_0} \dots \coprod\limits_{D_0} \Sigma(\alpha_k)
\end{tikzcd}$$

$\square$

Given $B \in \mathscr{L}(A)$, we denote by $i_B \colon B \to \mathbf{Cyl}(A)$ the colimit inclusion.

REMARK 7.25. If we consider the globular sum A as in the previous theorem, then $D_1 \underset{D_0}{\amalg} A$ and $A \underset{D_0}{\amalg} D_1$ both belongs to $\mathscr{L}(A)$ by construction. It is clear from the proof of the previous theorem that the colimit inclusion $i_{D_1 \underset{D_0}{\amalg} A} \colon D_1 \underset{D_0}{\amalg} A \to \mathbf{Cyl}(A)$ is given by $(\mathbf{Cyl}(\sigma^{\mathrm{ht}(A)}), \iota_1)$. In a completely analogous manner, $i_{A \underset{D_0}{\amalg} D_1} \colon A \underset{D_0}{\amalg} D_1 \to \mathbf{Cyl}(A)$ is equal to $(\iota_0, \mathbf{Cyl}(\tau^{\mathrm{ht}(A)}))$.

If the globular sum A decomposes as $A = S \underset{D_0}{\amalg} T$ then $S \underset{D_0}{\amalg} D_1 \underset{D_0}{\amalg} T$ belongs to $\mathscr{L}(A)$, and the colimit inclusion

$$i_{S \underset{D_0}{\amalg} D_1 \underset{D_0}{\amalg} T} \colon S \underset{D_0}{\amalg} D_1 \underset{D_0}{\amalg} T \to \mathbf{Cyl}(A) \cong \mathbf{Cyl}(S) \underset{\mathbf{Cyl}(D_0)}{\amalg} \mathbf{Cyl}(T)$$

is given, on each summand respectively, by the composites:

$$S \xrightarrow{\iota_0} \mathbf{Cyl}(S) \xrightarrow{i} \mathbf{Cyl}(S) \underset{\mathbf{Cyl}(D_0)}{\amalg} \mathbf{Cyl}(T)$$

$$D_1 \cong \mathbf{Cyl}(D_0) \xrightarrow{i} \mathbf{Cyl}(S) \underset{\mathbf{Cyl}(D_0)}{\amalg} \mathbf{Cyl}(T)$$

$$T \xrightarrow{\iota_1} \mathbf{Cyl}(T) \xrightarrow{i} \mathbf{Cyl}(S) \underset{\mathbf{Cyl}(D_0)}{\amalg} \mathbf{Cyl}(T)$$

where we denote with i the obvious colimit inclusions.

Finally, observe that if $B \in \mathscr{L}(A)$ and the new edge is attached at height $m > 0$, say to $\Sigma(\alpha_i)$, then the colimit inclusion $i_B \colon B \to \mathbf{Cyl}(A)$ factors through the natural map:

$$\Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma\mathbf{Cyl}(\alpha_i) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k) \to \mathbf{Cyl}(A)$$

whose existence is evident from the proof we have just presented, via the map:

$$1 \underset{D_0}{\amalg} \Sigma(i_{B'}) \underset{D_0}{\amalg} 1 \colon B \cong \Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma B' \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k) \to \Sigma(\alpha_1) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma\mathbf{Cyl}(\alpha_i) \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma(\alpha_k)$$

for a unique $B' \in \mathscr{L}(\alpha_i)$.

2. Operations on cylinders (overview)

Consider the globular sum preserving functor

$$\mathbf{Cyl}(\bullet) \colon \Theta_0 \to \mathbf{Mod}(\mathfrak{A}) \tag{34}$$

of which we have just given a more explicit definition. Constructing an extension of this functor to a cocontinuous endofunctor on $\mathbf{Mod}(\mathfrak{A})$ amounts to endowing (34) with the structure of a co-$\mathfrak{A}$-model in $\mathbf{Mod}(\mathfrak{A})$.

This means that we have to find an extension of the form:

$$\begin{tikzcd} \Theta_0 \arrow[r, "\mathbf{Cyl}(\bullet)"] \arrow[d] & \mathbf{Mod}(\mathfrak{A}) \\ \mathfrak{A} \arrow[ur, dashed, "\mathbf{Cyl}(\bullet)"'] & \end{tikzcd} \tag{35}$$

Thanks to the cellularity of $\mathfrak{A}$ and to Lemma 2.9, this becomes an inductive process, where we assume we have an operation $\varrho \colon D_n \to A$ in $\mathfrak{A}$, as well as interpretations of its boundary

$$\mathbf{Cyl}(\varrho \circ \sigma), \mathbf{Cyl}(\varrho \circ \tau) \colon \mathbf{Cyl}(D_{n-1}) \to \mathbf{Cyl}(A)$$

and we need to define a map $\mathbf{Cyl}(\varrho)\colon \mathbf{Cyl}(D_n) \to \mathbf{Cyl}(A)$ such that for $\varepsilon = \sigma, \tau$:

$$\mathbf{Cyl}(\varrho) \circ \mathbf{Cyl}(\varepsilon) = \mathbf{Cyl}(\varrho \circ \varepsilon)\colon \mathbf{Cyl}(D_{n-1}) \to \mathbf{Cyl}(A)$$

Given the fact that we have explained how to decompose of cylinders on globular sums into simpler pieces, we may try to use this fact to build maps representing a first approximation of these operations between cylinders.

In fact, we construct these first approximations only for categorical operations $\varrho\colon D_n \to A$. More precisely, we assume that $\mathfrak{C}$ is a (homogeneous) coherator for ∞-categories: for instance if $\mathfrak{A}$ is such then we can assume $\mathfrak{C} = \mathfrak{A}$, otherwise we have to consider a different coherator. Moreover, we will implicitly assume that a map $\mathfrak{C} \to \mathfrak{A}$ has been chosen once and for all (its existence is ensured by cellularity of $\mathfrak{C}$ and the fact that, without loss of generality, either $\mathfrak{C} = \mathfrak{A}$, $\mathfrak{A} = \mathfrak{C}^{\mathbf{W}}$ or $\mathfrak{A}$ is a coherator for ∞-groupoids and therefore it is contractible), and we identify maps in the domain with their image in the codomain. By doing so we manage to define in Definition 7.42, for every operation $\varrho\colon D_n \to A$ in $\mathfrak{C}$, a map of $\mathfrak{A}$-models

$$\hat{\varrho}\colon \mathbf{Cyl}(D_n) \to \mathbf{Cyl}(A)$$

satisfying two properties, that can be expressed in the following commutative diagrams:

$$\begin{tikzcd} D_n \amalg D_n \arrow[r, "\varrho \amalg \varrho"] \arrow[d, "\iota_0 \amalg \iota_1"'] & A \amalg A \arrow[d, "\iota_0 \amalg \iota_1"] \\ \mathbf{Cyl}(D_n) \arrow[r, "\hat{\varrho}"] & \mathbf{Cyl}(A) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Cyl}(S^{n-1}) \arrow[r, "(\widehat{\varrho\circ\sigma}, \widehat{\varrho\circ\tau})"] \arrow[d] & \mathbf{Cyl}(A) \\ \mathbf{Cyl}(D_n) \arrow[ur, "\hat{\varrho}"'] & \end{tikzcd} \tag{36}$$

This map is to be thought of as a first approximation of the "correct" functorial interpretation $\mathbf{Cyl}(\varrho)$. To remedy its non-functoriality (i.e. the fact that for composable operations $\varrho\colon A \to B, \varphi\colon B \to C$ one has, in general, $\widehat{\varphi \circ \varrho} \neq \hat{\varphi} \circ \hat{\varrho}$) and get a map $\mathbf{Cyl}(\bullet)\colon \mathfrak{C} \to \mathbf{Mod}(\mathfrak{A})$ it has to be inductively modified. We will consider an instance of this process in Chapter 8, where we extend $\mathbf{Cyl}$ to a functor $\mathbf{Cyl}\colon \mathfrak{C}_3^{\mathbf{W}} \to \mathbf{Mod}(\mathfrak{C}_3^{\mathbf{W}})$ ($\mathfrak{C}_3$ being a coherator for 3-categories). One then has to generalize the process outlined in said chapter to all higher dimensions in order to render these interpretations functorial, thus succeeding in extending $\mathbf{Cyl}$ to $\mathfrak{C}$.

The idea to obtain the map $\hat{\varrho}$ is to construct a vertical stack of $(n-1)$-dimensional (possibly degenerate) cylinders in the $\mathfrak{A}$-model $\mathbf{Cyl}(A)(x_0, x_m)$ for an appropriate pair of 0-cells (x_0, x_m) in $\mathbf{Cyl}(A)$. We then compose this vertical stack using a vertical composition operation, and the result is an $(n-1)$-cylinder in the space of paths $\mathbf{Cyl}(A)(x_0, x_m)$, which, by construction, transposes to give the desired map:

$$\widehat{\varrho}\colon \mathbf{Cyl}(D_n) \to \mathbf{Cyl}(A)$$

3. Vertical composition of cylinders

The goal of this section is to define an operation that performs the vertical composition of an m-tuple of compatible n-cylinders. This operation takes as input a sequence of n-cylinders $F_i\colon A_i \curvearrowright A_{i+1}$ in an ∞-groupoid X, and produces an n-cylinder denoted by:

$$F_m \otimes F_{m-1} \otimes \ldots \otimes F_1\colon A_1 \curvearrowright A_{m+1}$$

It is represented by a map:

$$\mathbf{Cyl}(D_n) \longrightarrow \mathbf{Cyl}(D_n) \otimes \ldots \otimes \mathbf{Cyl}(D_n)$$

where the codomain is defined to be the colimit of the following diagram:

$$\begin{tikzcd}
\mathbf{Cyl}(D_n) & & \mathbf{Cyl}(D_n) & & \ldots & & \mathbf{Cyl}(D_n) \\
& D_n \arrow[ul, "\iota_1"] \arrow[ur, "\iota_0"'] & & D_n \arrow[ul, "\iota_1"] \arrow[ur, "\iota_0"'] & & D_n \arrow[ul, "\iota_1"] \arrow[ur, "\iota_0"'] &
\end{tikzcd}$$

Moreover, it will have the property that:

$$\varepsilon(F_m \otimes F_{m-1} \otimes \ldots \otimes F_1) = \varepsilon(F_m) \otimes \varepsilon(F_{m-1}) \otimes \ldots \otimes \varepsilon(F_1) \tag{37}$$

for $\varepsilon = s, t$.

To begin with, we have to do some preliminary work, constructing some morphisms in $\mathbf{Mod}(\mathfrak{A})^{\mathbb{G}}$, by extending suitable maps into globular sums. The solution to these extension problems will produce cylinders that represent coherent rebracketings of certain composites of globular pasting diagrams in a given $\mathfrak{A}$-model.

For example, given an $\mathfrak{A}$-model X and a map $(f, \alpha, g)\colon D_1 \amalg_{D_0} D_2 \amalg_{D_0} D_1 \to X$, that can be represented as the following pasting diagram labelled by cells of X:

$$\begin{tikzcd}
\bullet \arrow[r, "f"] & \bullet \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet \arrow[r, "g"] & \bullet
\end{tikzcd}$$

we can consider two ways of composing this pasting diagram, namely $(g\alpha)f$ and $g(\alpha f)$, where binary composition may be interpreted, for instance, using the maps ${}_{D_2}w, w_{D_2}$. In general these two cells will differ, and also their boundary will, being given respectively by the pairs of parallel 1-cells $((gh)f, (gk)f)$ and $(g(hf), g(kf))$. Therefore, a comparison between the two 2-cells cannot be encoded by a 3-cell, but rather by a 2-cylinder whose boundary consists of a pair of 1-cylinders encoding a comparison between the (possibly) different 1-cells we have just described.

To obtain these cylinders in general, given $m \geq 0$, we consider the following map in $\mathfrak{A}^{\mathbb{G}}$

$$\Sigma^m D_\bullet \coprod_{S^{m-1}} \Sigma^m D_\bullet \xrightarrow{\Sigma^m(\iota)} \Sigma^m \mathbf{Cyl}(D_\bullet)$$

This map belongs to $\mathbb{I}^{\mathbb{G}}$, thanks to Remark 5.1 and Lemma 3.14.

In what follows, we assume we have chosen composition operations $\gamma\colon D_n \to D_1^{\otimes m} \coprod_{D_0} D_n \coprod_{D_0} D_1^{\otimes k}$ for $k, m, n > 0$, which are compatible with the source and target maps, i.e.

$$\gamma \circ \varepsilon = \left(1_{D_1^{\otimes m}} \coprod_{D_0} \varepsilon \coprod_{D_0} 1_{D_1^{\otimes k}}\right) \circ \gamma$$

There is no risk of confusion in referring to all such maps as γ, because the codomain uniquely determines such γ.

Definition 7.26. Given $q, m, k \geq 0$, we consider the following coglobular objects in $\mathbf{Mod}(\mathfrak{A})$:

$$D_1^{\otimes m} \coprod_{D_0} \Sigma D_\bullet \coprod_{D_0} D_1^{\otimes k}$$

and

$$D_1^{\otimes q} \coprod_{D_0} D_m \coprod_{D_{m-1}} \Sigma^m D_\bullet \coprod_{D_0} D_1^{\otimes k}$$

where the structural maps the obvious ones.

For $m, k \neq 0$ define maps:

$$\Sigma D_\bullet \coprod_{S^0} \Sigma D_\bullet \xrightarrow{\psi^{m,k}} D_1^{\otimes m} \coprod_{D_0} \Sigma D_\bullet \coprod_{D_0} D_1^{\otimes k}$$

by setting the first component in dimension n to be given by the composite:

$$D_{n+1} \xrightarrow{\gamma} D_1^{\otimes m-1} \coprod_{D_0} D_{n+1} \coprod_{D_0} D_1^{\otimes k} \xrightarrow{1_{D_1^{\otimes m-1}} \amalg w \amalg 1_{D_1^{\otimes k}}} D_1^{\otimes m} \coprod_{D_0} D_{n+1} \coprod_{D_0} D_1^{\otimes k}$$

and the second one to be:

$$D_{n+1} \xrightarrow{\gamma} D_1^{\otimes m} \coprod_{D_0} D_{n+1} \coprod_{D_0} D_1^{\otimes k-1} \xrightarrow{1_{D_1^{\otimes m}} \amalg w \amalg 1_{D_1^{\otimes k-1}}} D_1^{\otimes m} \coprod_{D_0} D_{n+1} \coprod_{D_0} D_1^{\otimes k}$$

This means that given an $\mathfrak{A}$-model X and a map

$$(f_1, \ldots, f_m, \alpha, g_1, \ldots, g_k) \colon D_1^{\otimes m} \coprod_{D_0} D_{n+1} \coprod_{D_0} D_1^{\otimes k} \to X$$

we get a pair of $(n+1)$-cells in X of the form $g_k \ldots g_1(\alpha f_m) f_{m-1} \ldots f_1$ and $g_k \ldots g_2(g_1\alpha) f_m f_{m-1} \ldots f_1$, where juxtaposition is the result of composition using the appropriate γ or w.

If $m = 0$ and $k \neq 0$ define:

$$\Sigma D_\bullet \coprod_{S^0} \Sigma D_\bullet \xrightarrow{\psi^{0,k}} \Sigma D_\bullet \coprod_{D_0} D_1^{\otimes k}$$

by setting the first component in dimension n to be given by the composite:

$$D_{n+1} \xrightarrow{w} D_{n+1} \coprod_{D_0} D_1 \xrightarrow{1_{D_{n+1}} \amalg_{D_0} \gamma} D_{n+1} \coprod_{D_0} D_1^{\otimes k}$$

and the second one to be:

$$D_{n+1} \xrightarrow{\gamma} D_{n+1} \coprod_{D_0} D_1^{\otimes k-1} \xrightarrow{w \amalg 1_{D_1^{\otimes k-1}}} D_{n+1} \coprod_{D_0} D_1^{\otimes k}$$

This means that given an $\mathfrak{A}$-model X and a map:

$$(\alpha, g_1, \ldots, g_k) \colon D_{n+1} \coprod_{D_0} D_1^{\otimes k} \to X$$

we get a pair of $(n+1)$-cells in X of the form $(g_k \ldots g_1)\alpha$ and $g_k \ldots g_2(g_1\alpha)$, where juxtaposition is the result of composition using the appropriate γ or w, as described above. Finally, if $k = 0$ and $m \neq 0$ define:

$$\Sigma D_\bullet \coprod_{S^0} \Sigma D_\bullet \xrightarrow{\psi^{m,0}} D_1^{\otimes m} \coprod_{D_0} \Sigma D_\bullet$$

by setting the first component in dimension n to be given by the composite:

$$D_{n+1} \xrightarrow{w} D_1 \coprod_{D_0} D_{n+1} \xrightarrow{\gamma \amalg_{D_0} 1_{D_{n+1}}} D_1^{\otimes m} \coprod_{D_0} D_{n+1}$$

and the second one to be:

$$D_{n+1} \xrightarrow{\gamma} D_1^{\otimes m-1} \coprod_{D_0} D_{n+1} \xrightarrow{1_{D_1^{\otimes m-1}} \coprod_{D_0} w} D_1^{\otimes m} \coprod_{D_0} D_{n+1}$$

This means that given an $\mathfrak{A}$-model X and a map:

$$(f_1, \ldots, f_m, \alpha) \colon D_1^{\otimes m} \coprod_{D_0} D_{n+1} \to X$$

we get a pair of $(n+1)$-cells in X of the form $\alpha(f_m f_{m-1} \ldots f_1)$ and $(\alpha f_m) f_{m-1} \ldots f_1$, where juxtaposition is the result of composition using the appropriate γ or w as described above.

For $m \geq 1$ also define:

$$\Sigma^m D_\bullet \coprod_{S^{m-1}} \Sigma^m D_\bullet \xrightarrow{\varphi^{q,m,k}} D_1^{\otimes q} \coprod_{D_0} D_{m+1} \coprod_{D_m} \Sigma^m D_\bullet \coprod_{D_0} D_1^{\otimes k}$$

where the first component is given by the central composite in

$$\begin{tikzcd}
 & D_{m+1} \arrow[r, "\gamma"] \arrow[dr, "i"] & D_1^{\otimes q} \coprod_{D_0} D_{m+1} \coprod_{D_0} D_1^{\otimes k} \arrow[dr, "i"] & \\
D_{n+m} \arrow[rr, "\Sigma^m(w)"] & & D_{m+1} \coprod_{D_m} D_{n+m} \arrow[r, "f"] & D_1^{\otimes q} \coprod_{D_0} D_{m+1} \coprod_{D_m} D_{n+m} \coprod_{D_0} D_1^{\otimes k} \\
 & D_{n+m} \arrow[r, "\gamma"] \arrow[ur, "i"] & D_1^{\otimes q} \coprod_{D_0} D_{n+m} \coprod_{D_0} D_1^{\otimes k} \arrow[ur, "i"] &
\end{tikzcd}$$

where, with a minor abuse of language, we let i denote the various colimit inclusions and f the map induced by the universal property of colimits.

The second component is given by:

$$D_{n+m} \xrightarrow{\gamma} D_1^{\otimes q} \coprod_{D_0} D_{n+m} \coprod_{D_0} D_1^{\otimes k} \xrightarrow{1 \coprod_{D_0} \Sigma^m(w) \coprod_{D_0} D_1} D_1^{\otimes q} \coprod_{D_0} D_{m+1} \coprod_{D_m} D_{n+m} \coprod_{D_0} D_1^{\otimes k}$$

This means that, given a map:

$$(f_1, \ldots, f_q, \alpha, \beta, g_1, \ldots, g_k) \colon D_1^{\otimes q} \coprod_{D_0} D_{m+1} \coprod_{D_m} D_{n+m} \coprod_{D_0} D_1^{\otimes k} \to X$$

we get a pair of $(n+m)$-cells $(g_k \ldots g_1 \beta f_q \ldots f_1)(g_k \ldots g_1 \alpha f_q \ldots f_1)$ and $g_k \ldots g_1 (\beta\alpha) f_q \ldots f_1$, where juxtaposition stands for the result of composing those cells using the appropriate operations described above. Notice that both these $(n+m)$-cells can be interpreted as n-cells in $\Omega^m(X, \varphi)$ for appropriate choices of $\varphi \colon S^{m-1} \to X$.

Similarly to $\varphi^{q,m,k}$, we define a map:

$$\Sigma^m D_\bullet \coprod_{S^{m-1}} \Sigma^m D_\bullet \xrightarrow{\vartheta^{q,m,k}} D_1^{\otimes q} \coprod_{D_0} \Sigma^m D_\bullet \coprod_{D_m} D_{m+1} \coprod_{D_0} D_1^{\otimes k}$$

with a completely analogous definition on both components. This time, given a map:

$$(f_1, \ldots, f_q, \alpha, \beta, g_1, \ldots, g_k) \colon D_1^{\otimes q} \coprod_{D_0} \Sigma^m D_\bullet \coprod_{D_m} D_{m+1} \coprod_{D_0} D_1^{\otimes k} \to X$$

we get back a pair of $(n+m)$-cells $g_k \dots g_1(\beta\alpha)f_q \dots f_1$ and $(g_k \dots g_1\beta f_q \dots f_1)(g_k \dots g_1\alpha f_q \dots f_1)$, where juxtaposition stands for the result of composing those cells using the appropriate operations described above. Again, both of these $(n+m)$-cells can be interpreted as n-cells in $\Omega^m(X,\varphi)$ for appropriate choices of $\varphi\colon S^m \to X$.

Let us now define maps $\Psi^{m,k}$, $\Phi^{m,k}$ and $\Theta^{q,m,k}$ by constructing fillers as follows. Their existence is ensured by contractibility of globular sums in case $\mathfrak{A}$ is a coherator for ∞-groupoids, by the results of Section 3 of Chapter 2 if $\mathfrak{A}$ is a coherator for ∞-categories and by applying these results and the free functor $F\colon \mathbf{Mod}(\mathfrak{C}) \to \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$ in case $\mathfrak{A} \cong \mathfrak{C}^{\mathbf{W}}$. The extension problems we consider are the ones depicted below:

$$\begin{tikzcd}
\Sigma D_\bullet \coprod\limits_{S^0} \Sigma D_\bullet \arrow[r, "\psi^{m,k}"] \arrow[d, "\Sigma(\iota)"'] & D_1^{\otimes m} \coprod\limits_{D_0} \Sigma D_\bullet \coprod\limits_{D_0} D_1^{\otimes k} \\
\Sigma\mathbf{Cyl}(D_\bullet) \arrow[ur, dashed, "\Psi^{m,k}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma^m D_\bullet \coprod\limits_{S^{m-1}} \Sigma^m D_\bullet \arrow[r, "\varphi^{q,m,k}"] \arrow[d, "\Sigma^m(\iota)"'] & D_1^{\otimes q} \coprod\limits_{D_0} D_{m+1} \coprod\limits_{D_m} \Sigma^m D_\bullet \coprod\limits_{D_0} D_1^{\otimes k} \\
\Sigma^m\mathbf{Cyl}(D_\bullet) \arrow[ur, dashed, "\Phi^{m,k}"']
\end{tikzcd} \tag{38}$$

$$\begin{tikzcd}
\Sigma^m D_\bullet \coprod\limits_{S^{m-1}} \Sigma^m D_\bullet \arrow[r, "\vartheta^{q,m,k}"] \arrow[d, "\Sigma^m(\iota)"'] & D_1^{\otimes q} \coprod\limits_{D_0} \Sigma^m D_\bullet \coprod\limits_{D_m} D_{m+1} \coprod\limits_{D_0} D_1^{\otimes k} \\
\Sigma^m\mathbf{Cyl}(D_\bullet) \arrow[ur, dashed, "\Theta^{q,m,k}"']
\end{tikzcd}$$

For example, this means that given a map:

$$(f_1, \dots, f_m, \alpha, g_1, \dots, g_k)\colon D_1^{\otimes m} \coprod_{D_0} D_{n+1} \coprod_{D_0} D_1^{\otimes k} \to X$$

we get an n-cylinder in $\Omega(X, s(F_1), t(g_k))$ of the form:

$$\Psi^{m,k}(f_1, \dots, f_m, \alpha, g_1, \dots, g_k)\colon g_k \dots g_1(\alpha f_m)f_{m-1} \dots f_1 \curvearrowright g_k \dots g_2(g_1\alpha)f_m f_{m-1} \dots f_1$$

and similarly for the other cases.

We are now ready to define vertical composition of cylinders. We define this recursively on n, simultaneously with an operation which, given:

$$\begin{cases} m \geq 0,\ g\colon S^m \to X,\ g_{|S^0} = (b,c) \text{ and an } n\text{-cylinder } F\colon A \curvearrowright B \text{ in } \Omega^{m+1}(X,g), \\ \text{1-cells } h_i\colon b_i \to b_{i+1} \text{ in } X,\ 1 \leq i \leq q, b_{q+1} = b \\ \text{1-cells } f_j\colon c_j \to c_{j+1} \text{ in } X,\ 1 \leq j \leq k, c_1 = c \end{cases}$$

produces an n-cylinder:

$$f_k \dots f_1 F h_q \dots h_1\colon f_k \dots f_1 A h_q \dots h_1 \curvearrowright f_k \dots f_1 B h_q \dots h_1 \tag{39}$$

in $\Omega^{m+1}(X, f_k \dots f_1 * g * h_q \dots h_1)$, where we define, for $m > 0$ and $g = (g_0, g_1)$:

$$f_k \dots f_1 * g * h_q \dots h_1 = (f_k \dots f_1 g_0 h_q \dots h_1, f_k \dots f_1 g_1 h_q \dots h_1)$$

where juxtaposition represents the result of composing cells using the appropriate operation γ. If $m = 0$ then $g = (b, c) \colon S^0 \to X$, and we define:

$$f_k \dots f_1 * g * h_q \dots h_1 = (s(h_1), t(f_k))$$

We begin with the case $n = 0$. A vertical stack of 0-cylinders consists of a string of composable 1-cells, which we compose using the appropriate choice of γ. Explicitly, given 0-cylinders $(F^i \colon x_i \curvearrowright x_{i+1})_{i \in \{1,\dots,n\}}$, i.e. 1-cells $F^i \colon x_i \to x_{i+1}$ in X, we define:

$$F^n \otimes \dots \otimes \ F^1 = F^n \dots F^1 \colon x_1 \curvearrowright x_{n+1}$$

To construct (39), note that a 0-cylinder $F \colon A \curvearrowright B$ in $\Omega^{m+1}(X, g)$ is just an $(m+2)$-cell in X. Therefore, we simply define the required 0-cylinder by:

$$f_k \dots f_1 F h_q \dots h_1 \colon f_k \dots f_1 A h_q \dots h_1 \curvearrowright f_k \dots f_1 B h_q \dots h_1$$

where, as usual, juxtaposition means the result of composing those cells using the appropriate operation γ

Turning to the recursive step, given $p > 0$ and $(n+1)$-cylinders $F^i \colon A_i \curvearrowright A_{i+1}$ in X for $1 \leq i \leq p$ we want to define an $(n+1)$-cylinder:

$$F^p \otimes \dots \otimes F^1 \colon A_1 \curvearrowright A_{p+1}$$

in a way that is compatible with the already-defined composition in lower dimensions. We can express the cylinders F^i in an equivalent way, by considering them as n-cylinders:

$$\bar{F}^i \colon A_{i+1} F^i_s \curvearrowright F^i_t A_i$$

in $X(s(F^i_s), t(F^i_t))$. We define the $(n+1)$-cylinder $F^p \otimes \dots \otimes F^1$ by setting:

$$(F^p \otimes \dots \otimes F^1)_\varepsilon = {F^p}_\varepsilon \dots {F^1}_\varepsilon$$

for $\varepsilon = s, t$, using γ to compose these 1-cells, and defining $\overline{F^p \otimes \ldots \otimes F^1}$ to be the vertical composition of the following sequence of n-cylinders in $X(s(F^1_s), t(F^p_t))$:

$$\begin{array}{c}
(F^p_t \ldots F^1_t)A_0 \\
\Big\downarrow {\scriptstyle \Psi^{0,p}(F^p_t,\ldots,F^1_t,A_0)} \\
F^p_t \ldots F^1_t(F^0_t A_0) \\
\Big\downarrow {\scriptstyle F^p_t \ldots F^1_t \overline{F^0}} \\
F^p_t \ldots F^1_t(A_1 F^1_s) \\
\Big\downarrow {\scriptstyle \Psi^{1,p-1}(F^p_t,\ldots,F^1_t,A_1,F^1_s)} \\
F^p_t \ldots (F^1_t A_1)F^1_s \\
\Big\downarrow \\
\ldots \\
\Big\downarrow {\scriptstyle \Psi^{p-1,1}(F^p_t,A_p,F^{p-1}_s,\ldots,F^1_s)} \\
(F^p_t A_p)F^{p-1}_s \ldots F^1_s \\
\Big\downarrow {\scriptstyle \overline{F^p} F^{p-1}_s \ldots F^1_s} \\
(A_{p+1}F^p_s)F^{p-1}_s \ldots F^1_s \\
\Big\downarrow {\scriptstyle \Psi^{p,0}(A_{p+1},F^p_s,\ldots,F^1_s)} \\
A_{p+1}(F^p_s \ldots F^1_s)
\end{array}$$

Let us now address the construction of (39).

The data are the following:

$$\begin{cases} m \geq 0,\ g\colon S^m \to X,\ g_{|S^0} = (b,c),\ \text{an } (n+1)\text{-cylinder } F\colon A \curvearrowright B \text{ in } \Omega^{m+1}(X,g), \\ \text{1-cells in } h_i\colon b_i \to b_{i+1} \text{ in } X,\ 1 \leq i \leq q, b_{q+1} = b \\ \text{1-cells in } f_j\colon c_j \to c_{j+1} \text{ in } X,\ 1 \leq j \leq k, c_{k+1} = c \end{cases}$$

View F as an n-cylinder:

$$\bar{F}\colon BF_s \curvearrowright F_t A \text{ in } \Omega(\Omega^{m+1}(X,g), s^{n+1}A, s^{n+1}B) \cong \Omega^{m+2}(X,\varphi)$$

where we set $\varphi\colon = (s^{n+1}A, s^{n+1}B)\colon S^{m+1} \to X$.

