ON TIME-INCONSISTENT INVESTMENT AND DIVIDEND PROBLEMS

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Signed Statement

I, Qian Zhao, declare that this thesis, submitted in fulfilment of the requirements for the award of Doctor of Philosophy in Applied Finance and Actuarial Studies at Macquarie University (MQU) and East China Normal University (ECNU), wholly represents my own work unless otherwise referenced or acknowledged. The document has not been previously included in a thesis, dissertation or report submitted to these two universities or any other institution for a degree, diploma or other qualifications.

The results in Chapter 2 and Chapter 4 have been published on Insurance: Mathematics and Economics (2014; 58, 1-13), European Journal of Operational Research (2014; 238(3), 824-835), respectively. Chapter 3 is an unpublished working paper. I finished the papers independently with necessary direction from my supervisors, Professor Rongming Wang (ECNU), Associate Professor Xian Zhou (MQU), Professor Tak Kuen Siu (MQU). I thank Dr. Jiaqin Wei and Dr. Yang Shen for their assistance.

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Dedication

To my parents, Jujiang Zhao and Jinying Liu.

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Abstract

In this thesis, we reconsider some continuous-time stochastic control problems in finance and insurance. To incorporate some well-documented behavioural features of human beings, we consider the situation where the discounting is non-exponential. This situation is far from trivial and renders the optimisation problem to be a non-standard one, namely, a time-inconsistent stochastic control problem since Bellman's principle of optimality does not hold. In this situation, the optimal control is time-inconsistent, namely, a strategy that is optimal for the initial time may not be optimal later. Three self-contained papers are included in this thesis, each of which is concerned with one specific optimisation problem. We analyse these problems within a game theoretic framework and try to find the time-consistent equilibrium strategies for each problem.

In Chapter 2, we study the dividend maximisation problem in a diffusion risk model and try to find an equilibrium strategy within the class of feedback controls. We assume that the dividends can only be paid at a bounded rate and consider the ruin risk in the dividend problem. We obtain an equilibrium HJB equation and verification theorem for a general discount function, and get closed-form solutions in two examples.

In Chapter 3, we investigate the defined benefit pension problem, where the aim of the decision-maker is to minimise two types of risks: the contribution rate risk and the solvency risk, by considering a quadratic performance criterion. In our model, we assume that the benefit outgo is constant and the pension fund can be invested in a riskfree asset and a risky asset whose return follows a geometric Brownian motion. We characterise the time-consistent equilibrium strategy and value function in terms of the solution of a system of integral equations. The existence and uniqueness of the solution is verified and the approximation of the solution is obtained.

In Chapter 4, we consider the consumption-investment problem with logarithmic utility in a non-Markovian framework. The coefficients in our model are assumed to be adapted stochastic processes. We first study an N-person differential game and adopt a martingale method to solve an optimisation problem of each player and characterise their optimal strategies and value functions in terms of the unique solutions of BSDEs. Then by taking the limit, we show that a time-consistent equilibrium consumption-investment strategy of the original problem consists of a deterministic function and the ratio of the market price of risk to the volatility, and the corresponding equilibrium value function can be characterised by the unique solution of a family of BSDEs parameterised by a time variable.

Keywords. Non-exponential discounting, time-inconsistency, BSDEs, equilibrium HJB equation

Chapter 1

Introduction

1.1 Non-Exponential Discounting

Exponential discounting has been widely used in the very rich literature. With an exponential discounting, the amount that people discount a future reward/cost depends only on the length of waiting and a constant rate of time preference. Such preference is time-consistent as the time preference of the decision maker for an earlier date over a later date is the same.

However, there is substantial evidence that people discount the future at a non-constant rate. More specifically, there is experimental evidence (see Frederick et al. (2002)) that people are more sensitive to a given time delay if it occurs earlier. When offered a larger reward in exchange for waiting a set amount of time, people act less impulsively (i.e., choose to wait) if the rewards happen further in the future. To illustrate, many people prefer to get \$1000 now to \$1100 in a year, but few people would prefer to get \$1000 in a year to \$1100 in two years. It appears that people would rather wait a year for \$100 if they have been waiting for a year from now. However, they prefer the opposite if they must wait right now. More generally, the discount rate decreases as the length of the delay increases.

This phenomenon is best described by *hyperbolic discounting*, which means that an individual has a declining discount rate. Luttmer and Mariotti (2003) suggested a hyperbolic discount function with a form

$$h(t) = (1+at)^{-\frac{b}{a}} e^{-ct}, \quad a,b,c > 0. \tag{1.1.1}$$

Particularly, when $c = 0$, it leads to a generalised hyperbolic discount function reported by Loewenstein and Prelec (1992). The corresponding discount rate of (1.1.1) is $r(t) = c + \frac{b}{1+at}$, which is a hyperbola and smoothly declining from $b+c$ (at $t = 0$) to c (at $t = \infty$). The coefficient a describes how close h is to the exponentials e^{-ct} and $e^{-(b+c)t}$.

Hyperbolic discounting leads an individual to consume or put off an onerous activity more than she would like from a prior perspective. Laibson (1997) investigated the role of illiquid assets, such as housing, concentrating on how a decision-maker would limit over-consumption by tying up her wealth in illiquid assets. The author showed how the hyperbolic discounting might explain some stylised empirical observations, such as the existence of asset-specific marginal propensities to consume, the excess co-movement of income and consumption, and the correlation of measured levels of patience with wealth, income, and age. Compared simulated data with real-world data, the author demonstrated how hyperbolic discounting could better explain a variety of empirical phenomenons in consumption-saving problems. For more information about hyperbolic discounting and its empirical studies, we refer the reader to Phelps and Pollak (1968), Loewenstein and Prelec (1992) and Barro (1999). Due to its empirical support, hyperbolic discounting has received a lot of attention in a wide range of arenas such as economics and behavioural finance.

Moreover, another two types of non-exponential discount functions will also be adopted in the thesis. One is called the *mixture of exponential discount functions*. It is a linear combination of two exponentials, i.e.,

$$h(t) = \delta e^{-\rho_1 t} + (1-\delta)e^{-\rho_2 t}, \quad \delta \in (0,1),\ 0 < \rho_2 < \rho_1.$$

Another one is called *pseudo-exponential discounting* which is proposed by Ekeland and Lazrak (2006). It is the product of an exponential by a polynomial of degree one and given by

$$h(t) = (1+\delta t)e^{-\rho t}, \quad \delta > 0,\ \rho > 0.$$

Note that the above two quasi-exponential discount functions, although mathematically convenient, are not realistic. As mentioned in Ekeland et al. (2012), there is another reason why quasi-exponential discounting is of interest. In standard portfolio theory, the investor is generally assumed to be an individual. But in most situations, investment decisions are made by a group, such as a management team in a financial company or a family. Then there should be one utility function and one discount rate per member of the group. As a result, it can be modelled by maximising a suitable convex combination of the member's utilities, and the weight conferred to each individual represents his/her power in the group. For more explanation on pseudo-exponential discounting, we refer the reader to Ekeland and Pirvu (2008).

1.2 Optimal Control and Time-Inconsistency

Optimisation theory has been well established and extensively used in a wide range of areas, such as physics, engineering, biology, economics, finance, etc. This theory can help decision makers to find the optimal strategy in related fields. For example, in a typical continuous-time stochastic optimal control problem, the goal of the decision maker is to maximise (or minimise) a performance functional of the form

$$J(t,x,u) := \mathsf{E}_{t,x}\left[\int_t^T e^{-\delta(s-t)}C(s,X(s),u(s))\,\mathrm{d}s + e^{-\delta(T-t)}F(X(T))\right], \tag{1.2.1}$$

where

- $X = \{X(s)\}_{s\in[t,T]}$ is a controlled stochastic process, usually called a *state process*,
- $u(s)$ is the control at time s, taking values in the space $\mathbb{U}$ (e.g., a convex subset of an Euclidean space),
- $\delta > 0$ is a constant discount rate, C and F are given measurable functions,
- $\mathsf{E}_{t,x}[\cdot] = \mathsf{E}[\cdot \mid X(t) = x]$.

The first term of the objective functional is referred to as an *intertemporal reward* (or *running cost*, correspondingly) and the second term is referred to as a *terminal reward* (or *terminal cost*, correspondingly). Let $\mathscr{U}[t,T]$ denote the set of all *admissible* controls. Then the optimal control problem is to find an admissible control such that, for any initial state

$(t,x) \in [0,T) \times \mathbb{R}$,

$$V(t,x) := J(t,x,\hat{u}) = \sup_{u \in \mathscr{U}[t,T]} J(t,x,u) \;\; (\text{or} \inf_{u \in \mathscr{U}[t,T]} J(t,x,u)). \tag{1.2.2}$$

Here $\hat{u}$ is the *optimal control* of this problem, and V is the *optimal value function*. The corresponding state process is denoted by $\hat{X}(\cdot) := \hat{X}(\cdot;t,x,\hat{u}(\cdot))$ with $\hat{X}(t) = x$.

A standard way of solving the above problem is by using the dynamic programming principle. This problem turns out to be *time-consistent* in the sense that the *Bellman's principle of optimality* holds. It means that if $\hat{u} \in \mathscr{U}[t,T]$ is optimal for an initial point (t,x), then for any later time $\tau > t$, $\hat{u}$ is also optimal for the corresponding initial pair $\left(\tau, \hat{X}(\tau)\right)$, i.e.,

$$J\left(\tau, \hat{X}(\tau), \hat{u}\big|_{[\tau,T]}\right) = V\left(\tau, \hat{X}(\tau)\right) = \sup_{u \in \mathscr{U}[\tau,T]} J\left(\tau, \hat{X}(\tau), u\right) \;\; (\text{or} \inf_{u \in \mathscr{U}[\tau,T]} J\left(\tau, \hat{X}(\tau), u\right)), \tag{1.2.3}$$

where $\hat{u}\big|_{[\tau,T]}$ denotes the restriction of $\hat{u}$ on the time interval $[\tau,T]$ with initial pair $\left(\tau, \hat{X}(\tau)\right)$. For more details, we refer the reader to Yong (2011, 2012a).

Given the Bellman optimality principle and some differentiability conditions, one can derive the Hamilton-Jacobi-Bellman (HJB) equation to determine the value function V and a verification theorem which shows that if V is a classical solution of the HJB equation, then V is indeed the optimal value function. The optimal control is the one that maximises (or minimises) the HJB equation.

Time consistency is very ideal in the sense that if a control $\hat{u}$ is optimal for any given initial pair (t,x), then it will henceforth stay optimal. However, the time consistency could be lost in the real world. For example, it is impossible for people to keep the commitments as they always change their minds. It can be reflected in the reality that people's consumption habit or the living standard keeps changing. Moreover, the environment we are living in is always changing due to the advances of technology (such as computer, internet, new material, etc) as well as new limits of resources like water, fuel, living space. Therefore, it is very difficult for people to make a long-term time-consistent plan.

Due to this fact, a number of researchers start to concern with the time-inconsistent problems in recent years, especially in the area of finance and economics. A typical example from financial economics with time inconsistency is the stochastic control problem under non-exponential discounting. Due to the possible subjectivity of people's preferences, the discount factors $e^{-\delta(s-t)}$ and $e^{-\delta(T-t)}$ appeared in (1.2.1) might be replaced by some general functions $h(s,t)$ and $h(T,t)$, or more generally, we may consider the following performance functional:

$$J(t,x,u) := \mathsf{E}_{t,x}\left[\int_t^T g(t,s,X(s),u(s))\,\mathrm{d}s + f(t,X(T))\right],$$

where $g(\cdot)$ and $f(\cdot)$ explicitly depend on the initial time t in some general way. In this situation, the optimal control is time-inconsistent, i.e., a strategy which is optimal for the initial time may not be optimal later. To illustrate, the planner will find an optimal strategy $\hat{u}$ for a given initial point (t,x) which maximises (or minimises) $J(t,x,u)$, whereas at some later point $\left(s,\hat{X}(s)\right)$ the control $\hat{u}$ will no longer be optimal for $J\left(s,\hat{X}(s),u\right)$. Thus the planner will not implement the strategy $\hat{u}$ at time s, unless there is some commitment mechanism. If there is none, then the strategy $\hat{u}$, which is optimal from the perspective of the planner at initial time t, is not implementable, and the planner at time t must seek a sub-optimal strategy. This is the so-called *time-inconsistent control problem* in the sense that the Bellman optimality principle does no longer hold. In other words, if we find some $\hat{u} \in \mathscr{U}[t,T]$ such that (1.2.2) is satisfied for initial state (t,x), we do not get (1.2.3), for any $\tau \in (t,T]$.

This thesis revisits some continuous-time optimisation problems in finance and insurance with non-exponential time preferences. To this end, we consider the following objective functional:

$$J(t,x,u) := \mathsf{E}_{t,x}\left[\int_t^T h(s,t)\,C(s,X(s),u(s))\,\mathrm{d}s + h(T,t)\,F(X(T))\right], \tag{1.2.4}$$

where h is a general discount function but not restricted to be exponential. This problem is different from the standard problem (1.2.1) since the initial time t enters the intertemporal reward (or running cost, correspondingly). As a result, it leads to a time-inconsistent control problem. Apparently, if $h(s,t) = e^{-\delta(s-t)}$, it turns out to be the standard problem (1.2.1) by factoring out $e^{\delta t}$.

Moreover, there are some other interesting performance functional that results in time-inconsistency. For example, one can consider an objective function with a constant discount factor while adjusting the utility function dynamically (i.e., the utility function depends on the initial time t). In this situation, the decision-maker has a consistent time preference but an inconsistent satisfaction and happiness from consumption or some service, according to personal willingness. Due to the limitation of the length, we do not discuss this case in the thesis. We will consider this kind of problem in our future work.

1.3 Approches to Handle Time Inconsistency

Difficulties arise when we solve time-inconsistent control problems by using the standard optimal control techniques. Since the Bellman optimality principle does not hold in this case, we can not apply the dynamic programming approach to derive the HJB equation.

In the literature, the idea to handle time inconsistency is to view the decision-maker at different time points as different players, and to analyse the optimisation problem within a game theoretic framework. In this case, we assume that future players choose strategies that are optimal for themselves, despite being suboptimal from the standpoint of the current player. Given this point of view, it is natural to seek *Nash sub-game perfect equilibria* for the game.

The game theoretic approach to addressing general time inconsistency via Nash equilibrium points has a long history starting with Strotz (1955) where a cake-eating problem was studied. This line of research has been followed by many others (see Pollak (1968), Peleg and Yaari (1973), Goldman (1980), Laibson (1997), Barro (1999) and Krusell and Smith (2003)) in both discrete and continuous time cases.

Recently, continuous-time time-inconsistent stochastic control problems have received a great deal of attention in economics and behavioural finance due to the fact that many practical problems in these fields can be formulated as this kind of control problems. Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008) considered an optimal consumption-investment problem under hyperbolic discounting and provided the precise definition of the equilibrium concept in a continuous-time model for the first time. The authors characterised the equilibrium policies through the solutions of a flow of backward stochastic differential equations (BSDEs), and showed that with special form of the discount factor, this system of BSDEs reduced to a system of two ordinary differential equations (ODEs) which had a solution. Following their definition of the equilibrium strategy, Björk and Murgoci (2010) studied the time-inconsistent control problem within a general Markovian framework, and derived the extended HJB equation together with the verification theorem. A modified HJB equation was derived in Marín-Solano and Navas (2010) which investigated a consumption-investment problem with non-constant discount rate for both naive and sophisticated agents. Yong (2011) investigated a time-inconsistent deterministic linear-quadratic (LQ) control problem and derived equilibrium controls by solving a multi-person differential game. Considering the hyperbolic discounting, Ekeland et al. (2012) studied the portfolio management problem for an investor who was allowed to consume and take out life insurance, and they characterised the equilibrium strategy by the solution of an integral equation. Considering a regime-switching model and with the assumption that the risk aversion depends on the state of the regime, Wei et al. (2013) investigated the equilibrium strategy for the mean-variance asset-liability management problem by using the extended HJB equation developed by Björk and Murgoci (2010).

The key point of solving a time-inconsistent control problem is how to define the equilibrium control in continuous time. In Ekeland and Lazrak (2006), Ekeland and Pirvu (2008) and the papers following their ideas, the equilibrium control was defined within the class of *closed-loop controls* (also called *Markov controls* or *feedback controls*). In this method, the equilibrium control is generally defined in the following way for a time-inconsistent control problem where the aim is to maximise the objective functional (1.2.4). Assume that at time $t \in [0,T]$, the control $u(t)$ is given by $u(t) = \pi(t,x)$, where x is the value of state at time t and $\pi : [0,T] \times \mathbb{R} \to \mathbb{U}$ is a Borel measurable function.

Definition 1.3.1. *Choose a control $\hat{\pi} \in \mathscr{U}[t,T]$ and a fixed real number $\varepsilon > 0$. For any fixed initial point $(t,x) \in [0,T] \times \mathbb{R}$, we define another control π^{ε} by*

$$\pi^{\varepsilon}(s,y) = \begin{cases} \pi(s,y), & \text{for } s \in [t, t+\varepsilon],\ y \in \mathbb{R}, \\ \hat{\pi}(s,y), & \text{for } s \in [t+\varepsilon, T],\ y \in \mathbb{R}, \end{cases}$$

where π is any control for which π^{ε} is admissible. If for any $(t,x) \in [0,T] \times \mathbb{R}$,

$$\liminf_{\varepsilon \downarrow 0} \frac{J(t,x,\hat{\pi}) - J(t,x,\pi^{\varepsilon})}{\varepsilon} \geq 0,$$

then we say that $\hat{\pi}$ is an equilibrium control for this problem, and the equilibrium value function V is defined by

$$V(t,x) = J(t,x,\hat{\pi}).$$

In Chapter 2 and Chapter 3, we will apply this method to deal with a dividend maximisation problem and a *defined benefit pension* (DB pension, for short) problem with non-exponential discounting, respectively.

Considering the time-inconsistent stochastic LQ control, Hu et al. (2012) defined the equilibrium control within the class of *open-loop controls*, and derived a general sufficient condition for equilibria through a flow of forward-backward stochastic differential equations (FBSDEs). However, the general existence of solutions to the flow of FBSDEs is an open problem. With the assumption that the state process is scalar valued and all the coefficients are deterministic, Hu et al. (2012) showed that the flow of FBSDEs could be reduced into several Riccati-like ordinary differential equations and the equilibrium control could be obtained explicitly. Also considering the scalar valued state process, Hu et al. (2012) dealt with the Markowitz's problem with state-dependent risk aversion and stochastic coefficients. Due to the difference between the definitions of equilibrium controls, their results were rather different from those obtained in Björk and Murgoci (2010) and Björk et al. (2014).

Another approach to define the equilibrium strategy in continuous-time time-inconsistent control problems is initially proposed by Yong (2011), whose research is based on a study of multi-person hierarchical differential games. Following this idea, Yong (2012a) considered a general time-inconsistent stochastic control problem for diffusion state processes with deterministic coefficients. In this paper, the author derived the so-called equilibrium HJB equation which can be used to characterise the equilibrium value function. Let us give a rough picture of this game in the following.

For any $N > 1$, we denote by Π a partition of the time interval $[0,T]$, i.e.,

$$\Pi: \ 0 = t_0 < t_1 < \cdots < t_N = T.$$

The mesh size $\|\Pi\|$ is given by

$$\|\Pi\| = \max_{1\leq k\leq N} (t_k - t_{k-1}).$$

We assume that, in this game, "the k-th player" controls the system on $[t_{k-1},t_k)$, $k = 1,\cdots,N$. These players can be treated as different incarnations of the decision-maker at different time points, and each player tries to find the optimal control for his/her own problem. Denote by $\hat{u}^k \in \mathscr{U}[t_{k-1},t_k]$ the optimal control that the k-th player will apply and by V^k defined on $[t_{k-1},t_k] \times \mathbb{R}$ the optimal value function for his/her own problem. Now we define

$$\hat{u}^\Pi(t) := \sum_{k=1}^{N} \hat{u}^k(t)\mathbf{1}_{[t_{k-1},t_k)}(t), \quad t \in [0,T),$$

and

$$V^\Pi(t,x) := \sum_{k=1}^{N} V^k(t,x)\mathbf{1}_{[t_{k-1},t_k)}(t), \quad (t,x) \in [0,T) \times \mathbb{R}.$$

We call $\hat{u}^\Pi$ and V^Π the *Nash equilibrium strategy* and *Nash equilibrium value function*, respectively. If there exists some $\hat{u} \in \mathscr{U}[0,T]$ and $V : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{\|\Pi\|\to 0} \hat{u}^\Pi(t) = \hat{u}(t), \quad t \in [0,T].$$

and

$$\lim_{\|\Pi\|\to 0} V^\Pi(t,x) = V(t,x), \quad (t,x) \in [0,T] \times \mathbb{R},$$

then we call $\hat{u}$ and V a *time-consistent equilibrium strategy* and a *time-consistent equilibrium value function*, respectively.

In Chapter 4, we will adopt the above approach to deal with a consumption-investment problem with non-exponential discounting and logarithmic utility in non-Markovian framework.

1.4 Structure of the Thesis

The aim of this thesis is to revisit some continuous-time stochastic control problems in finance and actuarial science with time-inconsistent preference in Markovian framework and non-Markovian framework. This thesis consists of three self-contained chapters, each of which is concerned with one specific portfolio optimisation problem. Rather than regurgitating, the background of each topic will be introduced at the beginning of each chapter.

In Chapter 2, we study the dividend optimisation problem with a non-constant discount rate in a diffusion risk model. We assume that the dividends can only be paid at a bounded rate and restrict ourselves to Markov strategies. In contrast to most of the existing literature which consider a fixed time horizon or an infinite time horizon, we consider the ruin risk in the dividend problem and assume that the time horizon is a random variable (the time of ruin). Following the idea of Yong (2012a), we obtain the extended HJB equation and prove the verification theorem for a general discount function. Moreover, the equilibrium dividend strategies for the mixture of exponential discount functions and the pseudo-exponential discounting are presented by solving the equilibrium HJB equation.

In Chapter 3, we investigate the DB pension plan, where the object of the manager is to minimise the contribution rate risk and the solvency risk by considering a quadratic performance criterion. To incorporate some well-documented behavioural features of human beings, we consider the situation where the discounting is non-exponential. In our model, we assume that the benefit outgo is constant and the pension fund can be invested in a risk-free asset and a risky asset whose return follows a geometric Brownian motion. We characterise the time-consistent strategies and value function in terms of the solution of a system of integral equations. The existence and uniqueness of the solution is verified and the approximation of the solution is obtained. Some numerical results of the equilibrium contribution rate and equilibrium investment policy are presented for three types of discount functions.

In Chapter 4, we revisit the consumption-investment problem with a general discount function and a logarithmic utility in a non-Markovian framework. The coefficients in our model are assumed to be adapted stochastic processes. Following Yong (2012a,b)'s method, we study an N-person differential game and adopt a martingale method to solve an optimisation problem of each player and characterise their optimal strategies and value functions in terms of the unique solutions of BSDEs. Then by taking the limit, we show that a time-consistent equilibrium consumption-investment strategy of the original problem consists of a deterministic function and the ratio of the market price of risk to the volatility, and the corresponding equilibrium value function can be characterised by the unique solution of a family of BSDEs parameterised by a time variable.

Chapter 2

On Dividend Strategies with Non-Exponential Discounting

2.1 Introduction

Since it was proposed by De Finetti (1957), the optimisation of dividend payments has been investigated by many researchers under various risk models. This problem is usually phrased as the management's problem of determining optimal timing and size of dividend payments in the presence of bankruptcy risk. For more literature on this problem, we refer the reader to a recent survey paper by Avanzi (2009).

In the very rich literature, a common assumption is that the discount rate is constant over time so that the discount function is exponential. However, some empirical studies of human behaviour suggest that the assumption of constant discount rate is unrealistic, see, e.g., Thaler (1981), Ainslie (1992) and Loewenstein and Prelec (1992). Indeed, there is experimental evidence that people are impatient about choices in the short term but more patient when choosing between long-term alternatives. More precisely, events in the near future tend to be discounted at a higher rate than events that occur in the long run. Considering such an effect, individual behaviour is best described by the hyperbolic discounting (see Phelps and Pollak (1968)), which has been extensively studied in the areas of microeconomics, macroeconomics, and behavioural finance, such as Laibson (1997) and Barro (1999) among others.

However, difficulties arise when we try to solve an optimal control problem with a non-constant discount rate by the standard dynamic programming approach. In fact, the standard optimal control techniques give rise to time inconsistent strategies, i.e., a strategy that is optimal for the initial time may be not optimal later. This is the so-called time inconsistent control problem and the classical dynamic programming principle does no longer hold. Strotz (1955) studied the time inconsistent problem within a game theoretic framework by using Nash equilibrium points. The author sought the equilibrium policy as the solution of a subgame-perfect equilibrium where the players are the agent and her future selves.

Recently, there is an increasing attention in the time inconsistent control problem due to the practical applications in economics and finance. A modified HJB equation was derived in Marín-Solano and Navas (2010) which solved an optimal consumption and investment problem with the non-constant discount rate for both naive and sophisticated agents. A similar problem was also considered by another approach in Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008), who provided the precise definition of the equilibrium concept in continuous time for the first time. They characterised the equilibrium policies through the solutions of a flow of a BSDE, and showed, for a special form of the discount factor, that this BSDE reduces to a system of two ODEs, which has a solution. Considering the hyperbolic discounting, Ekeland et al. (2012) studied the portfolio management problem for an investor who is allowed to consume and take out life insurance, and they characterised the equilibrium strategy by an integral equation. Following this definition of the equilibrium strategy, Björk and Murgoci (2010) studied the time-inconsistent control problem in a general Markovian framework, and derived the equilibrium HJB-equation together with the verification theorem. Björk et al. (2014) studied Markowitz's optimal portfolio problem with state-dependent risk aversion by utilising the equilibrium HJB-equation obtained in Björk and Murgoci (2010).

In this chapter, we revisit the dividend maximisation problem with a general discount function in a diffusion risk model. We assume that the dividends can only be paid at a bounded rate and restrict ourselves to Markov strategies. We use the equilibrium HJB-equation to solve this problem. In contrast to the papers mentioned above which considered a fixed time horizon or an infinite time horizon, in the dividend problem the ruin risk should be taken into account and the time horizon is a random variable (the time of ruin). Thus, the equilibrium HJB-equation given in this chapter looks different to the one obtained in Björk and Murgoci (2010). We first give the equilibrium HJB-equation, which is motivated by Yong (2012a) and the verification theorem for a general discount function. Then we solve the equilibrium HJB-equation for two special non-exponential discount functions: a mixture of exponential discount functions and a pseudo-exponential discount function. For more details about these discount functions, we refer the reader to Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008). Under the mixture of exponential discount functions, our results show that if the bound of the dividend rate is small enough, then the equilibrium strategy is to always pay the maximal dividend rate; otherwise, the equilibrium strategy is to pay the maximal dividend rate when the surplus is above a barrier and pay nothing when the surplus is below the barrier. Given some conditions, the results are similar under the pseudo-exponential discount function. These features of the equilibrium dividend strategies are similar to the optimal strategies obtained in Asmussen and Taksar (1997) who considered the exponential discounting in the diffusion risk model.

The remainder of this chapter is organised as follows. The dividend problem and the definition of an equilibrium strategy are given in Section 2.2. The equilibrium HJB-equation and a verification theorem are presented in Section 2.3. In Section 2.4, we study two cases with a mixture of exponential discount functions and a pseudo-exponential discount function. Section 2.5 concludes this chapter.

2.2 The Model

In the case of no control, the surplus process is assumed to follow

$$\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t, \qquad t \geq 0,$$

where μ, σ are positive constants and $\{W_t\}_{t\geq 0}$ is a one-dimensional standard Brownian motion on a filtered probability space $\left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathsf{P}\right)$ satisfying the usual hypotheses. The filtration $\{\mathscr{F}_t\}_{t\geq 0}$ is completed and generated by $\{W_t\}_{t\geq 0}$.

A dividend strategy is described by a stochastic process $\{l_t\}_{t\geq 0}$. Here, $l_t \geq 0$ is the rate of dividend payout at time t which is assumed to be bounded by a constant $M > 0$. We restrict ourselves to the feedback control strategies (Markov strategies), i.e. at time t, the control l_t is given by

$$l_t = \pi(t,x),$$

where x is the surplus level at time t and the control law $\pi : [0,\infty) \times [0,\infty) \to [0,M]$ is a Borel measurable function. In Section 2.4, we need to distinguish the cases with $M < \mu$ and $M \geq \mu$, and to be more careful with the former case when we verify the strategy conjectured is indeed an equilibrium strategy (see Corollaries 2.4.1 and 2.4.2).

When applying the control law π, we denote by $\{X_t^\pi\}_{t\geq 0}$ the controlled risk process. Considering the controlled system starting from the initial time $t \in [0,\infty)$, $\{X_s^\pi\}$ evolves according to

$$\begin{cases} \mathrm{d}X_s^\pi & = \mu \mathrm{d}s + \sigma \mathrm{d}W_s - \pi(s, X_s^\pi)\mathrm{d}s, \quad s \geq t, \\ X_t^\pi & = x. \end{cases} \tag{2.2.1}$$

Let

$$\tau_t^\pi := \inf\{s \geq t : X_s^\pi \leq 0\}$$

be the time of ruin under the control law π. Without loss of generality, we assume that $X_s^\pi \equiv 0$ for $s \geq \tau_t^\pi$.

Let $h : [0,\infty) \to [0,\infty)$ be a discount function which satisfies $h(0) = 1$, $h(t) \geq 0$ and $\int_0^\infty h(t)\mathrm{d}t < \infty$. Furthermore, h is assumed to be continuously differentiable on $[0,\infty)$ and $h'(t) \leq 0$.

Definition 2.2.1. *A control law π is said to be admissible if it satisfies: $0 \leq \pi(t,x) \leq M$ for all $(t,x) \in [0,\infty) \times [0,\infty)$, $\pi(t,0) \equiv 0$ for all $t \in [0,\infty)$. We denote by Π the set of all admissible control laws.*

For a given admissible control law π and an initial state $(t,x) \in [0,\infty) \times [0,\infty)$, we define the return function V^π by

$$V^\pi(t,x) = \mathsf{E}_{t,x}\left[\int_t^{\tau_t^\pi} h(z-t)\pi(z, X_z^\pi)\mathrm{d}z\right],$$

where $\mathsf{E}_{t,x}[\cdot]$ is the expectation conditioned on the event $\{X_t^\pi = x\}$. Note that for any admissible strategy $\pi \in \Pi$, we have

$$\mathsf{E}_{t,x}\left[\int_t^{\tau_t^\pi} \left|h(z-t)\pi(z,X_z^\pi)\right| \mathrm{d}z\right] \le M\int_0^\infty h(t)\mathrm{d}t < \infty, \quad \forall (t,x) \in [0,\infty)\times[0,\infty), \tag{2.2.2}$$

which means the performance functions $V^\pi(t,x)$ are well-defined for all admissible strategies.

In classical risk theory, the optimal dividend strategy, denoted by π^*, is an admissible strategy such that

$$V^{\pi^*}(t,x) = \sup_{\pi\in\Pi} V^\pi(t,x).$$

However, in our settings, this optimization problem is time-inconsistent in the sense that the Bellman optimality principle fails.

Similar to Ekeland and Pirvu (2008) and Björk and Murgoci (2010), we view the entire problem as a non-cooperative game and look for Nash equilibria for the game. More specifically, we consider a game with one player for each time t, where player t can be regarded as the future incarnation of the decision maker at time t. Given state (t,x), player t will choose a control action $\pi(t,x)$, and she/he wants to maximize the functional $V^\pi(t,x)$. In the continuous-time model, Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008) give the precise definition of this equilibrium strategy for the first time. Intuitively, equilibrium strategies are the strategies such that, given that they will be implemented in the future, it is optimal to implement them right now.

Definition 2.2.2. *Choose a control law $\hat{\pi} \in \Pi$, a fixed $l \in [0,M]$ and a fixed real number $\varepsilon > 0$. For any fixed initial point $(t,x) \in [0,\infty)\times[0,\infty)$, we define the control law π^ε by*

$$\pi^\varepsilon(s,y) = \begin{cases} 0, & \text{for } s \in [t,\infty),\ y = 0; \\ l, & \text{for } s \in [t,t+\varepsilon],\ y \in (0,\infty); \\ \hat{\pi}(s,y), & \text{for } s \in [t+\varepsilon,\infty),\ y \in (0,\infty). \end{cases}$$

If

$$\liminf_{\varepsilon\to 0} \frac{V^{\hat{\pi}}(t,x) - V^{\pi^{\varepsilon}}(t,x)}{\varepsilon} \geq 0,$$

for all $l \in [0,M]$, we say that $\hat{\pi}$ is an equilibrium control law. And the equilibrium value function V is defined by

$$V(t,x) = V^{\hat{\pi}}(t,x). \tag{2.2.3}$$

In the following section, we will first give the equilibrium HJB-equation for the equilibrium value function V, and then prove a verification theorem.

2.3 The Equilibrium Hamilton-Jacobi-Bellman Equation

In this section, we consider the objective function having the form

$$V^{\pi}(t,x) = \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\pi}} C\left(t,z,\pi(z,X_z^{\pi})\right) \mathrm{d}z\right], \tag{2.3.1}$$

where $C\left(t,z,\pi(z,X_z^{\pi})\right) = h(z-t)\pi(z,X_z^{\pi})$, for $z \geq t$.

For all $\pi \in \Pi$ and any real valued function $f(t,x) \in C^{1,2}\left([0,\infty)\times(0,\infty)\right)$, which means that the partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ exist and are continuous on $[0,\infty)\times(0,\infty)$, we define the infinitesimal generator $\mathscr{L}^{\pi}$ by

$$\mathscr{L}^{\pi} f(t,x) = \frac{\partial f}{\partial t}(t,x) + (\mu - \pi(t,x))\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x).$$

Let $\mathscr{D}[0,\infty) := \{(s,t) \mid 0 \leq s \leq t < \infty\}$ and $C^{0,1,2}\left(\mathscr{D}[0,\infty)\times[0,\infty)\right)$ be the set of all functions defined on $\mathscr{D}[0,\infty)\times[0,\infty)$ which are continuous with respective to the first variable, continuously differentiable with respective to the second variable and twice continuously differentiable with respective to the third variable. The following equilibrium HJB-equation is motivated by Equation (4.77) of Yong (2012a) and the proof of Theorem 2.3.1.

Definition 2.3.1. *For a function $c(s,t,x) \in C^{0,1,2}\left(\mathscr{D}[0,\infty)\times[0,\infty)\right)$, the equilibrium HJB-*

equation is given by

$$\begin{cases} \frac{\partial c}{\partial t}(s,t,x) + H\left(s,t,\phi\left(t,t,\frac{\partial c}{\partial x}(t,t,x),\frac{\partial^2 c}{\partial x^2}(t,t,x)\right),\frac{\partial c}{\partial x}(s,t,x),\frac{\partial^2 c}{\partial x^2}(s,t,x)\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall (s,t,x) \in \mathscr{D}[0,\infty)\times(0,\infty), \\ c(s,t,0) = 0, \quad \forall (s,t) \in \mathscr{D}[0,\infty), \end{cases} \tag{2.3.2}$$

where

$$\begin{cases} H(s,t,l,p,P) = \frac{1}{2}\sigma^2 P + (\mu - l)p + C(s,t,l), \\ \phi(s,t,p,P) = \arg\max H(s,t,\cdot,p,P), \end{cases} \tag{2.3.3}$$

for $(s,t,l,p,P) \in \mathscr{D}[0,\infty)\times[0,M]\times\mathbb{R}^2$.

Since the equilibrium HJB-equation given in Definition 2.3.1 is informal, we are now giving a verification theorem for motivating it.

Theorem 2.3.1. *(Verification Theorem) Assume that there exists a bounded function* $c(s,t,x) \in C^{0,1,2}\left(\mathscr{D}[0,\infty)\times[0,\infty)\right)$ *which solves the equilibrium HJB-equation in Definition 2.3.1. Let*

$$\hat{\pi}(t,x) := \phi\left(t,t,\frac{\partial c}{\partial x}(t,t,x),\frac{\partial^2 c}{\partial x^2}(t,t,x)\right), \tag{2.3.4}$$

and

$$V(t,x) := c(t,t,x). \tag{2.3.5}$$

If for any $(s,t,x) \in \mathscr{D}[0,\infty)\times[0,\infty)$ *it holds that*

$$\lim_{n\to\infty} c(s,\tau_n,X^{\hat{\pi}}_{\tau_n}) = 0, \quad a.s., \tag{2.3.6}$$

where $\tau_n = n \wedge \tau^{\hat{\pi}}_t, n \geq t,\ n = 1,2,\cdots$, *and* $X^{\hat{\pi}}$ *is the unique solution to the SDE (2.2.1) with* π *replaced by* $\hat{\pi}$ *and initial state* (t,x), *then* $\hat{\pi}$ *given by (2.3.4) is an equilibrium control law, and* V *given by (2.3.5) is the corresponding equilibrium value function.*

Proof. We give the proof in two steps: 1. We show that V is the value function corresponding to $\hat{\pi}$, i.e., $V(t,x) = V^{\hat{\pi}}(t,x)$; 2. We prove that $\hat{\pi}$ is indeed the equilibrium control law which is defined by Definition 2.2.2.

Step 1. With (2.3.4), we rewrite (2.3.2) as

$$\begin{cases} \mathscr{L}^{\hat{\pi}} c(s,t,x) + C(s,t,\hat{\pi}(t,x)) = 0, \quad (s,t,x) \in \mathscr{D}[0,\infty) \times (0,\infty), \\ c(s,t,0) = 0, \quad \forall (s,t) \in \mathscr{D}[0,\infty), \end{cases} \tag{2.3.7}$$

where the operator $\mathscr{L}^{\hat{\pi}}$ applies to the function $c(s,\cdot,\cdot)$.

By (2.3.7), applying Dynkin's formula to the function $c(s,\cdot,\cdot)$ yields that

$$\begin{aligned} c(s,t,x) &= \mathsf{E}_{t,x}\left[c\left(s,\tau_n,X_{\tau_n}^{\hat{\pi}}\right)\right] - \mathsf{E}_{t,x}\left[\int_t^{\tau_n} \mathscr{L}^{\hat{\pi}} c\left(s,z,X_z^{\hat{\pi}}\right) \mathrm{d}z\right] \\ &= \mathsf{E}_{t,x}\left[c\left(s,\tau_n,X_{\tau_n}^{\hat{\pi}}\right)\right] + \mathsf{E}_{t,x}\left[\int_t^{\tau_n} C(s,z,\hat{\pi}(z,X_z^{\hat{\pi}}))\mathrm{d}z\right]. \end{aligned}$$

Recalling Definition 2.2.1 of admissible strategies (see also (2.2.2)), for given $s \le t$, we have

$$\mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}} \left|C\left(s,z,\hat{\pi}(z,X_z^{\hat{\pi}})\right)\right| \mathrm{d}z\right] < \infty, \quad \forall (t,x) \in [0,\infty) \times [0,\infty).$$

Since $c(\cdot,\cdot,\cdot)$ is bounded, by (2.3.6), letting $n \to \infty$ and applying the dominated convergence theorem yields

$$c(s,t,x) = \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}} h(z-s)\hat{\pi}(z,X_z^{\hat{\pi}})\mathrm{d}z\right], \quad (s,t,x) \in \mathscr{D}[0,\infty) \times [0,\infty). \tag{2.3.8}$$

Thus, we have

$$V(t,x) := c(t,t,x) = V^{\hat{\pi}}(t,x).$$

Step 2. For a given $l \in [0,M]$, and a fixed real number $\varepsilon > 0$, we define π^{ε} by Definition 2.2.2. For simplicity, we denote by X^{ε} the path under the control law π^{ε}. Without loss of generality, we consider the case where ε is sufficiently small such that $t + \varepsilon < \tau_t^{\pi^{\varepsilon}} \wedge \tau_t^{\hat{\pi}}$ a.s. By the definition of $V^{\hat{\pi}}$ and $V^{\pi^{\varepsilon}}$, we obtain

$$\begin{aligned} V^{\hat{\pi}}(t,x) - V^{\pi^{\varepsilon}}(t,x) &= \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\hat{\pi}}} C\left(t,s,\hat{\pi}\left(s,X_s^{\hat{\pi}}\right)\right) \mathrm{d}s - \int_t^{\tau_t^{\pi^{\varepsilon}}} C\left(t,s,\pi^{\varepsilon}\left(s,X_s^{\varepsilon}\right)\right) \mathrm{d}s\right] \\ &= \mathsf{E}_{t,x}\left[\int_t^{t+\varepsilon} h(s-t)\left(\hat{\pi}\left(s,X_s^{\hat{\pi}}\right) - \pi^{\varepsilon}\left(s,X_s^{\varepsilon}\right)\right) \mathrm{d}s\right] \end{aligned}$$

$$
\begin{aligned}
&+\mathsf{E}_{t,x}\left[V^{\hat\pi}\left(t+\varepsilon,X^{\hat\pi}_{t+\varepsilon}\right)-V^{\hat\pi}\left(t+\varepsilon,X^{\varepsilon}_{t+\varepsilon}\right)\right]\\
&+\left(\mathsf{E}_{t,x}\left[\int_{t+\varepsilon}^{\tau_t^{\hat\pi}}\left(h\left(s-t\right)-h\left(s-t-\varepsilon\right)\right)\hat\pi\left(s,X^{\hat\pi}_s\right)\mathrm{d}s\right]\right.\\
&\left.-\mathsf{E}_{t,x}\left[\int_{t+\varepsilon}^{\tau_t^{\pi^\varepsilon}}\left(h\left(s-t\right)-h\left(s-t-\varepsilon\right)\right)\hat\pi\left(s,X^{\varepsilon}_s\right)\mathrm{d}s\right]\right). \qquad (2.3.9)
\end{aligned}
$$

Here $\hat\pi(s,X^{\varepsilon}_s)$ and $\hat\pi(s,X^{\hat\pi}_s)$ are the equilibrium control processes associated with the paths of X^{ε} and $X^{\hat\pi}$, respectively.

