Toward Theory Advancement in Mathematical Cognition and Teacher Cognition

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Dedicated to my family and my friends.

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Abstract

This thesis is concerned with mathematical cognition and teacher cognition, two of the subfields within mathematics education research. Within each there is a broad range of diverse theories that cultivate varied understandings of complex phenomena in mathematical thinking, learning, and teaching. However, the abundance and diversity of theories can polarize perspectives and foster the development of narrow and restricting theoretical accounts. This thesis uses existing theoretical tensions to stimulate the development of more powerful theoretical accounts by coordinating theoretical perspectives in mathematical cognition and teacher cognition.

The thesis consists of three articles, which aim to blend opposing theoretical perspectives to reveal complementarity in the field of mathematical knowing and learning, challenge assumptions to reveal restrictions in the field of teacher knowledge, and portray some complex phenomena that cannot be accounted for using intuitive models of teacher noticing. These articles link apparently disparate approaches, revealing the complexity of the phenomena under consideration and the limitations of existing theoretical accounts for them.

The first article blends theoretical perspectives from two local theories of mathematical cognition (abstraction-from-actions and abstraction-from-objects) to present a bi-directional, dynamic, non-linear view of mathematical concept formation. The second article examines teacher cognition, discussing existing conceptualizations of mathematics teacher knowledge, revealing their limitations, and offering alternative views that direct attention to underexplored issues. The third article examines teacher cognition from the perspective of the construct of teacher noticing, drawing on insights from cognitive science and the applied science of human factors to develop a model of teacher noticing which challenges intuitive assumptions and views individual and environment as interdependent and inseparable.

It is hoped that these contributions add value to the field by advancing knowledge, providing links between previous conceptualizations, and offering fresh insights and theoretical views.

Statement of Candidate

I certify that the research presented in this thesis is my original work and it has not been submitted as part of requirements for a degree to any other university or institution other than University of Hamburg and Macquarie University. I also certify that any help and assistance that I have received in my research work and the preparation of the thesis itself have been appropriately acknowledged. Additionally, I certify that all information sources and literature used are indicated in the thesis.

January 18, 2018 Thorsten Scheiner

Acknowledgments

I began working on this thesis in December 2013. It has evolved in many different forms, covering a wide diversity of topics, and some of the ideas explored, and insights gained, here have been presented and published in earlier incarnations elsewhere. Many academics have helped me in working through my ideas.

First and foremost, I would like to express my gratitude to Gabriele Kaiser and Joanne Mulligan for their wonderful joint doctoral supervision. Gabriele Kaiser got me started in mathematics education research, and encouraged and supported me throughout all the critical phases of my PhD. The intellectual roots and the intellectual freedom she gave me provided me with the seeds to grow and to travel my own academic track. Joanne Mulligan was key to conducting my PhD as a joint PhD, which allowed me to be engaged in, and learn from, two different academic environments. I am thankful for her interest and assistance in all aspects of my PhD. The thesis could not have been written without the ongoing advice, vast support, and immense intellectual generosity of my two supervisors, Gabriele Kaiser and Joanne Mulligan.

My PhD took me on a journey of visiting several international experts who helped me begin my academic career. My journey started with visiting David O. Tall at The University of Warwick, who inspired many of the early ideas in mathematical cognition that influenced the form and content of this thesis. I am thankful for his encouragement of my attempts to address fundamental issues in mathematics education. My journey continued with visiting Alan H. Schoenfeld and Andrea A. diSessa at the University of California at Berkeley, who with their writings and critical comments shaped my thinking in substantial ways. I am grateful for their many thought-provoking conversations and discussions on cognition that raised my awareness of my own limitations. My journey then took me to Chris Rasmussen at San Diego State University, who stimulated my thinking about the topic of teacher knowledge and teacher noticing by showing me where to look, but not what to see. I want to thank Chris Rasmussen for being an inspiring mentor. Finally, my journey took me to Hilda Borko at Stanford University, who helped me to develop a serious and thoughtful consideration of critical issues in mathematical cognition and teacher cognition. I wish to thank Hilda Borko for her extensive and thoughtful commentary on some of my later thoughts.

I have learned an immense amount from the many in-depth conversations and discussions with my supervisors, mentors, and colleagues, which have influenced the content and style of this thesis. I have benefited enormously from the thoughtful suggestions and challenging comments received. These scholars certainly had a significant impact on my views and have transformed my academic life in many ways. I also want to thank Zeid Ismail for language editing and proof-reading of an earlier version of this thesis.

I am deeply grateful to the Foundation of German Business (Stiftung der Deutschen Wirtschaft) for the Klaus Murmann Fellowship from 2013 to 2016, and Macquarie University for the Research Excellence Scholarship from 2016 to 2017.

Although I list them last, my family and my friends are first in my heart. They are persistent reminders to me of the indomitable character of the human spirit. This thesis is dedicated, with love and admiration, to them.

1 Introduction

Theory underlies most scholarly activity and is key to driving progress in both educational research and educational practice.[1] Despite empirical research being more common and often more prominent than theoretical research, theorizing is central to mathematics education research (Lester, 2005) and the deep understanding it fosters is often essential when confronting truly important problems (diSessa, 1991). A central issue met by this thesis is the prevalence of multiple and frequently conflicting theoretical perspectives and theoretical frameworks in mathematics education. This is the case because mathematics education is a very broad research field, divided into subfields that together encompass insights from fields such as anthropology, philosophy, psychology, semiotics, and sociology, among others. In each piece of research mathematics education is approached from a specific viewpoint, which determines to a large extent how mathematics education is understood as a research object. This thesis focuses on cognitive psychology, cognitive science, and complexity science, which provide theoretical insights that help account for complex phenomena in mathematical cognition and teacher cognition.[2]

1.1 Research Focus

The focus on mathematical cognition and teacher cognition relates directly to the complexity of learning and teaching in mathematics education, addressing critical questions regarding students' mathematical cognition and teachers' cognition that have been disputed about by scholars within both the mathematics education research community and the larger educational research community.

The learning-teaching environment is considered as a complex system in which teacher, learners, and subject matter are interrelated and in a state of flux, and the interacting agents are themselves regarded as complex systems (see Davis & Simmt, 2012). One way to capture the dimensions involved is by means of the didactic triangle, in which teacher, students, and subject matter represent the vertices of the triangle (see Figure 1).[3] Goodchild and Sriraman (2012) described the didactic triangle as "the classical *trivium* used to conceptualize teaching and learning in mathematics classrooms" (p. 581, italics in original), and further argued that:

> "Even though this representation may seem canonical to an extent and 'simplify' the complexity of what occurs within the classroom during a mathematics lesson, it serves as a starting point to theorize the dynamics of teaching-learning, as well as situating and contextualizing each element in relation to the others." (Goodchild & Sriraman, 2012, p. 581)

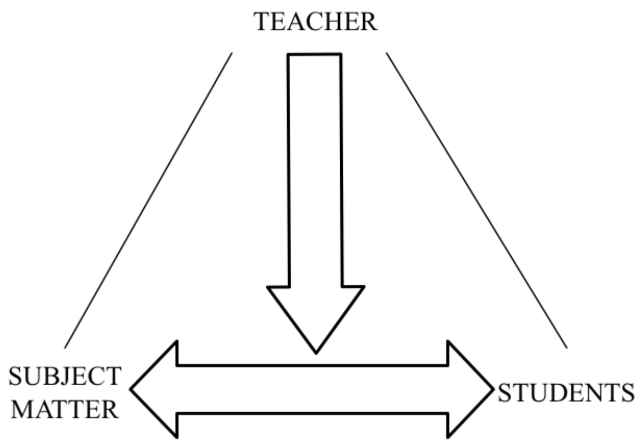

Figure 1: The didactic relation in the didactic triangle (modified from Kansanen, 2003, p. 230)

Here the didactic triangle functions as a heuristic (Ruthven, 2012) for foregrounding what Kansanen (2003) considered as two critical relations in the learning-teaching complexity that go beyond the pedagogical relation between teacher and students:

> "First, there is a relation between the student and the content. This is manifest as studying, and latent as learning and other changes. Secondly, the teacher has a relation to this relationship between the student and the content. In other words, the teacher has a relation to studying, and at the same time this relation is also to the learning. That may be called didactic relation […]. To highlight the importance of the didactic relation it may be emphasised that concentrating on the content makes the teacher an expert and concentrating on a student makes the teacher a caretaker of the pedagogical relation. To concentrate on the relation between the student and the content or on studying is, however, the core of a teacher's profession." (Kansanen, 2003, p. 230)

The didactic triangle speaks to the multidimensional concerns of mathematics education. The relations that Kansanen (2003) pointed out are of prime concern in this thesis: the relation between students and subject matter and the relation between teacher and the student-subject matter relation (see Figure 1). These two relations refer to two research areas in mathematics education: mathematical cognition and teacher cognition.

The thesis is a dissertation by publication, consisting of three journal articles in the area of mathematical cognition and teacher cognition (one article concerning mathematical cognition; the other two articles concerning teacher cognition). In the area of mathematical cognition, the thesis aims to advance theory on forms of abstraction and sense-making in mathematics. In the area of teacher cognition, the thesis intends to make theoretical contributions to the discussion of mathematics teacher knowledge and teacher noticing.

Research Focus 1: Abstraction and Sense-Making

Over recent years, various theoretical frameworks have arisen to account for cognitive development in mathematical knowing and learning. The focus here is explicitly on local theories of mathematical knowing and learning – in particular on two approaches (abstraction-from-actions and abstraction-from-objects) that have been previously construed as opposing – instead of global theories such as the embodied cognition approach (e.g. Lakoff & Nunez, 2000) or the situated learning approach (e.g. Lave & Wenger, 1991). The purpose here is to raise the debate beyond comparison of these seemingly opposing approaches, by identifying fundamental cognitive processes underlying both approaches in order to explore possibilities for coordinating them in a meaningful way that better speaks to the complexity of mathematical knowing. The purpose of coordinating these two approaches is not to attempt to build a unified theory, but to acknowledge the exquisite complexity of mathematical knowing and learning.

Research Focus 2: Teacher Knowledge

The last few decades have produced a considerable body of literature that conceptualizes, operationalizes, and measures mathematics teacher knowledge. The focus here is on general orientations and tendencies in conceptualizing mathematics teacher knowledge, and how the field currently conceives of what makes mathematics teacher knowledge specialized. The purpose here is to identify serious limitations of these orientations and tendencies and to provide alternative views to each of these orientations and tendencies that foreground topics in what makes mathematics teacher knowledge specialized that have only been partially investigated.

Research Focus 3: Teacher Noticing
The complexity of fields such as teacher noticing engenders difficulties when attempting to generate deep theoretical understanding. This thesis strives to generate such understanding by developing a theoretical perspective that borrows from other research disciplines. In doing so, it is hoped that researchers and educators will be provided with useful insights concerning the complexities of an individual's attentional engagement with the environment and her or his situation awareness. The approach taken here is an example of how a theoretical construct (namely teacher noticing) that is intensively discussed in mathematics teacher education can be re-conceptualized in light of rich conceptualizations of related phenomena discussed in cognitive science and the applied science of human factors. This may form a basis for reconsidering how to conceptualize the complexities involved in teacher noticing. Such a reframing may enable the identification of important questions that need to be addressed in the field.

1.2 Aims and Purposes

This thesis seeks to better understand the complexity of mathematical cognition and teacher cognition. Existing theories are considered to be restrictive or conflicting in their explanations of knowing, learning and teaching. Alternatives are sought that might contrast, link, and extend existing understandings. Hence, the mission of this thesis is to provide new theoretical insights that advance scholarly understanding of complex phenomena in student mathematical cognition and teacher cognition (phenomena that cannot be accounted for by deterministic accounts and cannot be understood strictly by means of analysis).[4]

The interest here is in generating, not testing, theoretical perspectives.[5] Theoretical perspectives can be generated through multiple modes of inquiry. Here the thesis follows critical and dialectical approaches in generating new theoretical perspectives and insights. The goal is to understand, criticize, and extend theoretical accounts of mathematical abstraction and sense-making, mathematics teacher knowledge, and teacher noticing, with the ultimate goal of transforming existing perspectives to provide greater insights and extend previous conceptualizations in new directions. Thus, the theoretical perspectives generated herein are intended to serve as catalysts for the development of more comprehensive theoretical accounts of the phenomena under consideration.

As such, the thesis intends to both extend current conversations and start new conversations in the areas of mathematical cognition and teacher cognition. Current conversations are extended by offering critical reflections and elaborations of existing views as well as making an attempt to suggest "how researchers […] can deal with the almost mystifying range of theories and theoretical perspectives that are being used" (Lester, 2005, pp. 176-177). New conversations are started by questioning existing conceptualizations and understandings of critical phenomena, and providing new points of view that move scholarly understanding closer toward better accounting for their complexity.[6]

1.3 Organization of the Thesis

The thesis is organized into six chapters, with different orientations and intentions, that deepen or extend the discussions provided in the three articles. Chapter 2 begins by acknowledging the broad diversity of (at times competing) theoretical perspectives in mathematics education research. The strategies employed by networking theories to deal with this abundance of theory are then discussed, with a particular emphasis on one form of coordination strategy, called blending. The chapter then states the three objectives of the thesis. Chapter 3 presents a new theoretical perspective on the acquisition of mathematical meaning by students, the dialogic framing, which is obtained by blending two existing theoretical perspectives that are often viewed as mutually exclusive, followed by the first article entitled

"New light on old horizon: constructing mathematical concepts, underlying abstraction processes, and sense making strategies" by T. Scheiner, published 2016 in *Educational Studies in Mathematics*, *91*(2), 165-183 (doi: 10.1007/s10649-015-9665-4). Chapter 4 examines teacher cognition, discussing existing conceptualizations of mathematics teacher knowledge and taking a critical stance toward the assumptions that underlie those conceptualizations, followed by the second article entitled "What makes mathematics teacher knowledge specialized? Offering alternative views" by T. Scheiner, M. A. Montes, J. D. Godino, J. Carrillo, & L. R. Pino-Fan, published online-first in *International Journal of Science and Mathematics Education* (doi: 10.1007/s10763-017-9859-6). Chapter 5 examines teacher cognition from the perspective of the construct of teacher noticing, drawing on insights from cognitive science and the applied science of human factors to develop a model of teacher noticing wherein individual and environment are inseparable and interdependent, followed by the third article entitled "Teacher noticing: enlightening or blinding?" by T. Scheiner, published 2016 in *ZDM Mathematics Education*, *48*(1-2), 227-238 (doi: 10.1007/s11858-016-0771-2). Chapter 6 concludes by summarizing the results of this thesis and discussing their wider significance.

Notes to Chapter 1

[1] The question of what theory is, is on its own a crucial question. There is a range of diverse answers to this question, but little consensus on a common definition of what theory is. For the purposes of this thesis, a broad, relatively general definition is used: theory is a collection of clearly defined concepts and their interrelationships that taken together offer an explanation for how and why a phenomenon occurs. It should be noted that the purpose of the thesis is not to develop new theories, but rather to make some theoretical contributions to the field that advance scholarly understanding of some complex phenomena in mathematics education.

[2] In response to the limitations of cognitive-oriented approaches to accounting for individual cognition, a number of theoretical perspectives have arisen in the past few decades that treat cognition as socially and culturally situated (see De Corte, Greer, & Verschaffel, 1996). While the distinction between cognitive and situated perspectives is important, the assumption that one needs to choose between them is misleading (for a discussion see e.g. diSessa, Levin, & Brown, 2016). It should be stressed that the focus on cognition in mathematics education herein is not arguing in favor of a cognitive orientation, but is simply an attempt to account for the cognitive structures and processes that seem to be involved.

[3] Certainly, the didactic triangle does not provide an all-embracing framing of the learning-teaching complexity in mathematics education, but it does foreground some central objects of mathematics education (or, more precisely, didactics of mathematics) and the important role that subject matter (mathematics) plays for learning and teaching processes. In the German-speaking countries especially, the didactic triangle has a long tradition, with an emphasis on didactical analyses of school mathematics, called *Stoffdidaktik* (see Steinbring, 1998; Strässer, 2007), and mathematics as a focal point of lessons (see Kaiser, 2002). For a more recent discussion on broadening the 'classical' version of the didactic triangle to view classroom activities from a socio-cultural perspective see Schoenfeld (2012).

[4] This thesis is not written from the assumption that there is only one paradigmatic way of thinking about cognition. It suggests instead an interpretation of certain phenomena in mathematical cognition and teacher cognition that might have value in its own right and that can put in dialogue, other, limiting, ways of thinking. As such, this thesis is not uniquely associated with any particular school of thought but acknowledges insights from other traditions without trapping itself in absolutes. Such an attitude recognizes that all accounts are partial, impermanent, and from a particular perspective, and that all theoretical framing, in consequence, changes and evolves over time.

[5] In testing theoretical perspectives, the primary interest is in the verification or falsification of those perspectives (with a focus on hypotheses testing and mainly quantitative analyses). Theory building in those cases typically occurs through the incremental adjustment or broadening (or occasionally, refusal) of the original theory.

[6] The author's standpoint is rooted in the conviction that what makes one theoretical construct or framework preferred over another is advancement toward "what is believed to be true" (Dubin, 1978, p. 13). The author believes that there is room for further theoretical contributions that reveal what was otherwise not seen, known, or conceived. These theoretical contributions may "allow us to see profoundly, imaginatively, unconventionally into phenomena we thought we understood. […] [A theoretical contribution] is of no use unless it initially surprises – that is, changes perceptions" (Mintzberg, 2005, p. 361).

2 Advancing Theory Building in Mathematics Education

In reviewing Sierpinska and Kilpatrick's (1998) edited book *Mathematics education as a research domain: A search for identity*, Steen (1999) stated that mathematics education is "a field in disarray, a field whose high hopes for a science of education have been overwhelmed by complexity and drowned in a sea of competing theories" (p. 236). Certainly, mathematics education is a 'pluralized field' (Jablonka & Bergsten, 2010) containing a broad range of diverse, at times competing, theoretical perspectives and theories.[1] These theories, in addressing issues about and related to the knowing, learning, and teaching of mathematics, sometimes borrow insights from many other disciplines (e.g. philosophy, psychology, sociology, cognitive science, etc.), while claiming to speak to both academics and practitioners.[2] The broad diversity of theoretical perspectives and theories is itself not a problem, but must be acknowledged to grasp the complexity of the objects of investigation (Lerman, 2006).[3] Theoretical perspectives and theories serve as lenses through which complex phenomena in mathematics education are looked at; the lenses being socially and culturally situated (Sierpinska & Kilpatrick, 1998) and relying on, and projecting, different philosophies and paradigms (Cobb, 2007).[4] Such a multi-perspective view is an attempt to account for a complex phenomenon by linking various theories to constitute a multi-dimensional account of the phenomenon.

2.1 Networking Theories: Fostering Deeper Insights

Recently, researchers working within the 'networking theories' group (Bikner-Ahsbahs et al., 2010; Bikner-Ahsbahs & Prediger, 2014) made substantial progress in dealing with the diversity of theories in mathematics education. Networking theories do not mean to remove diverse theoretical perspectives or theories through uniform assimilation, but instead to create a dialogue between theories in mathematics education (Radford, 2008). Networking theories aim at answering the question of "how to deal with the diversity of manifold, partly overlapping and partly contradictory theories and the connected diversity of conceptual descriptions for similar phenomena" (Bikner-Ahsbahs & Prediger, 2006, p. 52). The networking theories approach offers a systematic way of interacting with diverse theoretical positions and theories by using different strategies (Bikner-Ahsbahs & Prediger, 2006; Prediger, Bikner-Ahsbahs, & Arzarello, 2008). In this thesis, the strategies 'comparing' and 'contrasting', 'combining' and 'coordinating', as well as 'synthesizing' and 'integrating' are of primary concern.

The strategies of comparing and contrasting are useful when considering the diversity of existing theories. Comparing takes account of both similarities and differences, whereas contrasting stresses the differences and is less neutral. Using these strategies, the strengths and weaknesses of theoretical approaches can be highlighted. Prediger et al. (2008) identified three different aims associated with these two strategies: they can be used as an "inter-theoretical communication", a "competition strategy on the market of available theoretical approaches", and a "rational base for the choice of theories" (p. 171).

In contrast to comparing and contrasting, which aim to advance understanding of critical qualities and characteristics of theoretical perspectives for further theory development, the strategies of combining and coordinating are mainly used to looking at a particular phenomenon from different theoretical perspectives. Combining is described as a strategy that tries to combine a number of local theories, even those with incompatible background theories and conflicting perspectives, in order to get a multi-focal insight into the phenomenon under consideration. Coordinating is described as fitting elements from different theories to form a conceptual framework for making sense of the phenomenon. As coordinating is a strategy that can only be employed between theories with compatible core elements, using this strategy necessitates a careful analysis of the interconnections between and amongst components of each theory in order to determine the degree of their compatibility (see Prediger et al., 2008).

Whereas combining and coordinating primarily focus on a specific phenomenon with the aim of developing deeper insights into it, synthesizing and integrating aim at creating new theories by bringing together a small number of theoretical approaches into a new framework. These two strategies differ in their "degree of symmetry", or the extent to which constituents of both theories are utilized (Prediger et al., 2008, p. 173). Synthesizing describes a strategy used "when two (or more) equally stable theories are taken and connected in such a way that a new theory evolves" (Prediger et al., 2008, p. 173). Integrating is used where there is a lower degree of symmetry in related theoretical components, where a subset of the components of one theory are integrated into a more detailed and primary theory.

Networking theories can contribute to the empirical, the methodological, and the theoretical (Bikner-Ahsbahs et al., 2014). For the purpose of advancing theory development, this thesis focuses attention on the theoretical. In summary, networking theories attempt to explore ways of dealing with the increasing diversity of theories in mathematics education, by studying the insights offered by and limitations of each theory with the aim of advancing theory building in mathematics education. In particular, networking theories may advance theory building "by sharpening theoretical principles or constructs, extending theoretical approaches, building new concepts, posing new questions, or making explicit commonalities" (Bikner-Ahsbahs et al., 2014, p. 10).

2.2 Blending Theories: Fostering Novel Insights

Similarly to networking theories, this thesis uses various lenses to cultivate multifaceted understandings and diverse interpretations of critical phenomena in mathematical cognition and teacher cognition. Major efforts are made to look for theoretical tensions or oppositions and to recognize divides and bridges in existing theory, and to use them to stimulate the development of new insight and understanding. That is, the thesis aims not so much for an increasing degree of integration of different theories, but for a highlighting of contradictions and interdependencies in existing theoretical approaches.

The guiding philosophy of this thesis is based on the perspective that although at times there is a necessity to look at complex phenomena from different viewpoints, it might be even more productive to put in dialogue viewpoints (even competing ones) in order to generate novel and stronger theoretical insights. In a departure from the approach taken by previous attempts at networking theories, this thesis strives to coordinate conflicting perspectives and insights to generate new understanding. It is assumed that theoretical perspectives (or theoretical frameworks) can be 'blended' to provide novel insights and understanding that are absent when each individual theoretical perspective (or theoretical framework) is considered in isolation. As Tall (2013) stated:

> "[…] frameworks may benefit from a broader theory that is a blend of both, explicitly revealing the nature of aspects that are supportive in some contexts yet problematic in others, yet at the same time, these aspects may blend together so that an apparent dichotomy has the potential to offer new insights." (pp. 410-411).

Blending is a higher level of coordinating theoretical perspectives that does not imply synthesis (or unification) but, instead, seeks to view similarities, differences, inter-relationships and contradictions (between theories) in new light. The goal is a richer, comprehensive, and contextualized understanding. This is the level of coordinating perspectives that diSessa, Levin, and Brown (2016) described as 'deep synergy',

> "at which things pass beyond being 'interesting' to being 'fundamental for the field' […], where the intellectual support for at least some of the most important ideas comes from both

perspectives. This is the regime where retaining the identity of the two perspectives begins to become questionable. Genuinely new intellectual territory has been reached that is not construable from within only one perspective." (p. 5)

The term 'blending' has its origin in the work of Fauconnier and Turner (2002) on 'conceptual blending', who built a detailed framework of blending two knowledge domains from which novel elements result from in the blend that are not evident in either domain on its own. According to Fauconnier and Turner (2002): "In conceptual blending, frames from established domains (known as *inputs*) are combined to yield a hybrid frame (a *blend* or *blended model*) comprised of structures from each of the inputs, as well as unique structure of its own" (p. 115). As such, blending is a process of partial mapping or integration, called cross-space mapping, a mental operation of combining frames from (originally distinct) input spaces that leads to different meaning, novel insights, and conceptual compression.[5] Some scholars argue that the capacity for complex conceptual blending is essential for thought, and underlies the formation of meaning (for a comprehensive account, see Fauconnier & Turner, 2002).

Figure 2 shows the main features of conceptual blending: the four circles represent the mental spaces (two input spaces, a generic space containing structure common to the input spaces, and a blended space with unique structure). The solid lines designate the cross-space mapping between the input spaces, and the dashed lines designate links between input spaces and either generic or blended spaces. The rectangle inside the blended space designates emergent structure (along with selected aspects or structure from each input space).

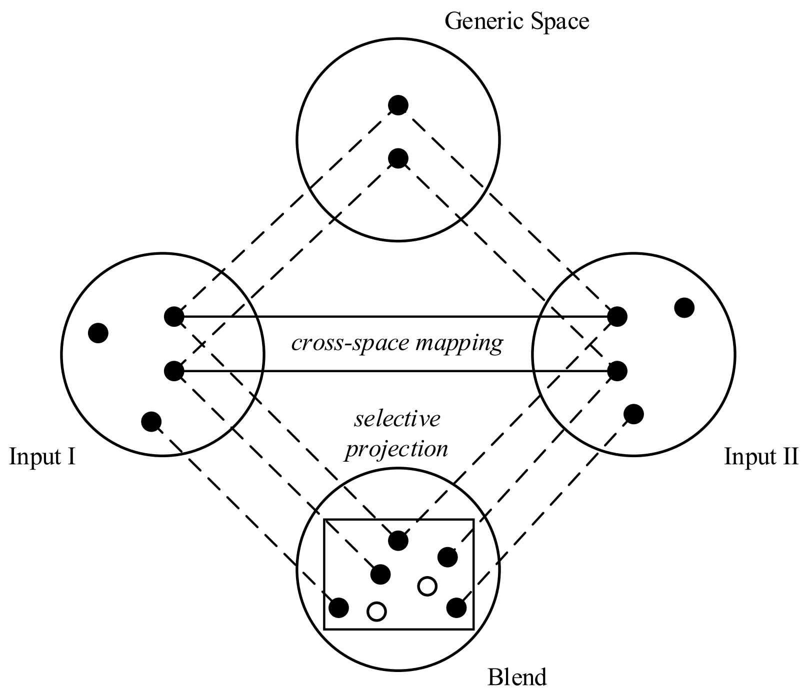

Figure 2: The basic diagram of conceptual blending (reproduced from Fauconnier & Turner, 2002, p. 46)[6]

Turner (2014) specified that

"The blend is not an abstraction, or an analogy, or anything else already named and recognized in common sense. A blend is a new mental space that contains some elements from different mental spaces in a mental web but that develops new meaning of its own that is not drawn from those spaces. This new meaning emerges in the blend." (p. 6)

Blending is considered here as a rich resource for networking theories that provides a productive way of producing novel insights that may not be manifest in the original theoretical frameworks from where critical components have been blended. What is important here is the recognition that different theoretical frameworks might have conflicting theoretical positions, but those conflicting positions can make central contributions to the blend, with the resulting blend being a conceptual system of higher explanatory power, flexibility, and greater insight. Theory development by conceptual blending can then be phrased as: putting in dialogue familiar (possibly mutually contradictory) ideas in an unfamiliar (possibly complementary) way, thereby producing novel (potentially enlightening) ideas.

2.3 Objectives for Theory Development

Blending is a recurrent theme in this thesis. The thesis consists of three major contributions, each having its specific objective for theory development in important areas in mathematics education. In particular, the thesis attempts to blend opposing theoretical perspectives to reveal complementarity in the field of mathematical knowing and learning, challenge taken-for-granted assumptions to reveal restrictions in the field of teacher knowledge, and portray some complex phenomena that cannot be accounted for using intuitive models of teacher noticing.

Objective 1: Transcending Dualisms in Mathematical Cognition

The first contribution (Scheiner, 2016a) of this thesis draws on various theoretical frameworks in mathematical concept formation to put in dialogue cognitive processes and sense-making strategies that are often considered to be in opposition to each other. This has the potential to move the discussion beyond simple comparison and offer new insights into the complexity of mathematical cognition and learning which cannot be appreciated by a taking a mono-logical vision of mathematical concept formation.

In particular, this thesis blends opposing theoretical perspectives to reveal complementarity. As such, the intention is to move beyond dualisms by examining supposedly conflicting views concurrently. The purpose here is not to dispute or strive to surpass previous ideas, but to give deeper meaning to such ideas and elaborate upon them in novel ways. As such, the theoretical contribution consists of presenting countervailing views and coordinating seemingly opposing theoretical perspectives via dialectical approaches.

Objective 2: Challenging Taken-For-Granted Assumptions in Conceptualizing Teacher Knowledge

The second contribution (Scheiner et al., 2017) of this thesis takes a critical stance toward existing conceptualizations of teacher knowledge by examining the assumptions that underlie them. In so doing the commonly accepted view of what lies at the core of the teaching profession, the transformation of subject matter for the purpose of teaching, is challenged.

The purpose here is to demonstrate how different 'frames of reference' (namely, the structure of a discipline vs. the structure of mind) foster certain conceptualizations of teacher knowledge and may lead to opposing views on subject matter (where subject matter is seen as an object of teaching vs. where subject matter is seen as an object of learning).

Objective 3: Going Beyond Intuitive Models of Teacher Noticing

The third contribution (Scheiner, 2016b) of this thesis intends to modify, extend, and redirect theoretical conceptualizations of the construct of teacher noticing by providing theoretical linkages with constructs discussed in cognitive science (inattentional blindness) and the applied science of human factors (situation awareness). Put differently and more provocatively, central constructs developed in cognitive science and the applied science of human factors are (re-)situated in the discussion of teacher noticing, to break away from the simplified and expected and explain complex phenomena involved in teacher noticing in new light.

More importantly, this situating brings a greater appreciation of the complex interdependencies of individual and environment that cannot be account for by intuitive models of teacher noticing. Explaining how and why teachers notice what they notice presents a major open scientific problem.

2.4 Summary

In summary, this chapter discusses strategies used by networking theories to deal with the diversity of theories in mathematics education and outlines the objectives of the thesis. Specific strategies employed by networking theories are presented, namely the strategies of comparing, contrasting, combining, coordinating, synthesizing, and integrating. This chapter uses the idea of blending to illuminate unspoken and unexamined practices for coordinating conflicting perspectives. Finally, the three objectives of this thesis are outlined: transcending dualisms in thinking about mathematical cognition, challenging critical assumptions implicit in most accounts of conceptualizing teacher knowledge, and going beyond intuitive models of teacher noticing by incorporating ideas from other fields. The following chapters discuss the presented objectives in detail, each chapter being self-standing and deepening or extending a particular aspect focused on in the respective article.

