The Role of Patterning Within and Beyond Mathematical Thinking

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Summary

This thesis is a theoretical inquiry into the construct of pattern across fields of knowledge to highlight elements of patterning that are intrinsically similar. In the first stage of this inquiry, a comparison of pattern recognition, a neuropsychological construct based on the work of Goldberg (2005), and pattern as defined in the field of mathematics education, generated a tentative generalised model of patterning (McCluskey, Mitchelmore, & Mulligan, 2013). This model was proposed to describe the nature of patterning across both domains of knowledge. A subsequent analysis of patterning across broader fields of cognitive neuroscience and the psychology of mathematics education supported a more generalised view of patterning within and beyond mathematical thinking.

In mathematics education, generalisations about concepts are formed through identification of patterns and relationships and the capacity to reason. Therefore in the next stage of this research I undertook a content analysis of key terms to highlight the incidence of the proficiency - reasoning developed throughout the *Australian Curriculum: Mathematics*.

A dynamic view of the role of reasoning within the proficiencies, connected with the authors' initial generalised model of patterning (McCluskey, Mitchelmore, & Mulligan, 2013) is therefore proposed to support the further development of generalised understandings in mathematics education.

Authorship Statement

I hereby certify that the research presented in this thesis is my original work and has not been submitted for a higher degree to any university or institution other than Macquarie University. I also certify that all sources of information used, and the extent to which the work of others has been referred to, are indicated in the thesis.

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Acknowledgments

I am grateful to Macquarie University for providing the opportunity to explore this research about the role of pattern within and beyond mathematical thinking. This would not have been possible without the guidance and ongoing support especially from my principal supervisor, Professor Joanne Mulligan who from our initial meeting saw the potential for engaging in this research inquiry. Joanne's encouragement along the way enabled me to keep on track and redirected me to new areas that my inquiry could connect with. Joanne also encouraged me to network with other researchers in the field and sourced feedback from her colleagues on various drafts along the way.

I am also deeply grateful to my adjunct supervisor, Associate Professor Mike Mitchelmore, who provided ongoing detailed feedback on drafts and guidance about the overall structure of my thesis. His and Joanne's patience, feedback, and encouragement supported me in pursuing and completing this research.

I would like to also thank Dr Penny Van Bergen who reviewed my initial proposal and suggested that I could propose a generalised model of patterning in the first phase of my research. Penny also provided valuable feedback by critically reading my thesis in the final stages.

I was very fortunate to secure professional editorial assistance from Dr Robert Trevethan in the final stages of compiling this thesis. Robert's attention to detail and support with formatting the chapters was timely and immensely informative. Accessing this service was a crucial factor in the completion of my thesis.

I would also like to thank my employer, Catholic Education South Australia, for the opportunity to attend research conferences and network with other researchers in the field. It was from this opportunity that this research inquiry was conceived. I am also grateful for the Catholic Education's Study Incentive Program, which supports employees to engage in research through funding study days. These leave days contributed to the formation of this research.

Finally I thank my partner and daughter who believed in me and have supported me in persevering to complete this research.

Researcher Background

As an educator I have greatly valued opportunities to reflect upon my practice and deepen my understanding of the learning process. I have completed postgraduate qualifications inquiring into the nature of learning. These include a Graduate Certificate in Neuroscience to understand the neurobiological basis for learning, memory formation, and visualisation, and a Master in Education (Gifted Ed.) to continue to explore the breadth and depth of learning, intelligence, and creativity. A description of previous unpublished papers I wrote through my previous postgraduate studies is included in Appendix A.

Before commencing my current master's research I had contributed to education for 25 years, primarily teaching children in the early years of school. My preferred pedagogy was as a "teacher researcher". I had engaged in professional learning, exploring action research, in the fields of both literacy and numeracy. There were elements of my classroom practice that I would systematically inquire into by observing, documenting, and reflecting upon children's learning in action. I analysed the data that I collected to inform future learning opportunities that I would design for the children. This ongoing collection of data provided a means of measuring growth in mathematical understandings over time.

As a practitioner, I began to notice that when children made connections with the "pattern" of their learning, their understandings deepened. The patterns of thinking I noticed children naturally engage with were "this is what I know about … it is similar to … when I do this, it is the same as ... what if I try this … I know what is going to happen because …". As children recognised similarities in concepts across different contexts, their understandings deepened, they became more fluent in their recall of information, and they were more willing and able to transfer information to a wider variety of contexts. I began to recognise patterns in the way children were learning across various curriculum areas. This recognition of everyday patterns led to the initial phase of this research, inquiring into the role of pattern within and beyond mathematical thinking.

From 2010 to 2013 I held a position as a numeracy consultant for Catholic Education in South Australia. In this role I was able to explore children's patterns of thinking, particularly in mathematics, in much greater depth. I designed and structured professional learning in mathematics education for early years teachers. The professional learning engaged teachers in exploring the "teacher researcher" pedagogy, whereby they inquired as a community of learners into elements of their practice and children's mathematical thinking, through collecting and reflecting upon documentation together. Over a 3-year period the early years group that I led collectively analysed children's levels of thinking as the children explored various concepts within and across the mathematical strands of number and algebra, and measurement and geometry, in the Australian Curriculum: Mathematics (ACM). A range of learning continua were constructed outlining the breadth and depth of thinking that children had revealed as they explored the various mathematical concepts. In leading this processes I found that children's thinking was deepened particularly through teachers' practice of noticing and questioning children to clarify current levels of understanding and inform future directions for learning. This process explicitly and systematically engaged the ACM proficiencies of understanding, problem solving, reasoning, and fluency through an emerging pedagogical structure that:

- used provocations to draw upon and connect with children's current levels of understanding;
- engaged children in a range of investigations, questioning them to bring them into conversation and identify similarities about the concept across contexts;
- embedded whole-group mathematical discourse to support and provide opportunities for the children to explain their thinking and hear different ways of exploring problems, revealing levels of reasoning about concepts explored; and
- re-engaged the children in further investigation to clarify, challenge, or stretch their understanding further.

Through this process of noticing and documenting children's understanding, I observed that the children's fluency strengthened as their patterns of knowing were re-engaged and they had opportunities to generalise concepts across contexts. Similarly, I observed from the teachers' experience that their ability to notice, identify, and question individual children's thinking also strengthened alongside their fluency in knowing the next step of learning for their students. The documentation of this research is currently being prepared for publication by Catholic Education South Australia to support teachers in identifying young children's mathematical thinking.

As a researcher, I was inspired to explore and make further connections with how the mathematical proficiencies worked together dynamically to support greater depth of children's conceptual understanding and their ability to generalise patterns of thinking in mathematics. This led to the second phase of my research: identifying the role of reasoning within the ACM.

My consultancy role enabled me to attend conferences in the field of mathematics education and network with various researchers in the field. I was fortunate to meet with Joanne Mulligan from Macquarie University in my initial year of consultancy and to have the opportunity to discuss my ideas for further research. Macquarie University's research into the Pattern and Structure Mathematics Awareness Program (PASMAP) captured my interest as it outlined the importance of children having an awareness of pattern and structure and the essential role of developing structural understandings in supporting children's future mathematical growth. Given that I had noticed that there was a pattern in the way that children build understandings generally and that research was affirming that an awareness of pattern and structure was essential in the development of mathematical concepts, my driving question was: What is the role of pattern and structure within and beyond mathematical thinking? Thus I pursued this master's research to primarily to explore the relationship of patterning across contexts.

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CHAPTER

1

INTRODUCTION

> Patterns are the very essence of mathematics, the language in which it is expressed. In recognizing and creating patterns of every conceivable type, mathematics is an art. It is also the science of analyzing and synthesizing such patterns.
>
> (Sandefur & Camp, 2004, p. 211)

1.1 Introduction

The term "pattern" is used widely across various domains of knowledge. A pattern is a template used to replicate something. There are patterns we follow to carry out procedures; patterns of thinking we engage in to problem solve; patterns we notice in the weather and in nature; and patterns that occur throughout our daily lives. In each instance, and across domains, patterns help us to make accurate predictions in both familiar and new situations. Patterns seem to emerge and be reflected in every aspect of our experiences, as Piaget (1950) claimed, "Life itself is a creator of patterns" (p. 167).

Pattern is viewed as both the language and science of mathematics (Sandefur & Camp, 2004; Steen, 1990). Broadly it is "defined as any predictable regularity" (Mulligan & Mitchelmore, 2009, p. 34). Generalising about the nature of patterns leads to the development of abstract understandings and a greater ability to apply generalisations in new situations (White & Mitchelmore, 2010). Patterning also supports the way we engage mathematically with concepts (Papic & Mulligan, 2005).

In the fields of neuropsychology and cognitive neuroscience, the term pattern is used to refer to the neural structures and the processes through which understandings are encoded in the brain. The recognition of these patterns in new situations allows stored information to be readily applied (Devlin, 2010; Goldberg, 2005; Willis, 2010).

Across domains, the ability to pattern is a recurring process in the building of understanding. This resonates with the view that mathematics is "seen as connected with … and part of the whole fabric of human knowledge" (Ernest, 1991, p. 26), implying that the way we learn, the way we engage, is experienced similarly across fields of knowledge. This thesis will explore this view of pattern across the domains of mathematics education, neuropsychology, the psychology of mathematics education, and cognitive neuroscience with the aim of proposing a generalised model of patterning.

1.2 Rationale

Prior knowledge is readily drawn upon when engaging in new learning. This knowledge is not necessarily domain specific, but connects with a generalised understanding about how to recognise a similar situation or solve that "type of problem". We recognise new situations as being similar if we have encountered something like this before. Goldberg refers to the formation of *generic memories*. These build neural structures that encase a pattern of experience (Goldberg, 2005). This capacity to recognise similarity supports new learning and is something we naturally do, but is not necessarily a conscious process. This instantaneous knowing "takes the form of pattern recognition rather than problem solving" (Goldberg, 2005, p. 20). Similarly, Franz (2003) acknowledged this type of "human intuition as subconscious pattern recognition" (pp. 265–266).

In the field of mathematics education, intentionally exploring similarity across contexts supports the development of abstract understandings (White & Mitchelmore, 2010). Mason, Drury, and Bills (2007) claimed that "experiencing and expressing generality is natural to human beings" (p. 42). They made recommendations that generalisation of structural understandings be embedded in the design of learning in mathematics education. However, they identify that not all types of generalisation are acquired intentionally, and refer to "*Enactive* generalisation … in which the body perceives a generality before the intellect becomes aware of it" (p. 51). This description of generalisation is similar to the pattern recognition that Goldberg refers to. Pattern recognition in this form reinforces the interplay between intentional and non-intentional processes in building understandings as well as the similarity in cross-domain perspectives in building generalisations.

1.3 Aims

I aim to investigate the relationship between patterning as a mathematical construct and patterning as a neuropsychological construct, highlighting any similarities that lend themselves to the development of a generalised model of patterning. I then endeavour to address educational implications that could emerge as a result of a generalised view of patterning within and beyond mathematical thinking.

1.4 Research questions

The central questions addressed within this thesis are:

1. What is similar about the construct of pattern across the fields of mathematics education and neuropsychology?
2. Is a generalised view of patterning applicable across broader fields of knowledge?
3. What are the educational implications of a generalised model of patterning? How is reasoning, the ability to express and justify generalised understandings, articulated in and developed through the proficiencies in the Australian Curriculum: Mathematics (ACM)?

Thus, the purpose of the thesis is to articulate how a generalised view of patterning is developed and applied to mathematics learning. The theoretical approach provides an integrated perspective that is cross-disciplinary.

I compare theoretical perspectives about the role of pattern initially across the fields of mathematics education/learning and neuropsychology, then across the broader fields of the psychology of mathematics education and cognitive neuroscience. I highlight what is intrinsically similar and different about the concept of patterning in each field and propose a generalised model of patterning (GMP) that is applicable to the mathematics education domain.

The implications of using a GMP are examined within mathematics education. In doing this, the role of the proficiencies in building a capacity to reason and construct generalised understandings within the ACM is discussed. The interaction of the proficiencies highlights the essential role of pattern and structure in the development of conceptual understanding within and beyond mathematical thinking.

In this thesis I present three publications that together support a proposed generalised model of patterning (GMP).

1.5 Structure of the thesis

This thesis is presented in a thesis by publication format, and contains three publications that systematically focus on the role of pattern within and beyond mathematical thinking.

1.5.1 Research phases

This thesis comprises two phases:

1. The initial phase is a theoretical inquiry drawing upon the theoretical perspectives in Chapter 2 to compare and contrast the construct of patterning across fields of mathematics education/learning and neuropsychology, then across the broader fields of the psychology of mathematics education and cognitive neuroscience. Publications 1 and 2 reflect the findings of this inquiry. A generalised model of patterning is proposed initially in Publication 1 and expanded on in Publication 2 to explain the similarity in the construct of patterning across four differing fields.

2. The second phase applies this generalised notion of patterning by examining the incidence of Key Proficiency Terms embedded in the ACM. This inquiry identifies the mathematical proficiency of reasoning as being inherent in the building of generalised understandings

Tables 1.1 and 1.2 provide a summary of the two phases of the research.

Table 1.1
First Research Phase

Research method	Research outcomes & activities
Compare and contrast the construct of pattern across: • The neuropsychological domain • Mathematics education Ascertain feedback from research community (MERGA36)	Publication 1: *Does an ability to pattern indicate that the way we think is mathematical?* Refer to Chapter 3, Section 3.3. This is a peer-reviewed publication presented at the MERGA36 conference and published in the conference proceedings. A generalised view of patterning is proposed to explain similarities in elements of patterning in the building of generalised understandings across both domains Refer to Table 1: Abstraction across domains, Chapter 3, Section 3.3.
Expand generalised view of patterning across broader fields of knowledge: • Psychology of mathematics education • Cognitive neuroscience	Publication 2: *The role of pattern within and beyond mathematical thinking.* This publication is prepared for submission to a mathematics education (philosophy) journal such as *For the Learning of Mathematics FLM* Refer to Chapter 4, Section 4.3. A revised generalised model of patterning is proposed. Refer to Table 2 in Chapter 4, Section 4.3.

Table 1.2

Second Research Phase

Research method	Research outcomes & activities
Explore educational implications of a generalised view of patterning. • Propose analysis of use of Key Proficiency Terms (KPTs) identifying the proficiency reasoning in the ACM • Feedback from research community (MERGA 38)	Short communication abstract: *The pattern and structure of the Australian Curriculum: Mathematics;* MERGA38 Refer to Chapter 5, Section 5.2, and Appendix C. Introduce generalised model of patterning as a means of further promoting reasoning in mathematics education.
Content analysis: • Identify use of KPTs which relate to reasoning embedded in content descriptions in the ACM	Analysis of KPTs in the ACM: F–9 Refer to Appendix E.
A comparison of KPTs identified in rationale and opening descriptions of proficiencies with KPTs embedded throughout content descriptions F-9. Ascertain feedback from research community	Publication 3: *The role of reasoning within the Australian Curriculum: Mathematics.* Refer to Chapter 5, Section 5.3. This paper was submitted for peer review for inclusion in MERGA39. An extended version of this paper is contained in Appendix D. It is anticipated that feedback obtained from presenting Publication 3 at MERGA39 will indicate modifications and future directions for this fourth publication.

The first publication was peer-reviewed and published in the conference proceedings of the Mathematics Education Research Group of Australasia (MERGA36) *Mathematics Education: Yesterday, Today and Tomorrow* (2013). Publication 3 has been submitted for peer-review for inclusion in the 2016 conference proceedings of the Mathematics Education Research Group of Australasia (MERGA39) *Opening up Mathematics Education Research*.

1.5.2 Overview of publications

The first publication of this thesis *Does an Ability to Pattern Indicate That We Think Mathematically?* was presented as a position paper at the MERGA36 conference in 2013. The aim of this paper was to gather initial feedback from researchers in mathematics education about the scope of the theoretical inquiry and the research questions.

Following the MERGA36 conference, this paper was rewritten and expanded for submission for publication in a mathematics education journal. As a result, *The Role of Pattern Within and Beyond Mathematical Thinking* became the second publication of this thesis. This second publication includes broader consideration across the fields of mathematics education, psychology of mathematics education, and cognitive neuroscience to propose a stronger argument for a generalised view of patterning that was proposed in the initial paper. This publication also includes a discussion of educational implications.

In the third publication, *The Role of Reasoning in the Australian Curriculum: Mathematic*s, I explored the role of the proficiencies in the ACM in developing generalised understandings. I inquired into an imbalance identified by other researchers about the incidence of key language terms pertaining to reasoning in the curriculum (Atweh, Miller, & Thornton, 2012). I conducted a content analysis to gather more explicit data about this proposed imbalance (see Appendix E). A lack of reference to key language terms pertaining to reasoning in the curriculum content descriptions F-9 was substantiated. This finding is discussed in terms of educational implications for adopting a generalised view of patterning to support the development of reasoning in mathematics education.

1.5.3 Overview of chapters

In Chapter 1 of this thesis (the current chapter) I provide an introduction to and rationale for the research, the aims and the purpose, and the key research questions. I outline the structure of the thesis, including a description of the research phases and an overview of publications and chapters. I conclude this chapter with acknowledgement of the coauthors' contributions.

In Chapter 2 I present theoretical perspectives about the role of patterning across domains of knowledge. I refer to literature that supports each phase of the research and underpins the three publications. In Chapters 3, 4, and 5 I present the three publications. These chapters include an introduction in which I describe the focus for each publication and the research question the publication is addressing as well as a preamble in which I describe the focus for and development of each publication. In the postscript in each chapter I summarise reflections about the publication, feedback gathered from presenting these publications and future directions for my research.

In Chapter 6 I summarise the findings of the research and identify areas for further investigation. In this chapter I also reflect upon the research process undertaken throughout this thesis.

1.6 Coauthors' contributions

All three publications were coauthored by my supervisors, Professor Joanne Mulligan and Associate Professor Michael Mitchelmore. I directed the theoretical approach and took the key responsibility for conducting, analysing, and reporting the findings. I developed the theoretical framework of the thesis based on Goldberg (2005); Professor Mulligan provided perspective on relevant theories on pattern and structure (Mulligan & Mitchelmore, 2009). A/P Mitchelmore provided insight into a comparative theory of abstraction (White & Mitchelmore, 2010). I integrated these perspectives into a generalised model of patterning, which represents my independent thought and insight. I initiated the content analysis of the Australian Curriculum: Mathematics (ACM) following a literature review of another study (Atweh et al. 2012). Professor Mulligan assisted primarily with ensuring that the methods used in the analysis were reliable and presented accurately.

My supervisors provided critical feedback on the structure of each publication to ensure that the argument reflected a balanced view from relevant literature, and they steered me toward appropriate research to expand my ideas. Both my supervisors supported me in preparing for conference presentations and developing publications appropriate for submission.

Michael Mitchelmore primarily assisted in preparing the paper for submission in the conference proceedings at MERGA36 in 2013. Joanne Mulligan primarily supported me in compiling the presentation of the position paper at MERGA36 and the short communication at MERGA38 in 2015, and with writing and the preparation of the research report for MERGA39 in 2016. The opportunity to present the first paper at MERGA36 and to be given critical feedback from participants at MERGA36 and MERGA38 was instrumental in monitoring and reviewing future directions of the thesis.

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CHAPTER

2

CROSS-DISCIPLINARY PERSPECTIVES

2.1 Introduction

In the initial phase of this thesis I inquire into cross-disciplinary perspectives from the fields of neuropsychology, mathematics education, the psychology of mathematics education, and cognitive neuroscience, concerning the role of pattern in the construction of mathematical understanding. In presenting these perspectives, I discuss theoretical considerations that were instrumental in the formation of this research inquiry. This involved a discussion on principles of quasi-empiricism and the embodiment of mathematical understandings. Finally I discuss the second phase of this thesis, that being the implications for mathematics education curricula and practice of adopting a cross-disciplinary view of patterning.

This chapter contains the following three sections:

1. Theoretical considerations,
2. Cross-disciplinary perspectives, and
3. Educational implications.

2.2 Theoretical considerations

In Sriraman and English's (2005, 2010) discussion about theories influencing mathematics education, they acknowledged the need for a synthesis of theories. In referring to a research forum convened at the annual conference of the psychology of mathematics educators (PME) in 2005, questions regarding "how theories from the general domain of cognition [could] contribute to mathematics education research" were raised (Sriraman & English, 2005, p. 452). Discussion about how "cognitively oriented theories have emphasized the mental structures that constitute and underlie mathematical learning, [and] how these structures develop" (p. 453) emerged from the forum.

Mathematics education directly involves the fields of mathematics and education. However, "numerous other disciplines [also] interact with these two fields" (Sriraman & English, 2010, p. 7). Understanding the interaction between fields of knowledge, and the possible fields involved, is needed for reconstructing an evolving view of mathematics education that enables "theoretical frameworks to interact systemically, eliminating dichotomies in discourse on thinking" (Kilpatrick, 2010, p. 3). Such a theory would view the common thread across fields of knowledge, and that we "should aspire to build such a theory. … This type of theory responds to a need for broad schemes of thought that can help us organize the field and relate our field to other fields" (Silver & Herbst, 2007, as cited in Sriraman & English, 2010, p. 17).

In Sriraman and English's discussion (2005) reference is also made to Lakoff and Nunez's (2000) embodiment of mathematical understanding whereby "the body and brain together with everyday experiences structure our conceptual systems" (Sriraman & English, 2005, p. 453). This suggests that there is still much to discover about how mathematical understanding is fully formed. Lakoff and Nunez (2000) asserted that

> "mathematics by itself does not and cannot empirically study human ideas; human cognition is simply not its subject matter. It is up to cognitive science and the neurosciences to do what mathematics itself cannot do — namely, apply the science of mind to human mathematical ideas" (pp. xi, xii).

This line of thinking moves towards a fallibilist view of mathematics, whereby mathematical ideas are viewed as not fully formed and are influenced by new fields of knowledge within the social context (Sriraman & English, 2005, 2010).

Sriraman and English (2005, 2010) refer to quasi-empiricism, a philosophical view of mathematics education developed by Imre Lakatos (1976, 1978). This view recognises that "mathematical activity is human activity" (Lakatos, 1976, p. 146) and like all human endeavours is fallible, uncertain, and therefore needs to be rigorously questioned. Mathematics knowledge "is seen as connected with, and … part of the whole fabric of human knowledge" (Ernest, 1991, p. 26). "Mathematics is what mathematicians do and have done, with all the imperfections inherent in any human activity or creation" (pp. 36–37) and "represents a new direction in the philosophy of mathematics" (pp. 34–35).

Ernest (1991) summarised the following principles of quasi-empiricism.

- Mathematics knowledge is fallible and like all knowledge should be questioned.
- Mathematics is hypothetico-deductive and is built upon hypothetical premises, meaning that potential falsifiers can be the informal theorems of pre-existing theory.
- Mathematical knowledge evolves, the history of mathematical knowledge being viewed as the "evolution of mathematical knowledge" (p. 36); new knowledge is therefore part of this ongoing process of knowledge creation.

- There is a "primacy of informal mathematics" (p. 36) in that all formal mathematics is derived from informal human experiences.
- There is a "genesis of mathematical knowledge" (p. 36) in that the creation of mathematical knowledge derives from the creation of human knowledge.

Interestingly, Ernest (1991) referred to a "pattern of mathematical discovery" or a "growth of informal mathematical theories" (p. 36) that moves through the following stages:

- primitive conjecture
- an argument, analysing the conjecture is proposed
- counter-examples to the conjecture are identified
- hidden examples may be built into the original idea to improve the conjecture.

