Riesz Transform Estimates in the Absence of a Preservation Condition and Applications to the Dirichlet Laplacian

by

Joshua Grahame Peate

B. Sc. (Hons) Macquarie University (2010)

This thesis is presented for the degree of

Doctor of Philosophy

Department of Mathematics

2015

Certification

This thesis entitled:

Riesz Transform Estimates in the Absence of a Preservation Condition and Applications to the Dirichlet Laplacian

written by: **Joshua Grahame Peate**

has been approved by the Department of Mathematics at Macquarie University.

Declaration

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of the requirements for a degree to any other university or institution other then Macquarie University.

I also certify that this thesis is an original piece of research and has been written by me. Any help and assistance that I have recieved in my research work and the preparation of the thesis itself has been appropriately acknowledged.

In addition, I certify that all information sources and literature used are indicated in the thesis.

Acknowledgements

First and foremost I would like to thank my supervisor Professor Xuan Duong for his help in guiding this thesis, for his advice and encouragement, and for his valuable skills at explaining the difficult aspects of analysis.

I would also like to express gratitude to the people of the Mathematics Department at Macquarie University for their environment of learning and support. Particular thanks go to my associate supervisor Dr Chris Meaney, and to Dr Ross Moore who provided the initial LaTeX template. Thanks also go to the various PhD and honours students that have crossed my path at Macquarie for their friendship and encouragement.

Final thanks go to family members for their support during my time of study.

Summary

R. Strichartz in [68] asked whether the $\mathcal{L}^p$ boundedness of the Riesz transform observed on $\mathbb{R}^n$ could be extended to a reasonable class of non-compact manifolds. Many partial answers have been given since. One such answer given by Auscher, Coulhon, Duong and Hofmann in [5] tied the $\mathcal{L}^p$ boundedness of the Riesz transform to the $\mathcal{L}^p$ boundedness of the Gaffney inequality. Their result was for $p > 2$ and held on non-compact manifolds satisfying doubling and Poincaré conditions, along with a stochastic completeness or preservation condition.

In this thesis the results of [5] are adapted to prove $\mathcal{L}^p$ bounds, $p > 2$, for Riesz transform variations in cases where a preservation condition does not hold. To compensate for the lack of a preservation condition, two new conditions are required. The results are general enough to apply in a large number of circumstances. Two extensions on this result are additionally presented.

The first extension is to non-doubling domains. This extension is specifically in the circumstance of a manifold with boundary and Dirichlet boundary conditions. An added benefit of this non-doubling extension is that the Poincaré inequality is no longer required near the boundary. The second extension shows that the weighted $\mathcal{L}^p$ boundedness of the Riesz transform observed on $\mathbb{R}^n$ can also be extended in some degree to a reasonable class of non-compact manifolds. This second extension includes generalised deriving of weight classes associated to skewed maximal functions and other operators.

This thesis also contains applications to the case of the Dirichlet Laplacian on various subsets of $\mathbb{R}^n$. The overall work and particularly the application are motivated by recent results from Killip, Visan and Zhang in [48].

Notation

n	Refers to the number of dimensions of the space, as in $\mathbb{R}^n$.
$x \wedge y$	Is the minimum of numbers either side, $x \wedge y = \min(x, y)$.
$f(x) \lesssim g(y)$	Indicates that there exists a constant $c > 0$ such that the inequality $f(x) \leq cg(y)$ holds for all x, y.
Ω	Is an arbitrary manifold possibly with boundary.
$\delta\Omega$	Indicates the boundary of the space Ω when it exists.
∇	Is the Riemannian gradient.
Δ_Ω	Is the Dirichlet Laplacian on the space Ω.
1_S	Is the characteristic function of a given set S.
$d(x, y)$	Is the distance from x to y in Ω.
$\rho(x)$	Is the minimal distance from a point $x \in \Omega$ to the boundary $\delta\Omega$, $\rho(x) = \inf_{z\in\delta\Omega} d(x, z)$.
$w(x)$	Indicates a weight. This is any positive function defined on Ω.
B	Is an open ball of radius r. If $B \subset \Omega$ then for some $x \in \Omega$ and $r > 0$, B is given by: $B = B(r) = B(x, r) = \{y \in \Omega : d(x, y) < r\}$.
$⨍_B$	Is an averaged integral over B, $⨍_B f(x)\,\mathrm{d}x = \frac{1}{\lvert B\rvert}\int_B f(x)\,\mathrm{d}x$.
$C_0^\infty(\Omega)$	The space of infinitely differentiable functions $f : \Omega \to \mathbb{R}$ that vanish on the boundary.
$\mathcal{L}^p(\Omega)$	Is the space of functions f that satisfy the following bound, $$\left[\int_\Omega \lvert f(x)\rvert^p\,\mathrm{d}x\right]^{1/p} < \infty.$$
p'	The conguate exponent p' of p is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.
$\mathcal{L}^p(w)$	Is the space of functions f that satisfy the following bound, $$\left[\int_\Omega \lvert f(x)\rvert^p w(x)\,\mathrm{d}x\right]^{1/p} < \infty.$$
A_p	Indicates the space of weights w that satisfy the following bound, $$\left[⨍_B w(x)\,\mathrm{d}x\right]\left[⨍_B w(x)^{-p'/p}\,\mathrm{d}x\right]^{p/p'} < \infty.$$
$A_p^{\alpha,\beta}$	Indicates the space of weights w that satisfy the following bound, $$\left[⨍_B \alpha(x)^p w(x)\,\mathrm{d}x\right]\left[⨍_B \beta(x)^{p'} w(x)^{-p'/p}\,\mathrm{d}x\right]^{p/p'} < \infty.$$

Table of Contents

List of Figures

Part I

Introduction and Main Results

Chapter 1:

Introduction

Slight alterations to the classical singular integral Riesz transform $\mathcal{L}^p$ estimate problem can create a plethora of issues, not all of which can be solved using the standard methods of the current techniques. Foremost among these issues is the loss of the so called *preservation condition.* This thesis extends the literature on Riesz transforms by dealing with this preservation condition loss.

The *classical singular integral Riesz transform* refers to the following singular integral operator, henceforth referred to as the classical Riesz transform.

$$Rf(x) = \nabla\Delta^{-1/2} f(x)$$

Here ∇ is the standard vector valued gradient operator on $\mathbb{R}^n$, and $\Delta^{-1/2}$ is the inverse square root of the standard Laplacian on $\mathbb{R}^n$, known to exist uniquely via spectral theory (see for example [39] or [56]). The $\mathcal{L}^p$ *estimate problem* refers to whether or not the Riesz transform satisfies the following $\mathcal{L}^p \to \mathcal{L}^p$ *norm* bound.

$$\||\nabla\Delta^{-1/2} f|\|_{\mathcal{L}^p(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{L}^p(\mathbb{R}^n)}$$

The class of all functions f that satisfy $\|f\|_{\mathcal{L}^p(\mathbb{R}^n)} < \infty$ is named $\mathcal{L}^p(\mathbb{R}^n)$. If $\mathrm{d}x$ is replaced by a general measure $\mathrm{d}\mu$, then this is known as a *weighted* $\mathcal{L}^p$ *estimate problem.* It is well known, (see for example [67]) that the classical Riesz transform satisfies an $\mathcal{L}^p$ norm estimate for all $1 < p < \infty$, and a weighted $\mathcal{L}^p$ norm estimate for all weights $\mathrm{d}\mu = w(x)\,\mathrm{d}x$ with w in the Muckenhoupt A_p class.

Possible alterations to the classical Riesz transform include replacing $\mathbb{R}^n$ with a new domain Ω, and substituting the Laplacian with a more general second order differential operator L (being careful to ensure $L^{-1/2}$ is well defined on Ω). The following question arises:

Does the Riesz transform altered in such a way still satisfy an $\mathcal{L}^p$ norm inequality?

$$\| |\nabla L^{-1/2} f| \|_{\mathcal{L}^p(\Omega)} \lesssim \|f\|_{\mathcal{L}^p(\Omega)} \tag{1.1}$$

This question was first posed by Strichartz in [68] and many partial solutions have been given since (see [5] and [17] and references therein. These are reviewed in section 1.3). Importantly for this thesis, cases where the preservation condition (also known as stochastic completeness) does not hold have not been dealt with in general. To say that there is no preservation condition is the same as to say,

$$e^{-tL}1 \neq 1.$$

The answers that this thesis provides are the following:

1. The Riesz transform in cases without a preservation condition is $\mathcal{L}^p$ bounded, $p > 2$, if $\mathcal{L}^p$ bounds for Gaffney and Hardy type operators hold (see Theorem 1.1).

2. Additional heat kernel decay allows the result above to extend to non-doubling domains (see Theorem 1.2).

3. The result also holds considering weighted $\mathcal{L}^p$ boundedness (see Theorem 1.3).

This thesis also contains applications to cases where Ω is a subset of $\mathbb{R}^n$ and L is the Dirichlet Laplacian Δ_Ω on Ω. In these cases specific weights for which weighted $\mathcal{L}^p$ boundedness hold are given. Some of these weights are outside the standard Muckenhoupt class. The main results of this thesis will be formally stated in section 1.4.

1.1 Motivation

The reasons for altering the classical Riesz transform and seeking $\mathcal{L}^p$ bounds are varied. Firstly, Riesz transforms are difficult operators to prove bounds for, as they hold a position near the gap between bounded and unbounded singular integrals. Studying Riesz transforms creates more knowledge of the general classification of singular integral operators. Further, Riesz transforms relate to other types of transforms, such as

Fourier transform multipliers, allowing comparative information between different operators. One final common use of Riesz transform boundedness is the implied relation between two generalised definitions of a first order differential operator. This allows the comparison of alternate definitions of Sobolev spaces. That is, the Riesz transform in equation (1.1) implies the following for all f in the appropriate Sobolev space.

$$\||\nabla f|\|_{\mathcal{L}^p(\Omega)} \lesssim \|L^{1/2} f\|_{\mathcal{L}^p(\Omega)} \tag{1.2}$$

Examples of variations on the Riesz transform that do not satisfy the preservation condition include the case where the Laplacian is replaced with the Schrödinger operator $L = -\Delta + V(x)$. Whilst the Schrödinger operator case is not investigated in this thesis, it has in part been investigated before, and the author plans to consider such operators from the perspective of this thesis as part of postdoctoral study. Another example already mentioned is the application of this thesis, where $\mathbb{R}^n$ is replaced with a subset Ω and L is the Dirichlet Laplacian Δ_Ω defined by Dirichlet boundary conditions on $\delta\Omega$. This second example confirms and extends results from [48] found by a different method.

In the case of [48], $\mathcal{L}^p$ boundedness of the Riesz transform cannot be achieved for $p \geq n$ showing the non-trivial nature of the examples. This assertion is reviewed in section 10.3 of this thesis. The other extreme of this thesis is a case where weighted $\mathcal{L}^p$ bounds hold for all $p \in (2, \infty)$ with weights exceeding the Muckenhoupt A_p class.

1.2 The Main Conditions

The process employed to prove Riesz transform bounds in this thesis is a variation of the good-λ method. This method is common in the literature (see for example [5] or [9]). To compensate for the lack of a preservation condition, two new conditions are required.

The first new condition is a bounded average on the derivatives of the *heat semigroup* e^{-tL}.

$$\sup_{B \subset \Omega} \left(\fint_B \varphi(x)^2 \, \mathrm{d}x \right) \left(\fint_B |\nabla e^{-r^2 L} 1_\Omega(x)|^2 \, \mathrm{d}x \right) \lesssim 1 \tag{1.3}$$

In this condition: $\varphi(x)$ is an arbitrary positive function, non-zero almost everywhere; $1_\Omega(x)$ is the characteristic function of the set Ω; and the supremum is over all balls

$B \subset \Omega$, which are the sets given by $B = B(x,r) = \{y \in \Omega : d(x,y) < r\}$ for some $x \in \Omega$ and $r > 0$. Hence r in condition (1.3) refers to the radius of the ball B. The distance $d(x,y)$ is the length of the shortest curve connecting x and y in Ω.

The second new condition is a Hardy type inequality,

$$\|\frac{1}{\varphi}L^{-1/2}f\|_{\mathcal{L}^p(\Omega)} \lesssim \|f\|_{\mathcal{L}^p(\Omega)} \tag{1.4}$$

where $\varphi(x)$ is the same function as in equation (1.3). In cases where the preservation condition holds and there are Gaussian lower bounds on the heat kernel, equation (1.4) would not be expected to hold for general $f \in \mathcal{L}^p(\Omega)$ for any $\varphi > 0$. However, the preservation condition holding case can sometimes be shown as a limit case ($\varphi \to 0$). Discussion on this idea is provided in section 10.3. In the application part of this thesis, where L is the Dirichlet Laplacian on $\Omega \subset \mathbb{R}^n$, φ is specifically chosen to be the minimal distance $\rho(x)$ from a point $x \in \Omega$ to the boundary $\delta\Omega$.

Additional to the two conditions above that replace the preservation condition, several other conditions are required to prove the generalised Riesz transform bound. These are all standard conditions usually required by proofs using a good-λ method. There are two conditions regarding the space Ω, the first is a *doubling* condition,

$$\mu(kB) \leq k^n \mu(B) \tag{1.5}$$

and the second is a *Poincaré inequality.*

$$\int_B |f(x) - f_B|^2 \, \mathrm{d}x \lesssim r^2 \int_B |\nabla f(x)|^2 \, \mathrm{d}x \tag{1.6}$$

It is well known that the Poincaré inequality holds for all balls $B \subset \Omega$ when Ω is $\mathbb{R}^n$, and further it was shown in [38] that the Poincaré inequality holds for all balls $B \subset \Omega$ when Ω satisfies an *inner uniform* condition. More comments on this will occur in the preliminaries chapter of this thesis. In some parts of this thesis, including Theorem 1.2 and its proof in chapter 4, these conditions will only need to hold locally.

In addition to these spatial conditions, there are several conditions on the differential operator L. These begin with the basic assumption that L is well defined on the space Ω, with appropriate spectral conditions to allow a holomorphic functional calculus, with the existence and uniqueness of $L^{-1/2}$.

$$L^{-1/2}f(x) = c\int_0^\infty e^{-tL}f(x)\frac{\mathrm{d}t}{\sqrt{t}}$$

The *heat semigroup* e^{-tL} is the integral operator with kernel $p_t^\Omega(x,y)$ solving the heat equation,

$$\left(L + \frac{\mathrm{d}}{\mathrm{d}t}\right)p_t^\Omega(x,y) = 0 \qquad \forall x,y \in \Omega$$

with initial condition given by the dirac-delta function, thought of in a distributional sense. When Ω has boundary the kernel $p_t^\Omega(x,y)$ is additionally defined by boundary conditions. The name of *semigroup* comes as e^{-tL} can be viewed as a semigroup with respect to the variable t. It is well known that appropriate spectral and semigroup properties hold for a large class of second order differential operators L. See chapter 2 for details and references.

The remaining conditions for the operator L are standard conditions for Riesz transform bound proofs. Three of these conditions apply to the heat semigroup and kernel.

The first remaining condition is that of Gaussian upper bounds for the heat kernel.

$$p_t^\Omega(x,y) \lesssim \frac{e^{-d(x,y)^2/ct}}{|B(x,\sqrt{t})|} \tag{1.7}$$

This is most often used in the form of an on-diagonal heat kernel bound, or an off-diagonal norm heat semigroup bound, both of which are implied by the above equation.

The next condition is an $\mathcal{L}^2$ Riesz transform bound.

$$\||\nabla L^{-1/2}f|\|_{\mathcal{L}^2(\Omega)} \lesssim \|f\|_{\mathcal{L}^2(\Omega)} \tag{1.8}$$

Such a condition is trivial for L as a Laplacian, but non-trivial in general case.

Another condition is the $\mathcal{L}^2$ off-diagonal Gaffney estimate,

$$\|\sqrt{t}|\nabla e^{-tL}f|\|_{\mathcal{L}^2(A)} \lesssim e^{-d(A,B)^2/ct}\|f\|_{\mathcal{L}^2(B)} \tag{1.9}$$

which is required to hold for all subsets A and B of Ω and all $t > 0$. The value $d(A,B)$ is the distance from A to B within Ω.

The final condition required is an $\mathcal{L}^q$ on-diagonal Gaffney estimate.

$$\|\sqrt{t}|\nabla e^{-tL} f|\|_{\mathcal{L}^q(\Omega)} \lesssim \|f\|_{\mathcal{L}^q(\Omega)} \tag{1.10}$$

This estimate is in terms of some $q > p$. This condition combined with the off-diagonal $\mathcal{L}^2$ Gaffney estimates implies an off-diagonal $\mathcal{L}^p$ Gaffney estimate for all $p \in (2, q)$. The on-diagonal Gaffney estimate is known to be a necessary condition for a Riesz transform bound (see for example [5]).

The six conditions (1.5 - 1.10) are well known to hold in a variety of applications and are common conditions for a Riesz transform problem. The difference here to previous proofs is, as already stated, due to the two new conditions (1.3) and (1.4) replacing the preservation condition. If L is self adjoint with Gaussian upper bounds, or has other similarities to the Laplacian operator on $\mathbb{R}^n$, some of the conditions become trivial.

It remains to discuss the application part of this paper. Consider the space Ω as an open subset of $\mathbb{R}^n$ with smooth boundary. The *Sobolev space* $W_0^{1,2}(\Omega)$ is the completion of the space of infinitely differentiable functions with compact support in Ω, under the following inner product.

$$\langle f\,,\,g\rangle_{W_0^{1,2}(\Omega)} = \int_\Omega \nabla f(x)\cdot\nabla g(x)\,\mathrm{d}x + \int_\Omega f(x)g(x)\,\mathrm{d}x$$

Next define the following quadratic form with domain $f \in W_0^{1,2}(\Omega)$.

$$Q(f,g) = \int_\Omega \nabla f(x)\cdot\nabla g(x)\,\mathrm{d}x$$

Associated with this quadratic form is a unique non-negative and self-adjoint operator that is symbolised as Δ_Ω and named the Dirichlet Laplacian. The square root of this operator is $\Delta_\Omega^{1/2}$ and has domain $W_0^{1,2}(\Omega)$ (see [46] or [56]). Due to the self-adjoint nature of the Dirichlet Laplacian, there is the equivalence,

$$\||\nabla f|\|_{\mathcal{L}^2(\Omega)} = \|\Delta_\Omega^{1/2} f\|_{\mathcal{L}^2(\Omega)},$$

which can be used to establish an $\mathcal{L}^2(\Omega)$ bound for the associated Riesz transform.

$$\||\nabla \Delta_\Omega^{-1/2} f|\|_{\mathcal{L}^2(\Omega)} = \|f\|_{\mathcal{L}^2(\Omega)}$$

Thus the Dirichlet Laplacian satisfies the $\mathcal{L}^2$ Riesz transform condition (1.8) from the main list of conditions. The Dirichlet Laplacian also satisfies all appropriate spectral conditions. Precise results regarding Riesz transforms based on the Dirichlet Laplacian are given for a variety of subsets Ω (see Theorem 1.6 and chapter 10). The main difficulty in extending the results found to a greater class of Ω is due to a lack of precise bounds on the heat kernels $p_t^\Omega(x, y)$. The choice of application was inspired by recent results from Killip, Visan and Zhang in [48].

1.3 Known Results

At the beginning of this thesis it was stated that the traditional Riesz transform $\nabla\Delta^{-1/2}$ is bounded $\mathcal{L}^p \to \mathcal{L}^p$ for all $1 < p < \infty$ and further satisfies a weighted bound $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ if and only if the weight w belongs to the Muckenhoupt class A_p (see Stein [67] chapter 5). It was additionally stated that there were partial answers to the question regarding whether such $\mathcal{L}^p \to \mathcal{L}^p$ boundedness extends to a larger class of operators or manifolds. This section is an overview of those answers. The first presented is due to Auscher, Coulhon, Duong and Hofmann, and is the main building block of this thesis.

THEOREM A (From [5]). *Let $\mathcal{M}$ be a manifold with measure μ that satisfies the doubling condition. Further, let ∇ be the Riemannian gradient and Δ be the Laplace-Beltrami operator. Suppose that the Poincaré inequality (1.6) holds, and an $\mathcal{L}^q$ Gaffney estimate (1.10) holds for some $q > 2$. Then the following inequality is true for all $2 \leq p < q$ and all $f \in \mathcal{L}^p(\mathcal{M})$.*

$$\| |\nabla\Delta^{-1/2} f| \|_{\mathcal{L}^p(\mathcal{M})} \lesssim \|f\|_{\mathcal{L}^p(\mathcal{M})}$$

The proof in [5] is general enough to apply to Riesz transforms of the form $\nabla L^{-1/2}$ that additionally satisfy Gaussian upper bounds on the heat kernel (1.7), an $\mathcal{L}^2$ Riesz transform bound (1.8), $\mathcal{L}^2$ Gaffney estimates (1.9) and the preservation condition.

The good-λ method used in this thesis is a variation on the good-λ method used to prove the above theorem in [5]. Another reference that uses such good-λ techniques is [4] which looks at Riesz transform bounds for elliptic operators. The next theorem presented is not specific to bounding Riesz transforms, but bounds general singular integrals based on kernel estimates. This is due to Duong and McIntosh.

THEOREM B (From [26]). *Let T be a linear operator bounded on $\mathcal{L}^2$ with associated kernel $k(x,y)$. Suppose that there exists operators A_t and B_t with kernels $a_t(x,y)$ and $b_t(x,y)$ respectively which decrease very quickly to zero, as t tends to infinity.*

$$|b_t(x,y)| + |a_t(x,y)| \leq c\, t^{-n/2} e^{-\alpha|x-y|^2/t}$$

Further suppose that for each $t > 0$, $T - TA_t$ has kernel $k(x,y) - k_t(x,y)$ which satisfies:

$$\int_{d(x,y) \geq ct^{1/m}} |k(x,y) - k_t(x,y)|\, \mathrm{d}\mu(x) \leq c$$

and that TB_t has kernel $K_t(x,y)$ that satisfies:

$$|K_t(x,y)| \leq \begin{cases} t^{-n/m} & \text{for } d(x,y) > t^{1/m} \\ \frac{t^{\alpha/m}}{d(x,y)^{n+\alpha}} & \text{for } d(x,y) < t^{1/m} \end{cases}$$

for some α and m. Then the supremum of the truncated operators,

$$T_* f(x) = \sup_{\epsilon > 0} \Big| \int_{d(x,y) \geq \epsilon} k(x,y) u(y)\, \mathrm{d}\mu(y) \Big|$$

is bounded $\mathcal{L}^p \to \mathcal{L}^p$ for all $1 < p < \infty$ and all $f \in \mathcal{L}^p$.

The filter $T(I - A_t)$ used in Theorem B, is similar to the filter used in this thesis when separating the Riesz transform into parts. Finding suitable conditions to bound the filter $T(I - A_t)$ is the key to both theorems above, and is also key in this thesis.

Theorem B is general in its approach, but does not hold in general Riesz transform applications. Theorem A only holds for Riesz transforms with a preservation condition and exponent $p > 2$. For $1 < p < 2$ the preservation and non-preservation cases are solved in the following from Coulhon and Duong.

THEOREM C (From [17]). *Let $\mathcal{M}$ be a manifold with measure μ that satisfies the doubling condition. Further, let ∇ be the Riemannian gradient and Δ be the Laplace-Beltrami operator. Suppose that for all $x \in \mathcal{M}$ and $t > 0$ the heat kernel satisfies an on-diagonal bound.*

$$p_t(x,x) \lesssim \frac{1}{|B(x,\sqrt{t})|}$$

Then the Riesz transform $\nabla\Delta^{-1/2}$ satisfies a weak type $(1,1)$ bound, and is norm bounded $\mathcal{L}^p \to \mathcal{L}^p$ for all $1 < p \leq 2$ and $f \in \mathcal{L}^p(\mathcal{M})$.

Due to Theorem C, $p > 2$ is the priority in finding new results. The papers [18] and [19] should also be mentioned as follow ons of the paper and theorem above with additional conditions for the case $p > 2$. However the authors of [18] and [19] have since contributed to the paper [5] already mentioned as a most general case.

The next theorem is a specific example of a case of Riesz transform related boundedness with $p > 2$ and no preservation condition. This is from Killip, Visan and Zhang.

THEOREM D (From [48])**.** *Let $\Omega \subset \mathbb{R}^n$ be the exterior of a compact convex object. Let Δ_Ω be the Dirichlet Laplacian on Ω. Then for all $1 < p < n$ there is a Sobolev space relation,*

$$\||\nabla f|\|_{\mathcal{L}^p(\Omega)} \lesssim \|\Delta_\Omega^{1/2} f\|_{\mathcal{L}^p(\Omega)}$$

that holds for all $f \in C_c^\infty(\Omega)$. The range of p is optimal.

The proof of Theorem D in [48] uses Littlewood-Paley functionals, as well as the boundedness of a Hardy type inequality similar to equation (1.4), but with $\varphi(x)$ named as the distance $\rho(x)$ from x to the boundary $\delta\Omega$. Similar techniques to those in [48] have recently been used in [49], [72] and others in pursuit of non-linear Schrödinger results. Theorem D does not imply that the Riesz transform is bounded $1 < p < n$, although it does imply the Riesz transform is unbounded for $p > n$. In the application part of this thesis it is shown that the Riesz transform is bounded $1 < p < n$ for all $f \in \mathcal{L}^p(\Omega)$ in the Theorem D case which does imply Theorem D. A similar result to Theorem D in a particular preservation case is considered in [16].

For the case of bounded Lipschitz domains with Dirichlet boundary conditions a more general theorem was proven by Jerison and Kenig.

THEOREM E (From [43], Theorem 7.5)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. If $n \geq 3$ then there exists $q > 3$ depending on the domain such that the following bound holds,*

$$\||\nabla f|\|_{\mathcal{L}^p(\Omega)} \lesssim \|\Delta_\Omega^{1/2} f\|_{\mathcal{L}^p(\Omega)}$$

for all $1 < p < q$ and $f \in W_0^{1,p}$. If $n = 2$ then q can be chosen greater then 4. If the *domain is C^1 then q can be chosen as infinity. These results are optimal in the sense that when $n \geq 3$, for every $q > 3$ there exists a domain Ω where the inequality fails.*

Shen proved a similar result in [60]. Shen's result was a theorem of the type above for the Riesz transform of symmetric and uniformly elliptic operators $\nabla L^{-1/2}$ satisfying certain conditions, and also included weighted results for $\mathcal{L}^2$. Theorem E and Shen's result show for Ω compact that a full range of Riesz transform boundedness may not occur if the boundary is not smooth, but does occur if the boundary is smooth. In this thesis the applications focus on non-compact Ω, where as seen from Theorem D a full range of boundedness may not occur even when the boundary is smooth.

1.4 Summary of New Results and Techniques

Whilst progress has been made on the problem of studying Riesz transforms on domains, a proof of the form of Theorem A has not been shown to hold for general domains without a preservation condition. Examples (see Theorem D or section 10.3 of this thesis) show that such a proof should be possible, though the final result will vary with Ω. These examples are backed up with the following main results.

THEOREM 1.1. *Let Ω be a doubling space satisfying the Poincaré inequality (1.6) for all balls $B \subset \Omega$. Let ∇ be the Riemannian gradient and L be a second order differential operator with well defined functional calculus and Gaussian upper bound on its heat kernel (1.7). Suppose L satisfies: an $\mathcal{L}^2$ Riesz transform bound (1.8); $\mathcal{L}^2$ Gaffney estimates (1.9); $\mathcal{L}^q$ Gaffney estimates (1.10) for some $q > 2$; a semigroup gradient bound of the type (1.3) with some non-negative $\varphi(x)$; and $\mathcal{L}^p$ Hardy type inequalities (1.4) for all $2 \leq p < q$ with the same $\varphi(x)$. Then the Riesz transform $\nabla L^{-1/2}$ is bounded $\mathcal{L}^p \to \mathcal{L}^p$ for all $2 \leq p < q$.*

Theorem 1.1 is proven in chapter 3 of this thesis. Discussion regarding the conditions continues throughout the application part of this thesis. Below is an outline of the technique used in this paper to prove Theorem 1.1, see chapter 3 for details.

The main idea is a split of the Riesz transform into parts according to Figure 1.1, where A_r is an operator given by $A_r = I - (I - e^{-r^2 L})^n$, and the notation $[f]_{B_i}$ indicates an averaging of the function f over the ball B_i. The rightmost branch in the diagram

Figure 1.1: Splitting of the Riesz transform in the proof of Theorem 1.1.

$$\begin{tikzcd}
 & & \nabla L^{-1/2} f(x) \arrow[dl] \arrow[dr] & \\
 & \nabla L^{-1/2} A_r f(x) \arrow[dl] \arrow[d] & & \nabla L^{-1/2}(I - A_r) f(x) \\
\nabla L^{-1/2} A_r f(x) - \nabla A_r [L^{-1/2} f]_{B_i}(x) & \nabla A_r [L^{-1/2} f]_{B_i}(x) & &
\end{tikzcd}$$

is bounded by standard estimates on the heat kernel and its derivative (these are conditions as outlined by equations (1.8) and (1.10)), and the leftmost branch is bounded by similar heat kernel estimates along with the Poincaré inequality (1.6). That all works comparably to a standard good-λ style proof. It is the middle branch of Figure 1.1 that is of chief concern. In a standard good-λ proof this part vanishes due to the preservation condition, but does not vanish in this case. Following the standard good-λ proof the middle branch of the diagram leaves us with the need to find a bound for the following term,

$$pK^p \int_0^\infty \lambda^{p-1} \sum_i |\{x \in B_i : |\nabla A_r [L^{-1/2} f]_{B_i}(x)| > \lambda\}| \, \mathrm{d}\lambda$$

which relies on two parts. These are $[L^{-1/2} f]_{B_i}$ and $|\nabla A_r 1_\Omega(x)|$. Once appropriately separated these parts match the new conditions given by equations (1.3) and (1.4) earlier in the introduction. See chapter 3 for details.

The next theorem is a variation on Theorem 1.1 designed to apply to non-doubling domains $\Omega \subset \mathbb{R}^n$. To compensate for no doubling condition, the value φ in equations (1.3) and (1.4) is specified to be $\rho(x)$, the minimal distance from x to the boundary $\delta\Omega$. Equations (4.3) and (4.7) mentioned in the theorem below are the same as equations (1.3) and (1.4) respectively with $\varphi(x)$ replaced by $\rho(x)$. Further there is a slight change to Gaussian upper bound, referenced by equation (4.8).

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^n$ be an open and connected subset that may or may not satisfy a doubling condition. Let ∇ be the Riemannian gradient. Further let L be a second order differential operator with well defined functional calculus and Gaussian upper bound on its heat kernel (4.8). Suppose on balls away from the boundary relative* *to their size ($c_0 r(B) < \rho(B)$ for some fixed c_0) L satisfies: a local $\mathcal{L}^2$ Riesz transform bound ($\||\nabla L^{-1/2} f|\|_{\mathcal{L}^2(B)} \lesssim \|f\|_{\mathcal{L}^2(\Omega)}$); a semigroup gradient bound of the type (4.3); and a local $\mathcal{L}^s$ Gaffney condition ($\sqrt{t}\||\nabla e^{-tL} f|\|_{\mathcal{L}^s(B)} \lesssim e^{-d(A,B)^2/ct}\|f\|_{L^s(A)}$) for both $s = 2$ and $s = q$ for some $q > 2$. Suppose also that a Hardy type inequality (4.7) holds for all $2 \le p < q$. Then the Riesz transform $\nabla L^{-1/2}$ is bounded $\mathcal{L}^p \to \mathcal{L}^p$ for all $2 \le p < q$.*

The proof of Theorem 1.2 uses that Ω satisfies a local doubling property and a local Poincaré inequality in the sense that such inequalities hold on balls B that do not intersect the boundary $\delta\Omega$. To keep balls away from the boundary a different covering then in Theorem 1.1 is used. For balls near the boundary the condition $\varphi(x) = \rho(x)$ means that $\nabla A_r L^{-1/2}$ is directly comparable to the Hardy operator by use of a Gaffney estimate. For balls away from the boundary an extra split is required as seen in Figure 1.2. The characteristic function 1_S ensures the Poincaré inequality is only used locally. The set S is empty for $x \in B_i$ near the boundary and is approximately $B(x, \rho(B_i)/2)$ for $x \in B_i$ away from the boundary. A ball B is said to be near the boundary if $\rho(B) \lesssim r(B)$ and away from the boundary otherwise. The proof of Theorem 1.2 is contained in chapter 4.

Figure 1.2: Splitting of the Riesz transform in the proof of Theorem 1.2.

$$\begin{tikzcd}
 & & \nabla\Delta_\Omega^{-1/2} f(x) \arrow[dl] \arrow[dr] & \\
 & \nabla\Delta_\Omega^{-1/2} A_r f(x) \arrow[dl] \arrow[ddr, "\text{This branch is only used if } x \text{ is contained in a ball away from the boundary}"] & & \nabla\Delta_\Omega^{-1/2}(I - A_r) f(x) \\
\nabla A_r 1_{S^c}\Delta_\Omega^{-1/2} f(x) & & & \\
 & & \nabla A_r 1_S \Delta_\Omega^{-1/2} f(x) \arrow[dl] \arrow[dr] & \\
 & \nabla A_r 1_S \Delta_\Omega^{-1/2} f(x) - \nabla A_r 1_S [\Delta_\Omega^{-1/2} f]_{B_i}(x) & & \nabla A_r 1_S [\Delta_\Omega^{-1/2} f]_{B_i}(x)
\end{tikzcd}$$

Theorem 1.2 is now extended to a weighted result. For the weighted theorems heat kernels will need to satisfy an upper bound of the form,

$$p_t(x, y) \lesssim \frac{\alpha_t(x)\beta_t(y) e^{-d(x,y)^2/ct}}{t^{n/2}} \tag{1.11}$$

for some α, β positive in Ω, that will affect the weights involved. It is common but not necessary for α and β to vanish on the boundary $\delta\Omega$. A new weight class is introduced known as the class of $A_p^{\alpha,\beta}$ weights. In the equation below and elsewhere, r is the radius of B.

$$\sup_{B\subset\Omega}\left(\fint_B \alpha_{r^2}(x)^p w(x)\,\mathrm{d}x\right)\left(\fint_B \beta_{r^2}(x)^{p'} w(x)^{-p'/p}\,\mathrm{d}x\right)^{p/p'} < \infty \tag{1.12}$$

A weight that satisfies equation (1.12) will be called an $A_p^{\alpha,\beta}$ weight.

THEOREM 1.3. *Fix $p > 2$. Let $\Omega \subset \mathbb{R}^n$ be a space that may or may not satisfy a doubling condition. Let ∇ be the Riemannian gradient and $w \in A_\infty$ be a weight. Further let L be a second order differential operator on Ω with well defined functional calculus, and heat kernel upper bounds of the form of equation (1.11) for some $\alpha_t(x) \lesssim 1$ and $\beta_t(y) \lesssim 1$. This α and β also must satisfy $\alpha_t(x) \leq 2\alpha_{2t}(x)$ and $\beta_t(x) \leq 2\beta_{2t}(x)$ for all $x \in \Omega$ and $t > 0$. The weight w must satisfy $w^{2/p} \in A_2^{\alpha,\beta}$ and $w^{q/p} \in A_R^{\alpha,\beta}$ for some $q > p$ where $R = 1 + \frac{q}{2}$. Further a weighted $\mathcal{L}^p(w)$ Hardy inequality must hold. Suppose for balls away from the boundary relative to their size (that is $c_0 r(B) < \rho(B)$ for some fixed c_0) that w satisfies: the Muckenhoupt A_2 condition; an $\mathcal{L}^2(w^{2/p})$ Riesz transform bound; an $\mathcal{L}^2(w^{2/p})$ Davies-Gaffney estimate; an $\mathcal{L}^q(w^{q/p})$ Gaffney estimate for the same $q > p$ as considered earlier; and a semigroup gradient bound of the form $\left(\fint_B \rho(x)^2 w^{-2/p}\,\mathrm{d}x\right)\left(\fint_B |\nabla e^{-kr^2L}1_\Omega|^2 w^{2/p}\,\mathrm{d}x\right) \lesssim 1$ for all $k \in [1,n]$. Then the Riesz transform $\nabla L^{-1/2}$ is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for all $2 \leq p < q$.*

Theorem 1.3 is proven in chapter 5 by the techniques of Theorem 1.2. A weighted Poincaré inequality is used in the proof, but only away from the boundary. This is what leads to the $w^{2/p} \in A_2$ away from the boundary condition. Near the boundary it is possible for w to be outside the A_p Muckenhoupt class. In the application part of this thesis, the class of weights that Theorem 1.3 finds for Riesz transform boundedness on Ω is not optimal (due to the $\mathcal{L}^2$ bound conditions), but Theorem 1.3 is required in the more complex applications on the way to achieving an optimal class of weights.

The next theorem is a result used during the application chapters to find weight classes for the various conditions listed in Theorem 1.3.

THEOREM 1.4. *For all $r > 0$ let $\alpha_{r^2}(x)$ and $\beta_{r^2}(x)$ be defined continuous and positive on the doubling space $\Omega \subset \mathbb{R}^n$. Suppose that $\alpha_{r^2}(x)$ and $\beta_{r^2}(x)$ satisfy the following four listed conditions. In each condition r is taken, as always, as the radius of the* *corresponding ball B.*

Firstly for all $p > 1$ there must exist $s \in (1, p)$ such that the following holds.

$$\sup_{B \subset \Omega} \left(\fint_B \alpha_{r^2}(x)^{s-p} \,\mathrm{d}x \right) \left(\fint_B \beta_{r^2}(x)^{(s-p)/(s-1)} \,\mathrm{d}x \right) \lesssim 1$$

The second condition is a estimate on local integrability.

$$\sup_{B \subset \Omega} \left(\fint_B \alpha_{r^2}(x) \,\mathrm{d}x \right) \left(\fint_B \beta_{r^2}(x) \,\mathrm{d}x \right) \lesssim 1$$

The third condition is as follows. For every $B \subset \Omega$ and $R > r > 0$ there must exist positive numbers $c_{B,R}$ and $C_{B,R}$ constant with respect to all $x \in B$ such that both estimates,

$$\alpha_{r^2}(x) \sim (c_{B,R})\, \alpha_{R^2}(x) \text{ and } \beta_{r^2}(x) \sim (C_{B,R})\, \beta_{R^2}(x)$$

hold for all $x \in B$. Further the values $c_{B,R}$ and $C_{B,R}$ must satisfy the inequalities $(r/R)^m \leq c_{B,R} \leq (R/r)^m$ and $C_{B,R} \leq (R/r)^m$ for some constant $m > 0$ and all $R > r$ and $B \subset \Omega$.

The fourth condition is that the set I of all balls $B \subset \Omega$ can be broken up into a finite collection of subsets I_i, where for each i there exists functions a_i and z_i such that the similarity $\alpha_{r^2}(x) \sim a_i(x) z_i(r)$ holds for all $x \in 5B$ whenever $B \in I_i$.

Now let T_t be an integral operator bounded on $\mathcal{L}^2$ with kernel of the form of equation (1.11) as both an upper and lower bound.

$$p_t(x, y) \sim \frac{\alpha_t(x)\beta_t(y)e^{-d(x,y)^2/ct}}{t^{n/2}} \tag{1.13}$$

Then $\sup_t T_t$ is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ if and only if $w \in A_p^{\alpha,\beta}$ (see equation 1.12).

Theorem 1.4 is proven in chapter 6. The idea behind the proof of Theorem 1.4 is a comparison to a maximal type function.

$$M^{\alpha,\beta} f(x) = \sup_{B \ni x} \alpha_{r^2}(x) \fint_B \beta_{r^2}(y) |f(y)| \,\mathrm{d}y$$

Weight classes for more general operators are also considered in chapter 6 by comparison to this maximal function. This maximal function is shown in chapter 6 to be bounded

$\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for all weights w in the class $A_p^{\alpha,\beta}$. The conditions on α and β are there to ensure that if w is in $A_p^{\alpha,\beta}$, then w is in $A_{p_1}^{\alpha,\beta}$ for some $p_1 > p$ and for some $p_1 < p$, both depending on w. This idea mirrors arguments from Stein in [67] for the case of standard Muckenhoupt weights and the standard Hardy-Littlewood Maximal function. Theorem 1.4 is used in chapter 8 to find bounds for the heat semigroup in various application cases. Cases where Ω is a non-doubling subset of $\mathbb{R}^n$ can still use Theorem 1.4 after an appropriate extension of the operators concerned to a doubling space. In all the application cases the class $A_p^{\alpha,\beta}$ is shown to contain weights outside the Muckenhoupt class A_p.

One of the applications of Theorem 1.4 is to prove the following result.

THEOREM 1.5. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be the exterior of a $C^{1,1}$ compact convex object. Let ∇ be the gradient operator and Δ_Ω be the Dirichlet Laplacian on Ω. Take $1 < p < \infty$. Then,*

- *The heat semigroup $\sup_t e^{-tL}$ is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for all weights in the following class,*

$$\sup_{B\subset\Omega} \left(\fint_B \Big(1 \wedge \frac{\rho(x)}{1\wedge r}\Big)^p w(x)\,\mathrm{d}x \right) \left(\fint_B \Big(1 \wedge \frac{\rho(x)}{1\wedge r}\Big)^{p'} w(x)^{-p'/p}\,\mathrm{d}x \right)^{p/p'} < \infty$$

  where r is the radius of B and $\rho(x)$ is the minimal distance from x to $\delta\Omega$.

- *The gradient of the heat semigroup $\sqrt{t}|\nabla e^{-t\Delta_\Omega}|$ is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for all weights in the following class,*

$$\sup_{B\subset\Omega} \left(\fint_B \Big(\frac{r}{\rho(x)\wedge r} \wedge \frac{\rho(x)}{r\wedge 1}\Big)^p w(x)\,\mathrm{d}x \right) \left(\fint_B \Big(1 \wedge \frac{\rho(x)}{r\wedge 1}\Big)^{p'} w(x)^{-p'/p}\,\mathrm{d}x \right)^{p/p'} < \infty$$

  where again r is the radius of B and $\rho(x)$ the minimal distance from x to $\delta\Omega$.

- *The Hardy operator $\frac{1}{\rho}L^{-1/2}f$ is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for all weights of the form $w(x) \sim \rho(x)^k$ with $\max(-1, p-n) < k < 2p-1$.*

Note that the first weight class above contains weights outside the Muckenhoupt class A_p, and in fact the class A_p is a subset of this first class. The Muckenhoupt A_p class is not a subset of the second weight class above but neither is that class a subset of A_p. For $p > n$ the second class above does not even contain the weight $w(x) = 1$, yet for all $p \geq 1$ the second class does contain $w(x) = \rho(x)^p$ which is outside the A_p class.

There are similar results for the case of Ω as both the area below a parabola in $\mathbb{R}^2$, and as the area above a $C^{1,1}$ Lipschitz curve in $\mathbb{R}^n$. The weight classes are even larger for those cases, for the area above a global $C^{1,1}$ Lipschitz curve in $\mathbb{R}^n$ the weight class $A_p^{\alpha,\beta}$ contains A_p for each $p > 1$. Good bounds on the heat kernel $p_t(x, y)$ from [74] allow the solution of this application theorem. Proving Theorem 1.5 is the main theme of chapters 8 and 9. It remains to consider the Riesz transform in the same application case.

THEOREM 1.6. *Suppose that Ω is the exterior of a $C^{1,1}$ compact convex object in $\mathbb{R}^n$, $n \geq 3$. Then the Riesz transform $\nabla\Delta_\Omega^{-1/2}$ is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$, $p \geq 2$, whenever $w(x) \sim \rho(x)^k$ with $\max(-1, p - n) < k < 2p - 1$.*

One of the reasons a more general class of weights (as seen in parts of Theorem 1.5) does not occur here is due to a condition on the gradient of the weight w which states that $|\nabla w| \lesssim w/\rho$.

Again similar results are proven for the case of Ω as the area above a $C^{1,1}$ Lipschitz curve in $\mathbb{R}^n$. Theorem 1.6 combines results from throughout this thesis. It will be formally proven in chapter 10. Any weight $\rho(x)^k$ with $k > p - 1$ is outside the Muckenhoupt A_p class, this is because such weights approach zero faster at the boundary than usually allowed. In the paper [48], Proposition 7.2 proves that for the exterior of a convex obstacle the Riesz transform is not bounded (case $w(x) = 1$) for $p > n$. This is the same as what is seen in the weight classes. The argument from [48] is visited in section 10.3 of this thesis.

Chapter 2:

Preliminaries

In this chapter the background theory required for the various results is covered. Basic knowledge of the techniques used in Analysis, found in any standard analysis textbook (for example Rudin [57]) are assumed known. Here more directed and technical ideas are contained.

The first section regards the idea of well defined operators. Given a second order differential operator L, conditions and theorems regarding the existence and uniqueness of the square root $L^{1/2}$ and inverse square root $L^{-1/2}$ are stated. This allows the definition of the associated Riesz transform $\nabla L^{-1/2}$.

The second section of this chapter considers when the heat kernel related to the differential operator L is known to satisfy reasonable upper bounds, namely Gaussian upper bounds.

The third section includes background regarding *inner uniform domains* and other domain types which satisfy the spatial conditions required for the application chapters. There is particular emphasis on Whitney coverings, the Poincaré inequality, and the Harnack principle for such domains.

The final part of this chapter outlines known results on Muckenhoupt weights in preparation for the later chapters on weighted results.

2.1 Some Functional Calculus

In this section emphasis is directed at the functional calculus of a differential operator. The basis is from books by Reed and Simon [56], Dunford and Swartz [25] and Kato [46], as well as lecture notes from Albrecht, Duong and McIntosh [2]. Other references include [27] and [28]. Consider the Banach space $\mathcal{L}^p(\Omega)$ and the second order differential operator L with domain $D(L) \subseteq \mathcal{L}^p(\Omega)$.

Definitions 2.1. The operator L is *bounded on* $E \subseteq D(L)$ if,

$$\|L\| = \sup\{\|Lu\|_p : u \in E, \|u\|_p = 1\} \lesssim 1$$

and is *closed* on E if the Graph of L given by $G(L) = \{(u, Lu) : u \in E\}$ is a closed subspace of $\mathcal{L}^p(\Omega) \times \mathcal{L}^p(\Omega)$. The *resolvent set* $\rho(L)$ is the set of all $\zeta \in \mathbb{C}$ such that the transformation $\zeta I - L$ is invertible. The *spectrum set* $\sigma(L)$ is the complement of $\rho(L)$, so $\sigma(L)$ contains the $\zeta \in \mathbb{C}$ that are solutions of the eigenvalue problem $Lu = \zeta u$.

As L is a second order differential operator, L is closed but not bounded on an associated Sobolev type subspace of $\mathcal{L}^p(\Omega)$. Let the space of all holomorphic functions on an open subset $S \subset \mathbb{C}$ be denoted by $H(S)$.

Proposition 2.2 (see chapter 7 of [25] or section 3 of [2] for details)**.** *Take L as a closed operator in $\mathcal{L}^p(\Omega)$ with non-empty resolvant set and with $\sigma(L) \subset S$ for some $S \subset \mathbb{C}$. Then for any $f \in H(S)$ the operator $f(L)$ exists and satisfies the following properties:*

- *If $f, g \in H(S)$ and $\alpha \in \mathbb{C}$ then $f(L) + \alpha g(L) = (f + \alpha g)(L)$*
- *If $f, g \in H(S)$ then $f(L)g(L) = (fg)(L)$*
- *If $f \in H(S)$ then $f(\sigma(L)) = \sigma(f(L))$*

There is in fact more to this theorem, see the references for details. Precise representations are constructed via the resolvant operator $(\zeta I - L)^{-1}$. In this thesis it is the operators given by e^{-tL} and $L^{-1/2}$ that are of interest. More results can be found for operators with greater control on the spectrum.

