Catalan Objects In Categories

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A thesis presented for the degree of Master of Research
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October 17, 2019

Abstract

The Catalan numbers $1, 2, 5, 14, 42, 132, \ldots$ count all sorts of interesting objects: for example, the nth Catalan number enumerates both the set of T_n of rooted finitely branching trees with $n+1$ vertices, and the set B_n of rooted binary trees with $n+1$ leaves. In fact for each n, the sets T_n and B_n are in bijection in a natural way.

The objective of this thesis is to study these bijections, and similar ones, through the lens of category theory. The set of rooted finitely branching trees can be characterised as an initial motor; where a motor is a monoid endowed with a endofunction (not preserving the monoid structure). The set of rooted binary trees can be characterised as an initial pointed magma; where a pointed magma is a set endowed with a constant and a binary operation.

We first give an account of the bijection between rooted trees and binary trees which uses only the universal properties of the initial motor and the initial pointed magma. We then generalise this in various directions; our most general result identifies the initial T-motor with the initial pointed T-magma, when T is an arbitrary endofunctor of a closed monoidal category $\mathcal{C}$; here, a T-motor is a monoid X endowed with a mapping $TX \to X$, while a pointed T-magma is an object with a map $I + TX \otimes X \to X$. We will also give numerous examples and applications.

I declare that this thesis is my own work and that any contribution by other people is referenced in the usual manner. This is an original thesis and has not been submitted for a higher degree at any other university or institution.

Signed: Geoff Edington-Cheater

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CHAPTER 1

Introduction and Motivation

The Catalan numbers are a sequence of numbers that arise as a solution to many counting problems in combinatorics; Stanley in [1] lists 214 such problems. The nth Catalan number is given by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

for which the first few terms are $C_n = 1, 1, 2, 5, 14, 42, 132, \ldots$ [2, A000108]

Consider the following examples of $\mathbb{N}$-indexed families of sets:

(1) The set of rooted, planar, finitely branching trees with n edges.

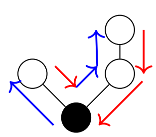
(2) The set of Dyck words of length $2n$. A Dyck word is a word on the symbols U and D which have the same number of Us and Ds, but never more Ds than Us when reading from left to right.

(3) The set of (full) binary trees with $n+1$ leaves (alternatively n internal vertices or $2n$ edges).

(4) The set of well bracketed expressions involving associating n applications of a binary operation.

In each case the nth set has cardinality C_n, so each of these structures is counted by the Catalan numbers.

There is a clear correspondence between the sets in the first two examples which is given by traversing along the tree in a fixed orientation (clockwise) from the root and denoting each "upwards" movement as a U, and each "downwards" movement as a D, as shown below.

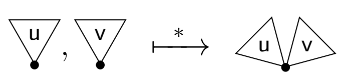


$\longmapsto UDUUDD$

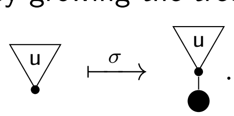
In fact both of these types of object can be readily built in a recursive fashion.

Observe the following structure which generates the set of rooted, finitely branching trees T:

- The trivial tree: $e = \bullet$
- An associative binary operation with unit e, which is given by merging two trees at the root:



- An endofunction which is given by growing the tree up from the root:



This motivates the following definition: A motor, $(A, e, *, \sigma)$, consists of a monoid, $(A, e, *)$, and an endofunction, $\sigma\colon A \to A$ (not necessarily preserving the monoid structure). The structure described above makes the T into a freely generated motor. The natural numbers also have an obvious motor structure, $(\mathbb{N}, 0, +, s)$, where s is the usual successor function, $s(n) = n + 1$.

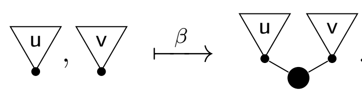
The free generation of the motor of rooted, finitely branching trees means that there is a unique function f from T to the natural numbers that preserves the motor structure, which is given by:

$$f(e) = 0$$
$$f(x * y) = f(x) + f(y)$$
$$f \circ \sigma(x) = s \circ f(x) = f(x) + 1.$$

This function counts the number of branches in the tree, and hence we see that the cardinality of the set given by the preimage of n under f is actually C_n.

There is also a clear correspondence between examples (3) and (4) on page 1, which is given by denoting each branching in the tree as performing the binary operation. The set of binary trees B can be recursively built up via:

- The trivial tree: $e = \bullet$
- A binary operation:



A pointed magma, $(M, m, \bullet)$, consists of a set, M, a designated point, $m \in M$, and a binary operation, $\_ \bullet \_\colon M \times M \to M$. The structure indicated above makes B into a freely generated pointed magma. The natural numbers form a pointed magma, $(\mathbb{N}, 0, t)$, where $t(x, y) = x + y + 1$. Like before, free generation of the pointed magma of binary trees gives us a unique grading, g, from the initial pointed magma, $(B, e, \bullet)$, to the natural numbers using the rules:

$$g(e) = 0,$$
$$g(x \bullet y) = t \circ \big(g(x), g(y)\big) = g(x) + g(y) + 1.$$

It is also again the case that the cardinality of the set given by the preimage of n under g is the nth Catalan number.

There is also a less clear, but nevertheless very interesting, correspondence between the first pair of examples, the ones with a motor structure, and the second pair, the ones with a pointed magma structure. This correspondence shows that while a given set which exhibits a Catalan counting scheme may be readily given either a pointed magma or a motor structure, we can always give it the other structure if we want to.

As this correspondence is between two different types of algebraic structure it is natural to use category theory and universal properties as a tool for exploring the correspondence. The benefit of using category theory becomes apparent when we generalise beyond the set-based situation.

In chapter 2 we give a category-theoretic proof that the freely generated motor and the freely generated pointed magma coincide. Specifically, defining $\mathbf{Motor}$ to be the category of motors together with their structure preserving morphisms, and $\mathbf{Magma}_*$ to be the category of pointed magmas together with their structure preserving morphisms, we prove:

Theorem 1.1. There exists a faithful functor $F\colon \mathbf{Motor} \to \mathbf{Magma}_*$ that both preserves and lifts initial objects.

In chapter 3 we give a generalisation of motors and magmas using the notion of algebras for an endofunctor T in $\mathbf{Set}$, and show that we get a similar correspondence between these generalised notions of initial "T-motor" and an initial "pointed T-magma". That is, we define categories T-$\mathbf{Motor}$ and T-$\mathbf{Magma}_*$, and then prove:

Theorem 1.2. There exists a faithful functor $F\colon T\text{-}\mathbf{Motor} \to T\text{-}\mathbf{Magma}_*$ that both preserves and lifts initial objects.

In chapter 4 we give a generalisation for T-motors and pointed T-magmas in a monoidal category with certain conditions, and show that the previous theorem still holds. Specifically, if $\mathcal{C}$ is a closed monoidal category with equalisers, and T is an endofunctor in $\mathcal{C}$, then we define categories T-$\mathbf{Motor}$ and T-$\mathbf{Magma}_*$, and prove:

Theorem 1.3. There exists a faithful functor $F\colon T\text{-}\mathbf{Motor} \to T\text{-}\mathbf{Magma}_*$ that both preserves and lifts initial objects.

We then conclude with some examples of where this general theorem is useful.

CHAPTER 2

A First Case

1. Introduction

In this chapter we explore the correspondence between the two given structures that exhibit a Catalan counting scheme; the initial motor and the initial pointed magma. This is done by first showing that the initial motor may be turned into a pointed magma in such a way as to preserve the Catalan counting scheme. Once we have established that the initial motor is a pointed magma, we then show it is the initial such, and therefore we have shown that there is a bijection between the two underlying sets.

Definition 2.1. If $(M, m, *)$ is a monoid, and $\sigma \colon M \to M$ is an endofunction, then we call the quadruple $\boldsymbol{M} = (M, m, *, \sigma)$ a *motor*. A motor morphism $f \colon \boldsymbol{M} \to \boldsymbol{N}$ between motors $\boldsymbol{M} = (M, m, *, \sigma)$ and $\boldsymbol{N} = (N, n, \star, \tau)$ is a function $f \colon M \to N$ such that

- $f(m) = n$
- $f(x * y) = f(x) \star f(y), \forall x, y \in M$
- $f\sigma(x) = \tau f(x), \forall x \in M$.

This yields a category of motors and motor morphisms which we call $\mathbf{Motor}$.

Recall that a Dyck word is a word on the symbols U and D which have the same number of Us and Ds, but never more Ds than Us when reading from left to right. The set of Dyck words on $\{U, D\}$ has a motor structure, $\boldsymbol{A} = (\mathfrak{D}, e, *, \sigma)$, where e is the empty word, $*$ is concatenation, and $\sigma \colon \mathfrak{D} \to \mathfrak{D}$ is given by $\sigma(w) = UwD$. The following shows that $\boldsymbol{A}$ is freely generated by its motor structure.

Given a Dyck word of length $2n$ we can define a height function $h \colon \{0, \dots, 2n\} \to \mathbb{N}$ where a U increases the height by 1 and a D decreases the height by 1. For example we have the following Dyck word and its associated sequence of heights:

$$\begin{array}{ccccccccc} 0 & & 1 & & 2 & & 1 & & 0 \\ & U & & U & & D & & D & \end{array}$$

We now define a Dyck word to be *connected* if it is non-empty, and $h(i) > 0$ for all $0 < i < 2n$. Using this we see that every non-empty Dyck word can be uniquely decomposed as $w = w_1 * \ldots * w_n$, where each w_i is a connected word. We also see that every connected word is uniquely the successor $\sigma(w)$ of another, possibly empty, word w. Hence every Dyck word can be built up from the empty word using repeated applications of $*$ and σ, uniquely up to the monoid axioms, i.e. $e, *$ and σ freely generate $\boldsymbol{A}$. We can make this precise via the category-theoretic notion of initiality:

Definition 2.2. Let $\mathcal{C}$ be a category. An object $I \in \mathcal{C}$ is called *initial* in $\mathcal{C}$ if given any object X in $\mathcal{C}$, there exists a unique morphism $f\colon I \to X$.

Any motor map out of $\boldsymbol{A}$ is determined by its action on $e, *$ and σ, and since the Dyck words in $\boldsymbol{A}$ are freely generated by $e, *$ and σ, any motor map out of $\boldsymbol{A}$ must be unique. Hence $\boldsymbol{A}$ is the initial motor.

We now relate this to the other kind of structure considered in the introduction.

Definition 2.3. Given a set M, an element $m \in M$, and a binary operation $\_\bullet\_\colon M \times M \to M$ on M, then we call the triple $\boldsymbol{M} = (M, m, \bullet)$ a *pointed magma*. The collection of pointed magmas forms a category, $\mathbf{Magma}_*$, with morphisms $f\colon \boldsymbol{M} \to \boldsymbol{N}$ between pointed magmas $\boldsymbol{M} = (M, m, \bullet)$ and $\boldsymbol{N} = (N, n, *)$ being functions $f\colon M \to N$ such that

- $f(m) = n$
- $f(x \bullet y) = f(x) * f(y), \forall x, y \in M.$

Earlier we saw that every Dyck word can be uniquely decomposed into connected words, and that every connected word is of the form UwD for some w. This means that every non-empty Dyck word is uniquely of the form $UwDx = \sigma(w) * x$ for some other Dyck words w and x. This gives us a binary operation on $\mathfrak{D}$, $x \bullet y = \sigma(x) * y$, and hence a freely generated pointed magma $\boldsymbol{B} = (\mathfrak{D}, e, \bullet)$. This observation plays a crucial role in [3].

Note that the pointed magma structure on $\mathfrak{D}$ is purely defined in terms of the motor structure on $\mathfrak{D}$. In fact this works for any motor as witnessed by:

Proposition 2.4. There exists a faithful functor $F\colon \mathbf{Motor} \to \mathbf{Magma}_*$ whose action on objects, $\boldsymbol{M} = (M, m, *, \sigma)$, is given by:

$$F\boldsymbol{M} = \Big(M, m, (x, y) \mapsto \sigma(x) * y\Big)$$

Proof. Given a motor morphism $f\colon \boldsymbol{M} \to \boldsymbol{N}$ between $\boldsymbol{M} = (M, m, *, \sigma)$ and $\boldsymbol{N} = (N, n, \star, \tau)$, we have that $f(m) = n$ and that

$$f\Big(\sigma(x) * y\Big) = f\sigma(x) \star f(y) = \tau f(x) \star f(y)$$

as required for $Ff\colon F\boldsymbol{M} \to F\boldsymbol{N}$ to be a pointed magma morphism. F preserves identities and compositions since Ff has the same action on elements as f, and so F is a functor. Lastly, F renders commutative the diagram

$$\begin{tikzcd} \mathbf{Motor} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathbf{Magma}_* \arrow[dl, "V"] \\ & \mathbf{Set} & \end{tikzcd}$$

where U and V are the forgetful functors. hence F is faithful since U is. $\square$

2. The Initial Motor is the Initial Pointed Magma

The argument given above shows that F sends the initial motor $\boldsymbol{A}$ to an initial pointed magma $F\boldsymbol{A}$, or in other words, F preserves initial objects. We would now like to study this phenomenon and its generalisations using category theory.

An obvious reason why F might preserve initial objects would be that F has a right adjoint. However with some effort one may show that F doesn't preserve coproducts, so this argument does not apply.

We instead try the more direct approach of showing that given an arbitrary pointed magma $\boldsymbol{B}$, there exists a unique pointed magma morphism g from $F\boldsymbol{A}$ to $\boldsymbol{B}$. To construct g we transform $\boldsymbol{B}$ into a motor $\hat{\boldsymbol{B}}$, and then show that the unique motor map from $\boldsymbol{A}$ to $\hat{\boldsymbol{B}}$ induces the required unique map g.

To this end we require a way of constructing a motor out of an arbitrary pointed magma, and this is done as follows.

Let $\boldsymbol{B} = (B, b, \bullet)$ be an arbitrary pointed magma, and now consider the following stack machine, which for every Dyck word w gives instructions for a function $h_w \colon B \to B$.

(1) First initialise the empty stack by pushing the function input $x \in B$ onto the stack.

(2) Next read the Dyck word from right to left with the following instruction set:
- D: Push b onto the stack
- U: Take the top two elements off the stack, say y and z, and push their product $y \bullet z$ back onto the stack

(3) When every instruction is read, the remaining element on the stack is $h_w(x)$.

This machine gives us the data for a map from the initial motor to the set of functions from B to B. We might hope to induce this map using initiality of the motor $\boldsymbol{A}$, as long as we can find a suitable motor structure on B^B. To do so let us examine the behaviour of the stack machine on words of the form $e, w_1 * w_2, \sigma(w)$.

If we run the machine with the empty instruction set e, then the resulting function is the identity function 1_B. Suppose we have Dyck words w_1 and w_2, then running the machine with instruction set $w_1 * w_2$ will result in a function that first evaluates $h_{w_2}(x)$, and then will evaluate h_{w_1} with input $h_{w_2}(x)$, i.e. the resulting function will be $h_{w_1} \circ h_{w_2}$. Running the machine with instruction set $\sigma(w)$ will mean that the resulting function will now evaluate $h_w(b)$, and then take a product with x.

This tells us that if we define the motor of endomorphisms as $\hat{\boldsymbol{B}} = \left(B^B, 1_B, \circ, \Theta\right)$, where $\Theta \colon B^B \to B^B$ is the function given by

$$\Theta(f)(x) = f(b) \bullet x, \quad \forall f \in B^B, x \in B,$$

then the unique map $f \colon \boldsymbol{A} \to \hat{\boldsymbol{B}}$ sends w to h_w.

We now exploit the above construction to prove that if $\boldsymbol{A}$ is the initial motor, then $F\boldsymbol{A}$ is the initial pointed magma. As this is a purely algebraic construction we don't need a specific representation of $\boldsymbol{A}$, so from now on we write $\boldsymbol{A} = (A, e, *, \sigma)$ for the initial motor.

Proposition 2.5. Let $\boldsymbol{A}$ be the initial motor, let $\boldsymbol{B} = (B, b, \bullet)$ be an arbitrary pointed magma, and let $f\colon \boldsymbol{A} \to \hat{\boldsymbol{B}}$ be the unique motor map given by the universal property of the initial object, then $g\colon F\boldsymbol{A} \to \boldsymbol{B}$ given by $g(x) = f(x)(b)$ is a pointed magma map.

Proof. In order for g to be a pointed magma map we require that

$$g(e) = b \tag{1}$$

$$g\Big(\sigma(x) * y\Big) = g(x) \bullet g(y), \tag{2}$$

and since is f a motor map we have that

$$f(e) = 1_B \tag{3}$$

$$f(x * y) = f(x) \circ f(y) \tag{4}$$

$$f\Big(\sigma(x)\Big) = \Theta\Big(f(x)\Big). \tag{5}$$

By definition of g and (3) we get that

$$g(e) = f(e)(b) = 1_B(b) = b$$

and so (1) holds. As for (2) we have that

$$\begin{aligned}
g\Big(\sigma(x) * y\Big) &= f\Big(\sigma(x) * y\Big)(b) && \text{by definition of } g \\
&= f\Big(\sigma(x)\Big) \circ f(y)(b) && \text{by (4)} \\
&= \Theta\Big(f(x)\Big)\Big(f(y)(b)\Big) && \text{by (5)} \\
&= f(x)(b) \bullet f(y)(b) && \text{by definition of } \Theta \\
&= g(x) \bullet g(y) && \text{by definition of } g
\end{aligned}$$

as required. $\square$

We have now established that for every pointed magma, $\boldsymbol{B}$, there is a pointed magma map, $g\colon F\boldsymbol{A} \to \boldsymbol{B}$, so proving this map is unique will establish that $F\boldsymbol{A}$ is the initial pointed magma.

The next proof is done by induction in the initial motor $\boldsymbol{A}$. This will be discussed in more detail at a later stage, but for now it should be clear that if a statement is true for e, and if the statement being true for x and y in A implies that it is also true for $x * y$ and $\sigma(x)$, then the statement must be true for all x in A.