According to the equation (2.3.9), we now consider the limitation $\lim_{\varepsilon\downarrow 0}\frac{V^{\hat\pi}(t,x)-V^{\pi^\varepsilon}(t,x)}{\varepsilon}$ in three parts separately:

1. Noting that $\int_0^\infty h(t)\mathrm{d}t<\infty$, l and $\hat\pi$ are bounded and applying the dominated convergence theorem, we get

$$
\lim_{\varepsilon\downarrow 0}\frac{\mathsf{E}_{t,x}\left[\int_t^{t+\varepsilon}h(s-t)\left(\hat\pi\left(s,X^{\hat\pi}_s\right)-\pi^{\varepsilon}\left(s,X^{\varepsilon}_s\right)\right)\mathrm{d}s\right]}{\varepsilon}=\hat\pi\left(t,x\right)-\pi^{\varepsilon}(t,x).
$$

2. We rewrite the second part in the right-side of the equation (2.3.9) by

$$
\begin{aligned}
&\mathsf{E}_{t,x}\left[V^{\hat\pi}\left(t+\varepsilon,X^{\hat\pi}_{t+\varepsilon}\right)-V^{\hat\pi}\left(t+\varepsilon,X^{\varepsilon}_{t+\varepsilon}\right)\right]\\
=\;&\mathsf{E}_{t,x}\left[V^{\hat\pi}\left(t+\varepsilon,X^{\hat\pi}_{t+\varepsilon}\right)-V^{\hat\pi}\left(t,x\right)\right]-\mathsf{E}_{t,x}\left[V^{\hat\pi}\left(t+\varepsilon,X^{\varepsilon}_{t+\varepsilon}\right)-V^{\hat\pi}\left(t,x\right)\right]\\
=\;&\mathsf{E}_{t,x}\left[\int_t^{t+\varepsilon}\mathrm{d}V^{\hat\pi}\left(u,X^{\hat\pi}_u\right)\right]-\mathsf{E}_{t,x}\left[\int_t^{t+\varepsilon}\mathrm{d}V^{\hat\pi}\left(u,X^{\varepsilon}_u\right)\right].
\end{aligned}
$$

Applying the Itô's formula, we get

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{\mathsf{E}_{t,x}\left[\int_t^{t+\varepsilon}\mathrm{d}V^{\hat\pi}\left(u,X^{\hat\pi}_u\right)\right]}{\varepsilon}\\
=\;&\frac{\partial V^{\hat\pi}(t,x)}{\partial t}+\left(\mu-\hat\pi\left(t,x\right)\right)\frac{\partial V^{\hat\pi}(t,x)}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2V^{\hat\pi}(t,x)}{\partial x^2}\\
=\;&\left(\mathscr{L}^{\hat\pi}V^{\hat\pi}\right)(t,x)\\
=\;&\left(\mathscr{L}^{\hat\pi}V\right)(t,x),
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{\mathsf{E}_{t,x}\left[\int_t^{t+\varepsilon}\mathrm{d}V^{\hat{\pi}}\left(u,X_u^{\varepsilon}\right)\right]}{\varepsilon}\\
=&\ \frac{\partial V^{\hat{\pi}}(t,x)}{\partial t}+(\mu-l)\,\frac{\partial V^{\hat{\pi}}(t,x)}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 V^{\hat{\pi}}(t,x)}{\partial x^2}\\
=&\ \left(\mathscr{L}^{\pi^{\varepsilon}}V^{\hat{\pi}}\right)(t,x)\\
=&\ \left(\mathscr{L}^{\pi^{\varepsilon}}V\right)(t,x).
\end{aligned}
$$

3. If $\tau_t^{\hat{\pi}}\geq\tau_t^{\pi^{\varepsilon}}$, noting that $\hat{\pi}\left(s,X_s^{\varepsilon}\right)\equiv 0$ for $s\geq\tau_t^{\pi^{\varepsilon}}$, we have

$$
\mathsf{E}_{t,x}\left[\int_{t+\varepsilon}^{\tau_t^{\pi^{\varepsilon}}}\left(h\left(s-t\right)-h\left(s-t-\varepsilon\right)\right)\hat{\pi}\left(s,X_s^{\varepsilon}\right)\mathrm{d}s\right]=\mathsf{E}_{t,x}\left[\int_{t+\varepsilon}^{\tau_t^{\hat{\pi}}}\left(h\left(s-t\right)-h\left(s-t-\varepsilon\right)\right)\hat{\pi}\left(s,X_s^{\varepsilon}\right)\mathrm{d}s\right].
$$

Thus,

$$
\begin{aligned}
&\mathsf{E}_{t,x}\left[\int_{t+\varepsilon}^{\tau_t^{\hat{\pi}}}\left(h\left(s-t\right)-h\left(s-t-\varepsilon\right)\right)\hat{\pi}\left(s,X_s^{\hat{\pi}}\right)\mathrm{d}s\right]\\
&-\mathsf{E}_{t,x}\left[\int_{t+\varepsilon}^{\tau_t^{\pi^{\varepsilon}}}\left(h\left(s-t\right)-h\left(s-t-\varepsilon\right)\right)\hat{\pi}\left(s,X_s^{\varepsilon}\right)\mathrm{d}s\right]\\
=&\mathsf{E}_{t,x}\left[\int_{t+\varepsilon}^{\tau_t^{\hat{\pi}}}\left(h\left(s-t\right)-h\left(s-t-\varepsilon\right)\right)\left[\hat{\pi}\left(s,X_s^{\hat{\pi}}\right)-\hat{\pi}\left(s,X_s^{\varepsilon}\right)\right]\mathrm{d}s\right].
\end{aligned}
$$

If $\tau_t^{\hat{\pi}}\leq\tau_t^{\pi^{\varepsilon}}$, it follows from $h\left(s-t\right)-h\left(s-t-\varepsilon\right)\leq 0$ and $\hat{\pi}\left(s,X_s^{\varepsilon}\right)\geq 0$ that

$$
\begin{aligned}
&\mathsf{E}_{t,x}\left[\int_{t+\varepsilon}^{\tau_t^{\hat{\pi}}}\left(h\left(s-t\right)-h\left(s-t-\varepsilon\right)\right)\hat{\pi}\left(s,X_s^{\hat{\pi}}\right)\mathrm{d}s\right]\\
&-\mathsf{E}_{t,x}\left[\int_{t+\varepsilon}^{\tau_t^{\pi^{\varepsilon}}}\left(h\left(s-t\right)-h\left(s-t-\varepsilon\right)\right)\hat{\pi}\left(s,X_s^{\varepsilon}\right)\mathrm{d}s\right]\\
\geq&\mathsf{E}_{t,x}\left[\int_{t+\varepsilon}^{\tau_t^{\hat{\pi}}}\left(h\left(s-t\right)-h\left(s-t-\varepsilon\right)\right)\left[\hat{\pi}\left(s,X_s^{\hat{\pi}}\right)-\hat{\pi}\left(s,X_s^{\varepsilon}\right)\right]\mathrm{d}s\right].
\end{aligned}
$$

Therefore, we always have

$$
\mathsf{E}_{t,x}\left[\int_{t+\varepsilon}^{\tau_t^{\hat{\pi}}}\left(h\left(s-t\right)-h\left(s-t-\varepsilon\right)\right)\hat{\pi}\left(s,X_s^{\hat{\pi}}\right)\mathrm{d}s\right]
$$

$$
\begin{aligned}
&- \mathsf{E}_{t,x}\left[\int_{t+\varepsilon}^{\tau_t^{\pi^\varepsilon}} \left(h\left(s-t\right)-h\left(s-t-\varepsilon\right)\right)\hat{\pi}\left(s,X_s^\varepsilon\right)\mathrm{d}s\right] \\
\geq &\mathsf{E}_{t,x}\left[\int_{t+\varepsilon}^{\tau_t^{\hat{\pi}}} \left(h\left(s-t\right)-h\left(s-t-\varepsilon\right)\right)\left[\hat{\pi}\left(s,X_s^{\hat{\pi}}\right)-\hat{\pi}\left(s,X_s^\varepsilon\right)\right]\mathrm{d}s\right].
\end{aligned}
$$

Furthermore, noting that $\hat{\pi}$ is bounded and $\int_0^\infty h(s)\mathrm{d}s < \infty$, by the dominated convergence theorem, we get

$$
\lim_{\varepsilon\downarrow 0}\frac{\mathsf{E}_{t,x}\left[\int_{t+\varepsilon}^{\tau_t^{\hat{\pi}}}\left[h\left(s-t\right)-h\left(s-t-\varepsilon\right)\right]\left(\hat{\pi}\left(s,X_s^{\hat{\pi}}\right)-\hat{\pi}\left(s,X_s^\varepsilon\right)\right)\mathrm{d}s\right]}{\varepsilon}=0.
$$

From (2.3.9) and the above three steps, we obtain that

$$
\lim_{\varepsilon\downarrow 0}\frac{V^{\hat{\pi}}(t,x)-V^{\pi^\varepsilon}(t,x)}{\varepsilon}\geq\left[\mathscr{L}^{\hat{\pi}}V\left(t,x\right)+C\left(t,t,\hat{\pi}(t,x)\right)\right]-\left[\mathscr{L}^{\pi^\varepsilon}V\left(t,x\right)+C\left(t,t,\pi^\varepsilon\left(t,x\right)\right)\right]. \tag{2.3.10}
$$

It follows from (2.3.3) and (2.3.4) that

$$
\left(\mathscr{L}^{\hat{\pi}}V\right)(t,x)+C\left(t,t,\hat{\pi}(t,x)\right)=\sup_{\pi\in\Pi}\left\{\left(\mathscr{L}^{\pi}V\right)(t,x)+C\left(t,t,\pi(t,x)\right)\right\}. \tag{2.3.11}
$$

Therefore, (2.3.10) and (2.3.11) imply that

$$
\lim_{\varepsilon\downarrow 0}\frac{V^{\hat{\pi}}(t,x)-V^{\pi^\varepsilon}(t,x)}{\varepsilon} \quad\geq\quad 0.
$$

This completes the proof. □

2.4 Solutions to Two Special Cases

In this section, we try to find a solution of the equilibrium HJB-equation in Definition 2.3.1 for specific discount functions. First of all, we make a conjecture of an equilibrium strategy for a general discount function. Since

$$
H(s,t,l,p,P) \quad = \quad \frac{1}{2}\sigma^2 P+(\mu-l)p+C(s,t,l)
$$

$$= \frac{1}{2}\sigma^2 P + \mu p + [h(t-s) - p]l,$$

we have

$$\phi(s,t,p,P) = \begin{cases} 0, & \text{if } p \geq h(t-s), \\ M, & \text{if } p < h(t-s). \end{cases}$$

We assume that there exists a constant $b \geq 0$ such that $\frac{\partial c}{\partial x}(t,t,x) \geq 1$, if $0 \leq x < b$, and $\frac{\partial c}{\partial x}(t,t,x) < 1$, if $x \geq b$. Thus, from Theorem 2.3.1, the equilibrium strategy would be given by

$$\hat{\pi}(t,x) = \phi\left(t,t,\frac{\partial c}{\partial x}(t,t,x),\frac{\partial^2 c}{\partial x^2}(t,t,x)\right) = \begin{cases} 0, & \text{if } 0 \leq x < b, \\ M, & \text{if } x \geq b. \end{cases} \tag{2.4.1}$$

Then the equilibrium HJB-equation (2.3.2) becomes

$$\begin{cases} \frac{\partial c}{\partial t}(s,t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2 c}{\partial x^2}(s,t,x) + \mu \frac{\partial c}{\partial x}(s,t,x) = 0, & (s,t,x) \in \mathscr{D}[0,\infty) \times (0,b), \\ \frac{\partial c}{\partial t}(s,t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2 c}{\partial x^2}(s,t,x) + (\mu - M)\frac{\partial c}{\partial x}(s,t,x) + h(t-s)M = 0, & (s,t,x) \in \mathscr{D}[0,\infty) \times [b,\infty), \\ c(s,t,0) = 0, \quad \forall (s,t) \in \mathscr{D}[0,\infty). \end{cases} \tag{2.4.2}$$

2.4.1 A Mixture of Exponential Discount Functions

Let us consider a case where the dividends are proportionally paid to N inhomogenous shareholders. The term *inhomogenous* refers to the assumption that the shareholders have different discount rates. Then given a control law π, the return function is

$$V^{\pi}(t,x) = \sum_{i=1}^{N} \mathsf{E}_{t,x}\left[\int_t^{\tau_t^{\pi}} \omega_i e^{-\delta_i(z-t)}\pi(z,X_z^{\pi})\mathrm{d}z\right],$$

where $\omega_i > 0$ satisfying $\sum_{i=1}^{N}\omega_i = 1$ is the proportion at which the dividends are paid to the shareholders, $\delta_i > 0$, $i = 1,2,\cdots,N$, are the constant discount rates of the shareholders, respectively.

In fact, a mixture of exponential discount functions is used in the above example. We consider a discount function defined by

$$h(t)=\sum_{i=1}^{N}\omega_i e^{-\delta_i t},\quad t\geq 0, \tag{2.4.3}$$

where $\delta_i>0$, $\delta_i\neq\delta_j$, for $i\neq j$, and $\omega_i>0$ satisfies $\sum_{i=1}^{N}\omega_i=1$.

We consider the following ansatz:

$$c(s,t,x) \;=\; \sum_{i=1}^{N}\omega_i e^{-\delta_i(t-s)}V_i(x),\quad (s,t,x)\in\mathscr{D}[0,\infty)\times[0,\infty), \tag{2.4.4}$$

where the functions $V_i(x)$, $i=1,2,\cdots,N$, are given by the system of ODEs

$$\begin{cases}\frac{1}{2}\sigma^2\frac{\partial^2 V_i}{\partial x^2}(x)+\mu\frac{\partial V_i}{\partial x}(x)-\delta_i V_i(x)=0, & x\in[0,b),\\ \frac{1}{2}\sigma^2\frac{\partial^2 V_i}{\partial x^2}(x)+(\mu-M)\frac{\partial V_i}{\partial x}(x)-\delta_i V_i(x)+M=0, & x\in[b,\infty),\\ V_i(0)=0.\end{cases} \tag{2.4.5}$$

Denote by $\theta_1(\eta,c)$ and $-\theta_2(\eta,c)$ the positive and negative roots of the equation $\frac{1}{2}\sigma^2y^2+\eta y-c=0$, respectively. Then

$$\begin{cases}\theta_1(\eta,c) &= \frac{-\eta+\sqrt{\eta^2+2\sigma^2c}}{\sigma^2},\\ \theta_2(\eta,c) &= \frac{\eta+\sqrt{\eta^2+2\sigma^2c}}{\sigma^2}.\end{cases}$$

Thus a general solution of the equation (2.4.5) has the form

$$V_i(x)=\begin{cases}C_{i1}e^{\theta_1(\mu,\delta_i)x}+C_{i2}e^{-\theta_2(\mu,\delta_i)x}, & x\in[0,b),\\ \frac{M}{\delta_i}+C_{i3}e^{\theta_1(\mu-M,\delta_i)x}+C_{i4}e^{-\theta_2(\mu-M,\delta_i)x}, & x\in[b,\infty),\end{cases} \tag{2.4.6}$$

for $i=1,2,\cdots,N$. From Theorem 2.3.1, we need to find a function $c(s,t,x)\in C^{0,1,2}\left(\mathscr{D}[0,\infty)\times[0,\infty)\right)$. Thus, in the following we shall find $V_i(x), i=1,2,\cdots,N$, which are C^2 functions.

Since $V_i(0)=0$, and $V_i(x)>0$, for all $x>0$, we have $C_{i1}=-C_{i2}:=C_i>0$, $i=1,2,\cdots,N$. Since we are looking for a bounded function $c(\cdot,\cdot,\cdot)$ (see Theorem 2.3.1), we have $C_{i3}=0$, $i=1,2,\cdots,N$. To simplify the notation, let $C_{i4}:=-d_i, i=1,2,\cdots,N$. Until now, we are not sure whether d_i is positive or not.

Now to find the value of $C_i, d_i, i = 1, 2, \cdots, N$ and b, we use "the principle of smooth fit" to get

$$\begin{cases} V_i(b+) & = V_i(b-), \quad i = 1, 2, \cdots, N, \\ V_i'(b+) & = V_i'(b-), \quad i = 1, 2, \cdots, N, \\ \frac{\partial c}{\partial x}(t, t, b+) & = 1 \ \left(\text{or equivalently, } \frac{\partial c}{\partial x}(t, t, b-) = 1\right). \end{cases} \tag{2.4.7}$$

Therefore by denoting

$$\theta_{i1} = \theta_1(\mu, \delta_i),\ \theta_{i2} = \theta_2(\mu, \delta_i),\ \theta_{i3} = \theta_2(\mu - M, \delta_i), \quad i = 1, 2, \cdots, N,$$

we can rewrite (2.4.7) as for $i = 1, 2, \cdots, N$,

$$C_i \left(e^{\theta_{i1} b} - e^{-\theta_{i2} b}\right) = \frac{M}{\delta_i} - d_i e^{-\theta_{i3} b}, \tag{2.4.8}$$

$$C_i \left(\theta_{i1} e^{\theta_{i1} b} + \theta_{i2} e^{-\theta_{i2} b}\right) = d_i \theta_{i3} e^{-\theta_{i3} b}, \tag{2.4.9}$$

and

$$\sum_{i=1}^{N} \omega_i C_i \left(\theta_{i1} e^{\theta_{i1} b} + \theta_{i2} e^{-\theta_{i2} b}\right) = 1. \tag{2.4.10}$$

From (2.4.8) - (2.4.9) we can get C_i and d_i in the expression of b:

$$C_i = \frac{M\theta_{i3}}{\delta_i} \left[(\theta_{i1} + \theta_{i3}) e^{\theta_{i1} b} + (\theta_{i2} - \theta_{i3}) e^{-\theta_{i2} b}\right]^{-1}, \tag{2.4.11}$$

$$d_i = \frac{M}{\delta_i} e^{\theta_{i3} b} \frac{\theta_{i1} e^{\theta_{i1} b} + \theta_{i2} e^{-\theta_{i2} b}}{(\theta_{i1} + \theta_{i3}) e^{\theta_{i1} b} + (\theta_{i2} - \theta_{i3}) e^{-\theta_{i2} b}}, \tag{2.4.12}$$

for $i = 1, 2, \cdots, N$. Since $C_i > 0$, we can directly see that $d_i > 0$, for $i = 1, 2, \cdots, N$.

Substituting C_i into (2.4.10), we obtain

$$\sum_{i=1}^{N} \omega_i \frac{M\theta_{i3}}{\delta_i} \frac{\theta_{i1} e^{\theta_{i1} b} + \theta_{i2} e^{-\theta_{i2} b}}{(\theta_{i1} + \theta_{i3}) e^{\theta_{i1} b} + (\theta_{i2} - \theta_{i3}) e^{-\theta_{i2} b}} = 1.$$

Let

$$F(b) := \sum_{i=1}^{N} \omega_i \frac{M\theta_{i3}}{\delta_i} \frac{\theta_{i1} e^{\theta_{i1} b} + \theta_{i2} e^{-\theta_{i2} b}}{(\theta_{i1} + \theta_{i3}) e^{\theta_{i1} b} + (\theta_{i2} - \theta_{i3}) e^{-\theta_{i2} b}} - 1.$$

Lemma 2.4.1. *If* $\sum_{i=1}^{N} \omega_i \frac{M\theta_{i3}}{\delta_i} > 1$*, then* $F(b) = 0$ *has a unique positive solution.*

Proof. The condition $\sum_{i=1}^{N} \omega_i \frac{M\theta_{i3}}{\delta_i} > 1$ implies that $F(0) > 0$. From Lemma 2.1 of Asmussen and Taksar (1997), we know that

$$\frac{M}{\delta_i} - \frac{1}{\theta_{i3}} - \frac{1}{\theta_{i1}} < 0, \quad i = 1, 2, \cdots, N.$$

Thus,

$$\begin{aligned}
F(+\infty) &= \sum_{i=1}^{N} \omega_i \frac{M\theta_{i3}}{\delta_i} \frac{\theta_{i1}}{\theta_{i1} + \theta_{i3}} - 1 \\
&= \sum_{i=1}^{N} \omega_i \left(\frac{M\theta_{i3}}{\delta_i} \frac{\theta_{i1}}{\theta_{i1} + \theta_{i3}} - 1 \right) \\
&= \sum_{i=1}^{N} \omega_i \frac{\theta_{i1}\theta_{i3}}{\theta_{i1} + \theta_{i3}} \left(\frac{M}{\delta_i} - \frac{1}{\theta_{i3}} - \frac{1}{\theta_{i1}} \right) \\
&< 0.
\end{aligned}$$

Furthermore, we have

$$F'(b) = \sum_{i=1}^{N} \omega_i \frac{M\theta_{i3}}{\delta_i} \frac{\Delta_i}{\left[(\theta_{i1} + \theta_{i3}) e^{\theta_{i1} b} + (\theta_{i2} - \theta_{i3}) e^{-\theta_{i2} b} \right]^2},$$

where

$$\begin{aligned}
\Delta_i &= \left(\theta_{i1}^2 e^{\theta_{i1} b} - \theta_{i2}^2 e^{-\theta_{i2} b} \right) \left[(\theta_{i1} + \theta_{i3}) e^{\theta_{i1} b} + (\theta_{i2} - \theta_{i3}) e^{-\theta_{i2} b} \right] \\
&\quad - \left(\theta_{i1} e^{\theta_{i1} b} + \theta_{i2} e^{-\theta_{i2} b} \right) \left[\theta_{i1} (\theta_{i1} + \theta_{i3}) e^{\theta_{i1} b} - \theta_{i2} (\theta_{i2} - \theta_{i3}) e^{-\theta_{i2} b} \right] \\
&= \left[\theta_{i1}^2 (\theta_{i2} - \theta_{i3}) - \theta_{i2}^2 (\theta_{i1} + \theta_{i3}) + \theta_{i1}\theta_{i2} (\theta_{i2} - \theta_{i3}) - \theta_{i2}\theta_{i1} (\theta_{i1} + \theta_{i3}) \right] e^{(\theta_{i1} - \theta_{i2}) b} \\
&= \left[\theta_{i1} (\theta_{i2} - \theta_{i3}) - \theta_{i2} (\theta_{i1} + \theta_{i3}) \right] (\theta_{i1} + \theta_{i2}) e^{(\theta_{i1} - \theta_{i2}) b} \\
&= -\theta_{i3} (\theta_{i1} + \theta_{i2})^2 e^{(\theta_{i1} - \theta_{i2}) b} \\
&< 0.
\end{aligned}$$

Therefore, the equation $F(b) = 0$ admits a unique solution on $(0, \infty)$. □

Theorem 2.4.1. *Given the discount function (2.4.3), there exists a bounded function* $c(\cdot, \cdot, \cdot) \in$

$C^{0,1,2}(\mathscr{D}[0,\infty)\times[0,\infty))$ *satisfying the equilibrium HJB-equation (2.3.2).*

(i) If $\sum_{i=1}^N \omega_i \frac{M\theta_{i3}}{\delta_i}\leq 1$, *then* $b=0$ *and the function* $c(\cdot,\cdot,\cdot)$ *is given by*

$$c(s,t,x)=\sum_{i=1}^N \omega_i e^{-\delta_i(t-s)}\frac{M}{\delta_i}\left(1-e^{-\theta_{i3}x}\right),\quad x\in[0,\infty). \tag{2.4.13}$$

(ii) If $\sum_{i=1}^N \omega_i \frac{M\theta_{i3}}{\delta_i}>1$, *then*

$$c(s,t,x)=\begin{cases}\sum_{i=1}^N \omega_i e^{-\delta_i(t-s)}C_i\left(e^{\theta_{i1}x}-e^{-\theta_{i2}x}\right), & x\in[0,b),\\ \sum_{i=1}^N \omega_i e^{-\delta_i(t-s)}\left(\frac{M}{\delta_i}-d_ie^{-\theta_{i3}x}\right), & x\in[b,\infty),\end{cases} \tag{2.4.14}$$

where $C_i,d_i,i=1,2,\cdots,N$, *and* b *is the unique solution to the system (2.4.8)-(2.4.10).*

Proof. (i) It is easy to check the function $c(s,t,x)$ given by (2.4.13) and $b=0$ satisfy the system of ODEs (2.4.2). Obviously, we have

$$\begin{aligned}\frac{\partial c}{\partial x}(s,t,0) &= \sum_{i=1}^N \omega_i e^{-\delta_i(t-s)}\frac{M}{\delta_i}\theta_{i3}\leq 1,\quad (s,t)\in\mathscr{D}[0,\infty),\\ \frac{\partial^2 c}{\partial x^2}(s,t,x) &= -\sum_{i=1}^N \omega_i e^{-\delta_i(t-s)}\frac{M}{\delta_i}\theta_{i3}^2e^{-\theta_{i3}x}<0,\quad (s,t,x)\in\mathscr{D}[0,\infty)\times[0,\infty).\end{aligned}$$

Thus, $\frac{\partial c}{\partial x}(t,t,x)<1$, for $x\geq 0$, which implies $c(\cdot,\cdot,\cdot)$ satisfies the equilibrium HJB-equation (2.3.2).

(ii) Similarly, it is easy to check that b and $c(\cdot,\cdot,\cdot,)$ given by (2.4.8)-(2.4.10) and (2.4.14) satisfy the system of ODEs (2.4.2). It is sufficient to show

$$\begin{cases}\frac{\partial c}{\partial x}(t,t,x)\geq 1, & x\in[0,b),\\ \frac{\partial c}{\partial x}(t,t,x)<1, & x\in[b,\infty).\end{cases} \tag{2.4.15}$$

The first and second derivatives of $c(s,t,x)$ given by (2.4.14) with respective to x are

$$\frac{\partial c}{\partial x}(t,t,x)=\begin{cases}\sum_{i=1}^N \omega_iC_i\left(\theta_{i1}e^{\theta_{i1}x}+\theta_{i2}e^{-\theta_{i2}x}\right), & (t,x)\in[0,\infty)\times[0,b),\\ \sum_{i=1}^N \omega_id_i\theta_{i3}e^{-\theta_{i3}x}, & (t,x)\in[0,\infty)\times[b,\infty),\end{cases}$$

and

$$\frac{\partial^2 c}{\partial x^2}(t,t,x) = \begin{cases} \sum_{i=1}^N \omega_i C_i \left(\theta_{i1}^2 e^{\theta_{i1}x} - \theta_{i2}^2 e^{-\theta_{i2}x}\right), & (t,x) \in [0,\infty)\times[0,b), \\ -\sum_{i=1}^N \omega_i d_i \theta_{i3}^2 e^{-\theta_{i3}x}, & (t,x) \in [0,\infty)\times[b,\infty), \end{cases}$$

respectively.

It is easy to check that $\frac{\partial c}{\partial x}(t,t,x) > 0$, for all $(t,x) \in [0,\infty)\times[0,\infty)$, which implies that $c(t,t,\cdot)$ is strictly increasing. Next we show that $c(t,t,\cdot)$ is a concave function on $[0,\infty)$, i.e. $\frac{\partial^2 c}{\partial x^2}(t,t,x) < 0$, for all $(t,x) \in [0,\infty)\times[0,\infty)$. First we show that $\frac{\partial^2 c}{\partial x^2}(t,t,x)$ is continuous at $x = b$. Apparently, $\frac{\partial^2 c}{\partial x^2}(t,t,x) < 0$, for all $(t,x) \in [0,\infty)\times[b,\infty)$. Recalling (2.4.4), (2.4.5) and (2.4.7), we have

$$\frac{1}{2}\sigma^2 \frac{\partial^2 c}{\partial x^2}(t,t,b-) = -\mu \frac{\partial c}{\partial x}(t,t,b) + \sum_{i=1}^N \omega_i \delta_i V_i(b),$$
$$\frac{1}{2}\sigma^2 \frac{\partial^2 c}{\partial x^2}(t,t,b+) = -(\mu - M)\frac{\partial c}{\partial x}(t,t,b) + \sum_{i=1}^N \omega_i \delta_i V_i(b) - M.$$

Since $\frac{\partial c}{\partial x}(t,t,b) = 1$, we get $\frac{\partial^2 c}{\partial x^2}(t,t,b-) = \frac{\partial^2 c}{\partial x^2}(t,t,b+) = \frac{\partial^2 c}{\partial x^2}(t,t,b)$.

Obviously, for all $0 \le x \le b$, we have

$$\frac{\partial^3 c}{\partial x^3}(t,t,x) = \sum_{i=1}^N \omega_i C_i \left(\theta_{i1}^3 e^{\theta_{i1}x} + \theta_{i2}^3 e^{-\theta_{i2}x}\right) > 0,$$

which means that $\frac{\partial^2 c}{\partial x^2}(t,t,x) \le \frac{\partial^2 c}{\partial x^2}(t,t,b) < 0$, for all $0 \le x \le b$. Thus, we proved (2.4.15). $\square$

Corollary 2.4.1. *Consider the discount function (2.4.3).*

(i) If $\sum_{i=1}^N \omega_i \frac{M\theta_{i3}}{\delta_i} \le 1$, then for $t \in [0,\infty)$

$$\hat{\pi}(t,x) = \phi\left(t,t,\frac{\partial c}{\partial x}(t,t,x),\frac{\partial^2 c}{\partial x^2}(t,t,x)\right) = M, \quad x \in [0,\infty),$$

is an equilibrium dividend strategy, and

$$V(t,x) = c(t,t,x) = \sum_{i=1}^{N} \omega_i \frac{M}{\delta_i}\left(1 - e^{-\theta_{i3}x}\right), \quad x \in [0,\infty),$$

is the corresponding equilibrium value function.

(ii) If $\sum_{i=1}^{N} \omega_i \frac{M\theta_{i3}}{\delta_i} > 1$, *then for* $t \in [0,\infty)$

$$\hat{\pi}(t,x) = \phi\left(t,t,\frac{\partial c}{\partial x}(t,t,x),\frac{\partial^2 c}{\partial x^2}(t,t,x)\right) = \begin{cases} 0, & x \in [0,b), \\ M, & x \in [b,\infty), \end{cases}$$

is an equilibrium dividend strategy, and

$$V(t,x) = c(t,t,x) = \begin{cases} \sum_{i=1}^{N} \omega_i C_i\left(e^{\theta_{i1}x} - e^{-\theta_{i2}x}\right), & x \in [0,b), \\ \sum_{i=1}^{N} \omega_i \left(\frac{M}{\delta_i} - d_i e^{-\theta_{i3}x}\right), & x \in [b,\infty), \end{cases}$$

is the corresponding equilibrium value function. Here $C_i, d_i, i = 1,2,\cdots,N$, *and* b *is the unique solution to the system (2.4.8)-(2.4.10).*

Proof. By Theorem 2.3.1 and Theorem 2.4.1, it is sufficient to verify (2.3.6). If $M \geq \mu$, in both cases (i) and (ii), it is well known that $\mathsf{P}\left(\tau_t^{\hat{\pi}} < \infty\right) = 1$ (see, e.g. Gerber and Shiu (2006)). Since $c(s,t,0) = 0$ for all $(s,t) \in \mathscr{D}[0,\infty)$,we get (2.3.6). If $M < \mu$, in both cases (i) and (ii), we have $\mathsf{P}\left(\tau_t^{\hat{\pi}} = \infty\right) > 0$ and $X_{\tau_t^{\hat{\pi}}}^{\hat{\pi}} = +\infty$ on $\{\tau_t^{\hat{\pi}} = \infty\}$. However, for any $s \in [0,\infty)$ we have $\lim_{t\to\infty,x\to\infty} c(s,t,x) = 0$. Thus, we still have (2.3.6). □

Under the mixture of exponential discount functions, Corollary 2.4.1 shows that if the bound of the dividend rate is small enough then the equilibrium strategy is to always pay the maximal dividend rate; otherwise, the equilibrium strategy is to pay the maximal dividend rate when the surplus is above a barrier while paying nothing when the surplus is below this barrier. These features of the equilibrium dividend strategies are similar to the optimal strategies obtained in Asmussen and Taksar (1997) which considered an exponential discount function in a diffusion risk model.

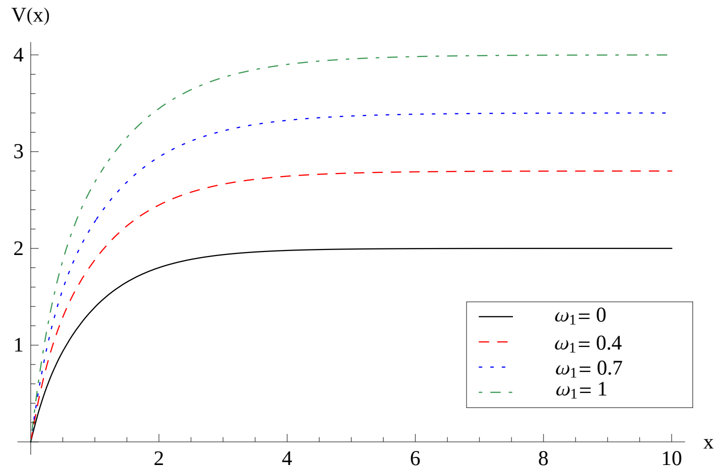

Figure 2.4.1: Equilibrium value functions with a mixture of exponential discount functions

Example 2.4.1. *Let* $N = 2$*,* $\mu = 1$*,* $\sigma = 1$*,* $M = 0.8$*,* $\delta_1 = 0.2$*,* $\delta_2 = 0.4$*. Figure 2.4.1 illustrates the equilibrium value functions for the mixture of exponential discount functions with* $\omega_1 = 0$*, 0.4, 0.7 and 1 and* $\omega_2 = 1 - \omega_1$*. The barriers are* 0.6525*,* 0.8781*,* 1.0207 *and* 1.1452*, respectively. The cases with* $\omega_1 = 0$ *and 1 are time consistent and the equilibrium strategies are optimal.*

2.4.2 A Pseudo-Exponential Discount Function

We now consider a pseudo-exponential discount function defined as

$$h(t) = (1 + \lambda t)e^{-\delta t}, \quad t \geq 0, \tag{2.4.16}$$

where $\lambda > 0$, $\delta > 0$ are parameters. We refer the reader to Ekeland and Pirvu (2008) for explanations of this discount function. To ensure h is decreasing, we assume that $\lambda < \delta$. To simplify the calculations, we shall impose more conditions on λ in the following.

We consider the following ansatz:

$$c(s,t,x) = e^{-\delta(t-s)}\{\lambda(t-s)V_3(x) + V_4(x)\}, \quad (s,t,x) \in \mathscr{D}[0,\infty) \times [0,\infty), \tag{2.4.17}$$

where $V_3(\cdot)$ and $V_4(\cdot)$ are given by

$$\begin{cases} \frac{1}{2}\sigma^2 \frac{\partial^2 V_3}{\partial x^2}(x) + \mu \frac{\partial V_3}{\partial x}(x) - \delta V_3(x) = 0, & x \in [0,b), \\ \frac{1}{2}\sigma^2 \frac{\partial^2 V_3}{\partial x^2}(x) + (\mu - M) \frac{\partial V_3}{\partial x}(x) - \delta V_3(x) + M = 0, & x \in [b,\infty), \\ V_3(0) = 0, \end{cases} \tag{2.4.18}$$

and

$$\begin{cases} \frac{1}{2}\sigma^2 \frac{\partial^2 V_4}{\partial x^2}(x) + \mu \frac{\partial V_4}{\partial x}(x) - \delta V_4(x) + \lambda V_3(x) = 0, & x \in [0,b), \\ \frac{1}{2}\sigma^2 \frac{\partial^2 V_4}{\partial x^2}(x) + (\mu - M) \frac{\partial V_4}{\partial x}(x) - \delta V_4(x) + \lambda V_3(x) + M = 0, & x \in [b,\infty), \\ V_4(0) = 0, \end{cases} \tag{2.4.19}$$

respectively. It is easy to check that the function $c(\cdot,\cdot,\cdot)$ given by (2.4.17)-(2.4.19) satisfies the system (2.4.2).

Recalling the situation we discussed in Subsection 2.4.1, the equation (2.4.18) has a general solution

$$V_3(x) = \begin{cases} C\left(e^{\theta_1(\mu)x} - e^{-\theta_2(\mu)x}\right), & x \in [0,b), \\ \frac{M}{\delta} - de^{-\theta_2(\mu-M)x}, & x \in [b,\infty), \end{cases} \tag{2.4.20}$$

where $C > 0,\ d > 0$ are two unknown constants to be determined, $\theta_1(\eta)$ and $-\theta_2(\eta)$ are the positive and negative roots of the equation $\frac{1}{2}\sigma^2 y^2 + \eta y - \delta = 0$, respectively.

According to "the principle of smooth fit", we have

$$\begin{cases} V_3(b+) &= V_3(b-), \\ V_3'(b+) &= V_3'(b-), \end{cases} \tag{2.4.21}$$

which yields that

$$C = \frac{M\theta_3}{\delta}\left[(\theta_1 + \theta_3)e^{\theta_1 b} + (\theta_2 - \theta_3)e^{-\theta_2 b}\right]^{-1}, \tag{2.4.22}$$

$$d = \frac{M}{\delta}e^{\theta_3 b}\frac{\theta_1 e^{\theta_1 b} + \theta_2 e^{-\theta_2 b}}{(\theta_1 + \theta_3)e^{\theta_1 b} + (\theta_2 - \theta_3)e^{-\theta_2 b}}, \tag{2.4.23}$$

where

$$\theta_1 = \theta_1(\mu), \quad \theta_2 = \theta_2(\mu), \quad \theta_3 = \theta_2(\mu - M).$$

After obtaining V_3, solving ODE (2.4.19) yields that

$$V_4(x) = \begin{cases} (D_1 - B_1 x)\, e^{\theta_1 x} + (D_2 + B_2 x)\, e^{-\theta_2 x}, & 0 \le x < b, \\ \frac{M}{\delta}\left(1 + \frac{\lambda}{\delta}\right) + (D_3 + B_3 x)\, e^{-\theta_3 x}, & x \ge b, \end{cases} \tag{2.4.24}$$

where

$$B_1 = \frac{\lambda C}{\mu + \sigma^2 \theta_1} > 0, \quad B_2 = \frac{\lambda C}{\mu - \sigma^2 \theta_2} < 0, \quad B_3 = \frac{\lambda d}{\mu - M - \sigma^2 \theta_3} < 0. \tag{2.4.25}$$

Since $V_4(0) = 0$, we have $D_1 = -D_2 := \hat{C}$. Also noting that $B_1 + B_2 = 0$, we rewrite (2.4.24) as

$$V_4(x) = \begin{cases} \left(\hat{C} - B_1 x\right) e^{\theta_1 x} - \left(\hat{C} + B_1 x\right) e^{-\theta_2 x}, & 0 \le x < b, \\ \frac{M}{\delta}\left(1 + \frac{\lambda}{\delta}\right) + (D_3 + B_3 x)\, e^{-\theta_3 x}, & x \ge b. \end{cases} \tag{2.4.26}$$

Applying the principle of smooth fit for determining a candidate b, we obtain

$$\begin{cases} V_4(b+) & = V_4(b-), \\ V_4'(b+) & = V_4'(b-), \\ \frac{\partial c}{\partial x}(t,t,b+) & = 1 \ \left(\text{or equivalently, } \frac{\partial c}{\partial x}(t,t,b-) = 1\right). \end{cases} \tag{2.4.27}$$

From the first two equations in (2.4.27), we obtain

$$\hat{C} = \frac{[(\theta_1 + \theta_3)\, b + 1]\, B_1 e^{\theta_1 b} - [(\theta_2 - \theta_3)\, b - 1]\, B_1 e^{-\theta_2 b} + B_3 e^{-\theta_3 b} + \theta_3 \left(1 + \frac{\lambda}{\delta}\right) \frac{M}{\delta}}{(\theta_1 + \theta_3)\, e^{\theta_1 b} + (\theta_2 - \theta_3)\, e^{-\theta_2 b}}, \tag{2.4.28}$$

$$D_3 = e^{\theta_3 b}\left[\left(\hat{C} - B_1 b\right) e^{\theta_1 b} - \left(\hat{C} + B_1 b\right) e^{-\theta_2 b} - \left(1 + \frac{\lambda}{\delta}\right)\frac{M}{\delta}\right] - B_3 b. \tag{2.4.29}$$

Furthermore, for a b such that $\frac{\partial c}{\partial x}(t,t,b+) = \frac{\partial c}{\partial x}(t,t,b-) = 1$, we have

$$\left[\left(\hat{C} - B_1 b\right)\theta_1 - B_1\right] e^{\theta_1 b} + \left[\left(\hat{C} + B_1 b\right)\theta_2 - B_1\right] e^{-\theta_2 b} - 1 = 0, \tag{2.4.30}$$

i.e.,

$$\hat{C} = \frac{1 + (\theta_1 b + 1) B_1 e^{\theta_1 b} - (\theta_2 b - 1) B_1 e^{-\theta_2 b}}{\theta_1 e^{\theta_1 b} + \theta_2 e^{-\theta_2 b}},$$

and

$$(-\theta_3 D_3 - \theta_3 B_3 b + B_3)\, e^{-\theta_3 b} - 1 = 0,$$

i.e.,

$$D_3 = \frac{1}{\theta_3}\left(B_3 - e^{\theta_3 b}\right) - B_3 b. \tag{2.4.31}$$

Putting (2.4.28) and (2.4.29) into the left-hand-side of (2.4.30), it can be rewritten as

$$\left[(\theta_1 + \theta_3)\, e^{\theta_1 b} + (\theta_2 - \theta_3)\, e^{-\theta_2 b}\right]^{-1} G(b),$$

where

$$\begin{aligned} G(b) \quad := \quad & -\theta_3 B_1 e^{2\theta_1 b} + \theta_3 B_1 e^{-2\theta_2 b} + \theta_1 B_3 e^{(\theta_1 - \theta_3) b} \\ & + \theta_2 B_3 e^{-(\theta_2 + \theta_3) b} + 2\,(\theta_1 + \theta_2)\,\theta_3 B_1 b e^{(\theta_1 - \theta_2) b} \\ & + \left[\theta_1 \theta_3 \left(1 + \frac{\lambda}{\delta}\right)\frac{M}{\delta} - (\theta_1 + \theta_3)\right] e^{\theta_1 b} \\ & + \left[\theta_2 \theta_3 \left(1 + \frac{\lambda}{\delta}\right)\frac{M}{\delta} - (\theta_2 - \theta_3)\right] e^{-\theta_2 b}, \end{aligned} \tag{2.4.32}$$

and

$$G(0) = (\theta_1 + \theta_2)\left\{\left[\frac{\lambda}{\mu - M - \sigma^2 \theta_3} + \theta_3\left(1 + \frac{\lambda}{\delta}\right)\right]\frac{M}{\delta} - 1\right\}.$$

Lemma 2.4.2. *If*

$$\frac{\frac{\delta}{M} - \theta_3}{\frac{1}{\mu - M - \sigma^2 \theta_3} + \frac{\theta_3}{\delta}} < \lambda < \frac{(\theta_1 + \theta_3)\left[\frac{\delta}{M}(\theta_1 + \theta_3) - \theta_1 \theta_3\right]}{\theta_1^2\left(\frac{1}{\mu - M - \sigma^2 \theta_3} + \frac{\theta_3}{\delta}\right) + \theta_3^2\left(\frac{\theta_1}{\delta} - \frac{1}{\mu + \sigma^2 \theta_1}\right)}, \tag{2.4.33}$$

then $G(b) = 0$ *has a positive solution.*

Lemma 2.4.3. *If*

$$\lambda \leq \frac{\theta_1+\theta_2}{\theta_1+3\theta_2}\frac{\delta^2}{M\theta_3} \wedge \frac{(\theta_1+\theta_3)(\theta_1+\theta_2)}{2\theta_1(\theta_1+2\theta_2)}\frac{\delta^2}{M\theta_3}, \tag{2.4.34}$$

then

$$\theta_1\hat{C} - 3B_1 - \theta_1 B_1 b > 0,$$

where $b > 0$ such that $G(b) = 0$.