Definitions 2.3. Define the set S_{w+} for $0 \le w < \pi$ as,

$$S_{w+} = \{\zeta \in \mathbb{C} : |\arg(\zeta)| \le w\} \cup \{0\} \tag{2.1}$$

Note that S^0_{w+} is the interior of this set. A closed operator is said to be of *type* S_{w+} if there exists w where $\sigma(L) \subset S_{w+}$ and for every $\mu \in (w, \pi)$ there exists c_μ where for each

$\zeta \in S_{\mu+}^c$ the following bound holds.

$$\|(L - I\zeta)^{-1}\| \leq c_\mu |\zeta|^{-1} \tag{2.2}$$

Proposition 2.4 (an example from [2])**.** *Suppose that L is a self-adjoint operator on a Hilbert space, and that $\langle Lu\,,\,u\rangle \geq 0$ for every u in a Banach space on which L is defined. Then L is of type S_{0+}.*

Example 2.5 (motivated by [48])**.** Consider the Dirichlet Laplacian defined on the exterior of the ball of radius 1 and centre 0 in $\mathbb{R}^n$, $n \geq 3$. The spectrum $\sigma(\Delta_\Omega)$ contains solutions to $\Delta_\Omega u = \lambda u$ with u satisfying Dirichlet boundary conditions. In radial coordinates,

$$u(r,\lambda) = -r^{n/2-1}\Big[J_{n/2-1}(\sqrt{\lambda}r)Y_{n/2-1}(\sqrt{\lambda}) - J_{n/2-1}(\sqrt{\lambda})Y_{n/2-1}(\sqrt{\lambda}r)\Big]$$

is the solution, where J and Y are Bessel functions, and λ is any real positive number. The Dirichlet Laplacian Δ_Ω is self-adjoint on the Hilbert space $H_0^1(\Omega)$, and satisfies $\langle \Delta_\Omega u\,,\,u\rangle \geq 0$. Hence by Proposition 2.4, Δ_Ω is of type S_{0+}, meaning all the eigenvalues of Δ_Ω must be real and positive. In fact every real positive number is an eigenvalue in this case. Further, as Δ_Ω is of type S_{0+}, Definition 2.3 implies that for every $\zeta \in S_{0+}^c$ the following bound holds.

$$\|(\Delta_\Omega - I\zeta)^{-1}\| \lesssim |\zeta|^{-1}$$

The operators of type S_{w+} are important in this thesis. Of great interest is the relationship between such operators and semigroups. Good references for semigroups include Pazy [54] for the first sections below and Stein [66] for the later sections.

Definition 2.6. For $t > 0$, let T_t be a linear operator bounded $\mathcal{L}^p(\Omega) \to \mathcal{L}^p(\Omega)$. Then the collection $\{T_t\}_{t>0}$ is a *semigroup of bounded linear operators* if T_t satisfies for all $t_1, t_2 > 0$ both a semigroup property and an identity preservation.

$$T_{t_1+t_2} = T_{t_1} \cdot T_{t_2} \qquad \text{and,} \quad T_0 = I \tag{2.3}$$

Definitions 2.7. A family T_z of bounded linear operators is called a *holomorphic semi-*

group if T_z satisfies a semigroup property for $z \in S^0_{w+}$ (naturally extending the semigroup property of the previous definition) and T_z is holomorphic on S^0_{w+}. The *infinitesimal generator* of a semigroup T_t, is an operator $-L$ that maps from its domain $D(-L) \subset \mathcal{L}^p(\Omega)$ to $\mathcal{L}^p(\Omega)$, and is defined by,

$$D(-L) = \{f \in \mathcal{L}^p(\Omega) : -Lf = \lim_{t\to 0^+} \frac{T_t f - f}{t} \text{ exists}\}$$

Definitions 2.8. A semigroup is said to be *strongly continuous* in the Banach space $\mathcal{L}^p(\Omega)$ if for all $f \in \mathcal{L}^p(\Omega)$,

$$\lim_{t\to 0^+} T_t f = f \tag{2.4}$$

and a semigroup is said to satisfy an $\mathcal{L}^p(\Omega)$ *contraction* if,

$$\|T_t(f)\|_{\mathcal{L}^p(\Omega)} \lesssim \|f\|_{\mathcal{L}^p(\Omega)}. \tag{2.5}$$

for all $f \in \mathcal{L}^p(\Omega)$ and $t \in (0, \infty)$.

Proposition 2.9 (Hille-Yosida)**.** *A linear operator $-L$ is the infinitesimal generator of a semigroup $\{T_t\}_{t\geq 0}$ which satisfies the strongly continuous property (2.4) and the contraction (2.5) property if and only if $-L$ is a closed operator, the closure of the domain $D(-L)$ of $-L$ is $\mathcal{L}^p(\Omega)$ and the resolvent set $\rho(L)$ contains $\mathbb{R}^+$ and for all $\lambda \in \mathbb{R}^+$,*

$$\|(-L - I\lambda)^{-1}\| \leq \frac{1}{\lambda}$$

holds. Note that $\rho(-L)$ containing $\mathbb{R}^+$ is equivalent to $\rho(L)$ containing $\mathbb{R}^-$.

The differential operators L considered in this thesis are of type S_{w+} for some $0 < w < \frac{\pi}{2}$ and have domain dense in $\mathcal{L}^p(\Omega)$, so that $-L$ is the infinitesimal generator of a semigroup with the appropriate properties.

The final step of this section improves the contraction property (equation 2.5). Define the following maximal type function.

$$M_{T_t} f(x) = \sup_{R>0} \frac{1}{R} \left| \int_0^R T_t(f)(x) \, \mathrm{d}t \right|$$

Then the contraction property can be improved to hold with such a maximal function, proven by the following generalisation given by Dunford and Schwartz in [25] to a theorem on ergodic means by Hopf in [42].

Lemma 2.10 (Hopf-Dunford-Schwartz ergodic lemma). *Let $\{T_t\}_{t\geq 0}$ be a strongly measurable semigroup (meaning $T_t f$ is measurable for each $f \in \mathcal{L}^p(\Omega)$) that satisfies an $\mathcal{L}^p(\Omega)$ contraction (2.5), let f be a measurable function on Ω. Then for all $f \in L^1(\Omega)$, the Maximal function satisfies a weak $(1,1)$ inequality.*

$$|\{x \in \Omega : M_{T_t}(f)(x) > \lambda\}| \leq \frac{2}{\lambda}\|f\|_{L^1(\Omega)}$$

Further if $f \in \mathcal{L}^p(\Omega)$ for some $1 < p < \infty$ then the Maximal function M_{T_t} satisfies a strong $\mathcal{L}^p$ bound.

$$\|M_{T_t} f\|_{\mathcal{L}^p(\Omega)} \leq 2\left(\frac{p}{p-1}\right)^{1/p}\|f\|_{\mathcal{L}^p(\Omega)}$$

This lemma is used to prove the next proposition of Stein. See [64] for the original paper, or [66] chapter 3 for a more comprehensive approach. First though, it is necessary to define the *semigroup maximal function* as,

$$f^*(x) = \sup_{t>0} |T_t(f)(x)| \tag{2.6}$$

as well as to have the following definition.

Definition 2.11. A semigroup $\{T_t\}_{t\geq 0}$ is *symmetric* if T_t is self adjoint on the Hilbert space $\mathcal{L}^2(\Omega)$. The kernel $p_t(x,y)$ of a symmetric semigroup satisfies $p_t(x,y) = p_t(y,x)$.

Not all operators that are considered in this thesis will satisfy a symmetry property. However, the Dirichlet Laplacians considered in the application chapters are symmetric.

Proposition 2.12 (Stein semigroup maximal theorem). *Suppose the semigroup $\{T_t\}_{t\geq 0}$ satisfies symmetry (Definition 2.11), $\mathcal{L}^p(\Omega)$ contraction (2.5) for $p = 2$ and $p = p_0$, and is a strongly continuous semigroup (2.4) with respect to the Banach space $\mathcal{L}^2(\Omega)$. Then the semigroup maximal function (2.6) satisfies a strong $\mathcal{L}^p(\Omega)$ bound for all p between 2 and p_0.*

$$\|f^*\|_{\mathcal{L}^p(\Omega)} \lesssim \|f\|_{\mathcal{L}^p(\Omega)}$$

Further, the semigroup is strongly continuous (see equation 2.4) with respect to the Banach spaces $\mathcal{L}^p(\Omega)$ for each p between 2 and p_0.

Finish this section with another theorem from Stein [64].

Proposition 2.13. *Let $(\mathcal{M}, \mu)$ be a σ-finite measure space, and let the semigroup given by $\{T_t\}_{t\geq 0}$ satisfy for every $t \geq 0$ the symmetric property (Definition 2.11) and an $\mathcal{L}^p(\Omega)$ contraction (equation 2.5) for some $1 < p < \infty$. Then $\{T_t\}_{t\geq 0}$ can be extended to a holomorphic operator T_z defined in the sector $\{z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{2}(1 - |\frac{2}{p} - 1|)\}$*

The result still holds if the $\mathcal{L}^p$ bounds of the semigroups are replaced with weighted bounds. The holomorphic property of a symmetric heat semigroup $T_t = e^{-tL}$ can be used in the construction of time and space derivative bounds (see Davies [22]).

In chapters 3, 4 and 5 it is supposed that the differential operator L has well defined functional calculus so that $L^{-1/2}$ and e^{-tL} are well defined. The symmetry property discussed is not involved until the application part of this thesis. In the application part L is specified as Δ_Ω which is symmetric and ideas stemming from Proposition 2.13 are used.

2.2 Heat Kernel Bounds

Emphasis on heat semigroups will have been noted. The bounds on the kernels of these heat semigroups is the focus of this section. The heat semigroup e^{-tL} is an operator that acts on functions f on some domain Ω, and satisfies a heat equation,

$$\begin{cases} \left(L + \frac{\mathrm{d}}{\mathrm{d}t}\right) e^{-tL} f(x) = 0 \\ \lim\limits_{t\to 0} e^{-tL} f(x) = f(x) \end{cases}$$

for all $x \in \Omega$ and $t > 0$. The initial condition mirrors the identity property (seen in equation 2.3) of the semigroup. The heat semigroup operator can be written as an integral operator in terms of a kernel $p_t(x, y)$. Formally this is,

$$e^{-tL} f(x) = \int_\Omega p_t(x, y) f(y)\, \mathrm{d}y$$

where $p_t(x,y)$ is the positive solution to the partial differential heat equation given by,

$$\begin{cases} \left(L + \frac{\mathrm{d}}{\mathrm{d}t}\right) p_t(x,y) = 0 \\ \lim\limits_{t\to 0} p_t(x,y) = \delta(x-y) \end{cases}$$

where $\delta(x-y)$ is the Dirac delta function understood in a distributional sense. In cases without boundary the solution is required to vanish tending towards extremities. In cases with boundary, boundary conditions are required for $p_t(x,y)$ to be defined uniquely. The kernel $p_t(x,y)$ for the semigroup is named the *heat kernel.* It is a standard assumption in the theory to have Gaussian upper bounds for the heat kernel.

$$|p_t(x,y)| \lesssim \frac{e^{-d(x,y)^2/ct}}{t^{n/2}}$$

The decay from such kernels allows off-diagonal semigroup estimates such as in the following proposition.

Proposition 2.14. *Suppose that the heat kernel $p_t(x,y)$ has Gaussian upper bounds. Then the following $\mathcal{L}^2$ off-diagonal bound holds,*

$$\|e^{-tL} f\|_{\mathcal{L}^2(A)} \lesssim e^{-d(A,B)^2/2ct} \|f\|_{\mathcal{L}^2(B)}$$

for any two subsets $A, B \subset \Omega$, all $t > 0$, and all functions f compactly supported on B. The term $d(A,B)$ is the distance from A to B in Ω.

This proposition is well known and the proof is by Hölder's inequality, followed by inserting the Gaussian upper bounds and evaluating the integrals. Similar $\mathcal{L}^p$ bounds are possible. The most basic heat kernel, the kernel of the usual Laplace heat equation on $\mathbb{R}^n$, is the Gaussian precisely. It was proven by Nash in [53] that the kernels of quite general elliptic operators satisfy Gaussian upper bounds. Similarly Schrödinger operators have Gaussian upper bounds by the Feynman-Kac formula (see Simon [61]).

The main theorems of this thesis also require reasonable $\mathcal{L}^p$ bounds on the gradient of the heat semigroup, and for the Hardy inequality $\frac{1}{\varphi}L^{-1/2}$ to be $\mathcal{L}^p$ bounded. To achieve these conditions in the application part of this thesis the heat kernels used are sub-Gaussian. Methods by Davies [22], Li and Yau [50], and Nash [53] are useful to gain bounds on the derivatives of these heat kernels. Particularly Davies techniques, which use the analytic property of the heat kernel to gain time derivatives, and results of Li and Yau, which give pointwise gradient bounds on subsets, will be used in this thesis. A detailed exposition on Davies idea is in the book [21].

2.2.1 Maximum Principles

Concentrating now on the application part of this thesis, consider the Dirichlet Laplacian on a subset $\Omega \subset \mathbb{R}^n$, and the following maximum principle.

Proposition 2.15 (Maximum Principle 1)**.** *Suppose that $u_t(x)$ is a C^2 solution of the heat equation on some domain $\Omega \times (0, \infty)$. Then the maximum of $u_t(x)$ occurs on the boundary of the domain. Similarly the minimum of $u_t(x)$ also occurs on the boundary of the domain.*

This proposition is well known so will not be proven here. This maximum principle is applied to prove the following proposition, also referred to as the maximum principle.

Proposition 2.16 (Maximum Principle 2)**.** *Suppose that A and B are open subsets of $\mathbb{R}^n$ which satisfy $A \subset B$. Let $p_t^A(x,y)$ and $p_t^B(x,y)$ be the heat kernels for the Dirichlet problem in A and B respectively. Then,*

$$p_t^A(x,y) \leq p_t^B(x,y)$$

for all $x, y \in A$.

Proof. Fix $y \in A$. Given $p_t^A(x,y)$ and $p_t^B(x,y)$ as in the proposition, observe that $u_t = p_t^B - p_t^A$ solves the heat equation for $x \in A$ with initial condition $\lim_{t \to 0} u_t = 0$ and boundary condition $u_t(x,y) = p_t^B(x,y) \geq 0$ for all $x \in \delta A$. Thus by the first maximum principle (Proposition 2.15), for every fixed $y \in A$, the function $u_t(x,y)$ has minimum with respect to $x \in A$ on the boundary δA. This minimum is non-negative. Thus,

$$p_t^B(x,y) - p_t^A(x,y) \geq 0$$

for every $x, y \in A$, which concludes the proof. $\square$

Proposition 2.16 allows the comparison of heat kernels of two sets whenever one set is a subset of the other. This is considered in section 7.3 of this thesis.

Thus far in this preliminaries chapter the existence of the operators $L^{-1/2}$ and e^{-tL} has been studied for a large class of L, and the existence of reasonable bounds on the heat kernel associated to these operators has been discussed. Here this discussion is ended, and there is a switch to consider in what spaces Ω it is reasonable to work with these operators on.

2.3 Regarding Appropriate Subsets

In this section definitions are given regarding the set Ω. This mostly refers to the subsets $\Omega \subset \mathbb{R}^n$ dealt with in the application part of this thesis. Such domains are open and connected subsets of $\mathbb{R}^n$. The boundary of Ω is smooth and named $\delta\Omega$. There is no requirement on compactness. A ball in Ω means the set:

$$B(x, r) = \{y \in \Omega : d(x, y) < r\} \tag{2.7}$$

where $d(x, y)$ is the length of the shortest curve connecting x and y that lies entirely in Ω. The centre x of the ball must also lie in Ω. Two properties required of Ω were named in the introduction of this thesis.

- Firstly Ω is a doubling space, see equation (1.5).
- Secondly Ω satisfies the Poincaré inequality, see equation (1.6).

In some parts of this thesis these conditions are only required to hold locally. The next few definitions classify domains that are important in the application chapters of this thesis. In Definitions 2.18 and 2.19 below the balls referred to are subsets of $\mathbb{R}^n$, unlike the majority of this thesis where the balls referred to will be subsets of the relevant Ω, as in equation (2.7).

Definition 2.17. A domain $\Omega \subset \mathbb{R}^n$ is said to be a *local Lipschitz domain* if there exists constants $r_0 > 0$ and $m > 0$ such that for every $\bar{x} \in \delta\Omega$ there exists a local coordinate system $(x_1, \ldots, x_n)$ where the boundary $\delta\Omega$ is a Lipschitz function with regards to the direction $\hat{x}_1$, whilst within the closure of the ball $B(\bar{x}, r)$ in $\mathbb{R}^n$. That is,

$$\||\nabla\psi(x_2, \ldots, x_n)|\|_{L^\infty(B(\bar{x},r_0))} \leq m$$

where for $x = (x_1, \ldots, x_n) \in B(\bar{x}, r_0)$ the function $x_1 = \psi(x_2, \ldots, x_n)$ describes the boundary of Ω in terms of a local coordinate system $(x_1, \ldots, x_n)$. The coordinates can be any rotation of the standard axes, and are chosen to minimise $\||\nabla\psi|\|_{L^\infty(B(\bar{x},r_0))}$. Such a domain is a *global Lipschitz domain* if r_0 can be made as large as wished whilst m remains fixed.

Definition 2.18. A domain $\Omega \subset \mathbb{R}^n$ is said to be $C^{1,1}$ if there exists constants $r > 0$ and $m > 0$ such that for every $\bar{x} \in \delta\Omega$ there exists a local coordinate system $(x_1, \ldots, x_n)$ such that the boundary $\delta\Omega$ can be written as $x_1 = \psi(x_2, \ldots, x_n)$ where ψ satisfies:

$$|\nabla\psi(x_2, \ldots, x_n) - \nabla\psi(y_2, \ldots, y_n)| \leq m\|(x_2 - y_2, \ldots, x_n - y_n)\|$$

for all $x, y \in \delta\Omega \cap B(\bar{x}, r)$.

Definition 2.19. An *inner uniform* domain Ω is any open and connected subset of $\mathbb{R}^n$ where there exists positive constants c and C, such that for any two points $x, y \in \Omega$, there exists a curve γ from x to y where,

$$\text{length}(\gamma) < c\, d(x, y) \text{ and } d(z, \delta\Omega) \leq C\frac{d(x, z)d(y, z)}{d(x, y)}$$

for all $z \in \gamma$. This means that a domain is inner uniform if a line can be drawn between any two points in the domain, and that line stays away from the boundary relative to its length.

Examples 2.20. The exterior of a ball in $\mathbb{R}^n$ is a doubling, locally Lipschitz, $C^{1,1}$ and inner uniform domain. However the domain between two parabolas given by the set $S = \{(x_1, x_2) \in \mathbb{R}^2 : -(1 + x_2^2) < x_1 < 1 + x_2^2\}$ is doubling, locally Lipschitz and $C^{1,1}$ but is not inner uniform. The domain between the exponential curve and the x_1 axis, given by $S = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < e^{x_1}\}$ is non-doubling, and not inner uniform, but has Lipschitz and $C^{1,1}$ boundary.

Lemma 2.21. *All inner uniform domains are doubling.*

Proof. Let Ω be an inner uniform domain, let x_0 be a point in Ω, and let $B(x_0, r)$ in Ω be a ball (as in the type defined equation (2.7)) centred at x_0. Under the assumption that $B(x_0, r)$ does not cover Ω there is a point $x_1 \in \delta B(x_0, r)$ that lies in Ω. Under the inner uniform domain condition, a curve can be drawn from x_0 to x_1 of length cr for some constant c, where every z on the curve satisfies,

$$d(z, \mathrm{d}\Omega) \leq C \frac{d(x_0, z) d(x_1, z)}{r}$$

for some $C > 0$. Suppose that z is a fractional amount $s \in (0, 1)$ along the curve so that $d(x_0, z) = csr$ and $d(x_1, z) = c(1 - s)r$. Then $d(z, \mathrm{d}\Omega) \lesssim s(1 - s)r$ holds. This equation establishes that around the path there is a volume comparable to r^n, and this volume lies inside both Ω and $B(x_0, r)$. A doubling condition results. $\square$

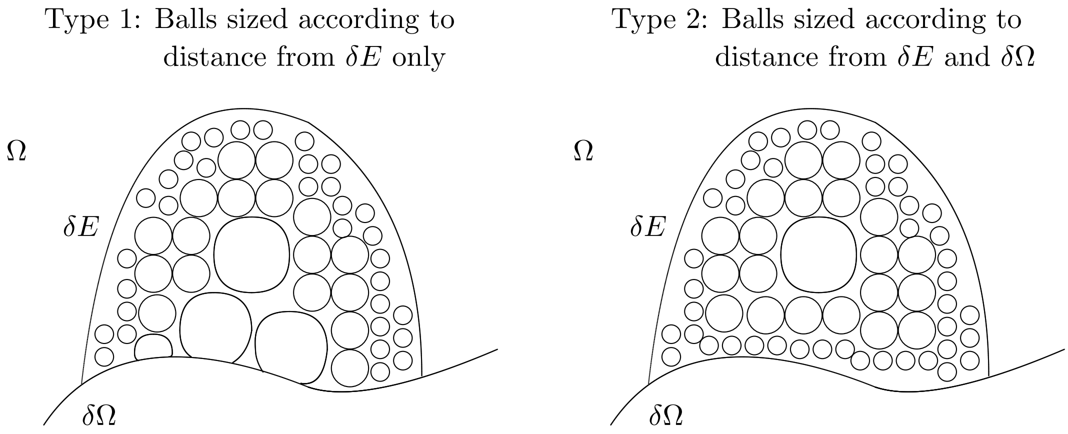
Inner uniform domains are studied in depth in [70]. The reason inner uniform domains are interesting in this context is that the Poincaré inequality is known to hold on such domains. The complete proof of this can be found in [38], a summary of the proof is included in subsection 2.3.2 below for the reader's convenience. First Whitney decompositions are discussed.

2.3.1 Whitney decomposition

There are two generalisations of Whitney decomposition used in this thesis. These are both based on the traditional Whitney decomposition on $\mathbb{R}^n$ details of which can be found in [67].

The first type of Whitney decomposition considered is when a subset $E \subset \Omega$ is covered by balls $B_i \subset \Omega$ that are sized according to their distance from the boundary of E. The balls in this covering may touch the boundary of Ω if the set E does. The benefit of this covering is that for each B_i in the covering, $4B_i$ contains points outside E. The problem with this covering is that the Poincaré inequality only holds for general balls B_i if Ω is an inner uniform domain. There are even more problems if Ω is non-doubling. This first type of Whitney decomposition is used in chapter 3 of this thesis.

Figure 2.1: Comparison of two generalisations of Whitney decomposition near the boundary of Ω.



The second type of Whitney decomposition considered is when a subset $E \subset \Omega$ is covered by balls $B_i \subset \Omega$ that are sized according to their distance from the boundaries of both E and Ω. Now the Poincaré inequality holds for general balls in the decomposition. The problem this time is that only for balls away from $\delta\Omega$ does $4B_i$ contain points outside E. This decomposition is used in chapters 4 and 5 of this thesis. If E does not touch the boundary of Ω then the two types of Whitney decomposition are identical.

The first type of Whitney decomposition is proven in the following lemma. The second type of Whitney decomposition is equivalent to the standard Whitney decomposition of [67] (with the boundary in two parts), so will not be proven here.

Lemma 2.22 (Whitney decomposition type 1). *Suppose $\Omega \subset \mathbb{R}^n$ is a doubling space with boundary $\delta\Omega$. Further suppose that $E \subset \Omega$ is a set with a closed non-empty complement. Then there exists a collection of pairwise disjoint balls $\{B_i\}_{i \in I}$, each with centre in Ω, and constants c_1 and c_2 such that:*

$$1).\ E = \cup_i c_1 B_i \qquad \textit{and;} \quad 2).\ c_2 B_i \cap E^c \neq \emptyset \textit{ for every } B_i$$

where E^c is the complement within Ω. That is, $E^c = \{x \in \Omega : x \notin E\}$.

Proof. This proof is based on the proof of a traditional Whitney decomposition given by Stein in [67] section I.3.2. The extra parts here are due to $\Omega \subset \mathbb{R}^n$ having its own boundaries interfere. Recall that a ball $B \subset \Omega$ is the set $B(x, r) = \{y \in \Omega : d(x, y) < r\}$ where $d(x, y)$ is the length of the shortest line that connects x and y and lies in Ω. Given a ball $B(x,r)$ define $B^* = B(x, c_1 r)$ and $B^{**} = B(x, c_2 r)$ where $1 < c_1 < c_2$. Consider an open set $E \subset \Omega$ and let $d(x)$ be the distance from $x \in E$ to δE. Choose $\epsilon \in (0,1)$ and observe the collection $\Psi = \{B(x, \epsilon d(x))\}_{x \in E}$ covers E. Select a maximal disjoint sub-collection of Ψ: $\{B\}_k = B_1, B_2, \ldots, B_k, \ldots$ and define $c_1 = 1/2\epsilon$, $c_2 = 2/\epsilon$. By construction the elements in $\{B\}_k$ are pairwise disjoint and $B_k^{**} \cap E^c \neq \emptyset$.

It remains to show $\cup B_k^* = E$. Choose $x \in E$, then as $\{B_k\}$ is maximal there exists $B(x_k, \epsilon d(x_k)) \in \{B_k\}$ such that $B(x_k, \epsilon d(x_k)) \cap B(x, \epsilon d(x)) \neq \emptyset$. Let $c_3 \geq 1$ be a constant such that $B(x_k, \delta) \subset B(x, c_3 \delta)$ whenever $B(x_k, \delta) \cap B(x, \delta) \neq \emptyset$. If $d(x_k) < d(x)/4c_3$ and $\epsilon < 1/2c_3$ then $B(x_k, 2d(x_k)) \cap B(x, d(x)/2c_3) \neq \emptyset$. So $d(x_k) < d(x)/4c_3$ implies $B(x_k, 2d(x_k)) \subset B(x, d(x)/2)$, which is a contradiction as $B(x_k, 2d(x_k))$ intersects E^c.

Hence $\epsilon < 1/2c_3$ implies $d(x_k) \geq d(x)/4c_3$ so $x \in B(x_k, 4c_3^2 \epsilon d(x_k))$ by replacing $d(x)$ with $4c_3 d(x_k)$. Take $B_k^* = B(x_k, 4c_3^2 \epsilon d(x_k))$ with ϵ small enough, so that $x \in E$ implies $x \in B_k^*$ for some k, and $\cup_k B_k^* = E$ to complete the proof. □

A vital consequence of this lemma is the following corollary.

Corollary 2.23 (Finite intersection lemma)**.** *Let W be a disjoint Whitney covering of $E \subset \Omega$ with associated constant c_1 as in Lemma 2.22. Then for each ball B_1 in the decomposition there exists only a finite number of other balls B_i such that $c_1 B_1 \cap c_1 B_i \neq \emptyset$. This number is uniformly bounded above.*

Proof. Choose $y \in E$. By the previous lemma there exists $B_1(x_1, r_1) \in W$ with $y \in c_1 B_1$. By the properties of the Whitney ball construction, and use of the triangle inequality the radius r_1 satisfies, $r_1 = \epsilon d(x_1, \mathrm{d}E) \leq \epsilon(c_1 r_1 + d(y, \mathrm{d}E))$. Take $\epsilon < 1/c_1$, and rearrange the inequality for r_1 to get, $r_1 \leq \frac{\epsilon}{1 - c_1 \epsilon} d(y, \mathrm{d}E))$. Suppose next there exists a second ball $B_2(x_2, r_2) \in W$ where $y \in c_1 B_2$. The triangle inequality on the $d(y, \mathrm{d}E)$ term implies $r_1 \leq \frac{\epsilon}{1 - c_1 \epsilon}(\frac{r_2}{\epsilon} + c_1 r_2) \leq 3r_2$ taking ϵ small. Similarly $r_2 \leq 3r_1$, implying $B_2 \subset 6c_1 B_1 \subset 6B_1/\epsilon$. There exists only a finite number of disjoint balls in $6B_1/\epsilon$ of radius minimum $r_1/3$. This number is uniformly bounded above. □

This completes the properties of the Whitney decomposition needed.

2.3.2 Poincaré Inequality

The following proposition is proven for inner uniform domains as subsets of very general spaces by Gyrya and Saloff–Coste in [38].

Proposition 2.24. *Suppose that Ω is an inner uniform domain. Then for all balls $B \subset \Omega$ the following inequality holds,*

$$\int_B |f(x) - f_B|^2 \, \mathrm{d}x \leq cr^2 \int_B |\nabla f(x)|^2 \, \mathrm{d}x \tag{2.8}$$

where r is the radius of B.

An outline of their proof is provided for the reader's benefit. The Poincaré inequality is well known to be true for all balls in $\mathbb{R}^n$. The issue here regards balls $B \subset \Omega$ that touch the boundary of Ω. The proof covers a ball touching the boundary of Ω by balls that do not touch the boundary of Ω (using the second type of Whitney decomposition mentioned in subsection 2.3.1). The proof is then based around that covering.

Lemma 2.25. *Suppose B_1 and B_2 are balls in a traditional Whitney decomposition. Further suppose B_1 and B_2 are neighbours, $3B_1 \cap 3B_2 \neq \emptyset$. Then the following bound holds.*

$$|f_{4B_1} - f_{4B_2}| \leq cr(B_1) \left(\frac{1}{|B_1|} \int_{16B_1} |\nabla f(x)|^2 \, \mathrm{d}x \right)^{1/2} \tag{2.9}$$

Lemma 2.25 is proven in [38] by straight calculation, using that the Poincaré inequality is true for B_1 and B_2 as balls in $\mathbb{R}^n$, and that $B_1 \subset cB_2$. See [38] for details. Lemma 2.25 allows adjacent balls in a traditional Whitney covering to be compared.

Outline of Poincaré inequality proof in [38]. Consider a ball $B \subset \Omega$ touching the boundary of Ω. Cover Ω with a Whitney decomposition (of the traditional type, that is, not touching $\delta\Omega$) named $\mathcal{R}$, and let $\mathcal{R}(B)$ be the collection of all balls A_i in the Whitney decompostion that satisfy $3A_i \cap B \neq \emptyset$. These A_i are small compared to B, as A_i are in a traditional Whitney decomposition and so are small near the boundary of Ω, whereas B is large near the boundary as B touches the boundary.

A 'central ball' A_0 in $\mathcal{R}(B)$ exists that has radius comparable to B. This central ball is used to replace B.

$$\begin{aligned}\int_B |f - f_B|^2 \,\mathrm{d}\mu &\lesssim \int_B |f - f_{4A_0}|^2 \,\mathrm{d}\mu \\ &\lesssim \sum_{A_i \in \mathcal{R}(B)} \int_{4A_i} |f - f_{4A_i}|^2 \,\mathrm{d}\mu + \int_{4A_i} |f_{4A_i} - f_{4A_0}|^2 \,\mathrm{d}\mu \\ &\lesssim \int_{4B} |\nabla f|^2 \,\mathrm{d}\mu + \sum_{A_i \in \mathcal{R}(B)} \int_{4A_i} |f_{4A_i} - f_{4A_0}|^2 \,\mathrm{d}\mu\end{aligned}$$

It remains to bound the right side above. If A_i and A_0 are neighbouring Whitney balls this is done by Lemma 2.25. Otherwise a string of comparable balls can be constructed so as to compare each ball to its neighbour down the string to eventually compare those at the ends. This bounds the second part above leading to the conclusion. □

Next variations on the Harnack inequality are discussed. The Harnack inequality is not used explicitly in this thesis, however it is used in several papers to prove heat kernel upper bounds that are required.

2.3.3 Harnack Inequality

Harnack inequalities compare variations in solutions to differential equations. The particular type of Harnack inequality used to prove the bounds needed in the application of this thesis is the heat equation boundary Harnack principle from [29]. The argument used for the development of this boundary Harnack principle mirrors similar arguments given for simpler operators in [14]. Boundary Harnack principles appeared originally in [47]. There is also a nice summary of boundary Harnack principles in the book by Cafferelli and Salsa [15]. More recently boundary Harnack principles have also been established for general inner uniform domains. Various proofs for different cases are given in [1], [3], [35] and [51] and most fully extended for the heat kernel case recently in [38].

Definition 2.26. A *uniform parabolic Harnack inequality* is satisfied by a domain Ω if for every cylinder $C = B(x, 2r) \times (t, t + 4r^2) \subset \Omega \times (0, \infty)$ and every non-negative solution u of the heat equation in C,

$$\sup_{Q-} u \lesssim \inf_{Q+} u$$

where,

$$Q- \stackrel{def}{=} B(x,r) \times (t+r^2, t+2r^2) \text{ and } Q+ \stackrel{def}{=} B(x,r) \times (t+3r^2, t+4r^2).$$

This is essentially the same as saying that if the heat u has been decaying for r^2 amount of time, then the supremum of u in $B(x,r)$ is comparable to the infimum of u in $B(x,r)$ after a further r^2 amount of decay time. The boundary version is simply the same idea holding near a specified boundary with boundary condition. The Harnack inequality has been traditionally important due to equivalences such as the following.

Proposition 2.27 (see papers by Gigor'yan [32] and Saloff-Coste [58])**.** *The parabolic Harnack inequality holding in Ω is equivalent to a two sided Gaussian heat kernel estimate for the Neumann heat kernel in Ω and is also equivalent to the Poincaré inequality with volume doubling estimates.*

Further papers by Gigor'yan on the same idea in other contexts are [33] and [34]. *Boundary Harnack principles* allow similar equivalences to various upper and lower bounds for Dirichlet heat kernels. Applications include Zhang [74] where upper and lower bounds for the Dirichlet heat kernel on the exterior of a compact convex object are found, using a local comparison form of a boundary Harnack principle. The boundary Harnack principle Zhang uses is from Fabes, Garofalo and Salsa [29]. This boundary Harnack principle is based on a Carleson estimate from Salsa [59]. Zhang's heat kernel bounds are used extensively in the application part of this thesis. The boundary Harnack principle from [29] will also be revisited in chapter 7 of the application part of this thesis, where more heat kernels are derived.

2.4 Muckenhoupt Weights

The preliminaries chapter is concluded by the following background on Muckenhoupt weights required for the weighted inequalities chapter. Let μ be a measure on $\mathbb{R}^n$, and take $f \in \mathcal{L}^p(\mu)$ for some $p \in [1, \infty]$. Define the *Hardy–Littlewood maximal function* to act on such f by,

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \,\mathrm{d}y. \tag{2.10}$$

Such a maximal function has many uses in harmonic analysis. It is well known that Mf is bounded $\mathcal{L}^p \to \mathcal{L}^p$ for all $p > 1$ and is bounded weak type when $p = 1$. The idea of a *Muckenhoupt weight* is to classify those μ for which Mf satisfies a weighted $\mathcal{L}^p(\mu)$ bound.

$$\|Mf\|_{\mathcal{L}^p(\mu)} \leq c\|f\|_{\mathcal{L}^p(\mu)} \tag{2.11}$$

The original solution to this question was provided by Muckenhoupt [52]. The reader is also referred to [8] and [67] for overviews.

Proposition 2.28. *For $1 < p < \infty$ the Hardy–Littlewood maximal function satisfies a weighted $\mathcal{L}^p$ bound of the form of equation (2.11) whenever:*

- $\mathrm{d}\mu$ *is absolutely continuous; and,*
- $\mathrm{d}\mu = w(x)\,\mathrm{d}x$ *where $w(x)$ satisfies the following Muckenhoupt weight condition for all balls $B \subset \mathbb{R}^n$,*

$$A_p(w) = \left(\fint_B w(x)\,\mathrm{d}x\right)\left(\fint_B w(x)^{-p'/p}\,\mathrm{d}x\right)^{p/p'} < \infty \tag{2.12}$$

where p' is the dual of p (meaning $\frac{1}{p} + \frac{1}{p'} = 1$).

There are also sets $A_1(w)$ and $A_\infty(w)$ for L^1 and L^∞ boundedness of the Hardy-Littlewood maximal function respectively, see references for details. The classes of weights w that satisfy the Muckenhoupt weight condition (2.12) for some $1 \leq p \leq \infty$ are denoted by A_p. A related class is the reverse Hölder class denoted by RH_p. A weight w is in the reverse Hölder class RH_p for some $1 < p < \infty$, if,

$$\left(\fint_B w(x)^p\,\mathrm{d}x\right)^{1/p} \lesssim \fint_B w(x)\,\mathrm{d}x$$

holds for all balls $B \subset \Omega$. If w is in an A_p class, then w is also in a RH_q class for some q depending on w. The converse also occurs (see [67] chapter 5 section 5.1). These two classes have many properties. For the properties in the proposition listed below and others, good outlines can be found in Stein [67], in the book by Garca-Cuerva and de Francia [31] and in Auscher and Martell [8].

Proposition 2.29. *The class A_p of weights w that satisfy the Muckenhoupt weight condition (equation 2.12) also satisfy the following*

1. *If $w(x)$ is in A_p, then the dilations $w(ax)$ and translations $w(x-a)$ are in A_p.*

2. *$A_1 \subset A_p \subset A_q \subset A_\infty$ for all $1 \leq p \leq q \leq \infty$.*

3. *If $w \in A_p$, then $w(x)\,\mathrm{d}x$ is a doubling measure.*

4. *For any $\alpha \in (0,1)$ and $w \in A_\infty$ there exists $\beta \in (0,1)$ such that for all balls $B \subset \mathcal{M}$, and all subsets $F \subset B$: $|F| \leq \alpha|B| \implies w(F) \leq \beta w(B)$.*

5. *If $w \in A_p$ for $1 < p < \infty$ then there exists $1 < q < p$ such that $w \in A_q$.*

6. *If $1 \leq q \leq \infty$ and $1 \leq s < \infty$ then $w \in A_q \cap RH_s$ if and only if $w^s \in A_{s(q-1)+1}$.*

The last property is from a paper by Johnson and Neugebauer [44]. The fact that any $w \in A_p$ is also in a reverse Hölder class RH_q is used to prove condition 5 in Proposition 2.29, which is what leads to the maximal function being bounded for weights in the Muckenhoupt class. The result for maximal functions can be extended to apply to more complex operators.

Proposition 2.30. *The standard Laplacian Riesz transform on $\mathbb{R}^n$ is bounded $\mathcal{L}^p(\mu) \to \mathcal{L}^p(\mu)$ if and only if $\mathrm{d}\mu = w(x)\,\mathrm{d}x$ and $w \in A_p$. An identical result can be shown for the standard Laplacian heat semigroup maximal function.*

See Stein's book [67] for a good overview on this. In fact this theorem applies to any operator that is approximately the shape of those outlined in the above proposition. Similar results for more general operators are sought in chapter 6 of this thesis. This concludes the preliminaries section. The following chapters prove the main results.

Chapter 3:

A Riesz Transform Bound Part 1:

A General Result in the Absence of a Preservation Condition

In this chapter Theorem 1.1 is proven. Theorem 1.1 gives conditions leading to $\mathcal{L}^p \to \mathcal{L}^p$ boundedness of the Riesz transform in the absence of a preservation condition. The proof is by use of a good-λ inequality in the style of [5]. Theorem 1.1 is reiterated below as a combination of Theorem 3.1 and Corollary 3.2. A non-doubling variation on this result is considered in chapter 4, and a weighted variation is considered in chapter 5.

THEOREM 3.1. *Let Ω be an open and connected doubling space where the Poincaré inequality (1.6) is satisfied for all balls $B \subset \Omega$. Let ∇ be the Riemannian gradient. Suppose that L is a second order differential operator, with well defined functional calculus on Ω, and that L also satisfies: Gaussian upper bounds on the heat kernel (1.7); an $\mathcal{L}^2$ Riesz transform bound (1.8); $\mathcal{L}^2$ off-diagonal Gaffney estimates (1.9); and $\mathcal{L}^q$ Gaffney estimates (1.10) for some $q > 2$. Further suppose that there exists a strictly positive function $\varphi(x)$ to satisfy equation (1.3). Then the Riesz transform satisfies a bound of the form,*

$$\| |\nabla L^{-1/2} f| \|_p \lesssim \|f\|_p + \|\frac{1}{\varphi} L^{-1/2} f\|_p$$

for all $2 \leq p < q$ and all $f \in C_0^\infty(\Omega)$.

If Ω has a boundary $\delta\Omega$, then it is reasonable for $\varphi(x)$ to be zero on that boundary. There are two corollaries. The first extends Theorem 3.1 to the full result of Theorem 1.1.

Corollary 3.2. *Suppose that all the conditions of Theorem 3.1 above hold. Further suppose that a Hardy type inequality (1.4) holds for some $p \in (2, q)$ and all $f \in C_0^\infty(\Omega)$.*

Then the Riesz transform is bounded $\mathcal{L}^p \to \mathcal{L}^p$ for all $f \in \mathcal{L}^p(\Omega)$.

The second corollary relates to a Sobolev space equivalence.

Corollary 3.3. *Suppose again that all the conditions of Theorem 3.1 above hold as well as a Hardy type inequality (1.4) for some $p \in [2, q)$. Then there is a Sobolev bound equivalence of the form,*

$$\| |\nabla f| \|_p \lesssim \|L^{1/2} f\|_p$$

for all f in $W^{1,p} \cap Dom(L^{1/2})$.

The corollaries are trivial extensions of the theorem so will not be proven, although a comment as to the change from $f \in C_0^\infty$ to $f \in \mathcal{L}^p$ in Corollary 3.2 is given at the start of the next section. Theorem 3.1 and the corollaries apply to any Riesz transform that satisfies the necessary conditions. This includes the Riesz transforms of the form of the Dirichlet Laplacian $\nabla \Delta_\Omega^{-1/2}$ that are the focus of the application part of this thesis.

3.1 Proof of the Result

In this section Theorem 3.1 is proven after a series of lemmas. In particular Lemma 3.4 is new in the literature and replaces a term that vanished when a preservation condition held in [5]. The rest of the proof is motivated by the work of Auscher, Coulhon, Duong and Hofmann in [5] sections 2 and 3.

First it needs to be mentioned, that the proof uses the integral representation,

$$\nabla L^{-1/2} = c \int_0^\infty \nabla e^{-tL} \frac{\mathrm{d}t}{\sqrt{t}}$$

known via functional calculus. There is an implicit assumption during the proof (for example during the decomposition) that such an operator acts on $\mathcal{L}^p(\Omega)$. To deal with this observe that the integral representation above can be given as a limit value,

$$T_\epsilon = c \int_\epsilon^{1/\epsilon} \nabla e^{-tL} \frac{\mathrm{d}t}{\sqrt{t}}$$

as ϵ tends to 0. This operator T_ϵ is bounded $\mathcal{L}^2 \to \mathcal{L}^2$ and via a Gaffney estimate is seen to act on $\mathcal{L}^p$, with an $\mathcal{L}^p$ bound depending on ϵ. The proof of Theorem 3.1 along with

Corollary 3.2 then show that the $\mathcal{L}^p$ bound of T_ϵ holds and does not depend on ϵ for all $f \in C_0^\infty$. Through a limiting argument $\nabla L^{-1/2}$ can be shown to satisfy an $\mathcal{L}^p$ bound for all $f \in C_0^\infty$. Then by a density argument $\nabla L^{-1/2}$ satisfies an $\mathcal{L}^p$ bound for all $f \in \mathcal{L}^p$. Thus the operator $\nabla L^{-1/2}$ has its definition extended from an operator bounded on $\mathcal{L}^2$ to an operator bounded on $\mathcal{L}^p$. This argument is common and implied in the proof, and both ideas regarding the limits and density will be implicit from this point, and no longer mentioned.

Lemma 3.4. *Suppose the conditions of Theorem 3.1 hold (In particular the new semi-group gradient condition (1.3) is needed here with associated function $\varphi(x)$). Then the following bound is true,*

$$\int_B |\nabla e^{-kr^2 L} f_B(x)|^2 \, \mathrm{d}x \lesssim \int_B \left[\frac{f(x)}{\varphi(x)}\right]^2 \mathrm{d}x$$

for all $B \subset \Omega$, where $k \in [1, n]$ is an integer, r is the radius of B and $f_B = \fint_B f(x)\, \mathrm{d}x$.

This lemma will be used in the main proof of this chapter with $L^{-1/2} f$ taking the place of f above. The term on the right is then an averaging of the Hardy operator.

Proof. Fix B. The value of f_B is constant with respect to x. This means it can be separated from the semigroup operator and the gradient.

$$|\nabla e^{-kr^2 L} f_B(x)| = |f_B| |\nabla e^{-kr^2 L} 1_\Omega(x)|$$

Square the equation above and integrate over B.

$$\int_B |\nabla e^{-kr^2 L} f_B|^2 \, \mathrm{d}x = f_B^2 \int_B |\nabla e^{-kr^2 L} 1_\Omega(x)|^2 \, \mathrm{d}x$$

Separate out from f_B terms φ and $\frac{1}{\varphi}$ and use Hölder's inequality.

$$\int_B |\nabla e^{-kr^2 L} f_B|^2 \, \mathrm{d}x \lesssim \Big(\fint_B \varphi(x)^2 \, \mathrm{d}x\Big) \Big(\fint_B \left[\frac{f(x)}{\varphi(x)}\right]^2 \mathrm{d}x\Big) \int_B |\nabla e^{-kr^2 L} 1_\Omega(x)|^2 \, \mathrm{d}x$$

Then use the new condition (1.3). To match condition (1.3) precisely the ball B in certain parts above will need to be increased to $\sqrt{k}B$. This will involve the doubling condition. The result is,

$$\int_B |\nabla e^{-kr^2 L} f_B|^2 \,\mathrm{d}x \lesssim \int_B \Big[\frac{f(x)}{\varphi(x)}\Big]^2 \,\mathrm{d}x$$

which concludes the proof. $\square$

Lemma 3.5. *Suppose that the conditions of Theorem 3.1 hold. Then there exists $q > p$ such that the following bound holds,*

$$\frac{1}{|B|}\int_B \Big|\nabla e^{-kr^2 L}\big(f(x) - f_B\big)\Big|^q \,\mathrm{d}x \lesssim \lambda^q$$

for all f smooth enough, and all balls $B \subset \Omega$ where for some fixed $c > 1$, cB contains a point $x \in \Omega$ where $M(|\nabla f|^2)(x) < \lambda^2$. Here M is the uncentred Hardy-Littlewood maximal function, $k \in [1, n]$ is an integer, and r is the radius of B.

This lemma is used in the proof of the main theorem with $L^{-1/2} f$ taking the place of f as written in the lemma. This allows f in the lemma to be smooth enough. The condition on B holds in the main theorem proof as B comes from an appropriate Whitney covering.

