Restriction presheaves and restriction colimits

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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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Abstract

Restriction categories, as defined by Cockett and Lack, are an abstraction of the notion of partial functions between sets, and therefore, are important in furthering our understanding of what it means to be partial. This thesis builds upon the work of Cockett and Lack, by providing restriction analogues of notions from ordinary category theory. One such notion is that of free cocompletion. We show that every restriction category may be freely completed to a cocomplete restriction category, and that this free cocompletion can be described in terms of a restriction category of restriction presheaves. Indeed, a restriction presheaf is defined precisely so that this is the case. We then generalise free cocompletion to join restriction categories, which are categories whose compatible maps may be combined in some way. To do this, we introduce the notion of join restriction presheaf, and show that for any join restriction category, its join restriction category of join restriction presheaves is its free cocompletion.

The second half of this thesis explores the notion of restriction colimit. More precisely, we define the restriction colimit of a restriction functor weighted by a restriction presheaf. We also show that cocomplete restriction categories may be characterised as those having all such restriction colimits. Finally, we give applications of restriction colimits. Some examples of restriction colimits are gluings of atlases in a restriction category, and composition of restriction profunctors. We conclude this thesis with notions in category theory that have no analogue in the restriction setting.

Contents

No human being is constituted to know the truth, the whole truth, and nothing but the truth; and even the best of men must be content with fragments, with partial glimpses, never the full fruition.

William Osler

1 Introduction

Partial functions play a fundamental role in many areas of mathematics, including computability theory, algebraic topology and of course, in analysis, from which the notion originated. As such, they have been the subject of study by mathematicians from across different fields, including category theory. One way of trying to understand the notion of partiality is to generalise the notion beyond sets and partial functions, and indeed this has been the approach taken by category theorists. We therefore begin with a quick overview of the important developments in this area over the last few decades. We shall then proceed to outline the main ideas present in this thesis, before giving a brief summary of each of the chapters.

1.1 From partial functions to restriction categories

An early instance of partiality extending beyond the usual setting of sets and partial functions was considered by [Booth & Brown, 1978]; they defined a *parc map* from a space X to a space Y to be a continuous function from a closed subspace $A \subset X$ to Y. At a more abstract level of partiality, we have the notion of *category with partial morphisms*, given by [Longo & Moggi, 1984]. Every such category $\mathbf{C}$ has an object $t \in \mathbf{C}$ called a *singleton object*; a point of interest here being that every category with partial morphisms has a subcategory $\mathbf{C}_T$, the *category of total morphisms*, whose class of morphisms is not dependent on the choice of the singleton object t.

The next few years following Moggi was a period of active development in this area. [Carboni, 1987] studied the notion of bicategory with a partial map structure; effectively bicategories equipped with a tensor product and a unique cocommutative comonoid structure. A familiar example of such a bicategory is given. For any left exact category $\mathbf{E}$ with a choice of a product, define *partial morphisms* from X to Y to be an equivalence classes of spans $X \xleftarrow{m} Z \xrightarrow{f} Y$, where m is a monomorphism and composition is by pullback. Then the bicategory $\mathsf{Par}(\mathbf{E})$ formed is a bicategory with a partial map structure; establishing a connection between subobjects and partiality.

In the very same year in which Carboni published his paper, [Di Paola & Heller, 1987] approached the study of partiality in an entirely different manner, through their introduction of *dominical category*. This was the first time that the notion of partiality had been axiomatised, with this partiality being expressed through the notion of *near-product*. However, it was the *domain* maps which occupied the greatest interest; given a morphism $\varphi \colon X \to Y$, the domain of φ was an endomorphism on X, and not necessarily a subobject of X. [Rosolini, 1986] continued the work began by Di Paola and Heller, by introducing his own interpretation of partiality through the notion of p-category. These p-categories were effectively the same as dominical categories, except for the fact that these did not require the existence of *pointed* or *zero maps*. Nonetheless, at this stage, it was now clear that the study of partial maps had passed on from the study of subobjects to studying endomorphisms.

However, it would still be some time before the connection between partiality and the *restriction structure* of a category was established; this restriction structure being the assignment of an idempotent $\bar{f} \colon X \to X$ for every map $f \colon X \to Y$ in the same category. In the early 1990s, [Grandis, 1990] introduced the notion of *e-cohesive category*, which described all of the necessary conditions required to establish a restriction structure on a category. The only issue, though, was that he did not make the connection between partiality and this *cohesive structure* on a category explicit.

It would not be until the early 2000s that this explicit connection between *restriction structure* and partiality would be made. Indeed, [Cockett & Lack, 2002] made it clear that the introduction of *restriction idempotents* was an attempt in describing this notion of partiality. From this, they developed the notion of *restriction category*, a notion which will form the foundation for this thesis.

1.2 Main ideas

Given an ordinary category $\mathbf{C}$, a restriction structure on $\mathbf{C}$ assigns to every map $f \colon A \to B$ in $\mathbf{C}$, a map $\bar{f} \colon A \to A$ called the *restriction* of f, satisfying four axioms [Cockett & Lack, 2002]. A basic example of a restriction category is the category $\mathbf{Set}_p$ of sets and partial functions. In $\mathbf{Set}_p$, the restriction of a partial function $f \colon X \to Y$ is defined to be the partial identity on X whose domain of definition is the same as that of f.

As it turns out, these maps $\bar{f} \colon A \to A$ are idempotents, with the additional property that the restriction of $\bar{f}$ is also $\bar{f}$. We call such maps *restriction idempotents*. If these restriction idempotents $\bar{f}$ split, that is, there exist maps m and r such that $mr = \bar{f}$ and $rm = 1$, then we call these idempotents *split restriction idempotents*. If every restriction idempotent in the restriction category splits, then we call such a category, a *split restriction category*.

The reason for the interest in split restriction categories is because they are closely associated with $\mathcal{M}$-categories, which are essentially categories equipped a class of specified monomorphisms that are closed under composition and stable under pullbacks. More specifically, there is a 2-equivalence between the 2-category of split restriction categories, $\mathbf{rCat}_s$, and the 2-category of $\mathcal{M}$-categories, $\mathcal{M}\mathbf{Cat}$. How this equivalence works, at least in one direction, is as follows. Given an $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, where $\mathcal{M}$ is the given class of monics, we may form the category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ whose objects are the same as $\mathbf{C}$, but whose maps are equivalence classes of spans $X \xleftarrow{m} Z \xrightarrow{f} Y$, with $m \in \mathcal{M}$ and f completely arbitrary. The restriction of each map (m, f) in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is given by (m, m), each such (m, m) is also split. In this way, one can see that the notion of *restriction category* is really an extension or generalisation of the notion of partial map category, dispensing with the idea that the study of partiality and subobjects are inseparable.

1.2.1 Cocomplete $\mathcal{M}$-categories and cocomplete restriction categories

Our first goal is to define cocomplete restriction category. However, the fact that there is a 2-equivalence between $\mathbf{rCat}_s$ and $\mathcal{M}\mathbf{Cat}$ suggests that, in order to give the *free cocompletion* of a restriction category, it is enough to define the free cocompletion of an $\mathcal{M}$-category. Recall that the free cocompletion of an ordinary category $\mathbf{C}$ is given by its presheaf category $\mathsf{PSh}(\mathbf{C})$. For any $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, there is a natural candidate for its free cocompletion; namely, its $\mathcal{M}$-category of presheaves, $(\mathsf{PSh}(\mathbf{C}), \mathsf{PSh}(\mathcal{M}))$. A map $\mu\colon P \to Q$ is in $\mathsf{PSh}(\mathcal{M})$ if for every map $\mathbf{y}B \to Q$, there is a monic $m \in \mathcal{M}$ making the following square a pullback:

$$\begin{tikzcd} \mathbf{y}A \arrow[r] \arrow[d, "\mathbf{y}m"'] & P \arrow[d, "\mu"] \\ \mathbf{y}B \arrow[r] & Q. \end{tikzcd}$$

However, it only makes sense to describe the $\mathcal{M}$-category $(\mathsf{PSh}(\mathbf{C}), \mathsf{PSh}(\mathcal{M}))$ as the free cocompletion of $(\mathbf{C}, \mathcal{M})$ if we have a notion of *cocomplete* $\mathcal{M}$-category. As a first guess, one might insist that the base category $\mathbf{C}$ of a cocomplete $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$ itself be cocomplete in the ordinary sense. Indeed this is one part of the definition. The other part is more subtle though; if $(\mathbf{C}, \mathcal{M})$ is cocomplete, then colimits in $\mathbf{C}$ should be preserved by the inclusion into its partial map category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$. One way of understanding this second requirement, and which we shall make precise in this thesis, is that coproducts and coequalisers of maps in $\mathcal{M}$ should interact well with pullbacks, and that colimits should also be stable under pullback along $\mathcal{M}$-maps.

Given that we now have a definition of cocomplete $\mathcal{M}$-category, we can then use the equivalence between split restriction categories and $\mathcal{M}$-categories to give a definition of *cocomplete restriction category*. This implies that cocomplete restriction categories must be split, their subcategory of total maps (those with restriction being the identity maps) must be cocomplete, and the inclusion $\mathsf{Total}(\mathbb{X}) \hookrightarrow \mathbb{X}$ must preserve colimits. The result of all this is that every restriction category has a free cocompletion. However, there is a slight problem; this free cocompletion does not have the "nice" form that we would expect. This is where the notion of *restriction presheaf* comes in.

1.2.2 Restriction presheaves

A presheaf on an ordinary category $\mathbf{C}$ is nothing but a functor from $\mathbf{C}^{\mathrm{op}}$ to $\mathbf{Set}$. Likewise, a *restriction presheaf* on a restriction category $\mathbb{X}$ is an ordinary presheaf $P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$, but now equipped with a *restriction structure*. This restriction structure on the presheaf P assigns, for each object $A \in \mathbb{X}$ and element $x \in PA$, a restriction idempotent $\bar{x}\colon A \to A$ in $\mathbb{X}$, satisfying axioms which are almost identical to the restriction category axioms. One way of understanding the notion of restriction presheaf is through the idea of *collage* of a *profunctor*.

Recall that every presheaf P may be written as a profunctor, or a bifunctor $P\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{1} \to \mathbf{Set}$, where $\mathbf{1} = \{\star\}$ is the terminal category. The *collage* of P, denoted $\tilde{P}$, is the category whose object set is the disjoint union of $\mathsf{Ob}(\mathbf{C})$ with $\mathbf{1}$, and whose hom-sets are given as follows:

$$\begin{aligned}\tilde{P}(\star,\star) &= 1_\star;\\ \tilde{P}(A,B) &= \mathbf{C}(A,B);\\ \tilde{P}(A,\star) &= P(A,\star);\\ \tilde{P}(\star,A) &= \varnothing.\end{aligned}$$

Therefore, we may think of the restriction on a presheaf P, as a structure imposed upon P in such a way as to make its collage a restriction category.

Now taking maps between restriction presheaves as arbitrary natural transformations, we see that this makes the category of restriction presheaves, which we denote by $\mathsf{PSh}_r(\mathbb{X})$, a full subcategory of presheaves on $\mathbb{X}$. However, what is perhaps surprising is that $\mathsf{PSh}_r(\mathbb{X})$ may be given a restriction structure, making it a restriction category. In fact, as we will show, this restriction category of restriction presheaves is the free cocompletion of a small restriction category $\mathbb{X}$.

1.2.3 Join restriction presheaf

Closely related to the notion of restriction category, is that of *join restriction category*. As its name implies, the only additional structure imposed is that of join; that is given a *compatible family* of maps from the same hom-set, this family has a join which is compatible with restriction and composition. A basic example of a join restriction category is again, the category $\mathbf{Set}_p$ of sets and partial functions. In $\mathbf{Set}_p$, a parallel pair of partial functions f and g is compatible if they agree on their overlap, with their join being the partial function whose domain of definition is the union of the domains of definition of f and g, and which restricts back to f and g on these domains.

In the same way as we have defined restriction presheaf, we define a *join restriction presheaf* on a join restriction category $\mathbb{X}$ as an ordinary presheaf $P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ equipped with a join restriction structure. That is, for every object $A \in \mathbb{X}$ and set PA, we assign every element in PA to a restriction idempotent in $\mathbb{X}$ in such a way that not only does it satisfy the usual restriction presheaf axioms, but that every compatible family of elements in PA also has a join which is compatible with composition and restriction. Again, using the notion of collage which we have just discussed, we may think of the join restriction structure on a join restriction presheaf $P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$, as the structure necessarily imposed on P in order to make the collage of P into a join restriction category.

Now in the same way that we have defined a restriction category of restriction presheaves on a restriction category $\mathbb{X}$, we may define a join restriction category of join restriction presheaves on a join restriction category $\mathbb{X}$, which we shall denote as $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$. One of the aims in this thesis is then to show that the free cocompletion of any join restriction category $\mathbb{X}$, is indeed this category of join restriction presheaves, $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$.

The way to do so is by first recognising that if $\mathsf{Par}(\mathbf{C},\mathcal{M})$ is a partial map category with a join restriction structure, then its underlying category $\mathbf{C}$ may be a given a Grothendieck topology J. We call the corresponding $\mathcal{M}$-category *geometric*, since unions of $\mathcal{M}$-subobjects are computed via colimits which are stable under pullback by arbitrary maps. We then define the $\mathcal{M}$-category of sheaves on this site, and then show that this $\mathcal{M}$-category of sheaves is the free cocompletion of any geometric $\mathcal{M}$-category, similar to how the category of sheaves $\mathsf{Sh}(\mathbf{C})$ is the free cocompletion of any site $(\mathbf{C}, J)$. The final step is then to show that there is an equivalence between the category of join restriction presheaves on a join restriction category, and the partial map category of sheaves on the corresponding site. One way to think about this relation between join restriction presheaves and sheaves, is to think of a matching family (in the case of sheaf) as a compatible family of elements, and think of an amalgamation as the join of such a family of elements.

1.2.4 Restriction colimits

Our next goal is to develop a notion of restriction colimit. Our notion of restriction colimit will be a slight modification of the existing notion of weighted colimit, which is itself an extension of the notion of conical colimit. Recall that given a functor $F\colon \mathbf{C} \to \mathbf{D}$, the conical limit of F, if one exists, is a universal cocone under F. That is, it consists of an object $\operatorname{colim} F \in \mathbf{D}$, together with, for each $C \in \mathbf{C}$, a map $p_C\colon FC \to \operatorname{colim} F$ such that for all $f\colon C \to C' \in \mathbf{C}$, we have $p_{C'} \circ Ff = p_C$. Its universal property is then a statement that for all other cocones under F, say with vertex Z and maps $q_C\colon FC \to Z$, there exists a unique map $\alpha\colon \operatorname{colim} F \to Z$ such that $\alpha \circ p_C = q_C$ for all $C \in \mathbf{C}$.

The notion of weighted colimit extends the above notion of conical colimit by adding a weight functor $W\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. The difference here is that instead of a *single* coprojection map $p_C\colon FC \to \operatorname{colim} F$, we now have a *family* of maps for each $C \in \mathbf{C}$, where the family $\{p_{C,x}\colon FC \to \operatorname{colim} F\}_{x \in WC}$ is indexed over the set WC. These families of maps satisfy the condition that whenever $x = x' \cdot f$ (or $(Wf)(x') = x$), we have $p_{C,x} = p_{C',x'} \circ Ff$. Also, rather than writing $\operatorname{colim} F$ as in the case of conical colimits, we adopt the notation $\operatorname{colim}_W F$ to express that the colimit is weighted by the presheaf W.

For weighted restriction colimits, we consider restriction functors $F\colon \mathbb{X} \to \mathbb{Y}$, with weights given by restriction presheaves $P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$. Our definition of the restriction colimit of F weighted by P, denoted by $\operatorname{rcolim}_P F$, is then exactly the same as the definition in the case of ordinary weighted colimits, except we assign a restriction to each of the coprojection maps $p_{C,x}$, with the restriction dependent on the element $x \in WC$. We will develop applications of this theory to gluings of atlases in a join restriction category, as well as to restriction profunctors.

1.3 Chapter summaries

This thesis is comprised of four main distinct sections; the first being cocompletion of *restriction categories*, the second being cocompletion of *join restriction categories*, the third being a description of *restriction colimits* and the fourth being applications of restriction colimits.

In Chapter 2, we begin our discussions by recalling the notions of *restriction category* and $\mathcal{M}$-category. Importantly, we recall that the 2-categories $\mathbf{rCat}_s$ and $\mathcal{M}\mathbf{Cat}$ of split restriction categories and $\mathcal{M}$-categories respectively, are 2-equivalent. Indeed, this fact is what we use to give a free cocompletion of restriction categories, by first determining a free cocompletion of $\mathcal{M}$-categories. We therefore give a definition of cocomplete $\mathcal{M}$-category, and completely characterise them. We also give a definition of cocomplete restriction category. We conclude the chapter by characterising the presheaf category of $\mathcal{M}$-categories as the free $\mathcal{M}$-category cocompletion, and also give a free cocompletion of restriction categories via the 2-equivalence between $\mathbf{rCat}$ and $\mathcal{M}\mathbf{Cat}$. In Chapter 3, we extend the result from Chapter 2 to $\mathcal{M}$-categories which are may not be small, but locally small. The goal in this chapter was to present the $\mathcal{M}$-category of small presheaves as the free cocompletion of locally small $\mathcal{M}$-categories.

Chapter 4 is where the action really begins. We introduce the notion of restriction presheaf; essentially a presheaf on a restriction category equipped with axioms resembling those of restriction categories. As it turns out, there is a restriction category with objects the restriction presheaves, and the aim for the rest of the chapter is showing that this category of restriction presheaves provides the free cocompletion of restriction categories. This is analogous to the case with ordinary categories; that the presheaf category $\mathsf{PSh}(\mathbf{C})$ on $\mathbf{C}$ is the free cocompletion of the category $\mathbf{C}$. We end this chapter with a discussion on the notion of *small restriction presheaf*.

In Chapters 5 and 6, we move on from restriction categories to the notion of *join restriction category*. In Chapter 5, we recap the definition of a join restriction category, and introduce the notion of *geometric $\mathcal{M}$-category*, which are $\mathcal{M}$-categories whose partial map categories are join restriction categories. We again completely characterise these geometric $\mathcal{M}$-categories. Following this, we briefly recap the notion of Grothendieck topology and sheaves on a site; we recall that for any site $(\mathbf{C}, J)$, there is a category $\mathsf{Sh}(\mathbf{C}, J)$ whose objects are sheaves on this site, and that the inclusion of sheaves $\mathsf{Sh}(\mathbf{C}, J)$ into the presheaf category $\mathsf{PSh}(\mathbf{C})$ has a left adjoint, called the *associated sheaf functor*.

The reason for this minor diversion is to show that every geometric $\mathcal{M}$-category may be given a corresponding Grothendieck topology, which is in fact *subcanonical*; that is, every representable presheaf is a sheaf. We then define an $\mathcal{M}$-category of sheaves on this site, and proceed to show that this $\mathcal{M}$-category of sheaves is indeed the free cocompletion of any geometric $\mathcal{M}$-category. We end this chapter by using these results to give a free cocompletion of join restriction categories.

In Chapter 6, we introduce the notion of join restriction presheaf on a join restriction category. In an entirely analogous manner to our approach with the restriction category of restriction presheaves, we define the *join restriction category of join restriction presheaves*. The rest of the chapter is then devoted towards showing that for any join restriction category $\mathbb{X}$, this join restriction category of join restriction presheaves $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ is its free cocompletion.

In Chapter 7, we present the notion of restriction colimit weighted by a restriction presheaf. The main result in this chapter shows that restriction categories are cocomplete if and only if they admit all weighted restriction colimits; again completely analogous to the case with ordinary categories. We also see that the notion of restriction coproduct given by [Cockett & Lack, 2007] is a specific instance of restriction colimit.

In Chapter 8, we give an application of restriction colimit to the notion of atlas in a restriction category. In particular, we show that gluings of atlases in a (join) restriction category are another kind of restriction colimit. We also provide a characterisation of gluings of atlases in a join restriction category, and describe explicitly the gluing of any atlas in the join restriction presheaf category.

Finally, in Chapter 9, we revisit the notion of *restriction profunctor* as described by [DeWolf, 2017]. However, the approach we take is via the notions of restriction colimit and restriction presheaf, which were introduced in earlier chapters. We also make a final remark that not every notion pertaining to ordinary categories has a corresponding notion in the restriction setting. For example, we argue that there does not appear to be a good notion of restriction coend. Another fact which may come as a surprise to category theorists, is that the left extension of restriction functors along the Yoneda embedding has no right adjoint.

2

Cocompletion of restriction categories

In this chapter, we introduce some basic theory on restriction categories and on the closely related notion of $\mathcal{M}$-category; that is, a category $\mathbf{C}$ together with a pullback-stable, composition-closed class of monics $\mathcal{M}$. Category theorists are familiar with the fact that given a category $\mathbf{C}$, its presheaf category $\mathsf{PSh}(\mathbf{C}) = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is the free cocompletion of $\mathbf{C}$. One of the aims of this chapter is to give an analogous notion of free cocompletion in the restriction setting. To do this, we give a notion of cocomplete $\mathcal{M}$-category, and for any $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, we study its free cocompletion. Then, making use of the fact that restriction categories and $\mathcal{M}$-categories are equivalent, we give a corresponding notion of cocomplete restriction category, and give the free cocompletion of any restriction category. Along the way, we completely characterise those $\mathcal{M}$-categories which are cocomplete; this characterisation will prove useful in a later chapter where we consider locally small restriction categories and their free cocompletion. Note that everything in this chapter is material from [Lin, 2015], except for the discussions on $\mathcal{M}$-subobjects and on the characterisation of cocomplete $\mathcal{M}$-categories. Explicitly, Lemma 2.19, Example 2.21, Proposition 2.22 and Remark 2.23 form new material.

2.1 Restriction categories and $\mathcal{M}$-categories

We begin with a definition of restriction category. Unless otherwise stated, everything in this section is from [Cockett & Lack, 2002].

Definition 2.1. A *restriction category* is a category $\mathbb{X}$ together with, for each pair of objects $A, B \in \mathbb{X}$, a function $\mathbb{X}(A, B) \to \mathbb{X}(A, A)$ sending f to $\bar{f}$, with $\bar{f}$ satisfying the following conditions:

(R1) $f\bar{f} = f$;

(R2) $\bar{f}\bar{g} = \bar{g}\bar{f}$;

(R3) $\overline{f\bar{g}} = \bar{f}\bar{g}$; and

(R4) $\bar{f}g = g\overline{fg}$,

for suitably composable maps f and g. The functions sending each f to $\bar{f}$ are collectively known as the *restriction structure* on $\mathbb{X}$, and we call $\bar{f}$ the *restriction* of f.

The restriction of any map f satisfies the following properties, the proofs of which may be found in [Cockett & Lack, 2002].

Proposition 2.2. *Let $\mathbb{X}$ be a restriction category. Then for suitably composable maps f and g, we have:*

1. *$\bar{f}$ is idempotent;*
2. *$\bar{f} = \bar{\bar{f}}$;*
3. *$\bar{f} = 1$ if f is a monomorphism;*
4. *$f\overline{gf} = \overline{gf}$;*
5. *$\overline{\bar{g}f} = \overline{gf}$.*

By the fact $\bar{f}$ is idempotent and $\bar{f} = \bar{\bar{f}}$, it makes sense to call $f \in \mathbb{X}$ a restriction idempotent if and only if $f = \bar{f}$. Also, when $\bar{f} = 1$, we say that f is *total* since in $\mathbf{Set}_p$, the maps whose restriction are the identities are precisely the fully defined or total functions. Now the composite of two total maps f and g is also total, as $\overline{gf} = \overline{\bar{g}f} = \bar{f} = 1$, and since identities are always total, the total maps of any restriction category $\mathbb{X}$ form a subcategory, which we denote by $\mathsf{Total}(\mathbb{X})$.

In a restriction category, if a map f is such that there exists a g where $gf = \bar{f}$ and $fg = \bar{g}$, then we call f a *partial isomorphism*. In this instance, we also say that g is the *partial inverse* of f. Restriction categories have one other interesting property: if $\mathbb{X}$ is a restriction category and $A, B \in \mathbb{X}$ are objects of $\mathbb{X}$, then the hom-set $\mathbb{X}(A, B)$ may be given a partial order, with $f \leq g$ if and only if $f = g\bar{f}$.

Analogous with ordinary categories, there is a notion of functor and also of natural transformation within the restriction setting.

Definition 2.3. Given restriction categories $\mathbb{X}$ and $\mathbb{Y}$, a functor $F\colon \mathbb{X} \to \mathbb{Y}$ is called a *restriction functor* if $F(\bar{f}) = \overline{F(f)}$ for all maps $f \in \mathbb{X}$. If F and G are restriction functors, then a natural transformation $\alpha\colon F \Rightarrow G$ is a *restriction transformation* if the components of α are total.

The following are some examples of restriction categories.

Example 2.4. Consider the category of sets and partial functions, $\mathbf{Set}_p$. We can give $\mathbf{Set}_p$ the following restriction structure by defining the restriction of each partial function $f\colon A \rightharpoonup B$ to be:

$$\bar{f}(a) = \begin{cases} a & \text{if } f(a) \text{ is defined;} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

In other words, for each $f\colon A \to B$ in $\mathbf{Set}_p$, the restriction of f is the partial identity function on A. For the rest of this thesis, $\mathbf{Set}_p$ will denote the restriction category with the above restriction structure.

Example 2.5. The category of topological spaces and partial continuous functions, denoted $\mathbf{Top}_p$, is again a restriction category. Recall that a partial continuous function $f\colon X \rightharpoonup Y$ between two topological spaces is a continuous function which is defined on some open subset $A \subseteq X$. The restriction $\bar{f}$ on each map $f \in \mathbf{Top}_p$ is defined in exactly the same way as for $\mathbf{Set}_p$.

Example 2.6. The category $\mathbf{Pos}_\perp$ of partially ordered sets with a bottom element, and maps which preserve these bottom elements is yet another example of a restriction category. Here, the restriction $\bar{f}\colon A \to A$ on each $f\colon A \to B$ is defined to be

$$\bar{f}(a) = \begin{cases} a & \text{if } f(a) \neq \perp; \\ \perp & \text{otherwise.} \end{cases}$$

Example 2.7. Let $\mathbb{N}$ denote the monoid (considered as a one-object category) whose composition is given by $n \circ m = \max(n, m)$ for each pair $n, m \in \mathbb{N}$, and whose identity is given by 0. Then one restriction structure on $\mathbb{N}$ is given by

$$\bar{n} = \begin{cases} n & \text{if } n = 0 \text{ or n is odd;} \\ n-1 & \text{otherwise.} \end{cases}$$

Example 2.8. For any restriction category $\mathbb{X}$ and object $A \in \mathbb{X}$, the slice category $\mathbb{X}/A$ is also a restriction category, with the restriction structure defined to be the same as for $\mathbb{X}$.

Example 2.9. Any category $\mathbf{C}$ may be given the *trivial restriction structure*, by declaring each $\bar{f}$ to be the identity map. We shall denote the restriction category with the trivial restriction structure by $\mathsf{Triv}(\mathbf{C})$. We therefore deduce that it is possible for categories to be given more than one restriction structure.

It will not be difficult for the reader to verify that there is a 2-category **rCat** whose objects are restriction categories, whose 1-cells are restriction functors, and whose 2-cells are restriction transformations. This 2-category **rCat** has an important sub-2-category called $\mathbf{rCat}_s$, the objects of which are restriction categories whose restriction idempotents split. A restriction idempotent $\bar{f}$ is *split* if there exist maps m and r such that $mr = \bar{f}$ and $rm = 1$; we call such maps m, the *restriction monics*.

From any restriction category $\mathbb{X}$, we may construct a split restriction category called $\mathsf{K}_r(\mathbb{X})$. This split restriction category $\mathsf{K}_r(\mathbb{X})$ is the value at $\mathbb{X}$ of the left biadjoint K_r to the inclusion $\mathbf{rCat}_s \hookrightarrow \mathbf{rCat}$, and its data is given by the following:

Objects: Pairs (A, e), where A is an object of $\mathbb{X}$ and $e\colon A \to A$ is a restriction idempotent on A;

Morphisms: Morphisms $f\colon (A, e) \to (A', e')$ are morphisms $f\colon A \to A'$ in $\mathbb{X}$ satisfying the condition $e' f e = f$;

Restriction: Restriction on f is given by $\bar{f}$.

The above construction for $\mathsf{K}_r(\mathbb{X})$ is Freyd's splitting of idempotents [Freyd, 1964]. The unit at $\mathbb{X}$ of this biadjunction, $J\colon \mathbb{X} \to \mathsf{K}_r(\mathbb{X})$, takes an object A to $(A, 1_A)$ and a map $f\colon A \to A'$ to $f\colon (A, 1_A) \to (A', 1_{A'})$ in $\mathsf{K}_r(\mathbb{X})$; in fact J is an embedding.

Example 2.10. A particularly important example of a split restriction category is the category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ of partial maps, where $\mathbf{C}$ is any ordinary category, and $\mathcal{M}$ is a *stable system of monics* in $\mathbf{C}$. A stable system of monics $\mathcal{M}$ in $\mathbf{C}$ is defined to be a class of monomorphisms in $\mathbf{C}$, which:

1. contains all of the isomorphisms in $\mathbf{C}$;

2. is closed under composition; and such that

3. the pullback of any map $m \in \mathcal{M}$ along any map in $\mathbf{C}$ exists, and is also in $\mathcal{M}$. These pullbacks are sometimes called $\mathcal{M}$-pullbacks.

The objects of $\mathsf{Par}(\mathbf{C},\mathcal{M})$ are the same as the objects of $\mathbf{C}$. A map from A to B in $\mathsf{Par}(\mathbf{C},\mathcal{M})$ is a span $X \xleftarrow{m} Z \xrightarrow{f} Y$ (with $m \in \mathcal{M}$ and f arbitrary) identified up to an equivalence relation, where $(m, f) \sim (n, g)$ if and only if there exists an isomorphism φ such that $m\varphi = n$ and $f\varphi = g$. Composition in this category is by pullback, and the identity is given by $(1, 1)$. The restriction of any map (m, f) in $\mathsf{Par}(\mathbf{C},\mathcal{M})$ is defined to be (m,m), and the splitting of each restriction idempotent (m,m) is given by $(m, 1)$ and $(1,m)$.

Many restriction categories are of the form $\mathsf{Par}(\mathbf{C},\mathcal{M})$. In fact, the restriction categories $\mathbf{Set}_p$ and $\mathbf{Top}_p$ from Examples 2.4 and 2.5 are of the form $\mathsf{Par}(\mathbf{C},\mathcal{M})$, with $\mathbf{Set}_p \cong \mathsf{Par}(\mathbf{Set},\text{all monics})$ and $\mathbf{Top}_p \cong \mathsf{Par}(\mathbf{Top},\text{all open injections})$.

Definition 2.11. An $\mathcal{M}$-category is a category $\mathbf{C}$ together with a stable system of monics $\mathcal{M} \subseteq \mathbf{C}$, as defined in Example 2.10. We denote $\mathcal{M}$-categories as a pair $(\mathbf{C},\mathcal{M})$.

Example 2.12. The pairs $(\mathbf{Set},\text{all monics})$ and $(\mathbf{Top},\text{open injections})$ from Example 2.10 are $\mathcal{M}$-categories.

To avoid confusion, where there is more than one $\mathcal{M}$-category being discussed, we will denote the stable system of monics by the calligraphic font $\mathcal{C}$ of the same letter as the category. For example, we shall write $(\mathbf{C},\mathcal{C})$ instead of $(\mathbf{C},\mathcal{M})$ when we have to refer to another $\mathcal{M}$-category, say $(\mathbf{D},\mathcal{D})$, as in the case below.

Definition 2.13. If $(\mathbf{C},\mathcal{C})$ and $(\mathbf{D},\mathcal{D})$ are $\mathcal{M}$-categories, a functor F between them is called an *$\mathcal{M}$-functor* if $m \in \mathcal{C}$ implies $Fm \in \mathcal{D}$, and if F preserves pullbacks of monics in $\mathcal{C}$. If $F, G\colon (\mathbf{C},\mathcal{C}) \to (\mathbf{D},\mathcal{D})$ are $\mathcal{M}$-functors, a natural transformation between them is called *$\mathcal{M}$-cartesian* if the naturality square is a pullback for all $m \in \mathcal{C}$. We denote by $\mathcal{M}\mathbf{Cat}$ the 2-category of $\mathcal{M}$-categories (objects), $\mathcal{M}$-functors (1-cells) and $\mathcal{M}$-cartesian natural transformations (2-cells).

With these definitions in place, we can now make the assignation $(\mathbf{C},\mathcal{M}) \mapsto \mathsf{Par}(\mathbf{C},\mathcal{M})$ 2-functorial.

Proposition 2.14. *There is a* 2*-functor* $\mathsf{Par}\colon \mathcal{M}\mathbf{Cat} \to \mathbf{rCat}_s$ *taking $\mathcal{M}$-categories* $(\mathbf{C},\mathcal{M})$ *to split restriction categories* $\mathsf{Par}(\mathbf{C},\mathcal{M})$.

Proof. We will provide only the necessary data. The interested reader is referred to [Cockett & Lack, 2002] for the full proof.

Given an $\mathcal{M}$-functor $F\colon (\mathbf{C},\mathcal{C}) \to (\mathbf{D},\mathcal{D})$, define $\mathsf{Par}(F)$ on objects $A \in \mathsf{Par}(\mathbf{C},\mathcal{C})$ to be FA, and define $\mathsf{Par}(F)$ on maps $(m, f) \in \mathsf{Par}(\mathbf{C},\mathcal{C})$ to be (Fm, Ff). If $\alpha\colon F \Rightarrow G$ is $\mathcal{M}$-cartesian, define $\mathsf{Par}(\alpha)$ componentwise at $A \in \mathsf{Par}(\mathbf{C},\mathcal{C})$ to be the total map $\mathsf{Par}(\alpha)_A = (1_{FA}, \alpha_A)$. □

Theorem 2.15. *The* 2*-functor* $\mathsf{Par}\colon \mathcal{M}\mathbf{Cat} \to \mathbf{rCat}_s$ *is an equivalence of* 2*-categories.*

Proof. Our first step is to establish a 2-functor $\mathcal{M}\mathsf{Total}$ from $\mathbf{rCat}_s$ to $\mathcal{M}\mathbf{Cat}$. On objects, define $\mathcal{M}\mathsf{Total}$ to take a split restriction category $\mathbb{X}$ to the pair $(\mathsf{Total}(\mathbb{X}),\mathcal{M}_{\mathbb{X}})$, where $\mathcal{M}_{\mathbb{X}}$ are the restriction monics in $\mathbb{X}$. It turns out that $(\mathsf{Total}(\mathbb{X}),\mathcal{M}_{\mathbb{X}})$ is an $\mathcal{M}$-category [Cockett & Lack, 2002, Proposition 3.3]. If $F\colon \mathbb{X} \to \mathbb{Y}$ is a restriction functor between split restriction categories, define $\mathcal{M}\mathsf{Total}(F)$ to be the functor $\mathsf{Total}(F)\colon \mathsf{Total}(\mathbb{X}) \to \mathsf{Total}(\mathbb{Y})$, and for all restriction transformations $\alpha\colon F \Rightarrow G$, define $\mathcal{M}\mathsf{Total}(\alpha)$ to be the natural transformation $\mathsf{Total}(\alpha)\colon \mathsf{Total}(F) \Rightarrow \mathsf{Total}(G)$. [Cockett & Lack, 2002, Proposition 3.3] then says $\mathcal{M}\mathsf{Total}(F)$ is an $\mathcal{M}$-functor, and that $\mathcal{M}\mathsf{Total}(\alpha)$ is $\mathcal{M}_{\mathbb{X}}$-cartesian.

To see that $\mathcal{M}\mathsf{Total}$ and Par are part of an equivalence of 2-categories, let us define, for each split restriction category $\mathbb{X}$, $\Phi_{\mathbb{X}}\colon \mathbb{X} \to \mathsf{Par}(\mathcal{M}\mathsf{Total}(\mathbb{X}), \mathcal{M}_{\mathbb{X}})$, which is the identity on objects, and takes each map $f \in \mathbb{X}$ to the span (m, fm), where m is the restriction monic from the splitting of $\bar{f}$. It turns out that $\Phi_{\mathbb{X}}$ is not only functorial, but is a restriction functor, and is also invertible [Cockett & Lack, 2002]. From this, we deduce a 2-natural isomorphism $\Phi\colon 1 \Rightarrow \mathsf{Par} \circ \mathcal{M}\mathsf{Total}$. The other required isomorphism $\mathcal{M}\mathsf{Total} \circ \mathsf{Par} \Rightarrow 1$ may be easily defined and verified by the reader. $\square$

2.2 Cocomplete $\mathcal{M}$-categories

For any small category $\mathbf{C}$, the category of presheaves $\mathsf{PSh}(\mathbf{C})$ is the *free cocompletion* of $\mathbf{C}$. That is, for any small-cocomplete category $\mathcal{E}$, the following is an equivalence of categories:

$$(-) \circ \mathbf{y}\colon \mathbf{Cocomp}(\mathsf{PSh}(\mathbf{C}), \mathcal{E}) \to \mathbf{Cat}(\mathbf{C}, \mathcal{E})$$

where $\mathbf{y}$ is the Yoneda embedding, $\mathbf{Cat}$ is the 2-category of small categories and $\mathbf{Cocomp}$ is the 2-category of small-cocomplete categories and cocontinuous functors. (For the rest of this paper, we shall take *cocomplete* to mean *small-cocomplete*, and *colimits* to mean *small colimits* unless otherwise indicated). However, it is not immediately clear that there is an analogous notion of cocompletion for small restriction categories $\mathbb{X}$. Nonetheless, a clue is given to us in light of the 2-equivalence between $\mathcal{M}\mathbf{Cat}$ and $\mathbf{rCat}_s$. That is, it might be helpful to first define a notion of cocomplete $\mathcal{M}$-category, and study the free cocompletion of small $\mathcal{M}$-categories.

In this section, we define the $\mathcal{M}$-category of presheaves $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$ for any small $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$ and give a definition of cocomplete $\mathcal{M}$-category and cocontinuous $\mathcal{M}$-functor. (As it turns out, this $\mathcal{M}$-category of presheaves, $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$, will be the free cocompletion of any small $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$). Then using the 2-equivalence between $\mathcal{M}\mathbf{Cat}$ and $\mathbf{rCat}_s$, we define cocomplete restriction category and cocontinuous restriction functor. This in turn provides a candidate for free restriction cocompletion, namely the split restriction category $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))))$ described by [Cockett & Lack, 2002].

For any small $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, there are various ways of constructing an $\mathcal{M}$-category of presheaves on $\mathbf{C}$. One way is the following, and we denote the $\mathcal{M}$-category arising in this way by $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M}) = (\mathsf{PSh}(\mathbf{C}), \mathsf{PSh}(\mathcal{M}))$. We say a map $\mu\colon P \to Q$ is in $\mathsf{PSh}(\mathcal{M})$ if for all $\gamma\colon \mathbf{y}D \to Q$, there is an $m \in \mathcal{M}$ making the following a pullback square:

$$\begin{tikzcd}
\mathbf{y}C \arrow[r] \arrow[d, "\mathbf{y}m"'] & P \arrow[d, "\mu"] \\
\mathbf{y}D \arrow[r, "\gamma"'] & Q
\end{tikzcd}$$

where $\mathbf{y}\colon \mathbf{C} \to \mathsf{PSh}(\mathbf{C})$ is the usual Yoneda embedding. In fact, the maps in $\mathsf{PSh}(\mathcal{M})$ are those monics classified by the generic subobject induced by the stable system of monics in $\mathcal{M}$ [Rosolini, 1986, Proposition 3.1.1]. Importantly, note that as the Yoneda embedding $\mathbf{y}\colon \mathbf{C} \to \mathsf{PSh}(\mathbf{C})$ preserves all small limits, the Yoneda embedding extends to an $\mathcal{M}$-functor $\mathbf{y}\colon (\mathbf{C}, \mathcal{M}) \to (\mathsf{PSh}(\mathbf{C}), \mathsf{PSh}(\mathcal{M}))$ between $\mathcal{M}$-categories.

Now, it is well known that for any small category $\mathbf{C}$, the Yoneda embedding $\mathbf{y}\colon \mathbf{C} \to \mathsf{PSh}(\mathbf{C})$ exhibits $\mathsf{PSh}(\mathbf{C})$ as the free cocompletion of $\mathbf{C}$. Therefore it is natural to ask whether for any small $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, the Yoneda embedding $\mathbf{y}\colon (\mathbf{C}, \mathcal{M}) \to \mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$ likewise exhibits $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$ as the free cocompletion of $(\mathbf{C}, \mathcal{M})$. First we need to give a definition of cocomplete $\mathcal{M}$-category and cocontinuous $\mathcal{M}$-functor.

Definition 2.16. An $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$ is said to be *cocomplete* if $\mathbf{C}$ is itself cocomplete and its inclusion into $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ preserves colimits. An $\mathcal{M}$-functor $F\colon (\mathbf{C}, \mathcal{C}) \to (\mathbf{D}, \mathcal{D})$ between $\mathcal{M}$-categories is cocontinuous if the underlying functor $F\colon \mathbf{C} \to \mathbf{D}$ is cocontinuous. We denote by $\mathcal{M}\mathbf{Cocomp}$ the 2-category of cocomplete $\mathcal{M}$-categories, cocontinuous $\mathcal{M}$-functors and $\mathcal{M}$-cartesian natural transformations.

Example 2.17. The $\mathcal{M}$-category ($\mathbf{Set}$, all monics) is cocomplete since $\mathbf{Set}$ is cocomplete and $\mathbf{Set} \hookrightarrow \mathsf{Par}(\mathbf{Set}, \text{all monics}) = \mathbf{Set}_p$ has a right adjoint.

As a matter of fact, there are whole classes of examples of cocomplete $\mathcal{M}$-categories. Before we give their construction, it will be helpful to define what we mean by an $\mathcal{M}$-subobject.

Definition 2.18. Let $(\mathbf{C}, \mathcal{M})$ be an $\mathcal{M}$-category and D an object in $\mathbf{C}$. Then an $\mathcal{M}$-subobject of D is an isomorphism class of maps in $\mathcal{M}$ with codomain D. That is, if $m\colon C \to D$ and $m'\colon C' \to D$ are both in $\mathcal{M}$, then m and m' represent the same subobject of D if there exists an isomorphism $\varphi\colon C \to C'$ such that $m = m'\varphi$. We shall use the notation $\mathsf{Sub}_{\mathcal{M}}(D)$ to denote the set of $\mathcal{M}$-subobjects of D in the $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$.

It will be useful to observe the following lemma in relation to $\mathcal{M}$-subobjects of representables in the $\mathcal{M}$-category $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$.

Lemma 2.19. *Let $(\mathbf{C}, \mathcal{M})$ be an $\mathcal{M}$-category. Then there exists an isomorphism as follows:*

$$\mathsf{Sub}_{\mathsf{PSh}(\mathcal{M})}(\mathbf{y}C) \cong \mathsf{Sub}_{\mathcal{M}}(C).$$

Proof. See [Rosolini, 1986, Proposition 3.1.1]. □

Remark 2.20. Now consider an $\mathcal{M}$-category $(\mathcal{E}, \mathcal{M})$, where $\mathcal{M}$ is a stable system of monics and $\mathcal{E}$ is a cocomplete category with a terminal object 1 and a generic $\mathcal{M}$-subobject $\tau\colon 1 \to \Sigma$. By a generic $\mathcal{M}$-subobject (or an $\mathcal{M}$-subobject classifier), we mean an object $\Sigma \in \mathcal{E}$ and a map $\tau\colon 1 \to \Sigma$ in $\mathcal{M}$ such that for any map $m\colon A \to B$ in $\mathcal{M}$, there exists a unique map $\tilde{m}\colon B \to \Sigma$ making the following square a pullback:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "m"'] & 1 \arrow[d, "\tau"] \\ B \arrow[r, "\tilde{m}"'] & \Sigma. \end{tikzcd}$$

Suppose the induced pullback functor $\tau^*\colon \mathcal{E}/\Sigma \to \mathcal{E}$ has a right adjoint Π_τ. Then by an analogous argument in topos theory [Johnstone, 2002, Proposition 2.4.7], $\mathcal{E}$ has a *partial map classifier* for every object $C \in \mathcal{E}$, and this in turn implies that the inclusion $\mathcal{E} \hookrightarrow \mathsf{Par}(\mathcal{E}, \mathcal{M})$ has a right adjoint [Cockett & Lack, 2003, p. 65], and so $\mathcal{M}$-categories of this kind are cocomplete. In fact, the partial map category $\mathsf{Par}(\mathcal{E}, \mathcal{M})$ is equivalent to the Kleisli category of the monad induced by the adjunction above [Mulry, 1994, Lemma 2.10].

Example 2.21. (1) Let $\mathcal{E}$ be any cocomplete elementary topos and $\mathcal{M}$ all monics in $\mathcal{E}$. Then $(\mathcal{E}, \mathcal{M})$ is a cocomplete $\mathcal{M}$-category since $\mathcal{E}$ is locally cartesian closed and has a generic subobject.

(2) If $\mathcal{E}$ is any cocomplete quasitopos and $\mathcal{M}$ are all the regular monics in $\mathcal{E}$, then $(\mathcal{E}, \mathcal{M})$ is also a cocomplete $\mathcal{M}$-category as $\mathcal{E}$ is locally cartesian closed and has an object which classifies all the regular monics in $\mathcal{E}$.

(3) For any small $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, the $\mathcal{M}$-category $\mathsf{PSh}(\mathcal{M})$ is a cocomplete $\mathcal{M}$-category [Rosolini, 1986, Proposition 3.1.1].

We now give a complete characterisation of cocomplete $\mathcal{M}$-categories, via the following proposition.

Proposition 2.22. *Suppose* $(\mathbf{C}, \mathcal{M})$ *is an* $\mathcal{M}$*-category, and the underlying category* $\mathbf{C}$ *is cocomplete. Then the following statements are equivalent:*

(1) The inclusion $\mathbf{C} \hookrightarrow \mathsf{Par}(\mathbf{C}, \mathcal{M})$ *preserves colimits;*

(2) The following conditions hold:

(a) If $\{m_i \colon A_i \to B_i\}_{i \in \mathcal{I}}$ *is a family of maps in* $\mathcal{M}$ *indexed by a small set* $\mathcal{I}$*, then their coproduct* $\sum_{i \in \mathcal{I}} m_i$ *is in* $\mathcal{M}$ *and the following squares are pullbacks for every* $i \in \mathcal{I}$*:*

$$\begin{tikzcd} A_i \arrow[r, "\iota_{A_i}"] \arrow[d, "m_i"'] & \sum_{i \in \mathcal{I}} A_i \arrow[d, "\sum_{i \in \mathcal{I}} m_i"] \\ B_i \arrow[r, "\iota_{B_i}"'] & \sum_{i \in \mathcal{I}} B_i. \end{tikzcd}$$

(b) Suppose $m \in \mathcal{M}$ *and the pullback of* m *along maps* $f, g \in \mathbf{C}$ *can be taken to be the same map* h*. If* f', g' *are the pullbacks of* f, g *along* m*, and* c, c' *are the coequalisers of* f, g *and* f', g' *respectively, then the unique* n *making the right square commute is in* $\mathcal{M}$ *and also makes the right square a pullback:*

$$\begin{tikzcd} \bullet \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] \arrow[d, "h"'] & \bullet \arrow[r, "c'"] \arrow[d, "m"] & \bullet \arrow[d, "n"] \\ \bullet \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \bullet \arrow[r, "c"'] & \bullet \end{tikzcd}$$

(c) Colimits are stable under pullback along $\mathcal{M}$*-maps.*

Proof. For the proof of (1) $\implies$ (2), we will be using Lemma 2.27 and Corollary 2.29 (both to be proven later).