Proposition 2.6. Let $\boldsymbol{B} = (B, b, \bullet)$ be a pointed magma, let $h \colon F\boldsymbol{A} \to \boldsymbol{B}$ be a pointed magma map, and let $f \colon \boldsymbol{A} \to \hat{\boldsymbol{B}}$ be as before. For all $x \in A$, the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "x * (-)"] \arrow[d, "h"'] & A \arrow[d, "h"] \\ B \arrow[r, "f(x)"'] & B \end{tikzcd}$$

Proof. This shall be proved by induction on $x \in A$.

We have that $e * (-) \colon A \to A = 1_A$, and $f(e) = 1_B$, so

$$\begin{tikzcd} A \arrow[r, "e * (-)"] \arrow[d, "h"'] & A \arrow[d, "h"] \\ B \arrow[r, "f(e)"'] & B \end{tikzcd}$$

commutes.

Suppose $x \in A$, and that

$$\begin{tikzcd} A \arrow[r, "x * (-)"] \arrow[d, "h"'] & A \arrow[d, "h"] \\ B \arrow[r, "f(x)"'] & B \end{tikzcd}$$

commutes, i.e.

$$h(x * y) = f(x)\Big(h(y)\Big), \quad \forall y \in A$$

and in particular

$$h(x) = h(x * e) = f(x)\Big(h(e)\Big) = f(x)(b) \tag{6}$$

which gives us

$$\begin{aligned} f\Big(\sigma(x)\Big)\Big(h(y)\Big) &= \Theta\Big(f(x)\Big)\Big(h(y)\Big) && \text{by (5)} \\ &= f(x)(b) \bullet h(y) && \text{by definition of } \Theta \\ &= h(x) \bullet h(y) && \text{by (6)} \\ &= h\Big(\sigma(x) * y\Big) && \text{since } h \text{ is a pointed magma map} \end{aligned}$$

for all y in A. We have therefore shown that if the left square commutes, then so must the right:

$$\begin{tikzcd} A \arrow[r, "x * (-)"] \arrow[d, "h"'] & A \arrow[d, "h"] \\ B \arrow[r, "f(x)"'] & B \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} A \arrow[r, "\sigma(x) * (-)"] \arrow[d, "h"'] & A \arrow[d, "h"] \\ B \arrow[r, "f\left(\sigma(x)\right)"'] & B. \end{tikzcd}$$

Now suppose both

$$
\begin{tikzcd}
A \arrow[r, "x * (-)"] \arrow[d, "h"'] & A \arrow[d, "h"] \\
B \arrow[r, "f(x)"'] & B
\end{tikzcd}
\tag{7}
$$

and

$$
\begin{tikzcd}
A \arrow[r, "y * (-)"] \arrow[d, "h"'] & A \arrow[d, "h"] \\
B \arrow[r, "f\left(y\right)"'] & B
\end{tikzcd}
\tag{8}
$$

commute. Then for all z in A,

$$
\begin{aligned}
f(x*y)\Big(h(z)\Big) &= f(x)\Big(f(y) \circ h(z)\Big) && \text{by (4)} \\
&= f(x)\Big(h(y*z)\Big) && \text{by (8)} \\
&= h(x*y*z) && \text{by (7)}
\end{aligned}
$$

Therefore we have shown that if the leftmost two squares commute, then so must the right.

$$
\begin{tikzcd}
A \arrow[r, "x * (-)"] \arrow[d, "h"'] & A \arrow[d, "h"] \\
B \arrow[r, "f(x)"'] & B
\end{tikzcd}
\quad \wedge \quad
\begin{tikzcd}
A \arrow[r, "y * (-)"] \arrow[d, "h"'] & A \arrow[d, "h"] \\
B \arrow[r, "f\left(y\right)"'] & B
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
A \arrow[r, "x * y * (-)"] \arrow[d, "h"'] & A \arrow[d, "h"] \\
B \arrow[r, "f\left(x * y\right)"'] & B.
\end{tikzcd}
$$

Therefore by induction the square

$$
\begin{tikzcd}
A \arrow[r, "x * (-)"] \arrow[d, "h"'] & A \arrow[d, "h"] \\
B \arrow[r, "f(x)"'] & B
\end{tikzcd}
$$

commutes for all x in A. ☐

We now have the following:

Theorem 2.7. If $\boldsymbol{A}$ is the initial motor, then $F\boldsymbol{A}$ is the initial pointed magma.

Proof. Let $\boldsymbol{B}$ be a pointed magma. We have already shown that the map $g\colon F\boldsymbol{A} \to \boldsymbol{B}$ is a pointed magma map, and this map is unique since given any other pointed magma map, $h\colon F\boldsymbol{A} \to \boldsymbol{B}$, we must have

$$h(x) = h(x*e) = f(x)\Big(h(e)\Big) = f(x)(b) = g(x), \quad \forall x \in A$$

by Proposition 2.6. Therefore $F\boldsymbol{A}$ is the initial pointed magma. ☐

Note that the functor $F\colon \mathbf{Motor} \to \mathbf{Magma}_*$ sends the motor $(\mathbb{N}, 0, +, s)$ to the pointed magma $(\mathbb{N}, 0, t)$, where, as before, $t(x, y) = x + y + 1$. This shows that $F\boldsymbol{A}$ is also a Catalan family of sets.

3. The Initial Pointed Magma is the Initial Motor

Our aim in this section is to show that the initial pointed magma gets lifted via F to an initial motor. First, however, we look at several properties that a functor may have.

Definition 2.8. Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ we may have that:

- F *preserves initial objects*: If X is initial in $\mathcal{C}$, then FX is initial in $\mathcal{D}$.
- F *reflects initial objects*: If FX is initial in $\mathcal{D}$, then X is initial in $\mathcal{C}$.
- F *lifts initial objects*: If Y is initial in $\mathcal{D}$, then $\exists X \in \mathcal{C}$ such that $Y \cong FX$.
- F *creates initial objects*: If F both lifts and reflects initial objects. Explicitly If Y is initial in $\mathcal{D}$, then there exists an essentially unique $X \in \mathcal{C}$ such that $Y \cong FX$ and X is initial in $\mathcal{C}$.
- F *reflects isomorphisms*: If $f\colon X \to Y$ in $\mathcal{C}$ such that $Ff\colon FX \to FY$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.

Proposition 2.9. Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ we have the following:

(1) If $\mathcal{C}$ has an initial object and F preserves initial objects, then F lifts initial objects.

(2) If $\mathcal{C}$ has an initial object, F preserves initial objects, and F reflects isomorphisms, then F reflects initial objects.

(3) If $\mathcal{D}$ has an initial object and F creates initial objects, then F preserves initial objects.

Proof.

(1) Suppose $\mathcal{C}$ has an initial object X, and that F preserves initial objects. Now suppose Y is initial in $\mathcal{D}$. Since F preserves initial objects, FX must also be initial in $\mathcal{D}$ and hence $Y \cong FX$. Therefore F lifts initial objects.

(2) Suppose $\mathcal{C}$ has an initial object X, that F preserves initial objects, and that F reflects isomorphisms. Now suppose Y in $\mathcal{C}$ is such that FY is initial in $\mathcal{D}$, and let $f\colon X \to Y$ be the unique map given by initiality of X in $\mathcal{C}$. F preserves initial objects and hence FX is initial in $\mathcal{D}$ and so $Ff\colon FX \to FY$ must be a unique isomorphism. f is an isomorphism since F reflects isomorphisms and hence Y is initial in $\mathcal{C}$ as it is isomorphic to X. Therefore F reflects initial objects.

(3) Suppose $\mathcal{D}$ has an initial object Y, and that F creates initial objects. Now suppose Z is initial in $\mathcal{C}$, since F creates initial objects there exists an initial object X in $\mathcal{C}$ such that $Y \cong FX$. Now $X \cong Z$ since X and Z are both initial, and therefore $Y \cong FX \cong FZ$, and hence FZ is initial in $\mathcal{D}$. Therefore F preserves initial objects. ∎

Proposition 2.10. The functor $F\colon \mathbf{Motor} \to \mathbf{Magma}_*$ as given in Proposition 2.4 reflects isomorphisms.

Proof. It is sufficient to show that the functor $U\colon \mathbf{Motor} \to \mathbf{Set}$ reflects isomorphisms since if f is a motor morphism such that Ff is a magma isomorphism, then $Uf = VFf$ is an isomorphism and hence if U reflects isomorphisms, then f is an isomorphism.

Let $\boldsymbol{M} = (M, m, *, \sigma)$ and $\boldsymbol{N} = (N, n, \star, \tau)$ be motors, and $f\colon \boldsymbol{M} \to \boldsymbol{N}$ be such that $f\colon M \to N$ is an invertible function. By definition of a motor morphism, f is such that all three of the diagrams

$$\begin{tikzcd} & 1 \arrow[dl, "m"'] \arrow[dr, "n"] & \\ M \arrow[rr, "f"'] & & N \end{tikzcd} \qquad \begin{tikzcd} M \times M \arrow[r, "f \times f"] \arrow[d, "*"'] & N \times N \arrow[d, "\star"] \\ M \arrow[r, "f"'] & N \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\sigma"'] & N \arrow[d, "\tau"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

commute. In a commuting diagram we may replace invertible arrows with their inverse as long as the resulting diagram makes sense. That is, we have commuting diagrams

$$\begin{tikzcd} & 1 \arrow[dl, "m"'] \arrow[dr, "n"] & \\ M & & N \arrow[ll, "f^{-1}"] \end{tikzcd} \qquad \begin{tikzcd} M \times M \arrow[d, "*"'] & N \times N \arrow[l, "f^{-1} \times f^{-1}"'] \arrow[d, "\star"] \\ M & N \arrow[l, "f^{-1}"] \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d, "\sigma"'] & N \arrow[l, "f^{-1}"'] \arrow[d, "\tau"] \\ M & N. \arrow[l, "f^{-1}"] \end{tikzcd}$$

Note that $(f \times f)^{-1} = f^{-1} \times f^{-1}$ by bifunctorality of the cartesian product. Therefore $f^{-1}\colon \boldsymbol{N} \to \boldsymbol{M}$ is a motor morphism and hence f is a motor isomorphism. Therefore U reflects isomorphisms, and hence so must F. $\square$

We have shown that both $\mathbf{Motor}$ and $\mathbf{Magma}_*$ have initial objects, and that $F\colon \mathbf{Motor} \to \mathbf{Magma}_*$ reflects isomorphisms. Therefore we have the following:

Corollary 2.11. The functor $F\colon \mathbf{Motor} \to \mathbf{Magma}_*$ preserves initial objects if and only if it creates initial objects.

We have already proven that F preserves initial objects, and hence it must also create initial objects. So that we can generalise it later, here is a direct proof that F creates initial objects.

For the rest of this chapter let $\boldsymbol{B} = (B, e, \bullet)$ be the initial pointed magma. We want to construct a motor structure $\tilde{\boldsymbol{B}}$ in such a way that $F\tilde{\boldsymbol{B}} = \boldsymbol{B}$, that is, we need an associative operation $*$ on B with identity e, and an endofunction $\tau\colon B \to B$ such that $\tau(x) * y = x \bullet y$, for all x and y in B.

The associative operation $*$ is given by the following recursive definition:

$$e * z = z$$
$$(x \bullet y) * z = x \bullet (y * z).$$

To prove that $(B, e, *)$ is a monoid we use induction in the initial pointed magma, that is, we use the fact that if a statement is true for e, and if the statement being true for x and y implies it is also true for $x \bullet y$, then it must be true for all x in B.

Proposition 2.12. e is the identity for $*$.

Proof. By definition of $*$ we have that $e * x = x$ for all x in B, and hence we just need to prove that $x * e = x$ for all x in B. This is done by induction on x.

By definition of $*$ we have that $e * e = e$.

Now suppose that $x_1 * e = x_1$ and $x_2 * e = x_2$, then by the recursive definition of $*$ we have that

$$(x_1 \bullet x_2) * e = x_1 \bullet (x_2 * e) = x_1 \bullet x_2$$

as required.

Therefore by induction $x * e = x$ for all x in B, and hence e is the identity for $*$. □

Proposition 2.13. $*$ is associative.

Proof. This is also done by induction on x. For the base case we have by definition of $*$ that

$$(e * y) * z = y * z = e * (y * z)$$

for all y, z in B.

Now suppose that $(x_1 * y) * z = x_1 * (y * z)$ and $(x_2 * y) * z = x_2 * (y * z)$ for all y, z in B. Then

$$\begin{aligned}\big((x_1 \bullet x_2) * y\big) * z &= \big(x_1 \bullet (x_2 * y)\big) * z \\ &= x_1 \bullet \big((x_2 * y) * z\big) \\ &= x_1 \bullet \big(x_2 * (y * z)\big) \\ &= (x_1 \bullet x_2) * (y * z)\end{aligned}$$

as required.

Therefore by induction $(x * y) * z = x * (y * z)$ for all x, y, z in B, i.e. $*$ is associative. □

In order to get a motor structure on B we require an endofunction $\tau \colon B \to B$ and this is given by $\tau(x) = x \bullet e$. Hence we have the motor $\tilde{\boldsymbol{B}} = (B, e, *, \tau)$.

Proposition 2.14. $F\tilde{\boldsymbol{B}} = \boldsymbol{B}$, i.e. F lifts initial objects.

Proof. All we need to show is that $\tau(x) * y = x \bullet y$. Indeed we have:

$$\begin{aligned}\tau(x) * y &= (x \bullet e) * y \\ &= x \bullet (e * y) \\ &= x \bullet y.\end{aligned}$$

□

We now show that F reflects, and therefore creates, initial objects.

Proposition 2.15. Given a motor $\boldsymbol{M} = (M, m, \star, \mu)$, the unique pointed magma map given by the universal property of the initial pointed magma, $f \colon \boldsymbol{B} \to F\boldsymbol{M}$, lifts to a motor map $\tilde{f} \colon \tilde{\boldsymbol{B}} \to \boldsymbol{M}$.

Proof. We must show that for all x, y in B, f satisfies

$$f(e) = m$$
$$f\Big(\tau(x)\Big) = \mu\Big(f(x)\Big)$$
$$f(x * y) = f(x) \star f(y).$$

By definition of f and $F\boldsymbol{M}$ we have that for all x, y in B

$$f(e) = m \tag{9}$$
$$f(x \bullet y) = \mu f(x) \star f(y). \tag{10}$$

We hence see that for all x in B

$$\begin{aligned}
f\Big(\tau(x)\Big) &= f(x \bullet e) && \text{by definition of } \tau \\
&= \mu\Big(f(x)\Big) \star f(e) && \text{by (10)} \\
&= \mu\Big(f(x)\Big) \star m && \text{by (9)} \\
&= \mu\Big(f(x)\Big) && \text{since } (M, m, \star) \text{ is a monoid.}
\end{aligned}$$

Finally we have that

$$f(e * y) = f(y) = m \star f(y) = f(e) \star f(y),$$

and if both $f(x_1 * y) = f(x_1) \star f(y)$ and $f(x_2 * y) = f(x_2) \star f(y)$, then

$$\begin{aligned}
f\Big((x_1 \bullet x_2) * y\Big) &= f\Big(x_1 \bullet (x_2 * y)\Big) && \text{by definition of } * \\
&= \mu\Big(f(x_1)\Big) \star f(x_2 * y) && \text{by (10)} \\
&= \mu\Big(f(x_1)\Big) \star \Big(f(x_2) \star f(y)\Big) && \text{by assumption} \\
&= \bigg(\mu\Big(f(x_1)\Big) \star f(x_2)\bigg) \star f(y) && \text{by associativity of } \star \\
&= f(x_1 \bullet x_2) \star f(y) && \text{by (10)}
\end{aligned}$$

for all y in B, and hence by induction

$$f(x * y) = f(x) \star f(y)$$

for all x, y in B. $\square$

The lifted map $\tilde{f}\colon \tilde{\boldsymbol{B}} \to \boldsymbol{M}$ is unique since given any other motor map $g\colon \tilde{\boldsymbol{B}} \to \boldsymbol{M}$, we must have $Fg = f = F\tilde{f}$ by initiality of $\boldsymbol{B}$ in $\mathbf{Magma}_*$, and therefore $g = \tilde{f}$ since F is faithful.

We have therefore proven that F creates initial objects and hence constructively shown:

Theorem 2.16. If $\boldsymbol{B}$ is the initial pointed magma, then F lifts $\boldsymbol{B}$ to the initial motor $\tilde{\boldsymbol{B}}$.

CHAPTER 3

A Generalisation in Set

In this chapter we present a generalisation of the results of the previous chapter using the notion of an algebra for an endofunctor.

1. Algebras, Induction and Recursion

Definition 3.1. Let $\mathcal{C}$ be a category and let $T\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor. A T-*algebra*, (A, α), consists of an object, $A \in \mathcal{C}$, and a morphism, $\alpha\colon TA \to A$. We call the object the *carrier*, and the morphism the *structure* of the algebra.

A *homomorphism* between two T-algebras, (A, α) and (B, β), is a morphism, $f\colon A \to B$ in $\mathcal{C}$, such that the following square commutes:

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

We write T-$\mathbf{Alg}$ for the category of T-algebras and their homomorphisms.

Definition 3.2. An initial T-algebra is an initial object in T-$\mathbf{Alg}$. That is, an initial T-algebra (A, α) is one such that given any algebra (B, β), there exists a unique $f\colon A \to B$ in $\mathcal{C}$ making the following square commute:

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

When $\mathcal{C} = \mathbf{Set}$ and $TX = 1 + X$ the T-algebras are pointed sets with an endofunction, which are sometimes known as numerals. An initial T-algebra for this endofunctor is given by the natural numbers together with 0 and the usual successor function. We now generalise the familiar notions of induction and recursion in the natural numbers to an arbitrary initial algebra. These are fairly well known constructions, see for example [4], however for completeness and readability we present the arguments and proofs here.

