Tangent Bundles, Monoidal Theories, and Weil Algebras

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A thesis presented for the degree of
Doctor of Philosophy

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Contents

Abstract

The construction of the tangent bundle of a manifold lies at the very foundations of differential geometry. There are various approaches to characterise the tangent bundle, and two such approaches are through Synthetic Differential Geometry (SDG) and Tangent Structures (in the sense of Cockett-Cruttwell).

Here, we shall give a different perspective, that Tangent Structure can be viewed as a model of an appropriate theory. This theory arises as a certain full subcategory $\mathbf{Weil_1}$ of the category **Weil** of all Weil algebras. The connection between Weil algebras and SDG is well established, but their connection to Tangent Structure is not evident.

In this thesis, we shall exhibit $\mathbf{Weil_1}$ as the universal tangent structure and in fact the axioms of tangent structure actually form a presentation for $\mathbf{Weil_1}$. We shall then continue by describing how this perspective allows us to extend this theory in a canonical manner.

Declaration

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of the requirements for a degree to any other university or institution other then Macquarie University.

I also certify that this thesis is an original piece of research and has been written by me. Any help and assistance that I have received in my research work and the preparation of the thesis itself has been appropriately acknowledged.

In addition, I certify that all information sources and literature used are indicated in the thesis.

Poon Leung

Date

Acknowledgements

It goes without saying that I first and foremost thank my principal supervisor A/Prof. **Stephen Lack**. To say that none of this would have been possible without him would be the understatement of the century. His seemingly infinite patience, knowledge and willingness to help was simply awe inspiring, to say the least. I am truly grateful for all he has done.

I also thank my associate supervisor Dr. **Richard Garner**. His insightful observations and discussion have also helped shape not only this thesis, but also myself as a researcher. I also wish to thank the CoACT group for the nurturing research environment and welcoming atmosphere they have created.

More broadly, I thank the Department of Mathematics at Macquarie University. Through hard times, the department would always band together and endure whatever challenges came our way. With a few exceptions, the department has become like a second family, and it is a genuine honour to have been able to count myself amongst you. My greatest reservation is that my circumstances may force me to leave at a time when I may be needed the most.

On a lighter note, there are the people in my personal life to whom I owe my gratitude. Honourable mentions go (in alphabetical order) to:

Audrey Markowskei: A true friend and pivotal part of the department. The mother of Ph.D.'s. Without you, we would all still be living in isolation from the people across the hall.

Barbi Clendining: For all the fun memories (and some that I don't remember, you know why). There's never a dull moment with you around.

Chloé Braithwaite: To my younger (albeit non-biological) sister. May our best days still be ahead of us, and our worst behind us. Remember, neither of us is ever truly away from the other, we are just not there right now.

Jiehoon Lah: My training partner and brother. Keep the good times rolling.

Leon Beale: Fellow practitioner of the martial arts, comic book nerd and cinematic connoisseur. The department dad.

Paul Slevin: As requested, the official Ph.D. meal tester.

Ron Claassens: Friend, mentor, and role model. Your wisdom beyond your years is matched only by your youthful energy.

Victor Fox: Friend, life coach, and on occasion, the little devil on my shoulder. You let me get away with WAY too much, but at the same time, you tolerate all my nonsense and you also know when to stop me. Don't ever change.

Dishonourable mentions go to:

Branko Nicolić: Your refusal to adhere to local social conventions.

Craig Angus: For always having an inappropriate (albeit hilarious) joke ready to go.

Erik Hermo: You know why.

Kenny Wong: You also know why.

Ramón Abud Alcalá: We seriously need to have interventions for your chilli and avocado addictions.

Rémy Tuyéras: The local wines and cheeses are "never as good as the ones in France".

It goes without saying that there are many more individuals I would like to thank, but I am not able to so within any reasonable amount of space.

Last, but not least, I of course wish to thank my family. They seem to tolerate my presence.

Chapter 1

Introduction

1.1 Tangent bundles

A fundamental notion in differential geometry is that of the *tangent bundle*; given a smooth manifold M, we can construct the manifold TM, which to each point $x \in M$ attaches the vector space T_xM of all tangents to M at x. TM comes equipped with a projection $p_M \colon TM \to M$ sending a tangent vector at a point $x \in M$ to the underlying point x (for more details, see [26]).

Then, for a suitably differentiable map $f \colon M \to N$, differentiation induces a map $Tf \colon TM \to TN$ in such a way that the diagram

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "p_M"'] & TN \arrow[d, "p_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

commutes. This then makes T into an endofunctor on the category **Man** of (smooth) manifolds. The projections $p_M \colon TM \to M$ then become the components of a natural transformation $p \colon T \Rightarrow 1_{\mathbf{Man}}$. We may equivalently say that T is a copointed endofunctor.

Now, just a copointed endofunctor alone is not enough to describe the tangent structure of differentiation; indeed, most copointed endofunctors have nothing to do with this. There are various types of extra structure for which we may ask.

The exact conditions for which we may ask will depend on the exact structure we desire of the tangent bundles. For instance, there will be different conditions for real vector spaces as compared with complex vector spaces. Rosický [29] gave a set of conditions that gave the tangent bundles the structure of abelian groups.

However, we will be mainly interested in the conditions given in Cockett-Cruttwell [8] (which we shall detail in Chapter 3), in which the bundles have the structure of commutative monoids. Here, a technical definition is given, and this will be what we mean when we say *Tangent Structure* (in fact, we will soon refer to this as a *"Tangent Structure corresponding to* $\mathbf{Weil_1}$*"*, for reasons that will become apparent later).

Related work on the ideas of tangent structure may be seen in [5], [6], [7] and [10].

This is but one approach to characterising tangent structure. Another (richer, and more developed) approach is that of synthetic differential geometry.

1.2 Synthetic differential geometry and Weil algebras

Synthetic differential geometry (SDG) constructs the tangent bundle through the use of infinitesimal objects. We begin with the geometric line R with two chosen points 0 and 1, and then equip R with the structure of a commutative ring. We then define the infinitesimal object $D = \{x \in R \mid x^2 = 0\}$, and impose the fundamental axiom:

Axiom For any $g\colon D \to R$, there exists a unique $b \in R$ such that

$$\forall d \in D,\ g(d) = g(0) + d \cdot b\ .$$

Intuitively, this says the function g is part of the straight line l through $(0, g(0))$ with slope b. This then implies the "local linearity" of (smooth) maps between smooth spaces.

There are further infinitesimal objects and axioms, of course, but ultimately, the main focus is on this object D. In particular, for any (appropriate) space X, the tangent space may then be defined as the internal hom or function space $[D, X]$ (working in an appropriate category $\mathcal{M}$; in particular, it is a topos). The projection $p_X\colon [D, X] \to X$ is then the evaluation at 0; explicitly, it sends a map $g\colon D \to X$ to $g(0)$. The tangent functor then is the representable $[D, -]\colon \mathcal{M} \to \mathcal{M}$.

For a certain class of finite dimensional nilpotent algebras, namely the Weil algebras (which we will describe in more detail in Chapter 2), we can define the category **Weil** of such algebras (and appropriate maps).

We then have a functor Spec: $\mathbf{Weil}^{op} \to \mathcal{M}$, and this in turn induces a functor $\mathbf{Weil} \to \mathrm{End}\,(\mathcal{M})$ (where $\mathrm{End}(\mathcal{M})$ is the endofunctor category on $\mathcal{M}$) sending $A \in \mathbf{Weil}$ to the functor

$$[\mathrm{Spec}\,(A)\,, -] : \mathcal{M} \to \mathcal{M}\ .$$

In particular, there is a Weil algebra $W = k[x]/x^2$ with $\mathrm{Spec}(W) = D$. Then $[\mathrm{Spec}\,(W)\,, -]$ gives tangent structure. For more details on SDG, see [15], [16], [20] or [25].

So, the Weil algebras, in particular the fundamental Weil algebra $W = k[x]/x^2$, play a key role in the constructions used in SDG. However, they do not appear explicitly in the definition of Tangent Structures. One of the main goals of this thesis, then, is to detail a connection between Weil algebras and Tangent Structure. In order to do so, we shall first digress with a discussion on the general framework of sketches and theories.

1.3 Sketches and theories

Algebraic structures such as groups or rings are typically defined through their operations $X^n \to X$ and equations. A key insight of Lawvere [22] was that for each such type of structure, there is an associated category $\mathbb{T}$ with finite products, such that a particular instance of that structure is then a finite-product-preserving (f.p.p.) functor from $\mathbb{T}$ into **Set**.

$\mathbb{T}$ is then called a (Lawvere) theory and the f.p.p. functor a model of the theory. These models can also be taken in categories other than **Set**.

For example, there is a category $\mathbb{T}_{\mathbf{Grp}}$ for the theory of groups, for which the category of f.p.p. functors from $\mathbb{T}_{\mathbf{Grp}}$ to **Set** is equivalent to the category **Grp** of groups. We may also just as easily take an arbitrary category $\mathcal{C}$ and refer to f.p.p. functors $F\colon \mathbb{T}_{\mathbf{Grp}} \to \mathcal{C}$ as groups internal to $\mathcal{C}$.

The description of the original structure using operations and equations is a sort of *presentation* for the theory; we may call this a *sketch* of the theory (following Ehresmann [11], who introduced sketches in a more general context).

These are just the Lawvere theories, but there are other variants:

- Finite limit (or essentially algebraic) theories (Gabriel-Ulmer, [27]):

  Instead of asking for $\mathbb{T}$ to have finite products, one instead asks that it has all finite limits. A model of $\mathbb{T}$ is then a functor from $\mathbb{T}$ preserving all finite limits.

- PROPs - symmetric monoidal theories (Mac Lane, [23])

  Instead of asking $\mathbb{T}$ to have finite products, one instead asks that it is a symmetric monoidal category generated by one object. A model (or algebra) of $\mathbb{T}$ is then a symmetric monoidal functor from $\mathbb{T}$.

- Grothendieck topos - "geometric theories" (for more details, see [24]):

  Instead of asking $\mathbb{T}$ to have finite products, one instead asks that it has a geometric embedding
  $$\mathbb{T} \hookrightarrow \mathrm{PSh}(\mathcal{C}).$$
  A model of $\mathbb{T}$ is then a functor preserving colimits and finite limits, i.e. it is a geometric morphism (or equivalently again, it is a functor with a left exact left adjoint).

In fact, Dolan [14] has studied *algebro-geometric theories*. Models of these theories are functors preserving both the symmetric monoidal structure in geometric approaches and colimits. He compares these geometric approaches to their algebraic counterparts (symmetric monoidal structure and algebraic limits) in the context of their underlying theories. For more details on theories, see [2].

The type of theory described in this thesis involves monoidal categories (of Weil algebras) and certain limits (discussed in more detail from Chapter 3 onwards).

1.4 Coalgebras and Weil algebras

Let **Coalg** be the category of cocommutative, coassociative, and counital coalgebras over a given field k. Now, **Coalg** is cocomplete, Cartesian closed, and has products given by $\otimes$. For more details, see [1], [12] or [13]. **Coalg** is an example of the type of theory described by Dolan. It is possible then to consider cocontinuous strong monoidal functors from **Coalg** to $\mathcal{C}$ (some cocomplete monoidal category) as models of **Coalg** in $\mathcal{C}$.

Alternatively, we may also restrict to various subcategories of **Coalg**, and then consider strong monoidal functors that preserve suitable classes of colimits. In particular, $\mathbf{Weil}^{op}$ can be seen as a full subcategory of $\mathbf{Coalg}$;

$$\mathbf{Weil}^{op} \hookrightarrow \mathbf{Alg}_{fd}^{op} \simeq \mathbf{Coalg}_{fd} \hookrightarrow \mathbf{Coalg}\ ,$$

where $\mathbf{Alg}_{fd}$ is the category of finite dimensional commutative, associative, and unital algebras.

Then, in place of models of $\mathbf{Weil}^{op}$ in $\mathcal{C}$, we may equivalently consider models of $\mathbf{Weil}$ in $\mathcal{C}^{op}$ (as a functor $F\colon \mathbf{Weil} \to \mathcal{C}^{op}$), but now using limits rather than colimits. However, we shall in fact consider subcategories of $\mathbf{Weil}$, and discuss in depth a few such candidates.

1.5 This thesis

We begin our discussion in Chapter 2, where we introduce the notion of Weil algebras in Section 2.1 and establish several important facts we will require for this thesis. In Section 2.2, we introduce the notion of graphs and detail some of their properties we will also require in later discussion.

Chapter 3 is then exclusively concerned with a full subcategory $\mathbf{Weil_1}$, built up as tensors of product powers W^n of the Weil algebra W. This is a natural starting point for discussing Tangent Structure and differential geometry, as this captures the most fundamental aspects and structure of the tangent bundle construction.

We then show in Section 3.3 that the objects of $\mathbf{Weil_1}$ can be parametrised by (certain) graphs in a canonical manner. We take this idea further in Section 3.4 by showing that for $k = \mathbb{2}$ (Definition 3.4.1), we can view morphisms in $\mathbf{Weil_1}$ as morphisms in a particular Kleisli category $\mathbf{Gph}_\chi$ (Proposition 3.4.13) for a monad $\chi\colon \mathbf{Gph} \to \mathbf{Gph}$ (Proposition 3.4.12.

We will use this perspective in Section 3.5 to then show that each morphism $f\colon A \to B$ of $\mathbf{Weil_1}$ is canonically *constructible* (Definition 3.5.1) from a set $\{\varepsilon_W, +, \eta_W, l, c\}$ of maps using certain operations. This set of maps and the operations canonically correspond to natural transformations which are required in Tangent Structures.

This will allow us to prove the main result of Chapter 3, namely Theorem 3.6.17. Explicitly, Theorem 3.6.17 states that a Tangent Structure on a category $\mathcal{M}$ (in the sense of [8]) amounts to the same thing as a strong monoidal functor

$$F\colon \mathbf{Weil_1} \to \mathrm{End}(\mathcal{M})$$

preserving certain limits. This can be seen as a universal property of $\mathbf{Weil_1}$ amongst Tangent Structures. This is precisely the *"missing link"* between the Tangent Structures of [8] and Weil algebras as described at the end of Section 1.2.

Chapter 4 is then more exploratory in nature. We introduce the category $\mathbf{Weil}_\infty$, with objects all those of the form

$$k[x_1, \ldots, x_n]/\{\text{some collection } I \text{ of monomials in the } x_i\text{'s}\}\ ,$$

and show in Section 4.1 that each such object can be parametrised by down-sets of $\mathbb{N}^r$ (Definition 4.1.12); moreover, this can be done in a manner consistent with the graph parameterisation of $\mathbf{Weil_1}$. This then allows us to characterise each object of $\mathbf{Weil}_\infty$ as a canonical limit of tensors of W_ns through Theorem 4.1.16, and yields a potential meaning for a "Tangent Structure corresponding to $\mathbf{Weil}_\infty$" via Definition 4.1.18.

This is useful if our set of scalars k takes the form $\mathbb{N}$ or $\mathbb{Z}$. However, we can obtain a far richer result by imposing a mild condition on k, namely that k contains the positive rationals $\mathbb{Q}_{>0}$. Consider the fork

$$k[x]/x^3 \xrightarrow{s} k[y,z]/y^2, z^2 \underset{id}{\overset{c}{\rightrightarrows}} k[y,z]/y^2, z^2$$

where $s(x) = y + z$ and c interchanges the generators y and z. For $k = \mathbb{N}$ or $\mathbb{Z}$, this is cannot be an equaliser due to the absence of $\frac{1}{2}$ in k. However, it will indeed be an equaliser if k contains $\frac{1}{2}$. We discuss this in far more detail in Section 4.2.

As such, by imposing this mild condition on k, we can then express each W_m as an appropriate equaliser (4.2.1). We then use this fact to show that each object of $\mathbf{Weil}_\infty$ can be canonically expressed as a limit of mW's in Theorem 4.2.10. This in turn allows us to show that the obvious inclusion functor

$$J \colon \mathbf{Weil_1} \hookrightarrow \mathbf{Weil}_\infty$$

is in fact codense (Proposition 4.3.2).

This culminates in Theorem 4.3.13, which states that a functor $F_\infty \colon \mathbf{Weil}_\infty \to \mathcal{G}$ (in the sense of Definition 4.3.1) is precisely a right Kan extension of the analogous functor $F \colon \mathbf{Weil_1} \to \mathcal{G}$.

We conclude our discussion in Chapter 5 with a discussion on viable candidates other than $\mathbf{Weil}_\infty$ for notions of Tangent Structure, as well as possibilities for future work. In particular, we discuss a generalised notion of the addition of tangents in 5.1.1, and although this is no longer an internal commutative monoid, the resulting structure is described in Section 5.2.

Chapter 2

Weil algebras and graphs

2.1 Weil algebras

All the algebras discussed in this document will be associative, commutative and unital. Let **Alg** be the category of such algebras as its objects, and with structure-preserving morphisms as the maps.

Further, except for Section 2.1.2, all algebras in this section will be defined over some given field k.

We begin by recalling some definitions for algebras.

Definition 2.1.1. *Let A be a given k-algebra. An* augmentation *for A is an algebra map $\varepsilon_A \colon A \to k$. Explicitly, this means that it preserves multiplication and the diagram*

$$\begin{tikzcd} k \arrow[r, "\eta"] \arrow[dr, "id"'] & A \arrow[d, "\varepsilon"] \\ & k \end{tikzcd}$$

commutes. An algebra equipped with an augmentation is then an augmented algebra*.*

Given augmented algebras B and C, an augmented algebra homomorphism $f \colon B \to C$ is an algebra homomorphism that is compatible with the augmentations, i.e. the diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "\varepsilon_B"'] & C \arrow[d, "\varepsilon_C"] \\ & k \end{tikzcd}$$

commutes.

Definition 2.1.2. *Given an augmented algebra A, the* augmentation ideal *of A is the kernel* $\ker(\varepsilon_A)$ *of the augmentation.*

We can now define a Weil algebra.

Definition 2.1.3. *A* Weil algebra *B is an augmented algebra with a finite dimensional underlying k-vector space, for which all elements of the augmentation ideal are nilpotent (i.e.* $\ker(\varepsilon_B)$ *is a nil ideal).*

Equivalently, we can say that a Weil algebra is simply a finite dimensional local algebra with residue field k.

Remark The equivalence arises from the fact that the augmentation ideal $\ker(\varepsilon_B)$ (for augmentation $\varepsilon\colon B \to k$) is the unique maximal ideal of B.

Definition 2.1.4. *Let R be a ring, and let I be an ideal for this ring.*

I is said to be a nil ideal *if for each $x \in I$, there exists a natural number $n_x \in \mathbb{N}$ such that $x^{n_x} = 0$. Further, I is said to be a* nilpotent ideal *if there exists a natural number $n \in \mathbb{N}$ such that $x^n = 0$ for all $x \in I$.*

Corollary 2.1.5. *For a Weil algebra B, the augmentation ideal $\ker(\varepsilon_B)$ being a nil ideal is equivalent to it being a nilpotent ideal.*

Proof. This is an immediate result of B having a finite dimensional underlying k-vector space. ■

Proposition 2.1.6. *Let B and C be Weil algebras, and $f\colon B \to C$ an algebra morphism. Then, f is an augmented algebra homomorphism.*

Proof. Since f is an algebra morphism, then f must send nilpotent elements to nilpotent elements. As such, f restricts to the augmentation ideals.

Further, this map $f\colon \ker(\varepsilon_B) \to \ker(\varepsilon_C)$ then extends to $\overline{f}\colon B/\ker(\varepsilon_B) \to C/\ker(\varepsilon_C)$.

Noting that the augmentations commute with the units and that $B/\ker(\varepsilon_B) \cong C/\ker(\varepsilon_C) \cong k$, it is then easy to check that $\overline{f}$ is the identity on k. ■

A morphism between Weil algebras B and C is thus simply an (augmented) algebra homomorphism.

From here onwards, we shall simply refer to these augmented algebra homomorphisms as maps.

Definition 2.1.7. *Let* **Weil** *be the category with objects the Weil algebras and morphisms the maps described above.*

Remark The category **Weil** is a full subcategory of **AugAlg** (= **Alg**/k, the category of augmented algebras).

It is convenient to give a Weil algebra B by a presentation

$$B = k[b_1, \ldots, b_m]/Q_B,$$

where we quotient the free algebra $k\,[b_1, \ldots, b_m]$ by the list of terms in Q_B.

Remark This is always possible. Given a Weil algebra B, we can take a (finite) basis $\{b_0 = 1, b_1 \ldots, b_r\}$ for the underlying vector space, then form the free algebra $k[b_1, \ldots, b_r]$ (recall that $b_0 = 1$). Then, there is a unique surjective map

$$!\colon k[b_1, \ldots, b_r] \twoheadrightarrow B\ .$$

A presentation for B is then given by $B = k[b_1, \ldots, b_r]/\ker(!)$.

Example 2.1.8.

- *$k[x]/x^2$ is the Weil algebra with $\{1, x\}$ as a basis for the underlying vector space and equipped with the obvious multiplication, but with x^2 identified with 0.*

- *$k[x]/x^3$ is the Weil algebra with $\{1, x, x^2\}$ as a basis for the underlying vector space and equipped with the obvious multiplication, but with x^3 identified with 0.*

- *$k[x, y]/x^2, y^2$ is the Weil algebra with $\{1, x, y, xy\}$ as a basis for the underlying vector space and equipped with the obvious multiplication, but with x^2 and y^2 each identified with 0.*

We will also note the following:

- We shall always use presentations for which the augmentation $\varepsilon_B \colon B \to k$ sends each generator b_i to 0.

- Recall that for a linear map $h \colon X \to Y$ between vector spaces, it suffices to define how h acts on basis elements of V. Analogously, for an augmented algebra homomorphism $f \colon B \to C$, it suffices to define how f acts on generators (then check that it is suitably compatible with the relations).

- For Weil algebras $A = k[a_1, \ldots, a_m]/Q_A$ and $B = k[b_1, \ldots, b_n]/Q_B$ and a map $f \colon A \to B$, $f(a_i)$ is a polynomial in the generators $b_1, \ldots, b_n$ with no constant term.

We shall now detail some properties of **Weil**.

2.1.1 Properties of Weil

We begin with the following facts:

- The category **Alg** has all limits and colimits. This is true since **Alg** is a category of models of a Lawvere theory, and is thus complete and cocomplete.

- Coproducts in **Alg** are given by $\otimes$. This result appears in many texts and is a well known result. For instance, see Proposition 6.1 of [19].

It is routine to show that these facts are also true for **Alg**$/k$ = **AugAlg**.

Proposition 2.1.9. *The category* **Weil** *has all finite products.*

Proof. Since k is a zero object, it is the nullary product. For arbitrary Weil algebras A and B, begin by taking the pullback

$$\begin{tikzcd} A \times_k B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\varepsilon_B"] \\ A \arrow[r, "\varepsilon_A"'] & k \end{tikzcd}$$

(or equivalently, the product) in **AugAlg**. Since both A and B are finite dimensional and have nilpotent augmentation ideals, then the same is true of $A \times_k B$. Thus it is also a Weil algebra.

$\therefore$ **Weil** has all finite products. ■

Definition 2.1.10. *Let* **NilAugAlg** *be the full subcategory of* **AugAlg** *containing all augmented algebras whose augmentation ideals are nilpotent.*

Proposition 2.1.11. *The category* **NilAugAlg** *has all finite limits.*

Proof. Let $\mathcal{A}$ be a finite category and consider an arbitrary diagram

$$R\colon \mathcal{A} \to \mathbf{NilAugAlg}\ .$$

Since **AugAlg** has all limits, we can form a limiting cone

$$\begin{tikzcd} \mathcal{A} \arrow[r, "R"'] \arrow[rr, bend left, "\Delta X"] & \mathbf{NilAugAlg} \arrow[u, phantom, "\Downarrow\gamma"] \arrow[r, hook] & \mathbf{AugAlg} \end{tikzcd}$$

But since $\mathcal{A}$ is finite, the (finite) set $\{\gamma_a \mid a \in \mathcal{A}\}$ is jointly monic and each Ra is nilpotent, then the augmentation ideal of X is necessarily nilpotent, and so $X \in \mathbf{NilAugAlg}$.

Therefore **NilAugAlg** has all finite limits. ■

Proposition 2.1.12. *The set* $\{W_n \mid n \in \mathbb{N}\}$*, where* $W_n = k[x]/x^{n+1}$ *for all* $n \in \mathbb{N}$*, forms a strong generator for* **NilAugAlg**.

Proof. We want to show that the set of functors

$$\mathbf{NilAugAlg}(W_n, \_)\colon \mathbf{NilAugAlg} \to \mathbf{Set}$$

for all $n \in \mathbb{N}$ jointly reflect isomorphisms.

Let $f\colon A \to B$ be an arbitrary map of **NilAugAlg** for which

$$\mathbf{NilAugAlg}(W_n, f)\colon \mathbf{NilAugAlg}(W_n, A) \to \mathbf{NilAugAlg}(W_n, B)$$

is an isomorphism for all $n \in \mathbb{N}$.

Let α be an element of A with $f(\alpha) = 0$. In particular, α is an element of the augmentation ideal $\ker(\varepsilon_A)$. Since this is nilpotent, then we can define

$$r = \max\{s \in \mathbb{N} \mid \alpha^s \neq 0\}\ .$$

Note also that $\alpha^{r+1} = 0$. As such, we may define a map $g\colon W_r \to A$ given as $g(x) = \alpha$. Further, let $z\colon W_r \to A$ be the zero map (i.e. $z(x) = 0$).

Now, we have $g, z \in \mathbf{NilAugAlg}(W_r, A)$. Moreover, we clearly have $f \circ g = f \circ z$. But since $\mathbf{NilAugAlg}(W_r, f)$ is an isomorphism, then we must have $g = z$, i.e. $\alpha = 0$.

Therefore $\ker(f) = \{0\}$.

Now, let β be an arbitrary element of $\ker(\varepsilon_B)$. Since $\ker(\varepsilon_B)$ is nilpotent, then we can define

$$\rho = \max\{\sigma \in \mathbb{N} \mid \beta^\sigma \neq 0\}\ .$$

Note also that $\beta^{\rho+1} = 0$. As such, we may define a map $\gamma\colon W_\rho \to B$ given as $\gamma(x) = \beta$.

But now we have $\gamma \in \mathbf{NilAugAlg}(W_\rho, B)$, and since $\mathbf{NilAugAlg}(W_\rho, f)$ is an isomorphism, then there is a unique map $h\colon W_\rho \to A$ such that

$$\begin{tikzcd} W_\rho \arrow[r, "h"] \arrow[dr, "\gamma"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

commutes. This shows that f is surjective on elements. But this means that f is an isomorphism in **Vect**.

Thus f is an isomorphism in **NilAugAlg**. Since **NilAugAlg** has all equalisers, then the set $\{W_n \mid n \in \mathbb{N}\}$ forms a strong generator for **NilAugAlg**. ■

In particular, since each $W_n \in \mathbf{Weil}$, this then says that the inclusion $I \colon \mathbf{Weil} \hookrightarrow \mathbf{NilAugAlg}$ preserves and reflects any existing (finite) limits.

Proposition 2.1.13. *For an arbitrary object $A \in \mathbf{Weil}$, the functor $A \otimes \_ \colon \mathbf{Weil} \to \mathbf{Weil}$ preserves finite connected limits.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathbf{Weil} \arrow[r, "A\otimes_"] \arrow[d, hook] & \mathbf{Weil} \arrow[d, hook] \\
\mathbf{NilAugAlg} \arrow[r, "A\otimes"] \arrow[d, hook] & \mathbf{NilAugAlg} \arrow[d, hook] \\
\mathbf{AugAlg} \arrow[r, "A\otimes_"'] & \mathbf{AugAlg} \, .
\end{tikzcd}$$

The inclusions all preserve and reflect (finite) limits, and $A \otimes \_ \colon \mathbf{AugAlg} \to \mathbf{AugAlg}$ preserves connected limits. ■

Proposition 2.1.14. *The category **Weil** has all finite coproducts, and moreover, coproduct is given by $\otimes$.*

Proof. (Finite) coproducts in **AugAlg** are given by $\otimes$, and since **Weil** is a full subcategory of **AugAlg**, it remains only to show that **Weil** is closed under (finite) $\otimes$.

Further, as k is a zero object, then it is the nullary coproduct. Now, since Weil algebras are finite dimensional, then any finite coproduct of them must also be finite dimensional. The nilpotency of the augmentation ideal is immediate. ■

Lemma 2.1.15. *Let A and B be Weil algebras with presentations*

$$A = k[a_1, \ldots, a_m]/Q_A$$
$$B = k[b_1, \ldots, b_n]/Q_B.$$

Then:

- *The product $A \times B$ has presentation*

$$A \times B = k[a_1, \ldots, a_m, b_1, \ldots, b_n]/Q_A \cup Q_B \cup \{a_i b_j | \forall i, j\} \ ;$$

- *The coproduct $A \otimes B$ has presentation*

$$A \otimes B = k[a_1, \ldots, a_m, b_1, \ldots, b_n]/Q_A \cup Q_B \ .$$

Proof. The proof is immediate. ■

Finally, let us define W to be the Weil algebra $k[x]/x^2$ (we will use this notation for this chapter as well as Chapter 3). Then, the nth power and copower of W, denoted W^n and nW respectively, have presentations

$$\begin{aligned} W^n &= k[x_1, \ldots, x_n]/\{x_i x_j | \, \forall i \leqslant j\} \\ nW &= k[x_1, \ldots, x_n]/\{{x_i}^2 | \, \forall i\} \ . \end{aligned}$$

Definition 2.1.16. *For Weil algebras A, B and C, a* foundational pullback *in* **Weil** *is any pullback of the form*

$$\begin{tikzcd} A \otimes (B \times C) \arrow[r, "A\otimes\pi_B"] \arrow[d, "A\otimes\pi_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \otimes B \arrow[d, "A\otimes\varepsilon_B"] \\ A \otimes C \arrow[r, "A\otimes\varepsilon_C"'] & A \ , \end{tikzcd}$$

where π_B and π_C are the product projections.

Remark Foundational pullbacks are indeed pullbacks by a direct application of Proposition 2.1.13 to Proposition 2.1.9, with products regarded as pullbacks over the zero object k.

The operations of $\times$ and $\otimes$, their common unit as well as the foundational pullbacks play a pivotal role in discussing tangent structure (as we shall see in Chapter 3). We are thus interested in categories of the form $(\mathcal{W}, \times, \otimes, I)$ with the properties described.

Clearly, **Weil** itself is a candidate for such a category. However, we may wish to restrict the permissible Weil algebras in the discussion by restricting to (full) subcategories of **Weil**. In particular, we will define the subcategories $\mathbf{Weil_1}$ and k-$\mathbf{Weil}_\infty$ (which we will describe in subsequent chapters) and discuss why they might be more appropriate than **Weil** itself. We will also discuss some other viable candidates in Section 5.1.

2.1.2 If k is not a field

The facts established in Section 2.1.1 assume k is a field. However, we are more interested in $k = \mathbb{N}$, $\mathbb{Z}$ and $\mathbb{2}$ (which we define in Definition 3.4.1). The notion of Weil algebras in this slightly higher level of generality then becomes somewhat delicate.

In general, for a given arbitrary (commutative) ring R, an R-module (underlying some given R-algebra) does not have a notion of dimension. One could of course ask for the R-module to be free, but then quotients and sub-algebras are not necessarily also free, and this can affect the existence of limits and colimits.

However, we avoid these issues by restricting to subcategories of **Weil** (in particular $\mathbf{Weil_1}$ and $\mathbf{Weil}_\infty$) that always consist of objects having a presentation of the form

$$k[x_1, \ldots, x_n]/\{\text{some collection } I \text{ of monomials in the } x_i\text{'s}\} \ ,$$

with some $x_i^{r_i} \in I$ for each $i = 1, \ldots, n$; and we will still refer to these as Weil algebras. In particular, such Weil algebras all have finitely generated and free underlying k-modules.

Restricting to these subcategories in the case of these more general k, Proposition 2.1.13 still holds using the same arguments.

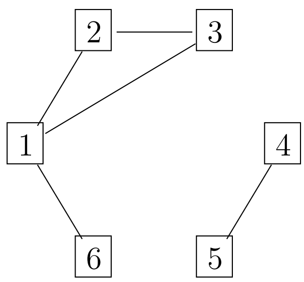
These more general k are needed in order to make our comparison with the definitions of [29] and [8]. For example, if we take $k = \mathbb{Z}$ (as a ring), then we will ultimately return the abelian group bundles of [29]. Of particular interest to us in this discussion, however, are the cases where k is $\mathbb{N}$ (as a rig, for the commutative monoid bundles of [8]) and $\mathbb{2}$ (again, we shall define this in Definition 3.4.1, and it will provide a convenient tool for our calculations).

2.2 Graphs

Here, we define some basic concepts relating to graphs that we will need to use. These are all, for the most part, standard definitions that can be found in any introductory graph theory textbook (for example, see [3]). The notation, however, seems to vary depending on the text.

Definition 2.2.1. *A* graph *G is a pair of sets (V, E), with V a finite set of "vertices" of G, and E a set of unordered pairs of distinct vertices, called the "edges" of G.*

Example 2.2.2. $G = \Big(\{1,2,3,4,5,6\}, \{(1,2),(1,3),(1,6),(2,3),(4,5)\}\Big)$ *is the graph*



Remark In more formal graph theory terms, we are actually describing simple (undirected edges, no loops and at most one edge between any pair of vertices) finite graphs.

Definition 2.2.3. *For graphs $G = (V, E)$ and $G' = (V', E')$, a* graph homomorphism *$h \colon G \to G'$ is a function $h \colon V \to V'$ such that for distinct $u, v \in V$,*

$$(u, v) \in E \Rightarrow (h(u), h(v)) \in E' \text{ or } h(u) = h(v).$$

Definition 2.2.4. *Let* **Gph** *be the category of graphs and graph homomorphisms.*

Definition 2.2.5. *For a non-empty graph $G = (V, E)$, we will say G is* connected *if for any two distinct vertices u and v, there exist vertices $v_1, \ldots, v_s \in V$ with $(v_i, v_{i+1}) \in E$ for each i, and $v_1 = u$, $v_s = v$.*

Definition 2.2.6. *Given a graph $G = (V, E)$, the* complement *of G is the graph $G^c = (V, E^c)$, where for any two distinct $u, v \in V$,*

$$(u, v) \in E \Leftrightarrow (u, v) \notin E^c.$$

We now define two important binary operations on graphs. Let graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be given.