(1) $\implies$ (2a) Let $\mathcal{I}$ be a small set considered as a discrete category, and let $H, K \colon \mathcal{I} \to \mathbf{C}$ be functors taking objects $i \in \mathcal{I}$ to A_i and B_i respectively. Let $\alpha \colon H \Rightarrow K$ be a natural transformation whose component at i is given by $m_i \colon A_i \to B_i$, and observe that all naturality squares are trivially pullbacks. Then by Lemma 2.27, the sum $\sum_{i \in \mathcal{I}} m_i$ is in $\mathcal{M}$ and for every $i \in \mathcal{I}$, the coproduct coprojection squares are pullbacks.

(1) $\implies$ (2b) Take $\mathbf{I}$ to be the category with two objects and a pair of parallel maps between them and apply Lemma 2.27.

(1) $\implies$ (2c) See Corollary 2.29.

(2) $\implies$ (1) Conversely, to show that the inclusion $\mathbf{C} \hookrightarrow \mathsf{Par}(\mathbf{C}, \mathcal{M})$ is cocontinuous, it is enough to show that it preserves all small coproducts and coequalisers.

So suppose c is a coequaliser of f and g in $\mathbf{C}$. To show the inclusion preserves this coequaliser, we need to show that for any map (m, k) such that $(m, k)(1, f) = (m, k)(1, g)$, there is a unique map (n, q) making the following diagram commute:

$$\begin{tikzcd}
\bullet \arrow[r, shift left, "(1{,}f)"] \arrow[r, shift right, "(1{,}g)"'] & \bullet \arrow[r, "(1{,}c)"] \arrow[dr, "(m{,}k)"'] & \bullet \arrow[d, dashed, "(n{,}q)"] \\
& & \bullet
\end{tikzcd}$$

Now the condition $(m, k)(1, f) = (m, k)(1, g)$ is precisely the condition that the pullbacks of m along f and g are the same map h,

$$\begin{tikzcd}
\bullet \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] \arrow[d, "h"'] & \bullet \arrow[d, "m"] \\
\bullet \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \bullet
\end{tikzcd}$$

and that $kf' = kg'$. Taking c' to be the coequaliser of f' and g', our assumption then implies there is a unique map $n \in \mathcal{M}$ making the following diagram a pullback:

$$\begin{tikzcd}
\bullet \arrow[r, "c'"] \arrow[d, "m"'] & \bullet \arrow[d, "n"] \\
\bullet \arrow[r, "c"'] & \bullet
\end{tikzcd}$$

Since c' is the coequaliser of f' and g' and $kf' = kg'$, there exists a unique map q such that $qc' = k$. This gives a map $(n, q) \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$ such that $(n, q)(1, c) = (m, k)$. To see it must be unique, suppose (n', q') also satisfies the condition $(n', q')(1, c) = (m, k)$. By assumption, as colimits are stable under pullback along $\mathcal{M}$-maps, the pullback of c along n' must be a coequaliser of f' and g', say c''.

$$\begin{tikzcd}
\bullet \arrow[rr, "c''"] \arrow[dr, "c'"'] \arrow[dd, "m"'] & & \bullet \arrow[dd, "n'"] \\
& \bullet \arrow[ur, "\varphi"'] \arrow[dr, "n"'] & \\
\bullet \arrow[rr, "c"'] & & \bullet
\end{tikzcd}$$

Now as coequalisers are unique up to isomorphism, there is an isomoprhism φ such that $c'' = \varphi c'$. But the fact

$$n'\varphi c' = n'c'' = cm = nc'$$

implies $n'\varphi = n$ as c' is an epimorphism. In other words, n and n' must be the same $\mathcal{M}$-subobject. Similarly, $q = q'\varphi$, which means $(n, q) = (n', q')$.

Next, suppose $\sum_{i \in \mathcal{I}} B_i$ is a small coproduct in $\mathbf{C}$, with coproduct coprojections $(\iota_{B_i} : B_i \to \sum_{i \in \mathcal{I}} B_i)_{i \in \mathcal{I}}$. Then $\sum_{i \in \mathcal{I}} B_i$ will be a small coproduct in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ if for any object

$D \in \mathsf{Par}(\mathbf{C},\mathcal{M})$ and family of maps $\big((m_i, f_i)\colon B_i \to D\big)_{i\in\mathcal{I}}$, there exists a unique map $(\mu,\gamma)\colon \sum_{i\in\mathcal{I}} B_i \to D$ making the following diagram commute for every $i \in \mathcal{I}$:

$$\begin{tikzcd} B_i \arrow[r, "{(1,\iota_{B_i})}"] \arrow[dr, "{(m_i,f_i)}"'] & \sum_{i\in\mathcal{I}} B_i \arrow[d, "{(\mu,\gamma)}"] \\ & D. \end{tikzcd}$$

By assumption, $\sum_{i\in\mathcal{I}} m_i$ is in $\mathcal{M}$, and so the map $(\sum_{i\in\mathcal{I}} m_i, f)\colon \sum_{i\in\mathcal{I}} B_i \to D$ is well-defined, where f is the unique map $\sum_{i\in\mathcal{I}} \mathsf{dom}(f_i) \to D$ induced by the universal property of the coproduct coprojections and the family of maps $\{f_i\}_{i\in\mathcal{I}}$. Since the coproduct coprojection squares are pullbacks, taking $\mu = \sum_{i\in\mathcal{I}} m_i$ and $\gamma = f$ certainly makes the above diagram commute, and the uniqueness of (μ,γ) follows by an analogous argument to the case of coequalisers by the stability of colimits under pullback. Therefore, if $\sum_{i\in\mathcal{I}} B_i$ is a small coproduct in $\mathbf{C}$, it must also be a small coproduct in $\mathsf{Par}(\mathbf{C},\mathcal{M})$.

Hence, as the inclusion $\mathbf{C} \hookrightarrow \mathsf{Par}(\mathbf{C},\mathcal{M})$ preserves all small coproducts and all coequalisers, it preserves all small colimits. □

Remark 2.23. There is yet another formulation for the condition that the inclusion $\mathbf{C} \hookrightarrow \mathsf{Par}(\mathbf{C},\mathcal{M})$ preserves all small colimits. That is, the inclusion is cocontinuous if and only if the presheaf $\mathsf{Sub}_{\mathcal{M}}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, which on objects takes C to the set of $\mathcal{M}$-subobjects of C, is continuous, and moreover, colimits are stable under pullback along maps in $\mathcal{M}$. The proof of this result is similar to the proof of Lemma 2.22.

Also, by conditions (2a) and (2c), observe that cocomplete $\mathcal{M}$-categories must be $\mathcal{M}$-extensive, meaning that for every $i \in \mathcal{I}$ (with $\mathcal{I}$ a small set), if the following square commutes with the bottom row being coproduct injections and $m, m_i \in \mathcal{M}$ (for all $i \in \mathcal{I}$), then the top row must be a coproduct diagram if and only if each square is a pullback:

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "m_i"'] & Z \arrow[d, "m"] \\ B_i \arrow[r, "\iota_{B_i}"'] & \sum_{i\in\mathcal{I}} B_i. \end{tikzcd}$$

In light of the previous proposition, we give an example of an $\mathcal{M}$-category which is not cocomplete.

Example 2.24. Consider the $\mathcal{M}$-category $(\mathbf{Ab}, \text{all monics})$ of small abelian groups and all monomorphisms in $\mathbf{Ab}$. Denote the trivial group by 0 and the group of integers by $\mathbb{Z}$. The coproduct of $\mathbb{Z}$ with itself is just the direct sum $\mathbb{Z}\oplus\mathbb{Z}$, along with coprojections $\iota_1\colon \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$ and $\iota_2\colon \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$ sending n to $(n,0)$ and $(0,n)$ respectively. Let $\Delta\colon \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$ denote the diagonal map, which is clearly a monomorphism. Now a pullback of Δ along ι_1 is the unique map $0 \to \mathbb{Z}$, and similarly for ι_2. This gives the following diagram, where both squares are pullbacks:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d, "\Delta"] & 0 \arrow[l] \arrow[d] \\ \mathbb{Z} \arrow[r, "\iota_1"'] & \mathbb{Z}\oplus\mathbb{Z} & \mathbb{Z}. \arrow[l, "\iota_2"] \end{tikzcd}$$

However, the top row is certainly not a coproduct diagram in $\mathbf{Ab}$. Therefore, the $\mathcal{M}$-category $(\mathbf{Ab}, \text{all monics})$ is not $\mathcal{M}$-extensive, and hence by Proposition 2.22, is not cocomplete as an $\mathcal{M}$-category.

2.3 Free cocompletion of $\mathcal{M}$-categories

Our goal in this section will be to show that for any small $\mathcal{M}$-category $(\mathbf{C}, \mathcal{C})$ and cocomplete $\mathcal{M}$-category $(\mathbf{D}, \mathcal{D})$, the following is an equivalence of categories:

$$(-) \circ \mathbf{y} \colon \mathcal{M}\mathbf{Cocomp}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{C}), (\mathbf{D}, \mathcal{D})) \to \mathcal{M}\mathbf{Cat}((\mathbf{C}, \mathcal{C}), (\mathbf{D}, \mathcal{D})).$$

This will require the next four lemmas.

Lemma 2.25. *Let $(\mathbf{C}, \mathcal{M})$ be an $\mathcal{M}$-category and let $m \in \mathcal{M}$. Then the following is a pullback square*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "n"'] & B \arrow[d, "m"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

if and only if the following diagram commutes in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$:

$$\begin{tikzcd} C \arrow[r, "{(1,f)}"] \arrow[d, "{(n,1)}"'] & D \arrow[d, "{(m,1)}"] \\ A \arrow[r, "{(1,g)}"'] & B. \end{tikzcd}$$

Proof. Diagram chase. □

Lemma 2.26. *Let $\mathbb{X}$ be a restriction category, $\mathbf{I}$ any small category and $L \colon \mathbf{I} \to \mathbb{X}$ a functor. Suppose* $\operatorname{colim} L$ *exists and its colimiting coprojections $(p_I \colon LI \to \operatorname{colim} L)_{I \in \mathbf{I}}$ are total. If $\varepsilon \colon L \Rightarrow L$ is a natural transformation such that each component is a restriction idempotent, then* $\operatorname{colim} \varepsilon$ *is also a restriction idempotent:*

$$\begin{tikzcd} LI \arrow[r, "p_I"] \arrow[d, "\varepsilon_I"'] & \operatorname{colim} L \arrow[d, "\operatorname{colim}\varepsilon"] \\ LI \arrow[r, "p_I"'] & \operatorname{colim} L. \end{tikzcd}$$

Proof. By the fact $\overline{p_I} = 1$ and $\varepsilon_I = \overline{\varepsilon_I}$, we have

$$\overline{\operatorname{colim}\varepsilon} \circ p_I = p_I \circ \overline{\operatorname{colim}\varepsilon \circ p_I} = p_I \circ \overline{p_I \circ \varepsilon_I} = p_I \circ \overline{\overline{p_I} \circ \varepsilon_I} = p_I \circ \overline{\varepsilon_I} = p_I \circ \varepsilon_I.$$

Therefore, $\operatorname{colim}\varepsilon = \overline{\operatorname{colim}\varepsilon}$ by uniqueness. □

Lemma 2.27. *Let $(\mathbf{C}, \mathcal{M})$ be a cocomplete $\mathcal{M}$-category, and let $H, K \colon \mathbf{I} \to \mathbf{C}$ be functors (with $\mathbf{I}$ small). Suppose $\alpha \colon H \Rightarrow K$ is a natural transformation such that for each $I \in \mathbf{I}$, α_I is in $\mathcal{M}$ and all naturality squares are pullbacks:*

$$\begin{tikzcd} HI \arrow[r, "Hf"] \arrow[d, "\alpha_I"'] & HJ \arrow[d, "\alpha_J"] \\ KI \arrow[r, "Kf"'] & KJ. \end{tikzcd}$$

Then $\operatorname{colim}\alpha$ *is in* $\mathcal{M}$*, and the following is a pullback for every* $I \in \mathbf{I}$*:*

$$\begin{tikzcd} HI \arrow[r, "p_I"] \arrow[d, "\alpha_I"'] & \operatorname{colim} H \arrow[d, "\operatorname{colim}\alpha"] \\ KI \arrow[r, "q_I"'] & \operatorname{colim} K \end{tikzcd}$$

where p_I, q_I *are colimit coprojections.*

Proof. Applying the inclusion $\iota\colon \mathbf{C} \to \mathsf{Par}(\mathbf{C}, \mathcal{M})$ gives the following commutative diagram for each $I \in \mathbf{I}$:

$$\begin{tikzcd} HI \arrow[r, "(1{,}p_I)"] \arrow[d, "(1{,}\alpha_I)"'] & \operatorname{colim} H \arrow[d, "(1{,}\operatorname{colim}\alpha)"] \\ KI \arrow[r, "(1{,}q_I)"'] & \operatorname{colim} K. \end{tikzcd}$$

Observe that there is a natural transformation $\beta\colon \iota K \Rightarrow \iota H$ whose components are given by $\beta_I = (\alpha_I, 1)$; simply apply Lemma 2.25 to our assumption that α_I is a pullback of α_J along Kf.

Now the fact that the inclusion preserves the colimits $(\operatorname{colim} H, p_I)_{i\in\mathbf{I}}$ and $(\operatorname{colim} K, q_I)_{i\in\mathbf{I}}$ implies the existence of a unique map $\operatorname{colim}\beta = (n, g)\colon \operatorname{colim} K \to \operatorname{colim} H$ making the following diagram commute for each $I \in \mathbf{I}$:

$$\begin{tikzcd} KI \arrow[r, "(1{,}q_I)"] \arrow[d, "(\alpha_I{,}1)"'] & \operatorname{colim} K \arrow[d, "(n{,}g)"] \\ HI \arrow[r, "(1{,}p_I)"] \arrow[d, "(1{,}\alpha_I)"'] & \operatorname{colim} H \arrow[d, "(1{,}\operatorname{colim}\alpha)"] \\ KI \arrow[r, "(1{,}q_I)"'] & \operatorname{colim} K. \end{tikzcd}$$

Observe that the left composite $(1, \alpha_I) \circ (\alpha_I, 1) = (\alpha_I, \alpha_I)$ is the component at I of a natural transformation $\varepsilon\colon \iota K \Rightarrow \iota K$ whose components are restriction idempotents. Therefore, by Lemma 2.26, the composite on the right $(1, \operatorname{colim}\alpha) \circ (n, g) = (n, (\operatorname{colim}\alpha)g)$ must be a restriction idempotent, and so $n = (\operatorname{colim}\alpha)g$.

On the other hand, the composite $(\alpha_I, 1) \circ (1, \alpha_I) = (1, 1)$ is the component of the identity natural transformation $\gamma\colon \iota H \Rightarrow \iota H$ at I, and so $\operatorname{colim}\gamma\colon \operatorname{colim} H \to \operatorname{colim} H$ must be $(1, 1)$. However, as the following diagram also commutes, we must have $(n, g) \circ (1, \operatorname{colim}\alpha) = (1, 1)$ by uniqueness:

$$\begin{tikzcd} HI \arrow[r, "(1{,}p_I)"] \arrow[d, "(1{,}\alpha_I)"'] & \operatorname{colim} H \arrow[d, "(1{,}\operatorname{colim}\alpha)"] \\ KI \arrow[r, "(1{,}q_I)"] \arrow[d, "(\alpha_I{,}1)"'] & \operatorname{colim} K \arrow[d, "(n{,}g)"] \\ HI \arrow[r, "(1{,}p_I)"'] & \operatorname{colim} H. \end{tikzcd}$$

So $(1, \operatorname{colim}\alpha) \circ (n, g) = (n, n)$ is a splitting of the restriction idempotent (n, n), which means that $(1, \operatorname{colim}\alpha)$ is a restriction monic. Therefore $\operatorname{colim}\alpha \in \mathcal{M}$, proving the first part of the lemma.

Regarding the second part of the lemma, observe that $(n,g) \circ (1, \operatorname{colim}\alpha) = (1,1)$ implies g is an isomorphism (as $n = (\operatorname{colim}\alpha)$). Therefore, $(n,g) = (\operatorname{colim}\alpha, 1)$ and so the following diagram commutes for all $I \in \mathbf{I}$:

$$\begin{tikzcd} KI \arrow[r, "{(1,q_I)}"] \arrow[d, "{(\alpha_I,1)}"'] & \operatorname{colim} K \arrow[d, "{(\operatorname{colim}\alpha,1)}"] \\ HI \arrow[r, "{(1,p_I)}"'] & \operatorname{colim} H. \end{tikzcd}$$

The result then follows by applying Lemma 2.25. □

Lemma 2.28. *Let $(\mathbf{C}, \mathcal{M})$ be a cocomplete $\mathcal{M}$-category, $H, K \colon \mathbf{I} \to \mathbf{C}$ functors (with $\mathbf{I}$ small), and $\alpha \colon H \Rightarrow K$ a natural transformation such that each $\alpha_I \in \mathcal{M}$ and all naturality squares are pullbacks (as in the previous lemma). Let $n \in \mathcal{M}$, and suppose $x \colon \operatorname{colim} H \to X$ and $y \colon \operatorname{colim} K \to Y$ make the right square commute and the outer square a pullback (for all $I \in \mathbf{I}$):*

$$\begin{tikzcd} HI \arrow[r, "p_I"] \arrow[d, "\alpha_I"'] & \operatorname{colim} H \arrow[r, "x"] \arrow[d, "\operatorname{colim}\alpha"] & X \arrow[d, "n"] \\ KI \arrow[r, "q_I"'] & \operatorname{colim} K \arrow[r, "y"'] & Y. \end{tikzcd}$$

Then the right square is also a pullback.

Proof. By Lemma 2.25, to show that the right square is a pullback is the same as showing $(1,x) \circ (\operatorname{colim}\alpha, 1) = (\operatorname{colim}\alpha, x) = (n,1) \circ (1,y)$ in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$. In other words, that the top-right square of the following diagram commutes:

$$\begin{tikzcd} KI \arrow[r, "{(1,q_I)}"] \arrow[d, "{(\alpha_I,1)}"'] & \operatorname{colim} K \arrow[r, "{(1,y)}"] \arrow[d, "{(\operatorname{colim}\alpha,1)}"] & Y \arrow[d, "{(n,1)}"] \\ HI \arrow[r, "{(1,p_I)}"'] \arrow[d, "{(1,\alpha_I)}"'] & \operatorname{colim} H \arrow[r, "{(1,x)}"'] \arrow[d, "{(1,\operatorname{colim}\alpha)}"] & X \arrow[d, "{(1,n)}"] \\ KI \arrow[r, "{(1,q_I)}"'] & \operatorname{colim} K \arrow[r, "{(1,y)}"'] & Y. \end{tikzcd}$$

Since $(\operatorname{colim}\alpha, x)$ and $(n,1)(1,y)$ are both maps out of $\operatorname{colim} K$, it is enough to show that

$$(\operatorname{colim}\alpha, x)(1, q_I) = (n,1)(1,y)(1,q_I)$$

for all $I \in \mathbf{I}$. But the left-hand side is equal to (α_I, xp_I) by commutativity of the top-left square, and the right-hand side is also (α_I, xp_I) by assumption. Hence the result follows. □

Corollary 2.29. *If $(\mathbf{C}, \mathcal{M})$ is a cocomplete $\mathcal{M}$-category, then colimits in $\mathbf{C}$ are stable under pullback along $\mathcal{M}$-maps.*

Proof. Let $K \colon \mathbf{I} \to \mathbf{C}$ be a functor, P any object in $\mathbf{C}$, and suppose $\mu \colon P \to \operatorname{colim} K$ is a $\mathcal{M}$-map. Since $\mu \in \mathcal{M}$, for each $I \in \mathbf{I}$, we may take pullbacks of μ along the colimiting coprojections of $\operatorname{colim} K$, $(k_I \colon K_I \to \operatorname{colim} K)_{I \in \mathbf{I}}$, and these we call $\alpha_I \colon HI \to KI$. This gives a functor $H\colon \mathbf{I} \to \mathbf{C}$, which on objects, takes I to HI, and on morphisms, takes $f\colon I \to J$ to the unique map making all squares in the following diagram pullbacks:

$$\begin{tikzcd}
HI \arrow[r, "Hf"'] \arrow[d, "\alpha_I"'] \arrow[rr, bend left, "p_I"] & HJ \arrow[r, "p_J"'] \arrow[d, "\alpha_J"] & P \arrow[d, "\mu"] \\
KI \arrow[r, "Kf"] \arrow[rr, bend right, "k_I"'] & KJ \arrow[r, "k_J"] & \operatorname{colim} K.
\end{tikzcd}$$

By construction, $(P, p_I)_{I\in\mathbf{I}}$ is a cocone in $\mathbf{C}$ and $\alpha\colon H \to K$ is a natural transformation. Now let $(h_I\colon HI \to \operatorname{colim} H)_{I\in\mathbf{I}}$ be the colimiting coprojections of $\operatorname{colim} H$. Then by the universal property of $\operatorname{colim} H$, there exists a unique $\gamma\colon \operatorname{colim} H \to P$ such that $p_I = \gamma h_I$ for all $I \in \mathbf{I}$, and by the universal property of $\operatorname{colim} K$, there is a $\operatorname{colim}\alpha\colon \operatorname{colim} H \to \operatorname{colim} K$ making the left square of the following diagram commute (for all $I \in \mathbf{I}$):

$$\begin{tikzcd}
HI \arrow[r, "h_I"'] \arrow[d, "\alpha_I"'] \arrow[rr, bend left, "p_I"] & \operatorname{colim} H \arrow[r, "\gamma"'] \arrow[d, "\operatorname{colim}\alpha"] & P \arrow[d, "\mu"] \\
KI \arrow[r] \arrow[rr, bend right, "k_I"'] & \operatorname{colim} K \arrow[r, equal] & \operatorname{colim} K.
\end{tikzcd}$$

It is easy to see that the right square commutes, and since the left square is a pullback for every $I \in \mathbf{I}$, the right square must be a pullback by Lemma 2.28. Therefore, because the pullback of the identity $1_{\operatorname{colim} K}$ is the identity, $P \cong \operatorname{colim} H$, and hence colimits are preserved by pullbacks along $\mathcal{M}$-maps. □

We now show that for any small $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, the Yoneda embedding $\mathbf{y}\colon (\mathbf{C}, \mathcal{M}) \to \mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$ exhibits the $\mathcal{M}$-category of presheaves $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$ as the free cocompletion of $(\mathbf{C}, \mathcal{M})$.

Theorem 2.30. *For any small $\mathcal{M}$-category $(\mathbf{C}, \mathcal{C})$ and cocomplete $\mathcal{M}$-category $(\mathbf{D}, \mathcal{D})$, the following is an equivalence of categories:*

$$(-) \circ \mathbf{y}\colon \mathcal{M}\mathbf{Cocomp}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{C}), (\mathbf{D}, \mathcal{D})) \to \mathcal{M}\mathbf{Cat}((\mathbf{C}, \mathcal{C}), (\mathbf{D}, \mathcal{D})). \tag{2.1}$$

Proof. We know that $(-) \circ \mathbf{y}\colon \mathbf{Cocomp}(\mathsf{PSh}(\mathbf{C}), \mathbf{D}) \to \mathbf{Cat}(\mathbf{C}, \mathbf{D})$ is an equivalence of categories; therefore, given a functor $F\colon \mathbf{C} \to \mathbf{D}$, there is a cocontinuous $G\colon \mathsf{PSh}(\mathbf{C}) \to \mathbf{D}$ such that $G\mathbf{y} \cong F$. So the functor in (2.1) will be essentially surjective on objects if this same G is an $\mathcal{M}$-functor.

To see that G takes monics in $\mathsf{PSh}(\mathcal{C})$ to monics in $\mathcal{D}$, let $\mu\colon P \to Q$ be a map in $\mathsf{PSh}(\mathcal{C})$. Since every presheaf is a colimit of representables, write $Q \cong \operatorname{colim} \mathbf{y}D$, where $D\colon \mathbf{I} \to \mathbf{C}$ is a functor (with $\mathbf{I}$ small). By definition of $\mu \in \mathsf{PSh}(\mathcal{C})$, for every $I \in \mathbf{I}$, there is a map $m_I\colon C_I \to D_I$ making the following a pullback:

$$\begin{tikzcd}
\mathbf{y}C_I \arrow[r, "p_I"] \arrow[d, "\mathbf{y}m_I"'] & P \arrow[d, "\mu"] \\
\mathbf{y}D_I \arrow[r, "q_I"'] & Q
\end{tikzcd}$$

(where q_I is a colimit coprojection). It follows there is a functor $C\colon \mathbf{I} \to \mathbf{C}$ which on objects takes I to C_I and on morphisms, takes $f\colon I \to J$ to the unique map Cf making the diagram below commute and the left square a pullback:

$$\begin{tikzcd} \mathbf{y}C_I \arrow[r, dashed, "\mathbf{y}Cf"'] \arrow[d, "\mathbf{y}m_I"'] \arrow[rr, bend left, "p_I"] & \mathbf{y}C_J \arrow[r, "p_J"'] \arrow[d, "\mathbf{y}m_J"] & P \arrow[d, "\mu"] \\ \mathbf{y}D_I \arrow[r, "\mathbf{y}Df"] \arrow[rr, bend right, "q_I"'] & \mathbf{y}D_J \arrow[r, "q_J"] & Q. \end{tikzcd} \tag{2.2}$$

The fact colimits in $\mathsf{PSh}(\mathbf{C})$ are stable under pullback implies $(p_I\colon \mathbf{y}C_I \to P)_{I\in\mathbf{I}}$ is colimiting. Now applying G to the above diagram gives

$$\begin{tikzcd} G\mathbf{y}C_I \arrow[r, dashed, "G\mathbf{y}Cf"'] \arrow[d, "G\mathbf{y}m_I"'] \arrow[rr, bend left, "Gp_I"] & G\mathbf{y}C_J \arrow[r, "Gp_J"'] \arrow[d, "G\mathbf{y}m_J"] & GP \arrow[d, "G\mu"] \\ G\mathbf{y}D_I \arrow[r, "G\mathbf{y}Df"] \arrow[rr, bend right, "Gq_I"'] & G\mathbf{y}D_J \arrow[r, "Gq_J"] & GQ. \end{tikzcd} \tag{2.3}$$

Since G is cocontinuous, both $(Gp_I)_{I\in\mathbf{I}}$ and $(Gq_I)_{I\in\mathbf{I}}$ are colimiting. Also, as $G\mathbf{y} \cong F$ and F is an $\mathcal{M}$-functor, the left square is a pullback for every pair $I, J \in \mathbf{I}$. Therefore, by Lemma 2.27, $G\mu$ must be in $\mathcal{D}$.

Observe that the same lemma (Lemma 2.27) says that for every $I \in \mathbf{I}$, the outer square in (2.3) is a pullback for every $I \in \mathbf{I}$. In other words, G preserves pullbacks of the form

$$\begin{tikzcd} \mathbf{y}C_I \arrow[r, "p_I"] \arrow[d, "\mathbf{y}m_I"'] & P \arrow[d, "\mu"] \\ \mathbf{y}D_I \arrow[r, "q_I"'] & Q. \end{tikzcd} \tag{2.4}$$

Now to see that G preserves $\mathsf{PSh}(\mathcal{C})$-pullbacks, consider the diagram below, where the right square is an $\mathsf{PSh}(\mathcal{C})$-pullback and the left square is a pullback for all $I \in \mathbf{I}$:

$$\begin{tikzcd} \mathbf{y}C_I \arrow[r, "p_I"] \arrow[d, "\mathbf{y}m_I"'] & P \cong \operatorname{colim} \mathbf{y}C \arrow[r] \arrow[d, "\mu"'] & P' \arrow[d, "\mu'"] \\ \mathbf{y}D_I \arrow[r, "q_I"'] & Q \cong \operatorname{colim} \mathbf{y}D \arrow[r] & Q'. \end{tikzcd}$$

The result then follows by applying G to the diagram and using Lemma 2.28. This proves the functor in (2.1) is essentially surjective on objects.

Finally, to show that the functor in (2.1) is fully faithful, we need to show for any cocontinuous pair of $\mathcal{M}$-functors $F, F'\colon \mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{C}) \to (\mathbf{D}, \mathcal{D})$ and $\mathcal{C}$-cartesian $\alpha\colon F\mathbf{y} \to F'\mathbf{y}$, there exists a unique $\mathsf{PSh}(\mathcal{C})$-cartesian $\tilde{\alpha}\colon F \to F'$ such that $\tilde{\alpha}\mathbf{y} = \alpha$. In other words, the following is an isomorphism of sets:

$$(-) \circ \mathbf{y}\colon \mathcal{M}\mathsf{Nat}(F, F') \to \mathcal{M}\mathsf{Nat}(F\mathbf{y}, F'\mathbf{y})$$

with $\mathcal{M}\mathsf{Nat}(F, F')$ being the $\mathsf{PSh}(\mathcal{C})$-cartesian natural transformations from F to F'. However, this condition may be reformulated as follows:

$$\text{For all natural transformations } \tilde{\alpha}\colon F \to F',\ \tilde{\alpha} \text{ is } \mathsf{PSh}(\mathcal{C})\text{-cartesian if } \tilde{\alpha}\mathbf{y}\colon F\mathbf{y} \Rightarrow F'\mathbf{y} \text{ is } \mathcal{C}\text{-cartesian.} \tag{2.5}$$

To see that these two statements are equivalent, observe that the second statement amounts to the following diagram being a pullback in **Set**:

$$\begin{tikzcd}
\mathcal{M}\mathsf{Nat}(F, F') \arrow[r] \arrow[d, hook] & \mathcal{M}\mathsf{Nat}(F\mathbf{y}, F'\mathbf{y}) \arrow[d, hook] \\
\mathsf{Nat}(F, F') \arrow[r, "(-)\circ\mathbf{y}"'] & \mathsf{Nat}(F\mathbf{y}, F'\mathbf{y})
\end{tikzcd}$$

where $\mathsf{Nat}(F, F')$ is the set of natural transformations from F to F'. However, as the bottom function is an isomorphism (ordinary free cocompletion), the top must also be an isomorphism and hence the two statements are equivalent. Therefore, we show the functor in (2.1) is fully faithful by proving (2.5).

So let $\mu\colon P \to Q$ be an $\mathsf{PSh}(\mathcal{C})$-map, and recall that the left square (diagram below) is a pullback for every $I \in \mathbf{I}$ as F preserves $\mathsf{PSh}(\mathcal{C})$-pullbacks:

$$\begin{tikzcd}
F\mathbf{y}C_I \arrow[r, "Fp_I"] \arrow[d, "F\mathbf{y}m_I"'] & FP \arrow[r, "\tilde{\alpha}_P"] \arrow[d, "F\mu"'] & F'P \arrow[d, "F'\mu"] \\
F\mathbf{y}D_I \arrow[r, "Fq_I"'] & FQ \arrow[r, "\tilde{\alpha}_Q"'] & F'Q.
\end{tikzcd} \tag{2.6}$$

To show that the right square is a pullback, we will show that the outer square is a pullback for every $I \in \mathbf{I}$ and apply Lemma 2.28. Now by naturality of $\tilde{\alpha}$, this outer square is the outer square of the following diagram:

$$\begin{tikzcd}
F\mathbf{y}C_I \arrow[r, "\tilde{\alpha}_{\mathbf{y}C_I}"] \arrow[d, "F\mathbf{y}m_I"'] & F'\mathbf{y}C_I \arrow[r, "F'p_I"] \arrow[d, "F'\mathbf{y}m_I"'] & F'P \arrow[d, "F'\mu"] \\
F\mathbf{y}D_I \arrow[r, "\tilde{\alpha}_{\mathbf{y}D_I}"'] & F'\mathbf{y}D_I \arrow[r, "F'q_I"'] & F'Q.
\end{tikzcd}$$

But $\tilde{\alpha} \circ \mathbf{y}$ being $\mathcal{C}$-cartesian implies the left square is a pullback, and the right square is also a pullback by the fact F' preserves pullbacks of the form (2.4). Therefore, by Lemma 2.28, each square on the right of (2.6) is a pullback, and so $\tilde{\alpha}$ is $\mathsf{PSh}(\mathcal{C})$-cartesian. Hence,

$$(-) \circ \mathbf{y}\colon \mathcal{M}\mathbf{Cocomp}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{C}), (\mathbf{D}, \mathcal{D})) \to \mathcal{M}\mathbf{Cat}((\mathbf{C}, \mathcal{C}), (\mathbf{D}, \mathcal{D}))$$

is an equivalence of categories. □

2.4 Cocompletion of restriction categories

Earlier, we explored the notion of cocomplete $\mathcal{M}$-category. Now, by the fact $\mathcal{M}\mathbf{Cat}$ and $\mathbf{rCat}_s$ are 2-equivalent, it makes sense to define a restriction category to be cocomplete in such a way that $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ will be cocomplete as a restriction category if and only if $(\mathbf{C}, \mathcal{M})$ is cocomplete as an $\mathcal{M}$-category.

Definition 2.31. A restriction category $\mathbb{X}$ is cocomplete if it is split, its subcategory $\mathsf{Total}(\mathbb{X})$ is cocomplete, and the inclusion $\mathsf{Total}(\mathbb{X}) \hookrightarrow \mathbb{X}$ preserves colimits. A restriction functor $F \colon \mathbb{X} \to \mathbb{Y}$ is cocontinuous if $\mathsf{Total}(F) \colon \mathsf{Total}(\mathbb{X}) \to \mathsf{Total}(\mathbb{Y})$ is cocontinuous. We denote by **rCocomp** the 2-category of cocomplete restriction categories, cocontinuous restriction functors and restriction transformations.

As we said earlier, we would like $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ to be cocomplete as a restriction category if and only if $(\mathbf{C}, \mathcal{M})$ is cocomplete as an $\mathcal{M}$-category, and as $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is always split, it makes sense to impose this as a condition of being cocomplete. Another reason why a cocomplete restriction category $\mathbb{X}$ ought to be split is because ordinary cocomplete categories have splittings of all idempotents, and so it makes sense for $\mathbb{X}$ to have splittings of all restriction idempotents. Observe that for any cocomplete restriction category $\mathbb{X}$, $\mathcal{M}\mathsf{Total}(\mathbb{X})$ is a cocomplete $\mathcal{M}$-category since $\mathsf{Total}(\mathbb{X})$ is cocomplete and $\mathsf{Total}(\mathbb{X}) \hookrightarrow \mathbb{X} \cong \mathsf{Par}(\mathcal{M}\mathsf{Total}(\mathbb{X}))$ preserves colimits. We now give examples of cocomplete restriction categories.

Example 2.32. For each class of examples from Example 2.21, $\mathsf{Par}(\mathbb{E}, \mathcal{M})$ is a cocomplete restriction category. In particular, the restriction category of sets and partial functions $\mathbf{Set}_p$ is a cocomplete restriction category since $\mathbf{Set}_p \cong \mathsf{Par}(\mathbf{Set}, \text{all monics})$.

Also note that since the $\mathcal{M}$-category $(\mathbf{Ab}, \text{all monics})$ of abelian groups and group monomorphisms is not cocomplete as an $\mathcal{M}$-category, $\mathsf{Par}(\mathbf{Ab}, \text{all monics})$ is also not a cocomplete restriction category.

We know that for any small $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$ is a cocomplete $\mathcal{M}$-category, and furthermore, $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M}))$ is a cococomplete restriction category. In particular, the split restriction category $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))))$ is a cocomplete restriction category for any small restriction category $\mathbb{X}$. We now show that the Cockett and Lack embedding below [Cockett & Lack, 2002, p. 252]

$$\Lambda \colon \mathbb{X} \xrightarrow{J} \mathsf{K}_r(\mathbb{X}) \xrightarrow{\Phi_{\mathsf{K}_r(\mathbb{X})}} \mathsf{Par}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))) \xrightarrow{\mathsf{Par}(\mathbf{y})} \mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))) \tag{2.7}$$

exhibits this split restriction category $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))))$ as the free restriction cocompletion of any small restriction category $\mathbb{X}$. (Recall Φ is the 2-natural isomorphism from Theorem 2.15).

Theorem 2.33. *For any small restriction category $\mathbb{X}$ and cocomplete restriction category $\mathbb{E}$, the following is an equivalence of categories:*

$$(-) \circ \Lambda \colon \mathbf{rCocomp}(\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))), \mathbb{E}) \to \mathbf{rCat}(\mathbb{X}, \mathbb{E})$$

where Λ is the Cockett and Lack embedding introduced in (2.7).

Proof. First note that $\mathbb{E} \cong \mathsf{Par}(\mathbf{D}, \mathcal{D})$ for some cocomplete $\mathcal{M}$-category $(\mathbf{D}, \mathcal{D})$ (as $\mathbb{E}$ is split), and that

$$\mathbf{rCocomp}(\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{C})), \mathsf{Par}(\mathbf{D}, \mathcal{D})) \simeq \mathcal{M}\mathbf{Cocomp}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{C}), (\mathbf{D}, \mathcal{D}))$$

since Par and $\mathcal{M}\mathsf{Total}$ are 2-equivalences. Therefore,

$$(-) \circ \mathsf{Par}(\mathbf{y}) \colon \mathbf{rCocomp}(\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{C})), \mathbb{E}) \to \mathbf{rCat}(\mathsf{Par}(\mathbf{C}, \mathcal{C}), \mathbb{E})$$

is an equivalence since

$$(-) \circ \mathbf{y} \colon \mathcal{M}\mathbf{Cocomp}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{C}), (\mathbf{D}, \mathcal{D})) \to \mathcal{M}\mathbf{Cat}((\mathbf{C}, \mathcal{C}), (\mathbf{D}, \mathcal{D})$$

is an equivalence (free cocompletion of $\mathcal{M}$-categories). Therefore the following composite is an equivalence:

$$\begin{tikzcd}
\mathbf{rCocomp}(\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))), \mathbb{E}) \arrow[d, "(-)\circ\mathsf{Par}(\mathbf{y})"] \\
\mathbf{rCocomp}(\mathsf{Par}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))), \mathbb{E}) \arrow[d, "(-)\circ\Phi_{\mathsf{K}_r(\mathbb{X})}\circ J"] \\
\mathbf{rCat}(\mathbb{X}, \mathbb{E})
\end{tikzcd}$$

as $\Phi_{\mathsf{K}_r(\mathbb{X})}$ is an isomorphism and J is the unit of the biadjunction $\mathsf{K}_r \dashv i$ at $\mathbb{X}$. $\square$

3

Free cocompletion of locally small restriction categories

So far in our discussions, we have considered the free cocompletion of a small $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$ and of a small restriction category $\mathbb{X}$, given by $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$ and $\mathsf{PSh}_r(\mathbb{X})$ respectively. We now turn our attention to cases where our categories may not necessarily be small, but locally small. When $\mathbf{C}$ is an ordinary locally small category, we understand the full subcategory $\mathcal{P}(\mathbf{C}) \subset \mathsf{PSh}(\mathbf{C})$ of small presheaves to be its free cocompletion [Day & Lack, 2007]. (A presheaf on $\mathbf{C}$ is called *small* if it can be written as a small colimit of representables [Day & Lack, 2007]). In an entirely analogous way, we would like to define, for each locally small $\mathcal{M}$-category, an $\mathcal{M}$-category of small presheaves which will be its free cocompletion, and then extend this result to locally small restriction categories. To begin, we define what we mean by a locally small $\mathcal{M}$-category.

Definition 3.1. An $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$ is called *locally small* if $\mathbf{C}$ is locally small and $\mathcal{M}$-well-powered. That is, for any object $C \in \mathbf{C}$, the $\mathcal{M}$-subobjects of C form a small partially ordered set.

Remark 3.2. Note that this definition is exactly what is required for $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ to be locally small when $\mathbf{C}$ is a locally small $\mathcal{M}$-category, as noted by [Robinson & Rosolini, 1988, p. 99].

By analogy with the case of locally small categories, we define for any locally small $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, the $\mathcal{M}$-category of small presheaves $\mathcal{P}_{\mathcal{M}}(\mathbf{C}, \mathcal{M}) = (\mathcal{P}(\mathbf{C}), \mathcal{P}(\mathcal{M}))$, where $\mathcal{P}(\mathcal{M})$ is defined in exactly the same way as for $\mathsf{PSh}(\mathcal{M})$. We begin by showing that $\mathcal{P}(\mathcal{M})$ is a stable system of monics.

Lemma 3.3. *Let $(\mathbf{C}, \mathcal{M})$ be a locally small $\mathcal{M}$-category, and let $\mu\colon P \to Q$ be a map in $\mathcal{P}(\mathcal{M})$. If $\gamma\colon Q' \to Q$ is a map in $\mathcal{P}(\mathbf{C})$, then the pullback of μ along γ calculated in $\mathsf{PSh}(\mathbf{C})$ is in $\mathcal{P}(\mathcal{M})$:*

$$\begin{tikzcd} P' \arrow[r] \arrow[d, "\mu'"'] & P \arrow[d, "\mu"] \\ Q' \arrow[r, "\gamma"'] & Q. \end{tikzcd}$$

Proof. Certainly μ' exists and is in $\mathsf{PSh}(\mathcal{M})$ by the fact $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C},\mathcal{M})$ is an $\mathcal{M}$-category. So all we need to show is that P' is a small presheaf. Since Q' is small, we may rewrite $Q' \cong \operatorname{colim} \mathbf{y}D$ for some functor $D\colon \mathbf{I} \to \mathbf{C}$ with $\mathbf{I}$ small, and denote the colimiting coprojections as $q_I\colon \mathbf{y}D_I \to Q'$. Now μ is a map in $\mathcal{P}(\mathcal{M})$, which means that for each $I \in \mathbf{I}$ and composite $\gamma \circ q_I$, there exists an $m_I\colon C_I \to D_I$ making the outer square a pullback:

$$\begin{tikzcd}
\mathbf{y}C_I \arrow[r, "p_I"] \arrow[d, "\mathbf{y}m_I"'] & P' \arrow[r] \arrow[d, "\mu'"'] & P \arrow[d, "\mu"] \\
\mathbf{y}D_I \arrow[r, "q_I"'] & Q' \arrow[r, "\gamma"'] & Q.
\end{tikzcd}$$

By the same argument as in the proof of Theorem 2.30, it follows that there is a functor $C\colon \mathbf{I} \to \mathbf{C}$ which on objects, takes I to C_I, and that there is a unique map $p_I\colon \mathbf{y}C_I \to P'$ making the left square a pullback for every $I \in \mathbf{I}$. However, because colimits are stable under pullback in $\mathsf{PSh}(\mathbf{C})$, this means $(p_I\colon \mathbf{y}C_I \to P')_{I\in\mathbf{I}}$ is colimiting, which ensures that P' is a small presheaf. □

Remark 3.4. Note that the previous result implies that $\mathcal{P}(\mathbf{C})$ admits pullbacks along $\mathcal{P}(\mathcal{M})$-maps, and that these are computed pointwise.

Having now shown that $\mathcal{P}(\mathcal{M})$ is a stable system of monics, and hence $\mathcal{P}_{\mathcal{M}}(\mathbf{C},\mathcal{M})$ is an $\mathcal{M}$-category, we claim that $\mathcal{P}_{\mathcal{M}}(\mathbf{C},\mathcal{M})$ is indeed the free cocompletion of $\mathbf{C}$. To do so however, it will first require showing that $\mathcal{P}_{\mathcal{M}}(\mathbf{C},\mathcal{M})$ is both locally small and cocomplete.

Lemma 3.5. *If* $(\mathbf{C},\mathcal{M})$ *is a locally small* $\mathcal{M}$*-category, then* $\mathcal{P}_{\mathcal{M}}(\mathbf{C},\mathcal{M})$ *is locally small.*

Proof. Since $\mathcal{P}(\mathbf{C})$ is a locally small category [Day & Lack, 2007], all we need to do is show that $\mathcal{P}_{\mathcal{M}}(\mathbf{C},\mathcal{M})$ is $\mathcal{M}$-well-powered. So let Q be a small presheaf, and rewrite $Q \cong \operatorname{colim} \mathbf{y}D$, where $D\colon \mathbf{I} \to \mathbf{C}$ is a functor with $\mathbf{I}$ small. Again denote the colimiting coprojections by $(q_I\colon \mathbf{y}D_I \to Q)_{I\in\mathbf{I}}$.

As before, if $\mu\colon P \to Q$ is an $\mathcal{M}$-subobject of Q, then μ induces a functor $C\colon \mathbf{I} \to \mathbf{C}$, which on objects, takes $I \mapsto C_I$, and takes maps $f\colon I \to J$ to the unique map Cf making the diagram in (2.2) commute and the left square of that diagram a pullback. Note that $P \cong \operatorname{colim} \mathbf{y}C$ as colimits are stable under pullback in $\mathcal{P}(\mathbf{C})$. There is also a natural transformation $\alpha\colon C \Rightarrow D$, given componentwise on I by $m_I \in \mathcal{M}$ and whose naturality squares are pullbacks for every $I \in \mathbf{I}$.

The assignation $\mu \mapsto \alpha$ gives a function $\mathsf{Sub}_{\mathcal{M}}\colon \mathsf{Sub}_{\mathcal{P}(\mathcal{M})}(Q) \to \mathsf{Sub}_{[\mathcal{I},\mathcal{C}]}(D)$, where we write $([\mathbf{I},\mathbf{C}],[\mathcal{I},\mathcal{C}])$ for the $\mathcal{M}$-category whose $\mathcal{M}$-maps are the natural transformations whose components are maps in $\mathcal{M}$. It is easy to see that $([\mathbf{I},\mathbf{C}],[\mathcal{I},\mathcal{C}])$ is locally small, so to show that $\mathcal{P}_{\mathcal{M}}(\mathbf{C},\mathcal{M})$ is $\mathcal{M}$-well-powered, it is enough to show that $\mathsf{Sub}_{\mathcal{M}}$ is injective. Let $\mu\colon P \to Q$ and $\mu\colon P' \to Q$ be two $\mathcal{M}$-subobjects of Q which are mapped to the same $\mathcal{M}$-subobject of D. That is, there is an isomorphism from C to C' making the following diagram commute:

$$\begin{tikzcd}
C \arrow[rr, "\cong"] \arrow[dr, "\alpha"'] & & C' \arrow[dl, "\alpha'"] \\
& D. &
\end{tikzcd}$$

But because $P \cong \operatorname{colim} \mathbf{y}C \cong \operatorname{colim} \mathbf{y}C' \cong P'$, this induces an isomorphism between P and

P' making the following diagram commute:

$$\begin{tikzcd}
\mathbf{y}C_I \arrow[rr, "p_I"] \arrow[dr, "\cong"] \arrow[dd, "\mathbf{y}\alpha_I"'] & & P \arrow[dr, dashed, "\cong"] \arrow[dd, "\mu"'] & \\
& \mathbf{y}C'_I \arrow[rr, "p'_I"] \arrow[dl, "\mathbf{y}\alpha'_I"] & & P' \arrow[dl, "\mu'"] \\
\mathbf{y}D_I \arrow[rr, "q_I"'] & & Q. &
\end{tikzcd}$$

In other words, μ and μ' are the same $\mathcal{M}$-subobject of Q, and so the function $\mathsf{Sub}_{\mathcal{M}}$ is injective. Hence, if $(\mathbf{C}, \mathcal{M})$ is a locally small $\mathcal{M}$-category, then so is $\mathcal{P}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$. $\square$

Next, to show that $\mathcal{P}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$ is cocomplete, we exploit Proposition 2.22 and the following two lemmas.