The proofs of Lemma 2.4.2 and Lemma 2.4.3 are shown in Appendix A.1 and Appendix A.2, respectively. Now we show the main result of this subsection in the following theorem.

Theorem 2.4.2. *Assume that $0 < \lambda < \delta$. Given the discount function (2.4.16), there exists a function $c(\cdot,\cdot,\cdot) \in C^{0,1,2}(\mathscr{D}[0,\infty) \times [0,\infty))$ satisfying the equilibrium HJB-equation (2.3.2).*

(i) If $\frac{\delta}{M} > \theta_3$ and $\lambda \leq \left(\frac{\delta}{M} - \theta_3\right)\left[\frac{1}{\mu-M-\sigma^2\theta_3} + \frac{\theta_3}{\delta}\right]^{-1}$, then $b = 0$ and $c(\cdot,\cdot,\cdot)$ is given by (2.4.17) with

$$\begin{aligned} V_3(x) &= \frac{M}{\delta}\left(1 - e^{-\theta_3 x}\right), \quad x \in [0,\infty), \\ V_4(x) &= \left(1+\frac{\lambda}{\delta}\right)\frac{M}{\delta} + \frac{M}{\delta}\left[\frac{\lambda}{\mu-M-\sigma^2\theta_3}x - \left(1+\frac{\lambda}{\delta}\right)\right]e^{-\theta_3 x}, \quad x \in [0,\infty). \end{aligned} \tag{2.4.35}$$

(ii) If (2.4.33) and (2.4.34) hold, and b is a positive solution to $G(b) = 0$, then $c(\cdot,\cdot,\cdot)$ is given by (2.4.17) with

$$\begin{aligned} V_3(x) &= \begin{cases} C\left(e^{\theta_1 x} - e^{-\theta_2 x}\right), & x \in [0,b), \\ \frac{M}{\delta} - de^{-\theta_3 x}, & x \in [b,\infty), \end{cases} \\ V_4(x) &= \begin{cases} \left(\hat{C} - B_1 x\right)e^{\theta_1 x} - \left(\hat{C} + B_1 x\right)e^{-\theta_2 x}, & x \in [0,b), \\ \left(1+\frac{\lambda}{\delta}\right)\frac{M}{\delta} + (D_3 + B_3 x)e^{-\theta_3 x}, & x \in [b,\infty), \end{cases} \end{aligned} \tag{2.4.36}$$

where $\left(b, C, d, \hat{C}, B_1, B_3, D_3\right)$ is a solution to (2.4.21) and (2.4.27).

Proof. It is easy to check that the function $c(\cdot,\cdot,\cdot)$ given by (2.4.17)-(2.4.19) satisfies the system (2.4.2). To prove $c(\cdot,\cdot,\cdot)$ satisfies the equilibrium HJB-equation (2.3.2), it is sufficient to show

$$\begin{cases} \frac{\partial c}{\partial x}(t,t,x) \geq 1, & x \in [0,b), \\ \frac{\partial c}{\partial x}(t,t,x) < 1, & x \in [b,\infty). \end{cases} \tag{2.4.37}$$

(i) Firstly, we show that the function V_4 defined by (2.4.35) is a concave function. Recalling Lemma A.1.1 and $\lambda > 0$, we obtain

$$\begin{aligned} V_4'(x) &= \frac{M}{\delta}\left(\frac{\lambda}{\mu - M - \sigma^2\theta_3} + \theta_3\left(1 + \frac{\lambda}{\delta}\right) - \theta_3 \frac{\lambda}{\mu - M - \sigma^2\theta_3} x\right) e^{-\theta_3 x} \\ &\geq \frac{M}{\delta}\left(\frac{\lambda}{\mu - M - \sigma^2\theta_3} + \theta_3\left(1 + \frac{\lambda}{\delta}\right)\right) e^{-\theta_3 x} \\ &= \frac{M}{\delta}\left[\left(\frac{1}{\mu - M - \sigma^2\theta_3} + \frac{\theta_3}{\delta}\right)\lambda + \theta_3\right] e^{-\theta_3 x} > 0. \end{aligned}$$

Also note that $V_3(0) = V_4(0) = 0$ and $V_4'(0) = \left[\frac{\lambda}{\mu - M - \sigma^2\theta_3} + \theta_3\left(1 + \frac{\lambda}{\delta}\right)\right]\frac{M}{\delta} \in (0,1]$. Recalling the second equation of (2.4.19), we have

$$\begin{aligned} \frac{1}{2}\sigma^2 V_4''(0) &= -(\mu - M)V_4'(0) + \delta V_4(0) - \lambda V_3(0) - M \\ &= -(\mu - M)V_4'(0) - M \\ &= -\mu V_4'(0) + M\left(V_4'(0) - 1\right) \\ &< 0. \end{aligned}$$

Thus,

$$\begin{aligned} V_4''(x) &= -\theta_3 \frac{M}{\delta}\left(\frac{2\lambda}{\mu - M - \sigma^2\theta_3} + \theta_3\left(1 + \frac{\lambda}{\delta}\right) - \theta_3 \frac{\lambda}{\mu - M - \sigma^2\theta_3} x\right) e^{-\theta_3 x} \\ &= \left[V_4''(0) + \theta_3^2 \frac{M}{\delta} \frac{\lambda}{\mu - M - \sigma^2\theta_3} x\right] e^{-\theta_3 x} < 0. \end{aligned}$$

Therefore $\frac{\partial c}{\partial x}(t,t,x) = V_4'(x) \leq 1$ for all $x > 0$.

(ii) For $x \geq b$, recalling (2.4.31),

$$\begin{aligned} V_4'(x) &= (-\theta_3 D_3 + B_3 - \theta_3 B_3 x)\, e^{-\theta_3 x} \\ &\geq (-\theta_3 D_3 + B_3 - \theta_3 B_3 b)\, e^{-\theta_3 x} \end{aligned}$$

$$= -\left[\theta_3 \left(D_3 + B_3 b\right) - B_3\right] e^{-\theta_3 x}$$
$$= e^{\theta_3 (b-x)} > 0,$$

and

$$\begin{aligned}
V_4''(x) &= \theta_3 \left(\theta_3 D_3 - 2B_3 + \theta_3 B_3 x\right) e^{-\theta_3 x} \\
&\leq \theta_3 \left[\theta_3 \left(D_3 + B_3 b\right) - 2B_3\right] e^{-\theta_3 x} \\
&= \theta_3 \left(-B_3 - e^{\theta_3 b}\right) e^{-\theta_3 x} \\
&< \theta_3 \left(-\frac{\lambda}{\mu - M - \sigma^2 \theta_3} \frac{M}{\delta} - 1\right) e^{\theta_3 (b-x)} \\
&\leq \theta_3 \left(\theta_3 \frac{M}{\delta} \frac{\lambda}{\delta} - 1\right) e^{\theta_3 (b-x)}.
\end{aligned}$$

The last inequality follows from Lemma A.1.1. Furthermore, by (2.4.34), we have $\theta_3 \frac{M}{\delta} \frac{\lambda}{\delta} - 1 \leq 0$. Therefore, $V_4''(x) < 0$, for $x \geq b$.

Now we deal with the case when $0 \leq x < b$. It follows from (2.4.19) and (2.4.27) that

$$\begin{aligned}
&\frac{1}{2}\sigma^2 V_4''(b-) = -\mu V_4'(b) + \delta V_4(b) - \lambda V_3(b), \\
&\frac{1}{2}\sigma^2 V_4''(b+) = -\left(\mu - M\right) V_4'(b) + \delta V_4(b) - \lambda V_3(b) - M \\
&\qquad\qquad = -\mu V_4'(b) + \delta V_4(b) - \lambda V_3(b),
\end{aligned}$$

which yields that $V_4''(b+) = V_4''(b-) = V_4''(b)$. Furthermore, for $0 \leq x < b$,

$$\begin{aligned}
V_4'''(x) &= \theta_1^2 \left[\theta_1 \hat{C} - 3B_1 - \theta_1 B_1 x\right] e^{\theta_1 x} + \theta_2^2 \left[\theta_2 \hat{C} - 3B_1 + \theta_2 B_1 x\right] e^{-\theta_2 x} \\
&> \theta_1^2 \left[\theta_1 \hat{C} - 3B_1 - \theta_1 B_1 b\right] e^{\theta_1 x} + \theta_2^2 \left[\theta_2 \hat{C} - 3B_1\right] e^{-\theta_2 x}.
\end{aligned}$$

It follows from Lemma 2.4.3 that if (2.4.34) holds, then $V_4'''(x) > 0$ for $0 \leq x < b$. Since $V_4''(x)$ is continuous at $x = b$ and $V_4''(b) < 0$, we get that $V_4''(x) < 0$, for $0 \leq x < b$. Therefore, $c(t,t,x) = V_4(x)$ is a concave function on $(0,\infty)$, which together with (2.4.27) implies (2.4.37). □

Similar to the proof of Corollary 2.4.1, it is easy to verify (2.3.6). We have the following corollary immediately by Theorem 2.3.1 and Theorem 2.4.2.

Corollary 2.4.2. *Assume that* $0 < \lambda < \delta$*. Consider the discount function (2.4.16).*

(i) If $\frac{\delta}{M} > \theta_3$ *and* $\lambda \leq \left(\frac{\delta}{M} - \theta_3\right)\left[\frac{1}{\mu - M - \sigma^2\theta_3} + \frac{\theta_3}{\delta}\right]^{-1}$*, then for* $(t,x) \in [0,\infty) \times [0,\infty)$*,*

$$\hat{\pi}(t,x) = \phi\left(t, t, \frac{\partial c}{\partial x}(t,t,x), \frac{\partial^2 c}{\partial x^2}(t,t,x)\right) = M,$$

is an equilibrium dividend strategy, and

$$V(t,x) = c(t,t,x) = \left(1 + \frac{\lambda}{\delta}\right)\frac{M}{\delta} + \frac{M}{\delta}\left[\frac{\lambda}{\mu - M - \sigma^2\theta_3}x - \left(1 + \frac{\lambda}{\delta}\right)\right]e^{-\theta_3 x},$$

is the corresponding equilibrium value function.

(ii) If (2.4.33) and (2.4.34) hold, then for $t \in [0,\infty)$*,*

$$\hat{\pi}(t,x) = \phi\left(t, t, \frac{\partial c}{\partial x}(t,t,x), \frac{\partial^2 c}{\partial x^2}(t,t,x)\right) = \begin{cases} 0, & x \in [0,b), \\ M, & x \in [b,\infty), \end{cases}$$

is an equilibrium dividend strategy, and

$$V(t,x) = c(t,t,x) = \begin{cases} \left(\hat{C} - B_1 x\right)e^{\theta_1 x} - \left(\hat{C} + B_1 x\right)e^{-\theta_2 x}, & x \in [0,b), \\ \left(1 + \frac{\lambda}{\delta}\right)\frac{M}{\delta} + \left(D_3 + B_3 x\right)e^{-\theta_3 x}, & x \in [b,\infty), \end{cases}$$

is the corresponding equilibrium value function. Here $\left(b, \hat{C}, B_1, B_3, D_3\right)$ *is the solution to (2.4.27).*

Remark 2.4.1. *(i) Since essentially we are looking for a Nash equilibrium which is not unique in general, we did not discuss the uniqueness of the solution of the equation* $G(b) = 0$ *in Lemma 2.4.2. If there is no unique positive solution, one may choose the "best" one by some other criteria, such as minimizing the ruin probability. Furthermore, given some proper conditions we can get the uniqueness of the solution, see Appendix A.1.*

(ii) Note that (2.4.33) and (2.4.34) are only sufficient conditions (not necessary). So we only showed the results for two special cases in Theorem 2.4.2, and we are not clear if the

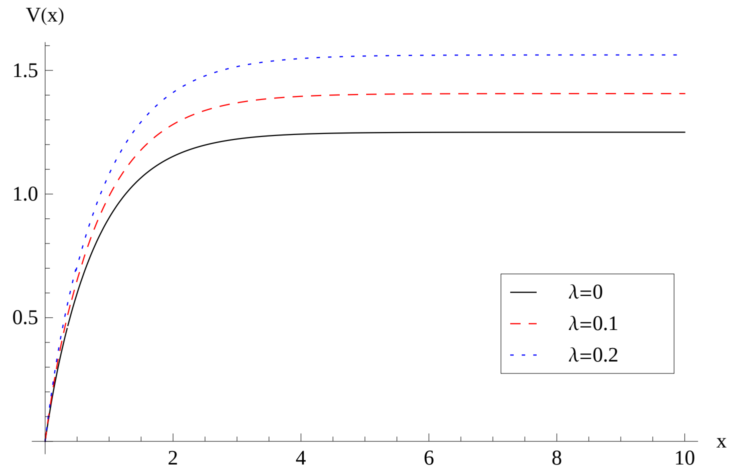

Figure 2.4.2: Equilibrium value functions with a pseudo-exponential discount function

results still hold in other cases. If λ is violating the restriction in (i) and (ii), one may still get the results under some other sufficient conditions. See an example at the end of Appendix A.2.

Example 2.4.2. *Let $\mu = 1$, $\sigma = 1$, $M = 1$, $\delta = 0.8$. Figure 2.4.2 shows the equilibrium value functions for pseudo-exponential discount functions with $\lambda = 0$, 0.1 and 0.2. The barriers b are 0.3470, 0.4141 and 0.4796, respectively. The case with $\lambda = 0$ is time consistent and the equilibrium strategy is optimal in the classical situation.*

2.5 Conclusion

In this chapter, we have studied the dividend optimisation problem with a general discount function in a diffusion risk model. Since the discount function is not restricted to be exponential, this problem turns out to be time-inconsistent. The goal of this problem is to find an equilibrium strategy within the class of feedback controls (or Markov controls). We assume that the dividends can only be paid at a bounded rate and consider the ruin risk in this dividend problem, i.e., the time horizon is a random variable (the time of ruin). We have obtained the equilibrium HJB equation, which is motivated by Yong (2012a), and the verification theorem for a general discount function. Finally, we have solved the equilibrium

HJB-equation for two special non-exponential discount functions: a mixture of exponential discount functions and a pseudo-exponential discount function. Our results imply that if the bound of the dividend rate is small enough, then the equilibrium strategy is to always pay the maximal dividend rate; otherwise, the equilibrium strategy is to pay the maximal dividend rate when the surplus is above a barrier and to pay nothing when the surplus is below the barrier.

Chapter 3

Minimization of Risks in Defined Benefit Pension Plan with Time-Inconsistent Preferences

3.1 Introduction

Defined benefit pension plan has received much attention due to its extensive use in a number of countries including the UK, USA, Canada and Australia. In a defined benefit occupational pension scheme, the members (employees) are guaranteed a regular income which is paid by the pension fund for their lifetime or for some pre-determined term. The fund receives income from the contributions (contributed by the employer and possibly the members) as well as investment returns, to make the promised benefit to be paid on various contingencies even including the failure of the sponsor's (or employer's) business. This object would be achieved by selecting a sufficient contribution rate and appropriate asset allocation strategies.

Along the lines of Haberman (1993), Haberman and Sung (1994), Haberman (1997) and Josa-Fombellida and Rincón-Zapatero (2001, 2004), there are two main types of risks with which the pension plan is confronted: i) the *contribution rate risk*, which is measured as the deviations of the contributions from the normal cost and is associated with the stability of the plan; ii) the *solvency risk*, which is measured with the unfunded actuarial liability (i.e., the deviations of the fund from the actuarial liability) and is related to the security of the plan. The characteristics of both stability and security can be incorporated by minimising some convex combination of the types of risks. In this way, the weight in the convex combination measures the relative significance of the risks.

Josa-Fombellida and Rincón-Zapatero (2001) investigated a dynamic model of the defined benefit pension fund in a quadratic cost criterion over an infinite time period. The assets of the fund were assumed to be invested in a bond and n risky securities following correlated geometrical Brownian motions. The investment strategy was constrained to be non-negative to avoid short-selling. They got a closed-form solution to this problem and proved strong stability properties of the solution. Ngwira and Gerrard (2007) considered an optimal funding and asset allocation control problem for a defined benefit pension scheme on a finite time horizon, by assuming that the pension fund could be invested in a risk-free asset and a risky asset whose return follows an exponential, jump-diffusion, Lévy process with log-normally distributed jumps. They obtained the optimal contribution and optimal asset allocation strategies in both constant and stochastic pension benefit outgo cases and analysed the effect of a jump magnitude on an asset allocation strategy. For other papers related to this subject, we refer the reader to Cairns (2000), Haberman and Sung (2002), Chang et al. (2003) and Owadally and Haberman (2004).

Note that in all the literature mentioned above it is assumed that the manager of the pension plan has a constant rate of time preference. Such preference is time-consistent in the sense that the preference of the manager for an earlier date over a later date is the same. However, it is unrealistic since in most situations the decisions are not made by an individual but a group. Thus there should be one discount factor per member of the group and the discounting can be described by a mixture of those exponential discount functions (see Ekeland and Lazrak (2006), Ekeland and Pirvu (2008) and Ekeland et al. (2012)). Furthermore, a number of empirical studies of human behaviour reveal that people are impatient about choices in the short term but more patient when choosing among long-term alternatives, see, e.g., Thaler (1981), Ainslie (1992) and Loewenstein and Prelec (1992). Particularly, cash flows in the near future tend to be discounted at a significantly higher rate than those occurring in the long run. Considering such behavioural features, economic decisions may be analysed using the hyperbolic discounting (see Phelps and Pollak (1968)). Indeed, the hyperbolic discounting has been extensively considered in economics and behavioural finance, see Laibson (1997) and Barro (1999) among others.

The aim of this chapter is to investigate the defined benefit pension problem with a general discounting in a finite time horizon. The pension fund can be invested in a risk-free asset and a risky asset whose return follows a geometric Brownian motion. Similar to Ngwira and Gerrard (2007), the goal of the sponsor and trustees is to minimise both the contribution rate risk and the solvency risk by considering a quadratic cost criterion. In this chapter, the only source of uncertainty is the investment returns.

However, an optimal control problem with a non-constant discount rate cannot be solved by the standard optimal control techniques such as dynamic programming principle. In fact, these techniques give rise to time inconsistent strategies, i.e, a strategy that is optimal for the initial time may be not optimal later. This is the so-called time inconsistent control problem and the classical dynamic programming principle is no longer valid. To obtain the time consistent strategies, Strotz (1955) studied the time inconsistent problem within a game theoretic framework using the Nash equilibria. He obtained an equilibrium strategy as the solution of a sub-game-perfect equilibrium where player t can be viewed as the future incarnation of the decision-maker at time t. This line of research has been followed by many others (see Pollak (1968), Peleg and Yaari (1973), Goldman (1980), Laibson (1997) and Barro (1999)), mostly in discrete-time framework.

More recently, the study of time inconsistent control problems has received more and more attention in economics, finance and insurance as many practical applications in these fields can be formulated as time inconsistent control problems. In a continuous-time model, a modified HJB equation was derived in Marín-Solano and Navas (2010), which solved an optimal consumption and investment problem with non-constant discount rate for both naive and sophisticated agents. A similar problem was also considered by using another approach in Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008), which provided the precise definition of the equilibrium concept in continuous-time model for the first time. Following their definition of the equilibrium strategy, Björk and Murgoci (2010) studied the time-inconsistent control problem in a general Markovian framework, and derived the extended HJB equation together with the verification theorem. Considering the hyperbolic discounting, Ekeland et al. (2012) studied the portfolio management problem for an investor who is allowed to consume and take out life insurance, and they characterised the equilibrium strategy by an integral equation. Björk et al. (2014) studied the Markowitz's problem with state-dependent risk aversion by utilising the extended HJB equation.

In this chapter, we employ the techniques in Ekeland and Pirvu (2008), Björk and Murgoci (2010) and Ekeland et al. (2012) and restrict ourselves to Markov strategies. However, in the general case in Björk and Murgoci (2010), it is almost impossible to find a general solution for the extended HJB equation. The solution is subject to different specified problems. In this chapter, an extended HJB equation is presented and a verification theorem is proved for a defined benefit pension plan. We characterise the time-consistent strategies and value function in terms of the solution of a system of integral equations. Explicit-form solution is obtained in a special case with constant discount rate. Following the numerical scheme in Ekeland et al. (2012) which is based on a Riemann sum approximation of the integral, we obtain an approximation of the solutions of the integral equations in general cases. In the numerical experiments, we compare the equilibrium strategies for an exponential discount function, a mixture of exponential functions and a hyperbolic discount function. The results show that the equilibrium contribution rate and the equilibrium investment strategy are gradually decreasing when the participating time is approaching the termination of the plan. Moreover, compared with the case of exponential discounting, the equilibrium contribution rate and equilibrium investment strategy are lower due to a declining discount factor in the other two non-exponential cases.

The remainder of this chapter is organised as follows. In Section 3.2, we introduce the model and the definition of the equilibrium strategy and value function. Section 3.3 presents the extended HJB equation and the verification theorem. The solution of the extended HJB equation is given in Section 3.4. Section 3.5 illustrates some numerical results. Section 3.6 concludes this chapter.

3.2 The Defined Benefit Pension Model

Consider a complete, filtered probability space $\left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathrm{P}\right)$ satisfying the usual conditions, where the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ is P-completed and generated by a one-dimensional standard Brownian motion denoted by $W = \{W(t)\}_{t\geq 0}$. Let T be a finite time horizon which can be seen as the term of the defined benefit pension plan.

Let $F(t)$ denote the value of the fund at time t. The dynamics of the pension fund is governed over time as:

$$\mathrm{d}F(t) = F(t)\mathrm{d}\psi(t) + (C(t) - D)\,\mathrm{d}t, \quad t \in [0, T]; \quad F(0) = x > 0,$$

where

- $\psi(t)$ is the instantaneous return on assets in the interval $(t, t + \mathrm{d}t)$
- $C(t)$ is the contribution rate at time t, which is made by the sponsor in order to accrue the amount of the defined benefit at the time of retirement. It is an adapted process with respect to $\{\mathscr{F}_t\}_{t\geq 0}$ such that

$$\int_0^\infty |C(s)|\,\mathrm{d}s < \infty, \quad \text{a.s.}$$

- D is the expected rate of the pension benefit outgo

We denote by NC the normal cost for all participants, by AL the actuarial liability, by $UAL(t) := AL - F(t)$ the unfunded actuarial liability and by $SC(t) := C(t) - NC$ the supplementary contribution rate amortising UAL at time t. Throughout this chapter, we consider constant values of D, NC and AL. As explained in Haberman and Sung (1994) and Josa-Fombellida and Rincón-Zapatero (2001), it is reasonable when the population in the pension plan is stationary from the start and there is no salary increase or there is a fixed rate of salary inflation. Here D, NC and AL would be determined in advance by the trustees, sponsor, members and advising actuary, based on the trust deed, rules and appropriate government regulations for the pension plan.

Assume that the valuation of the pension plan is done with a constant rate δ, called technical rate of interest. Then the main components of the plan are linked by the following equation (see Bowers et al. (1976)):

$$\delta AL + NC - D = 0. \tag{3.2.1}$$

How to choose the technical rate of interest δ is an important issue. In some literature, the valuation force of interest δ is assumed to be consistent with the riskless rate of interest r due to the fact that there is no uncertainty in the liability cash flows. This is also easily understood if the sponsor borrows money to satisfy his/her liabilities. Since the rate of borrowing is r, the "correct" valuation of the debt is $\delta = r$. However, the assumption $\delta = r$ would be inappropriate if there is some random element in the benefits (see Josa-Fombellida and Rincón-Zapatero (2001, 2004)). In industry, the technical interest rate is predetermined diversely by the insurance company according to various plans or rules. In this chapter, to compare our results with those in Ngwira and Gerrard (2007), we do not restrict $\delta = r$.

We consider a standard complete financial market comprised of a risk-free asset with price S^0 as well as a risky asset with price S. The employer manages the funding process by making a portfolio of these assets. Assume that the price of the risk-free asset evolves over time according to the following ordinary differential equation (ODE):

$$\mathrm{d}S^0(t) = rS^0(t)\mathrm{d}t, \quad t \in [0,T];\ S^0(0) = 1,$$

where r is the instantaneous interest rate which is assumed to be a constant.

The price of the risky asset is assumed to follow a geometric Brownian motion:

$$\mathrm{d}S(t) = S(t)\left(b\mathrm{d}t + \sigma \mathrm{d}W(t)\right), \quad t \in [0,T];\ S(0) = s_0 > 0,$$

where $b > r$ is the mean rate of return; $\sigma > 0$ is the volatility; the initial price of the risky asset s_0 is given.

An investment strategy is described by an $\{\mathscr{F}_t\}_{t\geq 0}$-adapted process λ such that

$$\int_0^\infty \lambda^2(t)\mathrm{d}t < \infty, \quad \text{a.s.}$$

$\lambda(t)$ represents the quantity of the fund invested in the risky asset by the sponsor at time t. Then the amount $F(t) - \lambda(t)$ is invested in the risk-free asset. $\lambda < 0$ means that the sponsor is selling short the stock. If $\lambda > F$, then the manager is borrowing money at rate r to invest in the risky asset.

Therefore, for all $t \in [0,T]$, the dynamics of the pension fund process $F := \{F(s)\}_{s\in[t,T]}$ is governed over time by the following stochastic differential equation (SDE):

$$\begin{aligned}
\mathrm{d}F(s) &= \lambda(s)\frac{\mathrm{d}S(s)}{S(s)} + (F(s) - \lambda(s))\frac{\mathrm{d}S^0(s)}{S^0(s)} + (C(s) - D)\,\mathrm{d}s \\
&= [rF(s) + (b-r)\lambda(s) + C(s) - D]\,\mathrm{d}s + \sigma\lambda(s)\mathrm{d}W(s) \\
&= [rF(s) + \theta\sigma\lambda(s) + SC(s) - \delta AL]\,\mathrm{d}s + \sigma\lambda(s)\mathrm{d}W(s), \quad s \in [t,T],
\end{aligned} \tag{3.2.2}$$

with $F(t) = x > 0$. Here $\theta := \frac{b-r}{\sigma}$ is the market price of risk and the last equation follows from (3.2.1).

We restrict ourselves to feedback controls (i.e. Markov strategies), which means that at each t, the pair $(SC, \lambda) := \{(SC(t), \lambda(t))\}_{t\in[0,T]}$ is given by

$$(SC(t), \lambda(t)) = \boldsymbol{\pi}(t,x) := (\pi_1(t,x), \pi_2(t,x)),$$

where x is the value of the pension fund at time t and the control $\boldsymbol{\pi} : [0,T] \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is a Borel measurable function in $(t,x) \in [0,T] \times \mathbb{R}$. We denote by Π the set of this class of controls.

Note that we choose SC as the control variable instead of C, leading to an equivalent control problem. The supplementary contribution rate $SC(t) = C(t) - NC$ is also an $\{\mathscr{F}_t\}_{t\geq 0}$-adapted process. Henceforth, we use the notation $F^{\boldsymbol{\pi}}$ instead of F to denote the controlled pension fund process.

In this chapter, we consider the following quadratic performance criterion.

Definition 3.2.1. *Given a control* $\boldsymbol{\pi} \in \Pi$ *and the corresponding pension fund process* $\{F^{\boldsymbol{\pi}}(s)\}_{s\in[t,T]}$ *(see (3.2.2)), we define the performance functional by*

$$J(t,x,\boldsymbol{\pi}) := \mathsf{E}_{t,x}\left[\int_t^T h(s-t)\left[\alpha_1\pi_1^2(s,F^{\boldsymbol{\pi}}(s)) + \alpha_2(F^{\boldsymbol{\pi}}(s)-AL)^2\right]\mathrm{d}s \right.$$
$$\left. + \alpha_3 h(T-t)(F^{\boldsymbol{\pi}}(T)-AL)^2\right], \qquad (3.2.3)$$

where $\alpha_1 > 0$, $\alpha_2, \alpha_3 \geq 0$, *and*

- $\mathsf{E}_{t,x}[\cdot]$ denotes the expectation conditioned on the event $\{F^{\boldsymbol{\pi}}(t) = x\}$ under P,
- $h : [0,\infty) \to \mathbb{R}$ is a discount function which is strictly positive on any finite time horizon $[0,T]$, continuously differentiable, decreasing with $h(0) = 1$, $\int_0^\infty h(s)\mathrm{d}s < \infty$ and the discount rate function $-\frac{h'}{h}$ is bounded on $[0,\infty)$.

To prove the convergency of the numerical scheme in Section 3.5, we also need a technical condition that the second derivative of the discount function h'' exists and is also bounded. For simplicity, we assume that there exists a constant $\rho > 0$ such that $\left|\frac{h'(t)}{h(t)}\right| \leq \rho$, for $t \geq 0$. In this chapter, three types of discount functions will be discussed: exponential discounting, a mixture of exponential discount functions of form $h(t) = \omega e^{-\rho_1 t} + (1-\omega)e^{-\rho_2 t}$, $\omega \in (0,1)$, $0 < \rho_2 < \rho_1$, and a general hyperbolic discounting of form $h(t) = (1+k_1 t)^{-\frac{k_2}{k_1}} e^{-\alpha t}$, $\alpha \geq 0$ and $k_2 > k_1 > 0$. It is easy to verify that these discount functions satisfy the above conditions on h. The discount rate of the mixture of exponential discount functions and the hyperbolic discounting are gradually declining over time.

The quadratic cost criterion (3.2.3) is similar to Ngwira and Gerrard (2007) which is a convex combination of the contribution rate risk and the solvency risk. The parameter α_1 and α_2 denote the weighting factor reflecting the relative importance for the employer of the contribution rate risk and the solvency risk, respectively.

The objective for the decision-maker is to find a strategy to minimise the quadratic cost criterion (3.2.3). Note that in classical risk theory, the optimal strategy, denoted by $\boldsymbol{\pi}^*$, is a strategy such that

$$V^{\boldsymbol{\pi}^*}(t,x) = \inf_{\boldsymbol{\pi}\in\Pi} J(t,x,\boldsymbol{\pi}).$$

However in the case when h is not exponential, this optimisation problem is time-inconsistent in the sense that the Bellman optimality principle fails.

Similar to Ekeland and Pirvu (2008) and Björk and Murgoci (2010), we view the entire problem as a non-cooperative game and look for Nash equilibria for the game. More specifically, we consider a game with one player for each time t, where player t can be regarded as the future incarnation of the decision maker at time t. Given state (t,x), player t will choose a control action $\boldsymbol{\pi}(t,x)$, and she/he wants to minimise the functional $J(t,x,\boldsymbol{\pi})$. In the continuous-time model, Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008) give the precise definition of this equilibrium strategy for the first time. Intuitively, equilibrium strategies are the strategies such that, given that they will be implemented in the future, it is optimal to implement them right now.

Definition 3.2.2. *Choose a control $\hat{\boldsymbol{\pi}}(\cdot,\cdot) = (\hat{\pi}_1(\cdot,\cdot), \hat{\pi}_2(\cdot,\cdot)) \in \Pi$. For any fixed initial point $(t,x) \in [0,T] \times \mathbb{R}$, and a fixed real number $\varepsilon > 0$, $\boldsymbol{\pi}^{\varepsilon}(\cdot,\cdot) = \left(\pi_1^{\varepsilon}(\cdot,\cdot), \pi_2^{\varepsilon}(\cdot,\cdot)\right)$ is another control defined by*

$$\boldsymbol{\pi}^{\varepsilon}(s,y) = \begin{cases} \boldsymbol{\pi}^{0}(s,y), & \text{for } s \in [t, t+\varepsilon], y \in \mathbb{R}, \\ \hat{\boldsymbol{\pi}}(s,y), & \text{for } s \in [t+\varepsilon, T], y \in \mathbb{R}, \end{cases}$$

where $\boldsymbol{\pi}^0$ is any strategy such that $\boldsymbol{\pi}^{\varepsilon} \in \Pi$.

If

$$\liminf_{\varepsilon \to 0} \frac{J(t,x,\boldsymbol{\pi}^{\varepsilon}) - J(t,x,\hat{\boldsymbol{\pi}})}{\varepsilon} \geq 0, \quad \forall (t,x) \in [0,T] \times \mathbb{R},$$

we say that $\hat{\boldsymbol{\pi}}$ is an equilibrium strategy for this control problem. The corresponding equilibrium value function $V(\cdot,\cdot) : [0,T] \times \mathbb{R} \to \mathbb{R}$ is then defined by

$$V(t,x) = J(t,x,\hat{\boldsymbol{\pi}}). \tag{3.2.4}$$

3.3 The Extended Hamilton-Jacobi-Bellman Equation

In this section we look for the equilibrium strategy and the equilibrium value function for the control problem proposed in Section 3.2.

For simplicity, we rewrite (3.2.3) as

$$J(t,x,\boldsymbol{\pi}) = \mathsf{E}_{t,x}\left[\int_t^T h(s-t)L(\pi_1(s,F^{\pi}(s)),F^{\pi}(s))\,\mathrm{d}s + h(T-t)G(F^{\pi}(T))\right], \text{ for } 0 \le t < T, \tag{3.3.1}$$

where $\boldsymbol{\pi} \in \Pi$,

$$L(\pi_1(s,F^{\pi}(s)),F^{\pi}(s)) := \alpha_1\pi_1^2(s,F^{\pi}(s)) + \alpha_2(F^{\pi}(s)-AL)^2,$$

and

$$G(F^{\pi}(T)) := \alpha_3(F^{\pi}(T)-AL)^2.$$

The corresponding controlled pension fund process $\{F^{\pi}(s)\}_{s\in[t,T]}$ is the solution of the following SDE:

$$\begin{cases} \mathrm{d}F^{\pi}(s) & = [rF^{\pi}(s) + \theta\sigma\pi_2(s,F^{\pi}(s)) + \pi_1(s,F^{\pi}(s)) - \delta AL]\,\mathrm{d}s + \sigma\pi_2(s,F^{\pi}(s))\,\mathrm{d}W(s), \\ F^{\pi}(t) & = x. \end{cases}$$

In the following, we will first give the extended HJB equation for the equilibrium value function V, and then prove a verification theorem.

For all $\boldsymbol{\pi} \in \Pi$ and any real valued function $f(t,x) \in C^{1,2}([0,\infty)\times\mathbb{R})$, which means that the partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ exist and are continuous on $[0,\infty)\times\mathbb{R}$, we define the infinitesimal generator $\mathscr{L}^{\pi}$ by

$$\mathscr{L}^{\pi}f(t,x) = \frac{\partial f}{\partial t}(t,x) + \{rx + \theta\sigma\pi_2(t,x) + \pi_1(t,x) - \delta AL\}\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2\pi_2^2(t,x)\frac{\partial^2 f}{\partial x^2}(t,x).$$

We assume that there exists an equilibrium strategy $\hat{\boldsymbol{\pi}} \in \Pi$ and we consider the extended HJB equation given in the following definition, see Björk and Murgoci (2010).

Definition 3.3.1. *Given the objective functional (3.3.1), the extended HJB equation for V is* *given by*

$$\begin{cases} \inf_{\pi\in\Pi}\left\{\mathscr{L}^{\pi}V(t,x)+L(\pi_1(t,x),x)\right\}+\int_t^T h'(s-t)\,\phi^s(t,x)\mathrm{d}s+h'(T-t)\,\gamma(t,x)=0, \\ V(T,x)=G(x), \end{cases} \tag{3.3.2}$$

for all $(t,x)\in[0,T)\times\mathbb{R}$, *where ϕ^s and γ are determined by*

$$\begin{cases} \mathscr{L}^{\hat{\pi}}\phi^s(t,x)=0, \quad 0\le t\le s\le T, \\ \mathscr{L}^{\hat{\pi}}\gamma(t,x)=0, \quad 0\le t\le T, \\ \phi^s(s,x)=L(\hat{\pi}_1(s,x),x), \quad 0\le s\le T, \\ \gamma(T,x)=G(x), \end{cases} \tag{3.3.3}$$

where $\hat{\pi}$ attains the infimum in (3.3.2).

Theorem 3.3.1. *(Verification Theorem) Assume that V solves the HJB equation given in Definition 3.3.1, and the infimum is attained for each $(t,x)\in[0,T)\times\mathbb{R}$ given a control $\hat{\pi}\in\Pi$. Then $\hat{\pi}$ is the equilibrium control, and V is the corresponding equilibrium value function.*

Proof. The proof of this theorem can be done by two steps: 1) we show that V is the value function corresponding to $\hat{\pi}$, i.e., $V(t,x)=J(t,x,\hat{\pi})$; 2) we prove that $\hat{\pi}$ is indeed the equilibrium control which is defined by Definition 3.2.2. We omit the second step since it is similar to the proof of (Björk and Murgoci, 2010, Theorem 4.1).

It follows from the Dynkin's formula that

$$\begin{aligned} V(t,x) &= \mathsf{E}_{t,x}\left[V\left(T,F^{\hat{\pi}}(T)\right)\right]-\mathsf{E}_{t,x}\left[\int_t^T \mathscr{L}^{\hat{\pi}}V\left(z,F^{\hat{\pi}}(z)\right)\mathrm{d}z\right] \\ &= \mathsf{E}_{t,x}\left[G\left(F^{\hat{\pi}}(T)\right)\right]-\mathsf{E}_{t,x}\left[\int_t^T \mathscr{L}^{\hat{\pi}}V\left(z,F^{\hat{\pi}}(z)\right)\mathrm{d}z\right]. \end{aligned} \tag{3.3.4}$$

From (3.3.2), we have

$$\mathscr{L}^{\hat{\pi}}V(t,x)=-\int_t^T h'(s-t)\phi^s(t,x)\mathrm{d}s-h'(T-t)\gamma(t,x)-L(\hat{\pi}_1(t,x),x). \tag{3.3.5}$$

It follows from (3.3.3) and the Dynkin's formula that

$$\phi^s(t,x) = \mathsf{E}_{t,x}\left[L\left(\hat{\pi}_1\left(s, F^{\hat{\pi}}(s)\right), F^{\hat{\pi}}(s)\right)\right],$$

and

$$\gamma(t,x) = \mathsf{E}_{t,x}\left[G\left(F^{\hat{\pi}}(T)\right)\right].$$

Then it follows from (3.3.5) and the above two equations that

$$\begin{aligned}\mathscr{L}^{\hat{\pi}}V(t,x) =& -\int_t^T h'(s-t)\mathsf{E}_{t,x}\left[L\left(\hat{\pi}_1\left(s,F^{\hat{\pi}}(s)\right),F^{\hat{\pi}}(s)\right)\right]\mathrm{d}s - h'(T-t)\mathsf{E}_{t,x}\left[G\left(F^{\hat{\pi}}(T)\right)\right]\\ &- L\left(\hat{\pi}_1(t,x),x\right)\\ =& -\mathsf{E}_{t,x}\left[\int_t^T h'(s-t)L\left(\hat{\pi}_1\left(s,F^{\hat{\pi}}(s)\right),F^{\hat{\pi}}(s)\right)\mathrm{d}s + h'(T-t)G\left(F^{\hat{\pi}}(T)\right)\right]\\ &- L\left(\hat{\pi}_1(t,x),x\right).\end{aligned}$$

Thus,

$$\begin{aligned}&\mathsf{E}_{t,x}\left[\int_t^T \mathscr{L}^{\hat{\pi}}V\left(z,F^{\hat{\pi}}(z)\right)\mathrm{d}z\right]\\ =& -\mathsf{E}_{t,x}\left[\int_t^T\int_z^T h'(s-z)L\left(\hat{\pi}_1\left(s,F^{\hat{\pi}}(s)\right),F^{\hat{\pi}}(s)\right)\mathrm{d}s\mathrm{d}z + \int_t^T h'(T-z)G\left(F^{\hat{\pi}}(T)\right)\mathrm{d}z\right]\\ &-\mathsf{E}_{t,x}\left[\int_t^T L\left(\hat{\pi}_1\left(z,F^{\hat{\pi}}(z)\right),F^{\hat{\pi}}(z)\right)\mathrm{d}z\right].\end{aligned} \tag{3.3.6}$$

Exchanging the order of integration in the first term of the right-hand-side of (3.3.6),

$$\begin{aligned}&\mathsf{E}_{t,x}\left[\int_t^T\int_z^T h'(s-z)L\left(\hat{\pi}_1\left(s,F^{\hat{\pi}}(s)\right),F^{\hat{\pi}}(s)\right)\mathrm{d}s\mathrm{d}z + \int_t^T h'(T-z)G\left(F^{\hat{\pi}}(T)\right)\mathrm{d}z\right]\\ =& -\mathsf{E}_{t,x}\left[\int_t^T (1-h(s-t))L\left(\hat{\pi}_1\left(s,F^{\hat{\pi}}(s)\right),F^{\hat{\pi}}(s)\right)\mathrm{d}s + G\left(F^{\hat{\pi}}(T)\right) - h(T-t)G\left(F^{\hat{\pi}}(T)\right)\right]\\ =&\mathsf{E}_{t,x}\left[\int_t^T h(s-t)L\left(\hat{\pi}_1\left(s,F^{\hat{\pi}}(s)\right),F^{\hat{\pi}}(s)\right)\mathrm{d}s + h(T-t)G\left(F^{\hat{\pi}}(T)\right)\right]\\ &-\mathsf{E}_{t,x}\left[\int_t^T L\left(\hat{\pi}_1\left(s,F^{\hat{\pi}}(s)\right),F^{\hat{\pi}}(s)\right)\mathrm{d}s + G\left(F^{\hat{\pi}}(T)\right)\right]\\ =&J(t,x,\hat{\boldsymbol{\pi}}) - \mathsf{E}_{t,x}\left[\int_t^T L\left(\hat{\pi}_1\left(s,F^{\hat{\pi}}(s)\right),F^{\hat{\pi}}(s)\right)\mathrm{d}s + G\left(F^{\hat{\pi}}(T)\right)\right].\end{aligned}$$

Therefore,

$$\mathsf{E}_{t,x}\left[\int_t^T \mathscr{L}^{\hat{\pi}} V\left(z, F^{\hat{\pi}}(z)\right) \mathrm{d}z\right] = -J(t,x,\hat{\pi}) + \mathsf{E}_{t,x}\left[G\left(F^{\hat{\pi}}(T)\right)\right]. \tag{3.3.7}$$

Putting (3.3.7) into (3.3.4), we obtain

$$V(t,x) = J(t,x,\hat{\pi}).$$

□

From the first order condition, we can easily get expression of an equilibrium strategy $\hat{\pi}$ for $t \in [0,T]$.