Definition 2.2.7. *The* disjoint union *of G_1 and G_2, denoted as $G_1 \otimes G_2$, is the graph*

$$G_1 \otimes G_2 = (V_1 \sqcup V_2, E_1 \sqcup E_2) \ ;$$

where $\sqcup$ denotes disjoint union of sets.

Or, put simply, it is the graph given by simply placing G_1 adjacent to G_2 without adding or removing any edges.

Definition 2.2.8. *The* graph join *of G_1 and G_2, denoted $G_1 \times G_2$, is the graph*

$$G_1 \times G_2 = (V_1 \sqcup V_2, \widetilde{E})$$

where $\widetilde{E} = E_1 \sqcup E_2 \sqcup (V_1 \times V_2)$.

Or, put simply, it is the graph given by taking $G_1 \otimes G_2$, then adding in an edge from each vertex in G_1 to each vertex in G_2. Equivalently, it can be defined as

$$G_1 \times G_2 = (G_1^c \otimes G_2^c)^c$$

Remark The notation $G_1 \times G_2$ is in no way intended to suggest the product of G_1 and G_2 in the category **Gph** of graphs.

Remark The use of $\otimes$ and $\times$ to denote the operations of disjoint union and graph join respectively do not coincide with the notation used in graph theory. Graph union is often denoted as $G_1 \cup G_2$ or $G_1 + G_2$. Further, the graph join, sometimes called "graph sum", is denoted $G_1 \vee G_2$, (to add to the confusion, some texts denote this as $G_1 + G_2$; moreover the meaning of "graph sum" can also vary depending on the literature). However, the notation $\{\otimes, \times\}$ was chosen in place of $\{\cup, \vee\}$ for consistency with the notation for Weil algebras, as we shall see in Section 3.3.

Definition 2.2.9. *A graph G is said to be* complete *if every pair (u, v) of distinct vertices has an edge joining them (i.e. $(u, v) \in E$ for all $u \neq v$).*

Equivalently, G is the graph join of an appropriate number of instances of the single point graph.

Definition 2.2.10. *A graph G is said to be* discrete *if the edge set E is empty.*

Equivalently, G is the disjoint union of an appropriate number of instances of the single point graph.

Equivalently again, G is discrete iff its complement G^c is complete.

Remark In graph theory literature, sometimes discrete graphs are also called "edgeless graphs" or "null graphs".

Definition 2.2.11. *We will give an iterative definition of* cograph *(complement-reducible graph) as follows:*

- *The empty graph (empty vertex set) and one point graph are cographs.*
- *If G_1 and G_2 are cographs, so are $G_1 \times G_2$ and $G_1 \otimes G_2$.*

Remark Cographs are not in any way a dual notion to graphs. The prefix “co-” is an abbreviation of “complement reducible”.

In fact, cographs have been studied extensively by graph theorists, and there are various equivalent characterisations of them (for instance, see [9]).

Remark For example, given a graph G, the following are equivalent:

1) G is a cograph;

2) G does not contain the graph P_4

$$\boxed{1} \;\text{———}\; \boxed{2} \;\text{———}\; \boxed{3} \;\text{———}\; \boxed{4}$$

(the path graph with four vertices) as a full subgraph.

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Chapter 3

The category $\mathbf{Weil}_1$

3.1 Tangent Structure

We begin with a formal definition of Tangent Structure. Tangent Structure was defined by Rosický [29] using (internal) bundles of abelian groups, but we will be following the definition of Cockett-Cruttwell [8] using (internal) bundles of commutative monoids. More explicitly, it requires that the tangent bundle TM sitting over a smooth manifold (or more generally, an object of the category $\mathcal{M}$ in question) M is a commutative monoid. We refer to this as an *additive bundle*.

3.1.1 Internal commutative monoid

Often, commutative monoids are considered in categories with all finite products, but we shall not be assuming this. For the definition below, we will only assume that $\mathcal{C}$ has finite powers of the object C in question (in particular, $\mathcal{C}$ has a terminal object t). We then have the obvious associativity isomorphism

$$\alpha\colon C \times (C \times C) \to (C \times C) \times C$$

and isomorphisms given by the projections $\pi\colon t \times C \to C$ and $\pi'\colon C \times t \to C$.

Definition 3.1.1. *Given a category $\mathcal{C}$, a* commutative monoid *in $\mathcal{C}$ consists of*

- *An object C such that finite powers of C exist;*
- *A pair of maps $\eta\colon t \to C$ and $\mu\colon C \times C \to C$ such that the following diagrams commute*

$$\begin{tikzcd}
C \times (C \times C) \arrow[r, "\alpha"] \arrow[d, "1\times\mu"'] & (C \times C) \times C \arrow[r, "\mu\times 1"] & C \times C \arrow[d, "\mu"] \\
C \times C \arrow[rr, "\mu"'] & & C
\end{tikzcd}
\qquad
\begin{tikzcd}
t \times C \arrow[r, "\eta\times 1"] \arrow[dr, "\cong"'] & C \times C \arrow[d, "\mu"] & C \times t \arrow[l, "1\times\eta"'] \arrow[dl, "\cong"] \\
& C &
\end{tikzcd}$$

and μ agrees with the symmetry map

$$sym\colon C \times C \to C \times C\ ,$$

so that the diagram

$$\begin{tikzcd}
C \times C \arrow[r, "sym"] \arrow[dr, "\mu"'] & C \times C \arrow[d, "\mu"] \\
& C
\end{tikzcd}$$

also commutes.

3.1.2 Additive bundles

Definition 3.1.2. *If A is an object in a category $\mathcal{M}$, then an* additive bundle over *A is a commutative monoid in the slice category $\mathcal{M}/A$. Explicitly, this consists of*

- *a map $p\colon X \to A$ such that pullback powers of p exist, the n^{th} pullback power denoted by $X^{(n)}$ and projections $\pi_i\colon X^{(n)} \to X$ for $i \in \{1, ..., n\}$;*
- *maps $+\colon X^{(2)} \to X$ and $\eta\colon A \to X$ with $p \circ + = p \circ \pi_1 = p \circ \pi_2$ and $p \circ \eta = id$ which are associative, commutative, and unital.*

Remark We will note here that the notation used in [8] for the n^{th} pullback power is X_n.

Definition 3.1.3. *Suppose $p\colon X \to A$ and $q\colon Y \to B$ are additive bundles. An* additive bundle morphism *is a pair of maps $f\colon X \to Y$ and $g\colon A \to B$ such that the following diagrams commute.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ A \arrow[r, "g"'] & B \end{tikzcd} \qquad \begin{tikzcd} X^{(2)} \arrow[r, "f\times f"] \arrow[d, "+"'] & Y^{(2)} \arrow[d, "+"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\eta"'] & B \arrow[d, "\eta'"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

3.1.3 Tangent Structure (in the sense of Cockett and Cruttwell [8])

Definition 3.1.4. *Given a category $\mathcal{M}$, a* Tangent Structure $\mathbb{T} = (T, p, \eta, +, l, c)$ *on $\mathcal{M}$ consists of*

- *(**tangent functor**) a functor $T\colon \mathcal{M} \to \mathcal{M}$ and a natural transformation $p\colon T \to 1_{\mathcal{M}}$ such that pullback powers $T^{(m)}$ of p exist and the composites T^n of T preserve these pullbacks for all $n \in \mathbb{N}$;*
- *(**tangent bundle**) natural transformations $+\colon T^{(2)} \Rightarrow T$ and $\eta\colon 1_{\mathcal{M}} \Rightarrow T$ making $p\colon T \to 1_{\mathcal{M}}$ into an additive bundle;*
- *(**vertical lift**) a natural transformation $l\colon T \Rightarrow T^2$ such that*

$$(l, \eta)\colon (p, +, \eta) \to (Tp, T+, T\eta)$$

  is an additive bundle morphism;
- *(**canonical flip**) a natural transformation $c\colon T^2 \to T^2$ such that*

$$(c, id_T)\colon \big(Tp, T+, T\eta\big) \to \big(pT, +T, \eta T\big)$$

  is an additive bundle morphism;

where the natural transformations l and c satisfy

- *(**coherence of** l **and** c) $c^2 = id$, $c \circ l = l$, and the following diagrams commute*

$$\begin{tikzcd} T \arrow[r, "l"] \arrow[d, "l"'] & T^2 \arrow[d, "Tl"] \\ T^2 \arrow[r, "lT"'] & T^3 \end{tikzcd} \quad \begin{tikzcd} T^3 \arrow[r, "Tc"] \arrow[d, "cT"'] & T^3 \arrow[r, "cT"] & T^3 \arrow[d, "Tc"] \\ T^3 \arrow[r, "Tc"'] & T^3 \arrow[r, "cT"'] & T^3 \end{tikzcd} \quad \begin{tikzcd} T^2 \arrow[r, "lT"] \arrow[d, "c"'] & T^3 \arrow[r, "Tc"] & T^3 \arrow[d, "cT"] \\ T^2 \arrow[rr, "Tl"'] & & T^3 \end{tikzcd} ;$$

- *(**universality of vertical lift**) the following is an equaliser diagram*

$$T^{(2)} \xrightarrow{(T+)\circ(l \times_T \eta T)} T^2 \underset{\eta\circ p\circ Tp}{\overset{Tp}{\rightrightarrows}} T \ ,$$

where $(T+) \circ (l \times_T \eta T)$ is the composite

$$\begin{tikzcd} & T \arrow[r, "l"] & T^2 & \\ T^{(2)} \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[rr, dashed] & & TT^{(2)} \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[r, "T+"] & T^2 \\ & T \arrow[r, "\eta T"'] & T^2 & \end{tikzcd} ;$$

the dotted arrow induced by the universal property of $TT^{(2)}$ as a pullback.

We may then refer to the pair $(\mathcal{M}, \mathbb{T})$ as a *tangent category.*

3.2 Tangent Structure and Weil algebras

The tangent functor T of Definition 3.1.4 is closely related to the Weil algebra $W = \mathbb{N}[x]/x^2$. In synthetic differential geometry (i.e. in the sense of [15]), T is the representable functor $(\_)^D$, where $D = \mathrm{Spec}(W)$ (although here, k would be taken to be $\mathbb{R}$).

Here, we will begin to describe a different relationship between **Weil** and Tangent Structure. Regard coproduct $\otimes$ as a monoidal operation on **Weil** (with unit k). For now, we shall take k to be an arbitrary field.

Proposition 3.2.1. *The (endo)functor*

$$W \otimes \_ \colon \mathbf{Weil} \to \mathbf{Weil}$$

can be used to define a Tangent Structure on **Weil***.*

Proof. With $T = W \otimes \_$, we first give the natural transformations required in order to have a tangent structure on **Weil**. The names for the morphisms used below will be deliberately chosen to coincide with those of Tangent Structure.

	Natural transformation	Explanation
Projection	$\varepsilon_W \otimes \_ \colon T \Rightarrow 1_{\mathbf{Weil}}$	$\varepsilon_W \colon W \to k$ is the augmentation for W
Addition	$+ \otimes \_ \colon T^{(2)} \Rightarrow T$	$T^{(2)}$ is the functor $W^2 \otimes \_$, $+\colon W^2 \to W$; $x_1, x_2 \mapsto x$
Unit	$\eta_W \otimes \_ \colon 1_{\mathbf{Weil}} \Rightarrow T$	$\eta_W \colon k \to W$ is the (multiplicative) unit for W
Vertical lift	$l \otimes \_ \colon T \Rightarrow T^2$	$T^2 = T \circ T$ is the functor $2W \otimes \_$ $l\colon W \to 2W$; $x \mapsto x_1x_2$
Canonical flip	$c \otimes \_ \colon T^2 \Rightarrow T^2$	$c\colon 2W \to 2W$; $x_i \mapsto x_{3-i}$, for $i = 1, 2$

With these choices of natural transformations as well as the facts established in Section 2.1.1 (so that $(W \otimes \_)^n = (nW \otimes \_)$ preserves the required pullbacks), it is a very routine exercise to verify that this does in fact define a Tangent Structure on **Weil**.

We will also note that the diagram

$$W^2 \xrightarrow{(W\otimes +)\circ(l\times_W(\eta_W\otimes W))} 2W \underset{\eta_W\circ(\varepsilon_W\otimes\varepsilon_W)}{\overset{W\otimes\varepsilon_W}{\rightrightarrows}} W$$

is an equaliser in **Weil** (the universality of vertical lift equaliser in Definition 3.1.4).

Note that the map $(W \otimes +) \circ (l \times_W (\eta_W \otimes W))$, which we will denote as v, is given as

$$\begin{aligned} k[x_1, x_2]/x_1^2, x_2^2, x_1x_2 &\to k[y_1, y_2]/y_1^2, y_2^2 \\ x_1 &\mapsto y_1y_2 \\ x_2 &\mapsto y_2 \ . \end{aligned}$$

The map $W \otimes \varepsilon_W : k[y_1, y_2]/y_1^2, y_2^2 \to k[z]/z^2$ sends y_1 to z and y_2 to 0, and $\eta_W \circ (\varepsilon_W \otimes \varepsilon_W) : k[y_1, y_2]/y_1^2, y_2^2 \to k[z]/z^2$ sends both y_1 and y_2 to 0. ■

This Tangent Structure on **Weil** arises from the object W, its (finite product) powers W^n and tensors of these. With this in mind, it makes sense to give the following definition:

Definition 3.2.2. *Let k-$\mathbf{Weil_1}$ be the category consisting of:*

- *Objects: For each $n \in \mathbb{N}$,*

  $$W^n = k[x_1, \ldots, x_n]/\left\{x_ix_j \mid \forall\ 1 \le i \le j \le n\right\}$$

  is an object of k-$\mathbf{Weil_1}$. Further, if A and B are objects of k-$\mathbf{Weil_1}$, then so is $A \otimes B$.

- *Morphisms: All algebra homomorphisms compatible with units and augmentations.*

Remark This definition is valid for $k = \mathbb{N}$, $\mathbb{Z}$ or $\mathbb{2}$ as well.

Recall that as a consequence of Lemma 2.1.15, the (finite product) power W^n would have presentation

$$k[x_1, \ldots, x_n]/\{x_ix_j|\forall i \le j\}\ ,$$

and that the presentation for a tensor $A \otimes B$ took a particular form. As such, a tensor $\bigotimes_{i=1}^{m} W^{n_i}$ of powers of W would have a certain presentation that we will not try to describe explicitly right now (we shall see this in Section 3.3).

In general, however, such objects will have a presentation

$$k[x_1, \ldots, x_n]/\{x_i x_j | \forall x_i \sim x_j\}$$

for some symmetric, reflexive relation $\sim$ (although not all symmetric, reflexive relations will yield an object of k-$\mathbf{Weil_1}$). Since we will always require $x_i^2 = 0$ in these presentations, there is no loss of information if we omit the corresponding relation $x_i \sim x_i$ and take $\sim$ to merely be symmetric (and in fact, anti-reflexive).

However, such symmetric relations can be thought of as graphs (as defined in Section 2.2).

Remark We treat such relations as anti-reflexive so that the corresponding graph will not have loops.

3.3 Graphs and Weil algebras

We first define a functor from the category **Gph** of graphs to **Weil** as:

Definition 3.3.1. *The functor*

$$\kappa \colon \mathbf{Gph} \to \mathbf{Weil}$$

is defined as follows:

- *On objects: For a graph $G = (V, E)$, $\kappa(G)$ is the Weil algebra $k[v_1, \ldots, v_m]/Q_E$, where $V = \{v_1, \ldots, v_m\}$, $v_i^2 \in Q_E$ for all i and for $i \neq j$, $v_i v_j \in Q_E \Leftrightarrow (v_i, v_j) \in E$.*
- *On morphisms: For a graph homomorphism $h \colon G \to G'$, $\kappa h \colon \kappa(G) \to \kappa(G')$ is given as*

  $$(\kappa h)(v_i) = h(v_i) \text{ for all } i \ ;$$

  where we use the underlying function $h \colon V \to V'$ on the vertex sets.

Remark We shall leave as an exercise to the reader to verify that κh is indeed a valid morphism of Weil algebras, and that this definition of κ is functorial, i.e. that it preserves identities and composition.

Conversely, we have the following:

Definition 3.3.2. *Given a Weil algebra X with a (specified) presentation of the form*

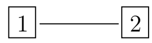
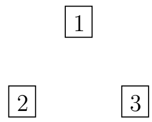
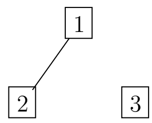
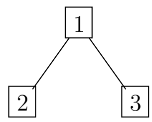
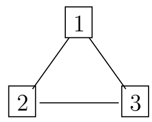
$$X = k[x_1, \ldots, x_n]/\{x_i x_j \mid \forall\ x_1 \sim x_j\}\ ,$$

let Γ_X denote the graph induced by $\sim$; namely the graph with vertices the generators $x_1, \ldots, x_n$ and an edge between x_i and x_j (for $i \neq j$) whenever $x_i \sim x_j$.

Remark There does not appear to be any clear way to define Γ as a functor from **Weil** (or any subcategory of **Weil**) to **Gph**. For instance, there is no canonical way to define Γl, for the map $l\colon W \to 2W$ as described in Section 3.2.

Remark With this convention, for a Weil algebra X with presentation as described above, it is easy to see that $\kappa(\Gamma_X) = X$, and for a graph G, we have $\Gamma_{\kappa(G)} = G$.

For example, we have

Weil algebra	Presentation	Graph
k	$k[\,]$	
W	$k[x]/x^2$	
$2W$	$k[x_1, x_2]/x_1^2, x_2^2$	
W^2	$k[x_1, x_2]/x_1^2, x_2^2, x_1x_2$	
$3W$	$k[x_1, x_2, x_3]/x_1^2, x_2^2, x_3^2$	
$W^2 \otimes W$	$k[x_1, x_2, x_3]/x_1^2, x_2^2, x_3^2, x_1x_2$	
$W \times 2W$	$k[x_1, x_2, x_3]/x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3$	
W^3	$k[x_1, x_2, x_3]/x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3$	

Remark The object $W \times 2W$ is not contained in the category k-$\mathbf{Weil_1}$, but we shall include it in the table anyway. Note that this is the path graph P_3 of length three.

Proposition 3.3.3. *For graphs G and G', we have:*

1) $\kappa(G) \otimes \kappa(G) = \kappa(G \otimes G')$*;*

1) $\kappa(G) \times \kappa(G) = \kappa(G \times G')$*.*

Proof. This is a direct consequence of Lemma 2.1.15 and Definitions 2.2.7 and 2.2.8. ■

To require precisely those Weil algebras given as the closure of $\{W^n\}_{n\in\mathbb{N}}$ under $\otimes$ is thus to ask for those that correspond to disjoint unions of complete graphs.

Definition 3.3.4. *A graph G is said to be* piecewise complete *(or a 'p.c. graph') if it can be expressed as a disjoint union of complete graphs.*

Remark The p.c. graphs are a subset of the cographs (as defined in Definition 2.2.11).

Remark Although we are interested in p.c. graphs in this chapter, we shall often speak in greater generality by discussing cographs.

As an aside, we also have the following:

Proposition 3.3.5. *For a graph $G = (V, E)$, the following are equivalent:*

1) *G is a p.c. graph.*

2) *G does not contain the path graph P_3 as a full subgraph.*

Proof. Let $G = (V, E)$ be a given graph.

- 1) $\Rightarrow$ 2): Suppose G is a p.c. graph. Let $a, b, c \in V$ be three distinct vertices of G with $(a, b), (b, c) \in E$.

  Since G is a disjoint union of complete graphs, then a, b and c must all belong to the same complete component of G, and so $(a, c) \in E$. Hence G cannot contain P_3 as a full subgraph.

- 2) $\Rightarrow$ 1): Suppose G does not contain the path graph P_3 as a full subgraph. Without loss of generality, let G be connected. We wish to show that G is complete.

  Let a and b be two distinct vertices of G. Since G is connected, there exist distinct vertices $v_1, \ldots, v_s \in V$ with $(v_i, v_{i+1}) \in E$ for each i, with $v_1 = a$ and $v_s = b$ (Definition 2.2.5).

  Now, since we have (v_1, v_2) and $(v_2, v_3) \in E$, and P_3 is not a full subgraph, then this says we must also have $(v_1, v_3) \in E$. Repeating this argument iteratively, we then conclude that $(a, b) \in E$, and so G is complete.

  Thus G is a p.c. graph.

■

Now that we have a description for the objects of our subcategory k-$\mathbf{Weil_1}$, we may now revisit the idea mentioned in Section 2.1.2; namely that k need not be a field. k being a field resulted in Weil algebras having underlying (finite dimensional) k-vector spaces.

Of course, different choices of k would lead to different structures for the tangent bundles; namely the structure of k-modules. For instance, as we noted in 2.1.2, the bundles of abelian groups in [29] would require k to be $\mathbb{Z}$, whereas the bundles of commutative monoids in [8] would require k to be $\mathbb{N}$ (so that the Weil algebras have finitely generated and free underlying $\mathbb{Z}$-modules and $\mathbb{N}$-modules respectively).

We now introduce a way to describe the morphisms.

3.4 Maps and graphs

Recall from Section 3.2 that there was a canonical Tangent Structure on **Weil** with tangent functor $W \otimes \_$. The natural transformations of this tangent structure arose from the maps $\{\varepsilon_W, +, \eta, l, c\}$ of **Weil** (in particular, these maps also exist in k-$\mathbf{Weil_1}$). These maps, along with W, will play a crucial role in our discussion.

For the remainder of this chapter, we shall predominantly focus our attention to the subcategory k-$\mathbf{Weil_1}$.

It is convenient for now to take k to be $\mathbb{2}$:

Definition 3.4.1. *Let* $\mathbb{2}$ *be the rig* $\{0, 1\}$*, with the usual multiplication, and addition given by* $\max$*; in particular* $1 + 1 = 1$.

We shall begin by showing that using the maps $\{\varepsilon_W, +, \eta, l, c\}$, composition, $\otimes$ and the universal property of foundational pullbacks (as given in Definition 2.1.16), we can construct (in some appropriate sense) any map of $\mathbb{2}$-$\mathbf{Weil_1}$.

Remark We will not need the universal property of $\otimes$ (the coproduct), but rather we shall consider $\mathbb{2}$-$\mathbf{Weil_1}$ as a monoidal category with respect to $\otimes$ (with k as the unit).

We will need some extra constructions relating to graphs before we begin. Let $G = (V, E)$ be a given graph.

Definition 3.4.2. *A* clique U *of* G *is a (possibly empty) subset of* V *for which any two distinct vertices in* U *have an edge between them (or equivalently, the full subgraph of* G *induced by* U *is complete).*

Definition 3.4.3. *Conversely, an* independent set U *of* G *is a (possibly empty) subset of* V *for which no two distinct vertices in* U *have an edge between them (or equivalently, the full subgraph of* G *induced by* U *is discrete).*

Remark A subset $U \subset V$ is an independent set of G iff it is a clique of G^c.

We can actually use these notions of cliques and independent sets to form new graphs from existing ones.

Definition 3.4.4. *Given a graph* $G = (V, E)$*, define* $\mathrm{Ind}(G)$ *to be the graph given by:*

- *Vertices: the independent sets of* G*;*
- *Edges: given any two distinct independent sets* U_1 *and* U_2 *of* G*, there is an edge between them in* $\mathrm{Ind}(G)$ *whenever there exist* $x \in U_1$ *and* $y \in U_2$ *such that either there is an edge between* x *and* y *in* G *or* $x = y$ *(so that* $U_1 \cap U_2 \neq \phi$*).*

Definition 3.4.5. *Given a graph* $G = (V, E)$*, define* $\mathrm{Cl}(G)$ *to be the graph given by:*

- *Vertices: the cliques of* G*;*
- *Edges: given any two distinct cliques* U_1 *and* U_2 *of* G*, there is an edge between them in* $\mathrm{Cl}(G)$ *whenever their union* $U_1 \cup U_2$ *is also a clique of* G *(note that there is no requirement for* U_1 *and* U_2 *to be disjoint).*

Remark In defining the graph $\mathrm{Cl}(G)$, there is often the additional requirement that cliques U_1 and U_2 are disjoint for there to be an edge between them. If that were the case, then we would have

$$\mathrm{Ind}(G) = (\mathrm{Cl}(G^c))^c$$

Remark As defined here, $\mathrm{Cl}\colon \mathbf{Gph} \to \mathbf{Gph}$ is functorial and moreover can be made into a monad. We shall not be needing this fact, so we shall not prove it.

Definition 3.4.6. *Given a graph $G = (V, E)$, define $\mathrm{Ind}_+(G)$ to be the full subgraph of $\mathrm{Ind}(G)$ where the vertices are the non-empty independent sets of G.*

3.4.1 Expressing maps using graphs

Recall that to define a map between (Weil) algebras, it suffices to define how the map acts on each of the generators. So, let a map $f\colon A \to B$ in $\mathbb{2}$-$\mathbf{Weil_1}$ be given, where A and B have presentations

$$A = \mathbb{2}[a_1, ..., a_m]/Q_A \text{ and } B = \mathbb{2}[b_1, ..., b_n]/Q_B.$$

Then, for each generator a_i of A, we can express $f(a_i)$ (uniquely) as a sum

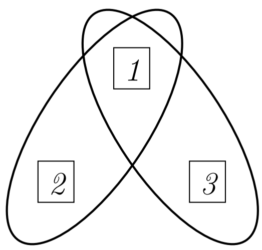
$$f(a_i) = \sum_{\underline{b} \in B} \alpha^{(i)}_{\underline{b}} \underline{b} \ ;$$

the sum being across all non-zero monomials $\underline{b}$ of B in the generators $\{b_1, \dots, b_n\}$, and $\alpha^{(i)}_{\underline{b}} \in \mathbb{2}$ is a constant (taking value 0 or 1).

In fact, since we are using a presentation for which $\varepsilon_A(a_i) = 0$ for all i, the sum can in fact skip the trivial monomial (i.e. the constant).

We may also try to express f pictorially.

Example 3.4.7. *Consider the map $f\colon W \to 3W$ given by $x \mapsto y_1y_2 + y_1y_3$. We can represent this pictorially as*



where each term of $f(x)$ is represented by circling the vertices that generate the term (so the term y_1y_2 is represented by the ellipse encompassing the vertices 1 and 2). Note in particular that $\{1, 2\}$ and $\{1, 3\}$ are independent sets of Γ_{3W}.

We also note that we label the vertices 1, 2 and 3 instead of y_1, y_2 and y_3 for convenience.

This suggests that we can express the map f using the language of graphs.

3.4.2 Expressing maps using cliques and independent sets

For this subsection, we shall not necessarily restrict the discussion to the p.c. graphs, but rather implicitly refer to all graphs.

Proposition 3.4.8. *For the Weil algebra $B = \mathbb{2}[b_1, ..., b_n]/Q_B$ with corresponding (p.c.) graph Γ_B, the set of non-zero monomials $\underline{b}$ of B in the generators $\{b_1, \ldots, b_n\}$ are (canonically) in bijection with the independent sets of Γ_B.*

Proof. Since each generator b_i of B squares to zero, then each non-zero monomial $\underline{b}$ can be expressed (uniquely) as

$$\prod_{i \in I} b_i \ ;$$

for some appropriate subset $I \subseteq \{b_1, \ldots, b_n\}$. Since $\underline{b} \neq 0$, then for distinct $b_i, b_j \in I$, we must have $b_i b_j \neq 0$, i.e. $b_i b_j \notin Q_B$. This equivalently means there is no edge between the vertices b_i and b_j in Γ_B. I is thus a (possibly empty) independent set of Γ_B.

The reverse direction for the bijection is then obvious. ∎

Remark Using Proposition 3.4.8, we can equivalently say that to give a non-constant monomial $\underline{b}$ is to give a vertex of $\mathrm{Ind}_+(\Gamma_B)$.

As such, we may now express $f(a_i)$ (uniquely) as

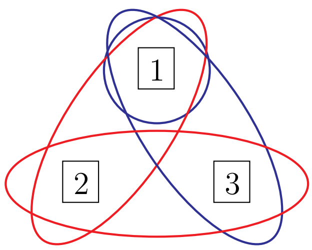
$$f(a_i) = \sum_{U \in \mathrm{Ind}_+(\Gamma_B)} \alpha_U^{(i)} b_U$$

over the non-empty independent sets U of Γ_B

Notation For a graph G, let a *circle* U of G simply mean an independent set of G, but regarded pictorially as some shape encompassing the relevant vertices.

We may use this idea to express $f\colon A \to B$ pictorially. Start by taking the generator a_1. Then take the graph Γ_B for B, and for each U with $\alpha_U^1 = 1$, we add onto Γ_B a circle corresponding to U, and we do this for all U with $\alpha_U^1 = 1$. Then repeat this process for each generator a_i, but (say) using a different colour for each different generator.

Example 3.4.9. *The map $f\colon 2W \to 3W$ given by $x_1 \mapsto y_1 y_2 + y_2 y_3$ and $x_2 \mapsto y_1 + y_1 y_3$ may be represented as*



where $f(x_1)$ is represented in red and $f(x_2)$ is represented in blue.

Notation For a map $f\colon A \to B$, let $\{U\}_f$ denote the graph Γ_B together with a set $\{(U,i) \mid \alpha_U^i = 1\}$, all of this regarded pictorially as a set of coloured circles on Γ_B.

Remark For a map $f\colon W \to B$, we will simply refer to a circle (U,i) of $\{U\}_f$ as U (i.e. we omit the index i).

So, to any map f we can associate a graph with coloured circles. However, not all sets of circles on the graph Γ_B are permissible.

In order to investigate this idea further, we begin with the following:

Proposition 3.4.10. *Consider maps of the form $f\colon W \to B$. To give such an f is to give a clique of* $\mathrm{Ind}_+(\Gamma_B)$.

Proof. Let x be the generator of W. Recall from Proposition 3.4.8 that each summand (monomial) of $f(x)$ is a (non-empty) independent set of Γ_B, i.e. a vertex of $\mathrm{Ind}_+(\Gamma_B)$. We may thus regard $f(x)$ as some subset X_f of the vertices of $\mathrm{Ind}_+(\Gamma_B)$.

Let distinct $U_1, U_2 \in X_f$ be given (i.e. two distinct monomials of $f(x)$). Then, since $x^2 = 0$, either

1) $U_1 \cap U_2 \neq \phi$ (so that they have a common vertex which becomes squared in the product $b_{U_1} b_{U_2}$), or

2) there exists $b_i \in U_1$ and $b_j \in U_2$ (with $i \neq j$) such that $(b_j b_{j'})$ is an edge of Γ_B.

In either case, each of the above conditions is equivalent to the independent sets U_1 and U_2 having an edge joining them in $\mathrm{Ind}_+(\Gamma_B)$. X_f is thus a clique of $\mathrm{Ind}_+(\Gamma_B)$. In particular, $f(x)$ corresponds to a vertex of $\mathrm{Cl}(\mathrm{Ind}_+(\Gamma_B))$.

Conversely, given a clique Y of $\mathrm{Ind}_+(\Gamma_B)$, there is the obvious polynomial $p_Y(b_1, \ldots, b_n)$ corresponding to Y, and it is routine to check that $f_Y(x) = p_X(b_1, \ldots, b_n)$ defines a valid morphism $f_Y\colon W \to B$. ∎

Notation For convenience, we shall let $\chi(\_)$ denote $\mathrm{Cl}(\mathrm{Ind}_+(\_))$.

We can take this one step further:

Proposition 3.4.11. *To give a map $f\colon A \to B$ is to give a graph homomorphism $\tilde{f}\colon \Gamma_A \to \chi(\Gamma_B)$.*

Proof. We know from Proposition 3.4.10 that each $f(a_i)$ corresponds to a vertex of $\chi(\Gamma_B)$.

This gives us a function from the set $\{a_1, \ldots, a_m\}$ of vertices of Γ_A to the set of vertices of $\chi(\Gamma_B)$. We now verify that this function yields a valid graph homomorphism.

Suppose a_i and a_j are two distinct vertices of Γ_A with an edge joining them.

$$\begin{aligned} & (a_i, a_j) \text{ is an edge of } \Gamma_A \\ \Rightarrow\ & a_i a_j = 0 \text{ in } A \\ \Rightarrow\ & f(a_i) f(a_j) = 0 \text{ in } B\ . \end{aligned}$$

This tells us that if $\underline{b}_i$ and $\underline{b}_j$ are each a monomial from $f(a_i)$ and $f(a_j)$ respectively, then $\underline{b}_i \underline{b}_j = 0$. Using the same idea as the proof for Proposition 3.4.10, this says that there is an edge joining $\underline{b}_i$ and $\underline{b}_j$ in $\mathrm{Ind}_+(\Gamma_B)$.

This is true for all such pairs of monomials, and so $f(a_i)$ and $f(a_j)$, viewed as cliques in $\mathrm{Ind}_+(\Gamma_B)$, together (i.e. taking the union of the two cliques) give a clique. As such, when viewed as vertices of $\chi(\Gamma_B)$, there is an edge joining $f(a_i)$ and $f(a_j)$.

Thus $f\colon A \to B$ yields a unique graph homomorphism $\tilde{f}\colon \Gamma_A \to \chi(\Gamma_B)$.

The reverse direction is then obvious. ∎

These ideas actually allow us to prove an interesting fact about χ.

Proposition 3.4.12. *χ defines an endofunctor on the category* **Gph**, *and moreover, χ is canonically a monad.*

Proof. We first exhibit χ as an endofunctor. It is already well-defined on objects. Let $G = (V, E)$ and $G' = (V', E')$ be arbitrary graphs and $h\colon G \to G'$ some chosen graph homomorphism.