Lemma 3.6. *Let $(\mathbf{C}, \mathcal{M})$ be a locally small $\mathcal{M}$-category and $\mathcal{I}$ a small set. If $\{\mu_i : P_i \to Q_i\}_{i \in \mathcal{I}}$ is a family of maps in $\mathcal{P}(\mathcal{M})$, then their coproduct $\sum_{i \in \mathcal{I}} \mu_i$ is also in $\mathcal{P}(\mathcal{M})$.*

Proof. Let $\{\mu_i : P_i \to Q_i\}_{i \in \mathcal{I}}$ be a family of maps in $\mathcal{P}(\mathcal{M})$, with $\mathcal{I}$ some small set. To show that $\sum_{i \in \mathcal{I}} \mu_i$ is also in $\mathcal{P}(\mathcal{M})$, we need to show that for any $h : \mathbf{y}D \to \sum_{i \in \mathcal{I}} Q_i$, there is a map $m : C \to D$ in $\mathcal{M}$ making the following diagram a pullback:

$$\begin{tikzcd}
\mathbf{y}C \arrow[r] \arrow[d, "\mathbf{y}m"'] & \sum_{i \in \mathcal{I}} P_i \arrow[d, "\sum_{i \in \mathcal{I}} \mu_i"] \\
\mathbf{y}D \arrow[r, "h"'] & \sum_{i \in \mathcal{I}} Q_i.
\end{tikzcd}$$

Since $\mathcal{P}(\mathbf{C})(\mathbf{y}D, \sum_{i \in \mathcal{I}} Q_i) \cong (\sum_{i \in \mathcal{I}} Q_i)(D)$ by the Yoneda lemma, and $(\sum_{i \in \mathcal{I}} Q_i)(D) \cong \sum_{i \in \mathcal{I}} Q_i D$ as coproducts in $\mathcal{P}(\mathbf{C})$ are taken pointwise, this means h corresponds uniquely with some element in $\sum_{i \in \mathcal{I}} Q_i D$. This, together with the naturality of the bijection $\mathcal{P}(\mathbf{C})(\mathbf{y}D, Q_i) \cong Q_i D$ for each $i \in \mathcal{I}$, implies that $h : \mathbf{y}D \to \sum_{i \in \mathcal{I}} Q_i$ factors through exactly one of the coproduct injections $\iota_{Q_j} : Q_j \to \sum_{i \in \mathcal{I}} Q_i$. By extensivity of the presheaf category $\mathsf{PSh}(\mathbf{C})$, the pullback of $\sum_{i \in \mathcal{I}} \mu_i$ along ι_{Q_j} must be μ_j. However, as μ_j is an $\mathcal{P}(\mathcal{M})$-map, there exists an $m : C \to D$ in $\mathcal{M}$ making the left square of the following diagram commute:

$$\begin{tikzcd}
\mathbf{y}C \arrow[r] \arrow[d, "\mathbf{y}m"'] & P_j \arrow[r, "\iota_{P_j}"] \arrow[d, "\mu_j"'] & \sum_{i \in \mathcal{I}} P_i \arrow[d, "\sum_{i \in \mathcal{I}} \mu_i"] \\
\mathbf{y}D \arrow[r, "h'"] \arrow[rr, bend right, "h"'] & Q_j \arrow[r, "\iota_{Q_j}"] & \sum_{i \in \mathcal{I}} Q_i.
\end{tikzcd}$$

Therefore, as both squares are pullbacks, $\mathbf{y}m$ is a pullback of $\sum_{i \in \mathcal{I}} \mu_i$ along h, which means $\sum_{i \in \mathcal{I}} \mu_i \in \mathcal{P}(\mathcal{M})$. $\square$

Lemma 3.7. *Let $(\mathbf{C}, \mathcal{M})$ be a locally small $\mathcal{M}$-category, and suppose m is a map in $\mathcal{P}(\mathbf{C})$. If the pullback of m along some epimorphism is an $\mathcal{P}(\mathcal{M})$-map, then m must also be in $\mathcal{P}(\mathcal{M})$.*

Proof. Let $m : P \to Q$ be a map in $\mathcal{P}(\mathbf{C})$, and suppose $m' : P' \to Q'$ is a pullback of m along some epimorphism $f : Q' \to Q$. To show that m is in $\mathcal{P}(\mathcal{M})$, let $g : \mathbf{y}D \to Q$ be any map in $\mathcal{P}(\mathbf{C})$. Again by Yoneda, there is a bijection $\mathcal{P}(\mathbf{C})(\mathbf{y}D, Q) \cong QD$, giving a corresponding element $\tilde{g} \in QD$. Since f is an epimorphism in $\mathcal{P}(\mathbf{C})$, its component at D, $f_D \colon Q'D \to QD$, must also be an epimorphism, which means there exists some element $\tilde{f'} \in Q'D$ such that $f_D(\tilde{f'}) = \tilde{g}$. The naturality of the bijection $\mathcal{P}(\mathbf{C})(\mathbf{y}D, Q) \cong QD$ then implies there is a map $f' \colon \mathbf{y}D \to Q'$ such that $g = ff'$. Now using the fact that m' is a $\mathcal{P}(\mathcal{M})$-map, there exists a map $n \in \mathcal{M}$ such that $\mathbf{y}n$ is the pullback of m' along f':

$$\begin{tikzcd}
\mathbf{y}C \arrow[r] \arrow[d, "\mathbf{y}n"'] & P' \arrow[r] \arrow[d, "m'"'] & P \arrow[d, "m"] \\
\mathbf{y}D \arrow[r, "f'"] \arrow[rr, bend right, "g"'] & Q' \arrow[r, "f"] & Q.
\end{tikzcd}$$

Then as both squares are pullbacks, $\mathbf{y}n$ must be the pullback of m along $g = ff'$, making m an $\mathcal{P}(\mathcal{M})$-map. □

Lemma 3.8. *Let $(\mathbf{C}, \mathcal{M})$ be a locally small $\mathcal{M}$-category. Then $(\mathcal{P}(\mathbf{C}), \mathcal{P}(\mathcal{M}))$ is a cocomplete $\mathcal{M}$-category.*

Proof. We begin by noting that the category of small presheaves on $\mathbf{C}$, $\mathcal{P}(\mathbf{C})$, is cocomplete. Therefore, it remains to show that the inclusion $\mathcal{P}(\mathbf{C}) \hookrightarrow \mathsf{Par}(\mathcal{P}(\mathbf{C}), \mathcal{P}(\mathcal{M}))$ is cocontinuous. However, by Proposition 2.22, it is enough to show that the following conditions hold:

(a) If $\{m_i \colon P_i \to Q_i\}_{i \in \mathcal{I}}$ is a family of maps in $\mathcal{P}(\mathcal{M})$ indexed by a small set $\mathcal{I}$, then $\sum_{i \in \mathcal{I}} m_i$ is also in $\mathcal{P}(\mathcal{M})$ and the following squares are pullbacks for each $i \in \mathcal{I}$:

$$\begin{tikzcd}
P_i \arrow[r, "\iota_{P_i}"] \arrow[d, "m_i"'] & \sum_{i \in \mathcal{I}} P_i \arrow[d, "\sum_{i \in \mathcal{I}} m_i"] \\
Q_i \arrow[r, "\iota_{Q_i}"'] & \sum_{i \in \mathcal{I}} Q_i.
\end{tikzcd}$$

(b) Given the following diagram,

$$\begin{tikzcd}
P' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] \arrow[d, "m'"'] & P \arrow[r, "c'"] \arrow[d, "m"] & G \arrow[d, "n"] \\
Q' \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Q \arrow[r, "c"'] & H
\end{tikzcd}$$

if $m \in \mathcal{P}(\mathcal{M})$ and the left two squares are pullbacks, and c, c' are the coequalisers of f, g and f', g' respectively, then the unique map n making the right square commute is in $\mathcal{P}(\mathcal{M})$ and the right square is also a pullback.

(c) Colimits in $\mathcal{P}(\mathbf{C})$ are stable under pullback along $\mathcal{P}(\mathcal{M})$-maps.

To see that (c) holds, recall that $\mathcal{P}(\mathbf{C})$ admits pullbacks along $\mathcal{P}(\mathcal{M})$-maps, and that these are calculated pointwise as in **Set** (Remark 3.4). The result then follows from the fact that colimits in $\mathcal{P}(\mathbf{C})$ are also calculated pointwise together with the fact that colimits are stable under pullback in **Set**.

For (b), it will be enough to show that the square on the right in (b) is a pullback (by Lemma 3.7). Now the right square is a pullback in $\mathcal{P}(\mathbf{C})$ if and only if componentwise for every $A \in \mathbf{C}$, it is a pullback in **Set**. So consider the diagram in (b) componentwise at $A \in \mathbf{C}$:

$$\begin{tikzcd} P'A \arrow[r, shift left, "f'_A"] \arrow[r, shift right, "g'_A"'] \arrow[d, "m'_A"'] & PA \arrow[r, "c'_A"] \arrow[d, "m_A"] & GA \arrow[d, "n_A"] \\ Q'A \arrow[r, shift left, "f_A"] \arrow[r, shift right, "g_A"'] & QA \arrow[r, "c_A"'] & HA. \end{tikzcd}$$

The two left squares remain pullbacks in **Set**, and c_A, c'_A remain coequalisers of f_A, g_A and f'_A, g'_A respectively since colimits in $\mathcal{P}(\mathbf{C})$ are calculated pointwise. Observe also that m_A is a monomorphism as maps between small presheaves in $\mathcal{P}(\mathbf{C})$ are monic if and only if they are componentwise monic for every $A \in \mathbf{C}$ (by a Yoneda argument). Now we know that the $\mathcal{M}$-category (**Set**, **Inj**) (where **Inj** are all the injective functions) is a cocomplete $\mathcal{M}$-category (Example 2.17), and since m_A is monic, the square on the right must be a pullback in **Set**. Therefore, as pullbacks in $\mathcal{P}(\mathbf{C})$ are calculated pointwise, the square on the right of (b) must also be a pullback.

For (a), we know that $\sum_{i \in I} m_I \in \mathcal{P}(\mathcal{M})$ from Lemma 3.6. Then, as (**Set**, **Inj**) is cocomplete and both pullbacks and colimits in $\mathcal{P}(\mathbf{C})$ are computed pointwise as in **Set**, the result follows by an analogous argument to (b).

Therefore, $(\mathcal{P}(\mathbf{C}), \mathcal{P}(\mathcal{M}))$ is a cocomplete $\mathcal{M}$-category. □

Theorem 3.9. *Let $(\mathbf{C}, \mathcal{C})$ be a locally small $\mathcal{M}$-category, and let $(\mathbf{D}, \mathcal{D})$ be a locally small, cocomplete $\mathcal{M}$-category. Then the following is an equivalence of categories:*

$$(-) \circ \mathbf{y} \colon \mathcal{M}\mathbf{Cocomp}(\mathcal{P}_{\mathcal{M}}(\mathbf{C}, \mathcal{C})), (\mathbf{D}, \mathcal{D})) \to \mathcal{M}\mathbf{CAT}((\mathbf{C}, \mathcal{C}), (\mathbf{D}, \mathcal{D}))$$

where $\mathcal{M}\mathbf{CAT}$ is the 2-category of locally small $\mathcal{M}$-categories.

Proof. The proof follows exactly the same arguments presented in the proof of Theorem 2.30. □

Corollary 3.10. *For any locally small restriction category $\mathbb{X}$ and locally small, cocomplete restriction category $\mathbb{E}$, the following is an equivalence of categories:*

$$(-) \circ \Lambda \colon \mathbf{rCocomp}(\mathsf{Par}(\mathcal{P}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))), \mathbb{E}) \to \mathbf{rCAT}(\mathbb{X}, \mathbb{E})$$

where Λ is the Cockett and Lack embedding introduced in (2.7) and $\mathbf{rCAT}$ is the 2-category of locally small restriction categories.

4

Restriction presheaves

We have now seen that for any small restriction category $\mathbb{X}$, the Cockett-Lack embedding from (2.7) exhibits the restriction category $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))))$ as a free cocompletion of $\mathbb{X}$, and an analogous result for locally small $\mathbb{X}$. However, this formulation of free cocompletion seems rather complex compared to the characterisation of $\mathsf{PSh}(\mathbf{C})$ and $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$ as the free cocompletions of ordinary categories and $\mathcal{M}$-categories respectively.

In this section, we give an alternate simpler definition of restriction cocompletion in terms of a restriction category $\mathsf{PSh}_r(\mathbb{X})$ of *restriction presheaves* and natural transformations. We will show that the underlying category of $\mathsf{PSh}_r(\mathbb{X})$ is a full subcategory $\mathsf{PSh}(\mathbb{X})$ and that the Yoneda embedding factors through a restriction functor $\mathbf{y}_r \colon \mathbb{X} \to \mathsf{PSh}_r(\mathbb{X})$ (strictly speaking through the underlying functor of $\mathbf{y}_r$). We conclude this section by showing that the category of restriction presheaves $\mathsf{PSh}_r(\mathbb{X})$ is equivalent to $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))))$, so that it gives another way of describing free cocompletion in the restriction setting.

Note that everything in this chapter, except for the last section on small restriction presheaves, contains material from [Lin, 2015].

4.1 Restriction category of restriction presheaves

We begin with the definition of restriction presheaf.

Definition 4.1. Let $\mathbb{X}$ be a restriction category. A *restriction presheaf* on $\mathbb{X}$ is an ordinary presheaf $P \colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ together with, for each object $A \in \mathbb{X}$, a map $PA \to \mathbb{X}(A, A)$ sending each element $x \in PA$ to a restriction idempotent $\bar{x} \colon A \to A$ in $\mathbb{X}$, with $\bar{x}$ satisfying the following three axioms:

(A1) $x \cdot \bar{x} = x$;

(A2) $\overline{x \cdot \bar{f}} = \bar{x} \circ \bar{f}$, where $\bar{f} \colon A \to A$ is a restriction idempotent in $\mathbb{X}$;

(A3) $\bar{x} \circ g = g \circ \overline{x \cdot g}$, where $g \colon B \to A$ in $\mathbb{X}$.

(For $x \in PA$ and $g \colon B \to A$ in $\mathbb{X}$, $x \cdot g$ denotes the element $P(g)(x) \in PB$). We call the class of maps above sending each $x \in PA$ (for each $A \in \mathbb{X}$) to $\bar{x}$ the *restriction structure* on P.

Unlike the restriction structure on a restriction category, the restriction structure on any restriction presheaf is unique, due to the following lemma.

Lemma 4.2. *Let $\mathbb{X}$ be a restriction category and $P\colon \mathbb{X}^{\mathrm{op}} \to$ **Set** a presheaf. Suppose P has two restriction structures given by $x \mapsto \bar{x}$ and $x \mapsto \tilde{x}$. Then $\bar{x} = \tilde{x}$ for all $A \in \mathbb{X}$ and $x \in PA$.*

Proof. We have
$$\bar{x} = \overline{x \cdot \tilde{x}} = \bar{x} \circ \tilde{x} = \tilde{x} \circ \bar{x} = \widetilde{x \cdot \bar{x}} = \tilde{x}$$
by the fact $\bar{x}$ and $\tilde{x}$ are restriction idempotents and using (A1),(A2). □

We also have the following analogues of basic results for restriction categories.

Lemma 4.3. *Suppose P is a restriction presheaf on a restriction category $\mathbb{X}$, and let $A \in \mathbb{X}$, $x \in PA$ and $g\colon B \to A$. Then*

(1) $\bar{g} \circ \overline{x \cdot g} = \overline{x \cdot g}$;

(2) $\overline{\bar{x} \circ g} = \overline{x \cdot g}$.

Proof. By (R2), (A2) and (R1),
$$\bar{g} \circ \overline{x \cdot g} = \overline{x \cdot g} \circ \bar{g} = \overline{(x \cdot g) \cdot \bar{g}} = \overline{x \cdot (g \circ \bar{g})} = \overline{x \cdot g}.$$
We also have
$$\overline{\bar{x} \circ g} = \overline{g \circ \overline{x \cdot g}} = \bar{g} \circ \overline{x \cdot g} = \overline{x \cdot g}$$
by (A3), (R3) and the previous result. □

The lemma above shows that (A2) and (A3) together imply $\overline{\bar{x} \circ g} = \overline{x \cdot g}$. However, what is perhaps surprising is that the converse is also true.

Lemma 4.4. *Suppose $P\colon \mathbb{X}^{\mathrm{op}} \to$ **Set** is a restriction presheaf, and let $A \in \mathbb{X}$, $x \in PA$. If $\overline{\bar{x} \circ g} = \overline{x \cdot g}$ is true for all maps $g\colon B \to A$, then $\overline{x \cdot e} = \bar{x} \circ e$ for all restriction idempotents $e\colon A \to A$, and also $\bar{x} \circ g = g \circ \overline{x \cdot g}$.*

Proof. The fact $\overline{\bar{x} \circ g} = \overline{x \cdot g}$ implies $\overline{x \cdot e} = \bar{x} \circ e$ is straightforward, and by assumption, we have $g \circ \overline{x \cdot g} = g \circ \overline{\bar{x} \circ g} = \bar{\bar{x}} \circ g = \bar{x} \circ g$. □

So in fact, we may replace restriction presheaf axioms (A2) and (A3) by the condition that $\overline{\bar{x} \circ g} = \overline{x \cdot g}$ for all maps $g\colon B \to A$. To explain what is happening, let us introduce *the* presheaf $O\colon \mathbb{X}^{\mathrm{op}} \to$ **Set**, sending each $A \in \mathbb{X}$ to the set $O(A)$ of restriction idempotents on A, and for each map $g\colon B \to A$, we have $O(g)(x) = x \cdot g = \overline{x \circ g}$ [Cockett & Lack, 2002, p. 253].

Now for each presheaf P on $\mathbb{X}$, there is an action by O on P in the following sense: there is a natural transformation $\alpha\colon P \times O \to P$ which on components, sends (x, e) to $x \cdot e$ (for each $A \in \mathbb{X}$, $x \in PA$ and each restriction idempotent $e\colon A \to A$). There is also another action on P given by $\pi\colon P \times O \to P$, which sends (x, e) to x (the first projection). As we now know that restriction structures are unique, we may characterise the restriction structure on any presheaf P in the following way.

Proposition 4.5. *Let $\mathbb{X}$ be a restriction category. Then a presheaf $P\colon \mathbb{X}^{\mathrm{op}} \to$ **Set** may be given a restriction structure if and only if there exists a (unique) section $\sigma\colon P \to P \times O$ to both the actions $\alpha, \pi\colon P \times O \to P$ described above.*

Proof. The condition $x \cdot \bar{x} = x$ is given by the section σ, and the other necessary and sufficient property that $\overline{\bar{x} \circ g} = \overline{x \cdot g}$ is simply restating the fact that σ is natural. □

Definition 4.6. The category of restriction presheaves on $\mathbb{X}$, $\mathsf{PSh}_r(\mathbb{X})$, is the restriction category whose objects are restriction presheaves and whose maps are ordinary natural transformations. The restriction of $\alpha \colon P \longrightarrow Q$ is the natural transformation $\bar{\alpha} \colon P \longrightarrow P$ given componentwise by $\bar{\alpha}_A(x) = x \cdot \overline{\alpha_A(x)}$ for every $A \in \mathbb{X}$ and $x \in PA$.

Note that $\bar{\alpha}$ is natural as

$$\bar{\alpha}_B(x \cdot f) = x \cdot \left(f \circ \overline{\alpha_B(x \cdot f)}\right) = x \cdot \left(f \circ \overline{\alpha_A(x) \cdot f}\right) = x \cdot \left(\overline{\alpha_A(x)} \circ f\right) = \bar{\alpha}_A(x) \cdot f$$

for all $f \colon B \longrightarrow A$. Also, the underlying category of $\mathsf{PSh}_r(\mathbb{X})$ is a full subcategory of $\mathsf{PSh}(\mathbb{X})$. The restriction category axioms are easily checked.

Now if $\mathbb{X}$ is a restriction category, then each representable $\mathbb{X}(-, A)$ has a restriction structure given by sending $f \in \mathbb{X}(B, A)$ to $\bar{f} \in \mathbb{X}$. In particular, this implies that the Yoneda embedding $\mathbf{y} \colon \mathbb{X} \longrightarrow \mathsf{PSh}(\mathbb{X})$ factors uniquely as a functor $\mathbf{y}_r \colon \mathbb{X} \longrightarrow \mathsf{PSh}_r(\mathbb{X})$:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathbf{y}_r"] \arrow[dr, "\mathbf{y}"'] & \mathsf{PSh}_r(\mathbb{X}) \arrow[d, hook] \\
& \mathsf{PSh}(\mathbb{X}).
\end{tikzcd}$$

Lemma 4.7. *For any restriction category $\mathbb{X}$, the functor $\mathbf{y}_r \colon \mathbb{X} \longrightarrow \mathsf{PSh}_r(\mathbb{X})$ is a restriction functor.*

Proof. Let $f \colon A \longrightarrow B$ be a map in $\mathbb{X}$. Then for all $X \in \mathbb{X}$ and $x \in \mathbb{X}(X, A)$, we have

$$\overline{\mathbf{y}_r f}_X(x) = x \cdot \overline{(\mathbf{y}_r f)_X(x)} = x \cdot \overline{f \circ x} = \bar{f} \circ x = (\mathbf{y}_r \bar{f})_X(x)$$

and so $\mathbf{y}_r \bar{f} = \overline{\mathbf{y}_r f}$. □

We can characterise the total maps in $\mathsf{PSh}_r(\mathbb{X})$ as those which are restriction preserving, due to the following proposition.

Proposition 4.8. *A map $\alpha \colon P \longrightarrow Q$ is total in $\mathsf{PSh}_r(\mathbb{X})$ if and only if $\overline{\alpha_A(x)} = \bar{x}$ for all $A \in \mathbb{X}$ and $x \in PA$.*

Proof. Suppose $\alpha \colon P \longrightarrow Q$ is total in $\mathsf{PSh}_r(\mathbb{X})$. Then $\bar{\alpha}_A(x) = 1_{PA}(x) = x$, or $x \cdot \overline{\alpha_A(x)} = x$. But this implies $\bar{x} \leq \overline{\alpha_A(x)}$ since

$$\bar{x} = \overline{x \cdot \overline{\alpha_A(x)}} = \bar{x} \circ \overline{\alpha_A(x)}.$$

On the other hand, $\overline{\alpha_A(x)} \leq \bar{x}$ as

$$\overline{\alpha_A(x)} = \overline{\alpha_A(x \cdot \bar{x})} = \overline{\alpha_A(x) \cdot \bar{x}} = \overline{\alpha_A(x)} \circ \bar{x}.$$

Therefore, α in $\mathsf{PSh}_r(\mathbb{X})$ is total if and only if α preserves restrictions. □

The restriction presheaf category has one more important property.

Proposition 4.9. *Let $\mathbb{X}$ be a restriction category. Then $\mathsf{PSh}_r(\mathbb{X})$ is a split restriction category.*

Proof. Let $\bar{\alpha}\colon P \to P$ be a restriction idempotent in $\mathsf{PSh}_r(\mathbb{X})$. Since all idempotents in $\mathsf{PSh}(\mathbb{X})$ split, we may write $\bar{\alpha} = \mu\rho$ for some maps $\mu\colon Q \to P$ and $\rho\colon P \to Q$ such that $\rho\mu = 1$. Componentwise, we may take μ_A to be the inclusion $QA \hookrightarrow PA$ with $QA = \{x \in PA \mid \bar{\alpha}_A(x) = x\}$. Therefore, to show $\mathsf{PSh}_r(\mathbb{X})$ is split, it is enough to show that Q is a restriction presheaf. However, P is a restriction presheaf and Q is a subfunctor of P. Therefore, imposing the restriction structure of P onto Q will make Q a restriction presheaf. Hence $\mathsf{PSh}_r(\mathbb{X})$ is a split restriction category. □

Before moving onto the main theorems in this chapter, let us recall the split restriction category $\mathsf{K}_r(\mathbb{X})$, whose objects are pairs (A, e) (with e a restriction idempotent on $A \in \mathbb{X}$). Also recall the unit of the biadjunction $i \dashv \mathsf{K}_r$ at $\mathbb{X}$, $J\colon \mathbb{X} \to \mathsf{K}_r(\mathbb{X})$, which sends objects A to $(A, 1_A)$ and morphisms $f\colon A \to B$ to $f\colon (A, 1_A) \to (B, 1_B)$.

Proposition 4.10. *$\mathsf{PSh}_r(\mathbb{X})$ and $\mathsf{PSh}_r(\mathsf{K}_r(\mathbb{X}))$ are equivalent as restriction categories.*

Proof. Since $\mathsf{K}_r(\mathbb{X})$ is a full subcategory of $\mathsf{Split}(\mathbb{X})$ (the idempotent completion of $\mathbb{X}$), the functor $(-) \circ J^{\mathrm{op}}\colon \mathsf{PSh}(\mathsf{K}_r(\mathbb{X})) \to \mathsf{PSh}(\mathbb{X})$ is an equivalence. Therefore, the result will follow if we can show this functor restricts back to an equivalence between $\mathsf{PSh}_r(\mathsf{K}_r(\mathbb{X}))$ and $\mathsf{PSh}_r(\mathbb{X})$. In other words, showing that the restriction of $(-) \circ J^{\mathrm{op}}$ to $\mathsf{PSh}_r(\mathsf{K}_r(\mathbb{X}))$ sends restriction presheaves on $\mathsf{K}_r(\mathbb{X})$ to restriction presheaves on $\mathbb{X}$, is essentially surjective on objects and is a restriction functor:

$$\begin{tikzcd}
\mathsf{PSh}_r(\mathsf{K}_r(\mathbb{X})) \arrow[r] \arrow[d, hook] & \mathsf{PSh}_r(\mathbb{X}) \arrow[d, hook] \\
\mathsf{PSh}(\mathsf{K}_r(\mathbb{X})) \arrow[r, "(-)\circ J^{\mathrm{op}}"'] & \mathsf{PSh}(\mathbb{X}).
\end{tikzcd}$$

So let P be a restriction presheaf on $\mathsf{K}_r(\mathbb{X})$. Then PJ^{op} will be a restriction presheaf on $\mathbb{X}$ if we define the restriction on $x \in (PJ^{\mathrm{op}})(A) = P(A, 1_A)$ to be the same as in $P(A, 1_A)$ for all $A \in \mathbb{X}$. Also, if $\bar{\alpha}\colon P \Rightarrow P$ is a restriction idempotent, then

$$(\overline{\alpha} \circ J^{\mathrm{op}})_A(x) = \overline{\alpha}_{(A,1_A)}(x) = x \cdot \overline{\alpha_{(A,1_A)}(x)} = x \cdot \overline{(\alpha \circ J^{\mathrm{op}})_A(x)} = \left(\overline{\alpha \circ J^{\mathrm{op}}}\right)_A(x)$$

implies $(-) \circ J^{\mathrm{op}}$ is a restriction functor. Therefore, all that remains is to show essential surjectivity.

Let Q be a restriction presheaf on $\mathbb{X}$, and define the presheaf Q' on $\mathsf{K}_r(\mathbb{X})$ by taking $Q'(A, e) = \{x \in QA \mid x \cdot e = x\}$ and $Q'f = Qf$. Note this is well-defined since for all $f\colon (A', e') \to (A, e)$ and $y \in Q'(A', e')$, we have

$$Q(f)(y) = Q(f)(y \cdot e') = Q(f)Q(e')(y) = Q(e'f)(y) = Q(fe)(y) = Q(f)(y) \cdot e.$$

Obviously $Q'(A, 1_A) = Q(A)$, and so $Q' \circ J^{\mathrm{op}} = Q$. Hence, $(-) \circ J^{\mathrm{op}}\colon \mathsf{PSh}(\mathsf{K}_r(\mathbb{X})) \to \mathsf{PSh}(\mathbb{X})$ is essentially surjective on objects, and therefore $\mathsf{PSh}_r(\mathbb{X})$ and $\mathsf{PSh}_r(\mathsf{K}_r(\mathbb{X}))$ are equivalent. □

4.2 An equivalence of $\mathcal{M}$-categories

In this section, we will prove an equivalence of $\mathcal{M}$-categories, which we will then use to show that $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))))$ and $\mathsf{PSh}_r(\mathbb{X})$ are, in fact, equivalent as restriction categories. However, in order to do this, we shall make use of the following lemma.

Lemma 4.11. *Let* $\mathbf{C}$ *be a category and let* m *be a monic in* $\mathbf{C}$. *Suppose the following is a pullback:*

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "n"'] & A \arrow[d, "m"] \\ B \arrow[r, "f"'] & C. \end{tikzcd}$$

Then n *is an isomorphism if and only if* $f = mh$ *for some* $h\colon B \to A \in \mathbf{C}$.

Proof. ($\Rightarrow$) Take $h = gn^{-1}$, and then use the fact that the pullback of m along itself is the identity. ($\Leftarrow$) Consider maps $1_B\colon B \to B$ and $h\colon B \to A$, and observe that $n^{-1}\colon B \to D$ is the unique induced map by the above pullback. □

We now give the following equivalence of $\mathcal{M}$-categories.

Theorem 4.12. *Suppose* $(\mathbf{C}, \mathcal{M})$ *is an* $\mathcal{M}$*-category. Then* $\mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M})))$ *and* $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$ *are equivalent as* $\mathcal{M}$*-categories.*

Proof. Our goal will be to find a pair of functors $F\colon \mathsf{PSh}(\mathbf{C}) \to \mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M})))$ and $G\colon \mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M}))) \to \mathsf{PSh}(\mathbf{C})$ (along with natural isomorphisms $\eta\colon 1 \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow 1$), and then show that F and G are in fact $\mathcal{M}$-functors. (Note that η and ε must necessarily be $\mathcal{M}$-cartesian).

So let P be a presheaf on $\mathbf{C}$, and define FP on objects as follows. If $X \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$, then $(FP)(X)$ is the set of equivalence classes

$$(FP)(X) = \{(m, f) \mid m\colon Y \to X \in \mathcal{M}, f \in PY\}$$

where $(m, f) \sim (n, g)$ if and only if there exists an isomorphism φ such that $n = m\varphi$ and $g = f \cdot \varphi$. To define FP on morphisms, given $(n, g)\colon Z \to X$ in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ and an element $(m, f) \in (FP)(X)$, define

$$((FP)(n, g))\,(m, f) = (nm', f \cdot g')$$

where (m', g') is the pullback of (m, g), as in:

$$\begin{tikzcd} \bullet \arrow[r, "g'"] \arrow[d, "m'"'] & \bullet \arrow[d, "m"] \\ \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$$

We shall sometimes denote the above informally as $(m, f)\cdot(n, g)$. Then defining the restriction on each $(m, f) \in (FP)(X)$ to be (m, m) makes $FP\colon \mathsf{Par}(\mathbf{C}, \mathcal{M})^{\mathrm{op}} \to \mathbf{Set}$ a restriction presheaf. This defines F on objects.

Now suppose $\alpha\colon P \to Q$ is a map in $\mathsf{PSh}(\mathbf{C})$. Define $F\alpha\colon FP \to FQ$ componentwise as follows:

$$(F\alpha)_X(m, f) = (m, \alpha_{\mathsf{dom}\, m}(f)).$$

Then $F\alpha$ is natural (by naturality of α) and also total, making F a functor from $\mathsf{PSh}(\mathbf{C})$ to $\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M})))$. We now give the data for the functor G from $\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M})))$ to $\mathsf{PSh}(\mathbf{C})$.

Let P be a restriction presheaf on $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, and define $GP\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ as follows. If $X \in \mathbf{C}$, then

$$(GP)(X) = \{x \mid x \in PX, \bar{x} = (1, 1)\}.$$

And if $f\colon Z \to X$ is a map in $\mathbf{C}$, define

$$(GP)(f) = P(1, f).$$

Note that $(GP)(f)$ is well-defined since for every $x \in (GP)(X)$,

$$\overline{P(1, f)(x)} = \overline{x \cdot (1, f)} = \overline{\bar{x} \circ (1, f)} = (1, 1),$$

and so $(GP)(f)$ is a function from $(GP)(X)$ to $(GP)(Z)$.

Finally, if $\alpha\colon P \to Q$ is a total map in $\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M}))$, define $G\alpha\colon GP \to GQ$ componentwise by

$$(G\alpha)_X(x) = \alpha_X(x)$$

for every $X \in \mathbf{C}$ and $x \in (GP)(X)$. Again, to see that $G\alpha$ is well-defined, note that α total implies $\overline{\alpha_X(x)} = \overline{x} = 1$ (Proposition 4.8) and so $\alpha_X(x) \in (GQ)(X)$. This makes G a functor from $\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M})))$ to $\mathsf{PSh}(\mathbf{C})$. The next step is defining isomorphisms $\eta\colon 1 \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow 1$.

To define η, we need to give components for every presheaf P on $\mathbf{C}$, and this involves giving isomorphisms $(\eta_P)_X\colon PX \to (GFP)(X)$. But $(GFP)(X) = \{(1, f) \mid f \in PX\}$. Therefore, defining $(\eta_P)_X(f) = (1, f)$ makes η an isomorphism, and naturality is easy to check.

Similarly, to define ε, we need to define isomorphisms $(\varepsilon_P)_X\colon (FGP)(X) \to PX$ for every restriction presheaf P on $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ and object $X \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$. Since

$$(FGP)(X) = \{(m, f) \mid m\colon Y \to X \in \mathcal{M}, f \in PY, \bar{f} = (1, 1)\},$$

define $(\varepsilon_P)_X(m, f) = f \cdot (m, 1)$. Its inverse $(\varepsilon_P)_X^{-1}\colon PX \to (FGP)(X)$ is then given by

$$(\varepsilon_P)_X^{-1}(x) = (n, x \cdot (1, n))$$

where $\bar{x} = (n, n)$ (as P is a restriction presheaf on $\mathsf{Par}(\mathbf{C}, \mathcal{M})$). Checking the naturality of ε is again straightforward. All that remains is to show that both $F\colon \mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M}) \to \mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C})))$ and $G\colon \mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M}))) \to \mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$ are $\mathcal{M}$-functors. However, as F and G are equivalences in $\mathbf{Cat}$, they necessarily preserve limits, and so all this will involve is showing that they preserve $\mathcal{M}$-maps. That is, $F\mu$ is a restriction monic in $\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M}))$ for all $\mu \in \mathsf{PSh}(\mathcal{M})$, and that $G\mu$ is in $\mathsf{PSh}(\mathcal{M})$ for all restriction monics $\mu \in \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M}))$.

So let $\mu\colon P \to Q$ be in $\mathsf{PSh}(\mathcal{M})$. To show $F\mu$ is a restriction monic, we need to show $F\mu$ is the equaliser of 1 and some restriction idempotent $\alpha\colon FQ \to FQ$. To define this α, let $X \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$ and $(n, g) \in (FQ)(X)$ (where $n\colon Z \to X$). Now as $g \in QZ$, there exists a corresponding natural transformation $\hat{g}\colon \mathbf{y}Z \to Q$ (Yoneda). However, as μ is in $\mathsf{PSh}(\mathcal{M})$, there exists an $m_g\colon B \to Z$ in $\mathcal{M}$ making the following a pullback:

$$\begin{tikzcd}
\mathbf{y}B \arrow[r] \arrow[d, "\mathbf{y}m_g"'] & P \arrow[d, "\mu"] \\
\mathbf{y}Z \arrow[r, "\hat{g}"'] & Q.
\end{tikzcd}$$

So define α by its components as follows,

$$\alpha_X(n, g) = (nm_g, g \cdot m_g).$$

It is then not difficult to show this α is well-defined, is a natural transformation and is a restriction idempotent.

Now to show that $F\mu$ equalises 1 and α, we need to show $(F\mu)_X : (FP)(X) \to (FQ)(X)$ is an equaliser of 1 and $\alpha_{(FQ)(X)}$ in **Set** for all $X \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$. In other words, that $(F\mu)_X$ is injective, and that:

$$\begin{aligned}&(n, g) \in (FQ)(X) \text{ satisfies } (n, g) = (F\mu)_X(m, f) = (m, \mu_{\mathsf{dom}\, m}(f)) \text{ for some}\\ &(m, f) \in (FP)(X) \text{ if and only if } \alpha_X(n, g) = (n, g).\end{aligned} \tag{4.1}$$

To show $(F\mu)_X$ is injective, suppose $(F\mu)_X(m, f) = (F\mu)_X(m', f')$, or equivalently, $(m, \mu_{\mathsf{dom}\, m}(f)) = (m', \mu_{\mathsf{dom}\, m'}(f'))$. That is, there exists an isomomorphism φ such that $m' = m\varphi$ and $\mu_{\mathsf{dom}\, m'}(f') = \mu_{\mathsf{dom}\, m}(f) \cdot \varphi$. But the naturality of μ implies $\mu_{\mathsf{dom}\, m'}(f \cdot \varphi) = \mu_{\mathsf{dom}\, m}(f) \cdot \varphi = \mu_{\mathsf{dom}\, m'}(f')$. Therefore, as μ is monic, we must have $f \cdot \varphi = f'$. Hence $(m, f) = (m', f')$, and so $(F\mu)_X$ is injective.

To prove (4.1), let $(n, g) \in (FQ)(X)$ and suppose $\mu_X(n, g) = (n, g)$. That is, $(nm_g, g \cdot m_g) = (n, g)$, or that m_g is an isomorphism. Now m_g is an isomorphism if and only if $\mathbf{y}m_g$ is an isomorphism, and by Lemma 4.11, $\mathbf{y}m_g$ is an isomorphism if and only if $\hat{g} = \mu\hat{h}$ for some $\hat{h} : \mathbf{y}Z \to P$:

$$\begin{tikzcd}
\mathbf{y}B \arrow[r] \arrow[d, "\mathbf{y}m_g"'] & P \arrow[d, "\mu"] \\
\mathbf{y}Z \arrow[r, "\hat{g}"'] \arrow[ur, dashed, "\hat{h}"'] & Q.
\end{tikzcd}$$

But by Yoneda, the statement $\hat{g} = \mu\hat{h}$ is equivalent to the statement that $g = \mu_Z(h)$ for some $h \in PZ$, which is the same as saying $(n, g) = (n, \mu_Z(h)) = (F\mu)_X(n, h)$, with $(n, h) \in (FP)(X)$. Therefore, $(F\mu)_X$ is an equaliser of 1 and $\alpha_{(FQ)(X)}$ in **Set** for all $X \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$, and hence, $F\mu$ equalises 1 and α.

Now to see that G is also an $\mathcal{M}$-functor, let $\mu : P \to Q$ be a restriction monic in $\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M}))$. To show $G\mu$ is in $\mathsf{PSh}(\mathcal{M})$, we need to show for any given $\hat{\theta} : \mathbf{y}C \to Q$, there exists a monic $m : D \to C$ in $\mathcal{M}$ and a map $\hat{\delta} : \mathbf{y}D \to P$ making the following a pullback:

$$\begin{tikzcd}
\mathbf{y}D \arrow[r, "\hat{\delta}"] \arrow[d, "\mathbf{y}m"'] & GP \arrow[d, "G\mu"] \\
\mathbf{y}C \arrow[r, "\hat{\theta}"'] & GQ.
\end{tikzcd}$$

Here we make two observations. First, commutativity says m and δ must satisfy $G\mu \circ \hat{\delta} = \hat{\theta} \circ \mathbf{y}m$. On the other hand, Yoneda tells us that $\hat{\theta} \circ \mathbf{y}m = \widehat{\theta \cdot m}$ and $G\mu \circ \hat{\delta} = \widehat{(G\mu)_D(\delta)}$, where $\theta \in QC$ and $\delta \in PD$ are the unique transposes of $\hat{\theta}$ and $\hat{\delta}$ respectively. Therefore, m and δ must satisfy the following condition:

$$(G\mu)_D(\delta) = \theta \cdot_{GQ} m. \tag{4.2}$$

That is, $\mu_D(\delta) = \theta \cdot_Q (1, m)$. Secondly, m and δ must make the following a pullback in **Set** (for all objects $X \in \mathbf{C}$):

$$\begin{tikzcd}
\mathbf{C}(X, D) \arrow[r, "\hat{\delta}_X = \delta \cdot_{GP} (-)"] \arrow[d, "m \circ (-)"'] & (GP)(X) \arrow[d, "(G\mu)_X"] \\
\mathbf{C}(X, C) \arrow[r, "\hat{\theta}_X = \theta \cdot_{GQ} (-)"'] & (GQ)(X).
\end{tikzcd}$$

In other words, for any $f \in \mathbf{C}(X, C)$ and $x \in (GP)(X)$ such that $\theta \cdot_{GQ} f = (G\mu)_X(x)$ (i.e., such that $\theta \cdot_Q (1, f) = \mu_X(x)$), there exists a unique $g \in \mathbf{C}(X, D)$ such that

$$\delta \cdot_{GP} g = x, \text{ and } mg = f. \tag{4.3}$$

Alternatively, $\delta \cdot_P (1, g) = x$ and $mg = f$. To find m, note that because μ is a restriction monic, there exists a ρ such that $\mu\rho = \bar{\rho}$ and $\rho\mu = 1$. Since $\theta \in QC$, applying ρ_C to θ and then taking its restriction gives $\overline{\rho_C(\theta)} = (m, m)$ for some $m \in \mathcal{M}$. This gives us m.

To define δ, observe that $P(1, m)$ is a function from PC to PD. So define

$$\delta = \rho_C(\theta) \cdot_P (1, m).$$

Then $\delta \in (GP)(D)$ since

$$\bar{\delta} = \overline{\overline{\rho_C(\theta)} \circ (1, m)} = \overline{(m, m) \circ (1, m)} = \overline{(1, m)} = (1, 1).$$

So all that remains is to show m and δ satisfy (4.2) and (4.3). To show m and δ satisfy (4.2), one simply substitutes the given values into the equation, using the fact $\mu\rho = \bar{\rho}$. To see that (4.3) is also satisfied, suppose there exist $f \in \mathbf{C}(X, C)$ and $x \in (GP)(X)$ such that $\theta \cdot_P (1, f) = \mu_X(x)$. Then applying ρ_X to both sides gives

$$\rho_C(\theta) \cdot_P (1, f) = x$$

since $\rho\mu = 1$. We need to show there exists a g such that $mg = f$ and $\delta \cdot_P (1, g) = x$. But $mg = f$ implies

$$x = \rho_C(\theta) \cdot_P (1, f) = \rho_C(\theta) \cdot_P (1, mg) = \rho_C(\theta) \cdot_P (1, m) \cdot_P (1, g) = \delta \cdot_P (1, g).$$

Therefore, we just need to find g.

Consider the composite $(m, m) \circ (1, f) = (m', mf')$, where (m', f') is the pullback of (m, f):

$$\begin{tikzcd} X \times_C D \arrow[r, "f'"] \arrow[d, "m'"'] & D \arrow[d, "m"] \\ X \arrow[r, "f"'] & C. \end{tikzcd}$$

Note that if m' is an isomorphism, then $g = f'(m')^{-1}$ will satisfy the condition $mg = f$. Now by restriction presheaf axioms and naturality of $\bar{\rho}$, we have $\theta \cdot_Q (m', mf') = \theta \cdot_Q (1, f)$. But $\theta \in (GQ)(C)$ implies

$$\overline{\theta \cdot_Q (m', mf')} = \overline{\bar{\theta} \circ (m', mf')} = \overline{(m', mf')} = (m', m')$$

and

$$\overline{\theta \cdot_Q (1, f)} = \overline{\bar{\theta} \circ (1, f)} = \overline{(1, f)} = (1, 1).$$

Therefore, m' must be an isomorphism, which means m and δ satisfy (4.3). Hence, G is also an $\mathcal{M}$-functor and $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})$ and $\mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M})))$ are equivalent. $\square$

4.3 An equivalence of restriction categories

We now make use of the previous theorem to prove the following result.

Proposition 4.13. *Let* $(\mathbf{C}, \mathcal{M})$ *be an* $\mathcal{M}$*-category. Then there exists an equivalence of restriction categories* $L\colon \mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})) \to \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M}))$ *satisfying the relation* $\mathbf{y}_r = L \circ \mathsf{Par}(\mathbf{y})$.

Proof. Since Par and $\mathcal{M}\mathsf{Total}$ are 2-equivalences, the following is an isomorphism of categories:

$$\mathcal{M}\mathbf{Cat}\Big(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M}), \mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M})))\Big) \cong \mathbf{rCat}\Big(\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M})), \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M}))\Big).$$

We know from Theorem 4.12 that $F\colon \mathsf{PSh}_{\mathcal{M}}(\mathbf{C}, \mathcal{M}) \to \mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M})))$ is an equivalence. So define $L = \tilde{F}$, the transpose of F. Explicitly, $\tilde{F} = \Phi^{-1}_{\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M}))} \circ \mathsf{Par}(F)$, where $\Phi_{\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M}))}$ is the unit of the Par and $\mathcal{M}\mathsf{Total}$ 2-equivalence.

Now define the functor $\widetilde{\mathbf{y}_r}\colon \mathbf{C} \to \mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M})))$ as the transpose of the Yoneda embedding $\mathbf{y}_r\colon \mathsf{Par}(\mathbf{C}, \mathcal{M}) \to \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M}))$. Explicitly, $\widetilde{\mathbf{y}_r}$ is the unique map whose underlying functor (also called $\widetilde{\mathbf{y}_r}$ by an abuse of notation) makes the following diagram commute:

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "\widetilde{\mathbf{y}_r}"] \arrow[d, hook] & \mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M}))) \arrow[d, hook] \\
\mathsf{Par}(\mathbf{C}, \mathcal{M}) \arrow[r, "\mathbf{y}_r"'] & \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M})).
\end{tikzcd}$$

Since $\widetilde{\mathbf{y}_r} = F\mathbf{y}$ will imply $\mathbf{y}_r = L \circ \mathsf{Par}(\mathbf{y})$, we prove the former. So let $A \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$. Then $\widetilde{\mathbf{y}_r}(A) = \mathsf{Par}(\mathbf{C}, \mathcal{M})(-, A)$ by definition. On the other hand, $(F\mathbf{y})(A)$ defined on objects $B \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$ is the following set:

$$(F\mathbf{y}A)(B) = \{(m, f) \mid m\colon Y \to B \in \mathcal{M}, f \in \mathbf{C}(Y, A)\}.$$

In other words, elements of $(F\mathbf{y}A)(B)$ are spans $B \xleftarrow{m} Y \xrightarrow{f} A$.

Clearly $(F\mathbf{y}A)(B) = \mathsf{Par}(\mathbf{C}, \mathcal{M})(B, A) = (\widetilde{\mathbf{y}_r}A)(B)$. Likewise, if $(n, g)\colon C \to B$ is a map in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, then $(F\mathbf{y}A)(n, g) = (-) \circ (n, g) = (\tilde{\mathbf{y}}_r A)(n, g)$, and so $\widetilde{\mathbf{y}_r}(A) = (F\mathbf{y})(A)$.