By recursion we can construct an n-cylinder:

$$f_k \ldots f_1 \bar{F} h_q \ldots h_1 \colon f_k \ldots f_1(BF_s)h_q \ldots h_1 \curvearrowright f_k \ldots f_1(F_t A)h_q \ldots h_1$$

in $\Omega^{m+2}(X, f_k \ldots f_1 * \varphi * h_q \ldots h_1)$, which is isomorphic to:

$$\Omega(\Omega^{m+1}(X, f_k \ldots f_1 * g * h_q \ldots h_1), f_k \ldots f_1 * s^{n+1}(A) * h_q \ldots h_1, f_k \ldots f_1 * t^{n+1}(B) * h_q \ldots h_1)$$

Finally, we define $f_k \ldots f_1 F h_q \ldots h_1$ by setting:

$$(f_k \ldots f_1 F h_q \ldots h_1)_\varepsilon = f_k \ldots f_1 F_\varepsilon h_q \ldots h_1$$

for $\varepsilon = s, t$, using γ to compose these 1-cells, and defining $\overline{f_k \dots f_1 F h_q \dots h_1}$ to be the vertical composition of the following sequence of n-cylinders in $\Omega^{m+2}(X, f_k \dots f_1 * \varphi * h_q \dots h_1)$:

$$\begin{array}{c}
(f_k \dots f_1 B h_q \dots h_1)(f_k \dots f_1 F_s h_q \dots h_1) \\
\big\downarrow {\scriptstyle \Phi^{q,m+2,k}(f_k,\dots,f_1,B,F_s,h_q,\dots h_1)} \\
f_k \dots f_1 (B F_s) h_q \dots h_1 \\
\big\downarrow {\scriptstyle f_k \dots f_1 \bar{F} h_q \dots h_1} \\
f_k \dots f_1 (F_t A) h_q \dots h_1 \\
\big\downarrow {\scriptstyle \Theta^{q,m+2,k}(f_k,\dots,f_1,F_t,A,h_q,\dots h_1)} \\
(f_k \dots f_1 F_t h_q \dots h_1)(f_k \dots f_1 A h_q \dots h_1)
\end{array}$$

This completes the recursion. Now, by an easy induction, one can check that this operation we have obtained is coglobular, i.e. it satisfies the condition expressed in (37).

4. Naive elementary interpretation of operations

We now describe how to define the map of $\mathfrak{A}$-models $\hat{\varrho} \colon \mathbf{Cyl}(D_n) \to \mathbf{Cyl}(A)$ for any given homogeneous operation in a coherator for ∞-categories $\mathfrak{C}$ in a way that will satisfy the first condition in (36) but not the second. The next section will then address and solve the problem of also satisfying the second condition. Let $p = |\mathscr{L}(A)|$, and $B_1, \dots, B_p$ the ordered list $\mathscr{L}(A)$. We recall here that the idea is to define $\hat{\varrho}$ as the transpose of the vertical composite of a stack of $(n-1)$-cylinders in $\mathbf{Cyl}(A)\,(a, b)$, for $a = \mathbf{Cyl}(\partial^m_\sigma) \circ \sigma$ and $b = \mathbf{Cyl}(\partial^m_\tau) \circ \tau$.

First, we specify a sequence of n-cells $(\alpha_0, \dots, \alpha_p)$ in $\mathbf{Cyl}(A)$ that appear as top and bottom cells of the cylinders in the stack. The first and the last are thus forced by the requirement that the vertical composite $(n-1)$-cylinder in the space of paths between a and b is the transpose of an actual n-cylinder in $\mathbf{Cyl}(A)$ satisfying the first of the conditions expressed in (36). Therefore, α_0 must be given by the composite:

$$D_n \xrightarrow{D_n w} D_n \coprod_{D_0} D_1 \xrightarrow{(\iota_0 \circ \varrho, \mathbf{Cyl}(\partial^m_\tau))} \mathbf{Cyl}(A)$$

where $m = \mathrm{ht}(A)$. Similarly, α_p must be defined to be the composite

$$D_n \xrightarrow{w_{D_n}} D_1 \coprod_{D_0} D_n \xrightarrow{(\mathbf{Cyl}(\partial^m_\sigma), \iota_1 \circ \varrho)} \mathbf{Cyl}(A)$$

For $1 \leq i \leq p-1$, we define $\alpha_i \colon D_n \to \mathbf{Cyl}(A)$ as the following composite

$$D_n \xrightarrow{\varrho} A \xrightarrow{z^A_{B_i}} B_i \xrightarrow{i_{B_i}} \mathbf{Cyl}(A) \tag{40}$$

Notice that the α_i's all transpose under the adjunction $\Sigma \dashv \Omega$ to give $(n-1)$-cells $\overline{\alpha_0}, \dots, \overline{\alpha_p}$ in $\mathbf{Cyl}(A)\,(a, b)$, where $a = \mathbf{Cyl}(\partial^m_\sigma) \circ \sigma$ and $b = \mathbf{Cyl}(\partial^m_\tau) \circ \tau$. In addition, by construction, $\overline{\alpha_{i-1}}$ and $\overline{\alpha_i}$ factor through $B_i(a, b)$, where, for every $B_i \in \mathscr{L}(A)$, we denote the endpoints of this globular sum (i.e. the 0-cells $\partial^{\mathrm{ht}(B_i)}_\sigma, \partial^{\mathrm{ht}(B_i)}_\tau \colon D_0 \to B_i$) with a and b, committing a slight abuse of language since they are all sent to a and b in $\mathbf{Cyl}(A)$ by the maps i_{B_i}.

Example 7.27. Let $A = D_2 \amalg_{D_0} D_1$ and consider a homogeneous map $\varrho \colon D_2 \to A$. This operation may represent, for instance, the whiskering of a 2-cell with a 1-cell.

Thanks to diagram (31), if we follow the algorithm explained above we find that the cells in the list we have to provide are given by the transposes of:

$$\begin{tikzcd}
 & & D_2 \underset{D_0}{\amalg} D_1 \underset{D_0}{\amalg} D_1 \arrow[dddr] & \\
D_2 \arrow[r, "\varrho"] & D_2 \underset{D_0}{\amalg} D_1 \arrow[ur, "1 \amalg w"] & D_2 \underset{D_0}{\amalg} D_2 \arrow[ddr] & \\
D_2 \arrow[r, "\varrho"] & D_2 \underset{D_0}{\amalg} D_1 \arrow[ur, "1 \amalg \tau"] & D_2 \underset{D_0}{\amalg} D_1 \underset{D_0}{\amalg} D_1 \arrow[dr] & \\
D_2 \arrow[r, "\varrho"] & D_2 \underset{D_0}{\amalg} D_1 \arrow[ur, "w \amalg 1"] & D_2 \underset{D_1}{\amalg} D_2 \underset{D_0}{\amalg} D_1 \arrow[r] & \mathbf{Cyl}(A) \\
D_2 \arrow[r, "\varrho"] & D_2 \underset{D_0}{\amalg} D_1 \arrow[ur, "w \amalg 1"] & D_3 \underset{D_0}{\amalg} D_1 \arrow[ur] & \\
D_2 \arrow[r, "\varrho"] & D_2 \underset{D_0}{\amalg} D_1 \arrow[ur, "\tau \amalg 1"] & D_2 \underset{D_1}{\amalg} D_2 \underset{D_0}{\amalg} D_1 \arrow[uur] & \\
D_2 \arrow[r, "\varrho"] & D_2 \underset{D_0}{\amalg} D_1 \arrow[ur, "{(i_1, i_2)}"] & &
\end{tikzcd}$$

together with the transpose of the composite

$$D_2 \xrightarrow{w} D_2 \underset{D_0}{\amalg} D_1 \xrightarrow{\varrho \underset{D_0}{\amalg} 1} A \underset{D_0}{\amalg} D_1 \xrightarrow{(\iota_0, \mathbf{Cyl}(\partial^2_\tau))} \mathbf{Cyl}(A)$$

as the first cell of the list, and of the composite

$$D_2 \xrightarrow{w} D_2 \underset{D_0}{\amalg} D_1 \xrightarrow{1 \underset{D_0}{\amalg} \varrho} D_1 \underset{D_0}{\amalg} A \xrightarrow{(\mathbf{Cyl}(\partial^2_\sigma), \iota_1)} \mathbf{Cyl}(A)$$

as the last.

We can now define a stack of $(n-1)$-cylinders $C_i^\varrho \colon \overline{\alpha_{i-1}} \curvearrowright \overline{\alpha_i}$ in $\mathbf{Cyl}(A)\,(a,b)$, for $1 \leq i \leq p$, such that C_i^ϱ factors through $B_i(a,b)$.

We do so by solving the following lifting problems:

$$\begin{tikzcd}
D_{n-1} \amalg D_{n-1} \arrow[r, "(\overline{\alpha_{i-1}}, \overline{\alpha_i})"] \arrow[d, "\partial"'] & B_i(a,b) \arrow[r, "\Omega(i_{B_i})"] & \mathbf{Cyl}(A)\,(a,b) \\
\mathbf{Cyl}(D_{n-1}) \arrow[ur, dashed] \arrow[urr, "C_i^\varrho"'] & &
\end{tikzcd}$$

In detail, we first transpose along the adjunction $\Sigma \dashv \Omega$ to get diagrams of the form:

$$\begin{tikzcd} \Sigma(D_{n-1} \amalg D_{n-1}) \arrow[r, "(\alpha_{i-1},\alpha_i)"] \arrow[d, "\partial"'] & B_i \\ \Sigma\mathbf{Cyl}(D_{n-1}) \arrow[ur, dashed] & \end{tikzcd}$$

The lift is now obtained by contractibility of globular sums in case $\mathfrak{A}$ is a coherator for ∞-groupoids, by the results of Section 3 of Chapter 2 if $\mathfrak{A}$ is a coherator for ∞-categories and by applying these results plus the free functor $F\colon \mathbf{Mod}(\mathfrak{C}) \to \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$ in case $\mathfrak{A} \cong \mathfrak{C}^{\mathbf{W}}$. Finally, set $\hat{\varrho}$ as the n-cylinder induced by the transpose of $C^{\varrho}_p \otimes \ldots \otimes C^{\varrho}_1 \colon \alpha_0 \curvearrowright \alpha_p$, viewed as a map $\Sigma\mathbf{Cyl}(D_{n-1}) \to \mathbf{Cyl}(A)$. It is now straightforward to check that, in general, this definition satisfies only the first condition of (36), essentially because we have no control on what happens to the boundary of $\hat{\varrho}$, i.e. $\hat{\varrho} \circ \mathbf{Cyl}(\varepsilon)$, in relation to $\widehat{\varrho \circ \varepsilon}$ for $\varepsilon = \sigma, \tau$. To rectify this, we will need to be a bit more careful with the definition of the stack of cylinders.

5. Degenerate Cylinders

In this section we want to define cylinders whose iterated source or target are degenerate, in a suitable sense. We will also extend the operation of vertical compositions to this more general setting, as it will be needed later to construct the "correct" elementary interpretation of a homogeneous operation, i.e. the one satisfying both conditions expressed in (36).

Definition 7.28. Let $n > 0$. Define the $\mathfrak{A}$-model $\mathbf{Cyl}^0_{-1}(D_n) = \mathbf{Cyl}^0(D_n)$ as the colimit of the following diagram:

$$\begin{tikzcd} & D_n \coprod_{D_0} D_1 \\ D_n \arrow[ur, "w"] \arrow[dr, "\Sigma(\iota_0)"'] & \\ & \Sigma\mathbf{Cyl}(D_{n-1}) \end{tikzcd}$$

Similarly, define $\mathbf{Cyl}^{-1}_0(D_n) = \mathbf{Cyl}_0(D_n)$ as the colimit of the following diagram:

$$\begin{tikzcd} & D_1 \coprod_{D_0} D_n \\ D_n \arrow[ur, "w"] \arrow[dr, "\Sigma(\iota_1)"'] & \\ & \Sigma\mathbf{Cyl}(D_{n-1}) \end{tikzcd}$$

Also set $\mathbf{Cyl}^0_0(D_n) = \Sigma\mathbf{Cyl}(D_{n-1})$.

Finally, given $0 < p, q < n$ with $|p - q| \leq 1$, define inductively:

$$\mathbf{Cyl}^p_q(D_n) = \Sigma\mathbf{Cyl}^{p-1}_{q-1}(D_{n-1})$$

We call $\mathbf{Cyl}^p_q$ the n-cylinder with degenerate p-source and degenerate q-target.

It is clear that all these cylinders come equipped with maps $\iota_0, \iota_1 \colon D_n \to \mathbf{Cyl}^p_q(D_n)$.

Definition 7.29. Given a pair of n-cells α, β in X and integers $0 \leq p, q < n$ as above, an n-cylinder in X from α to β with degenerate p-source and degenerate q-target is a map:

$$C \colon \mathbf{Cyl}^p_q(D_n) \to X$$

such that $C \circ \iota_0 = \alpha$ and $C \circ \iota_1 = \beta$. We will denote it by $C\colon \alpha \curvearrowright^p_q \beta$.

REMARK 7.30. Notice that a cylinder $C\colon \alpha \curvearrowright^p_q \beta$ exists only if $s^p(\alpha) = s^p(\beta)$ and $t^q(\alpha) = t^q(\beta)$.

To describe these data explicitly, we need to distinguish between cases.
If $p = 0$ and $q = -1$ then it consists of:

- a 1-cell $c\colon t^n(\alpha) \to t^n(\beta)$;
- an $(n-1)$-cylinder $\bar{C}\colon c\alpha \curvearrowright \beta$ in $X\,(s^n(\alpha), t^n(\beta))$.

If $p = -1$ and $q = 0$ then it consists of:

- a 1-cell $c\colon s^n(\alpha) \to s^n(\beta)$;
- an $(n-1)$-cylinder $\bar{C}\colon \alpha \curvearrowright \beta c$ in $X\,(s^n(\alpha), t^n(\beta))$.

If $p, q > 0$ then it consists of:

- an $(n-1)$-cylinder with degenerate $(p-1)$-source and $(q-1)$-target $\bar{C}\colon \alpha \curvearrowright^{p-1}_{q-1} \beta$ in $X\,(s^n(\alpha), t^n(\beta))$.

DEFINITION 7.31. Let $p, q \geq -1$ be integers such that $|p - q| \leq 1$. We define the category $\mathbb{G}^p_q$ as the full subcategory of $\mathbb{G}$ generated by:

- $\mathbb{G}_{\geq p+1}$ if $p = q$;
- $\mathbb{G}_{\geq p}$ and $\tau\colon q \to p$ if $q = p - 1$;
- $\mathbb{G}_{\geq q}$ and $\sigma\colon p \to q$ if $p = q - 1$;

Clearly, the direct category structure on $\mathbb{G}$ restricts to one on $\mathbb{G}^p_q$, and we can extend the previous construction to a functor

$$\mathbf{Cyl}^p_q(D_\bullet)\colon \mathbb{G}^p_q \to \mathbf{Mod}(\mathfrak{A})$$

5.1. Boundary of degenerate cylinders. As we did for normal cylinders, we will construct a map of diagrams indexed by a direct category, with codomain $\mathbf{Cyl}^p_q(D_\bullet)$, whose latching maps will represent the inclusion of the boundary of a degenerate cylinder. This construction will be fundamental to perform inductive constructions involving cylinders. Given such p, q as before, we construct a functor $B^p_q\colon \mathbb{G}^p_q \to \mathbf{Mod}(\mathfrak{A})$ by defining:

$$B^p_q(n) \cong \operatorname{colim} \begin{tikzcd} D_p \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & D_n \\ D_n & D_q \arrow[l, "\tau"'] \arrow[u, "\tau"'] \end{tikzcd}$$

where we set $D_{-1} = \emptyset$, the initial object of $\mathbf{Mod}(\mathfrak{A})$.

For each such pair of integers we get a natural transformation:

$$\iota\colon B^p_q \to \mathbf{Cyl}^p_q(D_\bullet)$$

induced by $\iota_0, \iota_1 : D_n \to \mathbf{Cyl}^p_q(D_n)$.

DEFINITION 7.32. We define the boundary of the n-cylinder with degenerate p-source and q-target to be the domain of $\hat{L}_n(\iota)$, and we denote it by $\partial\mathbf{Cyl}^p_q(D_n)$. Given an $\mathfrak{A}$-model X and an n-cylinder with degenerate p-source and q-target in X, represented by a map

$C : \mathbf{Cyl}^p_q(D_n) \to X$, we call the boundary of C the map we get by precomposing C with the natural map from the boundary, as displayed below:

$$\partial\mathbf{Cyl}^p_q(D_n) \rightarrowtail \mathbf{Cyl}^p_q(D_n) \xrightarrow{C} X$$

By definition, given an $\mathfrak{A}$-model X, specifying the boundary of a degenerate n-cylinder $\mathbf{Cyl}^p_q(D_n) \to X$ is equivalent to providing the following data:

- a pair of parallel $(n-1)$-cylinders $C\colon A \curvearrowright^p_q B$, $D\colon A' \curvearrowright^p_q B'$ in X (except in the case where $n-1 = p, q$);
- a pair of n-cells $\alpha\colon A \to A', \beta\colon B \to B'$ in X.

Example 7.33. A 1-cylinder with degenerate 0-source in an $\mathfrak{A}$-model X is represented by a map $C : \mathbf{Cyl}^0(D_1) \to X$, which consists of specifying the following data:

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, equal] \arrow[dr, phantom, "C\Downarrow"] & b \arrow[d, "g"] \\ a \arrow[r, "\beta"'] & c \end{tikzcd}$$

by which we mean a 2-cell $C\colon g\alpha \to \beta$. In this case, the pair (α, β) represents the natural map $\iota\colon B^0_{-1}(1) \to \mathbf{Cyl}^0(D_1)$.

Proposition 7.34. *The map*

$$\iota\colon B^p_q \to \mathbf{Cyl}^p_q(D_\bullet)$$

is a direct cofibration in $\mathbf{Mod}(\mathfrak{A})^{\mathbb{G}^p_q}$.

Proof. If $p, q \geq 0$, we have

$$B^p_q \cong \Sigma B^{p-1}_{q-1} \text{ and } \mathbf{Cyl}^p_q \cong \Sigma\mathbf{Cyl}^{p-1}_{q-1}$$

and the map $B^p_q \to \mathbf{Cyl}^p_q$ results from applying Σ to $B^{p-1}_{q-1} \to \mathbf{Cyl}^{p-1}_{q-1}$. Therefore, since Σ preserves cofibrations, it is enough to prove the result for $p = 0, q = -1$ and $p = -1, q = 0$. Let's consider $\mathbf{Cyl}^0(D_\bullet)$. Consider the following square where $n > 1$, which is cocartesian thanks to Proposition 5.9:

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D_n \arrow[d] \\ \partial\mathbf{Cyl}(D_{n-1}) \arrow[r, "\hat{L}_{n-1}(\iota)"] & \mathbf{Cyl}(D_{n-1}) \end{tikzcd}$$

We can prove representably (as we did for Proposition 5.9) that the following square is cocartesian, thus concluding the proof thanks to Remark 5.1:

$$\begin{tikzcd} \Sigma\partial\mathbf{Cyl}(D^{n-1}) \arrow[r, "\Sigma(\hat{L}_{n-1}(\iota))"] \arrow[d] & \Sigma\mathbf{Cyl}(D^{n-1}) \arrow[d] \\ \partial\mathbf{Cyl}^0(D_n) \arrow[r, "\hat{L}_n(\iota)"] & \mathbf{Cyl}^0(D^n) \end{tikzcd}$$

A similar argument shows that also $B_0 \to \mathbf{Cyl}_0(D_\bullet)$ is a direct cofibration of $\mathbb{G}_0$-diagrams in $\mathbf{Mod}(\mathfrak{A})$. $\square$

5.2. Vertical composition of degenerate cylinders. We now want to define an operation of vertical composition that generalizes the one we already have to the case of a vertical stack of (possibly) degenerate cylinders.

To do so, assume given a k-tuple of pairs of integers $(p_i, k_i)_{1\leq i\leq k}$, with $|p_i - q_i| \leq 1$ for each $1 \leq i \leq k$, and let $p = \min\{p_i\}_{1\leq i\leq k},\ q = \min\{q_i\}_{1\leq i\leq k}$. This operation is represented by a map:

$$\mathbf{Cyl}_q^p(D_n) \longrightarrow \mathbf{Cyl}_{q_1}^{p_1}(D_n) \otimes \ldots \otimes \mathbf{Cyl}_{q_k}^{p_k}(D_n) \tag{41}$$

where the codomain is defined to be the colimit of the following diagram:

$$\begin{tikzcd}[column sep=small]
\mathbf{Cyl}_{q_1}^{p_1}(D_n) & & \mathbf{Cyl}_{q_2}^{p_2}(D_n) & & \ldots & & \mathbf{Cyl}_{q_k}^{p_k}(D_n) \\
& D_n \arrow[ul, "\iota_1"] \arrow[ur, "\iota_0"'] & & D_n \arrow[ul, "\iota_1"] \arrow[ur, "\iota_0"'] & & D_n \arrow[ul, "\iota_1"] \arrow[ur, "\iota_0"'] &
\end{tikzcd}$$

We will adapt the construction we already have for the case $p_i = q_i = -1$, and again we proceed to construct it recursively on n, simultaneously with an operation of whiskerings of degenerate cylinders with 1-cells. Given:

$$\begin{cases} m \geq 0,\ g\colon S^m \to X,\ g_{|S^0} = (b, c) \text{ an } n\text{-cylinder } C\colon A \curvearrowright_q^p B \text{ in } \Omega^{m+1}(X, g) \\ \text{1-cells } h_i\colon b_i \to b_{i+1} \text{ in } X,\ 1 \leq i \leq q,\ b_{q+1} = b \\ \text{1-cells } f_j\colon c_j \to c_{j+1} \text{ in } X,\ 1 \leq j \leq k,\ c_1 = c \end{cases}$$

this whiskering operation produces an n-cylinder:

$$f_k \ldots f_1 F h_q \ldots h_1\colon f_k \ldots f_1 A h_q \ldots h_1 \curvearrowright_q^p f_k \ldots f_1 B h_q \ldots h_1$$

in $\Omega^{m+1}(X, f_k \ldots f_1 * g * h_q \ldots h_1)$, where we define (for $m > 0$):

$$f_k \ldots f_1 * g * h_q \ldots h_1 = (f_k \ldots f_1 g_0 h_q \ldots h_1, f_k \ldots f_1 g_1 h_q \ldots h_1)$$

if $g = (g_0, g_1)$. Once we have constructed these whiskerings, the rest of the proof follows just by adapting the one for normal cylinders, omitting the use of the Ψ's when no rebracketing is needed.

Let us start with the case $p = 0,\ q = -1$. By definition, C induces an $(n-1)$-cylinder:

$$\bar{C}\colon C_t A \curvearrowright B \text{ in } \Omega^{m+2}(X, \varphi)$$

where $\varphi = (s^n(A), t^n(B))$. Because we already know how to whisker normal cylinders with 1-cells, we get an $(n-1)$-cylinder:

$$f_k \ldots f_1 \bar{C} h_q \ldots h_1\colon f_k \ldots f_1 (C_t A) h_q \ldots h_1 \curvearrowright f_k \ldots f_1 B h_q \ldots h_1$$

We now define:

$$f_k \ldots f_1 C h_q \ldots h_1\colon f_k \ldots f_1 A h_q \ldots h_1 \curvearrowright^0 f_k \ldots f_1 B h_q \ldots h_1$$

as the vertical composition of the following cylinders in $\Omega^{m+2}(X, f_k \dots f_1 * \varphi * h_q \dots h_1)$:

$$\begin{array}{c} (f_k \dots f_1 c\ h_q \dots h_1)(f_k \dots f_1 A h_q \dots h_1) \\ \Big\downarrow {\scriptstyle \Phi^{q,m+2,k}(f_k,\dots,f_1,c,A,h_q,\dots h_1)} \\ f_k \dots f_1 (C_t A) h_q \dots h_1 \\ \Big\downarrow {\scriptstyle f_k \dots f_1 \bar{C} h_q \dots h_1} \\ f_k \dots f_1 B h_q \dots h_1 \end{array}$$

In a completely analogous way, if $p = -1,\ q = 0$, we obtain:

$$f_k \dots f_1 C h_q \dots h_1 \colon f_k \dots f_1 A h_q \dots h_1 \curvearrowright_0 f_k \dots f_1 (B C_s) h_q \dots h_1$$

in $\Omega^{m+2}(X, f_k \dots f_1 * \varphi * h_q \dots h_1)$.

The case $C\colon A \curvearrowright_0^0 B$ in $\Omega^{m+1}(X, g)$ is even simpler, because we simply define the whiskering as:

$$\overline{f_k \dots f_1 C h_q \dots h_1} = f_k \dots f_1 \bar{C} h_q \dots h_1$$

This concludes the base case of the recursion.

Finally, let us consider the case $C\colon A \curvearrowright_{p-1}^{p} B$ in $\Omega^{m+1}(X, g)$ with $p > 0$ (the remaining case $C\colon A \curvearrowright_{p}^{p-1} B$ in $\Omega^{m+1}(X, g)$ can be treated similarly). By definition, we have a cylinder:

$$\bar{C}\colon A \curvearrowright_{p-2}^{p-1} B \text{ in } \Omega^{m+2}(X, \varphi)$$

where $\varphi = (s^n(A), t^n(B))$. By inductive hypothesis, we obtain:

$$f_k \dots f_1 \bar{C} h_q \dots h_1 \colon f_k \dots f_1 A h_q \dots h_1 \curvearrowright_{p-2}^{p-1} f_k \dots f_1 B h_q \dots h_1$$

so that we can set:

$$\overline{f_k \dots f_1 C h_q \dots h_1} = f_k \dots f_1 \bar{C} h_q \dots h_1$$

Given vertically composable (possibly degenerate) n-cylinders $C_1, \dots, C_k$ in an $\mathfrak{A}$-model X, we denote by $C_1 \otimes \dots \otimes C_k$ the n-cylinder in X that results as their vertical composition.

6. Elementary interpretation of operations

In this section we finally define, for every homogeneous operation $\varrho : D_m \to A$ in a fixed coherator for ∞-categories $\mathfrak{C}$ (endowed with a map $\mathfrak{C} \to \mathfrak{A}$), a map:

$$\hat{\varrho}\colon \mathbf{Cyl}(D_m) \to \mathbf{Cyl}(A)$$

in $\mathbf{Mod}(\mathfrak{A})$ satisfying both properties depicted in (36). To achieve the goal we set for this section, given $\varepsilon = \sigma, \tau$ we need a description of the map:

$$\mathbf{Cyl}(\partial_\varepsilon)\colon \mathbf{Cyl}(\partial A) \to \mathbf{Cyl}(A) \tag{42}$$

in terms of the globular decomposition of both its domain and its target, where $\partial_\varepsilon : \partial A \to A$ are the maps in Θ_0 defined in 11.

By construction, we know that the bottom row (see (27)) of the globular decomposition of $\mathbf{Cyl}(A)$ is obtained by sticking a new branch at the bottom right of the tree associated with A and then letting this new branch traverse the tree counterclockwise.

We let $n = \mathrm{ht}(A)$. To begin with, we explain how to relate the list of trees appearing on the bottom row of the globular decomposition of $\mathbf{Cyl}(\partial A)$ with the one associated with

$\mathbf{Cyl}(A)$. In fact, this can be done in two ways, depending on whether we are interested in $\varepsilon = \sigma$ or $\varepsilon = \tau$ in (42). We will focus on $\varepsilon = \sigma$ in what follows. We have three possible cases for the newly added branch in each tree belonging to the set $\mathscr{L}(A)$:

- it is attached at height n, i.e. the new vertex is added at height $n + 1$;
- it is attached at height $k < n - 1$;
- it is attached at height $n - 1$.

Of these trees, we discard all of those in the first class, and we keep all the trees in the second class, chopping off everything above height $k = n - 1$. In the third case, if we focus on the strip between height $n - 1$ and n, the newly added branch has to appear in a certain corolla. If the newly added branch is at the far left of the corolla it belongs to, then we chop off everything above height $n - 1$ except this new branch, and we keep the resulting tree. If not, we discard the tree.

By doing so we get a new list that can be easily proven to correspond exactly to the one associated with ∂A, and we have maps in Θ_0 between each tree in the list for $\mathbf{Cyl}(\partial A)$ and the corresponding one in the list for $\mathbf{Cyl}(A)$, induced by the source maps or appropriate colimit inclusions.

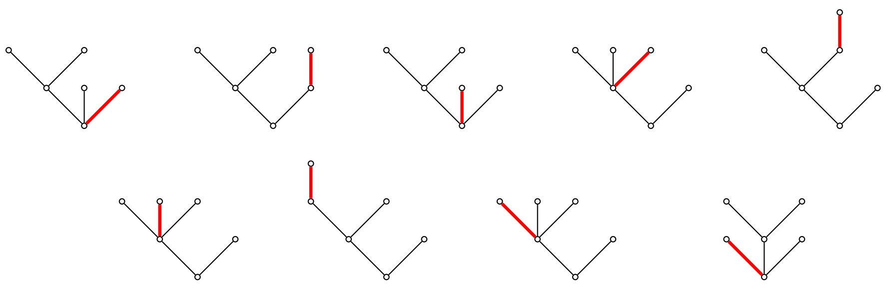
This can be reformulated in the following way. Consider the two diagrams of $\mathfrak{A}$-models $\underline{\mathbf{Cyl}}(\partial A) : I_{|\mathscr{L}(\partial A)|} \to \mathbf{Mod}(\mathfrak{A})$ and $\underline{\mathbf{Cyl}}(A) : I_{|\mathscr{L}(A)|} \to \mathbf{Mod}(\mathfrak{A})$. The previous analysis informally specifies a cocone under $\underline{\mathbf{Cyl}}(\partial A)$, whose vertex is $\mathbf{Cyl}(A)$, such that the map induced on the colimit is precisely $\mathbf{Cyl}(\partial_\sigma)$.

A similar analysis, replacing every occurence of "left" with "right", gives an analogous result for the map $\mathbf{Cyl}(\partial_\tau) : \mathbf{Cyl}(\partial A) \to \mathbf{Cyl}(A)$.

Example 7.35. Let's have a look at a specific example to clarify the situation.

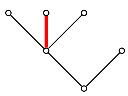
Consider the globular sum given by $A = D_2 \amalg_{D_1} D_2 \amalg_{D_0} D_1$. We have $\partial A = D_1 \amalg_{D_0} D_1$. According to the abovementioned rule, to describe the map $\mathbf{Cyl}(\partial_\sigma) : \mathbf{Cyl}(\partial A) \to \mathbf{Cyl}(A)$ we have to consider each of the trees in $\mathscr{L}(A)$, and check where the new edge is. We have to discard all those in which this special edge is attached at height $n = 2$, and keep those in which it is attached at height $n = 0$, chopping off everything above height $n = 1$. Moreover, whenever it is attached at height $n = 1$, we select only those in which the newly added edge is at the far left of the corolla it belongs to, and we chop everything above height $n = 1$ except for this edge.

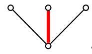
The list appearing on the bottom row in the zig-zag expressing the globular decomposition of $\mathbf{Cyl}(A)$ is given by:



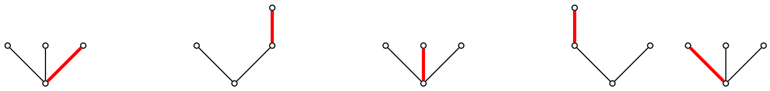
(43)

For example, let's consider the sixth tree of this list, namely:



The corolla which the special edge belongs to is , and the red edge is not at the far left of it, so we discard this tree.