Proof. Let $C_0 = 2B$ and $C_s = 2^{s+1}B \setminus 2^s B$ for $s \geq 1$. Split $e^{-kr^2 L} = e^{-kr^2 L/2} e^{-kr^2 L/2}$. Interpolation between the Gaffney estimates (1.9) and (1.10) gives an off-diagonal L^q Gaffney estimate for some $q > p$ (smaller than the q in condition 1.10). Use this on the first $e^{-kr^2 L/2}$.

$$\Big(\int_B \Big|\nabla e^{-kr^2 L}\big(f(x) - f_B\big)\Big|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} \lesssim \sum_{s=0}^{\infty} \frac{e^{-c4^s}}{r}\Big(\int_{C_s} \Big|e^{-kr^2 L/2}\big(f(x) - f_B\big)\Big|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} \quad (3.1)$$

Two bounds are applied to the above equation. The first bound is applied to $q - 2$ of the terms and is given through Hölder's inequality and Gaussian heat kernel bounds,

$$|e^{-kr^2 L}[1_{C_m} g](x)| \lesssim \|p_{kr^2}(x, \cdot)\|_{\mathcal{L}^2(C_m)} \|g\|_{\mathcal{L}^2(C_m)} \lesssim \frac{\|g\|_{\mathcal{L}^2(C_m)}}{|B|^{1/2}}$$

and holds for any function $g \in \mathcal{L}^2(C_m)$. On the two remaining terms use the $\mathcal{L}^2$ bound of e^{-tL},

$$\|e^{-kr^2 L/2}[1_{C_m} g]\|_{\mathcal{L}^2(C_s)} \lesssim e^{-c4^{|s-m|}} \|g\|_{\mathcal{L}^2(C_m)}$$

which again holds for any function g, and some positive constant c due to the Gaussian heat kernel upper bound (1.7). Apply both these bounds to equation (3.1).

$$\begin{aligned}\Big(\int_B \Big|\nabla e^{-kr^2L}\big(f(x)-f_B\big)\Big|^q \,\mathrm{d}x\Big)^{1/q} \\ \lesssim \sum_{s=0}^{\infty}\sum_{m=0}^{\infty} \frac{e^{-c4^s}e^{-c4^{|s-m|}}}{r|B|^{(q-2)/2q}} \Big[\int_{C_m}\big(f(x)-f_B\big)^2\,\mathrm{d}x\Big]^{1/2} \end{aligned} \tag{3.2}$$

Next Poincaré's inequality (1.6) is used in the following fashion. First split up the difference term.

$$\begin{aligned}|f(x)-f_B| &= |f(x)-f_{2^{m+1}B}+\sum_{l=0}^{m} f_{2^{l+1}B}-f_{2^lB}| \\ &\lesssim |f(x)-f_{2^{m+1}B}| + \sum_{l=0}^{m}\frac{1}{|2^lB|}\int_{2^lB}|f(y)-f_{2^{l+1}B}|\,\mathrm{d}y\end{aligned}$$

Take $\mathcal{L}^2$ norms on the space C_m on both sides of the inequality above. Then apply Minkowski's inequality.

$$\begin{aligned}\Big[\int_{C_m}|f(x)-f_B|^2\,\mathrm{d}x\Big]^{1/2} &\lesssim \Big[\int_{2^{m+1}B}|f(x)-f_{2^{m+1}B}|^2\,\mathrm{d}x\Big]^{1/2} \\ &+ 2^n\sum_{l=0}^{m}\frac{|2^{m+1}B|^{1/2}}{|2^{l+1}B|^{1/2}}\Big[\int_{2^{l+1}B}|f(x)-f_{2^{l+1}B}|^2\,\mathrm{d}x\Big]^{1/2}\end{aligned}$$

Then apply Poincaré's inequality (1.6) to all parts.

$$\Big[\int_{C_m}|f(x)-f_B|^2\,\mathrm{d}x\Big]^{\frac12} \lesssim 2^n\sum_{l=0}^{m+1}\frac{2^lr|2^{m+1}B|^{\frac12}}{|2^{l+1}B|^{\frac12}}\Big[\int_{2^{l+1}B}|\nabla f|^2\,\mathrm{d}x\Big]^{\frac12}$$

Insert this back into equation (3.2).

$$\begin{aligned}\Big(\int_B \Big|\nabla e^{-kr^2L}\big(f(x)-f_B\big)\Big|^q \,\mathrm{d}x\Big)^{1/q} \\ \lesssim \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{e^{-c4^s}e^{-c4^{|s-m|}}}{r|B|^{(q-2)/2q}}\sum_{l=0}^{m+1}\frac{2^lr|2^{m+1}B|^{\frac12}}{|2^{l+1}B|^{\frac12}}\Big[\int_{2^{l+1}B}|\nabla f|^2\,\mathrm{d}x\Big]^{\frac12}\end{aligned}$$

Recall that cB contains points in the set where $M(|\nabla f|^2)$ is less than λ^2. Further, for the balls of size $2B$ and larger up to cB, their size can be increased to cB for the same effect (with appropriate changes of the weightings to $|cB|^{-1}$ compensated for by doubling condition (1.5)). This means there exists x_0 not in the set where $M(|\nabla f|^2) > \lambda^2$ such that the following bound holds.

$$\Big(\int_B \Big|\nabla e^{-kr^2L}\big(f(x) - f_B\big)\Big|^q \,\mathrm{d}x\Big)^{1/q} \lesssim \sum_{s=0}^{\infty}\sum_{m=0}^{\infty} \frac{e^{-c4^s}e^{-c4^{|s-m|}}}{|B|^{(q-2)/2q}} \sum_{l=0}^{m+1} 2^l |2^{m+1}B|^{\frac{1}{2}} \big(M|\nabla f|^2(x_0)\big)^{\frac{1}{2}}$$

As discussed above, here $M(|\nabla f|^2)(x_0) < \lambda^2$. Use also that $l \le m+1$ to simplify.

$$\Big(\int_B \Big|\nabla e^{-kr^2L}\big(f(x) - f_B\big)\Big|^q \,\mathrm{d}x\Big)^{1/q} \lesssim \lambda |B|^{1/q} \sum_{s=0}^{\infty}\sum_{m=0}^{\infty} e^{-c4^s}e^{-c4^{|s-m|}} 2^{(m+1)(1+\frac{n}{2})}$$

These summations converge to a constant (depending only on n) and leaving $\lambda|B|^{1/q}$. This concludes the proof. □

The next lemma is an argument from the paper [5]. The proof has been reworded and presented below for the convenience of the reader. Note that the proof of this lemma uses only local estimates, in the sense that the operators involved are only considered over a local ball B.

Lemma 3.6 (Lemma 3.1 in [5]). *Suppose that $\nabla L^{-1/2}$ is an appropriately defined operator on the space Ω satisfying the various conditions of Theorem 3.1 (in particular the $\mathcal{L}^2$ Riesz transform bound (1.8) and $\mathcal{L}^2$ Gaffney estimates (1.9) are needed here). Then the following bound holds for all $B \subset \Omega$ and $x \in B$,*

$$\frac{1}{|B|}\int_B |\nabla L^{-1/2}(I - e^{-r^2L})^n f(y)|^2 \,\mathrm{d}y \lesssim M(|f|^2)(x)$$

where M is again the uncentred Hardy-Littlewood maximal function, and r the radius of B.

Proof. Similar to the previous proof, let $C_0 = 2B$ and $C_j = 2^{j+1}B \setminus 2^jB$ for $j \ge 1$.

$$\Big(\fint_B |\nabla L^{-1/2}(I - e^{-r^2L})^n f(y)|^2 \,\mathrm{d}y\Big)^{\frac{1}{2}} \lesssim \sum_{j=0}^{\infty} \Big(\fint_B |\nabla L^{-1/2}(I - e^{-r^2L})^n [1_{C_j} f](y)|^2 \,\mathrm{d}y\Big)^{\frac{1}{2}}$$

When $j = 0$ use that $\nabla L^{-1/2}$ is $\mathcal{L}^2$ bounded (1.8), as is $(I - e^{-r^2L})^n$ due to the Gaussian upper bound on the heat kernel (1.7). Also use the doubling condition (1.5). Then the following holds for any $x \in B$.

$$\Big(\fint_B |\nabla L^{-1/2}(I-e^{-r^2L})^n[1_{C_0}f](y)|^2\,\mathrm{d}y\Big)^{1/2} \lesssim \Big(\fint_{2B} |f(y)|^2\,\mathrm{d}y\Big)^{1/2} \lesssim \big[M(|f|^2)(x)\big]^{1/2}$$

Thus the lemma is true in the case $j=0$. For $j \geq 1$ expand the term $(1-e^{-r^2L})^n$.

$$|\nabla L^{-1/2}(I-e^{-r^2L})^n[1_{C_j}f](y)| = |\nabla L^{-1/2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}e^{-kr^2L}[1_{C_j}f](y)|$$

Expand to the integral representation of $L^{-1/2}$ and use a change of variables.

$$\begin{aligned}|\nabla L^{-1/2}(I-e^{-r^2L})^n[1_{C_j}f](y)| &\lesssim \int_0^\infty \sum_{k=0}^{n}(-1)^k\binom{n}{k}|\nabla e^{-(t+kr^2)L}[1_{C_j}f](y)|\frac{\mathrm{d}t}{\sqrt{t}} \\ &\lesssim \int_0^\infty \Big[\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k 1_{\{s>kr^2\}}(s)}{\sqrt{s-kr^2}}\Big]|\nabla e^{-sL}[1_{C_j}f](y)|\,\mathrm{d}s\end{aligned}$$

Square and integrate over B, then use Minkowski's integral inequality and $\mathcal{L}^2$ Gaffney estimates (1.9).

$$\begin{aligned}&\int_B |\nabla L^{-1/2}(I-e^{-r^2L})^n[1_{C_j}f](y)|^2\,\mathrm{d}y \\ &\qquad\lesssim \Big(\int_0^\infty \Big|\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k 1_{\{s>kr^2\}}(s)}{\sqrt{s-kr^2}}\Big|\frac{e^{-c4^jr^2/s}}{\sqrt{s}}\,\mathrm{d}s\Big)^2\int_{C_j}|f(y)|^2\,\mathrm{d}y \qquad (3.3)\end{aligned}$$

The integral with respect to s is evaluated in several parts. Firstly, for the integral from 0 to r^2, only one part of the sum over k is involved.

$$\int_0^{r^2}\frac{e^{-c4^jr^2/s}}{s}\,\mathrm{d}s \lesssim c\int_0^{r^2}\frac{s^{n-1}}{4^{jn}r^{2n}}\,\mathrm{d}s \lesssim \frac{c}{4^{jn}}$$

Next the integral from ar^2 to $(a+1)r^2$ with $1 \leq a \leq n$ is evaluated.

$$\int_{ar^2}^{(a+1)r^2}\frac{e^{-c4^jr^2/s}}{\sqrt{s}}\left|\sum_{k=0}^{a}\binom{n}{k}\frac{(-1)^k}{\sqrt{s-kr^2}}\right|\mathrm{d}s \lesssim c\int_{ar^2}^{(a+1)r^2}\frac{c_n}{4^{jn}\sqrt{s-ar^2}}\,\mathrm{d}s \lesssim \frac{c}{4^{jn}}$$

Lastly evaluate the integral from $(n+1)r^2$ to ∞. Here use a Taylor expansion on $(s-kr^2)^{-1/2}$ around the point $k=0$, then use that $\sum_{k=0}^{n}\binom{n}{k}(-1)^kk^l = 0$ for all integers l from 0 to $n-1$.

$$\int_{(n+1)r^2}^{\infty} \frac{e^{-c4^j r^2/s}}{\sqrt{s}} \left| \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\sqrt{s-kr^2}} \right| \mathrm{d}s \lesssim c \int_{(n+1)r^2}^{\infty} \frac{e^{-c4^j r^2/s}}{\sqrt{s}} \frac{r^{2n}}{s^{n+1/2}} \mathrm{d}s \lesssim \frac{c}{4^{jn}}$$

Place these integral values back into equation (3.3).

$$\int_B |\nabla L^{-1/2}(I-e^{-r^2L})^n[1_{C_j}f](y)|^2 \,\mathrm{d}y \lesssim \frac{1}{4^{jn}} \int_{C_j} |f(y)|^2 \,\mathrm{d}y$$

Use the doubling condition (1.5) and sum over j for the result. □

Now Theorem 3.1 is ready to be proven.

Proof of Theorem 3.1. The proof splits the Riesz transform into 3 parts, then finds a bound for each part. Given the integral representation of the $\mathcal{L}^p$ norm,

$$\||\nabla L^{-1/2}f|\|_p = p \int_0^\infty \lambda^{p-1} \mu(\{x \in \Omega : |\nabla L^{-1/2}f(x)| > \lambda\}) \,\mathrm{d}\lambda, \tag{3.4}$$

split up the set $E = \{x \in \Omega : M(|\nabla L^{-1/2}f|^2)(x) > \lambda^2\}$ into a set of balls $\{B_i\}_{i\in I}$ by a Whitney covering lemma. This is done so that:

1. There exists a constant $c_1 < 1$ such that the set $\{c_1 B_i\}_{i\in I}$ is pairwise disjoint; and,

2. The collection $\{B_i\}_{i\in I}$ of balls in the Whitney covering satisfies $\cup_{i\in I} B_i = E$;

3. There exists a constant $c_2 > 1$ where for each ball B_i in the covering: $c_2 B_i \cap E^c \neq \emptyset$.

The constants depend on n. The balls in this covering lemma all have centre in Ω and have size dependent on the distance to E^c. Note that E^c is considered as a complement inside Ω. That is, $E^c = \{x \in \Omega : x \notin E\}$. Hence if E touches the boundary of Ω (in cases where Ω has boundary) then some balls in the covering will also touch the boundary of Ω. See preliminaries section 2.3 for more discussion on this. Split equation (3.4) by this covering,

$$\||\nabla L^{-1/2}f|\|_p \leq p \int_0^\infty \lambda^{p-1} \sum_{i\in I} \mu(\{x \in B_i : |\nabla L^{-1/2}f(x)| > \lambda\}) \,\mathrm{d}\lambda$$

and then scale λ by some $K > 2$ to be chosen later.

$$\||\nabla L^{-1/2} f|\|_p \leq pK^p \int_0^\infty \lambda^{p-1} \sum_{i \in I} \mu(\{x \in B_i : |\nabla L^{-1/2} f(x)| > K\lambda\}) \,\mathrm{d}\lambda$$

Next define the operator A_r,

$$A_r = I - (I - e^{-r^2 L})^n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} e^{-kr^2 L} \tag{3.5}$$

where $r = r(B_i)$ is the radius of the ball B_i. Split the Riesz transform in the manner of $\nabla L^{-1/2} = \nabla L^{-1/2} A_r + \nabla L^{-1/2}(1 - A_r)$. Also split the sets containing these terms.

$$\begin{aligned}\||\nabla L^{-1/2} f|\|_p \leq pK^p \int_0^\infty \lambda^{p-1} \sum_{i \in I} \Big[&\mu(\{x \in B_i : |\nabla L^{-1/2} A_r f(x)| > (K-1)\lambda\}) \\ &+ \mu(\{x \in B_i : |\nabla L^{-1/2}(I - A_r) f(x)| > \lambda\})\Big] \,\mathrm{d}\lambda\end{aligned}$$

The first part on the right of the equation above needs to be split further. As A_r and $L^{1/2}$ commute due to the semigroup condition, $\nabla L^{-1/2} A_r f$ can be split in the following way: $\nabla L^{-1/2} A_r f(x) = \nabla A_r \big(L^{-1/2} f(x) - [L^{-1/2} f]_{B_i}\big) + \nabla A_r [L^{-1/2} f]_{B_i}$. Then the bound for the $\mathcal{L}^p$ norm of $\nabla L^{-1/2} f$ is given by,

$$\begin{aligned}\||\nabla L^{-1/2} f|\|_p \leq pK^p \int_0^\infty \lambda^{p-1} \sum_{i \in I} \Bigg[&\mu(\{x \in B_i : |\nabla A_r [L^{-1/2} f]_{B_i}(x)| > \lambda\}) \\ &+ \mu(\{x \in B_i : |\nabla A_r \big(L^{-1/2} f(x) - [L^{-1/2} f]_{B_i}\big)| > (K-2)\lambda\}) \\ &+ \mu(\{x \in B_i : |\nabla L^{-1/2}(I - A_r) f(x)| > \lambda\})\Bigg] \,\mathrm{d}\lambda. \end{aligned} \tag{3.6}$$

where there are three parts to bound separately. For the first part (that is the part involving $|\nabla A_r [L^{-1/2} f]_{B_i}(x)|$), consider whether the equation given by,

$$\frac{1}{|B_i|} \int_{B_i} |\nabla A_r [L^{-1/2} f]_{B_i}(x)|^2 \,\mathrm{d}x > \delta^2 \lambda^2 \tag{3.7}$$

is true or false, where $\delta > 0$ is small and specified later[1]. If equation (3.7) is false use a

[1]This idea of whether an equation is true or false for some small δ is the same idea as motivates good-λ inequalities, and is used here instead of a good-λ inequality to keep equations more compact.

weak $(2,2)$ inequality,

$$\mu(\{x \in B_i : |\nabla A_r[L^{-1/2}f]_{B_i}(x)| > \lambda\}) \lesssim \frac{1}{\lambda^2}\int_{B_i} |\nabla A_r[L^{-1/2}f]_{B_i}(x)|^2 \,\mathrm{d}x \lesssim \delta^2|B_i| \quad (3.8)$$

and if equation (3.7) is true use Lemma 3.4.

$$\begin{aligned}
&\mu(\{x \in B_i : |\nabla A_r[L^{-1/2}f]_{B_i}(x)| > \lambda\}) \\
&\qquad \le \mu(\{x \in B_i : \frac{1}{|B_i|}\int_{B_i} |\nabla A_r[L^{-1/2}f]_{B_i}(x)|^2 \,\mathrm{d}x > \delta^2\lambda^2\}) \\
&\qquad \le \mu(\{x \in B_i : \frac{c_n}{|B_i|}\int_{B_i} |\frac{1}{\varphi(x)}L^{-1/2}f(x)|^2 \,\mathrm{d}x > \delta^2\lambda^2\}) \\
&\qquad \le \mu(\{x \in B_i : c_n M|\frac{1}{\varphi}L^{-1/2}f|^2(x) > \delta^2\lambda^2\}) \qquad (3.9)
\end{aligned}$$

This completes a bound for the first part of equation (3.6). For the second part of equation (3.6), use a weak (q,q) bound along with Lemma 3.5.

$$\begin{aligned}
&\mu(\{x \in B_i : |\nabla A_r\big(L^{-1/2}f(x) - [L^{-1/2}f]_{B_i}\big)| > (K-2)\lambda\}) \\
&\quad \lesssim \frac{1}{(K-2)^q\lambda^q}\int_{B_i} |\nabla A_r\big(L^{-1/2}f(x) - [L^{-1/2}f]_{B_i}\big)|^q \,\mathrm{d}x \lesssim \frac{1}{(K-2)^q}|B_i| \qquad (3.10)
\end{aligned}$$

Finally, to bound the last part of equation (3.6), consider whether the equation given by,

$$\frac{1}{|B_i|}\int_{B_i} |\nabla L^{-1/2}(I - A_r)f(x)|^2 \,\mathrm{d}x > \delta^2\lambda^2 \qquad (3.11)$$

is true or false. If equation (3.11) is false, use a weak $(2,2)$ inequality,

$$\begin{aligned}
&\mu(\{x \in B_i : |\nabla L^{-1/2}(I - A_r)f(x)| > \lambda\}) \\
&\qquad \lesssim \frac{1}{\lambda^2}\int_{B_i} |\nabla L^{-1/2}(I - A_r)f(x)|^2 \,\mathrm{d}x \lesssim \delta^2|B_i| \qquad (3.12)
\end{aligned}$$

and if equation (3.11) is true, use Lemma 3.6.

$$\begin{aligned}
&\mu(\{x \in B_i : |\nabla L^{-1/2}(I - A_r)f(x)| > \lambda\}) \\
&\qquad \le \mu(\{x \in B_i : \frac{1}{|B_i|}\int_{B_i} |\nabla L^{-1/2}(I - A_r)f(x)|^2 \,\mathrm{d}x > \delta^2\lambda^2\}) \\
&\qquad \le \mu(\{x \in B_i : c_n M|f|^2(x) > \delta^2\lambda^2\}) \qquad (3.13)
\end{aligned}$$

Place all five bounds found (3.8, 3.9, 3.10, 3.12, 3.13) for the various parts of equation (3.6) back into equation (3.6).

$$\begin{aligned}\||\nabla L^{-1/2}f|\|_p \lesssim pK^p \int_0^\infty \lambda^{p-1} \sum_{i\in I} \Bigg[&\mu(\{x \in B_i : c_n M|\frac{1}{\varphi}L^{-1/2}f|^2(x) > \delta^2\lambda^2\}) \\ &+ \mu(\{x \in B_i : c_n M|f|^2(x) > \delta^2\lambda^2\}) + \Big[2\delta^2 + \frac{1}{(K-2)^q}\Big]|B_i|\Bigg]\,\mathrm{d}\lambda.\end{aligned}$$

Sum over the balls B_i using that they are from a Whitney decompostion of the set given by $E = \{x \in \Omega : M(|\nabla L^{-1/2}f|^2)(x) > \lambda^2\}$.

$$\begin{aligned}\||\nabla L^{-1/2}f|\|_p \lesssim pK^p \int_0^\infty \lambda^{p-1} \Bigg[&\mu(\{x \in \Omega : c_n M|\frac{1}{\varphi}L^{-1/2}f|^2(x) > \delta^2\lambda^2\}) \\ &+ \mu(\{x \in \Omega : c_n M|f|^2(x) > \delta^2\lambda^2\}) \\ &+ \Big[2\delta^2 + \frac{1}{(K-2)^q}\Big]\mu(\{x \in \Omega : M|\nabla L^{-1/2}f(x)|^2(x) > \lambda^2\})\Bigg]\,\mathrm{d}\lambda.\end{aligned}$$

Change out of the integral representations of the $\mathcal{L}^p$ norms.

$$\begin{aligned}\||\nabla L^{-1/2}f|\|_p \lesssim \frac{K^p}{\delta^p}\|(M|\frac{1}{\varphi}L^{-1/2}f|^2)^{1/2}\|_p + \frac{K^p}{\delta^p}\|(M|f|^2)^{1/2}\|_p \\ + K^p\Big[2\delta^2 + \frac{1}{(K-2)^q}\Big]\|(M|\nabla L^{-1/2}f|^2)^{1/2}\|_p.\end{aligned}$$

Choose δ small enough and K large enough to ensure $K^p\Big[2\delta^2 + \frac{1}{(K-2)^q}\Big]$ is small, using also that $q > p$. In addition use that the uncentred Hardy-Littlewood maximal function given by $[M(|f|^2)]^{1/2}$ is bounded $\mathcal{L}^p \to \mathcal{L}^p$ for all $p > 2$.

$$\||\nabla L^{-1/2}f|\|_p \lesssim \|\frac{1}{\varphi}L^{-1/2}f\|_p + \|f\|_p$$

This completes the proof of Theorem 3.1, and by corollary completes the proof of Theorem 1.1 from the introduction. □

The next two chapters look at variations on this proof.

Chapter 4:
A Riesz Transform Bound Part 2:
A Non-Doubling Variation

In this chapter Theorem 1.2 is proven. Theorem 1.2 gives conditions leading to $\mathcal{L}^p \to \mathcal{L}^p$ boundedness of the Riesz transform in the absence of preservation and global doubling conditions. Theorem 1.2 is revisited below as a combination of Theorem 4.2 and Corollary 4.3.

Let Ω be a subset of a doubling space. A ball $B \subset \Omega$ is said to touch the boundary $\delta\Omega$ if the minimal distance $\rho(B)$ from B to $\delta\Omega$ is 0. In this chapter general balls $B \subset \Omega$ are not required to satisfy global doubling or Poincaré conditions. Instead there are local estimates: every ball $B(x,r) \subset \Omega$ that does not touch the boundary of Ω satisfies a local doubling condition,

$$|B(x, 2^k r)| \lesssim 2^{nk} |B(x,r)| \tag{4.1}$$

and a local Poincaré condition,

$$\int_B |f(x) - f_B|^2 \, \mathrm{d}\mu \lesssim r^2 \int_B |\nabla f(x)|^2 \, \mathrm{d}\mu \tag{4.2}$$

but there is no guarantee that any ball $B \subset \Omega$ that touches the boundary of Ω will satisfy either condition. In fact the local Poincaré condition will not be used unless $r(B) \lesssim \rho(B)$. It does not matter whether the ball $B(x, 2^k r)$ in the doubling equation touches $\delta\Omega$, only that $B(x,r)$ does not touch $\delta\Omega$. It is not difficult to see that if $\Omega \subset \mathcal{M}$ is open and $\mathcal{M}$ satisfies the global doubling (1.5) and Poincaré (1.6) conditions then the local conditions (4.1) and (4.2) hold on Ω.

Example 4.1. Consider the domain given by $\{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < e^{x_1}\}$. Such a domain is not globally doubling but does satisfy the local conditions (4.1) and (4.2).

To compensate for the lack of a global doubling condition, $\varphi(x)$ from conditions (1.3) and (1.4) is chosen as specifically as the minimal distance $\rho(x)$ from x to the boundary $\delta\Omega$. A benefit of this choice is that other conditions from chapter 3 can now also be replaced by local estimates in this chapter. Let B be a ball away from the boundary (this means $c_0 r(B) < \rho(B)$ for some fixed c_0). Then the following conditions are required for such balls. Firstly a local version of equation (1.3),

$$\sup_{B\subset\Omega}\left(\fint_B \rho(x)^2\,\mathrm{d}x\right)\left(\fint_B |\nabla e^{-kr^2L}1_\Omega(x)|^2\,\mathrm{d}x\right)\lesssim 1 \tag{4.3}$$

where r remains the radius of B and $k\in[1,n]$ is an integer. Secondly a local $\mathcal{L}^2$ Riesz transform.

$$\||\nabla L^{1/2}f|\|_{\mathcal{L}^2(B)}\lesssim \|f\|_{\mathcal{L}^2(\Omega)} \tag{4.4}$$

Thirdly a local $\mathcal{L}^2$ off-diagonal Gaffney estimate for any subset $C\subset\Omega$ and f supported on C.

$$\||\nabla e^{-tL}f|\|_{\mathcal{L}^2(B)}\lesssim \frac{e^{-d(B,C)^2/ct}}{\sqrt{t}}\|f\|_{\mathcal{L}^2(C)} \tag{4.5}$$

Lastly a local $\mathcal{L}^q$ on-diagonal Gaffney estimate for some $q>p$.

$$\||\nabla e^{-tL}f|\|_{\mathcal{L}^q(B)}\lesssim \frac{1}{\sqrt{t}}\|f\|_{\mathcal{L}^q(\Omega)} \tag{4.6}$$

The above conditions are not required to hold on balls where $c_0 r(B)\not\lesssim \rho(B)$. The Hardy inequality (1.4) is not a local estimate,

$$\|\frac{1}{\rho}L^{-1/2}f\|_{\mathcal{L}^p(\Omega)}\lesssim \|f\|_{\mathcal{L}^p(\Omega)} \tag{4.7}$$

and needs to hold for all $f\in\mathcal{L}^p(\Omega)$. Further, the heat kernel must satisfy the following variation on Gaussian upper bounds.

$$p_t(x,y)\lesssim \frac{e^{-d(x,y)^2/ct}}{\max[|B(x,\sqrt{t})|,|B(y,\sqrt{t})|]} \tag{4.8}$$

Following the theorem below are remarks regarding the conditions above.

THEOREM 4.2. *Let Ω be a space satisfying the local doubling (4.1) and the local Poincaré (4.2) conditions, and let ∇ be the Riemannian gradient. Suppose L is a second order differential operator with well defined functional calculus and Gaussian upper bounds on its heat kernel (4.8). Further suppose on balls away from the boundary ($c_0 r(B) < \rho(B)$ for some fixed c_0) that L satisfies: a local $\mathcal{L}^2$ Riesz transform (4.4); local Gaffney conditions (4.5) and (4.6) for some $q > 2$; as well as the semigroup gradient condition (4.3). Then the Riesz transform satisfies the following $\mathcal{L}^p$ bound for every $2 < p < q$ and $f \in C_0^\infty(\Omega)$.*

$$\||\nabla L^{-1/2} f|\|_p \lesssim \|f\|_p + \|\frac{1}{\rho} L^{-1/2} f\|_p$$

Corollary 4.3. *Suppose further to the conditions of the above theorem that the Hardy operator $\frac{1}{\rho} L^{-1/2} f$ is bounded $\mathcal{L}^p \to \mathcal{L}^p$ (4.7) for some $p \in (2, q)$ as in Theorem 4.2. Then the following Riesz transform bound holds for that same p, and every $f \in \mathcal{L}^p(\Omega)$.*

$$\||\nabla L^{-1/2} f|\|_p \lesssim \|f\|_p$$

Further this leads to a Sobolev space comparison,

$$\||\nabla f|\|_p \lesssim \|L^{1/2} f\|_p$$

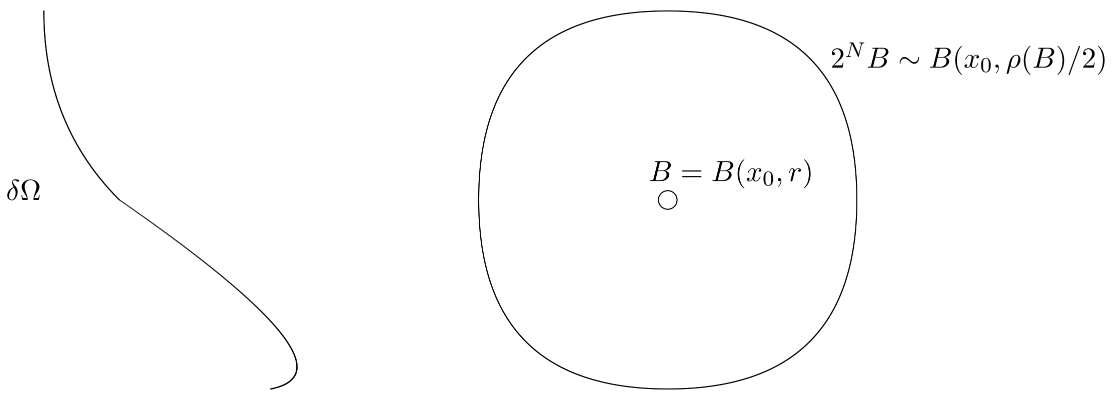
for every f in $W^{1,p} \cap Dom(L^{1/2})$.

Remarks 4.4. The proof of Theorem 4.2 uses a different Whitney decomposition than used in chapter 3. In this chapter the decomposition involves balls sized according to distance to both δE and $\delta\Omega$, whichever is closest (recall Ω as the underlying space on which our functions are defined, and $E = \{x \in \Omega : M|\nabla L^{-1/2} f|^2(x) > \lambda\}$). This change in the decomposition is to ensure every ball in the covering satisfies the local conditions outlined earlier. The property $3B \cap E^c \neq \emptyset$ used in chapter 3 is lost for balls closer to $\delta\Omega$, but this is compensated for by choice of $\varphi(x)$ as $\rho(x)$. The two types of covering were compared in the preliminaries section 2.3. Another main difference from chapter 3 is an extra characteristic function that ensures the Poincaré inequality is only used on balls away from the boundary ($c_0 r(B) < \rho(B)$).

Definition 4.5. For each ball $B \subset \Omega$ let $r(B)$ be the radius of B and $\rho(B)$ be the minimal distance from B to the boundary $\delta\Omega$. For each ball B away from the boundary (meaning $c_0 r(B) < \rho(B)$), let $N \geq 0$ be an integer such that $r(2^N B) \sim \rho(2^N B)$. The idea is that $N = 0$ for balls close to $\delta\Omega$ relative to their size, and $N \geq 1$ otherwise. See Figure 4.1.

In the proof of the main theorem of this chapter the function $f(x)$ is split according to whether x is in $C_0 = 2B$ or in $C_j = 2^{j+1}B \setminus 2^j B$ for some $j \geq 1$, and a different approach is taken depending on whether $j < N$ (x is inside $2^N B$), or if $j \geq N$ (x is outside $2^N B$).

Figure 4.1: Comparative sizings of B and $2^N B$ relative to the boundary.



When $j \geq N$, this implies $\rho(B) \lesssim 2^{j+1} r(B)$. Hence for f supported on $2^{j+1}B \setminus 2^j B$ with $j \geq N$, the local off-diagonal $\mathcal{L}^2$ Gaffney estimate (4.5), leads to the following bound.

$$|||\nabla e^{-kr^2 L} f|||_{\mathcal{L}^2(B)} \lesssim \frac{e^{-c4^j}}{r} \|f\|_{\mathcal{L}^2(2^{j+1}B \setminus 2^j B)} \lesssim 2^{j+1} e^{-c4^j} \left\| \frac{f}{\rho} \right\|_{\mathcal{L}^2(2^{j+1}B \setminus 2^i B)} \tag{4.9}$$

As usual r is the radius of B and $k \in [1, n]$ is an integer. Thus the benefit of fixing $\varphi(x)$ as the distance to the boundary $\rho(x)$, is the direct comparison to the Hardy operator near the boundary as seen in the above equation. The drawback of the choice $\varphi(x)$ as $\rho(x)$, is that the $\frac{1}{\rho}$ part of the Hardy operator will blow up at the boundary. To compensate for this, Dirichlet boundary conditions are needed for the heat kernel $p_t(x, y)$ in applications. When $j < N$ the ball $2^j B$ is away from the boundary and similar techniques to the proof in chapter 3 are used.

4.1 Proof of the Result

Before the main proof there are three lemmas.

Lemma 4.6. *Suppose the conditions of Theorem 4.2 hold. Let $B \subset \Omega$ be any ball far enough away from the boundary $\delta\Omega$ so that $N \geq 1$, where N is from Definition 4.5. Then the following holds for all integers $k \in [1, n]$.*

$$\left(\fint_B \rho(x)^2 \,\mathrm{d}x\right)\left(\fint_B |\nabla e^{-kr^2L} 1_{2^N B}|^2 \,\mathrm{d}x\right) \lesssim 1$$

Here $1_{2^N B}$ is the characteristic function of $2^N B$, r is the radius of B, and $\rho(x)$ is the distance from x to $\delta\Omega$.

Proof. Let $C_j = 2^{j+1}B \setminus 2^j B$ for $j \geq 1$. Split $1_{2^N B} = 1 - \sum_{j=N}^{\infty} 1_{C_j}$. Use this to split the term $|\nabla e^{-kr^2L} 1_{2^N B}|$ into parts where different conditions can be used on each part.

$$\left(\fint_B |\nabla e^{-kr^2L} 1_{2^N B}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \lesssim \left(\fint_B |\nabla e^{-kr^2L} 1_{\Omega}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} + \sum_{j=N}^{\infty} \left(\fint_B |\nabla e^{-kr^2L} 1_{C_j}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$$

For the rightmost term above use local $\mathcal{L}^2$ Gaffney estimates (4.5) and that $1 \leq k \leq n$.

$$\left(\fint_B |\nabla e^{-kr^2L} 1_{2^N B}|^2 \,\mathrm{d}x\right)^{1/2} \lesssim \left(\fint_B |\nabla e^{-kr^2L} 1_{\Omega}|^2 \,\mathrm{d}x\right)^{1/2} + \sum_{j=N}^{\infty} \frac{e^{-c4^j}}{r} \frac{|C_j|^{1/2}}{|B|^{1/2}}$$

By construction B does not touch the boundary of Ω so satisfies the local doubling condition (4.1). Further by construction $2^{-N}\rho(B) \lesssim r$. These properties are both used on the rightmost term of the above equation.

$$\left(\fint_B |\nabla e^{-kr^2L} 1_{2^N B}|^2 \,\mathrm{d}x\right)^{1/2} \lesssim \left(\fint_B |\nabla e^{-kr^2L} 1_{\Omega}|^2 \,\mathrm{d}x\right)^{1/2} + \sum_{j=N}^{\infty} \frac{2^{N+jn/2} e^{-c4^j}}{\rho(B)}$$

Evaluate the sum (the result will not depend on N) and multiply both sides of the equation by $(\fint \rho(x)^2 \,\mathrm{d}x)^{1/2}$, to get the left side resembling that in the lemma.

$$\begin{aligned}
&\left(\fint_B \rho(x)^2 \,\mathrm{d}x\right)^{1/2} \left(\fint_B |\nabla e^{-kr^2L} 1_{2^N B}|^2 \,\mathrm{d}x\right)^{1/2} \\
&\quad \lesssim \left(\fint_B \rho(x)^2 \,\mathrm{d}x\right)^{1/2} \left(\fint_B |\nabla e^{-kr^2L} 1_{\Omega}|^2 \,\mathrm{d}x\right)^{1/2} + \frac{1}{\rho(B)} \left(\fint_B \rho(x)^2 \,\mathrm{d}x\right)^{1/2}
\end{aligned}$$

The first term on the right above is bounded by a constant according to condition (4.3). The second term on the right above is bounded by a constant due to the set of inequalities given by $\rho(x) < \rho(B) + 2r < (1 + 2c_0)\rho(B)$ by choice of B away from the boundary in the lemma. This concludes the proof. $\square$

The next lemma replaces Lemma 3.4 from chapter 3. The difference here is due to the characteristic function present in the left side of the bound.

Lemma 4.7. *Suppose the various conditions of Theorem 4.2 hold. Let $B \subset \Omega$ be any ball far enough away from the boundary $\delta\Omega$ so that $N \geq 1$, where N is from Definition 4.5. Then the following inequality is true for all integers $k \in [1, n]$.*

$$\int_B |\nabla e^{-kr^2 L}(1_{2^N B} f_B)(x)|^2 \, \mathrm{d}x \lesssim \int_B \left[\frac{f(x)}{\rho(x)}\right]^2 \mathrm{d}x$$

Again $1_{2^N B}$ is the characteristic function of $2^N B$, f_B is the average of f on B, r is the radius of B, and $\rho(x)$ is the minimal distance from x to $\delta\Omega$.

Proof. Fix B. Then N and r depend on B and the value of f_B is constant with respect to x. This means f_B can be separated from the semigroup operator and the gradient.

$$|\nabla e^{-kr^2 L}(1_{2^N B} f_B)(x)| = |f_B||\nabla e^{-kr^2 L} 1_{2^N B}(x)|$$

Similar to the proof of Lemma 3.4, square the result above and integrate with respect to x. Regarding the f_B term, separate out ρ and $\frac{1}{\rho}$ and use Hölder's inequality.

$$\int_B |\nabla e^{-kr^2 L}(1_{2^N B} f_B)|^2 \, \mathrm{d}x \lesssim \left(\fint_B \rho(x)^2 \, \mathrm{d}x\right)\left(\fint_B \left[\frac{f(x)}{\rho(x)}\right]^2 \mathrm{d}x\right)\left(\int_B |\nabla e^{-kr^2 L} 1_{2^N B}(x)|^2 \, \mathrm{d}x\right)$$

Lastly use Lemma 4.6,

$$\int_B |\nabla e^{-kr^2 L}(1_{2^N B} f_B)(x)|^2 \, \mathrm{d}x \lesssim \int_B \left[\frac{f(x)}{\rho(x)}\right]^2 \mathrm{d}x$$

to conclude the proof. $\square$

The next lemma replaces Lemma 3.5 from chapter 3. The difference is again due to a characteristic function present in the left side of the bound. This time it is clear why the characteristic function is present: to ensure that Poincaré's inequality is only used on balls that do not touch $\delta\Omega$. Hence only the local Poincaré inequality (4.2) is required.

Lemma 4.8. *Suppose that L is an appropriately defined operator satisfying the various conditions of Theorem 4.2 (in particular the local Poincaré inequality (4.2), and local Gaffney estimates (4.5) and (4.6) are needed here). Let $B \subset \Omega$ be a ball far enough away from the boundary so that $N \geq 1$ (N is from Definition 4.5), where for some fixed c_n, $c_n B$ includes points outside the set where $M(|\nabla f|^2)(x)$ is greater than λ^2. Then there exists $q > p$ such that the following bound holds,*

$$\frac{1}{|B|}\int_B \left|\nabla e^{-kr^2L} 1_{(2^N B)}\big(f(x) - f_B\big)\right|^q \mathrm{d}x \lesssim \lambda^q$$

for all f smooth enough and all integers $k \in [1, n]$. Here M is the uncentred Hardy Littlewood maximal function, f_B is the average of f on B, $1_{(2^N B)}$ is the characteristic function of the set $2^N B$, and r is the radius of B.

Proof. Let $C_0 = 2B$ and let $C_j = 2^{j+1}B \setminus 2^j B$ for $j \geq 1$. Split the heat semigroup by the equation $e^{-kr^2L} 1_{(2^N B)} = \sum_{j=0}^{N-1}\sum_{s=0}^{\infty} e^{-kr^2L/2} 1_{C_s} e^{-kr^2L/2} 1_{C_j}$. Interpolation between the Gaffney estimates (4.5) and (4.6) gives a local off-diagonal L^q Gaffney estimate for some $q > p$ (smaller than the q in condition 4.6). Use this on the first part of the split.

$$\begin{aligned}\Big(\fint_B |\nabla e^{-kr^2L} 1_{(2^N B)}(f(x) - f_B)|^q \,\mathrm{d}x\Big)^{1/q} \\ \lesssim \sum_{j=0}^{N-1}\sum_{s=0}^{\infty} \frac{e^{-c4^s}}{r}\Big(\int_{C_s} |e^{-kr^2L/2} 1_{C_j}(f(x) - f_B)|^q \,\mathrm{d}x\Big)^{1/q}\end{aligned}$$

Next use heat kernel bounds as in the proof of Lemma 3.5 from chapter 3. It is important here that the ball B is away from the boundary and so local doubling holds.

$$\begin{aligned}\Big(\fint_B |\nabla e^{-kr^2L} 1_{(2^N B)}(f(x) - f_B)|^q \,\mathrm{d}x\Big)^{1/q} \\ \lesssim \sum_{s=0}^{\infty}\sum_{j=0}^{N-1} \frac{e^{-c4^s} e^{-c4^{|s-j|}}}{r|B|^{(q-2)/2q}}\left(\int_{C_j} |f(x) - f_B|^2 \,\mathrm{d}x\right)^{1/2}\end{aligned} \tag{4.10}$$

The difference term is now split up so that the local Poincaré inequality can be used.

$$\begin{aligned}|f(x) - f_B| &= |f(x) - f_{2^{j+1}B} + \sum_{l=0}^{j}(f_{2^{l+1}B} - f_{2^l B})| \\ &\lesssim |f(x) - f_{2^{j+1}B}| + \sum_{l=0}^{j} \frac{1}{|2^l B|}\int_{2^l B} |f(y) - f_{2^{l+1}B}|\,\mathrm{d}y\end{aligned}$$

Take $\mathcal{L}^2$ norms on the space C_j on both sides of the inequality above. Then apply Minkowski's inequality.

$$\begin{aligned}\Big[\int_{C_j} \left|f(x) - f_B\right|^2 \mathrm{d}x\Big]^{1/2} \leq &\Big[\int_{2^{j+1}B} |f(x) - f_{2^{j+1}B}|^2 \,\mathrm{d}x\Big]^{1/2} \\ &+ 2^n \sum_{l=0}^{j} \frac{|2^{j+1}B|^{1/2}}{|2^{l+1}B|^{1/2}}\Big[\int_{2^{l+1}B} |f(x) - f_{2^{l+1}B}|^2 \,\mathrm{d}x\Big]^{1/2}\end{aligned}$$

Then apply Poincaré's inequality to all parts. In this proof we have $l \leq j < N$ where N was chosen so that $r(2^N B) \sim \rho(2^N B)$ implying that the ball $2^l B$ will not touch the boundary of Ω. Thus only the local Poincaré condition (4.2) is used, which is where this proof differs from the proof of Lemma 3.5 in chapter 3.

$$\left[\int_{C_j} \left|f(x) - f_B\right|^2 \mathrm{d}x\right]^{\frac{1}{2}} \lesssim 2^n \sum_{l=0}^{j+1} \frac{2^l r |2^{j+1}B|^{\frac{1}{2}}}{|2^{l+1}B|^{\frac{1}{2}}}\left[\int_{2^{l+1}B} |\nabla f|^2 \,\mathrm{d}x\right]^{\frac{1}{2}}$$

Insert this result into equation (4.10).

$$\begin{aligned}&\Big(\int_B \left|\nabla e^{-kr^2 L} 1_{(2^N B)}\big(f(x) - f_B\big)\right|^q \mathrm{d}x\Big)^{1/q} \\ &\qquad\lesssim \sum_{s=0}^{\infty}\sum_{j=0}^{N-1} \frac{e^{-c4^s} e^{-c4^{|s-j|}}}{r|B|^{(q-2)/2q}} \sum_{l=0}^{j+1} \frac{2^l r |2^{j+1}B|^{\frac{1}{2}}}{|2^{l+1}B|^{\frac{1}{2}}}\left[\int_{2^{l+1}B} |\nabla f|^2 \,\mathrm{d}x\right]^{\frac{1}{2}}\end{aligned}$$

Recall that $c_n B$ contains a point x_0 where $M(|\nabla f|^2)(x_0) < \lambda^2$. Further, the balls of size $2B$ and larger up to $c_n B$, can be increased to $c_n B$ for the same effect (using local doubling condition (4.1)). The following bound now holds.

$$\begin{aligned}&\Big(\int_B \left|\nabla e^{-kr^2 L} 1_{(2^N B)}\big(f(x) - f_B\big)\right|^q \mathrm{d}x\Big)^{1/q} \\ &\qquad\lesssim \sum_{s=0}^{\infty}\sum_{j=0}^{N-1} \frac{e^{-c4^s} e^{-c4^{|s-j|}}}{|B|^{(q-2)/2q}} \sum_{l=0}^{j+1} 2^l |2^{j+1}B|^{\frac{1}{2}}\big(M|\nabla f|^2(x_0)\big)^{\frac{1}{2}}\end{aligned}$$

As discussed above, here $M(|\nabla f|^2)(x_0) < \lambda^2$. Use also that $l < j+1$ to simplify.

$$\Big(\int_B \Big|\nabla e^{-kr^2L} 1_{(2^N B)}\big(f(x) - f_B\big)\Big|^q \,\mathrm{d}x\Big)^{1/q} \lesssim \lambda |B|^{1/q} \sum_{s=0}^{\infty} \sum_{j=0}^{N-1} e^{-c4^s} e^{-c4^{|s-j|}} 2^{(j+1)(1+\frac{n}{2})}$$

These summations are bounded above by a constant (depending only on n) and leave only $\lambda|B|^{1/q}$. This concludes the proof. $\square$

Now Theorem 4.2 is proven. This proof uses the lemmas above along with Lemma 3.6 from chapter 3. The proof is similar to the proof of Theorem 3.1 from chapter 3.

Proof of Theorem 4.2. The proof splits the Riesz transform into parts, then finds a bound for each part. Given the integral representation of the $\mathcal{L}^p$ norm,

$$|||\nabla L^{-1/2} f|||_p = p \int_0^{\infty} \lambda^{p-1} \mu(\{x \in \Omega : |\nabla L^{-1/2} f(x)| > \lambda\}) \,\mathrm{d}\lambda, \tag{4.11}$$

split up the set $E = \{x \in \Omega : M(|\nabla L^{-1/2} f|^2)(x) > \lambda^2\}$ by a Whitney type covering lemma. This is arranged so that;

1. There exists a constant $c_1 < 1$ such that the collection of balls $\{c_1 B_i\}_{i \in I}$ in the covering of E are pairwise disjoint;

2. The collection $\{B_i\}_{i\in I}$ of balls in the Whitney covering of E satisfy $\cup_{i\in I} B_i = E$; and,

3. There exists a constant $c_2 > 1$ such that for every ball B_i in the covering of E $\min\big(d(B_i, \delta E), \rho(B_i)\big) \sim c_2 r(B_i)$ (meaning the balls in the decomposition are sized according to the distance to the boundaries $\delta\Omega$ and δE, whichever is closest).