Now let $h\colon B \to C$ be a map in $\mathbf{C}$. Then $(F\mathbf{y})(h)\colon \mathsf{Par}(\mathbf{C}, \mathcal{M})(-, B) \Rightarrow \mathsf{Par}(\mathbf{C}, \mathcal{M})(-, C)$ has components given by

$$(F\mathbf{y}h)_D(n, g) = (n, (\mathbf{y}h)_{\mathsf{dom}\, n}(g)) = (n, hg) = (1, h) \circ (n, g)$$

for all $D \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$ and $(n, g) \in \mathsf{Par}(\mathbf{C}, \mathcal{M})(D, C)$. But $\widetilde{\mathbf{y}_r}(h) = \mathbf{y}_r(1, h)$ also has components given by $(\mathbf{y}_r(1, h))_D = (1, h) \circ (-)$ at $D \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$. Therefore, $(F\mathbf{y})(h) = \widetilde{\mathbf{y}_r}(h)$ and so $F\mathbf{y} = \widetilde{\mathbf{y}_r}$. Hence, $\mathbf{y}_r = L \circ \mathsf{Par}(\mathbf{y})$. □

We now prove the main result of this section.

Theorem 4.14. *Let* $\mathbb{X}$ *be a restriction category. Then*

$$\mathsf{PSh}_r(\mathbb{X}) \simeq \mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))))$$

and the following diagram commutes up to isomorphism:

$$\begin{tikzcd}
& \mathbb{X} \arrow[dl, "\mathbf{y}_r"'] \arrow[dr, "\Lambda"] \arrow[d, phantom, "\cong"] & \\
\mathsf{PSh}_r(\mathbb{X}) & & \mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))) \arrow[ll, "\simeq"]
\end{tikzcd}$$

where Λ *is the Cockett and Lack embedding introduced in* (2.7).

Proof. Consider the following diagram, where $\mathbf{C} = \mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))$ and the top composite is the Cockett-Lack embedding Λ from (2.7):

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\Phi_{\mathsf{K}_r(\mathbb{X})}\circ J"] \arrow[d, "\mathbf{y}_r"'] & \mathsf{Par}(\mathbf{C},\mathcal{M}) \arrow[r, "\mathsf{Par}(\mathbf{y})"] \arrow[d, "\mathbf{y}_r"'] & \mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C},\mathcal{M})) \arrow[d, "L"] \\ \mathsf{PSh}_r(\mathbb{X}) & \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C},\mathcal{M})) \arrow[l, "(-)\circ(\Phi_{\mathsf{K}_r(\mathbb{X})}\circ J)^{\mathrm{op}}"] \arrow[r, equal] & \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C},\mathcal{M})). \end{tikzcd}$$

By Proposition 4.13, the right square commutes up to isomorphism. However, the left square also commutes up to isomorphism as $\Phi_{\mathsf{K}_r(\mathbb{X})} \circ J$ is fully faithful. Hence the result follows. □

Corollary 4.15. *For any small restriction category $\mathbb{X}$, the embedding $\mathbf{y}_r \colon \mathbb{X} \to \mathsf{PSh}_r(\mathbb{X})$ exhibits $\mathsf{PSh}_r(\mathbb{X})$ as the* free restriction cocompletion *of $\mathbb{X}$.*

4.4 Small restriction presheaves

Given that a small presheaf on an ordinary category is one that can be written as a colimit of small representables, it is natural to ask whether there is a similar notion of small restriction presheaf. So let $\mathbb{X}$ be a locally small restriction category. Denoting the $\mathcal{M}$-category $\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))$ by $(\mathbf{C},\mathcal{M})$, Corollary 3.10 then says that $\mathsf{Par}(\mathcal{P}_{\mathcal{M}}(\mathbf{C},\mathcal{M}))$ is the free cocompletion of $\mathbb{X}$. Since $\mathcal{P}(\mathbf{C})$ is a full replete subcategory of $\mathsf{PSh}(\mathbf{C})$ and $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C},\mathcal{M})) \simeq \mathsf{PSh}_r(\mathbb{X})$, there exists a full subcategory $\mathcal{P}_r(\mathbb{X}) \subset \mathsf{PSh}_r(\mathbb{X})$ which is equivalent to $\mathsf{Par}(\mathcal{P}_{\mathcal{M}}(\mathbf{C},\mathcal{M}))$:

$$\begin{tikzcd} \mathcal{P}_r(\mathbb{X}) \arrow[r, "\simeq"] \arrow[d, hook] & \mathsf{Par}(\mathcal{P}_{\mathcal{M}}(\mathbf{C},\mathcal{M})) \arrow[d, hook] \\ \mathsf{PSh}_r(\mathbb{X}) \arrow[r, "\simeq"'] & \mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C},\mathcal{M})) \end{tikzcd}$$

where the above square is a pullback and the bottom map is the equivalence from Theorem 4.14.

To see what objects should be in $\mathcal{P}_r(\mathbb{X})$, it is enough to apply Total to the above diagram, giving the following pullback:

$$\begin{tikzcd} \mathsf{Total}(\mathcal{P}_r(\mathbb{X})) \arrow[r] \arrow[d, hook] & \mathcal{P}(\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))) \arrow[d, hook] \\ \mathsf{Total}(\mathsf{PSh}_r(\mathbb{X})) \arrow[r, "G"'] & \mathsf{PSh}(\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))) \end{tikzcd}$$

where G is an equivalence. Since the above diagram is a pullback, an object P will be in $\mathsf{Total}(\mathcal{P}_r(\mathbb{X}))$ (and hence in $\mathcal{P}_r(\mathbb{X})$) if GP is an object in $\mathcal{P}(\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))$; that is, $GP \cong \operatorname{colim} \mathbf{y}C_I$, where $C \colon \mathbf{I} \to \mathsf{Total}(\mathsf{K}_r(\mathbb{X}))$ is a functor with $\mathbf{I}$ small. If we define H to be a pseudo-inverse for G, then an object will be in $\mathcal{P}_r(\mathbb{X})$ if it is of the form $P \cong \operatorname{colim} H\mathbf{y}C_I$, for some small $\mathbf{I}$ and functor $C \colon \mathbf{I} \to \mathsf{Total}(\mathsf{K}_r(\mathbb{X}))$. We call these P the *small restriction presheaves*.

We also give an explicit description of a small restriction presheaf as follows. Since GP is an object in $\mathcal{P}(\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))$, it will be the colimit of a small diagram whose vertices are of the form $\mathbf{y}(A, e)$, where (A, e) is an object in $\mathsf{K}_r(\mathbb{X})$. Now given $(A, e) \in \mathsf{K}_r(\mathbb{X})$, note the following splitting in $\mathsf{PSh}_r(\mathbb{X})$:

$$\begin{tikzcd} & Q(A,e) \arrow[dr] & \\ \mathbf{y}_r A \arrow[ur, two heads] \arrow[rr, "\mathbf{y}_r e"'] & & \mathbf{y}_r A. \end{tikzcd}$$

This gives a functor $Q \colon \mathsf{K}_r(\mathbb{X}) \to \mathsf{PSh}_r(\mathbb{X})$. Then a restriction presheaf is called *small* if it is the colimit of some functor $D \colon \mathbf{I} \to \mathsf{PSh}_r(\mathbb{X})$ ($\mathbf{I}$ small), where each DI is of the form $Q(A, e)$ for some $(A, e) \in \mathsf{K}_r(\mathbb{X})$, and each $D(f \colon I \to J)$ is total. We denote by $\mathcal{P}_r(\mathbb{X})$ the restriction category whose objects are small restriction presheaves on $\mathbb{X}$. By construction, it is also the free cocompletion of $\mathbb{X}$. It is not difficult to check that when $\mathbb{X}$ is a small restriction category, all restriction presheaves on $\mathbb{X}$ are small, and so $\mathcal{P}_r(\mathbb{X}) = \mathsf{PSh}_r(\mathbb{X})$.

5

Cocompletion of join restriction categories

We have seen that restriction categories have a free cocompletion given by the Cockett-Lack embedding, or equivalently, by the category of restriction presheaves. In this chapter, we extend this result to join restriction categories, which are restriction categories whose compatible maps may be patched together. To do this, we first define join restriction category, and characterise $\mathcal{M}$-categories whose associated partial map categories are join restriction categories. Doing so leads to the notion of geometric $\mathcal{M}$-category.

We shall see that every geometric $\mathcal{M}$-category may be given a subcanonical Grothendieck topology, and that the $\mathcal{M}$-category of sheaves on this site is also geometric. Moreover, this $\mathcal{M}$-category of sheaves is the free cocompletion of any geometric $\mathcal{M}$-category. Then, using this fact, we conclude by giving the free cocompletion of any join restriction category.

5.1 Join restriction categories and geometric $\mathcal{M}$-categories

Recall that in the restriction category $\mathbf{Set}_p$, the restriction idempotent on a partial function $f\colon A \rightharpoonup B$ is given by the identity map on the domain of definition of f. If $g\colon A \rightharpoonup B$ is another partial function with f and g agreeing on the intersection of their domains of definition, then $g\bar{f} = f\bar{g}$. Note that the converse is also true. More generally in any restriction category, we may represent such agreements between maps from the same hom-set.

Definition 5.1. Let $\mathbb{X}$ be a restriction category, and $f, g \in \mathbb{X}(A, B)$. We say that f and g are *compatible* if $f\bar{g} = g\bar{f}$, and denote this by $f \smile g$. For any set $S \subset \mathbb{X}(A, B)$, we say that S itself is *compatible* if maps in S are pairwise compatible.

The following lemma is a direct consequence of the definition of compatibility from [Cockett & Guo, 2006].

Lemma 5.2. *Let $\mathbb{X}$ be a restriction category and suppose $f, g \in \mathbb{X}(A, B)$. Then:*

1. *if $f \leq g$, then $f \smile g$, and*
2. *if $f \smile g$ and $\bar{f} = \bar{g}$, then $f = g$.*

Proof. If $f \leq g$, then $f = g\bar{f}$, which implies $f\bar{g} = (g\bar{f})\bar{g} = g\bar{g}\bar{f} = g\bar{f}$, or $f \smile g$.
If $f \smile g$ and $\bar{f} = \bar{g}$, then $f = f\bar{f} = f\bar{g} = g\bar{f} = g\bar{g} = g$. □ □

We observed in $\mathbf{Set}_p$ that two partial functions $f, g\colon A \rightharpoonup B$ satisfied the condition $g\bar{f} = f\bar{g}$ if and only if f and g agreed on the intersection of their domains of definition. If this is the case, we can define a new partial function $f \vee g\colon A \rightharpoonup B$ called the *join* of f and g, whose domain of definition is the union of the domains of definition of f and g. More generally in any *join restriction category*, if a family of maps from the same hom-set is compatible, then its join exists and satisfies the conditions below.

Definition 5.3 (Cockett-Guo)**.** A join restriction category $\mathbb{X}$ is a restriction category such that for each pair $A, B \in \mathbb{X}$ and compatible set $S \subset \mathbb{X}(A, B)$, the join $\bigvee_{s \in S} s$ exists with respect to the partial ordering on $\mathbb{X}(A, B)$, and satisfies the following conditions:

(J1) $\overline{\bigvee_{s\in S} s} = \bigvee_{s \in S} \bar{s}$;

(J2) $(\bigvee_{s\in S} s) \circ g = \bigvee_{s\in S}(s \circ g)$

for any map $g\colon Z \to A$.

Proposition 5.4 (Guo, Lemma 3.18)**.** *Let $\mathbb{X}$ be a join restriction category and let $S \subset \mathbb{X}(A, B)$ be a compatible set. Then*

$$f \circ (\textstyle\bigvee_{s\in S} s) = \bigvee_{s\in S}(f \circ s)$$

for any map $f\colon B \to C$.

Example 5.5. The restriction categories $\mathbf{Set}_p$ and $\mathbf{Top}_p$ from Examples 2.4 and 2.5 respectively are also join restriction categories. For any two compatible partial continuous functions $f\colon A \subseteq X \to Y$ and $g\colon B \subseteq X \to Y$ in $\mathbf{Top}_p$, their join is defined in exactly the same way as in $\mathbf{Set}_p$; that is, their join $f \vee g$ is given by the new partial continous function whose domain of definition is the union $A \cup B \subseteq X$.

Example 5.6. A similar example to the one above is given by the category **fdCts**. An object in this category is a natural number $n \in \mathbb{N}$, and a map $f\colon n \to m$ in **fdCts** is a partial continuous function $\mathbb{R}^n \to \mathbb{R}^m$ on an open subset of $U \subseteq \mathbb{R}^n$.

Example 5.7. For a different example, consider the category $\mathbf{Loc}_p$ of locales and partial locale homomorphisms. Recall that a locale is a partially ordered set with finite meets and arbitrary joins which satisfy the infinite distribute law $x \wedge (\bigvee_{i\in\mathcal{I}} a_i) = \bigvee_{i\in\mathcal{I}} x \wedge a_i$, and that a map from A to B in $\mathbf{Loc}_p$ is a function $f\colon A \leftarrow B$ preserving binary meets and arbitrary joins in B. We can make $\mathbf{Loc}_p$ a restriction category by declaring the restriction on $f\colon A \leftarrow B$ to be $\bar{f}(a) = a \wedge f(\top)$, where $\top$ here denotes the top element in B.

In fact, we can go further by making $\mathbf{Loc}_p$ a join restriction category as follows. Using the definition of compatibility, one can show that a family of maps $\{f_i\colon A \leftarrow B\}_{i\in\mathcal{I}}$ in $\mathbf{Loc}_p$ is compatible if and only if these maps satisfy the condition $f_i(b) \wedge f_j(\top) = f_j(b) \wedge f_i(\top)$ for every pair $i, j \in \mathcal{I}$. Then defining the join of $\{f_i\colon A \leftarrow B\}_{i\in\mathcal{I}}$ pointwise to be $(\bigvee_{i\in\mathcal{I}} f_i)(b) = \bigvee_{i\in\mathcal{I}} f_i(b)$ makes $\mathbf{Loc}_p$ a join restriction category.

If $\mathbb{X}$ and $\mathbb{Y}$ are join restriction categories, then a join restriction functor $F\colon \mathbb{X} \to \mathbb{Y}$ is a restriction functor which preserves the joins in $\mathbb{X}$. There is a 2-category **jrCat** of join restriction categories, join restriction functors and restriction transformations. Note that **jrCat** is a locally full sub-2-category of **rCat**.

Given that $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a restriction category for any $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, it is natural to ask what conditions $(\mathbf{C}, \mathcal{M})$ must satisfy for $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ to be a join restriction category.

Definition 5.8. An $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$ is called *geometric* if $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a join restriction category. An $\mathcal{M}$-functor $F\colon (\mathbf{C}, \mathcal{C}) \to (\mathbf{D}, \mathcal{D})$ between geometric $\mathcal{M}$-categories is also called *geometric* if $\mathsf{Par}(F)$ is a join restriction functor.

There exists a large class of examples of geometric $\mathcal{M}$-categories. As we shall see shortly, any Grothendieck topos together with all monics is, in fact, a geometric $\mathcal{M}$-category (see Example 5.13).

As one would hope, there is also a 2-category $\mathbf{g\mathcal{M}Cat}$ of geometric $\mathcal{M}$-categories, geometric $\mathcal{M}$-functors and $\mathcal{M}$-cartesian natural transformations. With the previous definition, the 2-equivalence between $\mathbf{rCat}_s$ and $\mathbf{\mathcal{M}Cat}$ restricts back to a 2-equivalence between the 2-category of split join restriction categories $\mathbf{jrCat}_s$ and the 2-category of geometric $\mathcal{M}$-categories:

$$\begin{tikzcd} \mathbf{jrCat}_s \arrow[r, "\simeq"] \arrow[d, hook] & \mathbf{g\mathcal{M}Cat} \arrow[d, hook] \\ \mathbf{rCat}_s \arrow[r, "\simeq"'] & \mathbf{\mathcal{M}Cat}. \end{tikzcd}$$

In [Guo, 2012, Theorems 3.3.3, 3.3.5], the author gave a characterisation of these geometric $\mathcal{M}$-categories. However, we will give a different characterisation using only elementary notions of pullbacks and colimits. In proving this theorem, we shall first define the *matching diagram* for any family of $\mathcal{M}$-subobjects in $\mathbf{C}$, and also use a restatement of [Guo, 2012, Lemma 1.6.20].

Definition 5.9. Let $(\mathbf{C}, \mathcal{M})$ be an $\mathcal{M}$-category, and let $M = \{m_i\colon A_i \to A\}_{i\in\mathcal{I}}$ be a family of $\mathcal{M}$-subobjects of A, indexed by the set $\mathcal{I}$. Denote the pullback of m_i along m_j by $m_j^*(m_i)$, as in the following diagram:

$$\begin{tikzcd} A_iA_j \arrow[r, "m_i^*(m_j)"] \arrow[d, "m_j^*(m_i)"'] & A_i \arrow[d, "m_i"] \\ A_j \arrow[r, "m_j"'] & A. \end{tikzcd}$$

We define the *matching diagram* for M as a diagram in $\mathbf{C}$ on the objects $\{A_i \mid i \in \mathcal{I}\} \cup \{A_iA_j \mid i \neq j\}$, and with morphisms the family $\{m_j^*(m_i) \mid i, j \in \mathcal{I}\}$.

Observe that in any $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, any family of $\mathcal{M}$-subobjects in $\mathbf{C}$ forms a cocone under its matching diagram.

Lemma 5.10 (Guo)**.** *Suppose $(m, f), (n, g)\colon A \to B$ are two morphisms in the partial map category* $\mathsf{Par}(\mathbf{C}, \mathcal{M})$*, with $m\colon C \to A$ and $n\colon D \to A$. Then $(m, f) \leq (n, g)$ if and only if there exists a (unique) arrow $\varphi\colon C \to D$ such that $n\varphi = m$ and $g\varphi = f$.*

Theorem 5.11. *An $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$ is geometric if and only if:*

1. *for any family of $\mathcal{M}$-subobjects $\{m_i\colon A_i \to A\}_{i\in\mathcal{I}}$, the colimit $\bigcup_{i\in\mathcal{I}} A_i$ of its matching diagram exists,*
2. *the induced map $\bigvee_{i\in\mathcal{I}} m_i\colon \bigcup_{i\in\mathcal{I}} A_i \to A$ is in $\mathcal{M}$, and*
3. *the colimit from (1), $\bigcup_{i\in\mathcal{I}} A_i$, is stable under pullback.*

Proof. We begin by proving the *if* direction. Let $\{(m_i, f_i)\}_{i\in\mathcal{I}}$ be a compatible family of maps from A to B in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, and let $\mu = \bigvee_{i\in\mathcal{I}} m_i \colon \bigcup_{i\in\mathcal{I}} A_i \to A$ be the unique induced map in $\mathcal{M}$. Now compatibility of $\{(m_i, f_i)\}_{i\in\mathcal{I}}$ means that the family $\{f_i\}_{i\in\mathcal{I}}$ is a cocone to the matching diagram for $\{m_i\}_{i\in\mathcal{I}}$ [Guo, 2012, Lemma 3.1.4]. This induces a unique map $\gamma \colon \bigcup_{i\in\mathcal{I}} A_i \to B$.

$$\begin{tikzcd}
A_iA_j \arrow[rr, "m_i^*(m_j)"] \arrow[dd, "m_j^*(m_i)"'] & & A_i \arrow[dl, "a_i"'] \arrow[dd, "m_i"] \arrow[dddr, bend left, "f_i"] & \\
& \bigcup_{i\in\mathcal{I}} A_i \arrow[dr, dashed, "\mu"'] \arrow[ddrr, dashed, "\gamma"] & & \\
A_j \arrow[ur, "a_j"] \arrow[rr, "m_j"'] \arrow[drrr, bend right, "f_j"'] & & A & \\
& & & B.
\end{tikzcd}$$

We claim that $(\mu, \gamma) = \bigvee_{i\in\mathcal{I}}(m_i, f_i)$. To see this, first observe that $(m_i, f_i) \le (\mu, \gamma)$ for all $i \in \mathcal{I}$ by applying Lemma 5.10, since $\mu a_i = m_i$ and $\gamma a_i = f_i$ by construction. Now suppose for each $i \in \mathcal{I}$, we have $(m_i, f_i) \le (u, v)$, where $u \colon D \to A$ is a map in $\mathcal{M}$. This means that for each i, there is a unique $\beta_i \colon A_i \to D$ such that $m_i = u\beta_i$ and $f_i = v\beta_i$. Since $m_i \circ m_i^*(m_j) = m_j \circ m_j^*(m_i)$ by construction, this implies that $\beta_i \circ m_i^*(m_j) = \beta_j \circ m_j^*(m_i)$ (as u is monic). In other words, the family $\{\beta_i\}_{i\in\mathcal{I}}$ is a cocone to the matching diagram for $\{m_i\}_{i\in\mathcal{I}}$. Therefore, there exists a unique map $\delta \colon \bigcup_{i\in\mathcal{I}} A_i \to D$ such that $\beta_i = \delta a_i$, for all $i \in \mathcal{I}$.

Since $m_i = u\beta_i = (u\delta)a_i$ and $f_i = v\beta_i = (v\delta)a_i$, by uniqueness, we must have $\mu = u\delta$ and $\gamma = v\delta$ (as μ and γ are the only maps satisfying the conditions $\mu a_i = m_i$ and $\gamma a_i = f_i$). Hence, $(\mu, \gamma) \le (u, v)$ by Lemma 5.10.

To see that our definition of (μ, γ) satisfies (J1), note that by construction, $(\mu, \mu) = \bigvee_{i\in\mathcal{I}}(m_i, m_i)$, which means

$$\overline{\bigvee_{i\in\mathcal{I}}(m_i, f_i)} = \overline{(\mu, \gamma)} = (\mu, \mu) = \bigvee_{i\in\mathcal{I}}(m_i, m_i) = \bigvee_{i\in\mathcal{I}} \overline{(m_i, f_i)}.$$

It remains to show that (μ, γ) also satisfies (J2). So let $(x, y) \colon X \to A$ be a map in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$. We need to show $\bigvee_{i\in\mathcal{I}}[(m_i, f_i)(x, y)] = (\mu, \gamma)(x, y)$, or alternatively, $\bigvee_{i\in\mathcal{I}}[(m_i, f_i)(1, y)] = (\mu, \gamma)(1, y)$ and $\bigvee_{i\in\mathcal{I}}[(m_i, f_i)(x, 1)] = (\mu, \gamma)(x, 1)$ since $(x, y) = (1, y)(x, 1)$. Now as composition in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is the same as pulling back in $\mathbf{C}$, the statement $\bigvee_{i\in\mathcal{I}}[(m_i, f_i)(1, y)] = (\mu, \gamma)(1, y)$ is equivalent to $y^*(\bigvee_{i\in\mathcal{I}} m_i) = \bigvee_{i\in\mathcal{I}} y^*(m_i)$, which is true as colimits of this form are stable under pullback by assumption. To show $\bigvee_{i\in\mathcal{I}}[(m_i, f_i)(x, 1)] = (\mu, \gamma)(x, 1)$, simply note that the family $\{xm_i\}_{i\in\mathcal{I}}$ gives rise to the same matching diagram as for $\{m_i\}_{i\in\mathcal{I}}$.

In the *only if* direction, let $\{m_i \colon A_i \to A\}_{i\in\mathcal{I}}$ be a family of $\mathcal{M}$-subobjects of A. As $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a join restriction category, denote $(\mu, \mu) = \bigvee_{i\in\mathcal{I}}(m_i, m_i)$, where $\mu \colon \bigcup_{i\in\mathcal{I}} A_i \to A$. Note that $\mu \in \mathcal{M}$ by definition. Also, since $(m_i, m_i) \le (\mu, \mu)$ for all $i \in I$, there exists a unique $a_i \colon A_i \to \bigcup_{i\in\mathcal{I}} A_i$ (for each $i \in \mathcal{I}$) such that $m_i = \mu a_i$. Observe that each $a_i \in \mathcal{M}$ as a_i is a pullback of m_i along μ. We now show that the family $\{a_i\}_{i\in\mathcal{I}}$ is a colimit to the matching diagram for $\{m_i\}_{i\in\mathcal{I}}$.

Clearly $\{a_i\}_{i\in\mathcal{I}}$ is a cocone to the matching diagram. Now let $\{b_i \colon A_i \to B\}_{i\in\mathcal{I}}$ be a cocone to the same matching diagram; that is, $b_i \circ m_i^*(m_j) = b_j \circ m_j^*(m_i)$ for each pair $i, j \in \mathcal{I}$. But as this implies that the family $\{(m_i, b_i)\}_{i\in\mathcal{I}}$ is compatible, we may take their join, which we denote by $(s, t) = \bigvee_{i\in\mathcal{I}}(m_i, b_i)$. By join restriction axioms,

$$(s, s) = \overline{(s, t)} = \overline{\bigvee_{i\in\mathcal{I}}(m_i, b_i)} = \bigvee_{i\in\mathcal{I}}(m_i, m_i) = (\mu, \mu),$$

which means $s = \mu$ (up to isomorphism). But because $(m_i, b_i) \leq (s,t) = (\mu,t)$, there exists an α_i such that $m_i = \mu\alpha_i$ and $b_i = t\alpha_i$ (for every $i \in \mathcal{I}$). However, a_i is the only map with the property $m_i = \mu a_i$. Therefore, we must have $\alpha_i = a_i$, which in turn implies that $b_i = ta_i$ for all $i \in \mathcal{I}$. We need to show that t is in fact the unique map with this property.

So suppose t' also satisfies the condition $b_i = t'a_i$. Then $(m_i, b_i) \leq (\mu, t')$ for all $i \in \mathcal{I}$, which means $(\mu,t) = \bigvee_{i\in\mathcal{I}}(m_i, b_i) \leq (mu, t')$. Therefore, $t = t'$ and so $\{a_i\}_{i\in\mathcal{I}}$ is indeed the required colimit to the matching diagram for $\{m_i\}_{i\in\mathcal{I}}$.

Observe that by the previous argument, the family $\{m_i\}_{i\in\mathcal{I}}$ will be a colimit to its matching diagram if and only if $\bigvee_{i\in\mathcal{I}}(m_i, m_i) = (1,1)$ if and only if $\mu = 1$. With this observation, it is easy to show that the colimit $\bigcup_{i\in\mathcal{I}} A_i$ is stable under pullback by noting that pullbacks in $\mathbf{C}$ are the same as composition in $\mathsf{Par}(\mathbf{C},\mathcal{M})$ and applying join restriction axioms. □

Remark 5.12. Substituting $\mathcal{I}$ to be the empty set in the above theorem tells us that if $(\mathbf{C},\mathcal{M})$ is a geometric $\mathcal{M}$-category, then $\mathbf{C}$ must have a strict initial object 0, and that maps $0 \to A$ are in $\mathcal{M}$ (for all $A \in \mathbf{C}$). Also, by the previous characterisation, it should now be clear why we have called such $\mathcal{M}$-categories *geometric*; instead of posets of subobjects having unions which are stable under pullback, we have posets of $\mathcal{M}$-subobjects.

Example 5.13. Every Grothendieck topos together with all monomorphisms is a geometric $\mathcal{M}$-category. This follows from a generalisation of [Johnstone, 2002, Proposition 1.4.3]. In particular, for every category $\mathbf{C}$ and site $(\mathbf{C}, J)$, the $\mathcal{M}$-categories $(\mathsf{PSh}(\mathbf{C}), \text{all monics})$ and $(\mathsf{Sh}(\mathbf{C}), \text{all monics})$ are geometric. We shall revisit the notions of Grothendieck topology and sheaf in the next section.

The following result follows immediately from Theorem 5.11.

Proposition 5.14. *An $\mathcal{M}$-functor $F\colon (\mathbf{C},\mathcal{C}) \to (\mathbf{D},\mathcal{D})$ between geometric $\mathcal{M}$-categories is geometric if and only if F preserves colimits of matching diagrams.*

By Theorem 5.11, if $(\mathbf{C},\mathcal{M})$ is a geometric $\mathcal{M}$-category, then any family of $\mathcal{M}$-subobjects $\{m_i\colon A_i \to A\}_{i\in\mathcal{I}}$ has a join given by the unique induced map $\bigvee_{i\in\mathcal{I}} m_i\colon \bigcup_{i\in\mathcal{I}} A_i \to A$. In fact:

Proposition 5.15. *If $(\mathbf{C},\mathcal{M})$ is a geometric $\mathcal{M}$-category, then for all $C \in \mathbf{C}$, $\mathsf{Sub}_{\mathcal{M}}(C)$ is a complete Heyting algebra and for each $f\colon D \to C$, the function $f^*\colon \mathsf{Sub}_{\mathcal{M}}(C) \to \mathsf{Sub}_{\mathcal{M}}(D)$ preserves joins.*

Proof. Let $\{m_i\colon A_i \to C\}_{i\in\mathcal{I}}$ be a family of $\mathcal{M}$-subobjects of C and define the join of the family of $\mathcal{M}$-subobjects to be the induced map $\bigcup_{i\in\mathcal{I}} A_i \to C$. As the colimit $\bigcup_{i\in\mathcal{I}} A_i$ is stable under pullback, it follows that $f^*\colon \mathsf{Sub}_{\mathcal{M}}(C) \to \mathsf{Sub}_{\mathcal{M}}(D)$ preserves all joins. Furthermore, since $\mathsf{Sub}_{\mathcal{M}}(C)$ has all joins, it also has all meets, and so it remains to show that joins distribute over finite meets.

So let $m\colon B \to C$ be any $\mathcal{M}$-subobject. Then $m^*\colon \mathsf{Sub}_{\mathcal{M}}(C) \to \mathsf{Sub}_{\mathcal{M}}(B)$ has a left adjoint given by $m \circ (-)\colon \mathsf{Sub}_{\mathcal{M}}(B) \to \mathsf{Sub}_{\mathcal{M}}(C)$. But their composite is $m^* \circ \Big(m \circ (-)\Big) = m \wedge (-)$, and both m^* and $m \circ (-)$ preserve joins. Therefore joins distribute over finite meets, and $\mathsf{Sub}_{\mathcal{M}}(C)$ is a complete Heyting algebra. □

5.2 Grothendieck topologies and sheaves

5.2.1 Grothendieck topology and basis

Before continuing with our discussion on geometric $\mathcal{M}$-categories and their free cocompletion, we shall briefly recall the notion of Grothendieck topology, and the notion of sheaf. Indeed, these two notions will prove critical in our understanding of cocomplete geometric $\mathcal{M}$-categories. We begin with the definition of a sieve.

Definition 5.16. Given a category $\mathbf{C}$, a *sieve* S on $C \in \mathbf{C}$ is a family of morphisms with codomain C such that if $g \in S$, then for all f such that gf is a composable pair, the composite gf is also in S.

Another way to describe a sieve on $C \in \mathbf{C}$, is as a subobject of $\mathbf{y}C$ in the category $\mathsf{PSh}(\mathbf{C})$. To see why this is the case, suppose we are given a sieve S on C in some category $\mathbf{C}$. Then there is a presheaf $Q\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ which on objects, takes $A \in \mathbf{C}$ to the set

$$Q(A) = \{f\colon A \to C \mid f \in S\}.$$

Clearly $QA \subset \mathsf{Hom}(A, C)$ for all $A \in \mathbf{C}$, which makes Q a subobject of $\mathbf{y}C$. Conversely, given a presheaf $Q\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ which is a subobject of $\mathbf{y}C$, then we may define a family of morphisms S with codomain C by

$$S = \{f \mid \mathsf{cod}(f) = C, f \in QA \text{ for some } A \in \mathbf{C}\}.$$

For any $f\colon B \to A$ in $\mathbf{C}$, Qf is precomposition by f, as Q is a subobject of $\mathbf{y}C$. Therefore, S is also a sieve on C. Clearly these two processes are inverses of another, and so sieves on C are really subobjects of $\mathbf{y}C$ in $\mathsf{PSh}(\mathbf{C})$. We will be using this characterisation of sieves extensively in the next section.

Also, notice that in a category $\mathbf{C}$, for any sieve S on C and map $f\colon A \to C$, the family of morphisms

$$f^*(S) = \{h \mid \mathsf{cod}(h) = A, fh \in S\}$$

is also a sieve. This is because in the presheaf category $\mathsf{PSh}(\mathbf{C})$, the pullback of any sieve $S \rightarrowtail \mathbf{y}C$ along the map $\mathbf{y}f\colon \mathbf{y}A \to \mathbf{y}C$ is a subobject of $\mathbf{y}A$, and hence explains the use of the notation $f^*(S)$. We now define a Grothendieck topology.

Definition 5.17. Given a category $\mathbf{C}$, a *Grothendieck topology* J on $\mathbf{C}$ is a function which assigns, for each object $C \in \mathbf{C}$, a collection $J(C)$ of sieves on C such that:

1. the maximal sieve $\{f \mid \mathsf{cod}(f) = C\}$ is in $J(C)$;
2. if $S \in J(C)$ and $h\colon D \to C$ is any map in $\mathbf{C}$, then $h^*(S) \in J(D)$; and
3. if $S \in J(C)$ and R is *any sieve* on C, if $h^*(R) \in J(D)$ for all $h\colon D \to C$, then R is also in $J(C)$.

The sieves in $J(C)$ are called *covering sieves*. We call a category equipped with a Grothendieck topology, a *site*, and denote such a pair as $(\mathbf{C}, J)$.

Example 5.18. Let $\mathbf{X}$ be a topological space, and consider the poset of open subsets $\mathcal{O}(\mathbf{X})$ of $\mathbf{X}$ as a category. In particular, there is a map from U to V in $\mathcal{O}(\mathbf{X})$ if and only if $U \subseteq V$. So then a sieve S on U is simply a collection of downward closed subsets of U; that is, $V' \subset V \in S$ implies $V' \in S$. We may then define a Grothendieck topology J on $\mathcal{O}(\mathbf{X})$ in the obvious way, so that a sieve S is in $J(U)$ if and only if U is contained in the union of the open subsets of S. In other words, $S \in J(U)$ if and only if the open subsets of S cover U.

The above example leads to our next point of discussion. Recall that with point-set topology, rather than specifying the topology on a given set in terms of the open sets, we may instead opt to describe the topology in terms of its basis. Indeed, there is a corresponding notion of basis for a Grothendieck topology.

Definition 5.19. A basis for a Grothendieck topology on a category $\mathbf{C}$ is a function K which assigns to each object $C \in \mathbf{C}$, a family of morphisms $K(C)$ with codomain C, such that:

1. if $f\colon A \to C$ is an isomorphism in $\mathbf{C}$, then $\{f\colon A \to C\} \in K(C)$;
2. if the family $\{f_i\colon C_i \to C\}_{i\in I}$ is in $K(C)$, then for all maps $g\colon D \to C$, the family of pullbacks $\{\pi_2\colon C_i \times_C D \to D\}_{i\in I}$ along g exist and are in $K(D)$; and
3. if $\{f_i\colon C_i \to C\}_{i\in I} \in K(C)$ and for each $i \in I$, $\{g_{ij}\colon C_{ij} \to C_i\}_{j\in J_i}$ is in $K(C_i)$, then the composite family $\{f_i \circ g_{ij}\colon C_{ij} \to C\}_{i\in I, j\in J_i}$ is in $K(C)$.

If $R \in K(C)$, then we say that R is a *basic cover* of C.

Given a basis K on a category $\mathbf{C}$, the topology J *generated* by K is as follows: for all objects $C \in \mathbf{C}$, we have $S \in J(C)$ whenever there exists an $R \in K(C)$ with $R \subseteq S$. It is then a matter of verifying that the collection of sieves for each $C \in \mathbf{C}$ satisfies the topology axioms. Conversely, given a topology J on $\mathbf{C}$, there is a *maximal* basis K which generates J. In more detail, if R is some basic cover of $C \in \mathbf{C}$, then we say that $R \in K(C)$ if and only if the sieve on C generated by R,

$$S = \{fg \mid f \in K(C), \mathsf{dom}(f) = \mathsf{cod}(g)\},$$

is in $J(C)$.

Before moving on to the notion of sheaf, observe that our definition of basis involves categories with certain pullbacks. Although there is an alternative definition of basis which avoids this assumption, this will not be necessary as the categories with which we will be working have “sufficient” pullbacks.

5.2.2 Sheaves on a site and the associated sheaf functor

Let us begin with a site $(\mathbf{C}, J)$, and let $P\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf on $\mathbf{C}$. Suppose S is a covering sieve on C. Then a *matching family* for S of elements of P is a function which sends every element $f\colon D \to C$ of S to an element $x_f \in P(D)$ such that for every $g\colon E \to D$, we have $x_f \cdot g = x_{fg}$. If M is a matching family for S, then an *amalgamation* of M is a single element $x \in P(C)$ satisfying the condition $x \cdot f = x_f$ for every $f \in S$.

Definition 5.20. Let $(\mathbf{C}, J)$ be a site and suppose $P\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ is a presheaf on $\mathbf{C}$. Then P is a *sheaf* (for J) if and only if every matching family for all covering sieves S has a unique amalgamation. On the other hand, if P is such that every matching family has *at most one amalgamation*, then P is called a *separated presheaf*.

An equivalent way to express the sheaf condition is as follows. We know that every sieve S on C is a subobject of $\mathbf{y}C$ in the presheaf category $\mathsf{PSh}(\mathbf{C})$. If $P\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ is a presheaf on $\mathbf{C}$, then P is also a sheaf if and only if for all natural transformations $S \Rightarrow P$, there is a unique extension $\mathbf{y}C \Rightarrow P$ making the following diagram commute:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & P \\ \mathbf{y}C. \arrow[ur, dashed] & \end{tikzcd}$$

Now suppose our topology J has a basis K, and that our category $\mathbf{C}$ has pullbacks. Then a sheaf for J may be described in terms of only its basis. Let $R = \{f_i \colon C_i \to C\}_{i \in I} \in K(C)$, and let $\{x_i \mid x_i \in P(C_i)\}_{i \in I}$ be a family of elements (picking exactly one element from each $P(C_i)$). We say such a family of elements is matching for R if and only if for all $i, j \in I$, we have $x_i \cdot f_i^*(f_i) = x_j \cdot f_j^*(f_i)$, where $f_i^*(f_i)$ and $f_j^*(f_i)$ are the pullbacks below:

$$\begin{tikzcd} C_i \times_C C_j \arrow[r, "f_j^*(f_i)"] \arrow[d, "f_i^*(f_i)"'] & C_j \arrow[d, "f_j"] \\ C_i \arrow[r, "f_i"'] & C. \end{tikzcd}$$

If the family $\{x_i \mid x_i \in P(C_i)\}_{i \in I}$ is a matching family for R, then an *amalgamation* for this matching family is a single element $x \in P(C)$ satisfying the conditions $x \cdot f_i = x_i$ for all $i \in I$.

Proposition 5.21. *Let* $(\mathbf{C}, J)$ *be a site, and let* K *be the maximal basis generating the topology* J*. Then the presheaf* $P \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ *is a sheaf for* J *if and only if for all basic covers* $R = \{f_i \colon C_i \to C\}_{i \in I} \in K(C)$*, every matching family* $\{x_i \mid x_i \in P(C_i)\}_{i \in I}$ *for* R *has a unique amalgamation.*

Proof. See [Mac Lane & Moerdijk, p.123]. □

For any site $(\mathbf{C}, J)$, we know that every sheaf P on this site is a presheaf $P \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. So we may define a category of sheaves on this site as a full subcategory $\mathsf{PSh}(\mathbf{C})$ of presheaves on $\mathbf{C}$. The inclusion $\mathsf{Sh}(\mathbf{C}, J) \hookrightarrow \mathsf{PSh}(\mathbf{C})$ has a left adjoint, called the *associated sheaf functor*. We shall not say much more about this associated sheaf functor, which we shall denote by the letter $\mathbf{a}$, other than stating the following fact.

Proposition 5.22. *Let* $(\mathbf{C}, J)$ *be a site, and let* S *be a sieve on* $C \in \mathbf{C}$*, considered as a subobject* $S \rightarrowtail \mathbf{y}C$*. Then* S *is a covering sieve if and only if* $\mathbf{a}(S \rightarrowtail \mathbf{y}C)$ *is an isomorphism in* $\mathsf{Sh}(\mathbf{C}, J)$*.*

Proof. See [Borceux, 1994, Lemma 3.5.1]. □

5.3 Free cocompletion of geometric $\mathcal{M}$-categories

In this section, we continue our discussion of geometric $\mathcal{M}$-categories. The goal of this section will be to show that every small geometric $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$ may be freely completed to a cocomplete geometric $\mathcal{M}$-category, where *cocomplete*, as defined in a previous chapter, means that $\mathbf{C}$ is cocomplete and the inclusion $\mathbf{C} \hookrightarrow \mathsf{Par}(\mathbf{C}, \mathcal{M})$ preserves colimits. The way we will show this is as follows.

First, we show that for every small geometric $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, its underlying category $\mathbf{C}$ may be given a Grothendieck topology J. This allows us to form an $\mathcal{M}$-category of sheaves $(\mathsf{Sh}(\mathbf{C}), \mathsf{Sh}(\mathcal{M}))$ on this site $(\mathbf{C}, J)$, for some class of monics $\mathsf{Sh}(\mathcal{M})$ in $\mathsf{Sh}(\mathbf{C})$. We then show that this $\mathcal{M}$-category $(\mathsf{Sh}(\mathbf{C}), \mathsf{Sh}(\mathcal{M}))$ is the free cocompletion of the geometric $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$.

5.3.1 Partial map category of sheaves

We begin with the following proposition.

Proposition 5.23. *Let* $(\mathbf{C}, \mathcal{M})$ *be a geometric* $\mathcal{M}$*-category and let* $C \in \mathbf{C}$*. Then there is a Grothendieck topology on* $\mathbf{C}$ *whose basic covers of* $C \in \mathbf{C}$ *are given by families of the following form:*

$$\{a_i \colon C_i \to C \mid a_i \in \mathcal{M}, \bigvee_{i \in \mathcal{I}} a_i = 1 \textit{ in } \mathsf{Sub}_{\mathcal{M}}(C)\}_{i \in \mathcal{I}}\,.$$

Equivalently, by Theorem 5.11, $\{a_i\}_{i\in\mathcal{I}}$ *is a basic cover of* C *if* C *is the colimit of a matching diagram for some family of* $\mathcal{M}$*-subobjects.*

Proof. Clearly $\{1_C \colon C \to C\}$ is a basic cover of C as $1_C \in \mathcal{M}$. If $f^*(a_i)$ is the pullback of a_i along $f \colon D \to C$ for each $i \in \mathcal{I}$, then $\{f^*(a_i)\}_{i\in\mathcal{I}}$ is also a basic cover of D as unions of $\mathcal{M}$-subobjects are stable under pullback.

Finally, for each $i \in \mathcal{I}$ and C_i, suppose $\{b_{ij}\}_{j\in\mathcal{J}_i}$ is a basic cover of C_i. We need to show that $\{a_i \circ b_{ij}\}_{i\in\mathcal{I}, j\in\mathcal{J}_i}$ is a cover of C. First note that $a_i \circ b_{ij} \in \mathcal{M}$ for each $i \in \mathcal{I}, j \in \mathcal{J}_i$ as $\mathcal{M}$ is closed under composition. Then since

$$\bigvee_{i\in\mathcal{I}, j\in\mathcal{J}_i} a_i b_{ij} = \bigvee_{i\in\mathcal{I}} \left(a_i \circ \left(\bigvee_{j\in\mathcal{J}_i} b_{ij}\right)\right) = \bigvee_{i\in\mathcal{I}} a_i = 1,$$

the family $\{a_i \colon C_i \to C\}_{i\in\mathcal{I}}$ above describes a basic cover of C for each $C \in \mathbf{C}$. □

Lemma 5.24. *Suppose* $(\mathbf{C}, \mathcal{M})$ *is a small geometric* $\mathcal{M}$*-category and let* J *be the topology generated by the basis described in Proposition 5.23. Then* J *is subcanonical.*

Proof. We need to show all representable presheaves on $\mathbf{C}$ are sheaves on the site $(\mathbf{C}, J)$. So let $D \in \mathbf{C}$ and consider the representable $\mathbf{y}D$. By Proposition 1 in [Mac Lane & Moerdijk, p. 123], $\mathbf{y}D$ is a sheaf if and only if for any basic cover $R = \{a_i \colon C_i \to C\}_{i\in\mathcal{I}}$, any matching family $\{x_i \in (\mathbf{y}D)(C_i)\}_{i\in\mathcal{I}}$ for R has a unique amalgamation. Consider the following pullback square (for some $i, j \in \mathcal{I}$):

$$\begin{tikzcd} C_iC_j \arrow[r, "a_i^*(a_j)"] \arrow[d, "a_j^*(a_i)"'] & C_i \arrow[d, "a_i"] \\ C_j \arrow[r, "a_j"'] & C. \end{tikzcd}$$

Let $\{x_i \in (\mathbf{y}D)(C_i)\}_{i\in\mathcal{I}}$ be a matching family for R. By definition, this implies that $x_i \circ a_i^*(a_j) = x_j \circ a_j^*(a_i)$, or that $\{x_i\}_{i\in\mathcal{I}}$ is a cocone to the diagram for which $\{a_i\}_{i\in\mathcal{I}}$ is a colimit. This means there exists a unique $x \colon C \to D$ such that $x \circ a_i = x_i$ for all $i \in \mathcal{I}$. In other words, this x is the unique amalgamation of $\{x_i\}_{i\in\mathcal{I}}$. Hence, the representable $\mathbf{y}D$ is a sheaf, and J is subcanonical. □

Now recall that if $(\mathbf{C}, \mathcal{M})$ is an $\mathcal{M}$-category, then there is an $\mathcal{M}$-category of presheaves over $\mathbf{C}$, denoted by $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$, or $(\mathsf{PSh}(\mathbf{C}), \mathsf{PSh}(\mathcal{M}))$, and that a map $\mu \colon P \Rightarrow Q$ is in $\mathsf{PSh}(\mathcal{M})$ if for every $\alpha \colon R \Rightarrow Q$, there is an $m \colon A \to B$ in $\mathcal{M}$ making the following a pullback square:

$$\begin{tikzcd} \mathbf{y}A \arrow[r] \arrow[d, "\mathbf{y}m"'] & P \arrow[d, "\mu"] \\ \mathbf{y}B \arrow[r, "\alpha"'] & Q. \end{tikzcd}$$

We now define the $\mathcal{M}$-category of sheaves on $\mathbf{C}$.

Definition 5.25. Suppose $(\mathbf{C}, \mathcal{M})$ is a small geometric $\mathcal{M}$-category. Denote the $\mathcal{M}$-category of sheaves on $\mathbf{C}$ by $\mathsf{Sh}_{\mathcal{M}}(\mathbf{C})$ or $(\mathsf{Sh}(\mathbf{C}), \mathsf{Sh}(\mathcal{M}))$, where a map $\mu \colon P \Rightarrow Q$ is in $\mathsf{Sh}(\mathcal{M})$ if and only if for every map $\alpha \colon \mathbf{ay}D \Rightarrow Q$ in $\mathsf{Sh}(\mathbf{C})$, there is a map $m \colon C \to D$ in $\mathcal{M}$ making the following a pullback square:

$$\begin{tikzcd} \mathbf{ay}C \arrow[r] \arrow[d, "\mathbf{ay}m"'] & P \arrow[d, "\mu"] \\ \mathbf{ay}D \arrow[r, "\alpha"'] & Q \end{tikzcd}$$

where $\mathbf{a} \colon \mathsf{PSh}(\mathbf{C}) \to \mathsf{Sh}(\mathbf{C})$ is the associated sheaf functor.