This view is in contrast to absolutist theories which view mathematics as "being fully formed and perfectly finished knowledge" requiring only "effective transmission of mathematical knowledge" or because of its pre-existence "just needs to be actively noticed and understood (Ernest, 1994, p. 1). However, there are many times when one does not notice or understand the meaning behind concepts being explored. Absolutist theories therefore do not account for those who are still developing their interpretation or understanding, and consequently infer that one needs to look beyond definitive terms to ones that are evolving. There is a known tension, recognised by many in education, that needs to be acknowledged. As a community of educators, we need to become familiar, and comfortable with, "not knowing" and open to discovering newness in learning, stepping into this "pattern of mathematical discovery" (Ernest, 1991, p. 36).

Constructivists' theories of learning affirm the processes through which understanding is built rather than transmitted, in "which the evolving organism must adapt to its environment in order to survive … personal theories are constructed as constellations of concepts" (Ernest, 1994, p. 1). Theorists such as Piaget (1936, 1937, 1975), Vygotsky (1978, 1986, 1987), von Glasersfeld (1989, 1995a, 1995b), Davis, Maher, and Noddings (1990), Davis (1992), Steffe (1991a, 1991b, 1995), and Ernest (1991, 1994) have been instrumental in building this view of mathematics as evolving, and therefore fallible and reconstructable knowledge.

2.3 Cross-disciplinary perspectives

This thesis draws upon research about the role of patterning from the following fields of knowledge: neuropsychology, mathematics education, the psychology of mathematics education, and cognitive neuroscience.

2.3.1 Neuropsychology and patterning

Elkhonon Goldberg, a renowned neuropsychologist, through his text *The Wisdom Paradox* (2005), proposed that in building our understandings we essentially pattern our experiences, whereby “common aspects of the situations are learned much faster that distinguishing aspects” (p. 124). The overlapping elements become encoded as a generic pattern that is “the shared properties of a whole class of similar things or events” (p. 125) and results from an accumulation of similar experiences.

Goldberg (2005) likened the ability to instantaneously read a situation as engaging pattern recognition. This process involves being able to understand the elements involved, knowing what action to take, and what outcomes could emerge. Therefore, “decision making takes the form of pattern recognition rather than of problem solving” (p. 20). Goldberg drew upon the words of Herbert Simon, founder of the theory of artificial intelligence (1966) in substantiating the role of pattern recognition as “the most powerful mechanism of human cognition” (Goldberg, 2005, p. 20). Herbert Simon has theorised about our ability to instantaneously know, describing “human intuition as subconscious pattern recognition … and complementary to analytical thinking” (Frantz, 2003, pp. 265–266).

Goldberg (2005) explicitly refers to pattern recognition in terms of demonstrating the coveted attribute of wisdom. These resilient patterns accumulate over a lifetime to create neural structures, cognitive templates, that Goldberg described as generic memories. Goldberg referred to these generic memories as an “abstract representation” of a set of similar experiences that is much more resilient than “concrete representations corresponding to unique things” (p. 125). Generic memories “capture the essence of a wide range of specific situations and the most effective actions associated with them” (p. 79) leading to high levels of competent, efficient decision making. Goldberg proposed that generic memories draw upon a network of common neural pathways related to the similar attribute of the experience they share, alluding to their inherent structure. He acknowledged the role of language in “shaping our cognition by imposing certain patterns on the world” (p. 91). However Goldberg did not refer to the role of mathematical thinking in the encoding of, and retrieval of, patterns of understanding. Also the possibility that the structure of these neural patterns are mathematical objects that similarly shape our cognition is left unnoticed.

2.3.2 Mathematics education and patterning

In the field of mathematics education, “a mathematical pattern may be described as any predictable regularity, usually involving numerical, spatial or logical relationships” and its structure is defined “as the way a pattern is organised” (Mulligan & Mitchelmore, 2009, p. 34).

Patterning is the process of recognising, identifying, manipulating, and generalising about the nature of patterns, and is therefore "critical to the abstraction of mathematical ideas and relationships, and the development of mathematical reasoning" (Papic & Mulligan, 2005, p. 609). The process of generalising is intrinsically connected to patterning as "generalising starts when you sense an underlying pattern, even if you cannot articulate it" (Mason, Burton, & Stacey, 2010, p. 8). The process of generalising about the nature of patterns is described as both a conscious and intuitive one, similar to the patterning process described in the neuropsychological domain.

Mathematics has been referred to as the "science of patterns … Seeing and revealing hidden patterns are what mathematicians do best" (Steen, 1990, p.1). In his text, *On the Shoulders of Giants: New Approaches to Numeracy*, Steen (1990) suggested that the field of mathematics has significantly expanded because of this innate desire to "search for pattern" (p. 1). He elaborated on the significance of patterning in the work of mathematicians. "Mathematics seeks to understand every kind of pattern—patterns that occur in nature, patterns invented by the human mind, and even patterns created by other patterns" (p. 8). Steen acknowledged the integral role of patterning in all aspects of mathematical study, stating how identifying and understanding the nature of patterns, and interestingly also the patterns in nature, is actually the process through which mathematical knowledge is created.

Structural understanding is an outcome of the generalisation and abstraction of patterns and has been linked to the development of pre-algebraic thinking (Papic, Mulligan, & Mitchelmore, 2011). "Young children are capable of abstraction and generalization of mathematical ideas. An intuitive awareness of patterning concepts and structural relationships … [is] … critical to fundamental mathematics learning" (Papic et al., 2011, p. 237). In simplest terms, structural awareness is a recognition of structures within a pattern that are "the same … Every pattern is a type of generalisation in that it involves a relationship that is 'everywhere the same' " (p. 240) and involves "a relationship that holds over the entire class of values, not only in isolated instances" (p. 239). The importance of structural awareness in the development of prealgebraic thinking in the early years of schooling and the later development of wider mathematical concepts is substantiated through a growing body of research (Mason, 1996; Mason, Dury, & Bills, 2007; Mason, Graham, & Johnston-Wilder, 2005; Mason, Stephens, & Watson, 2009; Papic et al., 2011). This research raises questions about how generalisations of mathematical concepts are embodied in contemporary mathematical education practice.

Succinctly described, "the power of mathematics lies in relations and transformations which give rise to patterns and generalisations. Abstracting patterns is the basics of structural knowledge, the goal of mathematics learning" (Warren, 2005, p. 26). An awareness of mathematical structure has the power to transform and engage mathematical thinking, but is dependent upon teachers' awareness of "structural relationships … [and strategies for] … bringing structural relationships to the fore" (Mason, Stephens, & Watson, 2009, p. 29).

Abstraction arises from opportunities to generalise about patterns, and the degree of abstraction relates to the extent that a concept is connected to a particular context (Skemp, 1986; White & Mitchelmore, 2010). There is a direct relationship between the degree of abstractness and the extent to which it is removed from specific situations, leading to a greater ability to generalise similarity about the concept across a range of contexts: "Knowledge is more general and its applicability to different situations is increased" (White & Mitchelmore, 2010, p. 2). In the model of teaching for abstraction (White & Mitchelmore, 2010), the role of the teacher in drawing students' attention to the similarities within and between contexts is highlighted through various stages. This deliberate process engages students in operating more deeply with abstract concepts. This type of mathematical thinking, guided through intentional instruction, is described as a conscious process. In forming this model, White and Mitchelmore drew from the psychology of mathematics education literature, particularly the thinking of Richard Skemp.

2.3.3 The psychology of mathematics education and patterning

In his book *The Psychology of Learning Mathematics*, Skemp (1986) described the process of abstraction as "becoming aware of similarities ... among our experiences … resulting in lasting mental change" (p. 21). Essentially the concept represents the similarities and is the product of abstraction. Hence, explicitly engaging with the elements of similarity across contexts will lead to a greater structural understanding of concepts explored. Thinking mathematically is primarily viewed as an intentional process involving rigorous thinking about one's thinking. Mueller, Yankelewitz, and Maher (2010) refer to Ball and Bass' (2003) claim that understanding is "meaningless without a serious emphasis on reasoning … to understand the relationships and make connections to new ideas" (p. 308). Therefore the process of building mathematical understanding is not a passive one.

In defining mathematical thinking, David Tall (2009) reflected on the work of John Mason, describing the teacher's role in supporting students' thinking to "discriminate between and see similarity across objects, to conjecture and inquire" (p. 17). Tall also acknowledged that our thinking is developed "through refining our knowledge structures … so that we can talk about them" and further referred to "Mason's insight" into thinking as "a delicate shift in attention "which involves "the discipline of noticing" (p. 23).

David Tall (2009) commented on Skemp's exploration into "how the human mind works through perception, action and reflection" (p. 23), clarifying that reflection is the vital element in creating mental links between perception and action. Tall developed a construct about how mathematical thinking develops and named it as " recognition, repetition and language" which leads to "thinkable concepts" (p. 24). These concepts can be further developed by engaging them as mathematical objects. Essentially these operate as "mental objects of attention to work at higher levels" of thinking (p. 24).

In exploring the structure of conceptual understanding, Skemp (1986) described the process of the expansion and reconstruction of schemas. For him, "a schema is a conceptual structure", a mental tool that "assimilates new knowledge (expands) and also reconstructs to adapt and make sense of new situations. … It integrates existing knowledge, acts as a tool for future learning and makes possible understanding" (pp. 37, 62, 41). The term understanding is complex. Paul Ernest (1994) explained the relationship between "knowledge development and acquiring understanding" (p. 156) as being an interaction between the application of mathematical operations and concepts in new contexts. If the concept "can be integrated into the open situation naturally … [it] … produces nearly no problem with understanding" (p. 157). Ernest was referring to the process of, and end product of, understanding, which involves the ease of transferring and assimilating mathematical concepts. However, the development of new understanding involves additional processes to reconstruct and adapt the understanding of the concept further or differently.

Robert Davis (1992) stressed the importance of new understanding being dependent upon the breadth and depth of previous understanding: "One gets a feeling of understanding when a new idea can be fitted into a larger framework of previously assembled ideas" (p. 228). The framework Davis described is in essence a structural pattern of understanding that has emerged through engagement with similar ideas. This process is comparable to Goldberg's description of pattern recognition, whereby the sense of similarity experienced across ideas connects with a previously encoded pattern. Mueller, Yankelewitz, and Maher (2010) elaborated on the work of Davis, referring to the process of understanding involving "representational structures that a learner builds as a collection of assimilation paradigms" (pp. 308–309). Skemp (1986) further explained the relationship between assimilation and understanding by pointing out that "to understand something is to assimilate it into an appropriate schema", and that "better organisation of a schema may improve understanding" (pp. 43–44). In this instance, schema can be compared with the term pattern, and assimilate with the term connect. Therefore, rephrasing Skemp's point reveals a similarity with the neuropsychological construct of pattern recognition: to understand something is to connect it into an appropriate pattern. This again affirms similarities in the description of the construction of across these domains. Skemp was also indicating the importance of challenging and expanding current understanding. Developing further understanding involves the processes of assimilation, accommodation, and expansion of the current schema. Through these processes, understanding is actively constructed and reconstructed to accommodate growing awareness and insight. In this way, the psychology of mathematics education perspective connects visibly with the view of intentional instruction in mathematics education in supporting the capacity to reason and develop generalised patterns of understanding.

2.3.4 Cognitive neuroscience and patterning

Cognitive neuroscience "informs our understanding of cognitive behaviours relevant to education" (Geake & Cooper, 2003, p. 8). This field eventuated from the merging of two academic fields of knowledge, neuroscience and cognitive psychology, in the late 1970s in an endeavour to "understand how the brain enabled cognition" (Gazzangia, Ivry, & Mangun, 2002, p. 21). Neuropsychology is a specialisation within the broader field of cognitive neuroscience (Gazzangia et al., 2002; Goldberg, 2005). Researchers within and across this broad field describe the construction of understanding using various terms such as neural networks, cognitive templates, internal maps, mental representations, and patterns of thinking (Devlin, 2010; Dispenza, 2007; Gazzangia et al., 2002; Geake, 1997). These terms all describe the neural structures involved through the process of building cognitive patterns. "Our minds are very good at recognising patterns, seeing connections, and making rapid judgements and inferences" (Devlin, 2010 p. 171). The processes Devlin refers to reaffirm the elements of pattern recognition identified earlier (Goldberg, 2005): that is, recognising patterns, seeing similarities and relationships between the elements, making connections, and using this to generalise and apply understanding in new situations.

Human memory relies heavily on the associations it develops with and between patterns of conceptual understanding (Devlin, 2010). When new experiences are connected with prior understandings, this "enables the brain to link new information with well encoded ideas" (Wiles & Wiles, 2003 p. 18). Hebbian learning, proposed by Donald Hebb (1949), describes the process through which we engage our associative memory to learn new information (Dispenza, 2007). It asserts that a weak association, something novel or unfamiliar, can be made strong by attaching known connections to the learning (Dispenza, 2007). This results in a change in the synaptic response of the cells involved (Gazzangia et al., 2002). It follows the notion that "cells that fire together, wire together" (Dispenza, 2007, p. 184) and consequently make strong, lasting, and resilient connections. These neural networks are a cognitive representation of an understanding and "develop as a result of continuous neural activation" (Dispenza, 2007,

p. 185) through reengagement with similar experiences. This is the result of the neuroplasticity of the brain, which describes “the capacity of the brain to change at a neurophysiological level in response to changes in the cognitive environment” (Geake & Cooper, 2003, p. 14). The similarity recognised across experiences enables patterns to be actively engaged and restructured through this process of neural plasticity.

Neural networks are both locally and globally encoded in the brain (Geake, 1997). Thoughts of one thing lead to another, and connections between those thought patterns are stimulated along with all the other connections present in the various neuronal groups. This activity leads to a creation of internal maps. When these maps overlap there is a mapping of maps, a making of meaning, and interpretation is possible. “The interconnections allow each cluster to correlate their information” (Geake, 1997, p. 28). Edelman, the founder of the theory of neuronal group selection (Edelman, 1992) proposed that the brain selects an appropriate interpretation of these maps through “natural selection”. This form of neural Darwinism “leads to superior contextual fitness” (Geake, 1997, pp. 28, 32). “The most useful correlated clustering … is selected … determining consequent behaviour … in a Darwinian adaptive sense” (Geake 1997, p. 28). Over time these maps accumulate, becoming resilient, fluently recalled patterns that have not succumbed to the process of natural selection.

Others in the field of cognitive neuroscience concur. “Successful, extensive patterning leads to more accurate predictions … extending and strengthening neural networks” (Willis, 2010, p. 61). Similarly, Elkhonon Goldberg (2005) described the formation of generic patterns through “exposure to the same or similar thing ... [this] … will breathe life into the reverberating loop supporting the formation of the memory about it … The more frequently encountered information usually wins” (p. 123). Patterns that are engaged more regularly have a greater chance of being encoded in long-term memory stores. Frequency and repetition of experience seem to be dominating factors in the formation of generic memories and patterns of thinking. Intentional instruction supporting the structure of patterns fosters the development of generalisations in mathematics education. This line of inquiry leads to questioning the role of patterning within the Australian Curriculum: Mathematics (ACM).

2.4 Educational implications of patterning within the ACM

In the rationale of the ACM, the field of mathematics is described as “composed of multiple but inter-related and inter-dependent concepts and systems” (Australian Curriculum and Assessment Reporting Authority [ACARA], 2015, p. 4). This description anticipates that schools will engage with the curriculum in a dynamic and symbiotic way. The goal of mathematics is also clearly articulated in the rationale statement: “It aims to instil in students an appreciation of the elegance and power of mathematical reasoning” (ACARA, 2015).

Mathematical reasoning is one of the four proficiencies named in the ACM, the other three being understanding, fluency, and problem solving. These proficiencies together describe the mathematical thinking and actions that students are engaged in: “They describe how the content is explored or developed; that is, the thinking and doing of mathematics” (ACARA, 2015, p. 5). Reasoning is described as the “capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising” (ACARA, 2015). Mathematical reasoning invites students into the process of enquiring and generalising about the nature of mathematical concepts.

The role of patterning within the teaching of mathematics is highlighted in the ACM through the interaction of the four mathematical proficiencies. However, throughout the ACM the proficiencies are named and described individually at the beginning of each year level without any indication about how they interact. In *Engaging the Curriculum-Mathematics: Perspectives from the Field*, Atweh, Miller, and Thornton (2012) identified challenges that schools and educators could face in interpreting and implementing the curriculum due to a “possible lack of cohesion between the aims and rationale, the content and its articulation” (p. 2). In particular, they noted inconsistency between the weighting of individual proficiencies and they expressed concern about the underrepresentation of reasoning identified in the content descriptions.

An interrelated view of the proficiencies is described in the United States report to the National Research Council (Kilpatrick, Swafford, & Findell, 2001). In this report, Kilpatrick et al. (2001) refer to a model, focused on mathematical proficiency, that is based on five strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. In this model, the term mathematical proficiency is used to “capture what we think it means for anyone to learn mathematics successfully … The most important observation we make about these five strands is that they are interwoven and interdependent… [and] … represent different aspects of a complex whole” (Kilpatrick et al., 2001, pp. 5, 116). Kilpatrick et al. (2001) stressed the importance of the relationship between all strands in building resilient understandings that can be fluently applied in new situations.

An integrated view of the proficiencies acknowledges their interrelationship in building and deepening mathematical understanding. Engaging mathematical reasoning naturally draws students into greater levels of fluency as they connect their understanding in new problem-solving contexts, and have opportunities to explain and justify their thinking. Sullivan (2012) proposed that teacher learning should focus on "ways of identifying tasks that can facilitate student engagement with all four of these proficiencies" (p. 183) as the "intention is that the full range of mathematical actions apply to each aspect of the content" (Sullivan, 2011, p. 8).

What remains unclear is how the role of patterning, that underpins the development of reasoning, abstraction, and generalisation is developed in a general sense through an interrelated view of the proficiencies.

2.5 Making connections to inform the research inquiry

Abstraction and generalisation in mathematics arise from opportunities to generalise about patterns, and the degree of abstraction relates to the extent to which a concept is connected to a particular context. Therefore, the degree of abstraction is directly related to the breadth and depth of the generalisation (Skemp, 1986; White & Mitchelmore, 2010). From this process of generalising, structural understandings can emerge. The intentional practice of reasoning, promoted through mathematics education, aims to support this goal of mathematics education (ACARA, 2015; Warren, 2005, 2008).

2.5.1 Cross-disciplinary role of patterning

From the neuropsychological field, mathematical abstraction resonates with Goldberg's description of patterning as a neural construction of generalised understandings, a generic memory that can be fluently recalled and applied generally in new contexts. This process of accessing generic memory supports the theoretical perspective that mathematical knowledge cannot be separated from human knowledge.

2.5.2 Conclusion

In concluding this chapter, pattern recognition can be described as a "superior form of contextual fitness" and "neuronal efficiency" (Geake, 1997, p. 32), from which Goldberg (2005) proposed the "cognitive dimension of wisdom" emerges (p. 11) as an ability to instantly recognise new situations as familiar ones. Familiarity and re-engagement with a concept allows similarity to be reexperienced and understood. Over time, similarities are encoded in the conceptual structure of patterns. These patterns are activated when similarity is again recognised, and strong resilient patterns of understanding support new connections to be made with less familiar contexts. Over time, a "patterning of patterns" accumulates, creating resilient generic memories that are in essence intuitive forms of recognising and understanding new situations as familiar ones. These patterns result from an ability to generalise about the properties of similar experiences and concepts. Interestingly, there is similarity in the processes through which understandings are constructed as patterns across domains of knowledge.

CHAPTER

3

PUBLICATION 1

Does an Ability to Pattern Indicate We Think Mathematically?

3.1 Introduction

In this chapter I present the first publication of this thesis, a position paper presented to the Mathematics Education Research Group Australasia (MERGA) in July 2013. This publication addressed the first research question in this thesis: What is similar about the construct of pattern across the fields of mathematics education and neuropsychology? (In this publication I refer to the neurological field rather than neuropsychology. This discrepancy is addressed in the postscript.)

There were two key themes in the paper:

1. comparing the role of pattern as a mathematical construct with pattern as a neurological construct, and
2. introducing an emerging view of a generalised model of patterning across the mathematical and neurological domains of knowledge.

The chapter begins with a preamble outlining the development and purpose of the paper. The postscript provides a discussion addressing key issues arising through the feedback from colleagues and my supervisors following the presentation of the paper. Through this feedback I reviewed my approach by providing a broader view of patterning across additional domains of knowledge to build a stronger theoretical position for the generalised model of patterning. Technical and structural aspects of the paper that needed further attention are addressed. I also identified areas for future research that were drawn upon to develop Publication 2: *The Role of Patterning Within and Beyond Mathematical Thinking*. This publication is provided in Chapter 4.

3.2 Preamble

> Intelligence as a whole, takes the form of a structuring, which impresses certain patterns on the interaction between the subject … and … surrounding objects.
>
> (Piaget, 1950, p. 167)

This quotation from Piaget promotes a view of intelligence as comprising patterns that encode meaningful elements of experiences. Similarities between certain patterns therefore would be encoded as a relationship, an overlapping of experience that forms a structure enabling that type of experience to be more readily understood. Hence, Piaget is describing the relationship between patterning, structuring, and intelligence.

It is the role of patterning in building mathematical understanding that is the focus of this chapter. I explore how the process of forming patterns could be mathematical. Pattern and structure underlie the development of a broad range of mathematical concepts. However, the concept of pattern also occurs in other fields. Therefore, in this paper I compare pattern recognition, a neurological construct based on the work of Goldberg (2005), with pattern as defined in the field of mathematics education, to highlight what is intrinsically similar about the concept in these fields. An emerging model of patterning is proposed to describe this relationship.

3.3 Publication 1: Does an ability to pattern indicate that our thinking is mathematical?

The following publication appeared in the MERGA36 conference proceedings. Its full reference is the following:

McCluskey, C., Mitchelmore, M. C., & Mulligan, J. T. (2013). Does an ability to pattern indicate that our thinking is mathematical? In V. Steinle, L. Ball, & C. Bandini (Eds.), *Mathematics education: Yesterday, today and tomorrow: Proceedings of the 36th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 482–489). Melbourne, VIC: MERGA.
https://merga.net.au/Public/Public/Publications/Annual_Conference_Proceedings/2013_MERGA_CP.aspx

The text reproduced below retains the wording of the original publication, but font pitch, line spacing, page margins, and other aspects of formatting have been altered to conform to Macquarie University regulations for theses as well as other text within this thesis.

Does an Ability to Pattern Indicate that Our Thinking is Mathematical?

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Research affirms that pattern and structure underlie the development of a broad range of mathematical concepts. However, the concept of pattern also occurs in other fields. This theoretical paper explores pattern recognition, a neurological construct based on the work of Goldberg (2005), and pattern as defined in the field of mathematics to highlight what is intrinsically similar about the concept in these domains. An emerging model of patterning is proposed to describe this relationship.

In contemplating the term *pattern,* what comes to mind? A number pattern, lyrics in a song, patterns in a design, rhythms of nature, a sequence of events, chords in a tune, or a template to make an outfit? What about the use of the term pattern to describe how we learn from our experiences? The term pattern has distinct meanings across differing domains; what is intrinsically similar about the concept of patterning in each instance?

This paper will explore the concept of patterning across two domains, exploring the relationship between patterning as a neurological construct, the processes through which our understandings are encoded, and patterning as defined in the field of mathematics. Is there any similarity in the concept of patterning across these two domains? What is fundamentally different about patterning in each context, and is it viable to create a generalised model of patterning across these domains?