Proceding as described by the rule, we are left with the following list of trees, identified with (respectively) the first, second, third, eighth and ninth tree of the previous list



(44)

Clearly, this is the list of trees appearing on the bottom row of the globular decomposition of $\mathbf{Cyl}(\partial A) = \mathbf{Cyl}(D_1 \amalg_{D_0} D_1)$, and there are maps induced by the source maps and colimit inclusions (i.e. maps in Θ_0) from each of these tree into the tree associated with it in $\mathscr{L}(A)$. For instance the first tree of the list (44) is associated with the first one in the list (43), and the map between them is induced by the map $\partial_\sigma\colon D_1 \to D_2 \coprod_{D_1} D_2$.

Let's formalize this: we start by defining a map of sets $\varphi_A^\sigma\colon \mathscr{L}(\partial A) \to \mathscr{L}(A)$ (resp. $\varphi_A^\tau\colon \mathscr{L}(\partial A) \to \mathscr{L}(A)$) by sending $B \in \mathscr{L}(\partial A)$, obtained by adjoining a new vertex to ∂A in the fiber over $x \in \partial A_{m-1}$, to the globular sum corresponding to the tree $\varphi_A^\sigma(B)$ (resp. $\varphi_A^\tau(B)$), whose underlying presheaf of sets (viewing globular sums as trees) is given by adjoining a new vertex to A in the fiber over the image of x in A, and whose linear order is defined next.

We need to consider two possible cases. First, if $m < n = \mathrm{ht}(A)$ then one has $(\iota_{m-1}^{\partial A})^{-1}\{x\} = (\iota_{m-1}^{A})^{-1}\{x\}$. Therefore, we endow the fiber over x in $\varphi_A^\sigma(B)$ (resp. $\varphi_A^\tau(B)$) with the linear order transported from the one in $(\iota_{m-1}^{\partial A})^{-1}\{x\}$. If $m = n$, we impose that the newly adjoined vertex in $\varphi_A^\sigma(B)$ (resp. $\varphi_A^\tau(B)$) is the least element (resp. the greatest element) in the fiber over x .

Given any $B \in \mathscr{L}(\partial A)$, let us define a map $j_B^\sigma : B \to \varphi_A^\sigma(B)$ (resp. $j_B^\tau : B \to \varphi_A^\tau(B)$) in Θ_0. Following the notation in the previous paragraph, if $m < n$, we define j_B^σ to be $\partial_\sigma\colon B \to \varphi_A^\sigma(B)$, and j_B^τ to be $\partial_\tau\colon B \to \varphi_A^\tau(B)$. If $m = n$, then the maps $j_B^\sigma : B \to \varphi_A^\sigma(B)$ and $j_B^\tau : B \to \varphi_A^\tau(B)$ are induced by the universal property of pushouts as depicted in the diagrams below, where the front and back faces of the cubes are cocartesian squares:

$$
\begin{tikzcd}
 & D_n^{\otimes k} \arrow[rr, "d^1"] \arrow[dd, "\beta"' near start] & & D_n^{\otimes k+1} \arrow[dd] \\
D_{n-1} \arrow[ur, "i_1\circ\sigma"] \arrow[rr, "\tau" near end] \arrow[dd, "\alpha"'] & & D_n \arrow[ur, "i_1"'] \arrow[dd] & \\
 & A \arrow[rr, "z_B^A" near start] & & \varphi_A^\sigma(B) \\
\partial A \arrow[ur, "\partial_\sigma"] \arrow[rr, "\partial_\tau"] & & B \arrow[ur, dashed, "j_B^\sigma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & D_n^{\otimes k} \arrow[rr, "d^{k+1}"] \arrow[dd, "\beta'"' near start] & & D_n^{\otimes k+1} \arrow[dd] \\
D_{n-1} \arrow[ur, "i_k\circ\tau"] \arrow[rr, "\sigma" near end] \arrow[dd, "\alpha'"'] & & D_n \arrow[ur, "i_{k+1}"'] \arrow[dd] & \\
 & A \arrow[rr, "v_B^A" near start] & & \varphi_A^\tau(B) \\
\partial A \arrow[ur, "\partial_\tau"] \arrow[rr, "\partial_\sigma"] & & B \arrow[ur, dashed, "j_B^\tau"'] &
\end{tikzcd}
\tag{45}
$$

Here, $D_n^{\otimes k} = \Sigma^{n-1}(D_1^{\otimes k})$, d^r is the map that skips the r-th summand, i_k denotes the inclusion of the k-th summand, α, α' represent the target (resp.source) of the leaf we are adjoining to ∂A and β, β' are the corresponding inclusion of the fiber of A over x.

Proposition 7.36. *Given a globular sum A with $\mathrm{ht}(A) = n > 0$, we have the following commutative square for each $B \in \mathscr{L}(\partial A)$ and $\varepsilon = \sigma, \tau$:*

$$\begin{tikzcd} B \arrow[r, "j_B^\varepsilon"] \arrow[d, "i_B"'] & \varphi_A^\varepsilon(B) \arrow[d, "i_{\varphi_A^\varepsilon(B)}"] \\ \mathbf{Cyl}(\partial A) \arrow[r, "\mathbf{Cyl}(\partial_\varepsilon)"] & \mathbf{Cyl}(A) \end{tikzcd}$$

Proof. We only prove the case $\varepsilon = \sigma$, the other one being entirely dual. The statement is clear if $A = D_1^{\otimes m}$, in which case $\partial A = D_0$. Otherwise, let $A = \Sigma A_1 \amalg_{D_0} \ldots \amalg_{D_0} \Sigma A_k$, as in (5.3), and assume the result holds for all the A_i's. Define A'_i to be A_i if $\mathrm{ht}(A_i) < n - 1$, or ∂A_i otherwise. Let us subdivide the set $\mathscr{L}(\partial A)$ as the union of the set of globular sums for which the new edge is joined at the root, and the sets of the form

$$\{\Sigma A'_1 \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma B' \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma A'_k\}_{1 \leq i \leq k, B' \in \mathscr{L}(A'_i)}$$

If B belongs to the first set, i.e. $B = \Sigma A'_1 \amalg_{D_0} \ldots \Sigma A'_i \amalg_{D_0} D_1 \amalg_{D_0} \Sigma A'_{i+1} \amalg_{D_0} \Sigma A'_k$ for some $0 \leq i \leq k+1$, then thanks to Remark 7.25 one has the following commutative square

$$\begin{tikzcd} B \arrow[r, "\partial_\sigma"] \arrow[d, "i_B"'] & \Sigma A_1 \underset{D_0}{\amalg} \ldots \Sigma A_i \underset{D_0}{\amalg} D_1 \underset{D_0}{\amalg} \Sigma A_{i+1} \underset{D_0}{\amalg} \Sigma A_k \arrow[d, "i_{\varphi_A(B)}"] \\ \mathbf{Cyl}(\partial A) \arrow[r, "\mathbf{Cyl}(\partial_\sigma)"] & \mathbf{Cyl}(A) \end{tikzcd}$$

To conclude the proof for this case, just observe that the upper horizontal map coincides with $j_B^\sigma \colon B \to \varphi_A^\sigma(B)$.

Next, suppose $B = \Sigma A'_1 \amalg_{D_0} \ldots \amalg_{D_0} \Sigma B' \amalg_{D_0} \ldots \amalg_{D_0} \Sigma A'_k$ for some $B' \in \mathscr{L}(A'_i)$. By construction, we have that the natural transformation $\mathbf{Cyl}(\partial_\sigma)$ restricted to the sub zig-zag

$$\Sigma A'_1 \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma A'_{i-1} \underset{D_0}{\amalg} \Sigma \underline{\mathbf{Cyl}}(A'_i) \underset{D_0}{\amalg} \Sigma A'_{i+1} \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma A'_k$$

coincides with

$$\Sigma \partial'_1 \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma \mathbf{Cyl}(\partial'_i) \underset{D_0}{\amalg} \Sigma \partial'_{i+1} \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma \partial'_k$$

where we set ∂'_i to be $\partial_\sigma : \partial A_i = A'_i \to A_i$ if $\mathrm{ht}(A_i) = n - 1$, and the identity otherwise.

Thanks to Remark 7.25 and the inductive hypothesis,we get the following commutative square

$$\begin{tikzcd} B \arrow[r, "\Sigma \partial'_1 \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma j_{B'} \underset{D_0}{\amalg} \Sigma \partial'_{i+1} \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma \partial'_k"] \arrow[d, "i_B"'] & \Sigma A_1 \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma \varphi^\sigma_{A'_i}(B') \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma A_k \arrow[d, "i"] \\ \mathbf{Cyl}(\partial A) \arrow[r, "\mathbf{Cyl}(\partial_\sigma)"] & \mathbf{Cyl}(A) \end{tikzcd}$$

We conclude by observing that

$$\Sigma A_1 \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma \varphi^\sigma_{A'_i}(B') \underset{D_0}{\amalg} \ldots \underset{D_0}{\amalg} \Sigma A_k = \varphi_A(B)$$

and

$$\Sigma\partial'_1 \underset{D_0}{\amalg} \dots \underset{D_0}{\amalg} \Sigma j_{B'} \underset{D_0}{\amalg} \Sigma\partial'_{i+1} \underset{D_0}{\amalg} \dots \underset{D_0}{\amalg} \Sigma\partial'_k = j_B$$

□

We also record here the following result, for future use (and generalization).

Proposition 7.37. *The following square commutes for $\varepsilon = \sigma, \tau$*

$$\begin{tikzcd} \partial A \arrow[r, "\partial^A_\varepsilon"] \arrow[d, "z^{\partial A}_B"'] & A \arrow[d, "z^A_{\varphi^\varepsilon_A(B)}"] \\ B \arrow[r, "j^\varepsilon_B"] & \varphi^\varepsilon_A(B) \end{tikzcd}$$

Proof. We only prove the case $\varepsilon = \sigma$, the other one being entirely dual. Let $*_B$ be the vertex adjoined to ∂A to get B, and denote by F the fiber of $*_B$.

If $*_B = \min F$ and $F \neq \{*_B\}$ (i.e. case (1) of Definition 7.20), then $m < n$ and the square in the statement is given by:

$$\begin{tikzcd} \partial A \arrow[r, "\partial_\sigma"] \arrow[d] & A \arrow[d] \\ B \arrow[r, "\partial_\sigma"] & \varphi^\sigma_A(B) \end{tikzcd}$$

where the unlabelled maps are the unique globular maps with such domain and codomain. This square obviously commutes.

If $F = \{*_B\}$ (i.e. case (2) of Definition 7.20) and $m < n$, the square is given by:

$$\begin{tikzcd} \partial A \arrow[r, "\partial_\sigma"] \arrow[d, "z^{\partial A}_B"'] & A \arrow[d, "z^A_{\varphi^\sigma_A(B)}"] \\ B \arrow[r, "\partial^\sigma_B"] & \varphi^\sigma_A(B) \end{tikzcd}$$

where $z^{\partial A}_B$ and $z^A_{\varphi^\sigma_A(B)}$ are both induced by target maps. In this case too, it is not hard to check commutativity.

If $m > n$, the square we obtain the bottom face of the cube appearing in Definition 45, and therefore commutes by construction.

If $*_B \neq \min F$, then $m < n$ and so the square is given by:

$$\begin{tikzcd} \partial A \arrow[r, "\partial_\sigma"] \arrow[d, "z^{\partial A}_B"'] & A \arrow[d, "z^A_{\varphi^\sigma_A(B)}"] \\ B \arrow[r, "\partial_\sigma"] & \varphi^\sigma_A(B) \end{tikzcd}$$

where both $z^{\partial A}_B$ and $z^A_{\varphi^\sigma_A(B)}$ are as in case (3) of Definition 7.20, so that the commutativity of the square above follows immediately from the globularity of the generalized whiskering w's. □

We now describe how to extend the results presented thus far, in order to study the maps $\mathbf{Cyl}(\partial_\varepsilon^k) : \mathbf{Cyl}(\partial^k A) \to \mathbf{Cyl}(A)$ for $k > 1$. This time too, we can obtain the list of trees appearing on the bottom row of the globular decomposition of $\mathbf{Cyl}(\partial^k A)$ from those in the decomposition of $\mathbf{Cyl}(A)$. Again, we concentrate on the case $\varepsilon = \sigma$.

Firstly, we have to discard all the trees in which the new edge has been attached at height $m \geq n-k+1$. If it has been attached at height $m < n-k$, we keep the trees after having chopped off everything above height $n-k$. Finally, if the new edge is attached at height $m = n-k$, we consider the strip comprised between height $m = n-k$ and $m' = n-k+1$. The new edge belongs to a corolla in here, and we only keep the trees in which it is at the far left of the corolla it belongs to. Again, in this case, we chop everything above height $n-k$, except for the newly added edge.

Each of the trees (representing a globular sum) we thus obtain comes equipped with a map (induced by a k-fold iteration of the source maps or by a colimit inclusion) towards the one appearing in the decomposition of $\mathbf{Cyl}(A)$ which it is associated to. In this way one gets a cocone under the diagram $\underline{\mathbf{Cyl}}(\partial^k A)\colon I_m \to \mathbf{Mod}(\mathfrak{A})$, whose vertex is $\mathbf{Cyl}(A)$, such that the map induced on the colimit is precisely $\mathbf{Cyl}(\partial_\sigma^k)$.

A similar argument, replacing every occurence of "left" with "right", yields an analogous result for the map $\mathbf{Cyl}(\partial_\tau^k)\colon \mathbf{Cyl}(\partial^k A) \to \mathbf{Cyl}(A)$. To make this precise we have to generalize the work already done for $k = 1$.

Definition 7.38. Let A be a globular sum, and $k > 0$ a positive integer. Define (for $\varepsilon = \sigma, \tau$) the map of sets $(\varphi_A^\varepsilon)^k\colon \mathscr{L}(\partial^k A) \to \mathscr{L}(A)$ inductively, by setting $(\varphi_A^\varepsilon)^1 = \varphi_A^\varepsilon$ and $(\varphi_A^\varepsilon)^k = \varphi_A^\varepsilon \circ (\varphi_{\partial A}^\varepsilon)^{k-1}$ for $k > 1$.

Unraveling the previous definition we see that $(\varphi_A^\varepsilon)^k = \varphi_A^\varepsilon \circ \varphi_{\partial A}^\varepsilon \circ \ldots \circ \varphi_{\partial^k A}^\varepsilon$. We also define, given $B \in \mathscr{L}(\partial^k A)$, a map $(j_B^\varepsilon)^k\colon B \to (\varphi_A^\varepsilon)^k(B)$ by setting $(j_B^\varepsilon)^1 = j_B^\varepsilon$ and $(j_B^\varepsilon)^k = j^\varepsilon_{(\varphi_{\partial A}^\varepsilon)^{k-1}(B)} \circ (j_B^\varepsilon)^{k-1}$ if $k > 1$.

The following result is an immediate consequence of Proposition 7.36.

Proposition 7.39. *Given a positive integer k and a globular sum A with $\mathrm{ht}(A) = n \geq k$, we have the following commutative square for each $B \in \mathscr{L}(\partial^k A)$:*

$$\begin{tikzcd}
B \arrow[r, "(j_B^\varepsilon)^k"] \arrow[d, "i_B"'] & (\varphi_A^\varepsilon)^k(B) \arrow[d, "i_{(\varphi_A^\varepsilon)^k(B)}"] \\
\mathbf{Cyl}(\partial^k A) \arrow[r, "\mathbf{Cyl}(\partial_\varepsilon^k)"] & \mathbf{Cyl}(A)
\end{tikzcd}$$

It is easy to give an explicit description of $(\varphi_A^\varepsilon)^k(B)$, similar to what we did for the case $k = 1$.

Lemma 7.40. *Consider a globular sum $B \in \mathscr{L}(\partial^k A)$, so that B is obtained by adjoining a new vertex $*_B$ to $\partial^k A$. Let $m = \mathrm{ht}(*_B)$, define F to be the fiber of $*_B$.*

If $1 \leq m \leq n-k$ then $(\varphi_A^\varepsilon)^k(B)$ is obtained by adding a new vertex to A in the fiber of $*_B$ in the unique way that makes it linearly isomorphic to F.

If $m = n-k+1$, then $(\varphi_A^\varepsilon)^k(B)$ is obtained by adding a new vertex to A in the fiber of $*_B$, where we extend the linear order by imposing that the newly added vertex is the least element in this fiber if $\varepsilon = \sigma$, and the greatest if $\varepsilon = \tau$.

PROOF. We prove this lemma by induction, the case $k = 1$ being already proven. We also assume $\varepsilon = \sigma$, the other case being entirely dual, and we drop the superscripts since we have just clarified any possible ambiguity. Let $k > 1$ and assume the claim holds for every $k' < k$. By definition, $\varphi^k_A(B) = \varphi_A\left(\varphi^{k-1}_{\partial A}(B)\right)$.

If $1 \leq m \leq n-k$ then $m \leq (n-1)-(k-1)$, so that, by inductive hypothesis, $\varphi^{k-1}_{\partial A}(B)$ is obtained by adding a new vertex to the fiber of ∂A over x (this fiber coincides with F, and the order is the transported one). Because $m \leq n-k < n$, φ_A sends $\varphi^{k-1}_{\partial A}(B)$ to a tree obtained by adding a new vertex to A, over x, with the order induced once again by that of F.

If $m = n-k+1 = (n-1)-(k-1)+1$ then $\varphi^{k-1}_{\partial A}(B)$ is obtained from ∂A by adding a new least element to its fiber over x. Since $k > 1$, this fiber is the same as that of A over x, and φ_A sends $\varphi^{k-1}_{\partial A}(B)$ to the tree obtained by adding a new vertex over x to A, with the order transported from that of $\varphi^{k-1}_{\partial A}(B)$. Therefore, the newly added vertex is going to be the minimum in the fiber over x, which concludes the proof. $\square$

We also generalize Proposition 7.37.

PROPOSITION 7.41. *Let k be a positive integer and A be a globular sum with $\mathrm{ht}(A) = n \geq k$. Given $B \in \mathscr{L}(\partial^k A)$ the following square commutes for $\varepsilon = \sigma, \tau$*

$$\begin{tikzcd}
\partial^k A \arrow[r, "\partial^k_\varepsilon"] \arrow[d, "z^{\partial^k A}_B"'] & A \arrow[d, "z^A_{(\varphi^\varepsilon_A)^k(B)}"] \\
B \arrow[r, "(j^\varepsilon_B)^k"] & (\varphi^\varepsilon_A)^k(B)
\end{tikzcd}$$

PROOF. The square is obtained by gluing together two squares, as displayed below

$$\begin{tikzcd}
\partial^k A \arrow[r, "\partial^k_\varepsilon"] \arrow[d, "z^{\partial^k A}_B"'] & \partial A \arrow[r, "\partial_\varepsilon"] \arrow[d, "z^{\partial A}_{(\varphi^\varepsilon_{\partial A})^{k-1}(B)}"] & A \arrow[d, "z^A_{(\varphi^\varepsilon_A)^k(B)}"] \\
B \arrow[r, "(j^\varepsilon_B)^{k-1}"] & (\varphi^\varepsilon_{\partial A})^{k-1}(B) \arrow[r, "j^\varepsilon_{(\varphi^\varepsilon_{\partial A})^{k-1}(B)}"] & (\varphi^\varepsilon_A)^k(B)
\end{tikzcd}$$

Therefore, the claim follows by induction from the case treated in Proposition 7.37. $\square$

We now get back to the task of defining $\hat{\varrho}\colon \mathbf{Cyl}(D_n) \to \mathbf{Cyl}(A)$, for a homogeneous map $\varrho\colon D_n \to A$, with $m = \mathrm{ht}(A)$. The goal is to define it as the vertical composition of a suitable stack of $(n-1)$-cylinders in the $\mathfrak{A}$-model $\mathbf{Cyl}(A)\,(a,b)$. Therefore, we just need to define this vertical stack so that its vertical composition has the desired properties. We assume this construction has been performed for every $k < n$, in the same way as in what follows. So far we have $(n-1)$-cells $\overline{\alpha_0}, \ldots, \overline{\alpha_p}$, as in (40). We now build possibly degenerate $(n-1)$-cylinders $C^\varrho_i\colon \overline{\alpha_{i-1}} \curvearrowright^{r_i}_{q_i} \overline{\alpha_i}$ in $\mathbf{Cyl}(A)\,(a,b)$ for $1 \leq i \leq p$, such that C^ϱ_i factors through $B_i(a,b)$.

We do so by defining the boundary of each of these cylinders, and then we extend these data to cylinders by applying Propositions 5.2 and 7.34 to diagrams of the form:

$$\begin{tikzcd}
\partial\mathbf{Cyl}^{r_i}_{q_i}(D_{n-1}) \arrow[r] \arrow[d, "\partial"'] & B_i(a,b) \arrow[r, "\Omega(i_{B_i})"] & \mathbf{Cyl}(A)\,(a,b) \\
\mathbf{Cyl}^{r_i}_{q_i}(D_{n-1}) \arrow[ur, dashed] \arrow[urr, "C^{\varrho}_i"'] & &
\end{tikzcd}$$

To begin with, we have to define r_i and q_i for every $0 \leq i \leq p-1$. B_i is obtained, by construction, by adding a new vertex $*_{B_i}$ to A. Letting $d = \mathrm{ht}(*_{B_i})$, we have three possibilities: if $d = 1$ then $r_i = q_i = -1$; otherwise either $*_{B_i}$ is the least (resp. greatest) element in its fiber or it is not. In the first case we set $r_i = d-3$ (resp. $q_i = d-3$), otherwise $r_i = d-2$ (resp. $q_i = d-2$).

If $n > m$ then we define the source (resp. the target) of C^{ϱ}_i to be $C^{\varrho\circ\sigma}_i$ (resp. $C^{\varrho\circ\tau}_i$), which have already been defined, since $\varrho\circ\varepsilon$ is a homogeneous map with domain D_{n-1}. It is easy to check that this is well defined, since the indexing set for the i's is given in both cases by $\mathscr{L}(A)$.

If $n = m$ then, for $\varepsilon = \sigma, \tau$, there exists a unique factorization of $\varrho\circ\varepsilon$, due to the homogeneity of the coherator for ∞-categories $\mathfrak{C}$, of the form:

$$\begin{tikzcd}
D_{n-1} \arrow[r, "\varepsilon"] \arrow[d, "\varrho_\varepsilon"'] & D_n \arrow[d, "\varrho"] \\
\partial A \arrow[r, "\partial^A_\varepsilon"] & A
\end{tikzcd}$$

where ϱ_ε is a homogeneous map. If $r_i = n-2$ (resp. $q_i = n-2$) there is nothing to do. If $r_i < n-2$ (resp. $q_i < n-2$), i.e. the source (resp. the target) is not collapsed, then either $d = n$ and $*_{B_i}$ is the least (resp. greatest) element in its fiber or $d \leq n-1$. In both cases there exists a unique $1 \leq k \leq |\mathscr{L}(\partial A)|$ such that $B_i = \varphi^{\sigma}_A(E_k)$ (resp. $B_i = \varphi^{\tau}_A(E_k)$), where E_k is the k-th element of $\mathscr{L}(\partial A)$. We now define the source (resp. target) of C^{ϱ}_i to be $C^{\varrho_\sigma}_k$ (resp. $C^{\varrho_\tau}_k$) This assignment is well defined, as the commutative squares below ensure that the cylinder $C^{\varrho_\sigma}_k$ (resp. $C^{\varrho_\tau}_k$) has, as top and bottom cells, the source (resp. target) of the top and bottom cells of C^{ϱ}_i

$$\begin{tikzcd}
D_{n-1} \arrow[r, "\varepsilon"] \arrow[d, "\varrho_\varepsilon"'] & D_n \arrow[d, "\varrho"] \\
\partial A \arrow[r, "\partial^A_\varepsilon"] \arrow[d, "z^{\partial A}_{E_k}"'] & A \arrow[d, "z^A_{\varphi^\varepsilon_A(E_k)}"] \\
E_k \arrow[r, "j_{E_k}"] & \varphi^{\varepsilon}_A(E_k)
\end{tikzcd}$$

Here, the bottom square commutes thanks to Proposition 7.37. The extensions are obtained as in the non-degenerate case, thanks to Proposition 7.34.

The vertical composition of the stack of cylinders given by:

$$C^{\varrho}_1 \otimes \ldots \otimes C^{\varrho}_p \colon \mathbf{Cyl}^{r_1}_{q_1}(D_{n-1}) \otimes \ldots \otimes \mathbf{Cyl}^{r_p}_{q_p}(D_{n-1}) \to \Omega\left(\mathbf{Cyl}(A), a, b\right)$$

produces an $(n-1)$-cylinder $C^{\varrho}\colon \overline{\alpha_0} \curvearrowright \overline{\alpha_p}$ in $\mathbf{Cyl}(A)\,(a,b)$, since $\min\{r_i\}_{1\leq i\leq p} = -1$ and $\min\{q_i\}_{1\leq i\leq p} = -1$ by construction.

DEFINITION 7.42. Let $\varrho\colon D_n \to A$ be a homogeneous operation in $\mathfrak{C}$. Using the notation established so far, we let $\hat{\varrho}\colon \mathbf{Cyl}(D_n) \to \mathbf{Cyl}(A)$ be the n-cylinder consisting of the following data:

- $\hat{\varrho}_\varepsilon = \mathbf{Cyl}(\partial^n_\varepsilon)\colon D_1 \cong \mathbf{Cyl}(D_0) \to \mathbf{Cyl}(A)$ for $\varepsilon = s,t$;
- $\overline{\hat{\varrho}}$ is given by $C^{\varrho}\colon \mathbf{Cyl}(D_{n-1}) \to \mathbf{Cyl}(A)\,(a,b)$.

We are now left with checking the compatibility with the coglobular structure, i.e. we have to prove that, for $\varepsilon = \sigma,\tau$, $\hat{\varrho} \circ \mathbf{Cyl}(\varepsilon) = \widehat{\varrho\circ\varepsilon}$ if $n > m$ and $\hat{\varrho} \circ \mathbf{Cyl}(\varepsilon) = \mathbf{Cyl}(\partial_\varepsilon) \circ \hat{\varrho}_\varepsilon$ if $n = m$.

The first case is straightforward by construction, since the operation of vertical composition is compatible with the coglobular structure.

The proof of the second case is accomplished by using the following lemma, which essentially says that the collapsed pieces of the boundaries do not contribute to the result of the vertical composition.

LEMMA 7.43. *Let q be a positive integer, and suppose given a sequence of n-cylinders $C_i\colon \alpha_i \curvearrowright^{p_i}_{q_i} \alpha_{i+1}$ in an $\mathfrak{A}$-model X. Consider the ordered set $\{p_i\}_{1\leq i\leq q}$, where $p_i < p_j$ if and only if $i < j$, and let $\{\bar{p}_{i_1},\ldots,\bar{p}_{i_k}\}$ be the (ordered) subset spanned by those $p_i < n-1$. Then the cylinders $(C_{i_j} \circ \mathbf{Cyl}(\sigma))_{1\leq j\leq k}$ are again composable, and moreover we have*

$$(C_1 \otimes \ldots \otimes C_q) \circ \mathbf{Cyl}(\sigma) = (C_{i_1} \circ \mathbf{Cyl}(\sigma)) \otimes \ldots \otimes (C_{i_k} \circ \mathbf{Cyl}(\sigma))$$

An analogous result holds true if we replace p with q and σ with τ.

PROOF. The fact that the C_{i_j} are again composable is obvious. We prove the second statement by induction on n, the base case $n = 1$ being straightforward. We know that $C_1 \otimes \ldots \otimes C_q$ is obtained by transposing the result of vertically composing a stack obtained from whiskerings of the $(n-1)$-cylinders $\bar{C}_i$ with appropriate 1-cells together with $(n-1)$-cylinders of the form $\psi^{c,d}$ that witness the rebracketing of cells when needed (as explained in Section 5.2). Whenever $p_i = n-1$ these Ψ's do not appear, so that the claim follows from the inductive assumption and the coglobularity of the remaining Ψ's. ☐

We conclude this section with an extension of Definition 7.42. Recall from Remark 2.15 that a map $\varphi\colon A \to B$ in $\mathfrak{C}$ is homogeneous if the homogeneous-globular factorizations $D_{i_k} \to B_k \to B$ of the composites $D_{i_k} \to A \to B$ for every i_k in the table of dimensions of A are such that the induced canonical map

$$\operatorname{colim}_k B_k \to B$$

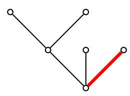
is an isomorphism.

DEFINITION 7.44. If $\varphi\colon A \to B$ is a homogeneous map in $\mathfrak{C}$, we can obtain an elementary interpretation of it which still satisfies the properties expressed in (36) simply by coglobularity of the construction recorded in Definition 7.42. Indeed, we can consider the induced homogeneous maps $\varphi_k\colon D_{i_k} \to B_k$ and define $\widehat{\varphi}\colon \mathbf{Cyl}(A) \to \mathbf{Cyl}(B)$ as the map induced by passing to the colimit the family of maps $\hat{\varphi_k}\colon \mathbf{Cyl}(D_{i_k}) \to \mathbf{Cyl}(B_k)$.

For a general map $h\colon A \to B$ in $\mathfrak{C}$, we factor h as $h = i \circ \varrho$, using homogeneity of $\mathfrak{C}$, with $\varrho\colon A \to C$ a homogeneous map and $i\colon C \to B$ globular map, i.e. a map in Θ_0. Now define its elementary interpretation as $\hat{h} = \mathbf{Cyl}(i) \circ \hat{\varrho}$, where we have used the fact that we do have a functor $\mathbf{Cyl}\colon \Theta_0 \to \mathbf{Mod}(\mathfrak{A})$.

We will now describe this in detail for a homogeneous operation $\varrho\colon D_2 \to A$. We proceed representably, so assume given a map $C\colon \mathbf{Cyl}(A) \to X$, with $C\colon U \curvearrowright V$.

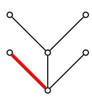
To each of the globular sums in $\mathscr{L}(A)$ we associate a (possibly degenerate) 1-cylinder in $X(x, y)$, where $x = s^2\left(X(\varrho)(U)\right),\ y = t^2\left(X(\varrho)(V)\right)$. These 1-cylinders will be vertically composable in the order induced by that of $\mathscr{L}(A)$, and the composite will produce the desired 2-cylinder $C\hat{\varrho}\colon \mathbf{Cyl}(D_2) \to X$ upon transposing along the adjunction $\Sigma \dashv \Omega$ (here, we make use of the fact that an n-cylinder is defined to be an $(n-1)$-cylinder in the space of paths between two objects, with the appropriate top and bottom $(n-1)$-cells, see Definition 5.5). The first 1-cylinder, i.e. the one associated with the globular sum $B \in \mathscr{L}(A)$ where the new vertex $*_B$ has been added at height 1 as the maximal element over the root (i.e. the right most one) is given by:



$$\begin{tikzcd} \cdot \arrow[r, "C_t\varrho(U)"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\Swarrow"] & \cdot \arrow[d, "\simeq"] \\ \cdot \arrow[r, "\varrho(U_{<p}{,}C_tU_p)"'] & \cdot \end{tikzcd} \tag{46}$$

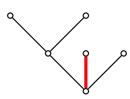
Here, $U_{<p}$ and U_p respectively denote the restriction of U to $\Sigma\alpha_1 \coprod\limits_{D_0} \ldots \coprod\limits_{D_0} \Sigma\alpha_{p-1}$ and $\Sigma\alpha_p$, where we have considered the decomposition of A as $\Sigma\alpha_1 \coprod\limits_{D_0} \ldots \coprod\limits_{D_0} \Sigma\alpha_p$ as in Lemma 5.3. Furthermore, juxtaposition is the result of composing using the maps introduced in Definition 2.12, and given a map $W\colon A \to X$, which we think as an A-shaped pasting diagram in X, we denote $X(\varrho)(W)$ with $\varrho(W)$. Finally, we denoted $C \circ \mathbf{Cyl}(\partial^2_\tau)$ with C_t. Both the sides and the interior of the square are obtained by solving extension problems in the globular sum B, using the algorithm outlined in this section, and the same holds for all the other cases to follow.