Remark 5.26. Note that $\mathsf{Sh}(\mathcal{M}) = \mathsf{PSh}(\mathcal{M}) \cap \mathsf{Sh}(\mathbf{C})$.

Recall that $(\mathsf{Sh}(\mathbf{C}), \text{all monics})$ and $(\mathsf{PSh}(\mathbf{C}), \text{all monics})$ are geometric $\mathcal{M}$-categories, with $\mathsf{Sh}(\mathbf{C})$ being the category of sheaves on some small site $(\mathbf{C}, J)$, and that there is a functor $\mathbf{a} \colon \mathsf{PSh}(\mathbf{C}) \to \mathsf{Sh}(\mathbf{C})$ called the associated sheaf functor. This associated sheaf functor is then also a geometric $\mathcal{M}$-functor between the $\mathcal{M}$-categories $(\mathsf{PSh}(\mathbf{C}), \text{all monics})$ and $(\mathsf{Sh}(\mathbf{C}), \text{all monics})$, as it not only preserves all colimits, but also all finite limits.

Lemma 5.27. *Suppose $(\mathbf{C}, \mathcal{M})$ is a small geometric $\mathcal{M}$-category. Then the $\mathcal{M}$-functor $\mathbf{ay} \colon (\mathbf{C}, \mathcal{M}) \to (\mathsf{Sh}(\mathbf{C}), \textit{all monics})$ is geometric.*

Proof. Let $\{C_i \to D\}_{i \in \mathcal{I}}$ be a family of $\mathcal{M}$-subobjects of D in $\mathbf{C}$, and consider the basic cover $\{C_i \to \bigcup_{i \in \mathcal{I}} C_i\}_{i \in \mathcal{I}}$. The associated covering sieve (as a subfunctor) is given by

$$\textstyle\bigcup_{i \in \mathcal{I}} \mathbf{y}C_i \to \mathbf{y}\left(\bigcup_{i \in \mathcal{I}} C_i\right) \tag{5.1}$$

in $\mathsf{PSh}(\mathbf{C})$, and the associated sheaf functor $\mathbf{a}$ takes this map to an isomorphism in $\mathsf{Sh}(\mathbf{C})$. Hence, as $\mathbf{a}$ is a left adjoint, we have $\mathbf{ay}\left(\bigcup_{i \in \mathcal{I}} C_i\right) \cong \bigcup_{i \in \mathcal{I}} \mathbf{ay}C_i$. □

Theorem 5.28. *If $(\mathbf{C}, \mathcal{M})$ is a small geometric $\mathcal{M}$-category, then the $\mathcal{M}$-category $\mathsf{Sh}_{\mathcal{M}}(\mathbf{C})$ is also geometric.*

Proof. First note that $\mathsf{Sh}(\mathcal{M})$ is a subset of all the monics in $\mathsf{Sh}(\mathbf{C})$. So to show that the $\mathcal{M}$-category $(\mathsf{Sh}(\mathbf{C}), \mathsf{Sh}(\mathcal{M}))$ is geometric, if suffices to prove that if $\{\alpha_i \colon P_i \to Q\}_{i \in \mathcal{I}}$ are $\mathsf{Sh}(\mathcal{M})$-subobjects of Q, then the induced monic $\bigvee_{i \in \mathcal{I}} \alpha_i \colon \bigcup_{i \in \mathcal{I}} P_i \to Q$ is also in $\mathsf{Sh}(\mathcal{M})$.

Denote the pullback of each α_i along $\gamma \colon \mathbf{ay}D \to Q$ by $\mathbf{ay}m_i$, with $m_i \in \mathcal{M}$:

$$\begin{tikzcd} \mathbf{ay}C_i \arrow[r] \arrow[d, "\mathbf{ay}m_i"'] & P_i \arrow[d, "\alpha_i"] \\ \mathbf{ay}D \arrow[r, "\gamma"'] & Q. \end{tikzcd}$$

Since $(\mathsf{Sh}(\mathbf{C}), \text{all monics})$ is geometric, by stability, the pullback of $\bigvee_{i \in \mathcal{I}} \alpha_i$ along the same γ is $\bigvee_{i \in \mathcal{I}} \mathbf{ay}m_i$:

$$\begin{tikzcd} \bigcup_{i \in \mathcal{I}} \mathbf{ay}C_i \arrow[r] \arrow[d, "\bigvee_{i \in \mathcal{I}} \mathbf{ay}m_i"'] & \bigcup_{i \in \mathcal{I}} P_i \arrow[d, "\bigvee_{i \in \mathcal{I}} \alpha_i"] \\ \mathbf{ay}D \arrow[r, "\gamma"'] & Q. \end{tikzcd}$$

But observe that the following diagram commutes by Lemma 5.27:

$$\begin{tikzcd}
\bigcup_{i \in \mathcal{I}} \mathbf{ay} C_i \arrow[rr, "\cong"] \arrow[dr, "\bigvee_{i \in \mathcal{I}} \mathbf{ay} m_i"'] & & \mathbf{ay}\left(\bigcup_{i \in \mathcal{I}} C_i\right) \arrow[dl, "\mathbf{ay}(\bigvee_{i \in \mathcal{I}} m_i)"] \\
& \mathbf{ay} D. &
\end{tikzcd}$$

Hence, the pullback of $\bigvee_{i \in \mathcal{I}} \alpha_i$ along γ is $\mathbf{ay}\left(\bigvee_{i \in \mathcal{I}} m_i\right)$, and so $\bigvee_{i \in \mathcal{I}} \alpha_i \in \mathsf{Sh}(\mathcal{M})$, which means $\mathsf{Sh}_{\mathcal{M}}(\mathbf{C})$ is geometric. $\square$

We have shown that $(\mathsf{Sh}(\mathbf{C}), \mathsf{Sh}(\mathcal{M}))$ is geometric if $(\mathbf{C}, \mathcal{M})$ is small geometric. For the same reason that $\mathsf{Sh}(\mathbf{C})$ is a cocomplete category, we would like to show $(\mathsf{Sh}(\mathbf{C}), \mathsf{Sh}(\mathcal{M}))$ is cocomplete as an $\mathcal{M}$-category.

5.3.2 $\mathcal{M}$-category of sheaves is cocomplete

Definition 5.29. A cocomplete geometric $\mathcal{M}$-category is one which is both cocomplete and geometric. This gives the 2-category $\mathbf{g\mathcal{M}Cocomp}$ of cocomplete geometric $\mathcal{M}$-categories, cocontinuous $\mathcal{M}$-functors and $\mathcal{M}$-cartesian natural transformations.

Observe that by Proposition 5.14, if $F\colon (\mathbf{C}, \mathcal{C}) \to (\mathbf{D}, \mathcal{D})$ is a cocontinuous $\mathcal{M}$-functor between geometric $\mathcal{M}$-categories, then it is also geometric.

Proposition 5.30. *If* $(\mathbf{C}, \mathcal{M})$ *is a small geometric $\mathcal{M}$-category, then the $\mathcal{M}$-category of sheaves* $(\mathsf{Sh}(\mathbf{C}), \mathsf{Sh}(\mathcal{M}))$ *is cocomplete.*

Proof. By Remark 2.20, it suffices to show $\mathsf{Sh}_{\mathcal{M}}(\mathbf{C})$ has an $\mathcal{M}$-subobject classifier.

From [Rosolini, 1986, Proposition 3.1.1], $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$ has an $\mathcal{M}$-subobject classifier Σ taking objects $C \in \mathbf{C}$ to $\mathsf{Sub}_{\mathcal{M}}(C)$, and morphisms f to f^* (by pullback along f). Moreover, the map $\tau\colon 1 \Rightarrow \Sigma$ is in $\mathsf{PSh}(\mathcal{M})$. We will show that this Σ is a sheaf, and then because $\mathsf{Sh}(\mathcal{M}) = \mathsf{PSh}(\mathcal{M}) \cap \mathsf{Sh}(\mathbf{C})$, it follows that $\tau\colon 1 \Rightarrow \Sigma$ is an $\mathcal{M}$-subobject classifier in $\mathsf{Sh}_{\mathcal{M}}(\mathbf{C})$.

We begin by showing that Σ is a separated presheaf. Let $R = \{a_i\colon A_i \to C\}_{i \in \mathcal{I}}$ be a basic cover of $C \in \mathbf{C}$, and let $M = \{m_i\colon B_i \to A_i\}_{i \in \mathcal{I}}$ be a matching family for R. Now suppose $x, y \in \mathsf{Sub}_{\mathcal{M}}(C)$ are two amalgamations for M. Then pulling either x or y back along a_i gives m_i (for all $i \in \mathcal{I}$). That is, $a_i^*(x) = a_i^*(y)$, and so post-composing both sides by a_i yields $a_i \wedge x = a_i \wedge y$.

The families $\{a_i \wedge x\}_{i \in \mathcal{I}}$ and $\{a_i \wedge y\}_{i \in \mathcal{I}}$ are families of monics, so we may take joins over all $i \in \mathcal{I}$, giving $\bigvee_{i \in \mathcal{I}}(a_i \wedge x) = \bigvee_{i \in \mathcal{I}}(a_i \wedge y)$. However, since $\mathsf{Sub}_{\mathcal{M}}(C)$ is a Heyting algebra, we get

$$x \wedge \bigvee_{i \in \mathcal{I}} a_i = y \wedge \bigvee_{i \in \mathcal{I}} a_i$$

by distributivity, and so $x = y$ as $\bigvee_{i \in \mathcal{I}} a_i = 1$ by definition. Therefore, Σ is a separated presheaf. It remains to show that any matching family for R has an amalgamation.

Again, let R be a basic cover of C and M a matching family for R as above, and consider the following diagram:

$$\begin{tikzcd}
& Z_i \arrow[r, "b_i"] \arrow[d, "n_i"'] & B_i \arrow[d, "m_i"] \\
Z_j \arrow[r, "n_j"] \arrow[d, "b_j"'] & A_i A_j \arrow[r, "a_i^*(a_j)"'] \arrow[d, "a_j^*(a_i)"'] & A_i \arrow[d, "a_i"] \\
B_j \arrow[r, "m_j"'] & A_j \arrow[r, "a_j"'] & C.
\end{tikzcd}$$

The above squares are all pullback squares. To say that m_i and m_j belong to the same matching family is to say that $n_i = n_j$ in $\mathsf{Sub}_{\mathcal{M}}(A_i A_j)$. Then because all squares are pullbacks, this implies that $a_i m_i b_i = a_j m_j b_j$ (by a quick diagram chase), or alternatively,

$$a_j \wedge a_i m_i = a_i \wedge a_j m_j \tag{5.2}$$

by writing compositions as intersections. So if m_i and m_j come from the same matching family for R, then they must satisfy (5.2).

We claim that the (unique) amalgamation for M is $\bigvee_{i\in\mathcal{I}} a_i m_i$. In other words, we need to show that $a_j^*(\bigvee_{i\in\mathcal{I}} a_i m_i) = m_j$ for all $j \in I$. However, as a_j is monic for all $j \in \mathcal{I}$, the previous equality holds if and only if $a_j \circ a_j^*(\bigvee_{i\in\mathcal{I}} a_i m_i) = a_j \circ m_j$, which is true if and only if

$$a_j \wedge (\textstyle\bigvee_{i\in\mathcal{I}} a_i m_i) = a_j \circ m_j, \quad \forall j \in \mathcal{I}.$$

Examining the left hand side, we have (by the distributive law),

$$\begin{aligned} a_j \wedge (\textstyle\bigvee_{i\in\mathcal{I}} a_i m_i) = \textstyle\bigvee_{i\in\mathcal{I}}(a_j \wedge a_i m_i) &= (a_j \wedge a_j m_j) \vee \left(\textstyle\bigvee_{i\neq j} a_j \wedge a_i m_i\right) \\ &= a_j m_j \vee \left(\textstyle\bigvee_{i\neq j} a_i \wedge a_j m_j\right) \end{aligned}$$

using (5.2) and the fact $a_j m_j \le a_j$ (as an $\mathcal{M}$-subobject). But for all $i \neq j$, $a_i \wedge a_j m_j \le a_j m_j$ (since there is an arrow b_j from the domain of $a_i \wedge a_j m_j$ to $a_j m_j$). Therefore, this means that $\left(\bigvee_{i\neq j} a_i \wedge a_j m_j\right) \le a_j m_j$. Hence,

$$a_j \wedge (\textstyle\bigvee_{i\in I} a_i m_i) = a_j m_j$$

as required.

So every matching family for R has an amalgamation, implying that Σ is indeed a sheaf. Therefore, when $(\mathbf{C}, \mathcal{M})$ is a geometric $\mathcal{M}$-category, $(\mathsf{Sh}(\mathbf{C}), \mathsf{Sh}(\mathcal{M}))$ is cocomplete as an $\mathcal{M}$-category. □

5.3.3 Free cocompletion of geometric $\mathcal{M}$-categories

We have now established that if $(\mathbf{C}, \mathcal{M})$ is a small geometric $\mathcal{M}$-category, then $(\mathsf{Sh}(\mathbf{C}), \mathsf{Sh}(\mathcal{M}))$ is a geometric and cocomplete $\mathcal{M}$-category. The next step is to show that $(\mathsf{Sh}(\mathbf{C}), \mathsf{Sh}(\mathcal{M}))$ is the free geometric cocompletion of $(\mathbf{C}, \mathcal{M})$. The following lemmas will be useful.

Lemma 5.31. *[Kelly, 1982, Theorem 5.56] Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor, where $\mathbf{C}$ is a small site and $\mathbf{D}$ is cocomplete. Denote the left Kan extension of F along $\mathbf{y}\colon \mathbf{C} \to \mathsf{PSh}(\mathbf{C})$ by $\tilde{F}\colon \mathsf{PSh}(\mathbf{C}) \to \mathbf{D}$, and suppose the right adjoint to $\tilde{F}$ factors through the inclusion $\mathbf{i}\colon \mathsf{Sh}(\mathbf{C}) \hookrightarrow \mathsf{PSh}(\mathbf{C})$. Denote by $\mathbf{Cat}_i(\mathbf{C}, \mathbf{D})$, the category of such functors $F\colon \mathbf{C} \to \mathbf{D}$ where $\tilde{F}$ factors through $\mathbf{i}$. Then the following is an equivalence of categories:*

$$(-) \circ \mathbf{ay}\colon \mathbf{Cocomp}(\mathsf{Sh}(\mathbf{C}), \mathbf{D}) \to \mathbf{Cat}_i(\mathbf{C}, \mathbf{D}),$$

with pseudo-inverse given by left Kan extension along **ay**.

Lemma 5.32. *Let $F\colon \mathbf{C} \to \mathbf{D}$ and $\tilde{F}\colon \mathsf{PSh}(\mathbf{C}) \to \mathbf{D}$ be functors as above, and denote the right adjoint to $\tilde{F}$ by $G\colon \mathbf{D} \to \mathsf{PSh}(\mathbf{C})$. Then for each $D \in \mathbf{D}$, $G(D)$ is a sheaf if and only if for all $C \in \mathbf{C}$, $\tilde{F}$ takes covering sieves $S \rightarrowtail \mathbf{y}C$ in $\mathsf{PSh}(\mathbf{C})$ to isomorphisms in* $\mathbf{D}$.

Proof. By definition, S is a covering sieve if and only if for all $D \in \mathbf{D}$ and $S \to G(D)$, there exists a unique extension $\mathbf{y}C \to G(D)$ making the following diagram commute:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "i"'] & G(D) \\ \mathbf{y}C. \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

In other words, if and only if there is an isomorphism

$$\mathsf{PSh}(\mathbf{C})(i, G(D)) \colon \mathsf{PSh}(\mathbf{C})(S, G(D)) \to \mathsf{PSh}(\mathbf{C})(\mathbf{y}C, G(D)).$$

As $\tilde{F} \dashv G$, the above is an isomorphism if and only if $\mathbf{D}(\tilde{F}i, D)$ is invertible for all $D \in \mathbf{D}$, and this in turn is true if and only if $\tilde{F}i$ is invertible. □

Lemma 5.33. *If $F \colon (\mathbf{C}, \mathcal{C}) \to (\mathbf{D}, \mathcal{D})$ is an $\mathcal{M}$-functor with $(\mathbf{D}, \mathcal{D})$ cocomplete, then $\tilde{F} = \mathsf{Lan}_{\mathbf{y}} F$ is an $\mathcal{M}$-functor.*

Proof. This is Theorem 2.30 from earlier. □

Lemma 5.34. *Let $(\mathbf{C}, \mathcal{C})$ and $(\mathbf{D}, \mathcal{D})$ be geometric $\mathcal{M}$-categories, with $(\mathbf{C}, \mathcal{C})$ small and $(\mathbf{D}, \mathcal{D})$ cocomplete. Let $F \colon (\mathbf{C}, \mathcal{C}) \to (\mathbf{D}, \mathcal{D})$ be an $\mathcal{M}$-functor. Then F preserves unions of $\mathcal{M}$-subobjects if and only if for all $C \in \mathbf{C}$, $\tilde{F}$ takes covering sieves $S \rightarrowtail \mathbf{y}C$ in $\mathsf{PSh}(\mathbf{C})$ to isomorphisms in $\mathbf{D}$.*

Proof. By the previous lemma, $\tilde{F}$ is a cocontinuous $\mathcal{M}$-functor. This means that for any covering sieve $S \rightarrowtail \mathbf{y}C$, and in particular, the covering sieve $\mu \colon \bigcup_{i \in \mathcal{I}} \mathbf{y}C_i \to \mathbf{y}\left(\bigcup_{i \in \mathcal{I}} C_i\right)$ from (5.1), we have

$$\tilde{F}\left(\bigcup_{i \in \mathcal{I}} \mathbf{y}C_i\right) \cong \bigcup_{i \in \mathcal{I}} \tilde{F}\mathbf{y}C_i \cong \bigcup_{i \in \mathcal{I}} FC_i,$$

and so $\tilde{F}\mu \colon \bigcup_{i \in \mathcal{I}} FC_i \to F\left(\bigcup_{i \in \mathcal{I}} C_i\right)$ is an isomorphism if and only if F preserves unions of $\mathcal{M}$-subobjects. □

Theorem 5.35. *Suppose $(\mathbf{C}, \mathcal{C})$ and $(\mathbf{D}, \mathcal{D})$ are geometric $\mathcal{M}$-categories, with $(\mathbf{D}, \mathcal{D})$ cocomplete. Then the following is an equivalence of categories:*

$$(-) \circ \mathbf{ay} \colon \mathbf{g}\mathcal{M}\mathbf{Cocomp}(\mathsf{Sh}_{\mathcal{M}}(\mathbf{C}), (\mathbf{D}, \mathcal{D})) \to \mathbf{g}\mathcal{M}\mathbf{Cat}((\mathbf{C}, \mathcal{C}), (\mathbf{D}, \mathcal{D})).$$

Proof. We first show that $(-) \circ \mathbf{ay}$ is essentially surjective on objects. Since following is an equivalence of categories,

$$(-) \circ \mathbf{y} \colon \mathcal{M}\mathbf{Cocomp}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}), (\mathbf{D}, \mathcal{D})) \to \mathcal{M}\mathbf{Cat}((\mathbf{C}, \mathcal{C}), (\mathbf{D}, \mathcal{D}),$$

it means that for every $F \colon (\mathbf{C}, \mathcal{C}) \to (\mathbf{D}, \mathcal{D})$, there is a cocontinuous $\tilde{F} \colon \mathsf{PSh}_{\mathcal{M}}(\mathbf{C}) \to (\mathbf{D}, \mathcal{D})$. But applying Lemmas 5.34, 5.32 and 5.31 in succession gives a cocontinuous functor $\tilde{F}\mathbf{i} \colon \mathsf{Sh}(\mathbf{C}) \to \mathbf{D}$ such that $\tilde{F}\mathbf{iay} \cong F$, since F is geometric. Now $\tilde{F} \circ \mathbf{i}$ is the composite of $\mathcal{M}$-functors, and so $(-) \circ \mathbf{ay}$ is essentially surjective on objects. The fact $(-) \circ \mathbf{ay}$ is fully faithful follows from Lemma 5.31, and therefore $(-) \circ \mathbf{ay}$ is an equivalence of categories. □

5.4 Free cocompletion of join restriction categories

In light of the 2-equivalence between **g$\mathcal{M}$Cat** and **jrCat**$_s$, we may now use the previous result to give the free cocompletion of any join restriction category. Indeed, this is what we will do in this section. But let us begin with the definition of cocomplete join restriction category.

Definition 5.36. We say a join restriction category $\mathbb{X}$ is *cocomplete* if it is cocomplete as a restriction category. Also, a join restriction functor between join restriction categories $F\colon \mathbb{X} \to \mathbb{Y}$ is called *cocontinuous* if $\mathsf{Total}(F)$ is cocontinuous. There is a 2-category **jrCocomp** of cocomplete join restriction categories, cocontinuous restriction functors and restriction transformations.

Observe that we have omitted the term "join" in describing the 1-cells of **jrCocomp**. The reason for this is as follows. As Par and $\mathcal{M}\mathsf{Total}$ are 2-equivalences, every split join restriction category $\mathbb{X}$ may be rewritten as $\mathbb{X} \simeq \mathsf{Par}(\mathsf{Total}(\mathbb{X}), \mathcal{M}_{\mathbb{X}})$, with $\mathcal{M}_{\mathbb{X}}$ being the restriction monics in $\mathbb{X}$ [Cockett & Lack, 2002]. So if $F\colon \mathbb{X} \to \mathbb{Y}$ is a cocontinuous restriction functor between split join restriction categories, then $\mathcal{M}\mathsf{Total}(F)$ from $(\mathsf{Total}(\mathbb{X}), \mathcal{M}_{\mathbb{X}})$ to $(\mathsf{Total}(\mathbb{Y}), \mathcal{M}_{\mathbb{Y}})$ is a cocontinuous $\mathcal{M}$-functor. But since cocontinuous $\mathcal{M}$-functors preserve joins of $\mathcal{M}$-subobjects, it follows that $F \cong \mathsf{Par}(\mathcal{M}\mathsf{Total}(F))$ is a join restriction functor by Proposition 5.14.

We now describe the free cocompletion of any join restriction category. Recall from earlier that the inclusion $\mathbf{rCat}_s \hookrightarrow \mathbf{rCat}$ has a left biadjoint K_r, and the unit of this biadjoint at $\mathbb{X}$ is a restriction functor J from $\mathbb{X}$ to $\mathsf{K}_r(\mathbb{X})$. It is easy to check that if $\mathbb{X}$ is a join restriction category, then so is $\mathsf{K}_r(\mathbb{X})$. Also, the fact **jrCocomp** and **g$\mathcal{M}$Cocomp** are 2-equivalent follows from their definitions. So consider the following solid diagram:

$$
\begin{tikzcd}
\mathbf{jrCat} \arrow[r, shift left, "\mathsf{K}_r"] \arrow[r, phantom, "\perp"] & \mathbf{jrCat}_s \arrow[l, shift left, hook'] \arrow[r, shift left, "\mathcal{M}\mathsf{Total}"] \arrow[r, phantom, "\simeq"] \arrow[d, shift left, dashed] & \mathbf{g}\mathcal{M}\mathbf{Cat} \arrow[l, shift left, "\mathsf{Par}"] \arrow[d, shift left, dashed, "\mathsf{Sh}_{\mathcal{M}}"] \\
& \mathbf{jrCocomp} \arrow[u, shift left, "U"] \arrow[r, shift left, "\mathcal{M}\mathsf{Total}"] \arrow[r, phantom, "\simeq"] & \mathbf{g}\mathcal{M}\mathbf{Cocomp}. \arrow[l, shift left, "\mathsf{Par}"] \arrow[u, shift left, "V"]
\end{tikzcd}
$$

Now let $\mathbb{X}$ be a small join restriction category. By Theorem 5.35, the forgetful 2-functor V has a left biadjoint at any *small* geometric $\mathcal{M}$-category, as indicated by the dotted arrow above. It follows that U also has a left biadjoint at any small join restriction category $\mathbb{X}$ given by $\mathsf{Par}(\mathsf{Sh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathbb{X})))$. Therefore, the following exhibits the codomain as the free join restriction cocompletion of $\mathbb{X}$:

$$\eta_{\mathbb{X}}\colon \mathbb{X} \xrightarrow{J} \mathsf{K}_r(\mathbb{X}) \xrightarrow{\cong} \mathsf{Par}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))) \xrightarrow{\mathsf{Par}(\mathbf{ay})} \mathsf{Par}(\mathsf{Sh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))), \tag{5.3}$$

in the sense that the following is an equivalence of categories:

$$(-) \circ \eta_{\mathbb{X}}\colon \mathbf{jrCocomp}(\mathsf{Par}(\mathsf{Sh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))), \mathbb{E}) \to \mathbf{jrCat}(\mathbb{X}, \mathbb{E}). \tag{5.4}$$

However, as we shall see in the next section, we may express the free cocompletion of any join restriction category in a simpler form via the notion of join restriction presheaf.

6 Join restriction presheaves

We saw in the previous chapter that the free cocompletion of any join restriction category $\mathbb{X}$ may be given by the partial map category of sheaves on some site. The aim of this section will be to present an equivalent category which is also the free cocompletion of any join restriction category. In order to do this, we need objects in this category to correspond with sheaves on $\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))$. In particular, we need a corresponding notion of matching family, and also that of amalgamation.

As it turns out, the corresponding object we need is a presheaf over a join restriction category (Definition 6.3). Instead of a matching family for a covering sieve, we have a compatible family of elements of the restriction presheaf, and instead of a unique amalgamation of such a matching family, we have a join of compatible families. These join restriction presheaves form a join restriction category, and we will show that this category is equivalent to some partial map category of sheaves, and hence show that it is indeed the free cocompletion of any join restriction category. In short, what we will do in this chapter is generalise what we have done with restriction presheaves and restriction categories, to join restriction presheaves and join restriction categories.

6.1 Presheaf over a join restriction category

Recall that if $P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ is a restriction presheaf over a restriction category, then for all $A \in \mathbb{X}$, the set PA also has a partial ordering given by $x \leq y$ if and only if $x = y \cdot \bar{x}$. Therefore, as with the case of join restriction categories, we may define compatibility between elements of the same set PA.

Definition 6.1. Let $\mathbb{X}$ be a restriction category and P be a restriction presheaf over $\mathbb{X}$. For any $A \in \mathbb{X}$, we say that $x, y \in PA$ are compatible if $x \cdot \bar{y} = y \cdot \bar{x}$, and denote this by $x \smile y$. A subset $S \subset PA$ is called *compatible* if elements in S are pairwise compatible.

Lemma 6.2. *Let $\mathbb{X}$ be a restriction category and P a restriction presheaf over $\mathbb{X}$. Let $A \in \mathbb{X}$ and $x, y \in PA$. Then*

1. *$x \leq y$ implies $x \smile y$, and*

2. *$x \smile y$ and $\bar{x} = \bar{y}$ implies $x = y$.*

Proof. Essentially the same proof as for Lemma 5.2. □

It is this inherent partial ordering of elements within the same set which will allow us to give a notion of presheaf over a join restriction category.

Definition 6.3. Let $\mathbb{X}$ be a join restriction category. A *join restriction presheaf* on $\mathbb{X}$ is a restriction presheaf $P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ such that for all $A \in \mathbb{X}$ and all compatible subsets $S \subset PA$, the join $\bigvee_{s\in S} s$ exists with respect to the partial ordering on PA, and satisfies the following conditions:

(JRP1) $\overline{\bigvee_{s\in S} s} = \bigvee_{s\in S} \bar{s}$;

(JRP2) $(\bigvee_{s\in S} s) \cdot g = \bigvee_{s\in S}(s \cdot g)$

for all $g\colon B \to A$ and $x \in PA$. Denote by $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$, the full subcategory of $\mathsf{PSh}_r(\mathbb{X})$ with join restriction presheaves as its objects.

These join restriction presheaves satisfy one other important property, analogous to Proposition 5.4 in the case of join restriction categories.

Proposition 6.4. *Let $\mathbb{X}$ be a join restriction category, and let P be a join restriction presheaf. Then for all $A \in \mathbb{X}$, $x \in PA$ and compatible $T \subset \mathbb{X}(B, A)$,*

$$x \cdot (\textstyle\bigvee_{t\in T} t) = \bigvee_{t\in T}(x \cdot t).$$

Proof. See discussion below. □

Instead of replicating essentially the same proof as found in Proposition 5.4, let us see why the above proposition is true by considering the notion of *collage*. Recall that any presheaf P over an ordinary category $\mathbf{C}$ may be regarded as a profunctor (or module or distributor) from the terminal category $\mathbf{1} \nrightarrow \mathbf{C}$, or as a bifunctor $P\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{1} \to \mathbf{Set}$. Further recall that the *collage* of this bifunctor $P\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{1} \to \mathbf{Set}$, denoted here by $\tilde{P}$, is a category whose objects are the disjoint union of the objects of $\mathbf{C}$ and $\mathbf{1} = \{\star\}$ [Street, 2004a]. Its hom-sets are defined as follows:

$$\begin{aligned}
\tilde{P}(\star,\star) &= \mathbf{1}(\star,\star) = 1_\star;\\
\tilde{P}(A,B) &= \mathbf{C}(A,B);\\
\tilde{P}(A,\star) &= P(A,\star);\\
\tilde{P}(\star,A) &= \emptyset.
\end{aligned}$$

(In fact, the collage of P is more than just a category; it is the lax colimit of the profunctor $\mathbf{1} \nrightarrow \mathbf{C}$ in the bicategory of profunctors).

It is easy to see that if $P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ is a restriction presheaf, then its collage may be given a canonical restriction structure. Conversely, if the collage of $P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ is a restriction category, then P may also be given a restriction structure, making it a restriction presheaf. The same is also true if $\mathbb{X}$ were a join restriction category, and $P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ a join restriction presheaf. Therefore, by construction, the previous proposition is true because Proposition 5.4 is true for join restriction categories.

Before moving on, we make a quick remark here that there is a bicategory **rProf** of restriction categories, restriction profunctors, and natural transformations [DeWolf, 2017], and that a restriction presheaf on $\mathbb{X}$ is simply a restriction profunctor from the terminal restriction category to $\mathbb{X}$ [DeWolf, 2017]. We shall look at this in more detail in Chapter 9.

6.2 Category of join restriction presheaves

We know that for any restriction category $\mathbb{X}$, the category of restriction presheaves on $\mathbb{X}$, $\mathsf{PSh}_r(\mathbb{X})$, is a restriction category. So if $\mathbb{X}$ is a *join* restriction category, then $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ must also be a restriction category since it is a fully subcategory of $\mathsf{PSh}_r(\mathbb{X})$. However, we will show that $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ is not only a restriction category, but is equipped with a natural notion of join, making it a join restriction category.

Lemma 6.5. *Let $\mathbb{X}$ be a join restriction category, and let P and Q be join restriction presheaves on $\mathbb{X}$. Let S be a compatible set of pairwise natural transformations from P to Q. Then the natural transformation $\bigvee_{\alpha \in S} \alpha$ defined as follows:*

$$\left(\bigvee_{\alpha \in S} \alpha\right)_A (x) = \bigvee_{\alpha \in S} \alpha_A(x)$$

is the join of S, and furthermore, satisfies conditions (J1) and (J2).

Proof. We first have to show that $\bigvee_{\alpha \in S} \alpha$ is well-defined. That is, for all $\alpha, \beta \in S$, $\alpha_A(x) \cdot \overline{\beta_A(x)} = \beta_A(x) \cdot \overline{\alpha_A(x)}$ (for all $A \in \mathbb{X}$ and $x \in PA$). But this follows by definition of restriction in $\mathsf{PSh}_r(\mathbb{X})$ and the naturality of α and β. Also, $\bigvee_{\alpha \in S} \alpha$ is natural since for all $g\colon B \to A$,

$$\begin{aligned}\left(\left(\bigvee_{\alpha \in S} \alpha\right)_A (x)\right) \cdot g &= \left(\bigvee_{\alpha \in S} \alpha_A(x)\right) \cdot g = \bigvee_{\alpha \in S} (\alpha_A(x) \cdot g)\\ &= \bigvee_{\alpha \in S} \alpha_A(x \cdot g) = \left(\bigvee_{\alpha \in S} \alpha\right)_A (x \cdot g)\end{aligned}$$

using the fact Q is a join restriction presheaf, and the naturality of $\alpha \in S$. To show that $\bigvee_{\alpha \in S} \alpha$ really is the join, we have to show $\alpha' \leq \bigvee_{\alpha \in S} \alpha$ for all $\alpha' \in S$, or equivalently, $\alpha'_A(x) = \left(\bigvee_{\alpha \in S} \alpha\right)_A \left(\overline{\alpha'}_A(x)\right)$. But this is true as

$$\begin{aligned}\left(\bigvee_{\alpha \in S} \alpha\right)_A \left(\overline{\alpha'}_A(x)\right) &= \left(\bigvee_{\alpha \in S} \alpha\right)_A \left(x \cdot \overline{\alpha'_A(x)}\right)\\ &= \left(\left(\bigvee_{\alpha \in S} \alpha\right)_A (x)\right) \cdot \overline{\alpha'_A(x)}\\ &= \left(\bigvee_{\alpha \in S} \alpha_A(x)\right) \cdot \overline{\alpha'_A(x)}\\ &= \left(\alpha'_A(x) \cdot \overline{\alpha'_A(x)}\right) \vee \bigvee_{\alpha \neq \alpha'} \alpha_A(x) \cdot \overline{\alpha'_A(x)}\\ &= \alpha'_A(x) \vee \bigvee_{\alpha \neq \alpha'} \alpha'_A(x) \cdot \overline{\alpha_A(x)}\\ &= \alpha'_A(x)\end{aligned}$$

by compatibility and the fact $\alpha'_A(x) \cdot \overline{\alpha_A(x)} \leq \alpha'_A(x)$. Also, if $\alpha \leq \beta$ for all $\alpha \in S$, then $\bigvee_{\alpha \in S} \alpha \leq \beta$ since

$$\begin{aligned}\beta_A \left(\overline{\bigvee_{\alpha \in S} \alpha}_A(x)\right) = \beta_A \left(x \cdot \overline{\bigvee_{\alpha \in S} \alpha_A(x)}\right) &= \beta_A(x) \cdot \bigvee_{\alpha \in S} \overline{\alpha_A(x)}\\ &= \bigvee_{\alpha \in S} \beta_A(x) \cdot \overline{\alpha_A(x)} = \bigvee_{\alpha \in S} \alpha_A(x) = \left(\bigvee_{\alpha \in S} \alpha\right)_A (x).\end{aligned}$$

Therefore, for any compatible set of natural transformations S, $\bigvee_{\alpha \in S} \alpha$ as defined previously is the join of S.

To see that this join satisfies (J1), simply replace β above by the identity. To see that (J2) is satisfied, let $\gamma\colon R \Rightarrow P$ be a natural transformation and observe that

$$\left(\bigvee_{\alpha \in S} \alpha\right)_A (\gamma_A(x)) = \bigvee_{\alpha \in S} \alpha_A(\gamma_A(x)) = \bigvee_{\alpha \in S} (\alpha\gamma)_A(x) = \left(\bigvee_{\alpha \in S} \alpha\gamma\right)_A (x).$$

Therefore, the natural transformation $\bigvee_{\alpha \in S} \alpha$ defined as above really is the join of any compatible $S \subset \mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})(P, Q)$, and furthermore, satisfies conditions (J1) and (J2). $\square$

The following proposition follows directly from Lemma 6.5.

Proposition 6.6 (Category of join restriction presheaves)**.** *Let $\mathbb{X}$ be a join restriction category. Then $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ is a join restriction category, with joins defined componentwise as in Lemma 6.5 for any compatible subset $S \subset \mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})(P,Q)$.*

The following are some properties of maps in $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$.

Proposition 6.7. *Let $\mathbb{X}$ be a join restriction category, and let $\alpha\colon P \Rightarrow Q$ be a map in $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$. Let $A \in \mathbb{X}$ and $S \subset PA$ be compatible. Then the set $\alpha_A(S) = \{\alpha_A(x) \mid x \in PA\}$ is also compatible. In addition, if $x, y \in PA$ with $x \leq y$, then $\alpha_A(x) \leq \alpha_A(y)$.*

Proof. Let $x, y \in S$, and observe that it is enough to show that $\alpha_A(x) \cdot \overline{\alpha_A(y)} = \alpha_A(y \cdot \bar{x})$ (by interchanging x and y and using the fact $x \smile y$). Since $\overline{\alpha_A(y)} \leq \bar{y}$ (as $\alpha_A(y) = \alpha_A(y \cdot \bar{y})$), we have

$$\begin{aligned}\alpha_A(x) \cdot \overline{\alpha_A(y)} &= \alpha_A\left(x \cdot \overline{\alpha_A(y)}\right) = \alpha_A\left(x \cdot \left(\bar{y} \circ \overline{\alpha_A(y)}\right)\right) = \alpha_A\left((y \cdot \bar{x}) \cdot \overline{\alpha_A(y)}\right)\\ &= \alpha_A\left(\bar{\alpha}_A(y)\right) \cdot \bar{x} = \alpha_A(y) \cdot \bar{x} = \alpha_A(y \cdot \bar{x}).\end{aligned}$$

Hence, $\alpha_A(S)$ is compatible if S is compatible. Now if $x \leq y$, then

$$\alpha_A(x) = \alpha_A(x) \cdot \overline{\alpha_A(x)} = \alpha_A(y \cdot \bar{x}) \cdot \overline{\alpha_A(x)} = \alpha_A(y) \cdot \left(\overline{\alpha_A(x)} \circ \bar{x}\right) = \alpha_A(y) \cdot \overline{\alpha_A(x)}$$

since $\overline{\alpha_A(x)} \leq \bar{x}$. Therefore, $x \leq y$ implies $\alpha_A(x) \leq \alpha_A(y)$. □

Proposition 6.8. *Let $\alpha\colon P \Rightarrow Q$ be a map in $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$. Let $A \in \mathbb{X}$, and let $S \subset PA$ be compatible. Then*

$$\alpha_A\left(\textstyle\bigvee_{x\in S} x\right) = \textstyle\bigvee_{x \in S} \alpha_A(x).$$

In other words, components of natural transformations preserve joins.

Proof. To prove equality, we will show they are compatible, and then show that their restrictions are equal. Now by definition, $x \leq \bigvee_{x\in S} x$, which means $\alpha_A(x) \leq \alpha_A\left(\bigvee_{x\in S} x\right)$ by Proposition 6.2. Therefore, $\bigvee_{x\in S} \alpha_A(x) \leq \alpha_A\left(\bigvee_{x\in S} x\right)$, and hence $\bigvee_{x\in S} \alpha_A(x) \smile \alpha_A\left(\bigvee_{x\in S} x\right)$.

To show their restrictions are equal, we first show $\bar{\alpha}_A\left(\bigvee_{x\in S} x\right) = \bigvee_{x\in S} \bar{\alpha}_A(x)$. But this is true since

$$\begin{aligned}\bar{\alpha}_A\left(\textstyle\bigvee_{x\in S} x\right) &= \left(\textstyle\bigvee_{x\in S} x\right) \cdot \overline{\alpha_A\left(\bigvee_{y\in S} y\right)} = \textstyle\bigvee_{x\in S} x \cdot \overline{\alpha_A\left(\bigvee_{y\in S} y\right)}\\ &= \textstyle\bigvee_{x\in S} x \cdot \left(\overline{\alpha_A\left(\bigvee_{y\in S} y\right)} \circ \bar{x}\right) = \textstyle\bigvee_{x\in S} x \cdot \overline{\alpha_A\left(\bigvee_{y\in S} y\right) \cdot \bar{x}}\\ &= \textstyle\bigvee_{x\in S} x \cdot \overline{\alpha_A\left(\bigvee_{y\in S} y \cdot \bar{x}\right)} = \textstyle\bigvee_{x\in S} x \cdot \overline{\alpha_A\left(x \cdot \bar{x} \vee \bigvee_{y\neq x} y \cdot \bar{x}\right)}\\ &= \textstyle\bigvee_{x\in S} x \cdot \overline{\alpha_A\left(x \vee \bigvee_{y\neq x} x \cdot \bar{y}\right)} = \textstyle\bigvee_{x\in S} x \cdot \overline{\alpha_A(x)}\\ &= \textstyle\bigvee_{x\in S} \bar{\alpha}_A(x).\end{aligned}$$

Observing that $\overline{\alpha_A(x)} = \overline{\bar{\alpha}_A(x)}$, we then have

$$\overline{\alpha_A\left(\textstyle\bigvee_{x\in S} x\right)} = \overline{\bar{\alpha}_A\left(\textstyle\bigvee_{x\in S} x\right)} = \overline{\textstyle\bigvee_{x\in S} \bar{\alpha}_A(x)} = \overline{\textstyle\bigvee_{x\in S} \alpha_A(x)},$$

which means the restrictions of $\alpha_A\left(\bigvee_{x\in S} x\right)$ and $\bigvee_{x\in S} \alpha_A(x)$ are equal. Therefore, as $\alpha_A\left(\bigvee_{x\in S} x\right)$ and $\bigvee_{x\in S} \alpha_A(x)$ are compatible, they must be equal. □

6.3 Join restriction presheaves and sheaves

Having introduced join restriction presheaves, our next goal is to show that for any small geometric $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, $\mathsf{Par}(\mathsf{Sh}_{\mathcal{M}}(\mathbf{C}))$ and $\mathsf{PSh}_{jr}(\mathsf{Par}(\mathbf{C}, \mathcal{M}))$ are equivalent as join restriction categories. Now recall that for any $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, there was an equivalence $F\colon \mathsf{PSh}_{\mathcal{M}}(\mathbf{C}) \to \mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M})))$ of $\mathcal{M}$-categories, which on objects, takes presheaves P on $\mathbf{C}$ to presheaves $\tilde{P}$ on $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, with $\tilde{P}(X) = \{(m, f) \mid m \in \mathcal{M}, f \in P(\mathsf{dom}\, m)\}$ for all $X \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$ (see Theorem 4.12). By the fact that $\mathcal{M}\mathbf{Cat}$ and $\mathbf{rCat}_s$ are 2-equivalent, we then have an equivalence of restriction categories $L\colon \mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})) \to \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M}))$ (the transpose of F). Explicitly, $L = \Phi^{-1}_{\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C},\mathcal{M}))} \circ \mathsf{Par}(F)$, where $\Phi_{\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C},\mathcal{M}))}$ is the unit of the 2-equivalence between $\mathcal{M}\mathbf{Cat}$ and $\mathbf{rCat}_s$. We will show that this equivalence L restricts back to an equivalence between join restriction categories $\mathsf{Par}(\mathsf{Sh}_{\mathcal{M}}(\mathbf{C}))$ and $\mathsf{PSh}_{\mathsf{jr}}(\mathsf{Par}(\mathbf{C}, \mathcal{M}))$.

However, let us first establish the following facts.

Lemma 6.9. *Let* $(\mathbf{C}, \mathcal{M})$ *be a small geometric* $\mathcal{M}$*-category. Then the restriction category* $\mathsf{Par}(\mathsf{Sh}(\mathbf{C}), \mathsf{Sh}(\mathcal{M}))$ *is a full subcategory of* $\mathsf{Par}(\mathsf{PSh}(\mathbf{C}), \mathsf{PSh}(\mathcal{M}))$.

Proof. Let P and Q be sheaves on $\mathbf{C}$, and consider a map $\mathbf{i}P \xleftarrow{\mu} R \xrightarrow{\tau} \mathbf{i}Q$ in $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}))$, where $\mathbf{i}\colon \mathsf{Sh}(\mathbf{C}) \hookrightarrow \mathsf{PSh}(\mathbf{C})$ is the inclusion. We need to find a map $P \xleftarrow{\mu'} R' \xrightarrow{\tau'} Q$ in $\mathsf{Par}(\mathsf{Sh}_{\mathcal{M}}(\mathbf{C}))$ such that $(\mathbf{i}\mu', \mathbf{i}\tau') = (\mu, \tau)$. However, as this will be true if $\mathsf{PSh}(\mathcal{M})$-subobjects of sheaves are sheaves, this is what we will prove.

So let $\{a_i\colon C_i \to C\}_{i \in I}$ be a basic cover of $C \in \mathbf{C}$, and let R be an $\mathsf{PSh}(\mathcal{M})$-subobject of P, where P is a sheaf. Consider the subfunctor $S \rightarrowtail \mathbf{y}C$, where S is the covering sieve generated by our basic cover, and let $\alpha\colon S \to R$ be any natural transformation. Since P is a sheaf, there exists a unique extension $\gamma\colon \mathbf{y}C \to P$ making the outer square of the following diagram commute:

$$\begin{tikzcd}
S \arrow[dr] \arrow[drr, bend left, "\alpha"] \arrow[ddr, tail, bend right] & & \\
& \mathbf{y}D \arrow[r, "\beta"] \arrow[d, tail, "\mathbf{y}m"'] & R \arrow[d, tail, "\mu'"] \\
& \mathbf{y}C \arrow[r, dashed, "\gamma"'] & P.
\end{tikzcd}$$

Now pulling back $\mu'\colon R \rightarrowtail P$ along this unique extension γ yields a square as shown, with $m \in \mathcal{M}$ and a unique induced map $S \to \mathbf{y}D$. But the fact S is the covering sieve generated by our basic cover means we have the following commutative diagram for every $i \in I$:

$$\begin{tikzcd}
C_i \arrow[rr, dashed] \arrow[dr, "a_i"'] & & D \arrow[dl, "m"] \\
& C. &
\end{tikzcd}$$

Since $\mathsf{Sub}_{\mathcal{M}}(C)$ is a complete Heyting algebra, taking the join of $\{a_i\}_{i \in I}$ means we have $1 = \bigvee_{i \in I} a_i \leq m$, and so $m = 1$. In other words, the map $\mathbf{y}m$ is invertible, and so for every natural transformation, $S \to R$, there exists an extension $\mathbf{y}C \to R$ given by the composite $\beta \circ (\mathbf{y}m)^{-1}$. However, as R is a subobject of a sheaf, and therefore separated, this implies that this extension is in fact unique. Hence, R is a sheaf and $\mathsf{Par}(\mathsf{Sh}_{\mathcal{M}}(\mathbf{C}))$ is a full subcategory of $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}))$. □

Theorem 6.10. *Let $(\mathbf{C}, \mathcal{M})$ be a small geometric $\mathcal{M}$-category. Then $\mathsf{Par}(\mathsf{Sh}_{\mathcal{M}}(\mathbf{C}))$ and $\mathsf{PSh}_{\mathsf{jr}}(\mathsf{Par}(\mathbf{C}, \mathcal{M}))$ are equivalent as join restriction categories.*

Proof. Since $\mathsf{Par}(\mathsf{Sh}_{\mathcal{M}}(\mathbf{C}))$ is a full subcategory of $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}))$ for any geometric $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$ (Lemma 6.9), let us consider the following solid diagram:

$$\begin{tikzcd}
\mathsf{Par}(\mathsf{Sh}_{\mathcal{M}}(\mathbf{C})) \arrow[r, dashed, "L'"] \arrow[d, hook] & \mathsf{PSh}_{\mathsf{jr}}(\mathsf{Par}(\mathbf{C}, \mathcal{M})) \arrow[d, hook] \\
\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})) \arrow[r, "L"'] & \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M})).
\end{tikzcd}$$

We wish to show L restricts to a functor $L' \colon \mathsf{Par}(\mathsf{Sh}_{\mathcal{M}}(\mathbf{C})) \to \mathsf{PSh}_{\mathsf{jr}}(\mathsf{Par}(\mathbf{C}, \mathcal{M}))$ making the above diagram commute, and that L' is an equivalence of join restriction categories. We will begin by showing that L' is well-defined; that is, given a sheaf $P \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, we have to show $\mathsf{Par}(F)(P) = F(P) = \tilde{P} \colon \mathsf{Par}(\mathbf{C}, \mathcal{M})^{\mathrm{op}} \to \mathbf{Set}$ is a join restriction presheaf.