Theoretical background

Quasi-empiricism, a philosophical view of mathematics developed by Imre Lakatos (1976, 1978), recognises that "mathematical activity is human activity" (Lakatos, 1976, cited in Ernest, 1991, p. 37) and like all human endeavours is fallible and uncertain and therefore needs to be rigorously questioned (Ernest, 1991). If mathematics knowledge "is seen as connected with, and ... part of the whole fabric of human knowledge" (Ernest, 1991, p. 26), is there any difference with how we construct our knowledge generally?

There are four aspects of quasi-empiricism that this paper builds upon:

- Mathematics knowledge is fallible and like all knowledge can and needs to be questioned.
- Mathematical knowledge evolves; new mathematical knowledge is then part of an on-going process of knowledge creation.
- The primacy of informal mathematics: all formal mathematics is derived from informal human experiences.

- The genesis of mathematical knowledge: the creation of mathematical knowledge cannot be separated from the creation of human knowledge (Ernest, 1991, pp. 35–36).

This paper asserts the view that mathematical knowledge is not created differently and should not be separated from human knowledge. In fact, this paper is essentially an inquiry into knowledge creation, identifying the possible mathematical elements involved in this process. I use the term *patterning* to describe the processes through which understandings are constructed.

The role of pattern has enjoyed a history of speculation in the field of mathematics. Steen in his groundbreaking text *On the shoulders of giants* (1990) has been widely cited as claiming that "mathematics is the science of patterns" (Steen, 1990, p. 1). Earlier, Piaget (1950) noted that "life itself is a creator of patterns" (cited in Lilejedahl, 2004, p. 24). Both are asserting the inherent role of pattern in the construction of knowledge, life knowledge and the formation of mathematical knowledge.

If "virtually all mathematics is based on pattern and structure" (Mulligan & Mitchelmore, 2009, p. 33), and "mathematical activity is human activity" which "produces mathematics" (Lakatos, 1976, cited in Ernest, 1991, p. 36), then is human activity also based upon pattern and structure? Are the elements of pattern and structure that are evident in mathematics also evident in human activity, and does this mean human activity is therefore mathematical? This question forms the basis of this paper. It is this relationship between mathematics and the nature of life itself that will be explored through the construct of patterning in both the mathematical and neurological domain. If this relationship can be substantiated, could a theory of mathematics evolve which views patterning, a mathematical construct, as the structure through which we create all our understandings?

Patterning as a neurological construct

In *The Wisdom Paradox* (2005), Elkhonon Goldberg, a world renowned neuropsychologist proposes that in building our understandings we are essentially patterning our experiences and retrieving them through the process of pattern recognition, which he refers to as an "ability to recognise a new problem as a member of an already familiar class of objects or problems" (p. 85). This pattern recognition eventuates from an accumulation of similar experiences, and "decision making takes the form of pattern recognition rather than problem solving" (p. 20). Goldberg cites the words of Herbert Simon (1966) in substantiating the role of pattern recognition as "the most powerful mechanism of human cognition" (p. 20). Goldberg refers to these patterns as "cognitive templates, each capturing the essence of a large number of pertinent experiences" which he relates to the acquisition of "wisdom … and a cognitive gain of aging" (pp. 21-22). These patterns enable us to rapidly recognise solutions to seemingly new situations as if they were familiar ones. The process involves being able to understand the elements involved, knowing what action to take and the possible outcomes that could result. Goldberg explicitly refers to this type of pattern recognition in terms of demonstrating the coveted attribute of wisdom and that these resilient patterns accumulate over a life time.

In everyday terms, Goldberg describes our ability to naturally sort and classify our experiences by relating what we know to past patterns of understanding; these being built through an accumulation of similar experiences over time. There are elements of new situations that have been experienced before and outcomes that have been previously tested. Goldberg claims that cognitive templates, which encase these familiar elements, are formed physically in the brain and are engaged through the process of pattern recognition.

The process of reconnecting to similar patterns over time creates neural structures, which Goldberg (2005) describes as "generic memories"; essentially these are "memories for patterns" (p. 125). "The more generic a pattern is … the vaster the set of experiences on whose overlap it emerged, the more robust … it is" (p. 125). Goldberg refers to these generic memories as an abstract representation of a set of similar experiences which is much more resilient than "concrete representations corresponding to unique things" (p. 125). Generic memories "capture the essence of a wide range of specific situations and the most effective actions associated with them," leading to high levels of competent, efficient decision making (p. 79).

Generic memory draws upon a network of common neural pathways related to the similar attribute of the experience they share. This overlapping of neural space becomes eventually "a shared network ... a mental representation of not any single thing or event, but rather the shared properties of a whole class of similar things or events" (p. 125), alluding to their inherent structure. The network not only represents the condensation of past experiences but also embodies information about "essential properties of class members" (p. 126), which can be added to, expanded upon and utilized in future situations, allowing rapid pattern recognition and application.

Goldberg acknowledges the role of language in "shaping our cognition by imposing certain patterns on the world" (p. 91); however he does not mention that pattern recognition, a neurological construct, could be mathematical. We have already explored though how this cognitive process involves sorting, classifying, identifying similarity, creating generic memories; all leading essentially to the development of abstract understandings and the application of these to predict in future situations. The purpose of this paper is to highlight how this everyday process is mathematical.

Pattern as a mathematical construct

Mathematics has been referred to as the "science of patterns … seeing and revealing hidden patterns are what mathematicians do best" (Steen, 1990, p. 1). In Steen's text *On the shoulders of giants: New approaches to numeracy* (1990), he suggests that the field of mathematics has significantly expanded because of this innate desire to "search for pattern" (p. 1). He elaborates on the significance of pattern in the field and work of mathematicians. "Mathematics seeks to understand every kind of pattern-patterns that occur in nature, patterns invented by the human mind, and even patterns created by other patterns" (p. 8). In both instances Steen acknowledges the integral role of pattern in all aspects of mathematical study, and alludes to the role of patterns in life. He states how identifying and understanding the nature of patterns; and interestingly also the patterns in nature; is actually the process through which mathematical knowledge is created. This affirms the philosophical view that mathematical knowledge is derived from human knowledge and the desire to understand the world we live in.

In the field of mathematics, "a mathematical pattern may be described as any predictable regularity, usually involving numerical, spatial or logical relationships" and its structure is defined as "the way a pattern is organised" (Mulligan & Mitchelmore, 2009, p. 34). Replicating regularity involves recognising, predicting, and repeating what is deemed similar. Structural understanding emerges from generalising about the similarity; it involves exploring "a relationship that holds over the entire class of values, not only in isolated instances" (Papic, Mulligan, & Mitchelmore, 2011 p. 239).

Recent research (Arcavi, 2003; Lilejedahl, 2004; Mulligan & Mitchelmore, 2009; Papic et al., 2011; Warren, 2008) confirms that it is the actual process of exploring pattern and structure and developing visualisation that builds broader mathematical understandings. This process involves identifying patterns and similarities between patterns, constructing generalisations, the creation of abstract mathematical objects and structural awareness, as "abstracting patterns is the basis for structural knowledge, the goal of mathematics learning" (Warren, 2008, p. 759). In simplest terms, structural awareness is a recognition of structures within a pattern which are "the same … every pattern is a type of generalisation in that it involves a relationship that is 'everywhere the same'" (Papic et al., 2011 p. 240).

When a prediction is made that is based upon a generalisation about the pattern, this type of mathematical thinking leads to the ability to abstract (White & Mitchelmore, 2010). Structural understandings, an identified goal of mathematics, are developed through this ability to engage abstractly with patterns (Warren, 2008). The term *abstraction*, as used in the mathematical context, refers to "the degree to which a unit of knowledge (or a relationship) is tied to a specific context" (White & Mitchelmore, 2010, p. 1). There is a direct relationship between the degree of abstractness and the extent to which it is removed from specific situations, leading to a greater ability to generalise about "relevant conceptual attributes" across a range of contexts, so that "knowledge is more general and its applicability to different situations is increased" (White & Mitchelmore, 2010, p. 2).

Richard Skemp (1986), known for his pioneering work into the psychology of mathematics education, describes the process of abstraction as becoming "aware of similarities (in the everyday, not the mathematical sense) among our experiences", resulting "in some kind of lasting mental change" (Skemp, 1986, p. 21). Skemp explains that the act of naming objects is a form of classification. This involves identifying that a particular object belongs to a category based upon some predetermined criteria which is satisfied by the whole class of objects. Skemp links this process of classification to conceptual development, "a concept therefore requires for its formation a number of experiences which have something in common" (p. 21). White and Mitchelmore (2010) elaborate further on the nature of the similarities that Skemp is referring to, not in terms of superficial appearances but of underlying structure "in a sense the concept embodies or reifies the similarities" (p. 206). Essentially the concept represents the similarities and is the end product of abstraction.

The relationship between mathematical and neurological patterning

In this description of mathematical pattern, we hear familiar words that Goldberg echoed in his definition of pattern recognition—the ability to identify similarity, distinguish difference, and essentially to understand and apply the predictable elements to new situations in a generalised way. Mathematical abstraction resonates visibly with Goldberg's description of patterning as a neural construction of our generalised understandings, further affirming the theoretical perspective that mathematical knowledge cannot be separated from human knowledge (Ernest, 1991). It is this relationship between mathematical abstraction and pattern recognition which will be explored further.

White and Mitchelmore (2010) outline a theory of how students develop generalised mathematical understandings through abstraction and propose an approach to teaching, called

Teaching for Abstraction, designed to support and strengthen this process. In this model, students engage in exploring a concept across a range of contexts and, as their sense of familiarity increases, learn to recognise similarity across contexts and develop generalised understandings, leading to a growing ability to predict and abstract.

The Teaching for Abstraction model has four phases, namely familiarity, similarity, reification, and application.

- In the familiarity phase, students explore a concept through engaging in a variety of contexts, becoming “familiar with the underlying structure of each context”.
- The similarity phase involves frequent matching and explicit attention to structural similarities within the varying contexts and differences with other contexts.
- The reification phase moves students into operating with and developing abstract concepts.
- The application phase allows students to consolidate their understanding of the abstract concept through application to new situations (White & Mitchelmore, 2010, p. 5)

Similarly, Goldberg (2005) describes pattern recognition as an ability to pattern our experiences and draw upon these patterns in future instances. In doing so, we naturally go about seeking similarity, discriminating difference, and creating varying measures of understanding by relating new experiences to what is already known and understood. What is common about the experience helps to reinforce and expand the pattern even further. Over time these patterns are encoded as cognitive templates, predicting possible solutions to future problems becomes a matter of pattern recognition.

Each phase of the development of mathematical abstraction, as summarised in the Instruction for Abstraction model, can be aligned with Goldberg’s description of pattern recognition, as shown in Table 1.

Goldberg describes how our everyday generalised understandings are constructed cognitively as generic memories and how in the long term this can lead to the rapid application of pattern recognition in new situations. This appears to be an intuitive process, as everyday understandings are constructed below our level of awareness. Situations can feel familiar because past experiences have merged to create a mental construct, a patterning of similar understandings, which is challenged and changed with each new experience.

In the mathematical domain the focus is on understanding and communicating the nature of patterns, which involves the ability to sort and classify; becoming familiar with the properties of mathematical objects across differing contexts; recognising similarities and using these to create, predict and generalise; leading to an ever increasing ability to deal with concepts in an abstract sense. Knowledge that is generalised across a range of contexts can be applied to new situations. Through this process of abstraction, freed from specific contexts. each generalisation “becomes a mathematical object in its own right” (White & Mitchelmore, 2010, p. 1). Increasing understanding leads to the development of “a point of view which guides our thinking” (Cassirer, 1923, cited in Van Oers, 2001, p. 284).

Table 1
Abstraction Across Domains[a]

Mathematical domain	Neurological domain
Familiarity: • explore a concept through a variety of contexts • become familiar with the underlying structure of each context	• Engagement with a range of situations set the scene for experiencing and sensing what is familiar about these situations.
Similarity: • frequent matching • explicit attention to similarities within and between varying contexts	• Further attention and engagement with familiar situations allows connections to emerge as we recognise what is similar about these experiences. • Engagement with these experiences enables similarity to be matched, measured, and understood. • Overlapping of neural networks encode the similarity experienced.
Reification: • moves students into operating with abstract concepts	• Over time a pattern is encoded as a generic memory, a mental representation of the similarities and shared properties of a 'type' of experience.
Application: • consolidation of the concept • application to new situations	• Pattern recognition refers to the ability to readily access this pattern in similar situations in the future.

[a]Information collated from White and Mitchelmore (2010) and Goldberg (2005), respectively.

At each stage, the mathematical structure described through the Teaching for Abstraction Model (White & Mitchelmore, 2010) mimics the process of pattern recognition proposed by Goldberg (2005). There appears to be mathematics inherent in the way we construct generalised everyday understandings which implicitly guide our thinking and our future choices in new situations. Could the generic memories that Goldberg refers to in the process of pattern recognition be viewed as mathematical objects, a by-product of our ability to generalise?

In their analysis of the process of reification, Thompson and Sfard (1994) also elaborate on the nature of mathematical objects. "Objects ... are in a sense, figments of our mind. They help put structure and order into our experience (p. 11), "objectness comes from possessing coordinated schemes" (p. 16) that are linked together "because we feel somehow they represent the same thing" (p. 2). By this definition, coordination of thinking is aligned through the structural awareness we put in place to understand a concept. Structural awareness is developed through the process of generalising and abstracting the patterns we encounter. This concept ... this structure ... this pattern ... this object is re-engaged each time we encounter a similar experience, whether this experience is classified in mathematical or everyday terms.

In both the mathematical and neurological domain, structural awareness emerges from engaging with familiar patterns; similar elements of patterns across varying contexts merge to form structural understandings which encase the conceptual experience. This process leads to generalised understandings which can be applied in new situations to further engage and deepen the concept at hand. There are noticeable similarities in the terms used to describe the elements, processes and outcomes of patterning in both the mathematical and neurological domains.

Summary

Exploring the relationship between mathematical learning and the formation of everyday understandings through the construct of patterning has highlighted the similarities between the two contexts. In both cases, learning follows a similar cognitive process, involving the seeking of similarity, understanding of differences, and iteration of what is common through the repetition of experience. Patterning also explores the relationship between the elements, leading to a structural awareness of how the concept is organised and what is the same in each instance. This structural awareness enables one to generalise the relationships inherent in the pattern and apply an abstract understanding of the concept to make predictions in new situations.

Essentially we experience patterning through our everyday encounters and patterning is a process through which we construct our understandings. If mathematical activity though is human activity (Lakatos, 1976 cited in Ernest 1991) and all formal mathematics is derived from informal human experience (Ernest 1991), and if mathematical knowledge "is seen as connected with, and ... part of the whole fabric of human knowledge" (Ernest, 1991, p. 26), then can the mathematical concept of pattern be viewed as the formal embodiment of the informal sense of patterning we encounter through our life experiences?

It appears that the very processes through which we construct our understandings and ultimately experience life itself are essentially mathematical. Could a generalised model of patterning evolve, one which views pattern, a mathematical construct, as the structure through which we create and build our understandings? How could a generalised model of patterning support the development of both mathematical and everyday understandings? What implications could such a model have for our educational practices?

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3.4 Postscript

Presenting the position paper at MERGA provided me with opportunity to gather feedback to ensure I was using correct terminology and presenting a coherent theoretical argument (Refer to Appendices B2 and B3). The feedback supported the expansion of the proposed generalised model of patterning and assisted in developing educational implications regarding the implementation of the model. Issues raised included:

1. Use of correct terminology

   Elkhonon Goldberg is a neuropsychologist, and therefore it was identified that this research is drawing upon the construct of pattern from a neuropsychological field not neurological field. The field of neuroscience is continually expanding to encompass specialised fields. Therefore, the correct use of terminology is reflected in my research and has been addressed in Publication 2.

2. Clarifying levels of abstraction across domains

   In Table 1 within the publication above, Goldberg's description of the development of pattern recognition (Goldberg, 2005) had been aligned with the stages of abstraction outlined in White and Mitchelmore's teaching for abstraction model (White & Mitchelmore, 2010). One participant questioned whether Goldberg's construct of pattern recognition aligned more closely with the reification stage of White and Mitchelmore's model, rather than the application stage. After discussion with Mike Mitchelmore, he clarified that

   - pattern recognition refers to the ability to readily access patterns in *similar* situations in the future, aligning more correctly with the *reification stage* in the teaching for abstraction model; and

- the *application stage* therefore involves recognising *familiar* patterns more fluently and *applying* patterns of knowing by recognising *new* problems as having *similar* characteristics to *familiar* ones.

Essentially, the application stage involves recognising the *same* pattern in *new* situations. As a result, Table 1: *Abstraction Across Domains* was adjusted and re-named the generalised model of patterning when Publication 2 was finalised in 2014 to reflect this difference and provide better clarification between the stages of abstraction.

3. Revising the title of the paper.
   The title of Publication 1, *Does an Ability to Pattern Indicate That We Think Mathematically?* came under question. In the feedback from colleagues I was asked to clarify what I meant by "thinking mathematically". I was questioned by participants whether this meant in a broad sense, namely that we think mathematically through our everyday encounters, or whether I was referring to thinking mathematically as occurring more specifically within the field of mathematics. Through the discussion it became clear that there was a distinct difference between these two contexts.

The argument in Publication 1 was structured around the proposition that if elements of mathematical pattern could be identified in the construction of everyday understandings, and if pattern and structure underlie the broad development of mathematical understandings, the construction of our generalised understandings could be viewed as mathematical. Comparing the role of patterning across the domains identified similarities in the stages of pattern and structure evident in the construction of understandings. In the title I was alluding to the analogy that the common elements of both mathematical and generic understandings were pattern and structure. Since pattern and structure are integral elements of mathematical thinking and "mathematics is an exploratory science that seeks to understand every kind of pattern" (Steen, 1990, p. 8), I felt my argument was justified: The way we think is mathematical because the elements of pattern have been identified and used.

However, in a mathematical sense, thinking mathematically is an intentional process involving metacognition, an awareness of one's thinking. Understanding is developed "when a new idea can be fitted into a larger framework of previously assembled ideas" (Davis, 1992, p. 228). The process of building mathematical understanding is not a passive one. Mueller, Yankelewitz, and Maher (2010) refer to Ball and Bass's (2003) claim that understanding is "meaningless without a serious emphasis on reasoning … to understand the relationships and make connections to new ideas" (p. 308). This view supports the notion that mathematical thinking is strengthened through capacity to generalise about relationships and transfer understandings in new contexts. In defining mathematical thinking, David Tall (2009) reflected on the work of John Mason, describing the teacher's role in supporting students' thinking as being to "discriminate between and see similarity across objects, to conjecture and inquire" (p. 17). Tall also argued that our thinking is developed "through refining our knowledge structures … so that we can talk about them", and further referred to Mason's insight into thinking as "a delicate shift in attention" … [involving] … "the discipline of noticing" (p. 23). Again, mathematical thinking is described as a conscious and deliberate process.

The role of the teacher in drawing students' attention to the similarities within and between contexts is highlighted through the various stages of abstraction in White and Mitchelmore's model of teaching for abstraction. This deliberate process engages students in operating with more abstract concepts. Conceptual understanding in mathematics therefore is deepened when awareness of similarities encountered in familiar and new situations is engaged. Again, this type of thinking is regarded as a conscious process.

However, In Goldberg's description of pattern recognition, the development of generic patterns occurs over time as an accumulation of experiences and is not always a conscious process. "Memories for patterns" are supported through "every new exposure to the same or similar thing" and this exposure "will breathe life into the reverberating loop supporting the formation of the memory about it … the more frequently encountered information wins" (Goldberg, 2005, p. 123). Every new exposure is not necessarily intentionally created, or consciously formed. Patterns that are engaged more regularly have a greater chance of being encoded in long-term memory stores. The frequency and repetition of experience seem to be dominating variables in the formation of generic memories. At this stage the involvement of conscious deliberation of concepts has yet to be explored in the formation of generic patterns. The involvement of conscious intent in the formation of memory is an area for further research.

Consequently, in consultation with my supervisors, it was decided to title the second publication *The Role of Pattern Within and Beyond Mathematical Thinking*. This title more accurately describes this research inquiry into the role of pattern rather than an argument about the "mathematical" way we think and construct understandings.

4. Discussion arose about the role of pedagogical interventions focused on pattern and structure in supporting children with special learning rights. If conceptual understanding develops through a greater sense of pattern and structure, how might the broader field of education capitalise upon the role of pattern in the building of generalised understandings? What could be the impact on learning when elements of pattern and structure are intentionally embedded in learning designs, both in a mathematical sense and through emphasising patterning generally across learning areas? The implications for learning are introduced in Publication 2 and further explored in Publication 3.

5. In the presentation at MERGA36, I introduced the potential relationship between the role of pattern across domains, illustrated in Table 1 within this chapter, and the development of the generalised understandings described through the proficiency strands in the Australian Curriculum: Mathematics. There was interest by participants in how this relationship could be developed further. In Publication 3, *The Role of Reasoning Within the Australian Curriculum: Mathematics*, the role of the mathematical proficiencies in building generalised understanding is explored in order to investigate this possible relationship. Two main questions emerged:

    - How could explicit use of the mathematical proficiencies build fluent, generalised understandings, and strengthen reasoning skills?
    - What are the implications for learning within and beyond mathematics education?

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CHAPTER

4

PUBLICATION 2

The Role of Pattern Within and Beyond Mathematical Thinking

> The human mind is a pattern recogniser. The ability to see patterns and similarities is one of the greatest strengths of the human mind.
>
> (Devlin, 2010, p. 169)

4.1 Introduction

In this chapter I present the second publication of this thesis: *The Role of Pattern Within And Beyond Mathematical Thinking.* This second publication is as an extension of Publication 1 and includes reference to the theoretical perspectives provided in Chapter 2. This publication addresses the second and third research questions for this thesis: Is a generalised view of patterning applicable across broader fields of knowledge? What are the educational implications of a generalised model of patterning?

This chapter comprises:

1. A preamble outlining the development of the paper, including:

    - the purpose of Publication 2;
    - modifications made from the original MERGA paper, including a discussion about the rationale for Publication 2; and
    - theoretical perspectives on the role of patterning.

2. Publication 2: *The Role of Pattern Within and Beyond Mathematical Thinking.*

3. A postscript that provides a reflection about how the theoretical perspectives in the paper can inform future research.

4.2 Preamble

4.2.1 Purpose of Publication 2

Publication 2 expands on the theoretical framework underpinning the generalised view of patterning proposed in Publication 1 and sets the scene for addressing educational implications explored in Publication 3: *The Role of Reasoning in the Australian Curriculum: Mathematics*.

4.2.2 Developing a broader view of the role of pattern across knowledge domains

In the following four points I address the feedback from presenting at MERGA36 to build a more comprehensive view about the role of pattern within Publication 2:

1. I revised Table 1 in Publication 1 to reflect similarity at each stage of the development of patterning across both fields. These changes were also identified in the postscript in Section 3.4 and have been incorporated in Publication 2. In order to reflect these changes, this table has been renamed the generalised model of patterning.

2. In Publication 2, I revisit the term "thinking mathematically". This required critical examination of what is known and understood by this term in the field of mathematics education. This has been discussed in Chapter 2 and is further addressed in Publication 2 to provide a more balanced argument. Discussion about this term also led to the change of title for Publication 2. This change indicated the need for further exploration of the role of patterning across domains without this being limited to, or indicative of, mathematical thinking.