Corollary 3.3.1. *Assume that V is smooth enough and $\frac{\partial^2 V}{\partial x^2}(t,x) \neq 0$, for all $(t,x) \in [0,T] \times \mathbb{R}$. Then an equilibrium strategy for the defined benefit pension plan for the given performance functional is given by*

$$\begin{cases} \hat{\pi}_1(t,x) & = -\frac{1}{2\alpha_1}\frac{\partial V}{\partial x}(t,x), \\ \hat{\pi}_2(t,x) & = -\frac{\theta}{\sigma}\frac{\partial V}{\partial x}(t,x) \setminus \frac{\partial^2 V}{\partial x^2}(t,x). \end{cases} \tag{3.3.8}$$

3.4 Solution of the Stochastic Control Problem

In this section, we look for the solution of the HJB equation (3.3.2). For simplicity, we use the notation $f_t := \frac{\partial f}{\partial t}$, $f_x := \frac{\partial f}{\partial x}$, $f_{xx} := \frac{\partial^2 f}{\partial x^2}$ hereafter.

Substituting the strategy (3.3.8) into the HJB equation (3.3.2), it is rewritten as

$$\begin{cases} V_t(t,x) + (rx - \delta AL) V_x(t,x) - \frac{1}{4\alpha_1} V_x^2(t,x) - \frac{1}{2}\theta^2 \frac{V_x^2(t,x)}{V_{xx}(t,x)} + \alpha_2 (x - AL)^2 \\ \qquad = -\int_t^T h'(s-t)\,\phi^s(t,x)\mathrm{d}s - h'(T-t)\,\gamma(t,x), \\ V(T,x) = \alpha_3 (x - AL)^2. \end{cases} \tag{3.4.1}$$

It follows from (3.3.3) that ϕ^s and γ are given by

$$\begin{cases} \phi_t^s(t,x)+\left[rx-\theta^2\frac{V_x(t,x)}{V_{xx}(t,x)}-\frac{1}{2\alpha_1}V_x(t,x)-\delta AL\right]\phi_x^s(t,x) \\ \qquad\qquad +\frac{1}{2}\theta^2\left(\frac{V_x(t,x)}{V_{xx}(t,x)}\right)^2\phi_{xx}^s(t,x)=0, \quad 0\le t\le s\le T, \\ \phi^s(s,x)=\frac{1}{4\alpha_1}V_x^2(s,x)+\alpha_2(x-AL)^2, \end{cases} \tag{3.4.2}$$

and

$$\begin{cases} \gamma_t(t,x)+\left[rx-\theta^2\frac{V_x(t,x)}{V_{xx}(t,x)}-\frac{1}{2\alpha_1}V_x(t,x)-\delta AL\right]\gamma_x(t,x) \\ \qquad\qquad +\frac{1}{2}\theta^2\left(\frac{V_x(t,x)}{V_{xx}(t,x)}\right)^2\gamma_{xx}(t,x)=0, \quad 0\le t\le T, \\ \gamma(T,x)=\alpha_3(x-AL)^2, \end{cases} \tag{3.4.3}$$

respectively.

We wish to look for the value function of the following form

$$V(t,x)=P(t)x^2-2Q(t)x+R(t),$$

where $P>0$, Q and R are differentiable functions to be determined. Then

$$\begin{cases} V_t(t,x) &= P'(t)x^2-2Q'(t)x+R'(t), \\ V_x(t,x) &= 2\left(P(t)x-Q(t)\right), \\ V_{xx}(t,x) &= 2P(t), \end{cases}$$

and hence the equilibrium strategy is given by

$$\begin{cases} \hat{\pi}_1(t,x) &= -\frac{1}{\alpha_1}\left(P(t)x-Q(t)\right), \\ \hat{\pi}_2(t,x) &= -\frac{\theta}{\sigma}\left(x-\frac{Q(t)}{P(t)}\right). \end{cases} \tag{3.4.4}$$

Assume that ϕ^s and γ have the form

$$\phi^s(t,x)=\beta(t,s)x^2+\zeta(t,s)x+\eta(t,s),$$

$$\gamma(t,x) = A(t)x^2 - 2B(t)x + E(t),$$

respectively, where β, ζ, η, A, B, E are all differentiable functions to be determined. Then (3.4.2) and (3.4.3) can be rewritten as

$$\begin{cases} \left[\left(r - \theta^2 - \frac{1}{\alpha_1}P(t)\right)x + \theta^2\frac{Q(t)}{P(t)} + \frac{1}{\alpha_1}Q(t) - \delta AL\right](2\beta(t,s)x + \zeta(t,s)) \\ \qquad + \beta'(t,s)x^2 + \zeta'(t,s)x + \eta'(t,s) + \theta^2\beta(t,s)\left(x - \frac{Q(t)}{P(t)}\right)^2 = 0, \quad 0 \le t \le s \le T, \\ \phi^s(s,x) = \left(\frac{1}{\alpha_1}P^2(s) + \alpha_2\right)x^2 - 2\left(\frac{1}{\alpha_1}P(s)Q(s) + \alpha_2 AL\right)x + \left(\frac{1}{\alpha_1}Q^2(s) + \alpha_2 AL^2\right), \end{cases} \tag{3.4.5}$$

and

$$\begin{cases} A'(t)x^2 - 2B'(t)x + E'(t) + \theta^2\left(x - \frac{Q(t)}{P(t)}\right)^2 A(t) \\ \qquad + 2\left[\left(r - \theta^2 - \frac{1}{\alpha_1}P(t)\right)x + \theta^2\frac{Q(t)}{P(t)} + \frac{1}{\alpha_1}Q(t) - \delta AL\right](A(t)x - B(t)) = 0, \quad 0 \le t \le T, \\ \gamma(T,x) = \alpha_3(x - AL)^2, \end{cases} \tag{3.4.6}$$

respectively. Here f' denotes the first-order partial derivative of a function $f(t,s)$ with respect to t.

It follows from (3.4.6) that

$$\begin{cases} x^2\left[A'(t) + \left(2r - \theta^2 - \frac{2}{\alpha_1}P(t)\right)A(t)\right] \\ \qquad - 2x\left\{B'(t) + \left(r - \theta^2 - \frac{1}{\alpha_1}P(t)\right)B(t) - \left(\frac{1}{\alpha_1}Q(t) - \delta AL\right)A(t)\right\} \\ \qquad + \left[E'(t) - 2\left(\theta^2\frac{Q(t)}{P(t)} + \frac{1}{\alpha_1}Q(t) - \delta AL\right)B(t) + \theta^2\left(\frac{Q(t)}{P(t)}\right)^2 A(t)\right] = 0, \quad 0 \le t \le T, \\ A(T) = \alpha_3, \quad B(T) = \alpha_3 AL, \quad E(T) = \alpha_3 AL^2, \end{cases}$$

which implies that for $0 \le t \le T$,

$$\begin{cases} A'(t) + \left(2r - \theta^2 - \frac{2}{\alpha_1}P(t)\right)A(t) = 0, \\ B'(t) + \left(r - \theta^2 - \frac{1}{\alpha_1}P(t)\right)B(t) - \left(\frac{1}{\alpha_1}Q(t) - \delta AL\right)A(t) = 0, \\ E'(t) - 2\left(\theta^2\frac{Q(t)}{P(t)} + \frac{1}{\alpha_1}Q(t) - \delta AL\right)B(t) + \theta^2\left(\frac{Q(t)}{P(t)}\right)^2 A(t) = 0, \\ A(T) = \alpha_3, \quad B(T) = \alpha_3 AL, \quad E(T) = \alpha_3 AL^2. \end{cases}$$

After solving the above system of ODEs, we obtain

$$\begin{cases} A(t) &= \alpha_3 e^{\int_t^T \left(2r-\theta^2-\frac{2}{\alpha_1}P(u)\right)\mathrm{d}u}, \\ B(t) &= \alpha_3 e^{\int_t^T \left(r-\theta^2-\frac{1}{\alpha_1}P(u)\right)\mathrm{d}u}\left[AL - \int_t^T \left(\frac{1}{\alpha_1}Q(s) - \delta AL\right)e^{\int_s^T \left(r-\frac{1}{\alpha_1}P(u)\right)\mathrm{d}u}\mathrm{d}s\right], \\ E(t) &= \alpha_3 AL^2 - 2\int_t^T \left(\theta^2\frac{Q(s)}{P(s)} + \frac{1}{\alpha_1}Q(s) - \delta AL\right)B(s)\mathrm{d}s + \theta^2\int_t^T \left(\frac{Q(s)}{P(s)}\right)^2 A(s)\mathrm{d}s, \end{cases} \tag{3.4.7}$$

for $0 \le t \le T$.

Similarly, it follows from (3.4.5) that

$$\begin{cases} x^2\left[\beta'(t,s) + \left(2r - \theta^2 - \frac{2}{\alpha_1}P(t)\right)\beta(t,s)\right] \\ \qquad + x\left[\zeta'(t,s) + \left(r - \theta^2 - \frac{1}{\alpha_1}P(t)\right)\zeta(t,s) + 2\beta(t,s)\left(\frac{1}{\alpha_1}Q(t) - \delta AL\right)\right] \\ \qquad + \eta'(t,s) + \left(\theta^2\frac{Q(t)}{P(t)} + \frac{1}{\alpha_1}Q(t) - \delta AL\right)\zeta(t,s) + \theta^2\beta(t,s)\left(\frac{Q(t)}{P(t)}\right)^2 = 0,\ 0 \le t \le s \le T, \\ \beta(s,s) = \frac{1}{\alpha_1}P^2(s) + \alpha_2, \quad \zeta(s,s) = -2\left(\frac{1}{\alpha_1}P(s)Q(s) + \alpha_2 AL\right), \\ \eta(s,s) = \frac{1}{\alpha_1}Q^2(s) + \alpha_2 AL^2, \end{cases}$$

which implies that for $0 \le t \le s \le T$,

$$\begin{cases} \beta'(t,s) + \left(2r - \theta^2 - \frac{2}{\alpha_1}P(t)\right)\beta(t,s) = 0, \\ \zeta'(t,s) + \left(r - \theta^2 - \frac{1}{\alpha_1}P(t)\right)\zeta(t,s) + 2\beta(t,s)\left(\frac{1}{\alpha_1}Q(t) - \delta AL\right) = 0, \\ \eta'(t,s) + \left(\theta^2\frac{Q(t)}{P(t)} + \frac{1}{\alpha_1}Q(t) - \delta AL\right)\zeta(t,s) + \theta^2\beta(t,s)\left(\frac{Q(t)}{P(t)}\right)^2 = 0, \\ \beta(s,s) = \frac{1}{\alpha_1}P^2(s) + \alpha_2, \quad \zeta(s,s) = -2\left(\frac{1}{\alpha_1}P(s)Q(s) + \alpha_2 AL\right), \\ \eta(s,s) = \frac{1}{\alpha_1}Q^2(s) + \alpha_2 AL^2. \end{cases}$$

Therefore, we give the solution of the above ODE system as follows:

$$\begin{cases} \beta(t,s) &= \left(\frac{1}{\alpha_1}P^2(s)+\alpha_2\right)e^{\int_t^s\left(2r-\theta^2-\frac{2}{\alpha_1}P(u)\right)\mathrm{d}u}, \\ \zeta(t,s) &= 2e^{\int_t^s\left(r-\theta^2-\frac{1}{\alpha_1}P(x)\right)\mathrm{d}x}\Bigg[\left(\frac{1}{\alpha_1}P^2(s)+\alpha_2\right)\int_t^s e^{\int_u^s\left(r-\frac{1}{\alpha_1}P(x)\right)\mathrm{d}x}\left(\frac{1}{\alpha_1}Q(u)-\delta AL\right)\mathrm{d}u \\ & \qquad -\left(\frac{1}{\alpha_1}P(s)Q(s)+\alpha_2 AL\right)\Bigg], \\ \eta(t,s) &= \frac{1}{\alpha_1}Q^2(s)+\alpha_2 AL^2+\int_t^s\left(\theta^2\frac{Q(u)}{P(u)}+\frac{1}{\alpha_1}Q(u)-\delta AL\right)\zeta(u,s)\mathrm{d}u \\ & \qquad +\theta^2\left(\frac{1}{\alpha_1}P^2(s)+\alpha_2\right)\int_t^s\left(\frac{Q(u)}{P(u)}\right)^2 e^{\int_u^s\left(2r-\theta^2-\frac{2}{\alpha_1}P(x)\right)\mathrm{d}x}\mathrm{d}u. \end{cases} \tag{3.4.8}$$

Now we rewrite (3.4.1) as

$$\begin{cases} P'(t)x^2-2Q'(t)x+R'(t)+2\left(rx-\delta AL\right)\left(P(t)x-Q(t)\right) \\ \qquad -\left(\frac{1}{\alpha_1}P^2(t)+\theta^2P(t)\right)\left(x-\frac{Q(t)}{P(t)}\right)^2+\alpha_2\left(x-AL\right)^2 \\ \qquad +\int_t^T h'\left(s-t\right)\left[\beta(t,s)x^2+\zeta(t,s)x+\eta(t,s)\right]\mathrm{d}s+h'\left(T-t\right)\left(A(t)x^2-2B(t)x+E(t)\right)=0, \\ P(T)=\alpha_3, \quad Q(T)=\alpha_3 AL, \quad R(T)=\alpha_3 AL^2, \end{cases}$$

i.e.,

$$\begin{cases} x^2\left[P'(t)-\frac{1}{\alpha_1}P^2(t)+\left(2r-\theta^2\right)P(t)+\int_t^T h'\left(s-t\right)\beta(t,s)\mathrm{d}s+h'\left(T-t\right)A(t)+\alpha_2\right] \\ \qquad -2x\Big\{Q'(t)+\left(r-\theta^2-\frac{1}{\alpha_1}P(t)\right)Q(t)+\delta ALP(t)+\alpha_2 AL \\ \qquad -\int_t^T h'\left(s-t\right)\frac{1}{2}\zeta(t,s)\mathrm{d}s+h'\left(T-t\right)B(t)\Big\} \\ \qquad +R'(t)+2\delta ALQ(t)-\left(\frac{1}{\alpha_1}+\frac{\theta^2}{P(t)}\right)Q^2(t)+\alpha_2 AL^2 \\ \qquad +\int_t^T h'\left(s-t\right)\eta(t,s)\mathrm{d}s+h'\left(T-t\right)E(t)=0, \\ P(T)=\alpha_3, \quad Q(T)=\alpha_3 AL, \quad R(T)=\alpha_3 AL^2. \end{cases} \tag{3.4.9}$$

Then it follows from (3.4.9) that

$$
\begin{cases}
P'(t) - \frac{1}{\alpha_1}P^2(t) + \left(2r - \theta^2\right)P(t) + \int_t^T h'(s-t)\beta(t,s)\mathrm{d}s + h'(T-t)A(t) + \alpha_2 = 0, \\
P(T) = \alpha_3; \\
Q'(t) + \left(r - \theta^2 - \frac{1}{\alpha_1}P(t)\right)Q(t) + \delta ALP(t) + \alpha_2 AL - \int_t^T h'(s-t)\frac{1}{2}\zeta(t,s)\mathrm{d}s \\
\qquad + h'(T-t)B(t) = 0, \quad Q(T) = \alpha_3 AL; \\
R'(t) + 2\delta ALQ(t) - \left(\frac{1}{\alpha_1} + \frac{\theta^2}{P(t)}\right)Q^2(t) + \alpha_2 AL^2 + \int_t^T h'(s-t)\eta(t,s)\mathrm{d}s \\
\qquad + h'(T-t)E(t) = 0, \quad R(T) = \alpha_3 AL^2.
\end{cases}
\tag{3.4.10}
$$

It is easy to verify that (3.4.10) is equivalent to the following system of integral equations

$$
\begin{cases}
P(t) = \int_t^T h(s-t)\beta(t,s)\mathrm{d}s + h(T-t)A(t), \quad P(T) = \alpha_3, \\
Q(t) = -\frac{1}{2}\int_t^T h(s-t)\zeta(t,s)\mathrm{d}s + h(T-t)B(t), \quad Q(T) = \alpha_3 AL, \\
R(t) = \int_t^T h(s-t)\eta(t,s)\mathrm{d}s + h(T-t)E(t), \quad R(T) = \alpha_3 AL^2,
\end{cases}
\tag{3.4.11}
$$

where (β,ζ,η) and (A,B,E) are given by (3.4.8) and (3.4.7), respectively.

Proposition 3.4.1. *There exists a unique continuously differentiable solution (P,Q,R) of the system of equations (3.4.11) (or equivalently, (3.4.10)).*

The above proposition is proved in Appendix B.1.We now obtain the main result in following theorem which follows from Theorem 3.3.1 and Proposition 3.4.1.

Theorem 3.4.1. *Let $(P(t),Q(t),R(t))$ be the solution of (3.4.11) (or equivalently, (3.4.10)), and $V(t,x) := P(t)x^2 - 2Q(t)x + R(t)$. Then $\hat{\boldsymbol{\pi}} = (\hat{\pi}_1,\hat{\pi}_2)$ given by (3.4.4) is an equilibrium strategy for this problem. The corresponding value function is given by $J(t,x,\hat{\pi}) = V(t,x)$.*

Remark 3.4.1. *Let $\hat{C}(t) := \hat{C}\left(t,F^{\hat{\pi}}(t)\right) = \hat{\pi}_1\left(t,F^{\hat{\pi}}(t)\right) + NC$, $\hat{\lambda}(t) := \hat{\lambda}\left(t,F^{\hat{\pi}}(t)\right) = \hat{\pi}_2\left(t,F^{\hat{\pi}}(t)\right)$. Then $\hat{C}(t)$ and $\hat{\lambda}(t)$ are the equilibrium contribution rate and the equilibrium investment* *strategy at time t, respectively, and given by*

$$\begin{cases} \hat{C}(t) &= NC + \frac{1}{\alpha_1} P(t) \left(UAL(t) - \mathscr{G}(t) \right), \\ \hat{\lambda}(t) &= \frac{\theta}{\sigma} \left(UAL(t) - \mathscr{G}(t) \right), \end{cases}$$

where

$$\mathscr{G}(t) := AL - \frac{Q(t)}{P(t)}, \quad \mathscr{G}(T) = 0.$$

This result is similar to Ngwira and Gerrard (2007) which considered a DB pension problem where the risky asset follows a jump diffusion process. In an exponential discounting paradigm, the equilibrium strategy we obtained is consistent with the one in Ngwira and Gerrard (2007) with no jump (see Example 3.4.1).

It is worth noting that the equilibrium investment in risky asset is an increasing function of the unfunded actuarial liability, which shows that the manager takes greater risks when the level of the fund is far below the actuarial liability than when it is closer. Similar counterintuitive situations are discussed in Cairns (2000), Josa-Fombellida and Rincón-Zapatero (2001, 2004) and Ngwira and Gerrard (2007). As explained in Josa-Fombellida and Rincón-Zapatero (2001), the sponsor is more concerned with the stabilisation of the funding process around the targets since the aim of the pension plan is to reduce the risks inherent to the funding process. As a result, the sponsor would like to take a greater risk to guide the fund towards the target. Moreover, the manager needs to sell the risky asset short whenever the unfunded actuarial liability is less than the adjustment $\mathscr{G}$*. It is also easy to verify that* $\mathscr{G} = 0$ *when* $\delta = r$*, and thus the equilibrium allocation in risky asset is a constant proportion of the unfunded actuarial liability in this case (similar to Ngwira and Gerrard (2007)).*

In terms of the equilibrium contribution rate, it is also increasing with the unfunded actuarial liability UAL since $P(t) > 0$*. It is reasonable for the sponsor to contribute more when the deficit is larger. The equilibrium contribution rate is less than its target value whenever UAL is less than the adjustment* $\mathscr{G}$*. Moreover, when* $\delta = r$*, the equilibrium contribution rate becomes a normal cost adjusted by a supplementary cost which is a product of a nonlinear function and the unfunded actuarial liability.*

To further analyse the equilibrium strategies, we note that in the case of full funding, i.e.,

$UAL = 0$, the equilibrium allocation in the risky asset can only be zero if $\mathscr{G} = 0$; whereas the equilibrium contribution rate can only equal to NC if $\mathscr{G} = 0$. This result can be achieved in a special case with $\delta = r$.

Example 3.4.1. *(A special case: exponential discounting) Assume that $h(x) = e^{-\kappa x},\ \kappa > 0$. Then our problem is consistent with a time-consistent control problem with constant discount rate. In this case, (3.4.10) is rewritten as*

$$\begin{cases} P'(t) - \frac{1}{\alpha_1}P^2(t) + \left(2r - \kappa - \theta^2\right)P(t) + \alpha_2 = 0, \quad P(T) = \alpha_3; \\ Q'(t) + \left(r - \kappa - \theta^2 - \frac{1}{\alpha_1}P(t)\right)Q(t) + \delta ALP(t) + \alpha_2 AL = 0, \quad Q(T) = \alpha_3 AL; \\ R'(t) - \kappa R(t) + 2\delta ALQ(t) - \left(\frac{1}{\alpha_1} + \frac{\theta^2}{P(t)}\right)Q^2(t) + \alpha_2 AL^2 = 0, \quad R(T) = \alpha_3 AL^2. \end{cases}$$

which admits a unique solution

$$\begin{cases} P(t) &= \frac{\omega_2(\alpha_3-\omega_1)-\omega_1(\alpha_3-\omega_2)e^{\omega_3(T-t)}}{(\alpha_3-\omega_1)-(\alpha_3-\omega_2)e^{\omega_3(T-t)}}, \\ Q(t) &= ALe^{\int_t^T\left(r-\kappa-\theta^2-\frac{1}{\alpha_1}P(u)\right)\mathrm{d}u}\left[\alpha_3 + \int_t^T e^{-\int_s^T\left(r-\kappa-\theta^2-\frac{1}{\alpha_1}P(u)\right)\mathrm{d}u}\left(\alpha_2 + \delta P(s)\right)\mathrm{d}s\right], \\ R(t) &= \alpha_3 AL^2 e^{-\kappa(T-t)} + \int_t^T e^{-\kappa(s-t)}\left[2\delta ALQ(s) - \left(\frac{1}{\alpha_1} + \frac{\theta^2}{P(s)}\right)Q^2(s) + \alpha_2 AL^2\right]\mathrm{d}s, \end{cases}$$

where

$$\begin{cases} \omega_1 &= \frac{\alpha_1}{2}\left[\left(2r-\kappa-\theta^2\right) + \sqrt{\left(2r-\kappa-\theta^2\right)^2 + \frac{4\alpha_2}{\alpha_1}}\right], \\ \omega_2 &= \frac{\alpha_1}{2}\left[\left(2r-\kappa-\theta^2\right) - \sqrt{\left(2r-\kappa-\theta^2\right)^2 + \frac{4\alpha_2}{\alpha_1}}\right], \\ \omega_3 &= \frac{\omega_1-\omega_2}{\alpha_1}. \end{cases}$$

This result is consistent with the one obtained in Ngwira and Gerrard (2007) in the special case without jumps.

3.5 Numerical Results

We employ a similar numerical scheme from Ekeland et al. (2012) to approximate the integral equation for P and Q in (3.4.11). We discretise the interval $[0, T]$ by introducing the points $t_n = T + \varepsilon n$, where $\varepsilon = -\frac{T}{N}$. Rewriting (3.4.11) for P and Q in a differential form, we obtain that

$$\begin{cases} P'(t) & = \frac{1}{\alpha_1}P^2(t) + \left(\theta^2 - 2r - \frac{h'(T-t)}{h(T-t)}\right)P(t) - \alpha_2 + \int_t^T \Psi(s,t)\left(\frac{1}{\alpha_1}P^2(s) + \alpha_2\right)\left(\frac{\varphi(s)}{\varphi(t)}\right)\mathrm{d}s, \\ Q'(t) & = \left(\theta^2 - r + \frac{1}{\alpha_1}P(t) - \frac{h'(T-t)}{h(T-t)}\right)Q(t) - AL(\delta P(t) + \alpha_2) \\ & \quad + \frac{1}{\sqrt{\varphi(t)}}\int_t^T \Phi(s,t)\sqrt{\varphi(s)}\left(\frac{1}{\alpha_1}P(s)Q(s) + \alpha_2 AL\right)\mathrm{d}s \\ & \quad - \frac{1}{\sqrt{\varphi(t)}}\int_t^T \frac{e^{-ru}}{\sqrt{\varphi(u)}}\left(\frac{1}{\alpha_1}Q(u) - \delta AL\right)\left[\int_u^T e^{rs}\Phi(s,t)\varphi(s)\left(\frac{1}{\alpha_1}P^2(s) + \alpha_2\right)\mathrm{d}s\right]\mathrm{d}u, \end{cases} \tag{3.5.1}$$

where

$$\Psi(s,t) := \left[\frac{h'(T-t)}{h(T-t)} - \frac{h'(s-t)}{h(s-t)}\right]h(s-t)e^{\left(2r-\theta^2\right)(s-t)},$$
$$\Phi(s,t) := \left[\frac{h'(T-t)}{h(T-t)} - \frac{h'(s-t)}{h(s-t)}\right]h(s-t)e^{\left(r-\theta^2\right)(s-t)},$$

and

$$\varphi(s) := \exp\left\{\frac{2}{\alpha_1}\int_s^T P(u)\mathrm{d}u\right\}.$$

Define $P_n,\ Q_n$ and φ_n, $n = 0, 1, \cdots, N$, by $P_0 = \alpha_3$, $Q_0 = \alpha_3 AL$, $\varphi_0 = 1$, and

$$\begin{cases} P_{n+1} & = P_n + \varepsilon\frac{1}{\alpha_1}(P_n)^2 + \varepsilon\left(\theta^2 - 2r - \frac{h'(T-t_n)}{h(T-t_n)}\right)P_n - \varepsilon\alpha_2 \\ & \quad - \varepsilon^2\sum_{j=0}^{n-1}\Psi(t_j,t_n)\left(\frac{1}{\alpha_1}\left(P_j\right)^2 + \alpha_2\right)\left(\frac{\varphi_j}{\varphi_n}\right), \\ Q_{n+1} & = Q_n + \varepsilon\left(\theta^2 - r + \frac{1}{\alpha_1}P_n - \frac{h'(T-t_n)}{h(T-t_n)}\right)Q_n - \varepsilon AL(\delta P_n + \alpha_2) \\ & \quad - \varepsilon^2\frac{1}{\sqrt{\varphi_n}}\sum_{j=0}^{n-1}\Phi(t_j,t_n)\sqrt{\varphi_j}\left(\frac{1}{\alpha_1}P_jQ_j + \alpha_2 AL\right) \\ & \quad - \varepsilon^2\frac{1}{\sqrt{\varphi_n}}\sum_{j=0}^{n-1}e^{-rt_j}\sqrt{\frac{1}{\varphi_j}}\left(\frac{1}{\alpha_1}Q_j - \delta AL\right)\left[\sum_{i=0}^{j-1}e^{rt_i}\Phi(t_i,t_n)\varphi_i\left(\frac{1}{\alpha_1}P_i^2 + \alpha_2\right)\right]. \\ \varphi_{n+1} & = \varphi_n - \varepsilon\frac{2}{\alpha_1}P_n\varphi_n. \end{cases}$$

We show the results in the following theorem. The proof of the theorem is illustrated in Appendix B.2.

Theorem 3.5.1. *Let $P_N(t)$, $Q_N(t)$ be the function obtained by the linear interpolation of the points ($t_n = T + \varepsilon n$, P_n, Q_n). Then there exists a positive constant C which is independent of*

N such that

$$|P_N(t) - P(t)| \leq C\,|\varepsilon|\,, \quad \forall t \in [0,T],$$

and

$$|Q_N(t) - Q(t)| \leq C\,|\varepsilon|\,, \quad \forall t \in [0,T].$$

In this numerical experiment, we aim to explore the performance of the equilibrium contribution rate and the equilibrium investment strategy during a fixed time horizon $[0,T]$ with three types of discount functions: the exponential discounting (which has been considered in Example 3.4.1), a mixture of exponential discounting of form $h(t) = \omega e^{-\rho_1 t} + (1-\omega)e^{-\rho_2 t},\ \omega \in (0,1),\ 0 < \rho_2 < \rho_1$, and the hyperbolic discounting of form $h(t) = (1+k_1 t)^{-\frac{k_2}{k_1}} e^{-\alpha t},\ \alpha \geq 0$ and $k_2 > k_1 > 0$.

Assume that a DB pension fund is established with a fixed initial value \$400 once an employee joins this plan. The pension fund can be invested in a bond with riskless interest rate $r = 0.05$ as well as a stock with an expected rate of return $b = 0.1$ and volatility $\sigma = 0.5$. The actuarial liability and the normal cost are set to be \$500 and \$60, respectively. Assume that $\delta = 0.08$. Then $\delta > r$, i,e, the technical rate of interest is larger than the riskless interest rate. It follows from (3.2.1) that a higher technical rate of actualisation leads to a higher benefit and might be more attractive to the employee. Thus the sponsor would contribute more to the fund and would invest more money in the risky asset to meet the higher need.

The values of the other parameters are given by: $T = 10$, $N = 1000$, $\alpha_1 = 0.8$, $\alpha_2 = 0.1$, $\alpha_3 = 0.1$, $k_1 = 0.01$, $\alpha = 0.01$, $\rho_1 = 0.08$, $\rho_2 = 0.02$. In order to show the differences among the results of three different discount functions, the triple of discount parameter (κ, ω, k_2) takes value of (0.05,0.5,0.04) to make all the three discount functions have the same discount rates at time $t = 0$.

In the figures, the horizontal axis t represents the initial time the employee joins the DB pension plan. Under our assumption, the unfunded actuarial liability is $UAL = AL - x = \$100$ for every t.

Figure 3.5.1 illustrates the equilibrium contribution rate $\hat{C}$ versus t for different discount functions. It is worth noting that the equilibrium contribution rate is gradually decreasing with time t. It shows that the equilibrium contribution rate is approaching the normal cost when the time of commencement is approaching the termination of the plan (T) to keep the stability of the plan. It is easily understood that the contribution rate is higher for the sponsor if the employee participates in this plan earlier in order to compensate the unfunded actuarial liability and keep the security of the plan. However when the joining time is getting closer to the termination, the sponsor would be more concerned with the stability of the plan (since $\alpha_1 > \alpha_2$) and reduce the contribution to avoid the contribution rate risk.

Figure 3.5.2 presents the equilibrium investment strategy $\hat{\lambda}$ versus t for various discount functions. Obviously, $\hat{\lambda}(t)$ decreases gradually as time goes by. It shows that more money would be allocated in the stock when the joining time is far from the termination. The sponsor would like to afford greater risk to win more benefit and to avoid solvency risk. When there is less time left, the sponsor would increase the investment in bond to avoid the uncertainty investment risk.

The above trend can also be analysed from the relationship between δ and r. From (3.2.2), we obtain

$$\mathrm{d}(AL-F(s)) = [r(AL-F(s)) - \theta\sigma\lambda(s) - SC(s) + (\delta - r)AL]\,\mathrm{d}s - \sigma\lambda(s)\mathrm{d}W(s), \quad s \in [t,T].$$

Note there is a positive drift term in the above equation, which implies that if $\delta > r$, UAL would increase definitely (if without any investment or supplementary contribution, and $AL - F(t) > 0$). Thus, to offset this deterministic increment in solvency risk, the sponsor would prefer to bear some financial risk and contribution rate risk. It is easy to see that this increment would be larger if the starting time of the plan is earlier, which implies a higher solvency risk. Therefore, the sponsor would increase the contribution and the investment in financial market to compensate the growth of the unfunded actuarial liability.

Furthermore, although the equilibrium strategies for non-exponential discounting have similar trends as the optimal strategy for the exponential discounting, there are some significant differences. In Figures 3.5.1 and 3.5.2, it shows that in the case with exponential discounting, the contribution rate and the amount of money invested in financial market are both higher than those in the other two cases. Here is an explanation. In the latter two cases, the discount rate function is decreasing over time, and the decision-maker takes his strategy with

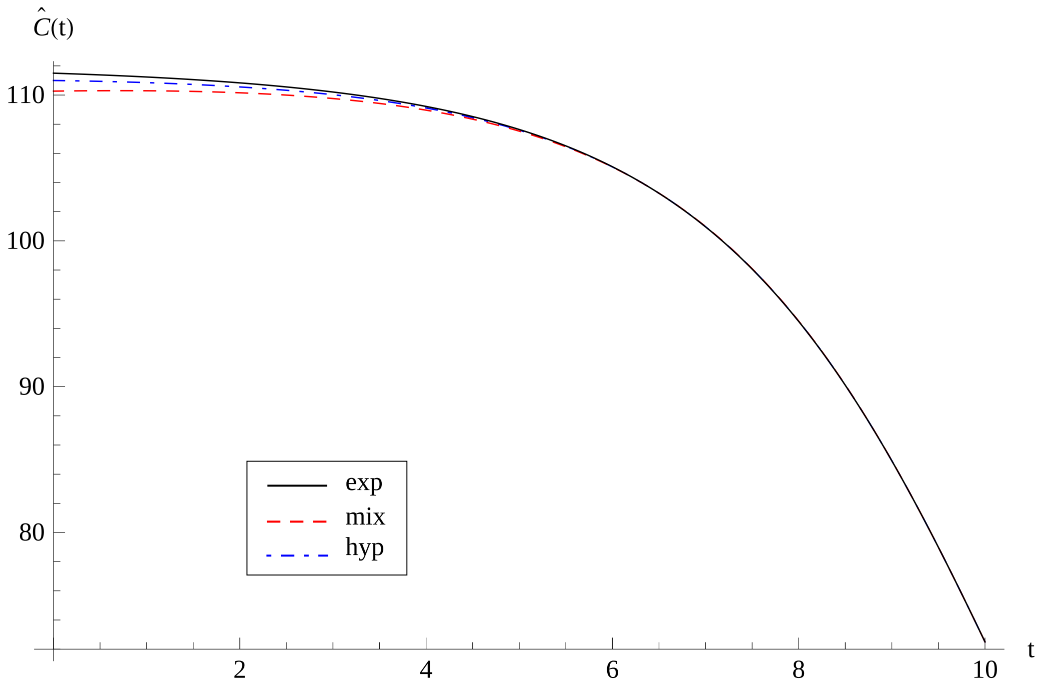

Figure 3.5.1: The equilibrium contribution rate $\hat{C}$ with different discount functions

taking into account such changes in his preference. At the initial time, the decision-maker knows that he will have a smaller discount rate in future, which means that he will become more sensitive to the contribution rate risk as well as the solvency risk (because a smaller discount rate implies a larger discounted cost). Thus, in these two cases, the decision-maker adopts a lower contribution rate which has smaller deviation to the normal cost (which is \$60 in the example), and allocates less money in the financial market to reduce the financial risk and consequently avoid larger unfunded actuarial liability. If the planing horizon becomes shorter, the effect of the change in the preference on the strategy is weakened and might be dominated by the other factors in the problem. Consequently, the gap between the strategies of exponential discounting and non-exponential discounting becomes smaller as the the time of commencement approaches the termination of the plan.

3.6 Conclusion

Exponential discount function has been widely used as the decision-makers' time-preferences in decision-making problems. Such preferences are time-consistent in the sense that the

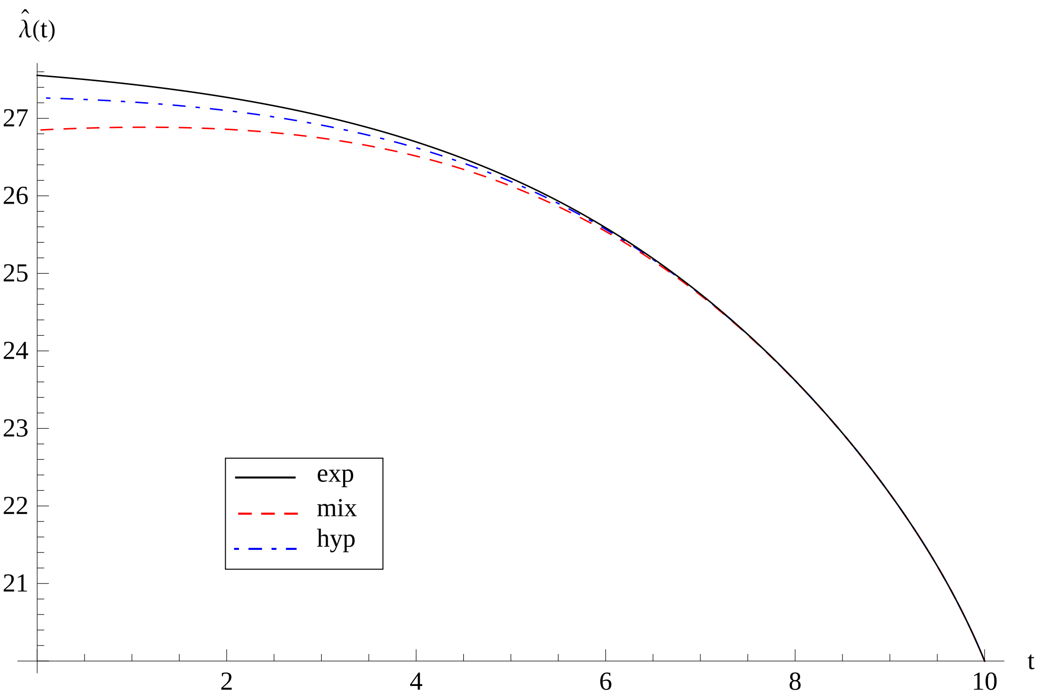

Figure 3.5.2: The equilibrium investment strategy $\hat{\lambda}$ with different discount functions

decision-makers' preferences for costs/rewards at an earlier date over a later date is the same whenever they are asked. However in general, a decision-maker cannot keep his preferences unchanged (due to subjective and/or objective reasons), especially in long-term decision-making problems, such as pension planning. In fact, a vast of experimental research suggest a decreasing discount rate so that economic decisions may be analysed using the hyperbolic discounting.

In this chapter, we have studied optimal contribution and asset allocation policies in a DB pension plan with a quadratic cost criterion and a general discount function. In the case with exponential discounting, the optimisation problems at each time $t \in [0,T]$ are the same (up to a multiplicator). Thus, the decision-maker only need to solve the problem at initial time 0, and apply the optimal strategy (obtained at time 0) at any time later. However, this is not appropriate for a general discount function. In fact, the decision-maker faces different optimisation problems at different times (note that (3.2.3) depends on t) in this case. Thus, the optimal strategy (minimising (3.2.3)) obtained at initial time may be suboptimal at any later time, which means that this optimal strategy may not be implemented in future (because only optimal strategy will be implemented). Consequently, in this case the optimal strategy that minimises (3.2.3) is meaningless since the optimal strategy should be obtained under the assumption that it would be implemented at any time. That is why for the decision-maker with non-exponential discounting, we need to define a new “optimal strategy”, i.e., an equilibrium strategy which is "optimal" (in the sense of Definition 3.2.2) at any time and thus can be implemented at any time.