Define $\chi(h)\colon \chi(G) \to \chi(G')$ as follows:

- For a vertex $v \in V$, regarded as the singleton clique of the singleton independent set (so that it is a vertex of $\chi(G)$), define $(\chi h)(v) = h(v)$ (where $h(v) \in V'$ is regarded as a vertex of $\chi(G')$ in the same way).

- For a non-empty independent set U of G (hence a vertex of $\mathrm{Ind}_+(G)$, and thus a singleton clique) viewed as a vertex of $\chi(G)$, define $(\chi h)(U)$ as

$$\begin{cases} \bigcup\limits_{v \in U} h(v) & ;\ \text{if the function } h \text{ restricted to domain } U \text{ is injective,} \\ & \quad \text{and this defines an independent set of } G' \\ \text{The empty clique} & ;\ \text{otherwise} \end{cases} ;$$

  if $\bigcup\limits_{v \in U} h(v)$ does indeed define an independent set of G', we again regard it as a singleton clique of $\mathrm{Ind}_+(G')$, hence a vertex in $\chi(G')$.

- For a clique C of $\mathrm{Ind}_+(G)$, define $\chi(C)$ as the clique of $\mathrm{Ind}_+(G')$ consisting of all $(\chi h)(U)$ not the empty clique, for all (non-empty) independent sets $U \in C$.

We leave as an exercise to the reader to show that this will preserve identities and composition, so that χ is functorial.

To show χ is a monad, we first give the unit $\eta\colon 1_{\mathbf{Gph}} \Rightarrow \chi$ by its components; $\eta_G\colon G \to \chi(G)$ sends each vertex $v \in V$ to the singleton clique of the singleton independent set $\{\{v\}\}$.

Using Proposition 3.4.11, it is easy to see that each $\eta_G\colon G \to \chi(G)$ corresponds to the identity $id_{\kappa(G)}\colon \kappa(G) \to \kappa(G)$.

The multiplication $\mu\colon \chi^2 \Rightarrow \chi$ has components $\mu_G\colon \chi^2(G) \to \chi(G)$ given as follows: Recall that

- Vertices of G correspond to generators of $\kappa(G)$ (Definition 3.3.1);

- Non-empty independent sets U of G correspond to non-constant, non-zero monomials of $\kappa(G)$ (Proposition 3.4.8), and an edge in $\mathrm{Ind}_+(G)$ is equivalent to the corresponding monomials multiply to zero;

- Cliques of such independent sets are polynomials squaring to zero (Proposition 3.4.10), and an edge in $\chi(G)$ means that the product of the two corresponding polynomials yields zero.

Using Definitions 3.4.5 and 3.4.6, we can then see that

- A non-empty independent set of such a clique (i.e. a vertex of $\mathrm{Ind}_+(\chi(G))$) then corresponds to a set X of polynomials for which the product of all polynomials in this set X is not zero, or X contains only the zero polynomial itself (taking the empty clique as a singleton). An edge between X and Y in this graph corresponds to there being polynomials $p \in X$ and $q \in Y$ such that $pq = 0$ in $\kappa(G)$;
- A (possibly empty) clique of such an independent set (i.e. a vertex of $\chi^2(G)$) is a family ϱ of such sets of polynomials such that for any two distinct sets X and Y of this family, there are polynomials $p \in X$ and $q \in Y$ such that $pq = 0$ in $\kappa(G)$, and an edge between vertices ϱ and σ says that the union of the two families is also such a family.

Then, to give $\mu_G \colon \chi^2(G) \to \chi(G)$ is to associate each family of sets of polynomials to a polynomial squaring to zero. Let ϱ be one such family. Let $X \in \varrho$, and suppose $X = \{p_1, \ldots, p_r\}$, where each p_i is a polynomial of $\kappa(G)$ squaring to zero.

With this notation, we define $\mu_G(\varrho)$ to be the polynomial

$$\sum_{X \in \varrho} \left(\prod_{p_i \in X} p_i \right) .$$

Explicitly, for each set $X \in \varrho$, multiply together all the polynomials in this set (recall that unless X contains only the zero polynomial, then this product is non-zero). Then add up all such resultant polynomials across all $X \in \varrho$.

Now, each polynomial p_i squares to zero, so each product

$$\prod_{p_i \in X} p_i$$

squares to zero. Since ϱ is a clique of $\mathrm{Ind}_+ \chi(G)$, then any two sets $X, Y \in \varrho$ therefore are joined by an edge. As such, there exists $p \in X$ and $q \in Y$ with $pq = 0$ in $\kappa(G)$. As such, the product

$$\left(\prod_{p_i \in X} p_i \right) \left(\prod_{q_j \in Y} q_j \right)$$

must be zero. This is true for all pairs $X, Y \in \varrho$.

Thus, the polynomial $\mu_G(\varrho)$ squares to zero (hence is a vertex of $\chi(G)$).

Finally, suppose σ is another vertex such that (ϱ, σ) is an edge of $\chi^2(G)$. This means that $\varrho \cup \sigma$ is another family. As such, $\mu_G(\varrho \cup \sigma)$ is well defined and moreover squares to zero. In particular, this means that $\mu_G(\varrho)\mu_G(\sigma) = 0$ in $\kappa(G)$.

Therefore there must be an edge between $\mu_G(\varrho)$ and $\mu_G(\sigma)$.

We shall leave verifying the axioms of the monad as an exercise for the reader.

■

Since χ is a monad, we can then consider the Kleisli category $\mathbf{Gph}_\chi$. Moreover, we can then define $\mathbf{Gph}'_\chi$ as the full subcategory whose objects are precisely the p.c. graphs.

Proposition 3.4.13. *There exists an equivalence of categories*

$$F\colon \mathbf{Gph}'_{\chi} \to \mathbb{2}\text{-}\mathbf{Weil_1}\ .$$

Proof. The functor F is defined as:

- On objects: $F(G) = \kappa(G)$
- On morphisms: A map $h\colon G \nrightarrow G'$ of $\mathbf{Gph}'_{\chi}$ is a graph homomorphism $h'\colon G \to \chi(G')$, and this corresponds to a unique map $\tilde{h}\colon \kappa(G) \to \kappa(G')$ of $\mathbb{2}$-$\mathbf{Weil_1}$ (Proposition 3.4.11). Thus, take $F(h) = \tilde{h}$.

Using Proposition 3.4.11, F is clearly full and faithful. Finally, from Definition 3.3.1, Definition 3.3.2, and the fact that $X = \kappa(\Gamma_X)$ for all $X \in \mathbb{2}$-$\mathbf{Weil_1}$, we can see that F is essentially surjective. ■

3.5 Construction of maps

We shall show in this section that using the set $\{\varepsilon_W, +, \eta_W, l, c\}$ (as defined in Section 3.2), composition, $\otimes$ and the universal property of foundational pullbacks (as given in Definition 2.1.16) of $\mathbb{2}$-$\mathbf{Weil_1}$, we are able to "construct" (in some appropriate sense) any map $f\colon A \to B$ of $\mathbb{2}$-$\mathbf{Weil_1}$. We begin by expressing the maps $\{\varepsilon_W, +, \eta_W, l, c\}$ in the form $\{U\}_f$ in Table 3.1 below:

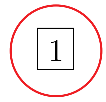
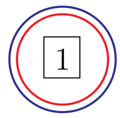
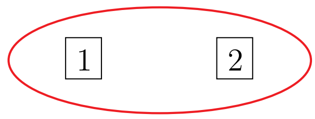
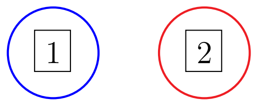
Map	**Action on Generators**	**Graph**
$\varepsilon_W : W \to \mathbb{2}$	$x_1 \mapsto 0$	(k corresponds to the empty graph)
$id_W : W \to W$	$x \mapsto x$	
$+ : W^2 \to W$	$x_1 \mapsto x$, $x_2 \mapsto x$	
$\eta_W : \mathbb{2} \to W$	($\mathbb{2}$ has no generators)	$\boxed{1}$
$l : W \to 2W$	$x \mapsto x_1x_2$	
$c : 2W \to 2W$	$x_1 \mapsto x_2$, $x_2 \mapsto x_1$	

Table 3.1:

Pictorially, given $\{U\}_f$ for some map $f : A \to B$, we can naively interpret 'post-composition' with the above maps as follows:

- ε_W corresponds to deleting a particular vertex in Γ_B as well as any circles that go through that vertex.
- $+$ corresponds to taking two vertices in Γ_B joined by an edge and collapsing them to a single vertex. Circles that had contained either vertex (but not both) now contain the collapsed vertex instead.
- η_W corresponds to adding a new vertex to Γ_B, but has no effect on any of the existing circles.
- l corresponds to taking a single vertex of Γ_B and splitting it into two vertices without an edge joining them, and any circle U that contained the original vertex now contain both of the new vertices.
- c corresponds to switching labels of (unjoined) vertices, and does nothing to the circles themselves.

These ideas will become clearer in subsequent discussion. We shall now precisely define what it means to say that a map $f\colon A \to B$ is "constructible".

Definition 3.5.1. *Let Ξ be a set of maps in $\mathbb{2}$-$\mathbf{Weil_1}$ given iteratively as follows:*

- *The maps $\varepsilon_W, +, \eta_W, l, c$ are contained in Ξ.*
- *Ξ contains all identities.*
- *For all $n \in \mathbb{N}$, each projection $\pi_i\colon W^n \to W$ is contained in Ξ.*
- *If $f\colon X \to Y$ and $g\colon Y \to Z$ are both in Ξ, then their composite $g \circ f\colon X \to Z$ is also in Ξ. Equivalently, Ξ is closed under composition.*
- *If $f\colon X \to Y$ and $g\colon A \to B$ are both in Ξ, then their tensor $f \otimes g\colon X \otimes A \to Y \otimes B$ is also in Ξ. Equivalently, Ξ is closed under tensor.*
- *For a foundational pullback*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathbb{2}$-$\mathbf{Weil_1}$, and for an arbitrary commuting square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

with $X \in \mathbb{2}$-$\mathbf{Weil_1}$ and $f, g \in \Xi$, then the uniquely induced map $h\colon X \to A$ is also in Ξ. Morphisms in Ξ will be called constructible*.*

Lemma 3.5.2. *For any Weil algebra $A \in \mathbb{2}$-$\mathbf{Weil_1}$, the unit η_A and augmentation ε_A are both constructible.*

Proof. The lemma is true by definition for $A = W$. We then simply note that $\varepsilon_{W^n} = \varepsilon_W \circ \pi_i$ (for any i) and η_{W^n} (induced using η_W and product diagrams regarded as foundational pullbacks) are both constructible, and for $X, Y \in \mathbb{2}$-$\mathbf{Weil_1}$ with $\eta_X, \eta_Y, \varepsilon_X, \varepsilon_Y$ constructible, then $\eta_{X\otimes Y} = \eta_X \otimes \eta_Y$, $\varepsilon_{X\otimes Y} = \varepsilon_X \otimes \varepsilon_Y$ are also constructible. ■

Corollary 3.5.3. *Any zero map $z\colon A \to B$ is constructible, for all $A, B \in \mathbb{2}$-$\mathbf{Weil_1}$.*

Proof. For given $A, B \in \mathbb{2}$-$\mathbf{Weil_1}$, the zero map $z\colon A \to B$ is the composite

$$A \xrightarrow{\varepsilon_A} \mathbb{2} \xrightarrow{\eta_B} B\ .$$

■

Lemma 3.5.4. *The only (non-trivial) products in $\mathbb{2}$-$\mathbf{Weil_1}$ are the product powers W^n.*

Proof. For arbitrary $X, Y \in \mathbb{2}$-$\mathbf{Weil_1}$, the graph $\Gamma_{X\times Y}$ for their product would need to be connected. The only connected p.c. graphs are the complete ones, and so $X \times Y = W^n$ for some n. ■

Lemma 3.5.5. *For arbitrary $0 < n' < n$ in $\mathbb{N}$, all projections*

$$\pi' \colon W^n \to W^{n'}$$

are constructible.

Proof. Let $\pi' \colon W^n \to W^{n'}$ be a given projection. Without loss of generality, suppose π' preserves the first n' generators of W^n. Since each product can be regarded as a foundational pullback (Definition 2.1.16), π' is then constructed as $id_{W^{n'}} \times \varepsilon_{W^{n-n'}}$. ■

Corollary 3.5.6. *Let*

$$\begin{tikzcd} A \otimes W^{m+n} \arrow[r, "A\otimes\pi_1"] \arrow[d, "A\otimes\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \otimes W^m \arrow[d, "A\otimes\varepsilon_{W^m}"] \\ A \otimes W^n \arrow[r, "A\otimes\varepsilon_{W^n}"'] & A \end{tikzcd}$$

be an arbitrary foundational pullback (recall from Lemma 3.5.4 *that the only products in* $\mathcal{2}$-**Weil**$_1$ *are product powers).*

Then, each of the four maps in this pullback diagram are constructible.

Proof. This is an immediate consequence of Definition 3.5.1, Lemma 3.5.2 and Lemma 3.5.5. ■

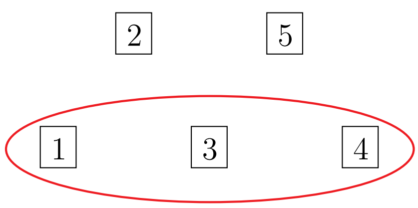
Clearly, any map that is constructible by definition must live in $\mathcal{2}$-**Weil**$_1$. We will now sequentially build up in a different manner the maps of Ξ and show that in fact all maps $f \colon A \to B$ of $\mathcal{2}$-**Weil**$_1$ are constructible.

3.5.1 Maps $W \to nW$ with one circle

Lemma 3.5.7. *Any map $f \colon W \to nW$ with precisely one circle is constructible.*

Let us begin with an example.

Example 3.5.8. *The map $f \colon W \to 5W$ given by $x \mapsto x_1x_3x_4$ may be represented as*



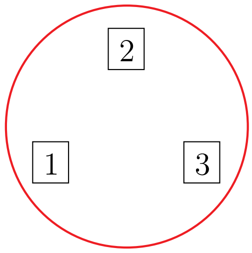
Define a map $\tilde{f}$ as the composite

$$W \xrightarrow{\;l\;} 2W \xrightarrow{\;W\otimes l\;} 3W$$

$$x \longmapsto x_1x_2 \longmapsto x_1x_2x_3\ .$$

Clearly $\tilde{f}$ is constructible.

Then $\{U\}_{\tilde{f}}$ is



i.e. the single circle includes all 3 vertices.

Now define a map g as the map

$$\begin{aligned} W \otimes \eta_W \otimes W \otimes W \otimes \eta_W \colon 3W &\to 5W \\ x_1 &\mapsto y_1 \\ x_2 &\mapsto y_3 \\ x_3 &\mapsto y_4 \ . \end{aligned}$$

Clearly, g is constructible.

Then the composite $g \circ \tilde{f}$ is precisely the original map f. Thus f is constructible.

We generalise this idea to prove Lemma 3.5.7.

Proof. Let $f \colon W \to nW$ with precisely one circle U be given. Let $r = |U|$. Define $\tilde{f}$ as the composite

$$W \xrightarrow{l} 2W \xrightarrow{W \otimes l} \dots \xrightarrow{(r-1)W \otimes l} rW$$

Clearly, $\tilde{f}$ is constructible.

In an analogous manner to Example 3.5.8, define a constructible map $g \colon rW \to nW$ with $g \circ \tilde{f} = f$. Thus f is constructible. ∎

3.5.2 Arbitrary maps $W \to nW$

Lemma 3.5.9. *All maps $f \colon W \to nW$ are constructible.*

Proof. If there are no circles in $\{U\}_f$ (i.e. $x \mapsto 0$), then the f is given by (say) the composite

$$W \xrightarrow{\varepsilon} k \xrightarrow{\eta} W \xrightarrow{W \otimes \eta} \dots \xrightarrow{(n-1)W \otimes \eta} nW$$

i.e. the zero map, hence f is constructible (alternatively, we may simply apply Corollary 3.5.3 directly).

If f has one circle, we apply Lemma 3.5.7.

If f has more than one circle, then we prove this by induction. Let $S(m)$ be the statement "All maps $f \colon W \to nW$ with m circles or fewer are constructible, for all $n \in \mathbb{N}$".

We know $S(1)$ is true. Suppose that $S(r)$ is true for some $r \in \mathbb{N}$.

Let a map $f \colon W \to nW$ with precisely $r+1$ circles be given. Explicitly, this means that $f(x)$ is a polynomial in the generators of nW (which we shall call $y_1, \dots, y_n$) with precisely $r+1$ monomial summands.

Recall that for f to be a valid map, since the codomain is nW (or equivalently, the corresponding graph Γ_{nW} is discrete), then any two distinct summands of $f(x)$

must have (at least) one generator y_i in common. Let t and t' be distinct summands, and without loss of generality, suppose y_n is a common generator.

Now define a map

$$f' \colon W \to (n-1)W \otimes W^2$$

where $W^2 = \mathbb{2}[y_n, \widetilde{y_n}]/y_n^2, \widetilde{y_n}^2, y_n\widetilde{y_n}$, with $f'(x)$ having the same expression as $f(x)$, except that the y_n in term t' is replaced with $\widetilde{y_n}$. It is a routine task to check that this is a valid map. Furthermore, the composite

$$W \xrightarrow{f'} (n-1)W \otimes W^2 \xrightarrow{(n-1)W\otimes +} nW$$

will return the original map f. Clearly, the map $(n-1)W \otimes +$ is constructible, so it suffices to show that f' is constructible.

But the codomain of f', $(n-1)W \otimes W^2$, is the pullback

$$\begin{tikzcd}
(n-1)W \otimes W^2 \arrow[r, "(n-1)W\otimes\pi_1"] \arrow[d, "(n-1)W\otimes\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & nW \arrow[d, "(n-1)W\otimes\varepsilon_W"] \\
nW \arrow[r, "(n-1)W\otimes\varepsilon_W"'] & (n-1)W
\end{tikzcd} ,$$

and moreover, this is a foundational pullback.

Thus, to prove that f' is constructible, it suffices to prove that each of the composites

$$\big((n-1)W \otimes \pi_i\big) \circ f' \colon W \to nW;\ i \in \{1,2\}$$

is constructible. But each of these composites has a number of circles strictly less than $r+1$. Since we assumed that $S(r)$ was true, then both these composites are constructible, hence f is constructible.

Thus $S(r+1)$ is true.

As such, all maps $f \colon W \to nW$ are constructible. ■

We can actually prove Lemma 3.5.9 more directly. Suppose we have an arbitrary map $f \colon W \to nW$ with $\{U_f\}$ given. For each $i \in \{1, \dots, n\}$, let m_i be the number of circles containing vertex i (or equivalently, the number of terms of $f(x)$ containing the generator y_i). Then, in a similar manner as before, we can define a map

$$f' \colon W \to W^{m_1} \otimes \cdots \otimes W^{m_n}$$

in such a way that $(+_{m_1} \otimes \cdots \otimes +_{m_n}) \circ f' = f$. Here, since $+$ is an associative and commutative operation, then $+_m \colon W^m \to W$ is well defined, and $+_0$ is the nullary sum η_W.

Clearly, the map $+_{m_1} \otimes \cdots \otimes +_{m_n}$ is constructible. Further, by iteratively using foundational pullbacks, it is relatively easy to show that $W^{m_1} \otimes \cdots \otimes W^{m_n}$ is a limit of an appropriate diagram of nW's.

As such, f decomposes immediately into a set of maps $\{f_j \colon W \to nW\}$, each of the type described in Section 3.5.1 (or a zero map), all of which are constructible (by Lemma 3.5.7), and so f is constructible.

Remark We explore the idea that $W^{m_1} \otimes \cdots \otimes W^{m_n}$ is a limit of a diagram of tensor powers of W's in more detail in Chapter 4.

3.5.3 Projection maps $A \to W$

Given an arbitrary object $A = \mathbb{2}[a_1, \ldots, a_n]/Q_A$ in $\mathbb{2}$-**Weil₁**, we wish now to consider maps of the form $f\colon A \to W$ with $a_i \mapsto x$ for some fixed i and $a_j \mapsto 0\ \forall\ j \neq i$.

Lemma 3.5.10. *Let an arbitrary object $A = \mathbb{2}[a_1, \ldots, a_n]/Q_A$ of $\mathbb{2}$-**Weil**$_\mathbf{1}$ be given. Then any map $f\colon A \to W$ given as $f(a_i) = x$ for some fixed i and $f(a_j) = 0$ for all $j \neq i$ is constructible.*

Proof. Since $A \in \mathbb{2}$-**Weil**$_\mathbf{1}$, then Γ_A is a p.c. graph, or more generally, a cograph. We then show that f is constructible recursively as follows:

1) If $\Gamma_A = \{\bullet\}$ (the one point graph), then f is the identity and is thus constructible.

2) If $\Gamma_A = G \otimes H$ with $a_i \in H$, then f is the composite

$$A = \kappa(G) \otimes \kappa(H) \xrightarrow{\varepsilon_{\kappa(G)} \otimes \kappa(H)} \kappa(H) \xrightarrow{f'} W\ ;$$

for a unique map f', and it thus suffices to show that f' is constructible.

3) If $\Gamma_A = G \times H$ with $a_i \in H$, then f is the composite

$$A = \kappa(G) \times \kappa(H) \xrightarrow{\pi_{\kappa(H)}} \kappa(H) \xrightarrow{f'} W\ ;$$

for a unique map f', and it thus suffices to show that f' is constructible. Noting Definition 2.2.11, the result becomes immediate.

■

3.5.4 Maps $A \to nW$ with no intersecting circles

Lemma 3.5.11. *Every map $f\colon A \to nW$ with no intersecting circles is constructible.*

Proof. Let $\mathcal{A}$ be the full subcategory of $\mathbb{2}$-**Weil**$_\mathbf{1}$ consisting of all objects A with the property that any map $A \to nW$ with no intersecting circles is constructible.

By Lemma 3.5.2, we have $\mathbb{2} \in \mathcal{A}$ (since $\mathbb{2}$ is a zero object, the only map to any A is the unit η_A), and by Lemma 3.5.9, we have $W \in \mathcal{A}$.

For arbitrary $m, n \in \mathbb{N}$ with $m \geq 2$, let an arbitrary map $f\colon W^m \to nW$ with no intersecting circles be given. If f is the zero map, then by Corollary 3.5.3, it is constructible. Suppose then that a is a generator of W^m for which $f(a) \neq 0$. Let a' be any other generator of W^m. Now, since $aa' = 0$ by construction, then $f(aa') = f(a)f(a') = 0$.

But since the codomain of f is nW and f has no intersecting circles, then we must have $f(a') = 0$. This is true for all generators of W^m (other than a). But this means that f factors through the appropriate projection $\pi\colon W^m \to W$ preserving a (the other map being one of the form described in Section 3.5.1), thus f is constructible. Thus $W^m \in \mathcal{A}$ for all $m \in \mathbb{N}$.

Now suppose that A_1 and A_2 are arbitrary objects of $\mathcal{A}$. Let an arbitrary map $f\colon A_1 \otimes A_2 \to nW$ with no intersecting circles be given. Then, with some appropriate post-composition with c's, we can write $f = f_1 \otimes f_2$, for an appropriate pair $f_1 \colon A_1 \to rW$ and $f_2 \colon A_2 \to (n-r)W$ neither of which have intersecting circles. Thus f is constructible. Thus we have $A_1 \otimes A_2 \in \mathcal{A}$.

Now, since $\mathcal{A}$ is a full subcategory of $\mathbb{2}$-$\mathbf{Weil_1}$ containing W^m $\forall$ $m \in \mathbb{N}$ and is closed under $\otimes$, then $\mathcal{A}$ is just $\mathbb{2}$-$\mathbf{Weil_1}$ itself. Thus any map $A \to nW$ with no intersecting circles is constructible. ■

3.5.5 Arbitrary maps $A \to nW$

Lemma 3.5.12. *Every map $f \colon A \to nW$ is constructible.*

Proof. Let an arbitrary map $f \colon A \to nW$ be given. Using an analogous idea to that described in Section 3.5.2, we can construct a map

$$f' \colon A \to W^{m_1} \otimes \cdots \otimes W^{m_n}$$

as follows:

1) For each generator a_i of A, take the polynomial $f(a_i)$ in the generators $z_1, \ldots, z_n$ of nW

2) Let m_j be the total number of terms across all the polynomials $f(a_1)$ containing z_j for $j = 1, \ldots, n$

3) Define the map $f' \colon A \to W^{m_1} \otimes \cdots \otimes W^{m_n}$ By specifying each $f'(a_i)$ to be $f(a_i)$, but in such a way that each generator of $W^{m_1} \otimes \cdots \otimes W^{m_n}$ is used exactly once (in a similar fashion to the proof for Lemma 3.5.9)

Example 3.5.13. *Consider the map $f \colon 2W \to 3W$ given as*

$$\begin{aligned} x_1 &\mapsto y_1y_2 + y_1y_3 \\ x_2 &\mapsto y_2y_3 \ . \end{aligned}$$

Noting that each generator y_i appears in exactly two monomials, then we have the map $f' \colon 2W \to W^2 \otimes W^2 \otimes W^2$ given as

$$\begin{aligned} x_1 &\mapsto y_1y_2 + y_1'y_3 \\ x_2 &\mapsto y_2'y_3' \end{aligned}$$

Then f is the composite

$$A \xrightarrow{f'} W^{m_1} \otimes \cdots \otimes W^{m_n} \xrightarrow{+_{m_1} \otimes \cdots \otimes +_{m_n}} nW \ ,$$

and so it suffices to show f' is constructible. But now, for each projection

$$\pi = \pi_{i_1} \otimes \cdots \otimes \pi_{i_n} \colon W^{m_1} \otimes \cdots \otimes W^{m_n} \to nW,$$

the composite $\pi \circ f' \colon A \to nW$ has no intersecting circles, and is thus constructible using Lemma 3.5.11, and we use a series of foundational pullbacks to recover f'. ■

3.5.6 Arbitrary maps $A \to B$

Recall that each p.c. graph is also a cograph. We begin with the following lemma:

Lemma 3.5.14. *Let G be a cograph with at least one edge (and hence at least two vertices). Then G can be expressed as $(G_1 \times G_2) \otimes H$, where G_1 and G_2 are non-empty cographs (H may be empty).*

Proof. Let e be a chosen edge of G. Let G' be the connected component of G containing the edge e. Clearly, we can express G as a disjoint union $G' \otimes H$ (with H possibly empty).

Now, since G' contains an edge, it cannot be the one point graph. Since it is connected, it cannot be expressed (non-trivially) as $G_1 \otimes G_2$. Since it is a cograph, then by Definition 2.2.11, it can be expressed non-trivially as $G_1 \times G_2$. ■

We now have the following:

Theorem 3.5.15. *Every map $f\colon A \to B$ in $\mathbb{2}$-**Weil₁** is constructible.*

Proof. Consider the Weil algebra B. If Γ_B has any edges then, using Lemma 3.5.14, it can be expressed (non-trivially) as $(G_1 \times G_2) \otimes H$ (with H possibly being the empty graph). Correspondingly, $B = (\kappa(G_1) \times \kappa(G_2)) \otimes \kappa(H)$ and we thus have the foundational pullback

$$\begin{tikzcd}
B \arrow[r, "\pi_1\otimes\kappa(H)"] \arrow[d, "\pi_2\otimes\kappa(H)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \kappa(G_1) \otimes \kappa(H) \arrow[d, "\varepsilon_1\otimes k[H]"] \\
\kappa(G_2) \otimes \kappa(H) \arrow[r, "\varepsilon_2\otimes\kappa(H)"'] & \kappa(H)
\end{tikzcd}$$

and so $f\colon A \to B$ is uniquely induced by the pair $(\pi_i \otimes \kappa(H)) \circ f$; $i = 1, 2$. As such, it suffices to show that each of these is constructible.

Note now that the graphs $G_i \otimes H$ for the codomains each have strictly fewer edges than Γ_B. As such, we repeat this process until the codomains are all of the form nW, then directly apply Lemma 3.5.12. ■

3.5.7 Obtaining coefficients beyond $\mathbb{2}$

We gave Theorem 3.5.15, which said that every map $f\colon A \to B$ is constructible. However, this was for the case of $\mathbb{2}$-**Weil₁**, and so we limit the permissible maps by restricting the coefficients to being either 0 or 1.

Consider k-**Weil₁** for an arbitrary rig k.

Definition 3.5.16. *For each $t \in k$, let $a_t\colon W \to W$ to be the map given as $a_t(x) = tx$. Note that a_0 is the zero map and a_1 is the identity id_W.*

Now, define Ξ_k in the same way as Definition 3.5.1, with the added condition that a_t is contained in Ξ_k for all $t \in k$. Define the notion of a *k-constructible* morphism in the same way.

Proposition 3.5.17. *Every map $g\colon A \to B$ of k-**Weil₁** is k-constructible.*

Proof. (Sketch) Consider first (the analogue of) Lemma 3.5.7. Suppose we had a map $f\colon W \to nW$ with $f(x)$ given by a single monomial (with some arbitrary coefficient $r \in k$). Let $f'\colon W \to nW$ be the map with $f'(x)$ being the same monomial, but with coefficient one. Clearly, f' is k-constructible.

Then, the composite

$$W \xrightarrow{a_r} W \xrightarrow{f'} W \quad ,$$

yields f, and so f is k-constructible.

From there, the proofs for (the analogues of) Lemma 3.5.9 through to Lemma 3.5.12 as well as Theorem 3.5.15 are identical. ■

Recall, however, that we are ultimately interested in $\mathbb{N}$-$\mathbf{Weil_1}$. We begin with the following:

Proposition 3.5.18. *Let $\psi^\dagger\colon \mathbb{N} \to \mathbb{2}$ be the rig morphism*

$$\psi^\dagger(n) = \begin{cases} 0 & ;\ n = 0 \\ 1 & ;\ \textit{otherwise} \end{cases} .$$

The canonical functor

$$\psi\colon \mathbb{N}\text{-}\mathbf{Weil_1} \to \mathbb{2}\text{-}\mathbf{Weil_1}$$

induced by the rig morphism above is bijective on objects and full.

Here, ψ sends each object $\mathbb{N}[x_1, \ldots, x_r]/Q$ of $\mathbb{N}$-$\mathbf{Weil_1}$ to its counterpart $\mathbb{2}[x_1, \ldots, x_r]/Q$ in $\mathbb{2}$-$\mathbf{Weil_1}$. There is analogous action of ψ on morphisms.

Proof. Bijectivity on objects follows immediately from the fact that Definition 3.2.2 defines the objects of k-$\mathbf{Weil_1}$ independently from the choice of k.

For any morphism $f\colon A \to B$ of $\mathbb{2}$-$\mathbf{Weil_1}$, there is a corresponding map $g\colon A \to B$ in $\mathbb{N}$-$\mathbf{Weil_1}$ given by the same action on generators as f. Clearly, we then have $\psi g = f$. ■

Let us now work with $\mathbb{N}$-$\mathbf{Weil_1}$. Let Ξ be defined as in Definition 3.5.1 (i.e. we do not explicitly include in Ξ the maps $\hat{g}_t$ for all $t \in \mathbb{N}$). We first have the following:

Lemma 3.5.19. *For each $t \in \mathbb{N}$, the map a_t is constructible.*

Proof. First we note again that a_0 is the zero map and a_1 is the identity, and thus both are constructible.

We shall show that all a_t's are constructible by induction. Let $S(t)$ be the statement "a_t is constructible". We have established that $S(0)$ and $S(1)$ are true. Suppose $S(r)$ is true.

We then have

$$\begin{tikzcd}
W \arrow[dr, dashed, "h"] \arrow[drr, bend left, "a_r"] \arrow[ddr, bend right, "id_W"'] & & \\
& W^2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "\varepsilon_W"] \\
& W \arrow[r, "\varepsilon_W"'] & k
\end{tikzcd} ,$$

so the map h is constructible. Then, the map a_{r+1} is clearly the composite

$$W \xrightarrow{h} W^2 \xrightarrow{+} W \ ,$$

so that a_{r+1} is also constructible, so that $S(r+1)$ is true. ■

Remark We may try to give a similar construction in $\mathbb{2}$-**Weil$_\mathbf{1}$**, but note that for all $t > 0$, we will have $a_t = id_W$, since $1 + 1 = 1$ in $\mathbb{2}$.

With these "coefficient maps" a_t being constructible along with Theorem 3.5.15, we now have the following:

Proposition 3.5.20. *Every map $g\colon A \to B$ of* $\mathbb{N}$ -**Weil$_\mathbf{1}$** *is constructible.*

Proof. This is a direct consequence of Proposition 3.5.17 and Lemma 3.5.19. ■

Remark For the case of (the rig) $k = \mathbb{Z}$, note that the axiomatisation of [29] requires a natural transformation

$$-\colon G \to G$$

that gives additive inverses. As such, there is the counterpart map

$$\begin{aligned} -\colon W &\to W \\ x &\mapsto -x \end{aligned}$$

in $\mathbb{Z}$-**Weil$_\mathbf{1}$**, and so we would include this map in the first point of Definition 3.5.1 and would again be able to conclude that all maps of $\mathbb{Z}$-**Weil$_\mathbf{1}$** were constructible. This is in direct analogy to $\mathbb{Z}$ requiring the operation $-$ to be generated as a rig.

More broadly, we may use a similar process for an arbitrary rig k; namely that the first point of Definition 3.5.1 would need to include sufficient maps to generate k as a rig so that k-**Weil$_\mathbf{1}$** is constructible (although this is beyond the scope of this thesis).

3.5.8 Instructions for assembly

In Sections 3.5.2 and 3.5.5, there was an element of choice involved; namely given a map $f\colon A \to nW$, the corresponding map $f'\colon A \to W^{m_1} \otimes \cdots \otimes W^{m_n}$ required a choice as to which circle would correspond to which projection. Ultimately, this choice is inconsequential as different choices are (up to isomorphism) equivalent.

However, for the purposes of what we wish to do, we will assume that for each $f\colon A \to nW$, there is some pre-determined choice that has already been made regarding the corresponding map f'.