3. Publication 2 addresses mathematical thinking more broadly across the fields of the psychology of mathematics education and cognitive neuroscience to substantiate the proposed generalised model of patterning.

4. The educational implications of a generalised model of patterning are discussed. Although implications were raised in the presentation at MERGA36 they were not discussed in Publication 1 (refer to postscript, Section 3.4).

In developing the structure of Publication 2, decisions were made regarding which elements of the original MERGA paper would be used and/or referred to in the second extended publication. There were two clear directions that were deliberated upon. These were to:

- build upon the theoretical position proposed in the original paper by adding a broader range of research from the fields of the psychology of mathematics education and cognitive neuroscience, and

- summarise feedback and recommendations from the position paper and use this to build a wider view of the generalised construct of patterning.

It was considered that presenting the same theoretical position in a second publication would appear repetitive. However, in this second paper the theoretical position is argued more coherently as a full paper using the background research that the proposed generalised model of patterning (GMP) was developed from. Consequently, in Publication 2 we have built a stronger view of the GMP that is the core of this research.

4.2.3 Structure of theoretical perspectives

Entries in Table 4.1 outline the theoretical perspectives from Chapter 2 that this second publication has drawn upon.

Table 4.1

Structure of Theoretical Perspectives

Field of research	Elements of research
Neuroscience Neuropsychology Cognitive neuroscience	• Clarify differences between disciplines • Identify how the broader field of neuroscience describes the process of patterning, including physiological changes in the brain
Mathematics Psychology of mathematics Mathematics education	• Reviewing the connection between the construction of conceptual schemas and the construct of patterning and how this aligns with the GMP.

4.3 Publication 2 The role of pattern within and beyond mathematical thinking

The following publication has not yet been submitted to a journal. However, it would be suitable for a philosophy of mathematics education journal such as *For the Learning of Mathematics* (FLM).

The text reproduced below contains the wording that is intended for the article that will be submitted, but font pitch, line spacing, and other aspects of formatting conform to Macquarie University regulations for theses as well as other text in this thesis. Appropriate alterations, according to the target journal's requirements, will be made prior to submission.

The Role of Pattern Within and Beyond Mathematical Thinking

Catherine McCluskey, Michael Mitchelmore, and Joanne Mulligan

A generalised model of patterning has been proposed to describe the relationship between the mathematical construct of pattern and the neuropsychological construction of patterning. The purpose of this paper is to further investigate this generalisation of patterning across the broader fields of cognitive neuroscience and the psychology of mathematics education. This paper expands upon the generalised model of patterning to describe the role of pattern within and beyond mathematical thinking, highlighting educational implications.

A pattern is a mathematical construct that can be defined as “any predictable regularity” and its structure is defined as “the way a pattern is organised” (Mulligan & Mitchelmore, 2009, p. 34). In the field of neuropsychology, the term pattern refers to a cognitive construct that encases and encodes a type of understanding. Through repetition, common elements of similar experiences become encoded as a pattern, a way of knowing and understanding the familiar elements of certain types of experiences. Over time, familiarity grows and key elements of the experience become instantly recognisable. Elkhonon Goldberg (2005), a world-renowned neuro-psychologist, refers to this intuitive knowing as pattern recognition.

McCluskey, Mitchelmore, and Mulligan (2013) explored the role of mathematical pattern in the construction of understanding, highlighting elements of pattern inherent in both the mathematical and neuropsychological fields. From this analysis a generalised view of patterning was proposed and identified as abstraction across domains. The purpose of this paper is to further explore the construct of patterning through the wider fields of the psychology of mathematics education and cognitive neuroscience. Two questions are addressed: Does a generalised view of patterning apply across wider fields of knowledge? and What is the role of patterning in the construction of understandings?

Theoretical background

Quasi-empiricism is a philosophical view of mathematics that asserts that all formal mathematics is derived from informal human experiences, that mathematics knowledge should not be separated from human knowledge, and that new knowledge is part of an ongoing process of knowledge creation (Ernest, 1991). With this in mind, mathematics could be viewed as a way and means of naming, describing, comparing, measuring, quantifying, and reflecting upon our experiences. Something is understood because it feels familiar and is similar to a past experience. This sense of familiarity draws us to engage in situations, possibly knowing how to respond because we already have a pattern of engaging with this type of experience before. Through this process of comparison, differences across situations become evident, challenging and expanding understanding further. Mathematics provides the means to compare, differentiate, and recognise what is familiar and known through our everyday experiences, and how to make predictions in less familiar experiences.

Identifying patterns and common elements across similar type patterns develops structural understandings. This involves exploring "a relationship that holds over the entire class of values, not only in isolated instances" (Papic, Mulligan, & Mitchelmore, 2011, p. 239). Structural understandings develop through a capacity to generalise what is similar and to predict in new situations, leading to the development of abstract understandings, which is a recognised goal for all mathematics education (Warren, 2008).

However this process of patterning and developing generalised understandings does not just occur in the field of mathematics. McCluskey, Mitchelmore, and Mulligan, (2013) proposed a generalised view of patterning through constructing the model that they called abstraction across domains. In this model they identified the similarities between Goldberg's theory of pattern recognition as outlined in his text *The Wisdom Paradox* (2005), and White and Mitchelmore's model of teaching for abstraction (White & Mitchelmore, 2010). White and Mitchelmore's model of teaching for abstraction outlines how, as a sense of familiarity about a concept is engaged, students learn to recognise similarity, and through guided instruction develop generalised understandings. This generalisation leads to a growing ability to predict and develop abstract understandings. Each stage of the development of mathematical abstraction, as summarised in the model of teaching for abstraction, can be aligned with Goldberg's description of pattern recognition. In both the neuropsychological and mathematics education domains, generalised understandings emerge from engaging with familiar patterns, and similar elements of the patterns across varying contexts merge to form structural understandings that encase the conceptual experience. These generalised understandings can be applied in new situations to predict, test, and reason. This type of thinking strengthens and expands conceptual understanding further.

The abstraction across domains model (McCluskey, Mulligan, & Mitchelmore, 2013) has been modified to better delineate stages in terms of similarity across both domains. Table 1 outlines these similarities in building generalised understandings across the mathematical and neuropsychological domains.

Table 1

Generalised Model of Patterning (GMP)[a]

Mathematical domain Teaching for abstraction model	Neuropsychological domain Goldberg: Pattern recognition
Familiarity:	
• Explore a concept through a variety of contexts • Become familiar with the underlying structure of each context	• Engagement with a range of situations set the scene for experiencing and sensing what is familiar about these situations.
Similarity:	
• Frequent matching • Explicit attention to similarities within and between varying contexts	• Further attention and engagement with familiar situations allows connections to emerge as what is similar about these experiences is recognised. • Engagement with these experiences enables similarity to be matched, measured, and understood. • Overlapping of neural networks encode the similarity experienced across different contexts.
Reification:	
• Moves students into operating with abstract concepts	• Over time a pattern is encoded as a generic memory, a mental representation of the similarities and shared properties of a "type" of experience. • Pattern recognition refers to the ability to readily access this pattern in similar situations in the future
Application:	
• Consolidation of the concept • Application to new situations	• Recognise familiar patterns more fluently. Automated, efficient, and competent decision making. • Applying patterns of knowing by recognising new problems as having similar characteristics to familiar ones … the same pattern in new situations.

[a] Collated from White and Mitchelmore (2010) and Goldberg (2005) respectively, and subsequently adapted from McCluskey, Mitchelmore, and Mulligan (2013).

In comparing the role of pattern across both domains, what is intrinsically different about the type of thinking is also discussed in both fields. Mathematics is a language of reason and therefore is a rigorous, explicit discipline. It involves a conscious process of theorising, testing, and restructuring ideas. In the instruction for abstraction model, there is intentional attention drawn to the attributes of mathematical objects to greater understand their similarities and differences. The process of constructing generalisations and consequent abstract mathematical understandings is conscious and deliberate. On the other hand, in the neuropsychological field pattern recognition occurs gradually, often happening below a level of conscious awareness. Situations feel familiar because over time and re-engagement with similar types of experiences, the inherent similarities emerge as a form of knowingness. Generalised understandings develop through repetition of experience and engaging familiar patterns of thinking that might or might not eventuate through intentional means.

The process of patterning is evident in both instances. However, the discriminating feature is the level of awareness brought to developing abstract understandings. The field of mathematics recognises the power of reasoning in the development and expansion of resilient conceptual understanding. In the field of neuropsychology, pattern recognition is strengthened through the formation of generic patterns produced by the repetition of similar types of experience. Other factors contributing to the development of resilient patterns that can be fluently recalled need to be explored.

Further inquiry/argument

A generalised model of patterning (GMP) has been proposed by McCluskey, Mitchelmore, and Mulligan (2013) to explain the similarities in the construct of patterning across the two fields of mathematics education and neuropsychology. An issue that arises is whether the GMP might be applicable across other fields of knowledge. Therefore, the role of patterning in the wider fields of the psychology of mathematics education and cognitive neuroscience is explored in this paper.

Conceptual understanding and the generalised model of patterning

White and Mitchelmore's model of teaching for abstraction (2010), referred to above in the GMP, was based upon the work of Richard Skemp (1986). Skemp is known for his pioneering work into the psychology of mathematics education. He described the process of abstraction as becoming "aware of similarities (in the everyday, not the mathematical sense) among our experiences", resulting "in some kind of lasting mental change" (p. 21) and he explained that the act of naming objects is actually a form of classification. This involves identifying that a particular object belongs to a category based upon some predetermined criteria that are satisfied by the whole class of objects. Skemp linked this process of ongoing classification to conceptual development: "A concept therefore requires for its formation a number of experiences which have something in common" (p. 21). A pattern, then, can be viewed as a conceptual structure that over time encodes the common elements of experiences.

In exploring the structure of conceptual understanding, Skemp described the process of the expansion and reconstruction of schemas: "A schema is a conceptual structure" … a mental tool that "assimilates new knowledge (expands) and also reconstructs to adapt and make sense of new situations … It integrates existing knowledge, acts as a tool for future learning and makes possible understanding" (Skemp, 1986, pp. 37, 62, & 41). It is not clear at what stage this process of assimilation moves from recognising similarity to the fluent, efficient pattern recognition that Goldberg refers to.

Describing the processes involved in building understanding is complex. In the field of mathematics, Paul Ernest (1994) explained the relationship between "knowledge development and acquiring understanding" (p. 156) as an interaction between the application of mathematical operations and concepts in new contexts. If the concept "can be integrated into the open situation naturally … [it] ... produces nearly no problem with understanding" (p. 157). Ernest is referring here to the process, and end product, of understanding that involves the ease of transferring and assimilating mathematical concepts. If understanding involves the process of assimilation, the development of understanding would bring in additional processes to reconstruct and adapt the understanding of the concept further.

Robert Davis (1992) stressed the importance of new understanding being dependent upon the breadth and depth of previous understanding: "One gets a feeling of understanding when a new idea can be fitted into a larger framework of previously assembled ideas" (p. 228). Mueller, Yankelewitz, and Maher (2010) have elaborated on the work of Davis, referring to the process of understanding involving "representational structures that a learner builds as a collection of assimilation paradigms" (pp. 308–9). Skemp (1986) had explained the relationship between assimilation and understanding in that "to understand something is to assimilate it into an appropriate schema", and that "better organisation of a schema may improve understanding" (pp. 43–44). In this instance, Skemp was indicating the importance of challenging and expanding current understanding. Developing further understanding involves the processes of assimilation and accommodation, of the current schema. In the process, understanding is actively engaged and reconstructed to accommodate growing awareness and insight.

These stages of schematic understanding can also be compared alongside the GMP in Table 1. Elements of patterning are evident in the development of schematic structures and move through similar stages to those identified also in the instruction for abstraction model

(White & Mitchelmore, 2010) and Goldberg's development of pattern recognition (Goldberg, 2005).

To further understand these connections across contexts, the field of cognitive neuroscience can offer insight into the role of patterning in the assimilation and accommodation of schematic understandings, shedding further light on the applicability of the proposed GMP.

The role of patterning as a neurological construct

In forming the GMP, McCluskey, Mitchelmore, and Mulligan (2013) drew primarily upon the work of Elkhonon Goldberg (2005) who described the process of intuitive knowing as pattern recognition. He is not alone in his view of patterning as a neuropsychological construct. The wider field of cognitive neuroscience "informs our understanding of cognitive behaviours relevant to education" (Geake & Cooper, 2003, p. 8). Researchers within and across this broad field describe the construction of understanding using various terms such as neural networks, cognitive templates, internal maps, mental representations, and patterns of thinking (Devlin, 2010; Dispenza, 2007; Gazzangia, Ivry, & Mangun, 2002; Geake, 1997). These terms describe neural structures, which are in essence the patterns of condensed understandings that Goldberg identified through the process of building pattern recognition.

Devlin (2010) described our minds as "very good at recognising patterns, seeing connections, and making rapid judgements and inferences" (p. 171). The processes Devlin referred to reaffirm the elements of pattern recognition identified earlier: recognising patterns, seeing similarities and relationships between the elements, making connections, and using these to generalise and apply understanding in new situations. This is also a result of the neuroplasticity of the brain, which "is the capacity of the brain to change at a neurophysiological level in response to changes in the cognitive environment" (Geake & Cooper, 2003, p. 14).

Human memory relies heavily on the associations it develops with and between patterns of conceptual understanding (Devlin, 2010). When new experiences are connected with prior understandings, this "enables the brain to link new information with well encoded ideas" (Wiles & Wiles, 2003, p. 18). This theory, Hebbian learning, was proposed by Donald Hebb (1949) and describes the process through which we engage our associative memory to acquire new information (Dispenza, 2007). It asserts that a weak association, something novel or unfamiliar, can be made strong by attaching known connections to the learning (Dispenza, 2007). This results in a change in the synaptic response of the cells involved (Gazzangia et al., 2002). It follows the notion that "cells that fire together, wire together" (Dispenza, 2007, p. 184) and consequently make strong, lasting, and resilient connections. These neural networks are a cognitive representation of an understanding, and "develop as a result of continuous neural activation" (Dispenza, 2007, p. 185). Adaptive plasticity is the capacity of the brain to accommodate and retain new information (Geake & Cooper, 2003) and in the process actively access familiar patterns recognised.

Neural networks are both locally and globally encoded in the brain (Geake, 1997). Thoughts of one thing lead to another, and connections between those thought patterns are stimulated along with all the other connections present in the various neuronal groups. This activity leads to a creation of internal maps. When these maps overlap there is a mapping of maps, a making of meaning—and interpretation is possible. "The interconnections allow each cluster to correlate their information" (Geake, 1997, p. 28).

This mapping of neuronal groups is similar to Goldberg's description of the formation of generic understandings as depicted in Figure 1. This image illustrates numerous networks representing information about various dogs. This visual mapping of a patterning of patterns highlights the common aspects of the experience. The common elements identify what is similarly encountered and understood about the concept of a dog across a range of contexts and experiences.

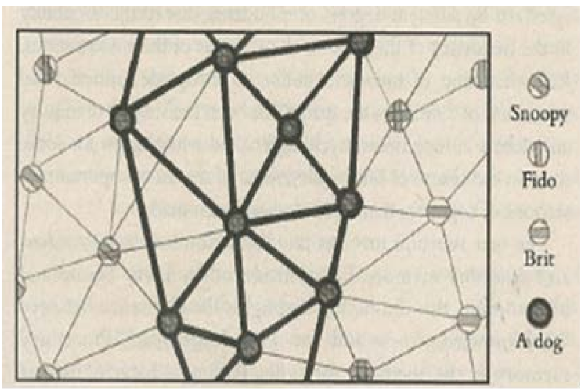

Figure 1. Network overlap of a dog (Goldberg, 2005, p. 125).

Edelman, the founder of the theory of neuronal group selection (Edelman, 1992) proposed that the brain selects an appropriate interpretation of these maps through "natural selection" (Geake, 1997, p. 28) and that this form of neural Darwinism "leads to superior contextual fitness" (Geake, 1997, p. 32). "The most useful correlated clustering … is selected … determining consequent behaviour … in a Darwinian adaptive sense" (Geake, 1997, p. 28). Over time, these maps accumulate, becoming resilient patterns that provide the structure to enable the effortless pattern recognition that Goldberg referred to. Others in the field of neuroscience concur: "Successful, extensive patterning leads to more accurate predictions … extending and strengthening neural networks" (Willis, 2010, p. 61). In essence, pattern recognition describes the "superior form of contextual fitness" and "neuronal efficiency" (Geake, 1997, p. 32), from which Goldberg (2005) proposed that the "cognitive dimension of wisdom" emerges (p. 11) as our ability to instantly recognise new situations as familiar ones.

These descriptions of the role that pattern plays in the building of conceptual understanding again affirm the stages outlined in the proposed GMP. Familiarity and re-engagement with a concept allow similarity to be experienced and understood. Over time, the similarity is encoded in the conceptual structure of the pattern. These patterns are activated when similarity is recognised, and strong resilient patterns of understanding support new connections to be made with less familiar contexts. Over time, patterning of patterns accumulates, creating resilient generic memories, which are in essence intuitive forms of knowing and understanding.

The GMP outlined in Table 1 can also be expanded to accommodate the construct of patterning identified through the psychology of mathematics domain as well as the wider field of cognitive neuroscience. This is demonstrated in Table 2.

The revised GMP, as outlined in Table 2, affirms the role of patterning within and beyond mathematical thinking in all stages of building generalised understandings. This raises questions about its applicability within mathematics education and across wider learning areas.

Educational implications

Highlighting the role of pattern in the construction of understandings leads to a consideration of the implications for learning and associated pedagogies. A greater understanding of the physical and neuropsychological changes occurring in the brain when we learn could have an impact on the design of learning within and beyond mathematics. Educational implications occur in three main areas. These are outlined below.

Ascertaining prior understanding and revisiting familiar concepts

Gathering and connecting with students' prior understandings is a valued educational practice as it allows opportunities for students to "view the new idea as 'just like' or 'similar to' an existing experience and use this to accommodate new knowledge" (Davis & Maher, 1993, as cited in Mueller, Yankelewitz, & Maher, 2010, p. 309). This process strengthens pathways to current and recognised patterns of knowing and supports links with making sense of new information. Geake and Cooper (2003) affirmed the importance of the "strength of synaptic functioning" and

Table 2

Revised Generalised Model of Patterning

Mathematical domain Instruction for abstraction model	Neuropsychological domain Goldberg: Pattern recognition	Psychology of mathematics domain Schematic understandings	Cognitive neuroscience domain Neural network patterns
Familiarity			
Explore a concept through a variety of contexts Become familiar with the underlying structure of each context	Engagement with a range of situations set the scene for experiencing and sensing what is familiar about these situations.	New understanding is dependent upon previous understanding. A sense of familiarity is engaged.	Weak associations can be made strong by attaching known connections to the learning; in this process the brain links new information with well encoded ideas
Similarity			
Frequent matching Explicit attention to similarities within and between varying contexts	Further attention and engagement with familiar situations allows connections to emerge as we recognise what is similar about these experiences. Engagement with these experiences enables similarity to be matched, measured, and understood. Overlapping of neural networks encode the similarity experienced	Assimilation Becoming aware of similarities. Naming objects is a form of classification. A new idea can be fitted into a larger framework of ideas.	Cells that fire together wire together and the synaptic response strengthens.
Reification			
Moves students into operating with abstract concepts	Over time, a pattern is encoded as a generic memory, a mental representation of the similarities and shared properties of a 'type' of experience. Pattern recognition refers to the ability to readily access this pattern in similar situations in the future.	Accessing an appropriate schema readily to assimilate familiar information Drawing connections between current schemas and similar situations	Maps accumulate, encoding well-known patterns of knowing and understanding. Natural selection leads to superior contextual fitness.
Application			
Consolidation of the concept Application to new situations	Recognise familiar patterns more fluently. Automated, efficient, and competent decision making. Applying patterns of knowing by recognising new problems as having similar characteristics to familiar ones … the same pattern in new situations.	Accommodation and expansion of current schemas would occur through the process of application and engagement in new situations.	Over time, patterns of patterns accumulate creating resilient generic memories, which act as intuitive forms of knowing and under-standing new situations as familiar ones.

the impact of "synchronised neural pathways" which "become more efficient in response to repeated coincident stimulation of the synapses along the route" (p. 14). Therefore, opportunities to regularly revisit familiar concepts, across varying contexts, with attentive instruction and guidance, supports the development of efficient neuronal pathways, depth of conceptual understanding, and fluent patterns of knowing (Geake & Cooper, 2003; White & Mitchelmore, 2010; Willis, 2010). These implications are not new in the field of education; however, they affirm and give importance to what is already known and accepted as effective practice.

Provocation and explicit instruction in supporting conceptual understanding

The educational issue of overfamiliarity with a concept can result in a narrowed perception of understanding. Designing provocations that challenge current modes of thinking, along with intentional instruction, is necessary in expanding understanding further. Skemp (1986) affirmed the importance of further developing students' conceptual understanding through the explicit process of expanding and reconstructing schemas. For him, "a schema is … a mental tool which assimilates new knowledge (expands) and also reconstructs to adapt and make sense of new situations" (p. 62). White and Mitchelmore's model of teaching for abstraction clearly outlines the intentional role of the educator in carefully constructing the design of learning to engage the current level of students' understanding and guide this through the stages of familiarity, similarity, and reification to support concepts to be more readily applied to new contexts, and therefore expand understanding further.

Teacher education and conceptual understanding

In the field of mathematics education, the term understanding has been used to describe one of the four mathematical proficiencies in the Australian Curriculum (ACARA, 2015, p. 5):

> Students build robust knowledge of adaptable and transferable mathematical concepts. They make connections between related concepts and progressively apply the familiar to develop new ideas … Students build understanding when they connect related ideas, when they represent concepts in different ways, when they identify commonalities and differences between aspects of content, when they describe their thinking mathematically and when they interpret mathematical information.

In mathematics education there is a need to know how understanding is built, particularly with regard to the role that other proficiencies play in developing the breadth and depth of students' understanding. Through explicit attention to instruction, the mathematical proficiencies have the potential to intertwine and thus continually engage deeper levels of reasoning that challenge and expand students' current schemas. In building resilient understanding, students require opportunities to engage in familiar contexts (problem solving), connect, explain, and justify their thinking (reasoning), and develop greater levels of fluency as they reveal patterns of knowing and (fluency). Teachers' design of learning needs to systematically engage the mathematical proficiencies in order to intentionally build and reconstruct students' current levels of thinking. As a result, there is a need for teachers' own conceptual understanding in mathematics to be engaged and developed, too, as educators can only lead the development of what they have an awareness and understanding of. Professional learning that leads educators through this process is recommended.

Summary

Essentially, we experience patterning through our everyday encounters, and patterning is a process through which we construct our understandings, often occurring below our level of awareness. In the mathematical domain, patterning is primarily recognised as a conscious and deliberate process we enter into as we explore and investigate mathematical concepts. Exploring patterns of similarity across mathematical contexts builds generalised understanding, an ability to predict, reason, generalise, and abstract. This process expands mathematical understanding further.

Pattern is a mathematical construct. However, it is also the structure through which we create and build general, everyday understandings. The GMP indicated the relationship between elements of patterning across mathematics education, the psychology of mathematics education, neuropsychology, and the wider field of cognitive neuroscience. At each stage there were similarities in the construct of patterning to describe how understanding is deepened and generalised across domains of knowledge. In this paper, we have proposed that a GMP highlights the role of pattern within and beyond mathematical thinking. We query how such a model of patterning could support the development of both mathematical and everyday understandings and discuss some implications that this model could have for our pedagogical practices and design of learning.