Dually, the last square in the stack is associated to the globular sum B obtained from A by adjoining a new vertex at height 1 as the minimal element over the root (i.e. the left most one). This time, the square is given by:



$$\begin{tikzcd} \cdot \arrow[r, "\varrho(V_1C_s{,}V_{>1})"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\Swarrow"] & \cdot \arrow[d, "\simeq"] \\ \cdot \arrow[r, "\varrho(V)C_s"'] & \cdot \end{tikzcd} \tag{47}$$

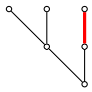
Suppose now the new vertex in $B \in \mathscr{L}(A)$ is adjoined at height 1 as the q-th element in the linear order on B_1. The associated square then looks like the one depicted here below:



$$\begin{tikzcd} \cdot \arrow[r, "\varrho(U_{<q}{,}V_qa{,}V_{>q})"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\Swarrow"] & \cdot \arrow[d, "\simeq"] \\ \cdot \arrow[r, "\varrho(U_{<q}{,}aU_q{,}V_{>q})"'] & \cdot \end{tikzcd} \tag{48}$$

Here, we have used a to denote the 1-cell in X corresponding to the restriction of the composite map $C \circ i_B \colon B \to \mathbf{Cyl}(A) \to X$ to the 1-cell in B associated with the newly added vertex.

Suppose now the vertex has been added to A at height 2, to get a globular sum $B \in \mathscr{L}(A)$. We need to consider all the vertices over the one at which the new edge has been adjoined, and again we distinguish according to the position of the newly added vertex. Firstly, let's consider the case in which it has been added over a copy of D_1 (i.e. it is the only vertex above the one to which the new edge is attached). This determines a decomposition $A = A_< \coprod_{D_0} D_1 \coprod_{D_0} A_>$. Precomposing U (resp. V) with the inclusion of $A_<$ (resp. $A_>$) we get an $A_<$-shaped (resp. $A_>$-shaped) pasting diagram in X which we call $U_<$ (resp.$V_>$). The square is then given by:



$$\begin{tikzcd} \cdot \arrow[r, "\varrho(U_<{,}V_> s(F))"] \arrow[d, "\varrho^*_\sigma(\partial_\sigma U_<{,}\partial_\sigma V_> F)"'] \arrow[dr, phantom, "\swarrow"] & \cdot \arrow[d, "\varrho^*_\tau(\partial_\tau U_<{,}\partial_\tau V_> F)"] \\ \cdot \arrow[r, "\varrho(U_<{,}V_> t(F))"'] & \cdot \end{tikzcd} \tag{49}$$

Here, $\partial_\varepsilon \circ \varrho_\varepsilon$ is the homogeneous-globular factorization of $\varrho \circ \varepsilon$ for $\varepsilon = \sigma, \tau$, F is the 2-cell that fills the 1-cylinder

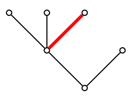
$$\mathbf{Cyl}(i) \colon \mathbf{Cyl}(D_1) \to \mathbf{Cyl}(A_< \coprod_{D_0} D_1 \coprod_{D_0} A_>) = \mathbf{Cyl}(A)$$

and $\partial_\varepsilon W$ denotes, given a map $W \colon A \to X$, the precomposition of W with $\partial_\varepsilon \colon \partial A \to A$. Finally, ϱ^*_ε is obtained as an extension of the following form:

$$\begin{tikzcd} S^1 \arrow[r, "(\partial_\sigma \circ \varrho_\varepsilon{,}\partial_\tau \circ \varrho_\varepsilon)"] \arrow[d] & \partial A^+ \\ D_2 \arrow[ur, "\varrho^*_\varepsilon"'] & \end{tikzcd}$$

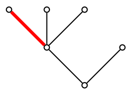
where ∂A^+ is obtained from ∂A by adjoining a new vertex in the same position as the one that was added to A in order to get B.

If the new vertex $*_B$ is not the only one in its fiber, then we have to distinguish three cases. If $*_B$ is the maximal element, and it has been added to $\Sigma\alpha_q$, then the 1-cylinder we get has degenerate source, and can be depicted as follows:



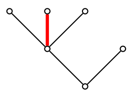
$$\begin{tikzcd} \cdot \arrow[r, "\varrho(U_{<q}{,}aU_q{,}V_{>q})"] \arrow[d, equal, "degenerate"'] \arrow[dr, phantom, "\swarrow"] & \cdot \arrow[d, "\varrho^*_\tau(\partial_\tau U_{<q}{,}\alpha{,}\partial_\tau V_{>q})"] \\ \cdot \arrow[r, "\varrho(U_{<q}{,}\alpha U_q{,}V_{>q})"'] & \cdot \end{tikzcd} \tag{50}$$

Here, we have denoted by α the 2-cell of the 1-cylinder $C_{|\Sigma\alpha_q} \circ \mathbf{Cyl}(\partial_\tau)$, and with a its target 0-cylinder (viewed as a 1-cell). Dually, if it is the minimal element, the 1-cylinder has degenerate target, and is of the following form:



$$\begin{tikzcd} \cdot \arrow[r, "\varrho(U_{<q}, V_q\alpha, V_{>q})"] \arrow[d, "\varrho^*_\sigma(\partial_\sigma U_{<q}, \alpha, \partial_\sigma V_{>q})"'] \arrow[dr, phantom, "\swarrow"] & \cdot \arrow[d, equal, "degenerate"] \\ \cdot \arrow[r, "\varrho(U_{<q}, V_q a, V_{>q})"'] & \cdot \end{tikzcd} \tag{51}$$

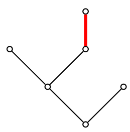
Here, we have denoted with α the 2-cell of the 1-cylinder $C_{|\Sigma\alpha_q} \circ \mathbf{Cyl}(\partial_\sigma)$, and with a its source 0-cylinder (viewed as a 1-cell). Finally, if the new vertex has been added as the r-th element in its fiber, then we get sub-globular sums of $\Sigma\alpha_q \cong \Sigma D_1^{\otimes m}$ of the form $\Sigma D_1^{\otimes r}$ and $\Sigma D_1^{\otimes m-r}$. Corresponding to this subdivision we have a $\Sigma D_1^{\otimes r}$-shaped diagram in X induced by U, that we denote with $U_q^{\leq r}$, and, similarly, a $\Sigma D_1^{\otimes m-r}$-shaped diagram induced by V, that we denote with $V_q^{\geq r}$. The corresponding 1-cylinder is essentially a 2-cell in $X(x, y)$, since its source and target are degenerate, as depicted here below:



$$\begin{tikzcd} \cdot \arrow[r, "\varrho\left(U_{<q}, U_q{}^{\leq r}, V_q{}^{\geq r}\alpha, V_{>q}\right)"] \arrow[d, equal, "degenerate"'] \arrow[dr, phantom, "\swarrow"] & \cdot \arrow[d, equal, "degenerate"] \\ \cdot \arrow[r, "\varrho\left(U_{<q}, \alpha U_q{}^{\leq r}, V_q{}^{\geq r}, V_{>q}\right)"'] & \cdot \end{tikzcd} \tag{52}$$

Here, we have denoted with α the 2-cell of the 1-cylinder given by the target of the r-th 2-cylinder in the image of $C_{|\Sigma\alpha_q}$.

The last case is that of a globular sum $B \in \mathscr{L}(A)$ in which the new vertex $*_B$ has been added to A at height 3. Say the 2-cell the new edge has been attached to is the r-th in $\Sigma\alpha_q \cong \Sigma D_1^{\otimes m}$, then the associated 1-cylinder has degenerate source and target, and is of the following form:



$$\begin{tikzcd} \cdot \arrow[r, "\varrho\left(U_{<q}, U_q{}^{<r}, s(F), V_q{}^{>r}a, V_{>q}\right)"] \arrow[d, equal, "degenerate"'] \arrow[dr, phantom, "\swarrow"] & \cdot \arrow[d, equal, "degenerate"] \\ \cdot \arrow[r, "\varrho\left(U_{<q}, U_q{}^{<r}, t(F), V_q{}^{>r}a, V_{>q}\right)"'] & \cdot \end{tikzcd} \tag{53}$$

Here, F denotes the 3-cell of the 2-cylinder in X, whose 0-dimensional source we denoted by a, picked out by precomposing C with $\mathbf{Cyl}(D_2 \to A)$, where the copy of D_2 in question is the one that corresponds to the vertex in A of height 2 over which $*_B$ has been added.

So far, we have described a stack of $|\mathscr{L}(A)|$ vertically composable (possibly degenerate) 1-cylinders in $X(x, y)$. Its (vertical) composite is a 1-cylinder $C_t\varrho(U) \curvearrowright \varrho(V)C_s$ in $X(x, y)$, that transposes under the adjunction $\Sigma \dashv \Omega$ to give the desired 2-cylinder $C \circ \hat{\varrho} \colon \varrho(U) \curvearrowright \varrho(V)$.

CHAPTER 8

Path object on Grothendieck 3-groupoids of type $\mathfrak{C}^{\mathbf{W}}$

So far, we have constructed the elementary interpretation of any given homogeneous operation $\varrho\colon D_k \to A$ of a given coherator for n-categories for $0 \leq n \leq \infty$. Unfortunately, as we observed at the beginning of Chapter 7, the assignment $\varrho \mapsto \hat{\varrho}\colon \mathbf{Cyl}(A) \to \mathbf{Cyl}(B)$ cannot be made into a functor of the form $\mathbf{Cyl}\colon \mathfrak{C} \to \mathbf{Mod}(\mathfrak{A})$, since it would not preserve composition of maps.

To address this problem, we introduce the concept of modifications, and we use these to interpret all the operations up to dimension 2. We then specialize to the 3-dimensional case, and further extend this construction to every map in a coherator for 3-categories, finally proving the existence of a semi-model structure on Grothendieck 3-groupoids of type $\mathfrak{C}^{\mathbf{W}}$ in Theorem 8.30 thanks to the results in Chapter 4.

1. Modifications

Assume given a globular theory $\mathfrak{A}$ among those considered in the previous section. Given an $\mathfrak{A}$-model X, a modification in X between n-cylinders $\Theta\colon C \Rrightarrow D$ will be defined inductively to consist of a pair of 2-cells $\Theta_s\colon C_s \to D_s$, $\Theta_t\colon D_t \to C_t$ together with a modification of $(n-1)$-cylinders in $X(x,y)$ of the form:

$$\bar{\Theta}\colon \Upsilon(\iota_0 C, \Theta_t) \otimes \bar{C} \otimes \Gamma(\Theta_s, \iota_1 C) \Rrightarrow \bar{D}$$

where $x = s^n(C) \circ \sigma$, $y = t^n(C) \circ \tau$, and Γ, Υ are cylinders we define below, and $\otimes$ denotes vertical composition of cylinders. We choose this asymmetric version (compare it to the one given for cylinders) since this will simplify the exposition later on (in particular, we want to prove Lemma 8.6).

Example 8.1. Before we formally give the definition of modification, we give an example of what modifications look like in low dimensions. If $n = 0$ then a modification is simply a 2-cell. If $n = 1$ then we can depict C and D as, respectively

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, "f"'] \arrow[dr, phantom, "\Gamma\,\swArrow"] & b \arrow[d, "g"] \\ c \arrow[r, "\beta"'] & d \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, "f'"'] \arrow[dr, phantom, "\Delta\,\swArrow"] & b \arrow[d, "g'"] \\ c \arrow[r, "\beta"'] & d \end{tikzcd}$$

A modification $\Theta\colon C \Rrightarrow D$ corresponds to the data of a pair of 2-cells $S\colon f \to f'$, $T\colon g' \to g$ in X and a 3-cell $\tilde{\Theta}\colon (\beta\Theta_s)(\Gamma(\Theta_t\alpha)) \to \Delta$ in X, where we denote by juxtaposition the result of the appropriate operations w involved in the definition.

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, "f"] \arrow[d, bend right=60, "f'"', ""{name=L}] \arrow[dr, phantom, "\Gamma\,\swArrow"] & b \arrow[d, "g"'] \arrow[d, bend left=60, "g'", ""{name=R}] \\ c \arrow[r, "\beta"'] & d \end{tikzcd} \overset{\tilde{\Theta}}{\Rrightarrow} \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, "f'"'] \arrow[dr, phantom, "\Delta\,\swArrow"] & b \arrow[d, "g'"] \\ c \arrow[r, "\beta"'] & d \end{tikzcd}$$

Notice that if $f = f'$ and $g = g'$, then a modification $\Theta\colon C \Rightarrow D$ such that Θ_s and Θ_t are identities can be equivalently thought of as a 3-cell between the 2-cells Γ and Δ.

We will construct a coglobular object $\mathbf{M}_\bullet\colon \mathbb{G} \to \mathbf{Mod}(\mathfrak{A})$, that will also come provided with a map $\Xi = (\Xi_0, \Xi_1)\colon \mathbf{Cyl}(D_\bullet) * \mathbf{Cyl}(D_\bullet) \to \mathbf{M}_\bullet$, where the domain denotes the colimit of the diagram:

$$\begin{tikzcd} D_\bullet \arrow[r, "\iota_0"] \arrow[d, "\iota_0"'] & \mathbf{Cyl}(D_\bullet) \\ \mathbf{Cyl}(D_\bullet) & D_\bullet \arrow[l, "\iota_1"'] \arrow[u, "\iota_1"'] \end{tikzcd}$$

Just like in the case of cylinders, this map will be proven to be a direct cofibration.

As a preliminary step, we need to construct cylinders that witness specific coherences between whiskerings, that will be used in the definition of modifications. We define these cylinders by solving the following extension problems in $\mathbf{Mod}(\mathfrak{A})^{\mathbb{G}}$:

$$\begin{tikzcd} \Sigma(D_\bullet \amalg D_\bullet) \arrow[r, "{(a,b)}"] \arrow[d, "{\Sigma(\iota_0,\iota_1)}"'] & D_2 \amalg_{D_0} \Sigma D_\bullet \\ \Sigma\mathbf{Cyl}(D_\bullet) \arrow[ur, dashed, "\Gamma"'] & \end{tikzcd} \qquad \begin{tikzcd} \Sigma(D_\bullet \amalg D_\bullet) \arrow[r, "{(a',b')}"] \arrow[d, "{\Sigma(\iota_0,\iota_1)}"'] & \Sigma D_\bullet \amalg_{D_0} D_2 \\ \Sigma\mathbf{Cyl}(D_\bullet) \arrow[ur, dashed, "\Upsilon"'] & \end{tikzcd}$$

where we define $a = (\sigma_1 \amalg_{D_0} 1) \circ w,\ b = (\tau_1 \amalg_{D_0} 1) \circ w,\ a' = (1 \amalg_{D_0} \sigma) \circ w,\ b' = (1 \amalg_{D_0} \tau) \circ w$. These extensions are obtained similarly to those of the case of constant cylinders in (24). In words, given an $\mathfrak{A}$-model X, these produce $(n-1)$-cylinders $\Gamma(c, B)\colon Bs(c) \curvearrowright Bt(c)$ in $X\left(s^2(c), t^n(B)\right)$ (resp. $\Upsilon(A, d)\colon s(d)A \curvearrowright t(d)A$ in $X\left(s^n(A), t^2(d)\right)$) out of an n-cell B and 2-cell c (resp. an n-cell A and a 2-cell d) in X which are suitably compatible .

We start with defining $M_0 = D_2$ and $(\Xi)_0$ to be simply the boundary inclusion $S^1 \to D_2$. Assuming we have defined $M_\bullet\colon \mathbb{G}_{\leq n-1} \to \mathbf{Mod}(\mathfrak{A})$ together with a direct cofibration of $(n-1)$-truncated coglobular objects $\Xi\colon \mathbf{Cyl}(D_\bullet) * \mathbf{Cyl}(D_\bullet) \to \mathbf{M}_\bullet$, we set $\mathbf{M}_n$ to be the colimit of the following diagram of $\mathfrak{A}$-models:

$$\begin{tikzcd} D_n \amalg_{D_0} D_2 & \Sigma\mathbf{M}_{n-1} & D_2 \amalg_{D_0} D_n \\ \Sigma\mathbf{Cyl}(D_{n-1}) \arrow[u, "\Upsilon_{n-1}"'] \arrow[dr, "\Sigma(i_1)"'] & \Sigma\mathbf{Cyl}(D_{n-1}) \arrow[u, "\Sigma(\Xi_0)"'] \arrow[d, "\Sigma(c)"] & \Sigma\mathbf{Cyl}(D_{n-1}) \arrow[u, "\Gamma_{n-1}"'] \arrow[dl, "\Sigma(i_3)"] \\ & \Sigma\left(\mathbf{Cyl}(D_{n-1}) \otimes \mathbf{Cyl}(D_{n-1}) \otimes \mathbf{Cyl}(D_{n-1})\right) & \end{tikzcd}$$

where c denotes the vertical composition of a stack of three $(n-1)$-cylinders and i_k is the inclusion on the k-th cylinder of the stack.

Define $\mathbf{M}_{\sigma_0}\colon \mathbf{M}_0 \to \mathbf{M}_1$ to be the composite $D_2 \to D_2 \amalg_{D_0} D_1 \to \mathbf{M}_1$, both maps being given by colimit inclusions. Analogously, we set $\mathbf{M}_{\tau_0}\colon \mathbf{M}_0 \to \mathbf{M}_1$ to be the composite $D_2 \to D_1 \amalg_{D_0} D_2 \to \mathbf{M}_1$.

Now suppose $n > 2$, and define $\mathbf{M}_{\varepsilon_{n-1}}\colon \mathbf{M}_{n-1} \to \mathbf{M}_n$ (for $\varepsilon = \sigma, \tau$) as the map obtained by applying the colimit functor to the natural transformation between the defining diagrams for $\mathbf{M}_{n-1}$ and $\mathbf{M}_n$ induced by ε_{n-1}, $\mathbf{Cyl}(\varepsilon_{n-2})$ and $\mathbf{M}_{\varepsilon_{n-2}}$.

We define $\Xi = (\Xi_0, \Xi_1)\colon \mathbf{Cyl}(D_n) * \mathbf{Cyl}(D_n) \to \mathbf{M}_n$ by setting Ξ_0 to be induced by the following cocone

$$
\begin{tikzcd}
D_n \arrow[r, "w"] \arrow[dr, "\iota_0"'] & D_n \coprod\limits_{D_0} D_1 \arrow[r, "1 \amalg \tau"] & D_n \coprod\limits_{D_0} D_2 \arrow[dr] & \\
& \Sigma\mathbf{Cyl}(D_{n-1}) \arrow[r, "\Sigma(i_1)"] & \Sigma\mathbf{Cyl}(D_{n-1})^{\otimes 3} \arrow[r] & \mathbf{M}_n \\
D_n \arrow[r, "w"] \arrow[ur, "\iota_1"] & D_1 \coprod\limits_{D_0} D_n \arrow[r, "\sigma \amalg 1"] & D_2 \coprod\limits_{D_0} D_n \arrow[ur] &
\end{tikzcd}
$$

Next, we set Ξ_1 to be induced by the following cocone

$$
\begin{tikzcd}
D_n \arrow[r, "w"] \arrow[dr, "\iota_0"'] & D_n \coprod\limits_{D_0} D_1 \arrow[r, "1 \amalg \tau"] & D_n \coprod\limits_{D_0} D_2 \arrow[dr] & \\
& \Sigma\mathbf{Cyl}(D_{n-1}) \arrow[r, "\Sigma(\Xi_1)"] & \Sigma\mathbf{M}_{n-1} \arrow[r] & \mathbf{M}_n \\
D_n \arrow[r, "w"] \arrow[ur, "\iota_1"] & D_1 \coprod\limits_{D_0} D_n \arrow[r, "\sigma \amalg 1"] & D_2 \coprod\limits_{D_0} D_n \arrow[ur] &
\end{tikzcd}
$$

In both cases the unlabeled maps denote the colimit inclusions.

Definition 8.2. Given an $\mathfrak{A}$-model X and a map $\Theta\colon M_n \to X$ such that $C = \Theta \circ \Xi_0$ and $D = \Theta \circ \Xi_1$ we say that Θ is a modification between the n-cylinders C and D. Notice that, by construction, $C_k = D_k$ for $k = 0, 1$.

We will also denote this by $\Theta\colon C \Rightarrow D$ or, pictorially, by:

$$
\begin{tikzcd}
\mathbf{Cyl}(D_n) \arrow[r, bend left, "C", ""{name=U, below}] \arrow[r, bend right, "D"', ""{name=D, above}] & X \arrow[Rightarrow, from=U, to=D, "\Theta"]
\end{tikzcd}
$$

$\Theta \circ \mathbf{M}_\sigma$ is called the source of Θ, and it is denoted by $s(\mathbf{M})$. Similarly, $\Theta \circ \mathbf{M}_\tau$ is called the target of Θ, and it is denoted by $t(\mathbf{M})$.

Given two modifications Θ_1, Θ_2 such that $\varepsilon(\Theta_1) = \varepsilon(\Theta_2)$ for $\varepsilon = \sigma, \tau$, we say that Θ_1 and Θ_2 are parallel.

We will also need the following notion, analogous to the one introduced in Definition 7.29 in the case of cylinders.

Definition 8.3. Given a pair of n-cylinders $C, D\colon \mathbf{Cyl}(D_n) \to X$ with $\varepsilon^{n-k}(C) = \varepsilon^{n-k}(D)$ for $\varepsilon = s, t$, we inductively define a k-collapsed modification $\Theta\colon C \Rrightarrow D$ for $-1 \leq k \leq n-1$ to be an ordinary modification if $k = -1$, and a $(k-1)$-collapsed modification $\overline{\Theta}\colon \bar{C} \Rrightarrow \bar{D}$ in $X\,(s^n(C_0), t^n(C_1))$ if $k \geq 0$.

Observe that an $(n-1)$-collapsed n-modification is an $(n+2)$-cell.

Lemma 8.4. *The map of coglobular objects* $\Xi\colon \mathbf{Cyl}(D_\bullet) * \mathbf{Cyl}(D_\bullet) \to \mathbf{M}_\bullet$ *is a direct cofibration.*

Proof. We will prove by induction on n that the n-th latching map $\hat{L}_n(\Xi)$ is a cofibration of $\mathfrak{A}$-models. For $n = 0$ this is just $(\Xi)_0$, i.e. the boundary inclusion $S^1 \to D_2$, and therefore it is a cofibration.

Assume by induction that $\hat{L}_k(\Xi)$ is the pushout of the boundary inclusion $S^{k+1} \to D_{k+2}$ for each $0 \leq q \leq n-1$ and let's prove the same holds true for $k = n$. We do this representably, as follows: let X be an $\mathfrak{A}$-model, and $C, D\colon \mathbf{Cyl}(D_n) \to X$ be two n-cylinders in X, such that $C_k = D_k$ for $k = 0, 1$. Assume given a pair of parallel modifications $\Theta\colon s(C) \Rightarrow s(D)$, $\Psi\colon t(C) \Rightarrow t(D)$. To extend this to a modification $C \Rightarrow D$ we have to give a modification of $(n-1)$-cylinders $\Upsilon(\iota_0 C, \Theta_t) \otimes \bar{C} \otimes \Gamma(\Theta_s, \iota_1 C) \Rightarrow \bar{D}$ in $X(a, b)$ where Θ_s, Θ_t are the 2-cells that are part of the data of both C and D, and $a = s^2(S)$, $b = t^2(T)$. Notice that we already have the source and target of this modification, so that (by inductive hypothesis), this extension amounts to filling in an n-sphere in $X(a, b)$. Upon transposing along the suspension-space of paths adjunction we see that the original extension problem is equivalent to extending along the boundary inclusion $S^{n+1} \to D_{n+2}$, which concludes the proof. $\square$

Thanks to this lemma, it is straightforward to prove the next result.

Lemma 8.5. *Let X be a contractible $\mathfrak{A}$-model, i.e. the map $X \to *$ is a trivial fibration of $\mathfrak{A}$-models. Given a pair of n-cylinders $C, D\colon A \curvearrowright B$ in X, there exists a modification $\Theta\colon C \Rightarrow D$ in X.*

We also record here, for future use, the following lemma

Lemma 8.6. *Given either an ∞-groupoid or a $\mathfrak{C}^{\mathbf{W}}$-model X, an n-cylinder $C\colon A \curvearrowright B$ in X, a pair of parallel $(n-1)$-cylinders $D_s, D_t\colon \mathbf{Cyl}(D_{n-1}) \to X$ and parallel modifications $\Theta_1\colon s(C) \Rightarrow D_s$, $\Theta_2\colon t(C) \Rightarrow D_t$ there exists an n-cylinder $D\colon \mathbf{Cyl}(D_n) \to X$ such that $s(D) = D_s$, $t(D) = D_t$ and a modification $\Theta\colon C \Rightarrow D$ such that $s(\Theta) = \Theta_1$ and $t(\Theta) = \Theta_2$.*

Proof. We prove this statement by induction, the base case being $n = 1$. We can use the 2-cells $\Theta_1\colon s(C) \to D_s$ and $\Theta_2\colon t(C) \to D_t$ and define the 2-cell filling D to be $(B\Theta_1)(C(\Theta_2^{-1}A))$. Clearly, it is possible to extend $(\Theta_1, \Theta_2^{-1})$ to a modification $\Theta\colon C \Rightarrow D$, thanks to the structure of X.

Now let $n > 1$ and assume the statement holds true for every integer $k < n$. The pair of parallel $(n-2)$-cylinders $\bar{D}_s, \bar{D}_t$ in $X\,(x, y)$ (where $x = s^n(C)\sigma$, $y = t^n(C) \circ \tau$), the $(n-1)$-cylinder $\Upsilon(\iota_0 C, \Theta_t) \otimes \bar{C} \otimes \Gamma(\Theta_s, \iota_1 C)$ in $X\,(x, y)$ and the modifications $\bar{\Theta}_1, \bar{\Theta}_2$ satisfy the assumptions of the lemma for $k = n - 1$. Therefore, we get an $(n-1)$-cylinder $\bar{D}$ and a modification $\bar{\Theta}\colon \Upsilon(\iota_0 C, \Theta_t) \otimes \bar{C} \otimes \Gamma(\Theta_s, \iota_1 C) \Rightarrow \bar{D}$, both in $X\,(x, y)$, which concludes the proof. $\square$

The content of the previous lemma can be pictorially represented by the following extension problem

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{Cyl}(S^{n-1}) \arrow[r, bend left, "{(s(C),t(C))}"] \arrow[r, bend right, "{(D_s,D_t)}"'] \arrow[r, phantom, "\Downarrow (\Theta_1,\Theta_2)"] \arrow[d, "{(\mathbf{Cyl}(\sigma),\mathbf{Cyl}(\tau))}"'] & X \\
\mathbf{Cyl}(D_n) \arrow[ur, bend left, "C"] \arrow[ur, dashed, bend right, "D"'] \arrow[ur, phantom, "\Downarrow \Theta"] &
\end{tikzcd}
$$

Remark 8.7. Note that all modifications of ∞-groupoids or $\mathfrak{C}^{\mathbf{W}}$-models are "invertible" in a sense that can be made precise, but here we content ourselves with the weaker statement that given n-cylinders C, D in an ∞-groupoid X, there exists a modification $\Theta\colon C \Rrightarrow D$ if and only if there exists a modification $\Theta'\colon D \Rrightarrow C$. This is proven in Lemma B.11.

2. Low dimensional operations in $\mathbb{P}X$

Given a coherator $\mathfrak{A}$ that admits inverses (meaning it is either a coherator for ∞-groupoids or it is of the form $\mathfrak{C}^{\mathbf{W}}$) and an $\mathfrak{A}$-model X, we will now endow the underlying globular set of $\mathbb{P}X$ with all the operations $\mathbb{P}X(\varrho)$ for $\varrho\colon D_n \to A$ in a homogeneous coherator for ∞-categories $\mathfrak{C}$, with $n \leq 2$. If we denote by $\mathbb{P}_2X$ the 2-globular set obtained from $\mathbb{P}X$ by identifying 2-cells connected by a 3-cell and keeping the same 0 and 1-cells, then we can construct a bicategory structure with weak inverses on $\mathbb{P}_2X$, thanks to this result and those of Chapter 6.

Remark 8.8. Thanks to Proposition 2.26 and a similar argument as the one used in the proof of the lemma below, it is not hard to show that one can assume, without loss of generality, that the coherator $\mathfrak{C}$ has been obtained in the following manner: there is a functor $\mathfrak{C}_\bullet\colon \omega \to \mathbf{GlTh}$ as in Definition 2.10, with $\mathfrak{C}_{n+1} = \mathfrak{C}_n[X]$, where $X = \{(h_1, h_2)\colon D_n \to A\}$, i.e. all the $(n+1)$-dimensional "basic" operations of $\mathfrak{C}$ are added at the $(n+1)$st step. Therefore, we may rephrase the goal of this section in terms of constructing an extension of the form:

$$
\begin{tikzcd}
\Theta_0 \arrow[r, "\mathbf{Cyl}"] \arrow[d] & \mathbf{Mod}(\mathfrak{A}) \\
\mathfrak{C}_2 \arrow[ur, dashed, "\mathbf{Cyl}"'] &
\end{tikzcd}
$$

We only need to interpret all the homogeneous operations of dimension $n \leq 2$ since we have already defined $\mathbf{Cyl}(\bullet)$ on globular maps. More precisely, we have to define $\mathbf{Cyl}(\varrho)\colon \mathbf{Cyl}(D_n) \to \mathbf{Cyl}(A)$ for every homogeneous map $\varrho\colon D_n \to A$ with $n \leq 2$ in $\mathfrak{C}$. Notice that this forces $m = \mathrm{ht}(A) \leq 2$. We will make use of the following fact, whose proof we only sketch not to disrupt the flow of this section

Lemma 8.9. *Given a cellular globular theory $\mathfrak{D}$, the inclusion $\Theta_0 \to \mathfrak{D}$ induces isomorphisms*

$$\Theta_0(D_0, A) \cong \mathfrak{D}(D_0, A)$$

PROOF. It is enough to prove that the unit map $\eta_A \colon A \to U \circ FA$ of the adjunction:

$$\begin{tikzcd} {[\mathbb{G}^{op}, \mathbf{Set}]} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}(\mathfrak{A}) \arrow[l, bend left, "U"] \end{tikzcd}$$

is sent to an isomorphism when we evaluate $[\mathbb{G}^{op}, \mathbf{Set}](D_0, -)$ at it. Thanks to Proposition 2.2 of [**Nik**], the unit is an $I_{\geq 1}$-cellular map (where we denote by $I_{\geq 1}$ the set $\{S^{k-1} \to D_k\}_{k \geq 1}$), and therefore it is 0-bijective. $\square$

To begin with, we start with operations of dimension 1, i.e. extending the functor **Cyl** to $\mathfrak{C}_1$.