The constant c_1 in the covering lemma depends only on the dimension n. The c_2 is chosen in part to match with the choice of c_0 in the localised conditions, and the constants involved in Definition 4.5. There are further comments on this below. Importantly none of the balls in the covering touch the boundary of Ω, so they all satisfy the local doubling (4.1) condition. Equation (4.11) is now given by the following.

$$|||\nabla L^{-1/2} f|||_p \le p \int_0^{\infty} \lambda^{p-1} \sum_{i\in I} \mu(\{x \in B_i : |\nabla L^{-1/2} f(x)| > \lambda\}) \,\mathrm{d}\lambda.$$

For each ball B_i choose N so that $r(2^N B_i) \sim \rho(2^N B_i)$ for some integer $N \geq 0$ (see Definition 4.5). It is necessary when fixing N and the constant c_2 in the Whitney decomposition, that they match with the constant c_0 from the conditions at the start of this chapter, so that $c_0 r(2^N B_i) < \rho(2^N B_i)$ for each B_i in the decomposition.

Split the group of balls $\{B_i\}_{i\in I}$ into two sets. The first set I_1 are those balls B_i with $N = 0$. These balls are close to $\delta\Omega$. The second set I_2 are the remaining balls, which satisfy $N \geq 1$ and are closer to δE. If E is away from the boundary the set I_1 may be empty, but that has no impact on the proof. In parts of the proof these two sets will be dealt with separately, that will be outlined when it occurs.

In the meantime the proof continues on the same lines as the proof from chapter 3. Scale λ by some $K > 3$ to be chosen later.

$$\||\nabla L^{-1/2} f|\|_p \leq pK^p \int_0^\infty \lambda^{p-1} \sum_{i\in I} \mu(\{x \in B_i : |\nabla L^{-1/2} f(x)| > K\lambda\})\, \mathrm{d}\lambda$$

Next define the operator A_r as,

$$A_r = I - (I - e^{-r^2 L})^n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} e^{-kr^2 L}. \tag{4.12}$$

where $r = r(B_i)$ is the radius of B_i. Split $\nabla L^{-1/2} = \nabla L^{-1/2} A_r + \nabla L^{-1/2}(1 - A_r)$ and also split the set containing these.

$$\begin{aligned}\||\nabla L^{-1/2} f|\|_p \leq pK^p \int_0^\infty \lambda^{p-1} \sum_{i\in I} \Big[&\mu(\{x \in B_i : |\nabla L^{-1/2} A_r f(x)| > (K-1)\lambda\}) \\ &+ \mu(\{x \in B_i : |\nabla L^{-1/2}(I - A_r) f(x)| > \lambda\})\Big]\, \mathrm{d}\lambda\end{aligned}$$

The next split differs from chapter 3, though the style is similar. Let $C_0 = 2B_i$ and $C_j = 2^{j+1} B_i \setminus 2^j B_i$ for $j \geq 1$. Then split up the sets,

$$\begin{aligned}\||\nabla L^{-1/2} f|\|_p \leq pK^p \int_0^\infty \lambda^{p-1} \Big[&\sum_{i\in I} \mu(\{x \in B_i : \sum_{j=N}^{\infty} |\nabla A_r 1_{C_j} L^{-1/2} f(x)| > \lambda\}) \\ &+ \sum_{i\in I_2} \mu(\{x \in B_i : |\nabla A_r 1_{(2^N B_i)} L^{-1/2} f(x)| > (K-2)\lambda\}) \\ &+ \sum_{i\in I} \mu(\{x \in B_i : |\nabla L^{-1/2}(I - A_r) f(x)| > \lambda\})\Big]\, \mathrm{d}\lambda\end{aligned}$$

where the second set above contains only terms in I_2 (these are the balls closer to δE). There is one more split that needs to occur before we can start bounding the various sets. This is a split of the form $L^{-1/2}f(x) = (L^{-1/2}f(x) - [L^{-1/2}f]_B) + [L^{-1/2}f]_B$ which is now used to split the second set above.

$$
\begin{aligned}
\||\nabla L^{-1/2}f|\|_p \leq pK^p \int_0^\infty \lambda^{p-1}\Big[&\sum_{i\in I}\mu(\{x\in B_i : \sum_{j=N}^{\infty}|\nabla A_r 1_{C_j} L^{-1/2}f(x)| > \lambda\}) \\
&+ \sum_{i\in I_2}\mu(\{x\in B_i : |\nabla A_r 1_{(2^N B_i)}[L^{-1/2}f]_{B_i}| > \lambda\}) \\
&+ \sum_{i\in I_2}\mu(\{x\in B_i : |\nabla A_r 1_{(2^N B_i)}(L^{-1/2}f(x) - [L^{-1/2}f]_{B_i})| > (K-3)\lambda\}) \\
&+ \sum_{i\in I}\mu(\{x\in B_i : |\nabla L^{-1/2}(I-A_r)f(x)| > \lambda\})\Big]\,\mathrm{d}\lambda
\end{aligned}
\tag{4.13}
$$

Each of these sets will be bound separately. To begin consider the first set on the right in equation (4.13) and whether (for some $\delta > 0$ chosen later) the equation given by,

$$
\fint_{B_i}\Big[\sum_{j=N}^{\infty}|\nabla A_r 1_{C_j} L^{-1/2}f(x)|\Big]^2\,\mathrm{d}x > \delta^2\lambda^2 \tag{4.14}
$$

is true or false. If equation (4.14) is false use a weak $(2,2)$ inequality.

$$
\begin{aligned}
&\mu(\{x\in B_i : \sum_{j=N}^{\infty}|\nabla A_r 1_{C_j} L^{-1/2}f(x)| > \lambda\}) \\
&\qquad \lesssim \frac{1}{\lambda^2}\int_{B_i}\Big[\sum_{j=N}^{\infty}|\nabla A_r 1_{C_j} L^{-1/2}f(x)|\Big]^2\,\mathrm{d}x \lesssim \delta^2|B_i|
\end{aligned}
$$

However if equation (4.14) is true, use equation (4.14) along with the definition of A_r.

$$
\begin{aligned}
&\mu(\{x\in B_i : \sum_{j=N}^{\infty}|\nabla A_r 1_{C_j} L^{-1/2}f(x)| > \lambda\}) \\
&\qquad \leq \mu(\{x\in B_i : \fint_{B_i}\Big[\sum_{j=N}^{\infty}|\nabla A_r 1_{C_j} L^{-1/2}f(x)|\Big]^2\,\mathrm{d}x > \delta^2\lambda^2\}) \\
&\qquad \leq \mu(\{x\in B_i : \sum_{k=1}^{n}\sum_{j=N}^{\infty} c_k\Big[\fint_{B_i}|\nabla e^{-kr^2L}1_{C_j}L^{-1/2}f(x)|^2\,\mathrm{d}x\Big]^{1/2} > \delta\lambda\})
\end{aligned}
$$

Due to the choice of N, the technique of equation (4.9) can be applied to get,

$$\mu(\{x \in B_i : \sum_{j=N}^{\infty} |\nabla A_r 1_{C_j} L^{-1/2} f(x)| > \lambda\})$$
$$\leq \mu(\{x \in B_i : c_n \sum_{j=N}^{\infty} \Big[\frac{2^{j+1} e^{-c4^j}}{|B_i|} \int_{C_j} \Big|\frac{L^{-1/2} f(x)}{\rho(x)}\Big|^2 \mathrm{d}x\Big]^{1/2} > \delta\lambda\})$$

and recall that the balls B_i in the decomposition satisfy the local doubling condition. Use the uncentred Hardy-Littlewood Maximal function as an upper bound, and evaluate the sum. The result does not depend on N.

$$\mu(\{x \in B_i : \sum_{j=N}^{\infty} |\nabla A_r 1_{C_j} L^{-1/2} f(x)| > \lambda\}) \leq \mu(\{x \in B_i : c_n M\Big(\Big|\frac{L^{-1/2} f}{\rho}\Big|^2\Big)(x) > \delta^2\lambda^2\})$$

This bound is now in terms of the Hardy inequality. Hence in total for the first part of equation (4.13), whether or not equation (4.14) is true, the following holds.

$$\mu(\{x \in B_i : \sum_{j=N}^{\infty} |\nabla A_r 1_{C_j} L^{-1/2} f(x)| > \lambda\})$$
$$\lesssim \mu(\{x \in B_i : c_n M\Big(\Big|\frac{L^{-1/2} f}{\rho}\Big|^2\Big)(x) > \delta^2\lambda^2\}) + \delta^2|B_i| \tag{4.15}$$

Next consider the the second part of equation (4.13). On this part the proof follows more closely that of chapter 3. For this part the balls B_i are in I_2, and $2^N B_i$, $N \geq 1$, does not touch $\delta\Omega$, and in fact satisfies $\rho(2^N B_i) \sim r(2^N B_i)$. Consider whether the equation given by,

$$⨍_{B_i} |\nabla A_r 1_{(2^N B_i)} [L^{-1/2} f]_{B_i}(x)|^2 \,\mathrm{d}x > \delta^2\lambda^2 \tag{4.16}$$

is true or false. If equation (4.16) is false use a weak $(2,2)$ inequality,

$$\mu(\{x \in B_i : |\nabla A_r 1_{(2^N B_i)} [L^{-1/2} f]_{B_i}(x)| > \lambda\})$$
$$\leq \frac{1}{\lambda^2} \int_B |\nabla A_r 1_{(2^N B_i)} [L^{-1/2} f]_{B_i}|^2 \,\mathrm{d}x \leq \delta^2 |B_i|$$

and if equation (4.16) is true then use Lemma 4.7.

$$\begin{aligned}
&\mu(\{x \in B_i : |\nabla A_r 1_{(2^N B_i)}[L^{-1/2}f]_{B_i}(x)| > \lambda\}) \\
&\qquad \leq \mu(\{x \in B_i : \fint_{B_i} |\nabla A_r 1_{(2^N B_i)}[L^{-1/2}f]_{B_i}(x)|^2 \,\mathrm{d}x > \delta^2\lambda^2\}) \\
&\qquad \leq \mu(\{x \in B_i : c_n M\Big(\Big|\frac{L^{-1/2}f}{\rho}\Big|^2\Big)(x) > \delta^2\lambda^2\})
\end{aligned}$$

So in total for the second part of equation (4.13), regardless of whether or not equation (4.16) is true, the following bound holds.

$$\begin{aligned}
&\mu(\{x \in B_i : |\nabla A_r 1_{(2^N B_i)}[L^{-1/2}f]_{B_i}(x)| > \lambda\}) \\
&\qquad \lesssim \mu(\{x \in B_i : c_n M\Big(\Big|\frac{L^{-1/2}f}{\rho}\Big|^2\Big)(x) > \delta^2\lambda^2\}) + \delta^2|B_i|
\end{aligned} \tag{4.17}$$

Next consider the third part of equation (4.13). Use a weak (q, q) inequality.

$$\begin{aligned}
&\mu(\{x \in B_i : |\nabla A_r 1_{(2^N B_i)}(L^{-1/2}f(x) - [L^{-1/2}f]_{B_i})| > (K-3)\lambda\}) \\
&\qquad \lesssim \frac{1}{(K-3)^q\lambda^q} \int_{B_i} |\nabla A_r 1_{(2^N B_i)}(L^{-1/2}f(x) - [L^{-1/2}f]_{B_i})|^q \,\mathrm{d}x
\end{aligned}$$

Due to the choice of $B_i \in I_2$, and the definitions of N and A_r, Lemma 4.8 can be applied.

$$\mu(\{x \in B_i : |\nabla A_r 1_{(2^N B_i)}(L^{-1/2}f(x) - [L^{-1/2}f]_{B_i})| > (K-3)\lambda\}) \lesssim \frac{|B_i|}{(K-3)^q} \tag{4.18}$$

This gives a bound for the third part of equation (4.13). It remains to bound the fourth part of equation (4.13). Here the method is identical to that of Chapter 3. Consider whether the equation given by,

$$\frac{1}{|B_i|} \int_{B_i} |\nabla L^{-1/2}(I - A_r)f(x)|^2 \,\mathrm{d}x > \delta^2\lambda^2 \tag{4.19}$$

is true or false. If equation (4.19) is false, use a weak $(2, 2)$ inequality,

$$\mu(\{x \in B_i : |\nabla L^{-1/2}(I - A_r)f(x)| > \lambda\}) \lesssim \frac{1}{\lambda^2} \int_{B_i} |\nabla L^{-1/2}(I - A_r)f(x)|^2 \,\mathrm{d}x \lesssim \delta^2|B_i|$$

and if equation (4.19) is true, use Lemma 3.6 from chapter 3. Note that this lemma uses only the local versions of the $\mathcal{L}^2$ Riesz and Gaffney estimates in its proof, and the $\mathcal{L}^2$ semigroup bound used (with f supported on a local ball B) follows from Gaussian upper bounds (4.8).

$$\begin{aligned}
&\mu(\{x \in B_i : |\nabla L^{-1/2}(I - A_r)f(x)| > \lambda\}) \\
&\qquad \le \mu(\{x \in B_i : \frac{1}{|B_i|}\int_{B_i} |\nabla L^{-1/2}(I - A_r)f(x)|^2 \,\mathrm{d}x > \delta^2\lambda^2\}) \\
&\qquad \le \mu(\{x \in B_i : c_n M(|f|^2)(x) > \delta^2\lambda^2\})
\end{aligned}$$

So in total in this fourth case of equation (4.13) the following bound holds.

$$\begin{aligned}
&\mu(\{x \in B_i : |\nabla L^{-1/2}(I - A_r)f(x)| > \lambda\}) \\
&\qquad \lesssim \mu(\{x \in B_i : c_n M(|f|^2)(x) > \delta^2\lambda^2\}) + \delta^2|B_i|
\end{aligned} \tag{4.20}$$

Combine all the bounds found for each part of equation (4.13) together, these are equations (4.15), (4.17), (4.18) and (4.20).

$$\begin{aligned}
\||\nabla L^{-1/2} f|\|_p^p \le pK^p \int_0^\infty \lambda^{p-1} \Bigg(&\sum_{i\in I} \mu(\{x \in B_i : c_n M\Big(\Big|\frac{L^{-1/2}f}{\rho}\Big|^2\Big)(x) > \delta^2\lambda^2\}) \\
&+ \frac{|B_i|}{(K-3)^q} + 3\delta^2|B_i| + \mu(\{x \in B_i : c_n M(|f|^2)(x) > \delta^2\lambda^2\})\Bigg) \,\mathrm{d}\lambda
\end{aligned}$$

The balls B_i, $i \in I$, cover E. Use also that E is the set where $M(|\nabla L^{-1/2} f|^2)(x) > \lambda^2$.

$$\begin{aligned}
\||\nabla L^{-1/2} f|\|_p \le &pK^p \int_0^\infty \lambda^{p-1} \Big[\mu(\{x \in \Omega : c_n M\Big(\Big|\frac{L^{-1/2}f}{\rho}\Big|^2\Big)(x) > \delta^2\lambda^2\}) \\
&+ \Big(3\delta^2 + \frac{1}{(K-3)^q}\Big)\mu(\{x \in \Omega : M|\nabla L^{-1/2} f|^2(x) > \lambda^2\}) \\
&+ \mu(\{x \in \Omega : c_n M(|f|^2)(x) > \delta^2\lambda^2\})\Big] \,\mathrm{d}\lambda
\end{aligned}$$

Change the integral representations back to $\mathcal{L}^p$ norm values.

$$\begin{aligned}
\||\nabla L^{-1/2} f|\|_p^p \le \frac{K^p}{\delta^p}\Big\|\Big(M\Big|\frac{L^{-1/2}f}{\rho}\Big|^2\Big)^{1/2}\Big\|_p^p + \frac{K^p}{\delta^p}\|(M|f|^2)^{1/2}\|_p^p \\
+ K^p\Big(3\delta^2 + \frac{1}{(K-3)^q}\Big)\|\big(M|\nabla L^{-1/2} f|^2\big)^{1/2}\|_p^p
\end{aligned}$$

The maximal function used is bounded on $\mathcal{L}^p$ for all $p > 2$. Chose δ small and K large for the result. $\square$

This concludes the proof of Theorem 4.2, and as a corollary concludes the proof of Theorem 1.2 from the introduction. The next chapter places weights throughout.

Chapter 5:

A Riesz Transform Bound Part 3:

A Weighted Result

In this chapter Theorem 1.3 is proven. Theorem 1.3 gives conditions for weighted $\mathcal{L}^p \to \mathcal{L}^p$ boundedness of the Riesz transform in the absence of a preservation condition. This is a weighted extension of chapters 3 and 4. Theorem 1.3 can be viewed as a combination of Theorem 5.1 and Corollary 5.2 below.

It is well known that A_p weights are those for which the standard Riesz transform on $\mathbb{R}^n$ is bounded (see for example [67]). In the case of elliptic operators on $\mathbb{R}^n$ or Riesz transforms on Manifolds weights are discussed in [7] and [10] respectively.

Take Ω as a possibly non-doubling open subset of $\mathbb{R}^n$. Consider balls $B \subset \Omega$ where $r = r(B)$ denotes the radius of B and $\rho(B)$ is the minimal distance from B to the boundary $\delta\Omega$. The heat kernel on Ω is required to satisfy,

$$p_t(x, y) \lesssim \alpha_t(x)\beta_t(y)\frac{e^{-d(x,y)^2/ct}}{t^{n/2}} \tag{5.1}$$

for some fixed constant $c > 0$ and a pair of continuous non-negative functions $\alpha \lesssim 1$ and $\beta \lesssim 1$ that satisfy $\alpha_t(x) \le 2\alpha_{2t}(x)$ and $\beta_t(y) \le 2\beta_{2t}(y)$ for all $t > 0$ and $x, y \in \Omega$. Say that a weight w satisfies an $A_p^{\alpha,\beta}$ condition ($p > 1$) if,

$$\left(\fint_B \alpha_{r^2}(x)^p w(x)\,\mathrm{d}x\right)\left(\fint_B \beta_{r^2}(x)^{p'} w(x)^{-p'/p}\,\mathrm{d}x\right)^{p/p'} \lesssim 1 \tag{5.2}$$

holds for all $B \subset \Omega$. This mirrors the standard Muckenhoupt A_p weight condition ($p > 1$)

which a weight w satisfies if,

$$\left(\fint_B w(x)\,\mathrm{d}x\right)\left(\fint_B w(x)^{-p'/p}\,\mathrm{d}x\right)^{p/p'} \lesssim 1 \tag{5.3}$$

for all balls $B \subset \Omega$. This standard A_p condition implies a weighted local doubling result,

$$\begin{aligned} w(2^jB) &\lesssim \frac{w(2^jB)}{r^n}\int_{2^jB} 1_B\,\mathrm{d}x \\ &\lesssim \left(\frac{1}{|B|}\int_{2^jB} 1_B^p w(x)\,\mathrm{d}x\right)\left(\int_{2^jB} w(x)\,\mathrm{d}x\right)\left(\int_{2^jB} w(x)^{-\frac{p'}{p}}\,\mathrm{d}x\right)^{\frac{p}{p'}} \lesssim 2^{jn}w(B) \end{aligned}$$

for all w satisfying equation (5.3), $j \geq 0$ and $B \subset \Omega$ away from the boundary so as to satisfy local doubling (4.1).

Continuing to restrict to balls B away from the boundary ($c_0 r(B) < \rho(B)$) there is also required in this chapter a weighted Poincaré inequality,

$$\int_B |f(x) - f_B|^2 w(x)^{2/p}\,\mathrm{d}x \lesssim \int_B |\nabla f(x)|^2 w(x)^{2/p}\,\mathrm{d}x \tag{5.4}$$

a weighted gradient semigroup unit bound,

$$\left(\fint_B \rho(x)^2 w(x)^{-2/p}\,\mathrm{d}x\right)\left(\fint_B |\nabla e^{-kr^2L}1_\Omega|^2 w(x)^{2/p}\,\mathrm{d}x\right) \lesssim 1 \tag{5.5}$$

a weighted local $\mathcal{L}^2$ Riesz transform estimate,

$$\||\nabla L^{-1/2}f|\|_{\mathcal{L}^2(B,w^{2/p})} \lesssim \|f\|_{\mathcal{L}^2(\Omega,w^{2/p})} \tag{5.6}$$

a weighted local $\mathcal{L}^2$ Davies-Gaffney estimate (with f supported on A),

$$\|\sqrt{t}|\nabla e^{-tL}f|\|_{\mathcal{L}^2(B,w^{2/p})} \lesssim e^{-d(A,B)^2/ct}\|f\|_{\mathcal{L}^2(A,w^{2/p})} \tag{5.7}$$

and a local weighted $\mathcal{L}^q$ Gaffney estimate (again f supported on A).

$$\||\nabla e^{-tL}f|\|_{\mathcal{L}^q(B,w^{q/p})} \lesssim \frac{e^{-d(A,B)^2/ct}}{\sqrt{t}}\|f\|_{\mathcal{L}^q(A,w^{q/p})} \tag{5.8}$$

The above conditions are not required to hold when considering balls near the boundary

$c_0 r(B) \not\lesssim \rho(B)$. The local Poincaré inequality (5.4) is well known to hold on Ω if $w^{2/p} \in A_2$ (5.3) for balls B away from the boundary $c_0 r(B) < \rho(B)$ (see [41]).

The final condition needed is a weighted Hardy inequality.

$$\|\frac{1}{\rho} L^{-1/2} f\|_{\mathcal{L}^p(w)} \leq \|f\|_{\mathcal{L}^p(w)} \tag{5.9}$$

The main theorem of this chapter is the following.

THEOREM 5.1. *Let Ω be an open subset of $\mathbb{R}^n$ with boundary $\delta\Omega$, and suppose that L is a second order differential operator with well defined functional calculus on Ω. Further suppose that the heat kernel of L on Ω has upper bounds of the form of equation (5.1) for some $\alpha \lesssim 1$ and $\beta \lesssim 1$ where $\alpha_t(x) \leq 2\alpha_{2t}(x)$ and $\beta_t(y) \leq 2\beta_{2t}(y)$ for all $t > 0$ and $x, y \in \Omega$. With the same α and β, let $w \in A_\infty$ be a weight that satisfies $w^{2/p} \in A_2^{\alpha,\beta}$ and $w^{q/p} \in A_R^{\alpha,\beta}$ for some $q > p > 2$ with $R = 1 + \frac{q}{2}$ (see equation 5.2). Considering now only balls B where $c_0 r(B) < \rho(B)$, suppose the general local weighted conditions (5.4), (5.5), (5.6), (5.7) and (5.8) all hold, with the q in condition (5.8) the same as the q already mentioned. Then the Riesz transform $\nabla L^{-1/2}$ satisfies the following weighted bound for all $f \in C_0^\infty$.*

$$\||\nabla L^{-1/2} f|\|_{\mathcal{L}^p(w)} \lesssim \|f\|_{\mathcal{L}^p(w)} + \|\frac{1}{\rho} L^{-1/2} f\|_{\mathcal{L}^p(w)}$$

Corollary 5.2. *In addition to the conditions of Theorem 5.1 suppose that the weighted Hardy inequality (5.9) is bounded $\mathcal{L}^p \to \mathcal{L}^p$ for the same weight w and exponent p. Then the Riesz transform is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for that same weight and all $f \in \mathcal{L}^p(\Omega)$.*

$$\||\nabla L^{-1/2} f|\|_{\mathcal{L}^p(w)} \lesssim \|f\|_{\mathcal{L}^p(w)}$$

Further there is a Sobolev space comparison of the form,

$$\||\nabla f|\|_{\mathcal{L}^p(w)} \lesssim \|L^{1/2} f\|_{\mathcal{L}^p(w)}$$

for all f in $W^{1,p}(w) \cap Dom(L^{1/2})$.

Remarks 5.3. There are three remarks on this theorem. Firstly, in the examples of chapter 10, the condition $w^{q/p} \in A_R^{\alpha,\beta}$ with $R = 1 + \frac{q}{2}$ is the most restrictive part of Theorem 4.2. This restriction could be weakened by using a higher power Poincaré inequality. The weight class $A_p^{\alpha,\beta}$ is studied in detail in the next chapter. For details on how $w^{q/p} \in A_R^{\alpha,\beta}$ restricts the range of weights in this theorem, and other methods to improve the range, see the application in chapter 10.

Secondly, this theorem and proof are based on the work in chapter 4. If instead one tried to extend to weights based on the work in chapter 3, one requirement would be a weighted Poincaré estimate for all balls B in Ω. If that requirement was satisfied, then the extension would work and give suitable weights. However, due to the weighted Poincaré inequality the weight classes that result would be smaller, because the Poincaré inequality only satisfies a weighted bound on balls in $\mathbb{R}^n$ for weights in the Muckenhoupt class A_2 (see chapter 15 of the book by Heinonen et al [41]). By using chapter 4, a more interesting result occurs by avoiding this restriction.

Thirdly, it was remarked in chapter 4 that the bound on the Hardy inequality in that chapter (4.7) implied that $L^{-1/2}$ needed to satisfy Dirichlet boundary conditions. Suppose now that the weight w considered in this chapter vanished on the boundary. Then w vanishing fast enough could still allow a weighted Hardy inequality to hold for at least some p without Dirichlet boundary conditions on $L^{-1/2}$.

5.1 Proof of the Result

The first lemma corresponds to Lemma 3.4 from chapter 3, and Lemmas 4.6 and 4.7 from chapter 4.

Lemma 5.4. *Suppose the conditions of Theorem 5.1 hold. Let B be a ball far enough from the boundary to ensure $N \geq 1$ (where N is defined in Definition 4.5 to ensure the similarity $\rho(2^N B) \sim r(2^N B)$). Then the following bound holds for such B.*

$$\int_B |\nabla e^{-kr^2 L}[1_{2^N B} f_B](x)|^2 w(x)^{2/p}\,\mathrm{d}x \lesssim \int_B \left[\frac{f(x)}{\rho(x)}\right]^2 w(x)^{2/p}\,\mathrm{d}x$$

Here f_B denotes the average $⨍_B f(x)\,\mathrm{d}x$, $1_{2^N B}$ is the characteristic function of $2^N B$, and r is the radius of B.

Proof. Fix B. The value of f_B is constant with respect to x. This means it can be separated from the gradient. Let $C_j = 2^{j+1}B \setminus 2^j B$ for integers $j \geq 1$ and separate out the characteristic function as $1_{2^N B} = 1 - \sum_{j=N}^{\infty} 1_{C_j}$.

$$|\nabla e^{-kr^2 L}[1_{2^N B} f_B(x)| \lesssim f_B |\nabla e^{-kr^2 L} 1_\Omega| + f_B \sum_{j=N}^{\infty} |\nabla e^{-kr^2 L} 1_{C_j}|$$

Apply a weight and integrate the result squared over B.

$$\begin{aligned}\Big(\int_B |\nabla e^{-kr^2 L}[1_{2^N B} f_B]|^2 w(x)^{\frac{2}{p}}\,\mathrm{d}x\Big)^{\frac{1}{2}} \lesssim{}& f_B \Big(\int_B |\nabla e^{-kr^2 L} 1_\Omega|^2 w(x)^{\frac{2}{p}}\,\mathrm{d}x\Big)^{\frac{1}{2}} \\ &+ f_B \sum_{j=N}^{\infty} \Big(\int_B |\nabla e^{-kr^2 L} 1_{C_j}|^2 w(x)^{\frac{2}{p}}\,\mathrm{d}x\Big)^{\frac{1}{2}}\end{aligned} \tag{5.10}$$

For the first term on the right above separate from f_B terms ρ and $\frac{1}{\rho}$ and use Hölder's inequality.

$$f_B^2 \lesssim \frac{1}{|B|^2} \Big(\int_B \rho(x)^2 w(x)^{-2/p}\,\mathrm{d}x\Big)\Big(\int_B \Big[\frac{f(x)}{\rho(x)}\Big]^2 w(x)^{2/p}\,\mathrm{d}x\Big)$$

Using such a bound for f_B along with condition (5.5), the first term on the right of equation (5.10) simplifies.

$$f_B \Big(\int_B |\nabla e^{-kr^2 L} 1_\Omega(x)|^2 w(x)^{2/p}\,\mathrm{d}x\Big)^{1/2} \lesssim \Big(\int_B \Big[\frac{f(x)}{\rho(x)}\Big]^2 w(x)^{2/p}\,\mathrm{d}x\Big)^{1/2}$$

For the second term on the right in equation (5.10) use the same expansion of f_B as used in bounding the first term. Also use that by the choice of N, $\rho(x) \lesssim 2^N r$ holds for all $x \in B$.

$$f_B^2 \lesssim \frac{1}{|B|^2} \Big(\int_B w(x)^{-2/p}\,\mathrm{d}x\Big)\Big(\int_B \Big[\frac{f(x)}{\rho(x)}\Big]^2 w(x)^{2/p}\,\mathrm{d}x\Big) 4^N r^2$$

Recall the equation (5.3) definition of A_p, and let $t \in (1, s)$ be given by $t = 1 + \frac{2s}{ps'}$. Using $w \in A_\infty$, which was a condition in Theorem 4.2, implies $w \in A_s$ for some s large. This further implies $w^{-s'/s} \in A_{s'}$, which implies $w^{-s'/s} \in A_{t'}$, which implies $w^{s't/st'} = w^{2/p} \in A_t$. This implies a local doubling condition on $w^{2/p}$. Use this local doubling along with weighted local $\mathcal{L}^2$ Gaffney estimates.

$$\sum_{j=N}^{\infty}\Big(\int_B |\nabla e^{-kr^2L}1_{C_j}|^2 w(x)^{2/p}\,\mathrm{d}x\Big)^{1/2} \lesssim \sum_{j=N}^{\infty}\frac{e^{-c4^j}}{r}\Big(\int_{C_j} w(x)^{2/p}\,\mathrm{d}x\Big)^{1/2}$$
$$\lesssim \sum_{j=N}^{\infty}\frac{2^{nj/2}e^{-c4^j}}{r}\Big(\int_{B} w(x)^{2/p}\,\mathrm{d}x\Big)^{1/2}$$

Combine with the f_B expansion and use that when considering balls B away from the boundary ($c_0 r(B) < \rho(B)$), then $w^{2/p}$ satisfies an A_2 condition (5.3). This follows from the condition that a local Poincaré inequality (5.4) holds, which requires $w^{2/p}$ to satisfy an A_2 condition on such balls [41]. Note that $w^{2/p}$ is certainly not required to satisfy an A_2 condition (5.3) with balls that do not satisfy $c_0 r(B) < \rho(B)$.

$$f_B\sum_{j=N}^{\infty}\Big(\int_B |\nabla e^{-kr^2L}1_{C_j}|^2 w^{2/p}\,\mathrm{d}x\Big)^{1/2} \lesssim \sum_{j=N}^{\infty} 2^N 2^{nj/2}e^{-c4^j}\Big(\int_B \Big[\frac{f(x)}{\rho(x)}\Big]^2 w(x)^{2/p}\,\mathrm{d}x\Big)^{1/2}$$

The sum is a constant that depends on n but does not depend on N. Add the bounds found for the two parts on the right of equation (5.10) to get,

$$\Big(\int_B |\nabla e^{-kr^2L}[1_{2^N B}f_B]|^2 w(x)^{2/p}\,\mathrm{d}x\Big)^{1/2} \lesssim \Big(\int_B \Big[\frac{f(x)}{\rho(x)}\Big]^2 w(x)^{2/p}\,\mathrm{d}x\Big)^{1/2}$$

which concludes the proof. $\square$

Lemma 5.5. *Suppose again that the various conditions of Theorem 5.1 hold. Let B be a ball far enough from the boundary so that $N \geq 1$ (where N is from Definition 4.5). Suppose there exists a constant $c > 1$ such that cB does not touch the boundary $\delta\Omega$ and cB contains a point x_0 where $M(|\nabla f|^2 w^{2/p})(x_0) \leq \lambda^2$. Then the following bound holds,*

$$\frac{1}{|B|}\int_B \Big|\nabla e^{-kr^2L}[1_{2^N B}\big(f(x) - f_B\big)]\Big|^q w(x)^{q/p}\,\mathrm{d}x \leq \lambda^q$$

for all f smooth enough and some $q > p$. Here M is the uncentred Hardy-Littlewood maximal function, $1_{2^N B}$ is the characteristic function of $2^N B$, and r is the radius of B.

Proof. Let $C_0 = 2B$, and $C_s = 2^{s+1}B \setminus 2^s B$ for $s \geq 1$. Split up the heat semigroup as $e^{-kr^2L}1_{2^N B} = \sum_{s=0}^{\infty}\sum_{m=0}^{N-1} e^{-kr^2L/2}1_{C_s}e^{-kr^2L/2}1_{C_m}$. Then use the local weighted

Gaffney estimate given by equation (5.8) on the first part of the split.

$$\Big(\int_B \Big|\nabla e^{-kr^2L}[1_{2^N B}(f-f_B)]\Big|^q w^{q/p}\,\mathrm{d}x\Big)^{1/q} \\ \lesssim \sum_{s=0}^{\infty}\sum_{m=0}^{N-1} \frac{e^{-c4^s}}{r}\Big(\int_{C_s} \Big|e^{-kr^2L/2}\big[1_{C_m}(f-f_B)\big]\Big|^q w^{q/p}\,\mathrm{d}x\Big)^{1/q} \tag{5.11}$$

Use Hölder's inequality and the upper bounds for the heat kernel from condition (5.1) to get the following for $x \in C_s$.

$$\big|e^{-kr^2L/2}\big[1_{C_m}(f-f_B)\big](x)\big|^q \\ \lesssim \frac{\alpha_{\frac{kr^2}{2}}(x)^q e^{-c4^{|s-m|}}}{r^{qn}} \left(\int_{C_m} \big(f(y)-f_B\big)^2 w(y)^{\frac{2}{p}}\,\mathrm{d}y\right)^{\frac{q}{2}} \left(\int_{C_m} \beta_{\frac{kr^2}{2}}(y)^2 w(y)^{-\frac{2}{p}}\,\mathrm{d}y\right)^{\frac{q}{2}}$$

Integrate this over all $x \in C_s$, and separate the parts.

$$\int_{C_s} \Big|e^{-kr^2L/2}\big[1_{C_m}(f-f_B)\big]\Big|^q w(x)^{q/p}\,\mathrm{d}x \\ \lesssim \frac{e^{-c4^{|s-m|}}}{r^{nq}} \left(\int_{C_m} \big(f-f_B\big)^2 w^{\frac{2}{p}}\,\mathrm{d}y\right)^{\frac{q}{2}} \left(\int_{C_s} \alpha^q_{\frac{kr^2}{2}} w^{\frac{q}{p}}\,\mathrm{d}x\right) \left(\int_{C_m} \beta^2_{\frac{kr^2}{2}} w^{-\frac{2}{p}}\,\mathrm{d}y\right)^{\frac{q}{2}}$$

The integrals over $\alpha^q w^{q/p}$ and $\beta w^{-2/p}$ will vanish due to the condition from Theorem 5.1 that states $w^{q/p} \in A_R^{\alpha,\beta}$ with $R = 1 + \frac{q}{2}$ and $A_R^{\alpha,\beta}$ formed according to equation (5.2). This is managed as follows. First define l by the equation $2^l = 2\max(2^s, 2^m)$. Then use local doubling and the condition that $\alpha_t(x) \le 2\alpha_{2t}(x)$ and $\beta_t(x) \le 2\beta_{2t}(x)$ for all $t > 0$ and $x \in \Omega$. Further $\alpha \lesssim 1$ and $\beta \lesssim 1$ implies $\alpha^q \lesssim \alpha^R$ and $\beta^2 \lesssim \beta^{R'}$ as $R = 1 + \frac{q}{2}$ implies $2 < R < q$. Finish this bound with the aforementioned condition that $w^{q/p} \in A_R^{\alpha,\beta}$.

$$\frac{1}{r^{nq}} \left(\int_{C_s} \alpha_{\frac{kr^2}{2}}(x)^q w(x)^{\frac{q}{p}}\,\mathrm{d}x\right) \left(\int_{C_m} \beta_{\frac{kr^2}{2}}(y)^2 w(y)^{-\frac{2}{p}}\,\mathrm{d}y\right)^{\frac{q}{2}} \\ \lesssim \frac{2^{l(n+2)(q+2)/2}}{r^{n(q-2)/2}} \left(\fint_{2^l\sqrt{k}B} \alpha^R_{4^l kr^2} w(x)^{\frac{q}{p}}\,\mathrm{d}x\right) \left(\fint_{2^l\sqrt{k}B} \beta^{R'}_{4^l kr^2} w(x)^{-\frac{2}{p}}\,\mathrm{d}y\right)^{\frac{q}{2}} \\ \lesssim \frac{2^{l(n+2)(q+2)/2}}{r^{n(q-2)/2}}$$

Put this information back into equation (5.11) after removing dependence on l.

$$\Big(\int_B \Big|\nabla e^{-kr^2L}[1_{2^N B}(f(x)-f_B)]\Big|^q w(x)^{q/p}\,\mathrm{d}x\Big)^{1/q} \lesssim \sum_{s=0}^{\infty}\sum_{m=0}^{N-1} \frac{2^{(n+2)(s+m)}e^{-c4^s}e^{-c4^{|s-m|}}}{r^{(2q+nq-2n)/2q}} \Big[\int_{C_m} \big(f(x)-f_B\big)^2 w(x)^{2/p}\,\mathrm{d}x\Big]^{1/2} \tag{5.12}$$

Next the weighted version of the local Poincaré's inequality (5.4) is used. First split up the difference term.

$$\begin{aligned}|f(x)-f_B| &= |f(x)-f_{2^{m+1}B}+\sum_{l=0}^{m}(f_{2^{l+1}B}-f_{2^l B})| \\ &\lesssim |f(x)-f_{2^{m+1}B}| + \sum_{l=0}^{m}\frac{1}{|2^l B|}\int_{2^l B}|f(y)-f_{2^{l+1}B}|\,\mathrm{d}y\end{aligned}$$

Take $\mathcal{L}^2$ norms on the space C_m on both sides of the inequality above. Then apply Minkowski's inequality.

$$\begin{aligned}&\Big[\int_{C_m}|f(x)-f_B|^2 w(x)^{2/p}\,\mathrm{d}x\Big]^{1/2} \\ &\quad\le \Big[\int_{2^{m+1}B}|f(x)-f_{2^{m+1}B}|^2 w(x)^{2/p}\,\mathrm{d}x\Big]^{1/2} \\ &\qquad + 2^n\sum_{l=0}^{m}\frac{|2^{m+1}B|^{1/2}}{|2^{l+1}B|^{1/2}}\Big[\int_{2^{l+1}B}|f(x)-f_{2^{l+1}B}|^2 w(x)^{2/p}\,\mathrm{d}x\Big]^{1/2}\end{aligned}$$

Then apply the weighted version of Poincaré's inequality (5.4) to all parts. Observe that due to the choice of N, and as $l < m+1 \le N$, the weighted Poincaré inequality is only applied to balls away from the boundary $r(2^{l+1}B) \lesssim \rho(2^{l+1}B)$.

$$\Big[\int_{C_m}|f(x)-f_B|^2 w(x)^{2/p}\,\mathrm{d}x\Big]^{\frac{1}{2}} \le 2^n\sum_{l=0}^{m+1}\frac{2^l r|2^{m+1}B|^{\frac{1}{2}}}{|2^{l+1}B|^{\frac{1}{2}}}\Big[\int_{2^{l+1}B}|\nabla f|^2 w(x)^{2/p}\,\mathrm{d}x\Big]^{\frac{1}{2}}$$

Insert this back into equation (5.12).

$$\begin{aligned}&\Big(\int_B \Big|\nabla e^{-kr^2L}[1_{2^N B}(f(x)-f_B)]\Big|^q w(x)^{q/p}\,\mathrm{d}x\Big)^{1/q} \\ &\lesssim \sum_{s=0}^{\infty}\sum_{m=0}^{N-1}\frac{2^{(n+2)(s+m)}e^{-c4^s}e^{-c4^{|s-m|}}}{r^{(nq-2n+2q)/2q}}\sum_{l=0}^{m+1}\frac{2^l r|2^{m+1}B|^{\frac{q}{2}}}{|2^{l+1}B|^{\frac{1}{2}}}\Big[\int_{2^{l+1}B}|\nabla f|^2 w(x)^{2/p}\,\mathrm{d}x\Big]^{\frac{1}{2}}\end{aligned}$$

Recall that cB contains a point x_0 where $M(|\nabla f|^2 w^{2/p})(x_0) < \lambda^2$ (supposed in the lemma). Further, for the balls of size $2B$ and larger up to cB, their size can be increased to cB for the same effect (with appropriate changes of the weightings compensated for by local doubling condition (4.1), balls up to size cB do not touch the boundary of Ω).

$$\Big(\int_B \Big|\nabla e^{-kr^2L}[1_{2^N B}\big(f(x)-f_B\big)]\Big|^q w(x)^{q/p}\,\mathrm{d}x\Big)^{1/q}$$
$$\lesssim \sum_{s=0}^{\infty}\sum_{m=0}^{N-1} \frac{2^{(n+2)(s+m)}e^{-c4^s}e^{-c4^{|s-m|}}}{r^{(nq-2n)/2q}} \sum_{l=0}^{m+1} 2^l |2^{m+1}B|^{\frac{1}{2}}\big(M(|\nabla f|^2 w^{2/p})(x_0)\big)^{\frac{1}{2}}$$

As discussed above, here $M(|\nabla f|^2 w^{2/p})(x_0) < \lambda^2$. Use also that $l < m+1$ and local doubling on $|2^{m+1}B|$ to simplify.

$$\Big(\int_B \Big|\nabla e^{-kr^2L}[1_{2^N B}\big(f(x)-f_B\big)]\Big|^q w(x)^{q/p}\,\mathrm{d}x\Big)^{1/q}$$
$$\lesssim \lambda r^{n/q} \sum_{s=0}^{\infty}\sum_{m=0}^{N-1} 2^{(n+2)(s+m)}e^{-c4^s}e^{-c4^{|s-m|}}2^{(m+1)(1+\frac{n}{2})}$$

These summations converge to a constant (depending only on n) leaving $\lambda|B|^{1/q}$ on the right side of the equation. This concludes the proof. □

Lemma 5.6. *Suppose that $\nabla L^{-1/2}$ is an appropriately defined operator on a doubling space Ω satisfying the various conditions of Theorem 5.1 (in particular the $\mathcal{L}^2$ Riesz transform bound (5.6) and $\mathcal{L}^2$ Davies-Gaffney estimates (5.7) are needed here). Then the following bound holds,*

$$\frac{1}{|B|}\int_B |\nabla L^{-1/2}(I-e^{-r^2L})^n f(y)|^2 w(y)^{2/p}\,\mathrm{d}y \lesssim M(|f|^2 w^{2/p})(x)$$

for all $x \in B$ away from the boundary of Ω ($c_0 r(B) < \rho(B)$). Here M is again the uncentred Hardy-Littlewood maximal function, and r the radius of B.

Proof. Once again let $C_0 = 2B$ and $C_j = 2^{j+1}B \setminus 2^j B$.

$$\Big(\fint_B |\nabla L^{-1/2}(I-e^{-r^2L})^n f(y)|^2 w(y)^{2/p}\,\mathrm{d}y\Big)^{1/2}$$
$$\lesssim \sum_{j=0}^{\infty}\Big(\fint_B |\nabla L^{-1/2}(I-e^{-r^2L})^n [1_{C_j} f](y)|^2 w(y)^{2/p}\,\mathrm{d}y\Big)^{1/2}$$

When $j = 0$ use that $\nabla L^{-1/2}$ is locally $\mathcal{L}^2(w^{2/p})$ bounded (5.6).

$$\Big(\fint_B |\nabla L^{-1/2}(I - e^{-r^2L})^n[1_{2B}f](y)|^2 w(y)^{2/p}\,\mathrm{d}y\Big)^{1/2} \lesssim \sum_{m=0}^{\infty}\Big(\int_{C_m}\big[(I - e^{-r^2L})^n[1_{2B}f](y)\big]^2 w(y)^{2/p}\,\mathrm{d}y\Big)^{1/2} \tag{5.13}$$

Regarding the term $(I - e^{-r^2L})^n$ use the heat kernel upper bound (5.1), Hölder's inequality and local doubling. Also use the conditions that $\alpha_t(x) \le 2\alpha_{2t}(x)$, $\beta_t(x) \le 2\beta_{2t}(x)$ and $w^{2/p} \in A_2^{\alpha,\beta}$. Then the following holds for any $k \in [1, n]$ and $x \in B$.

$$\begin{aligned}
&\Big(\int_{C_m} \big|e^{-kr^2L}[1_{2B}f](y)\big|^2 w(y)^{2/p}\,\mathrm{d}y\Big)^{1/2} \\
&\lesssim 2^{mn/2}e^{-c4^m}\Big(\fint_{C_m}\alpha_{kr^2}^2 w^{2/p}\,\mathrm{d}y\Big)^{\frac12}\Big(\fint_{2B}\beta_{kr^2}^2 w^{-2/p}\,\mathrm{d}y\Big)^{\frac12}\Big(\int_{2B} f^2 w^{2/p}\,\mathrm{d}y\Big)^{\frac12} \\
&\lesssim 2^{m(n+4)}e^{-c4^m}\Big(\fint_{2^m\sqrt{k}B}\alpha_{4^mkr^2}^2 w^{2/p}\,\mathrm{d}y\Big)^{\frac12}\Big(\fint_{2^m\sqrt{k}B}\beta_{4^mkr^2}^2 w^{-2/p}\,\mathrm{d}y\Big)^{\frac12} M\big(|f|^2w^{2/p}\big)^{\frac12} \\
&\lesssim 2^{m(n+4)}e^{-c4^m}\big[M(|f|^2w^{2/p})(x)\big]^{\frac12}
\end{aligned}$$

Hence $(1 - e^{r^2L})^n$ is $\mathcal{L}^2(w^{2/p})$ bounded. Put this into equation (5.13), sum over m and use local doubling.

$$\Big(\fint_B |\nabla L^{-1/2}(I - e^{-r^2L})^n[1_{2B}f](y)|^2 w(y)^{2/p}\,\mathrm{d}y\Big)^{1/2} \lesssim \Big(\fint_{2B} f(y)^2 w(y)^{2/p}\,\mathrm{d}y\Big)^{1/2}$$

Thus the lemma is true when $j = 0$. For $j \ge 1$ expand the term $(1 - e^{-r^2L})^n$.

$$|\nabla L^{-1/2}(I - e^{-r^2L})^n[1_{C_j}f](y)| = |\nabla L^{-1/2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}e^{-kr^2L}[1_{C_j}f](y)|$$

Expand to the integral representation of $L^{-1/2}$ and use a change of variables.

$$\begin{aligned}
|\nabla L^{-1/2}(I - e^{-r^2L})^n[1_{C_j}f](y)| &\lesssim \int_0^\infty \sum_{k=0}^{n}(-1)^k\binom{n}{k}|\nabla e^{-(t+kr^2)L}[1_{C_j}f](y)|\frac{\mathrm{d}t}{\sqrt{t}} \\
&\lesssim \int_0^\infty \Big[\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k 1_{\{s>kr^2\}}(s)}{\sqrt{s - kr^2}}\Big]|\nabla e^{-sL}[1_{C_j}f](y)|\,\mathrm{d}s
\end{aligned}$$

Square and integrate over B, then use Minkowski's integral theorem and the weighted local $\mathcal{L}^2$ Gaffney estimates (5.8).

$$\int_B |\nabla L^{-1/2}(I-e^{-r^2L})^n[1_{C_j}f](y)|^2 w(y)^{2/p}\,\mathrm{d}y \\ \lesssim \Big|\int_0^\infty \Big[\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k 1_{\{s>kr^2\}}(s)}{\sqrt{s-kr^2}}\Big]\frac{e^{-4^j r^2/s}}{\sqrt{s}}\,\mathrm{d}s\Big|^2 \int_{C_j}|f(y)|^2 w(y)^{2/p}\,\mathrm{d}y \tag{5.14}$$

The integral with respect to s is evaluated in several parts. This same integral is evaluated in detail in Lemma 3.6 so see that proof for the details. Place the integral value back into equation (5.14).

$$\int_B |\nabla L^{-1/2}(I-e^{-r^2L})^n[1_{C_j}f](y)|^2 w(y)^{2/p}\,\mathrm{d}y \lesssim \frac{1}{4^{2jn}}\int_{C_j}|f(y)|^2 w(y)^{2/p}\,\mathrm{d}y$$

Use the local doubling principle and sum over j for the result. □

Now conditions are ready for the proof of the main theorem of this chapter.