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4.4 Postscript

In this postscript I reflect on the positioning of this publication as the second component in the sequence of the three papers that together embody the structure of this thesis.

Publication 2 culminates the first research phase of this thesis (refer to Table 1.1, Section 1.5.1) and raises implications for education that are investigated in the next phase of the research.

Key outcomes of Publication 2 were:

- expanding the GMP to encompass broader areas of cognitive neuroscience and the psychology of mathematics education, and
- highlighting educational implications to direct future areas of research.

The GMP was expanded to explain the similarity in the construct of generalised understandings across domains. How, then, is an ability to generalise developed within mathematics education practice and curricula generally? How are mathematical understandings developed systematically and as patterns in mathematical thinking?

There are four mathematical proficiencies identified in the Australian Curriculum Mathematics: understanding, fluency, problem solving, and reasoning. These proficiencies refer to the thinking and actions that students are engaged in while learning the content. The ability to reason and generalise about concepts has been highlighted as a goal of mathematics education (ACARA, 2015). What evidence is there that indicates reasoning is emphasised in the curriculum?

This line of inquiry led to the formation of Publication 3: *The Role of Reasoning in the Australian Curriculum: Mathematics,* which is the focus of the next chapter.

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CHAPTER

5

PUBLICATION 3

The Role of Reasoning Within the Australian Curriculum: Mathematics

> The mathematics you learn, if you understand it, will teach you a way of thinking ... structural thinking. Thinking in structures, how structures fit into one another. How do they relate to each other and so on. … As long as you get the idea of what mathematical thinking is like, you can apply it to all sorts of other situations.
>
> (Sriraman & Lesh, 2007, p. 63).

5.1 Introduction

In this chapter I present the third publication of this thesis: *The Role of Reasoning in the Australian Curriculum: Mathematics (ACM)*. This third publication is the full version of the research report that was submitted for inclusion in the MERGA39 (2016) conference proceedings.

The theoretical perspectives addressed in Phase 1 of this research are connected with the discussion about the ACM. This discussion addresses the research questions: What are the educational implications of a generalised model of patterning? and How is reasoning, and the ability to express and justify generalised understandings, articulated in, and developed through, the proficiencies in the ACM? Therefore in this chapter I explore the role of reasoning through a content analysis of key proficiency terms (KPTs) that relate to reasoning embedded throughout the content descriptions in the ACM.

This chapter comprises:

1. A preamble outlining this second phase of the thesis — the application of the theoretical model to the ACM.

2. The third publication of this thesis: *The Role of Reasoning in the Australian Curriculum: Mathematics*. This publication reports the analysis of key proficiency terms (KPTs) identifying the incidence of reasoning articulated in the ACM.

3. A postscript that provides:
    - a reflection on the role of reasoning within the ACM,

- a discussion of the proposed proficiency cycle (pedagogical structure),
- a summary of areas identified for future research, and
- information about a publication that is proposed as a sequel to Publication 3.

5.2 Preamble

Purpose of Publication 3

The exploration of the relationship between patterning as a neuropsychological concept and patterning as defined in the field of mathematics have been the focus of Phase 1 of this thesis. From this, a generalised model of patterning has been proposed to describe the similarities encountered in the construction of patterns of thinking within and beyond the field of mathematics.

In Phase 2 of this thesis I turn to mathematics education curricula and practice. I inquire into how generalised understandings and a capacity to reason are developed within the ACM and the use of intentional pedagogies that support the development of reasoning. According to Mason (1996),

> "generalisation is the heartbeat of mathematics, and appears in many forms. If teachers are unaware of its presence and are not in the habit of getting students to work at expressing their own generalisations, then mathematical thinking is not taking place" (p 65).

This affirms the importance of teachers providing opportunities for students to construct generalisations about the similarities they are noticing within and between mathematical concepts. Just like the heartbeat that keeps a bodily system alive, developing a capacity to generalise is also self-perpetuating in connecting with and 'stretching' mathematical thinking further. Similarly, Wood (2002) proposed a shift in the learning of mathematics to acknowledging it as "a subject that consists of patterns and relationships that are understandable through mental activity that involves mathematical reasoning and logic" (p. 61). Wood acknowledges the role of reasoning in exploring patterns and constructing generalisations about the relationships between concepts. This type of thinking leads to structural thinking, which is the result of abstracting patterns generally. Thinking structurally is not limited to particular mathematical strands or topics but a means of engaging with mathematical concepts abstractly as "attention to structure runs through the whole of mathematics" (Mason, Stephens, & Watson, 2009, p. 12).

Therefore, this chapter addresses the third research question, initially explored by presenting a short communication at the Mathematics Education Research Group of Australasia conference in 2015. Refer to Appendix C1. At this presentation a handout showing the proposed generalised model of patterning (GMP) was circulated to gather feedback regarding the interaction of the proficiencies in building patterns of understanding in mathematics. This handout is provided in Appendix C2.

In Publication 3 I identify and describe how generalised understandings in mathematics are systematically built throughout the ACM through analysing KPTs that articulate the proficiency reasoning. I connect a cyclic view of the mathematical proficiencies with a generalised view of patterning. Thus, the purpose of this publication is to highlight the role of reasoning in teaching mathematics and connect it with the generalised view of patterning proposed in Phase 1.

5.3 Publication 3: The role of reasoning within the Australian Curriculum: Mathematics

The text reproduced below retains the wording of the original submission to MERGA for publication in its conference proceedings, but font pitch, line spacing, page margins, and other aspects of formatting have been altered to conform to Macquarie University regulations for theses as well as other text within this thesis.

The Role of Reasoning in the Australian Curriculum: Mathematics

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The mathematical proficiencies in the *Australian Curriculum: Mathematics* of understanding, problem solving, reasoning, and fluency are intended to be entwined actions that work together to build generalised understandings of mathematical concepts. A content analysis identifying the incidence of key proficiency terms (KPTs) embedded in the content descriptions from Foundation to Year 9 revealed a much lower representation of “actions” relating to the proficiency reasoning than to the other three proficiencies. A generalised model of patterning is proposed to provide an interrelated view of the proficiencies and to further support the development of generalised understandings in mathematics education.

Mathematics is widely accepted “as a subject that consists of patterns and relationships that are understandable through mental activity that involves mathematical reasoning and logic” (Wood, 2002, p. 61). The goal of mathematics education is clearly articulated in the Australian Curriculum: Mathematics (ACM) rationale statement: “It aims to instil in students an appreciation of the elegance and power of mathematical reasoning” (Australian Curriculum and

Assessment Reporting Authority [ACARA], 2015, p. 4). Reasoning is recognised as paramount in the development and growth of mathematical understanding (Ball & Bass, 2003; Mason, Stephens, & Watson, 2009). In the ACM reasoning is singled out as one of the four mathematical proficiencies: understanding, problem solving, reasoning, and fluency. These are identified as key processes that describe “the actions in which students can engage when learning and using the content” and similarly inform teachers “how the content is explored or developed” (ACARA, 2015, pp. 4, 5). The content knowledge in the ACM is structured around three strands that “describe what is to be taught and learnt” (p. 5) and the mathematical actions of the proficiencies are embedded in the content descriptions. Therefore it is the interaction within and between these content strands and the four proficiencies that builds conceptual understanding in mathematics.

Mathematical reasoning is described as the “capacity for logical thought and actions such as analysing, proving, evaluating, explaining, inferring, justifying and generalising” (ACARA, 2015, p. 5). Reasoning involves recognising similarity and differences encountered in concepts explored across multiple contexts leading to the development of abstract understandings. Explaining and justifying thinking enables knowledge to become “more general and its applicability to different situations … increased” (White & Mitchelmore, 2010, p. 2). Intentional instruction supports conceptual understanding to deepen, become more fluently recalled, and applicable in new learning contexts. Ball and Bass (2003) emphasise the role of the teacher in promoting reasoning, as “mathematical understanding is meaningless without a serious emphasis on reasoning” (p. 28). Engaging students in mathematical reasoning naturally draws students into greater levels of fluency as they connect their understandings in new problem-solving contexts.

Sullivan (2012) proposes that teacher learning should focus on “ways of identifying tasks that can facilitate student engagement with all four of these proficiencies” (p. 183) as the “intention is that the full range of mathematical actions apply to each aspect of the content” (Sullivan, 2011, p. 8). However, the organisational structure of the curriculum as three content strands comprising number and algebra, measurement and geometry, and statistics and probability, draws attention to content knowledge. How the proficiencies together build entwined conceptual understanding is well intended in the rationale of the ACM but not clearly articulated within the content strands. This raises key questions addressed in this paper: In what ways do the proficiencies in the ACM build generalised understandings and reasoning skills? Is this relationship between reasoning and generalised understanding of mathematics evident and transparent to teachers accessing the curriculum?

At a theoretical level, an interrelated view of the proficiencies will be discussed in light of a generalised model of patterning proposed by McCluskey, Mitchelmore, and Mulligan (2013) to highlight the importance of reasoning. An outcome of this paper is to identify how the proficiencies are articulated in the ACM through a content analysis of key language terms embedded in the content descriptions denoting the “actions” of the four proficiencies across Foundation to Year 9.

Background

In the rationale of the ACM the role of the mathematical proficiencies is highlighted: “The curriculum focuses on developing increasingly sophisticated and refined mathematical understanding, fluency, logical reasoning, analytical thought and problem solving skills”

(ACARA, 2015, p. 4). They are described as capabilities that “enable students to respond to familiar and unfamiliar situations by employing mathematical strategies to make informed decisions and solve problems efficiently” (ACARA, 2015, p. 4). The ACM describes the field of mathematics as “composed of multiple but inter-related and interdependent concepts and systems” (ACARA, 2015, p. 4), anticipating that teachers and students will engage with the ACM in a dynamic and symbiotic way and thus implying, similarly, that the proficiencies are also interrelated.

There is a clear intent in the introductory sections of the ACM to highlight the mathematical proficiencies as integral aspects of the curriculum. They are described in the *Key Ideas* section directly following the rationale and aims, and they are also outlined again in the next section, *Structure*, before the description of the content strands. Importantly, the proficiencies are embedded in the language of the content descriptions and achievement standards as verbs that describe the mathematical actions students engage with (Sullivan, 2012). This is demonstrated in the following content description: “Interpret and compare data displays” (ACARA, 2015, Section ACMS069): the verbs *interpret* and *compare* identify use of the mathematical proficiencies. Throughout the ACM the proficiencies are described individually, rather than an entwined system at the beginning of each year level. However, naming and identifying individual proficiencies may not encourage teachers to focus on the potential interrelationships between the proficiencies to build and deepen conceptual understanding. It is their connectedness that is not well articulated and thus does not resonate clearly with the rationale.

In *Engaging the Curriculum-Mathematics: Perspectives from the field*, Atweh, Miller, and Thornton (2012) identified challenges that schools and educators could face in interpreting and implementing the curriculum due to this “possible lack of cohesion between the aims and rationale, the content and its articulation” (p. 2). In particular, they noted inconsistencies in emphasis between the proficiencies, such as the role of reasoning which they argued was underrepresented in the content elaborations. Therefore, in this paper an interrelated view of the proficiencies is explored to address this imbalance and support the development of generalised understandings in mathematics.

Interrelationships Between Mathematical Proficiencies

Atweh et al. (2012) highlight the interrelationships between the proficiencies, explaining that these “proficiencies are not disjointed … [and that] … some content elaborations may relate to one or more of the proficiencies” (p. 8). They refer to a model, focused on mathematical proficiency, described in the United States report to the National Research Council (Kilpatrick, Swafford, & Findell, 2001). In this model, based on five strands, the term mathematical proficiency is used to “capture what we think it means for anyone to learn mathematics successfully … the most important observation we make about these five strands is that they are interwoven and interdependent” (Kilpatrick et al., 2001, p. 5) and “represent different aspects of a complex whole” (p. 116). For Kilpatrick et al., these strands are adaptive reasoning, strategic competence, conceptual understanding, productive disposition, and procedural fluency. The following descriptions explain the mathematical actions relating to these strands of proficiency from this model.

- Conceptual understanding “includes the comprehension of mathematical concepts, operations and relations”.

- Procedural fluency includes skill "in carrying out procedures flexibly, accurately, efficiently, and appropriately, and, in addition to these procedures, having factual knowledge and concepts that come to mind readily".
- Strategic competence is "the ability to formulate, represent and solve mathematical problems".
- Adaptive reasoning is "the capacity for logical thought, reflection, explanation and justification".
- Productive disposition is "a habitual inclination to see mathematics as sensible, useful and worthwhile, coupled with a belief in diligence and one's own efficacy" (Watson & Sullivan, 2008 as cited in Sullivan, 2011, pp. 6–7).

Kilpatrick et al. (2001) stressed the importance of the relationship between all strands in building resilient understandings that can be fluently applied in new situations. They refer to findings from cognitive science that indicate that "competence … depends upon knowledge that is not merely stored but represented mentally and organized (connected and structured) in ways that facilitate appropriate retrieval and application …. Organization improves retention, promotes fluency, and facilitates learning related material" (p. 118). Proficiency in mathematics involves the construction of effective neural networks that are structured in resilient and flexible ways to both connect understanding and accommodate new learning efficiently. This description proposes a view of the proficiencies in the ACM working interdependently to build conceptual understanding systematically. However, defining the proficiencies as individual strands still accentuates their separateness, not their integrated relationship in building patterns of thinking.

The ACM Proficiencies as an Opportunity for Changing Practice

Sullivan (2012) has asserted that the ACM provides an opportunity for educators to re-think and reshape mathematics learning for students by focussing on "the principles that underpin the structure of the curriculum and the use of these principles to inform teacher learning" (p. 175). These principles are that:

- the four proficiencies provide a framework for mathematical processes,
- the ACM has been designed to emphasise teacher decision making, and
- there is a focus on depth rather than breadth to address challenges of equity.

Sullivan identified the mathematical proficiencies as the first key principle that connects the other two principles, emphasising that engagement with the mathematical proficiencies encourages educators to make pedagogical decisions to explore not just the breadth but also importantly the depth of mathematical concepts. Incorporating learning experiences in relevant problem-based contexts creates opportunities for students to engage meaningfully with the mathematical proficiencies. "Mathematics … is more than following rules and procedures but can be about creating connections, developing strategies, effective communication … this view is not obvious in the content descriptions … it is part of the opportunity for those supporting teachers to communicate such views … [and is] … communicated through the proficiencies that underpin the curriculum" (Sullivan, 2012, p. 179). This raises the issue of whether the language identifying the proficiencies is visible to, and used by, teachers accessing the curriculum.

Identifying the Language of the Proficiencies

In describing this dynamic view of learning, Sullivan refers to the use of verbs identifying the actions of individual proficiencies. It is intended that teachers look within, across, and beyond the content descriptions to connect with the language that articulates the proficiencies. Taking up this point, Atweh et al. (2012) analysed the occurrence of the proficiencies stated in the ACM Year 8 content elaborations, finding that "53% relate to experiences to develop understanding … 56% relate to developing fluency … 12% relate to problem solving … and 7% refer to reasoning" (pp. 8–9). In this analysis, the proficiency of reasoning, an essential element in the development of generalised understandings, was rarely identified in the content elaborations.

However, reasoning may be represented in the use of language terms describing problem solving. Sullivan (2012) highlights the role of problem solving by engaging the proficiencies, in particular reasoning, through problem-based contexts. Investigating problem-based approaches assumes that "the teacher draws upon the various strategies used by the students" and that the learning "experience will communicate to students that there are many ways to approach mathematical tasks, they can choose their own approach, and that some approaches are more efficient than others" (p. 179). This type of thinking, authentically embedded in problem-solving contexts, builds a capacity to reason but is dependent upon teachers' awareness of "structural relationships … [and] strategies … [for]… bringing structural relationships to the fore" (Mason, Stephens, & Watson, 2009, p. 29). Structural relationships emerge from engaging in opportunities to reason. This involves generalising commonalities about concepts across contexts. Therefore the use of language terms in the ACM that relate to various proficiencies, in particular reasoning, requires further investigation.

Content Analysis: Reasoning

In the research reported here, an initial phase of a content analysis was used to identify the type of language used to describe the actions of the proficiencies. This was conducted to find evidence of terms related to reasoning that were articulated in the ACM. This content analysis extracted key proficiency terms (KPTs) that "can be thought of as verbs" (Sullivan, 2012, p. 179) from the content descriptions. (Note that some terms such as *efficiently*, *accurately,* and *appropriately* are adverbs and were included as KPTs if they modified a verb in the content description). The process occurred in the following four stages:

1. Each proficiency description in the key ideas section was analysed for KPTs.
2. A framework was constructed, identifying the KPTs that related to each proficiency. Table 1 indicates the KPTs by proficiency.
3. The KPTs embedded in the content descriptions from Foundation to Year 9 were extracted and categorised using the framework in Table 1 to compare the frequency of their use throughout the content descriptions from Foundation to Year 9. (Note, some KPTs recorded in Table 1 relate to more than one proficiency; however each KPT extracted from the content descriptions was counted to calculate the total number of occurrences relating to each proficiency.)
4. The KPTs embedded in the content descriptions from Foundation to Year 9 were counted and categorised using the framework in Table 1, to compare the frequency of

their use throughout the content descriptions from Foundation to Year 9. Table 2 contains entries that summarise the total number KPTs identified across F–2, 3–6, and 7–9 content descriptions.

Table 1

Key Proficiency Terms (KPTs)

Proficiency strand	Key proficiency terms (KPTs)
Understanding	Apply, build, connect, describe, develop, identify, interpret, make, represent
Fluency	Accurately, answering, appropriately, calculate, carrying, choose, choosing, develop, efficiently, find, manipulate, flexibly, recall, recalling, readily, recognise, regularly, use
Problem solving	Apply, communicate, design, develop, effectively, formulate, interpret, investigate, make, model, plan, represent, seek, solve, use, verify
Reasoning	Adapt, analysing, compare, contrast, deduce, develop, evaluating, explain, explaining, generalising, increasingly, inferring, justify, justifying, known, mathematically, prove, proving, reached, reasoning, transfer, thinking, used

Table 2

Frequencies and Percentages of Key Proficiency Terms (KPTs) Across the ACM [a]

Year level clusters	ACM proficiency strands				Total KPTs
	Under-standing	Fluency	Problem solving	Reasoning	
F–Year 2	33 (26)	36 (29)	32 (26)	24 (19)	125 (100)
Years 3–6	83 (29)	65 (22)	102 (35)	42 (14)	292 (100)
Years 7–9	33 (17)	50 (25)	89 (45)	25 (13)	197 (100)
F–Year 9	149 (24)	151 (25)	223 (36)	91 (15)	614 (100)

[a] Cell entries are frequencies (row percentages)

One could assume for each year level clustering (i.e., F–2, 3–6, and 7–9) that the individual proficiencies would be equally represented, with a similar proportion of KPTs relating to each of understanding, fluency, problem solving, and reasoning. However, this is not the case, with problem solving noticeably over-represented in Years 3–9: F–2: 26%, 3–6: 35%, and 7–9: 45%; and reasoning consistently under-represented across the year level clusters: F–2: 19%, 3–6: 14%, and 7–9: 13%.

Across the early years of school (F–2), a total of 125 terms were extracted from the F–2 content descriptions. From these, 19% related to reasoning, 29% related to fluency, and 26% each for KPTs relating to understanding and problem solving. This reflects the emphasis in the early years of developing conceptual understanding and fluency of procedural knowledge and processes through problem-solving contexts. However, reasoning is critical in the development of mathematical concepts. Further analysis will reveal if KPTs identifying reasoning are represented more in the later years of school. Throughout the primary years (3–6) there is an increasing incidence of KPTs embedded overall in the content descriptions. KPTs identifying understanding and problem solving were noted more frequently than were those identifying fluency and reasoning. KPTs relating to reasoning were identified 42 times from an overall count of 292 KPTs, resulting in only 14% of the total terms extracted. Similarly, in the middle years (7–9) an increasing focus on exploring content through problem-solving contexts is recognised, as 45% of the total KPTs identified across Years 7–9 related specifically to the proficiency problem solving. Fluency received 25% of the KPTs, understanding 17%, and reasoning 13%.

Overall, problem solving is predominantly represented in this content analysis, with 36% of total terms relating to developing this proficiency across years F–9. Understanding and fluency are similarly weighted, with 24% and 25% of the KPTs respectively. However, only 15% of KPTs from Foundation to Year 9 describe actions that relate specifically to students engaging in reasoning in their learning in mathematics. A higher representation of KPTs identifying problem solving could be attributed to the intent described in the ACM rationale "these proficiencies enable students to respond to familiar and unfamiliar situations by employing mathematical strategies to make informed decisions and solve problems efficiently" (ACARA, 2015, p. 4). It could be inferred in the ACM that reasoning would be built into this process of problem solving. However, this is not evident in the KPTs extracted. This is a limitation of the analytic process used here and the problem that differentiating the proficiencies individually presents. If reasoning is embedded in problem-solving contexts, this could be made explicit in the description of the proficiencies as an integrated system.

Integrating the Proficiencies

A generalised model of patterning (McCluskey, Mitchelmore, & Mulligan, 2013) has been proposed as a means of describing the abstraction of patterning across differing domains of knowledge. It was noted that patterning moves through a progressive cycle in building generalised understandings within and beyond mathematics in that:

- a sense of familiarity is experienced with known situations,
- similarity experienced across contexts is encoded in the conceptual structure of the pattern,
- patterns are activated when similarity is recognised, and
- familiar patterns are accessed more fluently and applied in new contexts.

Thus, we propose that, all four proficiency strands of *understanding, fluency, problem solving*, and *reasoning* in the ACM can naturally work together as an integrated whole, in a cyclic structure, building and deepening generalised patterns of mathematical understanding with a focus "on depth of learning rather than breadth" (Sullivan, 2012, p. 185). For example, as *understanding* is connected across *problem-solving* contexts, similarities about mathematical concepts are recognised, and students develop *reasoning* as they construct generalisations. Over time, *fluency* in recognising and engaging with similar problems is strengthened with an increasing capacity to transfer *understanding* to new contexts. The four proficiencies have a combined role in systematically building patterns of generalised understandings through this pedagogical cycle.

Summary and Recommendations

The ACM heralds in an opportunity for educators to focus on the interrelated development of the mathematical proficiencies, a key principle that underpins the curriculum (Sullivan, 2012). The importance of reasoning is clearly articulated in the rationale in the ACM. However, the KPTs that articulate reasoning appear to be noticeably under-represented in the content descriptions from Foundation–Year 9. In contrast, a clear emphasis on students engaging their thinking through problem-solving contexts was identified throughout the F–9 curriculum content descriptions. Sullivan (2011, 2012) has emphasised pedagogical use of relevant problem-solving contexts and approaches as a means of engaging a greater breadth and depth of proficiencies through teachers' choice of task design and consequent learning experiences for students. Similarly, the heavier weighting of KPTs relating to problem solving, identified through the content analysis, could encourage teachers to adopt practices and design learning experiences that will realise the intention of an integrated view of the proficiencies.

We propose a pedagogical cycle that could support teachers in engaging students' sense of reasoning systematically through problem-solving contexts. This structure acknowledges the mathematical proficiencies as being interrelated aspects that together build conceptual understanding through opportunities for students to:

- engage their current understandings through familiar experiences,
- identify and describe similarities in concepts,
- question and engage in mathematical discourse to communicate their thinking,
- generalise their conceptual understanding about concepts across contexts,
- develop fluent patterns of knowing how to engage with similar type problems,
- apply these patterns of understanding in new and unfamiliar contexts, and
- explain and justify their reasoning, which in turn re-shapes and strengthens conceptual understanding.