Induction

Let φ be a proposition about the natural numbers. In order to show that φ is true for all natural numbers we consider $P = \left\{ n \in \mathbb{N} : \varphi(n) = \top \right\}$. If we can show that both 0 is in P, and that n being in P implies that $n+1$ is also in P, then we can conclude by induction that $P = \mathbb{N}$. The hypotheses on P are equivalent to the requirement that the upper composite in

$$\begin{tikzcd} 1+P \arrow[r, "1+i"] \arrow[d, dashed] & 1+\mathbb{N} \arrow[d, "{[0,s]}"] \\ P \arrow[r, hook, "i"'] & \mathbb{N} \end{tikzcd}$$

factors through i as indicated. A triple consisting of a set, a point in that set, and an endofunction on that set is called a *numeral* by Freyd. We have hence shown that P is a sub numeral of $\mathbb{N}$. The conclusion that $P = \mathbb{N}$ now follows from initiality of $\mathbb{N}$ and:

Proposition 3.3. Let $\mathcal{C}$ be an arbitrary category and I be initial in $\mathcal{C}$. If X is in $\mathcal{C}$, and $m\colon X \rightarrowtail I$ is a monomorphism in $\mathcal{C}$, then m is an isomorphism.

Proof. I is initial and so there exists a map $f\colon I \to X$. By the universal property of the initial object, $mf\colon I \to I$ must equal 1_I. Additionally, $m(fm) = (mf)m = 1_I m = m$, and since m is a monomorphism we must have $fm = 1_X$. Therefore m is an isomorphism with inverse f. ☐

Using this result, if (A, α) is initial in T-$\mathbf{Alg}$, then we have a corresponding notion of induction: If given $i\colon P \rightarrowtail A$ for which there exists a factorisation ψ as indicated by

$$\begin{tikzcd} TP \arrow[r, "Ti"] \arrow[d, dashed, "\exists\psi"'] & TA \arrow[d, "\alpha"] \\ P \arrow[r, tail, "i"'] & A, \end{tikzcd}$$

then we can conclude that $i\colon P \rightarrowtail A$ is an isomorphism.

Recursion

Consider the first order recurrence relation

$$a_{n+1} = f(n, a_n)$$

where the a_i belong to some set X. If we are given an initial condition, a_0 in X, we know that this recurrence relation defines a unique sequence of elements of X, i.e. we have a unique function $g\colon \mathbb{N} \to X$ satisfying $g(0) = a_0$ and $g(n+1) = f\big(n, g(n)\big)$. This can be generalised to an arbitrary initial algebra as follows:

Theorem 3.4. Let (A, α) be the initial algebra for the endofunctor T on the category $\mathcal{C}$ with binary products. Given X in $\mathcal{C}$ and a morphism $f\colon T(A \times X) \to X$, there exists a unique morphism $g\colon A \to X$ such that the square

$$\begin{tikzcd} TA \arrow[r, "T\left(1_A, g\right)"] \arrow[d, "\alpha"'] & T(A \times X) \arrow[d, "f"] \\ A \arrow[r, "g"'] & X \end{tikzcd}$$

commutes in $\mathcal{C}$.

Proof. In order to use initiality of (A, α) we need an algebra structure on $A \times X$, and this is given by the pairing $(\alpha \circ T\pi_1, f)\colon T(A \times X) \to A \times X$, where π_1 is the projection map. By initiality of (A, α), there exists a unique morphism (h, g) making the square

$$\begin{tikzcd} TA \arrow[r, "T(h,g)"] \arrow[d, "\alpha"'] & T(A \times X) \arrow[d, "(\alpha \circ T\pi_1, f)"] \\ A \arrow[r, "(h,g)"'] & A \times X \end{tikzcd}$$

commute. Post-composing this square with the first projection map we get the commuting diagram

$$\begin{tikzcd} TA \arrow[r, "T(h,g)"] \arrow[d, "\alpha"'] & T(A \times X) \arrow[r, "T\pi_1"] \arrow[d, "(\alpha \circ T\pi_1, f)"] & TA \arrow[d, "\alpha"] \\ A \arrow[r, "(h,g)"'] & A \times X \arrow[r, "\pi_1"'] & A \end{tikzcd}$$

from which we conclude by initiality of A that $h = 1_A$. Post-composing with the second projection map we thus get

$$\begin{tikzcd} TA \arrow[r, "T(1_A,g)"] \arrow[d, "\alpha"'] & T(A \times X) \arrow[d, "f"] \\ A \arrow[r, "g"'] & X, \end{tikzcd}$$

for which g is the required unique morphism. $\square$

2. The Generalisation

For the rest of this chapter we fix $\mathcal{C} = \mathbf{Set}$, let $T\colon \mathbf{Set} \to \mathbf{Set}$ be a fixed endofunctor, and make the following definitions.

Definition 3.5. A *T-motor* is a quadruple $\boldsymbol{M} = (M, m, *, \sigma)$, where $(M, m, *)$ is a monoid and $\sigma\colon TM \to M$ is a function. A T-motor morphism $f\colon \boldsymbol{M} \to \boldsymbol{N}$ between $\boldsymbol{M} = (M, m, *, \sigma)$ and $\boldsymbol{N} = (N, n, \star, \tau)$ is a function such that all three of the diagrams

$$\begin{tikzcd} & 1 \arrow[dl, "m"'] \arrow[dr, "n"] & \\ M \arrow[rr, "f"'] & & N \end{tikzcd} \qquad \begin{tikzcd} M \times M \arrow[r, "f \times f"] \arrow[d, "*"'] & N \times N \arrow[d, "\star"] \\ M \arrow[r, "f"'] & N \end{tikzcd} \qquad \begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\sigma"'] & TN \arrow[d, "\tau"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

commute. We call the category of T-motors and their morphisms T-$\mathbf{Motor}$.

Definition 3.6. A *pointed T-magma* is a triple $\boldsymbol{M} = (M, m, \mu)$, where M is a set, m is an element of M, and $\mu\colon TM \times M \to M$ is a function. A pointed T-magma morphism $f\colon \boldsymbol{M} \to \boldsymbol{N}$ between

$\boldsymbol{M} = (M, m, \mu)$ and $\boldsymbol{N} = (N, n, \nu)$ is a function such that both of the diagrams

$$\begin{tikzcd} & 1 \arrow[dl, "m"'] \arrow[dr, "n"] & \\ M \arrow[rr, "f"'] & & N \end{tikzcd} \qquad \begin{tikzcd} TM \times M \arrow[r, "Tf \times f"] \arrow[d, "\mu"'] & TN \times N \arrow[d, "\nu"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

commute.

Our motivation for making these definitions becomes apparent upon observing that if T is the identity functor, then T-$\mathbf{Motor} = \mathbf{Motor}$ and T-$\mathbf{Magma}_* = \mathbf{Magma}_*$. We now seek to generalise the relationship between the initial motor and the initial pointed magma to the arbitrary T-algebra case.

Proposition 3.7. There exists a faithful functor $F\colon T\text{-}\mathbf{Motor} \to T\text{-}\mathbf{Magma}_*$ whose action on objects, $\boldsymbol{M} = (M, m, *, \sigma)$, is given by $F\boldsymbol{M} = (M, m, \tilde{\sigma})$, where $\tilde{\sigma}(x, y) = \sigma(x) * y$.

Proof. Given a morphism $f \in T\text{-}\mathbf{Motor}(\boldsymbol{M}, \boldsymbol{N})$ between $\boldsymbol{M} = (M, m, *, \sigma)$ and $\boldsymbol{N} = (N, n, \star, \tau)$, we have that

$$\begin{tikzcd} & 1 \arrow[dl, "m"'] \arrow[dr, "n"] & \\ M \arrow[rr, "f"'] & & N \end{tikzcd}$$

commutes by definition of f. We also have the commuting diagram

$$\begin{tikzcd} TM \times M \arrow[r, "Tf \times f"] \arrow[d, "\sigma \times \mathrm{id}"'] & TN \times N \arrow[d, "\tau \times \mathrm{id}"] \\ M \times M \arrow[r, "f \times f"] \arrow[d, "*"'] & N \times N \arrow[d, "\star"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

as required for $Ff\colon F\boldsymbol{M} \to F\boldsymbol{N}$ to be a morphism in T-$\mathbf{Magma}_*$. F preserves identities and compositions since Ff has the same action on elements as f, and so F is a functor. Lastly, F satisfies the composition

$$\begin{tikzcd} T\text{-}\mathbf{Motor} \arrow[rr, "F"] \arrow[dr, "U"'] & & T\text{-}\mathbf{Magma}_* \arrow[dl, "V"] \\ & \mathbf{Set} & \end{tikzcd}$$

where U and V are the forgetful functors, and hence F is faithful. $\square$

The major aim of this chapter is in proving the following:

Theorem 3.8. The functor $F\colon T\text{-}\mathbf{Motor} \to T\text{-}\mathbf{Magma}_*$ as given above both preserves and creates initial objects.

First however, we consider a few examples as motivation for why we made this generalisation.

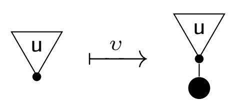
Example 3.9. Let A be a set, and let A^* be the set of finite words with alphabet A. It is a well established fact that $(A^*, \varepsilon, +)$ is the free monoid generated by A, where ε is the empty word and $+$ is word concatenation. As such if we let T be be the endofunctor $TX = A$ for all sets X with $Tf = 1_A$ for all functions f, then we have an initial T-motor $(A^*, \varepsilon, +, \sigma)$, where $\sigma\colon A \to A^*$ is the function which maps each element of A to its singleton list.

In computer science it is common to generate A^* via the empty word ε and the function $\alpha\colon A \times A^* \to A^*$, which appends an element x in A to the front of a word in A^*. This yields an initial pointed T-magma $(A^*, \varepsilon, \alpha)$, and the fact that these two descriptions of A^* agree is explained by Theorem 3.8.

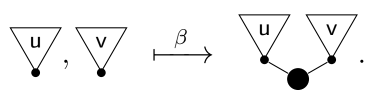
Example 3.10. A special case of the previous example occurs when T is the constant functor $TX = 1$ for all sets X. In this case a pointed T-magma is simply a numeral. The usual description of the natural numbers as being generated by the number 0 and the successor function $s(n) = n + 1$ tells us that $(\mathbb{N}, 0, s)$ is an initial pointed T-magma. An initial T-motor is a free monoid on a single generator, and Theorem 3.8 yields the fact that the natural numbers under addition is such an object.

Example 3.11. This example is closely related to the observations of [5]. A unary-binary tree is a rooted tree in which every node has at most two children. The set of unary-binary trees, $\mathcal{U}$, is generated by:

- The trivial tree: $e = \bullet$
- A unary operation:

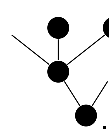


- A binary operation:



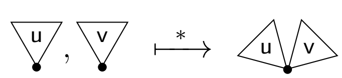
We hence have an initial pointed T-magma $\boldsymbol{\mathcal{U}} = \big(\mathcal{U}, e, [\upsilon, \beta]\big)$, where T is the endofunctor $TX = 1 + X$.

Now consider the set of all rooted trees having open or closed leaves, $\mathcal{H}$. An example of such a tree is

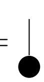


$\mathcal{H}$ is generated by:

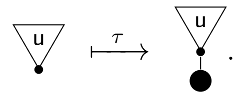
- The trivial tree: $e = \bullet$
- An associative binary operation with unit e:



- The tree consisting of a root and a single open leaf:

$a =$ 

- A unary operation:



In other words $\boldsymbol{\mathcal{H}} = \big(\mathcal{H}, e, *, [a, \tau]\big)$ is an initial T-motor.

Theorem 3.8 tells us that $F\boldsymbol{\mathcal{H}} \cong \boldsymbol{\mathcal{U}}$, and hence there must be a bijection between the two sets $\mathcal{H}$ and $\mathcal{U}$. A way to see this correspondence is via the Motzkin words, that is, words on the symbols U, D, and A which have the same number of Us and Ds, but never more Ds than Us when reading from left to right.

The set of Motzkin words, $\mathcal{M}$, is generated in a similar way to the set of Dyck words but with an extra generator, the symbol A. That is, if T is the endofunctor given by $TX = 1 + X$, then we have an initial T-motor $\boldsymbol{\mathcal{M}} = \big(\mathcal{M}, \varepsilon, +, [A, \sigma]\big)$, where ε is the empty word, $+$ is word concatenation, and $\sigma(x) = UxD$. Applying Theorem 3.8 we also have that $\mathcal{M}$ is generated by ε, the unary operation α which appends A to the front of a Motzkin word, and the binary operation given by $x \bullet y = UxDy$. That is, we have an initial pointed T-magma $F\boldsymbol{\mathcal{M}} = \big(\mathcal{M}, \varepsilon, [\alpha, \bullet]\big)$.

This gives us a way of finding a bijection between $\mathcal{H}$ and $\mathcal{U}$. If $f\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{M}}$ is the unique T-motor isomorphism given by initiality of both $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{M}}$, and $g\colon F\boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{U}}$ is the unique pointed T-magma isomorphism given by initiality of both $F\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{U}}$, then the composite $g \circ f$ is a bijection between $\mathcal{H}$ and $\mathcal{U}$.

The Motzkin numbers are a sequence with a close relationship to the Catalan numbers. The nth Motzkin number counts the number of Motzkin words of total length n. For Motzkin words x and y, the length of $\alpha(x)$ is 1 greater than the length of x, and the length of $x \bullet y$ is 2 greater then the sum of the lengths of x and y. Hence if we endow the natural numbers with a pointed T-magma structure $\boldsymbol{N} = \big(\mathbb{N}, 0, [s, \star]\big)$ where $s(n) = n + 1$, and $k \star m = k + m + 2$, then taking fibres over the unique pointed T-magma map $f\colon F\boldsymbol{\mathcal{M}} \to \boldsymbol{N}$ gives sets whose cardinality M_n is the number of Motzkin words of length n. This yields a recurrence relation for the Motzkin numbers

$$M_n = \sum_{k=0}^{n-2} M_k M_{n-k-2} + M_{n-1}, \quad M_0 = 1,$$

which has first terms: $1, 1, 2, 4, 9, 21, \ldots$ [2, A001006]

Looking back to the pointed T-magma structure on $\mathcal{U}$ we see that e has zero edges, and for trees u and v, $v(u)$ has one more edge than u, and the number of edges of $\beta(u, v)$ is two greater than the sum of the number of edges of u and v. Hence the nth Motzkin number also counts the number of unary-binary trees with n edges.

Looking at the T-motor structure on $\mathcal{M}$ we see that ε has length zero, A has length one, and for words x and y, the length of $x + y$ is the sum of the lengths of x and y, and the length of $\sigma(x)$ is two greater than the length of x. Similarly if we define a *half-edge* to be an incident pair of a vertex and an edge, then looking at the T-motor structure on $\mathcal{H}$ we see that e has zero half-edges, a has one half-edge, and for trees u and v, the number of half-edges of $u * v$ is the sum of the number of half-edges of u and v, and the number of half-edges of $\tau(u)$ is two greater than the number of half-edges of u. That is, the nth Motzkin number also counts the number of rooted trees having open or closed leaves with n half-edges.

Another closely related sequence is the Schröder numbers. The nth Schröder number counts the number of Motzkin words of length n where each U, D pair counts as length 1. To obtain the Schröder numbers we modify $\star$ to $k \star m = k + m + 1$, but leave the rest of the pointed T-magma structure on $\mathbb{N}$ unchanged. The nth Schröder number is then obtained through the same process as the Motzkin numbers yielding the recurrence relation

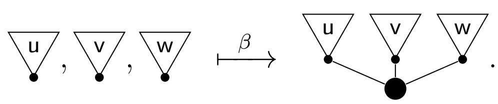
$$S_n = \sum_{k=0}^{n-1} S_k S_{n-k-1} + S_{n-1}, \quad S_0 = 1,$$

which has first terms: $1, 2, 6, 22, 90, 394, \ldots$ [2, A006318]

Following the same process as before we see that the nth Schröder number also counts the number of unary-binary trees with n internal vertices, and the number of rooted trees having open or closed leaves with n edges.

Example 3.12. The set of all full ternary trees, $\mathcal{T}$, is generated by:

- The trivial tree: $e = \bullet$
- A ternary operation:



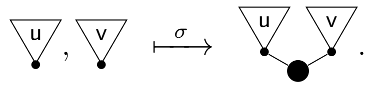
That is, if T is the endofunctor $TX = X^2$, then we have an initial pointed T-magma $\boldsymbol{\mathcal{T}} = (\mathcal{T}, e, \beta)$. Noting that the number of internal vertices of $\beta(u, v, w)$ is 1 greater than the sum of the internal vertices of u, v, and w, if we endow the natural numbers with the pointed T-magma structure $\boldsymbol{N} = \Big(\mathbb{N}, 0, (x, y, z) \mapsto x + y + z + 1\Big)$, then taking fibres over the unique pointed T-magma map $f \colon \boldsymbol{\mathcal{T}} \to \boldsymbol{N}$ gives sets $\mathcal{T}_n$ consisting of the full ternary trees with n internal vertices. From this we see that the recurrence relation

$$t_{n+1} = \sum_{l+k+m=n} t_l t_k t_m, \quad t_0 = 1$$

counts the number of full ternary trees with n internal vertices.

This recurrence relation has solution given by $t_n = \dfrac{1}{2n+1}\dbinom{3n}{n}$ [6, p. 361], and yields a sequence with first terms $1, 1, 3, 12, 55, \ldots$ This is sequence [2, A001764] and, as stated there, also counts the "number of rooted plane trees with $2n$ edges, where every vertex has even outdegree."

The set of rooted plane trees where every vertex has even outdegree, E has a T-motor structure $\boldsymbol{E} = (E, e, *, \sigma)$ where e and $*$ are as in the ordinary rooted tree case, and $\sigma\colon E^2 \to E$ is given by



E is in fact generated by this structure and so by Theorem 3.8 we again see a correspondence between the initial T-motor and the initial pointed T-magma.

The example on lists and the example on ternary trees show that is reasonable to suppose that the correspondence between the initial T-motor and the initial pointed T-magma holds whenever T is any polynomial functor. However, as we will prove, the correspondence holds for any endofunctor T.