Proof of Theorem 5.1. The proof splits the Riesz transform into parts, then finds a bound for each part. The proof is very similar to the proof of the main theorem in chapter 4. Given the integral representation of the $\mathcal{L}^p$ norm;

$$\||\nabla L^{-1/2}f|\|_{\mathcal{L}^p(\Omega,w)} = p\int_0^\infty \lambda^{p-1}|\{x\in\Omega : |\nabla L^{-1/2}f(x)|w(x)^{1/p} > \lambda\}|\,\mathrm{d}\lambda, \tag{5.15}$$

split up the set $E=\{x\in\Omega : M(|\nabla L^{-1/2}f|^2 w^{2/p})(x) > \lambda^2\}$ by a Whitney type covering lemma. This is arranged so that:

1. There exists a constant $c_1 < 1$ such that the collection of balls $\{c_1 B_i\}_{i\in I}$ for the covering of E is pairwise disjoint;
2. The collection $\{B_i\}_{i\in I}$ of balls in the Whitney covering of E satisfy $\cup_{i\in I}B_i = E$; and,
3. There exists a constant $c_2 > 1$ such that for every ball B_i in the covering of E $\min\big(d(B_i,\delta E),\rho(B_i)\big)\sim c_2 r(B_i)$.

The constant c_1 in the covering lemma depends only on the dimension n. The c_2 is chosen in part to match with the localised conditions and Definition 4.5. This is the second type of decomposition discussed in section 2.3. Importantly none of the balls in the covering touch the boundary of Ω. Equation (5.15) is replaced by the following.

$$\||\nabla L^{-1/2}f|\|_{\mathcal{L}^p(\Omega,w)} \leq p\int_0^\infty \lambda^{p-1}\sum_{i\in I}|\{x\in B_i : |\nabla L^{-1/2}f(x)|w(x)^{1/p} > \lambda\}|\,\mathrm{d}\lambda$$

Scale λ by some $K > 3$ to be chosen later.

$$\||\nabla L^{-1/2}f|\|_{\mathcal{L}^p(\Omega,w)} \leq pK^p\int_0^\infty \lambda^{p-1}\sum_{i\in I}|\{x\in B_i : |\nabla L^{-1/2}f(x)|w(x)^{1/p} > K\lambda\}|\,\mathrm{d}\lambda$$

Define the operator A_r as,

$$A_r = I - (I - e^{-r^2L})^n = \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}e^{-kr^2L} \tag{5.16}$$

where r is the radius of B_i. Split $\nabla L^{-1/2} = \nabla L^{-1/2}A_r + \nabla L^{-1/2}(1 - A_r)$.

$$\begin{aligned}
&\||\nabla L^{-1/2}f|\|_{\mathcal{L}^p(\Omega,w)} \\
&\quad\leq pK^p\int_0^\infty \lambda^{p-1}\sum_{i\in I}\Big[|\{x\in B_i : |\nabla L^{-1/2}A_rf(x)|w(x)^{1/p} > (K-1)\lambda\}| \\
&\qquad\qquad + |\{x\in B_i : |\nabla L^{-1/2}(I-A_r)f(x)|w(x)^{1/p} > \lambda\}|\Big]\,\mathrm{d}\lambda
\end{aligned}$$

Split the covering $\{B_i\}_{i\in I}$ into two sets as in the proof of chapter 4. This is aligned with Definition 4.5 where an exponent N is outlined for each B_i such that $\rho(2^NB_i) \sim r(2^NB_i)$. The first set I_1 in the split consists of those balls B_i that satisfy $\rho(B_i) \sim c_2r(B_i)$ so that $N = 0$. These balls are closer to $\delta\Omega$ then to δE. The second set I_2 are the remaining balls B_i (with $N \geq 1$), which due to the nature of the covering satisfy $c_2r(B_i) \sim d(B_i, \delta E)$, meaning they are closer to δE then to $\delta\Omega$. If E does not touch $\delta\Omega$ then the set I_1 is empty, but that has no impact on the proof. In parts of the proof I_1 and I_2 will be dealt with separately. Let $C_0 = 2B_i$ and $C_j = 2^{j+1}B_i \setminus 2^jB_i$ for $j \geq 1$. Then split up the sets further,

$$
\begin{aligned}
&\||\nabla L^{-1/2}f|\|_{\mathcal{L}^p(\Omega,w)} \\
&\quad\leq pK^p\int_0^\infty \lambda^{p-1}\Big[\sum_{i\in I}|\{x\in B_i:\sum_{j=N}^{\infty}|\nabla A_r 1_{C_j}L^{-1/2}f(x)|w(x)^{1/p}>\lambda\}| \\
&\quad+\sum_{i\in I_2}|\{x\in B_i:|\nabla A_r 1_{2^N B_i}L^{-1/2}f(x)|w(x)^{1/p}>(K-2)\lambda\}| \\
&\quad+\sum_{i\in I}|\{x\in B_i:|\nabla L^{-1/2}(I-A_r)f(x)|w(x)^{1/p}>\lambda\}|\Big]\,\mathrm{d}\lambda
\end{aligned}
$$

where the second set above contains only terms in I_2 (these are the balls closer to δE, they have $N\geq 1$). One more split needs to take place in the second set above.

$$
\begin{aligned}
&\||\nabla L^{-1/2}f|\|_{\mathcal{L}^p(\Omega,w)} \\
&\quad\leq pK^p\int_0^\infty \lambda^{p-1}\Big[\sum_{i\in I}|\{x\in B_i:\sum_{j=N}^{\infty}|\nabla A_r 1_{C_j}L^{-1/2}f(x)|w(x)^{1/p}>\lambda\}| \\
&\quad+\sum_{i\in I_2}|\{x\in B_i:|\nabla A_r 1_{2^N B_i}[L^{-1/2}f]_{B_i}|w(x)^{1/p}>\lambda\}| \\
&\quad+\sum_{i\in I_2}|\{x\in B_i:|\nabla A_r 1_{2^N B_i}(L^{-1/2}f(x)-[L^{-1/2}f]_{B_i})|w(x)^{1/p}>(K-3)\lambda\}| \\
&\quad+\sum_{i\in I}|\{x\in B_i:|\nabla L^{-1/2}(I-A_r)f(x)|w(x)^{1/p}>\lambda\}|\Big]\,\mathrm{d}\lambda \qquad (5.17)
\end{aligned}
$$

Each of these sets will be bound separately. Consider first the set,

$$S_1=\{x\in B_i:\sum_{j=N}^{\infty}|\nabla A_r 1_{C_j}L^{-1/2}f(x)|w(x)^{1/p}>\lambda\}$$

and whether the equation given by,

$$\sum_{j=N}^{\infty}\fint_{B_i}|\nabla A_r 1_{C_j}L^{-1/2}f(x)|^2 w(x)^{2/p}\,\mathrm{d}x>\delta^2\lambda^2 \qquad (5.18)$$

is true or false. If it is false apply a weak $(2,2)$ inequality to S_1 to get,

$$|S_1|\lesssim\frac{1}{\lambda^2}\sum_{j=N}^{\infty}\int_{B_i}|\nabla A_r 1_{C_j}L^{-1/2}f(x)|^2 w(x)^{2/p}\,\mathrm{d}x\lesssim\delta^2|B_i|$$

however if equation (5.18) is true then use the definition of A_r and weighted Gaffney estimates (5.7).

$$
\begin{aligned}
|S_1| &\leq |\{x \in B_i : \sum_{j=N}^{\infty} \fint_{B_i} |\nabla A_r 1_{C_j} L^{-1/2} f(x)|^2 w(x)^{2/p} \,\mathrm{d}x > \delta^2 \lambda^2\}| \\
&\leq |\{x \in B_i : c_n \sum_{j=N}^{\infty} \frac{e^{-4^j}}{r^2} \frac{1}{|B_i|} \int_{C_j} |L^{-1/2} f(x)|^2 w(x)^{2/p} \,\mathrm{d}x > \delta^2 \lambda^2\}|
\end{aligned}
$$

The ball B was chosen to satisfy a local doubling condition and N was chosen so that $r(2^N B) \sim \rho(2^N B)$ which implies $2^{-j}\rho(x) \lesssim r$ for all $x \in 2^{j+1}B$, as $j \geq N$. Use this before evaluating the sum.

$$
\begin{aligned}
|S_1| &\leq |\{x \in B_i : c_n \sum_{j=N}^{\infty} 2^{j(n+2)} e^{-4^j} \frac{1}{|2^{j+1} B_i|} \int_{2^{j+1} B_i} \frac{|L^{-1/2} f(x)|^2}{\rho(x)^2} w(x)^{2/p} \,\mathrm{d}x > \delta^2 \lambda^2\}| \\
&\leq |\{x \in B_i : c_n M\Big(\frac{|L^{-1/2} f|^2}{\rho^2} w^{2/p}\Big)(x) > \delta^2 \lambda^2\}|
\end{aligned}
$$

This leads to a bound for the first set S_1 in (5.17) which holds regardless of whether equation (5.18) is true or false.

$$
|S_1| \lesssim |\{x \in B_i : c_n M\Big(\frac{|L^{-1/2} f|^2}{\rho^2} w^{2/p}\Big)(x) > \delta^2 \lambda^2\}| + \delta^2 |B_i| \tag{5.19}
$$

Next treat the second set from equation (5.17). This set is given by the following equation.

$$
S_2 = \{x \in B_i : |\nabla A_r 1_{2^N B_i} [L^{-1/2} f]_{B_i}| w(x)^{1/p} > \lambda\}
$$

Consider whether the equation given by,

$$
\fint_{B_i} |\nabla A_r 1_{2^N B_i} [L^{-1/2} f]_{B_i}|^2 w(x)^{2/p} \,\mathrm{d}x > \delta^2 \lambda^2 \tag{5.20}
$$

is true or false. If false use a weak type $(2,2)$ inequality to get,

$$
|S_2| \lesssim \frac{1}{\lambda^2} \int_{B_i} |\nabla A_r 1_{2^N B_i} [L^{-1/2} f]_{B_i}|^2 w(x)^{2/p} \,\mathrm{d}x \lesssim \delta^2 |B_i|
$$

however if equation 5.20 is true use Lemma 5.4 to get the following.

$$\begin{aligned}|S_2| &\leq |\{x \in B_i : \fint_{B_i} |\nabla A_r 1_{2^N B_i}[L^{-1/2}f]_{B_i}|^2 w(x)^{2/p}\,\mathrm{d}x > \delta^2\lambda^2\}| \\ &\leq |\{x \in B_i : c_n \fint_{B_i} \frac{|L^{-1/2}f(x)|^2}{\rho(x)^2} w(x)^{2/p}\,\mathrm{d}x > \delta^2\lambda^2\}|\end{aligned}$$

So in total in this case, whether or not (5.20) is true or false, the following bounds the second set S_2 of equation (5.17).

$$|S_2| \lesssim |\{x \in B_i : c_n M\Big(\frac{|L^{-1/2}f|^2}{\rho^2} w^{2/p}\Big)(x) > \delta^2\lambda^2\}| + \delta^2|B_i| \tag{5.21}$$

The third set from equation (5.17) is next. This set is given by,

$$S_3 = \{x \in B_i : |\nabla A_r 1_{2^N B_i}(L^{-1/2}f(x) - [L^{-1/2}f]_{B_i})|w(x)^{1/p} > (K-3)\lambda\}$$

and is valid for each B_i with $i \in I_2$. These are the balls away from the boundary of Ω where $cB \cap E^c \neq \emptyset$ for some constant c. Use a weak type (q, q) inequality and Lemma 5.5.

$$|S_3| \lesssim \frac{1}{(K-3)^q\lambda^q} \int_B |\nabla A_r 1_{2^N B_i}(L^{-1/2}f(x) - [L^{-1/2}f]_{B_i})|^q w(x)^{q/p}\,\mathrm{d}x \lesssim \frac{|B_i|}{(K-3)^q} \tag{5.22}$$

It remains to find a bound for the fourth part of equation (5.17).

$$S_4 = |\{x \in B_i : |\nabla L^{-1/2}(I - A_r)f(x)|w(x)^{1/p} > \lambda\}|$$

Consider then whether the inequality given by,

$$\fint_{B_i} |\nabla L^{-1/2}(I - A_r)f(x)|^2 w(x)^{2/p}\,\mathrm{d}x > \delta^2\lambda^2 \tag{5.23}$$

is true or false. If equation (5.23) is false, then use a weak $(2, 2)$ inequality.

$$|S_4| \lesssim \frac{1}{\lambda^2} \int_{B_i} |\nabla L^{-1/2}(I - A_r)f(x)|^2\,\mathrm{d}x \lesssim \delta^2|B_i|$$

If however equation (5.23) is true then use Lemma 5.6.

$$\begin{aligned}
|S_4| &\le |\{x \in B_i : \fint_{B_i} |\nabla L^{-1/2}(I - A_r) f(x)|^2 w(x)^{2/p} \, \mathrm{d}x > \delta^2 \lambda^2\}| \\
&\le |\{x \in B_i : c_n M(|f|^2 w^{2/p})(x) > \delta^2 \lambda^2\}|
\end{aligned}$$

Thus in total, the following bound holds regardless of whether equation (5.23) is true or false.

$$|S_4| \lesssim |\{x \in B_i : c_n M(|f|^2 w^{2/p})(x) > \delta^2 \lambda^2\}| + \delta^2 |B_i| \tag{5.24}$$

Combine all cases together from above and put back into equation (5.17).

$$\begin{aligned}
|||\nabla L^{-1/2} f|||_{\mathcal{L}^p(w)} \lesssim pK^p \int_0^\infty \lambda^{p-1} \Big[&\sum_{i \in I} 3\delta^2 |B_i| + \frac{1}{(K-3)^q} |B_i| \\
&+ \sum_{i \in I} |\{x \in B_i : c_n M\Big(\frac{|L^{-1/2} f|^2}{\rho^2} w^{2/p}\Big)(x) > \delta^2 \lambda^2\}| \\
&+ \sum_{i \in I} |\{x \in B_i : c_n M(|f|^2 w^{2/p})(x) > \delta^2 \lambda^2\}| \Big] \, \mathrm{d}\lambda
\end{aligned}$$

Use that B_i cover the set E, with finite intersection by finite intersection lemma.

$$\begin{aligned}
&|||\nabla L^{-1/2} f|||_{\mathcal{L}^p(w)} \\
&\lesssim pK^p \int_0^\infty \lambda^{p-1} \Big[\Big(3\delta^2 + \frac{1}{(K-3)^q}\Big) |\{x \in \Omega : M(|\nabla L^{-1/2} f|^2 w^{2/p}) > \lambda^2\}| \\
&\quad + |\{x \in \Omega : c_n M\Big(\frac{|L^{-1/2} f|^2}{\rho^2} w^{2/p}\Big)(x) > \delta^2 \lambda^2\}| \\
&\quad + |\{x \in \Omega : c_n M(|f|^2 w^{2/p})(x) > \delta^2 \lambda^2\}| \Big] \, \mathrm{d}\lambda
\end{aligned}$$

Change out of the integral representations, and use that the maximal function is bounded $\mathcal{L}^p \to \mathcal{L}^p$ for all $p > 2$ without weight.

$$\begin{aligned}
|||\nabla L^{-1/2} f|||^p_{\mathcal{L}^p(w)} \lesssim &\Big(3K^p \delta^2 + \frac{K^p}{(K-3)^q}\Big) |||\nabla L^{-1/2} f|||^p_{\mathcal{L}^p(w)} \\
&+ \frac{K^p}{\delta^p} \|f\|^p_{\mathcal{L}^p(w)} + \frac{K^p}{\delta^p} \Big\| \frac{L^{-1/2} f}{\rho} \Big\|^p_{\mathcal{L}^p(w)}
\end{aligned}$$

Chose δ small and K large for the result. $\square$

This concludes the proof of Theorem 5.1, and as a consequence concludes the proof of Theorem 1.3 associated from the introduction chapter. This also concludes the general proofs of Riesz transform boundedness in this thesis. The next chapter provides tools to find weight classes needed for the conditions of this chapter. Later chapters apply Theorem 5.1 to cases involving the Dirichlet Laplacian.

Chapter 6:

Weighted Maximal Functions on Domains

In this chapter Theorem 1.4 is proven. Theorem 1.4 gives weights for the $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ boundedness of maximal type functions. Theorem 1.4 is restated as Theorem 6.5 below. The ideas of this chapter are based around variations on the traditional Hardy-Littlewood maximal function,

$$Mf(x) = \sup_{B \ni x} \fint_B |f(y)| \,\mathrm{d}y$$

which is known to be bounded $\mathcal{L}^p(\mu) \to \mathcal{L}^p(\mu)$ if and only if $\mathrm{d}\mu = w(x)\,\mathrm{d}x$ where w is in the Muckenhoupt A_p class (see Muckenhoupt [52] for the original idea or Stein [67] for an overview). The same result holds for the traditional Riesz transform. The motivation is from chapter 5 where weighted bounds for Riesz transform variations were developed through weight bounds for other operators.

For simplicity in this chapter it is presumed $\Omega \subset \mathbb{R}^n$. There is also a doubling condition in this chapter, operators on non-doubling spaces need to be extended onto doubling spaces for this chapter to be applied. The variations on the Maximal function and subsequent operators are controlled by the following.

Conditions 6.1. *Consider $\alpha_{r^2}(x)$ and $\beta_{r^2}(x)$ defined continuous and positive for all $x \in \Omega$ and $r > 0$. There are four conditions. Firstly: for every $p > 1$ an exponent $\gamma \in (1,p)$ is required such that the following holds for all balls $B \subset \Omega$, with $A = (p-\gamma)/\gamma$.*

$$\left(\fint_B \alpha_{r^2}(x)^{-A\gamma} \,\mathrm{d}x\right)^{1/\gamma} \left(\fint_B \beta_{r^2}(x)^{-A\gamma'} \,\mathrm{d}x\right)^{1/\gamma'} \lesssim 1 \tag{6.1a}$$

Usually but not necessarily γ is chosen close to p so that $A\gamma$ and $A\gamma'$ are small. The exponents γ and γ' are conjugates, and r is the radius of B.

The second condition is also required to hold for all balls $B \subset \Omega$, with r again the radius of B.

$$\left(\fint_B \alpha_{r^2}(x)\,\mathrm{d}x\right)\left(\fint_B \beta_{r^2}(x)\,\mathrm{d}x\right) \lesssim 1 \tag{6.1b}$$

The third condition follows: it is required for there to exist a constant $m > 0$ such that for every ball $B \subset \Omega$, and every $R > r$ (r is the radius of B), for there to exists values $c_{B,R}$ and $C_{B,R}$ constant with respect to $x \in B$ where the following holds for all $x \in B$.

$$\alpha_{r^2}(x) \sim (c_{B,R})\,\alpha_{R^2}(x) \text{ and } \beta_{r^2}(x) \sim (C_{B,R})\,\beta_{R^2}(x) \tag{6.1c}$$

Further the bounds $\left(\frac{r}{R}\right)^m < c_{B,R} < \left(\frac{R}{r}\right)^m$ and $0 < C_{B,R} < \left(\frac{R}{r}\right)^m$ must hold.

The final condition is: the set I of all balls $B \subset \Omega$ can be broken up into a finite number of disjoint sets I_i where for each I_i there exists functions $a_i(x)$ and $z_i(r)$ where,

$$\alpha_{r^2}(x) \sim a_i(x) z_i(r) \tag{6.1d}$$

holds for all $x \in 5B$ and $B \in I_i$. This final condition only applies to α. As usual, r is the radius of B.

The notation $f(x) \sim g(x)$ for two functions f and g means that there exists strictly positive constants c_1 and c_2 such that $f(x) \leq c_1 g(x)$ and $g(x) \leq c_2 f(x)$ for all x. This notation is used repeatedly in this chapter. With functions $\alpha_{r^2}(x)$ and $\beta_{r^2}(x)$ satisfying the conditions above consider the following weight class.

Definition 6.2. The class of $A_p^{\alpha,\beta}$ *adjusted Muckenhoupt weights* is given by the set of weights w such that,

$$A_p^{\alpha,\beta}(w) \stackrel{def}{=} \sup_{B\subset\Omega}\left(\fint_B \alpha_{r^2}(x)^p w(x)\,\mathrm{d}x\right)\left(\fint_B \beta_{r^2}(x)^{p'} w(x)^{-p'/p}\,\mathrm{d}x\right)^{p/p'} < \infty$$

where the value $A_p^{\alpha,\beta}(w)$ is known as the $A_p^{\alpha,\beta}$ constant of w and r is the radius of B.

Essentially the traditional Muckenhoupt weight class has been skewed by α and β. Properties of the weight class $A_p^{\alpha,\beta}$ are considered in section 6.1. If α and β are bounded above and below by positive constants, then $A_p^{\alpha,\beta}$ is equivalent to A_p. Further if $1 \lesssim \alpha, \beta$, then the inclusion $A_p^{\alpha,\beta} \subset A_p$ applies. The opposite inclusion occurs if $\alpha, \beta \lesssim 1$. If instead there is a relation $\alpha \sim \beta^{-1}$, then $A_p^{\alpha,\beta}$ is a direct scaling of the traditional A_p class.

Example 6.3. In the application section of this thesis one such α, β pair considered is given by $\alpha_{r^2}(x) = \beta_{r^2}(x) = \left(1 \wedge \frac{\rho(x)}{r}\right)$ where $\rho(x)$ is the minimal distance from x to the boundary of Ω and $\wedge$ refers to the minimum of the terms on either side. It is shown in chapter 8 that this α, β pair satisfy all of Conditions 6.1.

The following are the three main theorems of this chapter.

THEOREM 6.4. *Suppose $\alpha_{r^2}(x)$ and $\beta_{r^2}(x)$ satisfy all of Conditions 6.1 on the doubling space Ω. Then the maximal function,*

$$M^{\alpha,\beta} f(x) = \sup_{B \ni x} \frac{\alpha_{r^2}(x)}{|B|} \int_B \beta_{r^2}(y)|f(y)| \,\mathrm{d}y \tag{6.2}$$

is well defined and bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ if and only if $w \in A_p^{\alpha,\beta}$.

THEOREM 6.5. *Suppose $\alpha_{r^2}(x)$ and $\beta_{r^2}(x)$ satisfy all of Conditions 6.1 on the doubling space Ω. Consider integral operators T_t with kernels satisfying the following upper and lower bounds.*

$$p_t(x,y) \sim \frac{\alpha_t(x)\beta_t(y)e^{-d(x,y)^2/ct}}{t^{n/2}} \tag{6.3}$$

Then $\sup_t T_t$ is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ if and only if $w \in A_p^{\alpha,\beta}$.

THEOREM 6.6. *Suppose T is a sub-linear operator and that $\alpha_{r^2}(x)$ and $\beta_{r^2}(x)$ satisfy all of Conditions 6.1 on the doubling space Ω. Further suppose T satisfies the following inequalities for all balls $B \subset \Omega$, $x \in B$, $z \in B$ and $y \in 3B$.*

$$|\{x \in B : T[f 1_{3B}](x) > \lambda\}| \lesssim \frac{|B|}{\lambda} M^{\alpha,\beta} f(z) \text{ and } T[f 1_{(3B)^c}](x) \lesssim Tf(y) + M^{\alpha,\beta} f(z) \tag{6.4}$$

Lastly suppose that for some $w \in A_p^{\alpha,\beta} \cap A_\infty$, T satisfies $\|Tf\|_{\mathcal{L}^p(w)} < \infty$ whenever $\|M^{\alpha,\beta} f\|_{\mathcal{L}^p(w)} < \infty$. Then T is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$.

Theorem 6.4 is proven in section 6.2 and Theorems 6.5 and 6.6 are proven in section 6.3. Section 6.1 is where the properties of the $A_p^{\alpha,\beta}$ class are discussed. Theorem 6.5 is used in the application part of this thesis to find weight classes for heat semigroups and similar operators. Theorem 6.6 is included for the purpose of completeness. The weights $w(x)$ considered form continuous measures $w(x)\,\mathrm{d}x$. Singular measures can be possible but are not considered in general.

Remarks 6.7. Firstly, the conditions on α and β ensure the maximal operator $M^{\alpha,\beta}$ is well defined for all $f \in C_0$. The simplest way to observe this is to first consider the variation $M^{1,\alpha\beta}$. For this variation condition (6.1b) becomes $\fint \alpha_{r^2}(y)\beta_{r^2}(y)\,\mathrm{d}y \lesssim 1$. Hence for any $f \in C_0$,

$$M^{1,\alpha\beta}f(x) = \sup_{B\ni x} \fint_B \alpha_{r^2}(y)\beta_{r^2}(y)|f(y)|\,\mathrm{d}y \lesssim \|f\|_\infty$$

so that $M^{1,\alpha\beta}$ is well defined for such f. In the proof of Theorem 6.4, the maximal functions $M^{1,\alpha\beta}$ and $M^{\alpha,\beta}$ are transitioned between via a continuous and positive function $a_i(x)$. Hence $M^{\alpha,\beta}$ is also well defined for $f \in C_0$ by comparison.

The operator T_t from Theorem 6.5 is well defined by the comparison to $M^{\alpha,\beta}$ that occurs in the proof of Theorem 6.5. The operator in Theorem 6.6 is supposed well defined again by comparison to the Maximal function. Details are in section 6.2.

On non-doubling spaces or on spaces with difficult boundary, the contents of this chapter should be viewed by considering appropriate extensions to $\mathbb{R}^n$. The proofs from this chapter that $\|Tf\|_{\mathcal{L}^p(\mathbb{R}^n,w)} \lesssim \|f\|_{\mathcal{L}^p(\mathbb{R}^n,w)}$ then imply for f supported in $\Omega \subset \mathbb{R}^n$ that $\|Tf\|_{\mathcal{L}^p(\Omega,w)} \lesssim \|f\|_{\mathcal{L}^p(\Omega,w)}$ for the same class of weights. The only reason this chapter does not extend to $\mathbb{R}^n$ for all weights concerned, is because that approach is not necessary for the specific applications considered in chapters 8 - 10.

Variations on Muckenhoupt weights have been tried before. One example is [13] where weights for Schrödinger operator based semigroups and Riesz transforms were considered as part of the following class. Here x is the centre and r the radius of B,

$$\left(\int_B w\,\mathrm{d}x\right)^{1/p}\left(\int_B w^{-p'/p}\,\mathrm{d}x\right)^{1/p'} \lesssim |B|\left(1+\frac{r}{\gamma(x)}\right)^{\theta}$$

and $\gamma(x)$ is the *critial radius* of the Scrödinger operator. It was shown that such a class was sufficient for boundedness. The author would also like to acknowledge unpublished work by Xuan Duong, Lesley Ward and Ji Li on weight classes for Riesz transforms.

6.1 Properties of the New Weight Class

In this section important properties are proven for the $A_p^{\alpha,\beta}$ weight class. The main arguments follow from the development of the traditional A_p Muckenhoupt class in [67]. The difficulty lies in appropriate use of doubling and reverse Hölder inequalities. The following is the main proposition proven.

Proposition 6.8. *Suppose that $\alpha_{r^2}(x)$ and $\beta_{r^2}(x)$ are positive functions satisfying conditions (6.1a), (6.1b) and (6.1c). Then $w \in A_p^{\alpha,\beta}$ implies that there exists $q_1 < p$ and $q_2 > p$ such that $w \in A_{q_1}^{\alpha,\beta}$ and $w \in A_{q_2}^{\alpha,\beta}$. Further w is in $A_s^{\alpha,\beta}$ for all $s \in (q_1, q_2)$.*

The value of the $q_1 < p$ and $q_2 > p$ on which the proposition holds depend on the weight w. In the next section, Proposition 6.8 will be used to prove Theorem 6.4. Proposition 6.8 is proven by a series of lemmas, the key to which is the use of reverse Hölder inequalities. Any weight in A_∞ satisfies a reverse Hölder inequality,

$$\left(\fint_B w(x)^s \,\mathrm{d}x \right)^{1/s} \lesssim \fint_B w(x) \,\mathrm{d}x$$

for all balls B and some $s > 1$ depending on w. Many comparisons to the standard A_p classes are made throughout these proofs. Before proceeding with the lemmas, an apparent abuse of notation needs to be dealt with.

To say a function $\alpha_{r^2}(x)^p w(x)$ is in the space A_q is to say that,

$$A_q(\alpha_{r^2}^p w) = \sup_B \left(\fint_B \alpha_{r^2}(x)^p w(x) \,\mathrm{d}x \right) \left(\fint_B \alpha_{r^2}(x)^{-pq'/q} w(x)^{-q'/q} \,\mathrm{d}x \right)^{q/q'} < \infty \quad (6.5)$$

holds for all $B \subset \Omega$, where r is the radius of B. The problem is $\alpha_{r^2}(x)^p w(x)$ cannot truly be a weight as weights do not depend on the radius r of B. However, $\alpha_{r^2}(x)^p w(x)$ still satisfies a number of the properties of A_q weights. There are two mentioned below.

Proposition 6.9. *Suppose $\alpha_{r^2}(x)^p w(x) \in A_q$ for some α satisfying condition* (6.1c)*. Then $\alpha_{r^2}(x)^p w(x)$ satisfies a doubling statement and a reverse Hölder inequality. The same occurs for $\beta_{r^2}(x)^{p'} w(x)^{-p'/p} \in A_q$ with some β satisfying condition* (6.1c)*.*

Proof. Fix a cube Q centred in Ω. By the doubling principle, the definition of $A_p^{\alpha,\beta}$ can be changed to involve cubes rather then balls. Cubes are needed for the second half of the proof, the first half would still work using balls. Set r as the radius of the smallest ball containing Q. Use Hölder's inequality on an averaging f_Q.

$$f_Q^q \leq c\left(\fint_Q |f(x)|^q \alpha_{r^2}(x)^p w(x)\,\mathrm{d}x\right)\left(\fint_Q \alpha_{r^2}(x)^{-pq'/q} w(x)^{-q'/q}\,\mathrm{d}x\right)^{q/q'}$$

Multiply both sides by $\fint_Q \alpha_{r^2}^p w\,\mathrm{d}x$ and use that $\alpha_{r^2}(x)^p w(x) \in A_q$ as in equation (6.5). Set $f(x) = 1_E$ for some $E \subset Q$ to get the following statement.

$$|E|^q \int_Q \alpha_{r^2}(x)^p w(x)\,\mathrm{d}x \leq cA_q(\alpha_{r^2}^p w)|Q|^q \int_E \alpha_{r^2}(x)^p w(x)\,\mathrm{d}x$$

With B fixed and r fixed, the above equation is true for all $E \subset Q$ and can be rephrased,

$$\frac{|E|^q}{|Q|^q} \leq cA_q(\alpha_{r^2}^p w)\frac{(\alpha_{r^2}^p w)(E)}{(\alpha_{r^2}^p w)(Q)}$$

using notation $\alpha_{r^2}^p w(E) = \int_E \alpha_{r^2}(x)^p w(x)\,\mathrm{d}x$. Interpret this as: for any $0 < \varepsilon < 1$ then $\varepsilon|Q| \leq |E| \implies \varepsilon^q(\alpha_{r^2}^p w)(Q) \leq cA_q(\alpha_{r^2}^p w)\ (\alpha_{r^2}^p w)(E)$. Use equation (6.1c) to get for some constants c and C: $\varepsilon|Q| \leq |E| \implies \varepsilon^q(\alpha_{R^2}^p w)(Q) \leq cC^2 A_q(\alpha_{r^2}^p w)\ (\alpha_{R^2}^p w)(E)$, where $R \geq r$ is chosen later. This holds for any subset E of Q, so applying it to $Q \setminus E$ and using that $|E| + |Q\setminus E| = |Q|$ and $w(E) + w(Q\setminus E) = w(Q)$ allows for any $0 < \varepsilon < 1$ the following doubling type statement to hold.

$$|E| \leq (1-\varepsilon)|Q| \implies (\alpha_{R^2}^p w)(E) \leq \left(1 - \frac{\varepsilon^q}{cC^2 A_q(\alpha_{r^2}^p w)}\right)(\alpha_{R^2}^p w)(Q)$$

So for any $\gamma \in (0,1)$ and $R \geq r(B)$, there exists $\delta \in (0,1)$ such that,

$$|E| \leq \gamma|Q| \implies (\alpha_{R^2}^p w)(E) \leq \delta\,(\alpha_{R^2}^p w)(Q) \tag{6.6}$$

where δ does not depend on R or r. Equation (6.6) is a doubling statement for $\alpha_{r^2}(x)^p w(x)$. This doubling statement is now used to prove a reverse Hölder inequality for $\alpha_{r^2}^p w$ on any ball $B \subset \Omega$, following the techniques of Stein [67] chapter 5.

Let $B \subset \Omega$ be a ball of radius r. Cover B by disjoint cubes in a Whitney covering (the second type discussed in section 2.3, where cubes Q do not touch $\delta\Omega$, $\rho(Q) > 0$). Set $W(x) = (c_1\alpha_{r^2}^p w)(x)$ with c_1 chosen so that $W(B) = 1$. Dyadic cubes have side length 2^k

for some integer k and have the Cartesian coordinates of each vertex in a position $m2^k$ for the same integer k and some integer m. Let M^D be the dyadic maximal function,

$$M^D f(x) = \sup_Q \fint_Q |f(x)| \, \mathrm{d}x$$

where the supremum is only over dyadic cubes in Ω. Pick any cube Q from the covering of B. Scale Ω so that $|Q| = 1$ and arrange coordinates so that Q is dyadic. Let $E_k = \{x \in Q : M^D f(x) > 2^{Nk}\}$ for $k \geq 0$ (N is chosen so that $2^{n-N} = \gamma$ where $0 < \gamma < 1$ is a constant). By [67] chapter 4 section 3.1, each set E_k can be covered by disjoint dyadic cubes that satisfy the inequality given by $2^{Nk} < \frac{1}{|Q_i|} \int_{Q_i} |f(x)| \, \mathrm{d}x \leq 2^{n+Nk}$ (it is important here that $\rho(Q) > 0$, $|Q| = 1$, and $W(B) = 1$). Let Q_0 be a cube in such a covering of E_{k-1}, and $Q_j \subset Q_0$ be cubes in such a covering of E_k. Then,

$$|Q_0 \cap E_k| \leq \sum_j |Q_j| \leq 2^{-Nk} \sum_j \int_{Q_j} |f(x)| \, \mathrm{d}x \leq 2^{-Nk} \int_{Q_0} |f(x)| \, \mathrm{d}x \leq 2^{n-N} |Q_0| = \gamma |Q_0|$$

holds for all k. This implies by equation (6.6) that $W(Q_0 \cap E_k) \leq \delta W(Q_0)$ for some $0 < \delta < 1$. It is important that r in the definition of W is larger then the radius of the cubes involved. Union over all dyadic Q_0 that cover E_{k-1} to get $W(E_k) \leq \delta W(E_{k-1})$ which continues as a pattern to give $W(E_k) \leq \delta^k W(Q)$. Further,

$$\begin{aligned}
\int_Q W(x)^m \, \mathrm{d}x &\lesssim \int_Q [M^D W(x)]^{m-1} W(x) \, \mathrm{d}x \\
&\lesssim \int_{\{x \in Q : M^D W(x) \leq 1\}} [M^D W]^{m-1} W \, \mathrm{d}x + \sum_{k=0}^{\infty} \int_{E_k \setminus E_{k+1}} [M^D W]^{m-1} W \, \mathrm{d}x \\
&\lesssim W(Q) + \sum_{k=0}^{\infty} 2^{N(k+1)(m-1)} W(E_k) \\
&\lesssim W(Q) + \sum_{k=0}^{\infty} 2^{N(k+1)(m-1)} \delta^k W(Q) \lesssim W(Q)
\end{aligned}$$

where the series converges with m chosen close enough to 1, depending on $\delta < 1$, which itself depends on α and w only (and not c_1 or the dilation of Ω). This means m can be chosen the as the same value for every $Q \subset B$. Undo the dilation on Ω to get $|Q|^{-1}$ on both sides of the equation, which then cancel.

$$\int_B W(x)^m \,\mathrm{d}x \lesssim \sum_Q \int_Q W(x)^m \,\mathrm{d}x \lesssim \sum_Q W(Q) \lesssim W(B) = 1$$

To finish use that c_1 in the definition of W is given by $c_1 = (\fint_B \alpha_{r^2}(x)^p w(x)\,\mathrm{d}x)^{-1}$. This gives a reverse Hölder inequality for $\alpha_{r^2}(x)^p w(x)$.

$$\int_B \left[\alpha_{r^2}(x)^p w(x)\right]^m \mathrm{d}x \lesssim \left(\int_B \alpha_{r^2}(x)^p w(x)\,\mathrm{d}x\right)^m$$

An identical result holds for $\beta_{r^2}(x)^{p'} w(x)^{-p'/p}$. □

The next five lemmas together prove Proposition 6.8.

Lemma 6.10. *Suppose $w \in A_p^{\alpha,\beta}$ for some $p > 1$ and α, β pair continuous and positive and satisfying conditions (6.1a) and (6.1c). Then $\alpha_{r^2}(x)^p w(x)$ satisfies a reverse Hölder inequality.*

Proof. Given $p > 1$, let γ and A be from condition (6.1a). Choose $q = p + p/A > p$ and observe by this choice of q that $A = p(q-p)^{-1}$. Use Hölder's inequality and condition (6.1a) to get,

$$\fint_B \alpha_{r^2}^{-p/(q-p)} \beta_{r^2}^{-p/(q-p)} \,\mathrm{d}x \lesssim \left(\fint_B \alpha_{r^2}^{-p\gamma/(q-p)} \,\mathrm{d}x\right)^{1/\gamma} \left(\fint_B \beta_{r^2}^{-p\gamma'/(q-p)} \,\mathrm{d}x\right)^{1/\gamma'} \lesssim 1 \quad (6.7)$$

which will be used later in this proof. Next use Hölder's inequality on the following with exponent $s = qp'/pq' = (q-1)/(p-1) > 1$ (the conjugate of s is $s' = (q-1)/(q-p)$).

$$\left(\fint_B \alpha_{r^2}^{-pq'/q} w^{-q'/q} \,\mathrm{d}x\right)^{q/q'} \lesssim \left(\fint_B \beta_{r^2}^{p'} w^{-p'/p} \,\mathrm{d}x\right)^{p/p'} \left(\fint_B \alpha_{r^2}^{-\frac{p}{(q-p)}} \beta_{r^2}^{-\frac{p}{(q-p)}} \,\mathrm{d}x\right)^{q-p}$$

Multiply both sides by $\fint \alpha^p w \,\mathrm{d}x$ and use the condition developed as equation (6.7) earlier on the second integral on the right above. Then the equation formed is,

$$\left(\fint \alpha_{r^2}^p w \,\mathrm{d}x\right) \left(\fint_B \left[\alpha_{r^2}^p w\right]^{-q'/q} \mathrm{d}x\right)^{q/q'} \lesssim \left(\fint \alpha_{r^2}^p w \,\mathrm{d}x\right) \left(\fint_B \beta_{r^2}^{p'} w^{-p'/p} \,\mathrm{d}x\right)^{p/p'}$$

which is to say $A_q(\alpha_{r^2}^p w) \lesssim A_p^{\alpha,\beta}(w)$. This means for all $w \in A_p^{\alpha,\beta}$ then $\alpha_{r^2}^p w \in A_q \subset A_\infty$ follows. Hence $\alpha_{r^2}^p w$ satisfies a reverse Hölder inequality by Proposition 6.9. □

The next lemma is in a similar vein.

Lemma 6.11. *Suppose $p > 1$ and $w \in A_p^{\alpha,\beta}$ for some α, β pair continuous and positive and satisfying conditions (6.1a) and (6.1c). Then $\beta_{r^2}(x)^{p'} w(x)^{-p'/p}$ satisfies a reverse Hölder inequality.*

Proof. Given $p > 1$, let γ and A be from the first α, β condition (6.1a). Choose $q = \frac{pA+p}{A+p}$ so that $1 < q < p$ and observe by this choice of q that $A = p(q-1)/(p-q) > 0$. Use Hölder's inequality and condition (6.1a) to get,

$$\fint_B \alpha_{r^2}^{-\frac{p(q-1)}{p-q}} \beta_{r^2}^{-\frac{p(q-1)}{p-q}} \, \mathrm{d}x \lesssim \left(\fint_B \alpha_{r^2}^{-\frac{p(q-1)\gamma}{p-q}} \, \mathrm{d}x \right)^{1/\gamma} \left(\fint_B \beta_{r^2}^{-\frac{p(q-1)\gamma'}{p-q}} \, \mathrm{d}x \right)^{1/\gamma'} \lesssim 1$$

which will be used later in the proof. Again use Hölder's inequality, this time with exponent $s = (p-1)/(q-1) > 1$ (conjugate $s' = (p-1)/(p-q)$) on the following term.

$$\left(\fint_B \beta_{r^2}^{\frac{-p'q}{q'}} (w^{\frac{-p'}{p}})^{-\frac{q}{q'}} \, \mathrm{d}x \right)^{\frac{q'}{q}} \leq \left(\fint_B \alpha_{r^2}^p w \, \mathrm{d}x \right)^{\frac{p'}{p}} \left(\fint_B \alpha_{r^2}^{-\frac{p(q-1)}{(p-q)}} \beta_{r^2}^{-\frac{p(q-1)}{(p-q)}} \, \mathrm{d}x \right)^{\frac{q'}{qs'}}$$

Multiply both sides by $\fint \beta_{r^2}^{p'} w^{-p'/p} \, \mathrm{d}x$ and use the condition developed earlier.

$$\left(\fint_B \beta_{r^2}^{p'} w^{-p'/p} \, \mathrm{d}x \right) \left(\fint_B [\beta_{r^2}^{p'} w^{-p'/p}]^{-q/q'} \, \mathrm{d}x \right)^{q'/q} \lesssim \left(\fint_B \beta_{r^2}^{p'} w^{-p'/p} \, \mathrm{d}x \,)(\fint_B \alpha_{r^2}^p w \, \mathrm{d}x \right)^{p'/p}$$

Which is to say $A_{q'}(\beta_{r^2}^{p'} w^{-p'/p}) \lesssim A_p^{\alpha,\beta}(w)^{p'/p}$. Hence $w \in A_p^{\alpha,\beta}$ implies $\beta_{r^2}^{p'} w^{-p'/p} \in A_{q'}$. Hence $\beta_{r^2}^{p'} w^{-p'/p}$ satisfies a reverse Hölder inequality by Proposition 6.9. □

Now the two lemmas above are used to prove the q_1 and q_2 parts of Proposition 6.8.

Lemma 6.12. *Consider $w \in A_p^{\alpha,\beta}$. Suppose that $\alpha_{r^2}(x)^p w(x)$ satisfies a reverse Hölder inequality and further suppose the positive functions $\alpha_{r^2}(x)$ and $\beta_{r^2}(x)$ satisfy the second α, β condition (6.1b). Then $w \in A_q^{\alpha,\beta}$ for some $q > p$.*

Proof. Suppose that $w \in A_p^{\alpha,\beta}$. To show that $w \in A_q^{\alpha,\beta}$ start with the value below.

$$A_q^{\alpha,\beta}(w) = \left(\fint_B \alpha_{r^2}(x)^q w(x) \, \mathrm{d}x \right) \left(\fint_B \beta_{r^2}(x)^{q'} w(x)^{-q'/q} \, \mathrm{d}x \right)^{q/q'}$$

For the part involving $\beta_{r^2}(x)$ use Hölders inequality with exponent $t = (q-1)/(p-1) > 1$.

$$\left(\fint_B \beta_{r^2}(x)^{q'} w(x)^{-q'/q}\,\mathrm{d}x\right)^{q/q'} \lesssim \left(\fint_B \beta_{r^2}(x)^{p'} w(x)^{-p'/p}\,\mathrm{d}x\right)^{p/p'} \left(\fint_B \beta_{r^2}(x)\,\mathrm{d}x\right)^{q-p}$$

Then for the part involving $\alpha_{r^2}(x)$, use Hölder's inequality followed by a reverse Hölder inequality for some $s > 1$ depending on w ($q > p$ is chosen so that $q - p = 1/s'$).

$$\begin{aligned}\fint_B \alpha_{r^2}(x)^q w(x)\,\mathrm{d}x &\lesssim \left(\fint_B \alpha_{r^2}(x)^{ps} w(x)^s\,\mathrm{d}x\right)^{1/s} \left(\fint_B \alpha_{r^2}(x)^{(q-p)s'}\,\mathrm{d}x\right)^{1/s'} \\ &\lesssim \left(\fint_B \alpha_{r^2}(x)^p w(x)\,\mathrm{d}x\right) \left(\fint_B \alpha_{r^2}(x)^{(q-p)s'}\,\mathrm{d}x\right)^{1/s'}\end{aligned}$$

Then multiply the parts for $\alpha_{r^2}(x)$ and $\beta_{r^2}(x)$ together and use condition (6.1b).

$$\left(\fint_B \alpha_{r^2}^q w\,\mathrm{d}x\right)\left(\fint_B \beta_{r^2}^{q'} w^{-\frac{q'}{q}}\,\mathrm{d}x\right)^{\frac{q}{q'}} \lesssim \left(\fint_B \alpha_{r^2}^p w\,\mathrm{d}x\right)\left(\fint_B \beta_{r^2}^{p'} w^{-\frac{p'}{p}}\,\mathrm{d}x\right)^{\frac{p}{p'}}$$

Thus $A_q^{\alpha,\beta}(w) \lesssim A_p^{\alpha,\beta}(w)$ for some $q > p$ depending on w. □

The idea is now repeated for $q < p$. Even more reverse Hölder work is needed.