We have obtained an equilibrium strategy in terms of the solution of a system of integral equations. If the discount function is exponential, the equilibrium strategy is consistent with the optimal strategy obtained in Ngwira and Gerrard (2007) in a special case without jumps. Although the equilibrium strategies for non-exponential discounting have similar form as the optimal strategy for exponential discounting, there are some significant differences. The differences and similarities between the strategies for various discount functions have been shown by numerical results. It shows that the equilibrium contribution rate and equilibrium investment strategy are declining over time. And in the case with exponential discounting, the sponsor would contribute more and invest more money in financial market than with the other two non-exponential discount functions.

Chapter 4

Consumption-Investment Strategies with Non-Exponential Discounting and Logarithmic Utility

4.1 Introduction

In recent years, research on the time-inconsistent preferences has attracted increasing attentions. The empirical studies of human behaviour reveal that the constant discount rate assumption is unrealistic (see, for example, Thaler (1981), Ainslie (1992) and Loewenstein and Prelec (1992)). Experimental evidence shows that economic agents are impatient about choices in the short term but are more patient when choosing between long-term alternatives. Particularly, cash flows in the near future tend to be discounted at a significantly higher rate than those occur in the long run. Considering such behavioural feature, economic decisions may be analysed using the hyperbolic discounting (see Phelps and Pollak (1968)). Indeed, the hyperbolic discounting has been widely adopted in microeconomics, macroeconomics, and behavioural finance, such as Laibson (1997) and Barro (1999) among others.

However, difficulties arise when we attempt to solve an optimal control problem with a non-constant discount rate by some standard control techniques, such as dynamic programming approach. In fact, these techniques lead to time inconsistent strategies, i.e, a strategy that is optimal for the initial time may not be optimal later (see, for example, Ekeland and Pirvu (2008) and Yong (2012a)). In other words, the classical dynamic programming principle fails to solve the so-called time-inconsistent control problem. So, how to obtain a time-consistent strategy for time-inconsistent control problems becomes an interesting and challenging problem. In Strotz (1955), the author studied a cake-eating problem within a game theoretic framework where the players are the agent and his/her future selves, and seek a subgame perfect Nash equilibrium point for this game. Strotz's work has been pursued by many others, such as Pollak (1968), Peleg and Yaari (1973), Goldman (1980) and Laibson (1997) among others.

Recently, the time inconsistent control problems regain considerable attention in the continuous-time setting. A modified HJB equation was derived in Marín-Solano and Navas (2010), which solved the optimal consumption and investment problem with non-constant discount rate for both naive and sophisticated agents. The similar problem was also considered by another approach in Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008), which provided the precise definition of the equilibrium concept in continuous time for the first time. They characterised the equilibrium policies through the solutions of a flow of BSDEs, and showed that, with special form of the discount factor, this BSDE reduces to a system of two ODEs that has a solution. There are some literature following their definition of equilibrium strategy. In Björk and Murgoci (2010), the time-inconsistent control problem was considered in a general Markov framework, and an extended HJB equation together with the verification theorem were derived. Björk et al. (2014) investigated the Markowitz's problem with state-dependent risk aversion by utilising the extended HJB equation obtained in Björk and Murgoci (2010). Considering the hyperbolic discounting, Ekeland et al. (2012) studied the portfolio management problem for an investor who is allowed to consume and take out life insurance, and they characterised the equilibrium strategy by an integral equation.

Another approach to the time-inconsistent control problem is developed by Yong (2011, 2012a,b). In Yong's papers, a sequence of multi-person hierarchical differential games is studied first and then the time-consistent equilibrium strategy and equilibrium value function are obtained by taking limits. A brief description of the method is given as follows. Let

$T > 0$ be the fixed time horizon and $t \in [0, T)$ be the initial time. Take a partition $\Pi = \{t_k \mid 0 \le k \le N\}$ of the time interval $[t, T]$ with $t = t_0 < t_1 < \cdots < t_N = T$, and with the mesh size

$$\|\Pi\| = \max_{1 \le k \le N} (t_k - t_{k-1}).$$

Consider an N-person differential game: for $k = 1, 2, \cdots, N$, the k-th player controls the system on $[t_{k-1}, t_k)$, starting from the initial state $(t_{k-1}, X(t_{k-1}))$ which is the terminal state of the $(k-1)$-th player, and tries to maximise his/her own performance functional. Each player knows that the later players will do their best, and will modify their control systems as well as their cost functionals. In the performance functional, each player discounts the utility in his/her own way. Then for any given partition Π, a Nash equilibrium strategy is constructed to the corresponding N-person differential game. Finally, it can be shown that as the mesh size $\|\Pi\|$ approaches zero, the Nash equilibrium strategy to the N-person differential game approaches the desired time-consistent solution of the original time-inconsistent problem. By this method, Yong (2011, 2012b) considered a deterministic time-inconsistent linear-quadratic control problem. Considering a controlled stochastic differential equation with deterministic coefficients, Yong (2012a) investigated a time-inconsistent problem with a general cost functional and derived an equilibrium HJB equation.

In this chapter, we revisit the consumption-investment problem (Merton (1969, 1971)) with a general discount function and a logarithmic utility function. In contrast to the references cited above, we consider this problem in a non-Markovian framework. More specifically, the coefficients in our model, including the interest rate, appreciation rate and volatility of the stock, are assumed to be adapted stochastic processes. To our best knowledge, the literature on the time-inconsistent problem in a non-Markovian model is rather limited. A time-inconsistent stochastic linear-quadratic control problem is studied in a model with random coefficients by Hu et al. (2012). A time-consistent strategy is obtained for the mean-variance portfolio selection by Czichowsky (2013) in a general semimartingale setting. Following Yong's method, we first study an N-person differential game. Similar to Hu et al. (2005) and Cheridito and Hu (2011), we adopt a martingale method to solve an optimisation problem of each player and characterise their optimal strategies and value functions in terms of the unique solutions of BSDEs. Then by taking limits, we show that a time-consistent equilibrium consumption-investment strategy of the original problem consists of a deterministic function and the ratio of the market price of risk to the volatility, and the corresponding equilibrium value function can be characterised by the unique solution of a family of BSDEs parameterised by a time variable that can be understood as the initial time of each player in the N-person differential game.

The remainder of this chapter is organised as follows. Section 4.2 introduces the model. In Section 4.3, we study the N-person differential game. Section 4.4 gives a time-consistent equilibrium strategy and time-consistent equilibrium value function to the original problem. Section 4.5 concludes this chapter. Some proofs and technical results are collected in Appendix C.

4.2 The Model

Let $T > 0$ be a fixed finite time horizon, and $\{W(t)\}_{0\le t\le T}$ be a standard Brownian motion defined on the filtered probability space $\left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0\le t\le T}, \mathsf{P}\right)$. Here the filtration $\{\mathscr{F}_t\}_{0\le t\le T}$ is the augmentation under P of $\mathscr{F}_t^W := \sigma(W(s), 0\le s\le t)$, $t\in[0,T]$. We consider a market that consists of a bond and a stock. The price of the bond evolves according to the differential equation

$$\begin{cases} \mathrm{d}B(s) = r(s)B(s)\mathrm{d}s, \quad s\in[0,T], \\ B(0) = 1. \end{cases}$$

The price of the stock is modelled by the stochastic differential equation

$$\begin{cases} \mathrm{d}S(s) = \mu(s)S(s)\mathrm{d}s + \sigma(s)S(s)\mathrm{d}W(s), \quad s\in[0,T], \\ S(0) = s_0, \end{cases}$$

where $s_0 > 0$. The interest rate process $\{r(t)\}_{0\le t\le T}$ as well as the appreciation rate $\{\mu(t)\}_{0\le t\le T}$ and volatility $\{\sigma(t)\}_{0\le t\le T}$ of the stock are assumed to be $\{\mathscr{F}_t\}_{0\le t\le T}$-adapted and bounded uniformly in $(t,\omega)\in[0,T]\times\Omega$. In addition, we require that the volatility process $\{\sigma(t)\}_{0\le t\le T}$ is bounded away from zero.

A consumption-investment policy is a bivariate process $(c(t), u(t)) \in \mathbb{R}^+ \times \mathbb{R}$, where $c(t)$ is the consumption rate at time t as a proportion of the wealth and $u(t)$ is the proportion of wealth invested in the stock at time t.

Let

$$
\begin{aligned}
\mathscr{C}[t,T] &= \Big\{ c : [t,T] \times \Omega \to \mathbb{R}^+ \;\Big|\; c(\cdot) \text{ is a predictable process for which} \\
&\qquad \int_t^T |c(s)|\,\mathrm{d}s < \infty,\ a.s. \Big\}, \\
\mathscr{U}[t,T] &= \Big\{ u : [t,T] \times \Omega \to \mathbb{R} \;\Big|\; u(\cdot) \text{ is a predictable process for which} \\
&\qquad \int_t^T |u(s)\sigma(s)|^2\,\mathrm{d}s < \infty,\ a.s. \Big\}.
\end{aligned}
$$

For any initial time $t \in [0,T]$ and initial wealth $x > 0$, applying a consumption-investment policy $(c(s), u(s)) \in \mathscr{C}[t,T] \times \mathscr{U}[t,T]$, the wealth process of the investor, denoted by $X(\cdot)$, is governed by

$$
\begin{cases}
\mathrm{d}X(s) = [r(s) - c(s) + u(s)\sigma(s)\theta(s)]\,X(s)\mathrm{d}s + u(s)\sigma(s)X(s)\mathrm{d}W(s), \quad s \in [t,T], \\
X(t) = x,
\end{cases} \tag{4.2.1}
$$

where

$$
\theta(s) := \frac{\mu(s) - r(s)}{\sigma(s)}.
$$

To emphasise the dependence of the wealth process on the initial state and the policy, we also write the solution of (4.2.1) as $X(\cdot; t, x, c(\cdot), u(\cdot))$.

In this chapter, we focus on the logarithmic utility function. At any initial time $t \in [0,T]$ with initial wealth $x > 0$, the performance functional, i.e., the expected discounted utility from the consumption and terminal wealth is given by

$$
J(t,x;c(\cdot),u(\cdot)) = \mathsf{E}_t \left[\int_t^T h(s-t) \ln(c(s)X(s))\mathrm{d}s + h(T-t)\ln X(T) \right],
$$

where $\mathsf{E}_t[\cdot] = \mathsf{E}[\cdot|\mathscr{F}_t]$ and $h(\cdot)$ is a general discount function satisfying

$$
h(0) = 1, \quad h(\cdot) > 0, \quad h'(\cdot) \le 0, \quad \int_0^T h(s)\mathrm{d}s < \infty.
$$

We also impose a technical assumption on h.

Assumption 4.2.1. *There exists a constant $C > 0$ such that $|h(s) - h(t)| \leq C|s-t|$, for all $s,t \in [0,T]$.*

Note that Assumption 4.2.1 is satisfied by many discount functions, such as exponential discount functions, mixture of exponential functions and hyperbolic discount functions.

Definition 4.2.1. *Given an initial state $(t,x) \in [0,T] \times (0,\infty)$, a strategy $(c(\cdot),u(\cdot)) \in \mathscr{C}[t,T] \times \mathscr{U}[t,T]$ is said to be admissible if*

$$\mathsf{E}_t\left[\int_t^T |\ln(c(s)X(s))|\,\mathrm{d}s\right] < \infty.$$

We denote by $\mathscr{A}(t,x)$ the class of all such admissible strategies.

Since we understand $\ln x$ to be $-\infty$ for $x \leq 0$ by convention, an admissible strategy satisfies $c(\cdot) > 0$, $\mathrm{d}t \times \mathsf{P}$-a.e., where $\mathrm{d}t$ is the Lebesgue measure on $[0,T]$.

Problem (N)**.** For given $(t,x) \in [0,T] \times (0,\infty)$, find $(\hat{c}(\cdot),\hat{u}(\cdot)) \in \mathscr{A}(t,x)$ such that

$$J(t,x;\hat{c}(\cdot),\hat{u}(\cdot)) = \sup_{(c(\cdot),u(\cdot)) \in \mathscr{A}(t,x)} J(t,x;c(\cdot),u(\cdot)). \tag{4.2.2}$$

It is well-known that Problem (N) is time-inconsistent, i.e., if we find some $(\hat{c}(\cdot),\hat{u}(\cdot)) \in \mathscr{A}(t,x)$ such that (4.2.2) is satisfied for initial state (t,x), we do not have

$$\begin{aligned} &J(\tau,X(\tau;t,x,\hat{c}(\cdot),\hat{u}(\cdot));(\hat{c}(\cdot),\hat{u}(\cdot))\big|_{[\tau,T]}) \\ &\qquad = \sup_{(c(\cdot),u(\cdot)) \in \mathscr{A}(\tau,X(\tau;t,x,\hat{c}(\cdot),\hat{u}(\cdot)))} J(\tau,X(\tau;t,x,\hat{c}(\cdot),\hat{u}(\cdot);c(\cdot),u(\cdot)) \end{aligned}$$

in general, for any $\tau \in [t,T]$, where $(\hat{c}(\cdot),\hat{u}(\cdot))\big|_{[\tau,T]}$ is the restriction of $(\hat{c}(\cdot),\hat{u}(\cdot))$ on the time interval $[\tau,T]$. We refer the reader to Ekeland and Pirvu (2008) and Yong (2012a) for examples of time-inconsistent consumption-investment problem with more details.

We end this section by introducing two notations. For any $0 \leq T_1 < T_2 \leq T$, we denote by $\mathbb{H}^2_{T_1,T_2}(\mathbb{R})$ the space of all predictable processes $\phi : \Omega \times [T_1,T_2] \to \mathbb{R}$ such that $\|\phi\|^2 :=$

$\mathsf{E}_{T_1}\left[\int_{T_1}^{T_2}|\phi(t)|^2\,\mathrm{d}t\right] < +\infty$, and by $\mathbb{H}^{\infty}_{T_1,T_2}(\mathbb{R})$ the space of all essentially bounded predictable processes $Y:\Omega\times[T_1,T_2]\to\mathbb{R}$.

4.3 Multi-Person Differential Game

Let $t\in[0,T)$ and $\mathscr{P}[t,T]$ be the set of all partitions $\Pi=\left\{t_k \mid 0\le k\le N\right\}$ of $[t,T]$ with $t=t_0<t_1<\cdots<t_N=T$. The mesh size of Π is

$$\|\Pi\| = \max_{1\le k\le N}(t_k-t_{k-1}).$$

Given the initial state $(t,x)\in[0,T]\times(0,\infty)$, we consider the following N-person differential game associated with partition $\Pi\in\mathscr{P}[t,T]$.

Let us start with Player (N) who makes the consumption-investment strategy on $[t_{N-1},t_N]$. For $x_{N-1}\in(0,\infty)$, consider the following wealth process

$$\begin{cases}\mathrm{d}X_N(s)=[r(s)-c_N(s)+\theta(s)\sigma(s)u_N(s)]X_N(s)\mathrm{d}s+\sigma(s)u_N(s)X_N(s)\mathrm{d}W(s), \quad s\in[t_{N-1},t_N],\\ X_N(t_{N-1})=x_{N-1},\end{cases} \tag{4.3.1}$$

and the performance functional

$$J_N(t_{N-1},x_{N-1};c_N(\cdot),u_N(\cdot))=\mathsf{E}_{t_{N-1}}\left[\int_{t_{N-1}}^{t_N}h(s-t_{N-1})\ln\left(c_N(s)X_N(s)\right)\mathrm{d}s+h(t_N-t_{N-1})\ln X_N(t_N)\right]. \tag{4.3.2}$$

Note that,

$$J_N(t_{N-1},x_{N-1};c_N(\cdot),u_N(\cdot))=J(t_{N-1},x_{N-1};c_N(\cdot),u_N(\cdot)),$$

for $x_{N-1}>0$.

Definition 4.3.1. *A consumption-investment strategy $(c_N(\cdot),u_N(\cdot))\in\mathscr{C}[t_{N-1},t_N]\times\mathscr{U}[t_{N-1},t_N]$ is said to be admissible for Player (N) with initial state $x_{N-1}\in(0,\infty)$, if*

$$\mathsf{E}_{t_{N-1}}\left[\int_{t_{N-1}}^{t_N}|\ln\left(c_N(s)X_N(s)\right)|\,\mathrm{d}s\right]<\infty.$$

We denote by $\mathscr{A}_N(t_{N-1}, x_{N-1})$ the class of all such admissible strategies.

Problem (C_N)**.** For any $x_{N-1} \in (0,\infty)$, find a strategy $(\hat{c}_N(\cdot), \hat{u}_N(\cdot)) \in \mathscr{A}_N(t_{N-1}, x_{N-1})$ such that

$$
\begin{aligned}
J_N(t_{N-1}, x_{N-1}; \hat{c}_N(\cdot), \hat{u}_N(\cdot)) &= V_\Pi(t_{N-1}, x_{N-1}) \\
&:= \sup_{(c_N(\cdot), u_N(\cdot)) \in \mathscr{A}_N(t_{N-1}, x_{N-1})} J_N(t_{N-1}, x_{N-1}; c_N(\cdot), u_N(\cdot)).
\end{aligned} \tag{4.3.3}
$$

Theorem C.1.2 shows the expressions of the value function $V_\Pi(t_{N-1}, x_{N-1})$, the optimal strategy $(\hat{c}_N(\cdot), \hat{u}_N(\cdot))$ and the optimal wealth process $\hat{X}_N(\cdot) \equiv \hat{X}_N(\cdot; t_{N-1}, x_{N-1})$.

Next, we consider the optimal control problem for Player $(N-1)$ on $[t_{N-2}, t_{N-1})$. For each $x_{N-2} \in (0,\infty)$, consider the following wealth process

$$
\begin{cases}
\mathrm{d}X_{N-1}(s) &= [r(s) - c_{N-1}(s) + \theta(s)\sigma(s)u_{N-1}(s)]\,X_{N-1}(s)\mathrm{d}s \\
&\quad + \sigma(s)u_{N-1}(s)X_{N-1}(s)\mathrm{d}W(s), \quad s \in [t_{N-2}, t_{N-1}), \\
X_{N-1}(t_{N-2}) &= x_{N-2}.
\end{cases}
$$

Recall that Player $(N-1)$ can only control the system on $[t_{N-2}, t_{N-1})$ and Player (N) will take over at t_{N-1} to control the system thereafter. Moreover, Player $(N-1)$ knows that Player (N) will play optimally based on the initial pair $(t_{N-1}, X_{N-1}(t_{N-1}))$, which is the terminal pair of Player $(N-1)$. Hence, the performance functional of Player $(N-1)$ should be

$$
\begin{aligned}
& J_{N-1}(t_{N-2}, x_{N-2}; c_{N-1}(\cdot), u_{N-1}(\cdot)) \\
= \;& \mathsf{E}_{t_{N-2}} \left[\int_{t_{N-2}}^{t_{N-1}} h(s - t_{N-2}) \ln (c_{N-1}(s) X_{N-1}(s))\, \mathrm{d}s \right. \\
& + \int_{t_{N-1}}^{t_N} h(s - t_{N-2}) \ln \left(\hat{c}_N(s) \hat{X}_N(s; t_{N-1}, X_{N-1}(t_{N-1})) \right) \mathrm{d}s \\
& \left. + h(t_N - t_{N-2}) \ln \hat{X}_N (t_N; t_{N-1}, X_{N-1}(t_{N-1})) \right].
\end{aligned} \tag{4.3.4}
$$

Note that in (4.3.4) Player $(N-1)$ "discounts" the future utility in his/her own way, i.e., he/she uses the discount function $h(s - t_{N-2})$ for $s \in [t_{N-2}, t_N]$.

Let $g_{N-1}(\cdot)$ be a bounded positive function defined on $[t_{N-2},t_N]$ by the ODE

$$\begin{cases} g'_{N-1}(s) = -g_{N-1}(s)\frac{h'(s-t_{N-2})}{h(s-t_{N-2})} - 1, \quad s \in [t_{N-2},t_N], \\ g_{N-1}(t_N) = 1. \end{cases}$$

and $(\mathscr{Y}_N(\cdot),\mathscr{Z}_N(\cdot)) \in \mathbb{H}^{\infty}_{t_{N-1},t_N}(\mathbb{R}) \times \mathbb{H}^2_{t_{N-1},t_N}(\mathbb{R})$ be the unique solution of the BSDE

$$\mathscr{Y}_N(\tau) = \int_\tau^{t_N} \mathscr{F}_N(s,\mathscr{Y}_N(s))\mathrm{d}s + \int_\tau^{t_N} \mathscr{Z}_N(s)\mathrm{d}W(s), \quad \tau \in [t_{N-1},t_N],$$

where

$$\mathscr{F}_N(s,y) = -\frac{1}{g_{N-1}(s)}y - \frac{1}{2}\theta^2(s) + \frac{1}{g_{N-1}(s)}\ln g_N(s) + \frac{1}{g_N(s)} - r(s).$$

From Proposition C.1.1, we have

$$\begin{aligned} & J_{N-1}\left(t_{N-2},x_{N-2};c_{N-1}(\cdot),u_{N-1}(\cdot)\right) \\ =& \mathsf{E}_{t_{N-2}}\left[\int_{t_{N-2}}^{t_{N-1}} h(s-t_{N-2})\ln\left(c_{N-1}(s)X_{N-1}(s)\right)\mathrm{d}s \right. \\ & \left. + h(t_{N-1}-t_{N-2})g_{N-1}(t_{N-1})\left[\ln X_{N-1}(t_{N-1}) - \mathscr{Y}_N(t_{N-1})\right]\right]. \end{aligned}$$

Definition 4.3.2. *A consumption-investment strategy $(c_{N-1}(\cdot),u_{N-1}(\cdot)) \in \mathscr{C}[t_{N-2},t_{N-1}] \times \mathscr{U}[t_{N-2},t_{N-1}]$ is said to be admissible for Player $(N-1)$ with initial state $x_{N-2} \in (0,\infty)$, if*

$$\mathsf{E}_{t_{N-2}}\left[\int_{t_{N-2}}^{t_{N-1}} \left|\ln\left(c_{N-1}(s)X_{N-1}(s)\right)\right|\mathrm{d}s\right] < \infty.$$

We denote by $\mathscr{A}_{N-1}(t_{N-2},x_{N-2})$ the class of all such admissible strategies.

Problem (C_{N-1}). For any $x_{N-2} \in (0,\infty)$, find a strategy $(\hat{c}_{N-1}(\cdot),\hat{u}_{N-1}(\cdot)) \in \mathscr{A}_{N-1}(t_{N-2},x_{N-2})$ such that

$$\begin{aligned} & J_{N-1}\left(t_{N-2},x_{N-2};\hat{c}_{N-1}(\cdot),\hat{u}_{N-1}(\cdot)\right) \\ = \;& V_\Pi(t_{N-2},x_{N-2}) \\ := \;& \sup_{(c_{N-1}(\cdot),u_{N-1}(\cdot)) \in \mathscr{A}_{N-1}(t_{N-2},x_{N-2})} J_{N-1}\left(t_{N-2},x_{N-2};c_{N-1}(\cdot),u_{N-1}(\cdot)\right). \end{aligned}$$

In Theorem C.1.3, we get the value function $V_\Pi(t_{N-2},x_{N-2})$, the optimal strategy $(\hat{c}_{N-1}(\cdot),\hat{u}_{N-1}(\cdot))$ and the optimal wealth process $\hat{X}_{N-1}(\cdot)\equiv\hat{X}_{N-1}(\cdot;t_{N-2},x_{N-2})$.

Let

$$(\bar{c}_{N-1}(s),\bar{u}_{N-1}(s))=\begin{cases}(\hat{c}_{N-1}(s),\hat{u}_{N-1}(s)), & s\in[t_{N-2},t_{N-1}),\\ (\hat{c}_{N}(s),\hat{u}_{N}(s)), & s\in[t_{N-1},t_{N}],\end{cases}$$

and $\bar{X}_{N-1}(s)\equiv\bar{X}_{N-1}(s;t_{N-2},x_{N-2})$, $s\in[t_{N-2},t_N]$ be the unique solution to the SDE

$$\begin{cases}\mathrm{d}\bar{X}_{N-1}(s) & =[r(s)-\bar{c}_{N-1}(s)+\theta(s)\sigma(s)\bar{u}_{N-1}(s)(s)]\bar{X}_{N-1}(s)\mathrm{d}s\\ & \quad+\sigma(s)\bar{u}_{N-1}(s)\bar{X}_{N-1}(s)\mathrm{d}W(s),\quad s\in[t_{N-2},t_N],\\ \bar{X}_{N-1}(t_{N-2}) & =x_{N-2}.\end{cases}\tag{4.3.5}$$

Similarly, we can state the optimal control problem for Player $(N-2)$ on $[t_{N-3},t_{N-2})$. For each $x_{N-3}\in(0,\infty)$, consider the following wealth process

$$\begin{cases}\mathrm{d}X_{N-2}(s) & =[r(s)-c_{N-2}(s)+\theta(s)\sigma(s)u_{N-2}(s)]X_{N-2}(s)\mathrm{d}s\\ & \quad+\sigma(s)u_{N-2}(s)X_{N-2}(s)\mathrm{d}W(s),\quad s\in[t_{N-3},t_{N-2}],\\ X_{N-2}(t_{N-3}) & =x_{N-3}.\end{cases}$$

The performance functional of Player $(N-2)$ should be

$$\begin{aligned}&J_{N-2}\left(t_{N-3},x_{N-3};c_{N-2}(\cdot),u_{N-2}(\cdot)\right)\\ =\;&\mathsf{E}_{t_{N-3}}\left[\int_{t_{N-3}}^{t_{N-2}}h(s-t_{N-3})\ln\left(c_{N-2}(s)X_{N-2}(s)\right)\mathrm{d}s\right.\\ &+\int_{t_{N-2}}^{t_N}h(s-t_{N-3})\ln\left(\bar{c}_{N-1}(s)\bar{X}_{N-1}(s;t_{N-2},X_{N-2}(t_{N-2}))\right)\mathrm{d}s\\ &\left.+h(t_N-t_{N-3})\ln\bar{X}_{N-1}\left(t_N;t_{N-2},X_{N-2}(t_{N-2})\right)\right],\end{aligned}$$

where $\bar{X}_{N-1}(\cdot;t_{N-2},X_{N-2}(t_{N-2}))$ is the solution to (4.3.5) with initial state $(t_{N-2},X_{N-2}(t_{N-2}))$.

Let $g_{N-2}(\cdot)$ be a bounded positive function defined on $[t_{N-3}, t_N]$ by the ODE

$$\begin{cases} g'_{N-2}(s) = -g_{N-2}(s)\frac{h'(s-t_{N-3})}{h(s-t_{N-3})} - 1, \quad s \in [t_{N-3}, t_N], \\ g_{N-2}(t_N) = 1, \end{cases}$$

and $(\mathscr{Y}_{N-1}(\cdot), \mathscr{Z}_{N-1}(\cdot)) \in \mathbb{H}^{\infty}_{t_{N-2},t_{N-1}}(\mathbb{R}) \times \mathbb{H}^{2}_{t_{N-2},t_{N-1}}(\mathbb{R})$ be the unique solution of the BSDE

$$\mathscr{Y}_{N-1}(\tau) = \int_{\tau}^{t_N} \mathscr{F}_{N-1}(s, \mathscr{Y}_{N-1}(s))\mathrm{d}s + \int_{\tau}^{t_N} \mathscr{Z}_{N-1}(s)\mathrm{d}W(s), \quad \tau \in [t_{N-2}, t_N],$$

where

$$\mathscr{F}_{N-1}(s,y) = -\frac{1}{g_{N-2}(s)}y - \frac{1}{2}\theta^2(s) + \frac{1}{g_{N-2}(s)}\sum_{k=N-1}^{N} \ln g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(s) + \sum_{k=N-1}^{N} \frac{1}{g_k(s)}\mathbf{1}_{[t_{k-1},t_k)}(s) - r(s).$$

It follows from Proposition C.1.2 that

$$\begin{aligned} & J_{N-2}\left(t_{N-3}, x_{N-3}; c_{N-2}(\cdot), u_{N-2}(\cdot)\right) \\ =& \mathsf{E}_{t_{N-3}}\Bigg[\int_{t_{N-3}}^{t_{N-2}} h(s-t_{N-3})\ln\left(c_{N-2}(s)X_{N-2}(s)\right)\mathrm{d}s \\ & + h(t_{N-2}-t_{N-3})g_{N-2}(t_{N-2})\left[\ln X_{N-2}(t_{N-2}) - \mathscr{Y}_{N-1}(t_{N-2})\right]\Bigg]. \end{aligned}$$

Definition 4.3.3. *A consumption-investment strategy* $(c_{N-2}(\cdot), u_{N-2}(\cdot)) \in \mathscr{C}[t_{N-3}, t_{N-2}] \times \mathscr{U}[t_{N-3}, t_{N-2}]$ *is said to be admissible for Player* $(N-2)$ *with initial state* $x_{N-3} \in (0,\infty)$*, if*

$$\mathsf{E}_{t_{N-3}}\left[\int_{t_{N-3}}^{t_{N-2}} \left|\ln\left(c_{N-2}(s)X_{N-2}(s)\right)\right|\mathrm{d}s\right] < \infty.$$

We denote by $\mathscr{A}_{N-2}(t_{N-3}, x_{N-3})$ *the class of all such admissible strategies.*

Problem (C_{N-2})**.** For any $x_{N-3} \in (0,\infty)$, find a strategy $(\hat{c}_{N-2}(\cdot), \hat{u}_{N-2}(\cdot)) \in \mathscr{A}_{N-2}(t_{N-3}, x_{N-3})$ such that

$$\begin{aligned} & J_{N-2}\left(t_{N-3}, x_{N-3}; \hat{c}_{N-2}(\cdot), \hat{u}_{N-2}(\cdot)\right) \\ = \quad & V_{\Pi}(t_{N-3}, x_{N-3}) \end{aligned}$$

$$:= \sup_{(c_{N-2}(\cdot),u_{N-2}(\cdot))\in\mathscr{A}_{N-2}(t_{N-3},x_{N-3})} J_{N-2}\left(t_{N-3},x_{N-3};c_{N-2}(\cdot),u_{N-2}(\cdot)\right).$$

The value function $V_\Pi(t_{N-3},x_{N-3})$ and the optimal strategy $(\hat{c}_{N-2}(\cdot),\hat{u}_{N-2}(\cdot))$ are given by Theorem C.1.4.

The above procedure can be continued recursively. For $1\le k\le N$, we define a bounded positive function $g_k(\cdot)$ on $[t_{k-1},t_N]$ by the ODE

$$\begin{cases} g_k'(s) = -g_k(s)\frac{h'(s-t_{k-1})}{h(s-t_{k-1})} - 1, \quad s\in[t_{k-1},t_N], \\ g_k(t_N) = 1. \end{cases}$$

For $1\le k\le N$, let $(\mathscr{Y}_k(\cdot),\mathscr{Z}_k(\cdot))\in\mathbb{H}^\infty_{t_{k-1},t_k}(\mathbb{R})\times\mathbb{H}^2_{t_{k-1},t_k}(\mathbb{R})$ be the unique solution of the BSDE

$$\mathscr{Y}_k(\tau) = \int_\tau^{t_N}\mathscr{F}_k(s,\mathscr{Y}_k(s))\mathrm{d}s + \int_\tau^{t_N}\mathscr{Z}_k(s)\mathrm{d}W(s), \quad \tau\in[t_{k-1},t_N],$$

with the driver

$$\mathscr{F}_k(s,y) = -\frac{1}{g_{k-1}(s)}y - \frac{1}{2}\theta^2(s) + \frac{1}{g_{k-1}(s)}\sum_{n=k}^N \ln g_n(s)\mathbf{1}_{[t_{n-1},t_n)}(s) + \sum_{n=k}^N\frac{1}{g_n(s)}\mathbf{1}_{[t_{n-1},t_n)}(s) - r(s).$$

Recall that the optimal problem of Player (N) is described by (4.3.1)-(4.3.3). By induction, we know that for Player $(k), 1\le k\le N-1$, with initial state $x_{k-1}\in(0,\infty)$, the state process is

$$\begin{cases} \mathrm{d}X_k(s) = [r(s)-c_k(s)+\theta(s)\sigma(s)u_k(s)]X_k(s)\mathrm{d}s + \sigma(s)u_k(s)X_k(s)\mathrm{d}W(s), \quad s\in[t_{k-1},t_k), \\ X_k(t_{k-1}) = x_{k-1}, \end{cases}$$

and the performance functional is

$$\begin{aligned} J_k\left(t_{k-1},x_{k-1};c_k(\cdot),u_k(\cdot)\right) \;=\; & \mathsf{E}_{t_{k-1}}\bigg[\int_{t_{k-1}}^{t_k} h(s-t_{k-1})\ln\left(c_k(s)X_k(s)\right)\mathrm{d}s \\ & + \int_{t_k}^{t_N} h(s-t_{k-1})\ln\left(\bar{c}_{k+1}(s)\bar{X}_{k+1}(s;t_k,X_k(t_k))\right)\mathrm{d}s \\ & + h(t_N-t_{k-1})\ln\bar{X}_{k+1}(t_N;t_k,X_k(t_k))\bigg], \end{aligned}$$

where

$$(\bar{c}_{k+1}(s), \bar{u}_{k+1}(s)) = \sum_{i=k+1}^{N} (\hat{c}_i(s), \hat{u}_i(s)) \mathbf{1}_{[t_{i-1}, t_i)}(s),$$

and $\bar{X}_{k+1}(\cdot; t_k, x_k)$ is the unique solution of the SDE

$$\begin{cases} \mathrm{d}\bar{X}_{k+1}(s) & = [r(s) - c_{k+1}(s) + \theta(s)\sigma(s)\bar{u}_{k+1}(s)] \bar{X}_{k+1}(s)\mathrm{d}s \\ & \quad + \sigma(s)\bar{u}_{k+1}\bar{X}_{k+1}(s)(s)\mathrm{d}W(s), \quad s \in [t_k, t_N], \\ \bar{X}_{k+1}(t_k) & = x_k. \end{cases}$$

Let $\mathscr{Y}_{N+1}(t_N) := 0$. It follows form Proposition C.1.3 that the performance functional of Player $(k), 1 \le k \le N$, is given by

$$J_k(t_{k-1}, x_{k-1}; c_k(\cdot), u_k(\cdot)) = \mathsf{E}_{t_{k-1}} \bigg[\int_{t_{k-1}}^{t_k} h(s - t_{k-1}) \ln (c_k(s) X_k(s)) \, \mathrm{d}s \\ + h(t_k - t_{k-1}) g_k(t_k) (\ln X_k(t_k) - \mathscr{Y}_{k+1}(t_k)) \bigg]. \tag{4.3.6}$$

Definition 4.3.4. *For $k = 1, 2 \cdots, N$, a consumption-investment strategy $(c_k(\cdot), u_k(\cdot)) \in \mathscr{C}[t_{k-1}, t_k] \times \mathscr{U}[t_{k-1}, t_k]$ is said to be admissible for Player (k) with initial state $x_{k-1} \in (0, \infty)$, if*

$$\mathsf{E}_{t_{k-1}} \left[\int_{t_{k-1}}^{t_k} |\ln (c_k(s) X_k(s))| \, \mathrm{d}s \right] < \infty.$$

We denote by $\mathscr{A}_k(t_{k-1}, x_{k-1})$ the class of all such admissible strategies.

Problem (C_k)**.** For $k = 1, 2 \cdots, N$, for any $x_{k-1} \in (0, \infty)$, find a strategy $(\hat{c}_k(\cdot), \hat{u}_k(\cdot)) \in \mathscr{A}_k(t_{k-1}, x_{k-1})$ such that

$$\begin{aligned} J_k(t_{k-1}, x_{k-1}; \hat{c}_k(\cdot), \hat{u}_k(\cdot)) & = V_\Pi(t_{k-1}, x_{k-1}) \\ & := \sup_{(c_k(\cdot), u_k(\cdot)) \in \mathscr{A}_k(t_{k-1}, x_{k-1})} J_k(t_{k-1}, x_{k-1}; c_k(\cdot), u_k(\cdot)). \end{aligned}$$

Noting that (4.3.6) is in the form of (C.1.2), Problem (C_k) is a special case of Problem (C) studied in Appendix C.1.1. From Theorem C.1.1, we have the following solution to the Problem (C_k).

Theorem 4.3.1. *For $k = 1,2\cdots,N$, the value function of Problem (C_k) is given by*

$$V_\Pi(t_{k-1},x_{k-1}) = g_k(t_{k-1})\left[\ln x_{k-1} - Y_k(t_{k-1})\right], \quad x_{k-1}(0,\infty),$$

where $Y_k(\cdot)$ is defined by the unique solution $(Y_k(\cdot),Z_k(\cdot)) \in \mathbb{H}^\infty_{t_{k-1},t_k}(\mathbb{R}) \times \mathbb{H}^2_{t_{k-1},t_k}(\mathbb{R})$ of the BSDE

$$Y_k(\tau) = \mathscr{Y}_{k+1}(t_k) + \int_\tau^{t_k} f_k(s,Y_k(s))\mathrm{d}s + \int_\tau^{t_k} Z_k(s)\mathrm{d}W(s), \quad \tau \in [t_{k-1},t_k],$$

with the driver

$$f_k(s,y) = -\frac{1}{g_k(s)}y - \frac{1}{2}\theta^2(s) + \frac{1}{g_k(s)}\left(\ln g_k(s) + 1\right) - r(s).$$

The optimal trading strategy $(\hat{c}_k(\cdot),\hat{u}_k(\cdot)) \in \mathscr{A}_k(t_{k-1},x_{k-1})$ is given by

$$\hat{c}_k(s) = \frac{1}{g_k(s)}, \qquad \hat{u}_k(s) = \frac{\theta(s)}{\sigma(s)},$$

for $s \in [t_{k-1},t_k]$.

In summary, we state the following multi-person game.

Problem (G_Π). For $k = 1,2,\cdots,N$, the state process of Player (k) is given by

$$\begin{cases} \mathrm{d}X_k(s) = [r(s) - c_k(s) + \theta(s)\sigma(s)u_k(s)]X_k(s)\mathrm{d}s + \sigma(s)u_k(s)X_k(s)\mathrm{d}W(s), \quad s \in [t_{k-1},t_k), \\ X_k(t_{k-1}) = X_{k-1}(t_{k-1}), \end{cases}$$

with $X_0(t_0) = x$. He/she chooses a control $(c_k(\cdot),u_k(\cdot)) \in \mathscr{A}_k(t_{k-1},X_{k-1}(t_{k-1}))$ to maximise the performance functional

$$J^k_\Pi\left(x;(c_1(\cdot),u_1(\cdot)),(c_2(\cdot),u_2(\cdot)),\cdots,(c_N(\cdot),u_N(\cdot))\right) := J_k\left(t_{k-1},X_{k-1}(t_{k-1});c_k(\cdot),u_k(\cdot)\right).$$

Definition 4.3.5. *For a given partition $\Pi \in \mathscr{P}[t,T]$, an N-tuple of controls $((\hat{c}_1(\cdot),\hat{u}_1(\cdot)),\cdots,(\hat{c}_N(\cdot),\hat{u}_N(\cdot))) \in \mathscr{A}_1(t_0,x) \times \cdots \times \mathscr{A}_N(t_{N-1},X_N(t_{N-1}))$ is called an open-loop Nash equilibrium strategy of Problem (G_Π) if for all $k = 1,2,\cdots,N$, and for any $(c_k(\cdot),u_k(\cdot)) \in$*

$\mathscr{A}_k(t_{k-1}, X_k(t_{k-1}))$,

$$
\begin{aligned}
&J_\Pi^k\left(x; (\hat{c}_1(\cdot), \hat{u}_1(\cdot)), \cdots, (\hat{c}_{k-1}(\cdot), \hat{u}_{k-1}(\cdot)), (c_k(\cdot), u_k(\cdot)), (\hat{c}_{k+1}(\cdot), \hat{u}_{k+1}(\cdot)), \cdots, (\hat{c}_N(\cdot), \hat{u}_N(\cdot))\right) \\
\leq &J_\Pi^k\left(x; (\hat{c}_1(\cdot), \hat{u}_1(\cdot)), \cdots, (\hat{c}_{k-1}(\cdot), \hat{u}_{k-1}(\cdot)), (\hat{c}_k(\cdot), \hat{u}_k(\cdot)), (\hat{c}_{k+1}(\cdot), \hat{u}_{k+1}(\cdot)), \cdots, (\hat{c}_N(\cdot), \hat{u}_N(\cdot))\right).
\end{aligned}
$$

In this case, we call

$$
(\hat{c}_\Pi(s), \hat{u}_\Pi(s)) := \sum_{k=1}^N (\hat{c}_k(s), \hat{u}_k(s)) \mathbf{1}_{[t_{k-1}, t_k)}(s) \in \mathscr{A}(t, x)
$$

is a Nash equilibrium control of Problem (G_Π) with initial state $(t, x) \in [0, T] \times (0, \infty)$.