3. The Initial T-motor is the Initial Pointed T-magma

We now follow the construction of the previous chapter to prove that F preserves initial objects. This first requires a few definitions.

Definition 3.13. Given sets B and C we write C^B for the exponential object, that is, the set of functions from B to C. We indicate taking the *transpose* of a morphism f under the product-exponential adjunction in either direction by $\bar{f}$, i.e.

$$\frac{A \times B \xrightarrow{\quad f \quad} C}{A \xrightarrow[\quad \bar{f} \quad]{} C^B} \quad \text{and} \quad \frac{A \xrightarrow{\quad g \quad} C^B}{A \times B \xrightarrow[\quad \bar{g} \quad]{} C} \quad .$$

We use ev to denote the counit of the product-exponential adjunction, that is, the *evaluation* map $\mathrm{ev}\colon C^B \times B \to C$ such that

$$\mathrm{ev}(f, b) = f(b).$$

Given $b\colon 1 \to B$, we define $\mathrm{ev}_b\colon C^B \to C$ by the composite

$$\mathrm{ev}_b\colon C^B \xrightarrow{(\mathrm{id}, b)} C^B \times B \xrightarrow{\quad \mathrm{ev} \quad} C.$$

That is,

$$\mathrm{ev}_b(f) = f(b).$$

With these definitions in place we are now able to construct a T-motor from an arbitrary pointed T-magma as follows: Given a pointed T-magma $\boldsymbol{B} = (B, b, \beta)$, we define $\hat{\beta}$ to be the composite

$$\hat{\beta}\colon T\left(B^B\right) \xrightarrow{T\mathrm{ev}_b} TB \xrightarrow{\quad \bar{\beta} \quad} B^B,$$

and hence obtain a T-motor $\hat{\boldsymbol{B}} \coloneqq \left(B^B, 1_B, \circ, \hat{\beta}\right)$.

Proposition 3.14. If $\boldsymbol{B} = (B, b, \beta)$ is an arbitrary T-magma, then $\mathrm{ev}_b\colon F\hat{\boldsymbol{B}} \to \boldsymbol{B}$ is a pointed T-magma morphism.

Proof. We must show that both

$$\begin{tikzcd} & 1 \arrow[dl, "1_B"'] \arrow[dr, "b"] & \\ B^B \arrow[rr, "\mathrm{ev}_b"'] & & B \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} T\left(B^B\right) \times B^B \arrow[r, "T\mathrm{ev}_b \times \mathrm{ev}_b"] \arrow[d, "\tilde{\bar{\beta}}"'] & TB \times B \arrow[d, "\beta"] \\ B^B \arrow[r, "\mathrm{ev}_b"'] & B \end{tikzcd}$$

commute.

The triangle trivially commutes, and as for the square we have

$$\begin{tikzcd} T\left(B^B\right) \times B^B \arrow[d, "T\mathrm{ev}_b \times \mathrm{id}"'] \arrow[dr, "T\mathrm{ev}_b \times \mathrm{ev}_b"] \arrow[ddd, bend right=60, "\tilde{\bar{\beta}}"'] & \\ TB \times B^B \arrow[r, "\mathrm{id} \times \mathrm{ev}_b"'] \arrow[d, "\bar{\beta} \times \mathrm{id}"'] & TB \times B \arrow[d, "\bar{\beta} \times \mathrm{id}"] \arrow[dd, bend left=40, "\beta"] \\ B^B \times B^B \arrow[r, "\mathrm{id} \times \mathrm{ev}_b"'] \arrow[d, "\circ"'] & B^B \times B \arrow[d, "\mathrm{ev}"] \\ B^B \arrow[r, "\mathrm{ev}_b"'] & B \end{tikzcd}$$

as required. $\square$

Corollary 3.15. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be as above. If $f\colon \boldsymbol{A} \to \hat{\boldsymbol{B}}$ is the unique T-motor morphism given by initiality of $\boldsymbol{A}$, then the function $g\colon A \to B$ given by the composite

$$g\colon A \xrightarrow{\ f\ } B^B \xrightarrow{\ \mathrm{ev}_b\ } B$$

is a pointed T-magma morphism, that is, $g\colon F\boldsymbol{A} \to \boldsymbol{B}$.

Proof. F is a functor and hence $f\colon F\boldsymbol{A} \to F\hat{\boldsymbol{B}}$ is a T-$\mathbf{Magma}_*$ morphism, and therefore by the previous proposition so is the composite g. $\square$

Now that we have a pointed T-magma morphism $g\colon F\boldsymbol{A} \to \boldsymbol{B}$, we must show that it is unique for which we need the following:

Proposition 3.16. Let $\boldsymbol{A} = (A, e, *, \sigma)$ be the initial T-motor, let $\boldsymbol{B}$ be an arbitrary pointed T-magma, and let $f_A\colon \boldsymbol{A} \to \widehat{F\boldsymbol{A}}$ and $f_B\colon \boldsymbol{A} \to \hat{\boldsymbol{B}}$ be the unique T-motor morphisms given by initiality of $\boldsymbol{A}$. If $h\colon F\boldsymbol{A} \to \boldsymbol{B}$ is a pointed T-magma morphism, then the square

$$\begin{tikzcd} A \arrow[r, "f_A(a)"] \arrow[d, "h"'] & A \arrow[d, "h"] \\ B \arrow[r, "f_B(a)"'] & B \end{tikzcd}$$

commutes in $\mathbf{Set}$ for all a in A.

Proof. Let $C = \Big\{(u,v) \in A^A \times B^B : hu = vh\Big\}$, that is, C is the set of all pairs of functions u and v that make the following square commute:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "h"'] & A \arrow[d, "h"] \\ B \arrow[r, "v"'] & B. \end{tikzcd}$$

We now need to show that $\Big(f_A(a), f_B(a)\Big)$ is in C for all a in A. In order to do this we now consider the set of all pairs of functions $u\colon A \to A$ and $v\colon B \to B$, i.e. the set $A^A \times B^B$. This set has a T-motor structure given by the product

$$\widehat{F\boldsymbol{A}} \times \hat{\boldsymbol{B}} = \Big(A^A \times B^B, (1_A, 1_B), \star, \tau\Big)$$

where

$$(u_1, v_1) \star (u_2, v_2) = (u_1 \circ v_1, u_2 \circ v_2)$$

and

$$\tau\colon T\Big(A^A \times B^B\Big) \xrightarrow{\big(T\pi_1, T\pi_2\big)} T\Big(A^A\Big) \times T\Big(B^B\Big) \xrightarrow{\hat{\hat{\sigma}} \times \hat{\beta}} A^A \times B^B,$$

and therefore $(f_A, f_B)\colon \boldsymbol{A} \to \widehat{F\boldsymbol{A}} \times \hat{\boldsymbol{B}}$ is the unique T-motor morphism given by initiality of $\boldsymbol{A}$.

We claim that $C \subseteq A^A \times B^B$ is in fact a sub T-motor $\boldsymbol{C} \subseteq \widehat{F\boldsymbol{A}} \times \hat{\boldsymbol{B}}$, i.e. that there exists a factorisation

$$\begin{tikzcd} TC \arrow[r, "Ti"] \arrow[d, dashed] & T\Big(A^A \times B^B\Big) \arrow[d, "\tau"] \\ C \arrow[r, "i"'] & A^A \times B^B, \end{tikzcd}$$

where i is the inclusion of C into $A^A \times B^B$.

Note that C is the equaliser of the composites

$$A^A \times B^B \xrightarrow{\pi_1} A^A \xrightarrow{h^A} B^A$$

and

$$A^A \times B^B \xrightarrow{\pi_2} B^B \xrightarrow{B^h} B^A,$$

where h^A denotes post-composition by h, and B^h denotes pre-composition by h. Hence it suffices to prove that

$$\begin{tikzcd} TC \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & A^A \times B^B \arrow[r, "\pi_1"] & A^A \arrow[d, "h^A"] \\ A^A \times B^B \arrow[r, "\pi_2"'] & B^B \arrow[r, "B^h"'] & B^A \end{tikzcd}$$

commutes, or equivalently that

$$
\begin{tikzcd}
TC \times A \arrow[r, "\gamma \times \mathrm{id}"] \arrow[d, "\gamma \times \mathrm{id}"'] & \left(A^A \times B^B\right) \times A \arrow[r, "\pi_1 \times \mathrm{id}"] & A^A \times A \arrow[r, "\mathrm{ev}"] & A \arrow[d, "h"] \\
\left(A^A \times B^B\right) \times A \arrow[r, "\pi_2 \times \mathrm{id}"'] & B^B \times A \arrow[r, "\mathrm{id} \times h"'] & B^B \times B \arrow[r, "\mathrm{ev}"'] & B
\end{tikzcd}
$$

commutes, where γ is given by the composition:

$$
\gamma \colon TC \xrightarrow{Ti} T\left(A^A \times B^B\right) \xrightarrow{\tau} A^A \times B^B.
$$

By the definition of the pairing (id, e) we have that both the top and bottom trapeziums of

$$
\begin{tikzcd}
C \times A \arrow[r, "i \times \mathrm{id}"] & \left(A^A \times B^B\right) \times A \arrow[r, "\pi_1 \times \mathrm{id}"] & A^A \times A \arrow[drr, "\mathrm{ev}"] & & \\
C \arrow[u, "\left(\mathrm{id}, e\right)"] \arrow[r, "i"] \arrow[d, "\mathrm{id}"'] & A^A \times B^B \arrow[r, "\pi_1"] & A^A \arrow[u, "\left(\mathrm{id}, e\right)"] \arrow[r, "\left(\mathrm{id}, e\right)"] & A^A \times A \arrow[r, "\mathrm{ev}"'] & A \arrow[d, "h"] \\
C \arrow[r, "i"] \arrow[d, "\left(\mathrm{id}, e\right)"'] & A^A \times B^B \arrow[r, "\pi_2"] & B^B \arrow[r, "\left(\mathrm{id}, b\right)"] \arrow[d, "\left(\mathrm{id}, e\right)"'] & B^B \times B \arrow[r, "\mathrm{ev}"] & B \\
C \times A \arrow[r, "i \times \mathrm{id}"'] & \left(A^A \times B^B\right) \times A \arrow[r, "\pi_2 \times \mathrm{id}"'] & B^B \times A \arrow[r, "\mathrm{id} \times h"'] & B^B \times B \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd}
$$

commute. The outside commutes by definition of C and hence we conclude that the middle square

$$
\begin{tikzcd}
C \arrow[r, "i"] \arrow[d, "\mathrm{id}"'] & A^A \times B^B \arrow[r, "\pi_1"] & A^A \arrow[r, "\mathrm{ev}_e"] & A \arrow[d, "h"] \\
C \arrow[r, "i"'] & A^A \times B^B \arrow[r, "\pi_2"'] & B^B \arrow[r, "\mathrm{ev}_b"'] & B
\end{tikzcd}
$$

also commutes.

This square together with functoriality of T yields the commuting diagram

$$
\begin{tikzcd}
TC \times A \arrow[rr, "\left(\pi_1 \circ \gamma\right) \times \mathrm{id}"] \arrow[dr, "Ti \times \mathrm{id}"] \arrow[ddd, "\left(\pi_2 \circ \gamma\right) \times \mathrm{id}"'] & & A^A \times A \arrow[drr, "\mathrm{ev}"] & & \\
& T\left(A^A \times B^B\right) \times A \arrow[r, "T\pi_1 \times \mathrm{id}"] \arrow[d, "T\pi_2 \times \mathrm{id}"'] & T\left(A^A\right) \times A \arrow[u, "\hat{\hat{\sigma}} \times \mathrm{id}"] \arrow[r, "T\mathrm{ev}_e \times \mathrm{id}"'] & TA \times A \arrow[ul, "\bar{\bar{\sigma}} \times \mathrm{id}"'] \arrow[r, "\hat{\sigma}"'] \arrow[d, "Th \times h"'] & A \arrow[d, "h"] \\
& T\left(B^B\right) \times A \arrow[r, "T\mathrm{ev}_b \times \mathrm{id}"] & TB \times A \arrow[r, "\mathrm{id} \times h"] \arrow[dll, "\bar{\beta} \times \mathrm{id}"] & TB \times B \arrow[r, "\beta"] \arrow[d, "\bar{\beta} \times \mathrm{id}"'] & B \\
B^B \times A \arrow[rrr, "\mathrm{id} \times h"'] & & & B^B \times B, \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd}
$$

for which the top and leftmost squares commute by definition of γ, and the rightmost square commutes since h is a pointed T-magma morphism. The outside of this diagram is the desired square, and hence γ factors through C.

This gives us a sub T-motor $\boldsymbol{C} = \big(C, (1_A, 1_B), \star, \gamma\big)$ of $\widehat{F\boldsymbol{A}} \times \hat{\boldsymbol{B}}$. Therefore by initiality of $\boldsymbol{A}$ we get that the function (f_A, f_B) factors through the inclusion i, and hence for all a in A, $\big(f_A(a), f_B(a)\big)$ is in C, that is, the square

$$\begin{tikzcd}
A \arrow[r, "f_A(a)"] \arrow[d, "h"'] & A \arrow[d, "h"] \\
B \arrow[r, "f_B(a)"'] & B
\end{tikzcd}$$

commutes. $\square$

Proposition 3.17. The unique T-motor morphism $f_A\colon \boldsymbol{A} \to \widehat{F\boldsymbol{A}}$ given by initiality of $\boldsymbol{A}$ is $\overline{*}$. That is, $f_A(x)(y) = x * y$ for all x and y in A.

Proof. We must show that $\overline{*}$ is a T-motor morphism, i.e. that the following three diagrams all commute:

$$\begin{tikzcd}
 & 1 \arrow[dl, "e"'] \arrow[dr, "1_A"] & \\
A \arrow[rr, "\overline{*}"'] & & A^A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times A \arrow[r, "\overline{*} \times \overline{*}"] \arrow[d, "*"'] & A^A \times A^A \arrow[d, "\circ"] \\
A \arrow[r, "\overline{*}"'] & A^A
\end{tikzcd}
\qquad
\begin{tikzcd}
TA \arrow[r, "T\overline{*}"] \arrow[d, "\sigma"'] & T\big(A^A\big) \arrow[d, "\hat{\overline{\sigma}}"] \\
A \arrow[r, "\overline{*}"'] & A^A.
\end{tikzcd}$$

$(A, e, *)$ is a monoid and hence we have the following commuting triangle and its transpose:

$$\begin{tikzcd}
1 \times A \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "\pi_2"'] & A \times A \arrow[d, "*"] \\
 & A
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
1 \arrow[r, "e"] \arrow[dr, "1_A"'] & A \arrow[d, "\overline{*}"] \\
 & A^A.
\end{tikzcd}$$

We also have the commuting diagram

$$\begin{tikzcd}
A \times A \times A \arrow[r, "\overline{*} \times \overline{*} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times *"'] \arrow[dr, "\overline{*} \times *"] & A^A \times A^A \times A \arrow[d, "\mathrm{id} \times \mathrm{ev}"] \arrow[dd, bend left, "\overline{\circ}"] \\
A \times A \arrow[r, "\overline{*} \times \mathrm{id}"] \arrow[dr, "*"'] & A^A \times A \arrow[d, "\mathrm{ev}"] \\
 & A,
\end{tikzcd}$$

which by associativity of $*$ yields the following commuting square and its transpose:

$$\begin{tikzcd}
A \times A \times A \arrow[r, "\overline{*} \times \overline{*} \times \mathrm{id}"] \arrow[d, "* \times \mathrm{id}"'] & A^A \times A^A \times A \arrow[d, "\overline{\circ}"] \\
A \times A \arrow[r, "*"'] & A
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A \times A \arrow[r, "\overline{*} \times \overline{*}"] \arrow[d, "*"'] & A^A \times A^A \arrow[d, "\circ"] \\
A \arrow[r, "\overline{*}"'] & A^A.
\end{tikzcd}$$

Lastly, by function currying and the fact that $(A, e, *)$ is a monoid we get that

$$\begin{tikzcd}
A \arrow[r, "\overline{*}"] \arrow[d, "{\big(e, \mathrm{id}\big)}"'] \arrow[dr, "\mathrm{id}"] & A^A \arrow[d, "\mathrm{ev}_e"] \\
A \times A \arrow[r, "*"'] & A
\end{tikzcd}$$

must commute, i.e. that the composite

$$A \xrightarrow{\overline{*}} A^A \xrightarrow{\mathrm{ev}_e} A$$

is the identity map. Applying T and recalling the definition of $\hat{\hat{\sigma}}$ we get that

$$\begin{tikzcd}
TA \arrow[rr, "T\overline{*}"] \arrow[d, "\mathrm{id}"'] & & TA^A \arrow[d, "\hat{\hat{\sigma}}"] \arrow[dll, "T\mathrm{ev}_e"'] \\
TA \arrow[r, "\sigma"'] & A \arrow[r, "\overline{*}"'] & A^A
\end{tikzcd}$$

commutes, as required.

We have shown that $\overline{*}$ is a T-motor morphism, and therefore $f_A = \overline{*}$ by initiality of $\boldsymbol{A}$. $\square$

We are now able to prove the following:

Theorem 3.18. If $\boldsymbol{A} = (A, e, *, \sigma)$ is an initial T-motor, then $F\boldsymbol{A}$ is an initial pointed T-magma. That is, F preserves initial objects.