Lemma 6.13. *Suppose that $\alpha_{r^2}(x)^p w(x)$ and $\beta_{r^2}(x)^{p'} w(x)^{-p'/p}$ both satisfy reverse Hölder inequalities and further suppose the positive functions $\alpha_{r^2}(x)$ and $\beta_{r^2}(x)$ satisfy condition (6.1a). Then $w \in A_p^{\alpha,\beta}$ implies $w \in A_q^{\alpha,\beta}$ for some $q < p$.*

Proof. Fix $w \in A_p^{\alpha,\beta}$. Then there exists $s > 1$ for which both $\beta_{r^2}^{p'} w^{-p'/p}$ and $\alpha_{r^2}^p w$ satisfy a reverse Hölder inequality (if a weight satisfies a reverse Hölder inequality with exponent s_1, then that same weight satisfies a reverse Hölder inequality with exponent s for all $1 < s < s_1$). Choose $q = p - \frac{A\gamma}{s'}$ and $R = sp'q/pq' > 1$, given A and γ from condition (6.1a). By choice of q and R: $1 < q < p$; $1 < R < \infty$; $s'(q-p) = -A\gamma$; and $q'R'(q-p)/q = -A\gamma'$ all hold. Use Hölder's inequality with exponent R and a reverse Hölder inequality with exponent s.

$$\begin{aligned}\left(\fint_B \beta_{r^2}^{q'} w^{-q'/q}\,\mathrm{d}x\right)^{q/q'} &\lesssim \left(\fint_B \left(\beta_{r^2}^{p'} w^{-p'/p}\right)^s\,\mathrm{d}x\right)^{q/q'R} \left(\fint_B \beta_{r^2}^{q'R'-p'sR'/R}\,\mathrm{d}x\right)^{q/q'R'} \\ &\lesssim \left(\fint_B \beta_{r^2}^{p'} w^{-p'/p}\,\mathrm{d}x\right)^{p/p'} \left(\fint_B \beta_{r^2}^{q'R'(q-p)/q}\,\mathrm{d}x\right)^{q/q'R'}\end{aligned}$$

For the part of the $A_p^{\alpha,\beta}(w)$ constant involving $\alpha_{r^2}(x)$, similar to above use a classical

Hölder inequality, followed by a reverse Hölder inequality, both with exponent s.

$$\fint_B \alpha_{r^2}^q w \, \mathrm{d}x \lesssim \left(\fint_B (\alpha_{r^2}^p w)^s \, \mathrm{d}x\right)^{1/s} \left(\fint_B \alpha_{r^2}^{s'(q-p)} \, \mathrm{d}x\right)^{1/s'} \lesssim \left(\fint_B \alpha_{r^2}^p w \, \mathrm{d}x\right) \left(\fint_B \alpha_{r^2}^{s'(q-p)} \, \mathrm{d}x\right)^{1/s'}$$

Multiply the two parts together and use that the choice of q and R imply $-A\gamma = s'(q-p)$ and $-A\gamma' = q'R'(q-p)/q$ and $q/q'R' = \gamma/s'\gamma'$. Hence by equation (6.1a),

$$\left(\fint_B \alpha_{r^2}^q w \, \mathrm{d}x\right) \left(\fint_B \beta_{r^2}^{q'} w^{-q'/q} \, \mathrm{d}x\right)^{q/q'} \lesssim \left(\fint_B \alpha_{r^2}^p w \, \mathrm{d}x\right) \left(\fint_B \beta_{r^2}^{p'} w^{-p'/p} \, \mathrm{d}x\right)^{p/p'}$$

which concludes the proof. □

It remains to show the above implies $w \in A_s^{\alpha,\beta}$ for the full range of all $s \in (q_1, q_2)$.

Lemma 6.14. *Suppose that $w \in A_p^{\alpha,\beta}$ and $w \in A_q^{\alpha,\beta}$ where $1 < p < q < \infty$. Then $w \in A_s^{\alpha,\beta}$ for all $s \in (p, q)$.*

Proof. The idea is to bound $A_s^{\alpha,\beta}(w)$ by $A_p^{\alpha,\beta}(w)$ and $A_q^{\alpha,\beta}(w)$. Recall $p < s < q$ so that $s = tp + (1-t)q$ for some $t \in (0,1)$. Start with Hölders inequality applied to the $\alpha_{r^2}^s w$ part of $A_s^{\alpha,\beta}(w)$ with exponents $\theta = 1/t$ and $\theta' = 1/(1-t)$.

$$\begin{aligned}\fint \alpha_{r^2}(x)^s w(x) \, \mathrm{d}x &= \fint \alpha_{r^2}(x)^{tp} \alpha_{r^2}(x)^{(1-t)q} w(x)^t w(x)^{1-t} \, \mathrm{d}x \\ &\lesssim \left(\fint \alpha_{r^2}(x)^p w(x) \, \mathrm{d}x\right)^t \left(\fint \alpha_{r^2}(x)^q w(x) \, \mathrm{d}x\right)^{1-t}\end{aligned}$$

An identical method deals with the β part. Observe that for the same p, s, q as the α part $q' < s' < p'$, so that $s' = Tp' + (1-T)q'$ for some $T \in (0,1)$. For the same T $s'/s = T(p'/p) + (1-T)(q'/q)$ also holds. This relates to the t used in the α part by $T = s'pt/sp'$ and $(1-T) = s'q(1-t)/sq'$. Use this,

$$\begin{aligned}\left(\fint \beta_{r^2}(x)^{s'} w(x)^{-\frac{s'}{s}} \, \mathrm{d}x\right)^{\frac{s}{s'}} &= \left(\fint \beta_{r^2}(x)^{Tp'} \beta_{r^2}(x)^{(1-T)q'} w(x)^{-\frac{Tp'}{p}} w(x)^{-\frac{(1-T)q'}{q}} \, \mathrm{d}x\right)^{\frac{s}{s'}} \\ &\lesssim \left(\fint \beta_{r^2}(x)^{p'} w(x)^{-\frac{p'}{p}} \, \mathrm{d}x\right)^{\frac{tp}{p'}} \left(\fint \beta_{r^2}(x)^{q'} w(x)^{-\frac{q'}{q}} \, \mathrm{d}x\right)^{\frac{(1-t)q}{q'}}\end{aligned}$$

and put the results for α and β together to get $A_s^{\alpha,\beta}(w) \lesssim [A_p^{\alpha,\beta}(w)]^t [A_q^{\alpha,\beta}(w)]^{1-t}$. □

This concludes the proof of Proposition 6.8. To conclude this section some circumstances where $A_p^{\alpha,\beta}$ weights are similar to A_q weights are considered. Not every α, β pair that satisfies the main conditions of this chapter will satisfy these next few lemmas, however most examples of the Dirichlet Laplacian considered later in this thesis satisfy at least one of the lemmas below.

Lemma 6.15. *Let $f(r)$ be a positive function defined for all $r > 0$. Suppose that both $f(r) \lesssim \alpha_{r^2}(x)$ and $\fint_B \beta_{r^2}(x)^{-A\gamma'}\,\mathrm{d}x \lesssim f(r)^{A\gamma'}$ hold for all $B \subset \Omega$, where A and γ' are from condition (6.1a). Then $w \in A_p^{\alpha,\beta}$ implies $w \in A_q$ for some $q > p$.*

Proof. Given $p > 1$, then $\gamma > 1$ and $A > 0$ come from condition (6.1a). Choose $q = p/A\gamma' + p$ and $R = qp'/q'p$ and use Hölder's inequality on the β part with exponent R.

$$\left(\fint_B w\,\mathrm{d}x\right)\left(\fint_B w^{-\frac{q'}{q}}\,\mathrm{d}x\right)^{\frac{q}{q'}} \lesssim \frac{1}{f(r)^p}\left(\fint_B \alpha_{r^2}^p w\,\mathrm{d}x\right)\left(\fint_B \beta_{r^2}^{p'} w^{-\frac{p'}{p}}\,\mathrm{d}x\right)^{\frac{p}{p'}}\left(\fint_B \beta_{r^2}^{-\frac{R'p'}{R}}\,\mathrm{d}x\right)^{\frac{q}{q'R'}}$$

By choice of R and q, $R'p'/R = A\gamma'$ and $q/q'R' = p/A\gamma'$ both hold. Use the condition $\fint \beta_{r^2}(x)^{-A\gamma'}\,\mathrm{d}x \lesssim f(r)^{A\gamma'}$ to remove the extra β part. The end result is the inequality given by $A_q(w) \lesssim A_p^{\alpha,\beta}(w)$, which concludes the proof. □

Lemma 6.15 when it holds, implies the weight w is doubling and satisfies reverse Hölder estimates. The Hardy, Gaffney and Riesz transform weights found in the application of this thesis all satisfy the conditions of Lemma 6.15. As $q > p$ the weights in the $A_p^{\alpha,\beta}$ class here can still exceed the A_p class of weights and will in many application parts. There are also examples in the application chapters where the conditions of this lemma do not hold and the weight class found includes weights outside the A_∞ class. One example is the weight $w(x) = 1/\rho(x)$ which is outside A_∞ and occurs in heat semigroup weight classes. Rapid decay of the heat kernel near the boundary is what allows such weights to occur.

Lemma 6.16. *Suppose that $\alpha_{r^2}(x) \lesssim f(r)$ and $\beta_{r^2}(x) \lesssim f(r)^{-1}$ for some positive function f, and all $r > 0$, $x \in \Omega$. Then $w \in A_p$ implies $w \in A_p^{\alpha,\beta}$.*

Proof. The proof is a trivial substitution after which the two $f(r)$ parts cancel.

$$\left(\fint_B \alpha^p w\,\mathrm{d}x\right)\left(\fint_B \beta^{p'} w(x)^{-p'/p}\,\mathrm{d}x\right)^{p/p'} \lesssim \left(\fint_B w\,\mathrm{d}x\right)\left(\fint_B w(x)^{-p'/p}\,\mathrm{d}x\right)^{p/p'}$$

Hence $A_p^{\alpha,\beta}(w) \lesssim A_p(w)$ in this case. □

This lemma does not need to hold for the main theorems of this chapter to hold. The conditions for this lemma hold in most application parts, but not all.

6.2 Application to Maximal Functions

In this section Theorem 6.4 is proven by a series of lemmas and propositions. Theorem 6.4 described the following maximal function as being bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ if and only if $w \in A_p^{\alpha,\beta}$.

Definition 6.17. The *adjusted Maximal function* $M^{\alpha,\beta} f(x)$ is given by

$$M^{\alpha,\beta} f(x) = \sup_{B \ni x} \frac{\alpha_{r^2}(x)}{|B|} \int_B \beta_{r^2}(y) |f(y)| \,\mathrm{d}y$$

where r is the radius of the ball B.

The proofs mirror those found in [67] for the traditional Hardy-Littlewood maximal function and the Muckenhoupt weight class. The first proposition below is not satisfied for all maximal functions $M^{\alpha,\beta}$ considered in this chapter, but is included for completeness in the discussion.

Proposition 6.18. *Suppose that the adjusted Maximal function $M^{\alpha,\beta}$ satisfies a weighted bound $\mathcal{L}^p(\mu) \to \mathcal{L}^p(\mu)$ for some $1 < p < \infty$ and that α_{r^2} and β_{r^2} are continuous and positive in the interior of Ω. Further suppose there exists $\delta > 0$ such that for every $B \subset \Omega$ small enough both $\alpha_{r^2}(x) \geq \delta$ and $\beta_{r^2}(x) \geq \delta$ hold for all $x \in B$. Then $\mathrm{d}\mu$ is absolutely continuous and $\mathrm{d}\mu = w(x)\,\mathrm{d}x$.*

Proof. Separate $\mathrm{d}\mu$ into a continuous part and a singular part $\mathrm{d}\mu = w(x)\,\mathrm{d}x + \mathrm{d}\nu$. As $\mathrm{d}\nu$ is singular, if $\mathrm{d}\nu \neq 0$ then there must exist a compact set S of points where $|S| = 0$ but $\nu(S) > 0$. Cover such a set S with,

$$V_m = \{x \in \Omega : d(x, S) < \frac{1}{m}\}$$

and let 1_m represent the characteristic function of $V_m \setminus S$. Consider the maximal function $M^{\alpha,\beta}$ acting on 1_m. Suppose that $M^{\alpha,\beta}$ is bounded $\mathcal{L}^p(\mu) \to \mathcal{L}^p(\mu)$ then it is expected,

$$\int_\Omega |M^{\alpha,\beta}(1_m)(x)|^p \,\mathrm{d}\mu \lesssim \int_\Omega |1_m(x)|^p \,\mathrm{d}\mu = \int_{V_m\setminus S} w(x)\,\mathrm{d}x$$

where the rightmost part tends to 0 as m tends to ∞ by the dominated convergence theorem (for details on dominated convergence see for example Rudin [57]).

$$\lim_{m\to\infty} \int_\Omega |M^{\alpha,\beta}(1_m)(x)|^p \,\mathrm{d}\mu = 0$$

Next it is shown that this does not occur in the presence of a singular measure. Let B_0 be a ball that does not touch the boundary ($\rho(B_0) > \epsilon > 0$) and covers some subset S_0 of S so that $\nu(S_0) \neq 0$. For each $x \in S_0$ choose $B_m \subset B_0 \cap V_m$ for some m large. Use the lower bound of δ for α_{r^2} and β_{r^2} on B_m small enough.

$$\begin{aligned}
\int_\Omega |M^{\alpha,\beta}(1_m)(x)|^p \,\mathrm{d}\mu &= \int_\Omega \Big| \sup_{B\ni x} \frac{\alpha_{r^2}(x)}{|B|} \int_B \beta_{r^2}(y) 1_m(y)\,\mathrm{d}y \Big|^p \,\mathrm{d}\mu \\
&\geq \int_{S_0} \Big| \sup_{B\ni x} \frac{\alpha_{r^2}(x)}{|B|} \int_B \beta_{r^2}(y) 1_m(y)\,\mathrm{d}y \Big|^p \,\mathrm{d}\mu \\
&\geq \delta^2 \int_{S_0} \Big| \frac{1}{|B_m|} \int_{B_m} 1_m(y)\,\mathrm{d}y \Big|^p \,\mathrm{d}\mu \\
&\geq \delta^2 \int_{S_0} \Big| \frac{|B_m \cap (V_m \setminus S)|}{|B_m|} \Big|^p \,\mathrm{d}\mu = \delta^2 \nu(S_0) > 0
\end{aligned}$$

As m tends to ∞ this lower bound for $\|M^{\alpha,\beta} 1_m\|_p$ is fixed. This is a contradiction to the result of the assumed boundedness of M case considered earlier, which indicated 0 would be the limit as m tended to infinity. □

When the conditions of Proposition 6.18 do not hold then a singular measure $\mathrm{d}\nu$ may be possible. Note however that the heat kernels $e^{-t\Delta_\Omega}$ of the application chapters of this thesis do satisfy Proposition 6.18. Hence only continuous measures $w(x)\,\mathrm{d}x$ have been sought in general in this chapter. The next lemma is well known.

Lemma 6.19. *Suppose that for exponents $1 < r < q < \infty$ the sublinear operator T satisfies a weighted weak (r, r) bound and a weighted weak (q, q) bound for some given weight. Then T satisfies a $\mathcal{L}^p(w)$ bound for each $r < p < q$.*

Proof. The proof is for $f \geq 0$. The full case can be shown by combining cases for $f > 0$

and $-f > 0$. Use the integral representation of the weighted $\mathcal{L}^p$ norm of Tf.

$$\|Tf\|^p_{\mathcal{L}^p(w)} = p\int_0^\infty \lambda^{p-1}\mu(\{x\in\Omega : |Tf(x)| > \lambda\})\,\mathrm{d}\lambda$$

Let $f_1 = f$ whenever $f > \lambda$ and $f_1 = 0$ otherwise. Similarly let $f_2 = f - f_1$ and split the two parts. Also use the assumed weak boundedness of T.

$$\begin{aligned}\|Tf\|^p_{\mathcal{L}^p(w)} &\lesssim p\int_0^\infty \lambda^{p-1}\Big[\mu(\{x\in\Omega : |Tf_1(x)| > \frac{\lambda}{2}\}) + \mu(\{x\in\Omega : |Tf_2(x)| > \frac{\lambda}{2}\})\Big]\,\mathrm{d}\lambda\\ &\lesssim p\int_0^\infty \lambda^{p-1}\left[\frac{\|f_1\|^r_{L^r(w)}}{\lambda^r} + \frac{\|f_2\|^q_{L^q(w)}}{\lambda^q}\right]\mathrm{d}\lambda\end{aligned}$$

Consider the f_1 part. Use the integral representation of the $\mathcal{L}^p$ norm with the definition of f_1. To keep the equations compact use the notation $\Psi_\lambda = \{x\in\Omega : f(x) > \lambda\}$.

$$\begin{aligned}\|f_1\|^r_{L^r(w)} &= r\int_0^\infty s^{r-1}w(\{x\in\Omega : f_1(x) > s\})\,\mathrm{d}s\\ &= r\int_\lambda^\infty s^{r-1}w(\Psi_s)\,\mathrm{d}s + r\int_0^\lambda s^{r-1}w(\Psi_\lambda)\,\mathrm{d}s\end{aligned}$$

This part involving f_1 in the bound for T is controlled by,

$$\begin{aligned}\int_0^\infty \lambda^{p-r-1}\|f_1\|^r_{L^r(w)}\,\mathrm{d}\lambda &= r\int_0^\infty \lambda^{p-r-1}\int_\lambda^\infty s^{r-1}w(\Psi_s)\,\mathrm{d}s\,\mathrm{d}\lambda + r\int_0^\infty \lambda^{p-1}w(\Psi_\lambda)\,\mathrm{d}\lambda\\ &= r\int_0^\infty s^{r-1}w(\Psi_s)\int_0^s \lambda^{p-r-1}\,\mathrm{d}\lambda\,\mathrm{d}s + r\int_0^\infty \lambda^{p-1}w(\Psi_\lambda)\,\mathrm{d}\lambda\\ &\lesssim \int_0^\infty \lambda^{p-1}w(\Psi_\lambda)\,\mathrm{d}\lambda\end{aligned}$$

where $r < p$ was required for a finite integral with respect to λ. A similar result for f_2 holds (using upper bound of λ on f_2, and needing $q > p$ in that case) leads to,

$$\|Tf\|^p_{\mathcal{L}^p(w)} \lesssim p\int_0^\infty \lambda^{p-1}\mu(\{x\in\Omega : f(x) > \lambda\})\,\mathrm{d}\lambda \lesssim \|f\|^p_{\mathcal{L}^p(w)}$$

which concludes the proof. □

Now all conditions are ready for the proof of Theorem 6.4. Theorem 6.4 stated that the maximal function $M^{\alpha,\beta}$ is bounded $\mathcal{L}^p(w)\to\mathcal{L}^p(w)$ if and only if $w\in A_p^{\alpha,\beta}$ where α,β satisfy Conditions 6.1.

Proof of Theorem 6.4. ($\rightarrow$) Suppose $w \in A_p^{\alpha,\beta}$. Let I be the set of all balls $B \subset \Omega$, and I_i with $1 \leq i \leq k$ be the finite collection of disjoint sets from condition (6.1d) where $\cup_{i=1}^k I_i = I$. Let $M_i^{\alpha,\beta}$ be defined for all $x \in \cup_{B \in I_i} B$ and $f \in C_0^\infty$ by the following equation.

$$M_i^{\alpha,\beta} f(x) = \sup_{\substack{B \in I_i \\ B \ni x}} \alpha_{r^2}(x) \fint_B \beta_{r^2}(y)|f(y)| \, \mathrm{d}y$$

Again using condition (6.1d), let $W(x) = a_i(x)^p w(x)$ and observe that,

$$\|M_i^{\alpha,\beta} f\|_{\mathcal{L}^p(\cup_{B \in I_i} B, w)} = \|M_i^{1,\alpha\beta}(f/a_i)\|_{\mathcal{L}^p(\cup_{B \in I_i} B, W)}$$

holds.

Let $g = f/a_i$ and define $E_\lambda = \{x \in \cup_{B \in I_i} B : M_i^{1,\alpha\beta} g(x) > \lambda\}$. Let E be any compact subset of E_λ. For each $x \in E$ there exists a ball $B \ni x$ where $B \in I_i$ and equation (6.8) holds.

$$\lambda < \frac{1}{|B|} \int_B \alpha_{r^2}(y)\beta_{r^2}(y)|g(y)| \, \mathrm{d}y \tag{6.8}$$

Let J_1 be the set of all such balls. For each ball $B \in J_1$ either $E \subset B$ or the radius of B is less than twice the radius of the compact set E. Hence E can be covered by a subset J_2 of balls $B \in J_1$ with bounded radii. Then by a Vitali covering lemma there exists a further subset J_3 of disjoint balls $B \in J_2$ where $\cup 5B$ cover E (See [65] for details on such a covering). Using this covering, along with conditions (6.1c), (6.1d) and the doubling part of Proposition 6.9 the following holds.

$$\begin{aligned}
W(E) &\leq \sum_{B \in J_3} (a_i^p w)(5B) \\
&\lesssim \sum_{B \in J_3} \frac{1}{z_i(r)^p} \int_{5B} \alpha_{r^2}(x)^p w(x) \, \mathrm{d}x \\
&\lesssim \sum_{B \in J_3} \frac{1}{z_i(r)^p} \int_{5B} \alpha_{25r^2}(x)^p w(x) \, \mathrm{d}x \\
&\lesssim \sum_{B \in J_3} \frac{1}{z_i(r)^p} \int_{B} \alpha_{25r^2}(x)^p w(x) \, \mathrm{d}x \\
&\lesssim \sum_{B \in J_3} \frac{1}{z_i(r)^p} \int_{B} \alpha_{r^2}(x)^p w(x) \, \mathrm{d}x = \sum_{B \in J_3} W(B)
\end{aligned} \tag{6.9}$$

From equation (6.8) using Hölder's inequality and condition (6.1d) gives the following.

$$\lambda^p < \frac{A_p^{\alpha,\beta}(w)}{W(B)} \|g\|^p_{\mathcal{L}^p(B,W)}$$

Rearrange for $W(B)$ and place into equation (6.9).

$$W(E) \lesssim \frac{1}{\lambda^p} \sum_{B \in J_3} \|g\|^p_{\mathcal{L}^p(B,W)} \lesssim \frac{1}{\lambda^p} \|g\|^p_{\mathcal{L}^p(\cup_{B\in I_i} B,W)} \tag{6.10}$$

The constant involved does not depend on E so take the supremum over all compact $E \subset E_\lambda$.

$$W(E_\lambda) \lesssim \frac{1}{\lambda^p} \|g\|^p_{\mathcal{L}^p(\cup_{B\in I_i} B,W)}$$

From the results of the previous section, if $w \in A_p^{\alpha,\beta}$ then $w \in A_q^{\alpha,\beta} \cap A_s^{\alpha,\beta}$ for some $q < p$, and $s > p$ depending on w. By the working above this gives weak type inequalities for L^q and L^s which can be interpolated between by Lemma 6.19 to get an $\mathcal{L}^p$ norm bound for M_i. Then setting $g(x) = f(x)/a_i(x)$ for each i the following holds.

$$\begin{aligned}\|M^{\alpha,\beta} f\|_{\mathcal{L}^p(w)} &\lesssim \sum_{i=1}^m \|M_i^{\alpha,\beta} f\|_{\mathcal{L}^p(\cup_{B\in I_i} B,w)} \\ &= \sum_{i=1}^m \|M_i^{1,\alpha\beta} g\|_{\mathcal{L}^p(\cup_{B\in I_i} B,W)} \lesssim \sum_{i=1}^m \|g\|_{\mathcal{L}^p(\cup_{B\in I_i} B,W)} \lesssim \|f\|_{\mathcal{L}^p(w)}\end{aligned}$$

Hence $M^{\alpha,\beta} f(x)$ is $L^p(w)$ bounded for all $w \in A_p^{\alpha,\beta}$ which concludes the first half of the proof.

($\leftarrow$) Next it is shown if $M^{\alpha,\beta}$ satisfies an $\mathcal{L}^p(w)$ bound then $w \in A_p^{\alpha,\beta}$. Start with the supposition $M^{\alpha,\beta}$ satisfies an $\mathcal{L}^p(w)$ bound.

$$\int_\Omega |M^{\alpha,\beta} f(x)|^p w(x)\,\mathrm{d}x \lesssim \int_\Omega |f(x)|^p w(x)\,\mathrm{d}x$$

Fix $B \subset \Omega$ and set $f = \beta_{r^2}^{p'-1}(w+\epsilon)^{-p'/p} 1_B$ (the ϵ is to ensure local integrability).

$$\int_\Omega \left|\alpha_{r^2}(x) \fint_B \beta_{r^2}^{p'}(y)[w(y)+\epsilon]^{-p'/p}\,\mathrm{d}y\right|^p w(x)\,\mathrm{d}x \lesssim \int_B \beta_{r^2}(x)^{p'}[w(x)+\epsilon]^{-p'} w(x)\,\mathrm{d}x$$

Using then that $(w+\epsilon)^{-p'/p} = (w+\epsilon)^{-p'+1} \geq (w+\epsilon)^{-p'} w$ rearrange for,

$$\left(\fint_B (w+\epsilon)^{-p'} w(y)\beta_{r^2}(y)^{p'}\,\mathrm{d}y\right)^p \left(\int_B |\alpha_{r^2}(x)|^p w(x)\,\mathrm{d}x\right) \lesssim \int_B (w+\epsilon)^{-p'} \beta_{r^2}(x)^{p'} w(x)\,\mathrm{d}x$$

then divide both sides by the term on the right.

$$\frac{1}{|B|^p}\left(\int_B (w+\epsilon)^{-p'} w(y)\beta_{r^2}(y)^{-p'}\,\mathrm{d}y\right)^{p/p'} \left(\int_B |\alpha_{r^2}(x)|^p w(x)\,\mathrm{d}x\right) \lesssim 1$$

Take the limit as $\epsilon \to 0$ to get $A_p^{\alpha,\beta}(w) \lesssim 1$ which is the result. $\square$

This concludes the proof of Theorem 6.4.

6.3 Application to General Operators

In the final section of this chapter Theorems 6.5 and 6.6 are proven. Theorem 6.5 regards $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ boundedness of the following maximal operator for weights in $A_p^{\alpha,\beta}$.

$$\sup_t T_t f(x) \overset{def}{=} \sup_{t>0} \alpha_t(x) \int_\Omega \beta_t(y) \frac{e^{-d(x,y)^2/ct}}{t^{n/2}} f(y)\,\mathrm{d}y$$

Proof of Theorem 6.5. The result holds by comparison to the Maximal operator $M^{\alpha,\beta}$ of the previous section. First it is shown that $M^{\alpha,\beta} f(x) \lesssim \sup_t T_t f(x)$ for all positive functions f, using that if $x, y \in B$ then $d(x,y) < 2r$. The doubling condition on $\Omega \subset \mathbb{R}^n$ also helps by implying $|B| \sim r^n$.

$$\begin{aligned}
M^{\alpha,\beta} f(x) &= \sup_{B \ni x} \alpha_{r^2}(x) \fint_B \beta_{r^2}(y)|f(y)|\,\mathrm{d}y \\
&\lesssim \sup_{B \ni x} \alpha_{r^2}(x) \fint_B \beta_{r^2}(y)|f(y)| e^{-d(x,y)^2/cr^2}\,\mathrm{d}y \\
&\lesssim \sup_{B \ni x} \frac{\alpha_{r^2}(x)}{|B|} \int_\Omega \beta_{r^2}(y)|f(y)| e^{-d(x,y)^2/cr^2}\,\mathrm{d}y \\
&\lesssim \sup_{r>0} \frac{\alpha_{r^2}(x)}{r^n} \int_\Omega \beta_{r^2}(y)|f(y)| e^{-d(x,y)^2/cr^2}\,\mathrm{d}y \\
&= c_n \sup_{t>0} \alpha_t(x) \int_\Omega \beta_t(y)|f(y)| \frac{e^{-d(x,y)^2/ct}}{t^{n/2}}\,\mathrm{d}y = c_n \sup_t T_t f(x)
\end{aligned}$$

It remains to check the opposite inequality. The bound on the constant in the third condition on α, β (6.1c) is used here.

$$\begin{aligned}
\sup_t T_t f(x) &= \sup_{t>0} \alpha_t(x) \int_\Omega \beta_t(y) f(y) \frac{e^{-d(x,y)^2/ct}}{t^{n/2}} \, \mathrm{d}y \\
&= \sup_{r>0} \alpha_{r^2}(x) \int_\Omega \beta_{r^2}(y) f(y) \frac{e^{-d(x,y)^2/cr^2}}{r^n} \, \mathrm{d}y \\
&\lesssim \sup_{r>0} \alpha_{r^2}(x) \int_\Omega \beta_{r^2}(y) f(y) \sum_{i=0}^{\infty} \frac{1_{B(x,2^i r)} 2^{in} e^{-c4^i}}{2^{in} r^n} \, \mathrm{d}y \\
&= \sup_{r>0} \sum_{i=0}^{\infty} 2^{in} e^{-c4^i} \frac{\alpha_{r^2}(x)}{|2^i B|} \int_{B(x,2^i r)} \beta_{r^2}(y) f(y) \, \mathrm{d}y \\
&\lesssim \sup_{r>0} \sum_{i=0}^{\infty} 2^{i(m+n)} e^{-c4^i} \frac{\alpha_{4^i r^2}(x)}{|2^i B|} \int_{B(x,2^i r)} \beta_{4^i r^2}(y) f(y) \, \mathrm{d}y \\
&\lesssim \sup_{r>0} \sum_{i=0}^{\infty} 2^{i(m+n)} e^{-c4^i} M^{\alpha,\beta} f(x) = M^{\alpha,\beta} f(x)
\end{aligned}$$

The above comparisons can be generalised to hold in the form $|M^{\alpha,\beta} f(x)| \lesssim \sup_t T_t |f|(x)$ and $|\sup_t T_t f(x)| \lesssim |M^{\alpha,\beta} f(x)|$ for any function f by repeating the above proof for f negative (with appropriate absolute values) and combining the f positive and f negative cases. Thus $M^{\alpha,\beta} f(x)$ and $\sup_t T_t f(x)$ are bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ with identical weight classes, proving Theorem 6.5 via Theorem 6.4. Theorem 6.5 mirrors Theorem 1.4 from the introduction chapter, hence this also concludes the proof of Theorem 1.4. $\square$

In the application part of this thesis Theorem 6.5 is applied to heat semigroup maximal functions. The rest of this chapter looks to prove Theorem 6.6. Until this point the weights used could foreseeably be outside the Muckenhoupt class A_∞. For the next part there is an extra condition that the weights are in A_∞.

Lemma 6.20. *Suppose that T is a continuous sub-linear operator which satisfies the following weak $(1,1)$ bound for all balls $B \subset \Omega$ and all $z \in B$.*

$$|\{x \in B : |T[f 1_{3B}](x)| > \lambda\}| \lesssim \frac{|B|}{\lambda} M^{\alpha,\beta} f(z) \tag{6.11}$$

Further suppose that for all $x \in B$, $z \in B$, $y \in 3B$ and f positive, the operator T satisfies for some fixed $c_1 > 0$ the following regularity estimate.

$$|T(f 1_{(3B)^c})(x)| \leq c_1 [T f(y) + M^{\alpha,\beta} f(z)] \tag{6.12}$$

Then T satisfies the following good-λ inequality for all $w \in A_\infty$, where $b, c \in (0,1)$ can be chosen so that $a < b^p$.

$$w(\{x \in \Omega : |Tf(x)| > \lambda, M^{\alpha,\beta} f(x) \leq c\lambda\}) \leq aw(\{x \in \Omega : |Tf(x)| > b\lambda\}) \tag{6.13}$$

Proof. Take $f \geq 0$. The proof starts with the case $w(x) = 1$ which is extended to all $w \in A_\infty$ by properties of such weights. Cover the set $E = \{x \in \Omega : |Tf(x)| > b\lambda\}$ with balls $B \subset \Omega$ in a Whitney decomposition (the first type discussed in 2.3 where the balls may touch $\delta\Omega$). The value of $b \in (0,1)$ is chosen later. The set E is open as T is continuous. Consider any ball B in the covering that contains a point z where $M^{\alpha,\beta} f(z) \leq c\lambda$, for some $c \in (0,1)$ also chosen later. If there is no such point ignore B. Let y be a point outside E but within $3B$. Such a point exists due to the nature of the Whitney covering. Split $f = f1_{3B} + f1_{(3B)^c}$ and use the sub-linear property of T.

$$\begin{aligned} &|\{x \in B : |Tf(x)| > \lambda\}| \\ &\qquad \lesssim |\{x \in B : |T(f1_{3B})(x)| > b\lambda\}| + |\{x \in B : |T(f1_{(3B)^c})(x)| > (1-b)\lambda\}| \end{aligned} \tag{6.14}$$

For the first case use the weak $(1,1)$ bound supposed in the lemma.

$$|\{x \in B : |T(f1_{3B})(x)| > b\lambda\}| \leq \frac{C|B|}{b\lambda} M^{\alpha,\beta} f(z) \leq \frac{cC}{b}|B|$$

For the second case use equation (6.12) supposed in the lemma,

$$|T(f1_{(3B)^c})(x)| \lesssim |Tf(y)| + |M^{\alpha,\beta} f(z)| \lesssim b\lambda + c\lambda$$

so if b and c are chosen so that $b + c < 1 - b$, then the second set in equation (6.14) is empty. The proof concludes by summing over the relevant balls B using the finite intersection lemma. This gives,

$$|\{x \in \Omega : |Tf(x)| > \lambda, M^{\alpha,\beta} f(x) \leq c\lambda\}| \leq \frac{cC}{b}|\{x \in \Omega : |Tf(x)| > b\lambda\}|$$

where c can be chosen small enough so that $\frac{cC}{b} < b^p$ (needed in the next proposition). A weighted result follows for $w \in A_\infty$ as $|E| \leq \epsilon|B|$ implies $w(E) \leq \delta w(B)$. $\square$

Proposition 6.21. *Suppose that* $\|M^{\alpha,\beta} f\|_{\mathcal{L}^p(w)} < \infty \implies \|Tf\|_{\mathcal{L}^p(w)} < \infty$ *for all* $f \in \mathcal{L}^p(w)$ *and that* T *satisfies the good-*λ *inequality (6.13) for the same weight with* $a < b^p$*. Then* T *is bounded* $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ *for the same weight.*

This proposition is proven much more generally in Stein [67] section IV.3.5 lemma 2 so will not be proven here. This concludes the proof of Theorem 6.6.

Part II

Application to the Dirichlet Laplacian

Chapter 7:

Heat Kernel Bounds

At the core of the operators considered in this thesis is the heat kernel $p_t(x, y)$. In this chapter pointwise bounds for the heat kernel are found in several application cases.

The Dirchlet heat kernel of $\Omega \subset \mathbb{R}^n$ is the unique continuous solution of the partial differential equation $\left(\Delta_\Omega + \frac{\mathrm{d}}{\mathrm{d}t}\right)p_t(x, y) = 0$, with initial condition $\lim_{t\to 0} p_t(x, y) = \delta(x - y)$ (Δ_Ω was defined at the end of section 1.2). It is well known that the solution $p_t(x, y)$ is the same as the solution to the traditional heat equation on Ω with Dirichlet boundary conditions.

$$\begin{cases} \left(\Delta + \frac{\mathrm{d}}{\mathrm{d}t}\right)p_t(x, y) = 0 & \forall\, x, y \in \Omega \\ \lim_{t\to 0} p_t(x, y) = \delta(x - y) & \forall\, x, y \in \Omega \\ p_t(x, y)\Big|_{x\in\delta\Omega} = 0 & \forall y \in \Omega \end{cases}$$

The purpose of this chapter is to provide a combination of estimates for a variety of heat kernels. There is a mix of known results and new results. Various domains defined in section 2.3 are used.

THEOREM 7.1 (From [74]). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be the exterior of a compact $C^{1,1}$ domain. Then the heat kernel of Ω has upper and lower bounds of the form of equation (7.1).*

$$p_t(x, y) \sim c_1 \left(1 \wedge \frac{\rho(x)}{\sqrt{t} \wedge 1}\right)\left(1 \wedge \frac{\rho(y)}{\sqrt{t} \wedge 1}\right) \frac{e^{-d(x,y)^2/c_2 t}}{t^{n/2}} \tag{7.1}$$

Here $\rho(x)$ is the distance from x to $\delta\Omega$, $d(x, y)$ is the distance from x to y in Ω, and $\wedge$ indicates the minimum of the terms on either side.

Zhang proves this theorem in [74] using a *local comparison principle* from Fabes, Garofalo and Salsa [29]. Zhang's proof uses the local comparison principle to compare the Dirichlet heat kernel on Ω to the Green's function on Ω to achieve the desired bounds. The away from the boundary case was originally proven by Grigor'yan and Saloff-Coste in [35] who also prove the $n = 2$ case stated below in the same paper.

THEOREM 7.2 (From [35]). *Let $\Omega \subset \mathbb{R}^2$ be the exterior of a compact $C^{1,1}$ domain. Then the heat kernel of Ω has upper and lower bounds of the form of equation (7.2).*

$$p_t^{B^c}(x,y) \sim c_1 \left(1 \wedge \frac{\log(1+\rho(x))}{\log(1+\sqrt{t})}\right) \left(1 \wedge \frac{\log(1+\rho(y))}{\log(1+\sqrt{t})}\right) \frac{e^{-d(x,y)^2/c_2 t}}{t^{n/2}} \tag{7.2}$$

Zhang generalised Theorem 7.1 in [73] by showing that *inner uniform and* $C^{1,1}$ domains $\Omega \subset \mathbb{R}^n$, $n \geq 3$, satisfy for t small enough ($0 \leq t < T$ for some fixed $T > 0$) upper and lower bounds on the heat kernel of the form of equation (7.3) below for all $x, y \in \Omega$. Hence the Dirichlet heat kernels of all inner uniform $C^{1,1}$ domains act similar for small t.

Song in [62] applied Zhang's method to show that for Ω as the area above a bounded Lipschitz $C^{1,1}$ function, equation (7.3) holds for all $x, y \in \Omega$ as in the next theorem.

THEOREM 7.3. *Let $\Omega \subset \mathbb{R}^n$ be a $C^{1,1}$ global Lipschitz domain (this is the space above a $C^{1,1}$ globally Lipschitz and bounded curve $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$). Then the heat kernel of Ω has upper and lower bounds of the form of equation (7.3).*

$$p_t^{\mathbb{H}}(x,y) \sim c_1 \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) \frac{e^{-d(x,y)^2/c_2 t}}{t^{n/2}} \tag{7.3}$$

Song's proof and Zhang's method are visited in section 7.1 of this chapter. Similar results are contained in [71]. Next consider the special case of $\Omega \subset \mathbb{R}^n$, $n \geq 3$, as a bounded $C^{1,1}$ domain.

THEOREM 7.4 (Combining [20], [23] and [73]). *Let Ω be the interior of a $C^{1,1}$ bounded domain. Then the heat kernel of Ω has upper and lower bounds,*

$$p_t^{\mathbb{B}}(x,y) \sim c_1 \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) \frac{e^{-d(x,y)^2/c_2 t} e^{-\lambda_1 t}}{t^{n/2}}$$

where λ_1 is the first eigenvalue of the Dirichlet Laplacian on Ω.

For $0 < t < T$ the upper bound of Theorem 7.4 was originally shown by Davies [20] and Davies and Simon [23]. These papers also contained proofs showing upper and lower bounds for $t > T > 0$. The lower bound for $0 < t < T$ is from [73].

Back to the case of $\Omega \subset \mathbb{R}^n$ as a general *inner uniform domain* (here $n \geq 2$), Gyrya and Saloff-Coste prove in [37] and [38] heat kernel upper and lower bounds,

$$p_t(x,y) \sim \frac{h(x)h(y)e^{-d(x,y)^2/c_4 t}}{\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}}$$

where h is harmonic on Ω and satisfies Dirichlet boundary conditions. The $V_{h^2}(x,\sqrt{t})$ term means the volume of a ball weighted by h^2. The proof uses Doob's h-transform from Gaussian bounds (see [24] page 102 for details on this technique). Gyrya and Saloff-Coste's results include discussion on removing smoothness conditions. Applying this idea to the area below a parabola in $\mathbb{R}^2$ gives the final theorem of this chapter.

THEOREM 7.5. *Let Ω be the area below a parabola $x_1 = x_2^2$ in $\mathbb{R}^n$. Then the heat kernel of Ω has the following upper bound, where $R = \max(x_2, 1)$.*

$$p_t^\Omega(x,y) \lesssim \sqrt{\left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\rho(x)}{R \wedge \sqrt{t}}\right)\left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right)\left(1 \wedge \frac{\rho(y)}{R \wedge \sqrt{t}}\right)}\,\frac{e^{-d(x,y)^2/c_2 t}}{t^{n/2}}$$

Theorems 7.1, 7.2 and 7.4 were mentioned as having being proven elsewhere so will not be proven here. Theorem 7.3 is also a known result and is reviewed in section 7.1. Theorem 7.5 is new and is proven in section 7.2 via observations from [38]. Upper bounds for further cases via the maximum principle and symmetry, including non-doubling cases, are in section 7.3. The symmetry of the heat kernels can be used as the Dirichlet Laplacian is self-adjoint. Problems can occur when the boundary $\delta\Omega$ is not smooth, see comments in section 10.3.

A summary of the outcome of the above theorems is included in Figure 7.1.

7.1 Above a Lipschitz Function

In this section Song's proof in [62] giving bounds for the heat kernel above a bounded Lipschitz $C^{1,1}$ function is reviewed to emphasize how the method applies to the area above a $C^{1,1}$ global Lipschitz function as in Theorem 7.3. The following lemmas are required as part of both Zhang's [74] and Song's [62] proofs.

Figure 7.1: Coefficients of the Gaussians in Theorems 7.1 to 7.5.

	Interior of a Ball	Above a Lipschitz function	Exterior of a Parabola	Exterior of a Ball
t small	$1 \wedge \frac{\rho(x)}{\sqrt{t}}$	$1 \wedge \frac{\rho(x)}{\sqrt{t}}$	$1 \wedge \frac{\rho(x)}{\sqrt{t}}$	$1 \wedge \frac{\rho(x)}{\sqrt{t}}$
t large	$\rho(x)e^{-\lambda_1 t}$	$1 \wedge \frac{\rho(x)}{\sqrt{t}}$	$1 \wedge \frac{\sqrt{\rho(x)}}{t^{1/4}} \wedge \frac{\rho(x)}{R^{1/2}t^{1/4}}$	$1 \wedge \rho(x)$

Lemma 7.6. *Suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is the interior of a compact $C^{1,1}$ domain. Then the Green's function on Ω with Dirichlet boundary conditions on $\delta\Omega$ satisfies,*

$$G^{\Omega}(x,y) \sim \left(1 \wedge \frac{\rho(x)}{|x-y|}\right)\left(1 \wedge \frac{\rho(y)}{|x-y|}\right)\frac{1}{|x-y|^{n-2}} \tag{7.4}$$

where the constants depend only on n, the $C^{1,1}$ constant m and the ratio diameter$(\Omega)/r$.

See Definition 2.18 for details on the values of r and m. The upper bound was originally proven in [36] and the lower bound in [77]. The comment on the dependence of the constant is from [12]. The next lemma is Theorem 1.6 from [29] specified for our circumstance.

Lemma 7.7 (Local Comparison Theorem)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is the interior of a compact $C^{1,1}$ domain. Consider a point $(x_0, s) \in \delta\Omega \times (0,T)$ and the cylinder about (x_0, s) given by,*

$$\Psi(r) = \{(x,t) \in \Omega \times (0,\infty) : 0 < t < T, |x - x_0| < 2r, |t-s| < 4r^2\}$$

for some $r < min(\frac{1}{2}r_0, \frac{1}{2}\sqrt{s}, \frac{1}{2}\sqrt{T-s})$. Let u, v be two positive solutions of the heat equation $(\Delta + \frac{\mathrm{d}}{\mathrm{d}t})u = 0$ on Ψ that vanish continuously on $\delta\Omega$. Then the following bound holds for all $(x,t) \in \Psi(r/8)$,

$$\frac{u(x,t)}{v(x,t)} \leq c\frac{u(x_r, t+2r^2)}{v(x_r, t-2r^2)} \tag{7.5}$$

where $x_r \in \Omega$ is a point r away from the boundary point x_0. That is, $x_r = x_0 + \hat{x}_j r$ where $\hat{x}_j$ is a unit vector pointing away from the boundary.

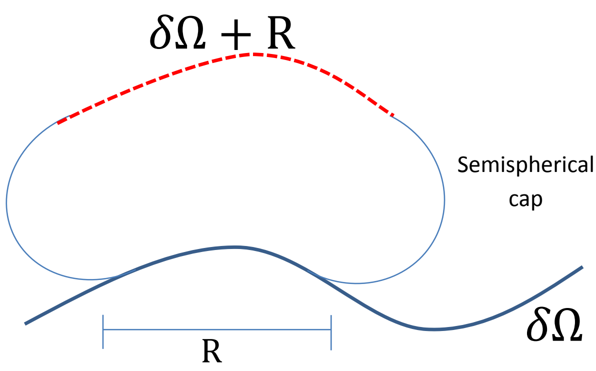
Lemma 7.7 allows a comparison between the Dirichlet heat kernel on Ω to other solutions of the heat equation, such as the Green's function from Lemma 7.6. It is proven in [29] using a Carleson estimate from [59]. It remains to determine how the area above a $C^{1,1}$ globally Lipschitz curve where the heat kernel is sought, and the area inside a compact $C^{1,1}$ domain where the Green's function is known, will be compared. The final lemma presented is Lemma 2.2 from the paper [62].

Lemma 7.8. *Consider a domain $\Omega \subset \mathbb{R}^n$ defined as the area above a smooth $C^{1,1}$ Lipschitz function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$. Then for any $x' \in \mathbb{R}^{n-1}$ and any $R \geq 1$ there exists a bounded $C^{1,1}$ domain $D(x, R) \subset \mathbb{R}^n$ such that the boundary δD includes the set:*

$$\{y = (y_1, y') \in \overline{\Omega} : |x' - y'| \leq R \text{ and either } y_1 = \psi(y') \text{ or } y_1 = \psi(y') + R\}.$$

Further the $C^{1,1}$ constant m of D is bounded above by a constant that does not depend on x or R, and the ratio R/r is also bounded above by a constant (where r is the other $C^{1,1}$ constant of D).

Figure 7.2: Cross-section of the construction of the bounded domain D to approximate the smooth Lipschitz domain Ω.



Proof. Fix R. The key is to join $\delta\Omega$ and $\delta\Omega + R$ by an approximate semi-spherical cap of radius cR for some universal constant c. Fix the $C^{1,1}$ constant r of D as cR, so that R/r is bounded above. The radius of the semi-spherical cap part of the boundary of D grows proportional to R so the $C^{1,1}$ constant m of this part is fixed with $r = cR$ (c small enough). The $C^{1,1}$ constant m of the $\delta\Omega$ and $\delta\Omega + R$ parts of the boundary of D is also fixed for all r due to the global Lipschitz condition. At the joins between $\delta\Omega$ and $\delta\Omega + R$

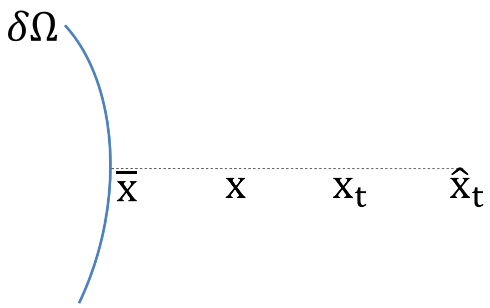
and the semi-spherical caps the contribution to the $C^{1,1}$ constant m will be smaller. The maximum of the m terms mentioned is the $C^{1,1}$ constant of D. See Figure 7.2.

This only works if the boundary of Ω can be described by either a bounded function, or a Lipschitz function. Otherwise the semi-spherical cap joining $\delta\Omega + R$ and $\delta\Omega$ may not have a radius that grows with R. $\square$

Now is the proof of Theorem 7.3.

Proof of Theorem 7.3. The boundary of Ω is given by $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ which is continuous, Lipschitz and $C^{1,1}$. Hence there exists $\kappa \in (0,1)$ where $\kappa(x_1 - \psi(x')) \leq \rho(x)$ for all $x = (x_1, x') \in \Omega$. Choose a point $(x, y, t) \in \Omega \times \Omega \times (0, \infty)$. There are several cases in this proof, start by considering the case when $x_1 - \psi(x') \leq \sqrt{a_1 t}$ and $y_1 - \psi(y') \geq \frac{4}{\kappa}\sqrt{a_1 t}$. Define $\bar{x} = (\psi(x'), x')$, $x_t = (\psi(x') + \sqrt{2a_1 t}, x')$ and $\hat{x}_t = (\psi(x') + \frac{4}{\kappa}\sqrt{a_1 t}, x')$ as in Figure 7.3.