So let $\{(m_i, f_i)\}_{i \in I}$ be a compatible family of maps in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$; that is, $f_i \cdot m_i^*(m_j) = f_j \cdot m_j^*(m_i)$ for any pair $i, j \in I$, where $m_i^*(m_j)$ is the pullback of m_j along m_i. Since $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a join restriction category by assumption, we may take the colimit $\{a_i\}_{i \in I}$ of the matching diagram for $\{m_i\}_{i \in I}$. Let μ be the induced map from this colimit.

Now the condition $f_i \cdot m_i^*(m_j) = f_j \cdot m_j^*(m_i)$ for all $i, j \in I$ implies that $\{f_i\}_{i \in I}$ is a matching family for the basic cover $\{a_i\}_{i \in I}$. But because P is a sheaf, this means there is a unique amalgamation γ such that $\gamma \cdot m_i = f_i$ for all $i \in I$. So define the join of $\{(m_i, f_i)\}_{i \in I}$ to be $(\mu, \gamma \cdot \mu)$. It is then easy but tedious to check that the join restriction presheaf axioms hold, which means that L' is well-defined.

Since $\mathsf{Par}(\mathsf{Sh}_{\mathcal{M}}(\mathbf{C}))$ and $\mathsf{PSh}_{jr}(\mathsf{Par}(\mathbf{C}, \mathcal{M}))$ are both full subcategories, it also follows that L' makes the above diagram commute. In addition, as L is an equivalence of categories, this makes L' fully faithful, and so it remains to show that L' is essentially surjective on objects, and that L' is a join restriction functor.

To show L' is essentially surjective, recall from Proposition 4.13 that there is an equivalence $G \colon \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}, \mathcal{M})) \to \mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}))$ of restriction categories, with $LG \cong 1$. On objects, G maps restriction presheaves $P \colon \mathsf{Par}(\mathbf{C}, \mathcal{M})^{\mathrm{op}} \to \mathbf{Set}$ to presheaves $\dot{P} \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, with $\dot{P}(X) = \{x \mid x \in PX, \bar{x} = (1,1)\}$ and $\dot{P}(f) = P(1, f)$. So if we can show that G maps join restriction presheaves to sheaves on $\mathbf{C}$, then L' will be essentially surjective on objects.

So let P be a join restriction presheaf on $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, and consider the presheaf $\dot{P} \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. Let $R = \{a_i \colon C_i \to C\}_{i \in I}$ be a basic cover of C, and let $\{f_i \in \dot{P}(C_i)\}_{i \in I}$ be a matching family for R. That is, $f_i \cdot \pi_j = f_j \cdot \pi_i$ for all $i, j \in I$, where π_i, π_j are the pullbacks below:

$$\begin{tikzcd}
C_i C_j \arrow[r, "\pi_j"] \arrow[d, "\pi_i"'] & C_i \arrow[d, "a_i"] \\
C_j \arrow[r, "a_j"'] & C.
\end{tikzcd}$$

Note that $\pi_i, \pi_j \in \mathcal{M}$. Now $\dot{P}$ will be a sheaf if we can find a unique $x \in \dot{P}(C)$ such that $x \cdot a_i = f_i$. We will show that $x = \bigvee_{i \in I} f_i \cdot (a_i, 1)$ is the unique amalgamation of $\{f_i\}_{i \in I}$. However, first we must show that such a join exists by showing $f_i \cdot (a_i, 1) \smile f_j \cdot (a_j, 1)$ for all

$i, j \in I$. Now using the fact $f_i \cdot \pi_j = f_j \cdot \pi_i$ if and only if $f_i \cdot (1, \pi_j) = f_j \cdot (1, \pi_i)$ and $\overline{f_i} = (1,1)$ for all $i \in I$, we have

$$\begin{aligned} f_i \cdot (a_i, 1) \cdot \overline{f_j \cdot (a_j, 1)} &= f_i \cdot (a_i, 1) \cdot \overline{\overline{f_j} \circ (a_j, 1)} = f_i \cdot (a_i, 1) \cdot (a_j, a_j) \\ &= f_i \cdot (a_j \pi_i, \pi_j) = \big(f_i \cdot (1, \pi_j)\big) \cdot (a_j \pi_i, 1) \\ &= \big(f_j \cdot (1, \pi_i)\big) \cdot (a_j \pi_i, 1) = f_j \cdot (a_j \pi_i, \pi_i) \\ &= f_j \cdot (a_i \pi_j, \pi_i) = f_j \cdot (a_j, 1) \cdot \overline{f_i \cdot (a_i, 1)}. \end{aligned}$$

So $x = \bigvee_{i \in I} f_i \cdot (a_i, 1)$ exists. To see that it is in $\dot{P}(C)$, we have

$$\bar{x} = \overline{\bigvee_{i \in I} f_i \cdot (a_i, 1)} = \bigvee_{i \in I} \overline{\overline{f_i} \circ (a_i, 1)} = \bigvee_{i \in I} (a_i, a_i) = (1,1).$$

We now check that x is an amalgation of $\{f_i\}_{i \in I}$. That is, $x \cdot a_j = f_j$ for all $j \in I$, or equivalently, $x \cdot (1, a_j) = f_j$. Now

$$\begin{aligned} x \cdot (1, a_j) &= (\bigvee_{i \in I} f_i \cdot (a_i, 1)) \cdot (1, a_j) \\ &= f_j \cdot (a_j, 1) \cdot (1, a_j) \vee \bigvee_{i \in I - \{j\}} f_i \cdot (a_i, 1) \cdot (1, a_j) \\ &= f_j \vee \bigvee_{i \in I - \{j\}} f_i \cdot (\pi_i, \pi_j). \end{aligned}$$

But $\bigvee_{i \in I - \{j\}} f_i \cdot (\pi_i, \pi_j) \leq f_j$ since

$$\begin{aligned} f_j \cdot \overline{\bigvee_{i \in I - \{j\}} f_i \cdot (\pi_i, \pi_j)} &= f_j \cdot \bigvee_{i \in I - \{j\}} \overline{\overline{f_i} \circ (\pi_i, \pi_j)} \\ &= f_j \cdot \bigvee_{i \in I - \{j\}} (\pi_i, \pi_i) \\ &= f_j \cdot \bigvee_{i \in I - \{j\}} (1, \pi_i)(\pi_i, 1) \\ &= \bigvee_{i \in I - \{j\}} \big(f_j \cdot (1, \pi_i)\big) \cdot (\pi_i, 1) \\ &= \bigvee_{i \in I - \{j\}} \big(f_i \cdot (1, \pi_j)\big) \cdot (\pi_i, 1) \\ &= \bigvee_{i \in I - \{j\}} f_i \cdot (\pi_i, \pi_j). \end{aligned}$$

So $x \cdot (1, a_i) = x \cdot a_i = f_i$ for all $i \in I$, making $x = \bigvee_{i \in I} f_i \cdot (a_i, 1)$ an amalgation of $\{f_i\}_{i \in I}$. It remains to show that such an x is unique.

Suppose $y \in \dot{P}(C)$ also satisfies the condition $y \cdot a_i = y \cdot (1, a_i) = f_i$ for all $i \in I$. Then $y \cdot (1, a_i) = x \cdot (1, a_i)$ implies $\bigvee_{i \in I} y \cdot (1, a_i)(a_i, 1) = \bigvee_{i \in I} x \cdot (1, a_i)(a_i, 1)$, which in turn implies $x = y$ since $\bigvee_{i \in I} (a_i, a_i) = (1,1)$. Therefore, if P is a join restriction presheaf on $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, then $\dot{P}$ is a sheaf, and so L' is essentially surjective on objects.

Finally, to show that L' is a join restriction functor, note that L' is a restriction functor as L is a restriction functor. Furthermore, $\mathsf{Total}(L') \colon \mathsf{Sh}(\mathbf{C}) \to \mathsf{Total}(\mathsf{PSh}_{\mathsf{jr}}(\mathsf{Par}(\mathbf{C}, \mathcal{M})))$ is an equivalence of categories, with pseudo-inverse given by $\mathsf{Total}(G)$ restricted back to $\mathsf{Total}(\mathsf{PSh}_{\mathsf{jr}}(\mathsf{Par}(\mathbf{C}, \mathcal{M})))$. Therefore, as $\mathsf{Total}(L')$ is cocontinuous, L' must be a join restriction functor, and hence $\mathsf{Par}(\mathsf{Sh}_{\mathcal{M}}(\mathbf{C}))$ and $\mathsf{PSh}_{\mathsf{jr}}(\mathsf{Par}(\mathbf{C}, \mathcal{M}))$ are equivalent as join restriction categories. $\square$

Corollary 6.11. *For any small join restriction category $\mathbb{X}$, the Yoneda embedding $\mathbf{y}_{\mathsf{jr}} \colon \mathbb{X}^{\mathrm{op}} \to \mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ exhibits the category of join restriction presheaves $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ as its free cocompletion, in the sense that the functor*

$$(-) \circ \mathbf{y}_{\mathsf{jr}} \colon \mathbf{jrCocomp}(\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X}), \mathbb{E}) \to \mathbf{jrCat}(\mathbb{X}, \mathbb{E})$$

is an equivalence of categories for any cocomplete join restriction category $\mathbb{E}$.

Proof. The composite

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "J"] & \mathsf{K}_r(\mathbb{X}) \arrow[d, "\cong"] & \\
& \mathsf{Par}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))) \arrow[r, "\mathsf{Par}(\mathbf{ay})"'] & \mathsf{Par}(\mathsf{Sh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))) \arrow[d, "\simeq"] \\
& & \mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})
\end{tikzcd}$$

from (5.3) is naturally isomorphic to $\mathbf{y}_{\mathsf{jr}}$ by the same argument as presented in Theorem 4.14. Since precomposition with (5.4) is an equivalence of categories, it follows that precomposition with $\mathbf{y}_{\mathsf{jr}}$ is also an equivalence of categories. □

7

Restriction colimits

So far in our discussions, we have looked at notions of cocomplete restriction category and join restriction category. In particular, we have seen that the free cocompletion of any (join) restriction category is cocomplete. Therefore, a natural question to ask is whether there is a notion of restriction colimit such that a cocomplete restriction category is one having all restriction colimits.

For ordinary categories, we may characterise cocomplete categories as categories with all *conical* colimits. So extending this characterisation to restriction categories would imply that cocomplete restriction categories were those admitting all restriction conical colimits. However, as it turns out, it is more natural to generalise a different view on cocomplete categories; as those having all *weighted* colimits. Indeed, this is what we will do in this chapter.

We will give a rather intuitive notion of *weighted restriction colimit*, with the *weights* being limited to restriction presheaves $\mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$. We then show that a restriction category is cocomplete if and only if it has all such weighted restriction colimits. In other words, cocomplete restriction categories may be characterised as having all weighted restriction colimits, in the same way that ordinary cocomplete categories are characterised as having all weighted colimits (with the weights being set-valued functors). We begin with a review of weighted colimits.

7.1 Weighted colimits and weighted restriction colimits

We know that given a functor $F\colon \mathbf{C} \to \mathbf{D}$, a cocone under F consists of a single object $V \in \mathbf{D}$, together with a family of morphisms $\{p_C\colon FC \to V\}_{C\in\mathbf{C}}$ such that for every $f\colon C \to C'$ in $\mathbf{C}$, we have $p_{C'} \circ Ff = p_C$. Another way of saying this is that a cocone under F is just a natural transformation $F \Rightarrow \Delta V$, where $\Delta V\colon \mathbf{C} \to \mathbf{D}$ is the constant functor sending every $C \in \mathbf{C}$ to $V \in \mathbf{D}$. A conical colimit of F is then a universal such cocone; that is, given any other cocone $Z \in \mathbf{D}$ with morphisms $\{q_C\colon FC \to Z\}_{C\in\mathbf{C}}$, there is a unique map $\alpha\colon V \to Z$ such that $\alpha \circ p_C = q_C$ for all $C \in \mathbf{C}$. Denoting the object component of this universal cocone by $\operatorname{colim} F$, we may express the universal property of $\operatorname{colim} F$ by the following natural isomorphism (in

Z):

$$\mathbf{D}(\operatorname{colim} F, Z) \cong [\mathbf{C}, \mathbf{D}](F, \Delta Z).$$

However, we may express the above natural isomorphism in yet another way, as follows:

$$\mathbf{D}(\operatorname{colim} F, Z) \cong [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}](1, \mathbf{D}(F-, Z)),$$

where 1 is the terminal presheaf. In particular, the presentation $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}](1, \mathbf{D}(F-, Z))$ expresses a cocone as a map out of a *single point*. The idea, then, behind weighted colimits is to replace the *single point* $1\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ by an arbitrary set-valued functor $W\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$.

Definition 7.1. Let $F\colon \mathbf{C} \to \mathbf{D}$ and $W\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ be functors. We say that the colimit of F *weighted* by W exists if there is an object $\operatorname{colim}_W F \in \mathbf{D}$ and a natural isomorphism

$$\mathbf{D}(\operatorname{colim}_W F, D) \cong [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}](W, \mathbf{D}(F-, D)) \tag{7.1}$$

which is natural in $D \in \mathbf{D}$.

Remark 7.2. Given the above definition, we may also think of weighted colimits as universal weighted cocones. That is, instead of a single morphism from FC to $\operatorname{colim} F$ (for each $C \in \mathbf{C}$) in the case of conical colimits, we now have, for each object $C \in \mathbf{C}$, a family of morphisms $\{p_{C,x}\colon FC \to \operatorname{colim}_W F\}_{x \in WC}$ which are indexed by the set WC. These families of morphisms interact as one would expect, with the property that for each $f\colon C \to C'$, we have $p_{C,x} = p_{C',x'} \circ Ff$ if $x = x' \cdot f$. Indeed, in the next section, we shall define weighted restriction colimits as universal weighted cocones, together with restrictions on the coprojection maps $\{p_{C,x}\colon FC \to \operatorname{colim}_W F\}_{x \in WC}$.

Before moving on to the topic of restriction colimits, we would like to make one more comment concerning the more general notion of weighted colimit. That is, for a closed symmetric monoidal category $\mathcal{V}$ and $\mathcal{V}$-categories C and $\mathcal{D}$, the colimit of a $\mathcal{V}$-functor $F\colon C \to \mathcal{D}$ with weight given by another $\mathcal{V}$-functor $W\colon C^{\mathrm{op}} \to \mathcal{V}$ is again an object $\operatorname{colim}_W F \in \mathcal{D}$, together with a $\mathcal{V}$-natural isomorphism

$$\mathcal{D}(\operatorname{colim}_W F, D) \cong [C^{\mathrm{op}}, \mathbf{Set}](W, \mathcal{D}(F-, D)).$$

Although [DeWolf, 2017] had proposed a notion of $\mathcal{V}$-enriched restriction category, he did not provide a corresponding notion of $\mathcal{V}$-enriched restriction functor, and the author is unaware of any such notion at the time of writing. Therefore, in this chapter, we have limited our discussions to weighted restriction colimits whose weights are given by restriction presheaves.

Definition 7.3 (Weighted restriction colimit)**.** Let $\mathbb{X}$ and $\mathbb{D}$ be restriction categories, $F\colon \mathbb{X} \to \mathbb{D}$ a restriction functor, and $P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ a restriction presheaf. The weighted restriction colimit of F by P is defined to be an object $\operatorname{rcolim}_P F \in \mathbb{D}$, together with a family of maps $(p_{A,x}\colon FA \to \operatorname{rcolim}_P F)_{A \in \mathbb{X}, x \in PA}$ satisfying the following two conditions:

1. for all $f\colon A \to A'$, we have $p_{A,x} = p_{A',x'} \circ Ff$ if $x = x' \cdot f$; and
2. the restrictions on each coprojection map $p_{A,x}$ satisfy $\overline{p_{A,x}} = F\bar{x}$.

Further, its universal property is such that for any other object $W \in \mathbb{D}$ and families of maps $(q_{A,x}\colon FA \to W)_{A \in \mathbb{X}, x \in PA}$ with $q_{A,x} = q_{A',x'} \circ Ff$ if $x = x' \cdot f$, there exists a unique map $\alpha\colon \operatorname{rcolim}_P F \to W$ such that $q_{A,x} = \alpha \circ p_{A,x}$ for all $A \in \mathbb{X}, x \in PA$.

Note the only difference between weighted restriction colimits and ordinary weighted colimits is the additional specification on the coprojection maps. Without this additional criterion, $\mathrm{rcolim}_P F$ reverts back to being an ordinary weighted colimit.

Another remark we shall make here is in relation to Equation 7.1. At first instance, it would seem natural to define weighted restriction colimits by simply replacing $F\colon \mathbf{C} \to \mathbf{D}$ and $W\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ from Definition 7.1 with their restriction counterparts; that is, by a restriction functor $F\colon \mathbb{C} \to \mathbb{D}$ and a restriction presheaf $W\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$. However, as we shall learn in Section 9.3.2, this is not possible due to the fact that the resulting presheaf $\mathbb{D}(F-, D)$ will not, in general, be a *restriction presheaf.*

We make the following observation on the universal property of restriction colimits.

Proposition 7.4. *Let $\mathbb{X}, \mathbb{D}$ be restriction categories, $F\colon \mathbb{X} \to \mathbb{D}$ a restriction functor, and $P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ a restriction presheaf, and suppose the weighted restriction colimit $\mathrm{rcolim}_P F$ exists. Let W be an object in $\mathbb{D}$, and let $(q_{A,x}\colon FA \to W)_{A\in\mathbb{X}, x\in PA}$ be a family of maps with the property $q_{A,x} = q_{A',x'} \circ Ff$ if $x = x' \cdot f$. If the maps $q_{A,x}$ have restrictions given by $F\bar{x}$, then the unique induced map $\alpha\colon \mathrm{rcolim}_P F \to W$ is total.*

Proof. To show that $\bar{\alpha} = 1$, it is sufficient to show that $\bar{\alpha} \circ p_{A,x} = p_{A,x}$ (by uniqueness). Now

$$\bar{\alpha} p_{A,x} = p_{A,x} \circ \overline{\alpha p_{A,x}} = p_{A,x}\overline{q_{A,x}} = p_{A,x} F\bar{x} = p_{A,x}$$

since $p_{A,x} = F\bar{x}$. Hence $\alpha = 1$ if $\overline{q_{A,x}} = F\bar{x}$. □

Before giving examples of restriction colimits, recall from Section 4.1 that for any restriction category $\mathbb{X}$, there is a presheaf $O\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$, defined on objects $A \in \mathbb{X}$ to be the set $O(A)$ of restriction idempotents on A, and on maps $f\colon A' \to A$ by $O(f)(e) = e \cdot f = \overline{ef}$ for any restriction idempotent $e \in O(A)$. That this presheaf O is in fact a restriction presheaf with the obvious restriction structure may be easily verified by the reader.

Example 7.5 (Binary restriction coproduct)**.** In [Cockett & Lack, 2007], they define the restriction coproduct of two objects in a restriction category as a binary coproduct in the usual sense, such that the coproduct coprojections are total. To see that this is a particular instance of a restriction colimit, let $\mathbb{2}$ be the discrete restriction category with two objects A, B, and consider $\mathrm{rcolim}_O(F\colon \mathbb{2} \to \mathbb{X})$. Explicitly, this is an object in $\mathbb{X}$ together with total maps $p_A\colon FA \to \mathrm{rcolim}_O F$ and $p_B\colon FB \to \mathrm{rcolim}_O F$, such that for any maps $q_A\colon FA \to W$ and $q_B\colon FB \to W$, there exists a unique $\alpha\colon \mathrm{rcolim}_O F \to W$ so that $\alpha p_A = q_A$ and $\alpha p_B = q_B$. This is the very definition of a binary restriction coproduct.

The above example of restriction colimit was for a specific restriction functor, with weight given by the restriction presheaf O. We now describe the restriction colimit for an arbitrary restriction functor F with the same weight O.

Example 7.6. Let $\mathbb{X}$ be any restriction category, $F\colon \mathbb{X} \to \mathbb{D}$ an arbitrary restriction functor, and O the restriction presheaf taking objects $A \in \mathbb{X}$ to its set $O(A)$ of restriction idempotents. Then the restriction colimit of F weighted by O, if it exists, consists of an object $\mathrm{rcolim}_O F \in \mathbb{D}$ together with a family of maps $(p_{A,e})_{A\in\mathbb{X}, e\in O(A)}$ satisfying the conditions $\overline{p_{A,e}} = Fe$ and $p_{A,e} = p_{A',e'} \circ Ff$ if $e = e' \cdot f$, or $e = \overline{e'f}$. But observe that to give the restriction colimit of F weighted by O, it is enough to specify only the *total* coprojection maps $p_{A,1_A}\colon FA \to \mathrm{colim}_O F$ together with the condition $p_{A,1_A} \circ F\bar{f} = p_{A',1_{A'}} \circ Ff$ for all $f\colon A \to A'$.

To see this, first note that the coprojection maps satisfy the condition $p_{A,e} = p_{A,1_A} \circ Fe$ for all $e \in O(A)$. Also note that for all $f\colon A \to A'$, as the coprojections satisfy $p_{A,\bar{f}} = p_{A',1_{A'}} \circ Ff$ and $p_{A,\bar{f}} = p_{A,1_A} \circ F\bar{f}$, we have $p_{A,1_A} \circ F\bar{f} = p_{A',1_{A'}} \circ Ff$.

In the converse direction, suppose given maps $(p_{A,1_A})_{A\in\mathbb{X}}$ which satisfy $p_{A,1_A}\circ F\bar{f} = p_{A',1_{A'}}\circ Ff$, we define for each $A\in\mathbb{X}$ and $e\in O(A)$, the coprojection maps $p_{A,e}$ to be the composite $p_{A,e} = p_{A,1_A}\circ Fe$. Then clearly the restriction of $p_{A,e}$ is $\overline{p_{A,e}} = \overline{p_{A,1_A}\circ Fe} = \overline{\overline{p_{A,1_A}}\circ Fe} = \overline{Fe} = Fe$, as each $p_{A,1_A}$ is total by assumption. It remains to show that the condition $p_{A,1_A}\circ F\bar{f} = p_{A',1_{A'}}\circ Ff$ implies $p_{A,e} = p_{A',e'}\circ Ff$ if $e = \overline{e'f}$. To do this, we use the fact that in a restriction category, $f = g$ if and only if $\overline{f} = \overline{g}$ and $f\leq g$.

First, it is clear that when $e = \overline{e'f}$, we have

$$\overline{p_{A,e}} = Fe = F(\overline{e'f}) = \overline{F(e'f)} = \overline{Fe'\circ Ff} = \overline{\overline{p_{A',e'}}\circ Ff} = \overline{p_{A',e'}\circ Ff}.$$

Now the condition $p_{A,1_A}\circ F\bar{f} = p_{A',1_{A'}}\circ Ff$ implies $p_{A',1_{A'}}\circ Ff\leq p_{A,1_A}$. But precomposing both sides of the inequality by Fe yields $p_{A',1_{A'}}\circ Ff\circ Fe\leq p_{A,1_A}\circ Fe = p_{A,e}$. When $e = \overline{e'f}$, we have

$$p_{A',1_{A'}}\circ Ff\circ Fe = p_{A',1_{A'}}\circ F(f\overline{e'f}) = p_{A',1_{A'}}\circ F(\overline{e'}f) = p_{A',1_{A'}}\circ Fe'\circ Ff = p_{A',e'}\circ Ff,$$

and so $p_{A',e'}\circ Ff\leq p_{A,e}$. Therefore, since $\overline{p_{A',e'}\circ Ff} = \overline{p_{A,e}}$, we conclude that $p_{A,e} = p_{A',e'}\circ Ff$ if $e = \overline{e'f}$. Hence, to give the restriction colimit of F weighted by O, it is enough to specify only the *total* coprojection maps $p_{A,1_A}\colon FA\to \mathrm{colim}_O F$ together with the condition $p_{A,1_A}\circ F\bar{f} = p_{A',1_{A'}}\circ Ff$ for all $f\colon A\to A'$.

Now to describe the universal property of $\mathrm{rcolim}_O F$, consider an object W together with a family of maps $(q_{A,e}\colon FA\to W)_{A\in\mathbb{X},e\in O(A)}$ which satisfy the condition $q_{A,x} = q_{A',x'}\circ Ff$ if $x = x'\cdot f$. But by the same argument as before, it is enough to give $W\in\mathbb{D}$ and maps $q_{A,1_A}\colon A\to W$ satisfying $q_{A,1_A}\circ F\bar{f} = q_{A',1_{A'}}\circ Ff$ for all $f\colon A\to A'$. However, note that these maps $q_{A,1_A}$ may or may not be total as the restriction of $q_{A,1_A}$ is not required to be equal to $F(1_A)$.

The universal property of $\mathrm{rcolim}_O F$ then amounts to the existence of a unique map $\alpha\colon \mathrm{rcolim}_O F\to W$ such that $q_{A,1_A} = \alpha\circ p_{A,1_A}$ for all $A\in\mathbb{X}$. Note that in particular, if the maps $q_{A,1_A}$ are total, then the condition $q_{A,1_A}\circ F\bar{f} = q_{A',1_{A'}}\circ Ff$ implies that $(q_{A,1_A})_{A\in\mathbb{X}}$ is a lax cocone.

7.2 Cocomplete restriction categories have all weighted restriction colimits

In this section, we prove that a restriction category is cocomplete if and only if it has all weighted restriction colimits. However, to assist readability, we divide up the proof into two separate theorems. We will also require the following lemma, as unlike ordinary presheaf categories, the restriction category $\mathsf{PSh}_r(\mathbb{X})$ of restriction presheaves does not generally have a terminal object.

Lemma 7.7. *Let $\mathbb{X}$ be a restriction category, and suppose that for every $A\in\mathbb{X}$, $O(A)$ (set of restriction idempotents on A) has a least element $\bot_A$, and this least element is preserved by precomposition. I.e., $\overline{\bot_A f} = \bot_B\in O(B)$ if $\bot_A\in O(A)$ for all $f\colon B\to A$. Then $\mathsf{PSh}_r(\mathbb{X})$ has a terminal object $\mathbf{1}$, with $\mathbf{1}(A) = \{\star\}$ and $\overline{\star} = \bot_A$.*

Proof. Let $P\in\mathsf{PSh}_r(\mathbb{X})$, and let $\alpha\colon P\Rightarrow\mathbf{1}$ be a natural transformation whose component at $A\in\mathbb{X}$ is given by $\alpha_A(x) = \star$ for all $x\in PA$. If α is a map in $\mathsf{PSh}_r(\mathbb{X})$, then α must satisfy $\overline{\alpha_A(x)} = \overline{\alpha_A(x\cdot\bar{x})} = \overline{\alpha_A(x)\cdot\bar{x}} = \overline{\alpha_A(x)}\circ\bar{x}\leq\bar{x}$ for all $A\in\mathbb{X}$ and $x\in PA$. So setting $\overline{\alpha_A(x)} = \bot_A$ satisfies this criterion. But α also needs to be natural and satisfy the condition $\alpha_B(x \cdot f) = \alpha_A(x) \cdot f$ for all $f\colon B \to A$. Taking restrictions on both sides yields the condition $\bot_B = \bot_A \circ f$. The first requirement $\overline{\alpha_A(x)} = \bot_A$ says that such an α must be unique, and hence $\mathsf{PSh}_r(\mathbb{X})$ has a terminal object if $O(A)$ has a least element which is preserved by composition. □

Theorem 7.8. *Suppose $\mathbb{D}$ is a restriction category with all weighted restriction colimits. Then $\mathbb{D}$ is a cocomplete restriction category. That is,* $\mathsf{Total}(\mathbb{D})$ *is cocomplete, $\mathbb{D}$ is split, and the inclusion* $\mathsf{Total}(\mathbb{D}) \hookrightarrow \mathbb{D}$ *preserves colimits.*

Proof. To show that $\mathsf{Total}(\mathbb{D})$ is cocomplete, let $G\colon \mathbf{C} \to \mathsf{Total}(\mathbb{D})$ be any ordinary functor, and denote by $\mathsf{Triv}(\mathbf{C})$, the restriction category whose underlying category is $\mathbf{C}$ and whose maps are all total. Define $\widetilde{G}$ to be the composite functor below:

$$
\begin{tikzcd}
\mathbf{C} = \mathsf{Triv}(\mathbf{C}) \arrow[r, "G"] \arrow[dr, "\widetilde{G}"'] & \mathsf{Total}(\mathbb{D}) \arrow[d, hook] \\
 & \mathbb{D}.
\end{tikzcd}
$$

Then $\widetilde{G}$ is a restriction functor $\mathsf{Triv}(\mathbf{C}) \to \mathbb{D}$. As all maps in $\mathsf{Triv}(\mathbf{C})$ are total, there exists a terminal presheaf $\mathbf{1} \in \mathsf{PSh}_r(\mathsf{Triv}(\mathbf{C}))$, and so by assumption, the weighted restriction colimit $\operatorname{rcolim}_{\mathbf{1}} \widetilde{G}$ exists. This involves giving an object $\operatorname{rcolim}_{\mathbf{1}} \widetilde{G} \in \mathbb{D}$ along with total maps $(p_A)_{A \in \mathbf{C}}$ such that for all $f \in \mathsf{Triv}(\mathbf{C})$ and hence in $\mathbf{C}$, the following diagram commutes:

$$
\begin{tikzcd}
\widetilde{G}A \arrow[rr, "\widetilde{G}f"] \arrow[dr, "p_A"'] \arrow[ddr, bend right, "q_A"'] & & \widetilde{G}B \arrow[dl, "p_B"] \arrow[ddl, bend left, "q_B"] \\
 & \operatorname{rcolim}_{\mathbf{1}} \widetilde{G} \arrow[d, dashed, "\alpha"] & \\
 & W. &
\end{tikzcd}
$$

Its universal property is such that for any cocone $(q_A)_{A \in \mathbf{C}}$ with $\overline{q_A} = 1$ and satisfying the condition $q_A = q_B \circ \widetilde{G}f$, there is a unique α making the above diagram commute, and α is total by Proposition 7.4. Since $\widetilde{G}A = GA$ and $\widetilde{G}f = Gf$ and $\mathsf{Total}(\mathbb{D}) \subset \mathbb{D}$, we must have $\operatorname{rcolim}_{\mathbf{1}} \widetilde{G} = \operatorname{colim} G$. In other words, $\mathsf{Total}(\mathbb{D})$ is cocomplete.

However, even if the maps $(q_A)_{A \in \mathbf{C}}$ are non-total, by definition of a weighted restriction colimit, there will still exist a unique map α (which will be non-total in this case) making the necessary triangles commute. In particular, this means that the inclusion $\mathsf{Total}(\mathbb{D}) \hookrightarrow \mathbb{D}$ preserves (ordinary) colimits. Therefore it remains to show that $\mathbb{D}$ is split.

Consider the one object restriction category $\mathbb{1} = \{\star\}$ whose only morphisms are 1 and $e\colon \star \to \star$ (with $\bar{e} = e$). Now let $\bar{f}\colon A \to A$ be a restriction idempotent in $\mathbb{D}$, and let $F\colon \mathbb{1} \to \mathbb{D}$ be the functor sending $\star$ to A and e to $\bar{f}$. Since $\mathbb{1}$ has the property that $O(\star)$ has a least element which is preserved under precomposition, the presheaf category $\mathsf{PSh}_r(\mathbb{1})$ has a terminal object $\mathbf{1}$. Therefore, as $\mathbb{D}$ is assumed to have all weighted restriction colimits, the restriction colimit, $\operatorname{rcolim}_{\mathbf{1}} F$, exists. The claim is then that $\operatorname{rcolim}_{\mathbf{1}} F$ is a splitting for $\bar{f} \in \mathbb{D}$.

To see this, observe that to give $\operatorname{rcolim}_{\mathbf{1}} F$ is to give a pair of maps $p, r\colon A \to \operatorname{rcolim}_{\mathbf{1}} F$ such that $\overline{p} = \overline{r} = Fe = \bar{f}$ with the property that $p = r \circ 1 = r$ and $p = r \circ \bar{f}$, or $r = r\bar{f}$.

Hence, it suffices to give a single map r:

$$\begin{tikzcd}
A \arrow[rr, shift left, "\bar{f}"] \arrow[rr, shift right, "1"'] \arrow[dr, "p"'] & & A \arrow[dl, "r"] \\
& \mathrm{rcolim}_{\mathbf{1}} F. &
\end{tikzcd}$$

Its universal property is then such that there is a unique morphism $m\colon \mathrm{rcolim}_{\mathbf{1}} F \to A$ with the property $\bar{f} = mr$.

$$\begin{tikzcd}
A \arrow[r, shift left, "\bar{f}"] \arrow[r, shift right, "1"'] & A \arrow[r, "r"] \arrow[dr, "\bar{f}"'] \arrow[ddr, bend right, "r"'] & \mathrm{rcolim}_{\mathbf{1}} F \arrow[d, "m"] \\
& & A \arrow[d, "r"] \\
& & \mathrm{rcolim}_{\mathbf{1}} F.
\end{tikzcd}$$

Then $rmr = r\bar{f} = r$, and so by uniqueness, $mr = 1$. Hence, every $\bar{f} \in \mathbb{D}$ has a splitting.

Therefore, if $\mathbb{D}$ has all weighted restriction colimits, then $\mathbb{D}$ is cocomplete as a restriction category. □

Theorem 7.9. *Suppose $\mathbb{D}$ is a cocomplete restriction category, and let $\mathbb{X}$ be any restriction category. If P is a restriction presheaf on $\mathbb{X}$ and $F\colon \mathbb{X} \to \mathbb{D}$ is a restriction functor, then the restriction colimit of F weighted by P exists.*

Proof. To show that the restriction colimit of F weighted by P exists, we will first define a functor $\widetilde{F}\colon \mathsf{el}(P) \to \mathsf{Total}(\mathbb{D})$, where $\mathsf{el}(P)$ is the category of elements of P. Then using the fact $\mathbb{D}$ is cocomplete as a restriction category, which means that the conical colimit of $\widetilde{F}$ exists, we express the data for this conical colimit in terms of F. We then show that this new colimit diagram is indeed the restriction colimit of F weighted by P.

So given a restriction presheaf $P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$, let $\mathsf{el}(P)$ be the category of elements of P; its objects are pairs (A, x) with $A \in \mathbb{X}$ and $x \in PA$, and a morphism $f\colon (A, x) \to (A', x')$ in $\mathsf{el}(P)$ is a morphism $f \in \mathbb{X}$ such that $x' \cdot f = x$. Next, let us define $\widetilde{F}\colon \mathsf{el}(P) \to \mathsf{Total}(\mathbb{D})$ to be a functor sending objects (A, x) to a chosen splitting of $F\bar{x}$ (as $\mathbb{D}$ is split by assumption); and on maps $f\colon (A, x) \to (A', x')$ in $\mathsf{el}(P)$, $\widetilde{F}f$ is defined to be the composite $r_{F\overline{x'}} \circ Ff \circ m_{F\bar{x}}$, where $m_{F\bar{x}}$ and $r_{F\bar{x}}$ are the splittings of $F\bar{x}$. That is, $m_{F\bar{x}} \circ r_{F\bar{x}} = F\bar{x}$ and $r_{F\bar{x}} \circ m_{F\bar{x}} = 1$. To see that this functor does indeed send maps in $\mathsf{el}(P)$ to total maps in $\mathbb{D}$, consider the composite $m_{F\overline{x'}} \circ \widetilde{F}f$, which has the same restriction as $\widetilde{F}f$ in $\mathbb{D}$ since $m_{F\overline{x'}}$ is a monomorphism. Now by definition of $\widetilde{F}$,

$$m_{F\overline{x'}} \circ \widetilde{F}f = m_{F\overline{x'}} \circ r_{F\overline{x'}} \circ Ff \circ m_{F\bar{x}} = F\overline{x'} \circ Ff \circ m_{F\bar{x}} = F(\overline{x'} \circ f) \circ m_{F\bar{x}}.$$

So taking restrictions and using restriction presheaf axioms, we get

$$\begin{aligned}
\overline{F(\overline{x'} \circ f) \circ m_{F\bar{x}}} &= \overline{F\left(\overline{\overline{x'} \circ f}\right) \circ m_{F\bar{x}}} = \overline{F(\overline{x' \cdot f}) \circ m_{F\bar{x}}} = \overline{F\bar{x} \circ m_{F\bar{x}}} \\
&= \overline{m_{F\bar{x}} \circ r_{F\bar{x}} \circ m_{F\bar{x}}} = \overline{m_{F\bar{x}}} = 1,
\end{aligned}$$

and so $\widetilde{F}$ is well-defined.

Now since $\mathbb{D}$ is a cocomplete restriction category, the conical colimit of $\widetilde{F}$ exists in $\mathsf{Total}(\mathbb{D})$, and moreover, is also a conical colimit in $\mathbb{D}$. Explicitly, this conical colimit involves an object $V \in \mathbb{D}$ along with total maps $\widetilde{p}_{A,x} \colon \widetilde{F}(A,x) \to V$ such that $\widetilde{p}_{A,x} = \widetilde{p}_{A',x'} \circ \widetilde{F}f$ for all $f \in \mathsf{el}(P)$. And as this is also a conical colimit in $\mathbb{D}$, its universal property is such that for any cocone under $\mathsf{el}(P) \xrightarrow{\widetilde{F}} \mathsf{Total}(\mathbb{D}) \hookrightarrow \mathbb{D}$, say $\widetilde{q}_{A,x} \colon \widetilde{F}(A,x) \to W$, there is a unique $\alpha \colon V \to W$ such that $\alpha \circ \widetilde{p}_{A,x} = \widetilde{q}_{A,x}$.

However, because the conical colimit of $\widetilde{F}$ is a diagram in terms of $\widetilde{F}$ and not F, we now attempt to express its colimiting data in terms of F. We will show that giving a cocone $\widetilde{q}_{A,x} \colon \widetilde{F}(A,x) \to W$ in $\mathbb{D}$ is the same as giving a family of maps $(q_{A,x} \colon FA \to W)_{A \in \mathbb{X}, x \in PA}$ such that $q_{A,x} = q_{A',x'} \circ Ff$ whenever $x' \cdot f = x$; that is, $(q_{A,x})$ is a weighted cocone under F. Finally, we show that under this correspondence, the universal cocone $\widetilde{p}$ corresponds to the family of maps $(p_{A,x} \colon FA \to V)_{A \in \mathbb{X}, x \in PA}$ such that $\overline{p_{A,x}} = F\bar{x}$.

So consider the following commutative diagram.

$$\begin{tikzcd}
 & \widetilde{F}(A,x) \arrow[rrr, bend left, "\widetilde{F}f"] \arrow[dr, hook, "m_{F\bar{x}}"] \arrow[ddr, dashed, "\widetilde{p}_{A,x}"] \arrow[dddr, dashed, "\widetilde{q}_{A,x}"'] & & & \widetilde{F}(A',x') \arrow[dr, hook, "m_{F\overline{x'}}"] \arrow[ddll, dashed, "\widetilde{p}_{A',x'}"] & \\
FA \arrow[ur, "r_{F\bar{x}}"] \arrow[rr, "F\bar{x}"] \arrow[drr, bend right, "p_{A,x}"] \arrow[ddrr, bend right, "q_{A,x}"'] & & FA \arrow[r, "Ff"'] & FA' \arrow[ur, "r_{F\overline{x'}}"] \arrow[rr, "F\overline{x'}"'] \arrow[dl, "p_{A',x'}"] & & FA' \\
 & & V \arrow[d, "\alpha"] & & & \\
 & & W & & &
\end{tikzcd}$$

Given the cocone $\widetilde{q}_{A,x} \colon \widetilde{F}(A,x) \to W$, we may precompose each $\widetilde{q}_{A,x}$ by $r_{F\bar{x}}$ to get $q_{A,x} \colon FA \to W$. We now have to show that the maps $(q_{A,x})_{A \in \mathbb{X}, x \in PA}$ satisfy $q_{A,x} = q_{A',x'} \circ Ff$ if $x = x' \cdot f$. So let us begin with the equality $\widetilde{q}_{A,x} = \widetilde{q}_{A',x'} \circ \widetilde{F}f$, as $\widetilde{q}$ is a cocone. Then precomposing both sides by $r_{F\bar{x}}$ gives

$$q_{A,x} = \widetilde{q}_{A',x'} \circ \widetilde{F}f \circ r_{F\bar{x}}, \tag{7.2}$$

since $\widetilde{q}_{A,x} \circ r_{F\bar{x}} = q_{A,x}$ by definition. Now, because $x = x' \cdot f$ implies $\bar{x} = \overline{\overline{x'} \circ f}$, and $F\overline{x'} = \overline{r_{F\overline{x'}}}$, the right hand side of (7.2) reduces to

$$\begin{aligned}
\widetilde{q}_{A',x'} \circ \widetilde{F}f \circ r_{F\bar{x}} &= \widetilde{q}_{A',x'} \circ r_{F\overline{x'}} \circ Ff \circ F\bar{x} = \widetilde{q}_{A',x'} \circ r_{F\overline{x'}} \circ Ff \circ F\left(\overline{\overline{x'} \circ f}\right) \\
&= \widetilde{q}_{A',x'} \circ r_{F\overline{x'}} \circ F(\overline{x'}) \circ Ff = \widetilde{q}_{A',x'} \circ r_{F\overline{x'}} \circ \overline{r_{F\overline{x'}}} \circ Ff \\
&= \widetilde{q}_{A',x'} \circ r_{F\overline{x'}} \circ Ff = q_{A',x'} \circ Ff.
\end{aligned}$$

This means that if $x = x' \cdot f$, then the maps $(q_{A,x})_{A \in \mathbb{X}, x \in PA}$ satisfy $q_{A,x} = q_{A',x'} \circ Ff$.

However, observe that this operation is invertible, meaning that given a family of maps $(q_{A,x} \colon FA \to W)_{A \in \mathbb{X}, x \in PA}$ satisfying $q_{A,x} = q_{A',x'} \circ Ff$ whenever $x = x' \cdot f$, we can get a cocone $\widetilde{q}_{A,x} \colon \widetilde{F}(A,x) \to W$ simply by precomposing each $q_{A,x}$ by $m_{F\bar{x}}$ (as $\mathbb{D}$ has splittings). To see that $\widetilde{q}_{A,x} = \widetilde{q}_{A',x'} \circ \widetilde{F}f$, suppose $q_{A,x} = q_{A',x'} \circ Ff$ whenever $x' \cdot f = x$. Then precomposing both sides by $m_{F\bar{x}}$ gives

$$\widetilde{q}_{A,x} = q_{A',x'} \circ Ff \circ m_{F\bar{x}}. \tag{7.3}$$

But note that the right hand side of the above equation may be reduced to

$$\begin{aligned} q_{A',x'} \circ Ff \circ m_{F\bar{x}} &= q_{A',x'} \circ Ff \circ m_{F\bar{x}} \circ (r_{F\bar{x}} \circ m_{F\bar{x}}) = q_{A',x'} \circ Ff \circ F\bar{x} \circ m_{F\bar{x}} \\ &= q_{A',x'} \circ Ff \circ F\left(\overline{\overline{x'} \circ f}\right) \circ m_{F\bar{x}} = q_{A',x'} \circ F\overline{x'} \circ Ff \circ m_{F\bar{x}} \\ &= q_{A',x'} \circ m_{F\overline{x'}} \circ \widetilde{F}f = \widetilde{q}_{A',x'} \circ \widetilde{F}f \end{aligned} \tag{7.4}$$

by a simple diagram chase. Therefore, from a family of maps $(q_{A,x} \colon FA \to W)_{A\in\mathbb{X},x\in PA}$ such that $q_{A,x} = q_{A',x'} \circ Ff$ whenever $x = x' \cdot f$, we get a cocone $\widetilde{q}_{A,x} \colon \widetilde{F}(A,x) \to W$ satisfying the condition $\widetilde{q}_{A,x} = \widetilde{q}_{A',x'} \circ Ff$. Hence, to give a cocone $\widetilde{q}_{A,x} \colon \widetilde{F}(A,x) \to W$ is the same as giving an object $W \in \mathbb{D}$ and a family of maps $(q_{A,x})_{A\in\mathbb{X},x\in PA}$ satisfying $q_{A,x} = q_{A',x'} \circ Ff$ if $x = x' \cdot f$. In other words, to give a cocone $\widetilde{q}$ under $\widetilde{F}$ is to give a cocone $(q_{A,x})_{A\in\mathbb{X},x\in PA}$ under F weighted by the elements of P.

So it remains to show that the universal total cocone $\widetilde{p}$ corresponds to the universal such weighted cocone $(p_{A,x})_{A\in\mathbb{X},x\in PA}$ with $\overline{p_{A,x}} = F\bar{x}$ satisfying $p_{A,x} = p_{A',x'} \circ Ff$ whenever $x = x' \cdot f$. Since $\widetilde{p}$ is a total cocone, this means

$$\overline{p_{A,x}} = \overline{\widetilde{p}_{A,x} \circ r_{F\bar{x}}} = \overline{\overline{\widetilde{p}_{A,x}} \circ r_{F\bar{x}}} = \overline{r_{F\bar{x}}} = F\bar{x}.$$

Conversely, suppose $p_{A,x} = F\bar{x}$. Then $\overline{\widetilde{p}_{A,x}} = 1$ since

$$\overline{\widetilde{p}_{A,x}} = \overline{p_{A,x} \circ m_{F\bar{x}}} = \overline{\overline{p_{A,x}} \circ m_{F\bar{x}}} = \overline{F\bar{x} \circ m_{F\bar{x}}} = \overline{\overline{r_{F\bar{x}}} \circ m_{F\bar{x}}} = \overline{r_{F\bar{x}} \circ m_{F\bar{x}}} = 1$$

Therefore, given any restriction presheaf $P \colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ and restriction functor $F \colon \mathbb{X} \to \mathbb{D}$, the restriction colimit of F weighted by P exists if $\mathbb{D}$ is cocomplete. □

7.3 Cocontinuous restriction functors preserve all weighted restriction colimits

Analogous with ordinary categories, we have the following alternative description of a cocontinuous restriction functor.

Proposition 7.10. *Let $\mathbb{D}$ and $\mathbb{E}$ be cocomplete restriction categories, and $F \colon \mathbb{D} \to \mathbb{E}$ a restriction functor. Then F preserves all weighted restriction colimits if and only if F is cocontinuous as a restriction functor; i.e.,* $\mathsf{Total}(F) \colon \mathsf{Total}(\mathbb{D}) \to \mathsf{Total}(\mathbb{E})$ *is cocontinuous.*

Proof. Suppose F preserves all weighted restriction colimits. Let $G \colon \mathbf{C} \to \mathsf{Total}(\mathbb{D})$ be a functor, and construct $\widetilde{G} \colon \mathsf{Triv}(\mathbf{C}) \to \mathbb{D}$ as in the previous proposition. So $\mathrm{rcolim}_1 \widetilde{G} = \mathrm{colim}\, G$, and since F preserves weighted restriction colimits and the colimiting injections are total, $\mathsf{Total}(F)$ must also preserve $\mathrm{colim}\, G$. Hence, F is cocontinuous.