PROPOSITION 8.10. *There exists an extension of the form:*

$$\begin{tikzcd} \Theta_0 \arrow[r, "\mathbf{Cyl}"] \arrow[d] & \mathbf{Mod}(\mathfrak{A}) \\ \mathfrak{C}_1 \arrow[ur, dashed, "\mathbf{Cyl}"'] & \end{tikzcd}$$

PROOF. Consider an operation h added as solutions of lifting problems of the following form, as in point (2) of Definition 2.10:

$$\begin{tikzcd} D_0 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, shift right, "\sigma_0"'] \arrow[d, shift left, "\tau_0"] & A \\ D_1 \arrow[ur, "h"'] & \end{tikzcd}$$

We know that, since $\mathfrak{C}$ is assumed to be homogeneous, this implies $\mathrm{ht}(A) \leq 1$, and therefore either $f = g = 1_{D_0}$, or $\mathrm{ht}(A) = 1$ and necessarily $f = \partial_\sigma, g = \partial_\tau$ thanks to the previous lemma. Therefore, setting $\mathbf{Cyl}(h) = \hat{h}$ as in Definition 7.42 is a well-defined choice. Doing so for all the 1-dimensional operations h added as fillers of pairs $(f, g) \in X_0$ with $\mathfrak{C}_1 = \mathfrak{C}_0[X_0]$, we get the desired extension. $\square$

Let us now address the problem of extending this to 2-dimensional operations, i.e. morphisms $D_2 \to A$ in $\mathfrak{C}$. This can be done using the following result, whose proof is fundamental to this section and will be subdivided into several lemmas.

PROPOSITION 8.11. *Given a map $\varrho \colon D_1 \to A$ in $\mathfrak{C}$, there exists a 0-collapsed modification:*

$$\vartheta_\varrho \colon \hat{\varrho} \Rightarrow \mathbf{Cyl}(\varrho)$$

where $\mathbf{Cyl}(\varrho)$ *is constructed in* (8.10). *Furthermore, these modifications can be built in such a way that if $\varrho_1, \varrho_2 \colon D_1 \to A$ are parallel maps then ϑ_{ϱ_1} and ϑ_{ϱ_2} are parallel.*

If we assume this result, we can now prove:

PROPOSITION 8.12. *Assume $\mathfrak{A}$ has inverses. Given any operation $\varrho \colon D_2 \to A$ in $\mathfrak{C}$ fitting into a diagram of the form:*

$$\begin{tikzcd} D_1 \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & A \\ D_2 \arrow[ur, "\varrho"'] & \end{tikzcd}$$

we can associate to it a map $\mathbf{Cyl}(\varrho)\colon \mathbf{Cyl}(D_2) \to \mathbf{Cyl}(A)$ *fitting into a diagram of the form*

$$\begin{tikzcd}
\mathbf{Cyl}(D_1) \arrow[r, shift left, "\mathbf{Cyl}(h_1)"] \arrow[r, shift right, "\mathbf{Cyl}(h_2)"'] \arrow[d, shift right, "\mathbf{Cyl}(\sigma)"'] \arrow[d, shift left, "\mathbf{Cyl}(\tau)"] & \mathbf{Cyl}(A) \\
\mathbf{Cyl}(D_2) \arrow[ur, "\mathbf{Cyl}(\varrho)"'] &
\end{tikzcd}$$

Moreover, this map also comes endowed with a modification $\vartheta_\varrho\colon \hat{\varrho} \Rightarrow \mathbf{Cyl}(\varrho)$ *whose boundary is given by* $(\vartheta_{\varrho\circ\sigma}, \vartheta_{\varrho\circ\tau})$.

Proof. Using Lemma 8.11, we can apply Lemma 8.6 to the following diagram, where the solid triangle at the back commutes by (36):

$$\begin{tikzcd}
\mathbf{Cyl}(S^1) \arrow[rr, bend left, "(\widehat{\varrho\circ\sigma},\widehat{\varrho\circ\tau})"] \arrow[rr, bend right, "(\mathbf{Cyl}(\varrho\circ\sigma),\mathbf{Cyl}(\varrho\circ\tau))"'] \arrow[rr, phantom, "\Downarrow(\vartheta_{\varrho\circ\sigma},\vartheta_{\varrho\circ\tau})"] \arrow[d, "(\mathbf{Cyl}(\sigma),\mathbf{Cyl}(\tau))"'] & & \mathbf{Cyl}(A) \\
\mathbf{Cyl}(D_2) \arrow[urr, bend left, "\hat{\varrho}"] \arrow[urr, dashed, "\mathbf{Cyl}(\varrho)"'] \arrow[urr, phantom, "\Downarrow\vartheta_\varrho"] & &
\end{tikzcd}$$

It is straightforward to observe that this is enough to conclude the proof. ☐

The following result follows immediately from the previous one and cellularity of $\mathfrak{C}$

Proposition 8.13. *There exists an extension of the form:*

$$\begin{tikzcd}
\Theta_0 \arrow[r, "\mathbf{Cyl}"] \arrow[d] & \mathbf{Mod}(\mathfrak{A}) \\
\mathfrak{C}_2 \arrow[ur, dashed, "\mathbf{Cyl}"'] &
\end{tikzcd}$$

We now prove Proposition 8.11 via a series of lemmas.

Lemma 8.14. *Given integers* $i, k, q > 0$, *a map* $\varrho\colon D_1 \to D_1^{\otimes k}$ *such that* $\varrho \circ \sigma = \partial_\sigma, \varrho \circ \tau = \partial_\tau$, *and a 0-collapsed modification* $\Theta\colon C \Rightarrow D\colon \mathbf{Cyl}(D_1) \to \mathbf{Cyl}(D_1^{\otimes q})$ *we get an induced 0-collapsed modification denoted by* $\hat{\varrho} \circ_i \Theta\colon \hat{\varrho} \circ_i C \Rightarrow \hat{\varrho} \circ_i D$ *between the following* 1*-cylinders:*

$$\begin{tikzcd}
\mathbf{Cyl}(D_1) \arrow[r, "\hat{\varrho}"] & \mathbf{Cyl}(D_1^{\otimes k}) \arrow[rr, bend left, "1 \amalg \ldots \amalg C \amalg \ldots \amalg 1"] \arrow[rr, bend right, "1 \amalg \ldots \amalg D \amalg \ldots \amalg 1"'] & & \mathbf{Cyl}(D_1^{\otimes k+q-1})
\end{tikzcd}$$

Proof. We prove this representably, i.e. we assume given compatible 1-cylinders $C_i\colon A_i \curvearrowright B_i$ in an ∞-groupoid X, with $s(C_i) = w_i$ and $t(C_i) = w_{i+1}$. We let $E = (C_i, \ldots, C_{i+q-1}) \circ C$

and $F = (C_i, \dots, C_{i+q-1}) \circ D$, and we observe that Θ essentially consists of a 2-cell $\overline{E} \to \overline{F} \colon w_{i+q}E_0 \to E_1 w_i$ in $X\,(s(E_0), t(E_1))$. The composite of the following pasting diagram in $X\,(s(A_1), t(B_{k+q-1}))$ transpose via the adjunction $\Sigma \dashv \Omega$ to give the 3-cell in X which $\hat{\varrho} \circ_i \Theta$ essentially corresponds to:

$$
\begin{tikzcd}
w_{k+q}\varrho(A_{k+q-1}, \dots,\ A_{i+q},\ E_0,\ A_{i-1}, \dots,\ A_1) \arrow[d] \\
\cdots \arrow[d] \\
\varrho(B_{k+q-1}, \dots,\ B_{i+q},\ (w_{i+q}E_0),\ A_{i-1}, \dots,\ A_1) \arrow[d, bend right=50, "{\varrho^*(B_{k+q-1},\dots,\ B_{i+q},\ \overline{E},\ A_{i-1},\dots,\ A_1)}"'] \arrow[d, bend left=50, "{\varrho^*(B_{k+q-1},\dots,\ B_{i+q},\ \overline{F},\ A_{i-1},\dots,\ A_1)}"] \arrow[d, phantom, "\overset{\dots\overline{\Theta}\dots}{\Longrightarrow}"] \\
\varrho(B_{k+q-1}, \dots,\ B_{i+q},\ (E_1 w_i),\ A_{i-1}, \dots,\ A_1) \arrow[d] \\
\cdots \arrow[d] \\
\varrho(B_{k+q-1}, \dots,\ B_{i+q},\ E_1,\ B_{i-1}, \dots,\ B_1)w_1
\end{tikzcd}
$$

where we have implicitly used the fact that $E_i = F_i$ for $i = 0, 1$ and we denoted by ϱ^* a homogeneous operation whose boundary is given by $(\partial_\sigma \circ \varrho, \partial_\tau \circ \varrho)$. Notice that, by definition, the left-hand side composite is $(C_1, \dots C_{i-1}, E, C_{i+q}, \dots, C_{k+q-1}) \circ \hat{\varrho}$, and the right-hand side one is $(C_1, \dots C_{i-1}, F, C_{i+q}, \dots, C_{k+q-1}) \circ \hat{\varrho}$, which concludes the proof. □

We will need something a bit stronger, namely the following generalization of the previous lemma, whose proof is left to the reader.

LEMMA 8.15. *Assume given integers $k, q_j > 0$ for $1 \leq j \leq k$, a map $\varrho \colon D_1 \to D_1^{\otimes k}$ such that $\varrho \circ \sigma = \partial_\sigma, \varrho \circ \tau = \partial_\tau$, and 0-collapsed modifications $\Theta_j \colon C_j \Rightarrow D_j \colon \mathbf{Cyl}(D_1) \to \mathbf{Cyl}(D_1^{q_j})$. We then get an induced 0-collapsed modification $(\Theta_1, \dots, \Theta_k) \circ \hat{\varrho} \colon (C_1, \dots, C_k) \circ \hat{\varrho} \Rightarrow (D_1, \dots, D_k) \circ \hat{\varrho}$, between the following 1-cylinders:*

$$
\begin{tikzcd}
\mathbf{Cyl}(D_1) \arrow[r, "\hat{\varrho}"] & \mathbf{Cyl}(D_1^{\otimes k}) \arrow[rr, bend left=15, "\coprod\limits_{1\leq j\leq k} C_j"] \arrow[rr, bend right=15, "\coprod\limits_{1\leq j\leq k} D_j"'] & & \mathbf{Cyl}\left(D_1^{\otimes(\sum_j q_j)}\right)
\end{tikzcd}
$$

LEMMA 8.16. *Given compatible operations $\varrho \colon D_1 \to D_1^{\otimes k}$, $\varphi_j \colon D_1 \to D_1^{\otimes q_j}$ for $1 \leq j \leq k$ similarly to the previous lemma, there is an induced 0-collapsed modification:*

$$
\begin{tikzcd}
 & \mathbf{Cyl}(D_1^{\otimes k}) \arrow[dr, "\coprod\limits_{1\leq j\leq k} \widehat{\varphi_j}"] \arrow[d, phantom, "\Downarrow"] & \\
\mathbf{Cyl}(D_1) \arrow[ur, "\hat{\varrho}"] \arrow[rr, "((\varphi_j)_{1\leq j\leq k} \circ \varrho)^\wedge"'] & & \mathbf{Cyl}\left(D_1^{\otimes(\sum_j q_j)}\right)
\end{tikzcd}
\tag{54}
$$

Proof. We prove this representably, thus we assume given an ∞-groupoid X and a family of compatible 1-cylinders C^m_r with $1 \leq m \leq k$ and $1 \leq r \leq q_m$. Denote with $\underline{C}^m$ the map $(C^m_1, \ldots, C^m_{q_m})\colon \mathbf{Cyl}(D_1^{\otimes q_m}) \to X$, and, for $\varepsilon = 0, 1$, let $\underline{C_\varepsilon}^m$ be the string of 1-cells in X given by $({C^m_1}_\varepsilon, \ldots, {C^m_{q_m}}_\varepsilon)\colon D_1^{\otimes q_m} \to X$. In this way, we have that $\widehat{\varphi_m}$ acts on $\underline{C}^m$, and φ_m acts on $\underline{C_\varepsilon}^m$. Define $n = \sum_j q_j$ and denote the target of the i-th cylinder in the list $(C^1_1, \ldots,\ C^1_{q_1}, \ldots,\ C^m_1, \ldots,\ C^m_{q_m})$ by w_i. Consider the following diagram in $X(a, b)$, where $a = s\left((C^1_1)_0\right)$ and $b = t\left((C^m_{q_m})_1\right)$, where the left and the right-hand side composites coincide with the upper and lower composites of (55):

$$
\begin{tikzcd}[column sep=tiny]
& w_n \varrho\left(\varphi_k(\underline{C_0}^k), \ldots,\ \varphi_1(\underline{C_0}^1)\right) \arrow[dl] \arrow[ddr] & \\
\varrho\left(\left(w_n\varphi_k(\underline{C_0}^k)\right),\ \varphi_{k-1}(\underline{C_0}^{k-1}), \ldots,\ \varphi_1(\underline{C_0}^1)\right) \arrow[drr, "(1)"] \arrow[dd, "g"'] & & \\
& & \varrho\left(\varphi_k\left((w_n(C^k_{q_k})_0), \ldots,\ (C^k_1)_0\right),\ \varphi_{k-1}(\underline{C_0}^{k-1}), \ldots,\ \varphi_1(\underline{C_0}^1)\right) \arrow[d] \arrow[d, phantom, "(2)"'] \\
\varrho\left(\left(\varphi_k(\underline{C_1}^k)w_{n-q_k}\right),\ \varphi_{k-1}(\underline{C_0}^{k-1}), \ldots,\ \varphi_1(\underline{C_0}^1)\right) \arrow[d] \arrow[ddrr] & & \cdots \arrow[dd] \\
\varrho\left(\varphi_k(\underline{C_1}^k),\ \left(w_{n-q_k}\varphi_k(\underline{C_0}^{k-1}\right), \ldots,\ \varphi_1(\underline{C_0}^1)\right) \arrow[dd] & & \\
& & \varrho\left(\varphi_k\left((C^k_{q_k})_1, \ldots,\ (C^k_1)_1 w_{n-q_k}\right), \varphi_{k-1}(\underline{C_0}^{k-1}), \ldots, \ldots \varphi_1(\underline{C_0}^1)\right) \arrow[d] \\
\cdots \arrow[d] & & \cdots \arrow[ddl] \\
\varrho\left(\varphi_k(\underline{C_1}^k), \ldots,\ \left(\varphi_1(\underline{C_1}^1)w_1\right)\right) \arrow[dr] & & \\
& \varrho\left(\varphi_k(\underline{C_1}^k), \ldots,\ \varphi_1(\underline{C_1}^1)\right) w_1 &
\end{tikzcd}
$$

where $g \overset{def}{=} \varrho^*\left(\left(\overline{\hat{\varphi}_k(\underline{C}^k)}\right),\ \varphi_{k-1}(\underline{C_0}^{k-1}), \ldots,\ \varphi_1(\underline{C_0}^1)\right)$ and $\varrho^*\colon D_2 \to D_2 \coprod_{D_0} D_1^{\otimes k}$ is an operation whose boundary is given by $(\partial_\sigma \circ \varrho, \partial_\tau \circ \varrho)$.

We now explain how to fill the part of the diagram labelled with (1) with a 2-cell, and the same argument will provide fillers for the other analogous subdivisions of the diagram, corresponding to the φ_j for $j < k$. Note the cell $\overline{\hat{\varphi}_k(\underline{C}^k)}$ appearing in g is a composite of the form

$$w_n\varphi_k(\underline{C_0}^k) \xrightarrow{a_1} \cdots \xrightarrow{a_p} \varphi_k(\underline{C_1}^k)w_{n-q_k}$$

Here, we denote the 1-cells which constitute the vertical stack of 0-cylinders appearing in the construction of $\widehat{\varphi_k}$ with $\{a_i\}_{1 \leq i \leq p}$.

By the structure of $\mathfrak{A}$, we see that there is a 2-cell $\alpha\colon g \to g'$, where g' is the composite of 2-cells of the form $\varrho^*\left(a_i\ , \varphi_{k-1}(\underline{C_0}^{k-1}), \ldots,\ \varphi_1(\underline{C_0}^1)\right)$. Furthermore, the target of $\varrho^*\left(a_1,\ \varphi_{k-1}(\underline{C_0}^{k-1}), \ldots,\ \varphi_1(\underline{C_0}^1)\right)$ is precisely given by

$$\varrho\left(\varphi_k\left(w_n(C^k_{q_k})_0 \ldots (C^k_1)_0\right),\ \varphi_{k-1}(\underline{C_0}^{k-1}), \ldots,\ \varphi_1(\underline{C_0}^1)\right)$$

and each of the 2-cells $\varrho^*\left(a_i,\ \varphi_{k-1}(\underline{C_0}^{k-1}),\ldots,\ \varphi_1(\underline{C_0}^1)\right)$ for $1 < i < p$ is parallel to (and appears in the same order as) one on the right-hand side composite labelled with (2). Each of this pair of parallel 2-cells factors, by construction, through the same globular sum. However, they could be obtained using possibly different operations in $\mathfrak{A}$. These have to be parallel operations by construction, and therefore there is a 3-cell between them (if $\mathfrak{A} = \mathfrak{C}^{\mathbf{W}}$, its existence is easily checked in $\mathbf{Mod}(\Theta)$ and so the usual argument applies). Using again the structure of $\mathfrak{A}$ for the triangle of the form:

$$\begin{tikzcd}[column sep=small]
w_n\varrho\left(\varphi_k(\underline{C_0}^k),\ldots,\ \varphi_1(\underline{C_0}^1)\right) \arrow[r] \arrow[d] & \varrho\left(\varphi_k\left(w_n(C^k_{q_k})_0\ldots(C^k_1)_0\right),\ \varphi_{k-1}(\underline{C_0}^{k-1}),\ldots,\ \varphi_1(\underline{C_0}^1)\right) \\
\varrho\left(\left(w_n\varphi_k(\underline{C_0}^k)\right),\ \varphi_{k-1}(\underline{C_0}^{k-1}),\ldots,\ \varphi_1(\underline{C_0}^1)\right) \arrow[ur, "\varrho\left(a_1,\ \varphi_{k-1}(\underline{C_0}^{k-1}),\ldots,\ \varphi_1(\underline{C_0}^1)\right)"'] &
\end{tikzcd}$$

and for the analogous one at the bottom we get the remaining 3-cell fillers needed to provide the desired filler for (1). Finally, we can compose this pasting diagram using the bicategorical structure on 0-cylinders and 0-modifications developed in the Appendix. $\square$

Finally, the last intermediate result before the proof of Lemma 8.11.

LEMMA 8.17. *Given compatible operations $\varrho\colon D_1 \to D_1^{\otimes k}$, $\varphi_j\colon D_1 \to D_1^{\otimes q_j}$ for $1 \le j \le k$ and 0-collapsed modifications $\mathbf{Cyl}(\varrho) \Rightarrow \hat{\varrho}, \mathbf{Cyl}(\varphi_j) \Rightarrow \widehat{\varphi_j}$, there is an induced 0-collapsed modification of $\mathfrak{A}$-models:*

$$\begin{tikzcd}
& \mathbf{Cyl}(D_1^{\otimes k}) \arrow[dr, "\amalg_{1\le j\le k}\mathbf{Cyl}(\varphi_j)"] \arrow[d, phantom, "\Downarrow \delta_{\varphi,\varrho}"] & \\
\mathbf{Cyl}(D_1) \arrow[ur, "\mathbf{Cyl}(\varrho)"] \arrow[rr, "((\varphi_j)_{1\le j\le k}\circ\varrho)^\wedge"'] & & \mathbf{Cyl}\left(D_1^{\otimes(\sum_j q_j)}\right)
\end{tikzcd} \tag{55}$$

PROOF. By Lemma 8.15 we get a modification $(\mathbf{Cyl}(\varphi_1),\ldots,\mathbf{Cyl}(\varphi_k))\circ\hat{\varrho} \Rightarrow (\hat{\varphi}_1,\ldots,\hat{\varphi}_k)\circ\hat{\varrho}$, and thanks to the previous lemma we get one of the form $(\hat{\varphi}_1,\ldots,\hat{\varphi}_k)\circ\hat{\varrho} \Rightarrow ((\varphi_j)_{1\le j\le k}\circ\varrho)^\wedge$. Moreover, we can precompose the former with:

$$\amalg_{1\le j\le k}\mathbf{Cyl}(\varphi_j)\circ\vartheta_\varrho\colon \amalg_{1\le j\le k}\mathbf{Cyl}(\varphi_j)\circ\mathbf{Cyl}(\varrho) \Rightarrow \amalg_{1\le j\le k}\mathbf{Cyl}(\varphi_j)\circ\hat{\varrho}$$

All of these modifications are 0-collapsed by construction, so that we can compose them as 2-cells in $\mathbf{Cyl}\left(D_1^{\otimes(\sum_j q_j)}\right)$ and conclude the proof. $\square$

COROLLARY 8.18. *Given operations $\varrho\colon A \to B, \varphi\colon B \to C$ in $\mathfrak{A}$ between 1-dimensional globular sums, and 0-collapsed modifications $\mathbf{Cyl}(\varrho) \Rightarrow \hat{\varrho}, \mathbf{Cyl}(\varphi) \Rightarrow \widehat{\varphi}$, one gets an induced 0-collapsed modification $\mathbf{Cyl}(\varphi)\circ\mathbf{Cyl}(\varrho) \Rightarrow \widehat{\varphi\circ\varrho}$.*

PROOF. The proof of this corollary is a straightforward generalization of that of the previous lemma. In fact, by precomposing $\mathbf{Cyl}(\varrho)$ with the various maps $\mathbf{Cyl}(i)\colon \mathbf{Cyl}(D_1) \to \mathbf{Cyl}(A)$ for every globe D_1 in the globular decomposition of $A \cong D_1^{\otimes k}$, we get 0-collapsed modifications $\delta_{\varphi,\varrho\circ i}$ as in the previous lemma, which can be glued together. More precisely, being 0-collapsed forces them to be compatible on their source and target, i.e. $t(\delta_{\varphi,\varrho\circ i_q}) = s(\delta_{\varphi,\varrho\circ i_{q+1}})$, where i_r is the colimit inclusion of the r-th copy of D_1 in $D_1^{\otimes k}$. Therefore, they induce the desired modification $\mathbf{Cyl}(\varphi)\circ\mathbf{Cyl}(\varrho) \Rightarrow \widehat{\varphi\circ\varrho}$ thanks to the universal property of colimits. $\square$

We now end this section with a proof of Proposition 8.11, which we recall here below.

PROPOSITION 8.19. *Given a map $\varrho\colon D_1 \to A$ in $\mathfrak{C}$, there exists a 0-collapsed modification:*

$$\vartheta_\varrho\colon \hat{\varrho} \Rightarrow \mathbf{Cyl}(\varrho)$$

where $\mathbf{Cyl}(\varrho)$ is constructed in (8.10). *Furthermore, these modifications can be built in such a way that if $\varrho_1, \varrho_2\colon D_1 \to A$ are parallel maps then ϑ_{ϱ_1} and ϑ_{ϱ_2} are parallel.*

PROOF. Define a category $\mathcal{C}'$ whose objects are globular sums of height less than or equal to 1, where a map $f\colon A \to B$ consists of a map $f\colon A \to B$ in $\mathfrak{C}$ such that for every globular map $i\colon A' \to A$ there exists a 0-collapsed modification $\mathbf{Cyl}(g) \to \hat{g}$, where g denotes the homogeneous part of the composite $f \circ i$, as depicted below:

$$\begin{tikzcd}
A' \arrow[r, two heads, "g"] \arrow[d, tail, "i"'] & B' \arrow[d, tail, "j"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Thanks to the hypothesis on the modifications being 0-collapsed, it is easy to show that this is a 1-globular theory, and that it contains every generator of $\mathfrak{C}_1$, thanks to the previous results. Therefore, it must coincide with $\mathfrak{C}_1$, which concludes the proof. $\square$

3. Path object on $\mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$

Given a coherator for 3-categories $\mathfrak{C}$, we are going to endow the globular set:

$$(\mathbb{P}X)_k = \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})\,(\mathbf{Cyl}(D_k), X)$$

with the structure of a $\mathfrak{C}$-model, which we can then extend to a $\mathfrak{C}^{\mathbf{W}}$-model thanks to the result of Chapter 7, thus providing a proof of Proposition 8.29. We can adapt the argument given in the previous section to find an extension of the form:

$$\begin{tikzcd}
\Theta_0 \arrow[r, "\mathbf{Cyl}"] \arrow[d] & \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}) \\
\mathfrak{C}_2 \arrow[ur, dashed, "\mathbf{Cyl}"'] &
\end{tikzcd}$$

where we denote by $\mathfrak{C}_2$ the step in the defining tower for the coherator $\mathfrak{C}$ in which all the 2-dimensional operations are introduced (see Remark 8.8).

We now focus on finding an extension of the form:

$$\begin{tikzcd}
\mathfrak{C}_2 \arrow[r, "\mathbf{Cyl}"] \arrow[d] & \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}) \\
\mathfrak{C}_3 \arrow[ur, dashed, "\mathbf{Cyl}"'] &
\end{tikzcd} \tag{56}$$

and this, in turn, amounts to defining a map $\mathbf{Cyl}(\varrho)\colon \mathbf{Cyl}(D_3) \to \mathbf{Cyl}(A)$ for every $\varrho\colon D_3 \to A$ added as a filler of a pair $(h_1, h_2) \in X_2$, in such a way that $\mathbf{Cyl}(\varrho) \circ \mathbf{Cyl}(\sigma) = \mathbf{Cyl}(h_1)$ and $\mathbf{Cyl}(\varrho) \circ \mathbf{Cyl}(\tau) = \mathbf{Cyl}(h_2)$. Note that these last 2 equations make sense, since $h_1, h_2 \in \mathfrak{C}_2$.

The strategy for constructing such maps will be the same as the one used to get the extension to $\mathfrak{C}_2$, namely to prove that we can endow every interpretation of a 2-dimensional operation $\mathbf{Cyl}(\varphi)\colon \mathbf{Cyl}(D_2) \to \mathbf{Cyl}(A)$ with a modification:

$$\Theta_{\varphi}\colon \widehat{\varphi} \Rrightarrow \mathbf{Cyl}(\varphi) \tag{57}$$

in a way that is compatible with source and target (as will be explained in more detail later on), so that we can then use Lemma 8.6 to produce the map we are after. In fact, we can apply this lemma to the situation depicted in the diagram below, thus getting the desired extension:

$$\begin{tikzcd}
\mathbf{Cyl}(S^2) \arrow[rr, bend left, "(\widehat{h_1},\widehat{h_2})"] \arrow[rr, bend right, "(\mathbf{Cyl}(h_1),\mathbf{Cyl}(h_2))"'] \arrow[rr, phantom, "\Downarrow(\Theta_{h_1},\Theta_{h_2})"] \arrow[d, "(\mathbf{Cyl}(\sigma),\mathbf{Cyl}(\tau))"'] & & X \\
\mathbf{Cyl}(D_3) \arrow[urr, bend left, "\widehat{\varrho}"] \arrow[urr, dashed, "\mathbf{Cyl}(\varrho)"'] \arrow[urr, phantom, "\Rrightarrow \Theta"] & &
\end{tikzcd}$$

We start with a lemma that allows us to "plug" modifications of globular sums of cylinders into the elementary interpretation of a 2-dimensional operation.

Lemma 8.20. *Assume given a homogeneous operation $\varrho\colon D_2 \to A$ in $\mathfrak{C}$, a $\mathfrak{C}^{\mathbf{W}}$-model X and a pair of cylinders $C, D\colon \mathbf{Cyl}(A) \to X$ that agree on the 0-cells of A (i.e. each inclusion $\mathbf{Cyl}\,(D_0 \to A)$ equalizes these maps), with $C, D\colon U \curvearrowright V$. Given a modification $\Theta\colon C \Rrightarrow D$ such that for each globular map $D_1 \to A$, the induced modification $\mathbf{M}_1 \to \mathbf{M}_A \xrightarrow{\Theta} X$ is 0-collapsed, we get an induced modification of the form $\Theta \circ \hat{\varrho}\colon C \circ \hat{\varrho} \Rrightarrow D \circ \hat{\varrho}$.*

Proof. The proof is structured in the following manner: since both cylinders $C \circ \hat{\varrho}$ and $D \circ \hat{\varrho}$ are built as the vertical composite of a stack of cylinders, we will construct compatible modifications from each of the cylinders that constitute the stack associated with $C \circ \hat{\varrho}$ towards the corresponding ones in the stack associated with $D \circ \hat{\varrho}$. We will then conclude by using the bicategorical structure described in part B of the appendix to compose up these modifications thus getting the desired map $\Theta \circ \hat{\varrho}$. Let

$$\begin{pmatrix} i_1 & & i_2 & \dots & i_{m-1} & & i_m \\ & i'_1 & & \dots & & i'_{m-1} & \end{pmatrix}$$

be the table of dimensions of A. Since ϱ is homogeneous, we have $\mathrm{ht}(A) \leq 2$, and therefore $i_k = 1, 2$ for every $1 \leq k \leq m$. By precomposing with the appropriate colimit inclusions we thus get cylinders $C_k, D_k\colon \mathbf{Cyl}(D_{i_k}) \to X$.