Notes to Chapter 2

[1] Not everything that is termed ‘theory’ is a theory. Instead, in discussing theoretical work, one might advocate more nuances. It might be useful to distinguish between ‘theoretical perspective’ (or ‘theoretical orientation’), ‘theoretical framework’, ‘theory’, and ‘model’. These notions, when considered as relative rather than absolute distinctions, can help to identify differences in how theories are interpreted. A *theoretical perspective* (or *theoretical orientation*) can be understood as a ‘worldview’ or a ‘background theory’ (in the sense of Mason & Waywood, 1996) that influences an individual’s approach to professional life. A *theoretical framework* might be understood as “a general pool of constructs for understanding a domain, but it is not tightly enough organized to constitute a predictive theory” (Anderson, 1983, p. 12). It is possible to generate a theory from this pool of constructs, one that makes unique empirical predictions that distinguish it from other frameworks. In this sense, “one judges a framework in terms of [the] success, or fruitfulness, of the theories it generates. If the theories lead to many accurate accounts of interesting phenomena, the framework is regarded as fruitful.” (Anderson, 1983, p. 12). On the other hand, a *theory* is “a precise deductive system that is more general than a model […]”, whereas a *model* is “the application of a theory to a specific phenomenon […]” (Anderson, 1983, p. 13).

[2] Another reason for the broad diversity of theories is the versatile use of theories (see Assude, Boero, Herbst, Lerman, & Radford, 2008; Sriraman & English, 2005). Theories function as “lens[es] to analyze data and produce results of research on a problem” (Silver & Herbst, 2007, p. 50); they are “the way in which we represent the knowledge and understanding that comes from any particular research study” (Bishop, 1992, p. 711). Theories are used “to direct action in ways more powerful than are possible

without the use of the theory, because they take account of qualities of the environment which are inaccessible to simple observation" (Skemp, 1979, p. 315). In addition, theories may also serve as "a language of descriptions of an educational practice" (Silver & Herbst, 2007, p. 56) and as a "tool which can help to design new [educational] practices" (ibid., p. 59).

[3] However, introducing isolated theoretical positions and theories rather than charting a discourse among them becomes a challenge for the research community and does not resolve researchers' continuing search for a disciplinary identity. On the other hand, the diversity of theoretical perspectives and theories can become an "eminently fruitful source for the development of a disciplinary identity" when different positions and traditions interact (Bikner-Ahsbahs & Prediger, 2006, p. 56).

[4] Instead of viewing phenomena through the lens of one particular theoretical orientation, we often act as *bricoleurs* (in the sense of Lévi-Strauss, 1966) by adopting ideas from a variety of theoretical sources to conform to our intentions and own biases (see e.g. Cobb, 2007; Gravemeijer, 1994). The thesis is to be understood as a bricolage, an emergent collage-like piece that brings together divergent views, understandings, and interpretations of some complex phenomena in mathematics education and the author's own analyses of these understandings and interpretations.

[5] Studies of conceptual blending conceive of it in terms of integration networks of mental spaces. "Mental spaces are small conceptual packets constructed as we think and talk, for purposes of local understanding and action" (Fauconnier & Turner, 2002, p. 40), they are partial collections containing elements that are organized by conceptual frames and mental models. In its simplest form, an integration network consists of two partially matched input spaces, a generic space, and a blended space. The generic space shares structure with both of the inputs; thus, it defines a partial mapping between representations in the input spaces. The blended space is constructed through (a) composition, (b) completion, and (c) elaboration, each of which provides for the possibility of emergent structure. In short, "*composition* of elements from the inputs makes relations available in the blend that do not exist in the separate inputs" (Fauconnier & Turner, 2002, p. 42, italics in original). In other words, composition involves identifying a relation between an element or elements of an input space and an element or elements from other input spaces. *Completion* is pattern-completion, which occurs when patterns extrapolated from the inputs fit with background knowledge that is brought into a blend. *Elaboration* is closely related to completion, and is a process whereby an event is performed and/or mentally simulated in the blend, and is constrained by the logic of the blended domain itself. In this sense, the blend has emergent dynamics – it can 'run', while remaining connected to the other spaces.

[6] Notice that this figure of conceptual blending is just a snapshot of a more complex process, presented in an order that does not necessarily correspond with the actual stages of blending as performed by an individual. More complex systems of connected mental spaces can have more input and blended spaces (where blends at one level can be inputs at another).

3 Transcending Dualisms in Mathematical Cognition: Toward a Dialogical Framing*

*This chapter refers to the first journal article, entitled "New light on old horizon: constructing mathematical concepts, underlying abstraction processes, and sense making strategies" by T. Scheiner, published 2016 in *Educational Studies in Mathematics*, *91*(2), 165-183. (doi: 10.1007/s10649-015-9665-4)

This chapter focuses on two strands of research concerning mathematical concept formation: abstraction-from-actions approaches and abstraction-from-objects approaches. The first article (Scheiner, 2016a) identifies cognitive processes and sense-making strategies underlying the two approaches and opens a new avenue to go beyond simply viewing the two approaches as being in opposition. The article draws on various theoretical frameworks to move the discussion from simple comparison towards a synergy of theoretical frameworks that acknowledges both the complementarity of the underlying cognitive processes and their respective sense-making strategies. This complementarity has been overlooked in previous approaches. Specifically, the article blends theoretical frameworks on two fundamental forms of abstraction (reflective abstraction and structural abstraction) and their respective sense-making strategies (extracting meaning and giving meaning).[1] This blending argues strongly against dismissing abstraction from objects as irrelevant for mathematical concept formation, and instead aims to overcome misleading dichotomies of abstraction from actions and abstraction from objects, as Piaget (1977/2001) put forth.[2] A detailed discussion of the particular cognitive processes, their respective sense-making strategies, and the new insights into the complexity of mathematical concept formation that emerged in blending the theoretical frameworks on reflective and structural abstraction can be found in Scheiner (2016a).

This article makes a theoretical contribution by discussing the dialogical framing of extracting meaning and giving meaning, which emerged from examination of the seemingly opposing approaches of abstraction from actions and abstraction from objects. This discussion focuses on the relation between extracting meaning and giving meaning and the potential of a blended theory to account for the complex dynamics involved in mathematical concept formation, dynamics which cannot be accounted for considering extracting meaning and giving meaning separately.

This chapter is structured in three parts: First, some theoretical assertions are outlined that oriented the theoretical framing put forth in Scheiner (2016a). Second, explicit and implicit assumptions underlying the respective sense-making strategies of extracting meaning and giving meaning are examined. Third, the dialogic framing of extracting meaning and giving meaning is outlined, revealing the complex dynamics involved in mathematical concept formation.

3.1 Theoretical Orientations and Orienting Assertions

The theoretical foundation for coordinating reflective and structural abstraction, as presented in Scheiner (2016a), relies on and projects several theoretical insights revealed by the German mathematician and philosopher Gottlob F. L. Frege (1848-1925) that have informed a variety of theoretical perspectives on mathematical knowing, thinking, and learning (see Arzarello, Bazzini, & Chiappini, 2001; Duval, 2006; Radford, 2002). In particular, the theoretical foundation in Scheiner (2016a) shares Frege's (1892a) assertion that a mathematical concept is not directly accessible through the concept itself but only through objects that act as proxies for it.[3]

However, mathematical objects (unlike objects of natural sciences) cannot be apprehended by human senses (we cannot, for instance, 'see' the object), but only via some 'mode of presentation' (Frege, 1892b) – that is, objects need to be expressed by using signs or other semiotic means such as a gestures, pictures, or linguistic expression (Radford, 2002). The 'mode of presentation' (or 'way of presentation') of an object is to be distinguished from the object that is represented, as individuals often confuse a $sense_F$ ('Sinn') of an expression (or representation) with the $reference_F$ ('Bedeutung') of an expression (or representation) (the subscript F indicates that these terms refer to Frege, 1892b).[4] The $\text{reference}_\text{F}$ of an expression is the object it refers to, whereas the sense_F is the way in which the object is given to the mind (Frege, 1892b), or in other words, it is the thought ('Gedanke') expressed by the expression (or representation) (Frege, 1918). The expression '$a = b$', for instance, is informative, in contrast to the expression '$a = a$', as the sense_F of 'a' differs from the sense_F of 'b'. Consider also Frege's (1892b) well-known example concerning the two different expressions of the planet Venus: 'the morning star' and 'the evening star'. The two expressions 'the morning star' and 'the evening star' have the same $\text{reference}_\text{F}$, that is the planet Venus, but have different ways the planet Venus is given to the mind: as a celestial body that shines in the east (morning sky) before sunrise or as a celestial body that shines in the west (evening sky) after sunset. Concerning mathematics, the two expressions '$3 + 2$' and '$7 - 2$', for instance, express different thoughts but have the same $\text{reference}_\text{F}$, the natural number 5. Thus, senses_F capture the epistemological and cognitive significance of expressions. This implies one of Frege's decisive assertions, that an object can only be apprehended via a sense_F of an expression (or representation): the sense_F orients how a person thinks of the object being referred to. Thus, it seems reasonable to understand Frege's (1892b) notion of an $idea_F$ ('Vorstellung') as the manner in which a person makes sense_F of the world. Ideas_F can interact with each other and form more compressed knowledge structures, called conceptions. A general outline of this view is provided in Figure 3, which is a slightly modified version of the original figure presented in Scheiner (2016a).

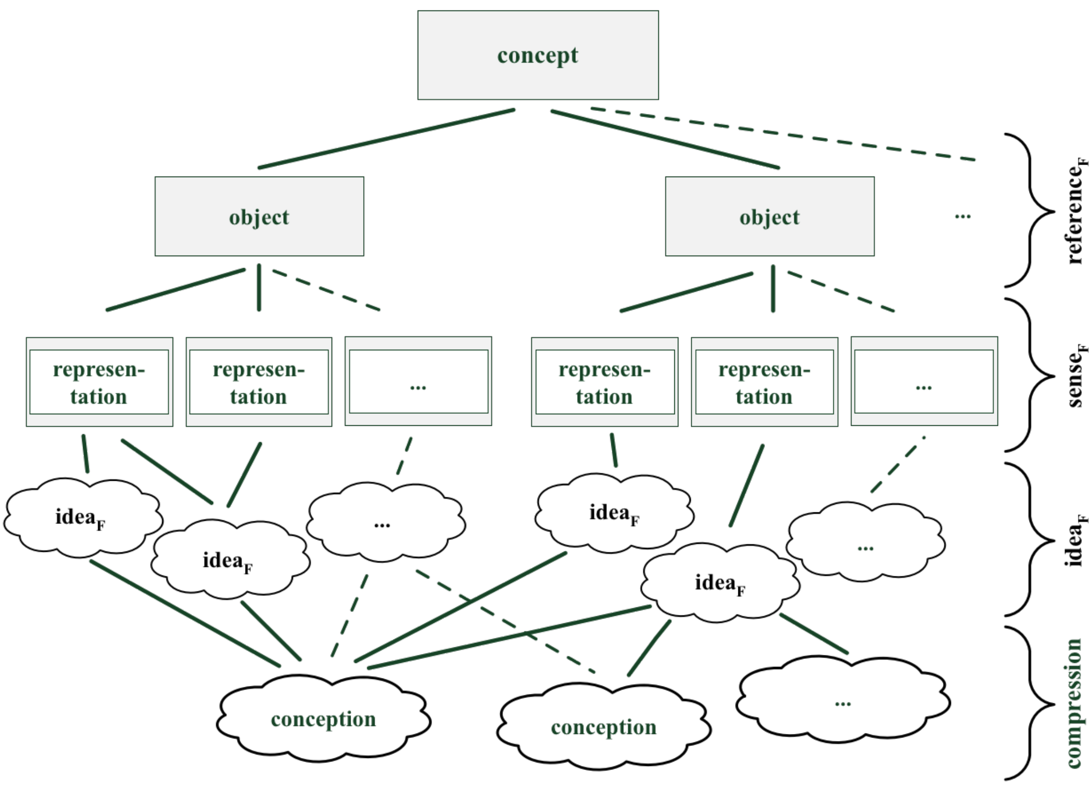

Figure 3: On $\text{reference}_\text{F}$, sense_F, and idea_F (modified from Scheiner, 2016a, p. 179)

3.2 On Extracting Meaning: Pointing to a Conception-to-Concept Direction of Fit

A common assumption is that the meaning of a mathematical concept is an inherent quality of objects that fall under a particular concept, and that this quality is to be extracted. This extraction of meaning is realized through the manipulation of objects and reflection of variations of senses$_F$ when objects are manipulated. These cognitive processes are often associated with reflective abstraction, that is, reflecting on the coordination of actions on mental objects (see Piaget, 1977/2001). Similarly, Duval (2006) argued that via systematic variation of one representation of an object and reflecting on resulting variations in another representation of the same object, an individual can recognize what is mathematically relevant and separate the sense$_F$ of a representation from the reference$_F$ of a representation. Such a view asserts that individuals internalize extracted mathematical structures and relations associated with their actions and reflections of their actions on objects. It gives the impression that individuals construct mental models (ideas$_F$ or conceptions) that correspond to an ideal realm (objects or concepts), though it might be read as taking a 'trivial constructivist' position (von Glasersfeld, 1989): the view that a necessary condition of knowledge is that individuals construct, constitute, make, or produce their own understanding (see Ernest, 2010). More importantly, such a view seems to suggest a 'conception-to-concept direction of fit' (Scheiner, 2017) that is, mathematical concept formation is regarded as individuals constructing conceptions that best reflect a (seemingly given) mathematical concept (see Figure 4).

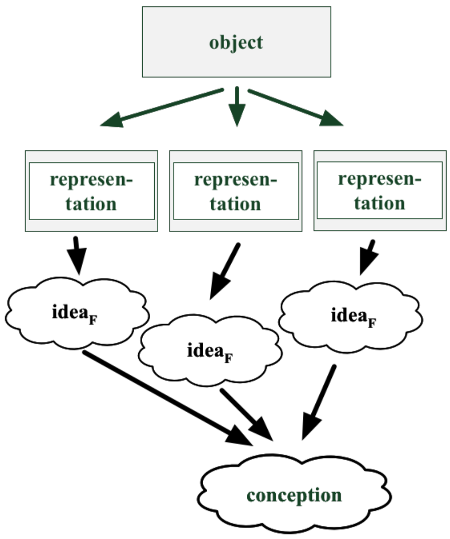

Figure 4: From object to idea to conception

3.3 On Giving Meaning: Pointing to a Concept-to-Conception Direction of Fit

In the attempt to coordinate abstraction-from-actions and abstraction-from-objects approaches, a new understanding of abstraction emerged: abstraction is not so much the extraction of a previously unnoticed meaning of a concept (or the recognition of structure common to various objects), but rather a process of giving meaning to the objects an individual interacts with from the perspective an individual has taken. Abstraction, as such, is more focused on "the richness of the particular [that] is embodied not in the concept as such but rather in the objects that falling under the concept [...]. This view gives primacy to meaningful, richly contextualized forms of (mathematical) structure over formal (mathematical) structures" (Scheiner, 2016, p. 175). This is to say, individuals give meaning to the objects they interact with by attaching ideas$_F$ to objects or, more precisely, by attaching ideas$_F$ to the senses$_F$ expressed by the representations in which an object actualizes. Recent research investigating the contextuality, complementarity, and complexity of this sense-making strategy (see Scheiner & Pinto, 2017) asserted that in contrast to Frege (1892b), who construed sense$_F$ in a disembodied fashion as a way an object is given to an individual, an individual assigns sense$_F$ to object. However, what sense$_F$ is assigned to an object is a function of what idea$_F$ is activated in the immediate context (see Figure 5). In this view, ideas$_F$ direct forming the modes of presentation under which an individual refers to an object. As such, it is a person's complex system of ideas$_F$ that directs forming a sense$_{F,}$ rather than merely the object a representation refers to.

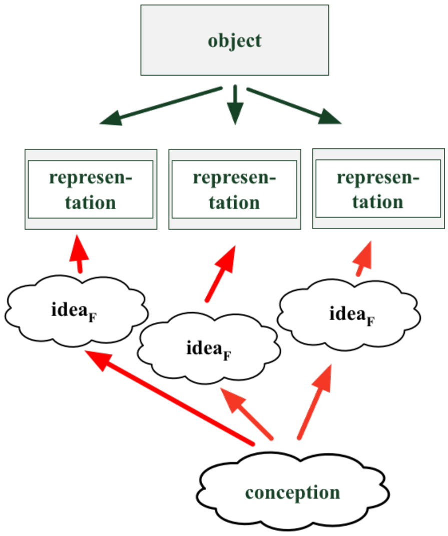

Figure 5: On activating ideas$_F$ and assigning senses$_F$

This research also indicated that individuals might even give meaning to objects that are yet to become. This means that although an object does not have a being prior to the individual's attempts to know it, an individual might create a new idea$_F$ that directs their thinking to potential objects, or more precisely: an individual might create an idea$_F$ that allows assigning a new sense$_F$ to objects that are yet to become (see Figure 6). That is, individuals might give meaning beyond what is apparent. It is proposed that the creation of such ideas$_F$ is of the nature of what Koestler (1964) described as 'bisociation', and Fauconnier and Turner (2002) elaborated as 'conceptual blending'.[5]

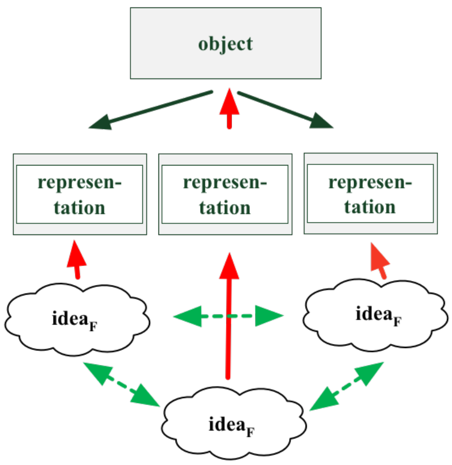

Figure 6: Transforming ideas$_F$ to give (new) meaning to an object

The key insight here is that unrelated ideas$_F$ can be transformed into new ideas$_F$ that allow 'setting the mind' not only to actual instances, but also to potential instances that might become 'reality' in the future. In such cases, conceptual development is not merely meant to reflect an actual concept, but rather to create a concept: a view that suggests a 'concept-to-conception direction of fit' (Scheiner, 2017) that is, mathematical concept formation is regarded as individuals creating a concept that best fits their conceptions. Similarly, Lakoff and Johnson (1980), drew attention to the power of (new) metaphors to create a (new) reality rather than simply to provide a way of conceptualizing a pre-existing reality: "changes in our conceptual system do change what is real for us and affect how we perceive the world and act upon those perceptions" (pp. 145-146.). It is reasonable to assume that students transform ideas$_F$ to express a yet-to-be-realized state of a concept. This accentuates Tall's (2013) assertion that the "whole development of mathematical thinking is presented as a combination of compression and blending of knowledge structures to produce crystalline concepts that can lead to imaginative new ways of thinking mathematically in new contexts" (p. 28).

3.4 On the Dialogical Framing of Extracting Meaning and Giving Meaning

Each of the previous two sections articulated a particular sense-making strategy: extracting meaning from objects (via manipulating objects and reflecting on the variations) and giving meaning to objects (via attaching existing and new ideas$_F$ to objects). These two sense-making strategies seem to differ in their directions of fit: extracting meaning involves individuals' attempts to construct conceptions that aim to fit a concept (conception-to-concept direction of fit), whereas giving meaning involves individuals' attempts to create a concept that aims to fit their conceptions (concept-to-conception direction of fit) (for a detailed discussion, see Scheiner, 2017).

In Scheiner (2016a), instead of construing extracting meaning and giving meaning as independent processes that point in two opposing directions, a bi-directional theoretical framing of mathematical concept formation was developed. Specifically, Scheiner (2016a) argued for a dialogical framing of extracting meaning and giving meaning, asserting that extracting meaning and giving meaning are interdependent (rather than independent): what meaning one extracts is very much a function of what meaning is given to, and vice versa (see Figure 7). This dialogical framing can better account for the complex emergence of evolving forms of meaning: meaning not only emerges (from Latin *emergere*, 'to become visible') via reflection on manipulations of objects, but also evolves (from Latin *evolvere*, 'to become more complex') via transforming previously constructed ideas$_F$ (see Scheiner, 2017).

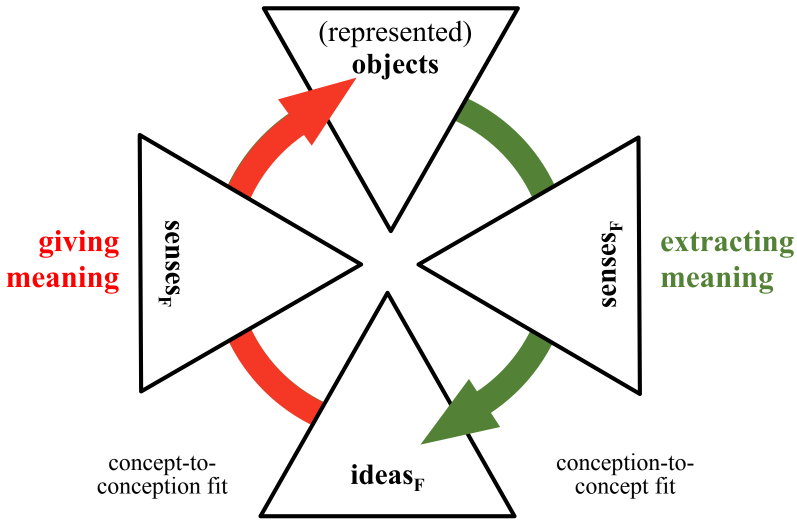

Figure 7: On the dialogue of extracting meaning and giving meaning

The dialogical framing of extracting meaning and giving meaning acknowledges the complex emergence of evolving forms of meaning that cannot be accounted for by viewing extracting meaning or giving meaning as separate. Extracting meaning and giving meaning, though they have value in their own right, are restricted, and restricting, in their accounts of mathematical concept formation. This is due to the 'hidden determinisms' inherent in the two approaches: extracting meaning assumes that what dictates meaning is the concept itself; while giving meaning advocates an individual's conceptions as the determinants of all meaning. The dialogical framing, in contrast, is not deterministic but bi-directional: mathematical concept formation involves processes that direct from conception to concept as much as it involves processes that direct from concept to conception. As such, the dialogical framing is more than a matter of recasting the concept-conception divide: it underlines that concept and conception are not static and apart but fluid and co-specifying (see Figure 8).

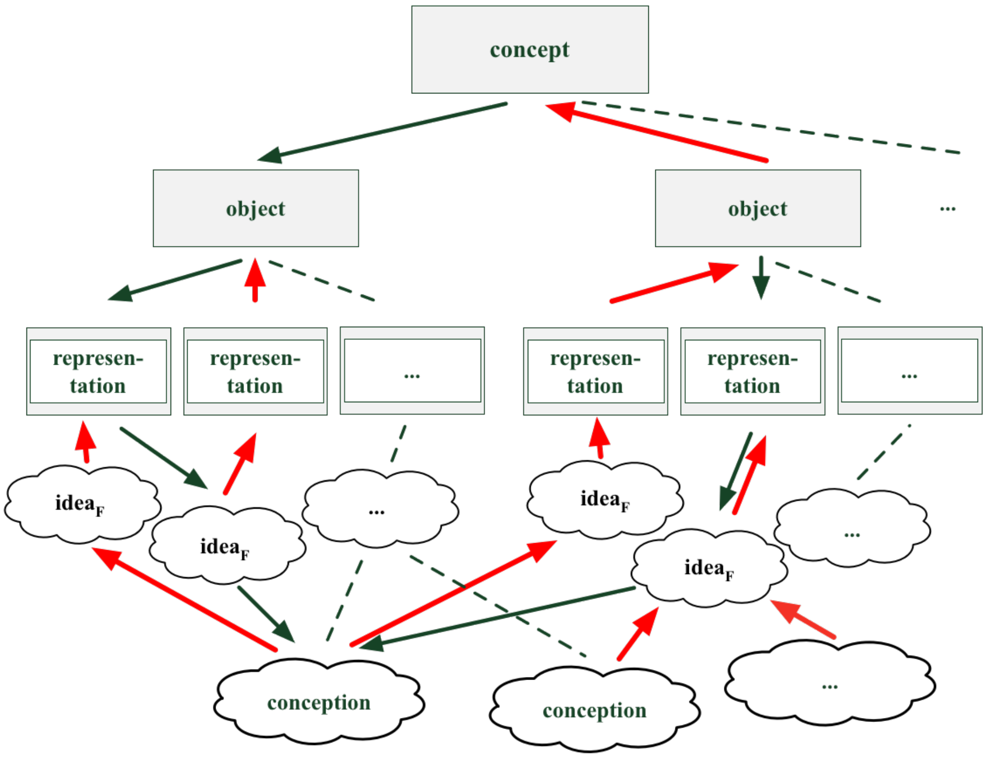

Figure 8: A complexivist frame: on the complex interaction between concept and conception

Figure 8 is an alternative to the reductionist view taken in respective approaches of extracting meaning (see Figure 4) and giving meaning (see Figure 5), both being rather uni-directional and deterministic in orientation. The dialogical framing provides new interpretative possibilities regarding the complex dynamics in mathematical concept formation, allowing for a move beyond simplistic assertions about linearity and determinism (that were transposed from analytical science and analytical philosophy onto discussions of mathematical concept formation). Figure 8 attends to the complexity in mathematical concept formation and speaks to the nonlinear, emergent characters of evolving forms of mathematical meaning (see e.g. Pirie & Kieren, 1994; Schoenfeld, Smith, & Arcavi, 1993).

3.5 Reflections

The theoretical contribution specified in Scheiner (2016a) makes the case that neither uni-directional framing of mathematical concept formation (whether involving extracting meaning or giving meaning) provides a comprehensive account of the complex emergence of evolving forms of meaning. It is argued for an alternative framing that acknowledges mathematical concept formation as both directed from concept to conception and from conception to concept. Mathematical concept formation, then, is construed as an ongoing, intertwined process of extracting meaning and giving meaning, in which conceptions shape, and are shaped by, the concepts with which an individual interacts.

This dialogical framing brings a greater insight: that any attempt to frame cognition in terms of mind over matter or matter over mind is misleading, as cognition is bi-directional: from the outside in (mind-to-world direction of fit) and from the inside out (world-to-mind direction of fit). That is, mind and world are engaged in a co-creative interaction: mind is shaped by the world and mind shapes the world. Such a world is subjectively articulated, in that its objectivity is relative to how it has been shaped by the knower (see Reason, 1998).

Such a dialogical framing is not so much a unification of any monism (that sees, for instance, mind as situated within its world), nor of any dualism (that sees mind apart from the world), but rather is an acknowledgment that mind is an integral part of the world, and as such both mind and world are in a constant state of flux, changing in the ever-unfolding process of extracting meaning and giving meaning.

3.6 Summary

In summary, this chapter presents a new theoretical perspective blended from the existing perspectives that mathematical meaning is extracted (from objects falling under a particular concept) and that mathematical meaning is given (to objects that an individual interacts with by that individual). This blending seeks to frame mathematical concept formation as bi-directional (where what meaning one extracts is a function of what meaning is given to, and vice versa) and to recast the concept-conception divide (by viewing concept and conception as fluid and co-specifying instead of static and apart). In doing so, the dialogical framing presents a view of mathematical concept formation that is complex, dynamic, non-linear, and possessed of emergent characteristics. After having focused on the subject matter–student axes of the didactic triangle in this chapter, attention is turned in the next chapter to the didactic relation between teacher and learning. In particular, the focus in the next chapter is on the underlying assumptions of existing conceptualizations of teacher knowledge.

Notes to Chapter 3

[1] There are several ways that individuals can make sense; here the focus is on Pinto's (1998) distinction between 'extracting meaning' and 'giving meaning' with respect to sense-making of a formal concept definition:

> "*Extracting meaning* involves working within the content, routinizing it, using it, and building its meaning as a formal construct. *Giving meaning* means taking one's personal concept imagery as a starting point to build new knowledge." (Pinto, 1998, pp. 298-299)

Gray, Pinto, Pitta, and Tall (1999) stated that in giving meaning a person attempts to build from their own perspective, trying to give meaning to mathematics from current cognitive structures. Tall (2013) elucidated that these two approaches are related to a 'natural approach', that builds on the concept image, and a 'formal approach', that builds formal theorems based on the formal definition.

In Scheiner (2016), extracting meaning was linked to the manipulation of objects and reflection on the variations in modes of presentation when objects are manipulated. These cognitive processes are often associated with Piaget's (1977/2001) reflective abstraction, that is, abstraction through coordination of actions on mental objects (see e.g. Dubinsky, 1991). Giving meaning, on the other hand, was related to attaching an idea$_F$ to a mode of presentation. That is, an individual gives meaning to the objects one interacts with from the perspective an individual has taken. These cognitive processes are more a manner of perspective-taking, grounded in the notion of structural abstraction that focuses on "the richness of the particular [that] is embodied not in the concept as such but rather in the objects that falling under the concept" (Scheiner, 2016a, p. 175).

[2] Piaget (1977/2001) dichotomized *abstraction à partir de l'action* (abstraction from actions) and *abstraction à partir de l'objet* (abstraction from objects), dismissing abstraction from object as irrelevant for conceptual knowing and learning in mathematics, due to a restricted understanding of empirical abstraction that "draws its information from objects" (p. 317) but "is limited to recording the most obvious and global perceptual characteristics of objects" (p. 319). Generally speaking, empirical abstraction is the extraction of superficial characteristics that individuals can observe in the environment. Skemp (1986), however, departed from the understanding of abstraction that focuses on underlying structures rather than superficial characteristics. Mitchelmore and White (2000) utilized Skemp's understanding of abstraction to develop an empirical abstraction approach in learning elementary mathematics, focusing on similarities of structures underlying objects or situations. The outline of structural abstraction provided in Scheiner (2016a) shares with Mitchelmore and White (2000) the focus on underlying structure but differs from them in that it accentuates the diversity rather than the similarity of structures underlying objects.

[3] While some have taken expressions or representations as objects (Font, Godino, & Gallardo, 2013), Scheiner (2016a) construed a mathematical object (e.g. a natural number, a linear function) as the reference$_F$ of a multiplicity of expressions (or representations), which are acknowledged as such by the scientific community. A mathematical object can be expressed (or represented) in the form of a linguistic element (e.g. expression, metaphor, notation), a definition (e.g. formal concept definition), or a proposition (e.g. statement), among others. The assumption that objects pre-exist or are given is not necessary, as an object comes into being in the representations in which it actualizes (Radford, 2013), and emerges in the recognition of referential equivalence of representations (the recognition that certain representations stand for the same object) (Duval, 2006).

[4] Duval (2006) provided a detailed description of students' confusion of a representation of an object with the object that is being represented, substantiated with what he called a 'cognitive paradox':

> "how can they [individuals] distinguish the represented object from the semiotic representation used if they cannot get access to the mathematical object apart from the semiotic representation?" (p. 107)

[5] Koestler's (1964) central idea is that any creative act is a *bisociation* of two (or more) unrelated (and seemingly incompatible) frames of thought (called matrices) into a new matrix of meaning by way of a process involving abstraction, categorization, comparison, and the use of analogies and metaphors. More recently, Fauconnier and Turner (2002) elaborated and formalized Koestler's idea of bisociation into what they called *conceptual blending*. Conceptual blending consists of constructing a partial match, called a cross-space mapping, in order to selectively project a set of inputs into a set of outputs. The inputs are frames from established domains, and the outputs are a novel hybrid frame (called blend), comprised of a structure from each of its inputs, and a unique structure of its own (or emergent structure). Fauconnier and Turner (2002) elucidated that individuals "are exceptionally adept at integrating two extraordinarily different inputs to create new emergent structures, which result in […] new ways of thinking" (p. 27).