This then implicitly equips each map $f\colon A \to B$ of $\mathbb{2}$-**Weil$_\mathbf{1}$** (or $\mathbb{N}$-**Weil$_\mathbf{1}$**) with a set of instructions for its construction.

3.5.9 The map Ω

We will now describe the construction of a certain map Ω, a map in $\mathbb{N}$ -$\mathbf{Weil_1}$, which we shall require in order to prove Proposition 3.6.14 later.

Let $s \in \mathbb{N}$ be given. For an arbitrary map $g\colon B \to sW$, recall that g decomposes (in the sense of Section 3.5.5) as

$$\begin{tikzcd} & W^{\beta_1} \otimes \cdots \otimes W^{\beta_s} \arrow[d, "+_\beta"] \\ B \arrow[ur, "g'"] \arrow[r, "g"'] & sW \end{tikzcd} .$$

By Section 3.5.8, the particular decomposition is fixed (i.e. g' is uniquely determined by g).

One way we can view this decomposition involves the slice category $\mathbb{N}$-$\mathbf{Weil_1}/sW$; the pair $(g, +_\beta)$ (again, g' is uniquely determined by g) can be seen as an object of the arrow category $(\mathbb{N}\text{-}\mathbf{Weil_1}/sW)^2$. We shall now extend this to a functor

$$\tau\colon \mathbb{N}\text{-}\mathbf{Weil_1}/sW \to (\mathbb{N}\text{-}\mathbf{Weil_1}/sW)^2$$

whose composite with the domain functor $d\colon\ (\mathbb{N}\text{-}\mathbf{Weil_1}/sW)^2 \to \mathbb{N}\text{-}\mathbf{Weil_1}/sW$ is the identity.

This amounts to giving, for each arrow

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & sW \end{tikzcd}$$

of $\mathbb{N}$-$\mathbf{Weil_1}/sW$, a morphism

$$\Omega\colon W^{\beta_1} \otimes \cdots \otimes W^{\beta_s} \to\ W^{\delta_1} \otimes \cdots \otimes W^{\delta_s}\ ,$$

such that the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "h'"'] & & B \arrow[d, "g'"] \\ W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \arrow[rr, "\Omega"'] \arrow[dr, "+_\delta"'] & & W^{\beta_1} \otimes \cdots \otimes W^{\beta_s} \arrow[dl, "+_\beta"] \\ & sW & \end{tikzcd}$$

commutes, and satisfying the evident functoriality conditions.

Remark We are, of course, taking $h\colon A \to sW$ to decompose as

$$\begin{tikzcd} & W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \arrow[d, "+_\delta"] \\ A \arrow[ur, "h'"] \arrow[r, "h"'] & sW \end{tikzcd} .$$

It now remains to specify Ω.

Since $W^{\beta_1} \otimes \cdots \otimes W^{\beta_s}$ is a limit (as discussed in Section 3.5.2), then it suffices to define each map $\Omega_{(r_1,\ldots,r_s)}$ as below

$$\begin{tikzcd}
W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \arrow[d, dashed, "\Omega"'] \arrow[drr, "\Omega_{(r_1,\ldots,r_s)}"] & & \\
W^{\beta_1} \otimes \cdots \otimes W^{\beta_s} \arrow[rr, "(r_1,\ldots,r_s)=\pi_{r_1}\otimes\cdots\otimes\pi_{r_s}"'] & & sW
\end{tikzcd},$$

where the $\Omega_{(r_1,\ldots,r_s)}$ are suitably compatible.

But to give $\Omega_{(r_1,\ldots,r_s)}$, it suffices to say where each generator of $W^{\delta_1} \otimes \cdots \otimes W^{\delta_s}$ is sent. Let y_1 be a generator of W^{δ_1} (without loss of generality, let $\alpha_1 \geq 1$). We shall refer to the generators of sW as $z_1, \ldots, z_s$. Observe that $+_\beta(y_1) = z_1$.

Recall from Section 3.5.8 the construction of $h' \colon A \to W^{\delta_1} \otimes \cdots \otimes W^{\delta_s}$. There is a unique circle (U_1, a) for some generator a of A with $y_1 \in U_1$ (and correspondingly, a unique circle (U_1, a) of h as well with $z_1 \in U_1$). Recall also that $h = g \circ f$. Let

$$h(a) = U_1 + U_2 + \ldots\ ,$$

where each U_i is a monomial in the generators $z_1, \ldots, z_s$. Similarly, let

$$f(a) = V_1 + V_2 + \ldots\ ,$$

where each V_i is a monomial in the generators $\{b_j\}$ of B.

Then (ignoring coefficients), since g preserves addition and multiplication, we can express $(g \circ f)(a)$ as

$$\begin{aligned}
(g \circ f)(a) &= g(f(a)) \\
&= g(V_1 + V_2 + \ldots) \\
&= g(V_1) + g(V_2) + \ldots \\
&= \left[\prod_{b_j \in V_1} g(b_j)\right] + \left[\prod_{b_j \in V_2} g(b_j)\right] + \ldots\ .
\end{aligned}$$

But this needs to be equal to $h(a)$. In particular, U_1 must be somewhere in the expression for $(g \circ f)(a)$. Without loss of generality, suppose U_1 is contained in the first term

$$\prod_{b_j \in V_1} g(b_j).$$

Now, for each $b_j \in V_1$, we must be able to choose precisely one circle Q_j in such a way that

$$\bigcup_{b_j \in V_1} Q_j = U_1$$

with the Q_j's pairwise distinct. This is because for each $b_j \in V_1$, $g(b_j)$ is a polynomial in the generators $z_1, \ldots, z_n$. Then, if the product of these polynomials (which in turn is another polynomial) is to contain a particular monomial (namely U_1), then this monomial must have arisen as the product of one monomial from each of the factor polynomials.

Moreover, since $z_1 \in U_1$, then we also have $z_1 \in Q_j$ for a unique j. Take $j = 1$ so that Q_1 is one of the terms of the polynomial $g(b_1)$.

$\Rightarrow Q_1$ is a circle of g corresponding to b_1

$\Rightarrow$ In $g' \colon B \to W^{\beta_1} \otimes \cdots \otimes W^{\beta_n}$, $\exists!$ generator v of W^{β_1} corresponding to the circle Q_1

$\Rightarrow$ Define $\Omega_{(r_1,\ldots,r_s)}(y_1) = \begin{cases} z_1 & ; \ (r_1, \ldots, r_s) \text{ preserves } v \text{ (in particular, } r_1 \text{ preserves } v) \\ 0 & ; \ \text{otherwise} \end{cases}$

and repeat for all generators of $W^{\beta_1} \otimes \cdots \otimes W^{\beta_n}$.

In particular, note that since Ω can only assign a generator from any W^{δ_i} to a generator of the corresponding W^{β_i}, then we have $\Omega = \Omega_1 \otimes \cdots \otimes \Omega_s$, for appropriate maps $\Omega_i \colon W^{\delta_i} \to W^{\beta_i}$.

Remark We shall note here that in full formality, we should use the label $\Omega_{f,g}$ (or something to this effect), but we shall not be doing this.

3.6 Linking back to Tangent Structure

We defined the category k-$\mathbf{Weil_1}$ in Definition 3.2.2, and in Section 3.5 we defined the notion of a *constructible* morphism (Definition 3.5.1) and showed that any map of $\mathbb{2}$-$\mathbf{Weil_1}$ was constructible (Theorem 3.5.15). We then said in Proposition 3.5.20 that in fact any map of $\mathbb{N}$-$\mathbf{Weil_1}$ was constructible, and moreover in Section 3.5.8 we noted that each map $g \colon A \to B$ was equipped with a set of instructions for its construction.

We shall conclude this chapter by linking these ideas about Weil algebras back to Tangent Structures in an explicit manner.

3.6.1 Preliminaries

Suppose that a category $\mathcal{M}$ is equipped with a Tangent Structure $\mathbb{T}$ (in the sense of Definition 3.1.4). Regard $\mathrm{End}(\mathcal{M})$ as a monoidal category with respect to composition $\circ$ with unit the identity functor $1_{\mathcal{M}}$.

We will be constructing a (strong monoidal) functor

$$F \colon \mathbb{N}\text{-}\mathbf{Weil_1} \to \mathrm{End}(\mathcal{M})$$

with certain properties soon, but we will need to establish some facts before doing so.

Notation To avoid confusion, when we want to regard composition as a monoidal operation in $\mathrm{End}(\mathcal{M})$, we will use concatenation if the meaning is clear (otherwise we will explicitly use $\otimes$), and save $\circ$ for actual composition. For example, if we have natural transformations $\alpha \colon R \Rightarrow S$ and $\beta \colon S \to U$ in $\mathrm{End}(\mathcal{M})$, then $\beta \circ \alpha$ denotes the composite

$$R \xRightarrow{\alpha} S \xRightarrow{\beta} U \ ;$$

whereas $\beta\alpha$ denotes the natural transformation

$$SR \xRightarrow{\beta\alpha} US \ .$$

We begin by specifying the action of our proposed functor on objects.

Definition 3.6.1. *Let*

$$F_0 \colon ob(\mathbb{N}\text{-}\mathbf{Weil_1}) \to ob(\mathrm{End}(\mathcal{M}))$$

be the function given as $F_0(\mathbb{N}) = 1_{\mathcal{M}}$, $F_0(W^m) = T^{(m)}$ *for all* $m \in \mathbb{N}$, *and then recursively, if* $A, B \in \mathbb{N}\text{-}\mathbf{Weil_1}$ *with* $F_0(A) = R$, $F_0(B) = S$, *then* $F_0(A \otimes B) = R \circ S$.

Proposition 3.6.2. *For any foundational pullback*

$$\begin{tikzcd} A \otimes (B \times C) \arrow[r, "A\otimes\pi_B"] \arrow[d, "A\otimes\pi_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \otimes B \arrow[d, "A\otimes\varepsilon_B"] \\ A \otimes C \arrow[r, "A\otimes\varepsilon_C"'] & A \end{tikzcd}$$

in $\mathbb{N}$-$\mathbf{Weil_1}$ *(recall from* Lemma 3.5.4 *that the only products in* $\mathbb{N}$ -$\mathbf{Weil_1}$ *are the powers* W^n *of* W*), we have a corresponding pullback*

$$\begin{tikzcd} F_0(A \otimes (B \times C)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_0(A \otimes B) \arrow[d] \\ F_0(A \otimes C) \arrow[r] & F_0(A) \end{tikzcd}$$

in $\mathrm{End}(\mathcal{M})$*, which we may also equivalently express as*

$$\begin{tikzcd} F_0(A)F_0(B \times C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_0(A)F_0(B) \arrow[d] \\ F_0(A)F_0(C) \arrow[r] & F_0(A)\ . \end{tikzcd}$$

We shall also refer to these as foundational pullbacks (in $\mathrm{End}(\mathcal{M})$*).*

Proof. The square in $\mathrm{End}(\mathcal{M})$ being a pullback is a direct consequence of the axioms of $\mathbb{T}$. ■

Now that we have given some treatment of the action of our proposed functor on objects, we turn our attention to the morphisms. We begin by defining a collection Ψ of pairs of morphisms, one of which comes from $\mathbb{N}$-$\mathbf{Weil_1}$ and the other from $\mathrm{End}(\mathcal{M})$.

We will then show that this collection Ψ will give the action of our proposed functor on morphisms. Namely, we will show that these pairings both "preserve" composition and that each morphism $f \colon A \to B$ of $\mathbb{N}$-$\mathbf{Weil_1}$ is paired with a natural transformation.

Definition 3.6.3. *Let* Ψ *be a collection of pairs* $(f, \widetilde{f})$*, where* $f \colon X \to Y$ *is a morphism in* $\mathbb{N}$-$\mathbf{Weil_1}$ *and* $\widetilde{f} : F_0(X) \Rightarrow F_0(Y)$ *is a morphism in* $\mathrm{End}(\mathcal{M})$ *(i.e. a natural transformation), given as follows:*

We begin with the following pairs:

- *Each element of* $\{\varepsilon_W, \eta_W, +, l, c\}$ *is paired with its obvious counterpart* $\{p, +, \eta, l, c\}$.

- *For each object $A \in \mathbb{N}$-$\mathbf{Weil_1}$, the pair $(id_A, id_{F_0(A)})$.*
- *For each $i, n \in \mathbb{N}$ with $n \geq 2$ and $1 \leq i \leq n$, the projections $\pi_i \colon W^n \to W$ and $\widetilde{\pi}_i \colon T^{(n)} \Rightarrow T$ form a pair.*

This gives us a starting point for Ψ. Recall from Section 3.5.8 *that any map $h \colon A \to B$ of $\mathbb{N}$-$\mathbf{Weil_1}$ is equipped with a (finite) sequential set of instructions for its construction. We then iteratively add to Ψ as follows:*
Let f, g and h be maps in $\mathbb{N}$-$\mathbf{Weil_1}$, and suppose we already have pairs

$$(f, \widetilde{f}), (g, \widetilde{g}) \in \Psi \ .$$

- *If the final step of the instructions of h was to obtain h as the composite $g \circ f$, then we add to Ψ the pair $(h, \widetilde{g} \circ \widetilde{f})$. That is, we close Ψ under certain compositions.*
- *If the final step of the instructions of h was to obtain h as the tensor $g \otimes f$, then we add to Ψ the pair $(h, \widetilde{g}\widetilde{f})$. That is, we close Ψ under certain tensors.*
- *If the final step of the instructions of h was to (uniquely) induce h using f and g as*

$$\begin{tikzcd}
A \arrow[dr, dashed, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[d] \\
& B_2 \arrow[r] & C \ ;
\end{tikzcd}$$

where the pullback square is a foundational one, then consider the diagram

$$\begin{tikzcd}
F_0(A) \arrow[drr, bend left, "\widetilde{f}"] \arrow[ddr, bend right, "\widetilde{g}"'] & & \\
& F_0(B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_0(B_1) \arrow[d] \\
& F_0(B_2) \arrow[r] & F_0(C)
\end{tikzcd}$$

in $\mathrm{End}(\mathcal{M})$ *(where we use the foundational pullback in* $\mathrm{End}(\mathcal{M})$ *corresponding to the one above).*

If the exterior commutes, then by the universal property of the pullback, a unique map $\widetilde{h} \colon F_0(A) \Rightarrow F_0(B)$ will be induced. In that case, add to Ψ the pair $(h, \widetilde{h})$.

If the exterior does not commute, then we will say that "h does not have a pairing in Ψ".

Notation When describing pairs in Ψ, if f is a map in $\mathbb{N}$-$\mathbf{Weil_1}$, we will use $\widetilde{f}$ to denote the corresponding natural transformation in $\mathrm{End}(\mathcal{M})$, i.e. we have $(f, \widetilde{f}) \in \Psi$.

Definition 3.6.4. *Let Φ be the collection of all maps $h\colon A \to B$ in $\mathbb{N}$-$\mathbf{Weil_1}$ which do not have a pairing in Ψ.*

Clearly, Ψ and Φ are mutually exclusive and exhaustive collections, in the sense that any map $f\colon A \to B$ of $\mathbb{N}$-$\mathbf{Weil_1}$ is either paired with some natural transformation $\widetilde{f}\colon F_0A \Rightarrow F_0B$ (and hence in Ψ) or not well defined (and hence in Φ, but clearly not both).

For now, let us focus on Ψ.

Lemma 3.6.5. *For arbitrary $0 < n' < n$ in $\mathbb{N}$, any projection*

$$\pi'\colon W^n \to W^{n'}$$

is paired with a (canonical) natural transformation $\widetilde{\pi'}\colon T^{(n)} \Rightarrow T^{(n')}$.

Proof. Using the fact that $T(n)$ is an n-fold pullback of p, $\widetilde{\pi'}$ is given in the obvious manner. ∎

Proposition 3.6.6. *For a foundational pullback*

$$\begin{tikzcd}
A \otimes (B \times C) \arrow[r, "A\otimes\pi_B"] \arrow[d, "A\otimes\pi_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \otimes B \arrow[d, "A\otimes\varepsilon_B"] \\
A \otimes C \arrow[r, "A\otimes\varepsilon_C"'] & A
\end{tikzcd}$$

in $\mathbb{N}$-$\mathbf{Weil_1}$, and the corresponding pullback

$$\begin{tikzcd}
F_0(A)F_0(B \times C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_0(A)F_0(B) \arrow[d] \\
F_0(A)F_0(C) \arrow[r] & F_0(A)
\end{tikzcd}$$

in $\mathrm{End}(\mathcal{M})$ *(*Proposition 3.6.2*), each map of the first pullback is paired with its counterpart in the second pullback.*

Proof. The proof becomes immediate once the maps in the second pullback are labelled:

$$\begin{tikzcd}
F_0(A)F_0(B \times C) \arrow[r, "F_0(A)\widetilde{\pi_B}"] \arrow[d, "F_0(A)\widetilde{\pi_C}"'] \arrow[dr, phantom, "\lrcorner", very near start] & F_0(A)F_0(B) \arrow[d, "F_0(A)\widetilde{\varepsilon_B}"] \\
F_0(A)F_0(C) \arrow[r, "F_0(A)\widetilde{\varepsilon_C}"'] & F_0(A)\ .
\end{tikzcd}$$

∎

Lemma 3.6.7. *For any Weil algebra A in $\mathbb{N}$-$\mathbf{Weil_1}$ the unit η_A and augmentation ε_A are paired with (canonical) natural transformations $\widetilde{\eta_A}\colon 1_{\mathcal{M}} \Rightarrow F_0A$ and $\widetilde{\varepsilon_A}\colon F_0A \Rightarrow 1_{\mathcal{M}}$.*

Proof. The statement is clearly true if A is W or $\mathbb{N}$.

We then simply note that if $A = W^n$, then $\widetilde{\eta_{W^n}}$ is canonically induced using the fact that $T^{(n)}$ is an $n-fold$ pullback of the natural transformation $p\colon T \Rightarrow 1_{\mathcal{M}}$ and the natural transformation $\eta\colon 1_{\mathcal{M}} \Rightarrow T$.

$\widetilde{\varepsilon_{W^n}}$ on the other hand is simply induced as the composite $p \circ \pi_i$, for an appropriate projection π_i of the pullback $T^{(n)}$ based on the instructions for $\varepsilon_{W^n} \colon A \to \mathbb{N}$ (although each projection π_j would yield the same result).

Finally, if X and Y are objects of $\mathbb{N}$-**Weil₁** with $(\eta_X, \widetilde{\eta_X})$, $(\varepsilon_X, \widetilde{\varepsilon_X})$, $(\eta_Y, \widetilde{\eta_Y})$, $(\varepsilon_Y, \widetilde{\varepsilon_Y}) \in \Psi$, then we have

$$\widetilde{\eta_{X\otimes Y}} = \widetilde{\eta_X}\widetilde{\eta_Y}$$
$$\widetilde{\varepsilon_{X\otimes Y}} = \widetilde{\varepsilon_X}\widetilde{\varepsilon_Y}$$

■

Corollary 3.6.8. *Any zero map $z \colon A \to B$ of $\mathbb{N}$-**Weil$_1$** is paired with a (canonical) natural transformation $\widetilde{z} \colon F_0 A \Rightarrow F_0 B$.*

Proof. Since z is constructed as the composite $\eta_B \circ \varepsilon_A$, then $\widetilde{z}$ is the composite $\widetilde{\eta_B} \circ \widetilde{\varepsilon_A}$. ■

Lemma 3.6.9. *Each coefficient map a_t is paired with some (canonical) natural transformation $\widetilde{a}_t$ in Ψ.*

Proof. We know that $\widetilde{a}_0$ is the composite

$$T \xRightarrow{p} 1_{\mathcal{M}} \xRightarrow{\eta} T$$

(since this was how a_0 was constructed) and $\widetilde{a}_1$ is the identity id_T (since a_1 was the identity).

Since each a_t is then constructed recursively as

$$\begin{tikzcd} & W & \\ W \arrow[ur, "a_{t-1}"] \arrow[dr, "id_W"'] \arrow[r, dashed] & W^2 \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[r, "+"] & W \\ & W & \end{tikzcd},$$

then each $\widetilde{a}_t$ is constructed recursively in the same way. ■

We will now sequentially show that the pairings of Ψ "preserve" (arbitrary) composition, i.e. if we have arbitrary pairings $(f, \widetilde{f}), (g, \widetilde{g}), (h, \widetilde{h}) \in \Psi$ such that $h = g \circ f$ in $\mathbb{N}$-**Weil₁**, then we have $\widetilde{h} = \widetilde{g} \circ \widetilde{f}$ in $\mathrm{End}(\mathcal{M})$.

Explicitly, suppose we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "h=g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

in $\mathbb{N}$-**Weil₁**. We wish to show that

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & F_0B \arrow[r, "\widetilde{g}"] & F_0C \end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$, for all $(f, \widetilde{f}), (g, \widetilde{g}), (h = g \circ f, \widetilde{h}) \in \Psi$.

3.6.2 If $A = qW$, $B = mW$, $C = nW$, and f, g have no intersecting circles

As with Section 3.5, we shall begin with the most basic case for "preservation" of composition by the pairings in Ψ, and then sequentially build our way up to the general case.

Proposition 3.6.10. *For all $f\colon qW \to rW$ and $g\colon rW \to sW$, with neither having intersecting circles, the diagram*

$$\begin{tikzcd} T^q \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & T^r \arrow[r, "\widetilde{g}"] & T^s \end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$.

Proof. First, since f and g have no intersecting circles, then h also has no intersecting circles.

Since f has domain qW and has no intersecting circles, it can (modulo some appropriate post-composition with $c's$) be expressed in the form

$$f = f_1 \otimes \cdots \otimes f_q \otimes \eta_{q'_W} : W \otimes \cdots \otimes W \otimes k \to \xi_1 W \otimes \cdots \otimes \xi_q W \otimes q'W$$

(and f is constructed as such); where each f_i has a single circle and is either given as ε_W if $\xi_i = 0$, or constructed as the composite

$$W \xrightarrow{\hat{g}_{a_i}} W \xrightarrow{l} \ldots \xrightarrow{(\xi_i - 1)W \otimes l} \xi_i W$$

(as described in Section 3.5.1), for an appropriate coefficient map $\hat{g}_{a_i}$.

Note that we also have

$$\left(\sum_{i=1}^{q} \xi_i \right) + q' = m \ .$$

An analogous fact is true for g and h. The natural transformations $\widetilde{f}, \widetilde{g}$ and $\widetilde{h}$ are then constructed in a corresponding manner.

Now, it can be shown that for all $c, d \in \mathbb{N}$, the diagram

$$\begin{tikzcd} T \arrow[r, "\widetilde{a}_c"] \arrow[d, "\widetilde{a}_{cd}"'] & T \arrow[r, "l"] & T^2 \arrow[d, "\widetilde{a}_d T"] \\ T \arrow[rr, "l"'] & & T^2 \end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$ (recall that each natural transformation $\widetilde{a}_t$ is paired with the coefficient map a_t in Ψ). Together with the fact that the diagram

$$\begin{tikzcd} T \arrow[r, "l"] \arrow[d, "l"'] & T^2 \arrow[d, "lT"] \\ T^2 \arrow[r, "Tl"'] & T^3 \end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$ (an axiom of $\mathbb{T}$), then we have $\widetilde{g} \circ \widetilde{f} = \widetilde{h}$. ∎

3.6.3 Making A arbitrary

Proposition 3.6.11. *For all $f\colon A \to rW$ and $g\colon rW \to sW$, with neither having intersecting circles, the diagram*

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & T^r \arrow[r, "\widetilde{g}"] & T^s \end{tikzcd}$$

commutes in End($\mathcal{M}$).

Proof. First, we note that if f and g do not have intersecting circles, then neither does h.

Consider $f\colon A \to rW$. We know from Section 3.5.4 that since f has no intersecting circles, it must factor through some particular projection $\pi\colon A \to qW$ of A (as the final step in its construction), and the same is true for h.

Correspondingly, α and γ both factor through the corresponding projection $\pi\colon F_0A \to T^q$.

As such, it suffices to assume $A = qW$, and so $F_0A = T^q$. Then, we can apply Proposition 3.6.10 directly. ■

3.6.4 Making f arbitrary

Proposition 3.6.12. *For all arbitrary $f\colon A \to rW$, and $g\colon rW \to sW$ with no intersecting circles, the diagram*

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & T^r \arrow[r, "\widetilde{g}"] & T^s \end{tikzcd}$$

commutes in End($\mathcal{M}$).

Proof. Let $\{y_1, \dots, y_r\}$ denote the generators of rW and $\{z_1 \dots, z_s\}$ denote the generators of sW.

By the same argument as used in the proof for Proposition 3.6.10, then modulo appropriate post-composition with c's, we can express g as

$$g = g_1 \otimes \cdots \otimes g_r \otimes \eta_{r'_W} \colon W \otimes \cdots \otimes W \otimes k \to \nu_1 W \otimes \cdots \otimes \nu_q W \otimes r'W$$

(and g is constructed as such); where each $g_i\colon W \to \nu_i W$ has a single circle.

Without loss of generality, we shall assume that $r' = 0$ and $\nu_i > 0$ for all i. This amounts to asking that no generator y_i of rW is sent by g to zero, and that each generator z_j of sW belongs to exactly one of the r circles of $\{U\}_g$.

This then defines a surjective function

$$\psi\colon \{z_1, \dots, z_n\} \to \{y_1, \dots, y_m\} .$$

Without loss of generality, suppose that $\psi(z_1) = y_1$.

Suppose the maps h and f factorise as the composites

$$\begin{tikzcd} & W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \arrow[d, "+_\delta"] \\ A \arrow[ur, "h'"] \arrow[r, "h"'] & sW \end{tikzcd} \qquad \begin{tikzcd} & W^{\vartheta_1} \otimes \cdots \otimes W^{\vartheta_r} \arrow[d, "+_\vartheta"] \\ A \arrow[ur, "f'"] \arrow[r, "f"'] & rW \end{tikzcd} .$$

(and are constructed as such, recall Lemma 3.5.12).

Note that correspondingly, $\widetilde{f}$ and $\widetilde{h}$ are given as composites

$$\begin{tikzcd} & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[d, "\widetilde{+_\delta}"] \\ F_0 A \arrow[ur, "\widetilde{h}'"] \arrow[r, "\widetilde{h}"'] & T^s \end{tikzcd} \qquad \begin{tikzcd} & T^{(\vartheta_1)} \otimes \cdots \otimes T^{(\vartheta_r)} \arrow[d, "\widetilde{+_\vartheta}"] \\ F_0 A \arrow[ur, "\widetilde{f}'"] \arrow[r, "\widetilde{f}"'] & T^r \end{tikzcd} ,$$

noting that since $+\colon T^{(2)} \Rightarrow T$ is an associative, commutative and unital map, then there is a well defined map

$$+_{(\delta_i)}\colon T^{(\delta_i)} \Rightarrow T$$

for each i, and finally, we define

$$\widetilde{+_\delta}\colon = +_{(\delta_1)} \otimes \cdots \otimes +_{(\delta_s)}$$

in $\mathrm{End}(\mathcal{M})$. The map $\widetilde{+_\vartheta}$ is defined in an analogous manner.

Firstly, this means that the map h has precisely δ_1 circles (say $U_1, \ldots, U_{\delta_1}$) containing the generator z_1. But given what we've established about g, and noting that $h = g \circ f$, then the z_1 term in each of these U_i must arise as a result of the generator y_1 (since $\psi(z_1) = y_1$). More explicitly, to each circle U_i of h containing z_1 we can associate a unique circle of f containing y_1.

Conversely, for each circle V_j of f containing y_1, we have $g(V_j) \neq 0$ (moreover, $g(V_j)$ is a single circle) and $z_1 \in g(V_j)$. Therefore the number of circles of f containing y_1 (namely ϑ_1) is the same as the number of circles of h containing z_1 (namely α_1). Thus, if $\psi(z_i) = y_j$, then $\delta_i = \vartheta_j$.

We then define a map

$$\Lambda\colon W^{\vartheta_1} \otimes \cdots \otimes W^{\vartheta_r} \to W^{\delta_1} \otimes \cdots \otimes W^{\delta_s}$$

induced using

$$\begin{tikzcd} & rW \arrow[rr, "g"] & & sW \\ W^{\vartheta_1} \otimes \cdots \otimes W^{\vartheta_r} \arrow[ur, "t"] \arrow[rr, dashed, "\exists!\ \Lambda"'] & & W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \arrow[ur, "\overline{a}"'] & \end{tikzcd} , \tag{3.1}$$

where, for each fixed projection $\overline{a} = (a_1, \ldots, a_s)$, t is determined as follows:

- Consider $\overline{a} \circ h'\colon A \to sW$. If this is the zero map, then t is also the zero map.
- If not, this means that there is at least one circle U of h (and hence h') with each of its generators preserved by $\overline{a}$. Moreover, if h has multiple circles, then they must be disjoint and each corresponds to a different generator of A (see Section 3.5.5).

  Without loss of generality, assume there is only one such circle U. Regard U as a subset of $\{z_1, \ldots, z_s\}$. Then we know $\psi(U)$ (the image of U under ψ) is the unique circle of f corresponding to U. Choose t (in the unique way) so that this circle $\psi(U)$ of f is preserved, but sends any $y_j \notin \psi(U)$ to 0.

We shall also note that there is a corresponding natural transformation $\widetilde{t}$ (i.e. $(t, \widetilde{t}) \in \Psi$). We shall not prove this in full. Instead simply note that for the second point above, t is constructed (modulo some appropriate post-composition with c's) as a tensor $t_1 \otimes \cdots \otimes t_r$, where each t_i is of the form

$$t_i \colon W^{\vartheta_i} \to \rho_i W$$

(and $\rho_1 + \cdots + \rho_r = r$). Each t_i has at most one circle, and so factors through an appropriate projection $\pi_j \colon W^{\vartheta_i} \to W$.

Now, it is fairly routine (albeit tedious) to show that Λ is paired with some unique natural transformation $\widetilde{\Lambda}$ in Ψ (i.e. that it exists). It can also be shown that the diagram

$$\begin{tikzcd}
W^{\vartheta_1} \otimes \cdots \otimes W^{\vartheta_r} \arrow[r, "\Lambda"] \arrow[d, "+_\vartheta"'] & W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \arrow[d, "+_\delta"] \\
rW \arrow[r, "g"'] & sW
\end{tikzcd}$$

commutes in $\mathbb{N}$-$\mathbf{Weil_1}$ (and that the corresponding diagram commutes in $\mathrm{End}(\mathcal{M})$). We now have the following diagram

$$\begin{tikzcd}
 & & T^r \arrow[ddrr, bend left, "\widetilde{g}"] & & \\
 & & T^{(\vartheta_1)} \otimes \cdots \otimes T^{(\vartheta_r)} \arrow[u, "\widetilde{+_\vartheta}"'] \arrow[d, "\widetilde{\Lambda}"] & & \\
F_0A \arrow[uurr, bend left, "\widetilde{f}"] \arrow[urr, "\widetilde{f'}"] \arrow[rr, "\widetilde{h'}"] \arrow[rrrr, bend right, "\widetilde{h}"'] & & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[rr, "\widetilde{+_\delta}"] & & T^n
\end{tikzcd},$$

and to show the exterior commutes, it suffices to show that

$$\begin{tikzcd}
 & T^{(\vartheta_1)} \otimes \cdots \otimes T^{(\vartheta_r)} \arrow[d, "\widetilde{\Lambda}"] \\
FA \arrow[ur, "\widetilde{f'}"] \arrow[r, "\widetilde{h'}"'] & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)}
\end{tikzcd}$$

commutes.

Since $T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)}$ is a limit with projections $(b_1, \ldots, b_s) = \pi_{b_1} \otimes \cdots \otimes \pi_{b_s}$, and since Λ (and hence $\widetilde{\Lambda}$) was given using (3.1), then it suffices to show that

$$\overline{b} \circ \widetilde{\Lambda} \circ \widetilde{f'} = \overline{b} \circ \widetilde{h'}$$

for all projections $\overline{b} = (b_1, \ldots, b_s)$.

Consider the diagram

$$\begin{tikzcd}
F_0A \arrow[rr, "\widetilde{f'}"] \arrow[dr, "\widetilde{t \circ f'}"] \arrow[ddrr, bend right, "\widetilde{\overline{b} \circ h'}"'] \arrow[dd, "\widetilde{h'}"'] & & T^{(\vartheta_1)} \otimes \cdots \otimes T^{(\vartheta_r)} \arrow[dl, "\widetilde{t}"'] \arrow[d, "\widetilde{\Lambda}"] \\
 & T^r \arrow[dr, "\widetilde{g}"] & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[d, "\overline{a}"] \\
T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[rr, "\overline{b}"'] & & T^s
\end{tikzcd}$$

(again, we leave as an exercise to the reader to verify that all the necessary pairs in Ψ exist and are well defined).

To show the commutativity of the exterior, we first note that the lower left triangle and right square commute by construction, and further that it is routine to check that the top triangle commutes. So all that remains is to verify the commutativity of the innermost triangle.

But since $t \circ f'$ by definition has no intersecting circles, then we can apply Proposition 3.6.11 directly. ■

3.6.5 Making B arbitrary

Proposition 3.6.13. *For all arbitrary $f\colon A \to B$, and $g\colon B \to sW$ with no intersecting circles, the diagram*

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & F_0B \arrow[r, "\widetilde{g}"] & T^s \end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$.

Proof. Using Lemma 3.5.14 and the results from Section 3.3, we can see that if the graph Γ_B contains any edges, then B is part of a (foundational) pullback

$$\begin{tikzcd} B = B' \otimes (B_1 \times B_2) \arrow[r, "B'\otimes\pi_1"] \arrow[d, "B'\otimes\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & B' \otimes B_1 \arrow[d, "B'\otimes\varepsilon_{B_1}"] \\ B' \otimes B_2 \arrow[r, "B'\otimes\varepsilon_{B_2}"'] & B' \end{tikzcd} .$$

Further, recall the proof of Lemma 3.5.11. Since $g\colon B \to sW$ has no intersecting circles, then if Γ_B has any edges, we know that g must then factorise through one of B's (foundational) projections, say as

$$B \xrightarrow{B'\otimes\pi_1} B' \otimes B_1 \xrightarrow{\gamma} sW$$

(and moreover, this would be in the instructions for its construction). The same is thus true of $\widetilde{g}$. We shall simply denote the projection as π_1 for convenience.