We end this section by showing a Nash equilibrium control of Problem (G_Π). First of all, we introduce the following notation. For any partition $\Pi \in \mathscr{P}[t, T]$ and $s \in [t_0, t_N]$, let

$$
l_\Pi(s) = \sum_{k=1}^N t_{k-1} \mathbf{1}_{[t_{k-1}, t_k)}(s).
$$

Let $\mathscr{D}[t, T] = \{(s, \tau) \,:\, t \leq \tau \leq s \leq T\}$. For any $(s, \tau) \in \mathscr{D}[t, T]$, define the bounded function $g_\Pi(s, \tau)$ by

$$
g_\Pi(s, \tau) = \sum_{k=1}^N g_k(s) \mathbf{1}_{[t_{k-1}, t_k)}(\tau).
$$

For $t \leq \tau \leq s \leq T$, let

$$
\begin{aligned}
Y_\Pi(s, \tau) &= \sum_{k=1}^N \left[Y_k(s) \mathbf{1}_{[t_{k-1}, t_k)}(s) + \mathscr{Y}_{k+1}(s) \mathbf{1}_{[t_k, t_N)}(s) \right] \mathbf{1}_{[t_{k-1}, t_k)}(\tau), \\
Z_\Pi(s, \tau) &= \sum_{k=1}^N \left[Z_k(s) \mathbf{1}_{[t_{k-1}, t_k)}(s) + \mathscr{Z}_{k+1}(s) \mathbf{1}_{[t_k, t_N)}(s) \right] \mathbf{1}_{[t_{k-1}, t_k)}(\tau),
\end{aligned}
$$

and

$$
f_\Pi(s, \tau, y) = \sum_{k=1}^N \left[f_k(s, y) \mathbf{1}_{[t_{k-1}, t_k)}(s) + \mathscr{F}_{k+1}(s, y) \mathbf{1}_{[t_k, t_N)}(s) \right] \mathbf{1}_{[t_{k-1}, t_k)}(\tau).
$$

It follows from Appendix C.2 that for $(s, \tau) \in \mathscr{D}[t, T]$ we have

$$
Y_\Pi(s, \tau) = \int_s^{t_N} f_\Pi(v, \tau, Y_\Pi(v, \tau)) \mathrm{d}v + \int_s^{t_N} Z_\Pi(v, \tau) \mathrm{d}W(v). \tag{4.3.7}
$$

From the construction of the multi-person game, we immediately get the following result.

Theorem 4.3.2. *For any partition $\Pi \in \mathscr{P}[t,T]$ and $\tau \in [t,T]$, let $g_\Pi(\cdot,\tau)$ be the unique solution of the ODE*

$$\begin{cases} g'_\Pi(s,\tau) = -g_\Pi(s,\tau)\frac{h'(s-l_\Pi(\tau))}{h(s-l_\Pi(\tau))} - 1, \quad t \le \tau \le s \le t_N, \\ g_\Pi(t_N,\tau) = 1, \end{cases}$$

where $g'_\Pi(s,\tau)$ is the partial derivative with respective to the first variable s. *For any $x \in (0,\infty)$, define $\hat{X}_\Pi(\cdot;t,x)$ by the SDE*

$$\begin{cases} \mathrm{d}\hat{X}_\Pi(s) = [r(s) - \hat{c}_\Pi(s) + \theta(s)\sigma(s)\hat{u}_\Pi(s)]\,\hat{X}_\Pi(s)\mathrm{d}s + \sigma(s)\hat{u}_\Pi(s)\hat{X}_\Pi(s)\mathrm{d}W(s), \quad s \in [t,t_N], \\ \hat{X}_\Pi(t) = x, \end{cases}$$

where

$$\hat{c}_\Pi(s) = \frac{1}{g_\Pi(s,s)}, \qquad \hat{u}_\Pi(s) = \frac{\theta(s)}{\sigma(s)}.$$

Then $\hat{X}_\Pi(\cdot;t,x)$ and $(\hat{c}_\Pi(\cdot),\hat{u}_\Pi(\cdot))$ are equilibrium state process and Nash equilibrium control for Problem (G_Π) with initial state $(t,x) \in [0,T] \times (0,\infty)$, respectively. Furthermore, for any $\tau \in [t,T]$, let $(Y_\Pi(\cdot,\tau), Z_\Pi(\cdot\tau))$ be the solution of the BSDE

$$\begin{cases} \mathrm{d}Y_\Pi(s,\tau) = -f_\Pi(s,\tau,Y_\Pi(s,\tau))\mathrm{d}s - Z_\Pi(s,\tau)\mathrm{d}W(s), \quad \tau \le s \le t_N, \\ Y_\Pi(t_N,\tau) = 0, \end{cases}$$

which is parameterised by τ. Then we have

$$V_\Pi(t,x) = g_{\Pi}(t,t)\,[\ln x - Y_\Pi(t,t)]\,,$$

for any $(t,x) \in [0,T) \times (0,\infty)$.

4.4 Time-Consistent Equilibrium Strategies

Now, we are going to derive a time-consistent solution to the original Problem (N) by letting $\|\Pi\| \to 0$ (i.e., $N \to \infty$). The proofs of the results in this section are given in Appendix C.3. The following definition is similar to Definition 4.3 of Yong (2012b).

Definition 4.4.1. *An adapted process $(\hat{c}(\cdot), \hat{u}(\cdot)) \in \mathscr{A}(t,x)$ is called a time-consistent equilibrium strategy of Problem (N) with initial state $(t,x) \in [0,T] \times (0,\infty)$ if the following hold:*

(i) The SDE

$$\begin{cases} \mathrm{d}\hat{X}(s) = [r(s) - \hat{c}(s) + \theta(s)\sigma(s)\hat{u}(s)]\hat{X}(s)\mathrm{d}s + \sigma(s)\hat{u}(s)\hat{X}(s)\mathrm{d}W(s), \quad s \in [t,T], \\ \hat{X}(t) = x, \end{cases}$$

admits a unique solution $\hat{X}(\cdot) \equiv \hat{X}(\cdot; t, x, \hat{c}(\cdot), \hat{u}(\cdot))$.

(ii) Approximate optimality: For any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any partition $\Pi \in \mathscr{P}[t,T]$ with $\|\Pi\| < \delta$, one has the following: For any $k = 1, 2, \cdots, N$, and $(c_k(\cdot), u_k(\cdot)) \in \mathscr{A}_k(t_{k-1}, \hat{X}(t_{k-1}))$,

$$J\left(t_{k-1}, \hat{X}(t_{k-1}); (\hat{c}(\cdot), \hat{u}(\cdot))\big|_{[t_{k-1}, t_N)}\right) \geq J\left(t_{k-1}, \hat{X}(t_{k-1}); (c_k(\cdot), u_k(\cdot)) \oplus (\hat{c}(\cdot), \hat{u}(\cdot))\big|_{[t_k, t_N)}\right) - \varepsilon, \tag{4.4.1}$$

where

$$\left[(c_k(\cdot), u_k(\cdot)) \oplus (\hat{c}(\cdot), \hat{u}(\cdot))\big|_{[t_k, t_N)}\right](s) = \begin{cases} (c_k(s), u_k(s)), & s \in [t_{k-1}, t_k), \\ (\hat{c}(s), \hat{u}(s)), & s \in [t_k, t_N). \end{cases}$$

For any $s \in [t,T)$, give a partition $\Pi \in \mathscr{P}[t,T]$ with $\|\Pi\| < \delta$ and $s = t_{k-1}$ for some $k = 1, 2, \cdots, N$. Then (4.4.1) implies that along the equilibrium state process $\hat{X}(\cdot)$, the time-consistent equilibrium strategy $(\hat{c}(\cdot), \hat{u}(\cdot))$ keeps the approximate optimality.

Definition 4.4.2. *A function $V(\cdot,\cdot) : [0,T] \times (0,\infty) \to \mathbb{R}$ is called an equilibrium value func-*

tion for Problem (N) if

$$V(t,x) \quad = \quad J(t,x;\hat{c}(\cdot),\hat{u}(\cdot)).$$

For $(s,\tau)\in\mathscr{D}[t,T]$, let

$$\begin{aligned} g(s,\tau) &= \exp\left\{\int_s^T \frac{h'(v-\tau)}{h(v-\tau)}\mathrm{d}v\right\}+\int_s^T \exp\left\{\int_s^z \frac{h'(v-\tau)}{h(v-\tau)}\mathrm{d}v\right\}\mathrm{d}z \\ &= \frac{1}{h(s-\tau)}\left[h(T-\tau)+\int_s^T h(v-\tau)\mathrm{d}v\right], \end{aligned} \tag{4.4.2}$$

and $(Y(\cdot,\tau),Z(\cdot,\tau))$ be the solution of the BSDE

$$\begin{cases} \mathrm{d}Y(s,\tau) = -f(s,\tau,Y(s,\tau))\mathrm{d}s - Z(s,\tau)\mathrm{d}W(s), \quad \tau\le s\le T, \\ Y(T,\tau)=0, \end{cases} \tag{4.4.3}$$

where

$$f(s,\tau,y) = -\frac{1}{g(s,\tau)}y+\frac{1}{g(s,\tau)}\ln g(s,s)+\frac{1}{g(s,s)}-\frac{1}{2}\theta^2(s)-r(s).$$

Note that (4.4.3) is a flow of BSDEs parameterised by the variable τ (see, e.g. El Karoui et al. (1997, Section 2.4)). Furthermore, for $s\in[t,T]$, we define

$$\hat{c}(s)=\frac{1}{g(s,s)}, \qquad \hat{u}(s)=\frac{\theta(s)}{\sigma(s)}. \tag{4.4.4}$$

Now we have the main results of this chapter.

Theorem 4.4.1. *We have*

$$\lim_{\|\Pi\|\to 0}\sup_{(s,\tau)\in\mathscr{D}[t,T]}|g_\Pi(s,\tau)-g(s,\tau)|=0,$$

and for any $\tau\in[t,T]$

$$\lim_{\|\Pi\|\to 0}\mathsf{E}\left[\sup_{s\in[\tau,T]}|Y_\Pi(s,\tau)-Y(s,\tau)|^2+\int_\tau^T|Z_\Pi(s,\tau)-Z(s,\tau)|^2\mathrm{d}s\right]=0.$$

Theorem 4.4.2. *The pair $(\hat{c}(\cdot), \hat{u}(\cdot))$ defined by (4.4.4) is a time-consistent equilibrium strategy of Problem (N) with initial state $(t,x) \in [0,T] \times (0,\infty)$, the corresponding equilibrium state process $\hat{X}(\cdot)$ and equilibrium value function are given by*

$$\begin{cases} \mathrm{d}\hat{X}(s) = \left[r(s) - \frac{1}{g(s,s)} + \theta^2(s)\right]\hat{X}(s)\mathrm{d}s + \theta(s)\hat{X}(s)\mathrm{d}W(s), \quad s \in [t,T], \\ \hat{X}(t) = x \end{cases} \tag{4.4.5}$$

and

$$V(t,x) = g(t,t)\left[\ln x - Y(t,t)\right], \quad \forall (t,x) \in [0,T] \times (0,\infty), \tag{4.4.6}$$

respectively.

Remark 4.4.1. *From the proof of Theorem 4.4.2 (see Appendix C.3), we know that the strategy given by (4.4.4) is admissible, and the equilibrium value function V is the limit of V_Π as $\|\Pi\|$ approaches 0.*

4.5 Concluding Remarks

We investigated a time-inconsistent consumption-investment problem with a general discount function and a logarithmic utility function in a non-Markovian framework. We followed Yong's approach to study an N-player differential game. Using the martingale method, we solved the optimisation problem of each player and characterised their optimal strategies and value functions. We obtained a time-consistent equilibrium consumption-investment strategy and the corresponding equilibrium value function of the original problem.

The same problem was studied in Marín-Solano and Navas (2010) under a model with deterministic coefficients. The authors solved this problem with general utility functions for both naive and sophisticated agents. Specifically, the authors showed that, for the logarithmic utility function, the naive strategy coincided with the one for the sophisticated agent. This special feature for the logarithmic utility function was also shown in Pollak (1968) and Marín-Solano and Navas (2009) which considered the cake-eating problem in deterministic models. Our results show that this coincidence is preserved for the model with stochastic coefficients. In fact, at any time $t \in [0,T]$ with wealth $x > 0$, a naive t-agent will solve Problem

(C) in Appendix C.1.1 with $T_1 = t, T_2 = T, a = 1$ and $F = 0$. In this special case, the function $g(\cdot)$ given by (C.1.3) (to emphasise the dependence on the time t, here we denote it by $g_t(\cdot)$) becomes

$$g_t(s) = \exp\left\{\int_s^T \frac{h'(v-t)}{h(v-t)} \mathrm{d}v\right\} + \int_s^T \exp\left\{\int_s^z \frac{h'(v-t)}{h(v-t)} \mathrm{d}v\right\} \mathrm{d}z, \quad s \in [t,T].$$

Thus, from Theorem C.1.1, the naive strategy (denoted by $(\tilde{c}(\cdot), \tilde{u}(\cdot))$) is given by

$$\tilde{c}(t) = \frac{1}{g_t(t)}, \qquad \tilde{u}(t) = \frac{\theta(t)}{\sigma(t)}, \qquad t \in [0,T].$$

Note that $g_t(t)$ equals to $g(t,t)$ (with an abuse of notation g), where the function $g(\cdot,\cdot)$ is defined by (4.4.2). Thus, in our case, the naive strategy coincides with the time-consistent strategy given in Section 4.4. It also shows that the time-consistent strategy in our non-Markovian framework is essentially the same with the one obtained in Marín-Solano and Navas (2010) (note that $h(0) = 1$, in their paper the consumption strategy is the dollar amount and in this chapter it is the proportion of the wealth).

Furthermore, similar to Marín-Solano and Navas (2010), the same problem can be considered with power and exponential utilities. However, there are some difficulties we wish to remark here. For logarithmic utilities, it follows from Theorem 4.3.1 that the optimal strategy of Player (k), i.e. $(\hat{c}_k(\cdot), \hat{u}_k(\cdot))$ is independent of $(Y_k(\cdot), Z_k(\cdot))$. Consequently, $(\mathscr{Y}_k(\cdot), \mathscr{Z}_k(\cdot))$, which depends on $(\hat{c}_n(\cdot), \hat{u}_n(\cdot)), n = k, \cdots, N$, is also independent of $(Y_n(\cdot), Z_n(\cdot)), n = k, \cdots, N$. Thus, by defining $Y_\Pi(\cdot,\cdot)$,$Z_\Pi(\cdot,\cdot)$ and $f_\Pi(\cdot,\cdot,\cdot)$, we get a standard BSDE (with a parameter), i.e. Equation (4.3.7). Then letting the mesh size approach zero, we get the limit equation of (4.3.7), i.e., Equation (4.4.3), which is also a standard BSDE. Unfortunately, this is not the case for power and exponential utilities. From the results obtained by Cheridito and Hu (2011), it is easy to see that for power and exponential utilities the optimal strategy of Player (k) depends on $(Y_k(\cdot), Z_k(\cdot))$, which is the unique solution of some BSDE. Similarly, we can define $(\mathscr{Y}_k(\cdot), \mathscr{Z}_k(\cdot))$ as the unique solution to some BSDE. But in these cases, $(Y_n(\cdot), Z_n(\cdot)), n = k, \cdots, N$, appear in the driver of the BSDE satisfied by $(\mathscr{Y}_k(\cdot), \mathscr{Z}_k(\cdot))$. Similarly, we define $Y_\Pi(\cdot,\cdot), Z_\Pi(\cdot,\cdot)$ and $f_\Pi(\cdot,\cdot,\cdot,\cdot)$. Unlike Equations (4.3.7) and (4.4.3), we can not get any standard equations for $(Y_\Pi(\cdot,\cdot), Z_\Pi(\cdot,\cdot))$ and the limit process $(Y(\cdot,\cdot), Z(\cdot,\cdot))$. At the moment, it is not clear to us what kind of (non-standard) equation it will lead to and how we can get the existence and uniqueness of the solution of the non-standard equation. Therefore, we leave the cases with power and exponential utilities to our future research.

Chapter 5

Conclusion

In this thesis, we discussed several continuous-time stochastic control problems with non-exponential discounting in finance and actuarial science. At first, we considered the dividend optimisation problem in a diffusion risk model. Then we investigated the defined benefit pension problem under a quadratic performance criterion. In the above two topics, we obtained the equilibrium HJB equations for a general discounting and found the equilibrium controls within the class of feedback controls in some special cases. Finally, we studied a multi-person differential game for the consumption-investment stochastic control problem with logarithmic utility in a non-Markovian model and obtained the open-loop time-consistent equilibrium consumption-investment strategy for the original problem.

We also discussed some extensions to the current research. In Chapter 4, we considered the consumption-investment optimisation problem for a logarithmic utility. The interest rate, appreciation rate and volatility of the stock are assumed to be adapted stochastic processes. The research on the time-inconsistent optimisation problems for the state processes with random coefficients can also be extended to the cases with power and exponential utilities. We remarked some difficulties regarding to these problems at the end of Chapter 4.

Another potential research is the time-inconsistent LQ control problems for SDEs and mean-field SDEs with random coefficients. We may consider a general performance functional including the non-exponential discounting as a special case. The mean-variance portfolio selection problem and mean-variance asset-liability management problem can be considered as applications.

Appendix A

Appendix A.1

Lemma A.1.1. $\frac{\theta_1}{\delta} - \frac{1}{\mu+\sigma^2\theta_1} > 0$ *and* $\frac{1}{\mu-M-\sigma^2\theta_3} + \frac{\theta_3}{\delta} > 0.$

Proof. Recall that θ_1 and θ_3 are given by

$$\theta_1 = \frac{-\mu + \sqrt{\mu^2+2\sigma^2\delta}}{\sigma^2}, \quad \theta_3 = \frac{\mu - M + \sqrt{(\mu-M)^2+2\sigma^2\delta}}{\sigma^2}.$$

Then it follows that

$$\begin{aligned}
\frac{\theta_1}{\delta} - \frac{1}{\mu+\sigma^2\theta_1} &= -\frac{1}{\sqrt{\mu^2+2\sigma^2\delta}} + \frac{-\mu + \sqrt{\mu^2+2\sigma^2\delta}}{\sigma^2\delta} \\
&= \frac{\left(-\mu + \sqrt{\mu^2+2\sigma^2\delta}\right)\sqrt{\mu^2+2\sigma^2\delta} - \sigma^2\delta}{\sigma^2\delta\sqrt{\mu^2+2\sigma^2\delta}} \\
&= \frac{-\mu\sqrt{\mu^2+2\sigma^2\delta} + \mu^2 + \sigma^2\delta}{\sigma^2\delta\sqrt{\mu^2+2\sigma^2\delta}} \\
&= \frac{\frac{1}{2}\mu^2 - 2\frac{\sqrt{2}}{2}\mu\sqrt{\frac{1}{2}\mu^2+\sigma^2\delta} + \left[\frac{1}{2}\mu^2+\sigma^2\delta\right]}{\sigma^2\delta\sqrt{\mu^2+2\sigma^2\delta}} \\
&= \frac{\left[\frac{\sqrt{2}}{2}\mu - \sqrt{\frac{1}{2}\mu^2+\sigma^2\delta}\right]^2}{\sigma^2\delta\sqrt{\mu^2+2\sigma^2\delta}} > 0,
\end{aligned}$$

and

$$\frac{1}{\mu-M-\sigma^2\theta_3} + \frac{\theta_3}{\delta} = -\frac{1}{\sqrt{(\mu-M)^2+2\sigma^2\delta}} + \frac{\mu-M+\sqrt{(\mu-M)^2+2\sigma^2\delta}}{\sigma^2\delta}$$

$$=\frac{\left(\mu-M+\sqrt{(\mu-M)^2+2\sigma^2\delta}\right)\sqrt{(\mu-M)^2+2\sigma^2\delta}-\sigma^2\delta}{\sigma^2\delta\sqrt{(\mu-M)^2+2\sigma^2\delta}}$$
$$=\frac{(\mu-M)\sqrt{(\mu-M)^2+2\sigma^2\delta}+(\mu-M)^2+\sigma^2\delta}{\sigma^2\delta\sqrt{(\mu-M)^2+2\sigma^2\delta}}$$
$$=\frac{\frac{1}{2}(\mu-M)^2+2\frac{\sqrt{2}}{2}(\mu-M)\sqrt{\frac{1}{2}(\mu-M)^2+\sigma^2\delta}+\left[\frac{1}{2}(\mu-M)^2+\sigma^2\delta\right]}{\sigma^2\delta\sqrt{(\mu-M)^2+2\sigma^2\delta}}$$
$$=\frac{\left[\frac{\sqrt{2}}{2}(\mu-M)+\sqrt{\frac{1}{2}(\mu-M)^2+\sigma^2\delta}\right]^2}{\sigma^2\delta\sqrt{(\mu-M)^2+2\sigma^2\delta}}>0.$$

□

Proof of Lemma 2.4.2. It is easy to check that

$$G(0) \;=\; (\theta_1+\theta_2)\left\{\lambda\left[\frac{1}{\mu-M-\sigma^2\theta_3}+\frac{\theta_3}{\delta}\right]\frac{M}{\delta}+\theta_3\frac{M}{\delta}-1\right\}$$

By Lemma A.1.1 and (2.4.33) we have $G(0)>0$. Now by (2.4.22), (2.4.23) and (2.4.25), we rewrite $G(b)$ as

$$\begin{aligned}
G(b) \;=\;& \theta_3\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\frac{1}{(\theta_1+\theta_3)e^{\theta_1 b}+(\theta_2-\theta_3)e^{-\theta_2 b}}\left(-e^{2\theta_1 b}+e^{-2\theta_2 b}\right)\\
&+\frac{\lambda}{\mu-M-\sigma^2\theta_3}\frac{M}{\delta}\frac{1}{(\theta_1+\theta_3)e^{\theta_1 b}+(\theta_2-\theta_3)e^{-\theta_2 b}}\left(\theta_1 e^{\theta_1 b}+\theta_2 e^{-\theta_2 b}\right)^2\\
&+2(\theta_1+\theta_2)\theta_3\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\frac{1}{(\theta_1+\theta_3)e^{\theta_1 b}+(\theta_2-\theta_3)e^{-\theta_2 b}}be^{(\theta_1-\theta_2)b}\\
&+\left[\theta_1\theta_3\left(1+\frac{\lambda}{\delta}\right)\frac{M}{\delta}-(\theta_1+\theta_3)\right]e^{\theta_1 b}+\left[\theta_2\theta_3\left(1+\frac{\lambda}{\delta}\right)\frac{M}{\delta}-(\theta_2-\theta_3)\right]e^{-\theta_2 b}\\
:=\;& \frac{1}{(\theta_1+\theta_3)e^{\theta_1 b}+(\theta_2-\theta_3)e^{-\theta_2 b}}g(b),
\end{aligned}$$

where

$$g(b):=2(\theta_1+\theta_2)\theta_3\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}be^{(\theta_1-\theta_2)b}$$

$$
\begin{aligned}
&+e^{2\theta_1 b}\left\{-\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3^2}{\delta}+\frac{\lambda}{\mu-M-\sigma^2\theta_3}\frac{M\theta_1^2}{\delta}\right.\\
&\left.+\left[\theta_1\theta_3\left(1+\frac{\lambda}{\delta}\right)\frac{M}{\delta}-(\theta_1+\theta_3)\right](\theta_1+\theta_3)\right\}\\
&+e^{(\theta_1-\theta_2)b}\left\{2\theta_1\theta_2\frac{\lambda}{\mu-M-\sigma^2\theta_3}\frac{M}{\delta}+\left[\theta_1\theta_3\left(1+\frac{\lambda}{\delta}\right)\frac{M}{\delta}-(\theta_1+\theta_3)\right](\theta_2-\theta_3)\right.\\
&\left.+\left[\theta_2\theta_3\left(1+\frac{\lambda}{\delta}\right)\frac{M}{\delta}-(\theta_2-\theta_3)\right](\theta_1+\theta_3)\right\}\\
&+e^{-2\theta_2 b}\left\{\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3^2}{\delta}+\frac{\lambda}{\mu-M-\sigma^2\theta_3}\frac{M\theta_2^2}{\delta}\right.\\
&\left.+\left[\theta_2\theta_3\left(1+\frac{\lambda}{\delta}\right)\frac{M}{\delta}-(\theta_2-\theta_3)\right](\theta_2-\theta_3)\right\}\\
=&2(\theta_1+\theta_2)\theta_3\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}be^{(\theta_1-\theta_2)b}\\
&+e^{2\theta_1 b}\left\{\lambda\frac{M}{\delta}\left[\theta_3^2\left(-\frac{1}{\mu+\sigma^2\theta_1}+\frac{\theta_1}{\delta}\right)+\theta_1^2\left(\frac{1}{\mu-M-\sigma^2\theta_3}+\frac{\theta_3}{\delta}\right)\right]\right.\\
&\left.+\theta_1\theta_3(\theta_1+\theta_3)\left(\frac{M}{\delta}-\frac{1}{\theta_3}-\frac{1}{\theta_1}\right)\right\}\\
&+e^{(\theta_1-\theta_2)b}\left\{\lambda\frac{M}{\delta}\left[2\theta_1\theta_2\frac{1}{\mu-M-\sigma^2\theta_3}+\theta_3(2\theta_1\theta_2-\theta_1\theta_3+\theta_2\theta_3)\frac{1}{\delta}\right]\right.\\
&\left.+\theta_3\frac{M}{\delta}(2\theta_1\theta_2-\theta_1\theta_3+\theta_2\theta_3)-2(\theta_2-\theta_3)(\theta_1+\theta_3)\right\}\\
&+e^{-2\theta_2 b}\left\{\lambda\frac{M}{\delta}\left[\theta_3^2\left(\frac{1}{\mu+\sigma^2\theta_1}-\frac{\theta_2}{\delta}\right)+\theta_2^2\left(\frac{1}{\mu-M-\sigma^2\theta_3}+\frac{\theta_3}{\delta}\right)\right]\right.\\
&\left.+(\theta_2-\theta_3)\left[\theta_2\theta_3\frac{M}{\delta}-(\theta_2-\theta_3)\right]\right\}.
\end{aligned}
$$

From Lemma 2.1 of Asmussen and Taksar (1997), we have

$$
\frac{M}{\delta}-\frac{1}{\theta_3}-\frac{1}{\theta_1}<0.
$$

By Lemma A.1.1 and (2.4.33), it is easy to see that the coefficient of $e^{2\theta_1 b}$ satisfies

$$
\lambda\frac{M}{\delta}\left[\theta_3^2\left(-\frac{1}{\mu+\sigma^2\theta_1}+\frac{\theta_1}{\delta}\right)+\theta_1^2\left(\frac{1}{\mu-M-\sigma^2\theta_3}+\frac{\theta_3}{\delta}\right)\right]+\theta_1\theta_3(\theta_1+\theta_3)\left(\frac{M}{\delta}-\frac{1}{\theta_3}-\frac{1}{\theta_1}\right)<0.
$$

Thus, $G(\infty)<0$. Therefore, the equation $G(b)=0$ admits a positive solution. □

In the following, we make some comments on the uniqueness of b. Let us consider the equation

$$\left[(\theta_1+\theta_3)e^{\theta_1 b}+(\theta_2-\theta_3)e^{-\theta_2 b}\right]^{-1}G(b)=0,$$

which is the original equation satisfied by b, i.e., (2.4.30). Denote by $\hat{G}(b)$ the left-hand side of the above equation. After tedious calculations, we have

$$\begin{aligned}
&\left[(\theta_1+\theta_3)e^{\theta_1 b}+(\theta_2-\theta_3)e^{-\theta_2 b}\right]^3\hat{G}'(b)\\
=&2\left\{-(\theta_1+\theta_3)e^{\theta_1 b}+(\theta_2-\theta_3)e^{-\theta_2 b}\right\}(\theta_1+\theta_2)^2\theta_3\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}be^{(\theta_1-\theta_2)b}\\
&+e^{(2\theta_1-\theta_2)b}\left\{\lambda\frac{M}{\delta}\theta_3(\theta_1+\theta_2)\left[2\left(\frac{\theta_1}{\delta}-\frac{1}{\mu+\sigma^2\theta_1}\right)\theta_3(\theta_2-\theta_1-2\theta_3)\right.\right.\\
&\left.-2\left(\frac{1}{\mu-M-\sigma^2\theta_3}+\frac{\theta_3}{\delta}\right)\theta_1(\theta_1+\theta_2)-\frac{\theta_3}{\delta}(\theta_1+\theta_3)(\theta_2-3\theta_1)\right]\\
&\left.-\frac{M}{\delta}(\theta_1+\theta_3)\theta_3^2(\theta_1+\theta_2)^2\right\}\\
&+e^{(\theta_1-2\theta_2)b}\left\{\lambda\frac{M}{\delta}\theta_3(\theta_1+\theta_2)\left[-2(\theta_2-\theta_3)\theta_3\left(\frac{\theta_1}{\delta}-\frac{1}{\mu+\sigma^2\theta_1}\right)\right.\right.\\
&-2\left(\frac{1}{\mu-M-\sigma^2\theta_3}+\frac{\theta_3}{\delta}\right)\theta_2(\theta_1+\theta_2)+2\theta_3\left(\frac{\theta_2}{\delta}-\frac{1}{\mu+\sigma^2\theta_1}\right)(\theta_1+\theta_3)\\
&\left.\left.+\frac{\theta_3}{\delta}(\theta_2-\theta_3)(\theta_1+\theta_2)\right]-\frac{M}{\delta}\theta_3^2(\theta_1+\theta_2)^2(\theta_2-\theta_3)\right\}.
\end{aligned}$$

Obviously, if $\lambda=0$, which is the case with exponential discounting, then $\hat{G}'(b)<0$. If $\lambda>0$ is sufficient small, then the coefficients of $e^{(2\theta_1-\theta_2)b}$ and $e^{(\theta_1-2\theta_2)b}$ are negative. If, in additional, assume that $\theta_2-\theta_1-2\theta_3<0$, then the decreasing function

$$-(\theta_1+\theta_3)e^{\theta_1 b}+(\theta_2-\theta_3)e^{-\theta_2 b}<0,$$

which means the coefficients of $be^{(\theta_1-\theta_2)b}<0$. Thus, we get $\hat{G}'(b)<0$ which guarantees the uniqueness of b.

Appendix A.2

Proof of Lemma 2.4.3. Assume that b is a positive solution to $G(x)=0$. It follows that

$$\begin{aligned}
&\theta_1\hat{C}-3B_1-\theta_1B_1b\\
=&\frac{\theta_1-2\theta_1\theta_2bB_1e^{-\theta_2b}-2B_1\theta_1e^{\theta_1b}+B_1\left(\theta_1-3\theta_2\right)e^{-\theta_2b}}{\theta_1e^{\theta_1b}+\theta_2e^{-\theta_2b}}\\
=&\frac{\theta_1+B_1\left[-2\theta_1e^{\theta_1b}+\left(-2\theta_1\theta_2b+\theta_1-3\theta_2\right)e^{-\theta_2b}\right]}{\theta_1e^{\theta_1b}+\theta_2e^{-\theta_2b}}\\
=&\frac{\theta_1\left[\theta_1+\theta_3-2\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\right]e^{(\theta_1+\theta_2)b}+\left[\theta_1\left(\theta_2-\theta_3\right)+\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\left(-2\theta_1\theta_2b+\theta_1-3\theta_2\right)\right]}{e^{\theta_2b}\left(\theta_1e^{\theta_1b}+\theta_2e^{-\theta_2b}\right)\left[\left(\theta_1+\theta_3\right)e^{\theta_1b}+\left(\theta_2-\theta_3\right)e^{-\theta_2b}\right]}.
\end{aligned} \tag{A.2.1}$$

Let

$$\begin{aligned}
q(b):=&\theta_1\left[\theta_1+\theta_3-2\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\right]e^{(\theta_1+\theta_2)b}\\
&+\left[\theta_1\left(\theta_2-\theta_3\right)+\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\left(-2\theta_1\theta_2b+\theta_1-3\theta_2\right)\right].
\end{aligned}$$

Then

$$\begin{aligned}
q'(b)&=\theta_1\left(\theta_1+\theta_2\right)\left[\theta_1+\theta_3-2\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\right]e^{(\theta_1+\theta_2)b}-2\theta_1\theta_2\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta},\\
q''(b)&=\theta_1\left(\theta_1+\theta_2\right)^2\left[\theta_1+\theta_3-2\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\right]e^{(\theta_1+\theta_2)b},
\end{aligned}$$

and

$$\begin{aligned}
q(0)&=\theta_1\left[\theta_1+\theta_3-2\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\right]+\theta_1\left(\theta_2-\theta_3\right)+\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\left(\theta_1-3\theta_2\right)\\
&=\theta_1\left(\theta_1+\theta_2\right)-\left(\theta_1+3\theta_2\right)\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta},\\
q'(0)&=\theta_1\left(\theta_1+\theta_2\right)\left[\theta_1+\theta_3-2\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\right]-2\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\theta_1\theta_2\\
&=\theta_1\left(\theta_1+\theta_2\right)\left(\theta_1+\theta_3\right)-2\theta_1\left(\theta_1+2\theta_2\right)\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}.
\end{aligned}$$

If $\lambda \leq \frac{\theta_1+\theta_2}{\theta_1+3\theta_2}\frac{\delta^2}{M\theta_3} \wedge \frac{(\theta_1+\theta_3)(\theta_1+\theta_2)}{2\theta_1(\theta_1+2\theta_2)}\frac{\delta^2}{M\theta_3}$ holds, then it follows from Lemma A.1.1 that

$$q(0) = (\theta_1+\theta_2)\left[\theta_1 - \frac{\theta_1+3\theta_2}{\theta_1+\theta_2}\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\right] \geq \delta(\theta_1+\theta_2)\left(\frac{\theta_1}{\delta} - \frac{1}{\mu+\sigma^2\theta_1}\right) > 0.$$

and similarly,

$$\begin{aligned} q'(0) &= (\theta_1+\theta_2)(\theta_1+\theta_3)\left[\theta_1 - \frac{2\theta_1(\theta_1+2\theta_2)}{(\theta_1+\theta_2)(\theta_1+\theta_3)}\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\right] \\ &\geq \delta(\theta_1+\theta_2)(\theta_1+\theta_3)\left(\frac{\theta_1}{\delta} - \frac{1}{\mu+\sigma^2\theta_1}\right) > 0. \end{aligned}$$

Thus, it follows from $q'(0) \geq 0$ that

$$\theta_1(\theta_1+\theta_2)\left[\theta_1+\theta_3 - 2\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\right] \geq 2\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta}\theta_1\theta_2 > 0.$$

Therefore, $q''(b) > 0$, $q'(b) > 0$, and then $q(b) > 0$. Finally, it follows from (A.2.1) that

$$\theta_1\hat{C} - 3B_1 - \theta_1 B_1 b > 0.$$

$\square$

In fact, the result of Lemma 2.4.3 may also hold under some other conditions rather than (2.4.34). For example, let us assume

$$\frac{\theta_1(\theta_1+\theta_3)}{\theta_1+3\theta_2} - \frac{M\theta_3}{\mu+\sigma^2\theta_1} > 0.$$

Recalling $0 < \lambda < \delta$ and $0 < \theta_1, \theta_3 < \theta_2$, then we have

$$\begin{aligned} \theta_1+\theta_3 - 2\frac{\lambda}{\mu+\sigma^2\theta_1}\frac{M\theta_3}{\delta} &> \theta_1+\theta_3 - 2\frac{M\theta_3}{\mu+\sigma^2\theta_1} \\ &> 2\left(\frac{\theta_1+\theta_3}{2} - \frac{M\theta_3}{\mu+\sigma^2\theta_1}\right) \\ &> 0, \end{aligned}$$

which means $q''(b) > 0$. Similarly,

$$\theta_1 (\theta_1 + \theta_2) - (\theta_1 + 3\theta_2) \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta} > \theta_1 (\theta_1 + \theta_3) - (\theta_1 + 3\theta_2) \frac{M\theta_3}{\mu + \sigma^2 \theta_1} > 0$$

and

$$\begin{aligned}
&(\theta_1 + \theta_2)(\theta_1 + \theta_3) - 2(\theta_1 + 2\theta_2) \frac{\lambda}{\mu + \sigma^2 \theta_1} \frac{M\theta_3}{\delta} \\
&> \theta_1 (\theta_1 + \theta_3) - (\theta_1 + \theta_2) \frac{M\theta_3}{\mu + \sigma^2 \theta_1} \\
&\quad + \theta_2 (\theta_1 + \theta_3) - (\theta_1 + 3\theta_2) \frac{M\theta_3}{\mu + \sigma^2 \theta_1} \\
&> 0,
\end{aligned}$$

which implies $q(0) > 0$ and $q'(0) > 0$, respectively. Thus, we still get the result.

Appendix B

Appendix B.1

Proof of Proposition 3.4.1. Step 1. We show the boundness and uniqueness of (3.4.11). Let us consider the first differential equation in (3.4.11) at first. Substituting β and A into this equation, we get

$$\begin{cases} P(t) & = \int_t^T h(s-t)\left(\frac{1}{\alpha_1}P^2(s)+\alpha_2\right)e^{\int_t^s\left(2r-\theta^2-\frac{2}{\alpha_1}P(u)\right)\mathrm{d}u}\mathrm{d}s \\ & \quad +\alpha_3 h(T-t)\,e^{\int_t^T\left(2r-\theta^2-\frac{2}{\alpha_1}P(u)\right)\mathrm{d}u}>0, \quad \forall t\in[0,T), \\ P(T) & =\alpha_3, \end{cases} \tag{B.1.1}$$

which is a integral equation for $P(t)$.

Since $-\rho \le \frac{h'(z)}{h(z)} \le \rho$, for $0 \le z \le T$, we have

$$-\rho P(t) \le P'(t) - \frac{1}{\alpha_1}P^2(t) + \left(2r-\theta^2\right)P(t) + \alpha_2 \le \rho P(t),$$

i.e.,

$$P'(t) \le \frac{1}{\alpha_1}P^2(t) + \left(\theta^2 - 2r + \rho\right)P(t) - \alpha_2, \tag{B.1.2}$$

$$P'(t) \ge \frac{1}{\alpha_1}P^2(t) + \left(\theta^2 - 2r - \rho\right)P(t) - \alpha_2. \tag{B.1.3}$$

From (B.1.3) and $\frac{1}{\alpha_1}P^2(t) \ge 0$, we know that

$$P'(t) - \left(\theta^2 - 2r - \rho\right)P(t) \ge -\alpha_2. \tag{B.1.4}$$

Similar to the proof of (Ekeland and Pirvu, 2008, Lemma 3.1), it follows from (B.1.4) that

$$\left(e^{-\left(\theta^2-2r-\rho\right)t}P(t)\right)' \geq -\alpha_2 e^{-\left(\theta^2-2r-\rho\right)t}.$$

Integrating this from t to T, we get the upper estimate on $P(t)$:

$$P(t) \leq \left(\alpha_3 - \frac{\alpha_2}{\theta^2-2r-\rho}\right)e^{-\left(\theta^2-2r-\rho\right)(T-t)} + \frac{\alpha_2}{\theta^2-2r-\rho}.$$

Similarly for (B.1.2), we have

$$P'(t) \leq \frac{1}{\alpha_1}P^2(t) + \left(\theta^2-2r+\rho\right)P(t). \tag{B.1.5}$$

It follows from (B.1.5) that

$$\vartheta'(t) + \left(\theta^2-2r+\rho\right)\vartheta(t) \geq -\frac{1}{\alpha_1},$$

where

$$\vartheta(t) := P^{-1}(t).$$

Therefore,

$$\left(e^{\left(\theta^2-2r+\rho\right)t}\vartheta(t)\right)' \geq -\frac{1}{\alpha_1}e^{\left(\theta^2-2r+\rho\right)t}.$$

After integrating this from t to T, we get the lower estimate on $P(t)$:

$$P(t) \geq \left[\left(\frac{1}{\alpha_3} + \frac{1}{\alpha_1\left(\theta^2-2r+\rho\right)}\right)e^{\left(\theta^2-2r+\rho\right)(T-t)} - \frac{1}{\alpha_1\left(\theta^2-2r+\rho\right)}\right]^{-1}.$$

The boundness together with the local existence show that (B.1.1) has a global solution.