Article 1

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New light on old horizon: Constructing mathematical concepts, underlying abstraction processes, and sense making strategies

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Abstract The initial assumption of this article is that there is an overemphasis on abstraction-from-actions theoretical approaches in research on knowing and learning mathematics. This article uses a critical reflection on research on students' ways of constructing mathematical concepts to distinguish between abstraction-from-actions theoretical approaches and abstraction-from-objects theoretical approaches. Acknowledging and building on research on knowing and learning processes in mathematics, this article presents a theoretical framework that provides a new perspective on the underlying abstraction processes and a new approach for interpreting individuals' ways of constructing concepts on the background of their strategies to make sense of a mathematical concept. The view taken here is that the abstraction-from-actions and abstraction-from-objects approaches (although different) are complementary (rather than opposing) frameworks. The article is concerned with the theoretical description of the framework rather than with its use in empirical investigations. This article addresses the need for more advanced theoretical work in research on mathematical learning and knowledge construction.

Keywords Cognition · Learner types · Reflective abstraction · Reflectural abstraction · Sense making strategies · Structural abstraction · Theory development

1 Introduction

It is widely acknowledged that the complex phenomena of knowing and learning processes in mathematics need pluralistic frameworks in order to adequately address the many facets of mathematical learning. The literature provides a variety of well-elaborated theoretical models and frameworks concerning mathematical concept construction such as Dubinsky and his

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colleagues' (Dubinsky, 1991; Cottrill et al., 1996) *APOS theory* using Piaget's (1977/2001) reflective abstraction as a point of departure, Tall's (2004, 2013) *Three Worlds of Mathematics* that cultivated Bruner's (1966) long-term development of the enactive-iconic-symbolic modes and van Hiele's (1986) levels of learning geometry and integrated them into a 'bigger picture' of mathematical learning, as well as Hershkowitz, Schwarz, and Dreyfus's (2001) *RBC model* (recognizing, building-with, & constructing). The latter has been further elaborated in later works (Hershkowitz, Hadas, Dreyfus, & Schwarz, 2007) into the RBC+C model (adding consolidation), its overall model grounded in an abstraction in context framework, suggesting that constructing new knowledge is largely based on vertical reorganization of existing knowledge elements. The latter two frameworks are remarkable examples as they integrate ideas from various 'schools of thought' that provide different perspectives on learning mathematics. For instance, the work of researchers within the RBC+C model uses the perspective of situated learning and apprenticeship, its main theoretical foundation lying in Vygotsky's Activity Theory. Tall's (2013) recent work, which presents a blending of numerous fundamental ideas underlying what Ernest (2006) called 'philosophies of learning', enabled him to identify three different worlds of mathematics: the (conceptual) embodied, the (operational) symbolic, and the (axiomatic) formal world.

These and further frameworks have shed light on important issues in research on learning mathematics and present strong cases showing that progress can and has been made in education research on learning and cognition in mathematics. It is a reasonable assertion that diSessa's (1991) description of the state of the art with respect to theory building in learning and knowledge construction as "quite poor" (p. 222) can no longer be considered to be accurate. Mathematics education research has progressed considerably since diSessa (1991) called for more advanced theories in learning and knowledge construction. Research has become more sophisticated in the sense that the same phenomena are looked at in more detail by using different perspectives. However, some areas need more emphasis. From the author's point of view, one of these areas falls within the dominant (if not the leading) perspective on how mathematics is learned, labeled as the cognitive constructivist perspective on learning. Research approaches that use this perspective have drawn on Piaget's work; in particular, Piaget's forms of abstraction have been influential in research on cognitive processes underlying mathematical concept construction. This article argues that in the past there has been an overemphasis on what in this article are called abstraction-from-actions theoretical approaches in research on constructing mathematical concepts. These approaches converge in an underlying cognitive process that Piaget described as reflective abstraction. However, Piaget (1977/2001) already distinguished between two forms of abstraction, namely *abstraction à partir de l'action* (abstraction from actions) and *abstraction à partir de l'objet* (abstraction from objects). While 'abstraction from actions' has been emphasized for several decades almost always as an exclusive way of concept construction, the focus on 'abstraction from objects' has been overlooked. Sfard (1998) reminds us that "giving full exclusivity to one conceptual framework would be hazardous. Dictatorship of a single metaphor […] may lead to theories that serve the interests of certain groups to the disadvantage of others" (p. 11). In this work, it is assumed that an overemphasis on 'abstraction from actions' cannot adequately represent the multi-faceted phenomena involved in constructing mathematical concepts.

In the following pages, a theoretical framework is described that (1) extends current perspectives on ways of constructing mathematical concepts, (2) identifies their underlying forms of abstraction, and (3) takes account of their interrelationships with strategies of making sense. A comprehensive reflection on the state of the art with respect to theoretical work on abstraction in mathematics education research is beyond the scope of this article. In addressing the issues mentioned, the following section (Section 2) focusses on two research strands on constructing mathematical concepts, namely abstraction-from-actions and abstraction-from-objects approaches. The article tries to identify the cognitive processes underlying mathematical concept construction within each strand. Based on the assumption that there is a need to better theoretically map out the cognitive processes that build the architecture for abstraction from objects, the second part of this section presents the outline of a theoretical reflection of this form of abstraction. Then, in Section 3, it is argued that research should take into account the interrelationships of cognitive processes underlying mathematical concept construction and students' strategies of making sense. Based on previous research on students' strategies of making sense, the article begins to examine this issue by providing a particular way to think about it. This effort is used to open a new avenue to go beyond simply viewing abstraction-from-actions and abstraction-from-objects approaches as being in opposition to each other. Section 4 provides some considerations of a particular way to think about the interrelationship between the two forms of abstraction highlighted in the previous sections.

This article draws on various theoretical frameworks that address local issues in concept construction to raise the discussion beyond simple comparison to move towards identifying deeper underlying themes that enable us to offer new insights into multiple cognitive processes and their interrelationships with strategies of making sense. The diversity of theoretical approaches and traditions can be a rich resource for theoretical and philosophical advancement when these approaches interact. In recent years, the work of researchers within the 'networking theories' group (Bikner-Ahsbahs et al., 2010; Bikner-Ahsbahs & Prediger, 2014) presents significant progress in how to deal with the richness of the diversity of theories. The 'networking' approach provides a systematic way of interacting with diverse theoretical approaches by using different strategies (Bikner-Ahsbahs & Prediger, 2006; Prediger, Bikner-Ahsbahs, & Arzarello, 2008). In this article, the strategies 'comparing' and 'contrasting' as well as 'integrating' and 'synthesizing' are of particular interest. In short, in order to consider the diversity of theories, the strategies comparing and contrasting are meaningful since the former one, comparing, takes account of both similarities and differences in a more neutral way of perceiving theoretical components, whereas the latter, contrasting, stresses only the differences. Especially by contrasting, the strengths and weaknesses of theoretical approaches can be highlighted. Synthesizing and integrating, on the other hand, aim at developing theories by putting together a small number of theoretical approaches into a new framework. These two strategies differ along the "degree of symmetry" (Prediger et al., 2008, p. 173) of the involved theoretical approaches. Synthesizing describes a strategy used "when two (or more) equally stable theories are taken and connected in such a way that a new theory evolves" (Prediger et al., 2008, p. 173). Integrating is used with a lower degree of symmetry of the linked theoretical components, in the sense, that only some components of a theory are integrated into an already more elaborate dominant theory.

With regard to this issue, one of the lessons learned from the recent work of Tall (2013) is that each theoretical framework has value in its own context and that, in addition, theoretical frameworks can be 'blended' to give new insights that were not available in each individual theoretical framework alone. Following Tall (2013), in this article, blending various frameworks is considered as a productive tool to produce emergent insights that may not be evident in the original theories. It is hoped this will shed light on aspects that have been overlooked in past approaches in research on cognitive processes underlying mathematical concept construction compatible with students' strategies of making sense.

The article is considered as an extended effort to advance theory building in research on cognition and learning mathematics which has the potential to overcome the predominance of a single way of constructing mathematical concepts. In doing so, the presented sketch of the framework calls for other researchers to broaden the perspective on the issue by addressing it in a complementary direction, and to extend and advance the explanatory power of the framework. The purpose of this article is not to challenge or explain ideas presented in an original work or compete with recent approaches in mathematics education but to theorize about, and provide deeper meaning to older ideas, and to take them forward in ways not conceived of originally. Thus, this article is intended to provide a further piece of the 'bigger picture' in research on abstraction in learning mathematics.

Notice that the choice of theoretical approaches commented on in this article represents a limited selection of approaches that fall within a cognitive constructivist orientation of learning. The author is aware that there are several versions of constructivism, including radical and social constructivism; the latter having two formulations: Piagetian and Vygotskian (Ernest, 1994). This article uses Piagetian formulation when referring to social constructivism.

2 Abstraction-from-actions and abstraction-from-objects theoretical approaches in mathematics education research

Several approaches, partly distinct and partly overlapping, shape the theoretical landscape in mathematics education research on abstraction. If taken as poles of a wide spectrum, two research strands can be distinguished: (1) an *abstraction-from-actions* strand and (2) an *abstraction-from-objects* strand. Each strand, as argued in this article, has a particular underlying cognitive process. While cognitive processes underlying the abstraction-from-actions strand have been extensively examined in the past two decades, cognitive processes underlying the abstraction-from-objects strand have been nearly overlooked. This article attempts to better theoretically map out the cognitive processes that build the architecture for abstraction from objects. The outline of a theoretical reflection on this kind of abstraction introduced under the same name by Tall (2013) is presented in the second part of this section.

Each strand, as argued in this article, has a particular underlying cognitive process that is also inextricably linked with a specific strategy for making sense of a mathematical concept. This is in line with a natural view of learning emphasizing that individuals' ways of concept construction and their strategies of making sense are inseparable from each other.

The following subsection provides an analysis of cognitive processes underlying the two research strands concerning mathematical concept construction.

2.1 Abstraction-from-actions theoretical approaches

Abstraction-from-actions theoretical approaches, in the literature often labelled as *process-object-encapsulation*, assume that learners first learn processes and procedures for solving problems in a particular domain and later extract domain-specific concepts through reflection on actions on known objects. This development of mental construction has been variously described as (1) interiorization, condensation, and reification (Sfard, 1991). (2) action, process, object, and schema (Dubinsky, 1991). or (3) procedure, process, and procept (Gray & Tall, 1994).

Mathematics education research has shown that there is an inherent process-object duality in the majority of mathematical concepts.

The basic tenet of Sfard's (1991) theory is that mathematical notions can be considered both *structurally* (as objects) and *operationally* (as processes). Sfard (1991) points out that these two approaches "although ostensibly incompatible […], are in fact complementary" (p. 4). Therefore, the process of learning is considered as the interplay between operational and structural conceptions of the same notions; whereas the operational conception emerges first and the structural conception develops afterwards through reification of the process (Sfard, 1991). However, the transition from an operational conception to a structural conception is a time-consuming process, subdivided into three hierarchically arranged phases (see Fig. 1), namely interiorization, condensation, and reification. In the phase of interiorization the learner becomes "skilled at performing processes", in the phase of condensation the learner becomes "more and more capable of thinking about a given process as a whole", without going into detail (Sfard, 1991, pp. 18–19). While interiorization and condensation occur gradually, *reification* requires "an ontological shift—a sudden ability to see something familiar in a totally new light" (Sfard, 1991, p. 19). This step turns out to be particularly complicated since the reification of an object is often associated with the interiorization of a higher-level process.

Dubinsky and his colleagues' (Cottrill et al., 1996; Dubinsky, 1991) approach of *actions becoming mental objects* as part of their APOS theory shows the same characteristics. The fundamental feature of the APOS theory is the assumption that objects are constructed by the *encapsulation* of processes. Encapsulation describes the conversion of a dynamic process into a static object (Dubinsky, 1991), in the sense that actions and processes become objects of thought by repeating them until the construction of structures is completed. As mentioned by Cottrill et al. (1996), encapsulation is started as an individual reflects on the transforming process and is achieved as an individual "becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations" (p. 170). During the encapsulation, a continuing oscillation between process and object conceptions is vital. It is assumed that the encapsulation is a reversible, often to be carried out act (Dubinsky & Harel, 1992, p. 85). The process of encapsulation is similar to reification (see Harel, Selden, & Selden, 2006; Sfard & Linchevski, 1994).

The same holds for Gray and Tall's (1994) progress from procedural thinking to *proceptual thinking*, where proceptual thinking means the ability to manipulate a mathematical symbol as both a process and a concept flexibly. Gray and Tall (1994) termed symbols that may be regarded as being a pivot between a process to compute or manipulate and a concept that may be thought of as a manipulable entity as *procepts*. The progression from doing a procedure to

Fig. 1 Abstraction-from-actions approaches in mathematics education research

	Sfard's theory of reification	**Dubinsky's APOS theory**	**Gray & Tall's theory of procept**
interiorization	process	action	procedure
coordination/ condensation	process	process	process
encapsulation/ reification	object	object/schema	procept

thinking about a procept can be briefly described in three stages (see Fig. 1): Students who know a specific procedure are able to do a specific computation or manipulation. Knowing more alternatives allows an individual to perform mathematics more flexibly because, for example, of the possibility to choose the most suitable route in solving routine problems. The shift from doing a process and performing a procedure to thinking about a symbol allows an individual to think about mathematics in a compressed and manipulable way, moving flexible between a process and a concept (Gray, Pinto, Pitta, & Tall, 1999; Gray & Tall, 1994; Tall et al., 2001).

Although these three theoretical approaches differ in detail, they are similar regarding their core assumptions. Comparing these approaches reveals the underlying cognitive process of concept construction within the abstraction-from-actions strand. Drawing on the work of Pegg and Tall (2005), who have already been thinking about these relationships, the underlying cognitive process of concept construction in the abstraction-from-actions strand is described in the progress from actions on known objects to thinking of those actions as manipulated mental objects. Although various terms (such as encapsulation and reification) have been introduced in research on mathematics concept construction, the cognitive process of forming a (structural) concept from an (operational) process is founded on Piaget's notion of *abstraction réfléchissante* (reflective abstraction).

Notice that contrasting the approaches mentioned above brings to light that both Dubinsky's (1991) and Sfard's (1991) approaches are uni-directional (processes become objects which are used at a 'higher' stage), while Gray and Tall's (1994) approach is bi-directional (moving flexibly between the process and the object); an important note made by one of the three reviewers. Further, in using the term 'encapsulation' Dubinsky and his colleagues as well as Tall and his colleagues explicitly refer to Piaget's reflective abstraction, while Sfard's theory of reification is not explicitly based on Piaget's reflective abstraction.

Encapsulation: The most powerful form of reflective abstraction Although the significance of reflection in thinking and learning processes was well highlighted, for instance, by von Humboldt's (see 1795/1908) work emphasizing that the essence of thinking consists in reflecting, the main expansion of the notion of reflection was done with Piaget's (1977/2001) *Recherches sur l' abstraction réfléchissante* (Studies in Reflective Abstraction). With his framework, labelled as genetic epistemology and understood as an intrinsically developmental theory of human knowledge, Piaget describes fundamentally operative knowledge as basically pragmatic or action-oriented. Operative knowledge, in contrast to figurative knowledge, is important to contribute to human development and consists of cognitive structures. For Piaget (1961/1969), the knowledge we get from perception is figurative, not operative. Thus, perception cannot be the source of any genuinely new construction. Thus, it is *abstraction réfléchissante* (reflective abstraction) that becomes one of the central ideas in Piaget's (1973) reissue of the *Introduction à l'épistémologie génétique*. Piaget describes that reflective abstraction "draws its information from the subject's *actions on objects* […] and particularly from the *coordination between these actions*" (Piaget, 1973, p. 11, italics added). The special function of reflective abstraction is, therefore, abstracting properties of an individual's action coordinations. That is, reflective abstraction is a mechanism for the isolation of particular properties of a mathematical structure that allows the individual to construct new pieces of knowledge. According to Piaget (1977/2001), reflective abstraction is *constructive* in the sense that it is linked to the elaboration of a new action on a higher level than the action from which the characteristic under consideration was constructed. In mathematics education research, the most important and powerful form of reflective abstraction is considered to be the process of encapsulation (or reification):

> "Reflective abstraction includes the act of reflecting on one's cognitive actions and coming to perceive a collection of thoughts as a structured whole. As a result, the subject can now encapsulate the structure, and can see it as an aliment for other structures" (Dubinsky & Lewin, 1986, p. 63).

While this process-object construction has been emphasized for several decades, almost always as an exclusive way of concept construction, other foci have been overlooked. The next subsection therefore provides an alternative account of cognitive processes underlying ways of constructing mathematical concepts. With abstraction-from-objects theoretical approaches an additional strand of concept construction is presented that is based on another kind of abstraction, namely *structural abstraction*.

2.2 Abstraction-from-objects theoretical approaches

Abstraction-from-objects theoretical approaches in mathematics education research assume that learners are first faced with specific objects that fall under a particular concept and acquire the meaningful components of the concept through studying the underlying mathematical structure of the objects. The guiding philosophy of the approach is rooted in the assumption that learners construct mathematical concepts in a domain initially using their backgrounds of existing domain-specific (conceptual) knowledge through progressive integration of previous concept images or by the insertion of a new discourse alongside existing concept images. The author's understanding concerning the abstraction-from-objects strand is rooted in a deeply constructivist view emphasizing that an individual's prior knowledge is the primary resource for acquiring new knowledge. Bruner's (1966) ideas concerning cognitive structures, for instance, are consistent with this view of learning that describes learning as an active process in which individuals construct new mathematical concepts based on their existing knowledge. Once conceptions and concept images are established, they become the vocabulary invoked to give meaning to later experiences. Moreover, individuals already have, in some contexts, substantial parts of the new conceptual structures in mind. Accordingly, they do not passively respond to new information but actively select parts of their concept images that are productive in a particular context. Consequently, individuals interpret (in the sense of Piaget's notion of assimilation) new concepts in terms of their prior knowledge. This article assumes that the underlying cognitive processes of concept construction in the abstraction-from-objects strand are founded in studying the underlying structure of a mathematical concept through a specific kind of abstraction, called *structural abstraction*, a notion that has already been used by Tall (2013) as a kind of abstraction focusing on the properties of objects. This kind of abstraction, as shown by Tall (2013), plays "a fundamental role at successive stages of increasing sophistication […] throughout the full development of mathematical thinking" (p. 39).

The study of the underlying structure has a long history in philosophy, psychology, and mathematics education. Skemp (1986), for instance, states the importance of the study of structures as followed:

> "The study of the structures themselves is an important part of mathematics, and the study of the ways in which they are built up and function is at the very core of the psychology of learning mathematics." (Skemp, 1986, p. 37)

If this is a correct reading of his work, it seems that Skemp's (1986) conception of abstraction as an "activity by which we become aware of similarities […] among our experiences" (p. 21) followed by embodiment of the similarities in a new mental entity is rooted in the idea of recognizing the similarity of the underlying (rather than superficial) structure. This view differs from Piaget's notion of empirical abstraction. In Piaget (1977/2001), the term empirical abstraction is described as the kind of abstraction that "ranges over physical objects or the material aspects of one's own activities" (p. 29) and "is limited to recording the most obvious and global perceptual characteristics of objects" (p. 319). Roughly speaking, empirical abstraction is abstraction of dimensions that individuals can perceive in the environment from experiential situations. As already noted by Mitchelmore and White (2007), Skemp's conception goes beyond Piaget's notion of empirical abstraction by providing a foundation of what Mitchelmore and White elaborated as empirical abstraction in learning elementary mathematical concepts. However, although Skemp's idea of seeing the underlying structure is considered as a building block for structural abstraction in this article, structural abstraction differs from Skemp's idea in two fundamental ways: (1) structural abstraction takes mental (rather than physical) objects as a point of departure and (2) the core of this kind of abstraction is complementarity (rather than similarity).

Notice that abstraction-from-objects theoretical approaches do not use an 'objectivist view' of knowledge, or one that views knowledge as grounded in objective reality. Instead, the idea of mathematical objectivity as a social construct is adopted.

Epistemological function of structural abstraction The underlying assumption of structural abstraction lies in the essence of Frege's (1892a) *Über Begriff und Gegenstand* emphasizing that the meaning of a mathematical concept (that differs both from the content of a concept and from the abstract general notion) is not directly accessible through the concept itself but through objects that (in Fregeian terminology) fall under that concept. In order to get access to the meaning of a concept, the concept "must first be converted into an object, or, more precisely, an object must go proxy for it […]" (Frege, 1892a, p. 197). These objects, which build the initial point for the cognitive processes underlying the abstraction-from-objects strand, may be either concrete or abstract. In this work, concreteness and abstractness are not considered as properties of an object in the classical sense, but rather as a property of an individual's *relatedness* to an object in the sense of the richness of an individual's representations, interactions, and connections with the object (Wilensky, 1991). A concrete object, then, is an object for which an individual has established rich representations and several ways of interacting with, as well as connections between it and other objects. This view differs from a classical perspective considering concrete objects as those objects that are mediated by the senses.

It is assumed that the essence of a concept is almost always contained in the unity of diverse meaningful components of a variety of specific objects that fall under the particular concept. However, structural abstraction requires the particularization of meaningful components as well as the underlying mathematical structure. The crucial aspect of this initial process is *contextualizing* that is, placing abstract objects in different specific contexts. The process of placing objects into different specific contexts may have to be guided by using a realistic model or by taking a specific perspective. The adjective 'realistic', relating to the Dutch *Realistic Mathematics Education* approach (rooted in Freudenthal and his colleagues' work), refers to the intention of making a mathematical problem imaginable for students. A realistic model, in its broad sense, can be, for instance, a metaphor or generic representation. Within this approach, a model is a tool for theoretically structuring the construction of mathematical concepts. It necessarily reflects the essential aspects of a mathematical concept but can have different manifestations (Van den Heuvel-Panhuizen, 2003). The crucial function of a model is considered to be bridging the gap between 'the abstract' and 'the concrete'. This means that in the beginning of a particular learning process a model is constituted that supports *ascending from the abstract to the concrete* as described by Davydov (1972/1990). Davydov's strategy of ascending from the abstract to the concrete describes the transition from the general to the particular as one where learners initially seek out the primary general 'kernel' and then deduce multiple particular features of the object using that 'kernel' as their mainstay. Taking Davydov's theory as a point of view, this implies the ascending from the abstract to the concrete in terms of this model. Similar to Davydov, Ilyenkov (1982) considers the course of ascent from the abstract to the concrete as basically related to looking at a concrete situation from a particular theoretical point of view. That means that "the concrete is realized in thinking through the abstract" (Ilyenkov, 1982, p. 37). In further learning processes both the context and the perspective may be shifted in the sense of looking at an object placed in different specific contexts from a particular point of view or looking at an object placed in a particular context from different specific points of views. This understanding is in line with van Oers (1998) perspective of abstraction as a process of *contextualization* arguing that abstraction is related to *re*contextualization instead of *de*contextualization. Thus, the abstract and the concrete sub-serve one another in thought through a dialectical interplay. This dialectic between the ascending from the abstract to the concrete and the ascending from the concrete to the abstract reflects Marx's original discussion of the abstract and the concrete in *Capital*. Drawing from Marx's work, the dialectic of the abstract and the concrete in thought and in theoretical processing is Ilyenkov's (1982) primary concern:

> "[…] the ascent from the concrete to the abstract and the ascent from the abstract to the concrete, are two mutually assuming forms of theoretical assimilation of the world, of abstract thinking. Each of them is realized only through its opposite and in unity with it." (Ilyenkov, 1982, p. 137).

According to this dialectical view, structural abstraction means (mentally) structuring the diverse aspects and the underlying structure of specific objects that have been particularized by placing the objects in a variety of different contexts. Whereas within the traditional (or empiricist) view conceptual unity relies on the commonality of elements, it is the interrelatedness of diverse elements that creates unity within the approach of structural abstraction. Thus, the essence of structural abstraction is *complementarity* rather than similarity. The overall framework is in line with Davydov's (1972/1990) description that the internal, essential relationships are detected "in mediations, *in a system*, within a whole, in its emergence" (p. 119, italics in original). This view has the advantage of escaping the weakness of concrete ideas in terms of their difficulty in combining or composing each other (since the specific features of their components may conflict with each other). In the case of constructing mathematical concepts through structural abstraction the abstract has primacy over the concrete.

The crucial puzzle lies in the observation that structural abstraction has a dual nature, namely (1) 'complementarizing' the aspects and structure underlying specific objects falling under a particular mathematical concept, and (2) facilitating the growth of coherent and complex knowledge structures through restructuring the constructed 'pieces of knowledge'.

From this point of view, structural abstraction takes place both on the objects-structure and on the knowledge-structure (see Fig. 2).

The former function, as it is described above, requires a concretizing process where the mathematical structure of an object is particularized by looking at the object in relation to itself or to other objects that fall under the particular concept. Through placing the object into different specific contexts with the 'structural advice' of a particular model or perspective as a framing instrument, the meaningful components of the object may be further highlighted. It must be emphasized that in psychology, for instance, these cognitive processes are regarded as particular kinds of activity. However, in the author's opinion, they are components of a general activity 'architecture' (rather than different forms of activity) that promote the realization of structural abstraction.

On the other hand, structural abstraction implies a process of restructuring 'pieces of knowledge' constructed through the mentioned process. Further, it also implies restructuring knowledge structures coming from already formed concept images that are essential for the construction of the new concept. The cognitive function of structural abstraction is to facilitate the assembly of more complex knowledge structures. It aims to establish highly coherent knowledge structures or – to put it in the words of Viholainen (2008) – to form concept images of a high level of coherence. The crucial aspect of structural abstraction, from the knowledge-structures perspective, is that structural abstraction moves from simple to complex knowledge structures (see Fig. 2).

Notice that, as described by Tall, Thomas, Davis, Gray, and Simpson (1999). the term 'structural' has multiple meanings in the literature: For instance, Sfard's notion of 'structural' can be subdivided into (a) whether the focus is on properties of observed or conceived objects and (b) whether some of these properties are specified as set-theoretic axioms and definitions to give a formal theory that is 'structural' in the sense of Bourbaki. The term 'structural' employed in this article, however, refers to both the structure of mental objects and the structure of knowledge. These two interrelated ideas are implicit in the notion of 'structural': Structures underlying the specific objects falling under a particular mathematical concept are represented as mental structures when placed into several contexts and situations, and those

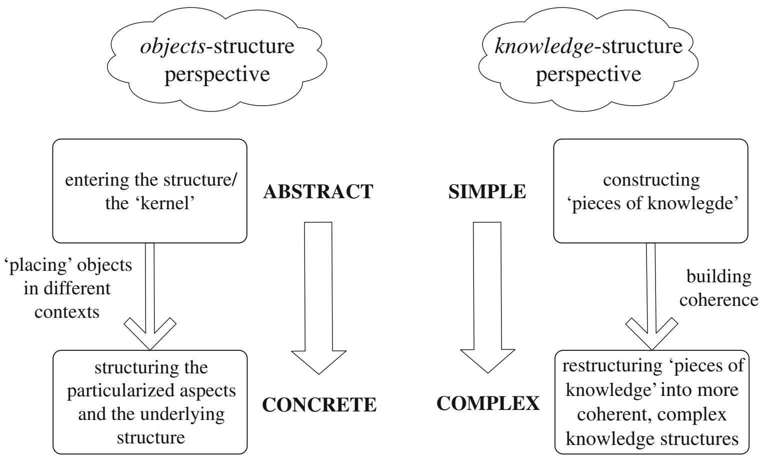

Fig. 2 The dual nature of structural abstraction

mental structures are further restructured (through complementarizing of meaningful components) with the aim to construct highly complex and coherent knowledge structures.

Further, notice that the former (structure of objects) relates to Sfard's (1991) understanding of structures, including, but not limited to, formal 'structures' of mathematical axiomatic systems. In addition to a formalist interpretation (in terms of set-theoretic definition and deduction), the notion of 'structural' includes the view that the richness of the particular is embodied not in the concept as such but rather in the objects falling under the concept—the particular is reflected in the context where the objects are placed in. This view gives primacy to meaningful, richly contextualized forms of (mathematical) structures over formal (mathematical) structures.

3 Shifting the perspective towards strategies of making sense

Learning mathematics is more than simply adding meaningful elements together in a relational way. The learner must make sense of the mathematical concept through restructuring knowledge structures that are built on previously constructed knowledge pieces. Moreover, as indicated by von Glasersfeld (1987), "to make sense of a given collection of experiences, […] means to have organized them in a way that permits us to make more or less reliable predictions" (p. 9).

Mathematics education research has identified various strategies used by students to make sense when they are involved in a mathematical activity. For the purposes of this work, the article builds upon the research by Pinto (1998), and Pinto and Tall (1999, 2002). In a study of novice mathematicians working through an introductory analysis course, Pinto (1998) identified two different strategies used by learners in their attempts to build up concepts given by formal definitions: (1) *extracting meaning* from a definition and (2) *giving meaning* to a definition.

> "*Extracting meaning* involves working within the content, routinizing it, using it, and building its meaning as a formal construct. *Giving meaning* means taking one's personal concept imagery as a starting point to build new knowledge." (Pinto, 1998, pp. 298–299)

Thus, the former strategy, extracting meaning, builds from the formal definitions, while the latter strategy, giving meaning, builds upon earlier experiences. In her work, Pinto (1998) highlighted two different types of learners based on work by Duffin and Simpson (1993), namely *natural* and *formal* learners, distinguished by whether they prefer extracting or giving meaning as the core strategy to make sense. There is a spectrum of performance in each of these types of learners; indicating that neither of these routes of learning necessarily lead to success or failure (Pinto, 1998).

Reflective and structural learners Pinto's (1998) work pays attention to extracting meaning from and giving meaning to the formal definition. However, the perspective on the two strategies identified by Pinto (1998) can be broadened in order to extend the focus to extracting meaning from and giving meaning to objects that fall under the mathematical concept in consideration. The notion of 'object' is used in a broad sense including, inter alia, the formal definition of the mathematical concept.

Based on research by Pinto (1998) and Pinto and Tall (1999, 2002), Scheiner (2013) further elaborates the distinction of the two strategies of making sense, claiming that according to the abstraction-from-actions approach *learners extract meaning of objects through reflective abstraction*. For instance, learners may focus on actions on known objects, where they begin with the manipulation of meaningful components. Learners preferring this approach are termed *reflective learners* (Scheiner, 2013). On the other hand, learners preferring the cognitive processes underlying the abstraction-from-objects approach *give meaning to a mathematical concept through structural abstraction*. Those learners build on a variety of previous knowledge structures and attribute meaning to the object of their thinking. This is consistent with Duffin and Simpson's (1993) description of a natural learner that has been adapted by Pinto (1998):

> "A *natural learner* always attempts to make sense of experiences by connecting them immediately to existing mental structures, looking for explanations and reasons based on those connections." (Simpson, 1995, p. 42, italics added)

Based on this description, Pinto (1998), later Pinto and Tall (1999), identified that natural thinkers reconstruct new knowledge from their concept images. They use their own imagery as a starting point to build on or modify through thought experiment. In Scheiner (2013), learners following this approach are termed *structural learners* (see Fig. 3).

Tall et al. (2001) presume that giving meaning and extracting meaning may be best used in sequence. They describe this in the following way:

> "First one gives meaning, by constructing examples and non-examples and building a range of possibilities that might be deduced from the definitions. Then one moves to the logical extraction of the hypothesized results by formulating them as theorems and proving them." (Tall et al., 2001, p. 100)

This is consistent with past research focusing on the dialectic between the development of procedural and conceptual knowledge. Past research showed that, on the one hand, learners have partial knowledge of both procedures and concepts and that, on the other hand, more knowledge of one type is related to more knowledge of the other (see Baroody & Gannon, 1984). Furthermore, Rittle-Johnson and Alibali (1999) state that improving knowledge of one type can lead to improvements of the other type.