We now have

$$\begin{tikzcd} & & F_0B'F_0B_1 \arrow[d, "\widetilde{\gamma}"] \\ F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & F_0B \arrow[ur, "\widetilde{\pi_1}"] \arrow[r, "\widetilde{g}"] & T^s \end{tikzcd} ,$$

and note that the map $\widetilde{\pi_1}\colon F_0B \to F_0B'F_0B_1$ is part of a foundational pullback in $\mathrm{End}(\mathcal{M})$ (Proposition 3.6.2).

As such, this tells us that $\widetilde{\pi_1} \circ \widetilde{f} = \widetilde{\pi_1 \circ f}$ (Definition 3.6.3). It thus suffices to show the commutativity of

$$\begin{tikzcd} & F_0B'F_0B_1 \arrow[d, "\widetilde{\gamma}"] \\ F_0A \arrow[ur, "\widetilde{\pi_1\circ f}"] \arrow[r, "\widetilde{h}"'] & sW \end{tikzcd} .$$

But we know that since g has no intersecting circles, then neither does γ, and we can thus repeat this iteratively until there are no more edges in B, i.e. we have $B = rW$ and apply Proposition 3.6.12 directly. ■

3.6.6 Making g arbitrary

Proposition 3.6.14. *For all arbitrary $f\colon A \to B$ and $g\colon B \to sW$, the diagram*

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & F_0B \arrow[r, "\widetilde{g}"] & T^s \end{tikzcd}$$

commutes in End($\mathcal{M}$)*.*

Proof. Recall from the proof of Lemma 3.5.12 that g factorises as the composite

$$\begin{tikzcd} & W^{\beta_1} \otimes \cdots \otimes W^{\beta_n} \arrow[d, "+_\beta"] \\ B \arrow[ur, "g'"] \arrow[r, "g"'] & sW \end{tikzcd}$$

(and is constructed as such). Thus, $\widetilde{g}$ is constructed as the corresponding composite

$$\begin{tikzcd} & T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_n)} \arrow[d, "+_\beta"] \\ F_0B \arrow[ur, "\widetilde{g'}"] \arrow[r, "\widetilde{g}"'] & T^n \end{tikzcd}$$

Recall that we also said $\widetilde{h}$ was constructed as the composite

$$\begin{tikzcd} & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[d, "+_\delta"] \\ F_0A \arrow[ur, "\widetilde{h'}"] \arrow[r, "\widetilde{h}"'] & T^s\ . \end{tikzcd}$$

We now have the following diagram

$$\begin{tikzcd} & F_0B \arrow[d, "\widetilde{g'}"] \arrow[ddr, bend left, "\widetilde{g}"] & \\ & T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_n)} \arrow[dr, "+_\beta"] & \\ F_0A \arrow[uur, bend left, "\widetilde{f}"] \arrow[ur, "\widetilde{g' \circ f}"] \arrow[r, "\widetilde{h'}"] \arrow[rr, bend right, "\widetilde{h}"'] & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[r, "+_\delta"] & T^s\ ; \end{tikzcd} \tag{3.2}$$

for which we wish to show the commutativity of the exterior.

We already know the bottom triangle as well as top right triangle commute by construction. We begin with the innermost square

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{g' \circ f}"] \arrow[d, "\widetilde{h'}"'] & T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_n)} \arrow[d, "+_\beta"] \\ T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[ur, dashed, "\widetilde{\Omega}"'] \arrow[r, "+_\delta"'] & FB\ , \end{tikzcd} \tag{3.3}$$

and introduce the map $\widetilde{\Omega}$ (constructed in a similar manner as Ω, as described in 3.5.9). Recall that

$$\Omega\colon W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \to W^{\beta_1} \otimes \cdots \otimes W^{\beta_n}$$

assigned each generator of the domain to a particular generator in the codomain, and moreover we have $\Omega = \Omega_1 \otimes \cdots \otimes \Omega_s$ for $\Omega_i\colon W^{\delta_i} \to W^{\beta_i}$.

First, it now becomes rather routine to show that

$$+_\delta = +_\beta \circ \widetilde{\Omega}$$

(i.e. the lower triangle in (3.3)). To show $\widetilde{g' \circ f} = \widetilde{\Omega} \circ \widetilde{h'}$ in $\mathrm{End}(\mathcal{M})$, note first that $g' \circ f = \Omega \circ h'$ in $\mathbb{N}$-$\mathbf{Weil_1}$ by design. So equivalently, we can show that

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{h'}"] \arrow[rr, bend right, "\widetilde{\Omega \circ h'}"'] & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[r, "\widetilde{\Omega}"] & T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_s)} \end{tikzcd}$$

commutes. But recall that $T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_n)}$ is a limit (constructed as iterations of foundational pullbacks in $\mathrm{End}(\mathcal{M})$) with projections $\overline{r} = (r_1, \ldots, r_s)$. As such, it suffices to show the commutativity of

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{h'}"] \arrow[rr, bend right, "\widetilde{\overline{r} \circ \Omega \circ h'}"'] & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \arrow[r, "\overline{r} \circ \widetilde{\Omega}"] & T^s \end{tikzcd}$$

for each $\overline{r}$.

But noting that $\Omega = \Omega_1 \otimes \cdots \otimes \Omega_s$ and $\overline{r} = \pi_{r_1} \otimes \cdots \otimes \pi_{r_s}$, and noting the form of each $\pi_{r_i} \circ \Omega_i\colon T^{(\alpha_1)} \otimes \cdots \otimes T^{(\alpha_n)} \to T$ from 3.5.9, then the commutativity of the upper triangle in (3.3) above is immediate.

Hence, all that remains is to show the commutativity of the upper left triangle of (3.2), namely the commutativity of

$$\begin{tikzcd} & F_0B \arrow[d, "\widetilde{g'}"] \\ F_0A \arrow[ur, "\widetilde{f}"] \arrow[r, "\widetilde{g' \circ f}"'] & T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_s)} . \end{tikzcd}$$

Again, since $T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_n)}$ is a limit, it suffices to show the commutativity of

$$\begin{tikzcd} & F_0B \arrow[d, "\overline{r} \circ \widetilde{g'} = \widetilde{\overline{r} \circ g'}"] \\ F_0A \arrow[ur, "\widetilde{f}"] \arrow[r, "\widetilde{\overline{r} \circ g' \circ f}"'] & T^s \end{tikzcd}$$

for each projection $\overline{r}$.

Finally, note that by definition, each map

$$\overline{r} \circ g'\colon B \to nW$$

has no intersecting circles, so we may apply Proposition 3.6.13 directly. ∎

3.6.7 Making C arbitrary

Proposition 3.6.15. *For all arbitrary $f\colon A \to B$ and $g\colon B \to C$, the diagram*

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & F_0B \arrow[r, "\widetilde{g}"] & F_0C \end{tikzcd}$$

commutes in $\mathrm{End}(\mathcal{M})$.

Proof. Using the same argument as in the proof of Proposition 3.6.13, if the graph Γ_C contains any edges, then C is part of a (foundational) pullback

$$\begin{tikzcd} C = C' \otimes (C_1 \times C_2) \arrow[r, "C'\otimes\pi_1"] \arrow[d, "C'\otimes\pi_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C' \otimes C_1 \arrow[d, "C'\otimes\varepsilon_{C_1}"] \\ C' \otimes C_2 \arrow[r, "C'\otimes\varepsilon_{C_2}"'] & C' . \end{tikzcd}$$

Correspondingly, F_0C is part of the pullback

$$\begin{tikzcd} F_0C \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & F_0C'F_0C_1 \arrow[d] \\ F_0C'F_0C_2 \arrow[r] & F_0C' . \end{tikzcd}$$

We now have

$$\begin{tikzcd} & & & F_0C_1 \\ F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{h}"'] & F_0B \arrow[r, "\widetilde{g}"] & F_0C \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & & & F_0C_2 \end{tikzcd}$$

in $\mathrm{End}(\mathcal{M})$, and using the fact that $\pi_i \circ \widetilde{g} = \widetilde{\pi_i \circ g}$ for $i = 1, 2$ (and a corresponding fact for h), it suffices to show the commutativity of

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[rr, bend right, "\widetilde{\pi_i\circ h}"'] & F_0B \arrow[r, "\widetilde{\pi_i\circ g}"] & F_0C_i \end{tikzcd}$$

for each i. Using this argument iteratively, it suffices to assume the graph Γ_C has no edges, i.e. $C = sW$ and apply Proposition 3.6.14 directly. ∎

We have now shown that the pairings of the collection Ψ "preserve" arbitrary compositions. We now need to consider the collection Φ (Definition 3.6.4).

3.6.8 The Problem with Pullbacks

As we mentioned in Definition 3.6.3, we may have maps $f, g, h \in \mathbb{N}$-$\mathbf{Weil_1}$ for which the final step of the instructions for h is to uniquely induce it using f and g as

$$\begin{tikzcd}
A \arrow[dr, dashed, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[d] \\
& B_2 \arrow[r] & C ,
\end{tikzcd}$$

but the exterior of

$$\begin{tikzcd}
F_0A \arrow[drr, bend left, "\widetilde{f}"] \arrow[ddr, bend right, "\widetilde{g}"'] & & \\
& F_0B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_0B_1 \arrow[d] \\
& F_0B_2 \arrow[r] & F_0C
\end{tikzcd}$$

does not commute, and so we cannot induce $\widetilde{h} \colon F_0A \to F_0B$ using the universal property of the relevant pullback, and said that $\widetilde{h}$ is "not well defined". We then defined Φ to be the set of such "undefined" maps (Definition 3.6.4). We will now show that this set Φ is in fact empty.

Proposition 3.6.16. *The collection Φ is empty.*

Proof. Suppose that Φ is non-empty. Then for each $f \in \Phi$ (with $f \colon A \to B$ a map in $\mathbb{N}$-$\mathbf{Weil_1}$), let $n(f)$ be the number of vertices in the graph Γ_B. Finally, let $N(\Phi) = \{n(f) \mid f \in \Phi\}$.

Since $N(\Phi)$ is a non-empty subset of $\mathbb{N}$, then by the well ordering principle, it has a least element n. Choose a map $h \colon A \to B$ corresponding to this least element n. Now, consider the cograph Γ_B. If Γ_B is discrete (so that $B = nW$), then we construct $\widetilde{h}$ directly using the ideas of Section 3.5.12. As such, let B have at least one edge.

We then have the diagram

$$\begin{tikzcd}
A \arrow[dr, dashed, "h"] \arrow[drr, bend left, "f=\pi_1\circ h"] \arrow[ddr, bend right, "g=\pi_2\circ h"'] & & \\
& B = (B_1 \times B_2) \otimes C \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \otimes C \arrow[d, "\varepsilon\otimes C"] \\
& B_2 \otimes C \arrow[r, "\varepsilon_2\otimes C"'] & C
\end{tikzcd}$$

and noting that since Γ_B has at least one edge, then $\Gamma_{B_1\otimes C}$ and $\Gamma_{B_2\otimes C}$ each have strictly fewer vertices in their respective cographs than Γ_B. Thus, $\widetilde{f}$ and $\widetilde{g}$ are both well defined.

We wish to show the commutativity of

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{g}"'] & F_0B_1 \otimes F_0C \arrow[d] \\ F_0B_2 \otimes C \arrow[r] & F_0C \end{tikzcd}$$

so that $\widetilde{h}$ can be induced using the foundational pullback in $\mathrm{End}(\mathcal{M})$.

Let $\psi = (\varepsilon_1 \otimes C) \circ f \colon A \to C$ in $\mathbb{N}$-$\mathbf{Weil_1}$, i.e. the composite

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\psi"'] & B_1 \otimes C \arrow[d, "\varepsilon_1 \otimes C"] \\ & C \end{tikzcd} .$$

Since Γ_C has strictly fewer vertices than Γ_B, then $\widetilde{\psi}$ is also well defined. But by Proposition 3.6.15, each of the triangles in the diagram

$$\begin{tikzcd} F_0A \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{g}"'] \arrow[dr, "\widetilde{\psi}"] & F_0B_1 \otimes F_0C \arrow[d] \\ F_0B_2 \otimes C \arrow[r] & F_0C \end{tikzcd}$$

commute in $\mathrm{End}(\mathcal{M})$, and thus the exterior commutes.

Therefore $\widetilde{h}$ is well defined. Thus the original assumption is incorrect, i.e. Φ is an empty set. ■

What we have shown then is that F_0 and the pairings of Ψ together define a functor.

3.6.9 The Functor F and the universality of $\mathbf{Weil_1}$

We now have the following:

Theorem 3.6.17. *Suppose we have a given category $\mathcal{M}$. Regard* $\mathrm{End}(\mathcal{M})$ *as a monoidal category with respect to composition and* $\mathbb{N}$-$\mathbf{Weil_1}$ *as monoidal with respect to coproduct.*

Then to give a Tangent Structure $\mathbb{T}$ to $\mathcal{M}$ is equivalent (up to isomorphism) to giving a strong monoidal functor $F \colon \mathbb{N}$-$\mathbf{Weil_1} \to \mathrm{End}(\mathcal{M})$ satisfying the following conditions:

1) Given a product $A = A_1 \times A_2$ in $\mathbb{N}$-$\mathbf{Weil_1}$, regarded as a pullback of the augmentations, and an arbitrary Weil algebra $B \in \mathbb{N}$-$\mathbf{Weil_1}$, then F preserves the pullback

$$\begin{tikzcd} B \otimes A \arrow[r, "B\otimes\pi_1"] \arrow[d, "B\otimes\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \otimes A_1 \arrow[d, "B\otimes\varepsilon_1"] \\ B \otimes A_2 \arrow[r, "B\otimes\varepsilon_2"'] & B \end{tikzcd}$$

i.e. it preserves all "foundational pullbacks" of $\mathbb{N}$-$\mathbf{Weil_1}$ *(as defined in* Definition 2.1.16*).*

2) The equaliser

$$W^2 \xrightarrow{v} 2W \underset{\eta_W \circ (\varepsilon_W \otimes \varepsilon_W)}{\overset{W \otimes \varepsilon_W}{\rightrightarrows}} W$$

as given in Section 3.2 *is preserved.*

Proof. Given such a functor F, the corresponding Tangent Structure is given as

$$\mathbb{T} = (FW, F\varepsilon_W, F\eta_W, F{+}, Fl, Fc) \ ,$$

and it can be readily verified that this satisfies all the necessary conditions to be a Tangent Structure.

Conversely, suppose we have a Tangent Structure $\mathbb{T}$. Then $F_0\colon ob(\mathbb{N}\text{-}\mathbf{Weil_1}) \to ob(\mathrm{End}(\mathcal{M}))$ and Ψ give us our assignations for objects and morphisms, and Propositions 3.6.15 and 3.6.16 together give functoriality.

Moreover, F_0 actually makes F monoidal (see Definition 3.6.1). F being strong monoidal as well as the preservation of foundational pullbacks is then a direct consequence of the fact that we are using composition as the monoidal structure of $\mathrm{End}(\mathcal{M})$ together with Proposition 3.6.2.

Finally, preservation of the equaliser

$$W^2 \xrightarrow{v} 2W \underset{\eta_W \circ (\varepsilon_W \otimes \varepsilon_W)}{\overset{W \otimes \varepsilon_W}{\rightrightarrows}} W$$

is trivial, since it is a condition of $\mathbb{T}$ that the corresponding fork in $\mathrm{End}(\mathcal{M})$ is also an equaliser. ■

We have thus shown that to equip a category $\mathcal{M}$ with a Tangent Structure $\mathbb{T}$ is equivalent to giving (up to a suitable isomorphism) a strong monoidal functor $F\colon \mathbb{N}\text{-}\mathbf{Weil_1} \to \mathrm{End}(\mathcal{M})$ satisfying some extra properties.

As such, $\mathbb{N}$-$\mathbf{Weil_1}$ becomes an initial Tangent Structure in the sense that it characterises any Tangent Structure $\mathbb{T}$ via this functor F.

We also note that this functor F only required that $\mathrm{End}(\mathcal{M})$ was a monoidal category (with respect to composition and with unit $1_{\mathcal{M}}$) and that certain pullbacks were preserved. As a result, we make the following generalisation.

Definition 3.6.18. *Let $(\mathcal{G}, \square, I)$ be a monoidal category. Regard the category* $\mathbb{N}$-$\mathbf{Weil_1}$ *as monoidal with respect to coproduct and having unit* $\mathbb{N}$*. A* Tangent Structure $\mathbb{G}$ internal to $\mathcal{G}$ *is a strong monoidal functor*

$$F\colon (\mathbb{N}\text{-}\mathbf{Weil_1}, \otimes, \mathbb{N}) \to (\mathcal{G}, \square, I)$$

satisfying the following conditions:

1) F preserves foundational pullbacks

2) The equaliser

$$W^2 \xrightarrow{v} 2W \underset{\eta_W \circ (\varepsilon_W \otimes \varepsilon_W)}{\overset{W \otimes \varepsilon_W}{\rightrightarrows}} W$$

is preserved

Corollary 3.6.19. *A Tangent Structure on $\mathcal{M}$ (in the sense of* Theorem 3.6.17*) is the same as a Tangent Structure internal to* $\mathrm{End}(\mathcal{M})$ *(in the sense of* Definition 3.6.18*).*

In fact, Definition 3.6.18 actually gives a universal property of the category $\mathbb{N}$-$\mathbf{Weil_1}$ in relation to Tangent Structures. One way we might express this is that Tangent Structures are simply models of $\mathbb{N}$-$\mathbf{Weil_1}$ (regarded as a theory).

Remark In the spirit of Definition 3.6.18 and Corollary 3.6.19, we will refer to $T = F(W)$ as a *tangent object* rather than a tangent functor.

Chapter 4

The category $\mathbf{Weil}_\infty$

In the previous chapter, we demonstrated that Tangent Structure was somehow "encoded" within the category $\mathbb{N}$-$\mathbf{Weil_1}$ (or more broadly, just $\mathbf{Weil_1}$) in some appropriate sense. It is only natural then that we may extend the world of Tangent Structure by extending our category of Weil algebras. Of course, there are many choices available when considering which Weil algebras to include.

One could try to work with **Weil** in its full entirety, and use the approach described in Section 1.4 in the introduction. This would certainly not be unreasonable, since functors are discussed in differential geometry without any restrictions imposed on the permissible Weil algebras (see Section 35 of [18], or [31] for more general discussions, or see [15] for discussion in the context of SDG). This is, however, beyond the scope of this thesis.

In place of **Weil**, we will begin this chapter by introducing a (sub)category $\mathbf{Weil}_\infty$ (more specifically, k-$\mathbf{Weil}_\infty$). The nature of our discussion here will be far more exploratory than that of Chapter 3. We shall first describe, for $k = \mathbb{2}$, $\mathbb{N}$, $\mathbb{Z}$ or an arbitrary field, a canonical way to equip each object X of k-$\mathbf{Weil}_\infty$ with a diagram $\Upsilon_X \colon \mathcal{B}_X^{op} \to k\text{-}\mathbf{Weil}_\infty$ (Definition 4.1.14) for which X is the limit (Theorem 4.1.16). This observation then yields one possible way to define what we may call a "Tangent Structure corresponding to k-$\mathbf{Weil}_\infty$ " through Definition 4.1.18; namely, as a strong monoidal functor preserving the limits in question.

However, we note that under a mild condition on k (namely that it contains the positive rationals $\mathbb{Q}_{>0}$), we can equip each object X of k-$\mathbf{Weil}_\infty$ with a diagram $\Upsilon'_X \colon \mathcal{B}'^{op}_X \to k\text{-}\mathbf{Weil}_\infty$ (Definition 4.2.9) which factors through k-$\mathbf{Weil_1}$. Once again, X will be the limit of this diagram (Theorem 4.2.10). This will ultimately allow us to show (in Theorem 4.3.13) that a strong monoidal functor $F_\infty \colon k\text{-}\mathbf{Weil}_\infty \to \mathcal{G}$ which preserves these limits (as in Definition 4.3.1) is a monoidal right Kan extension of $F \colon \mathbf{Weil_1} \to \mathcal{G}$ (in the sense of Definition 3.6.18.

Before we begin Section 4.1, we will also note that the "universality of vertical lift" equaliser (see Section 3.2) will be dropped from this chapter's discussion. The rationale here is that the counterpart equaliser in **Weil** is not strictly necessary to the process we will be describing. However, it is common practice in tangent category theory to require extra limits as necessary, and so one could easily ask that a functor $F_\infty \colon k\text{-}\mathbf{Weil}_\infty \to \mathcal{G}$ also preserve this equaliser (as well as other limits, as appropriate).

4.1 Characterising k-Weil$_\infty$

We begin with an explicit definition of k-**Weil**$_\infty$:

Definition 4.1.1. *k-**Weil**$_\infty$ is the full subcategory of **AugAlg** whose objects are all those algebras equipped with a chosen presentation*

$$k[x_1, \dots, x_n]/\{\text{some collection } I \text{ of monomials in the } x_i\text{'s}\}$$

in such a way that the resulting algebra is a Weil algebra, namely that for each $i = 1, \dots, n$, we have $x^{r_i} \in I$ for some $r_i \in \mathbb{N}_{>1}$ and, as usual, with $\varepsilon(x_i) = 0$ for all i.

We will also need the following:

Definition 4.1.2. *For each $n \in \mathbb{N}$, let W_n be the Weil algebra*

$$k[x]/x^{n+1}$$

Definition 4.1.3. *For each $m, n \in \mathbb{N}$, let $W_{m,n}$ be the Weil algebra*

$$k[x_1, \dots, x_m]/\{\text{all monomials of degree } n+1\}$$

Remark Note that $W_{1,n} = W_n$, $W = k[x]/x^2 = W_1$, $W_{m,1} = (W_1)^m$ and $W_{0,n} = W_{m,0} = k$.

Remark We shall still opt to use the notation W instead of W_1 for the Weil algebra $k[x]/x^2$.

We now introduce a canonical way to exhibit each object of k-**Weil**$_\infty$ as a limit of a diagram involving tensors of W_n's.

4.1.1 Partially ordered sets and inclusions

For the moment, let us work just with (partially ordered) sets. Consider the commutative monoid (not the rig) $\mathbb{N}$ under addition together with its total order. For any positive integer r, $\mathbb{N}^r$ is a monoid in the obvious manner, and we shall equip it with the product partial order $\leq$, namely

$$(a_1, \dots, a_r) \leq (b_1, \dots, b_r) \Leftrightarrow a_i \leq b_i \ \forall i$$

Definition 4.1.4. *A subset $S \subset \mathbb{N}^r$ is a* down-set *if it is finite, down-closed and non-empty.*

Definition 4.1.5. *A down-set $S \subset \mathbb{N}^r$ is* prismatic *if*

$$S = S_1 \times \cdots \times S_r$$

*(the product being taken in **Set**), where each S_i is a down-set of $\mathbb{N}$ (i.e. each S_i is of the form $\{0, 1, \dots, n_i\}$)*

Remark To say a down-set $S \subset \mathbb{N}^r$ is prismatic is to say that

$$S = \{x \in \mathbb{N}^r \mid x \leq y\}$$

for some $y \in \mathbb{N}^r$. Such a down-set is often called a *principal down-set* of $\mathbb{N}^r$ in order theory, and would refer to y as the *principal element* of this down-set. This is named in analogy with the principal ideal for a ring.

Lemma 4.1.6. *Any down-set $S \subset \mathbb{N}^r$ can be expressed as a union $\bigcup_{i=1}^{m} R_i$; where each R_i is a prismatic down-set.*

Proof. Since $S \subset \mathbb{N}^r$, then it is also a partially ordered set. Since S is finite and non-empty, then we can form the (non-empty) set $\{\underline{x}_1, \ldots, \underline{x}_m\}$ of maximal elements of S. Finally, for each i, let $R_i \subset \mathbb{N}^r$ be the prismatic down-set with $\underline{x}_i$ as its principal element.

Alternatively, we may (for each r) first take the poset $\mathbb{N}^r$ regarded as a category enriched in $\mathbf{2} = \{0 \to 1\}$. The down-set S then can be regarded as a functor

$$S\colon (\mathbb{N}^r)^{op} \to \mathbf{2}$$

where $s \in S \iff S(s) = 1$. As such, S is a presheaf, and all presheafs are colimits of representables. ■

Given $S = \bigcup_{i=1}^{m} R_i$ as above, let $Q_{i,j} = R_i \cap R_j$ whenever $1 \leq i < j \leq m$. Clearly, since R_i and R_j are both prismatic, then so is $Q_{i,j}$.

Consider now the diagram

$$\begin{tikzcd}
Q_{1,2} \arrow[d, hook] \arrow[dr, hook] & Q_{1,3} \arrow[dl, hook] \arrow[dr, hook] & & \cdots & & Q_{m-1,m} \arrow[dl, hook] \arrow[d, hook] \\
R_1 & R_2 & R_3 & \cdots & R_{m-1} & R_m
\end{tikzcd}$$

in **Set** (say). It can be readily shown that the colimit of this diagram is precisely S (with the obvious subset inclusions from the R_i's into S).

Notation For a given S, we shall say that the R_i's form the *first layer* and the $Q_{i,j}$'s form the *second layer* of the diagram for which S is the colimit.

We now have the following:

Definition 4.1.7. *For each $r \in \mathbb{N}$, let $\mathcal{N}_{(\mathbf{r})}$ be the poset consisting of all down-sets of $\mathbb{N}^r$, ordered by inclusion.*

Remark $\mathcal{N}_{(\mathbf{r})}$ is not the poset $\mathbb{N}^r$ itself.

4.1.2 The objects of k-$\mathbf{Weil}_\infty$

Definition 4.1.8. *Let $S \subset \mathbb{N}^r$ be a given down-set. Let $\kappa(S)$ denote the (finitely generated and free) k-module with (finite) generating set S. Further, define multiplication on basis elements $u, v \in \kappa(S)$ as*

$$\mu(u \otimes v) = u + v$$

(using the addition of $\mathbb{N}^r$) if $u+v \in S$, 0 otherwise, and extend linearly. This turns $\kappa(S)$ into an algebra. Finally, equip $\kappa(S)$ with an augmentation $\varepsilon_S \colon \kappa(S) \to k$ that sends the unit of $\mathbb{N}^r$ to 1 and all other elements of S to 0.

Defined this way, $\kappa(S)$ is then a Weil algebra of the form described in Definition 4.1.1.

Example 4.1.9. *Let $r = 2$, and let S be the down-set $\{(0,0),(1,0),(2,0),(0,1),(1,1)\}$. We first regard each (a,b) as an element $x^a y^b$. Then $\kappa(S)$ has $\{1, x, x^2, y, xy\}$ as the generating set for its underlying k-module. The multiplication is then obvious; noting that $y^2 = 0$, $x^2 y = 0$ and $x^3 = 0$.*

The augmentation is the map sending each of x, x^2, y, xy to zero. Finally, we extend the multiplication and augmentation linearly.

Quite clearly, we also have the following:

Corollary 4.1.10. *For a down-set $S = \{0, 1, \dots, n\} \subset \mathbb{N}$, $\kappa(S)$ is precisely W_n.*

Corollary 4.1.11. *For given down-sets $S_1 \in \mathcal{N}_{(r)}$ and $S_2 \in \mathcal{N}_{(s)}$, if $S = S_1 \times S_2$ (the product being taken in* **Set***), then (with some abuse of notation) we have $S \in \mathcal{N}_{(r+s)}$, and further $\kappa(S) = \kappa(S_1) \otimes \kappa(S_2)$.*

In particular, if S is a prismatic down-set (in the sense of Definition 4.1.5*), then $\kappa(S)$ is an appropriate tensor of W_n's.*

Definition 4.1.12. *For fixed $r \in \mathbb{N}$, let*

$$\kappa \colon \mathcal{N}_{(r)}^{op} \to k\text{-}\mathbf{Weil}_\infty$$

be the functor given as:

- *On objects: $\kappa(S)$ as given in* Definition 4.1.8.
- *On morphisms: For a subset inclusion $i \colon S \hookrightarrow S'$ in $\mathcal{N}_{(\mathbf{r})}$, define $\kappa(i)$ to be the obvious quotient map $q \colon \kappa(S') \to \kappa(S)$.*

Remark The choice to use κ to denote this functor was deliberate. A graph $G = (V, E)$ with

$$V = \{v_1, \dots, v_n\}$$

can be seen as a down-set $G \in \mathcal{N}_{(n)}$, where $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1) \in G$, and for $i \neq j$, we have $(e_i + e_j) \in G' \Leftrightarrow (v_i, v_j) \notin E$.

Then, $\kappa(G)$ (in the sense of Definition 3.3.1) is the same Weil algebra as $\kappa(G)$ (in the sense of Definition 4.1.12).

Definition 4.1.13. *For a Weil algebra $X \in k$-$\mathbf{Weil}_\infty$ with presentation*

$$k[x_1, \ldots, x_r]/\{\text{some collection } I \text{ of monomials in the } x_i\text{'s}\} \ ,$$

let S_X be the canonical down-set generated by the non-zero monomials of X. Explicitly, $x_1^{t_1}x_2^{t_2}\ldots x_r^{t_r} \neq 0$ if and only if $(t_1, \ldots, t_r) \in S_X$.

Definition 4.1.14. *Let $X \in k$-$\mathbf{Weil}_\infty$ be given. Recall also from* Section 4.1.1 *that the down-set $S_X \in \mathcal{N}_{(\mathbf{r})}$ (*Definition 4.1.13*) has a canonical diagram of R_i's and $Q_{i,j}$'s.*

Let $\mathcal{B}_X$ denote the category for this diagram (in full formality, this is a functor $D\colon \mathcal{B}_X \to \mathcal{N}_{(\mathbf{r})}$), and let $\Upsilon_X\colon \mathcal{B}_X^{op} \to k$-$\mathbf{Weil}_\infty$ be the composite

$$\mathcal{B}_X^{op} \xrightarrow{D^{op}} \mathcal{N}_{(r)}^{op} \xrightarrow{\kappa} k\text{-}\mathbf{Weil}_\infty \ .$$

Notation Recall from Section 4.1.1 that the objects of $\mathcal{B}_X$ are either a maximal element R_i of S_X (forming the first layer) or an intersection $Q_{i,j}$ of R_i and R_j (forming the second layer). We shall also refer to the $\Upsilon_X(R_i)$'s as the first layer and the $\Upsilon_X(Q_{i.j})$'s as the second layer of the diagram defined by Υ_X.

Example 4.1.15. *If we take X to be $k[x,y]/x^3, x^2y, y^2$, then we have*

$$S_X = \{(0,0), (1,0), (2,0), (0,1), (1,1)\}$$

(as we had in Example 4.1.9*). The diagram $\mathcal{B}_X$ is then*

$$\begin{tikzcd}
& \{(0,0),(1,0)\} \arrow[dl, hook] \arrow[dr, hook] & \\
\{(0,0),(1,0),(0,1),(1,1)\} & & \{(0,0),(1,0),(2,0)\}
\end{tikzcd} \ ,$$

and $\Upsilon_X\colon \mathcal{B}_X^{op} \to k$-$\mathbf{Weil}_\infty$ picks out the diagram

$$\begin{tikzcd}
k[x,y]/x^2, y^2 \arrow[dr] & & k[x]/x^3 \arrow[dl] \\
& k[x]/x^2 &
\end{tikzcd} \ .$$

Theorem 4.1.16. *For each $X \in k$-$\mathbf{Weil}_\infty$, we have*

$$\lim\left(\Upsilon_X\colon \mathcal{B}_X^{op} \to k\text{-}\mathbf{Weil}_\infty\right) \cong X \ .$$

Proof. (Sketch) Since $\operatorname{colim}\big(H\colon \mathcal{B}_S \to \mathcal{N}_{(\mathbf{r})}\big) = S$ and $\{W_n \mid n \in \mathbb{N}\}$ forms a strong generator for k-$\mathbf{Weil}_\infty$, it is an almost trivial exercise to show that the obvious cone

$$\begin{tikzcd}
\mathcal{B}_S^{op} \arrow[r, bend left, "\Delta X"] \arrow[r, "\Upsilon_X"'] \arrow[r, phantom, "\Downarrow"] & k\text{-}\mathbf{Weil}_\infty
\end{tikzcd}$$

is a limiting one. ∎

Remark Let $\underline{0}_r$ be the down-set $\{(0,\dots,0)\} \subset \mathbb{N}^r$, for each positive integer r. Then $\kappa(\underline{0}_r) = k$ for all $r \in \mathbb{N}$. We may also choose to think of $\kappa(\underline{0}_r)$ as $k \otimes \cdots \otimes k$, which is of course isomorphic to k.

Corollary 4.1.17. *Let an arbitrary Weil algebra $A \in k\text{-}\mathbf{Weil}_\infty$ be given. The functor*

$$A \otimes \_\colon k\text{-}\mathbf{Weil}_\infty \to k\text{-}\mathbf{Weil}_\infty$$

preserves the limits described in Theorem 4.1.16*.*

Proof. Noting that $k\text{-}\mathbf{Weil}_\infty$ is closed under $\otimes$, this is simply an application of Proposition 2.1.13, restricting to the category $k\text{-}\mathbf{Weil}_\infty$. ∎

We may then try to define a Tangent Structure corresponding to $k\text{-}\mathbf{Weil}_\infty$ as follows:

Definition 4.1.18. *For a monoidal category $(G, \Box, I)$, a* Tangent Structure corresponding to $k\text{-}\mathbf{Weil}_\infty$ *on $\mathcal{G}$ consists of a strong monoidal functor*

$$F\colon k\text{-}\mathbf{Weil}_\infty \to \mathcal{G}$$

which preserves each object as its canonical limit, in the sense of Theorem 4.1.16*.*

This is somewhat unwieldy. Indeed, the category $\mathbf{Weil_1}$ (regardless of the set of coefficients) has its objects generated as limits of tensors of $k[x]/x^2$. It is then understandable that to characterise $k\text{-}\mathbf{Weil}_\infty$ we would need to introduce the set $\{W_n \mid \forall n \in \mathbb{N}\}$.