Conversely, suppose F is cocontinuous. Let $G \colon \mathbb{C} \to \mathbb{D}$ and $P \in \mathsf{PSh}_r(\mathbb{C})$, and consider $\mathrm{rcolim}_P\, G$. By the proof of Theorem 7.9, we can construct $\mathrm{rcolim}_P\, G$ as the ordinary colimit of $\widetilde{G} \colon \mathsf{el}(P) \to \mathsf{Total}(\mathbb{D})$. But as F is cocontinuous, this ordinary colimit is preserved under $\mathsf{Total}(F)$, and unravelling this ordinary colimit in $\mathbb{E}$ gives $\mathrm{rcolim}_P\, FG$. □

Proposition 7.11. *Let $\mathbb{X}$ be a restriction category, and $P \in \mathsf{PSh}_r(\mathbb{X})$. Then $P \cong \mathrm{rcolim}_P\, \mathbf{y}_r$, where $\mathbf{y}_r \colon \mathbb{X} \to \mathsf{PSh}_r(\mathbb{X})$ is the Yoneda embedding.*

Proof. As ordinary presheaves, we have $P \cong \mathrm{colim}_P\, \mathbf{y}_r$, so the only thing we need to check is that the projection maps $p_{A,x} \colon \mathbf{y}_r A \to P$ have restrictions given by $\mathbf{y}_r \bar{x}$. By Yoneda, we have $x \in PA$ given by $x = (p_{A,x})_A(1_A)$. Now for $B \in \mathbb{X}$ and $y \in \mathbb{X}(B, A)$, by definition, we have $\overline{p_{A,x}}_B(y) = y \circ \overline{(p_{A,x})_B(y)}$. But

$$\bar{x} \circ y = y \circ \overline{x \cdot y} = y \circ \overline{(p_{A,x})_A(1_A) \cdot y} = y \circ \overline{(p_{A,x})_B(y)}.$$

Therefore, $\overline{p_{A,x}}_B(y) = \bar{x} \circ y$, or $\overline{p_{A,x}} = \bar{x} \circ (-) = \mathbf{y}_r \bar{x}$. And so $P \cong \mathrm{rcolim}_P\, \mathbf{y}_r$. □

Corollary 7.12. *Let $\mathbb{X}$ and $\mathbb{D}$ be restriction categories, with $\mathbb{D}$ cocomplete, and let $F \colon \mathbb{X} \to \mathbb{D}$ be a restriction functor. Then $\mathsf{Lan}_{\mathbf{y}_r} F(P) \cong \mathrm{rcolim}_P\, F$ for every restriction presheaf $P \in \mathsf{PSh}_r(\mathbb{X})$.*

Proof. By Proposition 7.10, we have $P \cong \mathrm{rcolim}_P\, \mathbf{y}_r$. Since $\mathsf{Lan}_{\mathbf{y}_r} F$ is cocontinuous and $\mathsf{PSh}_r(\mathbb{X})$ and $\mathbb{D}$ are both cocomplete, it preserves all weighted restriction colimits, and so

$$\mathsf{Lan}_{\mathbf{y}_r} F(P) \cong \mathsf{Lan}_{\mathbf{y}_r} F(\mathrm{rcolim}_P\, \mathbf{y}_r) \cong \mathrm{rcolim}_P \left(\mathsf{Lan}_{\mathbf{y}_r} F \circ \mathbf{y}_r\right) \cong \mathrm{rcolim}_P\, F$$

as $\mathsf{Lan}_{\mathbf{y}_r} F \circ \mathbf{y}_r \cong F$. □

7.4 Join restriction categories with restriction colimits are cocomplete

We conclude this chapter with a few statements about cocomplete join restriction categories and restriction colimits.

Definition 7.13. Suppose $\mathbb{X}$ and $\mathbb{D}$ are join restriction categories, $F \colon \mathbb{X} \to \mathbb{D}$ is a join restriction functor, and $P \colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ is a join restriction presheaf on $\mathbb{X}$. Then the restriction colimit of F weighted by P is defined as in Definition 7.3. That is, it consists of a single object $\mathrm{rcolim}_P\, F \in \mathbb{D}$ together with coprojection maps $(p_{A,x} \colon FA \to \mathrm{rcolim}_P\, F)_{A \in \mathbb{X}, x \in PA}$, such that the restrictions on each $p_{A,x}$ satisfy the condition $\overline{p_{A,x}} = F\bar{x}$.

Remark 7.14. Observe that in the above definition, there is no specification on the joins of compatible coprojection maps $p_{A,x}$. If we *were* to specify a condition on the joins, it would be that the coprojections should satisfy $p_{A,x \vee y} = p_{A,x} \vee p_{A,y}$ if x and y are compatible. However, the condition $p_{A,x\vee y} = p_{A,x} \vee p_{A,y}$ is actually implied by compatibility of x, y in PA. To see this, we first must show that $x \smile y$ implies $p_{A,x} \smile p_{A,y}$. But if $x \cdot \bar{y} = y \cdot \bar{x}$, then

$$p_{A,x} \circ \overline{p_{A,y}} = p_{A,x} \circ F\bar{y} = p_{A,x\cdot\bar{y}} = p_{A,y\cdot\bar{x}} = p_{A,y} \circ F\bar{x} = p_{A,y} \circ \overline{p_{A,x}},$$

which means $p_{A,x} \smile p_{A,y}$, and so their join $p_{A,x} \vee p_{A,y}$ exists. To show that this join is then given by $p_{A,x\vee y}$, we again use the fact that in a restriction category, $f = g$ if and only if $\bar{f} = \bar{g}$ and $f \leq g$.

Now clearly $\overline{p_{A,x\vee y}} = F(\overline{x \vee y}) = F\bar{x} \vee F\bar{y} = \overline{p_{A,x}} \vee \overline{p_{A,y}}$ by definition. Also,

$$\begin{aligned} p_{A,x\vee y} \circ \overline{p_{A,x} \vee p_{A,y}} &= p_{A,x\vee y} \circ (F\bar{x} \vee F\bar{y}) = (p_{A,x\vee y} \circ F\bar{x}) \vee (p_{A,x\vee y} \circ F\bar{y}) \\ &= p_{A,(x\vee y)\cdot\bar{x}} \vee p_{A,(x\vee y)\cdot\bar{y}} = p_{A,x\vee x\cdot\bar{y}} \vee p_{A,x\cdot\bar{y}\vee y} \\ &= p_{A,x} \vee p_{A,y} \end{aligned}$$

which implies $p_{A,x} \vee p_{A,y} \leq p_{A,x\vee y}$. Therefore, $p_{A,x\vee y} = p_{A,x} \vee p_{A,y}$. Hence, if x and y were compatible in PA, then the condition $p_{A,x\vee y} = p_{A,x} \vee p_{A,y}$ holds automatically, and so we dispense with this requirement in our definition.

Theorem 7.15. *Suppose $\mathbb{D}$ is a* join *restriction category with the property that for all join restriction functors $F\colon \mathbb{X} \to \mathbb{D}$ and restriction presheaves $P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$, the restriction colimit of F weighted by P exists. Then $\mathbb{D}$ is a cocomplete join restriction category. That is,* $\mathsf{Total}(\mathbb{D})$ *is cocomplete, all restriction idempotents in $\mathbb{D}$ are split, and the inclusion* $\mathsf{Total}(\mathbb{D}) \hookrightarrow \mathbb{D}$ *preserves ordinary colimits in* $\mathsf{Total}(\mathbb{D})$*.*

Proof. To show that $\mathsf{Total}(\mathbb{D})$ is cocomplete, again let $G\colon \mathbf{C} \to \mathsf{Total}(\mathbb{D})$ be arbitrary and consider $\mathsf{Triv}(\mathbf{C})$, the restriction category whose base category is $\mathbf{C}$ but whose morphisms are total. Denote by $\mathcal{I}\colon \mathbf{jrCat} \hookrightarrow \mathbf{rCat}$, the inclusion of join restriction categories into restriction categories. Now although this inclusion has a left adjoint $\mathbf{j}$ (see [Guo, 2012]), we will not be describing this adjoint in its full generality, except where it applies to $\mathsf{Triv}(\mathbf{C})$ and the one object restriction category $\mathbb{1}$, first introduced in Theorem 7.8. In those cases, we shall describe them explicitly, and the reader will have little difficulty in verifying that they are indeed the right definitions to take.

Returning to the adjunction $\mathbf{j} \dashv \mathcal{I}$, let us denote its unit at $\mathsf{Triv}(\mathbf{C})$ by $\eta_{\mathsf{Triv}(\mathbf{C})}$. This means that there is a unique join restriction functor $\widetilde{G}\colon \mathbf{j}(\mathsf{Triv}(\mathbf{C})) \to \mathbb{D}$, making the following diagram commute:

$$\begin{tikzcd}
\mathbf{C} = \mathsf{Triv}(\mathbf{C}) \arrow[r, "G"] \arrow[d, "\eta_{\mathsf{Triv}(\mathbf{C})}"'] & \mathsf{Total}(\mathbb{D}) \arrow[d, hook] \\
\mathcal{I}(\mathbf{j}(\mathsf{Triv}(\mathbf{C}))) \arrow[r, "\mathcal{I}(\widetilde{G})"'] & \mathcal{I}(\mathbb{D}).
\end{tikzcd}$$

This join restriction category $\mathbf{j}(\mathsf{Triv}(\mathbf{C}))$ has the exact same objects and maps as $\mathsf{Triv}(\mathbf{C})$, with an additional bottom map $\perp_{AB}$ for each hom-set $\mathsf{Hom}(A, B)$. The unique join restriction functor $\widetilde{G}$ is G on the objects and total maps in $\mathbf{j}(\mathsf{Triv}(\mathbf{C}))$, and takes each bottom map $\perp_{AB} \in \mathsf{Hom}(A, B)$ in $\mathbf{j}(\mathsf{Triv}(\mathbf{C}))$ to the bottom map $\perp\colon GA \to GB$ in $\mathbb{D}$.

Now let us define the join restriction presheaf $\mathbf{2}\colon \mathbf{j}(\mathsf{Triv}(\mathbf{C}))^{\mathrm{op}} \to \mathbf{Set}$, which on objects, takes $A \in \mathbf{j}(\mathsf{Triv}(\mathbf{C}))$ to the set $\mathbf{2}(A) = \{\perp_A, \star_A\}$, with the restriction on each element of $\mathbf{2}(A)$ given by $\overline{\perp_A} = \perp_{AA}$ and $\overline{\star_A} = 1_A$. On maps $f\colon B \to A$ in $\mathbf{j}(\mathsf{Triv}(\mathbf{C}))$ (with $f \neq \perp_{BA}$), $\mathbf{2}(f)$ is defined on $\mathbf{2}(A)$ by $\star_A \cdot f = \star_B$ and $\perp_A \cdot f = \perp_B$; and where $f = \perp_{BA}$, by $x \cdot f = \perp_B$ for all $x \in \mathbf{2}(A)$. Note that as $\perp_A$ and $\star_A$ are compatible, their join exists and is given by $\star_A$.

By assumption, the restriction colimit of $\widetilde{G}$ weighted by $\mathbf{2}$ exists. Explicitly, this restriction colimit consists of a single object $\mathrm{rcolim}_{\mathbf{2}}\, \widetilde{G}$ in $\mathbb{D}$, together with coprojections $p_{\star_A}, p_{\perp_A}\colon \widetilde{G}A \to \mathrm{rcolim}_{\mathbf{2}}\, \widetilde{G}$ (for each $A \in \mathbf{j}(\mathsf{Triv}(\mathbf{C}))$), such that for all maps $f \in \mathbf{j}(\mathsf{Triv}(\mathbf{C}))$, the following diagram commutes:

$$\begin{tikzcd}
\widetilde{G}A \arrow[rr, "\widetilde{G}f"] \arrow[dr, shift left, "p_{\star_A}"] \arrow[dr, shift right, "p_{\perp_A}"'] \arrow[ddr, bend right, shift left, "q_{\star_A}"] \arrow[ddr, bend right, shift right, "q_{\perp_A}"'] & & \widetilde{G}B \arrow[dl, shift right, "p_{\star_B}"'] \arrow[dl, shift left, "p_{\perp_B}"] \arrow[ddl, bend left, shift right, "q_{\star_B}"'] \arrow[ddl, bend left, shift left, "q_{\perp_B}"] \\
& \mathrm{rcolim}_{\mathbf{2}}\, \widetilde{G} \arrow[d, dashed, "\alpha"] & \\
& W. &
\end{tikzcd}$$

It is now easy to see that $\mathsf{Total}(\mathbb{D})$ is cocomplete, by taking the colimit of $G\colon \mathbf{C} \to \mathsf{Total}(\mathbb{D})$ to be the object $\mathrm{rcolim}_{\mathbf{2}}\, \widetilde{G}$, together with the total maps $p_{\star_A}$ for each $A \in \mathbf{j}(\mathsf{Triv}(\mathbf{C}))$. The fact that the inclusion $\mathsf{Total}(\mathbb{D}) \hookrightarrow \mathbb{D}$ preserves this colimit follows by the same argument as presented in Theorem 7.8.

Finally, to see that every restriction idempotent $\bar{f}: A \to A$ in $\mathbb{D}$ is split, consider the join restriction category $\mathbf{j}(\mathbb{1})$, where $\mathbb{1}$ is the one-object restriction category from Theorem 7.8. This join restriction category $\mathbf{j}(\mathbb{1})$ contains an additional bottom map $\perp_\star : \star \to \star$, which is clearly preserved by precomposition. So again from Lemma 7.7, $\mathsf{PSh}_r(\mathbf{j}(\mathbb{1}))$ has a terminal object $\mathbf{1}$. Now let $F: \mathbf{j}(\mathbb{1}) \to \mathbb{D}$ be the join restriction functor sending the object $\star \in \mathbf{j}(\mathbb{1})$ to $A \in \mathbb{D}$, and maps $e, \perp : \star \to \star$ in $\mathbf{j}(\mathbb{1})$ to $\bar{f}, \perp : A \to A$ in $\mathbb{D}$ respectively. Then, since $\mathbb{D}$ is assumed to have all weighted restriction colimits, the restriction colimit given by $\mathrm{rcolim}_{\mathbf{1}} F$ exists. It is then a matter of verifying that $\mathrm{rcolim}_{\mathbf{1}} F$ is again a splitting of $\bar{f}$, by the same reasoning as in Theorem 7.8.

Therefore, if the join restriction category $\mathbb{D}$ has all weighted restriction colimits, then $\mathbb{D}$ is a cocomplete join restriction category. $\square$

8

Atlases and their gluings

This chapter will focus on a particular example of a restriction colimit; namely gluings of atlases in a restriction category. The notion of atlas in a restriction category extends that of the notion of atlases for a topological n-manifold; for instance, instead of *transition functions* from one open subset of $\mathbb{R}^n$ to another, we have partial isomorphisms between objects of this atlas. This generalisation allows us to model different types of spaces by specifying them as atlases in some join restriction category. As an example, recall the join restriction category **fdCts** from Example 5.6, the objects of which are natural numbers $n \in \mathbb{N}$, and a map from n to m is a partial continuous function from $\mathbb{R}^n$ to $\mathbb{R}^m$ on an open subset $U \subseteq \mathbb{R}^n$. An atlas in this category then corresponds to a real topological manifold [Cockett & Cruttwell, 2014].

8.1 Atlases and gluings of atlases

Since the focus of this chapter will be on atlases, let us begin with the following definition.

Definition 8.1. [Grandis, 1990] Let $\mathbb{X}$ be a restriction category. An $\mathcal{I}$-object *total* atlas $(U_i, \varphi_{ij})_{i,j\in\mathcal{I}}$ in $\mathbb{X}$ consists of the following data:

1. a family of objects U_i in $\mathbb{X}$ indexed by the set $\mathcal{I}$;
2. a map $\varphi_{ij} \colon U_i \longrightarrow U_j$ for each $i, j \in \mathcal{I}$ with the property that $\overline{\varphi_{ij}} = \varphi_{ji}\varphi_{ij}$ if $i \neq j$ and $\varphi_{ii} = 1$ otherwise; and
3. $\varphi_{ik} \circ \overline{\varphi_{ij}} = \varphi_{jk} \circ \varphi_{ij}$ for all $i, j, k \in \mathcal{I}$.

Where it is clear that the indexing set is $\mathcal{I}$, we shall refer to $\mathcal{I}$-object total atlases in $\mathbb{X}$ as simply atlases.

Observe that the map φ_{ij} between objects U_i and U_j in any given atlas is, by definition, a partial isomorphism. (Recall from Chapter 2 that f is a partial isomorphism if there exists a map g such that $gf = \bar{f}$ and $fg = \bar{g}$). Now there is a more general definition of an atlas.

Definition 8.2. [Cockett & Cruttwell, 2014] Again, let $\mathbb{X}$ be a restriction category. An $\mathcal{I}$-object atlas $(U_i, \varphi_{ij})_{i,j\in\mathcal{I}}$ in $\mathbb{X}$ is given by the following data:

1. a family of objects U_i in $\mathbb{X}$ indexed by $\mathcal{I}$;
2. for each pair $i, j \in \mathcal{I}$ with $i \neq j$, a map $\varphi_{ij}\colon U_i \to U_j$ satisfying $\overline{\varphi_{ij}} = \varphi_{ji}\varphi_{ij}$; and
3. for each $i \in \mathcal{I}$, a restriction idempotent φ_{ii} with the property $\varphi_{ij}\varphi_{ii} = \varphi_{ij}$ for all $j \in \mathcal{I}$; and
4. $\varphi_{ik} \circ \overline{\varphi_{ij}} = \varphi_{jk} \circ \varphi_{ij}$ for all $i, j, k \in \mathcal{I}$.

Note the only difference between this more general atlas and a total atlas, is that instead of insisting $\varphi_{ii} = 1$ for all $i \in \mathcal{I}$, these φ_{ii} are allowed to be arbitrary restriction idempotents satisfying the condition $\varphi_{ij}\varphi_{ii} = \varphi_{ij}$. For the rest of the chapter, unless otherwise stated, our atlases will be of this more general type, with φ_{ii} arbitrary.

Definition 8.3 (Gluings of atlases)**.** If $\mathbb{X}$ is a restriction category and (U_i, φ_{ij}) is an atlas in $\mathbb{X}$, then a gluing $(G, g_i\colon U_i \to G)$ of this atlas is a lax cocone under the diagram $(U_i, \varphi_{ij})_{i,j\in\mathcal{I}}$, universal among lax cocones satisfying the condition $g_i\varphi_{ii} = g_i$. Explicitly, it consists of a family of maps $\{g_i\colon U_i \to G\}_{i\in\mathcal{I}}$ with $g_i\varphi_{ii} = g_i$ and $g_j\varphi_{ij} \leq g_i$ for each $i \neq j$, such that for any other family $\{h_i\colon U_i \to H\}_{i\in I}$ with $h_i\varphi_{ii} = h_i$ and $h_j\varphi_{ij} \leq h_i$, there exists a unique map $\alpha\colon G \to H$ such that $h_i = g_i \circ \alpha$ for all $i \in \mathcal{I}$.

Remark 8.4. Observe that the above gluing condition $g_j\varphi_{ij} \leq g_i$ for each pair $i, j \in \mathcal{I}$ may be expressed equivalently as $g_j\varphi_{ij} = g_i\overline{\varphi_{ij}}$. To see this, suppose $g_j\varphi_{ij} \leq g_i$, and precompose both sides of the inequality by $\overline{\varphi_{ij}}$ to give $g_j\varphi_{ij} \leq g_i\overline{\varphi_{ij}}$. Now all we have to show is that $g_i\overline{\varphi_{ij}} \leq g_j\varphi_{ij}$. But this follows from our assumption that $g_i\varphi_{ji} \leq g_j$, since precomposing by φ_{ij} on both sides gives $g_i\overline{\varphi_{ij}} \leq g_j\varphi_{ij}$, and because each φ_{ij} is a partial isomorphism. The other direction, that $g_j\varphi_{ij} = g_i\overline{\varphi_{ij}}$ implies $g_j\varphi_{ij} \leq g_i$, is trivial.

It is not hard to show that with the above definition, the gluing maps g_i are partial isomorphisms with partial inverses g_i^{-1}, and that $\varphi_{ij} = g_j^{-1} \circ g_i$; see Theorem 8.5 below. However, it turns out that gluings of atlases in a join restriction category may be completely characterised. The following statement is a slight modification of [Grandis, 1990, Proposition 6.2]

Theorem 8.5. *Let (U_i, φ_{ij}) be an atlas in a join restriction category $\mathbb{X}$. Then (G, g_i) is a gluing of (U_i, φ_{ij}) if and only if*

1. *each g_i is a partial isomorphism (with partial inverse g_i^{-1});*
2. *$\varphi_{ij} = g_j^{-1} \circ g_i$; and*
3. *$1_G = \bigvee_{i\in\mathcal{I}} \overline{g_i^{-1}}$.*

Proof. Suppose (G, g_i) is a gluing of (U_i, φ_{ij}). Now fix a $k \in \mathcal{I}$ and consider the lax cocone (U_k, φ_{ik}) to the same atlas (U_i, φ_{ij}), with vertex U_k and maps $\varphi_{ik}\colon U_i \to U_k$ $(i \neq k)$ and $\varphi_{kk}\colon U_k \to U_k$. Note that this lax cocone satisfies the required universal property. Because (G, g_i) is a gluing, define the partial inverse of g_k to be the unique map g_k^{-1} making the following diagram commute:

$$\begin{tikzcd}
U_i \arrow[rr, "\varphi_{ik}"] \arrow[dr, "g_i"] \arrow[ddr, bend right, "\varphi_{ik}"'] & & U_k \arrow[dl, "g_k"'] \arrow[ddl, bend left, "\varphi_{kk}"] \\
& G \arrow[d, dashed, "g_k^{-1}"'] & \\
& U_k &
\end{tikzcd}$$

To see that g_k^{-1} is indeed the partial inverse of g_k, we need to show $g_k^{-1}g_k = \overline{g_k}$ and $g_kg_k^{-1} = \overline{g_k^{-1}}$. We begin by showing the latter.

For every $i, j \in \mathcal{I}$, we have $g_k \circ \varphi_{jk} \circ \varphi_{ij} \le g_k \circ \varphi_{ik}$ as (U_i, φ_{ij}) is an atlas. Also, we have $g_k\varphi_{ik}\varphi_{ii} = g_k\varphi_{ik}$ (as $\varphi_{ik}\varphi_{ii} = \varphi_{ik}$), which means $(G, g_k \circ \varphi_{ik})_{i\in\mathcal{I}}$ is a lax cocone with the required property. Therefore, there is a unique map $h\colon G \to G$ such that $hg_i = g_k\varphi_{ik}$ for all $i \in \mathcal{I}$. Clearly, $g_k \circ g_k^{-1}$ is one such map.

Now by construction, $\overline{g_k^{-1}}g_k = g_k\overline{g_k^{-1}g_k} = g_k\varphi_{kk} = g_k$. But also

$$\overline{g_k^{-1}}g_i = g_i\overline{g_k^{-1}g_i} = g_i\overline{\varphi_{ik}} = g_k\varphi_{ik}.$$

Therefore, by uniqueness, $g_kg_k^{-1} = \overline{g_k^{-1}}$.

To show that $g_k^{-1}g_k = \overline{g_k}$, it suffices to show that $\varphi_{kk} = \overline{g_k}$ as by construction, $g_k^{-1}g_k = \varphi_{kk}$. By definition, $g_k\varphi_{kk} = g_k$, which implies $\overline{g_k} \le \varphi_{kk}$. But

$$\varphi_{kk} = \overline{g_k^{-1}g_k} = \overline{\overline{g_k^{-1}}g_k} = \overline{g_kg_k^{-1}g_k} = \overline{g_k\varphi_{kk}} = \overline{g_k}\varphi_{kk},$$

from which we conclude that $\varphi_{kk} \le \overline{g_k}$, and hence, $g_k^{-1}g_k = \overline{g_k}$ as required. This proves that each g_i is a partial isomorphism. And by construction of the partial inverses, we also have $g_j^{-1}g_i = \varphi_{ij}$. Therefore, it remains to show that $1_G = \bigvee_{i\in\mathcal{I}} \overline{g_i^{-1}}$.

But observe that for each i,

$$\left(\bigvee\nolimits_{j\in\mathcal{I}}\overline{g_j^{-1}}\right)g_i = \left(\bigvee\nolimits_{j\in\mathcal{I},j\neq i}\overline{g_j^{-1}}\right)g_i \vee \overline{g_i^{-1}}g_i = \bigvee\nolimits_{j\in\mathcal{I},j\neq i}\left(\overline{g_j^{-1}}g_i\right) \vee g_i\varphi_{ii} = g_i,$$

from which we deduce the result, by uniqueness.

Conversely, suppose all three conditions hold. We want to show that given any lax cocone (H, h_i) such that $h_i\varphi_{ii} = h_i$, there exists a unique map $G \to H$ making the following diagram commute:

$$\begin{tikzcd}
U_i \arrow[rr, "\varphi_{ij}"] \arrow[dr, "g_i"'] \arrow[ddr, bend right, "h_i"'] & & U_j \arrow[dl, "g_j"] \arrow[ddl, bend left, "h_j"] \\
& G \arrow[d, dashed] & \\
& H &
\end{tikzcd}$$

First, observe that the map $\bigvee_{i\in\mathcal{I}} h_ig_i^{-1}$ from G to H makes the above diagram commute since

$$\left(\bigvee\nolimits_{i\in\mathcal{I}}h_ig_i^{-1}\right)g_j = \left(\bigvee\nolimits_{i\in\mathcal{I},i\neq j}h_ig_i^{-1}\right)g_j \vee h_jg_j^{-1}g_j = \left(\bigvee\nolimits_{i\in\mathcal{I},i\neq j}h_i\varphi_{ji}\right) \vee h_j\varphi_{jj} = h_j,$$

as $h_j\varphi_{jj} = h_j$ by assumption. To show uniqueness, suppose $\alpha\colon G \to H$ satisfies the condition $\alpha g_i = h_i$ for all $i \in \mathcal{I}$, which implies $\alpha\overline{g_i^{-1}} = \alpha g_ig_i^{-1} = h_ig_i^{-1}$. Then taking joins over $\mathcal{I}$ and using the fact $\bigvee_{i\in\mathcal{I}} \overline{g_i^{-1}} = 1$, we have

$$\alpha = \alpha\bigvee\nolimits_{i\in\mathcal{I}}\overline{g_i^{-1}} = \bigvee\nolimits_{i\in\mathcal{I}}\alpha\overline{g_i^{-1}} = \bigvee\nolimits_{i\in\mathcal{I}}h_ig_i^{-1},$$

which means that such a map is unique. Hence (G, g_i) is a glueing of (U_i, φ_{ij}). □

Corollary 8.6. *If (G, g_i) is a gluing of an atlas (U_i, φ_{ij}) in a restriction category $\mathbb{X}$, and $F\colon \mathbb{X} \to \mathbb{D}$ is any join restriction functor, then $(F(G), F(g_i))$ is a gluing of the atlas $(FU_i, F\varphi_{ij})$ in $\mathbb{D}$. In other words, gluings of atlases in a join restriction category are absolute colimits.*

8.2 Gluings of atlases in the presheaf category, and maps between atlases

Earlier, we said that the main purpose of this chapter was to express gluings of atlases in a restriction category as a restriction colimit. From the previous chapter, we also know that cocomplete restriction categories have all restriction colimits. Therefore, *if* the gluing of an atlas is an example of a particular restriction colimit, then every cocomplete restriction category $\mathbb{X}$ should have gluings of all atlases in $\mathbb{X}$. In particular, as the join restriction category $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ is cocomplete for every join restriction category $\mathbb{X}$, we should expect all atlases in $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ to have gluings. Indeed, this is what we will show.

Lemma 8.7. *Let $\mathbb{X}$ be a join restriction category, and suppose $(P_i, \varphi_{ij})_{i\in\mathcal{I}}$ is an atlas in $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$. Then there is a join restriction presheaf $G\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ whose value at $A \in \mathbb{X}$ is given by*

$$GA = \{(x_i)_{i\in\mathcal{I}} \mid x_i \in P_i(A), (\varphi_{ij})_A(x_i) = x_j \cdot \overline{x_i}\},$$

and if $f\colon B \to A$ is a map in $\mathbb{X}$, the function $Gf\colon GA \to GB$ takes elements $(x_i)_{i\in\mathcal{I}} \in GA$ to $(x_i \cdot f)_{i\in\mathcal{I}}$.

Proof. First note that for all $f\colon B \to A$ in $\mathbb{X}$, Gf is well-defined since

$$\begin{aligned}(\varphi_{ij})_B(x_i \cdot f) &= (\varphi_{ij})_A(x_i) \cdot f = (x_j \cdot \overline{x_i}) \cdot f = x_j \cdot (\overline{x_i} \circ f)\\ &= x_j \cdot (f \circ \overline{x_i \cdot f}) = (x_j \cdot f) \cdot \overline{x_i \cdot f}.\end{aligned}$$

The restriction of each element $(x_i)_{i\in\mathcal{I}} \in GA$ is given by $\overline{(x_i)_{i\in\mathcal{I}}} = \bigvee_{i\in\mathcal{I}} \overline{x_i}$, using the fact each P_i is a restriction presheaf and that restriction idempotents are compatible. If $\{(x_{i,k})_{i\in\mathcal{I}}\}_{k\in\mathcal{K}}$ is a compatible family of elements in GA, then their join is given componentwise by $\left(\bigvee_{k\in\mathcal{K}} x_{i,k}\right)_{i\in\mathcal{I}}$. To see that this join is also well-defined, we need to show that if two elements are compatible, then they are componentwise compatible.

So suppose $(x_i)_{i\in\mathcal{I}}$ and $(y_i)_{i\in\mathcal{I}}$ are compatible; that is, $(x_i)_{i\in\mathcal{I}} \cdot \overline{(y_i)_{i\in\mathcal{I}}} = (y_i)_{i\in\mathcal{I}} \cdot \overline{(x_i)_{i\in\mathcal{I}}}$, or

$$(x_i \cdot \textstyle\bigvee_{i\in\mathcal{I}}\overline{y_i})_{i\in\mathcal{I}} = (y_i \cdot \bigvee_{i\in\mathcal{I}}\overline{x_i})_{i\in\mathcal{I}}\,. \tag{8.1}$$

But note that for all $i \in \mathcal{I}$, or componentwise, we have

$$(x_i \cdot \textstyle\bigvee_{i\in\mathcal{I}}\overline{y_i}) \cdot (\overline{x_i} \circ \overline{y_i}) = x_i \cdot (\overline{x_i} \circ \bigvee_{i\in\mathcal{I}}\overline{y_i} \circ \overline{y_i}) = x_i \cdot \left(\bigvee_{i\neq j}(\overline{y_j} \circ \overline{y_i}) \vee \overline{y_i}\right) = x_i \cdot \overline{y_i}$$

where the last equality follows from the fact $\overline{y_j} \circ \overline{y_i} \leq \overline{y_i}$. Similarly, we have $(y_i \cdot \bigvee_{i\in\mathcal{I}}\overline{x_i}) \cdot (\overline{y_i} \circ \overline{x_i}) = y_i \cdot \overline{x_i}$, and so by (8.1), $x_i \cdot \overline{y_i} = y_i \cdot \overline{x_i}$ for all $i \in \mathcal{I}$. Therefore, if $(x_i)_{i\in\mathcal{I}}$ and $(y_i)_{i\in\mathcal{I}}$ in GA are compatible, then they are componentwise compatible, and hence their join is well-defined. It is then straightforward to show that this join satisfies the required presheaf axioms. □

Proposition 8.8. *Let $\mathbb{X}$ be a join restriction category and let $(P_i, \varphi_{ij})_{i\in\mathcal{I}}$ be an atlas in $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$. Then the join restriction presheaf G described in the previous lemma is a gluing of the atlas $(P_i, \varphi_{ij})_{i\in\mathcal{I}}$, with the gluing maps $\gamma_i\colon P_i \to G$ defined componentwise at $A \in \mathbb{X}$ by*

$$(\gamma_i)_A(x_i) = \left((\varphi_{ij})_A(x_i)\right)_{j\in\mathcal{I}}\,.$$

Proof. We first must show that each gluing map γ_i is well-defined. That is, for every $i \in \mathcal{I}$, $(\varphi_{ij})_A(x_i) \in P_j(A)$, and $(\varphi_{jk})_A\big((\varphi_{ij})_A(x_i)\big) = (\varphi_{ik})_A(x_i)\cdot\overline{(\varphi_{ij})_A(x_i)}$. Clearly $(\varphi_{ij})_A(x_i) \in P_j(A)$ by definition of φ_{ij}. To see that the second condition is also satisfied, note

$$(\varphi_{ik})_A(x_i)\cdot\overline{(\varphi_{ij})_A(x_i)} = (\varphi_{ik})_A\big(x_i\cdot\overline{(\varphi_{ij})_A(x_i)}\big) = (\varphi_{ik})_A\big(\big(\overline{\varphi_{ij}}\big)_A(x_i)\big) = (\varphi_{jk})_A\big((\varphi_{ij})_A(x_i)\big)$$

since $\varphi_{jk}\circ\varphi_{ij} = \varphi_{ik}\circ\overline{\varphi_{ij}}$.

Now to show that G together with the family of maps $(\gamma_i)_{i\in\mathcal{I}}$ is the gluing of $(P_i,\varphi_{ij})_{i\in\mathcal{I}}$, we use will Theorem 8.5. We therefore begin by showing that each γ_i is a partial inverse. For each $i\in\mathcal{I}$, consider the map $\gamma_i^{-1}\colon G\to P_i$ defined componentwise at $A\in\mathbb{X}$ by

$$(\gamma_i^{-1})_A((x_i)_{i\in\mathcal{I}}) = x_i.$$

That is, each γ_i^{-1} is the projection from G to P_i. To see that γ_i^{-1} is indeed the partial inverse of γ_i, observe that for all $A\in\mathbb{X}$,

$$(\gamma_i^{-1})_A\big((\gamma_i)_A(x_i)\big) = (\gamma_i^{-1})_A\left(\big((\varphi_{ij})_A(x_i)\big)_{j\in\mathcal{I}}\right) = (\varphi_{ii})_A(x_i) = (\overline{\varphi_{ii}})_A(x_i),$$

as φ_{ii} is a restriction idempotent, and that

$$\begin{aligned}(\gamma_i)_A\big((\gamma_i^{-1})_A(x_i)_{i\in\mathcal{I}}\big) &= \big((\varphi_{ij})_A(x_i)\big)_{j\in\mathcal{I}} = (x_j\cdot\overline{x_i})_{j\in\mathcal{I}} = (x_i)_{i\in\mathcal{I}}\cdot\overline{x_i}\\ &= (x_i)_{i\in\mathcal{I}}\cdot\overline{(\gamma_i^{-1})_A(x_i)_{i\in\mathcal{I}}} = \overline{(\gamma_i^{-1})}_A(x_i)_{i\in\mathcal{I}}\end{aligned}$$

since $(x_i)_{i\in\mathcal{I}}\in GA$. This establishes γ_i as a partial isomorphism for each $i\in\mathcal{I}$. Clearly we also have $\varphi_{ik} = \gamma_k^{-1}\circ\gamma_i$, since

$$(\gamma_k^{-1})_A\left((\gamma_i)_A(x_i)\right) = (\gamma_k^{-1})_A\left(((\varphi_{ij})_A(x_i))_{j\in\mathcal{I}}\right) = (\varphi_{ik})_A(x_i).$$

Therefore, all that remains is to show $\bigvee_{i\in\mathcal{I}}\overline{\gamma_i^{-1}} = 1$. But for all $A\in\mathbb{X}$,

$$\begin{aligned}\left(\bigvee_{i\in\mathcal{I}}\overline{\gamma_i^{-1}}\right)_A((x_i)_{i\in\mathcal{I}}) &= \bigvee_{i\in\mathcal{I}}\left(\overline{\gamma_i^{-1}}\right)_A((x_i)_{i\in\mathcal{I}}) = \bigvee_{i\in\mathcal{I}}(x_i)_{i\in\mathcal{I}}\cdot\overline{(\gamma_i^{-1})_A\left((x_i)_{i\in\mathcal{I}}\right)}\\ &= \bigvee_{i\in\mathcal{I}}(x_j)_{j\in\mathcal{I}}\cdot\overline{x_i} = (x_j)_{j\in\mathcal{I}}\cdot\bigvee_{i\in\mathcal{I}}\overline{x_i}\\ &= \big(x_j\cdot\bigvee_{i\in\mathcal{I}}\overline{x_i}\big)_{j\in\mathcal{I}} = \left(x_j\cdot\overline{x_j}\vee\bigvee_{j\neq i}x_j\cdot\overline{x_i}\right)_{j\in\mathcal{I}}\\ &= (x_i)_{i\in\mathcal{I}},\end{aligned}$$

and so $\bigvee_{i\in\mathcal{I}}\overline{\gamma_i^{-1}} = 1$ as required. Therefore, the join restriction category $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ has gluings G of all atlases (P_i,φ_{ij}), with the gluing maps $\gamma_i\colon P_i\to G$ defined componentwise by $(\gamma_i)_A(x_i) = \big((\varphi_{ij})_A(x_i)\big)_{j\in\mathcal{I}}$. □

We have just witnessed the fact that for every join restriction category $\mathbb{X}$, its join restriction presheaf category $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ has all gluings. As a matter of fact, $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ has more than just gluings of all atlases; it actually has all weighted restriction colimits. It is therefore sensible to ask whether, given a join restriction category $\mathbb{X}$, there is a join restriction category $\mathsf{Gl}(\mathbb{X})$ with all gluings, and an embedding $\iota\colon\mathbb{X}\hookrightarrow\mathsf{Gl}(\mathbb{X})$. This embedding ι should also have the property that for any join restriction category $\mathbb{D}$ with all gluings and join restriction functor $F\colon\mathbb{X}\to\mathbb{D}$, there is a unique extension $\widetilde{F}\colon\mathsf{Gl}(\mathbb{X})\to\mathbb{D}$ such that $F\cong\widetilde{F}\circ\iota$. Indeed, such a join restriction category exists for every join restriction category $\mathbb{X}$, and the way we construct this category is as follows. We simply take the representables in $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$, and close them under gluings of all atlases in $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$. This category $\mathsf{Gl}(\mathbb{X}) \subset \mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ is then a join restriction category with the required property, by analogy with the notion of $\mathcal{F}$-free cocompletion in [Kelly, 1982].

Knowing that $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ has all gluings for every join restriction category $\mathbb{X}$, and that $\mathbb{X}$ embeds into $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ allows us to give a notion of map between atlases. Let $(U_i, \varphi_{ij})_{i,j\in\mathcal{I}}$ and $(W_k, \psi_{k\ell})_{k,\ell\in\mathcal{J}}$ be two atlases in a join restriction category $\mathbb{X}$. Now take each atlas into $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ via the Yoneda embedding, and then let G and H be the gluings of the respective atlases, with γ_i and ζ_k being the gluing maps:

$$\begin{tikzcd}
\mathbf{y}U_i \arrow[r, "\gamma_i"] \arrow[rr, bend left, "\mathbf{y}\varphi_{ij}"] \arrow[dr, "\alpha_i"'] & G \arrow[d, dashed] & \mathbf{y}U_j \arrow[l, "\gamma_j"'] \arrow[dl, "\alpha_j"] \\
\mathbf{y}W_k \arrow[r, "\zeta_k"'] & H & \mathbf{y}W_\ell. \arrow[l, "\zeta_\ell"]
\end{tikzcd}$$

Let $\{\alpha_i : \mathbf{y}U_i \to H\}_{i\in\mathcal{I}}$ be a lax cocone to the atlas $(\mathbf{y}U_i, \mathbf{y}\varphi_{ij})$ such that $\alpha_i \circ \mathbf{y}\varphi_{ii} = \alpha_i$ for all $i \in \mathcal{I}$. By Yoneda, each α_i corresponds with an element of HU_i; that is, a family of maps $(a_{ik} : U_i \to W_k)_{k\in\mathcal{J}}$ such that $\psi_{k\ell} \circ a_{ik} = a_{i\ell} \circ \overline{a_{ik}}$. The fact $\{\alpha_i\}_{i\in\mathcal{I}}$ is a lax cocone means that $\alpha_j \circ \mathbf{y}\varphi_{ij} = \alpha_i \circ \mathbf{y}\overline{\varphi_{ij}}$, or $a_{jk} \circ \varphi_{ij} = a_{ik} \circ \overline{\varphi_{ij}}$ for each $k \in \mathcal{J}$. And the condition $\alpha_i \circ \mathbf{y}\varphi_{ii} = \alpha_i$ on the family $\{\alpha_i\}_{i\in\mathcal{I}}$ means that $a_{ij} \circ a_{ii} = a_{ij}$ for all pairs $i, j \in \mathcal{I}$. This suggests the following definition of a map between atlases $(U_i, \varphi_{ij})_{i,j\in\mathcal{I}}$ and $(W_k, \psi_{k\ell})_{k,\ell\in\mathcal{J}}$, given by [Grandis, 1990] and modified by [Cockett & Cruttwell, 2014].

Definition 8.9 (Grandis-Cockett-Cruttwell)**.** Let $\mathbb{X}$ be a join restriction category, and let $(U_i, \varphi_{ij})_{i,j\in\mathcal{I}}$ and $(W_k, \psi_{k\ell})_{k,\ell\in\mathcal{J}}$ be atlases in $\mathbb{X}$. A map between atlases (U_i, φ_{ij}) and $(W_k, \psi_{k\ell})$ is defined to be a family of maps $(a_{ik} : U_i \to W_k)_{i\in\mathcal{I},k\in\mathcal{J}}$ satisfying the following conditions:

1. $a_{jk} \circ \varphi_{ij} = a_{ik} \circ \overline{\varphi_{ij}}$;
2. $\psi_{k\ell} \circ a_{ik} = a_{i\ell} \circ \overline{a_{ik}}$, and
3. $a_{ij} \circ \varphi_{ii} = a_{ij}$.

In the join restriction category $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$, we have seen how lax cocones to the atlas $(\mathbf{y}U_i, \mathbf{y}\varphi_{ij})$ satisfying $\alpha_i \circ \mathbf{y}\varphi_{ii} = \alpha_i$ correspond uniquely to maps out of G, the gluing of $(\mathbf{y}U_i, \mathbf{y}\varphi_{ij})$. Therefore, one way to think about maps between atlases in a join restriction category $\mathbb{X}$, is to think of them as maps between their gluings in $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$. Then, since maps between gluings in $\mathsf{PSh}_{\mathsf{jr}}(\mathbb{X})$ are composable (being just ordinary maps between join restriction presheaves), we can define composition of atlas maps in the following way.

Using the calculations of Theorem 8.5 and Proposition 8.8, given atlases (U_i, φ_{ij}), $(W_k, \psi_{k\ell})$ and an atlas map $(a_{ik} : U_i \to W_k)$, the induced map δ from G to H is defined componentwise at $B \in \mathbb{X}$ by

$$\delta_B\left((f_i : B \to U_i)_{i\in\mathcal{I}}\right) = \left(\textstyle\bigvee_{i\in\mathcal{I}} a_{ik} f_i\right)_{k\in\mathcal{J}}.$$

Now let $(Z_m, \xi_{mn})_{m,n\in\mathcal{K}}$ be another atlas with gluing K, and $(b_{km} : W_k \to Z_m)_{k\in\mathcal{J},m\in\mathcal{K}}$ another atlas map corresponding to the map $\varepsilon : H \to K$ defined by

$$\varepsilon_B\left((g_k : B \to W_k)_{k\in\mathcal{J}}\right) = \left(\textstyle\bigvee_{k\in\mathcal{J}} b_{km} g_k\right)_{m\in\mathcal{K}}.$$

The composite of ε with δ is then given at $B \in \mathbb{X}$ by

$$(\varepsilon\delta)_B\left((f_i : B \to U_i)_{i\in\mathcal{I}}\right) = \left(\textstyle\bigvee_{k\in\mathcal{J}} b_{km} \left(\bigvee_{i\in\mathcal{I}} a_{ik} f_i\right)\right)_{m\in\mathcal{K}} = \left(\textstyle\bigvee_{i\in\mathcal{I}} \left[\bigvee_{k\in\mathcal{J}} b_{km} a_{ik}\right] f_i\right)_{m\in\mathcal{K}}.$$

By uniqueness, the lax cocone corresponding to this map $\varepsilon\delta\colon G \to K$ corresponds to the family of maps $(\bigvee_{k\in\mathcal{J}} b_{km}a_{ik})_{i\in\mathcal{I},m\in\mathcal{K}}$. We therefore, deduce the composite of two atlas maps as follows.

Definition 8.10 (Grandis-Cruttwell)**.** Let $(a_{ik})_{i\in\mathcal{I},k\in\mathcal{J}}$ and $(b_{km})_{k\in\mathcal{J},m\in\mathcal{K}}$ be maps between atlases in a join restriction category $\mathbb{X}$. Their composite $(b_{km})\circ(a_{ik})$ is defined to be the atlas map

$$(\bigvee_{k\in\mathcal{J}} b_{km}a_{ik})_{i\in\mathcal{I},m\in\mathcal{K}}.$$

It is now not difficult to see that given a join restriction category $\mathbb{X}$, there is a category $\mathsf{Atl}(\mathbb{X})$ whose objects are atlases in $\mathbb{X}$, and maps are the atlas maps described above. In fact, [Grandis, 1990] shows that $\mathsf{Atl}(\mathbb{X})$ is a join restriction category with the same universal property as $\mathsf{Gl}(\mathbb{X})$, from which we deduce that $\mathsf{Atl}(\mathbb{X})$ and $\mathsf{Gl}(\mathbb{X})$ are in fact equivalent as categories.

8.3 Gluings as restriction colimits

Having discussed atlases and their gluings in quite some detail, we are now ready to describe gluings of altases as a particular restriction colimit. To do so, we introduce the following notion of *free restriction category* on an $\mathcal{I}$-object atlas.

Let $\mathcal{I}$ be any set and $\mathbb{X}$ a restriction category. Then we may form an ordinary category $\mathsf{Atl}_{\mathcal{I}}(\mathbb{X})$ whose objects are atlases in $\mathbb{X}$ indexed by $\mathcal{I}$, $(U_i,\varphi_{ij})_{i,j\in\mathcal{I}}$, and whose maps between $(U_i,\varphi_{ij})_{i,j\in\mathcal{I}}$ and $(W_i,\psi_{ij})_{i,j\in\mathcal{I}}$ are total maps $(\gamma_i\colon U_i \to W_i)_{i\in\mathcal{I}}$ in $\mathbb{X}$ making the following square commute for each pair i,j:

$$\begin{tikzcd} U_i \arrow[r, "\varphi_{ij}"] \arrow[d, "\gamma_i"'] & U_j \arrow[d, "\gamma_j"] \\ W_i \arrow[r, "\psi_{ij}"'] & W_j. \end{tikzcd}$$

In fact, for each set $\mathcal{I}$, $\mathsf{Atl}_{\mathcal{I}}\colon \mathbf{rCat} \to \mathbf{Cat}$ is a 2-functor, taking each restriction category $\mathbb{X}$ to $\mathsf{Atl}_{\mathcal{I}}(\mathbb{X})$. We now describe a representing object for this 2-functor.