In addition to the two previous paths of concept construction and types of learners, it is proposed that both paths may be interwoven and, as a consequence, that there is a third type of learner, namely *reflectural learners* (a blend of *reflec*tive and struc*tural*). Throughout concept construction, reflective abstraction and structural abstraction may interact with each other. In

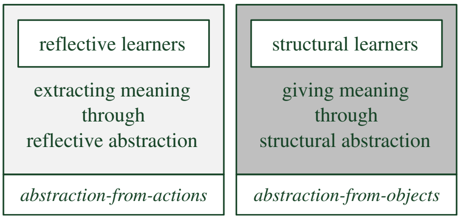

Fig. 3 Reflective and structural learners

this case, *reflectural learners* develop further knowledge in a hand-over-hand process of reflective and structural abstraction (see Fig. 4).

Based on these assumptions regarding the interrelationship between the two strategies of making sense, the following section goes beyond simply viewing abstraction-from-actions and abstraction-from-objects approaches as being in opposition to each other. Indeed, a way of viewing how reflective abstraction and structural abstraction may interact with each other is offered.

4 Reflectural abstraction: The dialectic between reflective and structural abstraction

In Section 2, two basic cognitive processes underlying mathematical concept construction have been polarized: (1) the process-object-encapsulation and (2) 'complementarizing' particularized meaningful components of the mathematical objects with simultaneous restructuring knowledge pieces intended to construct a highly complex knowledge system. The considerations in Section 3 open new avenues for theoretical reflection on the interaction between the cognitive processes underlying mathematical concept construction with students' strategies of making sense. In this section, a first attempt is made to blend the two above-mentioned frameworks with the intention of providing a potentially useful way to bring to light the interrelation between reflective and structural abstraction.

In his influential work *Über Sinn und Bedeutung*, the German philosopher and mathematician Gottlob Frege (1892b) introduced the distinction between *sense* and *reference* as two semantic functions of an expression (a name, sign, or description). In general, the former is the way that an expression refers to an object, whereas the latter is the object to which the expression refers. Duval (1995), for instance, applied Frege's (1892b) distinction to refer to the relation between a representation and its sense and a representation and its reference. Adapting the Fregeian terminology, a representation expresses its sense and stands for or designates its reference. Thereby, the sense depends on the selected representation system and the reference only on the represented object. As indicated in Section 2.2., in order to particularize the meaningful components or mathematical structure of an object, the object has to be placed into different specific contexts. The object may be 'exemplified' through a variety of representations, in which each representation has the same reference (the mathematical object) (see Fig. 5). This perspective, in line with the typology of primary mathematical objects described, for instance, by Font, Godino, and Gallardo (2013), allows the exemplified object to be a linguistic element (e.g., term, expression, notation), a definition (e.g., formal concept definition), or a proposition (e.g., statement), amongst others.

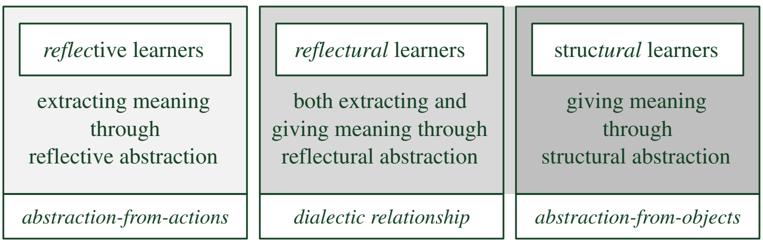

Fig. 4 Reflective, structural, and reflectural learners

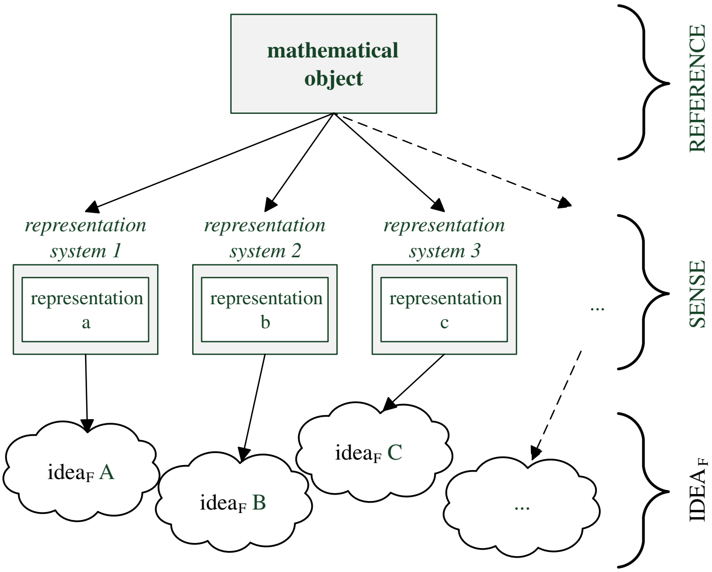

Fig. 5 Reference, sense, and idea$_F$

In accordance with Frege (1892b) and adapted by Duval (2006), each representation corresponds to a sense. As shown in Fig. 5, the corresponding sense may be connected with an *idea$_F$* (the subscript F indicates that the term 'idea' refers to Frege) that may differ amongst individuals since individuals may associate different senses with a given representation (see Frege, 1892b).

Connecting the idea$_F$ with a sense occurs when individuals give meaning to the corresponding sense by taking their previously constructed pieces of knowledge as a point of departure. At this level, an individual may create single or several competing mental representations of the same mathematical object. When various representations are being considered in parallel and when links between various representations are being established, then an individual may become aware of the underlying structure of a mathematical concept. However, in order to particularize the meaningful components or the mathematical structure underlying the representations that stand for a specific object, *systematic variation* within a representation system may have to take place; its importance has been indicated by Duval (2006) as followed.

> "It is only by investigating representation variations in the source register and representation variations in a target register, that students can at the same time realize what is mathematically relevant in a representation, achieve its conversion in another register, and dissociate the represented object from the content of these representations" (Duval, 2006, p. 125).

The systematic variation of a representation within a representation system leads to a variation of a representation within another representation system. Reflections of the variations of relevant variables within a representation system enable the extraction of the underlying structures and other relevant variables within the other representation system. Reflective abstraction, in this context, concretizes an individual's ideas$_F$ by extracting the underlying structures in the variations of representations of a specific object. At the heart of this level, reflective abstraction is considered as a mechanism for extracting particular meaningful components of a mathematical structure from the variations of various representations and allowing the individual to construct new pieces of knowledge. The extracted meaningful components become the subject of an individual's considerations that may lead to a reflective discourse across the ideas$_F$ representing the specific representations. However, the 'organization' of extracted structures into structured wholes does not take place by itself (as assumed by Piaget). Rather structural abstraction takes place in order to embed the extracted structures that subsequently become internalized and become the vocabulary invoked to give meaning to the structure as a whole. That is, this level is associated with the goal of 'packaging' ideas$_F$ into conceptions. Thus, an individual gives meaning to the extracted pieces of knowledge through structural abstraction. These conceptions, however, get connected by the learner through increasingly sophisticated structural abstraction to create more coherent and complex structures. From this point of view, conceptions can be considered as compressions of ideas$_F$. A conception is formed not just by one particular object but by a variety of objects falling under the same mathematical concept (see Fig. 6). Thus, a conception has a more complex character than an idea$_F$. In summary it can be stated that the individual may focus at first on a particular aspect, but then sees other meaningful aspects and links them to build not just various ideas$_F$, but also richer compressed conceptions that can operate as single entities in further learning processes (see Tall, 2013). This progress can be considered as evolving from simple to complex knowledge pieces; starting with the construction of uni-structural knowledge pieces that become multi-structural and relational through restructuring, and in further attempts highly coherent, compressed knowledge structures conceived as a single sophisticated entity that may in turn be an object of consideration in further learning processes. Such a development features as a local cycle of learning in the wide range of the above-mentioned theoretical frameworks.

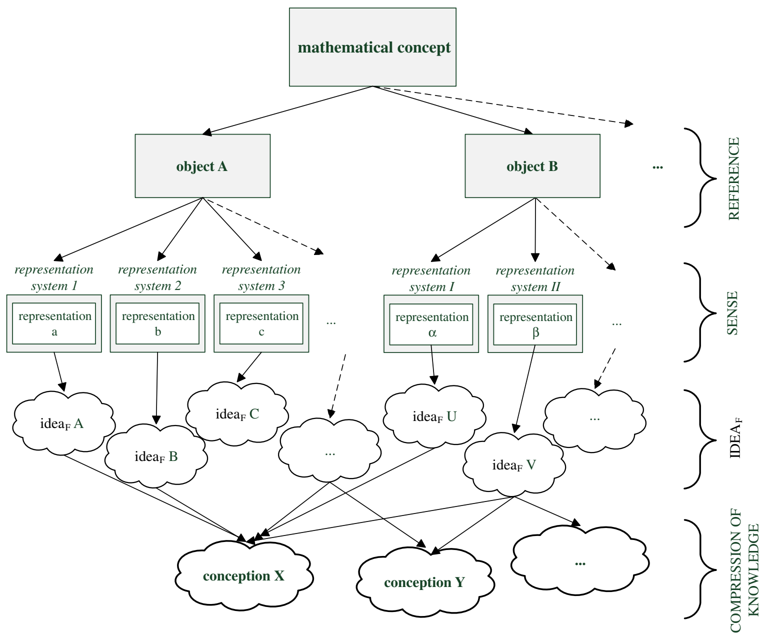

Fig. 6 The progress from simple to complex

Although the interaction of the two abstraction processes involves a developmental progression, this does not imply that they involve a strict order or hierarchy. Rather, it is to be understood that the highest impact can be considered in the dialectic interaction between reflective and structural abstraction. This hand-in-hand process is more of an inextricably combined activity of reflective and structural abstraction, called *reflectural* abstraction (a blend of *reflec*tive and struc*tural*).

5 Conclusion

This article provides a theoretical frame of various knowing and learning processes in mathematics that addresses several issues. Two research strands that are poles of a wide spectrum (the abstraction-from-actions and abstraction-from-objects approaches) are the objects of observation. In this article, it is argued that each research strand has an underlying cognitive process (reflective and structural abstraction); each of which is inextricably linked with a particular strategy of making sense (extracting meaning and giving meaning). The intention is to illuminate those aspects that have been overlooked in past approaches in research on cognitive processes underlying mathematical concept construction compatible with students' strategies of making sense. Of particular interest are the cognitive processes that build the architecture of structural abstraction since it is assumed that the cognitive processes underlying the abstraction-from-objects strand have nearly been overlooked in the past. With the outline of the theoretical framework on structural abstraction, it is hoped that abstraction is acknowledged as a movement across levels of complementarity and complexity (rather than levels of abstractedness). The author therefore asks that the term abstraction be freed from connotations that have been associated with it during several decades in numerous works.

In an attempt to move beyond various dichotomies in the psychology of learning mathematics, the last section provides the fundamentals of a framework incorporating the cognitive processes underlying the abstraction-from-actions and abstraction-from-objects approaches. It is argued that these frameworks (although different) are complementary (rather than opposing) frameworks.

With this theoretical article, the author intends to make a further contribution to more cutting-edge theoretical work in research on cognition and learning mathematics. The purpose of this article, however, is to provide a theoretical framework, not a theory. A theoretical framework is "a general pool of constructs for understanding a domain, but it is not tightly enough organized to constitute a predictive theory" (Anderson, 1983, p. 12). Thus, it is hoped that the above-mentioned outline is a building block for generating a theory that can be used to make unique empirical predictions that provide insights overlooked by other frameworks. The approach developed offers a plausible broadening of past and current perspectives on cognitive processes underlying ways of concept construction and their interrelationships with strategies of making sense. In addition, the article offers an initial departure point to rethink discussion and efforts in mathematics education concerning concept formation at both elementary and advanced mathematical levels. Instead of focusing on the ways of mathematical instruction (from operational growth to structural growth, or vice versa), we should shift our attention to the most natural ways of concept construction that, as considered here, are dependant on the individual rather than on the concept itself (or the instruction).

Theoretical frameworks such as this provide fertile ground for investigations. The article illustrated the complexity of the cognitive processes underlying concept construction and the necessity for clearly identifying and studying their interaction with sense-making strategies. Thus, this theoretical framework can be viewed as a source of ideas that scholars can appropriate and modify for their research purposes.

The question of paramount importance—yet to be answered—is whether this description is operational. That is, whether it is possible to specify the nature of the interrelationship between reflective and structural abstraction and the interconnections with the two strategies of making sense. Research attempts to address these questions are needed.

It is hoped that the validity of the framework will be confirmed with systematic investigations, both empirical and theoretical, whereby its status may change to a model or theory with explanatory power.

The author wants to note that a theoretically and philosophically based description of cognitive processes underlying mathematical concept construction and their interrelationships with strategies of making sense calls for reducing the number of objects of consideration in order to identify the most significant ones. This conveys the tension between complexity and simplicity. One can argue that additional aspects must be considered when theorizing and philosophizing about how students construct mathematical concepts while engaging in the act of making sense. The author is aware that only a glimpse of this fascinating issue has been provided in this article. The potential power and soundness of the mentioned theoretical reflection, together with its implications at the educational level, are to be investigated in future research.

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4 Challenging Conceptualizations of Teacher Knowledge: Toward Emerging Theoretical Perspectives*

*This chapter refers to the second journal article entitled "What makes mathematics teacher knowledge specialized? Offering alternative views" by T. Scheiner, M. A. Montes, J. D. Godino, J. Carrillo, & L. R. Pino-Fan, published (online first) in *International Journal of Science and Mathematics Education* (doi: 10.1007/s10763-017-9859-6).

This chapter focuses on the discussion of teacher professional knowledge, and the debate as to what makes mathematics teacher knowledge specialized in particular. The second article (Scheiner, Montes, Godino, Carillo, & Pino-Fan, 2017) critiques what the literature implies makes mathematics teacher knowledge specialized. In order to do so, the paper identifies the principal viewpoints that currently exist in the literature and points to the more severe limitations of their underlying assumptions. Specifically, the article takes a critical stance toward three general orientations that seem to be implicit in the present discussion on teacher knowledge:

(1) the field makes external comparisons (mathematics teachers vs. mathematicians; teaching mathematics vs. teaching other subjects) when discussing what makes mathematics teacher knowledge specialized;
(2) the field takes a disciplinary, reductionist perspective when considering teacher knowledge, arguing from the viewpoint of teaching mathematics; and
(3) the field has accumulated additional dimensions of teacher knowledge.

It should be emphasized that this article does not advocate a dismissal of empirical studies and approaches in measuring teacher knowledge. Such studies and approaches are relevant particularly when teacher knowledge is assessed on a large scale. The contribution made in Scheiner et al. (2017) is not a substitution for those empirical studies, but rather provides an alternative for exploring some of the complexity of teacher knowledge from disparate theoretical perspectives. Hence, alternative views to these three major orientations in the field of mathematics teacher knowledge are provided to focus on theoretical issues that have not been fully explored. The first such view argues that specialization is a process of becoming rather than a state of being, and argues that specialization should be accounted for via internal (or within-field) comparisons rather than external comparisons. The second view argues that reductionist conceptualizations of knowing and learning are inadequate, and calls for the recognition of the epistemological position inherent in mathematics teacher knowledge, one that entails the use by teachers of the historical and cognitive geneses of mathematical insights to unpack students' mathematical understandings. The third view argues for a holistic consideration of specialization, where specialization is considered to be the interaction of various pieces of knowledge that interact dynamically to form emergent structures. In summary, Scheiner et al. (2017) argue for an approach which is:

(1) intrinsic rather than extrinsic, dispensing with external reference points and accounting for specialization as a process of becoming rather than a state of being;
(2) anthropological-sociocultural rather than reductionist, eschewing reductionist approaches and instead underlining the epistemological thread inherent in mathematics teacher knowledge; and
(3) transformative rather than additive, construing teacher knowledge as complex interactions of knowledge facets within a dynamic structure rather than as an incremental accumulation of knowledge facets.

Collectively, these alternative views suggest that specialization in mathematics teacher knowledge requires an account of 'how' teachers' knowing comes into being, rather than an account of 'what' teachers know. It is concluded that it is not a kind of knowledge but a style of knowing that signifies specialization in mathematics teacher knowledge.

Here attention is focused on one of the three tendencies as outlined above in thinking about teacher knowledge: the reductionist orientation inherent in conceptualizations of teacher knowledge. This chapter intends to deepen the discussion given in Scheiner et al. (2017) in making explicit and visible what often remain implicit and invisible: the underlying assumptions of the taken-for-granted conceptualization of pedagogical content knowledge (PCK). The focus is directed to the theoretical construct of PCK as introduced by Shulman (1986, 1987), an accepted "academic construct" (Berry, Loughran, & van Driel, 2008, p. 1272) that has become "a powerful lexical item in the educational community" (Deng, 2007, p. 279). PCK has not only become mainstream but has become the conventional wisdom in thinking about "the category [of teacher knowledge] most likely to distinguish the understanding of the content specialist from that of the pedagogue" (Shulman, 1987, p. 8).

This chapter aims to articulate and critically reflect on the theoretical underpinnings of PCK, underpinnings that often go unrecognized and hence remain beyond the scrutiny of critical reflection. The chapter is structured in three parts. First, a relatively deep account is given of the guiding principles that shaped, and still shape, our thinking about PCK. To this end, the key to the notion of PCK is articulated, that is, the transformation of the subject matter in a way that is 'teachable', by identifying its underlying assumptions with some precision. Second, a critical stance is taken toward some of the assumptions underlying PCK. Third, potential contradictions with more recent understandings of students' knowing and learning are outlined, followed by a sketch of potential resolutions.

4.1 Making Visible the Invisible: Key Assumptions Underlying PCK

Shulman's (1986, 1987) most critical contribution in his research on teaching and teacher knowledge was his directing of attention to an issue absent from most studies within process-product approaches in research on teaching: the reference to subject matter. Shulman called this problem 'the missing paradigm' and argued that subject matter was a central and pivotal feature that needed to be included in any research program on teaching.

In this context, Shulman and his colleagues (Grossman, Wilson, & Shulman, 1989; Shulman, 1986, 1987; Wilson, Shulman, & Richert, 1987) foregrounded subject matter knowledge *for* teaching as a category of teacher knowledge, that "embodies the aspects of content most germane to its teachability" (Shulman, 1986, p. 9), as distinct from subject matter knowledge per se.

Shulman (1986, 1987) called this category of teacher knowledge pedagogical content knowledge (PCK), including "the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that makes it comprehensible for others" (Shulman, 1986, p. 9).

The conceptualization of PCK was developed in the context of, and informed by, Shulman and his colleagues' research program *Knowledge Growth in Teaching,* which attempted to articulate the interrelatedness of pedagogy and subject matter. The primary focus of this research program was on how novice secondary school teachers adapt their prior knowledge of the content of an academic discipline so that it becomes suitable for classroom teaching. They conceptualized this teaching task as a *transformation* of the content of an academic discipline to the content of a school subject – the latter considered as the kind of content appropriate for teaching in classrooms. Shulman (1987) argued that subject matter knowledge per se "must be transformed in some manner if they are to be taught. To reason one's way through an act of teaching is to think one's way from the subject matter as understood by the teacher into the minds and motivations of learners" (p. 16). In more general terms, the central intellectual task of teaching is considered to be transforming subject matter knowledge into a form in which it is teachable to particular learners (Geddis, 1993).

Shulman (1987) stated that

> "the key to distinguishing the knowledge base of teaching lies at the intersection of content and pedagogy, in the *capacity of a teacher to transform the content knowledge* he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students." (p. 15, italics added)

The recommended strategy is greatly, if not entirely, determined by the content of the discipline, as this forms the primary source of information for teaching and informs decisions about instruction.[1]

The transformation seems to concentrate on the structure and representation of disciplinary subject matter – in a word: the transformation takes place on the *structure of a discipline*. Gudmundsdottir (1991) described this transformation process as a "reorganization [of content knowledge] that derives from a disciplinary orientation" (p. 412), and Gudmundsdottir and Shulman (1987) designated it as a *re-definition* of subject matter knowledge to construct PCK. Grossman et al. (1989) described it as "translat[ing] [the] knowledge of subject matter into instructional representations" (p. 32). Marks (1990), on the other hand, portrayed it as a process of *interpretation* that means, "the content is examined for its structure and significance, then transformed as necessary to make it comprehensible and compelling to a particular group of learners" (p. 7). Although different terms for describing the transformation process were used, they are similar regarding their core assumption: the transformation is grounded in, and determined by, the structure of a discipline.

Although different scholars use various terms to describe the transformation process, they share the same understanding of the function of transformation, that is, to *make subject matter accessible* to the students. The primary goal of transformation is to re-structure the content of an (academic) discipline "into a form of knowledge that is appropriate for students and specific to the task of teaching" (Grossman et al., 1989, p. 32). Geddis and Wood (1997) stated that "[t]he end products of pedagogical transformations are the representations of subject matter and instructional strategies that enact specific instructional encounters" (p. 612).

4.2 Taking a Critical Stance Toward Assumptions Underlying PCK

That Shulman used the structure of a discipline as the foundation for his conceptualization of PCK is not surprising, given that in much research on learning and teaching subject matter disciplines were used as the organizing frame for investigation and implementation (see Steffe & Kieren, 1994). However, such an approach is reductionist in orientation as, in such an approach, subject matter is considered as a sort of package, where the quality of transferring subject matter into the minds of students depends on the quality of the vehicles of transformation. The mathematics education literature identifies various such discipline-specific practices, including, but not limited to, elementarizing, exemplification, explanation, decompression, and simplification, that require the capacity "to deconstruct one's own mathematical knowledge into a less polished and final form, where elemental components are accessible and visible" (Ball & Bass, 2000, p. 98). For instance, Ball (2000) highlighted the need for teachers to examine particular tasks to determine their utility for students. Ball (2000) then discussed the need for teachers to alter the task to make it easier or simpler, or to make it illuminate particular key components of a concept.[2]

This view points to a critical understanding: subject matter is mapped as an already existing object that is to be transferred from the mind of the teacher to the minds of students. In consequence, Shulman's idea of transforming subject matter seem to be narrowly and implicitly embedded in a transmission view of teaching, as already noted by McEwan and Bull (1991). Similarly, Meredith (1995) stated that "pedagogical reasoning, which leads to the transformation of subject knowledge, seems to be concerned primarily with the transmission of content" (p. 177).

In summary, taking the structure of a discipline as the determinant of the transformation process has contributed to an image of teaching that is deceptively simple and instrumental, an image driven by assumptions that are reductionist in orientation. One might take the didactic triangle as a lens to clarify the idea of transforming subject matter and analyze its relationship to three important elements of the teaching-learning process: subject matter, teacher, and students. It is argued that Shulman examined teacher practice merely from the perspective of teaching and focused his attention on the teacher-subject matter edge of the didactic triangle (see Figure 9).

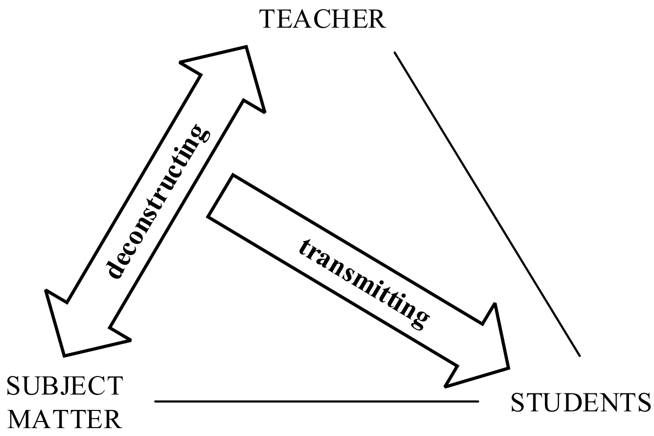

Figure 9: A transformation view: a top-down approach

4.3 Troubling Assertions and Potential Resolutions

The limitations of the underlying assumptions of PCK call attention to the problematic boundaries of our historical ways of thinking in conceptualizing teacher knowledge. One set of concerns relates to the issue that, in taking the structure of a discipline as the determinant of the transformation process, Shulman seems to have been, perhaps unintentionally, trapped in Cartesian epistemology, where our representations must conform to an object independent of the mind to constitute knowledge. Another set of concerns, and linked to the previous one, relates to the issue that Shulman's assertions ultimately advocate a position in which subject matter can be transferred to students by transforming subject matter in ways accessible to students. However, such views are troublesome in light of recent understandings of student knowing and sense-making, which portray a dynamic and complex view that contrasts with the linear, simplistic view of Shulman's model (see Chapter 3). It seems common sense, though not common practice, to suggest that the world does not harbor unambiguous 'truths' independent of the human mind, revealed to us through instruction; rather, the 'real' are (multiple) mental constructions, and 'truth' is a consensus construction arising in social interaction by negotiating personally constructed (subjective) realities into a socially shared (intersubjective) reality.

For instance, constructivism (both cognitive and social constructivism) has challenged 'transmission' views of teaching and 'absorptionist' views of learning; particularly the premise that subject matter is an object of teaching that can be transferred more or less directly from one party to another.[3] Cobb et al. (1991), for instance, reminded us that "from a constructivist perspective, […] learning is not a process of internalizing carefully packaged knowledge but is instead a matter of reorganizing activity, where activity is interpreted broadly to include conceptual activity or thought" (p. 5). Thus, it is not only the case that teachers cannot and do not have knowledge of subject matter 'in a form' for consumption by the students, but that knowledge has to be constructed by the learners themselves in order to be meaningful. This metaphor of 'knowledge construction' conveys the understanding that knowledge does not lie beyond the realm of human beings, but rather is something made by human beings, advocating the perspective that subject matter is an object of learning (rather than an object of teaching).

Though constructivism is a theory of knowing and learning, rather than a theory of teaching, constructivist assumptions about students' learning suggest a set of instructional commitments for teachers that differ from traditional discipline-centered approaches. If one subscribes to a constructivist view of learning, then instruction cannot be seen as translating the subject matter of the discipline downwards (à la Shulman) but as a process of co-construction upwards. From a constructivist view, the teacher has no longer only to develop pedagogical strategies to unpack the subject matter content and enable students to know objects and products of cultural development, but also has to attend to students' multiple, individual, subjective realities, which may differ from what has been socially constructed (Confrey, 1990). It is this call, for teachers to attend to students' mental structures by building models of students' thinking, that shape the constructivist view of teaching: "in the constructivist view, teachers should continually make a conscious attempt to 'see' both their own and the children's actions from the children's point of view" (Cobb & Steffe, 1983, p. 85).

The implication of this is to revise traditional views on learning and teaching. Rather than separating the student and the subject matter, the students' relation to the subject matter becomes the key to understanding the instructional process (see Figure 10). Subject matter is constructed individually and socially mediated, rather than passively received from authority (Driver, Asoko, Leach, Mortimer, & Scott, 1994; Tobin & Tippins, 1993); in the teaching-learning process multiple (subjective) realities then can become a temporarily, socially shared (intersubjective) reality.

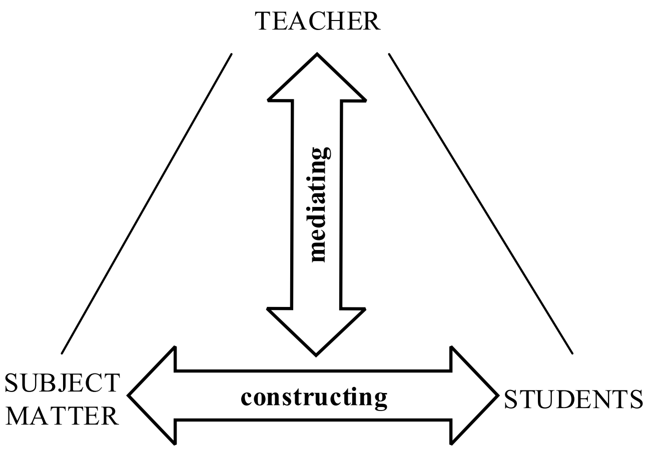

Figure 10: A constructivist view: a bottom-up approach

4.4 Reflections

Fundamental assumptions underlying PCK are in contention with more recent understandings of student knowing and learning concerning the critical issues of subject matter (object of teaching vs. object of learning) and the teaching-learning process (transmissive vs. constructive). This disagreement is grounded in the duality of the structure of a discipline (logic of a discipline) and the structure of mind (logic of students).

The persistence of the different frames of reference in thinking about structure (discipline vs. mind) reflects the power such oppositions have in shaping our thought and discussion. Once they are permitted to frame the debate, we are put in a position of having to choose between stark alternatives, a position from which it seems very difficult to extricate oneself. Often the structure of a discipline and the structure of mind are framed as competing perspectives between one must choose. Specifically, we tend to take either the structure of a discipline or the structure of mind as fundamental and as giving rise to the other. However, virtually no theoretical orientation or commitment can go unchallenged by proponents of contending movements of thought.[4] Not a single epistemological leitmotif is immune to fading away after a while. Consequently, it is more useful to reconcile paradigmatic differences through dialogue than to argue that the paradigmatic assumptions oppose one another.

However, those oppositions seem not to be evident when departing from an understanding of trivial constructivism (which reduces constructivism to the notion of students constructing their own understandings). Radford (2013) stated that:

> "It is now common in mathematics education discourse to talk about knowledge as something that you make or something that you construct. The fundamental metaphor behind this idea is that knowledge is somehow similar to the concrete objects of the world. You construct, build or assemble knowledge, as you construct, build or assemble chairs." (p. 8)

Inherent in such views is the assumption that subject matter is a 'regular thing' such as a chair (Brown, 2014). The acquisition of subject matter is framed in terminologies borrowed from architecture (Towers & Davis, 2002), such as building a house, constructing a wall, or more generally, putting things with static structure together to make or build something more complicated but also with static structure. Such views, however, project particularly linearized models of knowing and learning, in which subject matter can be both constructed and deconstructed, given that subject matter is considered "as regular things with static structure that react predictably to influences and that can be taken apart and put back together" (Brown, 2014, p. 1472). From these perspectives, subject matter is considered as an object in two rather complementary ways: as an object of teaching (to be deconstructed) as well as an object of learning (to be constructed).

Following the common practice of considering knowledge as a static structure, we can make the linear teaching-learning model bidirectional, indicating the complementarity of constructing and deconstructing subject matter (see Figure 11): a top-down approach in which teachers act upon subject matter (deconstructing) as well as a bottom-up approach in which students act upon subject matter (constructing).

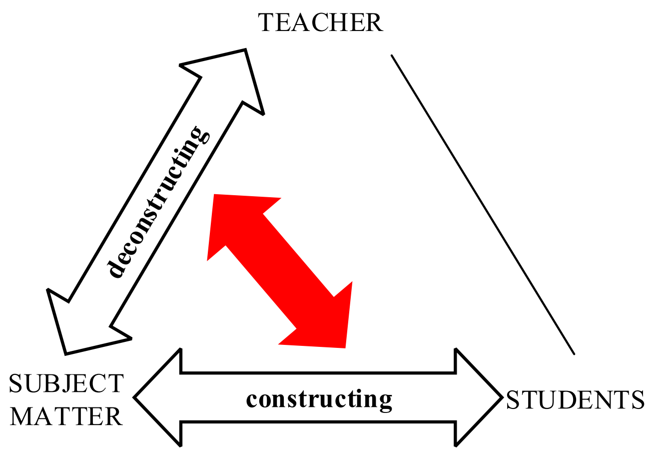

Figure 11: A complementary view: a bidirectional approach

Another way is to adopt the view of cognition and learning as dynamic and complex: dynamic in the sense that knowledge is fluid, changes, and expands; and complex in the sense that knowledge development is non-linear and often unpredictable due to multiple, mutually-influencing aspects of mind and contexts (see Chapter 3).