However, we shall show that under some mild conditions on k, we can express each object of $k\text{-}\mathbf{Weil}_\infty$ as a limit of a diagram involving nW's. We shall then see in Section 4.3 that this allows us to prove some useful properties relating to the perspective that Tangent Structures are given as strong monoidal functors from categories of Weil algebras to an arbitrary monoidal category $\mathcal{G}$.

4.2 $\mathbf{Weil}_\infty$ and the combinatorics of W_1

We begin with the following:

Definition 4.2.1. *Let $s_n\colon W_n \to \underbrace{W \otimes \cdots \otimes W}_{n \text{ copies}}$ be given as $s_n(x) = y_1 + \cdots + y_n$, where each y_i is the generator for the appropriate instance of W in the codomain.*

We leave as an exercise to the reader the verification that this is a valid map.

Notation Whilst we will not explicitly be considering maps using coloured circles as we did in Section 3.4, we will continue to refer to non-trivial monomials of a Weil algebra as circles. We will say that two circles *intersect* if they have a generator in common (even if they are of a different order). For example, xy^2 and yz are intersecting circles of the Weil algebra $k[x, y, z]/x^3, y^3, z^3$.

Proposition 4.2.2. *Let $f\colon W_n \to mW$ be an arbitrary map with no intersecting circles. Then f factors through s_n. Explicitly, there exists a map $f'\colon nW \to mW$ such that the diagram*

$$\begin{tikzcd} W_n \arrow[r, "s"] \arrow[dr, "f"'] & nW \arrow[d, "f'"] \\ & mW \end{tikzcd}$$

commutes.

Proof. Since f has no intersecting circles, then $f(x)$ (where x is the generator of W_n) is a polynomial in the generators $\{z_1, \ldots, z_m\}$ of mW with at most n monomial terms, since $f(x)^{n+1} = f\left(x^{n+1}\right) = 0$.

Further, since $z_i^2 = 0$ for all i, any monomial in the z_i's also squares to zero.

Then, there is clearly a map $f'\colon nW \to mW$ for which each generator $\{y_1, \ldots, y_n\}$ of nW picks out a different circle of f (or is sent to zero if there are fewer than n monomial terms in the polynomial $f(x)$). Note that f' will not be unique.

Then clearly we have $f = f' \circ s$. ∎

Proposition 4.2.3. *Let X be an arbitrary object of k-**Weil**$_\infty$ and $f\colon X \to mW$ be an arbitrary map with no intersecting circles. Then f factors through one of its projections $\pi\colon X \to \bigotimes_{i=1}^{n} W_{r_i}$ (of the kind described in* Theorem 4.1.16*). Explicitly, there exists a projection $\pi\colon X \to \bigotimes_{i=1}^{n} W_{r_i}$ and a map $f'\colon \bigotimes_{i=1}^{n} W_{r_i} \to mW$ such that the diagram*

$$\begin{tikzcd} W_n \arrow[r, "\pi"] \arrow[dr, "f"'] & \bigotimes_{i=1}^{n} W_{r_i} \arrow[d, "f'"] \\ & mW \end{tikzcd}$$

commutes.

Proof. Let X have presentation

$$k[x_1, \ldots, x_n]/\{\text{some list of monomials}\}$$

and, for each $i \in \{1, \ldots, n\}$, let f_i denote the polynomial in the generators $\{z_1, \ldots, z_m\}$ given by $f(x_i)$.

Then, let $\nu_i = \max\{r \in \mathbb{N} \mid (f_i)^r \neq 0\}$ (if $f_i = 0$, take $\nu_i = 0$). Since f has no intersecting circles, then ν_i is also precisely the number of monomial terms in f_i.

Now, let $Y = \bigotimes_{i=1}^{n} W_{\nu_i}$, and let $\{x_1, \ldots, x_n\}$ be its generators. Again, using a similar idea to that in the proof of Proposition 4.2.2, we can construct a map $g'\colon Y \to mW$ such that $g'(x_i) = f_i$ for each i.

In particular, we have

$$\tilde{x} = \prod_{i=1}^{n} x_i^{\nu_i} \neq 0$$

in Y, but this is also true in X. Because $\tilde{x}$ is a non-zero monomial of X, then from Section 4.1.2, there is some projection $\pi\colon X \to \bigotimes_{i=1}^{n} W_{r_i}$ for which $\pi(\tilde{x}) \neq 0$ (this implies that $n_i \geq \nu_i$ for all i).

Then there is a (unique) quotient map $q\colon \bigotimes_{i=1}^{n} W_{r_i} \to \bigotimes_{i=1}^{n} W_{\nu_i} = Y$ (possibly the identity) so that the diagram

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[drr, "f"] & & \\ \bigotimes_{i=1}^{n} W_{r_i} \arrow[r, "q"'] & Y \arrow[r, "g'"'] & mW \end{tikzcd}$$

commutes. Define f' to be the composite $g' \circ q$. ■

We will need Propositions 4.2.2 and 4.2.3 later in Section 4.3.

4.2.1 W_m as an equaliser

Let $\mathbb{F}$ be an arbitrary field of characteristic zero. For the remainder of this chapter, we shall take $k = \mathbb{F}$, i.e. we shall be working with $\mathbb{F}$-**Weil**$_\infty$.

Noting that the image of s_m is simply the symmetric polynomials in the generators $y_1, \ldots, y_m$, where no single generator has degree greater than 1, we then have the following:

Lemma 4.2.4. *The diagram*

$$W_m \xrightarrow{s_m} mW \underset{\longrightarrow}{\overset{m!\text{ permutations}}{\overset{\longrightarrow}{\cdots}}} mW$$

is a joint equaliser diagram in $\mathbb{F}$-**Weil**$_\infty$ *(although this is also true in* **AugAlg***) for all* $m \in \mathbb{N}$.

Proof. Let x be the generator of W_m and $y_1, \ldots, y_m$ be the generators of mW. Since $y_i^2 = 0$ for all i and $s_m(x) = y_1 + \cdots + y_m = e_1(y_1, \ldots, y_m)$ (the first elementary symmetric polynomial), it is routine to show that $s_m(x^j) = j!e_j(y_1, \ldots, y_m)$ (the j^{th} elementary symmetric polynomial).

Since we are working with a field $\mathbb{F}$ of characteristic zero, $j! \in \mathbb{F}$ is invertible, so that

$$s_m\left(\frac{1}{j!}x^j\right) = e_j(y_1, \ldots, y_m)\ .$$

The remainder of the proof then follows directly from the fundamental theorem of symmetric polynomials. ■

Remark Each of the permutation maps (we shall later refer to these as σ's) can be constructed from iterations of the map $c\colon 2W \to 2W$ (as given in Section 3.2) tensored with identity maps.

In particular, we have expressed each W_m as a (particular) limit of a diagram built up using W's and c's.

Although we took k to be a field of characteristic zero, it would have sufficed to take k to be a rig containing the non-negative rationals $\mathbb{Q}_{\geq 0}$. This is because the proof of Lemma 4.2.4 required $j!$ to be invertible.

To make the argument explicit, consider the fork

$$W_2 \xrightarrow{s_2} 2W \underset{id}{\overset{c}{\rightrightarrows}} 2W$$

(for the map c as defined in Section 3.2). Certainly, it is easy to verify that $c \circ s_2 = s_2$. However, if k is (say) $\mathbb{N}$, $\mathbb{Z}$ or the two element field, then this is not an equaliser diagram, since we have the fork

$$W \xrightarrow{l} 2W \underset{id}{\overset{c}{\rightrightarrows}} 2W$$

(for the map l as defined in Section 3.2), but l cannot factor through s_2 (this is a routine exercise).

The equaliser of c and id is not s_2 due to the absence of $\frac{1}{2}$ (by which we mean an element of the rig k for which $\frac{1}{2} + \frac{1}{2} = 1$) in $\mathbb{N}$, $\mathbb{Z}$ or (say) the two element field (in fact, there is no equaliser in k-**Weil**$_\infty$, but we shall not prove this).

Consider $s_2(x^2) = (y_1 + y_2)^2 = 2y_1y_2 = 2e_2(y_1, y_2)$. If $\frac{1}{2}$ is not an element of k, then there is no way to recover e_2 itself (i.e. with a coefficient of one), and so y_1y_2 is not in the image of s_2 (so that l cannot factor through s_2).

Remark This is not an issue if $k = \mathbb{2}$ (see Definition 3.4.1), since 1 is also $\frac{1}{2}$ (recall $1 + 1 = 1$). In this case, s_2 is indeed the equaliser of c and id.

4.2.2 $\bigotimes_{i=1}^{n} W_{m_i}$ as an equaliser

Consider the map

$$\bigotimes_{i=1}^{n} s_{m_i} \colon \bigotimes_{i=1}^{n} W_{m_i} \to \bigotimes_{i=1}^{n} m_i W = \left(\sum_{i=1}^{n} m_i \right) W$$

for some arbitrary values $m_1, \ldots, m_n \in \mathbb{N}_{>0}$. Let this map be denoted s.

It is easy to see that for an element (a polynomial) p in the image of this map s, p is symmetric in the generators of m_1W. Separately, it is symmetric in the generators of m_2W, and in fact this is true for all m_iW.

Lemma 4.2.5. *The diagram*

$$\bigotimes_{i=1}^{n} W_{m_i} \xrightarrow{s} \bigotimes_{i=1}^{n} m_i W \underset{}{\overset{\sigma_1 \otimes \cdots \otimes \sigma_n}{\underset{\cdots}{\rightrightarrows}}} \bigotimes_{i=1}^{n} m_i W$$

is an equaliser diagram (in $\mathbb{F}$-**Weil**$_\infty$*, or indeed* **AugAlg***), where we include all* $m_1! \ldots m_n!$ *maps of the form*

$$\sigma_1 \otimes \cdots \otimes \sigma_n \colon \bigotimes_{i=1}^{n} m_i W \to \bigotimes_{i=1}^{n} m_i W \ .$$

Proof. Let $x_1, \ldots, x_n$ be the generators for the domain $\bigotimes_{i=1}^{n} W_{m_i}$ of s, and for each $i \in \{1, \ldots, n\}$, let $y_{i,1}, \ldots, y_{i,m_i}$ be the generators for m_iW in the codomain $\bigotimes_{i=1}^{n} m_iW$.

Now, we clearly have $s(x_i) = e_1(y_{i,1}, \ldots, y_{i,m_i})$ for all i. Further, we also have $s(x_i^j) = j!e_j(y_{i,1}, \ldots, y_{i,m_i})$, so that

$$s\left(\frac{1}{j!}x^j\right) = e_j(y_{i,1}, \ldots, y_{i,m_i}) \ .$$

Again, the remainder of the proof then follows directly from the fundamental theorem of symmetric polynomials. ■

Each Weil algebra of the form $\bigotimes_{i=1}^{n} W_{m_i}$ can now be expressed canonically as a limit of a diagram built up using mW's and c's. Again, we stress that this process requires that k contains the positive rationals (so that the scalar $(j!)^{-1} \neq 0$ is well defined for each $j \in \mathbb{N}$).

4.2.3 Objects of $\mathbb{F}$-Weil$_\infty$ as limits of mW's

In Theorem 4.1.16, we said that each object of k-**Weil**$_\infty$ was canonically a limit of a diagram involving $\bigotimes_{i=1}^{n} W_{m_i}$'s, and in Lemma 4.2.5 (with $k = \mathbb{F}$), we said that any $\bigotimes_{i=1}^{n} W_{m_i}$ was the equaliser of a particular set of permutation maps between mW's.

This suggests that we may be able to characterise each object of $\mathbb{F}$-**Weil**$_\infty$ as the limit of some diagram involving mW's. For $0 \leq m' \leq m$, let $\varepsilon\colon W_m \to W_{m'}$ be given as $\varepsilon(x) = x$. Further, let $\varepsilon'\colon mW \to m'W$ be given as

$$\underbrace{id \otimes \cdots \otimes id}_{m' \text{ copies}} \otimes \underbrace{\varepsilon_W \otimes \cdots \otimes \varepsilon_W}_{(m-m') \text{ copies}} \colon mW \to m'W \ ;$$

i.e. it preserves the first m' instances of W and discards the rest.

Proposition 4.2.6. *For the maps ε and ε' as described above, the diagram*

$$\begin{tikzcd} W_m \arrow[r, "s_m"] \arrow[d, dashed, "\varepsilon"'] & mW \arrow[d, "\varepsilon'"] \\ W_{m'} \arrow[r, "s_{m'}"'] & m'W \end{tikzcd}$$

commutes. There is an analogous diagram for maps of the form

$$\varepsilon\colon \bigotimes_{i=1}^{n} W_{m_i} \to \bigotimes_{i=1}^{n} W_{m_i'} \ .$$

Proof. Obvious. ■

We wish to exhibit each object of $\mathbb{F}$-**Weil**$_\infty$ as the limit of a (canonical) diagram involving $mW's$ (in particular, a diagram in $\mathbb{F}$-**Weil$_1$**). We shall begin with an example to illustrate this process.

Remark Recall that ${}^nP_r$ denotes the number of r-permutations of n, and is given by the formula

$${}^nP_r = \frac{n!}{(n-r)!} \ .$$

Example 4.2.7. *Let X be the Weil algebra $k[x,y]/x^5, y^6, x^3y^4$. Using the ideas discussed in* 4.1.2, *it is routine to show that X, together with its canonical diagram, is the pullback*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & W_4W_3 \arrow[d] \\
W_2W_5 \arrow[r] & W_2W_3 .
\end{tikzcd}$$

We then add to this pullback square the equalisers described in Lemma 4.2.5, *as well as appropriate ε' maps as described in* Proposition 4.2.6*:*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & W_4W_3 \arrow[d] \arrow[dr, "s_4s_3"] & & \\
W_2W_5 \arrow[r] \arrow[dr, "s_2s_5"'] & W_2W_3 \arrow[dr, "s_2s_3"] & 4W3W \arrow[r, shift left, "4!3!\text{ permutations}"] \arrow[r, shift right, "\cdots"'] \arrow[d, "\varepsilon'1"] & 4W3W \\
& 2W5W \arrow[r, "1\varepsilon'"'] \arrow[d, shift right, "2!5!\text{ permutations}"'] \arrow[d, shift left, "\cdots"] & 2W3W \arrow[dr, shift left, "2!3!\text{ permutations}"] \arrow[dr, shift right, "\cdots"'] & \\
& 2W5W & & 2W3W
\end{tikzcd}$$

Since all the squares commute, then X is clearly the limit of the diagram

$$\begin{tikzcd}
& 4W3W \arrow[r, shift left, "4!3!\text{ permutations}"] \arrow[r, shift right, "\cdots"'] \arrow[d, "\varepsilon'1"] & 4W3W \\
2W5W \arrow[r, "1\varepsilon'"'] \arrow[d, shift right, "2!5!\text{ permutations}"'] \arrow[d, shift left, "\cdots"] & 2W3W \arrow[dr, shift left, "2!3!\text{ permutations}"] \arrow[dr, shift right, "\cdots"'] & \\
2W5W & & 2W3W .
\end{tikzcd}$$

But since the identity map is among the permutations, we may equivalently take the limit of

$$\begin{tikzcd}
& 4W3W \arrow[r, shift left, "4!3!\text{ permutations}"] \arrow[r, shift right, "\cdots"'] & 4W3W \arrow[dl, "\varepsilon'1"] \\
2W5W \arrow[d, shift right, "2!5!\text{ permutations}"'] \arrow[d, shift left, "\cdots"] & 2W3W \arrow[dr, shift left, "2!3!\text{ permutations}"] \arrow[dr, shift right, "\cdots"'] & \\
2W5W \arrow[ur, "1\varepsilon'"'] & & 2W3W .
\end{tikzcd}$$

From here, it is easy to see that we may equivalently take the limit of

$$\begin{tikzcd}
2W5W \arrow[dr, shift left, "\cdots"] \arrow[dr, shift right, "2!^5P_3\text{ maps}"'] & & 4W3W \arrow[dl, shift right, "\cdots"'] \arrow[dl, shift left, "3!^4P_2\text{ maps}"] \\
& 2W3W &
\end{tikzcd} ;$$

where the $2!^5P_3$ maps all permute between the generators of the $2W$ component, and preserve 3 of the 5 generators of $5W$ (possibly permuting them) and send the other 2 to zero (a similar statement holds for the $3!^4P_2$ maps).

This argument in fact applies to all objects of $\mathbb{F}$-**Weil**$_\infty$. We shall now formalise this. For a given Weil algebra $X \in \mathbb{F}$-**Weil**$_\infty$, we begin by defining a category $\mathcal{B}'_X$ as follows:

Definition 4.2.8. *Let $X \in \mathbb{F}$-**Weil**$_\infty$ be given. Recall from* Definition 4.1.14 *the category $\mathcal{B}_X$ as well as the functor Υ_X. Let $\mathcal{B}'_X$ be the category with:*

- *Objects: The same as those of $\mathcal{B}_X$*
- *Morphisms: Suppose we had a non-identity morphism $f \colon b \to c$ in $\mathcal{B}_X$ (i.e. b was in the second layer and c was in the first layer of $\mathcal{B}_X$) with*

$$\Upsilon_X f \colon \bigotimes_{i=1}^{r} W_{c_i} \to \bigotimes_{i=1}^{r} W_{b_i}$$

  (recall Υ_X is contravariant and that $S_X \in \mathcal{N}_{(\mathbf{r})}$). Note also that $b_i \leq c_i$ for all i.

  Then we define a morphism from b to c in $\mathcal{B}'_X$ to be an ordered r-tuple of injective functions

$$\{h_i \colon b_i \rightarrowtail c_i\}_{i=1,\ldots,r} \ .$$

  In particular, note that

$$|\mathcal{B}'_X(b,c)| = \prod_{i=1}^{r} {}^{c_i}P_{b_i} \ .$$

Remark We do not need to define composition for the category $\mathcal{B}'_X$ since (as with $\mathcal{B}_X$) the only non-identity morphisms must have an object of the second layer as its domain and an object of the first layer as its codomain.

We shall now define a functor $\Upsilon'_X \colon {\mathcal{B}'}_X^{op} \to \mathbb{F}$-**Weil**$_\infty$ which will serve a similar purpose to Υ_X (we shall explain this in more detail in a moment).

Definition 4.2.9. *Let a Weil algebra $X \in \mathbb{F}$-**Weil**$_\infty$ be given. The functor*

$$\Upsilon'_X \colon {\mathcal{B}'}_X^{op} \to \mathbb{F}\text{-}\mathbf{Weil}_\infty$$

is given as follows:

- *On objects: For an arbitrary object $b \in \mathcal{B}'_X$, if $\Upsilon_X b = \bigotimes_{i=1}^{r} W_{b_i}$, then we define*

$$\Upsilon'_X b = \bigotimes_{i=1}^{r} b_i W \ .$$

- *On morphisms: For a non-identity morphism $f \colon b \to c$ of ${\mathcal{B}'}_X^{op}$ (recall that this is an r-tuple $(h_i \colon b_i \rightarrowtail c_i)_{i=1,\ldots,r}$ of injective functions), define the map $\Upsilon'_X f \colon \bigotimes_{i=1}^{r} c_i W \to \bigotimes_{i=1}^{r} b_i W$ to be the map $\hat{h}_1 \otimes \cdots \otimes \hat{h}_r$, where $\hat{h}_i \colon c_i W \to b_i W$ is the map sending the j^{th} generator of $c_i W$ to:*

$$\begin{cases} \text{the } h^{-1}(j)^{th} \text{ generator of } b_i W & ;\ \text{if } j \text{ is in the image of } h_i \colon b_i \rightarrowtail c_i \\ 0 & ;\ \text{otherwise.} \end{cases}$$

Remark Such a diagram $\Upsilon'_X\colon \mathcal{B}'^{op}_X \to \mathbb{F}\text{-}\mathbf{Weil}_\infty$ factors through $\mathbb{F}\text{-}\mathbf{Weil_1}$.

Theorem 4.2.10. $\lim(\Upsilon'_X\colon \mathcal{B}'^{op}_X \to \mathbb{F}\text{-}\mathbf{Weil}_\infty) = X$.

Proof. Since $\mathcal{B}_X$ is a connected category, then clearly so is $\mathcal{B}'_X$, so we are forming a connected limit. Using Lemma 4.2.5, Proposition 4.2.6 and Theorem 4.1.16, the proof becomes immediate. ■

Remark For $X \in \mathbb{F}\text{-}\mathbf{Weil}_\infty$, the diagram Υ'_X of Definition 4.2.9 and the fact that X is the limit of this diagram is the analogue of what happens when considering the infinitesimal objects of SDG, for instance see [21]. The diagrams in Fig. 1 and Fig. 2 are precisely the sort of diagram we describe here, but with the arrows reversed.

Remark The canonical diagrams for each $X \in \mathbb{F}\text{-}\mathbf{Weil}_\infty$ can actually be simplified; for instance, the diagram

$$\begin{tikzcd} W_3 \arrow[r, "s_3"] & 3W \arrow[r, shift left=2, "1"] \arrow[r, "c1" description] \arrow[r, shift right=2, "1c"'] & 3W \end{tikzcd}$$

is an equaliser diagram, and we can recover the other three permutations from this.

The purpose of this combinatorial approach is actually rather simple. Whereas in Theorem 4.1.16, we characterised each object X of $\mathbb{F}\text{-}\mathbf{Weil}_\infty$ using W_n's, this process allows us to characterise X using only the object W, albeit through more elaborate diagrams. However, this deceptively simple observation yields a powerful result, which we shall describe in the remainder of this chapter.

4.3 Tangent Structure corresponding to $\mathbf{Weil}_\infty$

Using the perspective given by Theorem 4.1.16, we have already given one possible way to define Tangent Structure corresponding to $\mathbf{Weil}_\infty$, namely through Definition 4.1.18.

This is useful if our set of scalars k does not contain the (positive) rationals, such as $\mathbb{N}$, $\mathbb{Z}$ or finite fields. However, if we instead use the perspective offered by Theorem 4.2.10, there is still far more to say. As such, we shall not discuss Definition 4.1.18 any further beyond this point.

Taking $k = \mathbb{F}$ (or more generally, some appropriate structure containing the positive rationals, such as the rigs $\mathbb{Q}_{\geq 0}$ and $\mathbb{2}$), then we may instead use the perspective of Theorem 4.2.10 to give a different definition of Tangent Structure corresponding to $\mathbb{F}\text{-}\mathbf{Weil}_\infty$ as follows:

Definition 4.3.1. *For a monoidal category $(\mathcal{G}, \Box, I)$, a Tangent Structure corresponding to to $\mathbb{F}\text{-}\mathbf{Weil}_\infty$ on $\mathcal{G}$ consists of a strong monoidal functor*

$$F_\infty\colon \mathbb{F}\text{-}\mathbf{Weil}_\infty \to \mathcal{G}$$

which preserves each object as its canonical limit, in the sense of Theorem 4.2.10.

Let $J\colon \mathbb{F}\text{-}\mathbf{Weil_1} \hookrightarrow \mathbb{F}\text{-}\mathbf{Weil}_\infty$ be the obvious subcategory inclusion functor. We have the following:

Proposition 4.3.2. *The inclusion functor* $J\colon \mathbb{F}\text{-}\mathbf{Weil_1} \hookrightarrow \mathbb{F}\text{-}\mathbf{Weil}_\infty$ *is codense.*

Proof. We need to show that for arbitrary $X \in \mathbb{F}\text{-}\mathbf{Weil}_\infty$, we have

$$\lim\Big(\ (X \downarrow J) \xrightarrow{\ \pi\ } \mathbb{F}\text{-}\mathbf{Weil_1} \xhookrightarrow{\ J\ } \mathbb{F}\text{-}\mathbf{Weil}_\infty\ \Big) \cong X\ ,$$

where π is the forgetful functor sending an object $X \xrightarrow{f} A$ of $(X \downarrow J)$ to A.

More specifically, we shall show that the obvious cone

$$\begin{tikzcd} (X \downarrow J) \arrow[r, "\pi"'] \arrow[rr, bend left, "\Delta X"] & \mathbb{F}\text{-}\mathbf{Weil_1} \arrow[r, hook, "J"] \arrow[u, phantom, "\Downarrow\alpha"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \end{tikzcd}$$

is in fact a limiting cone. We also note here that this is trivial for $X \in \mathbb{F}\text{-}\mathbf{Weil_1}$.

Let an arbitrary cone

$$\begin{tikzcd} (X \downarrow J) \arrow[r, "\pi"'] \arrow[rr, bend left, "\Delta A"] & \mathbb{F}\text{-}\mathbf{Weil_1} \arrow[r, hook, "J"] \arrow[u, phantom, "\Downarrow\gamma"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \end{tikzcd}$$

be given. We first construct a (unique) map $h\colon A \to X$ as follows:

Recall from Theorem 4.2.10 that X is the limit of a canonical diagram in $\mathbb{F}\text{-}\mathbf{Weil_1}$. For each projection $\pi_i\colon X \to n_iW$ of this diagram, we have the corresponding component $\gamma_{\pi_i}\colon A \to n_iW$ of the natural transformation γ, and we use these to construct the unique map $h\colon A \to X$. It now remains to show that for arbitrary $f\colon X \to B$ (with $B \in \mathbb{F}\text{-}\mathbf{Weil_1}$), the diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "\gamma_f"'] & X \arrow[d, "f"] \\ & B \end{tikzcd}$$

commutes.

We first note the following:

1) Since B is a limit of an appropriate diagram of mW's, then to show that the maps $A \underset{\gamma_f}{\overset{f\circ h}{\rightrightarrows}} B$ are equal, it suffices to show their composites with any given projection $\pi\colon B \to mW$ are equal.

2) For any commuting diagram

$$\begin{tikzcd} X \arrow[d, "g_1"'] \arrow[dr, "g_2"] & \\ C \arrow[r, "g"'] & D \end{tikzcd}\ ,$$

(i.e. $g_2 = g \circ g_1$) with $C, D \in \mathbb{F}\text{-}\mathbf{Weil_1}$, the diagram

$$\begin{tikzcd} A \arrow[d, "\gamma_{g_1}"'] \arrow[dr, "\gamma_{g_2}"] & \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

commutes.

Using 1), we know that to show that $f \circ h = \gamma_f$, it suffices to show $\pi \circ f \circ h = \pi \circ \gamma_f$, for an arbitrary projection $\pi\colon B \to mW$ of B. From 2), we have $\pi \circ \gamma_f = \gamma_{\pi \circ f}$.

Equivalently, it suffices to assume $B = mW$ and show that the diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "\gamma_f"'] & X \arrow[d, "f"] \\ & mW \end{tikzcd}$$

commutes.

Now we can use the approach described in Section 3.5.5 to canonically express f as a composite

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[dr, "f"'] & W^{r_1} \otimes \cdots \otimes W^{r_m} \arrow[d, "+_{r_1} \otimes \cdots \otimes +_{r_m}"] \\ & mW \end{tikzcd},$$

and again using the arguments of Section 3.5.5 as well as those immediately above, it suffices to assume that f has no intersecting circles.

Now, by Proposition 4.2.3, such a map f factors through a projection

$$\pi\colon X \to \bigotimes_{i=1}^{n} W_{r_i}$$

(of the kind described in Theorem 4.1.16). So we now have

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dd, bend right, "\gamma_f"'] & X \arrow[d, "\pi"] \\ & \bigotimes_{i=1}^{n} W_{r_i} \arrow[d, "f'"] \\ & mW \end{tikzcd}$$

But note now that since f has no intersecting circles, then neither does f', and so the map f' is of the form $f_1 \otimes \cdots \otimes f_n$ (up to some appropriate composition with c's), where each

$$f_i\colon W_{r_i} \to m_i W;\ \text{with} \sum_{i=1}^{n} m_i = m,$$

has no intersecting circles.

By Proposition 4.2.2, since each $f_i\colon W_{r_i} \to m_i W$ has no intersecting circles, then each of them factors through $s_{r_i}\colon W_{r_i} \to r_i W$, i.e. for each i, there exists a map f_i' such that

$$\begin{tikzcd} W_{r_i} \arrow[d, "s_{r_i}"'] \arrow[dr, "f_i"] & \\ r_i W \arrow[r, "f_i'"'] & m_i W \end{tikzcd}$$

commutes. Let $s_\otimes = s_{r_1} \otimes \cdots \otimes s_{r_n}$ and $f'_\otimes = f'_1 \otimes \cdots \otimes f'_n$ for convenience.

We now have the diagram

$$\begin{tikzcd}
 & & A \arrow[dll, "h"'] \arrow[dr, "\gamma_{s_\otimes \circ \pi}"] \arrow[drr, "\gamma_f"] & & \\
X \arrow[r, "\pi"'] & \bigotimes_{i=1}^{n} W_{r_i} \arrow[rr, "s_\otimes"'] & & \bigotimes_{i=1}^{n} r_i W \arrow[r, "f'_\otimes"'] & \bigotimes_{i=1}^{n} m_i W = mW
\end{tikzcd} ,$$

for which the right triangle commutes by definition.

But now recall from Section 4.2.3 that the map $s_\otimes \circ \pi \colon X \to \bigotimes_{i=1}^{n} r_i W$ is precisely one of the projections in the canonical diagram (Theorem 4.2.10) in $\mathbb{F}$-**Weil**$_\mathbf{1}$ for which the limit (in $\mathbb{F}$-**Weil**$_\infty$) is X. It is precisely this diagram that was used to define h, thus the left triangle also commutes. ■

We then have:

Proposition 4.3.3. *Let $F_\infty \colon \mathbb{F}$-**Weil**$_\infty \to \mathcal{G}$ be a strong monoidal functor which preserves the limits described in* Theorem 4.2.10*. Further, let F be the composite $F_\infty \circ J$ (it is a routine exercise to then show that F is a strong monoidal functor of the type described in* Definition 3.6.18*, except with $\mathbb{N}$ replaced with $\mathbb{F}$).*

*For all $X \in \mathbb{F}$-**Weil**$_\infty$, we have*

$$\lim \Big(\; (X \downarrow J) \xrightarrow{\;\pi\;} \mathbb{F}\text{-}\mathbf{Weil_1} \xrightarrow{\;F\;} \mathcal{G} \; \Big) \cong F_\infty X \ .$$

Proof. The proof is similar to that of Proposition 4.3.2. ■

Thus, such a functor F_∞ will automatically be the right Kan extension of F along the (codense) inclusion J.

It is then only natural to ask the converse. Suppose we have a functor

$$F \colon \mathbb{F}\text{-}\mathbf{Weil_1} \to \mathcal{G}$$

defining Tangent Structure internal to $\mathcal{G}$ as in Definition 3.6.18 (except with $\mathbb{N}$ replaced with $\mathbb{F}$). Let $H \colon \mathbb{F}$-**Weil**$_\infty \to \mathcal{G}$ be the pointwise right Kan extension of F along the inclusion J. Then H will preserve the limits in question. Now it is only a matter of whether H is strong monoidal.

Remark To be able to take such a pointwise right Kan extension, we of course have to assume the existence of certain limits in the codomain category $\mathcal{G}$, and as such we shall make that assumption.

We begin by showing that H is necessarily monoidal.

4.3.1 H as a monoidal functor

We begin with the following (folklore) result:

Proposition 4.3.4. *Let $\mathcal{B}$ be a category with finite coproducts, thus making $\mathcal{B}$ a cocartesian monoidal category. Let $\mathcal{D}$ be an arbitrary monoidal category.*

Then, giving a monoidal functor $G \colon \mathcal{B} \to \mathcal{D}$ is equivalent to giving an ordinary functor $G' \colon \mathcal{B} \to \mathrm{Mon}(\mathcal{D})$.

Proof. (Sketch) Let a monoidal functor $G\colon \mathcal{B} \to \mathcal{D}$ be given. For an object $B \in \mathcal{B}$, there is the unique monoid

$$B + B \xrightarrow{\mu} B \xleftarrow{\eta} 0 \ ,$$

and since G is monoidal, then this gives a monoid in $\mathcal{D}$, and thus determines $G'(B)$.

Conversely, let an ordinary functor $G'\colon \mathcal{B} \to \mathrm{Mon}(\mathcal{D})$ be given, and let $G\colon \mathcal{B} \to \mathcal{D}$ be its composite with the forgetful functor $\mathrm{Mon}(\mathcal{D}) \to \mathcal{D}$. Define $G_0\colon I \to G0$ to be the unit of $G'0$. Define $G_2\colon GB \otimes GB' \to G(B + B')$ to be the composite

$$GB \otimes GB' \xrightarrow{Gi \otimes Gi'} G(B + B') \otimes G(B + B') \xrightarrow{\mu} G(B + B') \ ,$$

where 0 is the initial object of $\mathcal{B}$, the following

$$B \xrightarrow{i} B + B' \xleftarrow{i'} B'$$

is a coproduct diagram in $\mathcal{B}$ and μ is the multiplication for $G'(B + B')$. ∎

Proposition 4.3.5. *Let $\mathcal{B}$ and $\mathcal{C}$ be categories with finite coproducts, $J\colon \mathcal{B} \to \mathcal{C}$ be a functor preserving these coproducts. Let $\mathcal{D}$ be a monoidal category and let $F\colon \mathcal{B} \to \mathcal{D}$ be a monoidal functor.*

Then $\mathrm{Ran}_J F\colon \mathcal{C} \to \mathcal{D}$ (the pointwise right Kan extension, if it exists) is a monoidal functor.

Proof. (Sketch) Suppose $\mathrm{Ran}_J F\colon \mathcal{C} \to \mathcal{D}$ exists. The forgetful functor $U\colon \mathrm{Mon}(\mathcal{D}) \to \mathcal{D}$ creates limits. Form $F'\colon \mathcal{B} \to \mathrm{Mon}(\mathcal{D})$ as described in Proposition 4.3.4.

The limits needed for $\mathrm{Ran}_J F'$ are created by U, so that $U \circ \mathrm{Ran}_J F' = \mathrm{Ran}_J(UF') = \mathrm{Ran}_J F$. In particular, this means that $\mathrm{Ran}_J F'$ exists. Since $U \circ \mathrm{Ran}_J F' = \mathrm{Ran}_J F$, then $\mathrm{Ran}_J F$ has a lifting through U, and thus is monoidal. ∎

Similarly, the natural isomorphism

$$Ran_J F \circ J \cong F$$

is monoidal.