Definition 8.11. Let $\mathcal{I}$ be a small set. Then there is a restriction category $\mathbf{F}\mathcal{I}$, whose objects are the elements of $\mathcal{I}$, and whose hom-sets are given as follows. For every $i\in\mathcal{I}$, $\mathsf{Hom}(i,i) = S_{ii} \subset \mathcal{P}(\mathcal{I})$, where S_{ii} consists of the empty set together with the finite subsets of $\mathcal{I}$ containing i. For all $i\neq j$, we have $\mathsf{Hom}(i,j) = S_{ij} \subset \mathcal{P}(\mathcal{I})$, where S_{ij} are the finite subsets of $\mathcal{I}$ containing *both* i and j. Composition of maps is given by unions of subsets,

$$\mathsf{Hom}(i,j)\times\mathsf{Hom}(j,k) \to \mathsf{Hom}(i,k), \quad (S_{ij},S_{jk}) \mapsto S_{ij}\cup S_{jk},$$

and the empty set $\varnothing\in S_{ii}$ serves as the identity for each $i\in\mathcal{I}$. Finally, the restriction structure is given by the inclusions $\mathsf{Hom}(i,j) \hookrightarrow \mathsf{Hom}(i,i)$.

Note, by definition of the restriction structure on $\mathbf{F}\mathcal{I}$, that the only total maps in $\mathbf{F}\mathcal{I}$ are the identities $\varnothing\colon i\to i$. Also, for the purpose of notation, we shall on occasions denote arbitrary maps between i and j in $\mathbf{F}\mathcal{I}$ by $s_{ij}\colon i\to j$.

Proposition 8.12 (Free restriction category on an $\mathcal{I}$-object atlas)**.** *For any set $\mathcal{I}$, the restriction category $\mathbf{F}\mathcal{I}$ from Definition 8.11 is the* free restriction category *on an $\mathcal{I}$-object atlas. That is, $\mathbf{F}\mathcal{I}$ is the representing object which makes $\mathsf{Atl}_{\mathcal{I}}$ a representable 2-functor.*

Proof. Let $A_{\mathcal{I}} \in \mathsf{Atl}_{\mathcal{I}}(\mathbf{F}\mathcal{I})$ denote the atlas $(i, p_{ij})_{i,j\in\mathcal{I}}$, where p_{ij} is the set $\{i, j\}$ and $p_{ii} = \{i\}$. We will show that for any restriction category $\mathbb{X}$, the following functor

$$\Phi_{\mathbb{X}} \colon \mathbf{rCat}(\mathbf{F}\mathcal{I}, \mathbb{X}) \to \mathsf{Atl}_{\mathcal{I}}(\mathbb{X}), \quad G \mapsto \mathsf{Atl}_{\mathcal{I}}(G)(A_{\mathcal{I}}) = (Gi, Gp_{ij})_{i,j\in\mathcal{I}}$$

is an isomorphism. So consider the functor $\Psi_{\mathbb{X}} \colon \mathsf{Atl}_{\mathcal{I}}(\mathbb{X}) \to \mathbf{rCat}(\mathbf{F}\mathcal{I}, \mathbb{X})$ sending atlases $(U_i, \varphi_{ij})_{i,j\in\mathcal{I}}$ in $\mathbb{X}$ to restriction functors $\Psi_{\mathbb{X}}(U_i, \varphi_{ij}) \colon \mathbf{F}\mathcal{I} \to \mathbb{X}$, defined on objects to be $\Psi_{\mathbb{X}}(U_i, \varphi_{ij})(i) = U_i$. On maps, $\Psi_{\mathbb{X}}(U_i, \varphi_{ij})$ sends $\{i, j, k, \ldots, m\} \colon i \to j$ in $\mathbf{F}\mathcal{I}$ to $\varphi_{ij}\overline{\varphi_{ik}} \ldots \overline{\varphi_{im}}$, sends $\{i, a, \ldots, b\} \colon i \to i$ to $\varphi_{ii}\overline{\varphi_{ia}} \ldots \overline{\varphi_{ib}}$ and sends $\varnothing \colon i \to i$ to 1_{U_i}.

Now let $G \colon \mathbf{F}\mathcal{I} \to \mathbb{X}$ be a restriction functor, and apply $\Phi_{\mathbb{X}}$ to give the atlas $\Phi_{\mathbb{X}}(G) = (Gi, Gp_{ij})_{i,j\in\mathcal{I}}$ in $\mathbb{X}$. Then $\Psi_{\mathbb{X}}(\Phi_{\mathbb{X}}(G))$ is a restriction functor which sends objects $i \in \mathbf{F}\mathcal{I}$ to Gi, and sends maps $\{i, j\} \colon i \to j$ to $Gp_{ij} = G(\{i, j\})$. By induction and functoriality of G, we deduce that $\Psi_{\mathbb{X}}(\Phi_{\mathbb{X}}(G))$ must send maps of the form $\{i, j, k, \ldots, m\} \colon i \to j$ to $G(p_{ij}\overline{p_{ik}} \ldots \overline{p_{im}}) = G(\{i, j, k, \ldots, m\})$. Therefore, $1 = \Psi_{\mathbb{X}}\Phi_{\mathbb{X}}$ on objects, and similarly on maps.

On the other hand, let $(U_i, \varphi_{ij})_{i,j\in\mathcal{I}}$ be any atlas in $\mathbb{X}$ and apply $\Psi_{\mathbb{X}}$ to get the restriction functor $\Psi_{\mathbb{X}}(U_i, \varphi_{ij}) \colon \mathbf{F}\mathcal{I} \to \mathbb{X}$ described previously. This means

$$\Phi_{\mathbb{X}}(\Psi_{\mathbb{X}}(U_i, \varphi_{ij})) = \big(\Psi_{\mathbb{X}}(U_i, \varphi_{ij})(i), \Psi_{\mathbb{X}}(U_i, \varphi_{ij})(\{i, j\})\big) = (U_i, \varphi_{ij}),$$

and so $1 = \Phi_{\mathbb{X}}\Psi_{\mathbb{X}}$ on objects, and similarly on maps. That $\Phi \colon \mathbf{rCat}(\mathbf{F}\mathcal{I}, -) \to \mathsf{Atl}_{\mathcal{I}}$ is 2-natural follows from the 2-Yoneda lemma, with Φ corresponding to $A_{\mathcal{I}} \in \mathsf{Atl}_{\mathcal{I}}(\mathbf{F}\mathcal{I})$. $\square$

Having defined the free restriction category $\mathbf{F}\mathcal{I}$ on an $\mathcal{I}$-object atlas, we can now describe gluings of any atlas in a restriction category as a restriction colimit.

Theorem 8.13. *Let $(U_i, \varphi_{ij})_{i,j\in\mathcal{I}}$ be an atlas in a restriction category $\mathbb{X}$, and let $F \colon \mathbf{F}\mathcal{I} \to \mathbb{X}$ be the corresponding restriction functor. Let $\widetilde{O}$ be the restriction presheaf on $\mathbf{F}\mathcal{I}$ sending each object $i \in \mathbf{F}\mathcal{I}$ to the set of restriction idempotents on i, but excluding the identity $\varnothing \colon i \to i$. On maps $s_{ji} \colon j \to i$, $\widetilde{O}(s_{ji}) \colon \widetilde{O}(i) \to \widetilde{O}(j)$ is defined by sending each restriction idempotent s_{ii} to $\overline{s_{ji} \circ s_{ii}}$. Then the atlas $(U_i, \varphi_{ij})_{i,j\in\mathcal{I}}$ has a gluing in $\mathbb{X}$ if and only if the restriction colimit of F weighted by $\widetilde{O}$ exists.*

Proof. By definition, the restriction colimit of F weighted by $\widetilde{O}$ consists of an object $\mathrm{rcolim}_{\widetilde{O}} F \in \mathbb{X}$, and coprojection maps $\{g_{i,x} \colon U_i \to \mathrm{rcolim}_{\widetilde{O}} F\}_{i\in\mathbf{F}\mathcal{I}, x\in\widetilde{O}(i)}$. These maps $g_{i,x}$ satisfy the condition $g_{i,x} = g_{j,y} \circ F s_{ij}$ if and only if $x = y \cdot s_{ij}$, or $x = \overline{y \circ s_{ij}}$. Now observe that every restriction idempotent $x \in \widetilde{O}(i)$ may be written in the form $x = \overline{\{i\} \circ s_{ii}}$ for some finite subset of $\mathcal{P}(i)$ containing i. Therefore, it is enough to give, for each $i \in \mathbf{F}\mathcal{I}$, a *single* map $g_i \colon U_i \to \mathrm{rcolim}_{\widetilde{O}} F$.

So now it remains to show that the maps g_i satisfy the gluing condition, which following Remark 8.4, is equivalent to $g_j \circ \varphi_{ij} = g_i \circ \overline{\varphi_{ij}}$, or $g_j \circ F(\{i, j\}) = g_i \circ F(\overline{\{i, j\}})$. But this gluing condition clearly holds for all maps g_i, since $\overline{\{j\} \circ \{i, j\}} = \{i, j\}$ and $\overline{\{i\} \circ \overline{\{i, j\}}} = \{i, j\}$. Therefore, $(U_i, \varphi_{ij})_{i,j\in\mathcal{I}}$ has a gluing in $\mathbb{X}$ if and only if the restriction colimit of F weighted by $\widetilde{O}$ exists. $\square$

Remark 8.14. The previous theorem deals with gluings of general atlases (U_i, φ_{ij}) (with φ_{ii} arbitrary). However, we may also express gluings of total atlases as restriction colimits. To do this, we modify our restriction category $\mathbf{F}\mathcal{I}$ by removing the maps $\varnothing \colon i \to i$ for every $i \in \mathcal{I}$, and instead, declare maps $\{i\} \colon i \to i$ to be the new identities. Then a gluing of a total atlas (U_i, φ_{ij}) (with $\varphi_{ii} = 1$) may be given as the restriction colimit of F (from this modified $\mathbf{F}\mathcal{I}$) weighted by O, where O is the restriction presheaf from Examples 7.5 and 7.6.

9 Restriction profunctors and other restriction definitions

The aim of this last chapter will be to revisit the notion of *restriction profunctor* as described by [DeWolf, 2017], and provide further support as to why that notion appears to be the correct one (from the viewpoint of restriction presheaves). Closely following this discussion, we give reasons as to why the notion of *restriction coend* does not appear to be a good notion. This will be an example of a notion in category theory which is not replicated in the restriction setting. We will also give another example of a notion in category theory which does not have an analogue in the restriction world; namely, that left extension along the Yoneda embedding is not, in general, a left adjoint.

9.1 Profunctors

The notion of *profunctor* was first formally introduced in [Bénabou, 1973], although Bénabou called them *distributors* at the time. Subsequently, over the many years, they have been given different names by different authors; [Street, 2004b] used the term *bimodules*, and [Loregian, 2017] preferred to call them *relators*, seeing as how they generalise the notion of relations between sets. Despite the many names given to the same concept, we shall stick with term *profunctor* in these discussions.

Definition 9.1. A *profunctor* $P\colon \mathbf{C} \nrightarrow \mathbf{D}$ between two categories is a bifunctor $P\colon \mathbf{D}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$.

With the above definition, there is a natural notion of maps between profunctors, namely natural transformations between their corresponding bifunctors. Also note that if $x \in P(D,C)$, and $g\colon C \to C', f\colon D' \to D$ are maps in $\mathbf{C}$ and $\mathbf{D}$ respectively, then functoriality tells us that $P((P(f,C)(x)),g) = P(f,P(D,g)(x))$. If we abbreviate the function $P(f,C)(x)$ by xf and $P(D,g)(x)$ by gx, then the previous statement may be rewritten as $g(xf) = (gx)f$; that is, viewing maps in $\mathbf{C}$ and $\mathbf{D}$ as acting on elements in $P(D,C)$ for each $C \in \mathbf{C}$ and $D \in \mathbf{D}$ [Bénabou, 2000].

Now given two profunctors $P\colon \mathbf{A} \nrightarrow \mathbf{B}$ and $Q\colon \mathbf{B} \nrightarrow \mathbf{C}$, a natural question which arises is whether one may compose P and Q to get a profunctor $Q \circ P\colon \mathbf{A} \nrightarrow \mathbf{C}$. Indeed, there is a natural way of composing profunctors, using the following lemma.

Lemma 9.2. *Let $\mathbf{A}$ and $\mathbf{B}$ be categories, and denote by $\mathbf{Prof}(\mathbf{A}, \mathbf{B})$, the category whose objects are profunctors from $\mathbf{A}$ to $\mathbf{B}$, and maps are natural transformations $F \Rightarrow G\colon \mathbf{B}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Set}$. Then there is an equivalence of categories $\mathbf{Prof}(\mathbf{A}, \mathbf{B}) \simeq \mathbf{Cocomp}(\mathsf{PSh}(\mathbf{A}), \mathsf{PSh}(\mathbf{B}))$.*

Proof. We have the following series of natural isomorphisms:

$$\mathbf{Cat}(\mathbf{B}^{\mathrm{op}} \times \mathbf{A}, \mathbf{Set}) \cong \mathbf{Cat}(\mathbf{A} \times \mathbf{B}^{\mathrm{op}}, \mathbf{Set}) \cong \mathbf{Cat}(\mathbf{A}, [\mathbf{B}^{\mathrm{op}}, \mathbf{Set}]) \tag{9.1}$$

where the last isomorphism is due to the fact $\mathbf{Cat}$ is cartesian closed. Now by free cocompletion, there is also an equivalence of categories

$$\mathbf{Cat}(\mathbf{A}, \mathsf{PSh}(\mathbf{B})) \simeq \mathbf{Cocomp}(\mathsf{PSh}(\mathbf{A}), \mathsf{PSh}(\mathbf{B})), \tag{9.2}$$

which in one direction, takes functors $F\colon \mathbf{A} \to \mathsf{PSh}(\mathbf{B})$ to their left Kan extension along the Yoneda embedding, $\mathsf{Lan}_{\mathbf{y}}F\colon \mathsf{PSh}(\mathbf{A}) \to \mathsf{PSh}(\mathbf{B})$. Hence, the categories $\mathbf{Prof}(\mathbf{A}, \mathbf{B})$ and $\mathbf{Cocomp}(\mathsf{PSh}(\mathbf{A}), \mathsf{PSh}(\mathbf{B}))$ are equivalent. □

Corollary 9.3 (Bénabou)**.** *There is a bicategory* $\mathbf{Prof}$ *of categories, profunctors and natural transformations.*

Proof. The only thing we need to check is that we can define composition $\mathbf{Prof}(\mathbf{A}, \mathbf{B}) \times \mathbf{Prof}(\mathbf{B}, \mathbf{C}) \to \mathbf{Prof}(\mathbf{A}, \mathbf{C})$, and that this functor satisfies the associativity and unit isomorphisms. However, this follows from the fact $\mathbf{Cocomp}$ is a 2-category, and that we have the following equivalence of categories

$$\mathbf{Prof}(\mathbf{A}, \mathbf{B}) \simeq \mathbf{Cocomp}(\mathsf{PSh}(\mathbf{A}), \mathsf{PSh}(\mathbf{B}))$$

from Lemma 9.2. □

We shall now give an explicit description of profunctor composition. Given profunctors $P\colon \mathbf{A} \nrightarrow \mathbf{B}$ and $Q\colon \mathbf{B} \nrightarrow \mathbf{C}$, we obtain corresponding functors $\mathsf{Lan}_{\mathbf{y}}\hat{P}\colon \mathsf{PSh}(\mathbf{A}) \to \mathsf{PSh}(\mathbf{B})$ and $\mathsf{Lan}_{\mathbf{y}}\hat{Q}\colon \mathsf{PSh}(\mathbf{B}) \to \mathsf{PSh}(\mathbf{C})$ by applying (9.1) and (9.2) in sucession. The functors $\mathsf{Lan}_{\mathbf{y}}\hat{P}$ and $\mathsf{Lan}_{\mathbf{y}}\hat{Q}$ are clearly composable, and so we may define the profunctor composite $Q \circ P$ to be the transpose of $\mathsf{Lan}_{\mathbf{y}}\hat{Q} \circ \mathsf{Lan}_{\mathbf{y}}\hat{P}$, via the equivalence of categories as stated in Lemma 9.2.

More precisely, the transpose of $\mathsf{Lan}_{\mathbf{y}}\hat{Q} \circ \mathsf{Lan}_{\mathbf{y}}\hat{P}$ is calculated as follows:

$$\begin{tikzcd}
\mathsf{PSh}(\mathbf{A}) \arrow[r, "\mathsf{Lan}_{\mathbf{y}}\hat{P}"] & \mathsf{PSh}(\mathbf{B}) \arrow[r, "\mathsf{Lan}_{\mathbf{y}}\hat{Q}"] & \mathsf{PSh}(\mathbf{C}) \\
\mathbf{A}. \arrow[u, "\mathbf{y}"] \arrow[ur, "\hat{P}"'] \arrow[urr, bend right] & &
\end{tikzcd}$$

First precompose by $\mathbf{y}\colon \mathbf{A} \to \mathsf{PSh}(\mathbf{A})$, giving the composite $\mathsf{Lan}_{\mathbf{y}}\hat{Q} \circ \hat{P}$. Now for all $A \in \mathbf{A}$, $\hat{P}(A) = P(-, A)$ is a presheaf on $\mathbf{B}$, and given a such a presheaf $P(-, A)\colon \mathbf{B}^{\mathrm{op}} \to \mathbf{Set}$, we know that $\mathsf{Lan}_{\mathbf{y}}\hat{Q}$ takes $P(-, A)$ to the following equivalent values,

$$\mathsf{Lan}_{\mathbf{y}}\hat{Q}(P(-, A)) = \operatorname{colim} \hat{Q}\pi_{P(-,A)} = \operatorname{colim}_{P(-,A)} \hat{Q} = \int^{B \in \mathbf{B}} P(B, A) \cdot \hat{Q}(B),$$

where $\pi_{P(-,A)}$ is the projection from the category of elements of $P(-, A)$. This is a presheaf on $\mathbf{C}$, taking objects $C \in \mathbf{C}$ to $\left(\int^{B\in\mathbf{B}} P(B, A) \cdot \hat{Q}(B)\right)(C) = \int^{B\in\mathbf{B}} P(B, A) \cdot Q(C, B)$. Therefore, the transpose of $\mathsf{Lan}_{\mathbf{y}}\hat{Q} \circ \mathsf{Lan}_{\mathbf{y}}\hat{P}$, namely the profunctor $Q \circ P$, takes objects $(C, A) \in \mathbf{C}^{\mathrm{op}} \times \mathbf{A}$ to $\int^{B\in\mathbf{B}} P(B, A) \cdot Q(C, B)$.

Now in **Set**, $\int^{B\in\mathbf{B}} P(B, A) \cdot Q(C, B)$ takes the form $\coprod_{B\in\mathbf{B}} P(B, A) \times Q(C, B)/\sim$, where the equivalence relation may be described as follows: if $(x, y), (x', y') \in Q(C, B) \times P(B, A)$, then $(x, y) \sim (x', y')$ if there exists a map $h\colon B \longrightarrow B'$ such that $Q(C, h)(x) = x'$ and $P(h, A)(y') = y$, or $hx = x'$ and $y'h = y$ (by our previous abbreviation). We may view this relation diagrammatically.

$$
\begin{tikzcd}
 & B \arrow[dd, "h"] \arrow[dr, dashed, "y\in P(B,A)"] & \\
C \arrow[ur, dashed, "x\in Q(C,B)"] \arrow[dr, dashed, "x'\in Q(C,B)"'] & & A \\
 & B'. \arrow[ur, dashed, "y'\in P(B,A)"'] &
\end{tikzcd}
$$

We shall also make use of an alternative notation to describe elements of this quotient set. Instead of writing an element in $Q(C, B) \times P(B, A)$ as a pair (x, y), we shall denote this element by $y\square x$, where we view elements $x \in Q(C, B)$ and $y \in P(B, A)$ as *virtual* maps from C to B, and from B to A respectively. Then our equivalence relation is the condition that $(y'h)\square x = y'\square(hx)$. This notation will be useful later when we deal with *restriction profunctors*.

For each category $\mathbf{C}$, there is a profunctor $\mathsf{Hom}_{\mathbf{C}}\colon \mathbf{C} \nrightarrow \mathbf{C}$, the hom-functor on $\mathbf{C}$ which takes each object (C, C') to the set $\mathbf{C}(C, C')$. If we precompose a profunctor $P\colon \mathbf{C} \nrightarrow \mathbf{D}$ with $\mathsf{Hom}_{\mathbf{C}}$, then the resulting profunctor is not P, but only isomorphic to P. Similarly, composition of profunctors is not associative, but only associative up to isomorphism (essentially as products of sets are only associative up to ismorphism). This leads to the following proposition.

9.2 Restriction profunctors

In the same way in which we described restriction presheaves earlier (as a presheaf on a restriction category with a restriction structure), a *restriction profunctor* will be a profunctor P from a restriction category $\mathbb{A}$ to another restriction category $\mathbb{B}$ together with a restriction structure on $P(B, A)$ (with $A \in \mathbb{A}, B \in \mathbb{B}$). Our goal in this section will be to describe the bicategory of restriction profunctors.

Definition 9.4 (DeWolf)**.** A *restriction profunctor* P between restriction categories $\mathbb{A}$ and $\mathbb{B}$ is a profunctor $P\colon \mathbb{A} \nrightarrow \mathbb{B}$ together with, for each $B \in \mathbb{B}$ and $A \in \mathbb{A}$, a function $P(B, A) \longrightarrow \mathbb{B}(B, B)$ sending x to $\bar{x}$, where $\bar{x}\colon B \longrightarrow B$ is a restriction idempotent satisfying the following axioms:

(RProf1) $x\bar{x} = x$;

(RProf2) $\overline{x\bar{h}} = \bar{x} \circ \bar{h}$;

(RProf3) $\bar{x} \circ g = g \circ \overline{xg}$; and

(RProf4) $\bar{f}x = x\overline{fx}$,

for suitable composable maps f, g and h.

Remark 9.5. In dealing with restriction profunctors, we will sometimes informally refer to the first three axioms as the *presheaf* axioms, and the last axiom as the *restriction* axiom, for reasons which we will reveal shortly.

In order for there to be a bicategory of restriction categories and restriction profunctors, we will need the composite of two restriction profunctors to again be a restriction functor. So we shall adopt the same strategy as before; first by showing a natural isomorphism between restriction profunctors $P\colon \mathbb{A} \to \mathbb{B}$ and restriction functors $\hat{P}\colon \mathbb{A} \to \mathsf{PSh}_r(\mathbb{B})$.

Lemma 9.6. *Given two restriction categories $\mathbb{A}$ and $\mathbb{B}$, denote by* **rProf**$(\mathbb{A},\mathbb{B})$ *the category of restriction profunctors between $\mathbb{A}$ and $\mathbb{B}$, and arbitrary natural transformations. Then there is a natural isomoprhism* **rProf**$(\mathbb{A},\mathbb{B}) \cong$ **rCat**$(\mathbb{A},\mathsf{PSh}_r(\mathbb{B}))$.

Proof. We know that there is a bijective correspondence between profunctors $P\colon \mathbb{A} \to \mathbb{B}$ and functors $\hat{P}\colon \mathbb{A} \to \mathsf{PSh}(\mathbb{B})$ from (9.1). So all we need to show is that if P is a restriction profunctor, then the restriction structure on P induces a restriction presheaf structure on objects of $\mathsf{PSh}(\mathbb{B})$, and that $\hat{P}$ is a restriction functor. So let $x \in P(B, A)$. Then the restriction profunctor axioms tell us that there is an idempotent $\bar{x}\colon B \to B$ satisfying the conditions $x\bar{x} = x, \overline{xe} = \bar{x}e$ and $\bar{x}g = g \circ \overline{xg}$ for all maps $g \in \mathbb{B}$ and idempotents $e \in \mathbb{B}$. But these are precisely the restriction presheaf axioms, which tells us that $\hat{P}$ takes objects in $\mathbb{A}$ to restriction presheaves on $\mathbb{B}$. That is $\hat{P}(a) = P(-, a)\colon \mathbb{B}^{\mathrm{op}} \to$ **Set** is a restriction presheaf. Then defining $\hat{P}$ on maps in the usual way makes $\hat{P}$ a functor from $\mathbb{A}$ to $\mathsf{PSh}_r(\mathbb{B})$. It remains to show that $\hat{P}$ is a restriction functor.

Now for $f\colon A \to A'$, the component of $\hat{P}(f)$ at B takes $x \in P(B, A)$ to $fx \in P(B, A')$. In other words, $\hat{P}(f)$ is post-composition by f. But

$$\overline{\hat{P}(f)}_B(x) = x\overline{\hat{P}(f)(x)} = x\overline{fx} = \bar{f}x$$

by (RProf4), and so $\hat{P}$ preserves restrictions. Hence $\hat{P}$ is a restriction functor.

Conversely, suppose $\hat{Q}\colon \mathbb{A} \to \mathsf{PSh}_r(\mathbb{B})$ is a restriction functor. Then, as with ordinary categories, there is a functor $Q\colon \mathbb{B}^{\mathrm{op}} \times \mathbb{A} \to$ **Set** defined on objects by $Q(b, a) = (\hat{Q}(a))(b)$. That Q satisfies the *presheaf axioms* follows from the fact that $\hat{Q}(a)$ is a restriction presheaf for all $a \in \mathbb{A}$. To show that the *restriction axiom* is also satisfied, let $x \in Q(b, a) = \hat{Q}(a)(b)$ and let $f\colon A \to A'$ be a map in $\mathbb{A}$. Rewriting $(\hat{Q}(f))_B(x)$ as fx, and noting that $\hat{Q}\bar{f} = \overline{\hat{Q}f}$ as $\hat{Q}$ is a restriction functor, we have $(\hat{Q}\bar{f})_B(x) = \bar{f}x$ and

$$(\hat{Q}\bar{f})_B(x) = \overline{\hat{Q}f}_B(x) = x\overline{(\hat{Q}f)_B(x)} = x\overline{fx}$$

which means $\bar{f}x = x\overline{fx}$, and so Q satisfies (RProf4), making Q a restriction profunctor. It is easy to show that this bijection is natural. □

Proposition 9.7. *Given restriction categories $\mathbb{A}$ and $\mathbb{B}$, there is an equivalence of categories* **rProf**$(\mathbb{A},\mathbb{B}) \simeq$ **rCocomp**$(\mathsf{PSh}_r(\mathbb{A}),\mathsf{PSh}_r(\mathbb{B}))$.

Proof. By the previous lemma, we have an isomorphism **rProf**$(\mathbb{A},\mathbb{B}) \cong$ **rCat**$(\mathbb{A},\mathsf{PSh}_r(\mathbb{B}))$. But the restriction analogue of free cocompletion (Corollary 4.15)implies that there is an equivalence of categories **rCat**$(\mathbb{A},\mathsf{PSh}_r(\mathbb{B})) \simeq$ **rCocomp**$(\mathsf{PSh}_r(\mathbb{A}),\mathsf{PSh}_r(\mathbb{B})$. Hence the result follows. □

Corollary 9.8 (DeWolf)**.** *There is a bicategory* **rProf** *whose objects are restriction categories, 1-cells are restriction profunctors and 2-cells are arbitrary natural transformations.*

Proof. Same as for Corollary 9.3, but with **Prof** replaced by **rProf** and **Cocomp** replaced by **rCocomp**. □

We now give an explicit description of restriction profunctor composition. If $P\colon \mathbb{A} \nrightarrow \mathbb{B}$ and $Q\colon \mathbb{B} \nrightarrow \mathbb{C}$ are restriction profunctors, then applying the equivalence of categories from Proposition 9.7 gives restriction functors $\mathsf{Lan}_{\mathbf{y}_r}\hat{P}\colon \mathsf{PSh}_r(\mathbb{A}) \to \mathsf{PSh}_r(\mathbb{B})$ and $\mathsf{Lan}_{\mathbf{y}_r}\hat{Q}\colon \mathsf{PSh}_r(\mathbb{B}) \to \mathsf{PSh}_r(\mathbb{C})$. Again, as these restriction functors are clearly composable, we may define the composite restriction profunctor $Q \circ P$ to be the transpose of $\mathsf{Lan}_{\mathbf{y}_r}\hat{Q} \circ \mathsf{Lan}_{\mathbf{y}_r}\hat{P}$ under the equivalence from Proposition 9.7.

To calculate this transpose explicitly, we first precompose $\mathsf{Lan}_{\mathbf{y}_r}\hat{Q} \circ \mathsf{Lan}_{\mathbf{y}_r}\hat{P}$ by $\mathbf{y}_r\colon \mathbb{A} \to \mathsf{PSh}_r(\mathbb{A})$ to give $\mathsf{Lan}_{\mathbf{y}_r}\hat{Q} \circ \hat{P}$. By definition, for all $A \in \mathbb{A}$, $\hat{P}(A) = P(-,A)$ is a restriction presheaf on $\mathbb{B}$, and $\mathsf{Lan}_{\mathbf{y}_r}\hat{Q}$ takes $P(-,A)$ to the following restriction colimit,

$$\mathsf{Lan}_{\mathbf{y}_r}\hat{Q}(P(-,A)) = \mathrm{rcolim}_{P(-,A)}\,\hat{Q}.$$

Note that the above restriction colimit exists as $\mathsf{PSh}_r(\mathbb{C})$ is a cocomplete restriction category. Now recall that the *object* component of the weighted restriction colimit is the same as for ordinary weighted colimits. The presheaf underlying $\mathrm{rcolim}_{P(-,A)}\,\hat{Q}$ must take objects $C \in \mathbb{C}$ to the set $\int^{B\in\mathbf{B}} P(B,A)\cdot Q(C,B)$, which as we know takes the form $\coprod_{B\in\mathbf{B}} P(B,A)\times Q(C,B)/{\sim}$. However, as $\mathrm{rcolim}_{P(-,A)}\,\hat{Q}$ is a restriction presheaf, this implies a restriction structure on $\int^{B\in\mathbf{B}} P(B,A)\cdot Q(C,B)$. If we denote an element of the set $\coprod_{B\in\mathbf{B}} P(B,A)\times Q(C,B)/{\sim}$ by $y\Box x$ (with $y \in Q(C,B)$ and $x \in P(B,A)$), then giving $\coprod_{B\in\mathbf{B}} P(B,A)\times Q(C,B)/{\sim}$ a restriction structure amounts to defining $\overline{y\Box x}$ for every pair x and y. The clear candidate here is $\overline{\bar{y}x}$, and as restriction presheaf structures are unique, all that remains is to check that it satisfies the requisite restriction profunctor axioms.

Proposition 9.9 (DeWolf)**.** *If $P\colon \mathbb{A} \to \mathbb{B}$ and $Q\colon \mathbb{B} \to \mathbb{C}$ are restriction profunctors, and $A \in \mathbb{A}, B \in \mathbb{B}$ and $C \in \mathbb{C}$ are objects in their respective restriction categories, then for each $x \in Q(C,B)$ and $y \in P(B,A)$, the restriction of $y\Box x \in (Q\circ P)(C,A)$ is given by $\overline{y\Box x} = \overline{\bar{y}x}$.*

Proof. It is easy to see that the above restriction for composites is well-defined, since if $y\Box x = y'\Box x'$, then for some $h \in \mathbb{B}$, we have

$$\overline{y\Box x} = \overline{(y'h)\Box x} = \overline{\overline{y'h}x} = \overline{\overline{y'}hx} = \overline{\overline{y'}hx} = \overline{y'\Box(hx)} = \overline{y'\Box x'}.$$

So to check that this defines a restriction structure on the composite, we just have to check through each of the axioms.

Clearly (RProf1) is satisfied since

$$(y\Box x)\overline{y\Box x} = (y\Box x)\overline{\bar{y}x} = y\Box(x\overline{\bar{y}x}) = y\Box(\bar{y}x) = (y\bar{y})\Box x = y\Box x,$$

and also (RProf2) holds as

$$\overline{(y\Box x)\bar{h}} = \overline{y\Box(x\bar{h})} = \overline{\bar{y}(x\bar{h})} = \overline{(\bar{y}x)\bar{h}} = \overline{\bar{y}x}\circ\bar{h} = \overline{y\Box x}\circ\bar{h}.$$

In order to show that (RProf3) also holds, first observe that

$$\overline{(y\Box x)f} = \overline{y\Box(xf)} = \overline{\bar{y}(xf)} = \overline{(\bar{y}x)f} = \overline{\overline{\bar{y}x}f} = \overline{\overline{y\Box x}f}.$$

Then (RProf3) is satisfied as

$$\overline{y\Box x} \circ g = \overline{\bar{y}x} \circ g = g \circ \overline{(\bar{y}x)g} = g \circ \overline{\overline{\bar{y}x}g} = g \circ \overline{\overline{y\Box x}g} = g \circ \overline{(y\Box x)g}.$$

As the *presheaf axioms* hold, all that remains is to check that the *restriction axiom* is satisfied. But this is also true as

$$\begin{aligned}(y\Box x)\overline{f(y\Box x)} &= (y\Box x)\overline{(fy)\Box x} = (y\Box x)\overline{\overline{fy}x} = y\Box(x\overline{\overline{fy}x}) \\ &= y\Box(\overline{fy}x) = (y\overline{fy})\Box x = (\bar{f}y)\Box x = \bar{f}(y\Box x).\end{aligned}$$

Therefore, the restriction as defined above on the composite does indeed define a restriction structure on a profunctor. □

9.3 No restriction analogues

In these discussions, we have examined restriction analogues of notions associated with ordinary categories. This naturally begs the question of which notions in ordinary category theory do not have analogues within the restriction context. We now proceed to give two examples.

9.3.1 Restriction coend

We know that for a given bifunctor $F\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$, there is the notion of coend of F, which may be described in a number of ways; as a universal cowedge, as a coequaliser and also as a weighted colimit. However, transporting the notion of coend to the restriction setting appears problematic since the opposite category of a restriction category is not a restriction category in general; in other words, it does not even make sense to replace categories and bifunctors with restriction categories and restriction bifunctors.

However, if we were to replace the category $\mathbf{C}$ above with a restriction category and replace $\mathbf{D}$ with $\mathbf{Set}$, then F will be a profunctor from $\mathbb{C}$ to $\mathbb{C}$. If in addition F is given a restriction profunctor structure, then it appears that we may be able to use profunctor composition as a basis for defining the notion of restriction coend. So let us limit our attention to functors of the form $F\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Set}$ which are *restriction profunctors* $F\colon \mathbb{C} \nrightarrow \mathbb{C}$, however restrictive a notion of coend it may be.

Now one possible way to utilise restriction profunctor composition is to insist that such a profunctor $F\colon \mathbb{C} \nrightarrow \mathbb{C}$ has a factorisation $F \cong Q \circ P$, for some restriction profunctors $P\colon \mathbb{C} \nrightarrow \mathbb{1}$ and $Q\colon \mathbb{1} \nrightarrow \mathbb{C}$ (where $\mathbb{1}$ is the trivial restriction category). If this were the case, then we could define the *restriction coend* of F to be the weighted restriction colimit

$$\int^{C\in\mathbb{C}} F(C,C) = \mathrm{rcolim}_{P(-,C)}\, \hat{Q}(C).$$

However, the biggest issue with defining a restriction coend in this way is that it then loses its natural connection with the notion of extranatural transformations. Therefore, along with the rather restrictive class of functors to which the notion of restriction coend applies, as well as the somewhat arbitrary requirement that such a profunctor has a factorisation, it would appear that the notion of *restriction coend* is not a good one.

9.3.2 Left Kan extension has no right adjoint

Recall for ordinary categories that functors $F\colon \mathbf{C} \to \mathbf{D}$ (where $\mathbf{C}$ is small and $\mathbf{D}$ is cocomplete) have left extensions along the Yoneda embedding. If F is such a functor, then its left extension $\mathsf{Lan}_{\mathbf{y}}F\colon \mathsf{PSh}(\mathbf{C}) \to \mathbf{D}$ has a right adjoint $\mathbf{D}(F,1)$, which takes objects $D \in \mathbf{D}$ to $\mathbf{D}(F-,D)$, and morphisms $f \in \mathbf{D}$ to natural transformations whose components are post-composition by f.

Turning to the case of $\mathcal{M}$-categories, recall that if $F\colon (\mathbf{C},\mathcal{C}) \to (\mathbf{D},\mathcal{D})$ is an $\mathcal{M}$-functor with $(\mathbf{D},\mathcal{D})$ cocomplete, then F also has a left extension $\mathsf{Lan}_{\mathbf{y}}F\colon (\mathsf{PSh}(\mathbf{C}),\mathsf{PSh}(\mathcal{C})) \to (\mathbf{D},\mathcal{D})$ along the Yoneda embedding, and that the underlying functor of $\mathsf{Lan}_{\mathbf{y}}F$ is the same as with the case of ordinary categories. Further recall that there is a forgetful 2-functor $\mathcal{U}\colon \mathcal{M}\mathbf{Cat} \to \mathbf{Cat}$.

Therefore, if $\mathsf{Lan}_{\mathbf{y}}F$ has a right adjoint in $\mathcal{M}\mathbf{Cat}$, then by the fact that 2-functors preserve adjoint pairs, its right adjoint must be $\mathbf{D}(F,1)$. In other words, $\mathsf{Lan}_{\mathbf{y}}F$ is a left adjoint in $\mathcal{M}\mathbf{Cat}$ if and only if $\mathbf{D}(F,1)$ is an $\mathcal{M}$-functor.

Proposition 9.10. *Let $F\colon (\mathbf{C},\mathcal{C}) \to (\mathbf{D},\mathcal{D})$ be an $\mathcal{M}$-functor, with $(\mathbf{D},\mathcal{D})$ cocomplete. Then the left Kan extension of F along the Yoneda embedding, $\mathsf{Lan}_{\mathbf{y}}F$, has a right adjoint if and only if for all $C \in \mathbf{C}$ and $n\colon D \to FC$ in $\mathcal{D}$, there is an $m\colon C' \to C$ in $\mathcal{C}$ such that Fm factorises through n, with the property that for all maps $f\colon C'' \to C$ and $g\colon FC'' \to D$ such that $Ff = n \circ g$, there is a $k\colon C'' \to C'$ such that $f = m \circ k$ and g factorises through Fk:*

$$\begin{tikzcd}
FC'' \arrow[dr, dashed, "Fk"] \arrow[drr, bend left, "Ff"] \arrow[ddr, bend right, "g"'] & & \\
& FC' \arrow[r, tail, "Fm"] \arrow[d] & FC \arrow[d, "1"] \\
& D \arrow[r, tail, "n"'] & FC.
\end{tikzcd}$$

Proof. For $\mathbf{D}(F,1)$ to be an $\mathcal{M}$-functor, the only requirement is that it takes maps in $\mathcal{D}$ to maps in $\mathsf{PSh}(\mathcal{C})$, since $\mathbf{D}(F,1)$ preserves all limits (being a right adjoint in $\mathbf{Cat}$) and hence all $\mathcal{M}$-pullbacks. So suppose $n'\colon D'' \to D'$ is in $\mathcal{D}$. Now $\mathbf{D}(F,1)(n')\colon \mathbf{D}(F-,D'') \to \mathbf{D}(F-,D')$ is post-composition by n', and for this to be in $\mathsf{PSh}(\mathcal{C})$, it means that for all $\beta\colon \mathbf{y}C \to \mathbf{D}(F-,D')$, there needs to exist some $m\colon C' \to C$ in $\mathcal{C}$ and $\gamma\colon \mathbf{y}C' \to \mathbf{D}(F-,D'')$ making the following square a pullback:

$$\begin{tikzcd}
\mathbf{y}C' = \mathbf{C}(-,C') \arrow[r, "\gamma"] \arrow[d, "\mathbf{y}m = m\circ(-)"'] & \mathbf{D}(F-,D'') \arrow[d, "n'\circ(-)"] \\
\mathbf{y}C = \mathbf{C}(-,C) \arrow[r, "\beta"'] & \mathbf{D}(F-,D').
\end{tikzcd} \tag{9.3}$$

Let us denote the transpose of β and γ under Yoneda by $\tilde{\beta}\colon FC' \to D$ and $\tilde{\gamma}\colon FC' \to D''$ respectively. Then the diagram in (9.3) commutes if and only if there exists some $m\colon C' \to C$ in $\mathcal{C}$ and $\tilde{\gamma}\colon FC' \to D''$ making the following diagram commute:

$$\begin{tikzcd}
FC' \arrow[r, "Fm"] \arrow[d, "\tilde{\gamma}"'] & FC \arrow[d, "\tilde{\beta}"] \\
D'' \arrow[r, "n'"'] & D'.
\end{tikzcd}$$

Furthermore, the diagram in (9.3) is a pullback if and only if for all $C'' \in \mathbf{C}$, $f\colon C'' \to C$ and $h\colon FC'' \to D''$ which satisfy the condition $\tilde{\beta} \circ Ff = n' \circ h$, there is a unique $k\colon C'' \to C'$ making the following diagram commute:

$$\begin{tikzcd}
FC'' \arrow[dr, dashed, "Fk"] \arrow[drr, bend left, "Ff"] \arrow[ddr, bend right, "h"'] & & \\
& FC' \arrow[r, tail, "Fm"] \arrow[d, "\tilde{\gamma}"'] & FC \arrow[d, "\tilde{\beta}"] \\
& D'' \arrow[r, tail, "n'"'] & D'.
\end{tikzcd}$$

Therefore, $\mathbf{D}(F, 1)$ is an $\mathcal{M}$-functor if and only if the above two conditions are satisfied. But observe that the statement in the proposition is a specific case of the condition for $\mathbf{D}(F, 1)$ to be an $\mathcal{M}$-functor; namely, by replacing D' by FC and the map $\tilde{\beta}$ by the identity. So all we need to show is that, given β and n' as before, this specific case implies the more general condition.

So let β and n' be given. The step is to find an $m \in C$ and $\tilde{\gamma}$ such that $\tilde{\beta} \circ Fm = n' \circ \tilde{\gamma}$. But notice that because $n'\colon D'' \to D'$ is an $\mathcal{M}$-map, we can pull n' back along $\tilde{\beta}$ to get a map $n\colon D \to FC \in \mathcal{D}$.

$$\begin{tikzcd}
FC'' \arrow[dr, dashed, "Fk"] \arrow[drr, bend left, "Ff"] \arrow[dddr, bend right, "h"'] & & \\
& FC' \arrow[r, tail, "Fm"] \arrow[d, "m'"] & FC \arrow[d, "1"] \\
& D \arrow[r, tail, "n"'] \arrow[d, "\tilde{\beta}'"] & FC \arrow[d, "\tilde{\beta}"] \\
& D'' \arrow[r, tail, "n'"'] & D'.
\end{tikzcd}$$

Now using the fact that for all $n\colon D \to FC$, there is an $m \in C$ such that $Fm = n \circ m'$ for some m', we compose $\tilde{\beta}'$ and m' to give us our $\tilde{\gamma} = \tilde{\beta}' \circ m'$. Notice that the bottom square is actually an $\mathcal{M}$-pullback. All that remains is to show that given $h\colon FC'' \to D''$ and $f\colon C'' \to C$, there is a unique $k\colon C'' \to C'$ such that $\tilde{\gamma} \circ Fk = h$ and $m \circ k = f$. Now the fact the bottom square is a pullback means there is a unique $g\colon FC'' \to D$ such that $Ff = n \circ g$. Then using the condition as stated in the proposition, this means there is a unique $k\colon C'' \to C'$ with the required properties. Hence, $\mathsf{Lan}_{\mathbf{y}} F$ has a right adjoint if and only if the condition as stated in the proposition is satisfied. □

Corollary 9.11. *Let F be the same $\mathcal{M}$-functor as in the previous proposition. If $\mathsf{Lan}_{\mathbf{y}} F$ has a right adjoint, then the induced functor on $\mathcal{M}$-subobjects,*

$$F\colon \mathsf{Sub}_{C}(C) \to (\mathsf{Sub}_{\mathcal{D}}(FC)),$$

also has a right adjoint.

Proof. To show the induced $\mathcal{M}$-functor has a right adjoint G, we need to show that given $m'\colon C' \to C$ in C and $n\colon D \to FC$ in $\mathcal{D}$, that $Fm' \le n$ if and only if $m' \le Gn$. Now to define this G, recall the previous proposition that if $\mathsf{Lan}_{\mathbf{y}} F$ has a right adjoint, then for all

$n\colon D \to FC$ in $\mathcal{D}$, there is an $m\colon C' \to C$ in C such that Fm factorised through n:

$$\begin{tikzcd}
FC' \arrow[r, "Fm"] \arrow[d] & FC \arrow[d, "1"] \\
D \arrow[r, "n"'] & FC.
\end{tikzcd}$$

So define Gn to be such an m. Now suppose that $Fm' \leq n$; in other words, Fm' factorises through n. Then again by the previous proposition, the fact $\mathsf{Lan}_{\mathbf{y}}F$ has a right adjoint implies there is a unique k such that $Gn \circ k = m'$.

$$\begin{tikzcd}
FC'' \arrow[drr, bend left, "Fm'"] \arrow[dr, dashed, "Fk"] \arrow[ddr, bend right, "g"'] & & \\
& FC' \arrow[r, tail, "FGn"] \arrow[d] & FC \arrow[d, "1"] \\
& D \arrow[r, tail, "n"'] & FC.
\end{tikzcd}$$

In other words, whenever $Fm' \leq n$, we have $m' \leq Gn$ as subobjects. Conversely, if $m' \leq Gn$, then because $Fm' = FGn \circ Fk$ and the bottom square commutes, this means $Fm' \leq n$. Therefore, $G\colon (\mathsf{Sub}_{\mathcal{D}}(FC)) \to \mathsf{Sub}_{C}(C)$ is right adjoint to the induced functor F, and hence, if $\mathsf{Lan}_{\mathbf{y}}F$ has a right adjoint, then the induced functor F on posets must also have a right adjoint. $\square$

We now use the previous corollary to provide a quick counter-example as to why left extensions along the Yoneda embedding in $\mathcal{M}$**Cat** do not have right adjoints in general. Denote by $(\mathbb{1} = \{\star\}, \text{all})$ the trivial 1-object $\mathcal{M}$-category, and $(\{0 \leq 1\}, \text{all})$ the $\mathcal{M}$-category whose underlying category has two objects a single non-trival map between them. Observe that $(\{0 \leq 1\}, \text{all})$ is cocomplete. Now let F be the $\mathcal{M}$-functor taking $\star \in \mathbb{1}$ to $1 \in \{0 \leq 1\}$, and consider the induced functor $F\colon \mathsf{Sub}_{\text{all}}(\star) \to \mathsf{Sub}_{\text{all}}(1)$. But this functor $F\colon \mathsf{Sub}_{\text{all}}(\star) \to \mathsf{Sub}_{\text{all}}(1)$ clearly does not preserve all joins (in particular, the empty join), and so F cannot possibly be a left adjoint. Hence $\mathsf{Lan}_{\mathbf{y}}F$ has no right adjoint.

Therefore, as left extensions along the Yoneda embedding are not left adjoints in $\mathcal{M}$**Cat**, the same is true for **rCat**. Hence, left extensions being left adjoints have no analogue in the restriction setting.

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