Knowledge, in such a view, is neither seen as separated bits of knowledge nor as architectural structure with static interconnections among elements. A dynamic stance toward knowledge rejects the view of knowledge as an object: both that of subject matter as an object of teaching and that of subject matter as an object of learning.

Radford (2013) asked, "[…] if knowledge is neither something that you merely construct nor something that you transmit, what is it?" (p. 10). He suggested that knowledge is considered not as an object but as a *process*: "an ensemble of culturally and historically constituted embodied processes of reflection and action" (Radford, 2013, p. 10). Here knowledge is a moment of action (or process) rather than an entity that exists somehow in isolation. This theoretical re-orientation reflects Cobb's (1999) suggestion to "shift from the content metaphor to the emergence metaphor" (p. 31), that is, to understand mathematics not as existing content but as emergent knowledge. As Cobb (1999) explicated, "[t]he content metaphor entails the notion that mathematics is placed in the container of the curriculum, which then serves as the primary vehicle for making it accessible to students" (p. 31), whereas, when understood in emergent terms, a "mathematical idea […] [is] seen to emerge as the collective practices of the classroom community evolved" (p. 31).

The contribution made in Scheiner et al. (2017) takes the latter position: a complex, dynamic stance is adopted toward the discussion of mathematics teacher knowledge, and the debate regarding what makes it specialized. This highlights the complex, dynamic usage, function, and interaction of mathematics teacher knowing, and in doing so goes beyond considering only what teacher knowledge is about. It is argued that such an approach illuminates the conversation concerning the nature of mathematics teacher knowledge, allowing for a better integration of teacher knowledge and teacher action. Finally, such an approach frames mathematics teacher knowledge primarily as a style of knowing rather than as a set of static traits or dispositions.

4.5 Summary

In summary, this chapter critiques existing conceptualizations concerning mathematics teacher knowledge. After identifying some trends in the field, the chapter argues for an approach to understanding teacher knowledge which is: intrinsic rather than extrinsic, viewing specialization as a process of becoming rather than a state of being and rejecting out-of-field comparisons; anthropological-sociocultural rather than reductionist, highlighting the epistemological thread inherent in mathematics teacher knowledge; and transformative rather than additive, where teacher knowledge is conceived as a complex set of interacting knowledge facets within a dynamic structure rather than as an incremental accumulation of knowledge facets. The chapter then discusses the concept of PCK, giving an account of its guiding principles, critiquing its underlying assumptions, and providing a sketch of potential resolutions to possible contradictions. The view of subject matter (and the idea of transforming subject matter for the purposes of teaching) offered by Shulman works well for simplified understandings of knowing, learning, and teaching (where the teacher deconstructs disciplinary knowledge while the student constructs meaning); however, falls short with rather dynamic (where knowledge is considered more as a process than as an object) and complex views on knowing and learning (as it is often non-linear or unpredictable). In the next chapter, insights from other fields are used to develop a more comprehensive model of the phenomenon of teacher noticing.

Notes to Chapter 4

[1] The intended outcome of instruction based on the structure of a discipline is that students will learn what is taught in a way close to that of the logic of a discipline.

[2] These discipline-specific practices (elementarizing, exemplification, explanation, simplification, etc.) seem to be centered on ways to uncover the constituent elements of discipline-specific concepts and make subject matter more accessible to others. It requires the capacity to "work backward from mature

and compressed understanding of the content to unpack its constituent elements" (Cohen, in preparation, cit. in Ball & Bass, 2000, p. 98).

[3] The literature demonstrates many faces of constructivism (see Perkins, 1999; Phillips, 1995); the two most predominant ones in education research are *cognitive* constructivist perspectives and *social* constructivist perspectives (Cobb, 1994, Ernest, 1998). Within each (cognitive constructivism and social constructivism) there is also a range of positions. However, the various perspectives on constructivism are committed to a common theoretical assumption (Ernest, 2010): knowledge is not discovered but actively constructed, a theoretical position that von Glasersfeld (1989) called 'trivial constructivism'. Radical constructivism, on the other hand, is based on two principles: (1) knowledge is actively constructed by the subject through their cognition, not passively received from the environment (trivial constructivism); and (2) cognition is an adaptive process that organizes a person's experience; it does not discover an independent, pre-existing world that exists outside of the human mind (von Glasersfeld, 1989, p. 162). Von Glasersfeld's use of 'radical' is in the sense of fundamental, as already noted by Thompson (2014), that is, cognition is "a constitutive activity which, alone, is responsible for every type or kind of structure an organism comes to know" (von Glasersfeld, 1974, p. 10).

[4] Neither of the two positions contrasted here can be simply reduced to a single school of thought; in contrast, both are best referred to as movements of thought embodying a variety of forms (that are evolving and changing).

Article 2

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What Makes Mathematics Teacher Knowledge Specialized? Offering Alternative Views

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Abstract The purpose of this article is to contribute to the discussion of mathematics teacher knowledge, and the question of what makes it specialized. In the first part of the article, central orientations in conceptualizing mathematics teacher knowledge are identified and the more serious limitations of the grounds on which they stand are explicated. In the second part of the article, alternative views are offered to each of these orientations that direct attention to underexplored issues about what makes mathematics teacher knowledge specialized. Collectively, these alternative views suggest that specialization in mathematics teacher knowledge cannot be comprehensively accounted for by 'what' teachers know, but rather by 'how' teachers' knowing comes into being. We conclude that it is not a kind of knowledge but a style of knowing that signifies specialization in mathematics teacher knowledge.

Keywords Mathematical knowledge for teaching · Pedagogical content knowledge · Specialized knowledge · Teacher knowledge · Teacher professionalism

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Introduction

Mathematics teacher knowledge has become a fertile research field in mathematics education (see Ponte & Chapman, 2016). Scholars have considered mathematics teacher knowledge from multiple perspectives, using various constructs and frameworks to describe and explain what makes mathematics teacher knowledge specialized. [1] Despite the relatively short time that research on teacher knowledge has existed as a field, the literature is currently shaped by a diversity of conceptualizations of mathematics teacher knowledge (Petrou & Goulding, 2011; Rowland, 2014).

As research on teacher knowledge has moved to a more central role in mathematics education research (see Even & Ball, 2010; Fennema & Franke, 1992; Sullivan & Wood, 2008), the search for what signifies the specialization in mathematics teacher knowledge has been becoming an increasingly important enterprise in the research field. Recent research has addressed this issue by describing and identifying facets or types of teacher knowledge that have been considered as crucial for teaching mathematics, and in obtaining empirical evidence to support these (e.g., Ball, Thames & Phelps, 2008; Baumert, Kunter, Blum, Brunner, Voss, Jordan, Klusmann et al., 2010; Blömeke, Hsieh, Kaiser & Schmidt, 2014). As such, the focus tends to be on (seemingly distinct) facets of knowledge that an individual teacher possesses (knowledge for teaching) or uses in the classroom (knowledge in teaching). A number of scholars have pointed to inadequacies in such conceptualizations of teacher knowledge, arguing that they disregard the deep embeddedness of knowledge in professional activity (Hodgen, 2011) and ignore the dynamic interactions between different kinds or facets of teacher knowledge (Hashweh, 2005). Others have argued that the premises on which much research into teacher knowledge is based depend on assumptions that are rather aligned with transmission views of teaching (McEwan & Bull, 1991) and, in consequence, are rather asymmetrical to constructivist viewpoints (Cochran, DeRuiter & King, 1993). Thus, it is not surprising that scholars have called for making the assumptions underlying frameworks of teacher knowledge, teaching, and teacher learning explicit (Lerman, 2013) and for achieving coherence between research into teacher characteristics and teacher practice (Van Zoest & Thames, 2013).

This paper aims to make explicit the discussion of what makes mathematics teacher knowledge specialized, a question that has often been addressed implicitly by several scholars in various ways and with different emphases. The paper outlines further attempts that reflect theoretically on this important issue and try to articulate more explicitly what the specialization signifies, or may signify, in mathematics teacher knowledge. The purpose of this paper is, therefore, twofold: First, we try to elucidate central orientations currently available in the literature and point to the more serious limitations of the grounds on which they stand. Second, we provide alternative views that direct attention to underexplored issues about these orientations.

We begin this article by briefly discussing previous accounts on what mathematics teacher knowledge signifies and encompasses, and then take this retrospection as a

[1] We prefer using the term 'specialized' instead of 'special' with respect to mathematics teacher knowledge. The latter implies the assertion of a quality of teacher knowledge that is distinguishable from something. We use the term 'specialized' to indicate a quality of mathematics teacher knowledge that comes into being when enacted.

point of departure for outlining the limitations of these accounts. Afterwards, we articulate and draw a contrast with alternative viewpoints that provide a critical stance toward previous accounts but also provide new ways to think about the issues under consideration. The first perspective underlines the complex dynamics of the usage and function of mathematics teacher knowledge in context that calls for specialization as a process of becoming rather than a state of being. The second perspective points to the epistemological stance inherent in mathematics teacher knowledge, arguing for the sensitivity for the historical and cognitive geneses of mathematical insights. The third perspective accentuates the complex interactions of knowledge facets that generate dynamic structures. Then, we highlight underlying themes and convergences of these alternative views with regard to specialization in mathematics teacher knowledge. Finally, we conclude by proposing to construe specialization in mathematics teacher knowledge as a style of knowing rather than a kind of knowledge.

On the Evolution of Thinking About Conceptualizing Mathematics Teacher Knowledge

Research into mathematics teacher knowledge has evolved considerably, especially over the last three decades. The number of studies in this field has significantly increased, the nature and scope of the research have expanded, and the frameworks used to guide the study of mathematics teacher knowledge have become quite diverse. The growing diversity of frameworks for teacher knowledge testifies to the complexity and multidimensionality of the research field.

In the following, we try to outline the evolution of thinking within the field in conceptualizing mathematics teacher knowledge —with the explicit intention of identifying central orientations in the literature concerning what makes mathematics teacher knowledge specialized. We acknowledge that in any approach intending to identify central orientations in the literature a great deal of important detail is lost. More detailed accounts of this research can be found elsewhere (see e.g. Kaiser, Blömeke, König, Busse, Döhrmann & Hoth, 2017; Kunter, Baumert, Blum, Klusmann, Krauss & Neubrand, 2013; Rowland & Ruthven, 2011; Schoenfeld & Kilpatrick, 2008). A recent discussion of several research traditions is provided by Blömeke and Kaiser (2017), in which the same authors arrive at a complex framework of teacher competence and conceptualize the development of teacher competence as personally, situationally, and socially determined, as well as embedded in a professional context.

Our purpose here, however, is to foreground central orientations of what signifies mathematics teacher knowledge that have been provoked by scholars in the field. We start by portraying different dimensions of mathematical knowledge discussed in the literature as being essential for mathematics teachers. Then, we draw attention to selected contributions that articulate what particularizes subject matter knowledge for teaching, particularly in reference to mathematical knowledge for teaching, with an emphasis on the way specialization is considered. Afterwards, we focus on what is considered as the heart of teaching: the *transformation* of subject matter in ways accessible to students, an assumption that underlies several attempts in conceptualizing mathematics teacher knowledge.

Mathematical Knowledge

The literature foregrounds different aspects of mathematical knowledge as important for teachers. Shulman (1986), for instance, accredited "the amount and organization of the knowledge per se in the mind of the teacher" (p. 9), referring to Schwab's (1978) distinction between substantive and syntactic structures of a discipline. Substantive structures are the key concepts, principles, theories, and explanatory frameworks that guide inquiry in a discipline, while syntactic structures provide the procedures and mechanisms for the acquisition of knowledge, and include the canons of evidence and proof. Bromme (1994), then again, acknowledged that "school subjects have a 'life of their own' with their own logic; that is, the meaning of the concepts taught cannot be explained simply by the logic of the respective scientific disciplines" (p. 74). In recognizing school mathematics as a special kind of mathematics, Bromme (ibid.) suggested school mathematical knowledge and academic content knowledge as further dimensions of mathematical knowledge. Buchholtz, Leung, Ding, Kaiser, Park and Schwarz (2013) set forth a kind of knowledge "that comprises school mathematics, but goes beyond it and relates it to the underlying advanced academic mathematics" (p. 108). The same authors called this kind of knowledge, in homage to the pioneering work of Felix Klein, knowledge of elementary mathematics from an advanced standpoint.

This small selection of a fuller corpus of dimensions of mathematical knowledge already indicates a critical point to be expanded here: the contributions to dimensions of mathematical knowledge that teachers know, or should know, are accumulative (or incremental). However, as Monk (1994) reminds us, "a good grasp of one's subject areas is a necessary but not sufficient condition for effective teaching" (p. 142). We might interpret Monk's statement as a call for additional knowledge, but we might also understand it as a call for a *qualitatively* different kind of knowledge.

Subject Matter Knowledge for Teaching (Pedagogical Content Knowledge)

A critical advance in the field was the recognition that teaching entails a specialized kind of subject matter that is distinct from disciplinary subject matter. Shulman (1986) proposed a kind of knowledge "which goes beyond knowledge of subject matter *per se* to the dimension of subject matter knowledge *for teaching*" (p. 9, italics in original) that he labeled *pedagogical content knowledge* (PCK). Shulman (1986) described PCK as encompassing

> for the most regularly taught topics in one's subject area, the most useful forms of [external] representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others […] [and] an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. (p. 9)

In this view, PCK consists of two dimensions: 'knowledge of representations of subject matter' and 'knowledge of specific learning difficulties and students' conceptions'. These two dimensions often served as reference points in thinking about PCK, as Ball (1988), for instance, assumed "[…] 'forms of representation' […] to be the crucial substance of pedagogical content knowledge" (p. 166). She then explored the more dynamic aspects of this idea, examining preservice teachers' pedagogical reasoning in mathematics as the process whereby they build their knowledge of mathematics teaching and learning. Other scholars in mathematics education have delineated dimensions of PCK that extended or refined Shulman's original considerations. For instance, Marks (1990) clarified PCK in the context of mathematics by identifying four dimensions, including knowledge of students' understanding, knowledge of subject matter for instructional purposes, knowledge of media for instruction, and knowledge of instructional processes.

Shulman (1987) asserted that among multiple knowledge domains for teaching (e.g. content knowledge, general pedagogical knowledge, curriculum knowledge, knowledge of learners), it is PCK that is "the category most likely to distinguish the understanding of the content specialist from that of the pedagogue" (p. 8). As such, the existence of PCK relies on and projects the belief in a distinction between the subject matter knowledge of teachers and that of other subject specialists or scholars (e.g. mathematicians). While the notion of PCK advocated a position distinguishing teachers' and academics' subject matter knowledge, the concept of *mathematical knowledge for teaching* advocated a position distinguishing knowledge for teaching mathematics from knowledge for teaching other subjects (such as physics, biology, or the arts).

Mathematical Knowledge for Teaching

The notion of *mathematical knowledge for teaching* has become an important point of departure in thinking about what signifies the specialization in mathematics teacher knowledge. Various researchers have applied different emphases to this notion, as shall be seen below. In this realm, it is particularly the *Mathematical Knowledge for Teaching* (MKT) framework (e.g. Ball & Bass, 2000; Ball et al., 2008), that has attracted significant research attention. The MKT framework evolved through the application of a kind of job analysis (Ball et al., 2008) focusing on the use of knowledge in and for the work of teaching.

The MKT framework defines several sub-domains within two of Shulman's (1987) original knowledge domains: pedagogical content knowledge (PCK) and subject matter knowledge (SMK). PCK is divided into knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum, while SMK is divided into common content knowledge, specialized content knowledge, and knowledge at the mathematical horizon. We outline four of the six dimensions, excluding horizon content knowledge and knowledge of curriculum as they have so far not been the primary focus of studies into the area.

Within PCK, *knowledge of content and teaching* combines knowing about teaching and knowing about mathematics, including knowledge of the design of instruction, such as the knowledge governing the choice of examples to introduce a content item and those used to take students deeper into it. *Knowledge of content and students* is the knowledge that combines knowing about mathematics and knowing about students. It includes knowledge of common student conceptions and misconceptions about particular mathematical content as well as the interpretation of students' emerging and incomplete thinking.

Within the mathematical knowledge domain, *common content knowledge* refers to the mathematical knowledge and skill possessed by any well-educated adult, and certainly by all mathematicians, which is used in settings other than teaching. *Specialized content knowledge*, on the other hand, is defined as mathematical knowledge tailored to the specialized uses that come up in the work of teaching. It is described as being used by teachers in their work, but not held by well-educated adults, and is not typically needed for purposes other than teaching. Ball et al. (2008) noted that teaching may require "a *specialized* form of *pure* subject matter knowledge" (p. 396, italics added):

> *pure* because it is not mixed with knowledge of students or pedagogy and is thus distinct from the pedagogical content knowledge identified by Shulman and his colleagues and *specialized* because it is not needed or used in settings other than mathematics teaching. (Ball et al., 2008, p. 396, italics added)

Transforming Subject Matter

The previous two approaches support the assertion that a kind of subject matter knowledge *exists* that is qualitatively different from the subject matter knowledge of disciplinary scholars or teachers of other subjects. The nature of such knowledge, however, is not just a matter of mastering disciplinary subject matter. From the perspectives presented so far, teachers' primary concern is not with mathematics, but with teaching mathematics. The difference between disciplinary scholars and educators is, therefore, also seen in the different uses of their knowledge. This important recognition of the different purposes of disciplinary scholars and teachers highlights, as Shulman (1987) argued, a unique aspect of teachers' professional work: a teacher must "transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students" (Shulman, 1987, p. 15). It is this notion of *transforming* the subject matter of an (academic) discipline that highly impacted our thinking about teacher knowledge, but it seems to have been taken for granted once the picture of knowledge for teaching was defined. The primary purpose of transformation is to organize, structure, and represent the subject matter of an (academic) discipline in a form "that is appropriate for students and peculiar to the task of teaching" (Grossman, Wilson & Shulman, 1989, p. 32).

The literature on mathematical knowledge for teaching also identifies various discipline-specific practices of transformation, often described in terms of exemplifying, explaining, decompressing, or simplifying, that converge on teachers' core practice of unpacking mathematics content in ways that are accessible to students (Adler & Davis, 2006; Ball & Bass, 2000; Ma, 1999). It requires the capacity "to deconstruct one's own mathematical knowledge into a less polished and final form, where elemental components are accessible and visible" (Ball & Bass, 2000, p. 98). Hodgen (2011), for instance, takes this idea further arguing that the "essence of teacher knowledge involves an *explicit* recognition of this – 'unpacking' the mathematical ideas […], [whereas] doing mathematics only requires an *implicit* recognition of this." (pp. 34–35, italics in original).

More recently, the idea of transformation has also been further elaborated by scholars working in the *Knowledge Quartet* research program (Rowland, 2009; Rowland, Huckstep & Thwaites, 2005), as part of their conceptualization of the classification of situations in which mathematical knowledge surfaces in teaching. The research group considers transformation as concerning "knowledge in action as demonstrated both in planning to teach and in the act of teaching itself. A central focus is the representation of ideas to learners in the form of analogies, examples, explanations, and demonstrations" (Rowland, 2009, p. 237). This conceptualization concerns knowledge in action, focusing on teaching activity in the transmission of content.

Thinking About What Makes Mathematics Teacher Knowledge Specialized: Various Orientations, Different Responses

As innocent and straightforward as the question *What makes mathematics teacher knowledge specialized?* sounds, the research field has found it difficult to provide an explicit answer as there are various orientations toward teacher knowledge, each with a quite different response to the question. The previous section briefly outlined the following orientations regarding what mathematics teacher knowledge signifies: (1) identifying and describing multiple dimensions of mathematical knowledge (and pedagogical content knowledge), (2) declaring kinds of subject matter knowledge for teaching that are distinct from subject matter knowledge per se, and (3) asserting teachers' action upon subject matter (that is the transmission of subject matter in ways accessible to students) as the core task of teaching.

These three orientations seem to indicate different lines of thinking about what makes mathematics teacher knowledge specialized. Each focuses attention on particular aspects: the first considers additional knowledge dimensions (quantity), whereas the second turns the attention toward knowledge that is construed as qualitatively different. These different lines of thinking seem to be convolved in Shulman's idea of transforming subject matter, that is, the various orientations shape, and are shaped by, our interpretations of Shulman's idea of transforming subject matter.

One might interpret Shulman's (1986, 1987) initial writings on teacher knowledge as indicating a stance in which teachers' and disciplinary scholars' subject matter knowledge were differentiated, signifying the existence of a kind of subject matter knowledge for teaching (held by teachers) that is qualitatively different from subject matter knowledge per se (held by disciplinary scholars). On the other hand, Ball and her colleagues proposed a more nuanced differentiation in which subject matter content itself is considered in a way that only makes sense to *mathematics* teachers. In other words, while both notions of PCK and specialized content knowledge indicate the existence of a qualitatively different kind of knowledge, they differ in where to put emphasis: Shulman's notion of PCK puts emphasis on a kind of knowledge distinctive to *teachers* (and not to disciplinary scholars) and Ball and her colleagues' notion of specialized content knowledge puts emphasis on a kind of knowledge distinctive to *mathematics* teachers (and not to teachers of other subjects).

Each of these orientations provides a (partial) response to the question of what signifies mathematics teacher knowledge. The first orientation calls for the multidimensionality of mathematical knowledge in particular, and teacher knowledge in general. The second orientation argues for the qualitative differences between scholars' subject matter knowledge (per se) and teachers' subject matter knowledge (for teaching) or the qualitative differences between knowledge for teaching mathematics and knowledge for teaching other subjects. The third orientation, underlying and extending the previous one, points to teachers' actions upon subject matter, as manifested in notions such as transforming, unpacking, deconstructing, and decompressing subject matter.

Correspondingly, we can frame the responses of the three orientations concerning what makes mathematics teacher knowledge specialized as follows:

– mathematics teachers need to know more than the subject matter they teach (*additional knowledge*);
– mathematics teachers need to know subject matter in a qualitatively different way than other practitioners of mathematics (mathematicians, physicists, engineers, among others), and they need to hold a qualitatively different kind of knowledge than teachers of other subjects (physics teachers, biology teachers, history teachers, among others) (*qualitatively different knowledge*); and
– mathematics teachers need to know how to organize or structure the subject matter in ways accessible to students (*teaching-oriented action*).

These responses, taken together, seem to converge on an understanding that what mathematics teacher knowledge signifies depends on its *distinctiveness* or *exclusiveness*: mathematics teacher knowledge is construed as knowledge that is needed *only* for teaching mathematics, that is, knowledge not required for other purposes than teaching and not needed for teaching other subjects than mathematics.

Too often when we frame our thinking about what mathematics teacher knowledge signifies, we see ourselves getting caught in the mire of current debates without taking a critical stance toward the grounds on which they stand. In the present paper, it is intended to take a more critical stance toward the current state of what the literature implicitly represents as making mathematics teacher knowledge specialized. To this end, we explicitly identify the more significant boundaries demarking the outlined orientations and provide new ways of thinking about the issue under consideration. Our critique rests on at least three general tendencies that seem to have been implicit in the current discussion on teacher knowledge:

– the field brings up external references in justifying what makes teacher knowledge specialized (mathematics teachers vs. mathematicians; teaching mathematics vs. teaching other subjects);
– in its consideration of teacher knowledge, the field takes a disciplinary perspective which is reductionist in orientation, arguing from the viewpoint of teaching mathematics; and

– the field has been partly additive, that is, accumulating dimensions of teacher knowledge.

In the following sections, we adopt a critical stance to these general tendencies, around which we organize our understanding of the question of what makes knowledge for teaching mathematics specialized. As such, we argue for an approach which is:

– intrinsic: it dispenses with external reference points, and accounts for specialization as a process of becoming rather than a state of being;
– anthropological-sociocultural: it eschews a reductionist approach, and instead underlines the epistemological thread inherent in mathematics teacher knowledge; and
– transformative: rather than seeing teacher knowledge as an incremental accumulation of facets, it accentuates the complex interactions of knowledge within a dynamic structure.

In doing so, we draw on and debate different emerging perspectives that provide critical issues that are un- or under-addressed in the current literature, and, more importantly, that provide provocative new avenues for thinking about what makes mathematics teacher knowledge specialized in ways not yet explicitly articulated.

From an Extrinsic to an Intrinsic Approach

In this section, we adopt a critical stance to a tendency that seems to be common among scholars discussing mathematics teacher knowledge: the tendency of comparing mathematics teacher knowledge with the knowledge demanded of other professionals (such as mathematicians, teachers of subjects other than mathematics). Such an approach is extrinsically oriented (see Flores, Escudero & Carrillo, 2013) as it takes an external referent (e.g. mathematicians or teachers of other subjects) as a reference point for comparison. The explicit purpose of such an approach is to identify the distinctiveness of mathematics teacher knowledge in relation to someone else's knowledge.

Since Shulman (1986) acknowledged teachers as professionals, various scholars in mathematics education have attempted to identify the distinctiveness of knowledge for teaching mathematics in comparison with other forms of knowledge. This search took place primarily by looking outside of mathematics education to provide answers as to what mathematics teacher knowledge signifies. Researchers articulated ways in which mathematics teacher knowledge differs from mathematicians' knowledge, or how it differs from knowledge of those who teach subjects other than mathematics. This tendency to look beyond the discipline, we believe, is a very natural one, particularly when, at the same time, scholars were searching for an identity for the research field. In relating mathematics teachers to professionals of other disciplines, scholars were able to determine certain cognitive dispositions that seemed to be specific for mathematics teachers—aspects of teacher knowledge that have been referred to as being static, explicit, and objective (in the sense of being observable). However, it is one thing to make comparisons between mathematics teacher knowledge and the knowledge pertinent to other professionals, and quite another to interpret the seemingly distinctive features of teacher knowledge in terms of ‘specialization’. Whereas ‘specialization’ seems to have been understood in terms of distinctiveness, in this paper, we argue for a different meaning of specialization that allows us to focus our attention inside and not necessarily outside.

Flores et al. (2013), for instance, identified difficulties in defining the specialized nature of certain cognitive dispositions when analyzing the knowledge involved in assessing students’ subtraction strategies. They affirmed that it is debatable whether the knowledge used by a teacher is exclusive to him or her, or is shared with other practitioners of mathematics. They focus discussion on certain cognitive dispositions and wonder who else, other than a mathematics teacher might have such kind of knowledge, thus moving the focal point of the debate from mathematics teacher knowledge to that of other professionals.

The answers we might gain from such comparisons (mathematics teachers vs. mathematicians, mathematics teachers vs. teachers of other subjects, etc.) are external to mathematics education as a discipline, in that they offer justifications that are recognizable and measurable but neither cognitive (concerning the processes involved in knowledge) nor epistemological (regarding the nature of knowledge). External referents (such as mathematicians) might provide useful markers for identifying static traits that differ from mathematics teachers such as the content of teacher knowledge, that is, what teachers’ knowledge is about. However, they seem to be inappropriate in accounts of the complex dynamics of knowledge in use. Rather than framing the discussion of what makes mathematics teacher knowledge specialized in terms of external referents, we suggest an account of specialization understood in relation to mathematics teacher knowledge in action. That is to say, what makes mathematics teacher knowledge specialized is not so much “what” mathematics teachers know (which might indeed differ from other professionals), but “how” mathematics teachers know. This involves a shift away from the content of mathematics teacher knowledge to its usage and function, that is, how teacher knowledge comes into action (how it comes into being or how it actualizes). This shift in perspective foregrounds the context rather than the content.

Instead of an extrinsic perspective, we suggest taking an intrinsic view, that is, acknowledging the situatedness of mathematics teacher knowledge within the context of mathematics learning and teaching. Interestingly, Carrillo, Climent, Contreras and Muñoz-Catalán (2013) have already explicated a framework, termed the *Mathematics Teacher’s Specialized Knowledge* (MTSK) framework, which is constructed on, and projects, an intrinsic perspective whereby the idea of specialization is framed with regard to the inseparability of knowledge and context. The key to recognizing and making visible what makes mathematics teacher knowledge specialized lies, we argue, in the context in which the knowledge comes into being. Contextuality, then, becomes the central concern. Obviously, that context matters is hardly new nor provocative (see e.g. Fennema & Franke, 1992); however, the way in which the term is commonly used differs from the point we want to advance in this paper.

In our view, whether knowledge is specialized or not is a question of whether the knowledge is contextually adaptive (Hashweh, 2005), that is, functional on a moment-by-moment basis, and highly sensitive to the changing details of the situation as teachers interact with the environment and with the students around them. This means, rather than expecting differences in knowledge (concerning quantity, quality, etc.)

based on broad descriptions of context—such as school vs. scientific environment—the term “context” acquires a very different and deeper meaning than the ways it has been previously construed. This perspective assumes that context consists of situations and activities embedded in the learning-teaching complex in the immediate moment. In consequence, what signifies mathematics teacher knowledge might be better described (or can be better approached) from within the discipline. In this regard, mathematics teacher knowledge is treated not as static traits (that differ from other professions) but as interpretations of performances that are situated in the immediate context. In this regard, Putnam and Borko (2000) argued that “professional knowledge is developed in context, stored together with characteristic features of classrooms and activities, organised around the tasks that teachers accomplish in classroom settings, and accessed for use in similar situations” (p. 13). As such, a mathematics teacher’s action is not a simple display of a static system of some certain knowledge types, but rather a highly contingent and continually adaptive and proactive response that shapes, and is shaped by, the environment in which the teacher interacts.

In other words, it is not about *being* but about *becoming*, that is, it is less about static dispositions or traits differentiable from those of other professions and more about the complex dynamics of the usage and function of knowledge in context. Mathematics teacher knowledge becomes specialized in its adaptive function in response to the dynamics and complexities in which it comes into being.

From a Reductionist to an Anthropological-Sociocultural Approach

In this section, we adopt a critical stance to the disciplinary approach to teacher knowledge, an approach that is primarily reductionist in orientation and that argues from the viewpoint of teaching mathematics rather than from the standpoint of learning mathematics. We argue against a reductionist understanding of knowing and learning, in which knowledge is construed as independent of the knower. Instead, we argue for an anthropological-sociocultural perspective that accounts for the evolving nature of mathematical meaning in the learning process.

Shulman (1987) declared that subject matter knowledge per se “must be transformed in some manner if they are to be taught. To reason one’s way through an act of teaching is to think one’s way from the subject matter as understood by the teacher into the minds and motivations of learners” (p. 16). Generally speaking, the central task of teaching is considered as transforming subject matter knowledge into a form in which it is teachable to particular learners. This transformation of the subject matter is, according to Shulman (1987), heavily, if not wholly, determined by the disciplinary subject matter as the primary source of information for teaching and the principal route to informed decisions about instruction. Gudmundsdottir (1991) described this transformation as a “reorganization [of content knowledge] that derives from a disciplinary orientation” (p. 412) and Grossman et al. (1989) designated it as “translat[ing] knowledge of subject matter into instructional representations” (p. 32). As mentioned above, scholars in the field of mathematics education have recommended several discipline-specific practices of transformation that aim to unpack mathematics content in ways accessible to students. In this view, teachers must be able to take apart mathematical concepts, operations and strategies so as to enable students to gain access to the thought processes and ideas that they represent. Students, on the other hand, are considered as putting together the constituent pieces of those mathematical concepts, operation, and strategies. Such assertions rely on, and project, a reductionist understanding of the knowing and learning processes; an understanding in which the knowing and learning processes are construed as putting together what teachers intentionally picked apart. This view not only distorts the complexity of the processes of knowing and learning mathematics, but also advocates the assumption that knowledge is independent of the knower.