Recall now that tensor is actually coproduct in $\mathbb{F}$-**Weil**$_\infty$ (Proposition 2.1.14), and as $\mathbb{F}$ is the zero object, $\mathbb{F}$-**Weil**$_\infty$ then has all finite coproducts. Since $F\colon \mathbb{F}\text{-}\mathbf{Weil_1} \to \mathcal{G}$ is a (strong) monoidal functor, H is a monoidal functor. Before we discuss whether or not H is strong monoidal, we will first establish a few facts.

4.3.2 H, $\otimes$, and the preservation of limits

We said in Section 4.3.1 that H (the pointwise right Kan extension of F along J) is a monoidal functor. As such, there is the structure map

$$I \to H(\mathbb{F}) \ .$$

Since J is a codense inclusion, $F\colon \mathbb{F}\text{-}\mathbf{Weil_1} \to \mathcal{G}$ is strong monoidal and $\mathbb{F} \in \mathbb{F}\text{-}\mathbf{Weil_1}$, then this structure map $I \to H(\mathbb{F})$ is an isomorphism.

For arbitrary $X, Y \in \mathbb{F}\text{-}\mathbf{Weil}_\infty$, consider the structure map

$$H(X) \otimes H(Y) \to H(X \otimes Y) \ .$$

We wish to show this is an isomorphism.

Remark If X and Y are both objects of $\mathbb{F}$-$\mathbf{Weil_1}$, then because F is again strong monoidal, this is trivial.

Recall Definitions 4.2.8 and 4.2.9, as well as Theorem 4.2.10. Suppose the limiting cones for X and Y are

$$\begin{tikzcd} \mathcal{B}_X'^{\,op} \arrow[r, bend left, "\Delta X"] \arrow[r, "\Upsilon_X'"'] \arrow[r, phantom, "\Downarrow\rho"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}_Y'^{\,op} \arrow[r, bend left, "\Delta Y"] \arrow[r, "\Upsilon_Y'"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \end{tikzcd}$$

respectively.

Further, suppose X and Y have presentations

$$\mathbb{F}[x_1, \ldots, x_m]/Q_X \text{ and } \mathbb{F}[y_1, \ldots, y_n]/Q_Y$$

respectively, so that $X \otimes Y$ has presentation

$$\mathbb{F}[x_1, \ldots, x_m, y_i, \ldots, y_n]/Q_X \cup Q_Y$$

(Lemma 2.1.15). We shall use this convention for the remainder of this subsection.

Recall from Definition 4.1.13 that X, Y and $X \otimes Y$ have canonical corresponding down-sets S_X, S_Y and $S_{X\otimes Y}$ respectively. With some abuse of notation, we have

$$S_{X\otimes Y} = S_X \times S_Y$$

(the product being taken in **Set**).

Consider the down-set S_X. Suppose S_X had α maximal elements (i.e. $\mathcal{B}_X'$ had α maximal elements in its first layer). Since we take all distinct pairwise intersections of these maximal elements when forming the second layer of $\mathcal{B}_X'$ (see Section 4.1.1 for more details), there are ${}^{\alpha}C_2$ such intersections, and so $\mathcal{B}_X'$ has $\alpha + {}^{\alpha}C_2$ objects (recall that $\mathcal{B}_X'$ was defined to have the same objects as $\mathcal{B}_X$).

Notation For the current discussion, we shall refer to a general maximal element of S_X as $\underline{\mathbf{m}}_X$ (elements of the "first layer"). The intersection of two distinct such maximal elements $\underline{\mathbf{m}}_X$ and $\underline{\mathbf{m}}_X'$ (elements of the "second layer") will be denoted $\underline{\mathbf{m}}_X \cap \underline{\mathbf{m}}_X'$.

Similarly, suppose that S_Y has β maximal elements, so that $\mathcal{B}_Y'$ has $\beta + {}^{\beta}C_2$ objects.

Now consider $S_{X\otimes Y}$. Clearly, a maximal element $\underline{\mathbf{m}}$ (we shan't use $\underline{\mathbf{m}}_{X\otimes Y}$ for the sake of simplicity) of $S_{X\otimes Y}$ is simply (again, with some abuse of notation) a pair $(\underline{\mathbf{m}}_X, \underline{\mathbf{m}}_Y)$, where $\underline{\mathbf{m}}_X$ is a maximal element of S_X and $\underline{\mathbf{m}}_Y$ is a maximal element of S_Y. As such, we conclude that $\mathcal{B}_{X\otimes Y}'$ has $\alpha\beta + {}^{\alpha\beta}C_2$ objects (where $\alpha\beta$ is the number of maximal elements of $S_{X\otimes Y}$).

Consider one of the ${}^{\alpha\beta}C_2$ objects (i.e. a non-maximal element) of $\mathcal{B}_{X\otimes Y}'$. This must be of the form $\underline{\mathbf{m}} \cap \underline{\mathbf{m}}'$ (for maximal elements $\underline{\mathbf{m}}$ and $\underline{\mathbf{m}}'$). For $\underline{\mathbf{m}} = (\underline{\mathbf{m}}_X, \underline{\mathbf{m}}_Y)$ and $\underline{\mathbf{m}}' = (\underline{\mathbf{m}}_X', \underline{\mathbf{m}}_Y')$, we clearly have

$$\underline{\mathbf{m}} \cap \underline{\mathbf{m}}' = (\underline{\mathbf{m}}_X \cap \underline{\mathbf{m}}_X', \underline{\mathbf{m}}_Y \cap \underline{\mathbf{m}}_Y')\ .$$

Lemma 4.3.6. *Let distinct maximal elements* $\underline{\boldsymbol{m}} = (\underline{\boldsymbol{m}}_X, \underline{\boldsymbol{m}}_Y)$ *and* $\underline{\boldsymbol{m}}' = (\underline{\boldsymbol{m}}'_X, \underline{\boldsymbol{m}}'_Y)$ *of* $\mathcal{B}'_{X\otimes Y}$ *be given. There is a canonical bijection*

$$\mathcal{B}'_{X\otimes Y}(\underline{\boldsymbol{m}} \cap \underline{\boldsymbol{m}}', \underline{\boldsymbol{m}}) \cong (\mathcal{B}'_X \times \mathcal{B}'_Y)((\underline{\boldsymbol{m}}_X \cap \underline{\boldsymbol{m}}'_X, \underline{\boldsymbol{m}}_Y \cap \underline{\boldsymbol{m}}'_Y), (\underline{\boldsymbol{m}}_X, \underline{\boldsymbol{m}}_Y)) \ .$$

Proof. (Sketch) By definition of the product of categories, we have

$$\begin{aligned} & (\mathcal{B}'_X \times \mathcal{B}'_Y)((\underline{\mathbf{m}}_X \cap \underline{\mathbf{m}}'_X, \underline{\mathbf{m}}_Y \cap \underline{\mathbf{m}}'_Y), (\underline{\mathbf{m}}_X, \underline{\mathbf{m}}_Y)) \\ \cong\ & \mathcal{B}'_X(\underline{\mathbf{m}}_X \cap \underline{\mathbf{m}}'_X, \underline{\mathbf{m}}_X) \times \mathcal{B}'_Y(\underline{\mathbf{m}}_Y \cap \underline{\mathbf{m}}'_Y, \underline{\mathbf{m}}_Y) \ . \end{aligned}$$

Then, a morphism $f\colon \underline{\mathbf{m}}_X \cap \underline{\mathbf{m}}'_X \to \underline{\mathbf{m}}_X$ in $\mathcal{B}'_X$ consists of an ordered m-tuple $\{h_i\}_{i=1,\ldots,m}$ of injective functions (recall that the presentation of X had m generators, as well as Definition 4.2.8).

Similarly, a morphism $f'\colon \underline{\mathbf{m}}_Y \cap \underline{\mathbf{m}}'_Y \to \underline{\mathbf{m}}_Y$ in $\mathcal{B}'_Y$ consists of an ordered n-tuple $\{h'_i\}_{i=1,\ldots,n}$ of injective functions.

Finally, we shall note that a morphism $g\colon \underline{\mathbf{m}} \cap \underline{\mathbf{m}}' \to \underline{\mathbf{m}}$ in $\mathcal{B}_{X\otimes Y}$ is an ordered $(m+n)$-tuple of injective functions. ∎

We shall now need a functor which we shall call G which we shall later show is final.

Definition 4.3.7. *Let* $G\colon \mathcal{B}'_{X\otimes Y} \to \mathcal{B}'_X \times \mathcal{B}'_Y$ *be given as follows:*

- *On objects: Define* $G(\underline{\boldsymbol{m}}) = (\underline{\boldsymbol{m}}_X, \underline{\boldsymbol{m}}_Y)$ *and* $G(\underline{\boldsymbol{m}} \cap \underline{\boldsymbol{m}}') = (\underline{\boldsymbol{m}}_X \cap \underline{\boldsymbol{m}}'_X, \underline{\boldsymbol{m}}_Y \cap \underline{\boldsymbol{m}}'_Y)$.
- *On morphisms: Use the bijection of* Lemma 4.3.6 *to define* G *on morphisms.*

Lemma 4.3.8. *The functor* $G\colon \mathcal{B}'_{X\otimes Y} \to \mathcal{B}'_X \times \mathcal{B}'_Y$ *above is surjective on objects (but not necessarily injective).*

Proof. Let (c, d) be an arbitrary object of $\mathcal{B}'_X \times \mathcal{B}'_Y$.

Case 1: Suppose c and d are both maximal elements $\underline{\mathbf{m}}_X \in \mathcal{B}'_X$ and $\underline{\mathbf{m}}_Y \in \mathcal{B}'_Y$. Then clearly we have the maximal object $\underline{\mathbf{m}} = (\underline{\mathbf{m}}_X, \underline{\mathbf{m}}_Y)$ of $\mathcal{B}'_{X\otimes Y}$ with $G(\underline{\mathbf{m}}) = (c, d)$.

Case 2: Suppose c is a maximal element $\underline{\mathbf{m}}_X \in \mathcal{B}'_X$, and d is an intersection $\underline{\mathbf{m}}_Y \cap \underline{\mathbf{m}}'_Y \in \mathcal{B}'_Y$. Then we can take $\underline{\mathbf{m}} = (\underline{\mathbf{m}}_X, \underline{\mathbf{m}}_Y)$ and $\underline{\mathbf{m}}' = (\underline{\mathbf{m}}_X, \underline{\mathbf{m}}'_Y)$, and we clearly have $G(\underline{\mathbf{m}} \cap \underline{\mathbf{m}}') = (c, d)$.

A similar argument applies if c is an intersection and d is a maximal element.

Case 3: A similar argument works if both c and d are intersections.

∎

Proposition 4.3.9. *The functor* $G\colon \mathcal{B}'_{X\otimes Y} \to \mathcal{B}'_X \times \mathcal{B}'_Y$ *above is final.*

Proof. Let $(c, d) \in \mathcal{B}'_X \times \mathcal{B}'_Y$ be given. We need to show that the comma category $(c, d)/G$ is non-empty and connected for each choice of (c, d).

Being non-empty is a direct consequence of Lemma 4.3.8. Being connected is a direct consequence of Lemma 4.3.6. ∎

We shall also note that the composite

$$\mathcal{B}'^{\,op}_{X\otimes Y} \xrightarrow{G^{op}} \mathcal{B}'^{\,op}_{X} \times \mathcal{B}'^{\,op}_{Y} \xrightarrow{\Upsilon'_X \times \Upsilon'_Y} \mathbb{F}\text{-}\mathbf{Weil}_\infty \times \mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{\otimes} \mathbb{F}\text{-}\mathbf{Weil}_\infty$$

is precisely $\Upsilon'_{X\otimes Y} \colon \mathcal{B}'^{\,op}_{X\otimes Y} \to \mathbb{F}\text{-}\mathbf{Weil}_\infty$ (a routine calculation).

Proposition 4.3.10. *For all $m \in \mathbb{N}$, the functor $T^m \otimes H(\_) \colon \mathbb{F}\text{-}\mathbf{Weil}_\infty \to \mathcal{G}$ preserves the limits of* Theorem 4.2.10. *Explicitly, for all $X \in \mathbb{F}\text{-}\mathbf{Weil}_\infty$, we have*

$$\lim\left(\mathcal{B}'^{\,op}_{X} \xrightarrow{\Upsilon'_X} \mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{H} \mathcal{G} \xrightarrow{T^m\otimes\_} \mathcal{G}\right) \cong T^m \otimes H(X)\ .$$

Proof. First, since $mW \in \mathbb{F}\text{-}\mathbf{Weil_1}$ and $F(mW) = T^m$, we have

$$T^m \otimes H(\_) \cong H(mW \otimes \_) \colon \mathbb{F}\text{-}\mathbf{Weil}_\infty \to \mathcal{G}\ .$$

It thus suffices to show that $H(mW \otimes \_)$ preserves the required limits.

The canonical diagram of $mW \otimes X$ (i.e. $\Upsilon'_{mW\otimes X}$) can be expressed as

$$\mathcal{B}'^{\,op}_{X} \xrightarrow{\Upsilon'_X} \mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{mW\otimes\_} \mathbb{F}\text{-}\mathbf{Weil}_\infty\ .$$

As such, we have

$$\lim\left(\mathcal{B}'^{\,op}_{X} \xrightarrow{\Upsilon'_X} \mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{mW\otimes\_} \mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{H} \mathcal{G}\right) \cong H(mW \otimes X)\ .$$

■

Remark The same is true of $\_ \otimes T^m \colon \mathcal{G} \to \mathcal{G}$.

Proposition 4.3.11. *For X and Y as before, we have*

$$\lim\left(\mathcal{B}'^{\,op}_{Y} \xrightarrow{\Upsilon'_Y} \mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{X\otimes\_} \mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{H} \mathcal{G}\right) \cong H\left(X \otimes Y\right)\ .$$

Proof. Since

$$\lim\left(\mathcal{B}'^{\,op}_{Y} \xrightarrow{\Upsilon'_Y} \mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{X\otimes\_} \mathbb{F}\text{-}\mathbf{Weil}_\infty\right) \cong X \otimes Y$$

(using Proposition 2.1.13), we have the obvious cone

$$\begin{tikzcd} \mathcal{B}'^{\,op}_{Y} \arrow[r, "\Upsilon'_Y"'] \arrow[rrr, bend left, "\Delta H(X\otimes Y)"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[r, "X\otimes\_"'] \arrow[r, phantom, shift left=4, "\Downarrow H\alpha"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[r, "H"'] & \mathcal{G}\ . \end{tikzcd}$$

(where α is the limiting cone in $\mathbb{F}\text{-}\mathbf{Weil}_\infty$).

We wish to show $H\alpha$ is a limiting cone. Suppose we have an arbitrary cone

$$\begin{tikzcd} \mathcal{B}'^{\,op}_{Y} \arrow[r, "\Upsilon'_Y"'] \arrow[rrr, bend left, "\Delta A"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[r, "X\otimes\_"'] \arrow[r, phantom, shift left=4, "\Downarrow \gamma"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[r, "H"'] & \mathcal{G}\ . \end{tikzcd}$$

Now, we know that

$$\begin{aligned}
&H(X \otimes Y)\\
&= \lim\left(\mathcal{B'}^{op}_{X\otimes Y} \xrightarrow{\Upsilon'_{X\otimes Y}} \mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{H} \mathcal{G} \right)\\
&= \lim\left(\mathcal{B'}^{op}_{X} \times \mathcal{B'}^{op}_{Y} \xrightarrow{\Upsilon'_X \times \Upsilon'_Y} \mathbb{F}\text{-}\mathbf{Weil}_\infty \otimes \mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{\otimes} \mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{H} \mathcal{G} \right)
\end{aligned}$$

(using Proposition 4.3.9).

As such, we have a limiting cone

$$\begin{tikzcd}
\mathcal{B'}^{op}_{X} \times \mathcal{B'}^{op}_{Y} \arrow[r, "\Upsilon'_X \times \Upsilon'_Y"'] \arrow[rrr, bend left, "\Delta H(X\otimes Y)"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \times \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[r, "\otimes"'] \arrow[r, phantom, shift left=4, "\Downarrow H\beta"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[r, "H"'] & \mathcal{G} .
\end{tikzcd}$$

(where β is the limiting cone in $\mathbb{F}$-**Weil**$_\infty$).

Finally, we know that for each object $d \in \mathcal{B'}^{op}_{Y}$, we have $\Upsilon'_Y d = rW$ for some $r \in \mathbb{N}$, and by Proposition 4.3.10, we have (for each d) a limiting cone

$$\begin{tikzcd}
\mathcal{B'}^{op}_{X} \arrow[r, "\Upsilon'_X"'] \arrow[rrr, bend left, "\Delta H(X\otimes\Upsilon'_Y d)"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[r, "_\otimes\Upsilon'_Y d"'] \arrow[r, phantom, shift left=4, "\Downarrow H\delta"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[r, "H"'] & \mathcal{G} .
\end{tikzcd}$$

(where δ is the limiting cone in $\mathbb{F}$-**Weil**$_\infty$)

We first construct a unique map $h\colon A \to H(X \otimes Y)$ as follows: For each object $(c, d) \in \mathcal{B'}^{op}_{X} \times \mathcal{B'}^{op}_{Y}$, define $\iota_{(c,d)}$ as the composite

$$A \xrightarrow{\gamma_d} H(X \otimes \Upsilon'_Y d) \xrightarrow{H\delta_c} H\Upsilon'_X c \otimes H\Upsilon'_Y d \ .$$

These $\iota_{(c,d)}$'s induce a cone

$$\begin{tikzcd}
\mathcal{B'}^{op}_{X} \times \mathcal{B'}^{op}_{Y} \arrow[r, "\Upsilon'_X \times \Upsilon'_Y"'] \arrow[rrr, bend left, "\Delta A"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \times \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[r, "\otimes"'] \arrow[r, phantom, shift left=4, "\Downarrow \iota"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[r, "H"'] & \mathcal{G} .
\end{tikzcd}$$

(We note that this is a valid cone since for every morphism $(f, g)\colon (c, d) \to (c', d')$ of $\mathcal{B'}^{op}_{X} \times \mathcal{B'}^{op}_{Y}$, we can form the commuting diagram

$$\begin{tikzcd}
A \arrow[d, "\gamma_d"'] \arrow[dr, "\gamma_{d'}"] & & \\
H(X \otimes \Upsilon'_Y d) \arrow[r, "H(1\otimes\Upsilon'_Y g)"'] \arrow[d, "H\delta_c"] \arrow[dd, bend right=60, "H\delta_{c'}"'] & H(X \otimes \Upsilon'_Y d') \arrow[d, "H\delta'_c"] \arrow[ddr, bend left, "H\delta'_{c'}"] & \\
H\Upsilon'_X c \otimes H\Upsilon'_Y d \arrow[r, "1\otimes H\Upsilon'_Y g"'] \arrow[d, "H\Upsilon'_X f\otimes 1"] \arrow[drr, "H\Upsilon'_X f\otimes H\Upsilon'_Y g"] & H\Upsilon'_X c \otimes H\Upsilon'_Y d' \arrow[dr, "H\Upsilon'_X f\otimes 1"] & \\
H\Upsilon'_X c' \otimes H\Upsilon'_Y d \arrow[rr, "1\otimes H\Upsilon'_Y g"'] & & H\Upsilon'_X c' \otimes H\Upsilon'_Y d' ,
\end{tikzcd}$$

noting that $\Upsilon'_X c$ and $\Upsilon'_Y d$ are objects of $\mathbb{F}$-**Weil**$_\mathbf{1}$ so that $H(\Upsilon'_X c \otimes \Upsilon'_Y d) \cong H\Upsilon'_X c \otimes H\Upsilon'_Y d$, which shows the naturality of ι).

Then using the limiting cone β, we construct $h\colon A \to H(X \otimes Y)$. In particular, for each $(c,d) \in \mathcal{B'}_X^{op} \times \mathcal{B'}_Y^{op}$, the diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "\iota_{(c,d)}"'] & H(X \otimes Y) \arrow[d, "H\beta_{(c,d)}"] \\ & H\Upsilon'_X c \otimes H\Upsilon'_Y d \end{tikzcd}$$

commutes.

It now remains to show that for each $d \in \mathcal{B'}_Y^{op}$, the diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "\gamma_d"'] & H(X \otimes Y) \arrow[d, "H\alpha_d"] \\ & H(X \otimes \Upsilon'_Y d) \end{tikzcd}$$

commutes.

But remember that $H\delta$ gave a limiting cone. As such, it suffices to show that the diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\gamma_d"'] & H(X \otimes Y) \arrow[r, "H\alpha_d"] & H(X \otimes \Upsilon'_Y d) \arrow[d, "H\delta_c"] \\ H(X \otimes \Upsilon'_Y d) \arrow[rr, "H\delta_c"'] & & H\Upsilon'_X c \otimes H\Upsilon'_Y d \end{tikzcd}$$

commutes for each $c \in \mathcal{B}'^{\;op}_X$. Recall also that the limiting cones for X and Y are ρ and σ respectively. As such, we have

$$X \otimes Y \xrightarrow{\alpha_d} X \otimes \Upsilon'_Y d \;=\; X \otimes Y \xrightarrow{X \otimes \sigma_d} X \otimes \Upsilon'_Y d$$
$$X \otimes \Upsilon'_Y d \xrightarrow{\delta_c} \Upsilon'_X c \otimes \Upsilon'_Y d \;=\; X \otimes \Upsilon'_Y d \xrightarrow{\sigma_c \otimes \Upsilon'_Y d} \Upsilon'_X c \otimes \Upsilon'_Y d \;\;,$$

and so $H\delta_c \circ H\alpha_d = H(\sigma_c \otimes \rho_d)$. But we also have $\beta_{(c,d)} = \sigma_c \otimes \rho_d$. Finally, we have $\iota_{(c,d)} = H\delta_c \circ \gamma_d$ by definition. We thus have

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\gamma_d"'] \arrow[drr, "\iota_{(c,d)}"'] & H(X \otimes Y) \arrow[r, "H\alpha_d"] \arrow[dr, "H\beta_{(c,d)}"] & H(X \otimes \Upsilon'_Y d) \arrow[d, "H\delta_c"] \\ H(X \otimes \Upsilon'_Y d) \arrow[rr, "H\delta_c"'] & & H\Upsilon'_X c \otimes H\Upsilon'_Y d \end{tikzcd}$$

■

We saw in 4.3.1 that H was automatically monoidal. However, there seems to be no reason why H should automatically be strong monoidal. Equipped with these facts, however, we are able to instead give a condition on H equivalent to it being strong monoidal.

4.3.3 An equivalence to H being strong monoidal

For a monoidal functor $N\colon (\mathcal{A}, \otimes, I) \to (\mathcal{B}, \otimes, I')$ to be strong monoidal, we require the structure maps $N_{A,A'}\colon NA \otimes NA' \to N(A \otimes A')$ and $N_0\colon I' \to NI$ to be isomorphisms. Here we give an equivalent condition to this requirement. We have the following:

Theorem 4.3.12. *Let H be the pointwise right Kan extension of F along J (in the context of the above discussion). The following are equivalent:*

1) *H is strong monoidal.*

2) *For each $X \in \mathbb{F}\text{-}\mathbf{Weil}_\infty$, the composite*

$$\mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{H} \mathcal{G} \xrightarrow{H(X)\otimes\_} \mathcal{G}$$

preserves the limits of Theorem 4.2.10 *for all $Y \in \mathbb{F}\text{-}\mathbf{Weil}_\infty$.*

Proof. 1) $\Rightarrow$ 2) is routine: if $H(X \otimes Y) \cong H(X) \otimes H(Y)$, then we have a 2-cell

$$\begin{tikzcd} \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[r, "X\otimes\_"] \arrow[d, "H"'] \arrow[dr, phantom, "\cong"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[d, "H"] \\ \mathcal{G} \arrow[r, "H(X)\otimes\_"'] & \mathcal{G} \end{tikzcd}$$

and note that the top branch preserves Y as the limit of $\Upsilon'_Y\colon \mathcal{B}'^{op}_Y \to \mathbb{F}\text{-}\mathbf{Weil}_\infty$ (by Proposition 4.3.11).

2) $\Rightarrow$ 1) requires more work. As observed above, the unit map $I \to H\mathbb{F}$ is invertible.

To show that $H(X \otimes Y) \cong H(X) \otimes H(Y)$, we have the following:

$$\begin{aligned}
H(X\otimes Y) &\cong \lim\left(\mathcal{B}'^{op}_{X\otimes Y} \xrightarrow{\Upsilon'_{X\otimes Y}} \mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{H} \mathcal{G}\right)\\
&\cong \lim\left(\mathcal{B}'^{op}_X \times \mathcal{B}'^{op}_Y \xrightarrow{\Upsilon'_X\times\Upsilon'_Y} \mathbb{F}\text{-}\mathbf{Weil}_\infty \times \mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{\otimes} \mathbb{F}\text{-}\mathbf{Weil}_\infty \xrightarrow{H} \mathcal{G}\right)\\
&\cong \lim_{c,d} H\Upsilon'_X c \otimes H\Upsilon'_Y d \text{ (since } \Upsilon'_X c, \Upsilon'_Y d \in \mathbb{F}\text{-}\mathbf{Weil_1})\\
&\cong \lim_c \lim_d H\Upsilon'_X c \otimes H\Upsilon'_Y d\\
&\cong \lim_d\left((\lim_c H\Upsilon'_X c) \otimes H\Upsilon'_Y d\right) \ \text{ (using Proposition 4.3.10)}\\
&\cong \lim_d\left(HX \otimes H\Upsilon'_Y d\right)\\
&\cong HX \otimes (\lim_d H\Upsilon'_Y d) \ \text{ (using 2))}\\
&\cong HX \otimes HY
\end{aligned}$$

(these limits above being calculated in $\mathcal{G}$) ■

Remark This tells us that H is strong monoidal as long as $HX \otimes \_$ preserves certain limits. It is analogous to the requirement in the definition of Tangent Structure (as given in Definition 3.1.4) that T preserve the pullbacks $T^{(m)}$.

With Theorem 4.3.12 in mind, we shall now resume our discussion of the strong monoidal functor $F_\infty\colon \mathbb{F}\text{-}\mathbf{Weil}_\infty \to \mathcal{G}$. We have already noted that F_∞ is the right Kan extension of F. We will now show that it is in fact a monoidal right Kan extension.

4.3.4 F_∞ as a monoidal right Kan extension

Theorem 4.3.13. *F_∞ is a monoidal right Kan extension of F along J.*

Proof. We have $F\colon \mathbb{F}\text{-}\mathbf{Weil_1} \to \mathcal{G}$ and $F_\infty\colon \mathbb{F}\text{-}\mathbf{Weil}_\infty \to \mathcal{G}$ as previously discussed, and a natural isomorphism

$$\begin{tikzcd}
\mathbb{F}\text{-}\mathbf{Weil_1} \arrow[dr, hook, "J"'] \arrow[drr, bend left, "F"] & & \\
& \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[r, "F_\infty"'] \arrow[u, phantom, "\cong"] & \mathcal{G}
\end{tikzcd}$$

(we will not need this to necessarily be the identity natural isomorphism). Let a monoidal functor $G\colon \mathbb{F}\text{-}\mathbf{Weil}_\infty \to \mathcal{G}$ and a monoidal natural transformation $\beta\colon GJ \Rightarrow F$ be given. Explicitly, we have

$$\begin{tikzcd}
\mathbb{F}\text{-}\mathbf{Weil_1} \arrow[dr, hook, "J"'] \arrow[drr, bend left, "F"] & & \\
& \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[r, "G"'] \arrow[u, phantom, "\Uparrow\beta"] & \mathcal{G}
\end{tikzcd},$$

and moreover, the diagrams

$$\begin{tikzcd}
I \arrow[d] \arrow[dr, "\cong"] & \\
Gk \arrow[r, "\beta_k"'] & Fk
\end{tikzcd}
\qquad
\begin{tikzcd}
G(X\otimes Y) \arrow[r, "\beta_{X\otimes Y}"] \arrow[d] & F(X\otimes Y) \arrow[d, "\cong"] \\
G(X)\otimes G(Y) \arrow[r, "\beta_X\otimes\beta_Y"'] & F(X)\otimes F(Y)
\end{tikzcd}$$

commute for all $X, Y \in \mathbb{F}\text{-}\mathbf{Weil_1}$ (where I is the unit of $\mathcal{G}$).

Since F_∞ is the right Kan extension of F, we have a uniquely induced natural transformation $\alpha\colon G \Rightarrow F_\infty$ such that the composite

$$\begin{tikzcd}
\mathbb{F}\text{-}\mathbf{Weil_1} \arrow[dr, hook, "J"'] \arrow[drr, bend left, "F"] & & \\
& \mathbb{F}\text{-}\mathbf{Weil}_\infty \arrow[r, bend left, "F_\infty"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Uparrow\alpha"] \arrow[u, phantom, "\cong"] & \mathcal{G}
\end{tikzcd}$$

is equal to β.

We need only show that α is monoidal, i.e. that the diagrams

$$\begin{tikzcd}
I \arrow[d] \arrow[dr, "\cong"] & \\
Gk \arrow[r, "\alpha_k"'] & F_\infty k
\end{tikzcd}
\qquad
\begin{tikzcd}
G(U\otimes V) \arrow[r, "\alpha_{U\otimes V}"] \arrow[d] & F_\infty(U\otimes V) \arrow[d, "\cong"] \\
G(U)\otimes G(V) \arrow[r, "\alpha_U\otimes\alpha_V"'] & F_\infty(U)\otimes F_\infty(V)
\end{tikzcd}$$

commute for all $U, V \in \mathbb{F}\text{-}\mathbf{Weil}_\infty$.

The first diagram is immediate, since all the interior triangles of

$$\begin{tikzcd}
 & I \arrow[dl] \arrow[d, "\cong"] \arrow[dr, "\cong"] & \\
Gk \arrow[r, "\beta_k"] \arrow[rr, bend right, "\alpha_k"'] & Fk & F_\infty k \arrow[l, "\cong"]
\end{tikzcd}$$

commute.

As for the second diagram, recall from Theorem 4.2.10 that both U and V are limits of canonical diagrams with projections of the form $\pi_U \colon U \to mW$ and $\pi_V \colon V \to nW$, and these limits are preserved by F_∞. The same is true of $U \otimes V$, and moreover, from the description in Section 4.1.2, each projection of $U \otimes V$ (into some rW) is a tensor of a projection of U and one of V (i.e. $\pi_{U\otimes V} = \pi_U \otimes \pi_V$).

Thus, to show the commutativity of the second diagram, it suffices to show the commutativity of

$$\begin{tikzcd}
G(U \otimes V) \arrow[r, "\alpha_{U\otimes V}"] \arrow[d] & F_\infty(U \otimes V) \arrow[r, "\cong"] & F_\infty(U) \otimes F_\infty(V) \arrow[d, "\pi\pi"] \\
G(U) \otimes G(V) \arrow[r, "\alpha_U \otimes \alpha_V"'] & F_\infty(U) \otimes F_\infty(V) \arrow[r, "\pi\pi"'] & F_\infty(mW) \otimes F_\infty(nW)
\end{tikzcd}$$

for some arbitrary projection $\pi_U \otimes \pi_V \colon U \otimes V \to mW \otimes nW$ (denoted $\pi\pi$).

We simply fill in the interior as follows:

$$\begin{tikzcd}
 & & G(U \otimes V) \arrow[dll] \arrow[d, "\pi\pi"] \arrow[drr, "\alpha"] & & \\
G(U) \otimes G(V) \arrow[dd, "G(\pi)G(\pi)"] \arrow[ddd, bend right, "\alpha\alpha"'] & & G(mW \otimes nW) \arrow[d, "\beta"] \arrow[drr, "\alpha"] \arrow[ddll] & & F_\infty(U \otimes V) \arrow[d, "F_\infty(\pi\pi)"] \arrow[dd, bend left, "\cong"] \\
 & & F(mW \otimes nW) \arrow[d, "\cong"] & & F_\infty(mW \otimes nW) \arrow[ll, "\cong"] \\
G(mW) \otimes G(nW) \arrow[rr, "\beta\beta"] \arrow[drrrr, "\alpha\alpha"] & & F(mW) \otimes F(nW) & & F_\infty(U) \otimes F_\infty(V) \arrow[d, "F_\infty(\pi)F_\infty(\pi)"] \\
F_\infty(U) \otimes F_\infty(V) \arrow[rrrr, "F_\infty(\pi)F_\infty(\pi)"'] & & & & F_\infty(mW) \otimes F_\infty(nW) \arrow[ull, "\cong"]
\end{tikzcd}$$

(note we have omitted, for convenience, the labels for structure maps such as

$$G(U \otimes V) \to G(U) \otimes G(V) \ ,$$

as well as the subscripts identifying the components of the natural transformations α and β). ∎

We also note the following:

Proposition 4.3.14. *The limits for each object $X \in \mathbb{F}$-**Weil**$_\infty$ as described in* Theorem 4.2.10 *do not give a codensity representation.*

Proof. We simply need to show that for the inclusion $J\colon \mathbb{F}\text{-}\mathbf{Weil_1} \hookrightarrow \mathbb{F}\text{-}\mathbf{Weil}_\infty$, the functor

$$\begin{tikzcd}
\mathbb{F}\text{-}\mathbf{Weil}_\infty \to [\mathbb{F}\text{-}\mathbf{Weil_1}, \mathbf{Set}]^{op} \\
X \arrow[d, "f"'] & \mathbb{F}\text{-}\mathbf{Weil}_\infty\,[X, J(_)] \\
Y & \mathbb{F}\text{-}\mathbf{Weil}_\infty\,[Y, J(_)] \arrow[u, "\mathbb{F}\text{-}\mathbf{Weil}_\infty[f, J(_)]"']
\end{tikzcd}$$

does not preserve the limits described in Theorem 4.2.10.