Proof. We have already shown that given an arbitrary pointed T-magma $\boldsymbol{B} = (B, b, \beta)$ the composite

$$g\colon F\boldsymbol{A} \xrightarrow{f} F\hat{\boldsymbol{B}} \xrightarrow{\mathrm{ev}_b} \boldsymbol{B}$$

is a pointed T-magma morphism, and hence we now must show that it is the unique pointed T-magma morphism. Let $h\colon F\boldsymbol{A} \to \boldsymbol{B}$ be a pointed T-magma morphism. By the previous two propositions we have that the square

$$\begin{tikzcd}
A \arrow[r, "\overline{*}(a)"] \arrow[d, "h"'] & A \arrow[d, "h"] \\
B \arrow[r, "f(a)"'] & B
\end{tikzcd}$$

commutes in $\mathbf{Set}$ for all a in A. By currying this is equivalent to the square

$$\begin{tikzcd}
A \times A \arrow[r, "*"] \arrow[d, "\mathrm{id} \times h"'] & A \arrow[d, "h"] \\
A \times B \arrow[r, "\bar{f}"'] & B
\end{tikzcd}$$

also commuting. By definition of function pairing and the fact that $h(e) = b$, we therefore have the commuting diagram

$$\begin{tikzcd}
B^B \arrow[rrrd, "\big(\mathrm{id}, b\big)"] & & & \\
A \arrow[u, "f"] \arrow[r, "\big(\mathrm{id}, e\big)"] \arrow[dr, "\mathrm{id}"'] & A \times A \arrow[r, "\mathrm{id} \times h"] \arrow[d, "*"] & A \times B \arrow[r, "f \times \mathrm{id}"'] \arrow[d, "\bar{f}"'] & B^B \times B. \arrow[dl, "\mathrm{ev}"] \\
 & A \arrow[r, "h"'] & B &
\end{tikzcd}$$

Note that the top composite is g, so we conclude that $h = g$, i.e. that g is unique. $\square$

4. The Initial Pointed T-magma is the Initial T-motor

As in the previous chapter we have that F reflects isomorphisms, and hence we know that if T-$\mathbf{Motor}$ has an initial object, then F creates initial objects. We now give a proof that F creates initial objects irrespective of knowing whether T-$\mathbf{Motor}$ has an initial object or not.

For the rest of this chapter let $\boldsymbol{B} = (B, e, \beta)$ be an initial pointed T-magma. We now show that B also has a T-motor structure, and that B together with that T-motor structure is an initial T-motor. To this end we require a monoid structure $(B, e, *)$, and this is given by the recursive definition:

$$e * y = y$$
$$\beta(w, x) * y = \beta(w, x * y).$$

We now make this precise.

Proposition 3.19. There exists an operation $*\colon B \times B \to B$ such that the diagrams

$$\begin{tikzcd} & 1 \times B \arrow[dl, "e \times \mathrm{id}"'] \arrow[dr, "\pi_2"] & \\ B \times B \arrow[rr, "*"'] & & B \end{tikzcd} \tag{11}$$

and

$$\begin{tikzcd} TB \times B \times B \arrow[r, "\mathrm{id} \times *"] \arrow[d, "\beta \times \mathrm{id}"'] & TB \times B \arrow[d, "\beta"] \\ B \times B \arrow[r, "*"'] & B \end{tikzcd} \tag{12}$$

both commute.

Proof. To see that this is a valid recursive definition we consider the functions

$$1_B \colon 1 \to B^B$$

and

$$f\colon T\Big(B \times B^B\Big) \times B \times B^B \to B^B,$$

where f is the transpose of the composite

$$\bar{f}\colon T\Big(B \times B^B\Big) \times \Big(B \times B^B\Big) \times B \xrightarrow{T\pi_1 \times \pi_2 \times \mathrm{id}} TB \times \Big(B^B \times B\Big) \xrightarrow{\mathrm{id} \times \mathrm{ev}} TB \times B \xrightarrow{\beta} B.$$

By Theorem 3.4, taking $A = B$, $X = B^B$ and $T = 1 + T(-) \times (-)$, there exists a unique function φ making both of the digrams

$$\begin{tikzcd} & 1 \arrow[dl, "e"'] \arrow[dr, "1_B"] & \\ B \arrow[rr, "\varphi"'] & & B^B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} TB \times B \arrow[r, "T\left(\mathrm{id}, \varphi\right) \times \left(\mathrm{id}, \varphi\right)"] \arrow[d, "\beta"'] & T\Big(B \times B^B\Big) \times B \times B^B \arrow[d, "f"] \\ B \arrow[r, "\varphi"'] & B^B \end{tikzcd}$$

commute. Taking the transpose of the triangle yields

$$\begin{tikzcd}
& 1 \times B \arrow[dl, "e \times \mathrm{id}"'] \arrow[dr, "\pi_2"] & \\
B \times B \arrow[rr, "\bar{\varphi}"'] & & B,
\end{tikzcd}$$

and taking the transpose of the square yields

$$\begin{tikzcd}
TB \times B \times B \arrow[r, "T\left(\mathrm{id}, \varphi\right) \times \left(\mathrm{id}, \varphi\right) \times \mathrm{id}"] \arrow[dr, "\mathrm{id} \times \varphi \times \mathrm{id}"] \arrow[ddr, "\mathrm{id} \times \bar{\varphi}"'] \arrow[ddd, "\beta \times \mathrm{id}"'] & T\left(B \times B^B\right) \times \left(B \times B^B\right) \times B \arrow[d, "T\pi_1 \times \pi_2 \times \mathrm{id}"] \\
& TB \times \left(B^B \times B\right) \arrow[d, "\mathrm{id} \times \mathrm{ev}"] \\
& TB \times B \arrow[d, "\beta"] \\
B \times B \arrow[r, "\bar{\varphi}"'] & B,
\end{tikzcd}$$

from which we can conclude that $\bar{\varphi} = *$ satisfies the required properties. $\square$

Now that we've defined $*$ we must show that it is a monoid operation with identity element e, and this is done by induction.

Proposition 3.20. The element e is the identity for $*$.

Proof. By definition e is a left identity for $*$, so we just have to prove that e is a right identity for $*$. Let $C = \{x \in B : x * e = x\}$, and let i be the inclusion of C into B. By definition of $*$ we have $e * e = e$, and hence e is in C. We also have the commuting diagram

$$\begin{tikzcd}
TC \times C \arrow[r, "Ti \times \mathrm{id}"] \arrow[dr, "Ti \times i"'] & TB \times C \arrow[rr, "\mathrm{id} \times i"] \arrow[d, "\mathrm{id} \times i"] & & TB \times B \arrow[d, "\beta"] \\
& TB \times B \arrow[r, "\mathrm{id} \times \left(\mathrm{id}, e\right)"'] \arrow[d, "\beta"'] & TB \times B \times B \arrow[ur, "\mathrm{id} \times *"] \arrow[d, "\beta \times \mathrm{id}"'] & B \arrow[d, "\mathrm{id}"] \\
& B \arrow[r, "\left(\mathrm{id}, e\right)"'] & B \times B \arrow[r, "*"'] & B,
\end{tikzcd}$$

for which the top square is the definition of i, the bottom left square is given by function pairing, and the bottom right square is (12). This diagram shows that the composite

$$TC \times C \xrightarrow{Ti \times i} TB \times B \xrightarrow{\beta} B$$

equalises $B \xrightarrow{(\mathrm{id}, e)} B \times B \xrightarrow{*} B$ and $B \xrightarrow{\mathrm{id}} B$, and hence there exists a factorisation

$$\begin{tikzcd} TC \times C \arrow[r, "Ti \times i"] \arrow[d, dashed] & TB \times B \arrow[d, "\beta"] \\ C \arrow[r, hook, "i"'] & B. \end{tikzcd}$$

Therefore by induction we conclude that $C = B$, that is, e is a right identity for $*$. ☐

Proposition 3.21. $*$ is associative.

Proof. Let $C = \left\{x \in B : x * (y * z) = (x * y) * z, \forall y, z \in B\right\}$. Proving that $*$ is associative is now equivalent to showing that $C = B$. By definition of $*$ we have that

$$e * (y * z) = y * z = (e * y) * z,$$

and hence e is in C.

Now note that C is the equaliser

$$C \xhookrightarrow{i} B \underset{g}{\overset{f}{\rightrightarrows}} B^{(B \times B)},$$

where f and g are defined by the transposes of

$$\bar{f} \colon B \times B \times B \xrightarrow{\mathrm{id} \times *} B \times B \xrightarrow{*} B$$

and

$$\bar{g} \colon B \times B \times B \xrightarrow{* \times \mathrm{id}} B \times B \xrightarrow{*} B.$$

We therefore have the commuting diagram

$$C \times (B \times B) \xrightarrow{i \times \mathrm{id}} B \times B \times B \underset{\bar{g}}{\overset{\bar{f}}{\rightrightarrows}} B.$$

By bifunctorality of the product and (12) we get the commuting diagrams

$$\begin{tikzcd} TB \times B \times B \times B \arrow[r, "\beta \times \mathrm{id} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mathrm{id} \times *"] \arrow[dd, bend right, "\mathrm{id} \times \bar{f}"'] & B \times B \times B \arrow[d, "\mathrm{id} \times *"'] \arrow[dd, bend left, "\bar{f}"] \\ TB \times B \times B \arrow[r, "\beta \times \mathrm{id}"'] \arrow[d, "\mathrm{id} \times *"] & B \times B \arrow[d, "*"'] \\ TB \times B \arrow[r, "\beta"'] & B \end{tikzcd}$$

and

$$\begin{tikzcd}
TB \times B \times B \times B \arrow[r, "\beta \times \mathrm{id} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times * \times \mathrm{id}"] \arrow[dd, bend right, "\mathrm{id} \times \overline{g}"'] & B \times B \times B \arrow[d, "* \times \mathrm{id}"'] \arrow[dd, bend left, "\overline{g}"] \\
TB \times B \times B \arrow[r, "\beta \times \mathrm{id}"'] \arrow[d, "\mathrm{id} \times *"] & B \times B \arrow[d, "*"'] \\
TB \times B \arrow[r, "\beta"'] & B.
\end{tikzcd}$$

These diagrams together with the definition of C yield the commuting diagram

$$\begin{tikzcd}
 & & TB \times B \times B \times B \arrow[r, "\beta \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \overline{f}"] & B \times B \times B \arrow[d, "\overline{f}"] \\
TC \times C \times B \times B \arrow[r, "Ti \times \mathrm{id}"] \arrow[urr, bend left, "Ti \times i \times \mathrm{id}"] \arrow[drr, bend right, "Ti \times i \times \mathrm{id}"'] & TB \times C \times B \times B \arrow[ur, "\mathrm{id} \times i \times \mathrm{id}"] \arrow[dr, "\mathrm{id} \times i \times \mathrm{id}"'] & TB \times B \arrow[r, "\beta"] & B \\
 & & TB \times B \times B \times B \arrow[r, "\beta \times \mathrm{id}"'] \arrow[u, "\mathrm{id} \times \overline{g}"] & B \times B \times B, \arrow[u, "\overline{g}"']
\end{tikzcd}$$

which upon taking transposes across the clockwise and anticlockwise composites gives us the commuting diagram

$$TC \times C \xrightarrow{Ti \times i} TB \times B \xrightarrow{\beta} B \overset{f}{\underset{g}{\rightrightarrows}} B^{(B \times B)}.$$

This shows that the composite

$$TC \times C \xrightarrow{Ti \times i} TB \times B \xrightarrow{\beta} B$$

equalises f and g, and hence there exists a factorisation

$$\begin{tikzcd}
TC \times C \arrow[r, "Ti \times i"] \arrow[d, dashed] & TB \times B \arrow[d, "\beta"] \\
C \arrow[r, hook, "i"'] & B.
\end{tikzcd}$$

Therefore by induction we conclude that $C = B$, and hence $*$ is associative. ☐

We've now shown that $(B, e, *)$ is a monoid, and hence we have a T-motor $\tilde{\boldsymbol{B}} = \left(B, e, *, \mathrm{ev}_e \circ \overline{\beta}\right)$. Our aim now is to show that $\tilde{\boldsymbol{B}}$ is an initial T-motor.

Proposition 3.22. $F\tilde{\boldsymbol{B}} = \boldsymbol{B}$, i.e. F lifts initial objects.

Proof. By definition of F it is sufficient to show that the triangle

$$\begin{tikzcd}
TB \times B \arrow[d, "\overline{\beta} \times \mathrm{id}"'] \arrow[drr, "\beta"] & & \\
B^B \times B \arrow[r, "\mathrm{ev}_e \times \mathrm{id}"'] & B \times B \arrow[r, "*"'] & B
\end{tikzcd}$$

commutes. Indeed we have

$$
\begin{tikzcd}
TB \times B \arrow[r, "\mathrm{id} \times (e, \mathrm{id})", "(\mathrm{id}, e) \times \mathrm{id}"'] \arrow[d, "\overline{\beta} \times \mathrm{id}"'] \arrow[rr, bend left, "\mathrm{id}"] & TB \times B \times B \arrow[r, "\mathrm{id} \times *"] \arrow[d, "(\overline{\beta} \times \mathrm{id}) \times \mathrm{id}"'] \arrow[dr, "\beta \times \mathrm{id}"] & TB \times B \arrow[d, "\beta"] \\
B^B \times B \arrow[r, "(\mathrm{id}, e) \times \mathrm{id}"'] & B^B \times B \times B \arrow[r, "\mathrm{ev} \times \mathrm{id}"'] & B \times B \arrow[r, "*"'] & B
\end{tikzcd}
$$

for which the outside gives the required triangle. $\square$

Proposition 3.23. Given a T-motor $\boldsymbol{M} = (M, m, \star, \mu)$, the unique pointed T-magma map given by the universal property of the initial pointed T-magma, $f \colon \boldsymbol{B} \to F\boldsymbol{M}$, lifts to a T-motor map $\tilde{f} \colon \tilde{\boldsymbol{B}} \to \boldsymbol{M}$.

Proof. By definition of f we have that both

$$
\begin{tikzcd}
 & 1 \arrow[dl, "e"'] \arrow[dr, "m"] & \\
B \arrow[rr, "f"'] & & M
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
TB \times B \arrow[r, "Tf \times f"] \arrow[d, "\beta"'] & TM \times M \arrow[r, "\mu \times \mathrm{id}"] & M \times M \arrow[d, "\star"] \\
B \arrow[rr, "f"'] & & M
\end{tikzcd}
$$

commute. If we can show that f is such that the following three diagrams

$$
\begin{tikzcd}
 & 1 \arrow[dl, "e"'] \arrow[dr, "m"] & \\
B \arrow[rr, "f"'] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
TB \arrow[r, "Tf"] \arrow[d, "\mathrm{ev}_b \circ \overline{\beta}"'] & TM \arrow[d, "\mu"] \\
B \arrow[r, "f"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
B \times B \arrow[r, "f \times f"] \arrow[d, "*"'] & M \times M \arrow[d, "\star"] \\
B \arrow[r, "f"'] & M
\end{tikzcd}
$$

also commute, then we have shown that f lifts to a T-motor map $\tilde{f}$.

We already have the triangle, and by definition of f we get the commuting diagram

$$
\begin{tikzcd}
TM \arrow[rr, "\mu"] & & M \arrow[r, "(\mathrm{id}, e)"] & M \times B \arrow[d, "\mathrm{id} \times f"] \\
TB \arrow[u, "Tf"] \arrow[r, "(\mathrm{id}, e)"] \arrow[d, "\overline{\beta}"'] & TB \times B \arrow[r, "Tf \times f"] \arrow[d, "\overline{\beta} \times \mathrm{id}"'] \arrow[dr, "\beta"] & TM \times M \arrow[r, "\mu \times \mathrm{id}"'] & M \times M \arrow[d, "\star"] \\
B^B \arrow[r, "(\mathrm{id}, e)"'] & B^B \times B \arrow[r, "\mathrm{ev}"'] & B \arrow[r, "f"'] & M,
\end{tikzcd}
$$

which, since $m = f(e)$ and $(M, m, \star)$ is a monoid, is equivalent to the first commuting square

$$
\begin{tikzcd}
TB \arrow[r, "Tf"] \arrow[d, "\mathrm{ev}_b \circ \overline{\beta}"'] & TM \arrow[d, "\mu"] \\
B \arrow[r, "f"'] & M.
\end{tikzcd}
$$

We use induction to obtain the second square. Let $C = \Big\{x \in B : \forall y \in B, f(x * y) = f(x) \star f(y)\Big\}$, that is, we have the equaliser diagram

$$\begin{tikzcd} C \arrow[r, hook, "i"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & M^B \end{tikzcd}$$

where g and h are given by the transposes of

$$\bar{g} \colon B \times B \xrightarrow{\ *\ } B \xrightarrow{\ f\ } M$$

and

$$\bar{h} \colon B \times B \xrightarrow{f \times f} M \times M \xrightarrow{\ \star\ } M.$$

We hence have the commuting diagram

$$\begin{tikzcd} C \times B \arrow[r, "i \times \mathrm{id}"] & B \times B \arrow[r, shift left, "\bar{g}"] \arrow[r, shift right, "\bar{h}"'] & M. \end{tikzcd}$$

For all y in B we have

$$f(e * y) = f(y) = m \star f(y) = f(e) \star f(y),$$

and hence e is in C. By (12) we have the commuting diagram

$$\begin{tikzcd} TB \times B \times B \arrow[rr, "\beta \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times *"] \arrow[dd, bend right, "Tf \times \bar{g}"'] & & B \times B \arrow[d, "*"'] \arrow[dd, bend left, "\bar{g}"] \\ TB \times B \arrow[rr, "\beta"] \arrow[d, "Tf \times f"] & & B \arrow[d, "f"'] \\ TM \times M \arrow[r, "\mu \times \mathrm{id}"'] & M \times M \arrow[r, "\star"'] & M. \end{tikzcd}$$

By associativity of $\star$ and functorality of the product we also have the commuting diagram

$$\begin{tikzcd} TB \times B \times B \arrow[rr, "\beta \times \mathrm{id}"] \arrow[d, "Tf \times f \times f"] \arrow[dd, bend right, "Tf \times \bar{h}"'] & & B \times B \arrow[d, "f \times f"'] \arrow[dd, bend left, "\bar{h}"] \\ TM \times M \times M \arrow[r, "\mu \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \star"] & M \times M \times M \arrow[r, "\star \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \star"] & M \times M \arrow[d, "\star"'] \\ TM \times M \arrow[r, "\mu \times \mathrm{id}"'] & M \times M \arrow[r, "\star"'] & M. \end{tikzcd}$$

Putting these two diagrams we get

$$\begin{tikzcd} & & TB \times B \times B \arrow[rr, "\beta \times \mathrm{id}"] \arrow[d, "Tf \times \bar{g}"] & & B \times B \arrow[d, "\bar{g}"] \\ TC \times C \times B \arrow[r, "Ti \times \mathrm{id}"] \arrow[urr, bend left, "Ti \times i \times \mathrm{id}"] \arrow[drr, bend right, "Ti \times i \times \mathrm{id}"'] & TB \times C \times B \arrow[ur, "\mathrm{id} \times i \times \mathrm{id}"'] \arrow[dr, "\mathrm{id} \times i \times \mathrm{id}"] & TM \times M \arrow[r, "\mu \times \mathrm{id}"] & M \times M \arrow[r, "\star"] & M \\ & & TB \times B \times B \arrow[rr, "\beta \times \mathrm{id}"'] \arrow[u, "Tf \times \bar{h}"'] & & B \times B, \arrow[u, "\bar{h}"'] \end{tikzcd}$$

which upon taking transposes across the clockwise and anticlockwise composites gives us the commuting diagram

$$TC \times C \xrightarrow{Ti \times i} TB \times B \xrightarrow{\quad\beta\quad} B \underset{h}{\overset{g}{\rightrightarrows}} M^B.$$

This shows that the composite

$$TC \times C \xrightarrow{Ti \times i} TB \times B \xrightarrow{\quad\beta\quad} B$$

equalises g and h, and hence there exists a factorisation

$$\begin{tikzcd} TC \times C \arrow[r, "Ti \times i"] \arrow[d, dashed] & TB \times B \arrow[d, "\beta"] \\ C \arrow[r, hook, "i"'] & B. \end{tikzcd}$$

By induction we conclude that $C = B$, and therefore we have the commuting square

$$\begin{tikzcd} B \times B \arrow[r, "f \times f"] \arrow[d, "*"'] & M \times M \arrow[d, "\star"] \\ B \arrow[r, "f"'] & M. \end{tikzcd}$$

Therefore f lifts to a T-motor map $\tilde{f}\colon \tilde{\boldsymbol{B}} \to \boldsymbol{M}$. □

We now have a constructive proof of:

Theorem 3.24. If $\boldsymbol{B}$ is the initial pointed T-magma, then F lifts $\boldsymbol{B}$ to the initial T-motor $\tilde{\boldsymbol{B}}$. That is, F creates initial objects.