Some general approaches in mathematics education have challenged reductionist views on knowing and learning, including, but not limited to, Gestaltism, constructivism, problem-solving, socio-culturalism, and complexity thinking. Here, we follow anthropological-sociocultural perspectives, which, rather than consider knowledge as an object that exists apart from the individual, acknowledge the co-implicated nature of knowledge, knower, and context. In this perspective, particular emphasis is given to the genesis of mathematical knowing and learning by accounting for historical and cognitive evolutions, dynamics, and changes. In this view, knowledge is considered a process rather than an object—to acknowledge the complex dynamics in knowing mathematics.

For instance, the *Didactic Mathematical Knowledge* (DMK) framework (Pino-Fan, Assis & Castro, 2015) is grounded in an onto-semiotic perspective of mathematical knowledge and instruction (Font, Godino & Gallardo, 2013; Godino, Batanero & Font, 2007). As such, the framework is rooted in anthropological-sociocultural assumptions about mathematical knowledge (where mathematics is understood as a human activity), and takes up the ontological assumption of a diversity of mathematical objects as well as the semiotic assumption of a plurality of languages and meanings. The DMK framework, similar to other proposals (e.g. Ernest, 1989), relies on, and projects, assumptions that transcends realist-Platonic positions on the nature of mathematics and foregrounds an anthropological conception of mathematics. That is, teachers have to recognize the emergence of concepts, procedures, and propositions from mathematical practices, and attribute a central role to the various languages and artifacts involved in such practices. The applications—the use of mathematics as a cultural reality in itself to solve real-life or mathematical problems—promote a variety of meanings for mathematical objects, which must be progressively articulated in the learning process. Such a view acknowledges the embodied meanings of mathematical concepts that evolve in the learning process. The DMK framework particularly foregrounds an *epistemic facet* of teachers' didactical-mathematical knowledge which, according to Godino, Font, Wilhelmi and Lurduy (2011), interacts with other knowledge facets (affective, cognitive, ecological, interactional, and mediational). Consequently, the attentiveness (or mindfulness) to epistemological issues (such as the nature of mathematics and mathematics learning) is illuminated. From this perspective, teachers' sensitivity toward the epistemic genesis of mathematics and mathematics learning becomes a central aspect of what mathematics teacher knowledge signifies.

In short, an anthropological-sociocultural perspective acknowledges knowledge as an evolving process rather than a more or less static object that exists independent of the knower. In this view, not only the interaction between knowledge, knower, and context is highlighted, but also the historical and cognitive genesis of mathematical meanings. Thus, what makes mathematics teacher knowledge specialized is not the accumulation of distinct facets of knowledge, but the teachers' stance toward knowledge, in the light of the historical and cognitive geneses of mathematical insights. This perspective calls for a shift in thinking about teachers' core tasks: the teachers' focus should not be on acting upon subject matter by re-structuring, re-interpreting, re-configuring, and re-building mathematical concepts to make them accessible to students, but instead on the complex interactions between students and subject matter. That is, the key is not teachers' capacity to unpack mathematics, but their capacity to unpack students' ways of understanding in order to make students' ways of mathematical thinking visible.[2]

From an Additive to a Transformative Approach

In this section, we adopt a critical stance to another apparently widespread tendency that seems to have implicitly driven recent discussions on teacher knowledge: the tendency toward atomizing teacher knowledge for the sake of accumulating distinct and refined dimensions of teacher knowledge. We argue for a transformative approach that goes beyond a merely incremental approach to facets of knowledge by turning back to Shulman's idea of blending knowledge facets.

The last three decades have been colored by various attempts to capture what mathematics teacher knowledge is about and what it entails. Research studies started out by distinguishing, refining, and adding to various dimensions of knowledge regarded as critical for teaching mathematics. Since then, we have accumulated a considerable number of, often indistinguishable (see Silverman & Thompson, 2008), knowledge dimensions that, taken together, seem to provide a more refined picture of the multidimensionality of teacher knowledge. This undertaking allowed scholars to order, structure, and, most important, simplify the complexity of teacher knowledge, to reduce it to its observable and measurable parts.

The approach relies on the assumption that a full understanding of teacher knowledge should emerge from a detailed analysis of each of its parts. It is believed that the complexity of teacher knowledge can be studied by dissecting it into its smallest parts (knowledge facets, types, etc.), and that these knowledge units are the basis, or the fundamental particles, of what mathematics teacher knowledge signifies. Following these lines of thinking, reflections on mathematics teacher knowledge emphasize the nature of these parts—paying little attention to transformations that arise when knowledge elements are blended.

Instead of dividing and thinking in terms of multiple, distinct sub-categories of teacher knowledge, our disposition is to take a broader view that sees teacher knowledge as an organic whole. Interestingly, Shulman (1987) already described PCK as "that special *amalgam* of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (p. 8, italics added). Here, Shulman understood PCK not as the summation or accumulation of content knowledge and pedagogical knowledge: "[…] just knowing the content well was really important, just knowing general pedagogy was really important and yet when you add

[2] This is not to be understood as dichotomizing teachers' capacity for unpacking mathematics and their capacity for unpacking students' understandings, but to re-emphasize that teaching is not (merely) a top-down approach of transposing subject matter to the students but a bottom-up approach of students constructing mathematical ideas that are used as points of departure in the teaching-leaning complex.

the two together, you didn't get the teacher" (Shulman, cit. in Berry, Loughran & van Driel, 2008, p. 1274). Rather, the amalgamation of content and pedagogy means "the *blending* of content and pedagogy" (Shulman, 1987, p. 8, italics added) into a new kind of knowledge that is distinctively and qualitatively different from the knowledge dimensions from which it was constructed. However, by proposing PCK as the amalgam of content and pedagogy without accounting for the complex interactions between these and other knowledge facets, Shulman left the task of further clarifying the blending process to other scholars.

Surprisingly, though many scholars paraphrased Shulman's idea of amalgamation, they almost always took the result of blending knowledge domains (that is, according to Shulman, PCK) as given and often considered it as static (for a critique, see Hashweh, 2005). In other words, many scholars ignored the complex dynamics of blending, a high interaction of knowledge facets that forms new structure not evident in the previous facets.

To the best of our knowledge, blending seems to be an undertheorized phenomenon in research on teacher knowledge. Recently, Scheiner (2015) has suggested construing teacher knowledge as a complex, dynamic system of various knowledge atoms, which are understood as blends of different knowledge facets. The idea of 'knowledge atom' shares similarities with Sherin's (2002) idea of 'content knowledge complexes' construed as "tightly integrated structures containing [pieces of] both subject matter knowledge and pedagogical content knowledge" (p. 125) repeatedly accessed during instruction. Scheiner (2015) proposed that teacher knowledge is dynamic not simply because it evolves dynamically (which it does), but because it forms dynamically: teacher knowledge is dynamically emergent from the interactions of knowledge facets. This interaction of knowledge facets is in the nature of what Fauconnier and Turner (2002) described as *conceptual blending*. In technical terms, blending is a process of conceptual mapping and integration, a mental operation for combining frames or models in integration networks that leads to new meaning, global insights, and conceptual compression (see Fauconnier & Turner, 2002). The essence of conceptual blending is to construct a partial match, called cross-space mapping, between frames from established domains (known as inputs), to project selectively from those inputs into a novel hybrid frame (a blend or blended model), comprised of structure from each of its inputs, as well as a unique structure of its own (emergent structure). Crucially, the inputs are not just projected wholesale into the blend, but a combination of the processes of projection, completion, and elaboration (or 'running' the blend) "develops emergent structure that is not in the inputs" (Fauconnier & Turner, 2002, p. 42). The point we want to make here is that knowledge facets interact dynamically to form emergent structures. Not only do new elements arise in the blend that are not evident in either input domain on its own, but blending accounts also for the interdependencies of knowledge dimensions: the production of a blend is recursive, in the sense, that blends depend on previous blends.

Scheiner's (2015) proposal of teacher knowledge as a complex, dynamic system of various knowledge atoms attempts a dialectic between atomistic and holistic views of teacher knowledge. It puts the refinements of teacher knowledge identified and gained through atomistic approaches together into a complex system of blends that—as a whole—is more than the sum of its parts.

In a nutshell, a complex system perspective regards teacher knowledge as dynamically emergent and dimensions of teacher knowledge as being organically interrelated. It emphasizes that various knowledge facets are in constant dialog with each other, inform each other, and interact dynamically to form emergent structures. Thus, the key relies not on accumulating types of teacher knowledge but on blending knowledge facets that emerge dynamically. Accumulating teacher knowledge facets is additive (or complementary), but blending is transformative.

Discussion

In the three previous sections, we have critically appraised what the current literature implicitly represents as making mathematics teacher knowledge specialized. In each section, we have tried to make explicit the more serious limitations of the grounds on which at least three general tendencies stand, and which seem to have been inherent in the current discussion on teacher knowledge. Each section provides provocative new ways of thinking about the issue under consideration.

First, we called for an account of specialization that comes from the inside rather than the outside (such as comparisons with professionals working in other disciplines). In recognizing the situated nature of mathematics teacher knowledge in the immediate context, the complex dynamics of the usage and function of knowledge in the immediate context can be underlined. As such, specialization is not a state of being but a process of becoming: mathematics teacher knowledge becomes specialized in its adaptive function in response to the dynamics and complexities in which it comes into being.

Second, we argued that an account of specialization cannot be provided with itemization of mathematics teacher knowledge, but rather through teachers' epistemological stance toward knowledge and the sensitivity for the historical and cognitive geneses of mathematical insights. Going beyond a reductionist understanding of knowing and learning processes, in which the teacher's task is considered to be unpacking the subject matter of mathematics, we encouraged the view of teachers unpacking students' understandings to make students' ways of mathematical thinking explicit.

Third, we argued that an account of specialization lies not in the sum of the parts of mathematics teacher knowledge but in its organic whole, that is, various knowledge facets being constantly in dialog with each other, informing each other, and interacting dynamically to form emergent structures. We proposed a complex system perspective that construes teacher knowledge as blends of various knowledge facets that emerge dynamic structure.

On the one hand, these alternative views point to several aspects that scholars attempted to encompass in their use of the notion of *knowing* rather than *knowledge*: knowledge is usually treated as static, explicit, and objective, whereas what is described as knowing is seen as dynamic, tacit, and contextualized (see Adler, 1998; Ponte, 1994). However, the alternative views outlined above foreground aspects that might contribute further to the discussion of knowledge versus knowing. First, whereas knowledge has been debated as either existing independently of the knower (the realist viewpoint) or only existing in the mind of the knower (the relativist viewpoint), with the term knowing, we can signal the inseparability of knowledge and knower. That is, it makes no sense to talk about something being known without also talking about who knows it (and under which circumstances). Second, what is called knowledge is usually perceived as a state of being (or product), whereas what is described as knowing is seen as an emergent process—a process of becoming. However, this is not a call for a distinction between product and process, since the main point is seen in the complex dynamics underpinning the stability of established knowledge (see Davis & Simmt, 2006). It implies the dynamic character of knower, knowledge, and context such that all three are changing and evolving over time. This means knowing is not just situated in place—that is, it is contextual and embedded in the practices of teaching (Adler, 1998)—but also situated with respect to time and other factors, given that the context of knowing is similarly dynamic and changing over time. That knowing is situated with regard to time, place, and other factors implies that it cannot be reduced to some observable and measurable by-products. The whole venture is to understand mathematics teacher knowing as it is, as it comes into being, as it works in the immediate context; that is, to take a holistic (rather than a reductionist) view that acknowledges mathematics teacher knowing as highly personal, embodied, enacted, and performed. Any approach toward what makes teacher knowledge specialized must deal with this complex whole rather than with piecemeal facets or types of knowledge (see Beswick, Callingham & Watson, 2012).[3] Of course, such sensibilities are not entirely new. They might be argued to have been represented in the discourses of different movements of thought such as cognitive approaches and situated approaches (see Kaiser et al., 2017), as well as other discourses. However, the view advanced here takes the discussion to realms that often cast knowing and knowledge as oppositional.

On the other hand, and more importantly, the alternative viewpoints converge on the understanding that it is not a kind of knowledge but a *style of knowing* that accounts for specialization in mathematics teacher knowledge. To elaborate this aspect in more detail: In the past, the focus was primarily on knowledge about/of/for/in the discipline. This resulted in multiple descriptions and distinctions, such as knowledge about mathematics versus knowledge of mathematics, or mathematical knowledge for teaching as opposed to mathematical knowledge in teaching, and knowledge for teaching mathematics in contradistinction to knowledge in teaching mathematics, all primarily concerned with the question of 'what' mathematics teachers know. In this regard, comparisons such as mathematics teachers versus mathematicians or mathematics teachers versus teachers of other subjects were assumed to be decisive, as it was believed that it was the kind of knowledge—whether quantitatively or qualitatively different—that set mathematics teachers apart from other professionals. However, the alternative views discussed above consider the yet unsettled question of 'how' teachers' knowing comes into being rather than pointing to the question of 'what' teachers know. This brings to the fore the complex, dynamic usage, function, and interaction of mathematics teacher knowing, a position that goes beyond accounts that primarily

[3] Notice that we do not construe the relationship between knowing and knowledge as contradictory but rather as dialectical. In terms of the onto-semiotic approach, there is no mathematical practice without objects, or objects without practice, which is equivalent to the issues of knowing and knowledge discussed here.

address kinds of teacher knowledge. We intend to enunciate this shift in perspective by calling for attention to mathematics teachers' *styles of knowing* rather than merely teachers' *kinds of knowledge*. We believe that this shift in perspective is critical as it provides a new light on the discussion of the nature of mathematics teacher knowledge that allows us to better integrate knowledge and action. It articulates mathematics teacher knowledge more as a mindset rather than as some static traits or dispositions. To cast this idea in a term, we suggest a fine distinction in thinking about the issues under consideration: knowledge about/of/for/in a discipline and *disciplinary knowing*. Knowledge about/of/for/in the discipline prompts the question of different *kinds of knowledge*, while disciplinary knowing prompts the question of a *style of knowing* that is a function of particular activities, orientations, and dynamics recognizably disciplinary. From this perspective, we argue that it is *mathematics educational knowing* that signifies specialization in mathematics teacher knowledge.

Conclusions

Mathematics teacher knowing is a mysterious phenomenon indeed. To acknowledge this mystery is not to mystify mathematics teacher knowing, but to express our recognition of the exquisite complexity of how mathematics teacher knowing comes into being. Breaking up the complex nature of teacher knowledge for the sake of insights leads to atomizing our understanding, our thinking, of what makes mathematics teacher knowledge specialized. Such insights are themselves fragmented, not holistic. The piecemeal, atomistic, analytic approach (as advocated in the past) does not work in relation to the complex usage, function, and interaction of teacher knowing. Any approach toward what makes teacher knowledge specialized must deal with the complex whole rather than with some piecemeal facets or types of teacher knowledge.

In this paper, new avenues for theoretical reflection on some of the major orientations and tendencies in the field of mathematics teacher knowledge were outlined. These reflections were not intended to exhaust the object of consideration, but to include those approaches, initiatives, and theoretical insights that might prompt re-thinking about what mathematics teacher knowledge signifies.

We explained that the question of what makes teacher knowledge specialized cannot be comprehensively answered by only addressing "what" teachers know, but we need to account for "how" teachers' knowing comes into being. The alternative views discussed in the paper bring to the foreground that it is not a kind of knowledge but a style of knowing that accounts for specialization in mathematics teacher knowledge. Such style of knowing is not a state of being but a process of becoming—the becoming of a mathematics educational mindset.

This call for a style of knowing is rather different from what normally receives emphasis in discussion of mathematics teacher knowledge. We hypothesize that considering specialization as a style of knowing (rather than a kind of knowledge) can have far-reaching consequences not only for conceptualizing mathematics teacher knowledge.

With respect to mathematics teacher education programs, for instance, considering specialization as a style of knowing (rather than a kind of knowledge) advocates a holistic approach to mathematics teacher education, criticizing the separate acquirement of different kinds of knowledge (generally acquired from different academic departments such as mathematics, education, psychology, among others). Mathematics teacher education programs should be deliberately designed in an integrated fashion to support teachers in blending insights from various disciplines including, but not limited to, mathematics, education, and psychology, thereby creating novel styles of knowing that empower teachers to reshape the way they view their own profession. It is reasonable to assume that such styles of knowing develop gradually, rooted in authentic activities and within a community of individuals engaged in inquiry and practice (see Putnam & Borko, 2000). Further, a shift toward a style of knowing is expected to affect researchers' and educators' perceptions of teachers' professional identity, as the path to a mathematics educational mindset is a journey, not a proclamation. This would mean giving up deficit-oriented discussions on teacher knowledge in terms of identifying and fixing teachers' lack of knowledge (Askew, 2008). The central concern for future research, then, is to understand those mindsets, which underpin any authentic form of mathematics educational knowing. It is hoped that this call for a style of knowing offers a new vision of what makes mathematics teacher knowledge specialized.

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5 Going Beyond Intuitive Models of Teacher Noticing: Toward Emerging Theoretical Perspectives*

*This chapter refers to the third journal article, entitled "Teacher noticing: enlightening or blinding?" by T. Scheiner, published 2016 in *ZDM Mathematics Education*, *48*(1-2), 227-238. (doi: 10.1007/s11858-016-0771-2)

The focus in this chapter is expanded to the theoretical construct of teacher noticing, which has the potential to challenge reductionist assumptions that allowed teacher cognition and teacher performance to be parsed. The third article (Scheiner, 2017b), found in a selection of papers in a special issue of ZDM Mathematics Education, provides a commentary on the theoretical formulations and usage of the construct of teacher noticing. It is argued that current conceptualizations of teacher noticing are inadequate, as they seem to treat the phenomenon as self-evident in meaning. Furthermore, the phenomenon is sometimes regarded as an *explanans* (explanation of a phenomenon), rather than the *explanandum* (a phenomenon that needs to be explained) that it actually is. The contribution made in Scheiner (2016a) problematizes the theoretical construct of teacher noticing, drawing on phenomena discussed in and findings derived from cognitive science to challenge some intuitive assumptions on teacher noticing that restrict current conceptualizations of the phenomenon. The construct of teacher noticing is then compared and contrasted to the construct of situation awareness, a construct originating in the applied science of human factors that points to critical issues that seem to have been oversimplified in current conceptualizations of teacher noticing. Furthermore, the article sets the stage for a more comprehensive model that speaks to the complex interactions of perceptual and cognitive processes in dynamic situations that can hardly be appreciated with previous conceptualizations of teacher noticing. Here the focus is on new theoretical perspectives that emerged in the discussion provided in Scheiner (2016b). These perspectives have the potential to enrich our understanding of interactions and interdependencies in the context of teacher noticing and improve our understanding of the complexities involved.

This chapter is structured in two parts. First, a critical stance is taken toward some intuitive assumptions underlying the phenomenon of teacher noticing by drawing on phenomena from cognitive science and the applied science of human factors. Second, a more comprehensive model is outlined that speaks to the inseparability of individual and environment, and provides a promising perspective with which to better interpret the complex interactions involved in teacher noticing.

5.1 Problematizing the Theoretical Construct of Teacher Noticing

Scholars in the field of teacher research in mathematics education are increasingly interested in and cognizant of the dynamic interactions between teachers' cognitive and contextual resources, teachers' noticing, and teachers' classroom practice. The literature contains numerous contributions on the topic of teacher noticing, despite the recent nature of the field (Jacobs, Lamb, & Philipp, 2010; Kaiser, Busse, Hoth, König, & Blömeke, 2015; König et al., 2014; Mason, 2002; Sherin, Jacobs, & Philipp, 2011; Star & Strickland, 2008).

There is a consensus amongst researchers that the phenomenon of teacher noticing is comprised of a set of activities, skills, and processes. Despite attempts to determine the specific elements of teacher noticing, there is still scope for making definitions of terms more precise, clarifying the relationship of terms with other terms, and discussion of their appropriateness. The terms used seem to borrow much from the lexicon of cognitive psychology, but are often based on restrictive intuitive understandings (for instance, the intuitive assumption that we 'see' what we direct our eyes to). Similarly, scholars often treat the relationships between activities that are part of teacher noticing as given, despite holding diverse views on the subject. Scheiner (2016b) problematizes some of the terms used and the assumptions underlying how these activities relate to one another.

More importantly, it is argued that the complex interactions of perceptual and cognitive processes in dynamic situations (such as classrooms) can hardly be appreciated with current conceptualizations of teacher noticing. At a time when researchers seem directed by, or confined within, intuitive frames that cannot fully account for the phenomena under consideration, other disciplines such as cognitive science may allow for a better understanding of the complexity involved in teacher noticing. Research on attention capture and inattentional blindness[1], for instance, indicate that noticing is much more complex than previously considered. Attention does not a priori lead to awareness; even if teachers attend to certain events (for example, by directing their eyes to them), it is not certain that they become aware of them. Hence research findings on attention capture and inattentional blindness seem to indicate that it is not only with their eyes that people see, but also their minds. 'Blindness', then, is more a product of the absence of expectation (or anticipation), knowledge, or beliefs, than a simple absence of attention.

Further, Scheiner (2016b) uses the construct of situation awareness[2], a concept well-discussed in the applied science of human factors, to stress that any conceptualization of teacher noticing needs to account for the relevance of a given event with regard to the context and time it is bounded by. Most conceptualizations of teacher noticing lack both acknowledgment of the bounded nature (both in time and space) of the environment and consideration of the relevance of an event. Dynamic environments, such as classrooms, require that a person's situation awareness be continuously updated. When teachers interact in such environments, the relevance of a given event will depend on its context, and may change with time. Another aspect of situation awareness is the ability to project forward from current events to forecast future events and understand the implications of decisions (which allows for timely decision making).

Current conceptualizations of teacher noticing do not allow for a full appreciation of the complexities involved in attending to and becoming aware of events in dynamic situations. In the following, a theoretical model is described that incorporates insights derived from cognitive science and the applied science of human factors, and might offer a further step to Sherin and Star's (2011) call for "work[ing] toward the development of a more complete model of how teachers make sense, in the moment, of complex classroom events" (p. 77).

5.2 Toward a More Comprehensive Model of Teacher Noticing

Attempts to theorize the relation between individual and environment are too often oversimplified, when in reality this relation is a complex interplay between cognitive and contextual resources, perceptual and cognitive processes, and the actual situation. Such attempts are usually based on the assumption that either the individual, or her or his environment, determines what she or he will see. The contribution made in Scheiner (2016b) attempts to overcome this false dichotomy by developing a theoretical model of teacher noticing that acknowledges the mutual and recurring interaction between an individual and an environment. In particular, the contribution draws attention to the inseparability of the teacher and the environment in which the teacher is engaged when addressing issues such as perceiving, interpreting, and decision-making, amongst others. The approach taken in the article extends Neisser's (1976) perceptual cycle model that includes both top-down and bottom-up processes, and blends constructs and insights from cognitive science and the applied science of human factors.[3]

The model proposed in Scheiner (2016b) suggests (a) that perception and cognition reinforce each other and (b) that individual and environment are inseparable (see Figure 12). That is, how one perceives the world is very much a function of how one understands the world, and vice versa. This is not to say that reality is purely imagined, but rather that perception is filtered through a person's understanding of the world, which is affected by her or his knowledge, values, and intentions. Sherin and Star (2011) proposed that "what the teacher sees in the world is strongly driven by knowledge and expectations" (p. 73), and Schoenfeld (2011a) argued that "what you attend to [...] is in large measure a function of your orientations" (p. 232). Hence, cognitive resources such as knowledge, values and intentions cause some aspects of the world to be more important than others, thereby creating an understanding of the world with a particular bias.

Even more importantly, these cognitive resources may direct further perceptual exploration by creating expectations of certain events. A person notices information in her or his environment that may modify or extend their cognitive resources, and shifts her or his attention elsewhere. This cycle of attention guidance enriches the person's understanding of the situation in which she or he is engaged. Thus, perception and cognition reinforce one another: the perceptual processes relevant to situation awareness are directed or influenced by cognitive resources, and the outcome of perceptual exploration, in the form of sensory input, modifies or extends cognitive resources. These modified cognitive resources direct further perceptual exploration, determining what information will be noticed next. Hence, perception is an active rather than passive process, imposing expectation on experience.

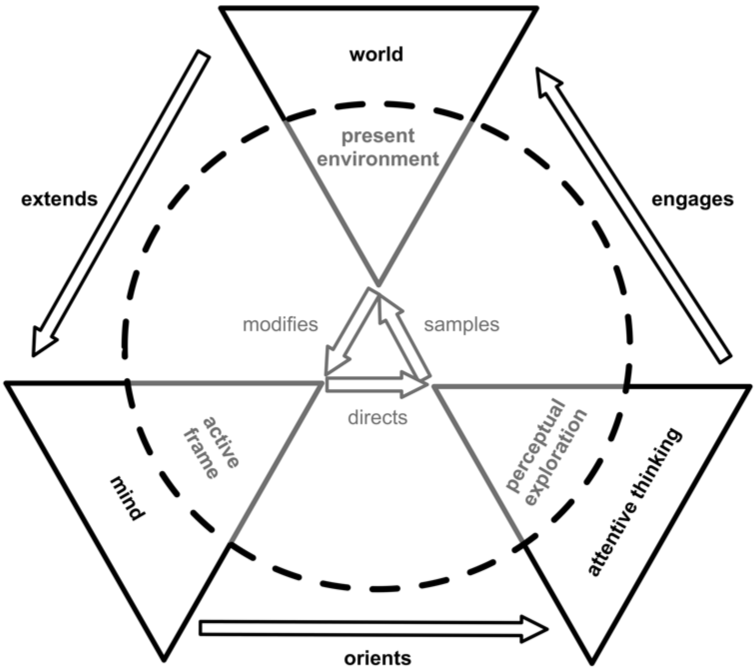

Figure 12: A cyclical model of the reciprocal relations of world, mind, and attention (modified from Scheiner, 2016b, p. 234)

Such a view advocates a position in which perceptual and conceptual processes are convoluted and oriented by one's conceptual frame – a frame that is cognitively affected, biologically enabled, and culturally infused. As such, mind orients one's attention to some aspects of the world, though attention is not so much an attempt of seeing or sensing anything, but rather *attentive thinking*, or attention as the direction of thinking (see Mole, 2011). In this respect, attention is intentional: it directs one's thinking to particular events in the world. The information of the world gained through attentive thinking, in turn, might extend one's conceptual frame, and, as such, shape one's mind in ways that direct one's attention to aspects of the world.

The interactive, reciprocal, and cyclical characteristic of the model outlined in Scheiner (2016b) provides a theoretical viewpoint that accounts for the interdependencies between an individual and her or his environment and a promising tool with which to interpret the phenomenon of situation awareness. Such a reciprocal and cyclical model between individual and environment – a model in which an individual is shaped by, but also shapes, the environment – provides a productive counterpart to any uni-directional model that is either based on the assumption that the individual determines what she or he sees, or that her or his environment determines it.

Scheiner's (2016b) model differs from Blömeke, Gustafsson, and Shavelson's (2015) more linear, unidirectional conceptualization of competence, in which competence was considered as "a continuum from traits (cognitive, affective, motivational) that underlie [...] perception, interpretation, and decision making that give rise to observed behavior in a particular real-world situation" (p. 11). In their conceptualization, situation-specific skills including perception, interpretation, and decision-making were considered as mediating the transformation of dispositions into practice. The model outlined in Scheiner (2016b) instead acknowledges a reciprocal and cyclical relationship between individual and environment that seems to be more aligned with Santagata and Yeh's (2016) suggestion that a knowledge of the specific context in which a teacher works is necessary to understand teacher competence (which is a complex interaction of situated knowledge, beliefs, and practices). Santagata and Yeh (2016) argued that it is through perception, interpretation, and decision-making that knowledge and beliefs become relevant in practice, and that these processes lay at the junction of knowledge, beliefs, and classroom practice. New knowledge and new beliefs (the enablers of changers in competence) are formed when decisions are made and interpreted based on practice. Thus, practice is a means of both refining perception, interpretation and decision-making, and of increasing knowledge and changing beliefs and therefore changing competence.

5.3 Reflections

Phenomena of perception and cognition are complex. Of course, this is not expected to be controversial nor to be ground-breaking. However obvious this may seem, it has had little impact on thinking about teacher noticing to date. Here the view was taken of teacher noticing as involving multiple, mutually influencing aspects of mind and world. A reciprocal and cyclical relationship was acknowledged between individual and environment, a relationship in which an individual is shaped by, but also shapes, the environment. The contribution made in Scheiner (2016b) argues against any uni-directional model that is either based on the assumption that the individual determines what she or he sees, or that her or his environment determines it. Scheiner (2016) puts forth a cyclical model that advocates a position in which perceptual and conceptual processes involved in developing situation awareness are directed by one's conceptual frame. The outcome of perceptual exploration and focused attention – the insights gained from the environment – modifies or extends the conceptual frame. Thus modified or extended, it directs further exploration and determines what will be noticed in the environment.

Current research on teacher noticing documents that over time teachers shift *what* they notice and *how* they talk about what they notice; however, the reason as to *why* this shift takes place remains elusive. The model proposed in Scheiner (2016b) enables one to generate a theoretical hypothesis with respect to this issue: any shift (or extension) of one's focus of attention is inspired by, and inspires, a shift (or extension) in one's conceptual frame. In the author's developmental research with pre-service secondary school mathematics teachers, he observed and documented shifts of teachers' foci of attention: from focusing on the deficits of student mathematical understanding to focusing on the productivity of student understanding. These shifts were quite remarkable and might be attributed to a mindset shift that has been encouraged by raising the pre-service teachers' awareness of their own understanding of student mathematical knowing and learning (specifically, suspending a fixed mindset that is primarily judgmental in order to encourage a growth mindset that is primarily based on understanding and interpreting student mathematical thinking).

5.4 Summary

In summary, this chapter identifies problems in current conceptualizations of teacher noticing, and draws on insights from cognitive science and the applied science of human factors to develop a more comprehensive account of the phenomenon of teacher noticing. In particular, the chapter highlights findings from research on attention capture, inattentional blindness, and situation awareness in order to challenge commonly held, intuitive assumptions about the phenomenon of teacher noticing. These insights are used to inform the development of a model of the phenomenon in which individual and environment are inextricably intertwined, one in which the individual both shapes and is shaped by her or his environment. In the next chapter, the contribution of this thesis and its wider significance is summarised.

Notes to Chapter 5

[1] Recent research into attention capture reveals a surprising degree of blindness to unusual events (that might be expected to capture attention). This phenomenon, known as 'inattentional blindness' (see Mack & Rock, 1998), consists of individuals failing to notice unexpected events directly in front of them when their attention is otherwise engaged. Inattentional blindness is particularly striking since it violates our intuition that people should see what is directly in their field of vision.

[2] Situation awareness is a construct used in the applied science of human factors to describe the level of awareness a person has of a situation being engaged in. Endsley (1995) described situation awareness as "the perception of the elements in the environment within a volume of time and space, the comprehension of their meaning and the projection of their status in the near future" (p. 36), and précised that situation awareness (SA)

> "is based on far more than simply perceiving information about the environment. It includes comprehending the meaning of that information in an integrated form, comparing it with operator goals, and providing projected future states of the environment. In this respect, SA is a broad construct that is applicable across a wide variety of application areas, with many underlying cognitive processes in common." (p. 37)

[3] A detailed discussion of Neisser's (1976) perceptual cycle model and its potential for better acknowledging the interactions between individuals' internal schemes, their perceptual exploration, and the situation in which they are engaged can be found in Scheiner (2016b). Here the focus is on the emerging theoretical model and how it may contribute to, and extend, the discussion on teacher noticing, giving primacy to the interdependencies between teacher and environment.