Now we show the uniqueness by contraposition. Let P_1 and P_2 be two solutions of (B.1.1). Without loss of generality, we assume that $P_2(t) > P_1(t)$. Then

$$|P_1(t) - P_2(t)|$$

$$
\begin{aligned}
&\leq \left| \int_t^T h(s-t)\, e^{\left(2r-\theta^2\right)(s-t)} \left(\frac{1}{\alpha_1} P_1^2(s) + \alpha_2 \right) e^{-\frac{2}{\alpha_1}\int_t^s P_1(u)\mathrm{d}u} \mathrm{d}s \right. \\
&\quad \left. - \int_t^T h(s-t)\, e^{\left(2r-\theta^2\right)(s-t)} \left(\frac{1}{\alpha_1} P_2^2(s) + \alpha_2 \right) e^{-\frac{2}{\alpha_1}\int_t^s P_2(u)\mathrm{d}u} \mathrm{d}s \right| \\
&+ \left| \alpha_3 h(T-t)\, e^{\left(2r-\theta^2\right)(T-t)} e^{-\frac{2}{\alpha_1}\int_t^T P_1(u)\mathrm{d}u} - \alpha_3 h(T-t)\, e^{\left(2r-\theta^2\right)(T-t)} e^{-\frac{2}{\alpha_1}\int_t^T P_2(u)\mathrm{d}u} \right|.
\end{aligned}
$$

It is easy to get that

$$
\begin{aligned}
&\left| \alpha_3 h(T-t)\, e^{\left(2r-\theta^2\right)(T-t)} e^{-\frac{2}{\alpha_1}\int_t^T P_1(u)\mathrm{d}u} - \alpha_3 h(T-t)\, e^{\left(2r-\theta^2\right)(T-t)} e^{-\frac{2}{\alpha_1}\int_t^T P_2(u)\mathrm{d}u} \right| \\
=&\alpha_3 h(T-t)\, e^{\left(2r-\theta^2\right)(T-t)} e^{-\frac{2}{\alpha_1}\int_t^T P_1(u)\mathrm{d}u} \left| 1 - e^{\frac{2}{\alpha_1}\int_t^T (P_1(u)-P_2(u))\mathrm{d}u} \right| \\
\leq&\alpha_3 h(T-t)\, e^{\left(2r-\theta^2\right)(T-t)} e^{-\frac{2}{\alpha_1}\int_t^T P_1(u)\mathrm{d}u} \frac{2}{\alpha_1} \int_t^T |P_1(u) - P_2(u)|\, \mathrm{d}u \\
\leq&M_1 \int_t^T |P_1(u) - P_2(u)|\, \mathrm{d}u,
\end{aligned}
$$

for some constant $M_1 > 0$, and

$$
\begin{aligned}
&\left| \int_t^T h(s-t)\, e^{\left(2r-\theta^2\right)(s-t)} \left(\frac{1}{\alpha_1} P_1^2(s) + \alpha_2 \right) e^{-\frac{2}{\alpha_1}\int_t^s P_1(u)\mathrm{d}u} \mathrm{d}s \right. \\
&\quad \left. - \int_t^T h(s-t)\, e^{\left(2r-\theta^2\right)(s-t)} \left(\frac{1}{\alpha_1} P_2^2(s) + \alpha_2 \right) e^{-\frac{2}{\alpha_1}\int_t^s P_2(u)\mathrm{d}u} \mathrm{d}s \right| \\
\leq&\frac{1}{\alpha_1} \int_t^T h(s-t)\, e^{\left(2r-\theta^2\right)(s-t)} \left| P_1^2(s) e^{-\frac{2}{\alpha_1}\int_t^s P_1(u)\mathrm{d}u} - P_2^2(s) e^{-\frac{2}{\alpha_1}\int_t^s P_2(u)\mathrm{d}u} \right| \mathrm{d}s \\
&+ \alpha_2 \int_t^T h(s-t)\, e^{\left(2r-\theta^2\right)(s-t)} \left| e^{-\frac{2}{\alpha_1}\int_t^s P_1(u)\mathrm{d}u} - e^{-\frac{2}{\alpha_1}\int_t^s P_2(u)\mathrm{d}u} \right| \mathrm{d}s. \qquad \text{(B.1.6)}
\end{aligned}
$$

For the first part and second part in the right-hand-side of (B.1.6), we have

$$
\begin{aligned}
&\frac{1}{\alpha_1} \int_t^T h(s-t)\, e^{\left(2r-\theta^2\right)(s-t)} \left| P_1^2(s) e^{-\frac{2}{\alpha_1}\int_t^s P_1(u)\mathrm{d}u} - P_2^2(s) e^{-\frac{2}{\alpha_1}\int_t^s P_2(u)\mathrm{d}u} \right| \mathrm{d}s \\
\leq&\frac{1}{\alpha_1} \int_t^T h(s-t)\, e^{\left(2r-\theta^2\right)(s-t)} P_1^2(s) \left| e^{-\frac{2}{\alpha_1}\int_t^s P_1(u)\mathrm{d}u} - e^{-\frac{2}{\alpha_1}\int_t^s P_2(u)\mathrm{d}u} \right| \mathrm{d}s \\
&+ \frac{1}{\alpha_1} \int_t^T h(s-t)\, e^{\left(2r-\theta^2\right)(s-t)} e^{-\frac{2}{\alpha_1}\int_t^s P_2(u)\mathrm{d}u} (P_1(s) + P_2(s))\, |P_1(s) - P_2(s)|\, \mathrm{d}s \\
\leq&M_2 \int_t^T |P_1(u) - P_2(u)|\, \mathrm{d}u,
\end{aligned}
$$

and

$$
\begin{aligned}
&\alpha_2 \int_t^T h(s-t)\, e^{\left(2r-\theta^2\right)(s-t)} \left| e^{-\frac{2}{\alpha_1}\int_t^s P_1(u)\mathrm{d}u} - e^{-\frac{2}{\alpha_1}\int_t^s P_2(u)\mathrm{d}u} \right| \mathrm{d}s \\
=&\alpha_2 \int_t^T h(s-t)\, e^{\left(2r-\theta^2\right)(s-t)} e^{-\frac{2}{\alpha_1}\int_t^s P_1(u)\mathrm{d}u} \left| 1 - e^{\frac{2}{\alpha_1}\int_t^s (P_1(u)-P_2(u))\mathrm{d}u} \right| \mathrm{d}s \\
\leq&\alpha_2 \int_t^T h(s-t)\, e^{\left(2r-\theta^2\right)(s-t)} e^{-\frac{2}{\alpha_1}\int_t^s P_1(u)\mathrm{d}u} \frac{2}{\alpha_1} \int_t^s |P_1(u)-P_2(u)|\, \mathrm{d}u\mathrm{d}s \\
=&\frac{2\alpha_2}{\alpha_1} \int_t^T |P_1(u)-P_2(u)| \int_u^T h(s-t)\, e^{\left(2r-\theta^2\right)(s-t)} e^{-\frac{2}{\alpha_1}\int_t^s P_1(u)\mathrm{d}u} \mathrm{d}s\mathrm{d}u \\
\leq&M_3 \int_t^T |P_1(u)-P_2(u)|\, \mathrm{d}u,
\end{aligned}
$$

for some positive constants M_2 and M_3. Therefore, there exists a positive constant M such that

$$
|P_1(t)-P_2(t)| \leq M \int_t^T |P_1(u)-P_2(u)|\, \mathrm{d}u.
$$

Then it follows from the Gronwal's inequality that $|P_1(t)-P_2(t)| = 0$. Now we know that there exists a unique global solution for the first equation of (3.4.11).

Similarly we obtain the boundness for Q. Since $-\rho \leq \frac{h'(z)}{h(z)} \leq \rho$, for $0 \leq z \leq T$, we have

$$
-\rho|Q(t)| \leq Q'(t) + \left(r - \theta^2 - \frac{1}{\alpha_1}P(t)\right) Q(t) + \delta ALP(t) + \alpha_2 AL \leq \rho|Q(t)|,
$$

i.e.,

$$
Q'(t) \leq \left(\theta^2 - r + \frac{1}{\alpha_1}P(t)\right) Q(t) + \rho|Q(t)| - (\delta P(t) + \alpha_2)AL, \tag{B.1.7}
$$

$$
Q'(t) \geq \left(\theta^2 - r + \frac{1}{\alpha_1}P(t)\right) Q(t) - \rho|Q(t)| - (\delta P(t) + \alpha_2)AL. \tag{B.1.8}
$$

If $Q(t) \geq 0$, then

$$
Q'(t) \leq \left(\theta^2 - r + \frac{1}{\alpha_1}P(t) + \rho\right) Q(t) - (\delta P(t) + \alpha_2)AL, \tag{B.1.9}
$$

$$
Q'(t) \geq \left(\theta^2 - r + \frac{1}{\alpha_1}P(t) - \rho\right) Q(t) - (\delta P(t) + \alpha_2)AL. \tag{B.1.10}
$$

Let us denote

$$C_1 := \max_{t\in[0,T]} \left(\theta^2 - r + \frac{1}{\alpha_1} P(t) + \rho\right),$$
$$C_2 := \min_{t\in[0,T]} \left(\theta^2 - r + \frac{1}{\alpha_1} P(t) - \rho\right),$$
$$C_3 := \min_{t\in[0,T]} (\delta P(t) + \alpha_2) AL \geq 0,$$
$$C_4 := \max_{t\in[0,T]} (\delta P(t) + \alpha_2) AL \geq 0.$$

Then (B.1.9) and (B.1.10) become

$$Q'(t) \leq C_1 Q(t) - C_3, \tag{B.1.11}$$
$$Q'(t) \geq C_2 Q(t) - C_4, \tag{B.1.12}$$

respectively. Similar to the proof of (Ekeland and Pirvu, 2008, Lemma 3.1), by integrating $\left(e^{-C_1 t} Q(t)\right)'$, $\left(e^{-C_2 t} Q(t)\right)'$, (B.1.11) and (B.1.12), we can get the lower and upper estimates of $Q(t)$:

$$0 \leq \alpha_3 A L e^{-C_1(T-t)} + C_3 \int_t^T e^{-C_1(s-t)} \mathrm{d}s \leq Q(t) \leq \alpha_3 A L e^{-C_2(T-t)} + C_4 \int_t^T e^{-C_2(s-t)} \mathrm{d}s.$$

If $Q(t) < 0$, then

$$Q'(t) \leq \left(\theta^2 - r + \frac{1}{\alpha_1} P(t) - \rho\right) Q(t) - (\delta P(t) + \alpha_2) AL, \tag{B.1.13}$$
$$Q'(t) \geq \left(\theta^2 - r + \frac{1}{\alpha_1} P(t) + \rho\right) Q(t) - (\delta P(t) + \alpha_2) AL. \tag{B.1.14}$$

Then

$$Q'(t) \leq C_2 Q(t) - C_3, \tag{B.1.15}$$
$$Q'(t) \geq C_1 Q(t) - C_4, \tag{B.1.16}$$

Similarly, we can get the lower and upper estimates on $Q(t)$:

$$0 \leq \alpha_3 ALe^{-C_2(T-t)} + C_3 \int_t^T e^{-C_2(s-t)} \mathrm{d}s \leq Q(t) \leq \alpha_3 ALe^{-C_1(T-t)} + C_4 \int_t^T e^{-C_1(s-t)} \mathrm{d}s,$$

which leads a contradiction.

The proof of uniqueness is also similar.

Step 2. We show the existence of a solution of (3.4.11) if the discount function is a linear combination of exponentials.

Let the discount function $h(t) = \sum_{n=1}^N \beta_n h_n(t)$, where $h_n(t) = e^{-\rho_n t}$ for some positive constant β_n such that $\sum_{n=1}^N \beta_n = 1$. Then

$$\begin{cases} P_n(t) & = \int_t^T h_n(s-t) \left(\frac{1}{\alpha_1} P^2(s) + \alpha_2 \right) e^{\int_t^s \left(2r - \theta^2 - \frac{2}{\alpha_1} P(u)\right) \mathrm{d}u} \mathrm{d}s + \alpha_3 h_n(T-t) e^{\int_t^T \left(2r - \theta^2 - \frac{2}{\alpha_1} P(u)\right) \mathrm{d}u}, \\ P_n(T) & = \alpha_3, \\ Q_n(t) & = -\int_t^T h_n(s-t) e^{\int_t^s \left(r - \theta^2 - \frac{1}{\alpha_1} P(x)\right) \mathrm{d}x} \left[\left(\frac{1}{\alpha_1} P^2(s) + \alpha_2 \right) \int_t^s e^{\int_u^s \left(r - \frac{1}{\alpha_1} P(x)\right) \mathrm{d}x} \left(\frac{1}{\alpha_1} Q(u) - \delta AL \right) \mathrm{d}u \right. \\ & \quad \left. - \left(\frac{1}{\alpha_1} P(s) Q(s) + \alpha_2 AL \right) \right] \mathrm{d}s \\ & \quad + \alpha_3 h_n(T-t) e^{\int_t^T \left(r - \theta^2 - \frac{1}{\alpha_1} P(u)\right) \mathrm{d}u} \left[AL - \int_t^T \left(\frac{1}{\alpha_1} Q(s) - \delta AL \right) e^{\int_s^T \left(r - \frac{1}{\alpha_1} P(u)\right) \mathrm{d}u} \mathrm{d}s \right], \\ Q_n(T) & = \alpha_3 AL, \end{cases}$$

solves the following ODE system

$$\begin{cases} P_n'(t) & = \frac{1}{\alpha_1} P^2(t) + \left(\theta^2 - 2r + \rho_n\right) P(t) - \alpha_2, \\ P_n(T) & = \alpha_3, \\ P(t) & = \sum_{n=1}^N \beta_n P_n(t); \\ Q_n'(t) & = \left(\theta^2 - r + \frac{1}{\alpha_1} P(t) + \rho_n \right) Q_n(t) - AL\left(\delta P(t) + \alpha_2\right) \\ Q_n(T) & = \alpha_3 AL, \\ Q(t) & = \sum_{n=1}^N \beta_n Q_n(t). \end{cases} \tag{B.1.17}$$

The existence of a solution for the above ODE system is granted locally by the Schauder Fixed Point Theorem. The uniqueness follows from the uniqueness of (3.4.11).

Now together with the boundness of P_n and Q_n, we know that this ODE system has a unique global solution.

Finally, the continuously differentiable solution of (3.4.11) can be constructed by the limit of the solution of the ODE system (B.1.17). We refer the reader to Ekeland and Pirvu (2008) and Ekeland et al. (2012) for more details. And this completes the proof. $\square$

Appendix B.2 Proof of Theorem 3.5.1

We follow the approach of approximating integral equations in Ekeland et al. (2012). It will be illustrated in the following lemmas step by step. Since the proofs of the following lemmas are very tedious and similar, we only provide the proof of Lemma B.2.2.

At first, we construct the sequence $P_n^{(1)}$ and $\varphi_n^{(1)}$ recursively by the following lemma.

Lemma B.2.2. *If $P_n^{(1)}$ and $\varphi_n^{(1)}$, $n=0,\cdots,N$, are defined by $P_0^{(1)}=\alpha_3$, $\varphi_0^{(1)}=1$, and*

$$\begin{cases} P_{n+1}^{(1)} & = P_n^{(1)} + \varepsilon\frac{1}{\alpha_1}\left(P_n^{(1)}\right)^2 + \varepsilon\left(\theta^2 - 2r - \frac{h'(T-t_n)}{h(T-t_n)}\right)P_n^{(1)} - \varepsilon\alpha_2 \\ & \quad + \varepsilon\int_{t_n}^{T}\Psi(s,t_n)\left(\frac{1}{\alpha_1}P^2(s)+\alpha_2\right)\left(\frac{\varphi(s)}{\varphi(t_n)}\right)\mathrm{d}s, \\ \varphi_{n+1}^{(1)} & = \varphi_n^{(1)} - \varepsilon\frac{2}{\alpha_1}P(t_n)\varphi_n^{(1)}, \end{cases}$$

then there exists a constant M such that

$$\left|P_n^{(1)} - P(t_n)\right| \le M|\varepsilon|, \quad \left|\varphi_n^{(1)} - \varphi(t_n)\right| \le M|\varepsilon|, \quad \forall n \in 0,1,\cdots,N.$$

Proof. First we show that $\{P_n^{(1)}\}$ is uniformly bounded. From the proof of Proposition 3.4.1, we know that $P(\cdot)$ and $P'(\cdot)$ are both bounded. Thus there exists some constant $c_1>0$ such that $|P(\cdot)|\le c_1$, $|P'(\cdot)|\le c_1$ and $\left|\theta^2-2r-\frac{h'(T-t_n)}{h(T-t_n)}\right|\le c_1$. Note that from (3.5.1),

$$\begin{aligned} P_{n+1}^{(1)} &= P_n^{(1)} + \varepsilon\frac{1}{\alpha_1}\left(P_n^{(1)}\right)^2 + \varepsilon\left(\theta^2 - 2r - \frac{h'(T-t_n)}{h(T-t_n)}\right)P_n^{(1)} \\ &\quad + \varepsilon\left\{P'(t_n) - \frac{1}{\alpha_1}P^2(t_n) - \left(\theta^2 - 2r - \frac{h'(T-t_n)}{h(T-t_n)}\right)P(t_n)\right\} \\ &= P_n^{(1)} + \varepsilon\frac{1}{\alpha_1}\left[\left(P_n^{(1)}\right)^2 - P^2(t_n)\right] + \varepsilon\left(\theta^2 - 2r - \frac{h'(T-t_n)}{h(T-t_n)}\right)\left(P_n^{(1)} - P(t_n)\right) + \varepsilon P'(t_n). \end{aligned}$$

If $n=0$, we have $P_0^{(1)}=\alpha_3$ and $\left|P_0^{(1)}-P(t_0)\right|=0\le M|\varepsilon|$. Now we assume that $\left|P_n^{(1)}\right|\le \alpha_3+\frac{n}{N}T(c_1+1)$ and $\left|P_n^{(1)}-P(t_n)\right|\le M|\varepsilon|$. Then, for $N\ge\left\{\frac{1}{\alpha_1}\left[\alpha_3+T(c_1+1)+c_1\right]+c_1\right\}MT$, we have

$$\begin{aligned}
\left|P_{n+1}^{(1)}\right| &= P_n^{(1)}+\varepsilon\frac{1}{\alpha_1}\left[\left(P_n^{(1)}\right)^2-P^2(t_n)\right]+\varepsilon\left(\theta^2-2r-\frac{h'(T-t_n)}{h(T-t_n)}\right)\left(P_n^{(1)}-P(t_n)\right)+\varepsilon P'(t_n)\\
&\le \alpha_3+\frac{n}{N}T(c_1+1)+\frac{1}{\alpha_1}\left[\alpha_3+\frac{n}{N}T(c_1+1)+c_1\right]M|\varepsilon|^2+Mc_1|\varepsilon|^2+\frac{1}{N}Tc_1\\
&\le \alpha_3+\frac{n}{N}T(c_1+1)+\left\{\frac{1}{\alpha_1}\left[\alpha_3+T(c_1+1)+c_1\right]+c_1\right\}M\frac{1}{N}T\frac{1}{N}T+\frac{1}{N}Tc_1\\
&\le \alpha_3+\frac{n+1}{N}T(c_1+1).
\end{aligned}$$

Thus, for any $n=0,1,\cdots,N$, $\left|P_n^{(1)}\right|\le\alpha_3+T(c_1+1)$. It shows that $\{P_n^{(1)}\}$ is uniformly bounded.

Next, we show $\left|P_{n+1}^{(1)}-P(t_{n+1})\right|\le M|\varepsilon|$. Let us define

$$e_n:=P_n^{(1)}-P(t_n).$$

Since $\frac{h'}{h}$ and its derivative is bounded, Ψ, Ψ', φ and φ' are all bounded. Thus $P''(t)$ exists and there exists a constant $c>0$ such that $P''(t)$ is bounded by c. Then it is easy to verify that

$$\left|P(t_{n+1})-P(t_n)-P'(t_n)\varepsilon\right|\le\frac{1}{2}c\varepsilon^2.$$

Consequently,

$$\begin{aligned}
|e_{n+1}| &= \left|P_{n+1}^{(1)}-P(t_{n+1})\right|\\
&\le \left|P_{n+1}^{(1)}-P(t_n)-P'(t_n)\varepsilon\right|+\left|P(t_n)+P'(t_n)\varepsilon-P(t_{n+1})\right|\\
&\le \left|P_{n+1}^{(1)}-P(t_n)-P'(t_n)\varepsilon\right|+\frac{1}{2}c\varepsilon^2\\
&= \Bigg|P_n^{(1)}+\varepsilon\frac{1}{\alpha_1}\left(P_n^{(1)}\right)^2+\varepsilon\left(\theta^2-2r-\frac{h'(T-t_n)}{h(T-t_n)}\right)P_n^{(1)}-\varepsilon\alpha_2\\
&\quad+\varepsilon\int_{t_n}^{T}\Psi(s,t_n)\left(\frac{1}{\alpha_1}P^2(s)+\alpha_2\right)\left(\frac{\varphi(s)}{\varphi(t_n)}\right)\mathrm{d}s-P(t_n)
\end{aligned}$$

$$
\begin{aligned}
&-\varepsilon\left[\frac{1}{\alpha_1}P^2(t_n)+\left(\theta^2-2r-\frac{h'(T-t_n)}{h(T-t_n)}\right)P(t_n)-\alpha_2\right.\\
&\left.\left.+\int_{t_n}^{T}\Psi(s,t_n)\left(\frac{1}{\alpha_1}P^2(s)+\alpha_2\right)\left(\frac{\varphi(s)}{\varphi(t_n)}\right)\mathrm{d}s\right]\right|+\frac{1}{2}c\varepsilon^2\\
=&\left|e_n+\varepsilon\frac{1}{\alpha_1}\left[\left(P_n^{(1)}\right)^2-P^2(t_n)\right]+\varepsilon\left(\theta^2-2r-\frac{h'(T-t_n)}{h(T-t_n)}\right)e_n\right|+\frac{1}{2}c\varepsilon^2\\
=&\left|e_n+\varepsilon\frac{1}{\alpha_1}\left(P_n^{(1)}+P(t_n)\right)\left(P_n^{(1)}-P(t_n)\right)+\varepsilon\left(\theta^2-2r-\frac{h'(T-t_n)}{h(T-t_n)}\right)e_n\right|+\frac{1}{2}c\varepsilon^2\\
\leq&\,|e_n|+|\varepsilon|\frac{1}{\alpha_1}\left(\alpha_3+T(c_1+1)+c_1\right)|e_n|+c_1|\varepsilon|\,|e_n|+\frac{1}{2}c\varepsilon^2.
\end{aligned}
$$

Therefore, there exists $C>0$ such that

$$
|e_{n+1}|\leq(1+C|\varepsilon|)\,|e_n|+\frac{1}{2}c\varepsilon^2,\quad \forall n\in 0,1,\cdots,N. \tag{B.2.1}
$$

By iterating (B.2.1) for $n=0,1,\cdots,N$, we obtain that

$$
|e_{n+1}|\leq\frac{1}{2}c\varepsilon^2\frac{(1+C|\varepsilon|)^{n+1}-1}{C|\varepsilon|}\leq\frac{1}{2}c\varepsilon^2\frac{e^{CT}-1}{C|\varepsilon|}\leq M|\varepsilon|,
$$

for some constant M independent of N.

Similar arguments show that

$$
\left|\varphi_n^{(1)}-\varphi(t_n)\right|\leq M|\varepsilon|,\forall n\in 0,1,\cdots,N.
$$

□

Secondly, we discretise the integral $\int_{t_n}^{T}\Psi(s,t_n)\left(\frac{1}{\alpha_1}P^2(s)+\alpha_2\right)\left(\frac{\varphi(s)}{\varphi(t_n)}\right)\mathrm{d}s$. This leads to the following lemma.

Lemma B.2.3. *If $P_n^{(2)}$ and $\varphi_n^{(2)}$, $n=0,1,\cdots,N$, are defined by $P_0^{(2)}=\alpha_3$, $\varphi_0^{(2)}=1$, and*

$$
\begin{cases}
P_{n+1}^{(2)} &= P_n^{(2)}+\varepsilon\frac{1}{\alpha_1}\left(P_n^{(2)}\right)^2+\varepsilon\left(\theta^2-2r-\frac{h'(T-t_n)}{h(T-t_n)}\right)P_n^{(2)}-\varepsilon\alpha_2\\
&\quad -\varepsilon^2\sum_{j=0}^{n-1}\Psi(t_j,t_n)\left(\frac{1}{\alpha_1}P^2(t_j)+\alpha_2\right)\left(\frac{\varphi(t_j)}{\varphi(t_n)}\right),\\
\varphi_{n+1}^{(2)} &= \varphi_n^{(2)}-\varepsilon\frac{2}{\alpha_1}P_n^{(2)}\varphi_n^{(2)},
\end{cases}
$$

then there exists a constant M such that

$$\left|P_n^{(2)} - P_n^{(1)}\right| \leq M\left|\varepsilon\right|, \quad \left|\varphi_n^{(2)} - \varphi_n^{(1)}\right| \leq M\left|\varepsilon\right|, \quad \forall n \in 0,1,\cdots,N.$$

Thirdly, we introduce an explicit scheme.

Lemma B.2.4. *If $P_n^{(3)}$ and $\varphi_n^{(3)}$, $n=0,1,\cdots,N$, are defined by $P_0^{(3)} = \alpha_3$, $\varphi_0^{(3)} = 1$, and*

$$\begin{cases} P_{n+1}^{(3)} & = P_n^{(3)} + \varepsilon\frac{1}{\alpha_1}\left(P_n^{(3)}\right)^2 + \varepsilon\left(\theta^2 - 2r - \frac{h'(T-t_n)}{h(T-t_n)}\right)P_n^{(3)} - \varepsilon\alpha_2 \\ & \qquad -\varepsilon^2\sum_{j=0}^{n-1}\Psi(t_j,t_n)\left(\frac{1}{\alpha_1}\left(P_j^{(3)}\right)^2 + \alpha_2\right)\left(\frac{\varphi_j^{(3)}}{\varphi_n^{(3)}}\right), \\ \varphi_{n+1}^{(3)} & = \varphi_n^{(3)} - \varepsilon\frac{2}{\alpha_1}P_n^{(3)}\varphi_n^{(3)}, \end{cases}$$

then there exists a constant M such that

$$\left|P_n^{(3)} - P_n^{(2)}\right| \leq M\left|\varepsilon\right|, \quad \left|\varphi_n^{(3)} - \varphi_n^{(2)}\right| \leq M\left|\varepsilon\right|, \quad \forall n \in 0,1,\cdots,N.$$

Similar to preceding steps, we construct the approximation of $Q(t)$. Now we construct the sequence $Q_n^{(1)}$ recursively by the following result.

Lemma B.2.5. *If $Q_n^{(1)}$, $n=0,\cdots,N$, is defined by $Q_0^{(1)} = \alpha_3 AL$, and*

$$\begin{aligned} Q_{n+1}^{(1)} = Q_n^{(1)} &+ \varepsilon\left(\theta^2 - r + \frac{1}{\alpha_1}P(t_n) - \frac{h'(T-t_n)}{h(T-t_n)}\right)Q_n^{(1)} - \varepsilon AL\left(\delta P(t_n) + \alpha_2\right) \\ &+ \varepsilon\int_{t_n}^{T}\Phi(s,t_n)\frac{\sqrt{\varphi(s)}}{\sqrt{\varphi(t_n)}}\left(\frac{1}{\alpha_1}P(s)Q(s) + \alpha_2 AL\right)\mathrm{d}s \\ &- \varepsilon\int_{t_n}^{T}\frac{e^{-ru}}{\sqrt{\varphi(t_n)\varphi(u)}}\left(\frac{1}{\alpha_1}Q(u) - \delta AL\right)\left[\int_u^T e^{rs}\Phi(s,t_n)\varphi(s)\left(\frac{1}{\alpha_1}P^2(s) + \alpha_2\right)\mathrm{d}s\right]\mathrm{d}u, \end{aligned}$$

then there exists a constant M such that

$$\left|Q_n^{(1)} - Q\left(t_n\right)\right| \leq M\left|\varepsilon\right|, \quad \forall n \in 0,1,\cdots,N.$$

Discretising the integral $\int_{t_n}^{T}\Phi(s,t_n)\frac{\sqrt{\varphi(s)}}{\sqrt{\varphi(t_n)}}\left(\frac{1}{\alpha_1}P(s)Q(s) + \alpha_2 AL\right)\mathrm{d}s$ and the outer integral of

$\int_{t_n}^{T} \frac{e^{-ru}}{\sqrt{\varphi(t_n)\varphi(u)}} \left(\frac{1}{\alpha_1}Q(u) - \delta AL\right) \left[\int_u^T e^{rs}\Phi(s,t_n)\varphi(s)\left(\frac{1}{\alpha_1}P^2(s)+\alpha_2\right)\mathrm{d}s\right]\mathrm{d}u$, we obtain the following lemma.

Lemma B.2.6. *If $Q_n^{(2)}$, $n = 0,\cdots,N$, is defined by $Q_0^{(2)} = \alpha_3 AL$ and*

$$\begin{aligned}
Q_{n+1}^{(2)} = Q_n^{(2)} &+ \varepsilon\left(\theta^2 - r + \frac{1}{\alpha_1}P(t_n) - \frac{h'(T-t_n)}{h(T-t_n)}\right)Q_n^{(2)} - \varepsilon AL\left(\delta P(t_n)+\alpha_2\right)\\
&- \varepsilon^2\sum_{j=0}^{n-1}\Phi(t_j,t_n)\frac{\sqrt{\varphi(t_j)}}{\sqrt{\varphi(t_n)}}\left(\frac{1}{\alpha_1}P(t_j)Q(t_j)+\alpha_2 AL\right)\\
&+ \varepsilon^2\sum_{j=0}^{n-1}\frac{e^{-rt_j}}{\sqrt{\varphi(t_n)\varphi(t_j)}}\left(\frac{1}{\alpha_1}Q(t_j)-\delta AL\right)\left[\int_{t_j}^{T} e^{rs}\Phi(s,t_n)\varphi(s)\left(\frac{1}{\alpha_1}P^2(s)+\alpha_2\right)\mathrm{d}s\right],
\end{aligned}$$

then there exists a constant M such that

$$\left|Q_n^{(2)} - Q_n^{(1)}\right| \le M\left|\varepsilon\right|, \quad \forall n \in 0,1,\cdots,N.$$

Lemma B.2.7. *If $Q_n^{(3)}$, $n = 0,\cdots,N$, is defined by $Q_0^{(3)} = \alpha_3 AL$ and*

$$\begin{aligned}
Q_{n+1}^{(3)} = Q_n^{(3)} &+ \varepsilon\left(\theta^2 - r + \frac{1}{\alpha_1}P(t_n) - \frac{h'(T-t_n)}{h(T-t_n)}\right)Q_n^{(3)} - \varepsilon AL\left(\delta P(t_n)+\alpha_2\right)\\
&- \varepsilon^2\sum_{j=0}^{n-1}\Phi(t_j,t_n)\frac{\sqrt{\varphi(t_j)}}{\sqrt{\varphi(t_n)}}\left(\frac{1}{\alpha_1}P(t_j)Q_j^{(3)}+\alpha_2 AL\right)\\
&+ \varepsilon^2\sum_{j=0}^{n-1}\frac{e^{-rt_j}}{\sqrt{\varphi(t_n)\varphi(t_j)}}\left(\frac{1}{\alpha_1}Q_j^{(3)}-\delta AL\right)\left[\int_{t_j}^{T} e^{rs}\Phi(s,t_n)\varphi(s)\left(\frac{1}{\alpha_1}P^2(s)+\alpha_2\right)\mathrm{d}s\right],
\end{aligned}$$

then there exists a constant M such that

$$\left|Q_n^{(3)} - Q_n^{(2)}\right| \le M\left|\varepsilon\right|, \quad \forall n \in 0,1,\cdots,N.$$

In the next step, we discretise the integral $\int_{t_j}^{T} e^{rs}\Phi(s,t_n)\varphi(s)\left(\frac{1}{\alpha_1}P^2(s)+\alpha_2\right)\mathrm{d}s$.

Lemma B.2.8. *If $Q_n^{(4)}$, $n = 0,\cdots,N$, is defined by $Q_0^{(4)} = \alpha_3 AL$ and*

$$\begin{aligned}
Q_{n+1}^{(4)} = Q_n^{(4)} &+ \varepsilon\left(\theta^2 - r + \frac{1}{\alpha_1}P(t_n) - \frac{h'(T-t_n)}{h(T-t_n)}\right)Q_n^{(4)} - \varepsilon AL\left(\delta P(t_n)+\alpha_2\right)\\
&- \varepsilon^2\sum_{j=0}^{n-1}\Phi(t_j,t_n)\frac{\sqrt{\varphi(t_j)}}{\sqrt{\varphi(t_n)}}\left(\frac{1}{\alpha_1}P(t_j)Q_j^{(4)}+\alpha_2 AL\right)
\end{aligned}$$

$$
-\varepsilon^3 \sum_{j=0}^{n-1} \frac{e^{-rt_j}}{\sqrt{\varphi(t_n)\varphi(t_j)}} \left(\frac{1}{\alpha_1} Q_j^{(4)} - \delta AL \right) \left[\sum_{i=0}^{j-1} e^{rt_i} \Phi(t_i, t_n) \varphi(t_i) \left(\frac{1}{\alpha_1} P^2(t_i) + \alpha_2 \right) \right],
$$

then there exists a constant M such that

$$
\left| Q_n^{(4)} - Q_n^{(3)} \right| \leq M |\varepsilon|, \quad \forall n \in 0, 1, \cdots, N.
$$

Finally, we get the explicit form.

Lemma B.2.9. *If $Q_n^{(5)}$, $n = 0, \cdots, N$, is defined by $Q_0^{(5)} = \alpha_3 AL$ and*

$$
\begin{aligned}
Q_{n+1}^{(5)} = Q_n^{(5)} &+ \varepsilon \left(\theta^2 - r + \frac{1}{\alpha_1} P_n^{(3)} - \frac{h'(T - t_n)}{h(T - t_n)} \right) Q_n^{(5)} - \varepsilon AL \left(\delta P_n^{(3)} + \alpha_2 \right) \\
&- \varepsilon^2 \sum_{j=0}^{n-1} \Phi(t_j, t_n) \frac{\sqrt{\varphi_j^{(3)}}}{\sqrt{\varphi_n^{(3)}}} \left(\frac{1}{\alpha_1} P_j^{(3)} Q_j^{(5)} + \alpha_2 AL \right) \\
&- \varepsilon^3 \sum_{j=0}^{n-1} \frac{e^{-rt_j}}{\sqrt{\varphi_n^{(3)} \varphi_j^{(3)}}} \left(\frac{1}{\alpha_1} Q_j^{(5)} - \delta AL \right) \left[\sum_{i=0}^{j-1} e^{rt_i} \Phi(t_i, t_n) \varphi_i^{(3)} \left(\frac{1}{\alpha_1} \left(P_i^{(3)} \right)^2 + \alpha_2 \right) \right],
\end{aligned}
$$

then there exists a constant M such that

$$
\left| Q_n^{(5)} - Q_n^{(4)} \right| \leq M |\varepsilon|, \quad \forall n \in 0, 1, \cdots, N.
$$

Eventually by applying the preceding lemmas and the Lipschitz continuity of $P(t)$ and $Q(t)$, we summarise the results of this section by Theorem 3.5.1.

Appendix C

Appendix C.1

C.1.1 A Time-Consistent Control Problem

In this appendix, we consider a time-consistent consumption-investment problem and give explicit solutions to the optimal strategy and value function in terms of the unique solution to a BSDE. The optimal control problems of all players in Section 4.3 are special cases of the problem studied in this section. The results of this section will be used frequently in Chapter 4.

Given $0 \le T_1 \le T_2 \le T$, for any $t \in [T_1, T_2)$ we consider the wealth process

$$\begin{cases} \mathrm{d}X^{c,u}(s) = [r(s) - c(s) + u(s)\sigma(s)\theta(s)]X^{c,u}(s)\mathrm{d}s + u(s)\sigma(s)X^{c,u}(s)\mathrm{d}W(s), \quad s \in [t, T_2], \\ X^{c,u}(t) = x, \end{cases} \tag{C.1.1}$$

where $x > 0$. Note that the wealth process given by (C.1.1) is positive.

Assume that at time $s \in [T_1, T_2]$ the investor adopts the discount factor $h(s - T_1)$. Consider the performance functional

$$J_c(t, x; c(\cdot), u(\cdot)) = \mathsf{E}_t \left[\int_t^{T_2} h(s - T_1) \ln\left(c(s)X^{c,u}(s)\right) \mathrm{d}s + h(T_2 - T_1)a\left(\ln X^{c,u}(T_2) - F\right) \right], \tag{C.1.2}$$

where $a > 0$ is a bounded constant and F is a bounded, $\mathscr{F}_{T_2}$-measurable random variable.

Definition C.1.1. *For any $(t, x) \in [T_1, T_2) \times (0, \infty)$, an admissible strategy consists of a pair*

$(c(\cdot), u(\cdot)) \in \mathscr{C}[t, T_2] \times \mathscr{U}[t, T_2]$ *such that*

$$\mathsf{E}_t \left[\int_t^{T_2} |\ln(c(s)X^{c,u}(s))| \, \mathrm{d}s \right] < \infty.$$

We denote by $\mathscr{A}_c(t,x)$ *the class of all such admissible strategies.*

Problem (C)**.** For any $(t,x) \in [T_1, T_2) \times (0, \infty)$, find a strategy $(\hat{c}(\cdot), \hat{u}(\cdot)) \in \mathscr{A}_c(t,x)$ such that

$$\begin{aligned} J_c(t,x;\hat{c}(\cdot),\hat{u}(\cdot)) &= V_c(t,x) \\ &:= \sup_{(c(\cdot),u(\cdot)) \in \mathscr{A}_c(t,x)} J_c(t,x;c(\cdot),u(\cdot)). \end{aligned}$$

Since T_1 is fixed (i.e., it is viewed as a parameter), it is well-known that Problem (C) is a time-consistent control problem, i.e., if $(\hat{c}(\cdot), \hat{u}(\cdot)) \in \mathscr{A}_c(t,x)$ is optimal for the initial pair (t,x), then

$$\begin{aligned} J_c(\tau, X^{\hat{c},\hat{u}}(\tau); (\hat{c}(\cdot),\hat{u}(\cdot))\big|_{[\tau,T_2]}) &= V_c(\tau, X^{\hat{c},\hat{u}}(\tau)) \\ &:= \sup_{(c(\cdot),u(\cdot)) \in \mathscr{A}_c(\tau, X^{\hat{c},\hat{u}}(\tau))} J_c(\tau, X^{\hat{c},\hat{u}}(\tau); c(\cdot), u(\cdot)), \quad \forall \tau \in [t, T_2]. \end{aligned}$$

Define a bounded positive function $g(\cdot)$ on $[T_1, T_2]$ by the ODE

$$\begin{cases} g'(s) = -g(s)\frac{h'(s-T_1)}{h(s-T_1)} - 1, \quad s \in [T_1, T_2], \\ g(T_2) = a. \end{cases}$$

It is easy to see

$$g(s) = a \exp\left\{ \int_s^{T_2} \frac{h'(v-T_1)}{h(v-T_1)} \mathrm{d}v \right\} + \int_s^{T_2} \exp\left\{ \int_s^z \frac{h'(v-T_1)}{h(v-T_1)} \mathrm{d}v \right\} \mathrm{d}z, \quad s \in [T_1, T_2]. \quad \text{(C.1.3)}$$

For $s \in [T_1, T_2]$, consider the BSDE

$$Y(s) = F + \int_s^{T_2} f(v, Y(v)) \mathrm{d}v + \int_s^{T_2} Z(v) \mathrm{d}W(v) \quad \text{(C.1.4)}$$

with the driver

$$f(s,y) = -\frac{1}{g(s)}y - \frac{1}{2}\theta^2(s) + \frac{1}{g(s)}(\ln g(s) + 1) - r(s).$$

Note that $f(s,y)$ satisfies linear grow in y and all the other terms are bounded, it is well know that (C.1.4) admits a unique solution $(Y,Z) \in \mathbb{H}^2_{T_1,T_2}(\mathbb{R}) \times \mathbb{H}^2_{T_1,T_2}(\mathbb{R})$ (see, e.g. Pardoux and Peng (1990) and El Karoui et al. (1997)). Furthermore, it is easy to see that Y is bounded, i.e. $Y \in \mathbb{H}^\infty_{T_1,T_2}(\mathbb{R})$.

Theorem C.1.1. *For any $(t,x) \in [T_1,T_2) \times (0,\infty)$, the value function for Problem (C) is*

$$V_c(t,x) = h(t-T_1)g(t)\left[\ln x - Y(t)\right],$$

and the optimal consumption-investment strategy is given by

$$\hat{c}(s) = \frac{1}{g(s)}, \qquad \hat{u}(s) = \frac{\theta(s)}{\sigma(s)}, \tag{C.1.5}$$

for $s \in [t,T_2]$.

The proof of the above theorem is similar to Cheridito and Hu (2011, Section 4.1) and we omit the details here. The idea is to prove that the process

$$R^{c,u}(s) := h(s-T_1)g(s)\left[\ln X^{c,u}(s) - Y(s)\right] + \int_t^s h(v-T_1)\ln\left(c(v)X^{c,u}(v)\right)\mathrm{d}v,$$

is a supermartingale. Then $R^{c,u}(t) \geq \mathsf{E}_t\left[R^{c,u}(T_2)\right]$. Furthermore, it can be shown that the equality holds if and only if (C.1.5) is satisfied.

C.1.2 Player (N)

Let $g_N(\cdot)$ be a bounded positive function defined on $[t_{N-1},t_N]$ by the ODE

$$\begin{cases} g_N'(s) = -g_N(s)\frac{h'(s-t_{N-1})}{h(s-t_{N-1})} - 1, \quad s \in [t_{N-1},t_N], \\ g_N(t_N) = 1. \end{cases}$$

Note that Problem (C_N) is a special case of Problem (C). From Theorem C.1.1, we have the following result.

Theorem C.1.2. *For $x_{N-1} \in (0,\infty)$, the value function of Problem (C_N) is given by*

$$V_\Pi(t_{N-1},x_{N-1}) = g_N(t_{N-1})\left(\ln x_{N-1} - Y_N(t_{N-1})\right),$$

where $Y_N(\cdot)$ is defined by the unique solution $(Y_N(\cdot),Z_N(\cdot)) \in \mathbb{H}^\infty_{t_{N-1},t_N}(\mathbb{R}) \times \mathbb{H}^2_{t_{N-1},t_N}(\mathbb{R})$ of the BSDE

$$Y_N(\tau) = \int_\tau^{t_N} f_N(s,Y_N(s))\mathrm{d}s + \int_\tau^{t_N} Z_N(s)\mathrm{d}W(s), \quad \tau \in [t_{N-1},t_N],$$

with the driver

$$f_N(s,y) = -\frac{1}{g_N(s)}y - \frac{1}{2}\theta^2(s) + \frac{1}{g_N(s)}\left(\ln g_N(s)+1\right) - r(s).$$

The optimal trading strategy $(\hat{c}_N(\cdot),\hat{u}_N(\cdot)) \in \mathscr{A}_N(t_{N-1},x_{N-1})$ is given by

$$\hat{c}_N(s) = \frac{1}{g_N(s)}, \qquad \hat{u}_N(s) = \frac{\theta(s)}{\sigma(s)},$$

for $s \in [t_{N-1},t_N]$. The optimal wealth process $\hat{X}_N(\cdot) \equiv \hat{X}_N(\cdot;t_{N-1},x_{N-1})$ is given by

$$\begin{cases} \mathrm{d}\hat{X}_N(s) = \left[r(s) - \frac{1}{g_N(s)} + \theta^2(s)\right]\hat{X}_N(s)\mathrm{d}s + \theta(s)\hat{X}_N(s)\mathrm{d}W(s), \quad s \in [t_{N-1},t_N], \\ \hat{X}_N(t_{N-1}) = x_{N-1}. \end{cases}$$

C.1.3 Player $(N-1)$

Proposition C.1.1. *For $(\tau,x_\tau) \in [t_{N-1},t_N] \times (0,\infty)$, define*

$$\Theta_N(\tau,x_\tau) \quad := \quad \mathsf{E}_\tau\left[\int_\tau^{t_N} h(s-t_{N-2})\ln\left(\hat{c}_N(s)\hat{X}_N(s;\tau,x_\tau)\right)\mathrm{d}s + h(t_N-t_{N-2})\ln\hat{X}_N\left(t_N;\tau,x_\tau\right)\right].$$

We have

$$\Theta_N(\tau,x_\tau) = h(\tau-t_{N-2})g_{N-1}(\tau)\left[\ln x_\tau - \mathscr{Y}_N(\tau)\right].$$

Proof. Similar to the proof of Proposition C.1.3. □

Similarly, Problem (C_{N-1}) is also a special case of Problem (C), from Theorem C.1.1 we have the following result.