The cylinders associated with case (46) to (48) in both stacks coincide thanks to the assumptions, thus we can use identity modifications in these cases. We now consider case (49): i.e globular sums $B \in \mathscr{L}(A)$ in which we added a new vertex $*_B$ to A at height $\mathrm{ht}(*_B) = 2$, in such a way that this new vertex is the unique element of its fiber. Fix a globular sum B in this family, such that the vertex $*_B$ has been added to A over $D_{i_r} = D_1$, and consider the vertical stacks of 1-cylinders whose composites are the transpose of $C \circ \hat{\varrho}$ and $D \circ \hat{\varrho}$ respectively. The 2-cell in B corresponding to the vertex $*_B$ picks out the 2-cell associated with the 1-cylinder C_r (resp.D_r) via the composites:

$$B \xrightarrow{i_B} \mathbf{Cyl}(A) \xrightarrow{C} X \quad B \xrightarrow{i_B} \mathbf{Cyl}(A) \xrightarrow{D} X$$

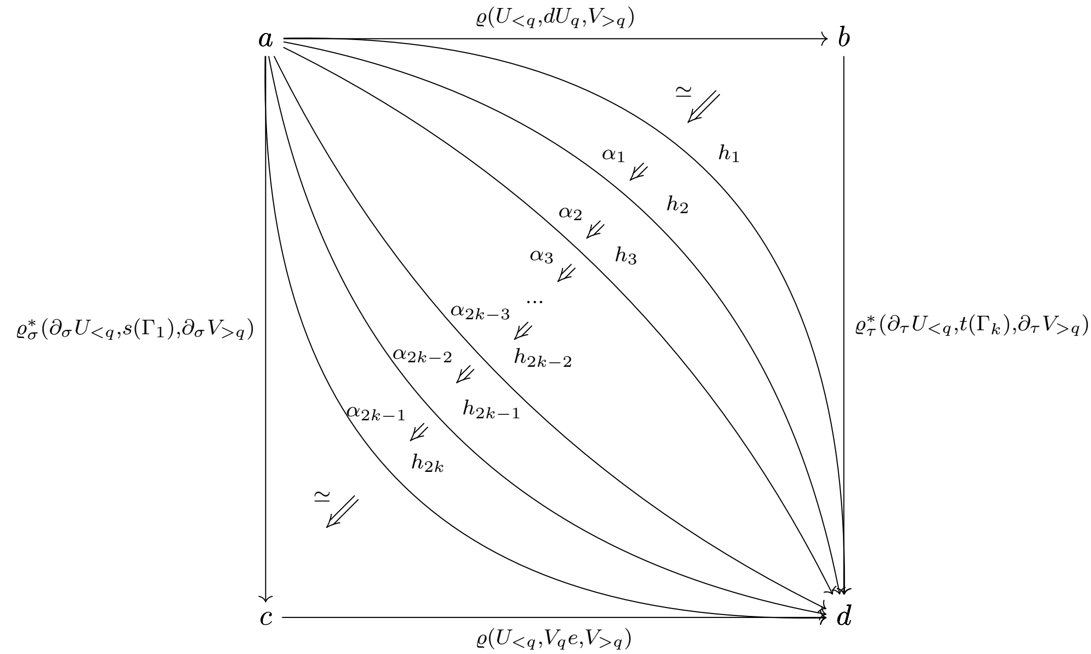
We can use the components of Θ to construct the following boundary of a 1-modification in $B(x,y)$ (for $x = s^{i_1}(C_1)_0,\ y = t^{i_m}(C_m)_1$), where the 1-cylinders Γ_B and Δ_B are the ones associated with the globular sum B in the two stacks (using the notation established at the end of Chapter 7):

$$\begin{tikzcd}
a \arrow[rr, "\varrho(U_{<r},V_{>r}s(C_r))"] \arrow[d, bend right=60, "\varrho^*_\sigma\left(U^\sigma_{<r},V^\sigma_{>r}\overline{C_r}\right)"'] \arrow[d, bend left=60, "\varrho^*_\sigma\left(U^\sigma_{<r},V^\sigma_{>r}\overline{D_r}\right)"] \arrow[d, phantom, "\overset{\varrho^+_\sigma\left(U^\sigma_{<r},V^\sigma_{>r}\overline{\Theta_r}\right)}{\Longleftarrow}"] & \arrow[d, phantom, "\overset{\Gamma_B}{\curvearrowleft}\ \underset{\Delta_B}{\curvearrowright}"] & b \arrow[d, bend right=60, "\varrho^*_\tau\left(U^\tau_{<r},V^\tau_{>r}\overline{D_r}\right)"'] \arrow[d, bend left=60, "\varrho^*_\tau\left(U^\tau_{<r},V^\tau_{>r}\overline{C_r}\right)"] \arrow[d, phantom, "\overset{\varrho^+_\tau\left(U^\tau_{<r},V^\tau_{>r}\overline{\Theta_r}\right)}{\Longleftarrow}"] \\
a' \arrow[rr, "\varrho(U_{<r},V_{>r}t(C_r))"'] & & b'
\end{tikzcd}$$

Here, we have committed a minor abuse of language in denoting by U^σ what we normally denote with $\partial_\sigma U$, and with $\varrho^+_\sigma\left(U^\sigma_{<r}, V^\sigma_{>r}\overline{\Theta_r}\right)$ the result of composing that pasting diagram with a chosen operation whose boundary is given by $(\partial_\sigma, \partial_\tau) \circ \varrho^*_\sigma$, and similarly for the analogues with τ. We can now use the fact that a filler certainly exists in ω-$\mathcal{C}at$ to extend this to a modification of 1-cylinders, and this concludes the construction for the first case.

Let us now address the case of globular sums $B \in \mathscr{L}(A)$ of case (50) to (53) that appear consecutively in $\mathscr{L}(A)$. We will build a modification involving the sub-stack associated with this subset of $\mathscr{L}(A)$, all at once rather than cylinder by cylinder. Let $A \cong \Sigma\alpha_1 \coprod_{D_0} \ldots \coprod_{D_0} \Sigma\alpha_p$ be the decomposition of A. Then, we are interested in those α_q such that $\alpha_q = D_1^{\otimes k}$ for $k > 0$. Pick any such. The globular inclusion $\Sigma D_1^{\otimes k} \to A$ picks out k composable 2-cylinders $\Gamma_1, \ldots, \Gamma_k$ via C and $\Delta_1, \ldots, \Delta_k$ via D. Notice that there exists an integer r such that $\Gamma_i = C_{r+i}$, and the same holds if we replace C and Γ with D and Δ, with the same r. Consider the vertical stacks of 1-cylinders whose composites are the transpose of $C \circ \hat{\varrho}$ and $D \circ \hat{\varrho}$ respectively. As in the proof of Proposition 7.24, we may identify $\Sigma\mathscr{L}(\alpha_q)$ with a subinterval of $\mathscr{L}(A)$. The sub-stack associated with the globular sums in the ordered set

$\Sigma\mathscr{L}(\alpha_q)$ is mapped under C to a pasting diagram in $X(x,y)$ of the form:



where $d = t^2(\Gamma_1)$, $e = s^2(\Gamma_1)$. Here, we have set:

$$h_{2m+1} = \varrho\left(U_{<q}, {U_q}^{<r}, s(\overline{\overline{\Gamma_{k-m}}}), {V_q}^{>r}a, V_{>q}\right)$$

$$h_{2m} = \varrho\left(U_{<q}, {U_q}^{<r}, t(\overline{\overline{\Gamma_{k-m}}}), {V_q}^{>r}a, V_{>q}\right)$$

where we have used $\overline{\overline{F}}$ to denote the 3-cell filling a 2-cylinder F, and $t(\Gamma_0)$ is defined to be $s(\Gamma_1)$. Obviously we get a similar one replacing every occurrence of C with D and of Γ with Δ. The 2-cells in $X(x,y)$ labelled with α's represent 1-cylinders whose source and target are degenerate. In particular, each α_{2m+1} is a whiskering of the 3-cell $\overline{\overline{\Gamma_{k-m}}}$ and each α_{2m} is an associativity constraint, for every $0 \leq m \leq k-1$, as explained in detail at the end of Chapter 7. We can use the components of Θ to find a modification between the vertical composites of these (degenerate) cylinders using the following lemma, which concludes the proof. $\square$

The following is a result that is needed in the proof of the previous lemma, but we only concern ourselves with a small simplification of it, leaving the (straightforward) proof of the generalization of the result to the interested reader. The simplification consists of restricting to the case $k = 2$, following the notation established above. Nevertheless, the proof of the general case is entirely similar and has no more genuine content than the one we present.

Lemma 8.21. *Assume given 2-cylinders $C, D\colon A \curvearrowright B$ and $C', D'\colon A' \curvearrowright B'$ in a $\mathfrak{C}^{\mathbf{W}}$-model X, with $t(C) = s(C'), t(D) = s(D')$, together with 2-dimensional pasting diagrams $\varepsilon\colon E \to X, \varphi\colon F \to X$ with $s^2(\varepsilon) = t^2(B), t^2(\varphi) = s^2(A)$ and an operation $\varrho\colon D_2 \to F \coprod_{D_0} D_2 \coprod_{D_1} D_2 \coprod_{D_0} E$ in $\mathfrak{C}$. This implies that, in particular, $t(A) = s(A')$ and $t(B) = s(B')$. Also, assume given modifications $\Theta\colon C \Rightarrow D, \Theta'\colon C' \Rightarrow D'$, with $t(\Theta) = s(\Theta')$, whose sources and targets, denoted respectively with $S\colon s(C) \Rightarrow s(D), S'\colon s(C') \Rightarrow s(D')$ and $T\colon t(D) \Rightarrow$*

$t(C), T'\colon t(D') \Rightarrow t(C')$, are 0-collapsed. Then we get an induced modification $\varepsilon\Theta'\Theta\varphi$ between the vertical composite depicted below for (C, C') and the corresponding one for (D, D').

$$\begin{tikzcd}[row sep=huge, column sep=huge]
x \arrow[r, "\varepsilon(C_tA_1)(C_tA_0)\varphi"] \arrow[d, "\varepsilon s(C)\varphi"'] \arrow[dr, bend left=50, "h_1"] \arrow[dr, bend left=20, "h_2"] \arrow[dr, bend right=20, "h_3"'] \arrow[dr, bend right=50, "h_4"'] & y \arrow[d, "\varepsilon t(G)\varphi"] \\
w \arrow[r, "\varepsilon(B_1C_s)(B_0C_s)\varphi"'] & z
\end{tikzcd}$$

The notation is defined as follows:

- $h_1 = \varrho(\varepsilon, (t(G)(C_tA_1))(C_tA_0), \varphi)$, $h_2 = \varrho(\varepsilon, ((B_1C_s)s(G))(C_tA_0), \varphi)$
- $h_3 = \varrho(\varepsilon, (B_1C_s)(s(G)(C_tA_0)), \varphi)$, $h_4 = \varrho(\varepsilon, (B_1C_s)((B_0C_s)s(C)), \varphi)$
- $\alpha_1 = \varrho^*(\varepsilon, \overline{\overline{G}}(C_tA_0), \varphi)$, $\alpha_3 = \varrho^*(\varepsilon, (B_1C_s)\overline{\overline{C}}, \varphi)$

where we have used $\overline{\overline{F}}$ to denote the underlying 3-cell of a 2-cylinder F, α_2 is simply an associativity constraint, and the 2-cells labelled with "$\simeq$" are also given by coherence constraints. Also, juxtaposition indicates the result of composing using the composition operations that appear in the definition of cylinders, and ϱ^* is an operation with boundary $(\partial_\sigma \circ \varrho, \partial_\tau \circ \varrho)$.

Proof. To begin with, we observe that the hypotheses imply $C_s = D_s$ and $C_t = D_t$, and we denote these 1-cells with a and b respectively. We consider the following pasting diagram in $\Omega^2(X, x, z)$ with x, z being the appropriate 1-cells of X depicted in the diagram above:

$$\begin{tikzcd}
(\varepsilon t(H)\varphi)(\varepsilon((bA_1)(bA_0))\varphi) \arrow[rr, "(\varepsilon T'\varphi)(\varepsilon((bA_1)(bA_0))\varphi)"] \arrow[d, "\simeq"'] & \arrow[d, phantom, "\simeq\Downarrow"] & (\varepsilon t(G)\varphi)(\varepsilon((bA_1)(bA_0))\varphi) \arrow[d, "\simeq"] \\
\varepsilon((t(H)(bA_1))(bA_0))\varphi \arrow[rr, "\varepsilon((T'(bA_1))(bA_0))\varphi"] \arrow[d, bend right, "\alpha'_1"'] \arrow[d, bend left, "\varepsilon(s(\Theta')(bA_0))\varphi"] \arrow[d, phantom, "\overset{\varepsilon(\underline{\Theta}'(bA_0))\varphi}{\Leftarrow}"] & \arrow[d, phantom, "\simeq\Downarrow"] & \varepsilon((t(G)(bA_1))(bA_0))\varphi \arrow[d, "\alpha_1"] \\
\varepsilon(((B_1a)s(H))(bA_0))\varphi \arrow[d, "\alpha'_2"'] & \arrow[d, phantom, "\simeq\Downarrow"] & \varepsilon(((B_1a)s(G))(bA_0))\varphi \arrow[ll, "\varepsilon(((B_1a)S')(bA_0))\varphi"'] \arrow[d, "\alpha_2"] \\
\varepsilon((B_1a)(s(H)(bA_0)))\varphi \arrow[d, bend right, "\alpha'_3"'] \arrow[d, bend left, "\varepsilon((B_1a)(s(\Theta')))\varphi"] \arrow[d, phantom, "\overset{\varepsilon((B_1a)\underline{\Theta})\varphi}{\Leftarrow}"] & \arrow[d, phantom, "\simeq\Downarrow"] & \varepsilon((B_1a)(s(G)(bA_0)))\varphi \arrow[ll, "\varepsilon((B_1a)(S'(bA_0)))\varphi"'] \arrow[d, "\alpha_3"] \\
\varepsilon((B_1a)((B_0a)s(D)))\varphi \arrow[d, "\simeq"'] & \arrow[d, phantom, "\simeq\Downarrow"] & \varepsilon((B_1a)((B_0a)s(C)))\varphi \arrow[ll, "\varepsilon((B_1a)((B_0a)S))\varphi"'] \arrow[d, "\simeq"] \\
(\varepsilon((B_1a)(B_0a))\varphi)(\varepsilon s(D)\varphi) & & (\varepsilon((B_1a)(B_0a))\varphi)(\varepsilon s(C)\varphi) \arrow[ll, "(\varepsilon((B_1a)(B_0a))\varphi)(\varepsilon S\varphi)"']
\end{tikzcd}$$

in which all the cells labelled by "$\simeq$" are obtained by verifying their existence in $\omega\text{-}\mathcal{C}at$, since their boundaries factor through appropriate globular sums, and $\underline{\Theta}, \underline{\Theta}'$ are the underlying 4-cell of the modifications. If we set $(\varepsilon\Theta\Theta'\varphi)_s = \varepsilon S\varphi$ and $(\varepsilon\Theta\Theta'\varphi)_t = \varepsilon T'\varphi$, we see that the composite of this pasting diagram gives the modification $\varepsilon\Theta\Theta'\varphi$ of the statement. $\square$

Given an operation $\varrho\colon D_2 \to A$ in $\mathfrak{C}$, we can consider the map $\tilde{\varrho}\colon \mathbf{Cyl}(D_2) \to \mathbf{Cyl}(A)$ obtained by applying Lemma 8.6 to the following diagram:

$$\begin{tikzcd}
\mathbf{Cyl}(S^1) \arrow[rr, bend left, "(\widehat{\varrho\circ\sigma},\widehat{\varrho\circ\tau})"] \arrow[rr, bend right, "(\mathbf{Cyl}(\varrho\circ\sigma){,}\mathbf{Cyl}(\varrho\circ\tau))"'] \arrow[rr, phantom, "\Downarrow(\vartheta_{\varrho\circ\sigma},\vartheta_{\varrho\circ\tau})"] \arrow[d, "(\mathbf{Cyl}(\sigma){,}\mathbf{Cyl}(\tau))"'] & & \mathbf{Cyl}(A) \\
\mathbf{Cyl}(D_2) \arrow[urr, bend left, "\widehat{\varrho}"] \arrow[urr, dashed, bend right, "\tilde{\varrho}"'] \arrow[urr, phantom, "\Downarrow\chi_\varrho"] & &
\end{tikzcd}$$

By construction, there is a modification $\chi_\varrho\colon \hat{\varrho} \Rightarrow \tilde{\varrho}$. Also, note that $\mathbf{Cyl}(\varrho)$ and $\tilde{\varrho}$, although potentially different, are parallel 2-cylinders.

LEMMA 8.22. *In the situation of Lemma 8.20 and in the context of $\mathfrak{C}^{\mathbf{W}}$-models, we can replace $\hat{\varrho}$ with $\tilde{\varrho}$.*

PROOF. Consider the following diagram:

$$\begin{tikzcd}
\mathbf{Cyl}(D_2) \arrow[r, bend left, "\tilde{\varrho}"] \arrow[r, bend right, "\hat{\varrho}"'] \arrow[r, phantom, "\Downarrow\chi_\varrho^{-1}"] & \mathbf{Cyl}(A) \arrow[r, bend left, "(C_j)_{1\leq j\leq m}"] \arrow[r, bend right, "(D_j)_{1\leq j\leq m}"'] & \mathbf{Cyl}\,(B)
\end{tikzcd}$$

where we denote by χ_ϱ^{-1} the modification obtained by applying Lemma B.11 to the modification χ_ϱ. It induces a modification $(C_j)_{1\leq j\leq m}\circ\chi_\varrho{}^{-1}\colon (C_j)_{1\leq j\leq m}\circ\tilde{\varrho}\Rightarrow(C_j)_{1\leq j\leq m}\circ\hat{\varrho}$, which we can compose with the modification $\Theta\circ\hat{\varrho}\colon (C_j)_{1\leq j\leq m}\circ\hat{\varrho}\Rightarrow(D_j)_{1\leq j\leq m}\circ\hat{\varrho}$ obtained in Lemma 8.6. Finally, we compose the result with $(D_j)_{1\leq j\leq m}\circ\chi_\varrho\colon (D_j)_{1\leq j\leq m}\circ\hat{\varrho}\Rightarrow(D_j)_{1\leq j\leq m}\circ\tilde{\varrho}$ using the results in the Appendix, to get the desired modification. $\square$

In what follows, we consider a homogeneous map $\varphi\colon A\to B$ and we use the notation $\tilde{\varphi}$ to denote the map $\mathbf{Cyl}(A)\to\mathbf{Cyl}(B)$ obtained by glueing the various maps $\tilde{\varphi_j}\colon \mathbf{Cyl}(D_{i_j})\to\mathbf{Cyl}(B_j)$ for $1\leq j\leq m$, where φ_j is the homogeneous part of the composite $D_{i_j}\to A\to B$ for every globe D_{i_j} in the globular decomposition of A, and $\operatorname{colim}_k B_k\cong B$.

LEMMA 8.23. *Assume given homogeneous operations $\varrho\colon D_2\to A$ and $\varphi\colon A\to B$. There is a 1-collapsed modification of $\mathfrak{C}^{\mathbf{W}}$-model of the form:*

$$\Lambda\colon \tilde{\varphi}\circ\tilde{\varrho}\Rightarrow\widetilde{\varphi\circ\varrho}$$

For dimensionality reasons, this is equivalent to $\tilde{\varphi}\circ\tilde{\varrho}=\widetilde{\varphi\circ\varrho}$.

The idea of the proof is to consider the following diagram of composable modifications, where the solid ones have already been constructed:

$$\tilde{\varphi}\circ\tilde{\varrho}\xRightarrow{\tilde{\varphi}\circ\chi_\varrho}\tilde{\varphi}\circ\hat{\varrho}\xRightarrow{\chi_\varphi{}^{-1}\circ\hat{\varrho}}\hat{\varphi}\circ\hat{\varrho}\overset{\eta}{=\,=\,\Rightarrow}\widehat{\varphi\circ\varrho}\xRightarrow{\chi_{\varphi\circ\varrho}}\widetilde{\varphi\circ\varrho}\tag{58}$$

We then have to construct the modification denoted with η, and then prove that the resulting composite can be adjusted so as to be rendered 1-collapsed. This is accomplished by making use of the following lemma, whose assumptions are satisfied in the case at hand.

LEMMA 8.24. *Assume given a pair of parallel n-cylinders $C,D\colon A\curvearrowright B$ in a $\mathfrak{C}^{\mathbf{W}}$-model X, together with a modification $\Theta\colon C\Rightarrow D$ between them. Assume further that $s(\Theta)$ and $t(\Theta)$ are $(n-1)$-collapsed, and that there are $(n+2)$-cells $\eta_s\colon s(\Theta)\to 1_{\widehat{s(C)}}$, $\eta_t\colon t(\Theta)\to 1_{\widehat{t(C)}}$, where $\widehat{E}$ denotes the n-cell filling an $(n-1)$-cylinder and 1_f the choice of an identity $(n+1)$-cell on an n-cell f. Then there exists an n-collapsed modification $\Theta'\colon C\Rightarrow D$.*

PROOF. We prove the statement by induction on $n>0$. Let $n=1$, and set $\Theta_\varepsilon=\varepsilon(\Theta)$ for $\varepsilon=s,t$. Consider the following pasting diagram in $X\,(s(C_s),t(C_t))$, where the unlabelled 2-cell comes from unitality of composition in $\mathfrak{C}^{\mathbf{W}}$:

$$\begin{tikzcd}
C_tA \arrow[r, bend left, "1"] \arrow[r, bend right, "\Theta_sA"'] \arrow[r, phantom, "(\eta_s)^{-1}A\Downarrow"] \arrow[rrr, bend left=40, "\hat{C}"] \arrow[rrr, bend right=40, "\hat{D}"'] & C_tA \arrow[r, "\hat{C}"] \arrow[r, phantom, shift left=5, "\Downarrow"] \arrow[r, phantom, shift right=5, "\Theta\Downarrow"] & BC_s \arrow[r, bend left, "1"] \arrow[r, bend right, "B\Theta_t"'] \arrow[r, phantom, "B(\eta_t)^{-1}\Downarrow"] & BC_s
\end{tikzcd}$$

The composite of this pasting diagram is the modification Θ' we are looking for.

Now let $n > 1$, we have a modification of $(n-1)$-cylinders $\overline{\Theta}\colon \overline{C} \Rrightarrow \overline{D}$ in $X\,(s(C_s), t(C_t))$. For $\varepsilon = s, t$ we have $\varepsilon(\overline{C}) = \overline{\varepsilon(C)} = \overline{\varepsilon(D)} = \varepsilon(\overline{D})$ and $\varepsilon(\overline{\Theta})$ is an n-cell between $(n-2)$-cylinders. Also, we can view η_s, η_t as $(n+1)$-cells in $X\,(s(C_s), t(C_t))$, so that we can apply the inductive hypothesis to get a modification $\overline{\Theta'}\colon \overline{C} \Rrightarrow \overline{D}$ which consists of an $(n+1)$-cell between $(n-1)$-cylinders in $X\,(s(C_s), t(C_t))$. Its transpose $\Theta'\colon C \Rrightarrow D$ is the modification we are looking for and this concludes the proof. $\square$

We refer the reader to Lemma 8.16 for the notation used in what follows. The proof is quite technical, but crucial to get the missing piece for this section.

LEMMA 8.25. *Assume given homogeneous operations $\varrho\colon D_2 \to A$ and $\varphi\colon A \to B$. There is a modification of the form:*

$$\eta\colon \widehat{\varphi} \circ \widehat{\varrho} \Rrightarrow \widehat{\varphi \circ \varrho}\colon \mathbf{Cyl}(D_2) \to \mathbf{Cyl}(B)$$

with source and target given by the modifications $\mathbf{Cyl}(j) \circ \delta_{(\varphi \circ i)_\varepsilon, \varrho_\varepsilon}$ for $\varepsilon = \sigma, \tau$, where $i \circ \varrho_\varepsilon$ is the homogeneous-globular factorization of $\varrho \circ \varepsilon$ and, similarly, $j \circ (\varphi \circ i)_\varepsilon$ is the homogeneous-globular factorization of $\varphi \circ i$.

$$\begin{tikzcd}
D_2 \arrow[r, "\varrho"] & A \arrow[r, "\varphi"] & B \\
D_1 \arrow[u, tail, "\varepsilon"] \arrow[r, two heads, "\varrho_\varepsilon"] & A' \arrow[u, tail, "i"] \arrow[r, two heads, "(\varphi\circ i)_\varepsilon"] & B' \arrow[u, tail, "j"]
\end{tikzcd}$$

Here, we have used the arrow $\twoheadrightarrow$ to denote homogeneous maps and $\rightarrowtail$ for globular ones.

PROOF. The proof proceeds very similarly as to that of Lemma 8.20, i.e. we construct the modification Δ as the composite of modifications from substacks of the stack defining $\widehat{\varphi} \circ \widehat{\varrho}$ towards substacks of the one defining $\widehat{\varphi \circ \varrho}$, parametrized by the globular sums in $\mathscr{L}(A)$ in an exhaustive fashion. We let:

$$\begin{pmatrix} i_1 & & i_2 & \dots & i_{m-1} & & i_m \\ & i'_1 & & \dots & & i'_{m-1} & \end{pmatrix} \qquad \begin{pmatrix} j_1 & & i_2 & \dots & j_{q-1} & & j_q \\ & j'_1 & & \dots & & j'_{q-1} & \end{pmatrix}$$

be the table of dimensions of A and B respectively. Notice that, by assumption on the homogeneity of ϱ and φ, we have $i_k, j_r \leq 2$ for every $1 \leq k \leq m$ and $1 \leq r \leq q$. We proceed representably. This means that we are given a $\mathfrak{C}^{\mathbf{W}}$-model X and a map $C\colon \mathbf{Cyl}(B) \to X$, with $C\colon U \curvearrowright V$. From this, we get cylinders $C_1, \dots, C_q$ in X, where C_k is a j_k-cylinder. The two cylinders $C \circ \hat{\varphi} \circ \hat{\varrho}$ and $C \circ \widehat{\varphi \circ \varrho}$ are both obtained as vertical composites of (different) stacks of 1-cylinders in $X(x, y)$ for $x = s^{i_1}(C_1)_0, y = s^{i_m}(C_m)_1$. Therefore, we need to provide a filler for this pair of composite 1-cells in the bicategory $\mathbf{hom}\,(D_1, X(x, y))$ (see the Appendix for clarifications on this notation), and we do so by decomposing both stacks into some subcomposite, and we then explain how to find fillers for each such piece. As in Lemma 8.20, we firstly consider the cases (46) to (48), where modifications can be constructed by using the fact that the corresponding cylinders factor through appropriate globular sums, and in ω-$\mathcal{C}at$ the boundary data of these modifications admits fillers in a way that is compatible with the modifications we already have for the boundary. We now address case (49), i.e. globular sums $D \in \mathscr{L}(A)$ which have been obtained by adding a new vertex $*_D$ to A at height $\mathrm{ht}(*_D) = 2$, in such a way that this new vertex is the unique element of its fiber. Given such globular sum D, the 2-cell in D represented by $*_D$ picks out via:

$$D \longrightarrow \mathbf{Cyl}(A) \xrightarrow{\hat{\varphi}} \mathbf{Cyl}(B) \xrightarrow{C} X$$

the 2-cell associated with the 1-cylinder $F_k = \widehat{\varphi_k}(C_{n_k}, \dots, C_{n_{k+1}-1})$, where $\varphi = (\varphi_i)_{1 \leq i \leq m}$, according to the globular decomposition of A, and each of the φ_k has the sub-globular sum $G_k \subset B$ spanned by $D_{j_{n_k}}, \dots, D_{j_{n_{k+1}}-1}$ as codomain. Corresponding to such D, we have a cylinder in the stack associated with $C \circ \hat{\varphi} \circ \hat{\varrho}$ of the form:

$$\begin{tikzcd}
a \arrow[rr, "\varrho\left(\varphi_{>k}\left(U_{|G_{>k}}\right), s\left(\overline{F_k}\right), \varphi_{<k}\left(V_{|G_{<k}}\right)\right)"] \arrow[d, "\varrho^*_\tau\left(\left(\varphi_{>k}\right)_\tau\left(U_{|G_{>k}}\right), \overline{F_k}, \left(\varphi_{<k}\right)_\tau\left(V_{|G_{<k}}\right)\right)"'] & \Downarrow & b \arrow[d, "\varrho^*_\tau\left(\left(\varphi_{>k}\right)_\tau\left(U_{|G_{>k}}\right), \overline{F_k}, \left(\varphi_{<k}\right)_\tau\left(V_{|G_{<k}}\right)\right)"] \\
a' \arrow[rr, "\varrho\left(\varphi_{>k}\left(U_{|G_{>k}}\right), t\left(\overline{F_k}\right), \varphi_{<k}\left(V_{|G_{<k}}\right)\right)"'] & & b'
\end{tikzcd}$$

We have used $\varphi_{>k}\left(U_{|G_{>k}}\right)$ to denote the result of composing, using $(\varphi_i)_{i \geq k}$, the restriction of U to the union of the sub-globular sum of B given by G_i for $i > k$. Similarly for the other piece of notation involving the indices smaller than k. We want to produce a modification having this cylinder as source, and having as target a sub-composite of the vertical stack of 1-cylinders associated to $C \circ \widehat{\varphi \circ \varrho}$. This sub-stack is the one parametrized by the family of globular sums of the form:

$$\{D_{j_1} \underset{D_{j'_1}}{\amalg} \dots \underset{D_{j'_{n_k-1}}}{\amalg} E \underset{D_{j_{n_{k+1}}-1}}{\amalg} \dots \underset{D_{j'_{q-1}}}{\amalg} D_{j_q}\}_{E \in \mathscr{L}(G_k)} \subset \mathscr{L}(B)$$

Notice that the respective boundaries of these cylinders are of the same form as the ones appearing in the proof of Lemma 8.16, and therefore we can use the modifications we produced there to compare the boundaries. These constitute the boundary of the modification we want to construct, whose existence follows, finally, from the fact that this boundary factors through the globular sum:

$$D_{j_1} \underset{D_{j'_1}}{\amalg} \dots \underset{D_{j'_{n_k-1}}}{\amalg} \underset{D_{j'_{n_k-1}}}{\amalg} \Sigma(D_1^{\otimes |\mathscr{L}(D_1^{\otimes n_{k+1}-n_k})|}) \underset{D_{j_{n_{k+1}}-1}}{\amalg} \dots \underset{D_{j'_{q-1}}}{\amalg} D_{j_q}$$

and a filler for it certainly exists in ω-$\mathcal{C}at$. We now turn to the case of globular sums $C \in \mathscr{L}(A)$ corresponding to case (50) to (53). We can thus consider the decomposition $A = \Sigma\alpha_1 \underset{D_0}{\amalg} \dots \underset{D_0}{\amalg} \Sigma\alpha_p$ and some α_k such that $\alpha_q = D_1^{\otimes k}$. The globular inclusion $\Sigma D_1^{\otimes k} \to A$ picks out k composable 2-cylinders $\Gamma_1, \dots, \Gamma_k$ in X, where we have $\Gamma_i = \varphi_{r+i}(C_{n_{r+i}}, \dots, C_{n_{r+i+1}-1})$ for a unique integer r. We will construct a modification whose source is given by a stack of (collapsed) cylinders of the form:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
a \arrow[r, "\varrho(U_{<q},dU_q,V_{>q})"] \arrow[d, "\varrho^*_\sigma(\partial_\sigma U_{<q},s(\Gamma_1),\partial_\sigma V_{>q})"'] \arrow[dr, bend left=40, "h_1"] \arrow[dr, bend left=25, "h_2"] \arrow[dr, bend left=10, "h_3"] \arrow[dr, bend right=10, "h_{2k-2}"] \arrow[dr, bend right=25, "h_{2k-1}"] \arrow[dr, bend right=40, "h_{2k}"] & b \arrow[d, "\varrho^*_\tau(\partial_\tau U_{<q},t(\Gamma_k),\partial_\tau V_{>q})"] \arrow[dl, phantom, "\simeq\ \Swarrow" near start] \\
c \arrow[r, "\varrho(U_{<q},V_q e,V_{>q})"'] \arrow[ur, phantom, "\simeq\ \Swarrow" near start] & d
\end{tikzcd}
$$

$$\alpha_1 \Swarrow,\ \alpha_2 \Swarrow,\ \alpha_3 \Swarrow,\ \dots,\ \alpha_{2k-3} \Swarrow,\ \alpha_{2k-2} \Swarrow,\ \alpha_{2k-1} \Swarrow$$