Article 3

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COMMENTARY PAPER

Teacher noticing: enlightening or blinding?

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Abstract This paper comments on the theoretical formulations and usage of the construct of teacher noticing in a selection of the papers in this special issue of ZDM Mathematics Education. The analysis of how the notion of teacher noticing is used in the papers suggests that it draws attention to several interdependencies involved that have not been attended to in the past. However, the contributions in this special issue have only partially accounted for the dynamic interactions in teacher noticing, suggesting that there is potential for enriching our understanding of the complexities involved in the realm of teacher noticing. The purpose of this commentary is to stimulate the current discussion on teacher noticing by providing insights from cognitive science and the applied science of human factors, which have the potential to challenge the current understanding of noticing. In doing so, the paper sets the stage for several related constructs from these research disciplines to raise awareness of aspects that recent conceptualizations of teacher noticing may have blinded rather than enlightened.

Keywords Attention · Perceptual cycle model · Situation awareness · Teacher cognition · Teacher noticing · Theory development

1 Introduction

Theoretical constructs are valued for their potential in advancing knowledge in a scientific discipline, in guiding research toward crucial questions, and in enlightening aspects that we otherwise had not seen or conceived.

> Constructs are not valued simply in terms of whether they are right or wrong; instead, they are valued by their usefulness to the field. Occasionally a construct emerges that transforms the field by enabling researchers to reconceptualize their endeavors and to shift, sometimes in subtle ways, the focus of their attention. (Sherin, Jacobs, & Philipp, 2011a, p. 3)

This potential has been attributed to the construct of *teacher noticing* by Sherin et al. (2011a); a construct that seems to cross the threshold of the mainstream of teacher research. Despite the relatively short time since teacher noticing has entered the vernacular of researchers and practitioners in mathematics education, there is actually quite a collection of contributions on the notion of teacher noticing (Jacobs, Lamb, & Philipp, 2010; Kaiser, Busse, Hoth, König, & Blömeke, 2015; König et al., 2014; Sherin, Jacobs, & Philipp, 2011b; Star & Strickland, 2008).

The papers in this special issue, taken together, offer a collection of important advancements of teacher noticing, and provide insight by presenting original approaches in integrating various research lines into the frame of noticing that may constitute a more advanced understanding of the observed phenomena. However, several papers seem to be guided by intuitive frames, speaking about teacher noticing as though its meaning were self-evident, or even treating teacher noticing as an explanatory construct for certain phenomena. As scientists, we cannot afford to be seduced by simple, intuitive, easy-to-understand answers. Instead we need to recognize that there may be more to this situation than meets our eyes. We need to recognize that teacher noticing is not an answer but a real and important question that invites us to enlighten (rather than blind) critical

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aspects in the field of teacher competence. This is exactly what is set as the goal for this commentary. In this paper, it is argued that there is room for further contributions in the process of indicating where to direct our eyes to—an opportunity for our field to question interactions and interdependencies in the realm of teacher noticing that we thought we understood.

This commentary examines the general contribution of various papers in this special issue. Yet, the very notion of selective looking (see Neisser, 1976) reminds us that we, as researchers, conduct research using a particular lens and that this focus has a bearing on what is noticed, that is, what is perceived and attended to, interpreted as significant, and ultimately reported. This use applies to commentators as well. The focus of this commentary will be on the usage of the construct of teacher noticing, and the postulated value of it in enlightening issues that otherwise have not been acknowledged. Then, some directions for future research will be developed by drawing on notions originating in cognitive science and the applied science of human factors that may allow us to see with greater perspective the complexities involved in the realm of teacher noticing.

This commentary aims at an interpretation and blending of several ideas gained from the various contributions to this issue with the goal of seeing profoundly and unconventionally into phenomena that are necessary to understand.

2 Teacher noticing: a critical and evolving theoretical construct

The merits of any body of research may be judged by how well it contributes to a current discussion and how well it represents an incremental advance in our understanding. Many papers in this special issue of ZDM Mathematics Education have certainly done so: they moved scholars in the field and advanced our understanding of many critical issues. Another way to judge the value of research is how well it contributes to seeing issues we thought we understood in a different way, how well it offers a critical redirection of existing views or provides a surprising advance in understanding, or even violates our intuition. The body of research considered in this paper has been evaluated based on these criteria. In this section, several contributions to this special issue that progressively advanced our field are highlighted. However, a more critical stance is also adopted in commenting on aspects that have been only partially considered and occasionally oversimplified.

The approach taken here will be more than usually assertional in the hope of raising issues provocatively. As the issues are deep and complex and simply cannot be elaborated in any great detail, they will mostly be defined rather than uncovered, explicated or settled.

In the following subsections, first a global focus is adopted on the ways in which contributions in this special issue enriched the emergent picture of teacher competence. Then the lens is focused on specific issues in the research on teacher noticing: what explicit and implicit assertions are made with regard to the various activities involved, their relation to each other, and how data has been analyzed.

2.1 Emerging insights in and new targets for research on teacher competence

Blömeke, Gustafsson, and Shavelson (2015) observed that, in the past, research on teacher competence focused primarily either on teacher dispositions in terms of cognition, affect, and motivation-volition or on teacher performance. In an attempt to overcome the ongoing tension in separating research on teacher dispositions from research on teacher performance, Blömeke et al. (2015) enunciated an integrated perspective articulating competence as a *continuum* of dispositions and performance. Blömeke et al. (2015) proposed to consider competence as "a continuum from traits (cognitive, affective, motivational) that underlie […] perception, interpretation, and decision making that give rise to observed behavior in a particular real-world situation" (p. 11). In this light, situation-specific skills including perception, interpretation, and decision-making were considered as mediating the transformation of dispositions into practice.

Dunekacke, Jenßen, Eilerts, and Blömeke (2016, this issue) supported this viewpoint on competence, arguing, based on their empirical findings, that special parts of knowledge and beliefs could predict preservice preschool teachers' perception and planning skills. Interestingly, when knowledge and beliefs have both been controlled, mathematical pedagogical content knowledge and application-related beliefs could predict the perception skills of prospective preschool teachers. Prospective preschool teachers' perception skills could then be used to predict their planning skills, while mathematical content knowledge was modeled as a precondition for mathematical pedagogical content knowledge.

On the other hand, Herbst, Chazan, Kosko, Dimmel, and Erickson (2016, this issue) made a case against a reductionist view of human action as only individual agency. They argued for going beyond the dominating account of the influence that individual cognitive factors have in decision-making by considering not only individual resources but also contextual resources. Herbst et al. (2016, this issue) hypothesized that decisions teachers make are "products of how individuals use personal resources to negotiate the demands of their institutional positions and the norms of the activities in which they play roles". They particularly paid attention to instructional norms and professional obligations as two sets of contextual resources that might help account for teachers' decision making. Similarly to the perspective proposed by Herbst et al. (2016, this issue), Lande and Mesa (2016, this issue) argued that not taking into account the working environment and other socio-cultural influences in understanding teacher action would be problematic. They argued that the societal and institutional contexts shape the role of teachers by establishing norms of professional behavior when individuals enact those roles and by defining obligations to which teachers respond. Lande and Mesa (2016, this issue) took a more ecological stance for understanding the work of mathematics teaching by recognizing that mathematics teaching is situated within classrooms (working environment), institutions (institutional environment), as well as social environments (society). In doing so, Herbst et al. (2016, this issue) and Lande and Mesa (2016, this issue) broadened the discussion on teacher competence by attending to both the psychological and socio-cultural influence and the interaction between them that may inform teachers' decision making.

To account for the influence of teacher communities on teachers' instructional decision making, Santagata and Yeh (2016, this issue) explicitly included communities in their conceptualization of teacher competence. These authors identified that the context in which teachers worked and other professional communities in which they engaged also served as lenses for attending to and interpreting their practices, and for making decisions. In their analysis consisting of a classroom video analysis survey, videotaped lessons, and post-lesson interviews, Santagata and Yeh (2016, this issue) came to a different conclusion than the view of competence Blömeke et al. (2015) suggested. Santagata and Yeh argued that perception, interpretation, and decision-making are at the center of the overlap of knowledge and beliefs with classroom practice. These situation-specific skills function as the processes through which knowledge and beliefs become relevant in practice. Conversely, the process of deliberately attending to, interpreting, and making decisions based on practice creates new knowledge and new beliefs, thus enabling changes in competence. Practice therefore functions as a means of refining perception, interpretation, and decision-making and of increasing knowledge and changing beliefs. This bi-directional relationship between knowledge, beliefs, skills, and practice differs from Blömeke et al.'s (2015) more linear, unidirectional conceptualization of competence. While Blömeke et al. (2015) proposed to consider competence as a continuum from dispositions to performance, Santagata and Yeh (2016, this issue) suggested considering teacher competence as a complex interaction of situated knowledge, beliefs, and practices that can be understood only in the specific context in which teachers work.

Overall, the merit of Santagata and Yeh's (2016, this issue) approach is the acknowledgement of the interdependence between an individual and the environment—an interdependence that surprisingly often remained unnoticed. Interactions between individual and contextual resources, situation-specific skills (such as perceiving, interpreting, and decision-making), and the environment have never been fully described in contemporary research, and often remain in the 'black box'.

2.2 Determining and defining activities in teacher noticing

The notion of teacher noticing has many faces, as previous contributions and the various contributions in this special issue revealed. Philipp, Jacobs, and Sherin (2014) asserted a range of conceptualizations of noticing in mathematics education. The same holds for many papers in this special issue. Descriptions of teacher noticing used in a selection of these papers are considered, such as Hoth et al. (2016, this issue), who used Kaiser et al.'s (2015) so-called PID-model comprising (a) *perceiving* particular events in an instructional setting, (b) *interpreting* the perceived activities in the classroom, and (c) *decision-making*, either as anticipating a response to students' activities or as proposing alternative instructional strategies, which is closely connected to the approach by Blömeke et al. (2015). Santagata and Yeh (2016, this issue) focused on (a) *attending* to the mathematics content at the center of the instruction, (2) *elaborating* on students' mathematical thinking and learning, and (c) *proposing improvements* in the form of alternative strategies teachers might adopt to enhance students' learning opportunities. These conceptualizations announce a variety of key activities: perceiving, attending, interpreting, elaborating, proposing improvements, and decision-making. These conceptualizations paint a picture fairly consistent with earlier approaches specifying activities involved in teacher noticing. For instance, Jacobs et al. (2010) conceptualized professional noticing of children's mathematical thinking as comprised of three skills: (a) *attending* to children's strategies, (b) *interpreting* children's understandings, and (c) *deciding how to respond* on the basis of children's understanding.

These contributions bring to the surface several critical activities (such as attending, interpreting, and decision-making) that allow the world to be seen in new and different terms. Although most authors tried to be quite specific in determining what the important elements of teacher noticing are, there is still room for making more precise the meaning of the terms used, clarifying how they are related to or differ from the ones used by other scholars, as well as for clarifying the appropriateness of their terms. The terms used in conceptualizing teacher noticing seem to bring into discussion much of the vocabulary of cognitive psychology, but apparently often based on intuitive, not necessarily appropriate, understanding. Almost all the effort in pursuing the meanings of terms, their integrity and general utility is left to the theoretically reflective reader. Section 3 provides a point of departure in thinking about the concern of perceiving and attending.

2.3 Relating activities involved in teacher noticing: continual, sequential, or interactional?

Currently researchers agree that teacher noticing is seen as a set of various activities, skills, or processes; however, they differ not only in the terms used but also in their assumptions of how these activities might be related to one another. Several scholars made explicit or implicit assertions concerning the relation between the various activities attributed to the construct of teacher noticing. Although these assertions were not the focus of their papers, they are important as they highlight a diversity of views about relationships which otherwise may be thought of as self-evident.

For instance, Bruckmaier, Krauss, Blum, and Leiss (2016, this issue) specified that "although the teachers investigated in the COACTIV video-study obviously had to perceive and interpret the video stimuli […], only the resulting *final continuation* ("decision") was assessed". The term 'final continuation' causes some kind of confusion. It raises the question of how something can be 'final' when it 'continues'. Is perceiving considered as one pole of a spectrum, and decision making as the other pole? In any case, this formulation carries the connotation that activities are ordered. One might think that the authors think in terms of a linear order or hierarchical order, or even that the various activities are embedded in one another. The point is that Bruckmaier et al.'s (2016, this issue) assertion allows much room for speculation. Santagata and Yeh (2016, this issue), on the other hand, hypothesized a "*cyclical process* of perception, interpretation, and decision making". The difference between a linear (or hierarchical) process and a cyclical process is that the latter implies an on-going process. Pankow et al. (2016, this issue) referred the identification of typical students' errors "to the first *phase* of noticing, namely the perception and anticipation of important classroom incidents". In doing so, they explicated that, in their opinion, noticing consists of several ordered phases, the first being the anticipation and perception. Similarly, Hoth et al. (2016, this issue) mentioned with regard to the PID-model that perception, interpretation, and decision-making are *phases*, whereas Dunekacke et al. (2016, this issue) hypothesized perception, interpretation, and planning action as being *steps*. One might think, based on these statements, that these activities take place sequentially or successively.

The diverse views presented in this special issue show that the relationship between the various activities is non-obvious. Interestingly, almost all mentioned papers treated the issue as given, considering the various activities as phases or steps in a continuum or in a cycle, among others. Yet reasonable clarity regarding how the activities are related to one another is still missing. Dyer and Sherin (2016, this issue) take a different stance, explicating that they do not mean to suggest that a teacher first develops an interpretation of student thinking and then reasons about it. Instead they propose a more dynamic relationship between the two processes. Their model of the way teachers make sense of student thinking treated interpretations and instructional reasoning as working in conjunction with one another, and could be iteratively revised and used flexibly. Similarly, Sherin et al. (2011a) suggested considering 'attending' and 'making sense' as "interrelated and cyclical" (p. 5). Based on empirical grounds, Dunekacke et al. (2016, this issue) stated a strong relation between perception and planning, indicating that the two activities cannot be distinguished empirically; however, despite their empirical finding, the authors suggested distinguishing between the two 'categories'—both in theory and in practice. This, obviously, raises more questions than it provides answers.

The argument is that more often neither theoretical nor empirical contributions justified the deduction and confirmation of the postulated relationship of the activities involved in teacher noticing. However, we need to be cautious about deducing the relational nature of the activities in order to avoid the risk of blinding the complexities involved. Section 5 provides the target to problematize the complexities involved more profoundly.

2.4 Theoretical and methodological issues in research on teacher noticing

Discussions of teacher noticing in this special issue have acknowledged the importance of theoretical frames in bounding problems of consideration. Bounding allows us to identify, from the many potential dimensions and interactions among dimensions that could be identified with a phenomenon, those aspects to which researchers should attend. Theoretical frames tell which details are relevant.

In many papers of this special issue, the theoretical frame of teacher noticing has been taken as a tool for analyzing the data that often took the form of teachers' comments (or responses) on classroom events: viewing video vignettes of classroom events (Bruckmaier et al., 2016, this issue; Dunekacke et al., 2016, this issue; Hoth et al., 2016, this issue) or drawing on teachers' own teaching in classrooms (Dyer & Sherin, 2016, this issue; Jacobs & Empson, 2016, this issue; Santagata & Yeh, 2016, this issue). To analyze the data, researchers often coded these comments, placing them either into categories (Bruckmaier et al., 2016, this issue; Kersting et al., 2016, this issue) or identifying new categories (Jacobs & Empson, 2016, this issue; Hoth et al., 2016, this issue). Sherin and Star (2011) reminded us that

> When we say that teachers are 'attending to pedagogy' in their comments, we are saying only what their comments are *about*, from a researcher's point of view, not what they were perceiving. […] These meters [coherent or topic meters] tell us something about emergent features of teacher reasoning. But they do not, in any direct way, tell us anything about the underlying noticing machinery that produced those emergent features. (p. 76, italics in original)

Kersting et al. (2016, this issue) concluded their contribution with the observation that a fundamental challenge is that our theoretical advances are limited by our measures and our measures are limited by our theoretical understanding. Thus, it is not surprising that we have focused our attention on the seemingly most observable aspects in teacher noticing, and that numerical scales have become the dominating measure in teacher noticing. However, quantitative instruments that symbolize teacher' noticing with a number on a scale provide a general orientation for, but fall short of, explaining phenomena of modest complexity. Inherent in a number system is an implication of a unidimensional continuum on which values (points) differ in degree rather than in kind. The use of an overall score for various dimensions or activities involved in teacher noticing (Bruckmaier et al., 2016, this issue; Santaga & Yeh, 2016, this issue), while a useful starting point, does not fully represent the phenomena being studied. As a measure of the extent to which teachers demonstrate the abilities defined by each rubric, the use of an overall score is justified. However, such a measure does not capture the interactions of activities and possible relationships between the dimensions being explored, thus omitting some qualitative detail.

The utility of Kersting et al.'s (2016, this issue) speculation that summating individual scores teachers obtained in various categories allows an interpretation in terms of a knowledge system perspective is unlikely. A knowledge system perspective is of value to provide insights in a structural description of teacher knowledge that accounts for the interactions of knowledge elements, the complex nature of the organization of the knowledge system, the dynamic and fluid nature of knowledge activation, and its non-linear development, amongst others (Scheiner, 2015). An overall score as a measure for the complexity and dynamics of a knowledge system is of limited value.

3 Looking at the black box: on vision and blindness

In the field of teacher noticing, we are guided by, or trapped in, intuitive frames that run the risk of blinding critical issues. As mentioned in Sect. 2.2, this becomes obvious with respect to perceiving and attending: Santagata and Yeh (2016, this issue), for instance, explicated that they used the terms attending and perceiving interchangeably. This may be grounded in the assumption that what we perceive we do attend to, and what we attend to we do perceive. Conversely, several scholars in cognitive psychology and cognitive science have clarified that not all perceived stimuli are attended, and not all attended stimuli are perceived (see Baars, 1997; Lamme, 2003). To illustrate this issue, in drawing reference to Lamme (2003), it is argued that we have various levels of processing that a stimulus can reach: unperceived or perceived, unattended or attended, and implicitly attended (without awareness) and explicitly attended (with awareness).

However, it is found that only in perceived stimuli that are attended and have the potential to be explicitly attended is there evidence of awareness (see Fig. 1). With this in mind, attention selects certain stimuli of a perceived scene for detailed analysis, while perception goes to build up a certain visual experience. Neisser (1976) clarified: "[o]nly the attended episode is involved in the cycle of anticipations, explorations, and information pick up" (p. 87), that is the way of gaining access to awareness. Thus, it is reasonable that Most, Scholl, Clifford, and Simons (2005) stated that "[p]erception is impoverished without attention" (p. 218). The central claim here is that attention is to be considered as selecting stimuli perceived in a scene but also as creating access to awareness. This is important as I believe that ultimately, awareness of the situation is all that matters in a teaching–learning situation. Simons (2000) argued that

> In most real-world settings, the critical question of interest is not whether an object will implicitly affect performance, but whether it will explicitly capture attention and reach awareness, thereby allowing us to modify our behavior and select new goals. Although, much, if not most, of perception and performance occurs without awareness, we feel that when salient events occur, we should become aware of them so that we can intentionally change our behavior. (p. 150)

Recent research on teacher noticing (including the many papers in this special issue) productively investigated *what* a teacher did or did not 'see', and whether a certain event affected a teacher's behavior; however, research is needed

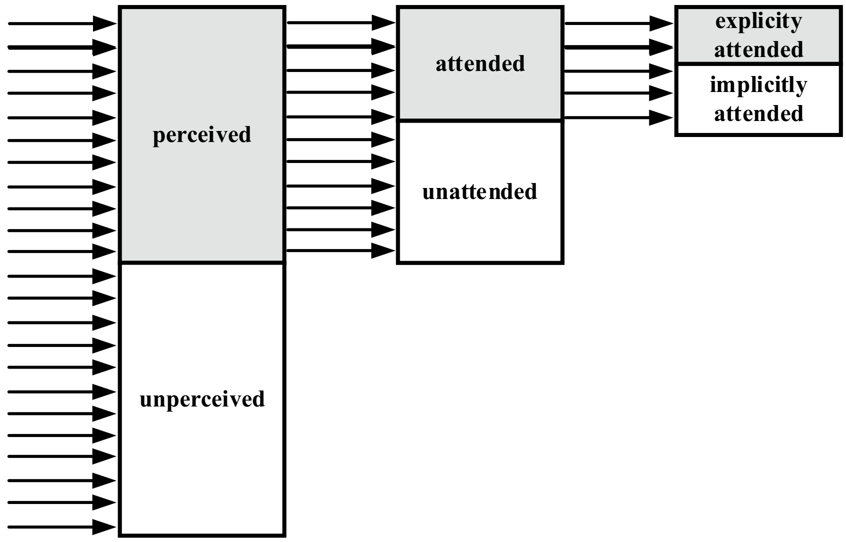

Fig. 1 Perceiving-attending-explicitly attending as the gateway to awareness (modified from Lamme, 2003, p. 13)

in determining the question of *why* a teacher did or did not 'see' a particular event.

At a time when it seems we are guided by, or trapped in, intuitive frames that are of limited explanatory power, we may turn to other research lines from cognitive science such as attention capture and inattentional blindness that may bring to light yet unaddressed issues in the teacher noticing literature: (a) how and why teachers tune into particular events and, at the same time, may remain sensitive to other important events; and (b) how and why different types of attention shifts do or do not give rise to awareness.

3.1 Setting the stage for attention capture and inattentional blindness

Research on attention capture showed that events that have been found to capture attention implicitly might not also capture awareness. Simons (2000), therefore, distinguished between instances in which events affect performance without necessarily impinging on awareness (*implicit attention capture*) from instances in which there is evidence of awareness (*explicit attention capture*). Recent studies of explicit attention capture reveal a surprising degree of blindness to unusual events that might be expected to capture attention. This blindness, known as *inattentional blindness* (Mack & Rock, 1998), is a phenomenon in which individuals fail to notice unexpected events appearing in front of their eyes when their attention is otherwise engaged. Inattentional blindness is particularly striking since it violates our intuition that people should see whatever they direct their eyes to (Mack & Rock, 1998). Several inattentional blindness experiments (see Most et al., 2005; Simons & Chabris, 1999) indicated that, although being engaged in a certain situation, a person may not necessarily explicitly attend to critical elements taking place in the situation. This phenomenon of inattentional blindness has been explained by Neisser (1976): that a person's own *expectations* (or anticipatory schemas) of what belongs in a scene determine where and how attention is directed.

In the following a way is described in which the discussion on teacher noticing may be productively extended, which accounts for a crucial, yet often unaddressed, issue: the teacher's awareness of the situation in which she or he is engaged. In doing so, the notion of situation awareness will be presented, a concept particularly important in the applied science of human factors.

4 Opening the black box: on attention and awareness

Implicit attention capture research and inattentional blindness research have illuminated different processes relevant to the noticing of critical objects. Important insights about the mechanisms of attention shifting can be drawn from the study of implicit attention capture that has focused primarily on measuring effects of certain events on task performance; however, it is still of limited practical value for research on teacher noticing since it primarily explored how well observers can ignore something they expect but know to be irrelevant. Ordinarily, the density of critical events taking place in the classroom raises a different question: how likely are teachers to notice something potentially relevant that they do not expect? Inattentional blindness research has been exploring this question, providing reviewed evidence that, quite often, unexpected events fail to capture attention. However, the literature on inattentional blindness has yielded only limited insights into the factors that determine whether an unexpected event in a dynamic scene captures awareness. This naturally raises the question of what accounts for developing and maintaining awareness of relevant events in a complex and dynamic situation like the classroom setting.

Most et al. (2005) recognized that the distinction between implicit and explicit attention capture reflects a "fundamental paradox concerning the nature of attention" (p. 218):

> On one hand, people engaging in challenging tasks must often maintain focus, effectively ignoring irrelevant information that might distract them from their goal. [...] On the other hand, attention must be distractible; if potentially dangerous or behaviorally relevant objects appear, they should divert cognitive resources. [...] A complete explication of attention must incorporate both these seemingly conflicting requirements (Allport, 1989). (Most et al., 2005, p. 218)

The same authors suggested theoretically bridging these two research fields by illuminating mechanisms of awareness and by "shifting the emphasis of the field from demonstrations of perceptual failure to investigations of factors underlying successful noticing" (Most et al., 2005, p. 237). This theoretical bridging of attention capture and inattentional blindness may be achieved by drawing on Neisser's (1976) perceptual cycle model that is discussed in Sect. 5. In accounting for the relation between attention and awareness, the stage will be set for the construct of situation awareness, a notion presented by scholars in the applied science of human factors that is highly relevant for the construct of teacher noticing.

4.1 Setting the stage for situation awareness

Situation awareness is the term used within the applied science of human factors to describe the level of awareness that a person has of the situation she or he is engaged in. Over the past two decades, the construct has become a fundamental theme within the human factors research community and has received considerable attention across a broad range of contexts, including aviation, air traffic, power plant operations, emergency services, and aircraft piloting, from whence the term originated. These contexts share many characteristics including "dynamism, complexity, high information load, variable workload, and risk" (Gaba, Howard, & Small, 1995, p. 20).

The human factors community has not settled on a single definition, or description, of situation awareness, but the most acknowledged one was given by Endsley (1995):

> Situation awareness is the perception of the elements in the environment within a volume of time and space, the comprehension of their meaning and the projection of their status in the near future. (p. 36)

Inherent in this description are three processes: First, it involves *perceiving* "the status, attributes, and dynamics of relevant elements in the [surrounding] environment" (Endsley, 1995, p. 36). This echoes scholars' understanding, working in the field of teacher noticing, arguing that a teacher must first be able to gather perceptual information from the environment, and, then, be able to selectively attend to those elements that are most relevant to the task at hand. Similarly to teacher noticing, situation awareness as a construct goes beyond mere perception. It also encompasses *comprehending* the current situation, which allows an individual to interpret its relevance in relation to the individual's task and goals. At first glance, one might argue that scholars working in the field of teacher noticing have stressed this issue in the same, or a similar, way. Of course, it echoes the main activity of "making sense of events [...] [that is] teachers necessarily interpret what they see, relating observed events to abstract categories and characterizing what they see in terms of familiar instructional episodes" (Sherin et al., 2011a, p. 5). However, comprehending means not only to "form a holistic picture of the environment" but also to determine the "significance of [...] elements in light of the pertinent operator goals" (Endsley, 1995, p. 37). This aspect places situation awareness squarely in the realm of ecological realism (Gibson, 1986). Situation awareness also includes the ability to *project* from current events and dynamics to forecast future situation events (and their implications). This ability to predict future events allows for timely decision-making and therefore seems to be of particular importance given the dynamic nature of the situations in which teachers are engaged. It is this aspect that sets situation awareness apart from teacher noticing. One might observe that the construct of situation awareness is similar to teacher noticing but uses different terms; however, a small shift in orientation might make a big difference in the contribution of our research to addressing important issues. For instance, Endsley's (1995) account of "within a volume of time and space" (p. 36) contained in the description of situation awareness points to a critical, yet often only implicitly assumed, aspect in the discussion on teacher noticing: the fact that the state of awareness of some environment is bounded in time and space. As environments change over time, the dynamic nature of situations (e.g., the ever-changing classroom situation) dictates that the person's situation awareness must be constantly maintained and kept up-to-date. Conversely, since people interact with the environment, a person constrains parts of the situation that are of interest to her or him. Thus, time and space become critical concerns in an individual's situation awareness. Attempts to define the essential components of teacher noticing in general suffer from the fact that, given the dynamic environment in which teachers are engaged, the relevance of events depends on the context, and will vary from time to time. Any conceptualization of teacher noticing needs to account for the relevance of a given event with regard to the context and time it is bounded by.

It is important to explicate that situation awareness is viewed here as theoretically distinct from decision-making, rather than as a single combined construct as many scholars in this special issue suggested with regard to teacher noticing. The argument made is that this distinction is important and real both in terms of models of human information processing and characterizations of dynamic systems (Endsley, 2000). Poor decisions may be made despite a high level of situation awareness for a variety of reasons, such as limited decision choices, lack of experience in similar situations, or unsuitable strategies guiding the decision-making process. Similarly, good decisions may occur despite low or absent situation awareness, particularly if decisions are affected by automaticity of cognitive processes. However, this distinction is not meant to dispute the significance of situation awareness in the decision-making process or the essential link between situation awareness and decision-making in many instances. On the contrary, in highly complex and dynamic environments, situation awareness and decision-making are necessarily highly interactive: decision making is often shaped by situation awareness and situation awareness is often shaped by decision making.

5 Looking inside the black box: on interdependencies between individual and environment

The complex interactions of cognitive and perceptual processes and activities in dynamic situations (such as classrooms) have never been fully described in research on teacher noticing, leaving many aspects of their interdependencies in the 'black box', unseen by researchers and educators and often understood only in isolation. This section intends to provide a first step towards a more comprehensive understanding of the interactions involved. As mentioned above, a more comprehensive stance for understanding attention and awareness may be achieved by blending various insights from cognitive science (attention capture and inattentional blindness) and the science of human factors (situation awareness). In framing this blending, the formulation is drawn on Neisser's (1976) perceptual cycle model that accounts for the interaction between an individual and an environment. Interestingly, other scholars have already been taking advantage of Neisser's (1976) perceptual cycle model in relating research on attention and awareness. In cognitive science, Most et al. (2005) utilized Neisser's perceptual cycle in theoretically bridging attention capture research and inattention blindness research. In the applied science of human factors, Adams, Tenney, and Pew (1995) and Smith and Hancock (1995) brought Neisser's model into the discussion on situation awareness.

5.1 Setting the stage for Neisser's (1976) perceptual cycle model

Neisser (1976) proposed an information-processing model that accounts for the interaction between a person's internal schemas (or mental models), the perceptual exploration, and the situation in which the individual is engaged. Neisser (1976) explicated that "[p]erception and cognition are usually not just operations in the head, but transactions with the world. These transactions do not merely inform the perceiver, they also transform him [or her]" (p. 11, italics in original). The model differs from linear models of information processing by acknowledging a reciprocal and cyclical relationship between a person and an environment. To concretize this position, Neisser's perceptual cycle model (see Fig. 2) suggests that perception is influenced and directed by a person's existing knowledge. This means existing knowledge (in the form of mental models or schemas) may lead to expectations or anticipations of certain events that in turn serve as the vehicle for perceptual exploration. As such, a person samples or picks up information available in the environment that may serve to modify and update schemas, and in turn shifts her or his attention to other critical elements in the environment. This cycle of attention guidance continuously enriches the emerging representation of the situation.

The perceptual cycle model may provide the key to unlock the black box of the complex interactions involved in developing and maintaining situation awareness. As such, the perceptual cycle model offers a promising theoretical perspective

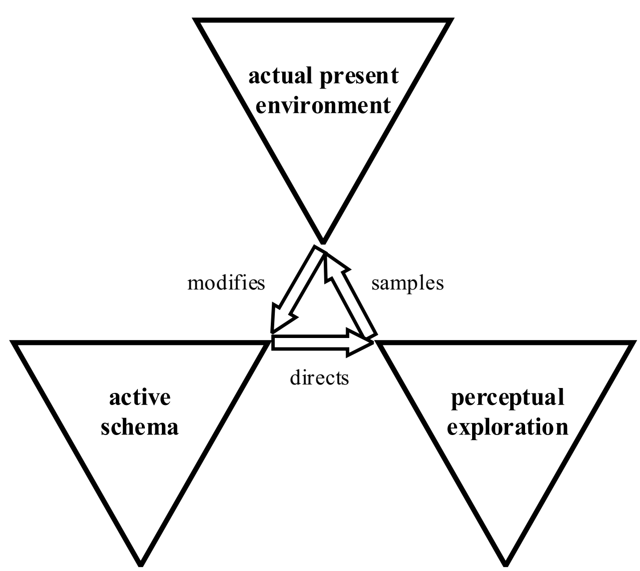

Fig. 2 Perceptual cycle model (adapted from Neisser, 1976, p. 21)

to account for the interdependencies between an individual and an environment in the process of situation awareness. It is a central thesis of this paper that the interactive nature of the perceptual cycle model is persuasive in explaining the dynamic aspects in developing situation awareness. We may argue that, according to the perceptual cycle model, situation awareness emerges through temporally recurrent and active engagement with the environment. Certain elements of the environment do not leap into awareness on initial attention engagement. Rather, the reciprocal and cyclical process proposed by Neisser (1976) is crucial in developing and maintaining an awareness of the actual situation.