Theorem C.1.3. *For any $x_{N-2} \in [t_{N-2}, t_{N-1})$, the value function of Problem (C_{N-1}) is given by*

$$V_{\Pi}(t_{N-2}, x_{N-2}) = g_{N-1}(t_{N-2}) \left[\ln x_{N-2} - Y_{N-1}(t_{N-2})\right],$$

where $Y_{N-1}(\cdot)$ is defined by the unique solution $(Y_{N-1}(\cdot), Z_{N-1}(\cdot)) \in \mathbb{H}^{\infty}_{t_{N-2},t_{N-1}}(\mathbb{R}) \times \mathbb{H}^{2}_{t_{N-2},t_{N-1}}(\mathbb{R})$ of the BSDE

$$\begin{aligned} Y_{N-1}(\tau) \;\; &= \;\; \mathscr{Y}_N(t_{N-1}) + \int_{\tau}^{t_{N-1}} f_{N-1}(s, Y_{N-1}(s))\mathrm{d}s \\ &\quad + \int_{\tau}^{t_{N-1}} Z_{N-1}(s)\mathrm{d}W(s), \quad \tau \in [t_{N-2}, t_{N-1}], \end{aligned}$$

with the driver

$$f_{N-1}(s,y) = -\frac{1}{g_{N-1}(s)} y - \frac{1}{2}\theta^2(s) + \frac{1}{g_{N-1}(s)} \left(\ln g_{N-1}(s) + 1\right) - r(s).$$

The optimal trading strategy $(\hat{c}_{N-1}(\cdot), \hat{u}_{N-1}(\cdot)) \in \mathscr{A}_{N-1}(t_{N-2}, x_{N-2})$ is given by

$$\hat{c}_{N-1}(s) = \frac{1}{g_{N-1}(s)}, \qquad \hat{u}_{N-1}(s) = \frac{\theta(s)}{\sigma(s)},$$

for $s \in [t_{N-2}, t_{N-1}]$. The optimal wealth process $\hat{X}_{N-1}(\cdot) \equiv \hat{X}_{N-1}(\cdot; t_{N-2}, x_{N-2})$ is given by

$$\begin{cases} \mathrm{d}\hat{X}_{N-1}(s) = \left[r(s) - \frac{1}{g_{N-1}(s)} + \theta^2(s)\right] \hat{X}_{N-1}(s)\mathrm{d}s + \theta(s)\hat{X}_{N-1}(s)(s)\mathrm{d}W(s), \quad s \in [t_{N-2}, t_{N-1}), \\ \hat{X}_{N-1}(t_{N-2}) = x_{N-2}. \end{cases}$$

C.1.4 Player $(N-2)$

Proposition C.1.2. *For $(\tau, x_\tau) \in [t_{N-2}, t_N] \times (0, \infty)$, define*

$$\begin{aligned} \Theta_{N-1}(\tau, x_\tau) \;\; &:= \;\; \mathsf{E}_\tau \Big[\int_{\tau}^{t_N} h(s - t_{N-3}) \ln\left(\bar{c}_{N-1}(s)\bar{X}_{N-1}(s; \tau, x_\tau)\right) \mathrm{d}s \\ &\qquad + h(t_N - t_{N-3}) \ln \bar{X}_{N-1}\left(t_N; \tau, x_\tau\right) \Big]. \end{aligned}$$

We have

$$\Theta_{N-1}(\tau,x_\tau)=h(\tau-t_{N-3})g_{N-2}(\tau)\left[\ln x_\tau-\mathscr{Y}_{N-1}(\tau)\right].$$

Proof. Similar to the proof of Proposition C.1.3. □

The following theorem follows from Theorem C.1.1.

Theorem C.1.4. *For any $x_{N-3}\in(0,\infty)$, the value function of Problem (C_{N-2}) is given by*

$$V_\Pi(t_{N-3},x_{N-3})=g_{N-2}(t_{N-3})\left[\ln x_{N-3}-Y_{N-2}(t_{N-3})\right],$$

where $Y_{N-2}(\cdot)$ is defined by the unique solution $(Y_{N-2}(\cdot),Z_{N-2}(\cdot))\in\mathbb{H}^\infty_{t_{N-3},t_{N-2}}(\mathbb{R})\times\mathbb{H}^2_{t_{N-3},t_{N-2}}(\mathbb{R})$ of the BSDE

$$Y_{N-2}(\tau)=\mathscr{Y}_{N-1}(t_{N-2})+\int_\tau^{t_{N-2}}f_{N-2}(s,Y_{N-2}(s))\mathrm{d}s+\int_\tau^{t_{N-2}}Z_{N-2}(s)\mathrm{d}W(s),\quad \tau\in[t_{N-3},t_{N-2}],$$

with the driver

$$f_{N-2}(s,y)=-\frac{1}{2}\theta^2(s)-\frac{1}{g_{N-2}(s)}y+\frac{1}{g_{N-2}(s)}\left(\ln g_{N-2}(s)+1\right)-r(s).$$

The optimal trading strategy $(\hat{c}_{N-2}(\cdot),\hat{u}_{N-2}(\cdot))\in\mathscr{A}_{N-2}(t_{N-3},x_{N-3})$ is given by

$$\hat{c}_{N-2}(s)=\frac{1}{g_{N-2}(s)},\qquad \hat{u}_{N-2}(s)=\frac{\theta(s)}{\sigma(s)},$$

for $s\in[t_{N-3},t_{N-2})$.

C.1.5 Player (k)

Proposition C.1.3. *For $(\tau,x_\tau)\in[t_k,t_N]\times(0,\infty)$, $k=1,2,\cdots,N-1$, define*

$$\Theta_{k+1}(\tau,x_\tau)\quad:=\quad\mathsf{E}_\tau\left[\int_\tau^{t_N}h(s-t_{k-1})\ln\left(\bar{c}_{k+1}(s)\bar{X}_{k+1}(s;\tau,x_\tau)\right)\mathrm{d}s+h(t_N-t_{k-1})\ln\bar{X}_{k+1}(t_N;\tau,x_\tau)\right].$$

We have

$$\Theta_{k+1}(\tau,x_\tau)=h(\tau-t_{k-1})g_k(\tau)\left[\ln x_\tau-\mathscr{Y}_{k+1}(\tau)\right].$$

Proof. Define the process

$$R(s) = h(s-t_{k-1})g_k(s)\left[\ln \bar{X}_{k+1}(s) - \mathscr{Y}_{k+1}(s)\right] + \int_\tau^s h(v-t_{k-1})\ln\left(\bar{c}_{k+1}(v)\bar{X}_{k+1}(v)\right)\mathrm{d}v,$$

for $s \in [\tau, t_N]$. It is easy to check that

$$\begin{aligned} R(\tau) &= h(\tau - t_{k-1})g_k(\tau)\left[\ln \bar{X}_{k+1}(\tau) - \mathscr{Y}_{k+1}(\tau)\right], \\ R(t_N) &= h(t_N - t_{k-1})\ln \bar{X}_{k+1}(t_N) + \int_\tau^{t_N} h(v-t_{k-1})\ln\left(\bar{c}_{k+1}(v)\bar{X}_{k+1}(v)\right)\mathrm{d}v, \end{aligned}$$

and

$$\begin{aligned} & \mathrm{d}\left(\ln \bar{X}_{k+1}(s) - \mathscr{Y}_{k+1}(s)\right) \\ = & \frac{1}{\bar{X}_{k+1}(s)}\mathrm{d}\bar{X}_{k+1}(s) - \frac{1}{2}\frac{1}{\bar{X}^2_{k+1}(s)}\left[\sigma(s)\bar{u}_{k+1}(s)\bar{X}_{k+1}(s)\right]^2\mathrm{d}s - \mathrm{d}\mathscr{Y}_{k+1}(s) \\ = & \left[r(s) - \bar{c}_{k+1}(s) + \theta(s)\sigma(s)\bar{u}_{k+1}(s) - \frac{1}{2}\sigma^2(s)\bar{u}^2_{k+1}(s) + \mathscr{F}_{k+1}(s,\mathscr{Y}_{k+1}(s))\right]\mathrm{d}s \\ & + \left[\sigma(s)\bar{u}_{k+1}(s) + \mathscr{Z}_{k+1}(s)\right]\mathrm{d}W(s), \end{aligned}$$

$$\begin{aligned} \mathrm{d}R(s) &= \left[h'(s-t_{k-1})g_k(s) + h(s-t_{k-1})g'_k(s)\right]\left[\ln \bar{X}_{k+1}(s) - \mathscr{Y}_{k+1}(s)\right]\mathrm{d}s \\ & \quad + h(s-t_{k-1})g_k(s)\mathrm{d}\left(\ln \bar{X}_{k+1}(s) - \mathscr{Y}_{k+1}(s)\right) + h(s-t_{k-1})\ln\left(\bar{c}_{k+1}(s)\bar{X}_{k+1}(s)\right)\mathrm{d}s \\ &= h(s-t_{k-1})g_k(s)\left\{\left[\sigma(s)\bar{u}_{k+1}(s) + \mathscr{Z}_{k+1}(s)\right]\mathrm{d}W(s) + A(s)\mathrm{d}s\right\}, \end{aligned}$$

where

$$\begin{aligned} A(s) &= \left[\frac{h'(s-t_{k-1})}{h(s-t_{k-1})} + \frac{g'_k(s)}{g_k(s)}\right]\left[\ln \bar{X}_{k+1}(s) - \mathscr{Y}_{k+1}(s)\right] \\ & \quad + r(s) - \bar{c}_{k+1}(s) + \theta(s)\sigma(s)\bar{u}_{k+1}(s) - \frac{1}{2}\sigma^2(s)\bar{u}^2_{k+1}(s) \\ & \quad + \mathscr{F}_{k+1}(s,\mathscr{Y}_{k+1}(s)) + \frac{1}{g_k(s)}\ln\left(\bar{c}_{k+1}(s)\bar{X}_{k+1}(s)\right) \\ &= \frac{1}{g_k(s)}\mathscr{Y}_{k+1}(s) + \theta(s)\sigma(s)\bar{u}_{k+1}(s) - \frac{1}{2}\sigma^2(s)\bar{u}^2_{k+1}(s) + \mathscr{F}_{k+1}(s,\mathscr{Y}_{k+1}(s)) \\ & \quad + \frac{1}{g_k(s)}\ln \bar{c}_{k+1}(s) - \bar{u}_{k+1}(s) + r(s) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{g_k(s)}\mathscr{Y}_{k+1}(s) + \frac{1}{2}\theta^2(s) + \mathscr{F}_{k+1}(s, \mathscr{Y}_{k+1}(s)) \\
&\quad -\frac{1}{g_k(s)}\sum_{n=k+1}^{N} \ln g_n(s)\mathbf{1}_{[t_{n-1},t_n)}(s) - \sum_{n=k+1}^{N}\frac{1}{g_n(s)}\mathbf{1}_{[t_{n-1},t_n)}(s) + r(s) \\
&= 0,
\end{aligned}
$$

which implies that R is a local martingale. Since $\mathscr{Z}_{k+1}(\cdot) \in \mathbb{H}^2_{t_k,t_{k+1}}(\mathbb{R})$ and the other terms in the coefficient of $\mathrm{d}W(s)$ in $\mathrm{d}R(s)$ are bounded, we conclude that $R(s)$ is a martingale, i.e. $R(\tau) = \mathsf{E}_\tau\left[R(t_N)\right]$. □

Appendix C.2

Obviously, for any $\tau \in [t, T]$, the function $g_\Pi(\cdot, \tau)$ satisfies the ODE

$$
\begin{cases}
g'_\Pi(s,\tau) = -g_\Pi(s,\tau)\frac{h'(s-l_\Pi(\tau))}{h(s-l_\Pi(\tau))} - 1, \quad t \le \tau \le s \le t_N, \\
g_\Pi(t_N, \tau) = 1,
\end{cases}
$$

where $g'_\Pi(s,\tau)$ is the partial derivative with respective to the first variable s. It is easy to see that for $t \le \tau \le s \le T$,

$$
\begin{aligned}
g_\Pi(s,\tau) &= \sum_{k=1}^{N} g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
&= \exp\left\{\int_s^{t_N} \frac{h'(v-l_\Pi(\tau))}{h(v-l_\Pi(\tau))}\mathrm{d}v\right\} + \int_s^{t_N} \exp\left\{\int_s^{z} \frac{h'(v-l_\Pi(\tau))}{h(v-l_\Pi(\tau))}\mathrm{d}v\right\}\mathrm{d}z \\
&= \frac{1}{h(s-l_\Pi(\tau))}\left[h(t_N - l_\Pi(\tau)) + \int_s^{t_N} h(z-l_\Pi(\tau))\mathrm{d}z\right]
\end{aligned}
$$

and

$$
g_\Pi(s,s)\mathbf{1}_{[t_{k-1},t_k)}(s) = g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(s). \tag{C.2.1}
$$

Note that

$$
\mathscr{F}_{k+1}(s,y) = -\frac{1}{g_k(s)}y - \frac{1}{2}\theta^2(s) + \frac{1}{g_k(s)}\sum_{n=k+1}^{N}\left[\ln g_n(s)\right]\mathbf{1}_{[t_{n-1},t_n)}(s)
$$

$$
\begin{aligned}
& \quad + \sum_{n=k+1}^{N} \left[\frac{1}{g_n(s)}\right] \mathbf{1}_{[t_{n-1},t_n)}(s) - r(s) \\
&= -\frac{1}{g_k(s)} y - \frac{1}{2}\theta^2(s) + \frac{1}{g_k(s)} \ln\left(\sum_{n=k+1}^{N} g_n(s)\mathbf{1}_{[t_{n-1},t_n)}(s)\right) \\
& \quad + \frac{1}{\sum_{n=k+1}^{N}\left(g_n(s)\mathbf{1}_{[t_{n-1},t_n)}(s)\right)} - r(s) \\
&= -\frac{1}{g_k(s)} y - \frac{1}{2}\theta^2(s) + \frac{1}{g_k(s)} \ln g_\Pi(s,s) + \frac{1}{g_\Pi(s,s)} - r(s), \quad t_{k-1} \le s < t_k.
\end{aligned}
$$

By (C.2.1), we have

$$
\begin{aligned}
f_\Pi(s,\tau,y) &= \sum_{k=1}^{N} \left[f_k(s,y)\mathbf{1}_{[t_{k-1},t_k)}(s) + \mathscr{F}_{k+1}(s,y)\mathbf{1}_{[t_k,t_N)}(s)\right] \mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
&= \left(\sum_{k=1}^{N} -\frac{1}{g_k(s)}\mathbf{1}_{[t_{k-1},t_k)}(\tau)\right) y - \frac{1}{2}\theta^2(s) - r(s) \\
& \quad + \sum_{k=1}^{N} \frac{1}{g_k(s)} \left[(\ln g_k(s))\,\mathbf{1}_{[t_{k-1},t_k)}(s) + (\ln g_\Pi(s,s))\,\mathbf{1}_{[t_k,t_N)}(s)\right] \mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
& \quad + \sum_{k=1}^{N} \left(\frac{1}{g_k(s)}\mathbf{1}_{[t_{k-1},t_k)}(s) + \frac{1}{g_\Pi(s,s)}\mathbf{1}_{[t_k,t_N)}(s)\right) \mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
&= -\frac{1}{\sum_{k=1}^{N} g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(\tau)} y - \frac{1}{2}\theta^2(s) - r(s) \\
& \quad + \sum_{k=1}^{N} \frac{1}{g_k(s)} \ln\left(g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(s) + g_\Pi(s,s)\mathbf{1}_{[t_k,t_N)}(s)\right) \mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
& \quad + \sum_{k=1}^{N} \frac{1}{g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(s) + g_\Pi(s,s)\mathbf{1}_{[t_k,t_N)}(s)} \mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
&= -\frac{1}{\sum_{k=1}^{N} g_k(s)\mathbf{1}_{[t_{k-1},t_k)}(\tau)} y - \frac{1}{2}\theta^2(s) - r(s) \\
& \quad + \sum_{k=1}^{N} \frac{1}{g_k(s)} \ln\left(g_\Pi(s,s)\mathbf{1}_{[t_{k-1},t_k)}(s) + g_\Pi(s,s)\mathbf{1}_{[t_k,t_N)}(s)\right) \mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
& \quad + \sum_{k=1}^{N} \frac{1}{g_\Pi(s,s)\mathbf{1}_{[t_{k-1},t_k)}(s) + g_\Pi(s,s)\mathbf{1}_{[t_k,t_N)}(s)} \mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
&= -\frac{1}{g_\Pi(s,\tau)} y + \sum_{k=1}^{N} \frac{1}{g_k(s)} \ln g_\Pi(s,s)\mathbf{1}_{[t_{k-1},t_k)}(\tau) \\
& \quad + \sum_{k=1}^{N} \frac{1}{g_\Pi(s,s)} \mathbf{1}_{[t_{k-1},t_k)}(\tau) - \frac{1}{2}\theta^2(s) - r(s)
\end{aligned}
$$

$$= -\frac{1}{g_\Pi(s,\tau)}y + \frac{1}{g_\Pi(s,\tau)}\ln g_\Pi(s,s) + \frac{1}{g_\Pi(s,s)} - \frac{1}{2}\theta^2(s) - r(s).$$

Then, for $\tau \in [t_{k-1}, t_k), s \in [\tau, t_k]$ and $k = 1, 2\cdots, N$, we have

$$\begin{aligned}
Y_\Pi(s,\tau) &= \mathscr{Y}_{k+1}(t_k) + \int_s^{t_k} f_k(v, Y_\Pi(v,\tau))\mathrm{d}v + \int_s^{t_k} Z_\Pi(v,\tau)\mathrm{d}W(v) \\
&= \int_{t_k}^{t_N} \mathscr{F}_{k+1}(v, Y_\Pi(v,\tau))\mathrm{d}v + \int_{t_k}^{t_N} Z_\Pi(v,\tau)\mathrm{d}W(v) \\
&\quad + \int_s^{t_k} f_k(v, Y_\Pi(v,\tau))\mathrm{d}v + \int_s^{t_k} Z_\Pi(v,\tau)\mathrm{d}W(v) \\
&= \int_s^{t_N} f_\Pi(v,\tau, Y_\Pi(v,\tau))\mathrm{d}v + \int_s^{t_N} Z_\Pi(v,\tau)\mathrm{d}W(v).
\end{aligned}$$

Obviously, the above equation holds for $\tau \in [t_{k-1}, t_k), s \in [t_k, t_N]$ and $k = 1, 2\cdots, N$.

Appendix C.3

Proof of Theorem 4.4.1. Recalling Assumption 4.2.1, for $(s,\tau) \in \mathscr{D}[t,T]$, it is easy to see that

$$\begin{aligned}
|g_\Pi(s,\tau) - g(s,\tau)| &\leq \frac{1}{h(s - l_\Pi(\tau))}\left(|h(T - l_\Pi(\tau)) - h(T-\tau)| + \int_s^T |h(v - l_\Pi(\tau)) - h(v-\tau)|\,\mathrm{d}v\right) \\
&\quad + \left|\frac{1}{h(s - l_\Pi(\tau))} - \frac{1}{h(s-\tau)}\right| \left[h(T-\tau) + \int_s^T h(v-\tau)\mathrm{d}v\right] \\
&\leq \frac{1}{h(T)}\left(1 + \frac{1}{h(T)}\right) C\,(1+T)\,|l_\Pi(\tau) - \tau| \\
&\leq \frac{1}{h(T)}\left(1 + \frac{1}{h(T)}\right) C\,(1+T)\,\|\Pi\|,
\end{aligned}$$

which implies

$$\lim_{\|\Pi\|\to 0} \sup_{(s,\tau)\in\mathscr{D}[t,T]} |g_\Pi(s,\tau) - g(s,\tau)| = 0.$$

Obviously, for any $(s,\tau) \in \mathscr{D}[t,T]$ we have

$$h(T) \leq g_\Pi(s,\tau), \quad g(s,\tau) \leq \frac{1}{h(T)}\,[1+T]\,.$$

Furthermore,

$$|\ln g_\Pi(s,\tau) - \ln g(s,\tau)| \le \frac{1}{h(T)} |g_\Pi(s,\tau) - g(s,\tau)| .$$

Consequently, for fixed y,

$$\begin{aligned}
&\int_\tau^T |f_\Pi(s,\tau,y) - f(s,\tau,y)|^2 \, \mathrm{d}s \\
=&\int_\tau^T \left| -\frac{1}{g_\Pi(s,\tau)} y + \frac{1}{g_\Pi(s,\tau)} \ln g_\Pi(s,s) + \frac{1}{g_\Pi(s,s)} + \frac{1}{g(s,\tau)} y - \frac{1}{g(s,\tau)} \ln g(s,s) - \frac{1}{g(s,s)} \right|^2 \mathrm{d}s \\
\le& 3 \int_\tau^T \left[\left| \frac{1}{g_\Pi(s,\tau)} - \frac{1}{g(s,\tau)} \right|^2 y^2 + \left| \frac{1}{g_\Pi(s,\tau)} \ln g_\Pi(s,s) - \frac{1}{g(s,\tau)} \ln g(s,s) \right|^2 + \left| \frac{1}{g_\Pi(s,s)} - \frac{1}{g(s,s)} \right|^2 \right] \mathrm{d}s \\
=& 3 \int_\tau^T \left[\left| \frac{g(s,\tau) - g_\Pi(s,\tau)}{g_\Pi(s,\tau) g(s,\tau)} \right|^2 y^2 + \left| \frac{g(s,s) - g_\Pi(s,s)}{g_\Pi(s,s) g(s,s)} \right|^2 \right] \mathrm{d}s \\
&+ 6 \int_\tau^T \left[|\ln g_\Pi(s,s) - \ln g(s,s)|^2 \frac{1}{g_\Pi^2(s,\tau)} + \left| \frac{g(s,\tau) - g_\Pi(s,\tau)}{g_\Pi(s,\tau) g(s,\tau)} \right|^2 [\ln g(s,s)]^2 \right] \mathrm{d}s \\
\le& 3 \frac{1}{h^2(T)} \int_\tau^T \left[|g(s,\tau) - g_\Pi(s,\tau)|^2 y^2 + |g(s,s) - g_\Pi(s,s)|^2 \right] \mathrm{d}s \\
&+ 6 \frac{1}{h^2(T)} \int_\tau^T \left[|g(s,s) - g_\Pi(s,s)|^2 \frac{1}{h^2(T)} + |g(s,\tau) - g_\Pi(s,\tau)|^2 C' \right] \mathrm{d}s \\
\le& C'' \|\Pi\|^2 ,
\end{aligned}$$

where

$$C' = \max \left\{ [\ln h(T)]^2 , \left[\ln \frac{1}{h(T)} [1+T] \right]^2 \right\}$$

is a positive constant depending on T and y. Thus, for any fixed y

$$\lim_{\|\Pi\| \to 0} \sup_{\tau \in [t,T]} \int_\tau^T |f_\Pi(s,\tau,y) - f(s,\tau,y)|^2 \, \mathrm{d}s = 0.$$

It follows from the stability of BSDEs that for any $\tau \in [t,T]$

$$\lim_{\|\Pi\| \to 0} \mathsf{E} \left[\sup_{s \in [\tau,T]} |Y_\Pi(s,\tau) - Y(s,\tau)|^2 + \int_\tau^T |Z_\Pi(s,\tau) - Z(s,\tau)|^2 \, \mathrm{d}s \right] = 0.$$

□

Lemma C.3.1. *Given any partition* $\Pi \in \mathscr{P}[t,T]$ *and any initial state* $(t_{k-1},x) \in [0,T) \times$

$(0,\infty)$, $k=1,2,\cdots N$, let $\hat{X}(\cdot;t_{k-1},x)$ and $\hat{X}_\Pi(\cdot;t_{k-1},x)$ be the unique solutions to the SDEs

$$\begin{cases} \mathrm{d}\hat{X}(s) = \left[r(s) - \frac{1}{g(s,s)} + \theta^2(s)\right]\hat{X}(s)\mathrm{d}s + \theta(s)\hat{X}(s)\mathrm{d}W(s), \quad s\in[t_{k-1},t_N], \\ \hat{X}(t_{k-1}) = x, \end{cases} \tag{C.3.1}$$

and

$$\begin{cases} \mathrm{d}\hat{X}_\Pi(s) = \left[r(s) - \frac{1}{g_\Pi(s,s)} + \theta^2(s)\right]\hat{X}_\Pi(s)\mathrm{d}s + \theta(s)\hat{X}_\Pi(s)\mathrm{d}W(s), \quad s\in[t_{k-1},t_N], \\ \hat{X}_\Pi(t_{k-1}) = x, \end{cases}$$

respectively. For $k=1,2,\cdots N$ and $n\le k$, let

$$\begin{aligned} \Theta(n;k,x) &:= \mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\ln\left(\hat{c}(s)\hat{X}(s;t_{k-1},x)\right)\mathrm{d}s + h(t_N-t_{n-1})\ln\hat{X}(t_N;t_{k-1},x)\right], \\ \Theta_\Pi(n;k,x) &:= \mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\ln\left(\hat{c}_\Pi(s)\hat{X}_\Pi(s;t_{k-1},x)\right)\mathrm{d}s + h(t_N-t_{n-1})\ln\hat{X}_\Pi(t_N;t_{k-1},x)\right]. \end{aligned}$$

Then for any $\varepsilon>0$, there exists a $\delta>0$ such that if $\|\Pi\|<\delta$, then

$$|\Theta(n;k,x) - \Theta_\Pi(n;k,x)| < \varepsilon, \quad \forall k=1,2,\cdots N,\, n\le k,\, x\in(0,\infty).$$

Particularly, when $n=k$, we have

$$\left|J\left(t_{k-1},x;(\hat{c}_\Pi(\cdot),\hat{u}_\Pi(\cdot))\big|_{[t_{k-1},t_N)}\right) - J\left(t_{k-1},x;(\hat{c}(\cdot),\hat{u}(\cdot))\big|_{[t_{k-1},t_N)}\right)\right| < \varepsilon,\ \forall k=1,2,\cdots N,\, x\in(0,\infty).$$

Proof. It is easy to see that the unique solution to SDE (C.3.1) is given by

$$\hat{X}(s) = x\exp\left\{\int_{t_{k-1}}^s \left[r(v) - \frac{1}{g(v,v)} + \frac{1}{2}\theta^2(v)\right]\mathrm{d}v + \int_t^s \theta(v)\mathrm{d}W(v)\right\}, \quad s\in[t_{k-1},T].$$

For all $k=1,2,\cdots,N$, and $n\le k$,

$$\begin{aligned} &\Theta(n;k,x) \\ =&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\ln\left(\hat{c}(s)\hat{X}(s)\right)\mathrm{d}s + h(t_N-t_{n-1})\ln\hat{X}(t_N)\right] \end{aligned}$$

$$
\begin{aligned}
=&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\left(\ln x+\int_{t_{k-1}}^{s}\left[r(v)-\frac{1}{g(v,v)}+\frac{1}{2}\theta^2(v)\right]\mathrm{d}v+\int_{t_{k-1}}^{s}\theta(v)\mathrm{d}W(v)-\ln g(s,s)\right)\mathrm{d}s\right.\\
&\left.+h(t_N-t_{n-1})\left(\ln x+\int_{t_{k-1}}^{t_N}\left[r(s)-\frac{1}{g(s,s)}+\frac{1}{2}\theta^2(s)\right]\mathrm{d}s+\int_{t_{k-1}}^{t_N}\theta(s)\mathrm{d}W(s)\right)\right]\\
=&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\left(\int_{t_{k-1}}^{s}\left[r(v)-\frac{1}{g(v,v)}+\frac{1}{2}\theta^2(v)\right]\mathrm{d}v-\ln g(s,s)\right)\mathrm{d}s\right.\\
&\left.+h(t_N-t_{n-1})\int_{t_{k-1}}^{t_N}\left[r(s)-\frac{1}{g(s,s)}+\frac{1}{2}\theta^2(s)\right]\mathrm{d}s\right]+\left[\int_{t_{k-1}}^{t_N}h(s-t_{n-1})\mathrm{d}s+h(t_N-t_{n-1})\right]\ln x
\end{aligned}
\tag{C.3.2}
$$

$$
\begin{aligned}
=&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\left(\int_{t_{k-1}}^{s}\left[r(v)+\frac{1}{2}\theta^2(v)\right]\mathrm{d}v\right)\mathrm{d}s+h(t_N-t_{n-1})\int_{t_{k-1}}^{t_N}\left[r(s)+\frac{1}{2}\theta^2(s)\right]\mathrm{d}s\right]\\
&-\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\left(\int_{t_{k-1}}^{s}\frac{1}{g(v,v)}\mathrm{d}v+\ln g(s,s)\right)\mathrm{d}s-h(t_N-t_{n-1})\int_{t_{k-1}}^{t_N}\frac{1}{g(s,s)}\mathrm{d}s\\
&+\left[\int_{t_{k-1}}^{t_N}h(s-t_{n-1})\mathrm{d}s+h(t_N-t_{n-1})\right]\ln x.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\Theta_{\Pi}(n;k,x)\\
=&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\left(\int_{t_{k-1}}^{s}\left[r(v)+\frac{1}{2}\theta^2(v)\right]\mathrm{d}v\right)\mathrm{d}s+h(t_N-t_{n-1})\int_{t_{k-1}}^{t_N}\left[r(s)+\frac{1}{2}\theta^2(s)\right]\mathrm{d}s\right]\\
&-\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\left(\int_{t_{k-1}}^{s}\frac{1}{g_{\Pi}(v,v)}\mathrm{d}v+\ln g_{\Pi}(s,s)\right)\mathrm{d}s-h(t_N-t_{n-1})\int_{t_{k-1}}^{t_N}\frac{1}{g_{\Pi}(s,s)}\mathrm{d}s\\
&+\left[\int_{t_{k-1}}^{t_N}h(s-t_{n-1})\mathrm{d}s+h(t_N-t_{n-1})\right]\ln x.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|\Theta(n;k,x)-\Theta_{\Pi}(n;k,x)|\\
=&\left|\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\left(\int_{t_{k-1}}^{s}\frac{1}{g(v,v)}\mathrm{d}v+\ln g(s,s)\right)\mathrm{d}s+h(t_N-t_{n-1})\int_{t_{k-1}}^{t_N}\frac{1}{g(s,s)}\mathrm{d}s\right.\\
&\left.-\int_{t_{k-1}}^{t_N} h(s-t_{n-1})\left(\int_{t_{k-1}}^{s}\frac{1}{g_{\Pi}(v,v)}\mathrm{d}v+\ln g_{\Pi}(s,s)\right)\mathrm{d}s-h(t_N-t_{n-1})\int_{t_{k-1}}^{t_N}\frac{1}{g_{\Pi}(s,s)}\mathrm{d}s\right|.
\end{aligned}
$$

By Theorem 4.4.1 (and similar arguments of its proof), for any $\varepsilon>0$, there exists a $\delta>0$

such that for any partition $\Pi \in \mathscr{P}[0,T]$ with $\|\Pi\| < \delta$, it holds that

$$|\Theta(n;k,x) - \Theta_\Pi(n;k,x)| < \varepsilon.$$

□

Proof of Theorem 4.4.2. It is easy to see that the unique solution to SDE (4.4.5) is given by

$$\hat{X}(s) = x\exp\left\{\int_t^s \left[r(v) - \frac{1}{g(v,v)} + \frac{1}{2}\theta^2(v)\right]\mathrm{d}v + \int_t^s \theta(v)\mathrm{d}W(v)\right\}, \quad s \in [t,T].$$

We are going to show the approximate optimality of $(\hat{c}(\cdot), \hat{u}(\cdot))$. From Lemma C.3.1, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any partition $\Pi \in \mathscr{P}[t,T]$ with $\|\Pi\| < \delta$, it holds that for all $k = 1, 2, \cdots, N$,

$$J\left(t_{k-1}, \hat{X}(t_{k-1}); (\hat{c}(\cdot), \hat{u}(\cdot))\big|_{[t_{k-1},t_N)}\right) \geq J\left(t_{k-1}, \hat{X}(t_{k-1}); (\hat{c}_\Pi(\cdot), \hat{u}_\Pi(\cdot))\big|_{[t_{k-1},t_N)}\right) - \varepsilon.$$

Pick any $(c_k(\cdot), u_k(\cdot)) \in \mathscr{A}(t_{k-1}, x)$, and let $X_k(\cdot) \equiv X_k(\cdot; t_{k-1}, \hat{X}(t_{k-1}), c_k(\cdot), u_k(\cdot))$ be the solution to the SDE

$$\begin{cases} \mathrm{d}X_k(s) = [r(s) - c_k(s) + \theta(s)\sigma(s)u_k(s)]X_k(s)\mathrm{d}s + \sigma(s)u_k(s)X_k(s)\mathrm{d}W(s), \quad s \in [t_{k-1}, t_k), \\ X_k(t_{k-1}) = \hat{X}(t_{k-1}). \end{cases}$$

By the construction of $(\hat{c}_\Pi(\cdot), \hat{u}_\Pi(\cdot))$, we have

$$J\left(t_{k-1}, \hat{X}(t_{k-1}); (\hat{c}_\Pi(\cdot), \hat{u}_\Pi(\cdot))\big|_{[t_{k-1},t_N)}\right) \geq J\left(t_{k-1}, \hat{X}(t_{k-1}); \tilde{c}_\Pi(\cdot), \tilde{u}_\Pi(\cdot)\right),$$

where

$$(\tilde{c}_\Pi(s), \tilde{u}_\Pi(s)) = \begin{cases} (c_k(s), u_k(s)), & s \in [t_{k-1}, t_k), \\ (\hat{c}_\Pi(s), \hat{u}_\Pi(s)), & s \in [t_k, t_N]. \end{cases}$$

Note that

$$J\left(t_{k-1}, \hat{X}(t_{k-1}); \tilde{c}_\Pi(\cdot), \tilde{u}_\Pi(\cdot)\right)$$

$$
\begin{aligned}
=&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_k} h(s-t_{k-1})\ln\left(c_k(s)X_k(s)\right)\mathrm{d}s\right]\\
&+\mathsf{E}_{t_{k-1}}\left[\int_{t_k}^{t_N} h(s-t_{k-1})\ln\left(\hat{c}_\Pi(s)\hat{X}_\Pi(s;t_k,X_k(t_k))\right)\mathrm{d}s+h(t_N-t_{k-1})\ln\hat{X}_\Pi(t_N;t_k,X_k(t_k))\right]\\
=&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_k} h(s-t_{k-1})\ln\left(c_k(s)X_k(s)\right)\mathrm{d}s\right]+\mathsf{E}_{t_{k-1}}\left[\Theta_\Pi(k;k+1,X_k(t_k))\right]\\
\geq&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_k} h(s-t_{k-1})\ln\left(c_k(s)X_k(s)\right)\mathrm{d}s\right]+\mathsf{E}_{t_{k-1}}\left[\Theta(k;k+1,X_k(t_k))\right]-\varepsilon\\
=&\mathsf{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_k} h(s-t_{k-1})\ln\left(c_k(s)X_k(s)\right)\mathrm{d}s\right]\\
&+\mathsf{E}_{t_{k-1}}\left[\int_{t_k}^{t_N} h(s-t_{k-1})\ln\left(\hat{c}(s)\hat{X}(s;t_k,X_k(t_k))\right)\mathrm{d}s+h(t_N-t_{k-1})\ln\hat{X}(t_N;t_k,X_k(t_k))\right]-\varepsilon\\
=&J\left(t_{k-1},x;\left(\tilde{c}(\cdot;t_{k-1},x),\tilde{u}(\cdot;t_{k-1},x)\right)\right)-\varepsilon,
\end{aligned}
$$

where the inequality follows from Lemma C.3.1. Thus we have

$$
J\left(t_{k-1},\hat{X}(t_{k-1});(\hat{c}(\cdot),\hat{u}(\cdot))\big|_{[t_{k-1},t_N)}\right)\geq J\left(t_{k-1},\hat{X}(t_{k-1});\tilde{c}(\cdot),\tilde{u}(\cdot)\right)-2\varepsilon,
$$

which implies the approximate optimality of $(\hat{c}(\cdot),\hat{u}(\cdot))$.

To obtain (4.4.6), note that $(Y(\cdot,\tau),Z(\cdot,\tau))$ is the unique solution to a linear BSDE for any $\tau\in[t,T]$. Let $\rho(\cdot,\tau)$ be the solution to the ODE

$$
\begin{cases}
\mathrm{d}\rho(s,\tau)=-\frac{1}{g(s,\tau)}\rho(s,\tau)\mathrm{d}s, & s\in[\tau,T],\\
\mathrm{d}\rho(\tau,\tau)=1,
\end{cases}
$$

then we have for $s\in[\tau,T]$,

$$
\rho(s,\tau)=e^{-\int_\tau^s\frac{1}{g(v,\tau)}\mathrm{d}v},
$$

and

$$
Y(s,\tau)=\frac{1}{\rho(s,\tau)}\mathsf{E}_s\left[\int_s^T\rho(v,\tau)\left(\frac{1}{g(v,\tau)}\ln g(v,v)+\frac{1}{g(v,v)}-\frac{1}{2}\theta^2(v)-r(v)\right)\mathrm{d}v\right].
$$

Noting that for any $\tau \in [t,T]$, the function $g(\cdot,\tau)$ satisfies the ODE

$$\begin{cases} g'(s,\tau) = -g(s,\tau)\frac{h'(s-\tau)}{h(s-\tau)} - 1, \quad 0 \le \tau \le s \le t_N, \\ g(T,\tau) = 1, \end{cases}$$

it follows that

$$\begin{aligned} \rho(s,\tau) &= e^{\int_\tau^s \left[\frac{h'(v-\tau)}{h(v-\tau)} + \frac{g'(v,\tau)}{g(v,\tau)}\right]\mathrm{d}v} \\ &= \frac{h(s-\tau)g(s,\tau)}{g(\tau,\tau)}. \end{aligned}$$

Thus,

$$Y(s,\tau) = \frac{g(\tau,\tau)}{h(s-\tau)g(s,\tau)}\mathsf{E}_s\left[\int_s^T \frac{h(v-\tau)g(v,\tau)}{g(\tau,\tau)}\left(\frac{1}{g(v,\tau)}\ln g(v,v) + \frac{1}{g(v,v)} - \frac{1}{2}\theta^2(v) - r(v)\right)\mathrm{d}v\right],$$

and

$$\begin{aligned} g(t,t)Y(t,t) &= \mathsf{E}_t\left[\int_t^T h(s-t)g(s,t)\left(\frac{1}{g(s,t)}\ln g(s,s) + \frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right] \\ &= \mathsf{E}_t\left[\int_t^T h(s-t)\ln g(s,s)\mathrm{d}s\right] \\ &\quad + \mathsf{E}_t\left[\int_t^T h(s-t)g(s,t)\left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right]. \end{aligned}$$

Since

$$\begin{aligned} &\mathsf{E}_t\left[\int_t^T h(s-t)g(s,t)\left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right] \\ =&\mathsf{E}_t\left[\int_t^T h(s-t)\frac{1}{h(s-t)}\left[h(T-t) + \int_s^T h(v-t)\mathrm{d}v\right]\left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right] \\ =&\mathsf{E}_t\left[\int_t^T \left[h(T-t) + \int_s^T h(v-t)\mathrm{d}v\right]\left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right] \\ =&h(T-t)\mathsf{E}_t\left[\int_t^T \left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right] \\ &+ \mathsf{E}_t\left[\int_t^T \int_s^T h(v-t)\mathrm{d}v\left(\frac{1}{g(s,s)} - \frac{1}{2}\theta^2(s) - r(s)\right)\mathrm{d}s\right] \end{aligned}$$

$$
\begin{aligned}
=& h(T-t)\mathsf{E}_t\left[\int_t^T\left(\frac{1}{g(s,s)}-\frac{1}{2}\theta^2(s)-r(s)\right)\mathrm{d}s\right] \\
&+\mathsf{E}_t\left[\int_t^T h(v-t)\int_t^v\left(\frac{1}{g(s,s)}-\frac{1}{2}\theta^2(s)-r(s)\right)\mathrm{d}s\mathrm{d}v\right],
\end{aligned}
$$

we have

$$
\begin{aligned}
g(t,t)Y(t,t) \quad = \quad & \mathsf{E}_t\left[\int_t^T h(s-t)\left(\int_t^s\left(\frac{1}{g(v,v)}-\frac{1}{2}\theta^2(v)-r(v)\right)\mathrm{d}v+\ln g(s,s)\right)\mathrm{d}s\right] \\
&+h(T-t)\mathsf{E}_t\left[\int_t^T\left(\frac{1}{g(s,s)}-\frac{1}{2}\theta^2(s)-r(s)\right)\mathrm{d}s\right].
\end{aligned}
$$

Similar to (C.3.2)

$$
\begin{aligned}
& J\left(t,x;\hat{c}(\cdot),\hat{u}(\cdot)\right) \\
= \quad & g(t,t)\ln x+\mathsf{E}_t\left[\int_t^T h(s-t)\left(\int_t^s\left[r(v)-\frac{1}{g(v,v)}+\frac{1}{2}\theta^2(v)\right]\mathrm{d}v-\ln g(s,s)\right)\mathrm{d}s\right. \\
& \left.+h(T-t)\int_t^T\left[r(s)-\frac{1}{\tilde{g}(s)}+\frac{1}{2}\theta^2(s)\right]\mathrm{d}s\right] \\
= \quad & g(t,t)\left(\ln x-Y(t,t)\right).
\end{aligned}
$$

□

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