Proof. All that remains to be shown is that the lifted map $\tilde{f}\colon \tilde{\boldsymbol{B}} \to \boldsymbol{M}$ is unique. This is the case since given any other motor map $g\colon \tilde{\boldsymbol{B}} \to \boldsymbol{M}$, we must have $Fg = f = F\tilde{f}$ by initiality of $\boldsymbol{B}$ in $T\text{-}\mathbf{Magma}_*$, and therefore, since F is faithful, $g = \tilde{f}$. □

Most of the proofs given in this chapter have been made without reference to elements of sets. It hence seems reasonable that the relationship between the initial T-motor and the initial pointed T-magma in the category of sets will generalise to any finitely complete cartesian closed category. In the next chapter we show that we can actually generalise the relationship to the case of a monoidal closed category with equalisers.

CHAPTER 4

A Generalisation for Monoidal Categories

In this chapter we extend the theorem that $F\colon T\text{-}\mathbf{Motor} \to T\text{-}\mathbf{Magma}_*$ preserves and creates initial objects to T-algebras where T is an endofunctor on a monoidal category. There are two major properties of $\mathbf{Set}$ that we used in our proof; having exponentials, and having all equalisers, and hence we require our monoidal category to have similar properties.

1. Internal Homs

Definition 4.1. Let $\mathcal{C} = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category, and let $A \in \mathcal{C}$. If the tensor product functor $(-) \otimes A$ has a right adjoint we denote it by $[A, -]$ and for any object $B \in \mathcal{C}$ we call $[A, B]$ the *internal hom* of A and B.

The counit for the adjunction is called *evaluation* and is denoted by ev. It has the universal property that given a map $f\colon C \otimes A \to B$ in $\mathcal{C}$, there exists a unique map $\bar{f}$ such that the following triangle commutes:

$$\begin{tikzcd}
{[A,B] \otimes A} \arrow[r, "\mathrm{ev}"] & B \\
C \otimes A. \arrow[u, "\bar{f} \otimes \mathrm{id}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

If all internal homs exist, then we say $\mathcal{C}$ is a *right closed monoidal category*. Note that many authors use the term *left closed* instead of right closed. This will later be abbreviated to simply *closed*.

Definition 4.2. Given internal homs $[A, B]$ and $[B, C]$ we can define a *composition* map by the following transpose:

$$\frac{\Big([B,C] \otimes [A,B]\Big) \otimes A \xrightarrow{\alpha^{-1}} [B,C] \otimes \Big([A,B] \otimes A\Big) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} [B,C] \otimes B \xrightarrow{\mathrm{ev}} C}{c_{A,B,C}\colon\ [B,C] \otimes [A,B] \xrightarrow[\overline{\mathrm{ev}\big(\mathrm{id} \otimes \mathrm{ev}\big)\alpha^{-1}}]{\hspace{8cm}} [A,C]}$$

When the objects that composition is defined upon are clear from context we will omit the subscripts of c.

In $\mathbf{Set}$ the collection of endofunctions of a set forms a monoid; this generalises to a closed monoidal category as follows:

Definition 4.3. A *monoid* (M, e, μ) in a monoidal category consists of a unit $e\colon I \to M$ and a multiplication $\mu\colon M \otimes M \to M$ satisfying the unit laws:

$$\begin{tikzcd}
I \otimes M \arrow[r, "e \otimes \mathrm{id}"] \arrow[dr, "\lambda"'] & M \otimes M \arrow[d, "\mu"] & M \otimes I \arrow[l, "\mathrm{id} \otimes e"'] \arrow[dl, "\rho"] \\
& M &
\end{tikzcd}$$

and the associative law:

$$\begin{tikzcd}
M \otimes (M \otimes M) \arrow[rr, "\mathrm{id} \otimes \mu"] \arrow[d, "\alpha"'] & & M \otimes M \arrow[d, "\mu"] \\
(M \otimes M) \otimes M \arrow[r, "\mu \otimes \mathrm{id}"'] & M \otimes M \arrow[r, "\mu"'] & M.
\end{tikzcd}$$

Definition 4.4. Let $\mathcal{C}$ be a closed monoidal category. The *monoid of endomorphisms* of M in $\mathcal{C}$ is given by $\big([M, M], i, c\big)$, where c is composition, and the unit is given by the following transpose:

$$\frac{I \otimes M \xrightarrow{\quad\lambda\quad} M}{i\colon I \xrightarrow[\bar{\lambda}]{\quad} [M, M]}\ .$$

An outline of a proof that the monoid of endomorphisms is actually a monoid is given in [7].

It is convenient to generalise the concept of an element and that of evaluating a function at an element by the following:

Definition 4.5. Given objects A, B and an *element* $a\colon I \to A$ we define the *evaluation at* a, written ev_a, by the following composition:

$$\mathrm{ev}_a\colon\ [A, B] \xrightarrow{\rho^{-1}} [A, B] \otimes I \xrightarrow{\mathrm{id} \otimes a} [A, B] \times A \xrightarrow{\mathrm{ev}} B.$$

We would like c and i to behave like composition and the identity function in the category of sets. This behaviour is exhibited in the following proposition which, due to space limitations, is left for the reader to verify.

Proposition 4.6. Given an element $a\colon I \to A$ both the diagrams

$$\begin{tikzcd}
I \arrow[r, "i"] \arrow[dr, "a"'] & {[A, A]} \arrow[d, "\mathrm{ev}_a"] \\
& A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
{[B, C] \otimes [A, B]} \arrow[r, "\mathrm{id} \otimes \mathrm{ev}_a"] \arrow[d, "c"'] & {[B, C] \otimes B} \arrow[d, "\mathrm{ev}"] \\
{[A, C]} \arrow[r, "\mathrm{ev}_a"'] & C
\end{tikzcd}$$

commute. Additionally, given $f\colon C \otimes A \to B$ the diagram

$$\begin{tikzcd}
C \arrow[r, "\bar{f}"] \arrow[d, "\rho^{-1}"'] & {[B, A]} \arrow[dr, "\mathrm{ev}_a"] & \\
C \otimes I \arrow[r, "\mathrm{id} \otimes a"'] & C \otimes A \arrow[r, "f"'] & B
\end{tikzcd}$$

also commutes.

Proposition 4.7. Given $f\colon A\otimes D\to E$ and $g\colon B\otimes C\to D$ the following diagram commutes:

$$\begin{tikzcd}
(A\otimes B)\otimes C \arrow[rr, "(\bar f\otimes\bar g)\otimes\mathrm{id}"] \arrow[d, "\alpha^{-1}"'] & & \big([D,E]\otimes[C,D]\big)\otimes C \arrow[d, "\bar c"] \\
A\otimes(B\otimes C) \arrow[r, "\mathrm{id}\otimes g"'] & A\otimes D \arrow[r, "f"'] & E.
\end{tikzcd}$$

Proof. By definition of composition and evaluation we have the commuting diagram

$$\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, "(\bar f\otimes\bar g)\otimes\mathrm{id}"] \arrow[d, "\alpha^{-1}"'] & \big([D,E]\otimes[C,D]\big)\otimes C \arrow[d, "\alpha^{-1}"] \\
A\otimes(B\otimes C) \arrow[r, "\bar f\otimes(\bar g\otimes\mathrm{id})"] \arrow[d, "\mathrm{id}\otimes g"'] \arrow[dr, "\bar f\otimes g"] & {[D,E]}\otimes\big([C,D]\otimes C\big) \arrow[d, "\mathrm{id}\otimes\mathrm{ev}"] \\
A\otimes D \arrow[r, "\bar f\otimes\mathrm{id}"] \arrow[dr, "f"'] & {[D,E]}\otimes D \arrow[d, "\mathrm{ev}"] \\
& E,
\end{tikzcd}$$

as required. □

2. The Generalisation

For the rest of this chapter let $\mathcal{C}$ be a monoidal closed category with all equalisers, and let $T\colon\mathcal{C}\to\mathcal{C}$ be a fixed endofunctor. As before we have the following:

Definition 4.8. A *T-motor* is a quadruple $\boldsymbol{M}=(M,m,\mu,\sigma)$, where (M,m,μ) is a monoid and $\sigma\colon TM\to M$. A T-motor morphism $f\colon\boldsymbol{M}\to\boldsymbol{N}$ between $\boldsymbol{M}=(M,m,\mu,\sigma)$ and $\boldsymbol{N}=(N,n,\nu,\tau)$ is a morphism such that all three of the diagrams

$$\begin{tikzcd}
& I \arrow[dl, "m"'] \arrow[dr, "n"] & \\
M \arrow[rr, "f"'] & & N
\end{tikzcd}
\qquad
\begin{tikzcd}
M\otimes M \arrow[r, "f\otimes f"] \arrow[d, "\mu"'] & N\otimes N \arrow[d, "\nu"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
TM \arrow[r, "Tf"] \arrow[d, "\sigma"'] & TN \arrow[d, "\tau"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$

commute in $\mathcal{C}$. We call the category of T-motors and their morphisms T-$\mathbf{Motor}$.

Definition 4.9. A *pointed T-magma* is a triple $\boldsymbol{M}=(M,m,\mu)$, where M is an object in $\mathcal{C}$, m is an element of M, and $\mu\colon TM\otimes M\to M$. A pointed T-magma morphism $f\colon\boldsymbol{M}\to\boldsymbol{N}$ between $\boldsymbol{M}=(M,m,\mu)$ and $\boldsymbol{N}=(N,n,\nu)$ is a morphism such that both of the diagrams

$$\begin{tikzcd}
& I \arrow[dl, "m"'] \arrow[dr, "n"] & \\
M \arrow[rr, "f"'] & & N
\end{tikzcd}
\qquad
\begin{tikzcd}
TM\otimes M \arrow[r, "Tf\times f"] \arrow[d, "\mu"'] & TN\otimes N \arrow[d, "\nu"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$

commute in $\mathcal{C}$. We call the category of pointed T-magmas and their morphisms T-$\mathbf{Magma}_*$.

Proposition 4.10. There exists a faithful functor $F\colon T\text{-}\mathbf{Motor} \to T\text{-}\mathbf{Magma}_*$ whose action on objects, $\boldsymbol{M} = (M, m, \mu, \sigma)$, is given by $F\boldsymbol{M} = \big(M, m, \mu(\sigma \otimes \mathrm{id})\big)$.

Proof. This proof is essentially the same as Proposition 3.7 with $\times$ replaced with $\otimes$. □

3. The Initial T-motor is the Initial Pointed T-magma

We now proceed as before to show that F preserves initial objects. Let $\boldsymbol{B} = (B, b, \beta)$ be an arbitrary pointed T-magma and define $\hat{\beta}$ as the composite

$$\hat{\beta}\colon T\,[B,B] \xrightarrow{T\mathrm{ev}_b} TB \xrightarrow{\;\bar{\beta}\;} [B,B]\,.$$

We now have a T-motor $\hat{\boldsymbol{B}} = \Big([B,B]\,, i, c, \hat{\beta}\Big)$, where $\Big([B,B]\,, i, c\Big)$ is the monoid of endomorphisms of B.

Proposition 4.11. If $\boldsymbol{B} = (B, b, \beta)$ is an arbitrary pointed T-magma, then $\mathrm{ev}_b\colon F\hat{\boldsymbol{B}} \to \boldsymbol{B}$ is a pointed T-magma morphism.

Proof. Using the identities given in Proposition 4.6 this proof is essentially the same as the proof given in Proposition 3.14. □

Corollary 4.12. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be as above. If $f\colon \boldsymbol{A} \to \hat{\boldsymbol{B}}$ is the unique T-motor morphism given by initiality of $\boldsymbol{A}$, then the map $g\colon A \to B$ given by the composite

$$g\colon A \xrightarrow{\;f\;} [B,B] \xrightarrow{\mathrm{ev}_b} B$$

is a pointed T-magma morphism, that is, $g\colon F\boldsymbol{A} \to \boldsymbol{B}$.

Proposition 4.13. The unique map $f_A\colon \boldsymbol{A} \to \widehat{F\boldsymbol{A}}$ given by initiality of $\boldsymbol{A}$ is $\bar{\mu}$.

Proof. Upon taking into account naturality of α, this proof is essentially the same as Proposition 3.17. □

We now require an induction principle for an initial T-motor, but for this we also need the following well known result:

Proposition 4.14. Let $\mathcal{C}$ be a monoidal category, (N, n, ν) be a monoid in $\mathcal{C}$, and $f\colon M \rightarrowtail N$ be a monomorphism such that n and $\nu \circ (f \otimes f)$ factor through f as follows

$$\begin{tikzcd} I \arrow[d, dashed, "m"'] \arrow[dr, "n"] & \\ M \arrow[r, tail, "f"'] & N \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \otimes M \arrow[r, "f \otimes f"] \arrow[d, dashed, "\mu"'] & N \otimes N \arrow[d, "\nu"] \\ M \arrow[r, tail, "f"'] & N. \end{tikzcd}$$

Then (M, m, μ) is a submonoid of (N, n, ν).

We now have the following induction principle for an initial T-motor:

Proposition 4.15. Suppose $\boldsymbol{A} = (A, e, \mu, \sigma)$ is an initial T-motor. Let $f, g \colon A \to X$ in $\mathcal{C}$, and let $\mathrm{eq} \colon E \rightarrowtail A$ be the equaliser of f and g. If all of $e \colon 1 \to A$, $\mu(\mathrm{eq} \otimes \mathrm{eq}) \colon E \otimes E \to A$, and $\sigma T(\mathrm{eq}) \colon TE \to A$ also equalise f and g, then $f = g$.

Proof. By our assumptions and the definition of an equaliser there exist factorisations

$$\begin{tikzcd} 1 \arrow[d, dashed, "h"'] \arrow[dr, "e"] & \\ E \arrow[r, tail, "\mathrm{eq}"'] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E \otimes E \arrow[r, "\mathrm{eq} \otimes \mathrm{eq}"] \arrow[d, dashed, "j"'] & A \otimes A \arrow[d, "\mu"] \\ E \arrow[r, tail, "\mathrm{eq}"'] & A, \end{tikzcd}$$

and hence E is a submonoid of (A, e, μ) by the previous proposition. There also exists a factorisation

$$\begin{tikzcd} TE \arrow[r, "T(\mathrm{eq})"] \arrow[d, dashed, "k"'] & TA \arrow[d, "\sigma"] \\ E \arrow[r, tail, "\mathrm{eq}"'] & A, \end{tikzcd}$$

and therefore (E, h, j, k) is a sub T-motor of $\boldsymbol{A}$. By initiality of $\boldsymbol{A}$ we have the commuting diagram

$$\begin{tikzcd} A \arrow[r, dashed, "\exists!"'] \arrow[rr, bend left, "\mathrm{id}"] & E \arrow[r, tail, "\mathrm{eq}"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X, \end{tikzcd}$$

from which we conclude that $f = g$. $\square$

Proposition 4.16. Let $\boldsymbol{B} = (B, b, \beta)$ be a pointed T-magma, and let $f \colon \boldsymbol{A} \to \hat{\boldsymbol{B}}$ be the unique T-motor map given by initiality of $\boldsymbol{A}$. If $h \colon F\boldsymbol{A} \to \boldsymbol{B}$ is a pointed T-magma morphism, then the following diagram commutes in $\mathcal{C}$

$$\begin{tikzcd} A \otimes A \arrow[r, "\mu"] \arrow[d, "\mathrm{id} \otimes h"'] & A \arrow[d, "h"] \\ A \otimes B \arrow[r, "\bar{f}"'] & B. \end{tikzcd}$$

Proof. We shall prove this by induction. Let j and k be the maps with transposes

$$\bar{j} \colon A \otimes A \xrightarrow{\mu} A \xrightarrow{h} B \quad \text{and} \quad \bar{k} \colon A \otimes A \xrightarrow{\mathrm{id} \otimes h} A \otimes B \xrightarrow{\bar{f}} B,$$

and let $\mathrm{eq} \colon E \to A$ be the equaliser of j and k. We hence have the following square and its transpose:

$$\begin{tikzcd} E \arrow[r, "\mathrm{eq}"] \arrow[d, "\mathrm{eq}"'] & A \arrow[d, "j"] \\ A \arrow[r, "k"'] & {[A, B]} \end{tikzcd} \rightsquigarrow \begin{tikzcd} E \otimes A \arrow[r, "\mathrm{eq} \otimes \mathrm{id}"] \arrow[d, "\mathrm{eq} \otimes \mathrm{id}"'] & A \otimes A \arrow[r, "\mu"] & A \arrow[d, "h"] \\ A \otimes A \arrow[r, "\mathrm{id} \otimes h"'] & A \otimes B \arrow[r, "\bar{f}"'] & B. \end{tikzcd}$$

By our T-motor induction principle it is now sufficient to show that $e \colon 1 \to A$, $\mu(\mathrm{eq} \otimes \mathrm{eq}) \colon E \otimes E \to A$, and $\sigma T(\mathrm{eq}) \colon TE \to A$ also equalise j and k.