Figure 7.3: The relative positions of the terms in the proof of Theorem 7.3.



Construct a bounded $C^{1,1}$ domain $D(x, \frac{8}{\kappa}\sqrt{a_1 t})$ as in Lemma 7.8. Inside such a domain the local comparison principle (Lemma 7.7) can be used to get,

$$p_t^{\Omega}(x,y) \lesssim \frac{G_{D(x,\frac{8}{\kappa}\sqrt{a_1 t})}(x, \hat{x}_t)}{G_{D(x,\frac{8}{\kappa}\sqrt{a_1 t})}(x_t, \hat{x}_t)} p_{2t}^{\Omega}(x_t, y)$$

where the constant does not depend on x, y or t. Use the bounds for the Green's function found in Lemma 7.6, and the constructed distances $|x - \hat{x}_t|$ and $|x_t - \hat{x}_t|$.

$$p_t^{\Omega}(x,y) \lesssim \frac{|x_t - \hat{x}_t|^{n-2}\rho(x)}{|x - \hat{x}_t|^{n-1}} p_{2t}^{\Omega}(x_t, y) \leq c\frac{\rho(x)}{\sqrt{t}} p_{2t}^{\Omega}(x_t, y)$$

To conclude the first case of this proof use that $p_t^\Omega(x,y)$ has Gaussian upper bounds by the maximal principle and that $|x-y| \geq 3|x_t - y|$ by construction.

$$p_t^\Omega(x,y) \lesssim \frac{\rho(x)}{\sqrt{t}} \frac{e^{-|x_t-y|^2/2ct}}{(2t)^{n/2}} \lesssim \frac{\rho(x)}{\sqrt{t}} \frac{e^{-|x-y|^2/18ct}}{t^{n/2}}$$

which concludes the case $x_n - \psi(x') \leq \sqrt{a_1 t}$ and $y_n - \psi(y') \geq \frac{4}{\kappa}\sqrt{a_1 t}$. Next suppose that $x_n - \psi(x') \leq \sqrt{a_2 t}$ and $y_n - \psi(y') \leq \sqrt{a_2 t}$. In the previous case the following equation occurred without using the condition on y.

$$p_t^\Omega(x,y) \lesssim \frac{\rho(x)}{\sqrt{t}} p_{2t}^\Omega(x_t, y) \tag{7.6}$$

Use the method of the previous case with $\bar{y}$, y_t and $\hat{y}_t$ replacing $\bar{x}$, x_t and $\hat{x}_t$ and with x_t replacing y and $2t$ replacing t.

$$p_{2t}^\Omega(y, x_t) \lesssim \frac{\rho(y)}{\sqrt{t}} p_{4t}^\Omega(y_t, x_t)$$

Use Gaussian upper bounds on the heat kernel and that $|x_t - y_t| \geq |x-y| - c\sqrt{a_2 t}$.

$$p_{2t}^\Omega(y, x_t) \lesssim \frac{\rho(y)}{\sqrt{t}} \frac{e^{-|x_t-y_t|^2/4ct}}{(4t)^{n/2}} \lesssim \frac{\rho(y)}{\sqrt{t}} \frac{e^{-|x-y|^2/4ct}}{t^{n/2}}$$

Put this into equation (7.6) to get,

$$p_t^\Omega(x,y) \lesssim \frac{\rho(x)\rho(y)}{t} \frac{e^{-|x-y|^2/4ct}}{t^{n/2}}$$

which concludes the second case. Gaussian upper bounds and symmetry prove the remaining cases. Take the diameter R of the bounded domain D to infinity so that D tends to Ω for the result. □

7.2 The Area Below a Parabola

In this section Theorem 7.5 is proven, which is upper and lower bounds for the heat kernel for Ω as the area below the parabola $x_2 = x_1^2$. This domain falls between the two cases already considered (that of an exterior domain, and that of a global Lipschitz domain), so results between these two is expected by the maximal principle. In the paper [38] by Gyrya and Saloff-Coste, upper and lower bounds for the Dirichlet heat kernel on inner uniform domains Ω are given as,

$$c_1 \frac{h(x)h(y)e^{-d(x,y)^2/ct}}{\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}} \le p_t^{\Omega}(x,y) \le c_2 \frac{h(x)h(y)e^{-d(x,y)^2/ct}}{\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}} \tag{7.7}$$

where $h(x)$ is harmonic on Ω with Dirichlet boundary conditions, and $V_{h^2}(x,\sqrt{t})$ is given by the following equation.

$$V_{h^2}(x,\sqrt{t}) = \int_{B(x,\sqrt{t})} |h(w)|^2 \, dw.$$

Further in [38], an example is given of h in the case of Ω as the area below the parabola $x_2 = x_1^2$ in $\mathbb{R}^2$.

$$h(x) = h(x_1,x_2) = \sqrt{2(\sqrt{x_1^2 + (x_2 - \frac{1}{4})^2} - x_2 + \frac{1}{4}) - 1}$$

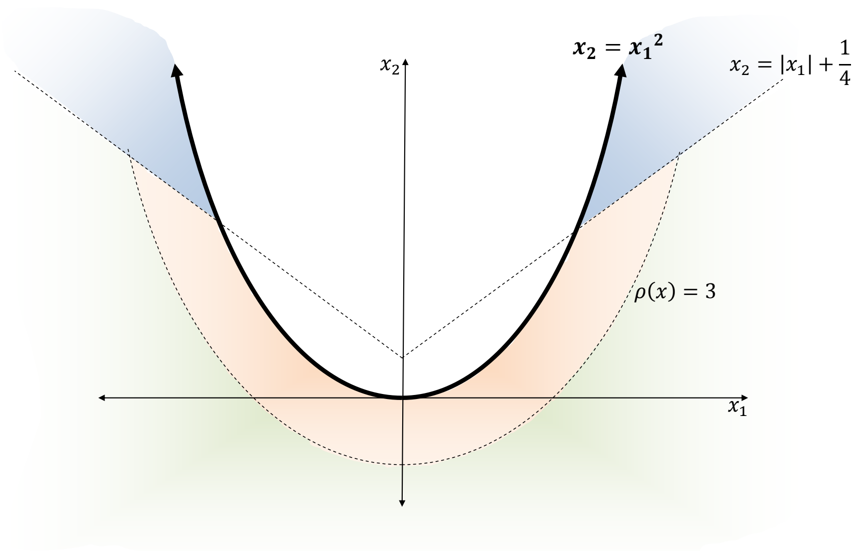
To find reasonable bounds for the parabola heat kernel, the above equations are examined on three separate exterior regions. These are formed according to whether a point is above or below the wedge $x_2 = |x_1| + \frac{1}{4}$, and for x below the wedge, whether near or away from the boundary (see Figure 7.4). Start by considering the area below the wedge, and away from the boundary, in the following lemma.

Lemma 7.9. *Let Ω denote the area below the parabola $x_2 = x_1^2$ in $\mathbb{R}^2$. Consider those $x = (x_1,x_2) \in \Omega$ where $x_2 < |x_1| + 1/4$, and $\rho(x) > 3$. Then the harmonic function on Ω satisfies for such x: $h(x) \sim \sqrt{\rho(x)}$. Further the coefficient $\frac{h(x)}{\sqrt{V_{h^2}(x,\sqrt{t})}}$ to the Gaussian in the heat kernel bound satisfies upper and lower estimates of the form:*

$$\frac{h(x)}{\sqrt{V_{h^2}(x,\sqrt{t})}} \sim \frac{1}{\sqrt{t}}\Big(1 \wedge \frac{\sqrt{\rho(x)}}{t^{1/4}}\Big)$$

Proof. For each x_1, x_2 pair satisfying $x_2 < |x_1| + 1/4$, either $x_1 = 0$ or there exists

Figure 7.4: Parabola in $\mathbb{R}^2$ showing the regions dealt with separately.



$a \in (-\infty, \infty)$ where $x_2 = ax_1 + 1/4$. Hence $h(x)$ has the following bound,

$$\begin{aligned} h(x) = h(x_1, x_2) &= \sqrt{2(\sqrt{x_1^2 + (x_2 - \frac{1}{4})^2} - x_2 + \frac{1}{4})} - 1 \\ &= \begin{cases} \sqrt{|x_1|}\sqrt{2(\sqrt{1+a^2} - a)} - 1 & \text{if } -1 \leq a \leq 1 \\ \sqrt{(\frac{1}{4} - x_2)}\sqrt{2(\sqrt{(1+a^{-2})}) + a^{-1}} - 1 & \text{if } 1 < |a| \\ \sqrt{(\frac{1}{4} - x_2)}\sqrt{2} - 1 & \text{if } x_1 = 0 \end{cases} \\ &\sim \left(x_1^2 + (x_2 - 1/4)^2\right)^{1/4} - 1 \\ &\sim \left(x_1^2 + x_2^2\right)^{1/4} \sim \sqrt{\rho(x)} \end{aligned}$$

which holds away from the boundary concluding the first part of the proof. For the second half consider the case where $x_2 < 0$ and $x_1 > 0$ (other cases are proven identically, this assumption only affects the limits used in the integrals). Integrate $h(x)^2$ to get V_{h^2}.

$$V_{h^2}(x, \sqrt{t}) \sim \int_{x_1}^{x_1+\sqrt{t}} \int_{x_2-\sqrt{t}}^{x_2} (\xi_1^2 + \xi_2^2)^{1/2} \, \mathrm{d}\xi_2 \, \mathrm{d}\xi_1$$

Then as $\rho(x) > 1$, and $x_2 < 0$,

$$
\begin{aligned}
V_{h^2}(x, \sqrt{t}) &\sim \int_{x_1}^{x_1+\sqrt{t}} \int_{x_2-\sqrt{t}}^{x_2} \xi_1 - \xi_2 \, \mathrm{d}\xi_2 \, \mathrm{d}\xi_1 \\
&\sim \frac{1}{2}\Big[(x_1 + \sqrt{t})^2\sqrt{t} - x_1^2\sqrt{t} - x_2^2\sqrt{t} + (-x_2 + \sqrt{t})^2\sqrt{t}\Big] \\
&\sim \big[x_1 t + 2t^{3/2} - x_2 t\big] \sim \rho(x)t + 2t^{3/2}
\end{aligned}
$$

so that, $\dfrac{h(x)}{\sqrt{V_{h^2}(x, \sqrt{t})}} \sim \dfrac{\sqrt{\rho(x)}}{\sqrt{t}\sqrt{\rho(x) \vee \sqrt{t}}} \sim \dfrac{1}{\sqrt{t}}\Big(1 \wedge \dfrac{\sqrt{\rho(x)}}{t^{1/4}}\Big)$ which concludes the proof.

□

The next lemma is the below the wedge and near the boundary case.

Lemma 7.10. *Let Ω be the area below the parabola $x_2 = x_1^2$, and consider the set of all $x = (x_1, x_2) \in \Omega$ satisfying $x_2 < |x_1| + 1/4$ and close to the origin $\rho(x) \le 3$. Then in this region $h(x) \sim \rho(x)$, and the following bound holds.*

$$
\frac{h(x)}{\sqrt{V_{h^2}(x, \sqrt{t})}} \sim \frac{1}{\sqrt{t}}\Big(1 \wedge \frac{\rho(x)}{\sqrt{t} \wedge t^{1/4}}\Big)
$$

Proof. Firstly it is a trivial exercise to calculate $|\frac{\mathrm{d}}{\mathrm{d}x_2} h(x_1, x_2)|$ and observe this value is bounded by positive constants above and below for $-4 \le x_2 \le 4$ which fits the appropriate region. This along with knowing that $h(x) = 0$ on the boundary $\delta\Omega$, but non-zero inside Ω, allows us to conclude $h(x) \sim \rho(x)$ in this case.

For the second half of the proof, first observe that as the area below a parabola is a $C^{1,1}$ and inner uniform domain, an upper and lower bound for $0 < t < 1$ is given by Zhang in [73].

$$
\frac{h(x)}{\sqrt{V_{h^2}(x, \sqrt{t})}} \sim \frac{1}{\sqrt{t}}\Big(1 \wedge \frac{\rho(x)}{\sqrt{t}}\Big)
$$

If $t > 1$ then the integration of the ball that $V_{h^2}(x, y)$ is integrated over, is dominated by values in the below the wedge, away from the boundary case. In that case it was shown there is a bound of the form $V_{h^2}(x, \sqrt{t}) \sim \rho(\tilde{x})t + 2t^{3/2}$ where in this case $\tilde{x}$ is the closest point to x in the away from boundary case. This means $\rho(\tilde{x}) \sim 1$. As $t > 1$ this means $V_{h^2}(x, \sqrt{t}) \sim t^{3/2}$. Thus the final upper bound in this case when $t > 1$ is given by the following.

$$\frac{h(x)}{\sqrt{V_{h^2}(x,\sqrt{t})}} \sim \frac{1}{\sqrt{t}}\Big(\frac{\rho(x)}{t^{1/4}}\Big)$$

Mix this with the $t < 1$ case to conclude the proof. $\square$

The final lemma considers the above the wedge case. This lemma continues on the same theme as the previous lemmas.

Lemma 7.11. *Let Ω be the area below the parabola $x_2 = x_1^2$, and consider the set of all $x = (x_1, x_2) \in \Omega$ where $x_2 > |x_1| + 1/4$. Then $h(x) = h(x_1, x_2) \sim \frac{\rho(x)}{\sqrt{x_2}}$, which leads to the following bound.*

$$\frac{h(x)}{\sqrt{V_{h^2}(x,\sqrt{t})}} \sim \frac{1}{\sqrt{t}}\left(1 \wedge \frac{\rho(x)}{\sqrt{t}\wedge t^{1/4}\sqrt{x_2}}\right)$$

Proof. The proof is for $x_1 > 0$, a similar proof holds for $x_1 < 0$. Start with the inequalities, $0 < \sqrt{x_2} \le x_1 \le x_2$ which hold for all x_1, x_2 in the region. Basic use of this ordering allows the determination of the truth of the following inequality.

$$\frac{x_1^2}{4x_2} - \frac{(2\sqrt{2}-5)}{2} + \frac{(2-3\sqrt{2})x_2}{x_1^2} - 4x_2 + \frac{4(2-\sqrt{2})x_2^2}{x_1^2} \le 1 \le \frac{x_1^2}{x_2}$$

Now run through some manipulations applying to all parts of the inequality: subtract 1 and multiply through by $x_1^2/4x_2$, then add $x_1^2 + (x_2 - \frac{1}{4})^2$. Take square root then add $-x_2 + 1/4$, multiply by 2, square root again and subtract 1 to get,

$$\frac{x_1}{\sqrt{2x_2}} - \frac{1}{\sqrt{2}} \le \sqrt{2(\sqrt{x_1^2 + (x_2 - \frac{1}{4})^2} - x_2 + 1/4)} - 1 \le \frac{x_1}{\sqrt{x_2}} - 1$$

which is more compactly expressed as the following bound.

$$\frac{x_1}{\sqrt{2x_2}} - \frac{1}{\sqrt{2}} \le h(x) \le \frac{x_1}{\sqrt{x_2}} - 1$$

In the region under consideration, and special case $x_1 > 0$, the distance to the boundary is majorised by the x_1 direction as $x_1 - f^{-1}(x_1) = x_1 - \sqrt{x_2}$ (where $f(x_1) = x_1^2$ describes the boundary). So $\rho(x) \sim x_1 - \sqrt{x_2}$ and the result of the lemma for $h(x)$ follows.

Use that h grows away from the boundary and $h \sim (\frac{x_1}{\sqrt{x_2}} - 1)$ to get a bound for V_{h^2}.

$$V_{h^2}(x, \sqrt{t}) \sim \int_{x_1}^{x_1+\sqrt{t}} \int_{x_2-\sqrt{t}}^{x_2} |h(\xi_1, \xi_2)|^2 \,\mathrm{d}\xi_1 \,\mathrm{d}\xi_2 \sim \int_{x_1}^{x_1+\sqrt{t}} \int_{x_2-\sqrt{t}}^{x_2} \frac{\xi_1^2}{\xi_2} - \frac{2\xi_1}{\sqrt{\xi_2}} + 1 \,\mathrm{d}\xi_2 \,\mathrm{d}\xi_1$$

Evaluate this integral.

$$V_{h^2} \sim [(x_1+\sqrt{t})^3 - x_1^3] \log(\frac{x_2}{x_2-\sqrt{t}}) - 2[(x_1+\sqrt{t})^2 - x_1^2](\sqrt{x_2} - \sqrt{x_2-\sqrt{t}}) + t$$

Now consider two separate cases. For the first take $0 < 2\sqrt{t} < x_2$, then use $\log(1+a) \sim a$ and $\sqrt{b} - \sqrt{b-a} \sim a/\sqrt{b}$ for small a and expand.

$$\begin{aligned} V_{h^2}(x, \sqrt{t}) &\sim [3x_1^2\sqrt{t} + 3x_1 t + t^{3/2}]\frac{\sqrt{t}}{x_2} - 2[2x_1\sqrt{t} + t]\frac{\sqrt{t}}{\sqrt{x_2}} + t \\ &\sim \frac{3x_1^2 t}{x_2} + \frac{3x_1 t^{3/2}}{x_2} + \frac{t^2}{x_2} - \frac{4x_1 t}{\sqrt{x_2}} - \frac{2t^{3/2}}{\sqrt{x_2}} + t \end{aligned}$$

Next use $\rho(x) \sim (x_1 - \sqrt{x_2})$, and that for any $a, b > 0$ then $a + \sqrt{ab} + b \sim a + b$.

$$V_{h^2}(x, \sqrt{t}) \sim \frac{\rho(x)^2 t}{x_2} + \frac{\rho(x) t^{3/2}}{x_2} + \frac{t^2}{x_2} \sim \max\left(\frac{\rho(x)^2 t}{x_2}, \frac{t^2}{x_2}\right)$$

Which a bound in this case.

$$\frac{h(x)}{\sqrt{V_{h^2}(x, \sqrt{t})}} \sim \frac{\rho(x)}{\sqrt{x_2}\sqrt{\max\left(\frac{\rho(x)^2 t}{x_2}, \frac{t^2}{x_2}\right)}} \sim \frac{1}{\sqrt{t}}\Big(1 \wedge \frac{\rho(x)}{\sqrt{t}}\Big)$$

It remains to check the case that $2\sqrt{t} > x_2$. In this case the box integrated over will go outside the wedge and tend to being dominated by the outside the wedge case. By the outside the wedge case there is an upper bound of the form $V_{h^2}(x, \sqrt{t}) \sim \rho(\tilde{x})t + 2t^{3/2}$ where $\tilde{x}$ is the closest point to x lying on the wedge. In this case $\rho(\tilde{x}) \sim x_2$. So then $2\sqrt{t} > x_2$ implies $V_{h^2}(x, \sqrt{t}) \sim t^{3/2}$ and there is an upper bound of the form,

$$\frac{h(x)}{\sqrt{V_{h^2}(x, \sqrt{t})}} \sim \frac{1}{\sqrt{t}}\Big(\frac{\rho(x)}{t^{1/4}\sqrt{x_2}}\Big)$$

which concludes the proof. □

The upper bound for the heat kernel in the parabola case is now fully determined in terms of t and $\rho(x)$. The result proven so far for the parabola is summarised below.

$$\frac{h(x)}{\sqrt{V_{h^2}}} \sim \frac{1}{\sqrt{t}} \begin{cases} \left(1 \wedge \frac{\rho(x)}{t^{1/4}(1\wedge t^{1/4})}\right) & x \text{ below wedge and } \rho(x) \leq 3 \\ \left(1 \wedge \frac{\sqrt{\rho(x)}}{t^{1/4}}\right) & x \text{ below wedge and } \rho(x) > 3 \\ \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right) & x \text{ above wedge and } \sqrt{t} \leq x_2 \\ \left(\frac{\rho(x)}{t^{1/4}\sqrt{x_2}}\right) & x \text{ above wedge and } \sqrt{t} > x_2 \end{cases} \tag{7.8}$$

Proof of Theorem 7.5. To prove this theorem all that needs to be done is to show that,

$$\frac{h(x)}{\sqrt{V_{h^2}}} \sim \frac{1}{\sqrt{t}} \sqrt{\left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\rho(x)}{\max(x_2, 1) \wedge \sqrt{t}}\right)} \tag{7.9}$$

then the proof is complete by the result of Gyrya and Saloff-Coste in [38] mentioned in equation (7.7) of this thesis. From the summary of the previous lemmas in equation (7.8) it is not difficult to see that equation (7.9) is true. The harmonic function does not change when considering the area below the parabola as a subset of $\mathbb{R}^n$, and hence the full result of Theorem 7.5 follows. □

This concludes the parabola example.

7.3 Extensions and Discussion

Here are collected various remarks and extensions on the theorems of this chapter.

7.3.1 General kernels

Consider what might be expected as an upper bound for a space between (in subset sense) the exterior of a $C^{1,1}$ compact object, and the area above a $C^{1,1}$ global Lipschitz curve. One such domain is the area below a parabola (as considered in section 7.2) where the heat kernel found was very similar to a multiplication average of the heat kernels of the global Lipschitz and exterior domain cases. With the motivation of Hardy and Riesz transform bounds in mind, hypothesize a multiplication average of the form,

$$\begin{aligned} p_t(x,y) &= p_t^H(x,y)^{1/q}\, p_t^{B^c}(x,y)^{1/q'} \qquad (7.10) \\ &\sim \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right)^{\frac{1}{q}} \left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right)^{\frac{1}{q}} \left(1 \wedge \frac{\rho(x)}{1 \wedge \sqrt{t}}\right)^{\frac{1}{q'}} \left(1 \wedge \frac{\rho(y)}{1 \wedge \sqrt{t}}\right)^{\frac{1}{q'}} \frac{e^{-d(x,y)^2/ct}}{t^{n/2}} \end{aligned}$$

for some $q > 1$, where p_t^H is the global Lipschitz heat kernel of Theorem 7.3 and $p_t^{B^c}$ is the exterior domain heat kernel of Theorem 7.1. When finding $\mathcal{L}^p \to \mathcal{L}^p$ bounds for the Hardy inequality in chapter 9 of this thesis:

- The Hardy inequality for the exterior of a ball is bounded for all $1 < p < n$; and,
- The Hardy inequality in the halfspace case is bounded for all $1 < p < \infty$.

Interestingly, the Hardy inequality for a weighted average kernel of the type of equation (7.10) would be bounded for all $1 < p < qn/(q-1)$. This leads to Riesz transform bounds of the same type. It is unknown what shapes would satisfy the averaged kernel.

For shapes between the halfspace and the interior of a ball, weighted averages are not as insightful due to domination by the exponential decay of the interior case. Some work has been done for the interior of a parabola in [11].

There are many spaces where there is little known about heat kernel upper bounds. The main emphasis of this discussion is to note that while only a small fraction of spaces have good known upper bounds for their heat kernel, if more kernel upper bounds are found for specific spaces that fit the pattern above, or are indeed even only of the general form,

$$p_t(x,y) = \alpha_t(x)\beta_t(y)\frac{e^{-|x-y|^2/t}}{t^{n/2}}$$

then the techniques in the previous and following chapters can still use this to determine $\mathcal{L}^p$ boundedness of the associated Riesz transform for certain p.

7.3.2 A Non doubling domain

Consider the following domain between the lines $x_2 = 0$ and $x_2 = e^{x_1}$.

$$\Omega = \{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_2 < e^{x_1}\}$$

This domain is clearly non-doubling and can be written as the intersection of two domains which are now analysed separately. Firstly the area in $\mathbb{R}^2$ above the line $x_2 = 0$ has heat kernel upper bound,

$$p_t^{\{x_2>0\}}(x,y) \lesssim \left(1 \wedge \frac{\rho_1(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\rho_1(y)}{\sqrt{t}}\right)\frac{e^{-d(x,y)^2/ct}}{t^{n/2}}$$

where $\rho_1(x) = x_2$ is the distance from the point x to the line $x_2 = 0$. Secondly, the area below the line $x_2 = e^{-x_1}$ is not a global Lipschitz and bounded function, but with the influence of the other boundary it is still expected for there to be a heat kernel upper bound of the form,

$$p_t^{\{x_2<e^{x_1}\}}(x,y) \lesssim \left(1 \wedge \frac{\rho_2(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\rho_2(y)}{\sqrt{t}}\right)\frac{e^{-d(x,y)^2/ct}}{t^{n/2}}$$

where $\rho_2(x)$ is the distance from the point x to the line $x_2 = e^{x_1}$. The heat kernel for the region Ω between the lines $x_2 = 0$ and $x_2 = e^{x_1}$ is bounded above by the minimum of these two kernels by the maximum principle. It is likely that better heat kernel upper bounds exist for Ω in this case, however this maximal principle idea is enough to invoke the theorems of chapters 4 or 5 to get Riesz transform bounds on the domain Ω, and is also used in section 9.3 to get bounds for the Hardy inequality on Ω.

Chapter 8:

Heat Semigroup and Related Bounds

In this chapter the first parts of Theorem 1.5 are proven, along with some other heat semigroup theorems. This relates to solving heat semigroup and gradient bounds corresponding to conditions (1.3), (1.9) and (1.10) from the introduction chapter, and their weighted versions (5.5), (5.7) and (5.8).

The space $\Omega \subset \mathbb{R}^n$ is assumed to be open and connected with smooth boundary. In some parts Ω is named specifically, in other parts the properties of Ω are kept general. The operator Δ_Ω continues as the Dirichlet Laplacian on $\Omega \subset \mathbb{R}^n$. The first section of this chapter proves the following extrapolation on the first part of Theorem 1.5.

THEOREM 8.1. *Consider the heat semigroup $e^{-t\Delta_\Omega}$ with Ω a $C^{1,1}$ global Lipschitz domain. Then $e^{-t\Delta_\Omega}$ is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for all weights w in the following class.*

$$A_p^{Lip1}(w) = \sup_{B\subset\Omega} \left(\fint_B \left[1 \wedge \frac{\rho(x)}{r}\right]^p w(x)\,\mathrm{d}x \right) \left(\fint_B \left[1 \wedge \frac{\rho(x)}{r}\right]^{p'} w(x)^{-p'/p}\,\mathrm{d}x \right)^{p/p'} < \infty \tag{8.1}$$

Next consider the heat semigroup $e^{-t\Delta_\Omega}$ with $\Omega \subset \mathbb{R}^n$, $n \geq 3$, as the exterior of a $C^{1,1}$ compact convex domain[1]. Then $e^{-t\Delta_\Omega}$ is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for all weights w in the class,

$$A_p^{Ext1}(w) = \sup_{B\subset\Omega} \left(\fint_B \left[1 \wedge \frac{\rho(x)}{1\wedge r}\right]^p w(x)\,\mathrm{d}x \right) \left(\fint_B \left[1 \wedge \frac{\rho(x)}{1\wedge r}\right]^{p'} w(x)^{-p'/p}\,\mathrm{d}x \right)^{p/p'} < \infty \tag{8.2}$$

Here, as always, r is the radius of B. The proof of this theorem is in section 8.1 of this chapter and uses results from chapters 6 and 7. The next theorem regards condition (5.5)

[1] The convex requirement is solely to ensure integration up to the boundary can be managed smoothly and efficiently and can be replaced with more general requirements if necessary.

for more general $\Omega \subset \mathbb{R}^n$.

THEOREM 8.2. *Given a weight w, suppose that there exists a function $u(x)$ and a constant $c > 0$ where $w(x) \leq u(x) \leq 2w(x)$ and $|\nabla u(x)| \leq c\frac{u(x)}{\rho(x)}$ both hold. Restricting to only balls away from the boundary $c_0 r(B) < \rho(B)$, suppose that the weight w satisfies an A_2 condition: $(\fint_B w \, \mathrm{d}x)(\fint_B w^{-1} \, \mathrm{d}x) \lesssim 1$. Then on such balls the operator $\nabla e^{-t\Delta_\Omega} 1_\Omega$ satisfies the following $\mathcal{L}^2$ norm bound,*

$$\left(\fint_B \rho(x)^2 w(x)^{-1} \, \mathrm{d}x \right) \left(\fint_B |\nabla e^{-r^2 \Delta_\Omega} 1_\Omega(x)|^2 w(x) \, \mathrm{d}x \right) \lesssim 1 \tag{8.3}$$

where again r is the radius of B.

This theorem is proven in section 8.2. In condition (5.5) the weight w is replaced by $w^{2/p}$. There is also a k term in condition (5.5), which follows from this theorem by a basic heat kernel scaling. The second part of Theorem 1.5 is extended in the following theorem.

THEOREM 8.3. *Suppose that $w(x)$ is a weight on $\Omega \subset \mathbb{R}^n$ for which the usual heat semigroup $e^{-t\Delta_\Omega}$, and a varied version $\frac{\sqrt{t}}{\rho(x)} e^{-t\Delta_\Omega}$, are both bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$. Further suppose that the heat kernel $p_t(x, y)$ satisfies $|t \frac{\mathrm{d}}{\mathrm{d}t} p_t(x, y)| \lesssim |p_t(x, y)|$. Then the heat semigroup of the Dirichlet Laplacian satisfies the following Gaffney estimate for all f supported on A and all balls $B \subset \Omega$.*

$$\| \sqrt{t} |\nabla e^{-tL} f| \|_{\mathcal{L}^p(B,w)} \lesssim e^{-d(A,B)^2/ct} \|f\|_{\mathcal{L}^p(A,w)} \tag{8.4}$$

Theorem 8.3 is proven in section 8.3. Equation (8.4) was required to hold without weight in chapters 3 and 4, and with weight in chapter 5. Techniques stemming from 'integration by parts' on Ω and from the analytic nature of the heat kernel are visited repeatedly throughout this chapter. In particular ideas of Davies [22] and Li and Yau [50] are used. In the first section close attention is paid to the heat kernel bounds derived in the previous chapter. These theorems (with others) are used in chapter 10 of this thesis as conditions in order to invoke the theorems of chapter 5.

8.1 The Heat Semigroup

In this section Theorem 8.1 is proven as a combination of Propositions 8.4 and 8.5. After the proofs there are remarks included regarding further cases.

Proposition 8.4. *Suppose $\Omega \subset \mathbb{R}^n$ is a global Lipschitz domain (this is the space above a $C^{1,1}$ globally Lipschitz curve). Then the heat semigroup $e^{-t\Delta_\Omega}$ satisfies a bound $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for all weights w in the class A_p^{Lip1} from Theorem 8.1.*

Proof. The kernel of the heat semigroup in this case was given in the previous chapter, and is stated again below for convenience.

$$p_t(x,y) \sim \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right)\frac{e^{-d(x,y)^2/ct}}{t^{n/2}}$$

Choose $\alpha_t(x) = \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right)$ and $\beta_t(x) = \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right)$. The idea is to show that this α, β pair satisfy all of Conditions 6.1 (see chapter 6). Continuity and strict positivity are trivially satisfied, leaving four remaining conditions: (6.1a), (6.1b), (6.1c) and (6.1d).

For condition (6.1a) consider first balls B away from the boundary $r(B) < \rho(B)$. In such a case $\alpha = \beta = 1$ everywhere in B and this condition holds trivially.

$$\left(\fint_B \alpha_{r^2}(x)^{-A\gamma}\,\mathrm{d}x\right)\left(\fint_B \beta_{r^2}(y)^{-A\gamma'}\,\mathrm{d}y\right) \lesssim \left(\fint_B 1\,\mathrm{d}x\right)\left(\fint_B 1\,\mathrm{d}y\right) \lesssim 1$$

Next consider balls B small and near the boundary $\rho(B) \lesssim r(B)$. Write $x = (x_1, \ldots, x_n)$ and let $x' = (x_2, \ldots, x_n)$. This coordinate system is chosen so that $x_1 = \psi(x')$ is a $C^{1,1}$ and Lipschitz function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ that describes the boundary of Ω. Suppose $z = (z_1, \ldots, z_n)$ is the centre of B and let B' be the $\mathbb{R}^{n-1}$ dimensional ball radius $r(B)$ and centre $(z_2, \ldots, z_n)$. Then the following holds.

$$\begin{aligned}
&\left(\fint_B \alpha_{r^2}(x)^{-A\gamma}\,\mathrm{d}x\right)\left(\fint_B \beta_{r^2}(y)^{-A\gamma'}\,\mathrm{d}y\right)\\
&\quad\lesssim \left(\fint_B \frac{r^{A\gamma}}{\rho(x)^{A\gamma}}\,\mathrm{d}x\right)\left(\fint_B \frac{r^{A\gamma'}}{\rho(y)^{A\gamma'}}\,\mathrm{d}y\right)\\
&\quad\lesssim \left(\frac{1}{r^n}\int_{B'}\int_{\psi(x')}^{cr+\psi(x')} \frac{r^{A\gamma}\,\mathrm{d}x_1\,\mathrm{d}x'}{(x_1-\psi(x'))^{A\gamma}}\right)\left(\frac{1}{r^n}\int_{B'}\int_{\psi(y')}^{cr+\psi(y')} \frac{r^{A\gamma'}\,\mathrm{d}y_1\,\mathrm{d}y'}{(y_1-\psi(y')^{A\gamma'}}\right) \lesssim 1
\end{aligned}$$

The value of the constant c in the limits depends on the Lipschitz constant of Ω. The integrals require $A\gamma$ and $A\gamma'$ to be small enough. This proves condition (6.1a).

Condition (6.1b) is trivial as $\alpha, \beta \lesssim 1$ implies: $\fint_B \alpha\, \mathrm{d}x \fint_B \beta\, \mathrm{d}y \lesssim \fint_B 1\, \mathrm{d}x \fint_B 1\, \mathrm{d}y \lesssim 1$.

For condition (6.1c), consider $x, y \in B$. Then for all $M \geq 0$,

- If $\rho(B) \leq r$ then: $\alpha_{r^2}(x) \sim \frac{\rho(x)}{r} \sim 2^M \left(\frac{\rho(x)}{2^M r}\right) \sim 2^M \alpha_{4^M r^2}(x)$;
- If $r \leq \rho(B) \leq 2^M r$ then $\alpha_{r^2}(x) \sim \frac{2^M r}{\rho(B)} \alpha_{4^M r^2}(x)$ where $1 \leq \frac{2^M r}{\rho(B)} \leq 2^M$;
- If $2^M r \leq \rho(B)$ then $\alpha_{r^2}(x) \sim 1 \sim \alpha_{4^M r^2}(x)$.

This proves (6.1c) for both α and β as they are the same in this case.

Condition (6.1d) holds by the split of the set of all balls I into the set I_1 where $\rho(B) \leq 5r(B)$ and $\alpha_{r^2}(x) \sim \rho(x)/r$ for $x \in 5B$; and the set I_2 where $5r(B) < \rho(B)$ and $\alpha_{r^2}(x) \sim 1$ for $x \in 5B$.

Theorem 6.5 in chapter 6 states that when α and β satisfy Conditions 6.1 the heat semigroup is bounded for all w in the weight class $A_p^{\alpha,\beta}$, which matches A_p^{Lip1} claimed. □

Proposition 8.5. *Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is the exterior of a compact convex $C^{1,1}$ domain. Then the heat semigroup $e^{-t\Delta_\Omega}$ is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for all weights w in the class A_p^{Ext1} from Theorem 8.1.*

Proof. The kernel of the heat semigroup in this case was stated in the previous chapter.

$$p_t(x,y) \sim \left(1 \wedge \frac{\rho(x)}{1 \wedge \sqrt{t}}\right)\left(1 \wedge \frac{\rho(y)}{1 \wedge \sqrt{t}}\right) \frac{e^{-d(x,y)^2/ct}}{t^{n/2}}$$

Choose $\alpha_t(x) = \left(1 \wedge \frac{\rho(x)}{1 \wedge \sqrt{t}}\right)$ and $\beta_t(x) = \left(1 \wedge \frac{\rho(x)}{1 \wedge \sqrt{t}}\right)$. These are continuous, strictly positive in Ω, and pointwise larger than their counterparts in the global Lipschitz case.

For condition (6.1a) observe small balls ($r \lesssim 1$) have locally identical geometry to the global Lipschitz case, so that condition (6.1a) is satisfied by comparison. Similarly when $1 \lesssim r$ and $\rho(B) \geq 1$ then $\alpha = \beta = 1$ so that condition (6.1a) is satisfied trivially. For large balls ($1 \lesssim r$) near the boundary ($\rho(B) < 1$), split B into a finite number of subsets S_i, $1 \leq i \leq m$ (m depends only on n and the Lipschitz constant of Ω), where: $S_1 = \{x \in B : \rho(x) \geq 1\}$ is the collection of $x \in B$ away from the boundary; and $S_2, \ldots, S_m$ cover those $x \in B$ near the boundary ($\rho(x) < 1$). As Ω is Lipschitz, each separate S_i near the boundary, $2 \leq i \leq m$, can be sized so that there exists a local coordinate system $(x_1, \ldots, x_n)$ where distance to the boundary for each $x \in S_i$ has approximation: $\rho(x) \sim x_1 - \psi(x_2, \ldots, x_n)$ for some Lipschitz and $C^{1,1}$ function ψ that locally describes $\delta\Omega$. Using these sets S_i construct the following bound.

$$\begin{aligned}
&\fint_B \alpha_{r^2}(x)^{-A\gamma}\,\mathrm{d}x \fint_B \beta_{r^2}(y)^{-A\gamma'}\,\mathrm{d}y \\
&\lesssim \frac{1}{r^{2n}}\left(|S_1| + \sum_{i=2}^m \int_{S_i} \frac{\mathrm{d}x}{[x_1-\psi(x')]^{A\gamma}}\right)\left(|S_1| + \sum_{i=2}^m \int_{S_i} \frac{\mathrm{d}y}{(y_1-\psi(y')^{A\gamma'}}\right) \\
&\qquad \lesssim \frac{1}{r^{2n}}\left(r^n + r^{n-1}\sum_{i=2}^m \int_0^c \frac{\mathrm{d}t}{t^{A\gamma}}\right)\left(r^n + r^{n-1}\sum_{i=2}^m \int_0^c \frac{\mathrm{d}s}{s^{A\gamma'}} + 1\right) \lesssim 1
\end{aligned}$$

The integrals require $A\gamma$ and $A\gamma'$ to be small enough. This proves condition (6.1a).

Further both α and β are less then 1 so must satisfy condition (6.1b).

The third condition (6.1c) is again by direct verification:

- If $2^M r \leq 1$ then the proof is the same as in the previous case;
- If $\rho(B) \leq r \leq 1 \leq 2^M r$ then $\alpha_{r^2}(x) \sim \frac{\rho(x)}{r} \sim \frac{1}{r}\alpha_{4^M r^2}(x)$ where $1 \leq \frac{1}{r} \leq 2^M$;
- If $r \leq \rho(B) \leq 1 \leq 2^M r$ then $\alpha_{r^2}(x) \sim 1 \sim \frac{\rho(x)}{\rho(B)} \sim \frac{\alpha_{4^M r^2}(x)}{\rho(B)}$ where $1 \leq \frac{1}{\rho(B)} \leq 2^M$;
- If $r \leq 1 \leq 2^M r \wedge \rho(B)$ or if $1 \leq r \wedge \rho(B)$ then $\alpha_{r^2}(x) \sim 1 \sim \alpha_{4^M r^2}(x)$;
- Last is the case $\rho(B) \leq 1 \leq r$ which gives $\alpha_{r^2}(x) \sim \rho(x) \sim \alpha_{4^M r^2}(x)$.

This concludes the verification of condition (6.1c), using that α and β are identical here. The final condition (6.1d) holds by splitting the set of all balls in Ω into the sets:

- I_1 where $1 \leq r(B)$ so that $\alpha_{r^2}(x) = 1 \wedge \rho(x)$ on $5B$;
- I_2 where $\rho(B) \leq 5r(B) < 5$ so that $\alpha_{r^2}(x) \sim \rho(x)/r$ on $5B$;
- I_3 where $5r(B) < \rho(B) \wedge 5$ so that $\alpha_{r^2}(x) \sim 1$ on $5B$.

This concludes the proof. □

Remarks 8.6. A similar conclusion holds for the exterior of the parabola $x_2 = x_1^2$ case, this time with weight class A_p^{Para} given below, where $R = \max(x_2, 1)$.

$$\begin{aligned}
A_p^{Para}(w) = \sup_B &\left(\fint_B \left[\left(1 \wedge \frac{\rho(x)}{R \wedge r}\right)\left(1 \wedge \frac{\rho(x)}{r}\right)\right]^{p/2} w(x)\,\mathrm{d}x\right) \\
&\cdot \left(\fint_B \left[\left(1 \wedge \frac{\rho(x)}{R \wedge r}\right)\left(1 \wedge \frac{\rho(x)}{r}\right)\right]^{p'/2} w(x)^{-p'/p}\,\mathrm{d}x\right)^{p/p'} \lesssim 1
\end{aligned}$$

The non-doubling case of $\{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < e^{x_1}\}$ can similarly be shown to have a weight class $A_p^{\alpha,\beta}$ with $\alpha = (1 \wedge \frac{\rho(x)}{\sqrt{t}})$ and $\beta = 1$ based on the heat kernel derived in section 7.3. This would require an extension of the heat kernel of that case to $\mathbb{R}^n$ (or another suitable doubling space) for correct use of the theorems of chapter 6.

Alternatively, the maximal principle can be used to determine weights by comparison to the heat semigroups on the domains given by $S_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ and $S_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 < e^{x_1}\}$ which have weight classes as global Lipschitz domains. This maximal principle idea is used in section 9.3 for Hardy operator bounds

Remarks 8.7. Weights of the form $w(x) = \rho(x)^k$ are in both the exterior and global Lipschitz case weight classes for all k in the range $-p-1 < k < 2p-1$. On balls away from the boundary ($r(B) \lesssim \rho(B)$) then $\alpha = \beta = 1$ in all examples. Hence away from the boundary the weights resemble the standard A_p Muckenhoupt weights. It is near the boundary ($\rho(B) \lesssim r(B)$) that the weights act outside the Muckenhoupt classes.

8.2 Gradient of Semigroup Unit

In this section Theorem 8.2 is proven. This theorem bounds $\fint_B |\nabla e^{-r^2\Delta_\Omega} 1_\Omega|^2 \,\mathrm{d}x$. It is not good enough to seek the L^1 norm bound $\| |\nabla p_{r^2}(x, \cdot)| \|_1$, as such a bound is much too large for our purposes. Some concept of cancellation inside the kernel $\nabla p_t(x, y)$ needs to be accounted for. Begin with some remarks.

Remarks 8.8. The idea behind the proof of Theorem 8.2 is to break the heat semigroup up into the terms $e^{-r^2\Delta_{\mathbb{R}^n}} 1_\Omega$ and $e^{-r^2\Delta_\Omega} 1_\Omega - e^{-r^2\Delta_{\mathbb{R}^n}} 1_\Omega$ ($\Delta_{\mathbb{R}^n}$ means the standard Laplacian on $\mathbb{R}^n$). The key is how quickly as r grows do the respective kernels $p^\Omega_{r^2}(x, y)$ and $p^{\mathbb{R}^n}_{r^2}(x, y)$ separate. In Lemma 8.9 this rate is shown comparable to $e^{-\rho(x)^2/r^2}$. Techniques of Davies [20] convert this to an estimate on $\Delta e^{-r^2\Delta_\Omega} 1_\Omega$. This is further converted to an estimate on $\int_B |\nabla e^{-r^2\Delta_\Omega} 1_\Omega|^2 \,\mathrm{d}x$ by an 'integration by parts' argument.

In fact for r large and small it is not difficult to see that Theorem 8.2 will hold. Firstly with appropriate smoothness on the heat kernel it would be reasonable to expect,

$$\lim_{r \to 0} \nabla e^{-r^2 L} 1_\Omega(x) = 0$$

holds for all $x \in \Omega$, due to approximation to the identity properties of the heat kernel. As r tends to 0 there is a preservation effect, as the heat has not had time to escape through the boundary. Secondly observe that the second integral in equation (8.3) resembles part of the $\mathcal{L}^2$ Gaffney estimate given in equation (1.9). In this regard consider 1_Ω

as a sum $1_\Omega = \sum 1_{C_i}$ where C_i are the shells $2^{i+1}B \setminus 2^i B$. Then by the Gaffney estimate,

$$\begin{aligned}\frac{1}{|B|}\int_B |\nabla e^{-r^2L}1_\Omega|^2\,\mathrm{d}x &\le \sum_i \frac{2^{in}}{|2^iB|}\int_B |\nabla e^{-r^2L}1_{C_i}|^2\,\mathrm{d}x\\ &\lesssim \sum_i \frac{2^{in}}{|2^iB|}\frac{e^{-c4^i}}{r^2}\int_{C_i} 1\,\mathrm{d}x \lesssim \sum_i \frac{2^{in}e^{-c4^i}}{r^2} \lesssim \frac{1}{r^2}\end{aligned}$$

So for r very small or very large Theorem 8.2 will hold. The proof will verify these observations and include bounds with r between these extremes.

Lemma 8.9. *Suppose that Ω is an open subset of $\mathbb{R}^n$. Then the Dirichlet heat semigroup in Ω satisfies the following bounds for all $x \in \Omega$.*

$$|1 - e^{-t\Delta_\Omega}1_\Omega(x)| \lesssim e^{-\rho(x)^2/72t} \qquad \textit{and} \qquad |\Delta e^{-t\Delta_\Omega}1_\Omega(x)| \lesssim \frac{e^{-\rho(x)^2/144t}}{t}$$

where $\rho(x)$ is the distance from x to the boundary $\delta\Omega$.