Adopting such an integrated view of the role of the mathematical proficiencies has implications for professional learning to ensure teachers' pedagogical content knowledge and promotion of reasoning enables their students' to develop generalised understandings of mathematical concepts.

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5.4 Postscript

5.4.1 Reflection on research questions

In this chapter I have inquired into the educational implications of a generalised model of patterning, and how reasoning was articulated in and developed through the mathematical proficiencies in the ACM. It was found that reasoning was highlighted throughout the introductory sections of the ACM and examples of reasoning were also offered at the beginning of each year level before the content descriptions. However, throughout the content analysis, the occurrence of KPTs identifying reasoning revealed an underrepresentation compared with the other proficiencies, particularly problem solving, from Foundation to Year 9.

It was considered that reasoning may well be represented through the proficiency problem solving. However, this connection is not explicitly drawn in the ACM. Direct references to individual proficiencies supported their “separateness” as opposed to promoting interrelationships. There were few examples of how the proficiencies were actually interrelated in terms of student learning. It is therefore recommended that an integrated, dynamic view of the proficiencies be considered as a means of explicitly highlighting reasoning and generalised understandings in mathematics education.

5.4.2 Application of proposed proficiency cycle

The pedagogical structure was proposed to support the development of generalised understandings, drawing upon an innate ability to reason throughout the cycle (refer to Figure 2, Appendix D). I implemented this proposed proficiency cycle with a group of 5- to 6-year-olds in a foundation year setting. In this context, the children were engaging in an outside learning space where they had previously planted broad bean seeds that were now growing, The teachers involved noticed children discussing how tall the broad bean plants in the garden had grown over the term break.

1. *Engaging students’ prior and current understandings and familiar experiences*
   The children were observed directly comparing themselves to the height of the plants, using nonverbal gestures and expressing a range of comparative statements through their conversations. The children were also observed building constructions that would fall down and they exhibited a growing interest in discovering how high they could build them. Again, a diverse range of language and gesture was noted as children used their bodies as reference points to compare the height of these constructions.

2. *Connect understanding through identifying similarities about concepts across problem-solving contexts*

   Digital images of the children engaged in both play contexts were displayed, and the children shared stories about what they remembered. Comparative language to describe size and relative "bigness" emerged from the children's description of the experiences. Similarities in key mathematical concepts about attributes of objects, comparison of length, and growing patterns emerged from the type of language the children revealed.

3. *Intentionally question to explain thinking and elicit levels of reasoning about concepts explored*

   The role of the teacher here was paramount in supporting the children to:

   - clarify their thinking,
   - connect their understanding across both contexts, and
   - challenge and stretch understanding further though engaging in new problem-solving contexts.

4. *Fluency is strengthened as patterns of knowing are frequently re-engaged and generalised*

   Children's use of language to describe their thinking and strategies for comparing were more readily observed and visibly transferred to new contexts. Again, the teacher's role in connecting children's prior understanding to new situations supports what is experienced as being similar across contexts, leading to generalisations about mathematical concepts.

5. *Patterns of understanding can be more readily applied in new problem-solving situations*

   A greater range of language and strategies to compare size was noticed in new learning contexts initiated by the children. Throughout this proficiency cycle, elements of reasoning are engaged at each stage, for example:

   - noticing and engaging with familiarity;
   - identifying similarity encountered in problems and experiences;
   - describing and explaining thinking, fluently predicting what could happen in new learning contexts; and
   - applying understanding and reasoning through self-and group initiated play.

5.4.3 Summary of areas identified for future research

5.4.3.1 Content analysis: Limitations

The analysis was not as simple as extracting the verbs and adverbs in the content descriptions, as the nouns and adjectives also provided important indicators of mathematical actions students could be engaging with. In the field of mathematics, nouns name the object of mathematical study:

> "mathematical structures … are somewhat more complex nouns, and consist of mathematical objects linked together by certain relationships or laws of combination. The symbols of combination or of relation … play a similar role to that of verbs … [and] … the existence of more complicated mathematical objects may likewise be experienced in terms of how one interacts with the objects" (Davis, Hersch, & Marchisotto, 2011, pp. 156, 159).

This interaction with the object is described by adjectives and their relationship with the object. Therefore, identifying and calculating the occurrence of proficiencies might also involve terms describing the depth of structural understandings of concepts (objects) explored and hence could refer to the identification of not just verbs/adverbs but also adjectives and nouns.

In summary, this process of identifying KPTs was challenging and there were some limitations needing addressing if a second phase of a content analysis were to be conducted. For these reasons, a second phase of a content analysis is recommended to identify

- complex nouns, in particular those that have a direct relationship to the proficiencies (e.g., understanding and reasoning are both nouns but are also both proficiencies)
- adjectives that qualify complex nouns (concepts), and
- the KPTs embedded in not just the content descriptions but content elaborations and achievement standards.

5.4.3.2 Educational implications

In describing an entwined view of the mathematical proficiencies, Kilpatrick et al. (2001) refer to findings from cognitive science that assert the importance of "connected and structured" (p. 118) neural networks in the encoding, storing, and retrieval of information. Therefore the type of thinking and processes involved in the construction of such effective networks needs further investigation.

Geake and Cooper (2003) emphasised the importance of reasoning to establish "clear relationships within learning contexts" (p. 15), thus supporting neuronal efficiency in actively recognising and selecting what is deemed to be familiar across patterns of understanding. Others in the field of neuroscience concur: "The human mind is a pattern recogniser" (Devlin, 2010, p. 169) and that "our brains perceive and generate patterns and use these patterned networks to predict the correct response to new stimuli. … Successful, extensive patterning leads to more accurate predictions … extending and strengthening neural networks" (Willis, 2010, pp. 59, 61). Edelman, the founder of the theory of neuronal group selection (TNGS, Edelman, 1992) proposed that the brain selects an appropriate interpretation of these patterns through "natural selection" (Geake, 1997 p. 28). Over time these maps of thinking accumulate, becoming resilient, fluently recalled patterns. This patterning is the result of "the meaningful organisation and categorisation of information" (Willis, 2010, p. 59).

The inter-related aspects of cognition work together to connect and stretch new understanding further. "When new information is recognised as prior knowledge [fluency] learning extends [understanding] and is available through transfer to create new predictions [reasoning] and solutions to problems [problem solving] in other areas beyond the classroom" (Willis, 2010, pp. 60–61). In each stage of Willis' description of patterning, the actions of the ACM's proficiencies could be associated, working together to connect with and build understanding in new contexts. For this reason connecting research from a cognitive neuroscience perspective to a generalised view of mathematical understanding could support an interrelated view of the proficiencies and the development of associated pedagogies. The emergence of cross-disciplinary research may be possible.

5.4.4 Proposal for Publication 4

Publication 3 was an edited, concise version of the full paper that is contained in Appendix D. The purpose for submitting a research paper to the MERGA39 conference proceedings was to gather feedback to modify the expanded version for publication in an appropriate mathematics education journal.

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CHAPTER

6

DISCUSSION AND CONCLUSIONS

6.1 Introduction

In this chapter I discuss the analyses across both phases of this research. In each research phase I address the research questions (Section 1.4) and connect these to the rationale and theoretical perspectives. I raise implications for pedagogy and practice. I reflect upon the challenges and limitations of this thesis. Finally, I consider the implications of the research and identify directions for future research.

6.2 Discussion

6.2.1 Research Phase 1

This phase focused on two initial research questions: What is similar about the construct of pattern across the fields of mathematics education and neuropsychology? and Is a generalised view of patterning applicable across broader fields of knowledge?

The outcome of this research phase was the development of an integrated theoretical perspective, to propose a generalised construct of patterning across fields of knowledge. An initial comparison identified similarities and differences in the use of the construct *patterning* across the fields of mathematics education and neuropsychology. This was summarised in Table 1: Abstraction across domains (Section 3.3). This analysis was then expanded to encompass the similarities across the wider fields of the psychology of mathematics education and cognitive neuroscience, contributing to the proposed generalised model of patterning that is presented in Table 2: Revised generalised model of patterning (Section 4.3).

The comparative analyses indicated that the notion of patterning moved through similar stages or aspects in the development of generalised understandings across all domains. A common aspect was that the repeated exposure to familiar experiences enabled a sense of similarity to emerge that reflected what was commonly experienced across contexts. Over time and repeated occurrences, an abstraction, or knowingness, about how to engage in these types of situations, was recognised. This led to a greater ability to reengage and apply this pattern in new situations. Thus this important insight, which supported a generalised view of patterning, was found to be common with the process of mathematical generalisation.

The comparison between domains also highlighted differences in the use of the construct of patterning. These differences centred on the level of awareness that the learner engaged in when constructing patterns of thinking. In mathematics education, engaging with a sense of reasoning to build generalisations is recognised as an intentional mathematical process. However, applying a sense of intuitive knowingness when engaging with mathematical concepts is also differentiated in the mathematics education research (Fischbein, 1999; Fischbein & Grossman, 1997; Gray & Tall, 2007; Mason, 2008).

6.2.2 Research Phase 2

This phase focused on the research questions: What are the educational implications of a generalised model of patterning? and How is reasoning, the ability to express and justify generalised understandings, articulated in and developed through the proficiencies in the ACM?

This phase addresses the second part of the research enquiry identified in Section 1.5.1, regarding the educational implications of adopting a generalised model of patterning. Reasoning is paramount in building generalised patterns of understanding. Therefore, the focus of Phase 2 was a content analysis to identify key proficiency terms (KPTs) that indicated the incidence of reasoning throughout the Australian Curriculum: Mathematics (ACM) from Foundation to Year 9.

From the content analysis, KPTs identifying reasoning were consistently underrepresented throughout the content descriptions with F–2 (19%), 3-6 (14%), 7–9 (13%), and overall KPTs indicating reasoning occurring F–9 (15%). However, it was also noted that the proficiency problem solving was overrepresented throughout years 3–9 in the content descriptions with F–2 (27%), 3–6 (34%), 7–9 (45%), and overall F-9 (36%). It was acknowledged that the proficiency reasoning could be inferred through engagement with problem solving contexts. However, this was not explicit in either the description of the proficiencies or the content descriptions.

6.2.3 Content analysis: Key proficiency terms

In Research Phase 2 of this thesis I engaged in an initial stage of a content analysis that involved extracting KPTs from the content descriptions in the ACM. KPTs comprised verbs and adverbs as they related specifically to the actions of the mathematical proficiencies embedded in the content. However, through this process I questioned my reasoning for not including adjectives and nouns as key terms, as they also provided relevant indicators to consider in relation to the action of the mathematical proficiencies. The following examples highlight the difficulties experienced in determining how KPTs should be defined.

1. In the content description “Identify symmetry in the environment” (ACARA, 2015, [ACMMG066]), *identify* is recognised as the KPT, and *symmetry* was discounted as it was classified as a noun. However, it names a mathematical relationship that would naturally engage students’ sense of reasoning to identify it.

2. Adjectives, such as *efficient*, *effective,* and *familiar*, were not included as KPTs. However building efficient and effective strategies would require a capacity to reason, as well as a capacity to explain and justify thinking and choice of strategies. Similarly, engaging in familiar contexts or with familiar materials and concepts is essential in the development of generalised understandings. Therefore, adjectives might also validly be included in a second phase of a content analysis.

A more thorough analysis is required to characterise the classification of KPTs to discern terms that also warrant inclusion in the content analysis. Also extracting the KPTs from content elaborations and the achievement strands would generate valuable data to further understand the role of reasoning in building and assessing patterns of thinking throughout the ACM.

6.3 Implications for pedagogy and practice

There were discrepancies between the goal of reasoning articulated in the rationale section of the curriculum and the lack of KPTs indicating reasoning throughout the content descriptions. The mathematical proficiencies were consistently described individually, minimising their possible interaction in the building of conceptual understanding. This could cause confusion for educators accessing the curriculum to design learning structures that maximise the role of reasoning in developing mathematical thinking.

6.3.1 Pedagogical cycle

A dynamic view of the proficiencies working together in an interrelated way was proposed to address the discrepancy between the view of reasoning articulated in the rationale section of the ACM and the occurrence of KPTs relating to reasoning embedded in the content descriptions. A supplementary pedagogical cycle was proposed to connect the role of all the proficiencies in systematically building a capacity to reason with the generalised model of patterning. Similarities in the construct of patterning were connected with a cyclic view of the proficiencies to illustrate the combined role the proficiencies play in connecting with and deepening mathematical thinking.

6.3.2 Professional learning

Sullivan (2011, 2012) advocated the use of pedagogies, immersed in problem-solving contexts that encompass all the mathematical proficiencies. This type of practice supports the development of structural understandings within mathematics education. The ACM provides an opportunity to reflect on current practice and adopt a dynamic view of teaching and learning in mathematics. In realising this intent of the ACM, professional learning that engages teachers in designing learning experiences that systematically build proficiency in mathematical reasoning is essential. Developing structures for noticing mathematical behaviours, in both formal and informal contexts, could support teachers in drawing on children's current understandings, developing discourse around the similarities they experience across contexts, and supporting children in making connections to new learning experiences. Over time patterns of knowing, in the sense of professional pedagogical behaviours, become more fluently recalled and applied. Structural elements of effective learning design for students, and associated pedagogical practice for teachers and pre-service teachers, are areas for further theoretical and applied research.

6.4 Reflection on the research process: Challenges and limitations

In Phase 1, I engaged in a theoretical analysis of the literature to compare and contrast the role and construct of patterning, I found it challenging to identify cross-disciplinary literature that focused on a generalised view of patterning. I was able to provide new insight into the levels of awareness involved in constructing and connecting patterns across domains. The analysis of intuitive forms of knowing in building patterns of thinking was evident in several domains and remains a cross-disciplinary research area that can be pursued.

Initially this thesis was a theoretical inquiry to establish a generalised perspective on the role of patterning within and beyond mathematics, and to highlight the consequent educational implications. In addressing educational implications I became interested in how the proficiency of reasoning was developed throughout the ACM. In the text, *Engaging the Curriculum Mathematics: Perspectives From the Field* (Atweh, Goos, Jorgensen, & Siemon, 2012), the chapters by both Sullivan and Atweh et al. referred to analysing the use of verbs that identify actions of the proficiencies. However, access to their research data and analysis was limited. This limitation provided me with the impetus to undertake an initial stage of a content analysis in order to identify the language related to the proficiency of reasoning. This process was time consuming but generated valuable data and insight into the structure of the ACM. As mentioned before, a more rigorous content analysis could explore a wider range of key terms and the relationship between them.

6.5 Implications and directions for further research

6.5.1 The embodiment of mathematical understandings

The role of intuitive forms of knowing in constructing patterns of mathematical understanding has been investigated in several studies with young children (Mulligan, English, & Mitchelmore, 2013; Warren & Cooper, 2008). However, the relationship between explicit and implicit mathematical knowledge, derived from both formal and informal contexts, in developing generalisation through the various stages of learning requires systematic, longitudinal investigation. In this way, evidence of patterns of mathematical thinking can be linked with more generic patterns demonstrated by the learner.

An example of this intuitive knowing in an informal context is included within the postscript, situated in Section 5.4.2. In this context, early years children were noticed engaging in a pattern of knowing about the concept of linear measurement. Informal play contexts that the children were freely engaging with were in essence connecting with children's innate sense of measurement. Comparative language, and nonverbal gestures used by the children revealed this type of thinking. This is an example of implicit knowledge that the children revealed in an informal setting that provides a valuable context to investigate reasoning and patterns of thinking.

The pedagogical choices educators make when connecting these informal understandings to more formal and explicit opportunities invite further research. This would involve a review of literature to synthesise research into the embodiment of mathematical understanding. Tall (2005) declared that our human facility to "observe one or more objects and to have a primitive sense of 'numerosity' [is] already set in our cognitive structure" (p. 3). This is witnessed through young children noticing changes in small sets of objects presented to them (Tall, 2005). He referred to "set-before" and "met-before" categories of embodiment. The term "met before" relates to prior knowledge that is accumulated through experience, whereas "set-before" relates to what is known and carried within the body genetically (p. 4). Therefore, "cognitive growth is revealed as a story of each individual born differently endowed with an underlying set-before structure and having a variety of experiences that construct met-befores used later to develop highly individual mental capacities" (p. 5). The relationship between set-before and met-before structures in the development of generalised mathematical knowledge requires further investigation to explore the influence of both formal and informal contexts.

6.5.2 Structures for noticing

I am interested in engaging in research that develops pedagogical structures for noticing young children's embodied mathematical understanding as emerging patterns of knowingness. Agnes Macmillan (2009) referred to "Bishop's (1988) six mathematical activities: counting, measuring, locating, designing, playing and explaining" (p. 21) that together support the analysis of mathematical discourse in early childhood. However, not all children articulate their understandings verbally: "There is much of children's thinking that is not captured when we only consider their verbal expressions. Before and without speech, children seem to already possess prelinguistic thinking processes … [and] exhibit conceptual knowing in and through their bodies" (Kim, Roth, & Thom, 2011, p. 209). Therefore, the development of pedagogical processes that intentionally notice children's nonverbal interaction with each other and their environment could direct new research about children's emerging *patterns* of mathematical understandings.

6.5.3 Generalised models of learning

Developing the generalised model of patterning has raised my awareness about similarities in the patterns of conceptual understanding across different learning content areas. This is currently investigated through national initiatives such as integrating science, technology, engineering, and mathematics (iSTEM, 2012). Research into educational initiatives that promote authentic integration across learning areas, needs to question whether learning is connected by *common patterns of understanding*. For example, as a homeroom educator in an early years context I have documented key experiences in a child's first year of school and mapped this to the Australian curriculum to uncover generalised patterns of understanding across different learning areas. This informal research could be developed to systematically connect areas of the curriculum and provide authentic contexts for integrative learning in the early and primary years of school.

This research could contribute to solving the difficulties of a crowded curriculum experienced by educators in early childhood and primary settings. The Australian Primary Principals Association (APPA) described the Australian curriculum as "a crowded curriculum that is impossible for primary teachers to implement successfully … too much, too soon, too complex" (Australian Primary Principals Association, 2014, p. 2). Researching the pedagogical cycle proposed in Publication 3 could be expanded to engage various learning areas simultaneously to identify common elements and simplify what has become complex to implement. This raises the question whether a *greater capacity to reason* could eventuate from *engaging with and building generalised patterns of understanding across learning areas*. In early childhood, measuring changes in generalised patterns would involve the development of pedagogical structures for noticing the embodiment of mathematical behaviours.

6.6 Concluding remarks

Adopting a cross-disciplinary view of the role patterning in developing mathematics understanding has been advantageous in broadening awareness of the need to identify common underlying aspects of human cognition. Taking a broader perspective based on the theory of Goldberg has highlighted the important similarities but explicit differences in attempting to integrate different domain-specific views into new models. Cognitive neuroscience describes the construction of understanding using various terms such as neural networks, cognitive templates, internal maps, mental representations, and patterns of thinking (Devlin, 2010; Dispenza, 2007; Gazzangia et al., 2002; Geake, 1997). These terms all describe the neural structures involved in the process of building cognitive patterns. "Our minds are very good at recognising patterns, seeing connections, and making rapid judgements and inferences" (Devlin, 2010 p. 171). The processes Devlin refers to generally are readily applicable in mathematics, showing the important connection between pattern formation within and across disciplines.

This thesis has raised questions about the application of pattern as a learning process both generically and in terms of mathematics learning. The possibilities for examining this construct across many aspects of knowledge development is unlimited and yet to be explored. The contribution of a mathematics education perspective may offer the most appropriate pathway for recognising the role of patterning in the development of generalised understandings. I conclude with the question raised in the thesis, yet to be fully answered:

Is an ability to form patterns fundamentally mathematical?

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APPENDIX A

Previous Postgraduate Research Papers

This appendix comprises a table that contains the titles and descriptions of research papers that I wrote as part of my previous postgraduate studies.

Table A.1

Previous Postgraduate Research Papers

Paper	Description
Imagine that! The science of visualisation April 2007	Literature review investigating the role of visualisation, how it compares with perceiving information visually. Highlights role of similar neural structures and processes in both contexts. Explores the role of visualisation and intention in setting goals.
Magic moments July 2007	Literature review outlining different types of memory patterns and the structure of their formation.
The nature of consciousness October 2007	Literature review exploring contemporary views about the role of consciousness in the building of understanding and formation of memories.
Is the sum greater than the whole? April 2008	Literature review comparing and contrasting the similarities and differences between intelligence and creativity in gifted education.
So you want to be creative? July 2008	A critical examination of creativity in the building and expression of understanding and intelligence. Includes implications for practice.
Visual thinking October 2008	Comparing the process of visualisation and mental imagery with visual perception. What are the creative elements of mental, visual synthesis? How does drawing and other forms of representation support the visualisation process?
Visual thinking and giftedness April 2009	Comparing models of visual thinking with Gagne's model of giftedness and Edelman's model of neuronal group selection. This analysis highlighted the role of visualisation in gifted education.

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APPENDIX B

Feedback on the Position Paper: MERGA 2013

This appendix contains information about the presentation of the position paper *Does an Ability to Pattern Indicate That We Think Mathematically?* at the Mathematics Education Research Group of Australasia (MERGA36) conference, *Mathematics Education: Yesterday, Today and Tomorrow* in 2013. It contains three sections.

Appendix B1

MERGA 2013 Review Panel Recommendations

Title: Does an ability to pattern indicate our thinking is mathematical?

Code: RP110

MERGA acknowledges many forms of scholarly inquiry and accepts a broad range of research approaches. This paper is essentially (delete irrelevant points):

- a position paper

Following refereeing, the recommendation of the review panel to the Editorial Committee is that a paper be (delete irrelevant point):

- **ACCEPTED** as a paper to be published in the proceedings and presented at this conference

Note that there is no option for rejection of a paper. A paper considered by a panel to be not suitable for inclusion in the conference (e.g. it is an advertisement for teaching materials, it is not related to mathematics education research) will be referred to the VP Conferences, Roberta Hunter <R.Hunter@massey.ac.nz>) for a final decision.

The criteria upon which papers will be considered relate to the quality of each of the following:

1. Statement of problem/issue and discussion of its significance:
2. Literature review and theoretical framework:
3. Methodology, and data analysis where applicable:
4. Results and discussion:
5. Conclusions and implications:

Recommendations for the author(s) will be made by panels in relation to:

- Oral presentation of the paper (e. g. request the author(s) to address specific points or supply more details in the conference presentation):

  Recommendations could also include

- Re-writing and publication as a journal article (e.g., what should be expanded, which journal could be approached, which sections need more detail and examples):

REQUIRED EDITS

Mismatch page 2 para 2: Steen (1991) whereas is Steen (1990) in reference list

Mismatch	page 1 states	Lakatos, 1976 cited in Ernest, 1991
	Page 2	Lakatos, 1976 directly quoted
	Page 6	Lakatos, 1976 cited in Ernest, 1991

If Lakatos (1976) is just cited then should not appear in reference list. If primary source has been used then should not use citing.

Mismatch page 3 final para: Lijedahl should read Lilejedahl

Mismatch	page 4	Skemp (1986) in text, Skemp (1990) in reference list

Cassirer is only cited in the paper, so should not appear in reference list.

Appendix B2

Reflections on MERGA 2013 Presentation

Presenting my position paper *Does the role of pattern indicate our thinking is mathematical?* at the Mathematics Education Research Group Australasia (MERGA36) conference, *Mathematics Education: Yesterday, Today and Tomorrow* in 2013, offered me an opportunity to gather feedback from researchers in the field of mathematics on my initial thesis question:

What is similar about the construct of pattern across the fields of mathematics education and neuropsychology?