(A, e, μ) is a monoid, and hence by naturality of λ we get the commuting diagram

$$\begin{tikzcd}
I \otimes A \arrow[r, "e \otimes \mathrm{id}"] \arrow[dr, "\lambda"'] \arrow[dd, "\mathrm{id} \otimes h"'] & A \otimes A \arrow[d, "\mu"'] \arrow[dd, bend left, "\bar{j}"] \\
& A \arrow[d, "h"'] \\
I \otimes B \arrow[r, "\lambda"'] & B.
\end{tikzcd}$$

By definition of f we have the commuting triangle and its transpose

$$\begin{tikzcd}
I \arrow[r, "e"] \arrow[dr, "\bar{\lambda}"'] & A \arrow[d, "f"] & {} \arrow[r, squiggly] & {} & I \otimes B \arrow[r, "e \otimes \mathrm{id}"] \arrow[dr, "\lambda"'] & A \otimes B \arrow[d, "\bar{f}"] \\
& {[B, B]} & & & & B,
\end{tikzcd}$$

and hence by bifunctorality of $\otimes$ we get the commuting diagram

$$\begin{tikzcd}
I \otimes A \arrow[r, "e \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes h"'] & A \otimes A \arrow[d, "\mathrm{id} \otimes h"'] \arrow[dd, bend left, "\bar{k}"] \\
I \otimes B \arrow[r, "e \otimes \mathrm{id}"] \arrow[dr, "\lambda"'] & A \otimes B \arrow[d, "\bar{f}"'] \\
& B.
\end{tikzcd}$$

We therefore have the commuting diagram

$$\begin{tikzcd}
& A \otimes A \arrow[dr, "\bar{j}"] & \\
I \otimes A \arrow[ur, "e \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes h"] \arrow[dr, "e \otimes \mathrm{id}"'] & I \otimes B \arrow[r, "\lambda"] & B \\
& A \otimes A, \arrow[ur, "\bar{k}"'] &
\end{tikzcd}$$

from which we conclude that $e \colon I \to A$ equalises j and k.

By associativity of μ, and the definition of eq we have the commuting diagram

$$\begin{tikzcd}
A \otimes (A \otimes A) \arrow[r, "\mathrm{id} \otimes \mu"] & A \otimes A \arrow[r, "\mathrm{id} \otimes h"] & A \otimes B \arrow[d, "\bar{f}"] \\
E \otimes (A \otimes A) \arrow[r, "\mathrm{id} \otimes \mu"] \arrow[u, "\mathrm{eq} \otimes \mathrm{id}"] \arrow[d, "\mathrm{eq} \otimes \mathrm{id}"'] & E \otimes A \arrow[u, "\mathrm{eq} \otimes \mathrm{id}"'] \arrow[d, "\mathrm{eq} \otimes \mathrm{id}"] & B \\
A \otimes (A \otimes A) \arrow[r, "\mathrm{id} \otimes \mu"] \arrow[d, "\alpha"'] & A \otimes A \arrow[r, "\mu"] & A \arrow[u, "h"'] \\
(A \otimes A) \otimes A \arrow[rr, "\mu \otimes \mathrm{id}"'] & & A \otimes A, \arrow[u, "\mu"']
\end{tikzcd}$$

which by definition of j is equivalent to the commuting diagram

$$
\begin{tikzcd}
E \otimes (A \otimes A) \arrow[r, "\mathrm{eq} \otimes \mathrm{id}"] \arrow[d, "\mathrm{eq} \otimes \bar{j}"'] & A \otimes (A \otimes A) \arrow[r, "\alpha"] & (A \otimes A) \otimes A \arrow[r, "\mu \otimes \mathrm{id}"] & A \otimes A \arrow[d, "\bar{j}"] \\
A \otimes B \arrow[rrr, "\bar{f}"'] & & & B.
\end{tikzcd}
\tag{13}
$$

By definition of f we have the commuting square and its transpose

$$
\begin{tikzcd}
A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\mu"'] & {[B,B] \otimes [B,B]} \arrow[d, "c"] \\
A \arrow[r, "f"'] & {[B,B]}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
(A \otimes A) \otimes B \arrow[r, "(f \otimes f) \otimes \mathrm{id}"] \arrow[d, "\mu \otimes \mathrm{id}"'] & \big([B,B] \otimes [B,B]\big) \otimes B \arrow[d, "\bar{c}"] \\
A \otimes B \arrow[r, "\bar{f}"'] & B,
\end{tikzcd}
$$

which by definition of eq and naturality of α yields the commuting diagram

$$
\begin{tikzcd}
E \otimes (A \otimes A) \arrow[r, "\mathrm{eq} \otimes \mathrm{id}"] \arrow[d, "\mathrm{eq} \otimes \left(\mathrm{id} \otimes h\right)"'] & A \otimes (A \otimes A) \arrow[r, "\alpha"] & (A \otimes A) \otimes A \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes h"'] & A \otimes A \arrow[d, "\mathrm{id} \otimes h"] \\
A \otimes (A \otimes B) \arrow[rr, "\alpha"] \arrow[dd, "\mathrm{id} \otimes \bar{f}"'] & & (A \otimes A) \otimes B \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "(f \otimes f) \otimes \mathrm{id}"'] & A \otimes B \arrow[dd, "\bar{f}"] \\
& & \big([B,B] \otimes [B,B]\big) \otimes B \arrow[dr, "\bar{c}"] & \\
A \otimes B \arrow[rrr, "\bar{f}"'] & & & B,
\end{tikzcd}
$$

where the bottom left pentagon commutes by by Proposition 4.7. We hence by definition of k have the commuting diagram

$$
\begin{tikzcd}
E \otimes (A \otimes A) \arrow[r, "\mathrm{eq} \otimes \mathrm{id}"] \arrow[d, "\mathrm{eq} \otimes \bar{k}"'] & A \otimes (A \otimes A) \arrow[r, "\alpha"] & (A \otimes A) \otimes A \arrow[r, "\mu \otimes \mathrm{id}"] & A \otimes A \arrow[d, "\bar{k}"] \\
A \otimes B \arrow[rrr, "\bar{f}"'] & & & B.
\end{tikzcd}
\tag{14}
$$

Combining (13) and (14) and applying the definition of eq once again we obtain the commuting diagram

$$
\begin{tikzcd}
 & & (A \otimes A) \otimes A \arrow[ddr, "\mu \otimes \mathrm{id}"] & \\
 & & A \otimes (A \otimes A) \arrow[u, "\alpha"'] & \\
 & E \otimes (E \otimes A) \arrow[r, "\mathrm{id} \otimes \big(\mathrm{eq} \otimes \mathrm{id}\big)"'] & E \otimes (A \otimes A) \arrow[u, "\mathrm{eq} \otimes \mathrm{id}"'] \arrow[d, "\mathrm{eq} \otimes \bar{\jmath}"'] & A \otimes A \arrow[d, "\bar{\jmath}"'] \\
(E \otimes E) \otimes A \arrow[uuurr, bend left, "\big(\mathrm{eq} \otimes \mathrm{eq}\big) \otimes \mathrm{id}"] \arrow[ur, "\alpha^{-1}"'] \arrow[dr, "\alpha^{-1}"] \arrow[dddrr, bend right, "\big(\mathrm{eq} \otimes \mathrm{eq}\big) \otimes \mathrm{id}"'] & & A \otimes B \arrow[r, "\bar{f}"] & B \\
 & E \otimes (E \otimes A) \arrow[r, "\mathrm{id} \otimes \big(\mathrm{eq} \otimes \mathrm{id}\big)"] & E \otimes (A \otimes A) \arrow[u, "\mathrm{eq} \otimes \bar{k}"'] \arrow[d, "\mathrm{eq} \otimes \mathrm{id}"'] & A \otimes A \arrow[u, "\bar{k}"'] \\
 & & A \otimes (A \otimes A) \arrow[d, "\alpha"'] & \\
 & & (A \otimes A) \otimes A, \arrow[uur, "\mu \otimes \mathrm{id}"'] &
\end{tikzcd}
$$

from which we conclude that $\mu(\mathrm{eq} \otimes \mathrm{eq})$ equalises j and k.

By definition of eq and naturality of ρ we obtain the commuting diagram

$$
\begin{tikzcd}
 & & & A \arrow[ddddd, "h"] \\
A \arrow[r, "\rho^{-1}"] & A \otimes I \arrow[r, "\mathrm{id} \otimes e"'] \arrow[urr, bend left, "\rho"] & A \otimes A \arrow[ur, "\mu"'] & \\
E \arrow[u, "\mathrm{eq}"] \arrow[r, "\rho^{-1}"'] \arrow[d, "\mathrm{eq}"'] & E \otimes I \arrow[u, "\mathrm{eq} \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes e"] \arrow[d, "\mathrm{eq} \otimes \mathrm{id}"'] & E \otimes A \arrow[u, "\mathrm{eq} \otimes \mathrm{id}"'] \arrow[d, "\mathrm{eq} \otimes \mathrm{id}"] & \\
A \arrow[r, "\rho^{-1}"'] \arrow[dd, "f"'] & A \otimes I \arrow[r, "\mathrm{id} \otimes e"] \arrow[dr, "\mathrm{id} \otimes b"'] & A \otimes A \arrow[d, "\mathrm{id} \otimes h"] & \\
 & & A \otimes B \arrow[dr, "\bar{f}"] & \\
{[B, B]} \arrow[rrr, "\mathrm{ev}_b"'] & & & B,
\end{tikzcd}
$$

where the bottom left pentagon commutes by Proposition 4.6. The outside of this diagram is

$$
\begin{tikzcd}
E \arrow[r, "\mathrm{eq}"] \arrow[d, "\mathrm{eq}"'] & A \arrow[r, "h"] & B \\
A \arrow[rr, "f"'] & & {[B, B]}, \arrow[u, "\mathrm{ev}_b"']
\end{tikzcd}
$$

which upon applying T and recalling the definitions of $\hat{\beta}$ and f we get the commuting diagram

$$\begin{tikzcd}
TE \arrow[r, "T\mathrm{eq}"] \arrow[d, "T\mathrm{eq}"'] & TA \arrow[r, "Th"] & TB \arrow[d, "\bar{\beta}"] \\
TA \arrow[r, "Tf"] \arrow[d, "\sigma"'] & T\left[B,B\right] \arrow[r, "\hat{\beta}"] \arrow[ur, "T\mathrm{ev}_b"] & \left[B,B\right] \\
A. \arrow[urr, "f"'] & &
\end{tikzcd}$$

By definition of h we now have the commuting diagram

$$\begin{tikzcd}
TE \otimes A \arrow[rrr, "T\mathrm{eq} \otimes \mathrm{id}"] \arrow[dr, "\mathrm{id} \otimes h"] \arrow[ddd, "T\mathrm{eq} \otimes \mathrm{id}"'] & & & TA \otimes A \arrow[r, "\sigma \otimes \mathrm{id}"] \arrow[d, "Th \otimes h"] & A \otimes A \arrow[d, "\mu"] \\
& TE \otimes B \arrow[r, "T\mathrm{eq} \otimes \mathrm{id}"] \arrow[d, "T\mathrm{eq} \otimes \mathrm{id}"'] & TA \otimes B \arrow[r, "Th \otimes \mathrm{id}"] & TB \otimes B \arrow[d, "\beta"'] & A \arrow[dl, "h"] \\
& TA \otimes B \arrow[r, "\sigma \otimes \mathrm{id}"] & A \otimes B \arrow[r, "\bar{f}"] & B & \\
TA \otimes A \arrow[ur, "\mathrm{id} \otimes h"] \arrow[rr, "\sigma \otimes \mathrm{id}"'] & & A \otimes A, \arrow[u, "\mathrm{id} \otimes h"'] & &
\end{tikzcd}$$

whose outside is

$$\begin{tikzcd}
TE \otimes A \arrow[r, "T\mathrm{eq} \otimes \mathrm{id}"] \arrow[d, "T\mathrm{eq} \otimes \mathrm{id}"'] & TA \otimes A \arrow[r, "\sigma \otimes \mathrm{id}"] & A \otimes A \arrow[r, "\mu"] & A \arrow[d, "h"] \\
TA \otimes A \arrow[r, "\sigma \otimes \mathrm{id}"'] & A \otimes A \arrow[r, "\mathrm{id} \otimes h"'] & A \otimes B \arrow[r, "\bar{f}"'] & B,
\end{tikzcd}$$

from which we conclude that $\sigma T\mathrm{eq}$ equalises j and k.

We have therefore shown by induction that the square

$$\begin{tikzcd}
A \otimes A \arrow[r, "\mu"] \arrow[d, "\mathrm{id} \otimes h"'] & A \arrow[d, "h"] \\
A \otimes B \arrow[r, "\bar{f}"'] & B
\end{tikzcd}$$

commutes in $\mathcal{C}$. $\square$

We are now able to prove:

Theorem 4.17. If $\boldsymbol{A} = (A, e, \mu, \sigma)$ is an initial T-motor, then $F\boldsymbol{A}$ is an initial pointed T-magma. That is, F preserves initial objects.

Proof. We must show that given an arbitrary pointed T-magma $\boldsymbol{B} = (B, b, \beta)$ the composite

$$g \colon F\boldsymbol{A} \xrightarrow{f} F\hat{\boldsymbol{B}} \xrightarrow{\mathrm{ev}_b} \boldsymbol{B}$$

is a unique pointed T-magma morphism. Let $h\colon F\boldsymbol{A} \to \boldsymbol{B}$ be a pointed T-magma morphism. By naturality of ρ and the previous proposition we have that the diagram

$$\begin{tikzcd}
A \arrow[r, "\rho^{-1}"] \arrow[dd, "\rho^{-1}"'] & A \otimes I \arrow[d, "\mathrm{id} \otimes a"'] \arrow[dr, "\rho"] & \\
 & A \otimes A \arrow[r, "\mu"'] \arrow[d, "\mathrm{id} \otimes h"'] & A \arrow[d, "h"] \\
A \otimes I \arrow[r, "\mathrm{id} \otimes b"'] & A \otimes B \arrow[r, "\bar{f}"] \arrow[dr, "f \otimes \mathrm{id}"'] & B \\
 & & {[B,B] \otimes B} \arrow[u, "\mathrm{ev}"']
\end{tikzcd}$$

must commute. Taking the anticlockwise path around the outside of the above diagram from A to B we see from

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left, "g"] \arrow[d, "\rho^{-1}"'] & {[B,B]} \arrow[r, "\mathrm{ev}_b"'] \arrow[d, "\rho^{-1}"] & B \\
A \otimes I \arrow[r, "f \otimes \mathrm{id}"'] & {[B,B] \otimes I} \arrow[r, "\mathrm{id} \otimes b"'] & {[B,B] \otimes B} \arrow[u, "\mathrm{ev}"']
\end{tikzcd}$$

and bifunctorality of the monoidal product, that this composite is simply g.

Therefore $g = h$, that is, g is unique. $\square$

4. The Initial Pointed T-magma is the Initial T-motor

We now prove that F creates initial objects. When we did this in the set based case we used recursion and this presents a problem in the monoidal category case as general recursion requires duplication and elimination of variables - something that can't be done in a general monoidal category. However we will see that there is another way of obtaining the structure which we found by recursion, and with a slight modification, the proof can be adapted to the monoidal category case.

Let $\boldsymbol{B} = (B, e, \beta)$ be an initial pointed T-magma. In order to show that $\boldsymbol{B}$ lifts to an initial T-motor we must first construct a T-motor structure on B. That is we require a multiplication $\mu\colon B \otimes B \to B$ for which e is the identity. In order to make this construction we once again employ the motor of endomorphisms $\hat{\boldsymbol{B}} = \left([B,B], i, c, \hat{\beta}\right)$, where $\left([B,B], i, c\right)$ is the monoid of endomorphisms of B and $\hat{\beta}$ is the composite:

$$\hat{\beta}\colon T[B,B] \xrightarrow{T\mathrm{ev}_e} TB \xrightarrow{\bar{\beta}} [B,B].$$

Applying F we obtain the pointed T-magma $F\hat{\boldsymbol{B}} = \left([B,B], i, \tilde{\hat{\beta}}\right)$, where $\tilde{\hat{\beta}}$ is the composite:

$$\tilde{\hat{\beta}}\colon T[B,B] \otimes [B,B] \xrightarrow{T\mathrm{ev}_e \otimes \mathrm{id}} TB \otimes [B,B] \xrightarrow{\bar{\beta} \otimes \mathrm{id}} [B,B] \otimes [B,B] \xrightarrow{c} [B,B].$$

By initiality of $\boldsymbol{B}$ we obtain a unique pointed T-magma morphism $\bar{\mu}\colon \boldsymbol{B} \to F\hat{\boldsymbol{B}}$, and we now show that its transpose $\mu\colon B \otimes B \to B$ satisfies the conditions of Proposition 3.19.

Proposition 4.18. The composite $B \xrightarrow{\bar{\mu}} [B, B] \xrightarrow{\mathrm{ev}_e} B$ is the identity for B.