We know that the equaliser

$$\begin{tikzcd}
W_2 \arrow[r, "s_2"] & 2W \arrow[r, shift left, "c"] \arrow[r, shift right, "1"'] & 2W
\end{tikzcd}$$

is one such limit. It suffices to show that the diagram

$$\begin{tikzcd}
\mathbb{F}\text{-}\mathbf{Weil}_\infty[2W, 5W] \arrow[r, shift left, "{\mathbb{F}\text{-}\mathbf{Weil}_\infty[c, 5W]}"] \arrow[r, shift right, "1"'] & \mathbb{F}\text{-}\mathbf{Weil}_\infty[2W, 5W] \arrow[r, "{\mathbb{F}\text{-}\mathbf{Weil}_\infty[s_2, 5W]}"] & \mathbb{F}\text{-}\mathbf{Weil}_\infty[W_2, 5W]
\end{tikzcd}$$

is not a coequaliser in **Set**.

Consider the map

$$\begin{aligned} h\colon W_2 &\to 5W \\ x &\mapsto y_1y_2 + y_2y_3 + y_3y_4 + y_4y_5 + y_1y_5 \ . \end{aligned}$$

It is rather routine to see that this map does not factor through s_2.

As such, the function $\mathbb{F}\text{-}\mathbf{Weil}_\infty[s_2, 5W]$ is not surjective, so the diagram above cannot be a coequaliser. ■

Chapter 5

Concluding Remarks

5.1 Other candidates for defining Tangent Structures

In Chapter 3, we exhibited the universal property of the category $\mathbf{Weil_1}$ with regards to Tangent Structure; to give a Tangent Structure (in the sense of [8]) is to give a strong monoidal functor

$$F : \mathbb{N}\text{-}\mathbf{Weil_1} \to \mathrm{End}(\mathcal{M})$$

preserving certain limits (Theorem 3.6.17). Moreover, we said, in the form of Definition 3.6.18, that we may more generally take an arbitrary monoidal category $\mathcal{G}$ instead of $\mathrm{End}(\mathcal{M})$.

In Chapter 4 we defined the category $\mathbf{Weil}_\infty$ and described a process for expressing each of its objects canonically as a limit of W_n's. We then showed that if k were a field $\mathbb{F}$ of characteristic zero, then each object of $\mathbb{F}\text{-}\mathbf{Weil}_\infty$ can in fact be expressed as a limit of nW's once again, and had a corresponding statement regarding right Kan extensions of F along an inclusion $J : \mathbb{F}\text{-}\mathbf{Weil_1} \hookrightarrow \mathbb{F}\text{-}\mathbf{Weil}_\infty$.

It certainly makes sense then to try to define a Tangent Structure corresponding to $\mathcal{W}$, where $\mathcal{W}$ is any full subcategory of $\mathbb{F}\text{-}\mathbf{Weil}_\infty$ containing $\mathbb{F}\text{-}\mathbf{Weil_1}$ as a full subcategory, i.e.

$$\mathbb{F}\text{-}\mathbf{Weil_1} \hookrightarrow \mathcal{W} \hookrightarrow \mathbb{F}\text{-}\mathbf{Weil}_\infty ,$$

as well as being closed under $\otimes$, and this would simply amount to a right Kan extension of F along $J_\mathcal{W} : \mathbb{F}\text{-}\mathbf{Weil_1} \hookrightarrow \mathcal{W}$ that is strong monoidal (an analogue of Theorem 4.3.12 would need to hold).

Of course, some choices for $\mathcal{W}$ are more appropriate than others. We will soon describe some arguably more viable candidates. However, before we haphazardly introduce higher dimensional candidates for $\mathcal{W}$ (in the sense that they involve W_n's for $n \geq 2$), we need to discuss a rather important idea: the addition of tangents.

5.1.1 Addition of tangents

Recall from Definition 3.1.4 the natural transformation

$$+ : T^{(2)} \Rightarrow T .$$

The purpose of this natural transformation is that it allows the addition of tangent vectors. We thus would like an analogue of this addition in higher dimensions.

Remark If the object $W = k[x]/x^2$ corresponds to giving tangent vectors (straight lines) to each point in a manifold M (in the form of TM), then (say) the object $W_2 = k[x]/x^3$ may correspond to giving tangent parabolas (with some measure of an appropriate notion of concavity, perhaps) to each point of M, and we obviously would like a notion of addition for these tangent parabolas.

Consider W_2. Naively, we may try to use the product

$$W_2 \times W_2 = k[x,y]/x^3, y^3, xy$$

and try to define a map

$$\begin{aligned} +_2 \colon W_2 \times W_2 &\to W_2 \\ x &\mapsto z \\ y &\mapsto z \ , \end{aligned}$$

but since $xy = 0$ and $z^2 \neq 0$, this is not a valid map.

Recall from Definition 4.1.3 that $W_{2,2}$ is the Weil algebra

$$k[x,y]/x^3, x^2y, xy^2, y^3 \ .$$

We now have a (valid) map

$$\begin{aligned} +_2 \colon W_{2,2} &\to W_2 \\ x &\mapsto z \\ y &\mapsto z \ . \end{aligned}$$

Just as $\mathbf{Weil_1}$ contained $W_{m,1}$ for all $m \in \mathbb{N}$, it is only natural that if $\mathcal{W}$ contains some W_n, then it should contain all $W_{m,n}$.

Moreover, if $\mathcal{W}$ contains a particular W_n (for $n > 1$), then it should also contain all $W_{n'}$ for which $n' < n$ (as well as all the corresponding $W_{m,n'}$'s). Intuitively, this simply says that if it is possible to take the n^{th} derivative, then it should also be possible to take any lower derivative.

We also have the following

Proposition 5.1.1. *The Weil algebra $W_{m,n}$ is the limit of the diagram*

$$\begin{tikzcd}
W_n \otimes (m-1)W_0 \arrow[d] & W_{n-1} \otimes W_1 \otimes (m-2)W_0 \arrow[dl] & \ldots & W_0 \otimes W_a \otimes W_{n-a} \otimes (m-3)W_0 \arrow[dl] \arrow[d] & \ldots \\
W_{n-1} \otimes (m-1)W_0 & \ldots & W_0 \otimes W_{a-1} \otimes W_{n-b} \otimes (m-3)W & W_0 \otimes W_a \otimes W_{n-b-1} \otimes (m-3)W_0 & \ldots
\end{tikzcd}$$

where there are always m factors in each tensor, and the subscripts of each object in the top row sum to n, the subscripts for each object in the bottom row sum to $n-1$, and all possible objects of this form are used.

Proof. These are precisely the diagrams of Theorem 4.2.10. ∎

Remark Since $W_{m,n}$ is no longer the m-fold product of W_n for $n > 1$, then the maps $+_n \colon W_{2,n} \to W_n$ do not define internal commutative monoids in the sense of Section 3.1.1. We will give an alternative to this in Section 5.2.

5.1.2 The category $\mathbf{Weil_{cog}}$

For the remainder of this chapter, we shall prefer to omit k as a prefix for our categories of Weil algebras when we wish deliberately to leave k unspecified. We note that we will always be referring to full subcategories of $\mathbf{Weil}_\infty$. Further, we shall discuss, where appropriate, the values k may take for the different cases we define.

In Chapter 3, we defined the category $\mathbf{Weil_1}$ to have, as its objects, precisely those Weil algebras corresponding to *p.c. graphs* (piecewise complete graphs), but quickly noted that these form a (proper) subset of the cographs.

Recall from Definition 2.2.11 that the set of cographs was given as the closure of the one point graph under finite graph joins and disjoint unions (Definition 2.2.11). Recall also that in Section 3.3, we said that

$$\kappa(G_A) \times \kappa(G_B) = \kappa(G_A \times G_B)$$
$$\kappa(G_A) \otimes \kappa(G_B) = \kappa(G_A \otimes G_B) \ .$$

As such, one natural extension of $\mathbf{Weil_1}$ is to take the all the cographs and form a (sub)category we shall call $\mathbf{Weil_{cog}}$. As such, to say that the objects of $\mathbf{Weil_{cog}}$ are precisely those Weil algebras corresponding to cographs is equivalent to saying that its objects are given as the closure of W under (finite iterations of) $\times$ and $\otimes$.

Moreover, each object is either of the form nW or part of a foundational pullback (Definition 2.1.16). This comes from Lemma 3.5.14.

5.1.3 The category $\mathbf{WEIL_1}$

We may continue to stay with W and not involve any higher W_n's. Recall from 4.1.2 that each object X of $\mathbf{Weil}_\infty$ corresponded to some down-set $S_X \subset \mathbb{N}^r$.

Definition 5.1.2. *Let* $\mathbf{WEIL_1}$ *be the full subcategory of* $\mathbf{Weil}_\infty$ *whose objects are given by all down-sets of the form* $S \subset \{0,1\}^r$ *(in the sense of* 4.1.2*).*

Remark The capitalisation is not in any way intended to suggest this is a large category. This will still be a full subcategory of $\mathbf{Weil}_\infty$.

Since $\mathbf{WEIL_1}$ is a full subcategory of $\mathbf{Weil}_\infty$, then each object $X \in \mathbf{WEIL_1}$ comes equipped with its canonical diagram (in the sense of Theorem 4.2.10). Moreover, it is easy to see that both $\mathbf{Weil_1}$ and $\mathbf{Weil_{cog}}$ are full subcategories of $\mathbf{WEIL_1}$. They are proper subcategories, since $\mathbf{WEIL_1}$ contains such Weil algebras as

$$k[x, y, z]/x^2, y^2, z^2, xyz \ .$$

Further, $\mathbf{WEIL_1}$ is the largest (full) subcategory of $\mathbf{Weil}_\infty$ with the condition that for any object $X \in \mathbf{WEIL_1}$, each generator x_i of X will always square to zero. A Tangent Structure corresponding to $\mathbf{WEIL_1}$ is then (in some sense) the largest Tangent Structure possible without involving the second (or higher) derivative(s).

Moreover, the idea of the right Kan extension of $F\colon \mathbf{Weil_1} \to \mathcal{G}$ along the inclusion $\mathbf{Weil_1} \hookrightarrow \mathbf{WEIL_1}$ does not require fractions as discussed in Section 4.2 (explicitly, we do not require $\frac{1}{n}$ to form the equaliser for W_n). As such, this idea is valid when k is $\mathbb{N}$, $\mathbb{Z}$ or an arbitrary field. The same is true for $\mathbf{Weil_{cog}}$.

Remark Recall in Section 3.5 that (ignoring coefficients), any map of $\mathbf{Weil_1}$ can be constructed using $\{\varepsilon, +, \eta, c, l\}$ as well as the limit diagram for each object. This is still true for $\mathbf{WEIL_1}$. We simply need to note that each object $X \in \mathbf{WEIL_1}$ is the limit of a diagram involving only mW's and apply Proposition 4.2.3.

Remark In the same way that $\mathbf{Weil_1}$ corresponded to the p.c. graphs and $\mathbf{Weil_{cog}}$ corresponded to the cographs, we may also say that the objects of $\mathbf{WEIL_1}$ correspond to hypergraphs (for which the definition can be found in numerous texts, for instance see [4]).

Remark We may also define $\mathbf{Weil_{Gph}}$ in the obvious manner. We would then have

$$\mathbf{Weil_1} \hookrightarrow \mathbf{Weil_{cog}} \hookrightarrow \mathbf{Weil_{Gph}} \hookrightarrow \mathbf{WEIL_1} \ .$$

5.1.4 The category $\mathbf{Weil_2}$

The next natural progression past $\mathbf{WEIL_1}$ is to include W_2 (as well as all $W_{m,2}$ as discussed in Section 5.1.1). One possibility is to take a "minimalistic" approach as we did with $\mathbf{Weil_1}$.

Definition 5.1.3. *Let $\mathbf{Weil_2}$ be the full subcategory of $\mathbf{Weil_\infty}$ with objects $W_{m,1}$, $W_{m',2}$ for all $m, m' \in \mathbb{N}$, as well as all (finite) tensors of these objects.*

Further, we have an analogy to Section 3.5. Of course, with the introduction of the $W_{m,2}$'s, the set $\{\varepsilon, +, \eta, c, l\}$ is no longer sufficient. We introduce the following (extended) set of generating maps:

- $\varepsilon_1 \colon W_1 \to W_0$ and $\varepsilon_2 \colon W_2 \to W_1$; ε_1 is the augmentation for W as in $\mathbf{Weil_1}$, and we have
$$\begin{aligned} \varepsilon_2 \colon k[x]/x^3 &\to k[x]/x^2 \\ x &\mapsto x \ . \end{aligned}$$
- $\eta_1 \colon W_0 \to W_1$, $\eta_2 \colon W_1 \to W_2$; η_1 is the unit for W as in $\mathbf{Weil_1}$, and we have
$$\begin{aligned} \eta_2 \colon k[x]/x^2 &\to k[x]/x^3 \\ x &\mapsto x^2 \ . \end{aligned}$$
- $+_1 \colon W_{2,1} \to W_1$ and $+_2 \colon W_{2,2} \to W_2$; $+_1$ is $+$ from $\mathbf{Weil_1}$, $+_2$ as given in 5.1.1.
- $l_{m,n} \colon W_m \to W_m \otimes W_n$, for $(m,n) \in \{(1,2),(2,2)\}$, given as $x \mapsto x \otimes y$
- $c_{m,n} \colon W_m \otimes W_n \to W_n \otimes W_m$, for $m, n \in \{1,2\}$ the obvious symmetry isomorphisms.
- $s_2 \colon W_2 \to W_1 \otimes W_1$ as given in Definition 4.2.1.

Remark The unit of W_2 is given as $\eta_2 \circ \eta_1$, the augmentation is given as $\varepsilon_1 \circ \varepsilon_2$. $l \colon W_1 \to W_1 \otimes W_1$ as given in Section 3.2 is the composite

$$W_1 \xrightarrow{l_{1,2}} W_1 \otimes W_2 \xrightarrow{1\otimes\varepsilon_2} W_1 \otimes W_1$$

Each map of $\mathbf{Weil_2}$ is then constructible (in an appropriate sense analogous to Definition 3.5.1). However, we shall omit the proof.

We may also define $\mathbf{Weil_{cog,2}}$, $\mathbf{Weil_{Gph}}_{,2}$ and $\mathbf{WEIL_2}$ in an analogous manner to $\mathbf{Weil_{cog}}$ and $\mathbf{WEIL_1}$. For these categories involving W_2, k need not be a field $\mathbb{F}$ of characteristic zero either. All that would be required is that k were a rig containing the fraction $\frac{1}{2}$. One example of such a k is the set of non-negative binary fractions (i.e. the rational numbers that can be expressed in the form $\frac{m}{2^n}$, for $m, n \in \mathbb{N}$).

Moreover, we can use these ideas to define $\mathbf{Weil_n}$, $\mathbf{Weil_{cog,n}}$, $\mathbf{Weil_{Gph}}_{,n}$ and $\mathbf{WEIL_n}$. In these cases, then k would need to be a rig containing the fractions $\{\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\}$ in order express the a Tangent Structure corresponding to $\mathcal{W}$ as a strong monoidal right Kan extension of $F\colon \mathbf{Weil_1} \to \mathcal{G}$.

5.2 In place of internal commutative monoids

Recall the map $+_n\colon W_{2,n} \to W_n$ discussed towards the end of Section 5.1.1. The domain $W_{2,n}$ of this map is not the product $W_n \times W_n$, but rather a limit of the form described in Proposition 5.1.1. As such, it $+_n$ does not define a commutative monoid in the sense of Section 3.1.1.

However, we still wish to think of $+_n$ a an "operation on W_n", and we shall see that there is a sense in which it is commutative, associative, and unital. We shall give a perspective here that does precisely this.

Definition 5.2.1. *Let* $\mathbf{Pol_f}$ *(defined in an almost identical manner to* [30]*, although this also appears in earlier texts) be the category given as*

- *Objects: Finite sets*
- *Morphisms: A morphism* $f\colon A \nrightarrow B$ *is a diagram*

$$\begin{tikzcd} & C \arrow[r, "f_2"] \arrow[dl, "f_1"'] & D \arrow[dr, "f_3"] & \\ A & & & B \end{tikzcd};$$

  where C *and* D *are finite sets,* f_1*,* f_2*,* f_3 *are functions and moreover* f_2 *is surjective. Such morphisms represent polynomial functions (with coefficients in* $\mathbb{N}$*). We give an example below.*

- *Composition is simply given as composition of the corresponding polynomial functions. Identities* $id\colon A \nrightarrow A$ *are given as*

$$\begin{tikzcd} & A \arrow[r, "id"] \arrow[dl, "id"'] & A \arrow[dr, "id"] & \\ A & & & A \end{tikzcd}.$$

Remark We note that our definition differs to that of [30] since we have the additional requirement that f_2 is surjective. However, this simply says that the corresponding polynomial function does not have (non-zero) constant terms. Clearly, this property is preserved by composition.

Example 5.2.2. *Consider the polynomial function*

$$(x, y, z) \mapsto (x^2 + xy + z, x + y + z^3)\ .$$

We first express the polynomial as

$$(x, y, z) \mapsto (xx + xy + z, x + y + zzz)\ .$$

We then label the terms

$$(\underbrace{x}_{1}\underbrace{x}_{2} + \underbrace{x}_{3}\underbrace{y}_{4} + \underbrace{z}_{5}, \underbrace{x}_{6} + \underbrace{y}_{7} + \underbrace{z}_{8}\underbrace{z}_{9}\underbrace{z}_{10})\ ,$$

the summands

$$(\underbrace{xx}_{a} + \underbrace{xy}_{b} + \underbrace{z}_{c}, \underbrace{x}_{d} + \underbrace{y}_{e} + \underbrace{zzz}_{f})\ ,$$

and the components

$$(\underbrace{xx + xy + z}_{\alpha}, \underbrace{x + y + zzz}_{\beta})\ .$$

of the right hand term of the polynomial function as shown above.

Then the polynomial function can be expressed as

$$\begin{tikzcd}
& \{1, \dots, 10\} \arrow[r, "f_2"] \arrow[dl, "f_1"'] & \{a, \dots, f\} \arrow[dr, "f_3"] & \\
\{x, y, z\} & & & \{\alpha, \beta\}\ ,
\end{tikzcd}$$

where f_1, f_2, f_3 are the obvious functions.

Now, let ω be the poset $\mathbb{N}$ regarded as a category.

Definition 5.2.3. *The functor $\Phi\colon \mathbf{Pol_f}^{op} \to [\omega^{op}, \mathbf{Weil}_\infty]$ is given as follows:*

- *On objects: $\Phi(m) = W_{m,-}$*
- *On morphisms: For $g\colon m \nrightarrow n\ \in \mathbf{Pol_f}$ given as*

  $$(x_1, \dots, x_m) \mapsto \big(p_1(x_1, \dots, x_m), \dots, p_n(x_1, \dots, x_m)\big)\ ,$$

  the natural transformation $\Phi(g)\colon W_{n,-} \Rightarrow W_{m,-}$ has as its component at $r \in \mathbb{N}$

  $$\begin{aligned} \Phi(g)_r\colon W_{n,r} &\to W_{m,r} \\ u_i &\mapsto p_i(v_1, \dots, v_m) \end{aligned}$$

  (it is routine to verify that this not only defines a valid morphism for each r, but that this is also natural).

Definition 5.2.4. *For F and $G \in [\omega^{op}, \mathbf{Weil}_\infty]$, define $(F * G) \in [\omega^{op}, k\text{-}\mathbf{Weil}_\infty]$ as the functor given as*

$$(F * G)(n) = \lim_{\substack{u+v \leq j \\ j \leq n}} F(u) \otimes G(v)\ .$$

Explicitly, $(F * G)(n)$ *is the limit of the diagram*

$$\begin{tikzcd}
F(0) \otimes G(n) \arrow[d] & F(1) \otimes G(n-1) \arrow[d] \arrow[dl] & \dots & F(n) \otimes G(0) \arrow[d] \\
F(0) \otimes G(n-1) \arrow[d] & F(1) \otimes G(n-2) \arrow[d] \arrow[dl] & \dots & F(n-1) \otimes G(0) \arrow[d] \\
\dots & \dots & \dots & \dots
\end{tikzcd}$$

However, note that in the diagram above, all squares commute, and so it suffices to instead take the limit of the diagram (omitting $\otimes$ as usual)

$$\begin{tikzcd}
F(0)G(n) \arrow[d] & F(1)G(n-1) \arrow[d] \arrow[dl] & \dots & F(n)G(0) \arrow[d] \\
F(0)G(n-1) & F(1)G(n-2) & \dots & F(n-1)G(0)
\end{tikzcd}$$

This operation is clearly symmetric and associative (since $\otimes$ is symmetric and associative also). Further, the unit of this operation is the constant functor K at k. Thus $[\omega^{op}, \mathbf{Weil}_\infty]$ is in fact a symmetric monoidal category.

Proposition 5.2.5. *For* $m, n \in \mathbb{N}$, $(W_{m,-} * W_{n,-}) \cong W_{m+n,-}$ *in* $[\omega^{op}, \mathbf{Weil}_\infty]$.

Proof. It suffices to show $(W_{m,-} * W_{n,-})(r) \cong W_{m+n,r}\ \forall\ r \in \mathbb{N}$. The proof is trivial if m or n are 0.

Let $r \in \mathbb{N}$ be given. Suppose further that $m = 1$. Then $(W_{1,-} * W_{n,-})(r)$ is the limit of the diagram

$$\begin{tikzcd}
W_r W_{n,0} \arrow[d] & W_{r-1} W_{n,1} \arrow[d] \arrow[dl] & \dots & W_0 W_{n,r} \arrow[d] \\
W_{r-1} W_{n,0} & W_{r-2} W_{n,1} & \dots & W_0 W_{n,r-1}\ .
\end{tikzcd}$$

But recall that the Weil algebra $W_{m,n}$ is the limit of a particular diagram (Proposition 5.1.1) and that for any Weil algebra A, the functor $A \otimes \_$ preserves this limit (Corollary 4.1.17). Then the diagram above is the same as that for $W_{n+1,r}$. ∎

Now, $\mathbf{Pol_f}$ is monoidal under $+$ (disjoint union), and $[\omega^{op}, \mathbf{Weil}_\infty]$ is monoidal under $*$. Φ is thus strong monoidal with respect to these structures.

Finally, we have the following monoid in $\mathbf{Pol_f}$:

- $\eta \colon 0 \nrightarrow 1$ given as

$$\begin{tikzcd}
& 0 \arrow[r, "id"] \arrow[dl, "id"'] & 0 \arrow[dr, "!"] & \\
0 & & & 1
\end{tikzcd}$$

  corresponding to the polynomial function $(\ ) \mapsto (0)$.

- $\mu \colon 2 \nrightarrow 1$ given as

$$\begin{tikzcd}
& 2 \arrow[r, "id"] \arrow[dl, "id"'] & 2 \arrow[dr, "!"] & \\
2 & & & 1
\end{tikzcd}$$

  corresponding to the polynomial function $(x, y) \mapsto x + y$.

Then, $\Phi(\eta)_r \colon k \to W_r$ is the unit of W_r, $\Phi(\mu)_r \colon W_{2,r} \to W_r$ is the map $+_r$.

5.3 Objects outside of $\mathbf{Weil}_\infty$ as limits of nW_1's

We saw in Section 4.2 that objects of $\mathbf{Weil}_\infty$ could be described through the combinatorics of mW's. A similar approach may yield objects beyond those in $\mathbf{Weil}_\infty$.

Consider the Weil algebra $X = k[x,y]/3x^2 = y^3, xy$ (we use $3x^2 = y^3$ rather than $x^2 = y^3$ for convenience, in that the calculations are slightly nicer). For k taking the form $\mathbb{2}$, $\mathbb{N}$, $\mathbb{Z}$ or a field at least, it is rather straightforward to show that X is not contained in $\mathbf{Weil}_\infty$ in that there cannot exist an isomorphism in $\mathbf{Weil}$ between X and any object of $\mathbf{Weil}_\infty$.

Proposition 5.3.1. *For $k = \mathbb{2}$, $\mathbb{N}$, $\mathbb{Z}$ or a field, the Weil algebra $X = k[x,y]/3x^2 = y^3, xy$ is not isomorphic to any object of* $\mathbf{Weil}_\infty$.

Proof. We note first that as a k-module, X is (freely) generated by the elements $\{1, x, x^2, y, y^2\}$, and is thus five dimensional.

The only five dimensional Weil algebras in k-$\mathbf{Weil}_\infty$ are:

- $W_4 = k[x]/x^5$,
- $W_3 \times W_1 = k[x,y]/x^4, y^2, xy$,
- $W_2 \times W_2 = k[x,y]/x^3, y^3, xy$,
- $k[x,y]/x^3, x^2y, y^3$,
- $W_2 \times W_1 \times W_1 = k[x,y,z]/x^3, y^2, z^2, xy, yx, xz$,
- $(W_1 \otimes W_1) \times W_1 = k[x,y,z]/x^2, y^2, z^2, xz, yz$,
- $(W_1)^4 = k[w,x,y,z]/\{\text{all degree two monomials}\}$

(we can see this by considering the down-sets of Definition 4.1.4).

We shall demonstrate the case for W_4 only, however the approach will be similar for the others.

Let $W_4 = k[u]/u^5$, and consider an arbitrary map

$$\begin{aligned} f\colon W_4 &\to X \\ u &\mapsto \alpha x + \beta x^2 + \gamma y + \delta y^2 \ . \end{aligned}$$

We then have

$$\begin{aligned} f(u^2) &= \alpha^2 x^2 + \gamma^2 y^2 \\ f(u^3) &= \gamma^3 y^3 = 3\gamma^3 x^2 \\ f(u^4) &= 0 \ . \end{aligned}$$

But $u^4 \neq 0$, and so this map f is not injective on elements. As such, it cannot be an isomorphism. ∎

Taking k to be once again a field $\mathbb{F}$ of characteristic zero, we shall exhibit X as a limit of nW's. There is an injective map

$$\begin{aligned} g\colon X &\to 6W \\ x &\mapsto z_1z_2z_3 + z_4z_5z_6 \\ y &\mapsto z_1z_4 + z_2z_5 + z_3z_6 \ . \end{aligned}$$

We may then try to use a combinatorial approach to describe this map and ultimately characterise the Weil algebra in question.

Consider the following diagram

$$\begin{tikzcd}
5W & 5W & 7W \\
4W & 6W \arrow[ul, shift left, "\lambda_2"'] \arrow[ul, shift right, "\lambda_1"] \arrow[u, shift right, "\lambda_3"] \arrow[u, shift left, "\lambda_4"'] \arrow[ur, shift left, "\gamma_1"] \arrow[ur, shift right, "\gamma_2"'] \arrow[l, shift right, "\rho'_{1,2,3,5}"'] \arrow[l, shift left, "\rho_{1,2,3,5}"] \arrow[r, shift left=2, "\phi_1"] \arrow[r, "\phi_2" description] \arrow[r, shift right=2, "id"'] \arrow[dl, shift right, "\rho'_{1,2,3,4}"'] \arrow[dl, shift left, "\rho_{1,2,3,4}"] \arrow[d, shift right=2, "0"'] \arrow[d, phantom, "\cdots"] \arrow[d, shift left=2, "h_{r,s,t}"] \arrow[dr, shift left=2, "g_{r,s}"] \arrow[dr, phantom, "\cdots"] \arrow[dr, shift right=2, "0"'] & 6W \\
4W & 3W & 2W
\end{tikzcd} ;$$

where:

- ϕ_1 is given by the permutation $(123)(456)$, ϕ_2 is given by the permutation $(14)(25)(36)$
- $g_{r,s} \colon 6W \to 2W$ is given as
$$\begin{aligned} z_r &\mapsto a_1 \\ z_s &\mapsto a_2 \\ z_i &\mapsto 0 \text{ otherwise,} \end{aligned}$$
for all $(r, s) \in \mathbb{N}^2$ with $0 < r < s < 7$, except for $(1, 4)$, $(2, 5)$ and $(3, 6)$
- $h_{r,s,t} \colon 6W \to 3W$ is given as
$$\begin{aligned} z_r &\mapsto a_1 \\ z_s &\mapsto a_2 \\ z_t &\mapsto a_3 \\ z_i &\mapsto 0 \text{ otherwise,} \end{aligned}$$
for $(r, s, t) \in \{(1, 2, 6), (1, 3, 5), (1, 5, 6), (2, 3, 4), (2, 4, 6), (3, 4, 5)\}$.
- $0 \colon 6W \to nW$ is the zero map
- $\rho_{1,2,3,4} \colon 6W \to 4W$ is given as
$$\begin{aligned} z_1 &\mapsto a_1 \\ z_2 &\mapsto a_2 \\ z_3 &\mapsto a_3 \\ z_4 &\mapsto a_4 \\ z_5, z_6 &\mapsto 0 \ , \end{aligned}$$
- $\rho'_{1,2,3,4} \colon 6W \to 4W$ is given as
$$\begin{aligned} z_1 &\mapsto a_1 \\ z_2 &\mapsto a_3 \\ z_3 &\mapsto a_2 \\ z_4 &\mapsto a_4 \\ z_5, z_6 &\mapsto 0 \end{aligned}$$

- $\rho_{1,2,3,5} \colon 6W \to 4W$ is given as

$$\begin{aligned} z_1 &\mapsto a_1 \\ z_2 &\mapsto a_2 \\ z_3 &\mapsto a_3 \\ z_5 &\mapsto a_4 \\ z_4, z_6 &\mapsto 0 \ , \end{aligned}$$

- $\rho'_{1,2,3,5} \colon 6W \to 4W$ is given as

$$\begin{aligned} z_1 &\mapsto a_3 \\ z_2 &\mapsto a_2 \\ z_3 &\mapsto a_1 \\ z_5 &\mapsto a_4 \\ z_4, z_6 &\mapsto 0 \end{aligned}$$

- $\lambda_1 \colon 6W \to 5W$ is given as

$$\begin{aligned} z_1 &\mapsto a_1 \\ z_2 &\mapsto a_2 \\ z_3 &\mapsto a_3 a_4 \\ z_4 &\mapsto a_2 a_5 \\ z_5 &\mapsto 0 \\ z_6 &\mapsto 0 \ , \end{aligned}$$

- $\lambda_2 \colon 6W \to 5W$ is given as

$$\begin{aligned} z_1 &\mapsto a_1 a_2 \\ z_2 &\mapsto a_3 \\ z_3 &\mapsto a_4 \\ z_4 &\mapsto a_5 \\ z_5 &\mapsto 0 \\ z_6 &\mapsto 0 \end{aligned}$$

- $\lambda_3 \colon 6W \to 5W$ is given as

$$\begin{aligned} z_1 &\mapsto a_1 \\ z_2 &\mapsto a_2 \\ z_3 &\mapsto 0 \\ z_4 &\mapsto a_2 a_4 \\ z_5 &\mapsto 0 \\ z_6 &\mapsto a_3 a_5 \ , \end{aligned}$$

- $\lambda_4 \colon 6W \to 5W$ is given as

$$\begin{aligned} z_1 &\mapsto a_1a_2 \\ z_2 &\mapsto a_3 \\ z_3 &\mapsto 0 \\ z_4 &\mapsto a_4 \\ z_5 &\mapsto 0 \\ z_6 &\mapsto a_5 \end{aligned}$$

- $\gamma_1 \colon 6W \to 7W$ is given as

$$\begin{aligned} z_1 &\mapsto a_1a_2 \\ z_2 &\mapsto a_3a_4 \\ z_3 &\mapsto a_5 \\ z_4 &\mapsto a_6 \\ z_5 &\mapsto a_7 \\ z_6 &\mapsto 0 \ , \end{aligned}$$

- $\gamma_2 \colon 6W \to 7W$ is given as

$$\begin{aligned} z_1 &\mapsto a_1 \\ z_2 &\mapsto a_3a_4 \\ z_3 &\mapsto a_2a_5 \\ z_4 &\mapsto a_2a_6 \\ z_5 &\mapsto a_7 \\ z_6 &\mapsto 0 \ . \end{aligned}$$

Although this looks rather complicated, the limit (joint equaliser) of this diagram is constructed in **Vect**, and is precisely $k[x, y]/3x^2 = y^3, xy$.

Now, the fact that g is an injective map also exhibits X as a subalgebra of $6W$. So one possibility then is to exhibit arbitrary Weil algebras (i.e. those outside of $\mathbb{F}$-**Weil**$_\infty$) as subalgebras of an appropriate nW.

In particular, there is what one might call the *Laplacian* Weil algebra (as seen in [17], say)

$$L = k[x, y]/x^2 = y^2, xy \ ,$$

and we have an injective map $f \colon L \to 4W$ given as

$$\begin{aligned} x &\mapsto z_1z_2 + z_3z_4 \\ y &\mapsto z_1z_3 + z_2z_4 \end{aligned}$$

One proposed approach to dealing with Weil algebras outside of **Weil**$_\infty$ (for general k) is to try to express such a Weil algebra X as a subalgebra of some nW, and then use the injective map $X \hookrightarrow nW$ (if it exists) to express X as some joint equaliser.

That being said, this is no easy task. In fact Poonen [28], taking k to be an algebraically closed field, actually classifies the Weil algebras by dimension (up to six) and shows that the set of isomorphism classes of dimension seven or greater is infinite. Without the assumption of k being algebraically closed, we would expect even more isomorphism classes.

Example 5.3.2. *Suppose $k = \mathbb{C}$. The map*

$$\begin{aligned} f\colon \mathbb{C}[x,y]/x^2 = y^2, xy &\to \mathbb{C}[u,v]/u^2, v^2 \\ x &\mapsto u+v \\ y &\mapsto i(u-v) \end{aligned}$$

is an isomorphism. It is rather simple to check first that this is a valid map. To see it is an isomorphism, note that $f(x^2) = f(y^2) = 2uv$, and moreover, note that $\{1, u+v, i(u-v), 2uv\}$ forms a basis for the complex vector space underlying $k[u,v]/u^2, v^2$, and so f defines an isomorphism in **Vect***, and thus is an isomorphism in* **Weil**.

Remark It is also routine to show that no isomorphism can exist between these two objects in the absence of i (e.g. if k were $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{R}$).

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