This paper explored the relationship between

- pattern as a neuropsychological construct, the processes through which our understandings are encoded, and
- pattern as defined in the field of mathematics.

The purpose of the position paper was to

- explore similarity in the concept of pattern across these two domains and
- to propose a generalised model of patterning applicable across these domains

This position paper was based on the premise that if "virtually all mathematics is based on pattern and structure" (Mulligan & Mitchelmore, 2009, p. 33), and "mathematical activity is human activity" which "produces mathematics" (Lakatos, 1976, p. 146), then is human activity also based upon pattern and structure?

Comments from the field:

- This inquiry was drawing upon the specific field of neuropsychology (Elkhonon Goldberg), not the neurological field. Correct use of terminology needs to be used for specific fields within the broader field of neuroscience.
- In comparing the similarities between White and Mitchelmore's model of teaching for abstraction and Goldberg's construct of pattern recognition the boundary between the reification and application stage was unclear and needed to be defined
- In reference to the neuropsychological field, how do the pattern networks start?

- The terms pattern and patterning have been used throughout the paper, what is specifically meant by both terms in each field?
- What is meant by the term 'Thinking Mathematically'? In the field of mathematics what is commonly understood about the term mathematical thinking?
- Is the position paper capturing the whole picture? I have highlighted the similarities between pattern and generalised thinking, but I haven't explored how mathematical thinking is different to generalised thinking. A comparison has been made across the fields but not a contrast.
- Interest into how a generalised model of patterning could describe the role of the mathematical proficiencies in building fluent conceptual understanding in the Australian Curriculum: Mathematics (ACM) was shown from the field at the end of the presentation.
- Could an empirical research component be designed to further explore the role of pattern in building conceptual understanding?

Exploring the questions…what actions need to be taken?

1. Use of terminology: Clarify the difference between neurology and neuropsychology and which field to refer to in respect to this research.
2. Reification vs application stage in *Abstraction across domains* table
3. Pattern…or patterning?
4. How does a pattern start?
5. Thinking mathematically…what do I mean?
6. Exploring the contrast…too one sided
7. The role of pattern/patterning in building the mathematical proficiencies
8. Could this research have an empirical component?

Appendix B3

Summary of Feedback From MERGA Presentation

- Use of correct terminology…this paper is comparing the construct of pattern from a neuro-psychological perspective/field as opposed to 'neurological' field which is too broad a term. When I draw in the findings from others in the field of neuroscience will need to consider which branch of neuroscience is being referred to

- In Table 1, *Abstraction across domains*, rethink and clarify Goldberg's construct of pattern recognition…does it sit in the reification stage or application stage alongside White/Mitchelmore's Model of Teaching for Abstraction? If it is in the reification stage then what would be in the application stage? Does it span both stages? Will need to clarify this with Mike Mitchelmore.

- I was asked what is meant by the term "we think 'mathematically'"…is this in a broad sense…we think mathematically in an everyday sense? Or, do I need to unpack the term 'thinking mathematically' more? What is 'thinking mathematically' mean in the mathematical field? In this paper I was exploring the sense of pattern, structure and abstraction inherent in the process of building conceptual understandings generally not just purely in the 'mathematical' sense, and not just in the 'mathematical field'. However because we draw upon this innateness to pattern, structure our thinking and develop abstract understandings generally; and that pattern, structure and abstraction underlie a broad range of mathematical understandings; this is what I meant by the term 'thinking mathematically.'

- Is this research purely theoretical or will there be an empirical component? This Masters research is exploring a theoretical position about the role of pattern in the construction of understandings to propose an emerging generalised view of pattern across domains of knowledge. This question however made me consider if there could be an empirical component. This led to Phase 2 of this research whereby I undertook a content analysis as a means of analysing use of language terms that indicated the proficiency reasoning embedded in the content descriptions in the ACM.

- Could exploring, collecting data on children's embodied mathematical understandings and categorising this into stages of AMPS alongside the GMP provide empirical evidence?

- Did I answer my original question? No, this research is exploring a generalised view of patterning. The term 'thinking mathematically' needs further investigation.

- If understanding though could be built through developing a greater sense of pattern and structure then how does the field of education capitalise upon this? There was some discussion at the end of the presentation about pattern and structure intervention in supporting children with special learning rights. What could be the impact on learning when elements of pattern and structure are embedded in the learning design, both in a mathematical sense and generally across learning areas?

- How could explicit use of the 'mathematical proficiency cycle' build fluent conceptual understanding? This could lead to a comparison of the mathematical proficiencies with Goldberg's pattern recognition and White/Mitchelmore's Model of teaching for Abstraction? Which proficiency could be compared with pattern recognition? Or is it a combination of the proficiencies that leads to pattern recognition and generalised understandings?

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APPENDIX C

Proposing the Generalised Model of Patterning: MERGA 2015

This appendix contains information about the presentation of a short communication at the Mathematics Education Research Group Australasia (MERGA38) conference, *Mathematics Education in the Margins* in 2015. It includes a short communication abstract that outlines the purpose of the presenting Phase 2 of this thesis research, and the proposed generalised model of patterning that was circulated during the presentation.

It contains 2 sections.

Appendix C1

MERGA 2015 Short Communication Abstract

The pattern and structure of the Australian Curriculum—Mathematics

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The mathematical proficiencies in the *Australian Curriculum—Mathematics* describe the processes students are engaged in while developing mathematical concepts (ACARA, 2014). This presentation focuses on how the proficiencies: understanding, problem solving, reasoning and fluency, may work together to build patterns of thinking which can lead to generalised understandings of mathematical concepts. The authors connect the combined role of these proficiencies with a proposed Generalised Model of Patterning (McCluskey, Mitchelmore, & Mulligan, 2013), highlighting the role of patterning in the development of conceptual understandings within and beyond mathematics.

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Appendix C2

Generalised Model of Patterning

Table 1
Generalised Model of Patterning[a]

Instruction for Abstraction Model Mathematical domain	Goldberg: Pattern Recognition Neuropsychological domain	Schematic understandings Psychology of Mathematics Education domain
Familiarity: • explore a concept through a variety of contexts • become familiar with the underlying structure of each context	• Engagement with a range of situations set the scene for experiencing and sensing what is familiar about these situations.	New understanding is dependent upon previous, understanding A sense of familiarity is engaged
Similarity: • frequent matching • explicit attention to similarities within and between varying contexts	• Further attention and engagement with familiar situations allows connections to emerge as we recognise what is similar about these experiences. • Engagement with these experiences enables similarity to be matched, measured, and understood. • Overlapping of neural networks encode the similarity experienced	Assimilation 'becoming aware of similarities' 'naming objects is form of classification 'a new idea can be fitted into a larger framework of ideas'
Reification: • moves students into operating with abstract concepts	• Over time a pattern is encoded as a generic memory, a mental representation of the similarities and shared properties of a 'type' of experience. • Pattern recognition refers to the ability to readily access this pattern in similar situations in the future	Accessing an appropriate schema readily to assimilate familiar information Drawing connections between current schemas and similar situations
Application: • consolidation of the concept • application to new situations	• Recognise familiar patterns more fluently. Automated, efficient and competent decision making • Applying patterns of knowing by recognising new problems as having familiar characteristics to familiar ones…the same pattern in new situations	Accommodation and expansion of current schemas would occur through the process of application and engagement in new situations

Table 1: Generalised Model of Patterning *Information gathered from Table 1: Abstraction across domains (McCluskey, Mitchelmore & Mulligan 2013, p. 487; Davis (1992), Skemp (1986),*

Table 2
Generalised Model of Patterning with Mathematical Proficiencies[a]

Instruction for Abstraction Model Mathematical domain	Goldberg: Pattern Recognition Neuropsychological domain	Schematic understandings Psychology of Mathematics Education domain	Australian Curriculum-Mathematical Proficiencies
Familiarity: • explore a concept through a variety of contexts • become familiar with the underlying structure of each context	• Engagement with a range of situations set the scene for experiencing and sensing what is familiar about these situations.	New understanding is dependent upon previous, understanding A sense of familiarity is engaged	*Explore concepts across range of* ***problem solving*** *contexts* *Connect with prior/current* ***understandings***
Similarity: • frequent matching • explicit attention to similarities within and between varying contexts	• Further attention and engagement with familiar situations allows connections to emerge as we recognise what is similar about these experiences. • Engagement with these experiences enables similarity to be matched, measured, and understood. • Overlapping of neural networks encode the similarity experienced	Assimilation 'becoming aware of similarities' 'naming objects is form of classification 'a new idea can be fitted into a larger framework of ideas'	*Explicitly* ***connect understanding*** *through identification of similarities within and across* ***problem solving contexts*** *Question to explain thinking and elicit levels of* ***reasoning*** *and justification* ***Fluency*** *is encouraged through frequency of matching and naming/identifying properties of* ***problems****/materials investigated*
Reification: • moves students into operating with abstract concepts	• Over time a pattern is encoded as a generic memory, a mental representation of the similarities and shared properties of a 'type' of experience. • Pattern recognition refers to the ability to readily access this pattern in similar situations in the future	Accessing an appropriate schema readily to assimilate familiar information Drawing connections between current schemas and similar situations	*Patterns of knowing and knowing how to are readily recalled* ***(fluency)*** *in a range of* ***problem solving*** *contexts* ***Problem solving*** *becomes a matter of pattern recognition* ***(fluency, reasoning and understanding)***
Application: • consolidation of the concept • application to new situations	• Recognise familiar patterns more fluently. Automated, efficient and competent decision making • Applying patterns of knowing by recognising new problems as having familiar characteristics to familiar ones…the same pattern in new situations	Accommodation and expansion of current schemas would occur through the process of application and engagement in new situations	*All four proficiencies interplay interdependently in new and unfamiliar contexts* ***understanding/problem solving/reasoning/fluency***

Table 2: Generalised Model of Patterning with Mathematical Proficiencies *Information gathered from Table 1: Abstraction across domains (McCluskey, Mitchelmore & Mulligan 2013, p. 487; Davis (1992), Skemp (1986), ACARA (2014)*

APPENDIX D

Expanded Version of Publication 3

This appendix contains the expanded version of Publication 3 that was submitted for inclusion in MERGA39 conference proceedings. The full version of this paper is intended to provide the basis for a fourth publication that is responsive to feedback generated from presenting Publication 3 at the MERGA conference and from the evaluation of research in this masters thesis. Appendix E contains data generated from the content analysis relevant to this publication.

Key Terms Identifying Reasoning Within the Australian Curriculum: Mathematics

Catherine McCluskey, Joanne Mulligan, and, Michael Mitchelmore

The mathematical proficiencies in the *Australian Curriculum: Mathematics* (ACM) of understanding, problem solving, reasoning, and fluency are intended to be entwined actions that work together to build generalised understandings of mathematical concepts. A content analysis identifying the incidence of key proficiency terms (KPTs) embedded in the content descriptions from Foundation to Year 9 revealed a much lower representation of "actions" relating to the proficiency reasoning than the other three proficiencies. However, reasoning is affirmed in the rational of the ACM, therefore a generalised model of patterning is proposed to provide an inter-related view of the proficiencies and to further support the development of generalised understandings in mathematics education.

Mathematics is widely accepted "as a subject that consists of patterns and relationships that are understandable through mental activity that involves mathematical reasoning and logic" (Wood, 2002, p. 61). The goal of mathematics education is clearly articulated in the Australian Curriculum: Mathematics (ACM) rationale statement: "It aims to instil in students an appreciation of the elegance and power of mathematical reasoning" (Australian Curriculum and Assessment Reporting Authority [ACARA], 2015, p. 4). Reasoning is recognised as paramount in the development and growth of mathematical understanding (Ball & Bass, 2003; Mason, Stephens, & Watson, 2009; Mueller & Maher, 2010). In the ACM, reasoning is singled out as one of the four mathematical proficiencies: understanding, problem solving, reasoning, and fluency. These are identified as key processes that describe "the actions in which students can engage when learning and using the content" and similarly inform teachers, "how the content is explored or developed" (ACARA, 2015, pp. 4, 5). The content knowledge in the ACM is structured around three strands that "describe what is to be taught and learnt" (p. 5) and the mathematical actions of the proficiencies are embedded in the content descriptions in the ACM. Therefore it is interaction within and between these content strands and the four proficiencies that builds conceptual understandings in mathematics.

Mathematical reasoning is described as the "capacity for logical thought and actions such as analysing, proving, evaluating, explaining, inferring, justifying and generalising" (ACARA 2015, p. 5). Reasoning involves recognising similarities and differences encountered in concepts explored across multiple contexts leading to the development of abstract understandings. Explaining and justifying thinking enables knowledge to become "more general and its applicability to different situations … increased" (White & Mitchelmore, 2010, p. 2). Intentional instruction supports conceptual understanding to deepen, become more fluently recalled, and applicable in new learning contexts. Ball and Bass (2003) emphasise the role of the teacher in promoting reasoning, as "mathematical understanding is meaningless without a serious emphasis on reasoning" (p. 28). Engaging mathematical reasoning naturally draws students into greater levels of fluency as they connect their understanding in new problem-solving contexts.

Sullivan (2012) proposes that teacher learning should focus on "ways of identifying tasks that can facilitate student engagement with all four of these proficiencies" (p. 183) as the "intention is that the full range of mathematical actions apply to each aspect of the content" (Sullivan, 2011, p. 8). However, the organisational structure of the curriculum as three content strands comprising number and algebra, measurement and geometry, and statistics and probability, draws attention to content knowledge. How the proficiencies together build entwined conceptual understanding is well intended in the rationale of the ACM but not clearly articulated within the content strands. This raises the key questions addressed in this paper: In what ways do the proficiencies build generalised understandings and reasoning skills? And is this relationship between reasoning and generalised understandings of mathematics evident and transparent to teachers accessing the curriculum?

At a theoretical level, an interrelated view of the proficiencies will be discussed in light of a generalised model of patterning proposed by McCluskey, Mitchelmore, and Mulligan (2013) to highlight the importance of reasoning. An outcome of this paper is to identify how the proficiencies are articulated in the ACM through a content analysis of key language terms embedded in the content descriptions denoting the "actions" of the four proficiencies across K–9.

Background

In the rationale of the ACM the role of the mathematical proficiencies is highlighted: "The curriculum focuses on developing increasingly sophisticated and refined mathematical understanding, fluency, logical reasoning, analytical thought and problem solving skills" (ACARA, 2015, p. 4). These mathematical proficiencies are described as capabilities that "enable students to respond to familiar and unfamiliar situations by employing mathematical strategies to make informed decisions and solve problems efficiently" (ACARA, 2015, p. 4). More generally, the curriculum describes the field of mathematics as "composed of multiple but interrelated and interdependent concepts and systems" (ACARA, 2015, p. 4), anticipating that schools will engage with the ACM in a dynamic and symbiotic way and thus implying, similarly, that the proficiencies are also interrelated.

There is a clear intent in the introductory sections of the ACM to highlight the mathematical proficiencies as integral aspects of the curriculum. They are described in the *Key Ideas* section directly following the rationale and aims, and they are outlined again under the next section, *Structure*, before the description of the content strands. Importantly, the proficiencies are embedded in the language of the content descriptions and achievement standards as verbs that describe the mathematical actions students engage with (Sullivan, 2012). This is demonstrated in the following content description: "Interpret and compare data displays" (ACARA, 2015, Section ACMSP070): the verbs *interpret, and compare* identify use of the mathematical proficiencies.

Throughout the ACM the proficiencies are described individually, rather than as an entwined system, at the beginning of each year level. The extract below illustrates examples of how the proficiencies are developed in the foundation year content. The verbs italicised in the text identify the actions students are engaged in as they explore the content. (Note, terms such as *readily* are adverbs and were included as KPTs if they modified a verb in the description)

At this year level:

1. **understanding** includes *connecting* names, numerals and quantities
2. **fluency** includes *readily counting* numbers in sequences, *continuing* patterns and *comparing* the lengths of objects
3. **problem-solving** includes *using* materials to *model* authentic problems, *sorting* objects, *using* familiar *counting* sequences to *solve* unfamiliar problems and *discussing* the reasonableness of the answer
4. **reasoning** includes *explaining* comparisons of quantities, *creating* patterns and *explaining* processes for indirect comparison of length. (ACARA, 2015)

Highlighting the interaction of the proficiencies individually within the content could support teachers noticing the type of learning intended and designing learning activities that engage that specific proficiency. However, naming and identifying individual proficiencies may not encourage teachers to focus on the potential interrelationships between the proficiencies to build and deepen conceptual understanding. It is their connectedness that is not well articulated and thus does not resonate clearly within the rationale.

In *Engaging the Curriculum-Mathematics: Perspectives from the field*, Atweh, Miller, and Thornton (2012) identified challenges that schools and educators could face in interpreting and implementing the curriculum due to this "possible lack of cohesion between the aims and rationale, the content and its articulation" (p. 2). In particular, they noted inconsistencies between the proficiencies, such as the role of reasoning, which they argued was underrepresented in the content elaborations. It may not be intended or necessary for the four proficiencies to be represented equally across the ACM. What is important is whether the reasoning proficiency is well articulated through the KPTs and whether teachers see the critical link with the problem solving proficiency. Therefore, in this paper an interrelated view of the proficiencies is explored. We aim to address this possible imbalance between the view proposed in the rationale and the incidence of key language terms articulated in the content, to support the development of generalised understandings in mathematics at the student level.

Inter-relationship between mathematical proficiencies

Atweh et al. (2012) highlight the inter-relationship between the proficiencies, that these "proficiencies are not disjointed … [and that] … some content elaborations may relate to one or more of the proficiencies" (p. 8). They refer to a model, mathematical proficiency, described in the United States report to the National Research Council (Kilpatrick, Swafford, & Findell, 2001) on how the U.S. national standards promote proficiencies in learning mathematics. (Note: The U.S. model of proficiency differs to the four strands of proficiency adopted by the ACM, whereby productive disposition has not been included.) In this model, the term mathematical proficiency is used to "capture what we think it means for anyone to learn mathematics successfully … The most important observation we make about these five strands is that they are interwoven and interdependent … [and] … represent different aspects of a complex whole" (Kilpatrick et al., 2001, pp. 5, 116). For Kilpatrick et al., these strands are adaptive reasoning, strategic competence, conceptual understanding, productive disposition, and procedural fluency. The inter-relationship of these strands is illustrated in Figure 1.

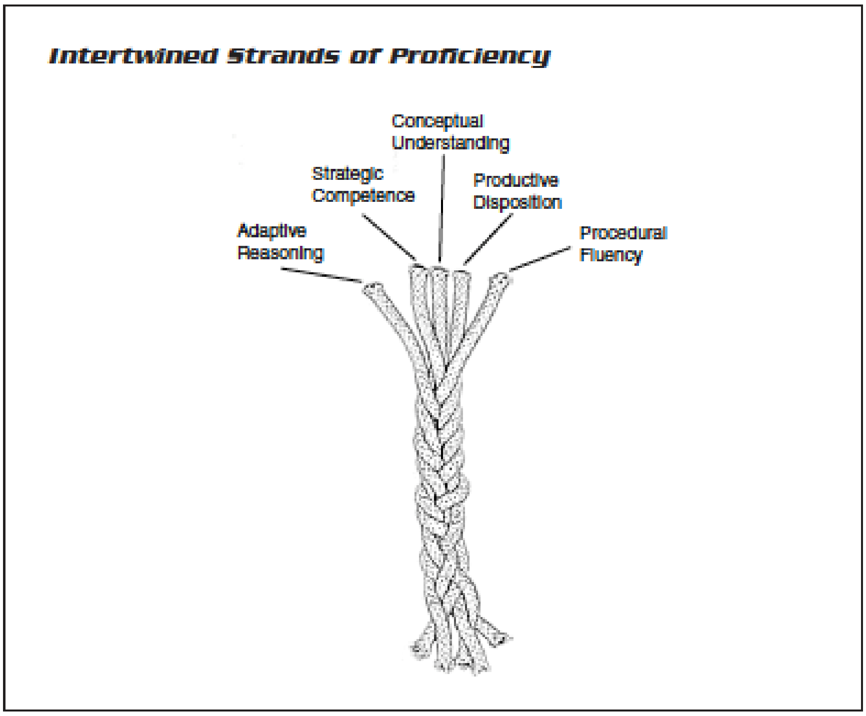

Figure 1. Intertwined strands of proficiency (Kilpatrick et al., 2001, p.117).

Watson and Sullivan (2008) succinctly describe the mathematical actions relating to these strands of proficiency from this model. They argue that:

- Conceptual understanding "includes the comprehension of mathematical concepts, operations and relations".
- Procedural fluency includes skill "in carrying out procedures flexibly, accurately, efficiently, and appropriately, and, in addition to these procedures, having factual knowledge and concepts that come to mind readily".
- Strategic competence is "the ability to formulate, represent and solve mathematical problems".
- Adaptive reasoning is "the capacity for logical thought, reflection, explanation and justification".
- Productive disposition is "a habitual inclination to see mathematics as sensible, useful and worthwhile, coupled with a belief in diligence and one's own efficacy" (Watson & Sullivan, 2008, as cited in Sullivan, 2011, pp. 6–7).

Kilpatrick et al. (2001) stress the importance of the relationship between all strands in building resilient understandings that can be fluently applied in new situations. They refer to findings from cognitive science that indicate that attention to structural features of mathematical concepts supports effective coding of information, "Organization improves retention, promotes fluency, and facilitates learning related material" (p. 118). Proficiency in mathematics involves the construction of effective neural networks that are structured in resilient and flexible ways to both connect understanding and accommodate new learning. This involves the capacity of the brain to adapt to and retain new information effectively. Findings from cognitive neuroscience indicate that we develop patterns of understandings, and encode these patterns for effective retrieval of information, "Patterning refers to the meaningful organisation and categorisation of information…the brain…connect[s] incoming information with stored patterns, categories of data, or past experiences, thereby extending existing patterns with new input" (Willis, 2010, p. 59). In a similar way, the image in Figure 1 illustrates the individual strands woven together in a pattern as an entwined whole. This illustrates the interconnectedness of the different strands. This description of the process of patterning from cognitive neuroscience proposes a view of the proficiencies in the ACM working interdependently to build conceptual understanding systematically. However, the definition and image of the proficiencies as individual strands still accentuates their separateness, not their integrated relationship with each other in building patterns of thinking.

The ACM proficiencies as an opportunity for changing practice

Sullivan (2012) has asserted that the ACM provides an opportunity for educators to rethink and reshape mathematics learning for students by focussing on "the principles that underpin the structure of the curriculum and the use of these principles to inform teacher learning" (p. 175). These principles are that:

- the four proficiencies provide a framework for mathematical processes
- the ACM has been designed to emphasise teacher decision making, and
- there is a focus on depth rather than breadth to address challenges of equity.

Sullivan identified the mathematical proficiencies as the first key principle that enacts the other two principles, emphasising that engagement with the mathematical proficiencies encourages educators to make pedagogical decisions to explore not just the breadth but also importantly the depth of mathematical concepts. Incorporating learning experiences in relevant problem-based contexts creates opportunities for students to engage meaningfully with the mathematical proficiencies:

> "Mathematics … is more than following rules and procedures but can be about creating connections, developing strategies, effective communication … This view is not obvious in the content descriptions … It is part of the opportunity for those supporting teachers to communicate such views" … [and is] … "communicated through the proficiencies that underpin the curriculum" (Sullivan, 2012, p. 179).