Here, we set $d = t^2(\Gamma_1)$, $e = s^2(\Gamma_1)$, and:

$$h_{2m+1} = \varrho\left(U_{<q}, U_q{}^{<r}, s(\overline{\overline{\Gamma_{k-m}}}), V_q{}^{>r}a, V_{>q}\right)$$

$$h_{2m} = \varrho\left(U_{<q}, U_q{}^{<r}, t(\overline{\overline{\Gamma_{k-m}}}), V_q{}^{>r}a, V_{>q}\right)$$

where, as before, we have used $\overline{\overline{F}}$ to denote the underlying 3-cell of a 2-cylinder F, and $t(\Gamma_0)$ is defined to be $s(\Gamma_1)$. The 2-cells in $X(x, y)$ labelled with α's represent 1-cylinders whose source and target are degenerate. In particular, α_{2m+1} is a whiskering of the 3-cell $\overline{\overline{\Gamma_{k-m}}}$ and α_{2m} is an associativity constraint. The target of the modification we want to construct is a composite of a sub-stack of the one associated with $\widehat{\varphi \circ \varrho}$, parametrized by the family of globular sums given by

$$\{D_{j_1} \underset{D_{j'_1}}{\amalg} \cdots \underset{D_{j'_{n_k-1}}}{\amalg} E \underset{D_{j_{n_{k+1}}-1}}{\amalg} \cdots \underset{D_{j'_{q-1}}}{\amalg} D_{j_q}\}_{E\in\mathscr{L}(G_p)} \subset \mathscr{L}(B)$$

To finish this construction, we introduce an intermediate step in this modification by taking into consideration Lemma A.2, and we focus on the square (59) that originates from it. By applying this to each of the (possibly degenerate) 1-cylinders in the sub-stack we are considering, we obtain a new stack of the same shape where all the new 1-cylinders are whiskering of the previous ones in the appropriate sense. The respective boundaries of these stacks we are comparing are of the same form as the ones appearing in the proof of Lemma 8.16, and therefore we can use the modifications we produced there to compare the boundaries. In the same way, the boundary of this new "whiskered" stack and the one of the source of the modification we want to build can also be compared using the modification of Lemma 8.16 (which was indeed the composite of two such). After having composed the boundary with such modifications, filling in the rest of the modification follows from a straightforward application of the classical result of coherence for pseudofunctors and bicategories. □

By construction, it is clear that the boundary of the composite modification in (58) satisfies the assumptions of Lemma 8.24. Hence, we get the abovementioned 1-collapsed modification which implies:

$$\tilde{\varphi} \circ \tilde{\varrho} = \widetilde{\varphi \circ \varrho}$$

for dimensionality reasons. Therefore, we see that an extension to $\mathfrak{C}_2$ is equivalently obtained by setting $\mathbf{Cyl}(\varrho) = \tilde{\varrho}$ for all homogeneous operations $\varrho\colon D_2 \to A$. Finally, we recall that, by definition, $\mathfrak{C}_3 \cong \mathfrak{C}_2[X_2]$ and so we can obtain the desired extension depicted in (56) by defining $\mathbf{Cyl}(\Phi)$, for every $\Phi\colon D_3 \to A$ added as a filler of $(\varphi_0, \varphi_1) \in X_2$, in the following manner:

$$\begin{tikzcd} D_2 \arrow[r, shift left, "\varphi_0"] \arrow[r, shift right, "\varphi_1"'] \arrow[d, shift right, "\sigma_2"'] \arrow[d, shift left, "\tau_2"] & A \\ D_3 \arrow[ur, "\Phi"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd}[column sep=huge, row sep=huge] \mathbf{Cyl}(S^2) \arrow[r, bend left, "(\widehat{\varphi_0},\widehat{\varphi_1})"] \arrow[r, bend right, "(\widetilde{\varphi_0},\widetilde{\varphi_1})"'] \arrow[r, phantom, "\Downarrow (\chi_{\varphi_0},\chi_{\varphi_1})"] \arrow[d, "(\mathbf{Cyl}(\sigma){,}\mathbf{Cyl}(\tau))"'] & \mathbf{Cyl}(A) \\ \mathbf{Cyl}(D_3) \arrow[ur, bend left, "\widehat{\Phi}"] \arrow[ur, bend right, dashed, "\mathbf{Cyl}(\Phi)"'] \arrow[ur, phantom, "\Rrightarrow"] & \end{tikzcd}$$

It turns out that extending along $i_3\colon \mathfrak{C}_3 \to \mathfrak{C}_4 = \mathfrak{C}$ is automatic, thanks to the following result.

Lemma 8.26. *Suppose given a pair of n-cylinders (F, G) in a $\mathfrak{C}^{\mathbf{W}}$-model X, where $\mathfrak{C}$ is a coherator for n-categories, such that $s(F) = s(G)$, $t(F) = t(G)$ and $F_0 = G_0 = A$, $F_1 = G_1 = B$. Then $F = G$.*

Proof. If $n = 0$ the result is clear. Assume $n > 0$, then we get $(n-1)$-cylinders $\overline{F}, \overline{G}$ in $X\,(s(f), t(g))$ where $f = F_s = G_s$ and $g = F_t = G_t$. By definition, we have $\overline{F}, \overline{G}\colon gA \curvearrowright Bf$ and $\varepsilon(\overline{F}) = \varepsilon(\overline{G})$ for $\varepsilon = s, t$. Therefore, by inductive assumption we get that $\overline{F} = \overline{G}$, which concludes the proof. □

We can now apply this lemma to the situation where we have a pair of parallel operations $\alpha, \beta\colon D_3 \to A$ in X_3, so that $\alpha = \beta$ in $\mathfrak{C}$, and interpretations $\mathbf{Cyl}(\alpha), \mathbf{Cyl}(\beta)\colon \mathbf{Cyl}(D_3) \to \mathbf{Cyl}(A)$ which are compatible with the map $\iota\colon D_3 \amalg D_3 \to \mathbf{Cyl}(D_3)$. We want to prove that $\mathbf{Cyl}(\alpha) = \mathbf{Cyl}(\beta)$, and we do so representably. Given $H\colon \mathbf{Cyl}(A) \to Y$, with $Y \in \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$, we see that

$$(H \circ \mathbf{Cyl}(\alpha))_\varepsilon = H_\varepsilon \circ \alpha = H_\varepsilon \circ \beta = (H \circ \mathbf{Cyl}(\beta))_\varepsilon$$

for $\varepsilon = 0, 1$. Moreover, $s(H \circ \mathbf{Cyl}(\alpha)) = H \circ \mathbf{Cyl}(\alpha \circ \sigma) = H \circ \mathbf{Cyl}(\beta \circ \sigma) = s(H \circ \mathbf{Cyl}(\beta))$, and similarly for the target. This implies $\mathbf{Cyl}(\alpha) = \mathbf{Cyl}(\beta)$.

This concludes the construction of the $\mathfrak{C}$-model $\mathbb{P}X$ associated with a $\mathfrak{C}^{\mathbf{W}}$-model X, as we record here below.

Theorem 8.27. *Let $\mathfrak{C}$ be a 3-coherator for categories. Then there is an extension of the form:*

$$\begin{tikzcd} \Theta_0 \arrow[r, "\mathbf{Cyl}"] \arrow[d] & \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}) \\ \mathfrak{C} \arrow[ur, dashed, "\mathbf{Cyl}"'] & \end{tikzcd}$$

Thanks to the results of Chapter 6, it is possible to extend the domain a bit further, thus providing more structure on cylinders, as follows.

COROLLARY 8.28. *Let $\mathfrak{C}$ be a 3-coherator for categories, then there exists an extension of the form:*

$$\begin{tikzcd} \Theta_0 \arrow[r, "\mathbf{Cyl}"] \arrow[d] & \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}) \\ \mathfrak{C}^{\mathbf{W}} \arrow[ur, dashed, "\mathbf{Cyl}"'] & \end{tikzcd}$$

PROOF. Such an extension amounts to endow the $\mathfrak{C}$-model $\mathbb{P}X$ obtained in the previous theorem with a system of inverses with respect to the chosen systems of compositions and identities. This was done in Theorem 6.8. □

As anticipated earlier, when $n = 3$ we can use the results of this chapter in conjunction with those of Chapter 6 to obtain an endofunctor $\mathbb{P}$ on $\mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$ with the desired properties.

PROPOSITION 8.29. *Given a coherator $\mathfrak{C}$ for 3-categories, there exists a functor:*

$$\mathbb{P}\colon \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}}) \to \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$$

equipped with a natural transformation $\mathbf{ev}\colon \mathbb{P} \Rightarrow \mathbf{Id} \times \mathbf{Id}$ which is a pointwise fibration.

Moreover the composites with the product projections $\mathbf{ev}_i \overset{def}{=} \pi_1 \circ \mathbf{ev}$ are trivial fibrations for $i = 0, 1$.

PROOF. It is enough to define $(\mathbb{P}X)_k \overset{def}{=} \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})\,(\mathbf{Cyl}(D_k), X)$ for every $\mathfrak{C}^{\mathbf{W}}$-model X, and then use the previous corollary to endow this globular set with the structure of a $\mathfrak{C}^{\mathbf{W}}$-model. □

It follows that, in the situation of the previous proposition, $\mathfrak{C}^{\mathbf{W}}$ is a coherator for 3-groupoids, and we can now present the central result of this work.

THEOREM 8.30. *There exists a cofibrantly generated semi-model structure on the category 3-$\mathcal{G}pd \cong \mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$ of Grothendieck 3-groupoids of type $\mathfrak{C}^{\mathbf{W}}$, whose set of generating cofibrations (resp. trivial cofibrations) consists of boundary inclusions $\{S^{k-1} \to D_k\}_{0\leq k\leq 4}$ (resp. source maps $\{\sigma_k\colon D_k \to D_{k+1}\}_{0\leq k\leq 2}$), where by definition we set $S^3 \to D_4$ equal to $(1,1)\colon S^3 \to D_3$. The weak equivalences coincide with the class $\mathbf{W}$ defined in (3.2), and all the objects are fibrant.*

PROOF. Thanks to the previous corollary, we have that $\mathfrak{C}^{\mathbf{W}}$ satisfies all the hypotheses of Theorem 4.2, and this concludes the proof. □

CHAPTER 9

Perspectives

To sum up, the goal of this work was to investigate the homotopy theory of globular models for higher structures, in the form of Grothendieck ∞-categories and ∞-groupoids. We defined the underlying globular set for a path object on the category of ∞-groupoids and proved how endowing it with the structure of an ∞-category is enough to get a semi-model structure on the category of ∞-groupoids and therefore prove the homotopy hypothesis thanks to the work of Henry (see [**Hen**]). Also, we characterized those globular theory whose category of models admits a semi-model structure of a specific form, and whose objects can be endowed with the structure of an ∞-groupoid. We have then addressed a finite-dimensional version of this problem and achieved the desired result in dimension $n = 3$.

At this point, there are two possible directions one could take. On the one hand, one could extend our construction of the path object on 3-dimensional $\mathfrak{C}^{\mathbf{W}}$-models to infinite-dimensional ones.

On the other hand, Grothendieck's homotopy hypothesis admits an adaptation to n-types, which can be roughly formulated as follows:

Weak n-groupoids model all homotopy n-types

As mentioned earlier, in [**Hen**] it was proved that if a (semi)-model structure on the category of Grothendieck ∞-groupoids exists, then the homotopy hypothesis holds true: this result is achieved by showing that the associated $(\infty, 1)$-category (i.e. quasi-category) has the same universal property as that of homotopy types, namely they are both the free cocompletion of the terminal $(\infty, 1)$-category. More precisely, if we consider the $(\infty, 2)$-category of presentable $(\infty, 1)$-categories and cocontinuous functors, then we have that:

$$\mathbf{Fun}^{\mathrm{L}}(\mathcal{H}, \mathcal{C}) \simeq \mathcal{C}$$

where $\mathcal{H}$ denotes the $(\infty, 1)$-category of homotopy types and $\mathbf{Fun}^{\mathrm{L}}$ is the $(\infty, 1)$-category of cocontinuous functors from $\mathcal{H}$ to $\mathcal{C}$.

One could adapt that result to the case of n-groupoids, to show that under similar conditions, i.e. the existence of a (semi)-model structure on the category of Grothendieck n-groupoids, the generalized homotopy hypothesis is valid. Thanks to Theorem 8.30, this would immediately imply that Grothendieck 3-groupoids model homotopy 3-types (a similar statement involving Gray-groupoids is usually credited to Joyal and Tierney, and was proven independently in [**Ber2**] and [**La2**]).

The key point is to formulate the correct universal property that the $(\infty, 1)$-category of n-types $\mathcal{H}_n$ satisfies. One possible way is to exploit, given any complete $(\infty, 1)$-category $\mathcal{C}$, the tensoring with objects of $\mathcal{H}$: given an object $x \in \mathcal{C}$ and a homotopy type $w \in \mathcal{H}$, the tensor $x \otimes w$ is defined by taking the colimit of the constant functor $W \to \mathcal{C}$ at x, where we identify the homotopy type w with the corresponding Kan complex (thought of as an ∞-groupoid) W. By functoriality of colimits, one gets a map $x \otimes S^n \to x$ for every $x \in \mathcal{C}$

and every $n \geq -1$ induced by $S^n \to *$. The universal property of $\mathcal{H}_n$ can then be stated as follows:

$\mathcal{H}_n$ is the free cocomplete $(\infty, 1)$-category containing an object x such that $x \otimes S^{n+1} \to x$ is an equivalence

Having formulated this universal property, it is then a matter of adapting the machinery developed in [**Hen**] to the finite case, guided by the case $n = 3$ where the semi-model structure has already been established.

On the semantics side, one may try to compare Grothendieck ∞-categories with Verity's (weak) complicial sets, whose theory is developed in [**Ve2**] in which, among other things, the author equips this category with a model structure. The first step is to define a globular sums-preserving functor $\mathfrak{C} \to \mathbf{Alg}_{\underline{\mathbf{Cs}}}$ whose codomain denotes the category of algebraically fibrant (see [**Nik**] for the notion of algebraically fibrant object) weak complicial sets.

By left Kan extension this would induce an adjunction of the form:

$$\begin{tikzcd} \infty\text{-}\mathcal{C}at \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathbf{Alg}_{\underline{\mathbf{Cs}}} \arrow[l, bend left] \end{tikzcd}$$

where ∞-$\mathcal{C}at \simeq \mathbf{Mod}(\mathfrak{C})$ denotes the category of models of a coherator for ∞-categories. The right adjoint should preserve weak equivalences, although we have not defined them yet in the case of Grothendieck ∞-categories. The conjecture is that this functor is an equivalence of relative categories.

APPENDIX A

Pseudofunctoriality of whiskerings

In this section we want to record some results and constructions that involve the Grothendieck 2-category of morphisms $X(x,y)$, where X is a Grothendieck 3-category and x,y are 0-cells in X. Clearly, this all applies to the case of globular theories with inverses. Having in mind that Grothendieck 2-categories essentially corresponds to unbiased bicategories, we will treat them as such.

Let's consider the following situation. We are given 1-dimensional globular pasting diagrams in X of the form $\alpha\colon A\to X, \beta\colon B\to X$, with $\partial_\sigma(\alpha)\overset{def}{=}\alpha\circ\partial_\sigma=w,\ \partial_\tau(\alpha)=x,\ \partial_\sigma(\beta)=y,\ \partial_\tau(\beta)=z$. Moreover, we are given a homogeneous operation $\varrho\colon D_1\to A\coprod_{D_0}D_1\coprod_{D_0}B$. We then have the following result:

Lemma A.1. *The data above extend to a pseudofunctor of bicategories of the form:*

$$(\alpha,-,\beta)\circ\varrho\colon X(x,y)\to X(w,z)$$

Proof. Choose operations ϱ^2,ϱ^3 as depicted in the following diagrams:

$$\begin{tikzcd} D_1 \arrow[r, shift left, "(1,\sigma,1)\circ\varrho"] \arrow[r, shift right, "(1,\tau,1)\circ\varrho"'] \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & A\coprod_{D_0}D_2\coprod_{D_0}B \\ D_2 \arrow[ur, "\varrho^2"'] & \end{tikzcd} \qquad \begin{tikzcd} D_2 \arrow[r, shift left, "(1,\sigma,1)\circ\varrho^2"] \arrow[r, shift right, "(1,\tau,1)\circ\varrho^2"'] \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] & A\coprod_{D_0}D_3\coprod_{D_0}B \\ D_3 \arrow[ur, "\varrho^3"'] & \end{tikzcd}$$

Next, define the underlying map of globular sets to be given by $(\alpha,-,\beta)\circ\varrho^{k+1}\colon X(x,y)_k\to X(w,z)_k$ on k-cells, where we implicitly use the isomorphism of sets $X(a,b)_k\cong\{f\in X_{k+1}|s^k(f)=a,t^k(f)=b\}$. The fact that this extends to a pseudofunctor is a simple exercise using the algebraic structure of globular sums, and is thus left to the interested reader. $\square$

If we go one dimension up, we can consider the following situation: we are given globular sums A,B with $\max\{\mathrm{ht}(A),\mathrm{ht}(B)\}=2$, and maps $\alpha\colon A\to X,\beta\colon B\to X$, with $\partial_\sigma^{\mathrm{ht}(A)}(\alpha)\overset{def}{=}\alpha\circ\partial_\sigma^{\mathrm{ht}(A)}=w,\ \partial_\tau^{\mathrm{ht}(A)}(\alpha)=x,\ \partial_\sigma^{\mathrm{ht}(B)}(\beta)=y,\ \partial_\tau^{\mathrm{ht}(B)}(\beta)=z$. Furthermore, assume given a homogeneous operation of the form $\varrho\colon D_2\to A+D_2+B$, where $+$ denotes either $\coprod_{D_0}$ or $\coprod_{D_1}$. We then have the following result, whose proof is analogous to that of the previous one.

Lemma A.2. *The previous data determine a pseudo-natural transformation of the form:*

$$\begin{tikzcd}[column sep=huge] X(x,y) \arrow[r, bend left, "\left(\partial_\sigma^{\mathrm{ht}(A)-1}(\alpha){,}-{,}\partial_\sigma^{\mathrm{ht}(B)-1}(\beta)\right)\circ\varrho_\sigma"] \arrow[r, bend right, "\left(\partial_\tau^{\mathrm{ht}(A)-1}(\alpha){,}-{,}\partial_\tau^{\mathrm{ht}(B)-1}(\beta)\right)\circ\varrho_\tau"'] \arrow[r, phantom, "(\alpha{,}-{,}\beta)\circ\varrho\Downarrow"] & X(w,z) \end{tikzcd}$$

where ϱ_ε denotes the homogeneous part of the composite $\varrho \circ \varepsilon$ for $\varepsilon = \sigma, \tau$.

Finally, we observe that given bicategories $\mathcal{K}, \mathscr{L}$, a square in $\mathcal{K}$ of the form:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\Theta\Downarrow"] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

and a pseudo-natural transformation:

$$\begin{tikzcd} \mathcal{K} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathscr{L} \end{tikzcd}$$

we get a filler for the square:

$$\begin{tikzcd} FA \arrow[r, "G(h)\alpha_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\exists\Phi \overset{\shortparallel}{\Downarrow}"] & GB \arrow[d, "G(g)"] \\ FC \arrow[r, "F(k)\alpha_D"'] & GD \end{tikzcd} \tag{59}$$

Indeed, it is enough to consider the following composite:

$$G(g)(G(h)\alpha_A) \xrightarrow{\cong} G(gh)\alpha_A \xrightarrow{G(\Theta)\alpha_A} G(kf)\alpha_A \xrightarrow{\alpha_{kf}} \alpha_D F(kf) \xrightarrow{\cong} (\alpha_D F(k))F(f)$$

It is clear that an analogous statement holds if we replace squares, i.e. 1-cylinders, with degenerate 1-cylinders.

APPENDIX B

A bicategory of cylinders and modifications

Given an ∞-groupoid X and an integer $n \geq 0$, we want to organize the collection of n-cells, n-cylinders and modifications between n-cylinders into an algebraic structure that allows us to perform calculations with them. This is a truncation of a (yet to be defined) internal hom ∞-groupoid of the form $\mathbf{hom}(D_n, X)$, which justifies the notation we establish here below.

Definition B.1. Given an ∞ groupoid X, we define a 2-truncated globular set $\mathbf{hom}(D_n, X)$ out of it, for each $n \geq 0$, as follows: its objects are n-cells in X; 1-cells are n-cylinders $C \colon A \curvearrowright B$ and there is a unique 2-cell $C \Rightarrow D$ if and only if there exists a modification from C to D.

Remark B.2. Everything that follows can be proven to hold true also in $\mathbf{Mod}(\mathfrak{C})$ (or $\mathbf{Mod}\left(\mathfrak{C}^{\mathbf{W}}\right)$) for any given coherator for ∞-categories $\mathfrak{C}$. Indeed, all the fillers obtained using contractibility can be obtained using the methods we described in Section 5, once we observe that the latching map of $\Xi \colon \mathbf{Cyl}(\bullet) * \mathbf{Cyl}(\bullet) \to \mathbf{M}_\bullet$ is a pushout of the boundary inclusion $S^{n+1} \to D_{n+2}$, as proven in Lemma 8.4.

Theorem B.3. *Given $n \geq 0$ and an ∞-groupoid X, $\mathbf{hom}(D_n, X)$ can be endowed with the structure of a bicategory.*

All the proof and construction that follow, can be adapted to the more general case of (possibly) degenerate cylinders as 1-cells and (possibly) collapsed modifications as 2-cells.

We already have some of the operations required to get a locally posetal bicategory out of it: composition of 1-cells is given by vertical composition of cylinders, and the identity 1-cell on an n-cell $A \in X_n$ is the trivial cylinder $\mathbf{C}_A$ defined as the composite $\mathbf{Cyl}(D_n) \xrightarrow{\mathbf{C}_n} D_n \xrightarrow{A} X$.

The existence of the rest of the structure in the case $n = 0$ is straightforward, and follows directly from the structure of the coherator $\mathfrak{C}$. In what follows, we fix an integer $n > 0$ and we assume as inductive hypotheses that $\mathbf{hom}(D_k, X)$ is a (locally posetal) bicategory for each $k < n$ together with the statements in Lemma B.10.

Let us now firstly address vertical composition of modifications. From here onwards, until the end of this section, whenever a 1-cell is labelled with Θ, Ψ or Φ, that refers to the coherence cylinders considered in Definition 38.

Lemma B.4. *Given a pair of composable modifications $\Theta \colon F \Rightarrow G$, $\Psi \colon G \Rightarrow H$ between n-cylinders $F, G, H \colon A \curvearrowright B$ in X, there exists a composite modification $\Psi \circ \Theta \colon F \Rightarrow H$*

Proof. Consider the following 2-dimensional pasting diagram in $\mathbf{hom}(D_{n-1}, X(x, y))$, where $x = s^2(\Theta_s)$, $y = t^2(\Theta_t)$ and the 2-cells $(\Psi \circ \Theta)_\varepsilon = \Psi_\varepsilon \circ \Theta_\varepsilon$ for $\varepsilon = s, t$ come from the base case $n=0$:

$$
\begin{tikzcd}
& H_tA \arrow[dl, "\Gamma(A{,}(\Psi\circ\Theta)_t)"'] \arrow[d, "\Gamma(A{,}\Psi_t)"] \arrow[ddd, bend left=70, "\bar{H}"] \arrow[dl, phantom, shift left=3ex, "\overset{\alpha}{\Rightarrow}"] & \\
F_tA \arrow[d, "\bar{F}"'] \arrow[dr, phantom, "\overset{\bar{\Theta}}{\Rightarrow}"] & G_tA \arrow[l, "\Gamma(A{,}\Theta_t)"'] \arrow[d, "\bar{G}"] \arrow[dr, phantom, "\underset{\bar{\Psi}}{\Rightarrow}"] & \\
BF_s \arrow[r, "\Upsilon(\Theta_s{,}B)"] \arrow[dr, "\Upsilon((\Psi\circ\Theta)_s{,}B)"'] & BG_s \arrow[d, "\Upsilon(\Psi_s{,}B)"] \arrow[dl, phantom, shift left=3ex, "\underset{\beta}{\Rightarrow}"] & \\
& BH_s &
\end{tikzcd}
$$

Here, the existence of α (resp. β) follows by an application of Lemma 8.5 to the contractible ∞-groupoid $D_n \coprod\limits_{D_0} D_2 \coprod\limits_{D_1} D_2$ (resp. $D_2 \coprod\limits_{D_1} D_2 \coprod\limits_{D_0} D_n$). The composite of this pasting diagram defines the modification claimed in the statement, thus concluding the proof. $\square$

We now construct whiskerings of 2-cells with 1-cells.

Lemma B.5. *Assume given n-cylinders $F\colon A \curvearrowright B$, $G\colon B \curvearrowright C$ together with a modification $\Theta\colon F \Rrightarrow F'$ in X. Then there is an induced modification $G * \Theta\colon G\circ F \to G \circ F'$.*

Proof. Consider the bicategory $\mathbf{hom}(D_{n-1}, X(s^n(A), t^n(C)))$, and the following 2-dimensional pasting diagram in it:

$$
\begin{tikzcd}
(G_tF_t)A \arrow[d, "\Psi"'] & (G_tF'_t)A \arrow[l, "\Gamma(A{,}G_t\Theta_t)"'] \arrow[d, "\Psi"] \arrow[dl, phantom, "\Rightarrow"] \\
G_t(F_tA) \arrow[d, "G_t\overline{F}"'] & G_t(F'_tA) \arrow[l, "G_t\Gamma(A{,}\Theta_t)"'] \arrow[d, "G_t\overline{F'}"] \arrow[dl, phantom, "\overset{ind.hyp.}{\Rightarrow}"] \\
G_t(BF_s) \arrow[r, "G_t\Upsilon(\Theta_s{,}B)"] \arrow[d, "\Psi"'] \arrow[dr, phantom, "\Rightarrow"] & G_t(BF'_s) \arrow[d, "\Psi"] \\
(G_tB)F_s \arrow[d, "\overline{G}F_s"'] & (G_tB)F'_s \arrow[l] \arrow[d, "\overline{G}F'_s"] \arrow[dl, phantom, "\overset{ind.hyp.}{\Rightarrow}"] \\
(CG_s)F_s \arrow[r] \arrow[d, "\Psi"'] \arrow[dr, phantom, "\Rightarrow"] & (CG_s)F'_s \arrow[d, "\Psi"] \\
C(G_sF_s) \arrow[r, "\Upsilon(G_s\Theta_s{,}C)"'] & C(G_sF'_s)
\end{tikzcd}
$$

The unlabeled cells come from the contractibility of the appropriate globular sums, while the remaining 2-cells are provided by the inductive hypothesis, as indicated. $\square$

Let us now address the problem of constructing identity 2-cells in $\mathbf{hom}(D_n, X)$.

Lemma B.6. *Given an n-cylinder $F\colon A \curvearrowright B$ in X, there exists a modification of n-cylinders in X of the form $1_F\colon F \Rrightarrow F$.*

Proof. Consider the following 2-dimensional pasting diagram in $\mathbf{hom}\,(D_{n-1}, X(x,y))$, with $x = s^n(A)$, $y = t^n(B)$ and where the pair of 2-cells $(1_F)_\varepsilon = 1_{F_\varepsilon}$ for $\varepsilon = s,t$ comes from the base case $n = 0$, where 1_f denotes the choice of an identity 2-cell on a 1-cell f:

$$
\begin{tikzcd}
 & F_tA \arrow[dl, bend right, "\Gamma(A,(1_{F_t}))"'] \arrow[dl, bend left, "\mathbf{C}_{F_tA}"] \arrow[dl, phantom, "\Downarrow\alpha"] \arrow[ddd, bend left=60, "\bar{F}"] \\
F_tA \arrow[d, "\bar{F}"'] \arrow[r, phantom, "\overset{\gamma}{\Longrightarrow}"] & \\
BF_s \arrow[dr, bend left, "\mathbf{C}_{BF_s}"] \arrow[dr, bend right, "\Upsilon((1_{F_s}),B)"'] \arrow[dr, phantom, "\Uparrow\beta"] & \\
 & BF_s
\end{tikzcd}
$$

Here, α (resp. β) is obtained by applying Lemma 8.5 to the contractible ∞-groupoid $D_n \coprod_{D_0} D_1$ (resp. $D_1 \coprod_{D_0} D_n$), and γ is a pasting of unit constraints in the bicategory $\mathbf{hom}(D_{n-1}, X(x,y))$. The composite of this pasting diagram provides the modification we are looking for, and thus we conclude the proof. $\square$

The next lemma provides the unit constraint for the bicategory structure on $\mathbf{hom}(D_n, X)$. We only prove one side of the unit constraint, the other one being analogous.

Lemma B.7. *Given an n-cylinder $C\colon A \curvearrowright B$ there exists a modification $\upsilon\colon C \circ \mathbf{C}_A \Rightarrow C$.*

Proof. Consider the following pasting diagram in $\mathbf{hom}(D_{n-1}, X(s^n(A), t^n(B)))$, where $a = s^n(A), b = t^n(B)$ and the pair of 2-cells υ_s, υ_t comes from the base case $n = 0$:

$$
\begin{tikzcd}
(c_t1_b)A \arrow[d, "\Psi"'] & & c_tA \arrow[ll, "\Gamma(A,\upsilon_t)"'] \arrow[dd, "\overline{C}"] \arrow[dddll, "\lambda_1"'] \\
c_t(1_bA) \arrow[d, "c_t\overline{\mathbf{C}_A}"'] \arrow[r, phantom, "\Longrightarrow"] & \arrow[r, phantom, "\overset{(1)}{\Uparrow}"] & \\
c_t(A1_a) \arrow[d, "\Psi"'] & & Bc_s \\
(c_tA)1_a \arrow[r, "\overline{C}1_a"'] & (Bc_s)1_a \arrow[ur, "\lambda_2"] \arrow[r, "\Psi"'] \arrow[r, phantom, shift left=4, "\Uparrow"] & B(c_s1_a) \arrow[u, "\Upsilon(\upsilon_t,A)"']
\end{tikzcd}
$$

The unlabeled 2-cells come from contractibility of appropriate globular sums, as well as λ_1 and λ_2, and the 2-cell labeled with (1) is provided by the inductive hypothesis. $\square$

We now turn to the final construction, that of the associator for the bicategory $\mathbf{hom}(D_n, X)$.