Proof. By definition $\bar{\mu}$ is a pointed T-magma morphism, and by Proposition 4.11 so is ev_e. Therefore by initiality of $\boldsymbol{B}$, their composite must be the identity for B. ☐

Proposition 4.19. μ renders commutative

$$\begin{tikzcd} & I \otimes B \arrow[dl, "e \otimes \mathrm{id}"'] \arrow[dr, "\lambda"] & \\ B \arrow[rr, "\mu"'] & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (TB \otimes B) \otimes B \arrow[r, "\alpha^{-1}"] \arrow[d, "\beta \times \mathrm{id}"'] & TB \otimes (B \otimes B) \arrow[r, "\mathrm{id} \times \mu"] & TB \otimes B \arrow[d, "\beta"] \\ B \times B \arrow[rr, "\mu"'] & & B. \end{tikzcd}$$

Proof. By definition of $\bar{\mu}$ we have the commuting triangle

$$\begin{tikzcd} & I \arrow[dl, "e"'] \arrow[dr, "i = \bar{\lambda}"] & \\ B \arrow[rr, "\bar{\mu}"'] & & {[B, B]} \end{tikzcd}$$

which upon taking transposes yields the required triangle.

For the square we consider the diagram

$$\begin{tikzcd} TB \otimes B \arrow[r, "T\bar{\mu} \otimes \bar{\mu}"] \arrow[dr, "\mathrm{id} \otimes \bar{\mu}"'] \arrow[ddd, "\beta"'] & T\,[B, B] \otimes [B, B] \arrow[d, "T\mathrm{ev}_e \otimes \mathrm{id}"] \arrow[ddd, bend left, "\tilde{\bar{\beta}}"] \\ & TB \otimes [B, B] \arrow[d, "\bar{\beta} \otimes \mathrm{id}"] \\ & {[B, B] \otimes [B, B]} \arrow[d, "c"] \\ B \arrow[r, "\bar{\mu}"'] & {[B, B]}, \end{tikzcd}$$

whose outside commutes by definition of $\bar{\mu}$, and the top triangle commutes by the previous proposition. We hence have the commuting square

$$\begin{tikzcd} TB \otimes B \arrow[r, "\bar{\beta} \otimes \bar{\mu}"] \arrow[d, "\beta"'] & {[B, B] \otimes [B, B]} \arrow[d, "c"] \\ B \arrow[r, "\bar{\mu}"'] & {[B, B]}, \end{tikzcd}$$

which upon taking transposes yields the equivalent square

$$\begin{tikzcd}
(TB \otimes B) \otimes B \arrow[r, "\left(\bar{\beta} \otimes \bar{\mu}\right) \otimes \mathrm{id}"] \arrow[d, "\beta \otimes \mathrm{id}"'] & \Big([B,B] \otimes [B,B]\Big) \otimes B \arrow[d, "\bar{c}"] \\
B \otimes B \arrow[r, "\mu"'] & B.
\end{tikzcd}$$

By Proposition 4.7 this in turn gives us the commuting diagram

$$\begin{tikzcd}
(TB \otimes B) \otimes B \arrow[r, "\alpha^{-1}"] \arrow[dr, "\left(\bar{\beta} \otimes \bar{\mu}\right) \otimes \mathrm{id}"] \arrow[dd, "\beta \otimes \mathrm{id}"'] & TB \otimes (B \otimes B) \arrow[r, "\mathrm{id} \otimes \mu"] & TB \otimes B \arrow[dd, "\beta"] \\
& \Big([B,B] \otimes [B,B]\Big) \otimes B \arrow[dr, "\bar{c}"] & \\
B \otimes B \arrow[rr, "\mu"'] & & B
\end{tikzcd}$$

as required. □

Now repeating the steps in Theorem 3.24, mutatis mutandis, gives:

Theorem 4.20. If $\boldsymbol{B}$ is an initial pointed T-magma, then F lifts $\boldsymbol{B}$ to an initial T-motor. That is, F creates initial objects.

5. Examples

We now present some interesting examples of some initial T-motors and T-magmas.

Example 4.21. Consider the category $\mathbf{Set}^{\mathbb{N}}$ of natural-number-indexed families of sets and functions. $\mathbf{Set}^{\mathbb{N}}$ can be made into a monoidal category $\left(\mathbf{Set}^{\mathbb{N}}, \otimes, I\right)$ with tensor product

$$(X \otimes Y)_n = \sum_{n=m+k} X_m \times Y_k,$$

and unit

$$I_n = \begin{cases} \emptyset, & \text{if } n \geq 0 \\ \{*\}, & \text{if } n = 0. \end{cases}$$

A monoid in $\left(\mathbf{Set}^{\mathbb{N}}, \otimes, I\right)$ is a graded monoid consisting of an $\mathbb{N}$-indexed family of sets X, together with an element $e \in X_0$, and a family of products

$$X_m \times X_k \xrightarrow{\;*\;} X_{m+k}$$

for every m and k in $\mathbb{N}$.

Now let $T\colon \mathbf{Set}^{\mathbb{N}} \to \mathbf{Set}^{\mathbb{N}}$ be the endofunctor given by

$$(TX)_n = \begin{cases} X_{n-1}, & \text{if } n \geq 0 \\ \emptyset, & \text{if } n = 0. \end{cases}$$

Setting C_n to be the set of all rooted finitely branching trees with n non-root vertices, then C is a graded monoid under joining trees at the root, and also a T-algebra under adding a new root. In fact C is freely generated by these structures, that is, C is an initial $(T, \otimes)$-motor, and the cardinality of each set C_n is the nth Catalan number. Correspondingly C is also an initial pointed $(T, \otimes)$-magma.

Example 4.22. An (m, k) shuffle σ is given by a diagram

$$\begin{tikzcd} \{1 < \cdots < m\} \arrow[dr, "i_\sigma"'] & & \{1 < \cdots < k\} \arrow[dl, "j_\sigma"] \\ & \{1 < \cdots < m+k\}, & \end{tikzcd}$$

where i_σ and j_σ are order preserving injections that are jointly surjective. The collection of (m, k) shuffles is denoted $\mathrm{Sh}(m, k)$. Note that $\mathrm{Sh}(m, k) \times \mathrm{Sh}(m + k, l) \cong \mathrm{Sh}(m, k + l) \times \mathrm{Sh}(k, l)$ via the unique function that sends a pair of shuffles (σ, τ) to the pair of shuffles $\left(\sigma', \tau'\right)$ for which

$$\begin{tikzcd}[column sep=small] m \arrow[dr, "i_\sigma"'] & & k \arrow[dl, "j_\sigma"] & & l \arrow[ddll, "j_\tau"] \\ & m+k \arrow[dr, "i_\tau"'] & & & \\ & & m+k+l & & \end{tikzcd} = \begin{tikzcd}[column sep=small] m \arrow[ddrr, "i_{\sigma'}"'] & & k \arrow[dr, "i_{\tau'}"'] & & l \arrow[dl, "j_{\tau'}"] \\ & & & k+l \arrow[dl, "j_{\sigma'}"] & \\ & & m+k+l & & \end{tikzcd} . \tag{15}$$

Consider now the monoidal category $\left(\mathbf{Set}^{\mathbb{N}}, \otimes_{\mathrm{Sh}}, I\right)$, with unit as in the previous example, and tensor product given by

$$(X \otimes_{\mathrm{Sh}} Y)_n = \sum_{\substack{n=m+k \\ \sigma \in \mathrm{Sh}(m,k)}} X_m \times Y_k.$$

For the associativity constraint, note that elements of $\left((X \otimes_{\mathrm{Sh}} Y) \otimes_{\mathrm{Sh}} Z\right)_n$ and $\left(X \otimes_{\mathrm{Sh}} (Y \otimes_{\mathrm{Sh}} Z)\right)_n$ are respectively tuples of the form

$$\left(\sigma \in \mathrm{Sh}(m, k), \tau \in \mathrm{Sh}(m + k, l), x \in X_m, y \in Y_k, z \in Z_l\right)$$

and

$$\left(\sigma' \in \mathrm{Sh}(m, k + l), \tau' \in \mathrm{Sh}(k, l), x \in X_m, y \in Y_k, z \in Z_l\right)$$

for some m, k, l with $m + k + l = n$. Thus $\alpha_{X,Y,Z}$ can be given by the function sending (σ, τ, x, y, z) to $\left(\sigma', \tau', x, y, z\right)$, where the assignation of (σ, τ) to $\left(\sigma', \tau'\right)$ is as in (15). The unit constraint is given in a similar manner.

A monoid in $\left(\mathbf{Set}^{\mathbb{N}}, \otimes_{\mathrm{Sh}}, I\right)$ now consists of an $\mathbb{N}$-indexed family of sets X, together with an element $e \in X_0$, and a family of products

$$X_m \times X_k \xrightarrow{*_\sigma} X_{m+k}$$

for every m and k in $\mathbb{N}$ and for every σ in $\mathrm{Sh}(m,k)$ such that

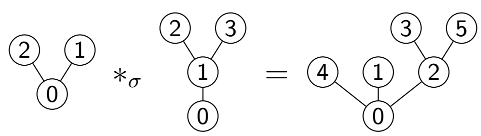
$$e *_! x = x = x *_! e$$

where ! is the unique shuffle with the empty set, and

$$(x *_\sigma y) *_\tau z = x *_{\sigma'} (y *_{\tau'} z)$$

where $x \in X_m, y \in X_k, z \in X_l$, and $\sigma, \tau, \sigma', \tau'$ are as in (15).

We call a tree with $n+1$ vertices *increasing-ordered* when all the vertices are labelled by a natural number less than or equal to n in such a way that every vertex's label is greater than its parent's. Let A_n denote the set of all increasing-ordered rooted trees with n non-root vertices. Given an (m,k) shuffle σ, we define the product $s *_\sigma t$ of increasing-ordered trees $s \in A_m$ and $t \in A_k$ to be the tree obtained by joining s and t at the root. Each non-root vertex labelled by l in s is labelled by $i_\sigma(l)$ in $s *_\sigma t$, and each non-root vertex labelled by l in t is labelled by $j_\sigma(l)$ in $s *_\sigma t$. An example for this product is given by



with $\mathrm{Im}(i_\sigma) = \{1,4\}$, and $\mathrm{Im}(j_\sigma) = \{2,3,5\}$. The trivial tree consisting of just a root is clearly a unit for this operation. A is also a T-algebra under adding a new root and increasing the labels by 1. A is in fact freely generated by these structures, that is, A is an initial $(T, \otimes_{\mathrm{Sh}})$-motor.

By Theorem 4.17 A is also an initial pointed $(T, \otimes_{\mathrm{Sh}})$-magma. A pointed $(T, \otimes_{\mathrm{Sh}})$-magma consists of an $\mathbb{N}$-indexed family of sets X, together with an element $e \in X_0$, and a family of functions

$$(TX)_m \times X_k \xrightarrow{\beta_\sigma} X_{m+k}$$

for every m and k in $\mathbb{N}$ and for every σ in $\mathrm{Sh}(m,k)$. By definition of T this is equivalent to having functions

$$X_{m-1} \times X_k \xrightarrow{\beta_\sigma} X_{m+k}$$

for every m and k in $\mathbb{N}$ with $m \neq 0$, and for every σ in $\mathrm{Sh}(m,k)$.

With this description of A it is easy to see that if a_n is the number of increasing-ordered rooted trees with n non-root vertices, i.e. $a_n = |A_n|$, then a_n is given by the recurrence relation

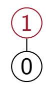
$$a_n = \sum_{m=1}^{n} \binom{n}{m} a_{m-1} a_{n-m}$$

with $a_0 = 1$. This yields the sequence $1, 1, 3, 15, 105, \ldots$ [2, A001147], which has solution given by the double factorial of odd numbers, that is $a_n = (2n-1)!! = 1 \times 3 \times \cdots \times (2n-1)$.

Example 4.23. Following the previous example if we modify T to be

$$(TX)_n = \begin{cases} X_{n-1}, & \text{if } n \geq 1 \\ X_0 + \{*\}, & \text{if } n = 1 \\ \emptyset, & \text{if } n = 0, \end{cases}$$

then we have the same operations as before, and also a tree consisting of the root and a single leaf of a different colour to the root, i.e. the tree



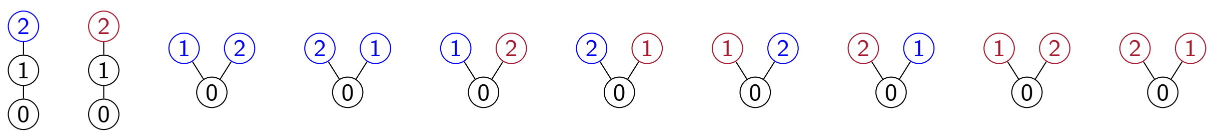
.

The resulting initial $(T, \otimes_{\mathrm{Sh}})$-motor now freely generates the set of all increasing-ordered trees with two possible colours of leaf.

Applying Theorem 4.17 we obtain the recurrence relation

$$a_n = \sum_{m=1}^{n} \binom{n}{m} a_{m-1} a_{n-m} + n \times a_{n-1}, \quad a_0 = 1,$$

which counts the number of increasing-ordered trees with $n+1$ vertices and 2 possible colours of non-root leaves. This is sequence [2, A112487] and has first terms $1, 2, 10, 82, \ldots$ For example the ten increasing ordered trees with three vertices and two possible colours of leaf are:



.

It is an easy matter to extend T to

$$(TX)_n = \begin{cases} X_{n-1}, & \text{if } n \geq 1 \\ X_0 + \mathbf{k}, & \text{if } n = 1 \\ \emptyset, & \text{if } n = 0, \end{cases}$$

where $\mathbf{k}$ is the set with k elements. The resulting initial $(T, \otimes_{\mathrm{Sh}})$-motor now freely generates the set of all increasing-ordered trees with *k+1* possible colours of leaf. Applying Theorem 4.17 we obtain the recurrence relation

$$a_n = \sum_{m=1}^{n} \binom{n}{m} a_{m-1} a_{n-m} + n(k-1) \times a_{n-1}, \quad a_0 = 1,$$

which counts the number of increasing-ordered trees with $n+1$ vertices and k possible colours of non-root leaves.

It is hoped that some further examples involving Joyal's category of species [8] could also be characterised by Theorem 4.17.

CHAPTER 5

Further Research

The major aim of abstracting the correspondence between the initial motor and the initial magma to more general structures in a monoidal category has now been achieved. We conclude by suggesting some possible avenues of further research.

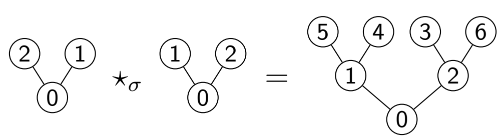
- Following the work of Kirby [9] there is a motor-magma type relationship with regards to the collection of hereditarily finite sets H. The usual description is that H is generated by the recursive definition:
  - $\emptyset$ is a hereditarily finite set
  - If $a_1, \cdots, a_k$ are hereditarily finite sets, then so is $\{a_1, \cdots, a_k\}$.

  Note that $\boldsymbol{H} = \Big(H, \emptyset, \cup, \sigma\Big)$ is a motor where H is the set of all hereditarily finite sets, and $\sigma(x) = \{x\}$. In fact, $\boldsymbol{H}$ is an initial *commutative idempotent motor*.

  We now have that $F\mathbf{H} = \Big(H, \emptyset, \bullet\Big)$ is a magma where $\bullet$ is the *adduction* operator $x \bullet y = \{x\} \cup y$. Note that $\bullet$ has the *skew-commutative* property $x \bullet (y \bullet z) = y \bullet (x \bullet z)$, and the *skew-idempotent* property $x \bullet (x \bullet y) = x \bullet y$. Kirby shows in his paper that the empty set together with adduction generates the collection of hereditarily finite sets. In fact, $F\mathbf{H}$ is an initial *skew-commutative skew-idempotent pointed magma*.

  This suggests the following definitions. A T-magma $\boldsymbol{B} = (B, e, \beta)$ is called *skew-commutative* if $\beta\Big(x, \beta(y, z)\Big) = \Big(y, \beta(x, z)\Big)$ for all $x, y \in TB$ and for all $z \in B$. $\boldsymbol{B}$ is called *skew-idempotent* if $\beta\Big(x, \beta(x, y)\Big) = \beta(x, y)$ for all $x \in TB$ and for all $y \in B$. One would hope that the results of chapter 3 would then extend to a theorem relating the initial commutative idempotent T-motors and the initial skew-commutative skew-idempotent pointed T-magma.

- Another collection of interest is the set of all Joyce trees J. Joyce trees are increasing ordered full binary trees and are generated by:
  - The trivial Joyce tree: 0
  - The binary operation given be growing both of the individual trees up from the root, and then performing the monoidal operation as given in Example 4.22. For example



  with $\mathrm{Im}(i_\sigma) = \{1, 4\}$, and $\mathrm{Im}(j_\sigma) = \{2, 3, 5\}$. J does not have a pointed T-magma structure for some T, but has in fact an initial $(1 + T \otimes_{\mathrm{Sh}} T)$-algebra structure in $\Big(\mathbf{Set}^{\mathbb{N}}, \otimes_{\mathrm{Sh}}, I\Big)$ where T is the same endofunctor as given in Example 4.21. One might now enquire as to whether there is an analogous T-motor like structure for which we get a similar correspondence to the one explored in this thesis.

- There is actually another way of generating the set of all rooted trees that differs to that of the motor construction that we gave. A *parengebra* is a set equipped with a single n-ary operation for each natural number n. Taking the nullary operation to be the trivial rooted tree, the unary operation to be the operation that grows a tree up from a root, and any higher order n-ary operation to be the operation that takes n trees and joins them at the root, we see that the set of all rooted trees forms an initial parengebra. Now we might ask: In general, how does the initial T-motor relate to the initial T-parengebra?
- Lastly one might wish to consider what happens if instead of looking at initial T-motors and initial pointed T-magmas, we instead look at terminal co-T-motors and terminal pointed co-T-magmas. We can then ask whether there are there any interesting instances of these, and whether we get any correspondence between these two notions.

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