Identifying the language of the proficiences

In describing this dynamic view of learning, Sullivan refers to the use of verbs identifying the actions of individual proficiencies. It is intended that teachers look within, across, and beyond the content descriptions to connect with the language that articulates the use of the proficiencies in connecting with and deepening mathematical thinking.

Taking up this point, Atweh et al. (2012) analysed the occurrence of the proficiencies stated in the Year 8 content elaborations, finding that "53% relate to experiences to develop understanding … 56% relate to developing fluency … 12% relate to problem solving … and 7% refer to reasoning" (pp. 8–9). In this analysis, the proficiency of reasoning, an essential element in the development of generalised understandings, was rarely identified in the content elaborations. However, reasoning may be well represented in the KPTs describing problem solving. For example *in The Australian Curriculum: Mathematics as an opportunity to support teachers and improve student learning,* Sullivan (2012) has previously outlined the value of engaging the proficiencies through problem-based contexts. Investigating problem-based approaches assumes that:

> "the teacher draws upon the various strategies used by the students … [and that the learning] … experience will communicate to students that there are many ways to approach mathematical tasks, they can choose their own approach, and that some approaches are more efficient than others" (pp. 178–179).

This type of thinking, authentically embedded in problem-solving contexts, builds a capacity to reason but is dependent on teachers' awareness of "structural relationships … [and] strategies … [for]… bringing structural relationships to the fore" (Mason, Stephens, & Watson, 2009, p. 29). Structural relationships emerge from engaging in opportunities to reason. This involves generalising commonalities about concepts across contexts. Therefore the use of language in the ACM that indicates the incidence of the proficiencies, in particular reasoning, requires further investigation.

Content analysis of reasoning: content descriptions

In the research reported here, an initial phase of a content analysis was used to identify the type of language used to describe the actions of the proficiencies. This was conducted to find evidence of terms related to reasoning that were articulated in the ACM. This content analysis extracted key proficiency terms (KPTs) that "can be thought of as verbs" (Sullivan, 2012, p. 179) from the content descriptions. (Note, adverbs were also included in this content analysis.) This process occurred in the following four stages:

1. Each proficiency description in the key ideas section was analysed for KPTs.

2. A framework was constructed identifying the KPTs that related to each proficiency.

3. The KPTs embedded in the content descriptions from Foundation to Year 9 were extracted and categorised using the framework in Table 1 to compare the frequency of their use throughout the content descriptions from Foundation to Year 9. (Note, some KPTs recorded in Table 1 relate to more than one proficiency; however each KPT extracted from the content descriptions was counted to calculate the total number of occurrences relating to each proficiency.)

4. Table 2 contains entries that summarise the total number KPTs identified across F-2, 3-6, and 7-9 content descriptions.

Table 1

Key Proficiency Terms (KPTs)

Proficiency strand	Key proficiency terms (KPTs)
Understanding	Apply build connect describe develop identify interpret make represent
Fluency	Accurately answering appropriately calculate carrying choose choosing develop efficiently find manipulate flexibly recall recalling readily recognise regularly use
Problem solving	Apply communicate design develop effectively formulate interpret investigate make model plan represent seek solve use verify
Reasoning	Adapt analysing compare contrast deduce develop evaluating explain explaining generalising increasingly inferring justify justifying known mathematically prove proving reached reasoning something transfer thinking used

Table 2

Frequencies and Percentages of Key Proficiency Terms (KPTs) Across the Curriculum[a]

Year level clusters	ACM proficiency strands				Total KPTs
	Under-standing	Fluency	Problem solving	Reasoning	
F–Year 2	33 (26)	36 (29)	32 (26)	24 (19)	125 (100)
Years 3–6	83 (29)	65 (22)	102 (35)	42 (14)	292 (100)
Years 7–9	33 (17)	50 (25)	89 (45)	25 (13)	197 (100)
F–Year 9	149 (24)	151 (25)	223 (36)	91 (15)	614 (100)

[a] Cell entries are frequencies (row percentages)

For each year level clustering (i.e., F–2, 3–6, and 7–9) it was questioned whether the individual proficiencies would be equally represented, with a similar proportion of KPTs relating to each of understanding, fluency, problem solving, and reasoning. However, this was not the case, with problem solving noticeably over-represented from years 3–9: F–2: 26%, 3–6: 35%, and 7–9: 45%; and reasoning consistently under-represented throughout the content descriptions across the year level clusters: F–2: 19%, 3–6: 14%, and 7–9: 13%.

Across the early years of school (F–2), a total of 125 terms were extracted from the F–2 content descriptions. From these 19% related to reasoning, with 29% relating to fluency, and 26% each for KPTs relating to understanding and problem solving. This reflects the emphasis in the early years of developing conceptual understanding and fluency of procedural knowledge and processes through problem-solving contexts. However, reasoning is critical in the development of mathematical concepts. Further analysis will reveal if KPTs identifying reasoning are represented more in the later years of school.

Throughout the primary years there is an increasing incidence of KPTs embedded overall in the content descriptions. KPTs identifying understanding and problem solving were noted more frequently than were those identifying fluency and reasoning. KPTs relating to reasoning were identified 42 times from an overall count of 292 KPTs, resulting in only 14% of the total terms extracted.

Similarly, in the middle years (7-9) an increasing focus on exploring content through problem-solving contexts is recognised, as 45% of the total KPTs identified across Years 7–9 related specifically to the proficiency problem solving. Fluency received 25% of the KPTs, understanding 17%, and reasoning 13%.

Overall, problem solving is predominantly represented in this analysis, with 36% of total terms relating to developing this proficiency across years F–9. Understanding and fluency are similarly weighted, with 25% and 24% of the KPTs respectively. However, only 15% of KPTs from Foundation to Year 9 describe actions that relate specifically to students engaging in reasoning in their learning in mathematics. A higher representation of KPTs identifying problem solving could be attributed to the intent described in the ACM rationale "that these proficiencies enable students to respond to familiar and unfamiliar situations by employing mathematical strategies to make informed decisions and solve problems efficiently" (ACARA, 2014, p. 4). It could be inferred in the ACM that reasoning would be built into this process of problem solving and this concurs with Sullivan's assertion that teachers use problem-based approaches to engage the proficiencies (Sullivan, 2012). However, this is not evident in the KPTs extracted. This is a limitation of the analytic process used here and the problem that differentiating the proficiencies individually presents. If reasoning is embedded in problem-solving contexts, this could be made explicit in the description of the proficiencies as an integrated system.

Integrating the proficiencies

In previous work, we proposed a generalised model of patterning (McCluskey, Mitchelmore, & Mulligan, 2013) as a means of describing the abstraction of patterning across differing domains of knowledge. We suggest that patterning moves through a progressive cycle in building generalised understandings within and beyond mathematics in that:

- a sense of familiarity is experienced with known situations,
- similarity experienced across contexts is encoded in the conceptual structure of the pattern
- patterns are activated when similarity is recognised, and
- familiar patterns are accessed more fluently when applied in new contexts.

Thus, we propose that, all four proficiency strands of *understanding, fluency, problem solving,* and *reasoning* in the ACM can naturally work together as an integrated whole, in a cyclic structure, building and deepening generalised patterns of mathematical understanding with a focus "on depth of learning rather than breadth" (Sullivan, 2012, p. 185). For example, as *understanding* is connected across *problem-solving* contexts, similarities about mathematical concepts are recognised, and students develop *reasoning* as they construct generalisations. Over time, *fluency* in recognising and engaging with similar problems is strengthened with an increasing capacity to transfer *understanding* to new contexts. The four proficiencies have a combined role in systematically building patterns of generalised understandings.

In Figure 2 this integrated view of the proficiencies is illustrated in the proposed pedagogical cycle. In the centre the four proficiencies are labelled as inter-related aspects of a greater whole with their inter-connectedness engaging with and deepening conceptual understanding. Surrounding this is a cycle of processes describing how the progressive cycle of the generalised model of patterning could highlight the systematic interaction of the proficiencies working in this dynamic way.

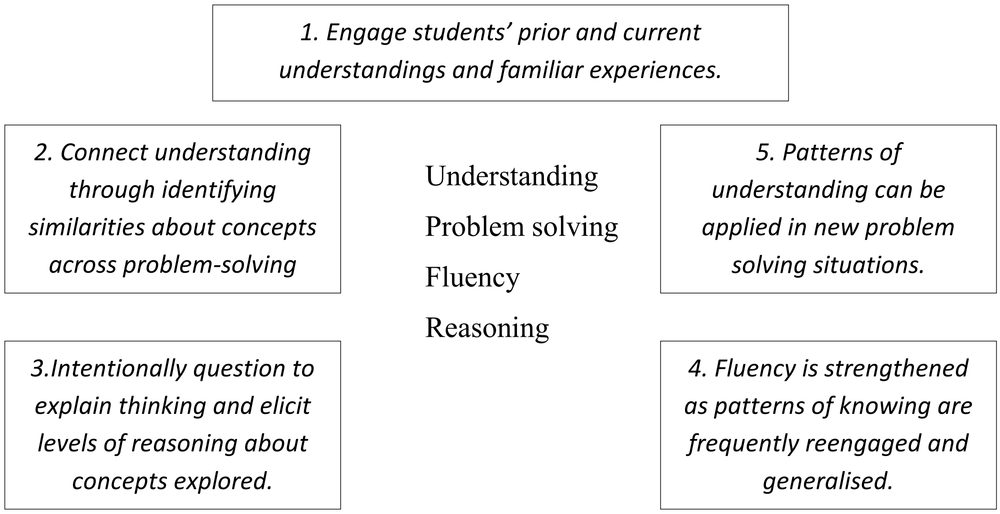

Figure 2. The proposed pedagogical cycle.

This pedagogical structure assumes that there is interaction and relationship of all four proficiencies in connecting with and deepening mathematical thinking, As learning is engaged, understanding is connected and stretched further across problem-solving contexts, similarities about mathematical concepts are recognised, and students reveal levels of reasoning as they construct generalisations about concepts and explore the applicability and use of problem-solving approaches. Over time, patterns of recognising and engaging with similar problems become more fluently recalled and readily transferred in new contexts.

Summary and recommendations

The ACM heralds in an opportunity for educators to focus on the interrelated development of the mathematical proficiencies, a key principle that underpins the curriculum (Sullivan, 2012). The importance of reasoning is clearly articulated in the rationale in the ACM. However, the KPTs that articulate reasoning appear to be noticeably under-represented in the content descriptions from Foundation–Year 9. In contrast, a clear emphasis on students engaging their thinking through problem-solving contexts was identified throughout the F–9 curriculum content descriptions.

Sullivan (2011, 2012) has emphasised pedagogical use of relevant problem-solving contexts and approaches as a means of engaging a greater breadth and depth of proficiencies through teachers' choice of task design and consequent learning experiences for students. Similarly, the heavier weighting of KPTs relating to problem solving, identified through the content analysis, could encourage teachers to adopt practices and design learning experiences that will realise the intention of an integrated view of the proficiencies.

If problem solving can be used to support the development of all the proficiencies then pedagogical cycles, including the one outlined in Figure 2, could in turn support teachers in engaging students' sense of reasoning, fluency and understanding systematically through problem solving contexts. Such pedagogical cycles acknowledge the mathematical proficiencies as being interrelated aspects of cognition that together build conceptual understanding through opportunities for students to:

- engage their current understandings through familiar experiences,
- identify and describe similarities in concepts,
- question and engage in mathematical discourse to communicate their thinking,
- generalise their conceptual understanding about concepts across contexts,
- develop fluent patterns of knowing how to engage with similar types of problems,
- apply these patterns of understanding in new and unfamiliar contexts, and
- explain and justify their reasoning, which in turn would reshape and strengthen conceptual understanding.

Adopting such an integrated view of the role of the mathematical proficiencies has implications for professional learning to ensure teachers' pedagogical content knowledge and promotion of reasoning enables their students' to develop generalised understandings of mathematical concepts.

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APPENDIX E

Content Analysis of Key Proficiency Terms (KPTs)

This appendix contains information about the content analysis of KPTs embedded in the content descriptions in the Australian Curriculum: Mathematics (ACM) from Foundation to Year 9. It contains three sections:

Section 1

Key Proficiency Terms: Foundation- Year 9

Proficiency Strand	Key Proficiency Terms (KPTs) Foundation	Addit (KPTs)	Freq. %
Understanding	apply build connect describe develop identify interpret make represent	*connecting*	*5* *21%*
Fluency	accurately answering appropriately calculate carrying choose choosing develop efficiently find manipulate flexibly recall recalling readily recognise regularly use	*counting continuing comparing*	*5* *21%*
Problem Solving	apply communicate design develop effectively formulate interpret investigate make model plan represent seek solve use verify	*using sorting counting discussing*	*5* *21%*
Reasoning	adapt analysing compare contrast deduce develop evaluating explain explaining generalizing increasingly inferring justify justifying known prove proving reached reasoning something transfer thinking used	*creating explaining*	*9* *37%*
Other verbs identified in content description	*establish including subitise order classify copy decide name answer collect moving*		
Total Key Proficiency Terms			24 100%

Proficiency Strand	Key Proficiency Terms (KPT) Year 1	Addit (KPT)	Freq. %
Understanding	apply build connect describe develop identify interpret make represent	*connecting partitioning*	10 29.5%
Fluency	accurately answering appropriately calculate carrying choose choosing develop efficiently find manipulate flexibly recall recalling readily recognise regularly use	*counting readily locating naming*	10 29.5%
Problem Solving	apply communicate design develop effectively formulate interpret investigate make model plan represent seek solve use verify	*using model giving receiving counting solve discussing*	11 32%
Reasoning	adapt analysing compare contrast deduce develop evaluating explain explaining generalizing increasingly inferring justify justifying known prove proving reached reasoning transfer used	*using created explaining*	3 9%
Other verbs identified in content descriptions	*starting read write order rearranging measure formed tell classify give follow gather*		
Total KPT			34 100%

Proficiency Strand	Key Proficiency Terms (KPT) Year 2	Addit (KPT)	Freq. %
Understanding	apply build connect describe develop identify interpret make represent	*connecting counting partitioning combining flexibly identifying describing*	18 27%
Fluency	accurately answering appropriately calculate carrying choose choosing develop efficiently find manipulate flexibly recall recalling readily recognise regularly use	*counting using iteratively compare comparing describe describing*	21 31%
Problem Solving	apply communicate design develop effectively formulate interpret investigate make model plan represent seek solve use verify	*formulating using matching*	16 24%
Reasoning	adapt analysing compare contrast deduce develop evaluating explain explaining generalizing increasingly inferring justify justifying known prove proving reached reasoning transfer used	*using derive comparing contrasting creating interpreting related*	12 18%
Other verbs identified in content descriptions	*moving order group rearrange facilitate explore written repeated grouping according based tell name determine draw involve gather collect check classify*		
Total KPT			65 100%

Proficiency Strand	Key Proficiency Terms (KPT) Year 3	Addit. (KPT)	Freq. %
Understanding	apply build connect describe develop identify interpret make represent	*connecting partitioning combining flexibly representing using identifying communicate*	23 28%
Fluency	accurately answering appropriately calculate carrying choose choosing develop efficiently find manipulate flexibly recall recalling readily recognise regularly use	*recalling using order compare identifying describing interpreting communicating*	22 27%
Problem Solving	apply communicate design develop effectively formulate interpret investigate make model plan represent seek solve use verify	*formulating modelling planning making using continue*	23 28%
Reasoning	adapt analysing compare contrast deduce develop evaluating explain explaining generalizing increasingly inferring justify justifying known prove proving reached reasoning transfer thinking used	*using generalizing comparing creating interpreting*	14 17%
Other verbs identified in content descriptions	*rearrange regroup involving written complete count performing measure tell show conduct collect organise*		
Total KPT			82 100 %

Proficiency Strand	Key Proficiency Terms (KPT) Year 4	Addit. (KPT)	Freq. %
Understanding	apply build connect describe develop identify interpret make represent	*making partitioning combining flexibly extending using describing*	17 27%
Fluency	accurately answering appropriately calculate carrying choose choosing develop efficiently find manipulate flexibly recall recalling readily recognise regularly use	*recalling communicating using measure creating collecting recording*	13 21%
Problem Solving	apply communicate design develop effectively formulate interpret investigate make model plan represent seek solve use verify	*formulating modelling recording comparing using continue*	26 41%
Reasoning	adapt analysing compare contrast deduce develop evaluating explain explaining generalizing increasingly inferring justify justifying known prove proving reached reasoning transfer thinking used	*deriving comparing communicating*	7 11%
Other verbs identified in content descriptions	*order rearrange regroup assist count locate explore involving find convert splitting contained classify happen select trial including construct illustrating*		
Total KPT			63 100 %

Proficiency Strand	Key Proficiency Terms (KPT) Year 5	Addit. (KPT)	Freq. %
Understanding	apply build connect describe develop identify interpret make represent	*making connections using represent comparing ordering representing describing identifying*	26 34%
Fluency	accurately answering appropriately calculate carrying choose choosing develop efficiently find manipulate flexibly recall recalling readily recognise regularly use	*using check measure*	16 21%
Problem Solving	apply communicate design develop effectively formulate interpret investigate make model plan represent seek solve use verify	*formulating solving using creating*	23 30%
Reasoning	adapt analysing compare contrast deduce develop evaluating explain explaining generalizing increasingly inferring justify justifying known prove proving reached reasoning transfer used	*investigating perform continuing involving interpreting posing*	12 15%
Other verbs identified in content descriptions	*including locate extended resulting convert construct explore estimate list collect including*		
Total KPT			77 100%

Proficiency Strand	Key Proficiency Terms (KPT) Year 6	Addit. (KPT)	Freq. %
Understanding	apply build connect describe develop identify interpret make represent	*describing using representing making*	17 24%
Fluency	accurately answering appropriately calculate carrying choose choosing develop efficiently find manipulate flexibly recall recalling readily recognise regularly use	*representing calculating using converting measuring interpreting*	14 20%
Problem Solving	apply communicate design develop effectively formulate interpret investigate make model plan represent seek solve use verify	*formulating solving using interpreting finding*	30 43%
Reasoning	adapt analysing compare contrast deduce develop evaluating explain explaining generalizing increasingly inferring justify justifying known prove proving reached reasoning transfer thinking used	*performing describing continuing differ*	9 13%
Other verbs identified in content descriptions	*select involving locate result add subtract check multiply divide create write explore construct conduct expected presented*		
Total KPT			70 100 %

Proficiency Strand	Key Proficiency Terms (KPT) Year 7	Addit. (KPT)	Freq. %
Understanding	apply build connect describe develop identify interpret make represent	*describing recognizing plotting identifying formed crossing connecting*	15 19%
Fluency	accurately answering appropriately calculate carrying choose choosing develop efficiently find manipulate flexibly recall recalling readily recognise regularly use	*calculating representing investigating finding*	20 26%
Problem Solving	apply communicate design develop effectively formulate interpret investigate make model plan represent seek solve use verify	*formulating solving using working identifying calculating interpreting*	32 41%
Reasoning	adapt analysing compare contrast deduce develop evaluating explain explaining generalizing increasingly inferring justify justifying known prove proving reached reasoning transfer thinking used	*applying interpreting*	11 14%
Other verbs identified in content descriptions	*written order add subtract locate including multiply divide express introduce create substituting extend establish draw classify construct assign determine involving collected demonstrate*		
Total KPT			78 100%

Proficiency Strand	Key Proficiency Terms (KPT) Year 8	Addit. (KPT)	Freq. %
Understanding	apply build connect describe develop identify interpret make y represent	*describing identifying connecting explaining*	7 13%
Fluency	accurately answering appropriately calculate carrying choose choosing develop efficiently find manipulate flexibly recall recalling readily recognise regularly use	*calculating recognizing factorizing simplifying evaluating including*	16 29%
Problem Solving	apply are communicate design develop effectively formulate interpret investigate make model plan represent seek solve use verify	*formulating modelling using calculate*	29 53%
Reasoning	adapt analysing are compare contrast deduce develop evaluating explain explaining generalizing increasingly inferring justify justifying known mathematically prove proving reached reasoning transfer used	*justifying deriving using finding deduce*	3 5%
Other verbs identified in content descriptions	*establish written involving extend plot verify convert define establish explore drawn*		
Total KPT			55 100%

Proficiency Strand	Key Proficiency Terms (KPT) Year 9	Addit. (KPT)	Freq. %
Understanding	apply build connect describe develop identify interpret make represent	*describing simplifying explaining estimate use*	11 17%
Fluency	accurately answering appropriately calculate carrying choose choosing develop efficiently find manipulate flexibly recall recalling readily recognise regularly use	*applying expressing listing developing involving calculating*	14 22%
Problem Solving	apply communicate design develop effectively formulate interpret investigate make model plan represent seek solve use verify	*formulating modelling involving applying solving collecting investigate*	28 44%
Reasoning	adapt analysing compare contrast deduce develop evaluating explain explaining generalizing increasingly inferring justify justifying known prove proving reached reasoning something transfer used	*following evaluating using clarify developing investigate sketching*	11 17%
Other verbs identified in content descriptions	*list assign determine how construct explore corresponding extend including collect located graphing including*		
Total KPT			64 100%

Section 2

Key Proficiency Terms Across Year Level Clusters

Key Proficiency Terms: F-2: Early Years Cluster					
Year Level	KPT: Proficiency Strands				Total KPT
	Understanding	Fluency	Problem Solving	Reasoning	
Foundation % total KPT	5 21%	5 21%	5 21%	9 37%	24 100%
Year 1 % total KPT	10 29.5%	10 29.5%	11 32%	3 9%	34 100%
Year 2 % total KPT	18 27%	21 31%	16 24%	12 18%	67 100%
F-2 Cluster % total KPT	33 26%	36 29%	32 26%	24 19%	125 100%

Key Proficiency Terms: Year 3-6: Primary Years Cluster					
Year Level	KPT: Proficiency Strands				Total KPT
	Understanding	Fluency	Problem Solving	Reasoning	
Year 3 % total KPT	23 28%	22 27%	23 28%	14 17%	82 100%
Year 4 % total KPT	17 27%	13 21%	26 41%	7 11%	63 100%
Year 5 % total KPT	26 34%	16 21%	23 30%	12 15%	77 100%
Year 6 % total KPT	17 24%	14 20%	30 43%	9 13%	70 100%
Yr 3-6 Cluster % total KPT	83 29%	65 22%	102 35%	42 14%	292 100%

Key Proficiency Terms: 7-9: Middle Years Cluster					
Year Level	KPT: Proficiency Strands				Total KPT
	Understanding	Fluency	Problem Solving	Reasoning	
Year 7 % total KPT	15 19%	20 26%	32 41%	11 14%	78 100%
Year 8 % total KPT	7 13%	16 29%	29 53%	3 5%	55 100%
Year 9 % total KPT	11 17%	14 22%	28 44%	11 17%	64 100%
7-9 Cluster % total KPT	33 17%	50 25%	89 45%	25 13%	197 100%

Section 3

Key Proficiency Terms Across Foundation – Year 9

Key Proficiency Terms: F-2; 3-6; 7-9					
Year Level Cluster	KPT: Proficiency Strands				Total KPT
	Understanding	Fluency	Problem Solving	Reasoning	
F- Year 2 % total KPT	33 26%	36 29%	32 26%	24 19%	125 100%
Year 3-6 % total KPT	83 29%	65 22%	102 35%	42 14%	292 100%
Year 7-9 % total KPT	33 17%	50 25%	89 45%	25 13%	197 100%
7-9 Cluster % total KPT	149 24%	151 25%	223 36%	91 15%	614 100%