It should be explicated that in drawing on Neisser's perceptual cycle model, perception, comprehension, and projection are neither considered as being cyclically related with each other nor as particular phases of the cycle as some scholars in this special issue assumed with regard to activities involved in teacher noticing (see Sect. 2.3). It is unlikely that a teacher sequentially perceives all elements of a situation, then interprets and understands their relevance in relation to her or his task and goals, and then predicts future situation events. In contrast, in naturalistic settings, it is more likely that perceiving, comprehending, and projecting take place concurrently (rather than successively) and are interwoven (rather than separated), and each of these processes apply to the entire cycle. Still, the question of how perception, comprehension, and projection interact remains unanswered. To draw this issue back to teacher noticing, 'attending' and 'making sense' (Sherin et al., 2011a) are, from this point of view, not to be considered as separated but rather interwoven and do not take place successively but concurrently. As Towers and Davis (2002) once indicated: "what we notice is completely framed by what we know. Perception and conception are inextricable. An event of noticing is always and already an event of interpretation" (p. 318).

6 Discussion

Research on teacher cognition and teacher decision-making has mainly focused on constraints internal to the human mind. A real value of the theoretical construct of teacher noticing is to draw attention to the inseparability of individual and environment when addressing issues such as perceiving, interpreting, and decision-making, amongst others. Gibson (1986) referred to this as the challenge of ecological validity. Teacher noticing as a theoretical framing calls attention to a lesson that Gibson (1986) tried to teach long ago: the correspondence between perception and action, and the demands of the environment.

The growing interest in teacher noticing illustrates that scholars in the field of teacher research in mathematics education are coming to recognize the dynamic interactions between teachers' cognitive and contextual resources, teachers' noticing, and teachers' classroom practice. The growing appreciation for these interactions can be seen in several papers in this special issue (Dyer & Sherin, 2016, this issue; Herbst et al., 2016, this issue; Jacobs & Empson, 2016, this issue; Lande & Mesa, 2016, this issue). These contributions make it clear that attention needs to be drawn to the complex interactions involved. In this light, teacher noticing is a theoretical construct that challenges the reductionist assumptions that permitted parsing of teacher cognition and teacher performance.

However, attempts to account for an individual attending to specific issues and becoming aware of them have too often been oversimplified. They are usually based on the assumption that either the individual herself or himself determines what she or he will see, or else her or his environment determines it. We overcome this false dichotomy by using an information processing model that encompasses both top-down and bottom-up processes and that acknowledges the reciprocal and cyclical interaction between an individual and an environment.

The perceptual cycle model might be relevant to the current discussion on teacher noticing for several reasons, including: (1) the model accounts for, and distinguishes between, attentional orienting and active, extended attentional engagement with the environment; and (2) the interactive, reciprocal and cyclical characteristic of the perceptual cycle offers a promising tool to interpret the dynamic aspects involved in situation awareness.

In more detail, the perceptual cycle model distinguishes between an orienting response and the more extended processing necessary for subjective awareness. That is, transient shifts of attention can be relatively automatic, but sustained shifts often involve significant cognitive resources. The question naturally arises as to what determines whether a transient shift is followed by sustained allocation of attention. Neisser (1976) proposed that a person's own expectations of what belongs in a scene determine how sustained attention is directed, stating that: "Because we can see only what we know how to look for, it is these [anticipatory] schemata (together with the information actually available) that determine what will be perceived" (p. 20). Similarly, Sherin and Star (2011) specified that "what the teacher sees in the world is strongly driven by knowledge and expectations" (p. 73). In addition to an individual's knowledge and expectations, Schoenfeld (2011a) reminded us that "what you attend to […] is in large measure a function of your orientations" (p. 232). In this light, noticing takes place within the context of knowledge, beliefs, intentions, goals, expectations, and experiences, amongst others (in short, individual resources). However, these assertions do not suggest that individual resources and the environment are uni-directionally related but instead bi-directionally related: Perceptual and conceptual processes involved in developing situation awareness are directed by the individual resources, and the outcome of perceptual exploration—the information picked up in the environment—modifies the original individual resources. Thus modified, they direct further exploration and determine what will be picked up in the environment next (see Neisser, 1976).

Dunekacke et al. (2016, this issue) argued that perception and interpretation provide the basis to activate teachers' knowledge and to make meaningful decisions. This assertion sounds reasonable; however, it is only half of the equation. Research on attentional capture and inattentional blindness (see Sect. 3) highlighted the importance of considering the potential impact of activated schemas for perceiving certain events. This issue has been addressed by Pankow et al. (2016, this issue) taking account of the relation between anticipation and identification of typical student errors. In short, in order to address issues of the interaction between cognition, perception, and environment both sides of the equation must be considered: the potential impact of individual resources on perception, and the potential impact of perceived information on individual resources, and their activation.

It is a central proposition of this paper that managing the perceptual and conceptual processes that permit situation awareness involve, and are shaped by, not only significant individual resources but also contextual resources. This position draws on Herbst et al.'s (2016, this issue) and Lande and Mesa's (2016, this issue) account for both individual and contextual resources in informing teachers' decision making. Individual characteristics such as knowledge, beliefs, goals, experiences, and intentions have been identified as having an impact for instructional actions (Borko, Roberts, & Shavelson, 2008; Schoenfeld, 2011b). Schoenfeld's (2011b) insightful investigations of in-the-moment decision-making posited that an individual's resources (including knowledge), orientations (including beliefs), and goals are critically important determinants in what teachers do, and why they do so. That is, according to Schoenfeld, one must know another person's resources, orientations, and goals well enough to predict what she or he will do in a given situation. However, Neisser (1976) reminded us that even then we cannot be sure what another person will do if we have an incomplete understanding of the situation in which the person is engaged. This is not in contradiction to Schoenfeld's (2011b) assertions but emphasizes the perspective that "perception and behavior are controlled interactively […] depend[ing] on the individual as well as the environment" (Neisser, 1976, p. 186). In this light, it is reasonable that Herbst et al. (2016, this issue) argued for going beyond the dominating account of individual cognitive factors by considering contextual resources as well. Attending to both the individual and the environment allows us to examine how the environment might affect the individual, and vice versa.

7 Concluding remarks

This paper draws on phenomena described in and findings gained from cognitive science and the applied science of human factors in the hope of finding a foundation for better understanding critical issues that have too often been overlooked in research on teacher noticing. The motivation for doing so was that although the notion of teacher noticing shows great promise for merging various research lines in mathematics education, we do not have access to the complexities involved in the processes involved, from attending to certain events, to becoming aware of these events in dynamic situations. Though turning to insights gained from cognitive science and the applied science of human factors might be beneficial to go beyond an intuitive model of teacher noticing (Sherin & Star, 2011), we need to be cautious about their ecological validity since they may not necessarily be approximations to what ordinarily takes place in classrooms and in classroom interactions.

At first glance, the accounts given in this paper seem to make the matter more mysterious: We cannot be sure that teachers 'see' certain events, though they direct their eyes to them. Even if they attend to certain events, we cannot be sure they become aware of them. And, even when they became aware of the events, we cannot be sure that the decisions they make are reasonable. This seems to be true as far as it goes; nevertheless, there are congruencies that the insights presented and briefly discussed in this paper point to. The bigger picture converges to the understanding that it is not only our eyes with which we see but also our minds. Our 'blindness' results not so much from our absence of attention but from our absence of expectation (or anticipation), knowledge, or beliefs. Even more importantly, the bigger picture converges to the understanding that it is all about the interdependencies between individual and environment, or, in more detail, the interactions between cognitive and contextual resources, perceptual and cognitive processes, and the actual situation. Thus, in this paper, teacher noticing—or more appropriately teacher situation awareness and teacher decision making—is treated as a construct that gives primacy to the interdependencies between teacher and environment.

Therefore, an important lesson to be learned from the inquiry thus far is that we need to step out of intuitive frames that hide the complexities involved in teacher noticing. With the above-mentioned arguments in mind, we may argue that both attending and developing situation awareness are mindful and cultural processes; however, attention does not a priori lead to awareness. Attention selection results from the convolution of cognition and processing inputs from the environment, a convolution that takes place in a broader socio-cultural context. On the other hand, situation awareness requires recurrent interactions between an individual's cognitive and contextual resources, perceptual and conceptual processes, and the environment (including a broader, societal environment).

This more global orienting frame for discussions of teacher noticing allows us to rephrase the well-known slogan in research on teacher noticing "teacher noticing: seeing through teachers' eyes" to "teacher noticing: teachers' seeing with their minds' eyes" that takes place in continuous interdependence with the environment. Referring to the colloquial proverb by Richard Bach it can be formulated:

> Don't believe what your eyes are telling you. All they show is limitation. Looking with your understanding, find out what you already know, and you'll see the way to fly.

The same principle applies to this commentary: What is 'seen' in the assertions and arguments in this paper will depend not only on what was said in this paper but also on the reader's knowledge and beliefs prior to reading it.

Certainly, in any field as complex as teacher noticing is, it is difficult to develop deep theoretical understanding; however, we will not achieve this if we do not set our minds to it. The purpose of this paper was to do so by cultivating a theoretical perspective in research on teacher noticing by drawing on other research disciplines that may provide researchers and educators with useful insights into the complexities of an individual's attentional engagement with the environment and the development and maintenance of an awareness of the actual situation the individual is engaged in. The approach taken in this commentary was more than usually assertional in the hope of providing some degree of foresight in identifying important coming issues that need to be conceptualized in our field. The many advances provided in this special issue provide viable grounds for reconsidering how we might think more profoundly about the complexities in teacher noticing.

This paper directed to Sherin and Star's (2011) call for the development of a more comprehensive model of teacher noticing: "as a field, we should work toward the development of a more complete model of how teachers make sense, in the moment, of complex classroom events" (p. 77). A 'first cut' has been taken in accounting for the complex interactions involved in teacher noticing, drawing on Neisser's (1976) perceptual cycle model and blending sound insights from cognitive science and the applied science of human factors. It is hoped that the discussion presented here offers a promising theoretical perspective to further explore the complex interactions underlying the interdependencies involved in teacher noticing. In particular, more ground-breaking theoretical and empirical research is needed on the nature and dynamics of the resources and processes involved in understanding teacher situation awareness and decision-making in real-time events. It is hoped that the discussion reinforces the intellectual framing of what we need to set our minds to in the future in order to enlighten the black box of teacher noticing.

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6 Conclusion

Mathematics education is certainly a vibrant field, replete with diverse theoretical views that cultivate varied understandings and interpretations of complex phenomena in mathematical thinking, learning, and teaching. However, the abundance of disparate theories encourages both the multiplication of perspectives and the division of thought into opposing schools. This frequently hinders dialogue across traditions and paradigms, biases theorists, and promotes the creation and growth of narrow, and at times restricting, theories. Recognizing this challenge, this thesis attempted to look for theoretical tensions or oppositions and use them to stimulate the development of more powerful theories. In particular, this thesis goes beyond previous attempts at coordinating theoretical perspectives in the context of mathematical cognition and teacher cognition that fostered unidimensional representations and privileged one side of a dualism (i.e., an either/or distinction such as extracting meaning or giving meaning, the structure of a discipline or the structure of mind, individual or environment). The thesis highlights the contradictions and interdependencies of critical issues in mathematical cognition and teacher cognition that might inspire other scholars to question dualisms. In exploring and encouraging both greater expansion and more interlinking of multiple, at times competing, theoretical positions and approaches, theory and scholarly debate may become more detailed and useful, moving beyond deceptive dualisms and enabling deeper and more accurate understandings for theoreticians, practitioners, and researchers.

More importantly, the different contributions in this thesis seek a perspective (or a multiplicity of perspectives) from which one can appreciate the interaction between seemingly conflicting, yet ultimately interconnected, insights. The theoretical perspectives generated in this thesis demonstrate that it is possible to construct bridges between seemingly dissimilar viewpoints, revealing that some conflicting viewpoints underscore interwoven (rather than contradictory) facets of complex phenomena.[1] In doing so, this thesis contributes to raising awareness that seemingly unambiguous phenomena and issues in mathematical cognition and teacher cognition are more nuanced than they have heretofore been considered and that existing theoretical conceptualizations and theories that attempt to account for them are in many ways restricting.

It is hoped that this thesis offers both a contribution of theory advancement and a contribution to theory advancement. It might offer a contribution of theory development as it: presents a novel theoretical perspective on mathematical concept formation, the dialogic framing, which blends aspects of the existing perspectives that mathematical meaning is extracted (from objects falling under a particular concept) and that mathematical meaning is given (by individuals to objects they interact with) in order to present a bi-directional, dynamic, non-linear view of meaning making; challenges the dominant view of subject matter transformation and its underlying assumptions by presenting knowledge as co-constructed by both student and teacher; and develops a model of teacher noticing which challenges intuitive assumptions and views individual and environment as interdependent and inseparable. It is hoped that these three contributions add value to the field by advancing understanding and contributing to progress in the field, providing novel insights and informative views regarding complex phenomena, as well as presenting new relations among previous conceptualizations and exploring potentials of alternative views.

This thesis might also offer a contribution to theory development because it articulates a yet-unaddressed strategy of networking theories, blending, which has the potential to generate new theoretical insights not present in previous theories. As such, it extends the space of possibilities of networking strategies and speaks to the complexity involved in theory-building processes, particularly when coordinating apparently conflicting theoretical accounts. Each contribution made in this thesis puts forth a specific approach for theory advancement: the first contribution (Scheiner, 2016a) blends seemingly opposing approaches to transcend dualisms by revealing complementarity among conflicting theoretical perspectives, the second contribution (Scheiner et al., 2017) questions taken-for-granted assumptions underlying existing conceptualizations of teacher knowledge and provides alternative views that have the potential to redirect the way one conceives of what makes mathematics teacher knowledge specialized, and finally, the third contribution (Scheiner, 2016b) moves away from simplified conceptions by considering phenomena discussed in cognitive science and the applied science of human factors to better account for the complex process of awareness, including the significance of cognitive and affective structures, and the many interdependencies involved in teacher noticing.[2]

Notes to Chapter 6

[1] The emerging theoretical perspectives discussed in this thesis are not substitutes for existing perspectives or conceptualizations in the field but, rather, alternatives or extensions for exploring complex phenomena from a new angle. Indeed, the theoretical perspectives may help extend existing theoretical constructs, a theme that the three articles in this thesis explicitly consider. In Scheiner (2016a), 'reflectural abstraction' is introduced as a new term to provide a vocabulary that speaks to the dialectic between reflective abstraction and structural abstraction and the interdependencies of the respective sense-making strategies of extracting meaning and giving meaning. In Scheiner et al. (2017), the construct of teacher knowledge is extended in a way that accounts for specialization in mathematics teacher knowledge as a style of knowing rather than a kind of knowledge. In Scheiner (2016b), the theoretical construct of teacher noticing is enriched with theoretical insights derived from cognitive science and the applied science of human factors, surpassing the limitations of intuitive models of teacher noticing.

[2] The purpose of this thesis was clearly not to offer actionable insights into how to blend conflicting theoretical frameworks; this is an undertaking for another time. Rather, it is to foster a new perspective on networking theories that may provide new insights and deeper meaning, and may take original ideas forward in ways not conceived of originally. Taking the idea of blending forward in the area of theory development may extend the lexicon for talking about these and related issues, and cast light upon the implicit and unexamined practices found in attempts to coordinate competing or conflicting theories.

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Appendix A: Summary/Zusammenfassung

Summary

Mathematics education is a very broad research field divided into several subfields, two of them being of primary concern in this thesis: mathematical cognition and teacher cognition. These subfields contain a broad range of diverse theories that cultivate varied understandings of complex phenomena in mathematical thinking, learning, and teaching. However, the now-pervasive diversity of theories encourages the development of conflicting viewpoints, frequently hindering dialogue across traditions and paradigms, biasing theorists, and promoting the formation and growth of narrow, and at times restricting, theoretical accounts of complex phenomena. Recognizing this challenge, this thesis attempts to look for theoretical tensions or oppositions and use them to stimulate the development of more powerful theoretical accounts. In particular, this thesis goes beyond previous attempts at coordinating theoretical perspectives in the context of mathematical cognition and teacher cognition that fostered unidimensional representations and privileged one side of a dualism.

This thesis consists of three articles, each having its specific objective for theory development in the areas of mathematical cognition and teacher cognition. In particular, the thesis attempts to blend opposing theoretical perspectives to reveal complementarity in the field of mathematical knowing and learning, challenge taken-for-granted assumptions to reveal restrictions in the field of teacher knowledge, and portray some complex phenomena that cannot be accounted for using intuitive models of teacher noticing. The articles, taken together, call for a perspective from which one can recognize the interplay of apparently conflicting, yet interdependent, insights. The theoretical perspectives generated in this thesis demonstrate possibilities for linking apparently disparate approaches, revealing that some conflicting approaches underscore interwoven (rather than contradictory) facets of complex phenomena. In doing so, this thesis contributes to raising awareness that seemingly unambiguous phenomena and issues in mathematical cognition and teacher cognition are far more complex than one might have imagined and that existing theoretical conceptualizations and theories that attempt to account for them are in many ways restricting.

The first article, and the respective chapter in the thesis, discuss local theories of mathematical cognition, in particular two approaches (abstraction-from-actions and abstraction-from-objects) that have been previously construed as opposing. The thesis blends theoretical perspectives from both approaches to present a bi-directional, dynamic, non-linear view of mathematical concept formation. The second article, and the respective chapter in the thesis, examine teacher cognition, discussing existing conceptualizations of mathematics teacher knowledge and taking a critical stance toward how the field currently conceives of what makes this knowledge specialized. The thesis reveals some limitations of current conceptualizations of teacher knowledge and offers alternative views that provide insights into underexplored issues regarding what makes mathematics teacher knowledge specialized. The third article, and the respective chapter in the thesis, examine teacher cognition from the perspective of the construct of teacher noticing, drawing on insights from cognitive science and the applied science of human factors to develop a model of teacher noticing which challenges intuitive assumptions and views individual and environment as interdependent and inseparable. The thesis provides viable grounds for reconsidering how to think about the complexities involved in teacher noticing, in the hope of anticipating crucial unanswered questions in this field.

It is hoped that these contributions add value to the field by advancing understanding and moving the field's thinking forward, offering fresh insights regarding complex phenomena, producing decidedly different and uniquely informative theoretical views of phenomena under study, providing new connections among previous conceptualizations and exploring the implications of alternative views.

Zusammenfassung

Mathematikdidaktik ist ein sehr weites Forschungsfeld, das in mehrere Teilbereiche unterteilt ist, von denen zwei in dieser Dissertation von vorrangiger Bedeutung sind: mathematische Kognition und Lehrerkognition. Diese Teilgebiete enthalten bzw. beziehen sich auf ein breites Spektrum unterschiedlicher Theorien, die komplexe Phänomene im mathematischen Denken, Lernen und Lehren unterschiedlich verstehen und interpretieren. Gleichzeitig erlaubt die heute verbreitete Vielfalt von Theorien die Entwicklung von Perspektiven, die den Diskurs über mathematische Kognition sowie über komplexe Phänomene des mathematischen Denkens, Lehrens und Lernens behindern – u.a. durch von Traditionen und engen Paradigmen geprägten Auffassungen, die die Entwicklung enger und bisweilen einschränkender theoretischer Darstellungen komplexer Phänomene fördern. Die vorliegende Arbeit stellt sich dieser Problematik und intendiert, theoretisch bedingte Spannungen oder Gegensätze zu identifizieren und diese zu nutzen, um die Entwicklung theoriebasierter Ansätze zu stimulieren. Insbesondere geht diese Dissertation über Ansätze hinaus, die theoretische Perspektiven im Bereich der mathematischen Kognition und Lehrerkognition im Hinblick auf eine eindimensionale Repräsentation bevorzugten und plurale Ansätze vernachlässigten.

Die Dissertation basiert auf drei Artikeln in wissenschaftlichen Zeitschriften, die jeweils einen spezifischen Aspekt der Theorieentwicklung in den Bereichen der mathematischen Kognition und der Lehrerkognition fokussieren. Dabei intendiert die Arbeit gegensätzliche theoretische Perspektiven zu kombinieren, die die Notwendigkeit komplementärer Ansätze auf dem Gebiet des mathematischen Wissens und Lernens aufzeigen, die als selbstverständlich vorausgesetzte theoriebasierte Annahmen hinterfragen, die Einschränkungen im Bereich des Lehrerwissens aufdecken und komplexe Phänomene darlegen, die in den Theorieansätzen nicht berücksichtigt wurden. Dabei dient die professionelle Unterrichtswahrnehmung als zentraler Ansatz. Die der Dissertation zugrundeliegenden Artikel fordern eine Perspektive, aus der man die Verknüpfung scheinbar widerstreitender, aber voneinander abhängiger Ansätze erkennen kann. Die theoretischen Perspektiven, die in dieser Arbeit generiert werden, zeigen Möglichkeiten auf, scheinbar unvereinbare Ansätze zu verknüpfen, und decken auf, dass widersprüchliche Ansätze eher miteinander verwobene und nicht widersprüchliche Facetten komplexer Phänomene darstellen. Damit trägt diese Arbeit dazu bei, das Bewusstsein zu schärfen, dass scheinbar augenfällige Phänomene der mathematischen Kognition und Lehrerkognition deutlich komplexer sind, als man es sich gemeinhin vorstellt und dass existierende theoretische Konzeptualisierungen und Theorien, die versuchen, diese zu erklären, in vielerlei Hinsicht einschränkend bzw. eingeschränkt sind.

Im Detail behandelt der erste Artikel lokale Theorien der mathematischen Kognition, insbesondere zwei Ansätze (Abstraktion von Aktionen und Abstraktion von Objekten), die zuvor als gegensätzlich konzeptualisiert wurden. Die Dissertation geht über die Diskussion eines Vergleichs dieser scheinbar gegensätzlichen Ansätze hinaus, indem theoretische Perspektiven grundlegender kognitiver Prozesse, die beiden Ansätzen zugrunde liegen, miteinander verschmolzen werden, um eine bi-direktional ausgerichtete, dynamische und nichtlineare Sicht der mathematischen Begriffsbildung zu entwickeln. Der zweite Artikel untersucht die Lehrerkognition, indem bestehende Konzeptualisierungen des Professionswissens von Mathematiklehrkräften diskutiert werden. Insbesondere wird eine kritische Haltung eingenommen gegenüber der Frage, wie das Forschungsfeld gegenwärtig Spezialisierung im Professionswissen von Mathematiklehrkräften konzeptualisiert. Die Arbeit zeigt Einschränkungen aktueller Konzeptualisierungen des Lehrerprofessionswissens auf, und bietet alternative Sichtweisen, die die Aufmerksamkeit auf wenig erforschte Fragen hinsichtlich der Spezialisierung des Lehrerprofessionswissens lenken. Der dritte Artikel untersucht Lehrerkognition aus der Perspektive des Konstrukts der professionellen Unterrichtswahrnehmung von Lehrkräften und stützt sich dabei auf Erkenntnisse der Kognitionsforschung, um ein Modell der professionellen Wahrnehmung von

Lehrkräften zu entwickeln, das intuitive Annahmen herausfordert und Individuum und Umwelt als voneinander abhängig und untrennbar betrachtet. Die Dissertation eröffnet eine Grundlage zur Reflektion der Komplexität der professionellen Wahrnehmung von Lehrkräften, um damit einen Beitrag zur Identifizierung wichtiger Fragen in diesem Bereich zu ermöglichen.

Es ist zu hoffen, dass es der Dissertation gelingt, einen Beitrag zur Fortentwicklung der Diskussion zur Kognition und insbesondere Lehrerkognition zu leisten, indem neue Erkenntnisse zur Interpretation komplexer Phänomene angeboten und Verbindungen zwischen bereits existierenden Konzeptualisierungen von Lehrerprofessionalität hergestellt und damit alternative Weiterentwicklungen der Diskussion ermöglicht werden.

Appendix B: Titles of Articles of the Dissertation

This dissertation by publications consists of three journal articles. The titles of the articles of the dissertation are:

Article 1 Scheiner, T. (2016). New light on old horizon: Constructing mathematical concepts, underlying abstraction processes, and sense making strategies. *Educational Studies in Mathematics*, *91*(2), 165-183. (doi: 10.1007/s10649-015-9665-4)

Article 2 Scheiner, T., Montes, M. A., Godino, J. D., Carrillo, J., & Pino-Fan, L. R. (2017). What Makes Mathematics Teacher Knowledge Specialized? Offering Alternative Views. *International Journal of Science and Mathematics Education.* (doi: 10.1007/s10763-017-9859-6) (online-first)

Article 3 Scheiner, T. (2016). Teacher noticing: enlightening or blinding?. *ZDM Mathematics Education*, *48*(1-2), 227-238. (doi: 10.1007/s11858-016-0771-2)

Appendix C of this thesis has been removed as it may contain sensitive/confidential content

Appendix D: Publication List

Peer-Reviewed Academic Journals

Scheiner, T., Montes, M. A., Godino, J. D., Carrillo, J., & Pino-Fan, L. (2017). What makes mathematics teacher knowledge specialized? Offering alternative views. *International Journal of Science and Mathematics Education*. (online first)

Scheiner, T. (2016). Teacher noticing: enlightening or blinding? *ZDM Mathematics Education*, *48*(1), 227-238.

Scheiner, T. (2016). New light on old horizon: constructing mathematical concepts, underlying abstraction processes, and sense making strategies. *Educational Studies in Mathematics*, *91*(2), 165-183.

Pinto, M. M. F., & Scheiner, T. (2015). Visualização e ensino de análise matemática [Visualization and the teaching of mathematical analysis]. *Educação Matemática Pesquisa*, *17*(3), 637-654.

Peer-Reviewed Conference Proceedings

Scheiner, T. (2017). Conception to concept or concept to conception? From being to becoming. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 145-152). Singapore, Singapore: PME.

Scheiner, T., & Pinto, M. M. F. (2017). Emerging insights from the evolving framework of structural abstraction. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 274-284). San Diego, CA: RUME.

Scheiner, T., & Pinto, M. M. F. (2016). Images of abstraction in mathematics education: contradictions, controversies, and convergences. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 155-162). Szeged, Hungary: PME.

Pinto, M. M. F., & Scheiner, T. (2016). Making sense of students' sense making through the lens of the structural abstraction framework. In E. Nardi, C. Winslow & T. Hausberger (Eds.), *Proceedings of the First Conference of the International Network for Didactic Research in University Mathematics* (pp. 474-483). Montpellier, France: INDRUM.

Scheiner, T. (2015). Shifting the emphasis toward a structural description of (mathematics) teachers' knowledge. In K. Bewick, T. Muir, & J. Wells (Eds.), *Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 129-136). Hobart, Australia: PME.

Scheiner, T. (2015). Theorizing about mathematics teachers' professional knowledge: The content, form, nature, and source of teachers' knowledge. In M. Marshman, V. Geiger, & A. Bennison (Eds.), *Mathematics education in the margins* (Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia, pp. 563-570). Sunshine Coast, Australia: MERGA.

Scheiner, T. (2015). Lessons we have (not) learned from past and current conceptualizations of mathematics teachers' knowledge. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Conference of the European Society for Research in Mathematics Education* (pp. 3248-3253). Prague, Czech Republic: ERME.

Scheiner, T., & Pinto, M. M. F. (2014). Cognitive processes underlying mathematical concept construction: The missing process of structural abstraction. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education* (Vol. 5, pp. 105-112). Vancouver, Canada: PME.

Peer-Reviewed Conference Papers

Scheiner, T. (2016). Are we trapped in old habits? Revisiting ways of thinking in conceptualizing teacher knowledge. *13th International Congress on Mathematical Education* (TSG 46: Knowledge in/for teaching mathematics at secondary level). Hamburg, Germany.

Scheiner, T., & Pinto, M. M. F. (2016). Abstraction in mathematics: taking account for the increasing complexity and context-sensitivity of the knowledge system. *13th International Congress on Mathematical Education* (TSG 27: Learning and cognition in mathematics). Hamburg, Germany.

Scheiner, T. (2016). Crossing the boundaries of our historical ways of thinking in conceptualizing teachers' knowledge. *2016 Annual Meeting of the American Education Research Association*. Washington, D.C.

Scheiner, T. (2015). Conceptualization of teachers' knowledge: Shifting the attention to the nature and form. *16th Biennial Conference of the European Association for Research on Learning and Instruction* (EARLI). Limassol, Cyprus.

Scheiner, T. (2015). Examining (mathematics) teachers' professional knowledge through various theoretical lenses: A research agenda. *19th Conference of the Junior Researchers of the European Association for Research on Learning and Instruction* (JURE). Limassol, Cyprus.

Scheiner, T., & Pinto, M. M. F. (2015). Examining a student's resource for reconstructing the limit concept at need: A structural abstraction perspective. *38th Annual Conference of the Mathematics Education Research Group of Australasia*. Sunshine Coast, Australia.

Scheiner, T. (2013). Mathematical concept acquisition: Reflective, structural, and reflectural learners. *Working Group 'Factors that Foster or Hinder Mathematical Thinking' of the 37th Conference of the International Group for the Psychology of Mathematics Education*. Kiel, Germany.

Research Report

Kaiser, G., Scheiner, T., & Buchholtz, N. (2013). *Evaluation des innovativen Projektes „MINT – Lehrerbildung Neu Denken" der Deutschen Telekom Stiftung an der FU Berlin. Abschlussbericht zu den qualitativen Ergebnissen*. [Final Report of the Evaluation of the Project "MINT – Rethinking Teachers' Education": Qualitative Results]. University of Hamburg, Germany.

Test Instrument

Buchholtz, N., Scheiner, T., Döhrmann, M., Suhl, U., Kaiser, G. & Blömeke, S. (2016). *TEDS-shortM: Teacher Education and Development Study – Short Test on Mathematics Content Knowledge (MCK) and Mathematics Pedagogical Content Knowledge (MPCK). Kurzfassung der mathematischen und mathematikdidaktischen Testinstrumente aus TEDS-M, TEDS-LT und TEDS-Telekom* (2nd ed.). Hamburg, Germany: University of Hamburg. (1st ed. published 2012)

Erklärung über die Eigenständigkeit der Dissertation

Eidesstattliche Versicherung

Hiermit versichere ich, dass die Dissertation von mir selbst verfasst wurde und keine anderen als die angegebenen Hilfsmittel von mir genutzt wurden.

18. Januar 2018

Datum Unterschrift

Erklärung

Ich erkläre hiermit, dass ich keine kommerzielle Promotionsberatung in Anspruch genommen habe und, dass ich mich bisher keiner weiteren Doktorprüfung unterzogen habe. Insbesondere habe ich die Dissertation in der gegenwärtigen oder einer anderen Fassung an keiner anderen Fakultät eingereicht.

18. Januar 2018

Datum Unterschrift