Lemma B.8. *Given a composable triple of n-cylinders $F\colon A \curvearrowright B$, $G\colon B \curvearrowright C$, $H\colon C \curvearrowright D$ in X, there exists a modification $\alpha\colon (H \circ G) \circ F \Rightarrow H \circ (G \circ F)$.*

Proof. The required 2-cells α_s, α_t come from the base case $n = 0$. The modification α is induced by composing the 2-dimensional pasting diagram in $\mathbf{hom}(D_{n-1}, X(s^n(A), t^n(C)))$ depicted in Figure 1 here below. In said diagram, the unlabelled 2-cells and the 1-cells η_i, μ_i and ν_i for $i = 1, 2$ all come from contractibility of suitable globular sums, others are obtained by inductive hypothesis (when indicated) and the 2-cells labelled with (0) and (1) are built in an analogous way. $\square$

Figure 1. Inductive step for the associator

$$
\begin{tikzcd}
 & (h_t(G_tF_t))A \arrow[dl, "\Gamma(A{,}\alpha_t)"'] \arrow[dr, "\Psi"] & & \\
((h_tG_t)F_t)A \arrow[d, "\Psi"] \arrow[rr, phantom, "\Longrightarrow"] & & h_t((G_tF_t)A) \arrow[dl, "h_t\Psi"'] \arrow[dd, "h_t\overline{G\circ F}"'] & \\
(h_tG_t)(F_tA) \arrow[dd, "(h_tG_t)\overline{F}"'] & h_t(G_t(F_tA)) \arrow[l, "\eta_1"'] \arrow[dd, "h_t(G_t\overline{F})"] & & \\
\arrow[r, phantom, "\overset{ind.hyp.}{\Longrightarrow}"] & \arrow[r, phantom, "\overset{ind.hyp.}{\Longrightarrow}"] & h_t(C(G_sF_s)) \arrow[ddr, "\Psi"] & \\
(h_tG_t)(BF_s) \arrow[r, "\eta_2"'] \arrow[ddd, "\Psi"'] & h_t(G_t(BF_s)) \arrow[d, "h_t\Psi"] & & \\
\arrow[r, phantom, "\Longrightarrow"] & h_t((G_tB)F_s) \arrow[r, "h_t(\overline{G}F_s)"] \arrow[dd, "\mu_1"'] & h_t((CG_s)F_s) \arrow[uu, "h_t\Psi"] & \\
 & \arrow[r, phantom, "\overset{(0)}{\Longrightarrow}"] & \arrow[r, phantom, "\Longrightarrow"] & (h_tC)(G_sF_s) \arrow[dl, "\nu_1"'] \arrow[dddll, bend left, "\overline{H}(G_sF_s)"] \\
((h_tG_t)B)F_s \arrow[r, "\Psi F_s"] \arrow[d, "\overline{H\circ G}F_s"'] & (h_t(G_tB))F_s \arrow[r, "(h_t\overline{G})F_s"] \arrow[d, phantom, "\overset{ind.hyp}{\Longrightarrow}"] & (h_t(CG_s))F_s \arrow[uu, "\mu_2"'] \arrow[d, "\Psi F_s"'] & \\
(D(h_sG_s))F_s \arrow[d, "\Psi"'] \arrow[dr, phantom, "\Longrightarrow"] & ((Dh_s)G_s)F_s \arrow[l, "\Psi F_s"'] \arrow[d, "\nu_2"'] & ((h_tC)G_s)F_s \arrow[l, "(\overline{H}G_s)F_s"'] \arrow[dl, phantom, "\overset{(1)}{\Longrightarrow}"] & \\
D((h_sG_s)F_s) \arrow[d, "\Upsilon(\alpha_s{,}D)"'] & (Dh_s)(G_sF_s) \arrow[dl, "\Psi"] & & \\
D(h_s(G_sF_s)) & & &
\end{tikzcd}
\tag{60}
$$

Lemma B.9. *Given a pair of composable modifications $\Theta\colon F \Rrightarrow G$, $\Psi\colon G \Rrightarrow H$ between n-cylinders $F, G, H\colon A \curvearrowright B$ in X, there exists a composite modification $\Psi\circ\Theta\colon F \Rrightarrow H$.*

Proof. The 2-cells $(\Psi\circ\Theta)_\varepsilon = \Psi_\varepsilon\circ\Theta_\varepsilon$ for $\varepsilon = s, t$ come from the base case $n = 0$. Consider the following 2-dimensional pasting diagram in $\mathbf{hom}(D_{n-1}, X(x, y))$, where $x = s^2(\Theta_s)$ and $y = t^2(\Theta_t)$:

$$
\begin{tikzcd}
 & H_tA \arrow[dl, "\Gamma(A{,}(\Psi\circ\Theta)_t)"'] \arrow[d, "\Gamma(A{,}\Psi_t)"] \arrow[ddd, bend left=70, "\bar{H}"] \arrow[dl, phantom, shift left=4, "\overset{\alpha}{\Longrightarrow}"] \\
F_tA \arrow[d, "\bar{F}"'] \arrow[dr, phantom, "\overset{\bar{\Theta}}{\Longrightarrow}"] & G_tA \arrow[l, "\Gamma(A{,}\Theta_t)"'] \arrow[d, "\bar{G}"'] \arrow[d, phantom, shift left=10, "\underset{\bar{\Psi}}{\Longrightarrow}"] \\
BF_s \arrow[r, "\Upsilon(\Theta_s{,}B)"] \arrow[dr, "\Upsilon((\Psi\circ\Theta)_s{,}B)"'] \arrow[dr, phantom, shift left=4, "\underset{\beta}{\Longrightarrow}"] & BG_s \arrow[d, "\Upsilon(\Psi_s{,}B)"] \\
 & BH_s
\end{tikzcd}
$$

The existence of α (resp. β) follows by contractibility of $D_n \amalg_{D_0} D_2 \amalg_{D_1} D_2$ (resp. $D_2 \amalg_{D_1} D_2 \amalg_{D_0} D_n$). The composite of this pasting diagram defines the modification claimed in the statement, thus concluding the proof. $\square$

The following lemma is a collection of auxiliary facts that we have included in the inductive assumptions in order to prove the existence of the bicategory structure.

Lemma B.10. *The following facts hold true:*

(1) *Given an n-cylinder $F\colon A \curvearrowright B$ in $\Omega_m(X,\varphi)$, a 1-cell g in $\Omega^m(X,\varphi)$ and a 1-cell $h\colon a \to s^{n+m}(A)$ in X, such that $s^2(g) = t^{n+1}(A) = t^{n+1}(B)$, there is a modification χ as displayed below, where the cylinders denoted by λ and μ are obtained by contractibility of the appropriate globular sums:*

$$\begin{tikzcd} (gh)(Ah) \arrow[d, "(gh)(Fh)"'] \arrow[r, phantom, "\overset{\chi}{\Rightarrow}" near end, shift right=6] & (gA)h \arrow[l, "\lambda"'] \arrow[d, "(gF)h"] \\ (gh)(Bh) \arrow[r, "\mu"'] & (gB)h \end{tikzcd}$$

(2) *Let $F\colon A \curvearrowright B$, $G\colon B \curvearrowright C$ be n-cylinders in $\Omega^m(X,\varphi_1,\varphi_2)$ where $(\varphi_1,\varphi_2) = \varphi\colon S^{m-1} \to X$. Given a 1-cell $h\colon a \to s^{n+m}(A) = s^{n+m}(B) = s^{n+m}(C)$ in X, we get a modification:*

$$\Theta^{g,f,h}\colon (Gh)\circ(Fh) \Rightarrow (G\circ F)h\colon Ah \curvearrowright Ch$$

where $(\bullet)h$ denotes the operation of whiskering defined in Section 3 and $\circ$ is the vertical composition of cylinders.

(3) *Given a pair of n-cylinders $F, G\colon A \curvearrowright B$ in $\Omega_m(X,\varphi)$, a modification $\Lambda\colon F \Rightarrow G$ and a 1-cell $c\colon b = t^{n+m}(B) \to b'$, there exists an induced modification $c\Lambda\colon cF \Rightarrow cG$ between the n-cylinders $cF, cG\colon cA \curvearrowright cB$ in $\Omega_m(X, c\varphi)$.*

(4) *Given an n-cylinder $G\colon A \curvearrowright B$ in $\Omega_m(X,\varphi)$ and a 2-cell in X of the form:*

$$\begin{tikzcd} a \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] & s^{n+m}(A) \end{tikzcd}$$

we get an induced modification:

$$\begin{tikzcd} Af' \arrow[r, "\Lambda_1"] \arrow[d, "Gf'"'] \arrow[dr, phantom, "\overset{\Delta}{\Rightarrow}"] & Af \arrow[d, "Gf"] \\ Bf' & Bf \arrow[l, "\Lambda_2"'] \end{tikzcd}$$

Here, Λ_1 and Λ_2 are obtained by contractibility of suitable globular sums, and the existence of Δ does not depend on the choice of these.

(5) *Assume given an n-cylinder $C\colon A \curvearrowright B$ in $\Omega_m(X,\varphi)$ and a choice of an identity 1-cell $1_a\colon a \to a$ in X, where $a = s^{n+m}(A)$. We then get a modification of the following form:*

$$\begin{tikzcd} A1_a \arrow[d, "C1_a"'] \arrow[dr, phantom, "\overset{\beta}{\Rightarrow}"] & A \arrow[l, "\Lambda_1"'] \arrow[d, "C"] \\ B1_a \arrow[r, "\Lambda_2"'] & B \end{tikzcd}$$

Again, Λ_1 and Λ_2 are obtained by contractibility of the appropriate globular sums and the existence of β does not depend on a choice of such.

(6) *Given an n-cylinder $F\colon A \curvearrowright B$ in $\Omega_m(X,\varphi)$, and a pair of composable 1-cells $h\colon t^{n+m}(A) \to b,\ g\colon b \to c$, there is a modification:*

$$\begin{tikzcd} (hg)A \arrow[d, "(hg)F"'] \arrow[r, phantom, "\overset{\zeta}{\Longrightarrow}" below, shift right=6] & h(gA) \arrow[l, "\lambda_1"'] \arrow[d, "h(gF)"] \\ (hg)B \arrow[r, "\lambda_2"'] & h(gB) \end{tikzcd}$$

Here, λ_1, λ_2 come from the contractibility of $D_n \coprod_{D_0} D_1 \coprod_{D_0} D_1$, and the existence of ζ does not depend on the choice of such cylinders.

Proof. Firstly, let us address point (1). Notice that the existence of such modification does not depend on the choice of λ and μ. By definition, given $\varepsilon = s, t$, we have (from the base case $n = 0$) that $(\mu \circ (gh)(Fh) \circ \lambda)_\varepsilon$ is given by the composite:

$$(g\varepsilon^n(A))h \xrightarrow{\lambda_t} (gh)(\varepsilon^n(A)h) \xrightarrow{(gh)(F_\varepsilon h)} (gh)(\varepsilon^n(B)h) \xrightarrow{\mu_t} (g\varepsilon^n(B))h$$

where the first and the third map arise from contractibility of suitable globular sums.

On the other hand, $((gF)h)_\varepsilon$ is given by $(gF_\varepsilon)h\colon (g\varepsilon^n(A))h \to (g\varepsilon^n(B))h$. The base case $n = 0$ provides a pair of two cells χ_s, χ_t as required in the definition of a modification. The rest of the modification is obtained by composing up the following pasting diagram, whose left-hand side composite is $\overline{\Upsilon(\chi_s,(gB)h)\otimes(\mu\otimes(gh)(Fh)\otimes\lambda)\otimes\Gamma((gA)h,\chi_t)}$ and whose right-hand composite is $\overline{(gF)h}$, where we let $C=\mu\circ(gh)(Fh)\circ\lambda$:

$$
\begin{tikzcd}
 & ((gF_t)h)((gA)h) \arrow[dl, "\Gamma((gA)h,\chi_t)"'] \arrow[drr, "\Psi"] & & \\
(\mu_tC_t\lambda_t)((gA)h) \arrow[d, "\Psi"'] & & & ((gF_t)(gA))h \arrow[ddl, "\Phi h"'] \arrow[dddddd, "\overline{gF}h"] \\
(\mu_tC_t)(\lambda_t((gA)h)) \arrow[d, "(\mu_tC_t)\overline{\lambda}"'] & \overset{\mathit{contr.}}{\Longrightarrow} & & \\
(\mu_tC_t)(((gh)(Ah))\lambda_s) \arrow[d, "\Psi"'] & & (g(F_tA))h \arrow[dl, "\lambda'"'] \arrow[ddd, "(g\overline{F})h"'] & \\
\mu_t(C_t((gh)(Ah)))\lambda_s \arrow[r, "\Phi"] \arrow[d, "\mu_t\overline{(gh)(Fh)}\lambda_s"'] \arrow[dr, phantom, "\overset{\mathit{ind.hyp.}}{\Longrightarrow}"] & \mu_t(C_t((gh)(Ah)))\lambda_s \arrow[d, "\mu_t(C_t((gh)(\overline{Fh})))\lambda_s"'] \arrow[dr, phantom, "\overset{\mathit{ind.hyp.}}{\Longrightarrow}"] & & \overset{\mathit{ind.hyp.}}{\Longrightarrow} \\
\mu_t(((gh)(Bh))C_s)\lambda_s \arrow[d, "\Psi"'] & \mu_t(C_t((gh)(Bh)))\lambda_s \arrow[l, "\Theta"'] \arrow[dr, "\mu'"] & & \\
(\mu_t((gh)(Bh)))(C_s\lambda_s) \arrow[d, "\overline{\mu}(C_s\lambda_s)"'] & & (g(BF_s))h \arrow[dr, "\Theta h"] & \\
(((gB)h)\mu_s)(C_s\lambda_s) \arrow[d, "\Psi"'] & \overset{\mathit{contr.}}{\Longrightarrow} & & ((gB)(gF_s))h \arrow[d, "\Theta"] \\
((gB)h)(\mu_sC_s\lambda_s) \arrow[rrr, "\Upsilon(\chi_s,(gB)h)"'] & & & ((gB)h)((gF_s)h)
\end{tikzcd}
$$

The 2-cells in this pasting diagram either come from contractibility of the appropriate globular sums (in this case we do not label them), or from the inductive hypotheses, as indicated. Composing up this pasting diagram using the bicategorical structure on $\mathbf{hom}(D_{n-1},X(x,y))$ for the approriate x,y in X gives the desired modification.

We now prove point (2). To begin with, the base case $n=0$ provides a pair of 2-cells:

$$\Theta_s^{g,f,h}\colon (G_sh)(F_sh)\to(G_sF_s)h \text{ and } \Theta_t^{g,f,h}\colon (G_tF_t)h\to(G_th)(F_th)$$

in $\Omega_m(X,\varphi_1h,\varphi_2h)$. These are obtained, in the target case, from the contractibility of the globular sum $D_1\coprod_{D_0}D_{m+1}\coprod_{D_m}D_{m+1}$. Indeed, one has the following string of equalities:

$$s^{n+m}(A)=s^m(s^n(A))=s^m(t^n(A))=s^m(s(F_t))=s^m(t(F_t))=s^m(s(G_t))$$

which implies that there is a map $(h,F_t,G_t)\colon D_1\coprod_{D_0}D_{m+1}\coprod_{D_m}D_{m+1}\to X$. The source case is treated similarly.

We then have the following diagram in $\mathbf{hom}(D_{n-1}, \Omega_{m+1}(X, s^n(Ah), t^n(Ch)))$:

$$
\begin{tikzcd}
 & & ((G_tF_t)h)(Ah) \arrow[dl, "\Gamma(Ah,\Theta_t^{g,f,h})"'] \arrow[dr, "\Phi"] & \\
 & ((G_th)(F_th))(Ah) \arrow[d, "\Psi"'] \arrow[rr, phantom, shift right=4ex, "\Longrightarrow"] & & ((G_tF_t)A)h \arrow[d, "\Psi h"] \arrow[ddddddd, bend left=60, "(\overline{G\circ F})h"] \\
 & (G_th)((F_th)(Ah)) \arrow[dl, "(G_th)\overline{Fh}"'] \arrow[dr, "(G_th)\Phi"] & & ((G_t)(F_tA))h \arrow[dl] \arrow[dd, "(G_t\overline{F})h"] \\
(G_th)((Bh)(F_sh)) \arrow[ddd, "\Psi"'] \arrow[rr, phantom, "\overset{\text{ind.hyp.}}{\Longrightarrow}"] & & (G_th)((F_tA)h) \arrow[dl, "(G_th)(\overline{F}h)"] \arrow[r, phantom, shift right=3ex, "\overset{\text{ind.hyp.}}{\Longrightarrow}"] & \\
 & (G_th)((BF_s)h) \arrow[ul, "(G_th)\Theta"] \arrow[rr] & & (G_t(BF_s))h \arrow[d, "\Psi h"] \arrow[r, phantom, "\overset{\text{ind.hyp.}}{\Longrightarrow}"] & {} \\
 & \Longrightarrow & & {} \\
((G_th)(Bh))(F_sh) \arrow[rr, "\Phi(F_sh)"] \arrow[d, "(\overline{Gh}(F_sh))"'] \arrow[r, phantom, shift right=4ex, "\overset{\text{ind.hyp.}}{\Longrightarrow}"] & & ((G_tB)h)(F_sh) \arrow[d, "((\overline{G}h)(F_sh))"'] \arrow[r, phantom, shift right=4ex, "\overset{\text{ind.hyp.}}{\Longrightarrow}"] & ((G_tB)F_s)h \arrow[l] \arrow[d, "(\overline{G}F_s)h"] \\
((Ch)(G_sh))(F_sh) \arrow[dr, "\Psi"'] & & ((CG_s)h)(F_sh) \arrow[ll, "\Theta(G_sh)"'] \arrow[r] \arrow[d, phantom, "\Longrightarrow"] & ((CG_s)F_s)h \arrow[d, "\Psi h"] \\
 & (Ch)((G_sh)(F_sh)) \arrow[rr] \arrow[dr, "\Upsilon(\Theta_s^{g,f,h},Ch)"'] & \Longrightarrow & (C(G_sF_s))h \arrow[dl, "\Theta"] \\
 & & (Ch)((G_sF_s)h) &
\end{tikzcd}
$$

The composite of this pasting diagram provides the 2-cell we are looking for, the left-hand side (resp. right-hand side) composite being (isomorphic to) $\Upsilon(\Theta_s^{g,f,h}, Ch) \circ \overline{(Gh) \circ (Fh)} \circ \Gamma(Ah, \Theta_t^{g,f,h})$ (resp.$\overline{(G \circ F)h}$).

The 2-cells filling this diagram either come from the inductive hypotheses (where specified) or from contractibility of the appropriate globular sums (the unlabeled 2-cells).

Let us now prove point (3). Consider the bicategory $\mathbf{hom}(D_{n-1}, \Omega_{m+1}(X, s^n(cA), t^n(cB))$, inside which we define the following 2-dimensional pasting diagram:

$$
\begin{tikzcd}
 & (cG_t)(cA) \arrow[dl, "\Gamma(cA,c\Lambda_t)"'] \arrow[d, "\varphi"] \\
(cF_t)(cA) \arrow[d, "\varphi"'] \arrow[r, phantom, "\Longrightarrow"] & c(G_tA) \arrow[dl, "c\Gamma(A,\Lambda_t)"'] \arrow[dd, bend right=50, "c(\Upsilon\circ\overline{F}\circ\Gamma)"'] \arrow[dd, bend left=50, "c\overline{G}"] \arrow[dd, phantom, "\overset{\text{ind.hyp.}}{\Longrightarrow}"] \\
c(F_tA) \arrow[d, "c\overline{F}"'] \arrow[r, phantom, shift right=3ex, "\overset{\text{ind.hyp.}}{\Longrightarrow}"] & \\
c(BF_s) \arrow[r, "c\Upsilon(\Lambda_s,B)"] \arrow[d, "\Theta"'] \arrow[dr, phantom, "\Longrightarrow"] & c(BG_s) \arrow[d, "\Theta"] \\
(cB)(cF_s) \arrow[r, "\Upsilon(c\Lambda_s,cB)"] & (cB)(cG_s)
\end{tikzcd}
$$

The 2-cells that fill the diagram either come from the inductive hypotheses or by contractibility of suitable globular sums when unlabeled. The composite of this pasting diagram is the 2-cell we are looking for, and so this concludes the proof of this inductive step.

Let us now prove the inductive step of point (4). The base case $n = 0$ provides us with pair of 2-cells Δ_s, Δ_t, obtained by contractibility of suitable globular sums, and the modification Δ is given by the composite of the following 2-dimensional pasting diagram in the bicategory $\mathbf{hom}(D_{n-1}, \Omega_m(X, \varphi)(s^n(Af'), t^n(Bf')))$:

$$
\begin{tikzcd}
((1(G_t f))1)(Af') \arrow[d, "\Psi"'] & & (G_t f')(Af') \arrow[ll, "\Gamma(Af',\Delta_t)"'] \arrow[d, "\Phi"] & & & & \\
(1(G_t f))(1(Af')) \arrow[d, "(1(G_t f))\overline{\Lambda}_1"'] \arrow[r, phantom, "\Longrightarrow"] & {} & (G_t A)f' \arrow[d, "\eta_1"] \arrow[rrrrdd, "\overline{G}f'"] & & & & \\
(1(G_t f))((Af)1) \arrow[d, "\Psi"'] & & (G_t A)f \arrow[d, "ind.hyp"] \arrow[rrd, "\overline{G}f"] \arrow[rrrd, phantom, "ind.hyp. \Longrightarrow"] & & & & \\
(1((G_t f)(Af)))1 \arrow[rr] \arrow[rd, "(1\overline{G}f)1"'] \arrow[rrd, phantom, "ind.hyp \Longrightarrow"] & & (1((G_t A)f))1 \arrow[rrd, "(1(\overline{G}f))1"] & & (BG_s)f \arrow[rr, "\eta_2"'] & & (BG_s)f' \arrow[dd, "\Theta"] \\
 & (1((Bf)(G_s f)))1 \arrow[d, "\Psi"'] & & & (1((BG_s)f))1 \arrow[lll] \arrow[u] \arrow[r, phantom, "\Longrightarrow"] & {} & \\
 & (1(Bf))((G_s f)1) \arrow[rrr, "\Lambda_2((G_s f)1)"] & & & ((Bf')1)((G_s f)1) \arrow[rd, "\Psi"'] & & (Bf')(G_s f') \\
 & & & & & (Bf')(1((G_s f)1)) \arrow[ru, "\Upsilon(\Delta_s,Bf')"'] &
\end{tikzcd}
$$

The unlabelled cells are obtained by contractibility of suitable globular sums, and the remaining 2-cells exist by inductive hypothesis, as indicated.

We now prove the inductive step of point (5). The 1-cells $(\Lambda_i)_\varepsilon$ in $\Omega_m(X, \varphi)$, for $\varepsilon = s, t$ and $i = 1, 2$, are obtained by contractibility of D_n and are therefore identity cells (having the same source and target). For this reason, we denote all of them by 1, since there is no risk of ambiguity.

The pair of 2-cells β_s, β_t provided by the base case $n = 0$ is obtained by contractibility, and we choose β to be induced by the composite of the following 2-dimensional pasting diagram in $\mathbf{hom}(D_{n-1}, \Omega_m(X, \varphi)(s^n(A), t^n(B)))$:

$$
\begin{tikzcd}
(1(c_t 1_a)1)A \arrow[d, "\Psi"'] & c_t A \arrow[l, "\Gamma(A,\beta_t)"'] \arrow[r, "\overline{C}"] \arrow[dd, "\gamma_1"] \arrow[rd, phantom, "\Uparrow ind.hyp."] & Ba & B((c_s 1_a)1) \arrow[l, "\Upsilon(\beta_s,B)"'] \\
(1(c_t 1_a))(1A) \arrow[d, "(1(c_t 1_a))\overline{\Lambda_1}"'] \arrow[r, phantom, "\Longrightarrow"] & {} & {} \arrow[r, phantom, "\Longleftarrow"] & (1(B1_a))((c_s 1_a)1) \arrow[u, "\overline{\Lambda_2}((c_s 1_a)1)"'] \\
(1(c_t 1_a))((A1_a)1) \arrow[r, "\Psi"] & (1((c_t 1_a)(A1_a)))1 \arrow[rr, "(1(\overline{C}1_a))1"] & & (1((B1_a)(c_s 1_a)))1 \arrow[u, "\Psi"'] \arrow[luu, "\gamma_2"]
\end{tikzcd}
$$

where the unlabelled 2-cells arise from contractibility of the appropriate globular sums, and the remaining one comes from the inductive hypothesis.

Finally, let us prove the inductive step of point (6). We denote the 1-cells $\varepsilon^n(\lambda_i)$, for $\varepsilon = s, t$ and $i = 1, 2$ with a, being instances of an associativity constraint.

The 2-cells ζ_s, ζ_t provided by the base case $n = 0$ come from contractibility of appropriate globular sums, and the modification we are looking for is given by the composite of the following 2-dimensional pasting diagram in $\mathbf{hom}(D_{n-1}, \Omega_m(X, \varphi)(s^n(h(gA)), t^n(h(gB))))$:

$$
\begin{tikzcd}
 & (h(gF_t))(h(gA)) \arrow[dl, "\Gamma(h(gA),\zeta_t)"'] \arrow[dr, "\Phi"] & & & \\
(a(((hg)F_t)a))(h(gA)) \arrow[d, "\Psi"'] & & h((gF_t)A) \arrow[r, "h(\overline{gF})"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\overset{ind.hyp.}{\Longrightarrow}"] & h((gB)(gF_s)) \arrow[ddddr, "\Theta"] & \\
(a((hg)F_t))(a(h(gA))) \arrow[dd, "(a((hg)F_t))\overline{\lambda_1}"'] & \Longrightarrow & h(g(F_tA)) \arrow[dr, "h(g(\overline{F}))"] \arrow[dd, "\delta_1"'] & & \\
 & & & h(g(BF_s)) \arrow[uu, "\Theta"'] & \\
(a((hg)F_t))(((hg)A)a) \arrow[dd, "\Psi"'] & & (a((hg)(F_tA)))a \arrow[dd, "(a((hg)\overline{F}))a"'] \arrow[r, phantom, "\overset{ind.hyp.}{\Longrightarrow}"] & & \\
 & & & & (h(gB))(h(gF_s)) \\
(a(((hg)F_t)((hg)A)))a \arrow[dd, "(a(\overline{(hg)F}))a"'] \arrow[uurr, "(a\Phi)a"] \arrow[rr, phantom, "\overset{ind.hyp.}{\Longrightarrow}"] & & (a((hg)(BF_s)))a \arrow[uuur, bend right, "\delta_2"'] \arrow[ddll, "(a\Theta)a"'] & & \\
 & & \Longrightarrow & & (h((gB)a))(((hg)F_s)a) \arrow[uu, "\Upsilon(\zeta_s,h(gB))"'] \\
(a(((hg)B)((hg)F_s)))a \arrow[drr, "\Psi"] & & & & \\
 & & (a((hg)B))(((hg)F_s)a) \arrow[uurr, "\overline{\lambda_2}(((hg)F_s)a)"'] & &
\end{tikzcd}
$$

Here, δ_1, δ_2 and the unlabelled 2-cells come from contractibility of suitable globular sums and the remaining ones comes from the inductive hypothesis, as indicated. ☐

This concludes the construction of the bicategory structure on $\mathbf{hom}(D_n, X)$ and thus proves Theorem B.3.

We end this section with the following result, which requires the existence of inverses and does not hold true in $\mathbf{Mod}(\mathfrak{C})$.

Lemma B.11. *Given a pair of n-cylinders $F, G \colon \mathbf{Cyl}(D_n) \to X$ in $\mathbf{Mod}(\mathfrak{C}^{\mathbf{W}})$ and a modification $\Theta \colon F \to G$ there exists a modification $\Theta' \colon G \to F$*

Proof. We denote by f^{-1} the result of promoting either a left or a right inverse for f to a two-sided inverse. If $n = 0$ then Θ' is obtained by inverting the 2-cell Θ. Let $n > 0$, we define $\Theta'_s = (\Theta_s)^{-1}$ and $\Theta'_t = (\Theta_t)^{-1}$. By definition, Θ induces a modification of $(n-1)$-cylinders of the form $\overline{\Theta} \colon \Upsilon(C_0, \Theta_t) \otimes \bar{C} \otimes \Gamma(\Theta_s, C_1) \Rightarrow \bar{D}$ (where $\otimes$ denotes the vertical composition operation). By inductive hypothesis this can be inverted, to give us $\overline{\Theta}' \colon \bar{D} \Rightarrow \Upsilon(C_0, \Theta_t) \otimes \bar{C} \otimes \Gamma(\Theta_s, C_1)$. Thereom B.3 implies that we get a modification:

$$
\begin{array}{c}
\Upsilon(C_0, (\Theta_t)^{-1}) \otimes \bar{D} \otimes \Gamma((\Theta_s)^{-1}, C_1) \\
{\scriptstyle \Upsilon(C_0,(\Theta_t)^{-1})\overline{\Theta}'\Gamma((\Theta_s)^{-1}}\Big\Downarrow \\
\Upsilon(C_0, (\Theta_t)^{-1}) \otimes \Upsilon(C_0, \Theta_t) \otimes \bar{C} \otimes \Gamma(\Theta_s, C_1) \otimes \Gamma((\Theta_s)^{-1}, C_1)
\end{array}
$$

by whiskering.

Now, the existence of 3-cells $\Theta_s \circ (\Theta_s)^{-1} \to 1_{t(\Theta_s)}$ and $(\Theta_t)^{-1} \circ (\Theta_t) \to 1_{s(\Theta_t)}$ implies that there is an induced modification (using the usual methods to produce such modification in

$\mathbf{Mod}(\mathfrak{C})$, applied to the globular sums $D_3 \coprod_{D_0} D_n$ and $D_n \coprod_{D_0} D_3$):

$$\Upsilon(C_0, (\Theta_t)^{-1}) \otimes \Upsilon(C_0, \Theta_t) \Rightarrow \Upsilon(C_0, 1_{t(\Theta_s)})$$

and

$$\Gamma(\Theta_s, C_1) \otimes \Gamma((\Theta_s)^{-1}, C_1) \Rightarrow \Gamma(1_{s(\Theta_t)}, C_1)$$

The usual methods can also be employed to construct modifications $\Gamma(1_{s(\Theta_t)}, C_1) \Rightarrow \mathbf{C}_{n-1}(C_1 C_s)$ and $\Upsilon(C_0, 1_{t(\Theta_s)}) \Rightarrow \mathbf{C}_{n-1}(C_t C_0)$, so that upon composing the $(n-1)$-modification defined so far we get one of the form:

$$\begin{array}{c} \Upsilon(C_0, (\Theta_t)^{-1}) \otimes \bar{D} \otimes \Gamma((\Theta_s)^{-1}, C_1) \\ \Big\Downarrow \\ \mathbf{C}_{n-1}(C_t C_0) \otimes \bar{C} \otimes \mathbf{C}_{n-1}(C_1 C_s) \end{array}$$

We can now finish the construction of $\overline{\Theta'}$ by using the unit constraints provided by Theorem B.3. $\square$

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