Proof. There are 3 parts to this proof. The first part of the proof finds a bound for the term $e^{-t\Delta_{\mathbb{R}^n}}1_\Omega(x)$. The heat kernel $p_t^{\mathbb{R}^n}(x,y)$ is well known to be the Gaussian function which is used to our advantage. The following holds for all $x \in \Omega$.

$$e^{-t\Delta_{\mathbb{R}^n}}1_\Omega(x) = \int_\Omega \frac{e^{-d(x,y)^2/4t}}{(4\pi t)^{n/2}}1\,\mathrm{d}y = \int_{\mathbb{R}^n} \frac{e^{-d(x,y)^2/4t}}{(4\pi t)^{n/2}}1\,\mathrm{d}y - \int_{\Omega^c} \frac{e^{-d(x,y)^2/4t}}{(4\pi t)^{n/2}}1\,\mathrm{d}y$$

The first integral on the right is 1, and for the second integral on the right use that as $x \in \Omega$, then $d(x,y) > \rho(x)$ (meaning $e^{-\rho(x)^2/8t} > e^{-d(x,y)^2/8t}$). The second integral keeps a $e^{-d(x,y)^2/8t}$ term, so is still over a Gaussian function, and can be evaluated as $2^{n/2}$.

$$e^{-t\Delta_{\mathbb{R}^n}}1_\Omega(x) \ge 1 - e^{-\rho(x)^2/8t}\int_{\Omega^c} \frac{e^{-d(x,y)^2/8t}}{(4\pi t)^{n/2}}1\,\mathrm{d}y \ge 1 - 2^{n/2}e^{-\rho(x)^2/8t}$$

Note that 0 is also a lower bound by the maximum principle, so the calculated lower bound only gives insight when $\rho(x)/\sqrt{t}$ is large. The maximum principle also gives an upper bound of 1. This result holds for all $x \in \Omega$.

$$1 - c_n e^{-\rho(x)^2/8t} \le e^{-t\Delta_{\mathbb{R}^n}}1_\Omega(x) \le 1 \tag{8.5}$$

Let $p_t^\Omega(x,y)$ be the Dirichlet heat kernel for Ω, and $p_t^{\mathbb{B}_y}(x,y)$ be the Dirichlet heat kernel for the ball with centre at y and radius $\rho(y)$. Whenever $x \in \mathbb{B}_y$ the inequalities $0 \leq p_t^{\mathbb{B}_y}(x,y) \leq p_t^{\Omega}(x,y) \leq p_t^{\mathbb{R}^n}(x,y)$ hold by the maximum principle. For the same x, y pair,

$$p_t^{\mathbb{R}^n}(x,y) - p_t^{\Omega}(x,y) \leq p_t^{\mathbb{R}^n}(x,y) - p_t^{\mathbb{B}_y}(x,y)$$

which means a lower bound for $p_t^{\mathbb{B}_y}(x,y)$ is needed. Again by the maximum principle,

$$p_t^{\mathbb{B}_y}(x,y) \geq p_t^{\mathbb{R}^n}(x,y) - \sum_{i=1}^{n} p_t^{\mathbb{R}^n}(x,y_i') - \sum_{i=1}^{n} p_t^{\mathbb{R}^n}(x,y_i'')$$

where $y_i' = (y_1, \ldots, y_{i-1}, y_i + \rho(y), y_{i+1}, \ldots, y_n)$ and y_i'' is similarly defined but with $\rho(y)$ subtracted from the ith coordinate rather then added. Let $\mathbb{B}_y/2$ be the ball with centre y and radius $\rho(y)/2$. Then the following holds for all $x \in \Omega$.

$$\begin{aligned} p_t^{\mathbb{R}^n}(x,y) - p_t^{\Omega}(x,y) &\leq \begin{cases} p_t^{\mathbb{R}^n}(x,y) & \text{if } x \notin \mathbb{B}_y/2 \\ \sum_{i=1}^n \left[p_t^{\mathbb{R}^n}(x,y_i') + p_t^{\mathbb{R}^n}(x,y_i'')\right] & \text{if } x \in \mathbb{B}_y/2 \end{cases} \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \begin{cases} e^{-\frac{d(x,y)^2}{4t}} & \text{if } x \notin \mathbb{B}_y/2 \\ \sum_{i=1}^n \left[e^{-\frac{d(x,y_i')^2}{4t}} + e^{-\frac{d(x,y_i'')^2}{4t}}\right] & \text{if } x \in \mathbb{B}_y/2 \end{cases} \end{aligned}$$

Then $\rho(x) \leq 3d(x,y)$ for all $x \notin \mathbb{B}_y/2$ and also both $\rho(x) \leq 3d(x,y_i')$ and $\rho(x) \leq 3d(x,y_i'')$ hold for all $x \in \mathbb{B}_y/2$ and integers $1 \leq i \leq n$. This means a $\rho(x)$ term can be separated from the heat kernels.

$$p_t^{\mathbb{R}^n}(x,y) - p_t^{\Omega}(x,y) \leq \frac{e^{-\rho(x)^2/72t}}{(4\pi t)^{n/2}} \begin{cases} e^{-d(x,y)^2/8t} & \text{if } x \notin \mathbb{B}_y/2 \\ \sum_{i=1}^n \left[e^{-\frac{d(x,y_i')^2}{8t}} + e^{-\frac{d(x,y_i'')^2}{8t}}\right] & \text{if } x \in \mathbb{B}_y/2 \end{cases}$$

An $e^{-\rho(y)^2/ct}$ term could also be separated if wished, but it is not necessary. Observe that $d(x,y) \leq d(x,y_i')$ and $d(x,y) \leq d(x,y_i'')$ are true for all $x \in \mathbb{B}_y/2$ and integers $1 \leq i \leq n$. Hence the following upper bound holds regardless of the position of x and y in Ω.

$$p_t^{\mathbb{R}^n}(x,y) - p_t^{\Omega}(x,y) \leq \frac{2ne^{-\rho(x)^2/72t}}{(4\pi t)^{n/2}} e^{-d(x,y)^2/8t}$$

Integrate the above equation over Ω with respect to y to get an inequality of the following form: $e^{-t\Delta_{\mathbb{R}^n}}1_\Omega(x) - e^{-t\Delta_\Omega}1_\Omega(x) \leq c_n e^{-\rho(x)^2/72t}$. Then combine with the lower bound for $e^{-t\Delta_{\mathbb{R}^n}}1_\Omega(x)$ from equation (8.5).

$$1 - e^{-t\Delta_\Omega}1_\Omega(x) \leq c_n e^{-\rho(x)^2/72t} \tag{8.6}$$

This holds for all $x \in \Omega$. This is the first result of the lemma. This result is trivial if $c_n e^{-\rho(x)^2/72t} \geq 1$ but the important part is when $\rho(x)^2/72t$ is large.

The final part of this proof is to find a bound for $\Delta e^{-t\Delta_\Omega}1$. To do this first a time derivative bound for the difference constructed above is needed. A method similar to that of Davies [22] proposition 1 part 2 is used. Let $h(z) = e^{\rho(x)^2/72z}(1 - e^{-z\Delta_\Omega}1_\Omega)$, so that $h(z)$ (which is a holomorphic function for $z \neq 0$ complex) is bounded above by a constant with respect to z. Call this constant C_0 and use Cauchy's integral formula with a loop centre t and radius $t/2$ to get a bound of the form: $|h'(t)| \leq \frac{2C_0}{t}$. Substitute in the definition of $h(t)$ and use the product rule to get,

$$\Big|\frac{\rho(x)^2}{72t^2}e^{\rho(x)^2/72t}(1 - e^{-t\Delta_\Omega}1_\Omega) + e^{\rho(x)^2/72t}\Big[\frac{\mathrm{d}}{\mathrm{d}t}e^{-t\Delta_\Omega}1_\Omega\Big]\Big| \leq \frac{2C_0}{t}$$

which can be rearranged for for a bound on $\frac{\mathrm{d}}{\mathrm{d}t}e^{-t\Delta_\Omega}1_\Omega$.

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}e^{-t\Delta_\Omega}1_\Omega\Big| \leq \frac{2C_0 e^{-\rho(x)^2/72t}}{t} + \frac{\rho(x)^2}{72t^2}|1 - e^{-t\Delta_\Omega}1_\Omega|$$

Then use the upper bound for $1 - e^{-t\Delta_\Omega}1_\Omega$ found in equation (8.6) (a lower bound for this term is 0 by the maximal principle).

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}e^{-t\Delta_\Omega}1_\Omega\Big| \lesssim \frac{e^{-\rho(x)^2/72t}}{t}\Big[1 + \frac{\rho(x)^2}{t}\Big] \lesssim \frac{e^{-\rho(x)^2/144t}}{t}$$

Use that $e^{-t\Delta_\Omega}1_\Omega$ satisfies the heat equation,

$$|\Delta_\Omega e^{-t\Delta_\Omega}1_\Omega(x)| \lesssim \frac{e^{-\rho(x)^2/144t}}{t}$$

where $\Delta_\Omega e^{-t\Delta_\Omega}f = \Delta e^{-t\Delta_\Omega}f$ for all $t > 0$ as $e^{-t\Delta_\Omega}f(x) \in C_0^2$. This bound holds for all $x \in \Omega$, the constant depends only on n. □

The bounds from the previous lemma are now used to prove Theorem 8.2: that the weighted equation (5.5) holds in the case of $L = \Delta_\Omega$ as the Dirchlet Laplacian on a subset $\Omega \subset \mathbb{R}^n$.

Proof of Theorem 8.2. Recall that the ball $B(x_0, r)$ is away from the boundary (meaning that $2r(B) < \rho(B)$) and let $\phi(x)$ be an additional weight function supported on $2B$. This weight function satisfies the following properties:

- ϕ is in $C_c^\infty(\mathbb{R}^n)$ and has support on $B(x_0, 2r)$;
- $\phi(x) \leq c|B(x_0, r)|^{-1/2}$ for all $x \in \Omega$;
- $\phi(x) \geq c|B(x_0, r)|^{-1/2}$ for all $x \in B(x_0, r)$; and,
- there exists a constant c such that $|\nabla\phi(x)| \leq cr^{-n/2-1}$ for all $x \in \mathbb{R}^n$.

A continuous and compactly supported approximation of $r^{-n/2}e^{-\sum|x_i - x_0|/r}$ will do. The proof will centre around the following constant A.

$$\begin{aligned}
A^2 &= \int_\Omega \phi(x)^2 |\nabla e^{-r^2\Delta_\Omega} 1_\Omega(x)|^2 u(x)\, \mathrm{d}x \\
&= -\int_\Omega \phi(x)^2 \nabla(1 - e^{-r^2\Delta_\Omega} 1_\Omega(x)) \cdot \nabla e^{-r^2\Delta_\Omega} 1_\Omega(x) u(x)\, \mathrm{d}x
\end{aligned}$$

Use integration by parts, using that $\phi(x)$ is smooth and vanishes on $\delta\Omega$ by construction. Also use the product rule of differentiation.

$$\begin{aligned}
A^2 &\leq \int_\Omega \big(1 - e^{-r^2\Delta_\Omega} 1_\Omega(x)\big) \nabla[\phi(x)^2 \nabla e^{-r^2\Delta_\Omega} 1_\Omega(x) u(x)]\, \mathrm{d}x \\
&\leq \int_\Omega \big(1 - e^{-r^2\Delta_\Omega} 1_\Omega(x)\big) \phi(x)^2 \Delta e^{-r^2\Delta_\Omega} 1_\Omega(x) u(x) \\
&\qquad + \big(1 - e^{-r^2\Delta_\Omega} 1_\Omega(x)\big) \phi(x) \nabla e^{-r^2\Delta_\Omega} 1_\Omega(x) \cdot \big[2u(x)\nabla\phi(x) + \phi(x)\nabla u(x)\big]\, \mathrm{d}x
\end{aligned}$$

Insert upper bounds for $1 - e^{-t\Delta_\Omega} 1_\Omega(x)$ and $\Delta e^{-t\Delta_\Omega} 1_\Omega(x)$ determined in previous lemmas. Also use the support of $\phi(x)$ and the upper bounds $|\nabla\phi(x)| \lesssim r^{-n/2-1}$ determined by construction, and $|\nabla u(x)| \lesssim u(x)/\rho(x)$ determined by assumption. The ball $2B$ does not touch the boundary $\delta\Omega$ so satisfies $r^{-n} \sim |2B|^{-1}$ by local doubling.

$$A^2 \lesssim \int_{2B} \phi(x)^2 \frac{e^{-\frac{c\rho(x)^2}{r^2}}}{r^2} u(x) + \phi(x) e^{-\frac{c\rho(x)^2}{r^2}} |\nabla e^{-r^2\Delta_\Omega} 1_\Omega(x)| u(x) \Big[\frac{1}{r^{n/2+1}} + \frac{\phi(x)}{\rho(x)}\Big]\, \mathrm{d}x$$

Use Hölder's inequality on second part on the right.

$$A^2 \lesssim \int_{2B} \phi(x)^2 \frac{e^{-c\rho(x)^2/r^2}}{r^2} u(x)\,\mathrm{d}x + \left[\int_{2B} \left[\frac{1}{r^{n+2}} + \frac{\phi(x)^2}{\rho(x)^2}\right] e^{-\frac{2c\rho(x)^2}{r^2}} u(x)\,\mathrm{d}x\right]^{\frac{1}{2}} \left[\int_{2B} \phi(x)^2 |\nabla e^{-r^2\Delta_\Omega} 1_\Omega(x)|^2 u(x)\,\mathrm{d}x\right]^{\frac{1}{2}}$$

Next substitute A in on the right and use that $\phi(x) \lesssim r^{-n/2}$ which translates to $|2B|^{-1/2}$ due to $2B$ away from the boundary. Also use here that $e^{-\rho(x)^2/r^2} \lesssim (1 \wedge \frac{r^2}{\rho(x)^2})$.

$$A^2 \lesssim \fint_{2B} \frac{1}{\rho(x)^2} u(x)\,\mathrm{d}x + \left[\fint_{2B} \frac{1}{\rho(x)^2} u(x)\,\mathrm{d}x\right]^{1/2} A$$

This is a quadratic in A which can be solved.

$$A \lesssim \frac{1}{2} \left[\fint_{2B} \frac{1}{\rho(x)^2} u(x)\,\mathrm{d}x\right]^{1/2}$$

Use $u(x) \leq 2w(x)$ and multiply the above equation squared by $\fint \rho(x)^2 w(x)^{-1}\,\mathrm{d}x$.

$$A^2 \fint_B \rho(x)^2 w(x)^{-1}\,\mathrm{d}x \lesssim \left(\fint_B \rho(x)^2 w(x)^{-1}\,\mathrm{d}x\right) \left(\fint_{2B} \frac{1}{\rho(x)^2} w(x)\,\mathrm{d}x\right)$$

Use that $\rho(B) > 2r$ implies $\rho(2B) > r$, so that for all $y \in B$ and $x \in 2B$ then the relations $\rho(y) \leq \rho(B) + 2r \leq 4\rho(2B) \leq 4\rho(x)$ hold. Hence the $\rho(x)$ terms in the two integrals will cancel to give,

$$A^2 \fint_B \rho(x)^2 w(x)^{-1}\,\mathrm{d}x \lesssim \left(\fint_B w(x)^{-1}\,\mathrm{d}x\right) \left(\fint_{2B} w(x)\,\mathrm{d}x\right)$$

which is itself bounded by a constant as $w(x)$ satisfies an A_2 condition for balls away from the boundary. A lower bound for A^2 is given by $\fint_B |\nabla e^{-r^2\Delta_\Omega} 1(x)|^2 w(x)\,\mathrm{d}x$ due to the lower bound of ϕ in the ball B, and the lower bound of $w(x)$ for the weight $u(x)$.

$$\left(\fint_B \rho(x)^2 w(x)^{-1}\,\mathrm{d}x\right) \left(\fint_B |\nabla e^{-r^2\Delta_\Omega} 1(x)|^2 w(x)\,\mathrm{d}x\right) \lesssim 1$$

This concludes the proof. □

A similar result would hold for balls near the boundary if $w \in A_2$ for such balls. It is only the away from boundary case that is needed to invoke chapter 5 theorems.

8.3 Gaffney Estimates

In this section Theorem 8.3 is proven. Similar ideas to those in the previous section of this chapter are used. This includes use of the techniques of Davies [22] and 'integration by parts' methods. The latter part of this section uses localised estimates from Li and Yau [50]. These estimates find an upper bound of the gradient $\sqrt{t}|\nabla p_t^\Omega(x,y)|$ in terms of the kernels of both the heat semigroup and the Hardy operator. To begin is a proposition well known in the literature, see for example [5], where this proposition and proof are contained as a comment.

Proposition 8.10. *Suppose that the Riesz transform is bounded $\mathcal{L}^p(\Omega) \to \mathcal{L}^p(\Omega)$. Then the on-diagonal Gaffney inequality also holds $\mathcal{L}^p(\Omega) \to \mathcal{L}^p(\Omega)$.*

Proof. This can be shown by: $\|\sqrt{t}|\nabla e^{-tL} f|\|_p \le \|\sqrt{t}L^{1/2}e^{-tL}f\|_p \le \|f\|_p$, where the first inequality is due to a Sobolev inequality as a result of a Riesz transform bound, and the second holds as the heat semigroup is analytic. □

So the $\mathcal{L}^p$ Gaffney condition is necessary for an $\mathcal{L}^p$ Riesz transform. The on-diagonal part of the $\mathcal{L}^2$ Gaffney estimate is similarly necessary, and thus follows from, the $\mathcal{L}^2$ Riesz transform (5.6). The following lemma looks towards getting the off-diagonal part of the $\mathcal{L}^2$ Gaffney estimate.

Lemma 8.11. *Suppose that $\Omega \subset \mathbb{R}^n$, and that the Dirichlet heat kernel $p_t(x,y)$ has upper bound,*

$$p_t(x,y) \lesssim \alpha_t(x)\alpha_t(y)\frac{e^{-d(x,y)^2/ct}}{t^{n/2}}$$

for some α, α pair satisfying all of Conditions 6.1. Then the heat semigroup $e^{-t\Delta_\Omega}$ satisfies the following Davies-Gaffney estimate,

$$\|e^{-t\Delta_\Omega} f\|_{\mathcal{L}^2(A,w)} + \|t\Delta e^{-t\Delta_\Omega} f\|_{\mathcal{L}^2(A,w)} \lesssim e^{-d(A,B)^2/ct}\|f\|_{\mathcal{L}^2(B,w)}$$

for all sets $A, B \subset \Omega$, all f supported on B, and all weights w in the class $A_2^{\alpha,\alpha}$ (see chapter 6 for the details of such a weight class).

Proof. Start by considering the inequality for $e^{-t\Delta_\Omega}$. If $d(A,B) = 0$ then the result is a corollary of Theorem 6.5 in chapter 6. If $d(A,B) \neq 0$, then the kernel of e^{-tL} will contain $e^{-d(x,y)^2/ct} = e^{-d(A,B)^2/2ct}e^{-d(x,y)^2/2ct}$ for all $x \in A$, $y \in B$. Extracting the $e^{-d(A,B)^2/2ct}$ part allows the lemma again to hold as a corollary of Theorem 6.5 in chapter 6.

The remainder of the proof is for the value $\Delta e^{-t\Delta}$. The techniques of Davies are used similar to as in the previous section. The difference $e^{-t\Delta_\Omega} - e^{-t\Delta_{\mathbb{R}^n}}$ cannot be used as in the previous section here as the term $e^{-t\Delta_{\mathbb{R}^n}}$ will not disappear in this case. Fix $f(x)$ and let $h(z) = e^{-z\Delta_\Omega}f(x)$, which is holomorphic and bounded above. Then by Cauchy's integral formula,

$$|h'(t)| \lesssim \frac{e^{-t\Delta_\Omega}f(x)}{t}$$

and as $h(t) = e^{-t\Delta_\Omega}f(x)$ satisfies the heat equation in Ω the following holds,

$$|\Delta e^{-t\Delta_\Omega}f(x)| \lesssim \frac{e^{-t\Delta_\Omega}f(x)}{t}$$

using $\Delta_\Omega e^{-t\Delta_\Omega}f(x)$ and $\Delta e^{-t\Delta_\Omega}f(x)$ are identical for all $t > 0$. So the result holds for $t\Delta e^{-t\Delta_\Omega}f(x)$ as a consequence of the result for $e^{-t\Delta_\Omega}f(x)$. □

It remains to prove a similar estimate for $|\nabla e^{-tL}f|$. The method is inspired by [5].

Proposition 8.12. *Suppose that $w(x)$ is a weight on Ω for which $e^{-t\Delta_\Omega}$ and $t\Delta e^{-t\Delta_\Omega}$ are off-diagonally $\mathcal{L}^2(w)$ bounded, and for which $\frac{1}{\rho(x)}\Delta_\Omega^{-1/2}$ and $\nabla\Delta_\Omega^{-1/2}$ are on-diagonally $\mathcal{L}^2(w)$ bounded. Further suppose there exists a function $u(x)$ and a constant $c > 0$ such that $w(x) < u(x) < 2w(x)$ and $|\nabla u(x)| \lesssim \frac{u(x)}{\rho(x)}$. Then the heat semigroup $e^{-t\Delta_\Omega}$ of the Laplace operator with Dirichlet boundary conditions on Ω satisfies the following Gaffney estimate for all f supported on B.*

$$\|\sqrt{t}|\nabla e^{-t\Delta_\Omega}f|\|_{\mathcal{L}^2(A,w)} \lesssim e^{-d(A,B)^2/ct}\|f\|_{\mathcal{L}^2(B,w)}$$

Proof. The bound when $d(A,B) < \sqrt{t}$ is a consequence of the Riesz transform being bounded on $\mathcal{L}^2(w)$, using in that case $e^{-d(x,y)^2/ct} > e^{-1/c}$. It remains to consider the case $d(A,B) > \sqrt{t} > 0$. Define ϕ as a continuous function satisfying the following properties:

- $\phi \in C_0^\infty$ and is supported on A_0, where $A \subset A_0$ and $d(A,B) < 2d(A_0,B)$;

- $\phi = 1$ on A and $\phi \leq 1$ for all other $x \in \Omega$; and,

- there exists c where $|\nabla \phi(x)| \leq cd(A,B)^{-1}$ for all $x \in \Omega$.

Next define the function Φ by,

$$\Phi^2 = \int_{A_0} t\,\phi(x)^2 |\nabla e^{-t\Delta_\Omega} f(x)|^2 u(x)\,\mathrm{d}x$$

then use integration by parts (it is important in that regard that $e^{-t\Delta_\Omega} f(x)$ vanishes on the boundary and $\phi(x)$ is supported on A_0), and differentiation by the product rule.

$$\begin{aligned}
\Phi^2 &= \int_{A_0} t\phi(x)^2 \left[\nabla e^{-t\Delta_\Omega} f(x) \cdot \nabla e^{-t\Delta_\Omega} f(x)\right] u(x)\,\mathrm{d}x \\
&= t \int_{A_0} \Big(\phi(x)^2\, \Delta e^{-t\Delta_\Omega} f(x)\, e^{-t\Delta_\Omega} f(x) w(x) \\
&\qquad\qquad - 2\phi(x)\big[\nabla\phi(x) \cdot \nabla e^{-t\Delta_\Omega} f(x)\big] e^{-t\Delta_\Omega} f(x) u(x) \\
&\qquad\qquad - \phi(x)^2 \big[\nabla u(x) \cdot \nabla e^{-t\Delta_\Omega} f(x)\big] e^{-t\Delta_\Omega} f(x)\Big)\,\mathrm{d}x
\end{aligned}$$

Use the constructed upper bound for $|\nabla\phi|$, and the assumed upper bound for $|\nabla u(x)|$.

$$\begin{aligned}
\Phi^2 \lesssim t \int_{A_0} \Big(&|\phi(x)^2 \Delta e^{-t\Delta_\Omega} f(x) e^{-t\Delta_\Omega} f(x) u(x)| + |\frac{c\phi(x)}{d(A,B)} |\nabla e^{-t\Delta_\Omega} f(x)|\, e^{-t\Delta_\Omega} f(x) u(x)| \\
&+ |\phi(x)^2 |\nabla e^{-t\Delta_\Omega} f(x)|\, e^{-t\Delta_\Omega} f(x) \frac{u(x)}{\rho(x)}|\Big)\,\mathrm{d}x
\end{aligned}$$

Next Hölder's inequality is used on each term, and also use $\phi \leq 1$.

$$\begin{aligned}
\Phi^2 \lesssim &\Big[\int_{A_0} |t\Delta e^{-t\Delta_\Omega} f(x)|^2 u(x)\,\mathrm{d}x\Big]^{1/2} \Big[\int_{A_0} |e^{-t\Delta_\Omega} f(x)|^2 u(x)\,\mathrm{d}x\Big]^{1/2} \\
&+ \frac{\sqrt{t}}{d(A,B)} \Big[\int_{A_0} t\phi(x)^2 |\nabla e^{-t\Delta_\Omega} f(x)|^2 u(x)\,\mathrm{d}x\Big]^{1/2} \Big[\int_{A_0} |e^{-t\Delta_\Omega} f(x)|^2 u(x)\,\mathrm{d}x\Big]^{1/2} \\
&+ \Big[\int_{A_0} t\phi(x)^2 |\nabla e^{-t\Delta_\Omega} f(x)|^2 u(x)\,\mathrm{d}x\Big]^{1/2} \Big[\int_{A_0} |\frac{\sqrt{t}}{\rho(x)} e^{-t\Delta_\Omega} f(x)|^2 u(x)\,\mathrm{d}x\Big]^{1/2}
\end{aligned}$$

Next use that $u(x) < 2w(x)$, that $\frac{1}{\rho}\Delta_\Omega^{-1/2}$ is bounded for the weight w, and that $e^{-t\Delta_\Omega}$ and $t\Delta e^{-t\Delta_\Omega}$ are off-diagonally bounded for the same weight w. Also use the definition of Φ to replace appropriate terms.

$$\Phi^2 \lesssim e^{-d(A,B)^2/ct}\|f\|^2_{\mathcal{L}^2(B,w)} + e^{-d(A,B)^2/ct}\|f\|_{\mathcal{L}^2(B,w)}\Big[\frac{\sqrt{t}}{d(A,B)} + 1\Big]\Phi$$

This can be solved as a quadratic inequality to get the bound,

$$\Phi \lesssim e^{-d(A,B)^2/ct}\|f\|_{\mathcal{L}^2(B,w)}$$

with $d(A,B) > \sqrt{t}$. The result follows as Φ is an upper bound for $\||\nabla e^{-t\Delta_\Omega} f|\|_{\mathcal{L}^2(A,w)}$.

□

The $\frac{\rho(x)}{\sqrt{t}}e^{-t\Delta_\Omega}$ part of the proof of Proposition 8.12 is only involved for arbitrary weights. If $w = 1$ was the weight considered, then $|\nabla w| = 0$ and the $\frac{\rho(x)}{\sqrt{t}}e^{-t\Delta_\Omega}$ part vanishes. An $\mathcal{L}^p(w)$ bound for the Hardy operator $\frac{1}{\rho}\Delta_\Omega^{-1/2}$ would imply an $\mathcal{L}^p(w)$ bound for $\frac{\rho(x)}{\sqrt{t}}e^{-t\Delta_\Omega}$. Weighted $\mathcal{L}^p$ bounds for $\frac{1}{\rho}\Delta_\Omega^{-1/2}$ and $\frac{\rho(x)}{\sqrt{t}}e^{-t\Delta_\Omega}$ are considered in chapter 9. The $\mathcal{L}^2$ weighted Riesz transform that was required in this proof is considered in chapter 10.

The proof of Proposition 8.12 does not work to prove $\mathcal{L}^p$ bounds. Hence a different approach is taken. This new approach is specific for the case $\Omega \subset \mathbb{R}^n$ and L the Dirichlet Laplacian. The result of Proposition 8.12 is implied by the proof below, but was proven separately above to keep the $\mathcal{L}^2$ Gaffney inequality method as general as possible.

This next part uses a localised result from Li and Yau [50]. For further detail see also Zhang [76] and for variations see Perelman [55] and Zhang [75] and Zhang and Souplet [63].

Lemma 8.13 (Lemma 1.1 in [50])**.** *Suppose f satisfies the following differential equation.*

$$\Big(\Delta + \frac{\mathrm{d}}{\mathrm{d}t}\Big)f = -|\nabla f|^2 \tag{8.7}$$

Then the function F given by $F = t|\nabla f|^2 - 2tf_t$ satisfies the following inequality.

$$\Big(\Delta + \frac{\mathrm{d}}{\mathrm{d}t}\Big)F \geq -2\nabla f \cdot \nabla F - \frac{F}{t} + \frac{2t}{n}(|\nabla f|^2 - f_t)^2$$

The proof is by direct computation and is contained in [50] so will not be proven here. The important part is how the above lemma is used in the proof of the next lemma, which is a version of Theorem 1.2 from [50].

Lemma 8.14. *Suppose that $p_t(x, y)$ is a heat kernel in Ω. Then $p_t(x, y)$ satisfies the following bound for all $x \in \Omega$, $t > 0$.*

$$t|\nabla p_t(x,y)|^2 \leq c(1 + \frac{t}{\rho(x)^2})p_t(x,y)\Big[p_t(x,y) + 2t\frac{\mathrm{d}}{\mathrm{d}t}p_t(x,y)\Big]$$

Proof. Choose a ball B centre x where $r(B) < \rho(B)$. Define the function $\phi \in C^2(\Omega)$ supported on $2B$ where $\phi(x) = 1$ for all $x \in B$ and ϕ satisfies the following upper bounds on its first two derivatives: $|\nabla\phi(x)| < r^{-1}\sqrt{\phi(x)}$ and $|\Delta\phi(x)| \leq cr^{-2}$. Let F be as in the previous lemma and let (x_0, t_0) be the point in $2B \times [0, t]$ at which ϕF is maximal. At such a point (x_0, t_0) there are the following three properties.

$$\nabla(\phi F) = 0 \text{ and } (\phi F)_t \geq 0 \text{ and } -\Delta(\phi F) \leq 0$$

Note that $t_0 = 0$ cannot occur as $F = 0$ at $t = 0$ by construction. Further if $t_0 \in (0, t)$ then $(\phi F)_t = 0$ would occur. The $(\phi F)_t \geq 0$ then takes into account the possibility $t_0 = t$. Expand out the third property in the equation above.

$$0 \geq (-\Delta\phi)F + 2\nabla F \cdot \nabla\phi + \phi(-\Delta F)$$

Apply Lemma 8.13 to ΔF along with the bound constructed earlier for $\Delta\phi$. For $\nabla\phi$ use the expansion given by $\nabla(\phi F) = \phi(\nabla F) + (\nabla\phi)F$ so that at the point (x_0, t_0), the equality $\phi(\nabla F) = -(\nabla\phi)F$ occurs.

$$0 \geq -c\frac{F}{r^2} - \frac{2F}{\phi}|\nabla\phi|^2 + \phi\Big(-2\nabla f \cdot \nabla F - \frac{F}{t_0} + \frac{2t_0}{n}(|\nabla f|^2 - f_t)^2 + F_t\Big)$$

Use the upper bound constructed for $|\nabla\phi|$ and use again that $\phi(\nabla F) = -(\nabla\phi)F$. Also use the lower bound stated at (x_0, t_0) for F_t earlier (the ϕ is independent of t).

$$0 \geq -c\frac{F}{r^2} + 2F\nabla f \cdot \nabla\phi - \frac{\phi F}{t_0} + \frac{2\phi t_0}{n}(|\nabla f|^2 - f_t)^2$$

Next multiply through by ϕt_0 and use the upper bound constructed for $|\nabla\phi|$.

$$0 \geq -c\frac{\phi t_0 F}{r^2} - \frac{2\phi^{3/2} t_0 F}{r}|\nabla f| - \phi^2 F + \frac{2\phi^2 t_0^2}{n}(|\nabla f|^2 - f_t)^2$$

Use that for any positive number A then $A \leq 1 + A^2$. So let A be given by the value $A = \phi^{1/2}|\nabla f| r(2n)^{-1}$ and then apply this principle (using also that F is positive at its maximum point (x_0, t_0)).

$$0 \geq -c\frac{\phi t_0 F}{r^2} - \frac{4n\phi t_0 F}{r^2} - \frac{\phi^2 t_0}{n}F|\nabla f|^2 - \phi^2 F + \frac{2\phi^2 t_0^2}{n}(|\nabla f|^2 - f_t)^2$$

Move some terms to the other side of the equation and substitute in the definition of F.

$$\begin{aligned} c\frac{\phi t_0 F}{r^2} &\geq -\frac{\phi^2 t_0}{n}F|\nabla f|^2 - \phi^2 F + \frac{\phi^2}{2n}(2F + t_0|\nabla f|^2)^2 \\ &\geq -\phi^2 F + \frac{2\phi^2 F^2}{n} + \frac{\phi^2 t_0 F|\nabla f|^2}{n} + \frac{\phi^2 t_0^2|\nabla f|^4}{2n} \end{aligned}$$

Remove two of the positive terms to get,

$$c\frac{\phi t_0 F}{r^2} + \phi^2 F \geq \frac{2\phi^2 F^2}{n}$$

which implies $\phi F \leq c(1 + \frac{t_0}{r^2})$. Choose then for f the value $f = \log(p_t(x, y))$ which satisfies the requirement (8.7) from Lemma 8.13. The value F can then be written in terms of $p_t(x, y)$.

$$\phi\left(\frac{t|\nabla p_t(x, y)|^2}{p_t(x, y)^2} - 2t\frac{\frac{\mathrm{d}}{\mathrm{d}t}p_t(x, y)}{p_t(x, y)}\right) \leq c(1 + \frac{t_0}{r^2})$$

Remove ϕ and arrange the equation using $t_0 \leq t$ for,

$$t|\nabla p_t(x, y)|^2 \leq c(1 + \frac{t}{r^2})p_t(x, y)\Big[p_t(x, y) + 2t\frac{\mathrm{d}}{\mathrm{d}t}p_t(x, y)\Big]$$

which holds for all $x \in B$ and $t \in (0, \infty)$. The radius $r = r(B)$ can be made as large as $\rho(x)/2$. This substitution in the above equation concludes the proof. $\square$

Proof of Theorem 8.3. From the previous lemma the following holds.

$$t|\nabla p_t(x,y)|^2 \lesssim p_t^2 + \frac{t}{\rho(x)^2}p_t^2 + p_t\Big|t\frac{\mathrm{d}}{\mathrm{d}t}p_t\Big| + \frac{t}{\rho(x)^2}p_t\Big|t\frac{\mathrm{d}}{\mathrm{d}t}p_t\Big|$$

Further in the exterior of a convex object case, and in the area above a Lipschitz function case, the work of Davies [22] implies the value $\left|t\frac{\mathrm{d}}{\mathrm{d}t}p_t\right|$ has an upper bound of the heat kernel $p_t(x,y)$ (identical to the idea of Lemma 8.9 where bounds for $\Delta e^{-t\Delta_\Omega}f$ were found). This means an upper bound for $\nabla p_t(x,y)$ is then,

$$\sqrt{t}|\nabla p_t(x,y)| \lesssim p_t(x,y) + \frac{\sqrt{t}}{\rho(x)}p_t(x,y)$$

which for f supported on the set $A \subset \Omega$ gives the following.

$$\begin{aligned}
&\|\sqrt{t}|\nabla e^{-t\Delta_\Omega}f|\|^p_{\mathcal{L}^p(B,w)} \\
&\lesssim \int_B \Big[\int_A \sqrt{t}|\nabla p_t(x,y)|f(y)\,\mathrm{d}y\Big]^p w(x)\,\mathrm{d}x \\
&\lesssim \int_B \Big[\int_A p_t(x,y)f(y) + \frac{\sqrt{t}}{\rho(x)}p_t(x,y)f(y)\,\mathrm{d}y\Big]^p w(x)\,\mathrm{d}x \\
&\lesssim e^{-d(A,B)/2ct}\int_B \Big[\int_A p_{2t}(x,y)f(y) + \frac{\sqrt{t}}{\rho(x)}p_{2t}(x,y)f(y)\,\mathrm{d}y\Big]^p w(x)\,\mathrm{d}x \\
&\lesssim e^{-d(A,B)/2ct}\int_B \Big[e^{-2t\Delta_\Omega}f(x)\Big]^p w(x)\,\mathrm{d}x + e^{-d(A,B)/2ct}\int_B \Big[\frac{\sqrt{t}}{\rho(x)}e^{-2t\Delta_\Omega}f(x)\Big]^p w(x)\,\mathrm{d}x \\
&\lesssim e^{-d(A,B)/2ct}\|e^{-2t\Delta_\Omega}f\|^p_{\mathcal{L}^p(B,w)} + e^{-d(A,B)/2ct}\|\frac{\sqrt{t}}{\rho}e^{-2t\Delta_\Omega}f\|_{\mathcal{L}^p(B,w)}
\end{aligned}$$

So the operator $\sqrt{t}\nabla e^{-tL}$ is bounded $\mathcal{L}^p \to \mathcal{L}^p$ for all weights w for which the heat semigroup and a variation by $\frac{\sqrt{t}}{\rho}$ are bounded $\mathcal{L}^p \to \mathcal{L}^p$ for. Precisely what this class of weights looks like in the exterior and global Lipschitz cases is considered in the next chapter. □

Chapter 9:

Weighted Hardy Estimates

In this chapter the latter parts of Theorem 1.5 are proven. This is achieved via Theorems 9.1 and 9.3 below. The traditional Hardy inequality is the result,

$$\int_0^\infty \left[\frac{1}{x}\int_0^x |f(t)|\,\mathrm{d}t\right]^p \mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(x)|^p\,\mathrm{d}x$$

which holds for all $f \in \mathcal{L}^p(\mathbb{R}^+)$. This is based on a discrete inequality first considered by Hardy in [40]. This chapter determines the values p and weights w for which a generalised version of the Hardy inequality holds,

$$\|\frac{1}{\rho}\Delta_\Omega^{-1/2} f\|_{\mathcal{L}^p(w)} \lesssim \|f\|_{\mathcal{L}^p(w)} \tag{9.1}$$

and for which a semigroup inequality of a similar form holds.

$$\|\frac{\sqrt{t}}{\rho} e^{-t\Delta_\Omega} f\|_{\mathcal{L}^p(w)} \lesssim \|f\|_{\mathcal{L}^p(w)} \tag{9.2}$$

In the case of $\Omega \subset \mathbb{R}^n$, $n \ge 3$, as the exterior of a compact convex object, it is known from Killip, Visan and Zhang [48], that equation (9.1) holds with weight $w = 1$ for $1 < p < n$. It could further be argued that [48] implies in the case of Ω as a $C^{1,1}$ global Lipschitz domain that equation (9.1) holds for $w = 1$ and all $1 < p < \infty$.

Theorem 6.5 from Chapter 6 is used in parts of this chapter. There is also discussion regarding how the weight classes found compare to the standard A_p classes. To begin is a summary of the main results, some repeat Theorem 1.5.

THEOREM 9.1. *Let $\Omega \subset \mathbb{R}^n$ be the exterior of a $C^{1,1}$ compact convex domain with* *locally Lipschitz boundary*[1]*. Then the Hardy operator on such a domain is bounded* $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ *for weights* w *of the form* $w(x) \sim \rho(x)^k$ *with* $\max(-1, p-n) < k < 2p-1$.

There is an even better range for the global Lipschitz case.

THEOREM 9.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a global Lipschitz domain*[2]*. Then the Hardy operator on such a domain is bounded* $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ *for all weights* w *of the form* $w(x) \sim \rho(x)^k$ *with* $-1 < k < 2p - 1$.

The final theorem considers the operator $\frac{\sqrt{t}}{\rho(x)} e^{-t\Delta_\Omega}$. As always r is the radius of B.

THEOREM 9.3. *Let* $\Omega \subset \mathbb{R}^n$ *be the exterior of a* $C^{1,1}$ *compact convex domain with locally Lipschitz boundary. Then the operator* $\frac{\sqrt{t}}{\rho(x)} e^{-t\Delta_\Omega}$ *is bounded* $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ *for all weights* w *satisfying* $A_p^{Ext2}(w) < \infty$ *with* $A_p^{Ext2}(w)$ *defined by the following equation.*

$$A_p^{Ext2}(w) \stackrel{def}{=} \sup_{B \subset \Omega} \left(\fint_B \left[\frac{r}{r \wedge \rho(x)} \wedge \frac{r}{r \wedge 1} \right]^p w(x) \,\mathrm{d}x \right) \left(\fint_B \left[1 \wedge \frac{\rho(y)}{r \wedge 1} \right]^{p'} w(x)^{-\frac{p'}{p}} \,\mathrm{d}x \right)^{\frac{p}{p'}}$$

Similarly let $\Omega \subset \mathbb{R}^n$ *be a global Lipschitz domain. Then the operator* $\frac{\sqrt{t}}{\rho(x)} e^{-t\Delta_\Omega}$ *is bounded* $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ *for all weights* w *satisfying* $A_p^{Lip2}(w) < \infty$ *where* $A_p^{Lip2}(w)$ *is defined by the following equation.*

$$A_p^{Lip2}(w) \stackrel{def}{=} \sup_{B \subset \Omega} \left(\fint_B w(x) \,\mathrm{d}x \right) \left(\fint_B \left[1 \wedge \frac{\rho(y)}{r} \right]^{p'} w(x)^{-p'/p} \,\mathrm{d}x \right)^{p/p'} \tag{9.3}$$

Remarks 9.4. The Gaffney inequality is bounded for the same weight classes as in Theorem 9.3 above due to Theorem 8.3. It is important that there are local coordinate systems that allow smooth integration up to boundaries. Comments are given in the last section of this chapter regarding the area below a parabola in $\mathbb{R}^2$ and the non-doubling example $\{(x_1, x_2) : 0 < x_2 < e^{x_1}\}$. The better than Gaussian upper bounds from chapter 7 are used extensively in this chapter, and directly determine the range of weights found. Away from boundaries (and away from extreme points x with $\rho(x) \to \infty$) the weights found act similar to those in the traditional Muckenhoupt weight classes.

[1] As in the previous chapter the convexity condition is solely to ensure neat integration up to the boundary in the proofs, and can be weakened if necessary.

[2] This is the space above a $C^{1,1}$ globally Lipschitz and bounded $\mathbb{R}^{n-1}$ curve

For the sake of comparison consider the following example regarding when the weight $w(x) = \rho(x)^k$ is in the traditional A_p Muckenhoupt weight class.

Example 9.5. Let $\Omega \subset \mathbb{R}^n$ be a domain with Lipschitz and $C^{1,1}$ boundary. This example is completed with the global Lipschitz and exterior type domains in mind. Consider the traditional Muckenhoupt class defined according to balls in this domain.

$$A_p(w) = \sup_{B\subset\Omega} \left(\fint_B w(x)\, \mathrm{d}x \right) \left(\fint_B w(x)^{-p'/p}\, \mathrm{d}x \right)^{p/p'} < \infty$$

Let $w(x) = \rho(x)^k$ and consider 2 cases. Firstly if $r(B) \lesssim \rho(B)$ then the relation given by $\rho(B) \sim \rho(x)$ holds due to the set of inequalities: $\rho(B) < \rho(x) < \rho(B) + r < 2\rho(B)$.

$$\left(\fint_B \rho(x)^k\, \mathrm{d}x \right) \left(\fint_B \rho(x)^{-kp'/p}\, \mathrm{d}x \right)^{p/p'} \lesssim \rho(B)^k \left(\rho(B)^{-kp'/p} \right)^{p/p'} \lesssim 1$$

So there is no restriction on k from this part. Next consider the case $\rho(B) \lesssim r(B)$. Suppose there is a local coordinate system $x = (x_1, \ldots, x_n)$ such that $x_1 = \psi(x')$ describes $\delta\Omega$ where $x' = (x_2, \ldots, x_n)$ and $\rho(x) \sim x_1 - \psi(x')$ for all x in B. Then,

$$\left(\fint_B \rho(x)^k\, \mathrm{d}x \right) \left(\fint_B \rho(x)^{-kp'/p}\, \mathrm{d}x \right)^{p/p'} \lesssim \left(\frac{1}{r} \int_0^{3r} \lambda^k\, \mathrm{d}\lambda \right) \left(\frac{1}{r} \int_0^{3r} \lambda^{-kp'/p}\, \mathrm{d}\lambda \right)^{p/p'} \lesssim 1$$

holds with k restricted $-1 < k < p-1$. This covers all cases for a global Lipschitz domain, and all cases where $r(B) \lesssim 1$ for an exterior domain. For the remaining exterior case, when $\rho(B) \lesssim r(B)$ and $1 \lesssim r(B)$, use that the ball B can be split into a finite number of regions S_i, $1 \le i \le m$, where $S_1 = \{x \in B : \rho(x) \ge 1\}$, and $S_2, \ldots, S_m$ cover $x \in B$ where $\rho(x) < 1$. Additionally each $S_2, \ldots, S_m$ have local coordinate systems $(x_1, \ldots x_n)$ where $\rho(x) \sim x_1 - \psi(x')$ for some Lipschitz and $C^{1,1}$ function ψ that locally describes $\delta\Omega$. For $x \in S_1$ there exists $z \in \mathbb{R}^n$ such that $\rho(x) \sim |x - z| > 1$. With such regions in mind the following is a summary of the integrals involved, with sums over the regions S_i.

$$\left(\fint_B \rho(x)^k\, \mathrm{d}x \right) \left(\fint_B \rho(x)^{-\frac{kp'}{p}}\, \mathrm{d}x \right)^{p/p'}$$
$$\lesssim \left(\int_1^{3r} \frac{\lambda^{k+n-1}}{r^n}\, \mathrm{d}\lambda + \sum_{i=2}^{m} \int_0^1 \frac{\lambda^k}{r}\, \mathrm{d}\lambda \right) \left(\int_1^{3r} \frac{\lambda^{-\frac{kp'}{p}+n-1}}{r^n}\, \mathrm{d}\lambda + \sum_{i=2}^{m} \int_0^1 \frac{\lambda^{\frac{-kp'}{p}}}{r}\, \mathrm{d}\lambda \right)^{\frac{p}{p'}} \lesssim 1$$

The integrability near the boundary is the biggest restriction on k. That restriction is given by $-1 < k < p-1$.

Before the proofs of the main results of this chapter, there is a proposition regarding circumstances when a Hardy inequality may be necessary for a Riesz transform bound.

Proposition 9.6. *Suppose for f in the range of $L^{-1/2}$ acting on $\mathcal{L}^p$, a different generalised Hardy inequality held for some $1 < p < \infty$.*

$$\|\frac{f}{\varphi}\|_{\mathcal{L}^p(\Omega)} \lesssim \||\nabla f|\|_{\mathcal{L}^p(\Omega)} \tag{9.4}$$

Then the Riesz transform bounded for that p implies that the Hardy inequality (1.4) is bounded for that p ($\varphi(x)$ as a general function is replacing $\rho(x)$ here).

Proof. The proof is one line. If both equation (9.4) and the Riesz transform are bounded then $\|\frac{L^{-1/2}f}{\varphi}\|_p \lesssim \|\nabla L^{-1/2}f\|_p \lesssim \|f\|_p$ holds for the same p. □

Notice equation (9.4) does not directly mention the operator L, and so will hold for a domain Ω and function $\varphi(x)$ with a class of functions f independent of L. Equation (9.4) is not expected to hold for all functions f, certainly some condition of smoothness on f as well as boundary conditions are expected.

When $\varphi(x) = |x|$ and $\Omega = \mathbb{R}^n$ equation (9.4) is classical and known to hold for all $f \in C_c^\infty$ when $p < n$. Similar equations to (9.4) with $\varphi(x) = \rho(x)$ are proven in the cases of Ω as an interior domain in [69] and as an exterior domain in [48] (again with $p < n$).

9.1 Case of the Exterior of a Compact Convex Object

In this section Theorem 9.1 is proven along with the first part of Theorem 9.3. The proof of Theorem 9.1 uses Schur's test motivated by techniques from [48]. For convenience the diameter of the compact object is assumed to be 1.

Lemma 9.7 (Schur's test)**.** *Let $k(x,y) > 0$ be the associated kernel of an integral operator T. Then $\|Tf\|_p \leq \|f\|_p$ if and only if there exists $v(x,y)$ such that the following holds.*

$$\int_\Omega k(x,y)v(x,y)^{1/p}\,\mathrm{d}y \lesssim 1 \text{ and } \int_\Omega k(x,y)v(x,y)^{1/p'}\,\mathrm{d}x \lesssim 1$$

The *only if* part is due to Gagliardo [30] and works as $k(x,y) > 0$. Schur's test is well known so will not be proven here. The next lemma calculates the kernel of the Hardy operator for the exterior domain case.

Lemma 9.8 (Lemma 5.2 in [48]). *The Hardy operator in the exterior of a compact convex object case can be written as an integral operator with associated kernel $k(x,y)$ bounded by the following.*

$$k(x,y) \lesssim \frac{|x-y|}{\rho(x)} \left(1 \wedge \frac{\rho(x)}{1 \wedge |x-y|}\right) \left(1 \wedge \frac{\rho(y)}{1 \wedge |x-y|}\right) \frac{1}{|x-y|^n}$$

Proof. Write the Hardy inequality as an integral over the heat semigroup. This gives the following equation.

$$\frac{1}{\rho(x)} \Delta_\Omega^{-1/2} f(x) = \frac{c}{\rho(x)} \int_0^\infty e^{-t\Delta_\Omega} f(x) \frac{\mathrm{d}t}{\sqrt{t}}$$

The kernel of the Hardy inequality is therefore given by,

$$k(x,y) = \frac{c}{\rho(x)} \int_0^\infty p_t^{B^c}(x,y) \frac{\mathrm{d}t}{\sqrt{t}}$$

which is a well defined value, integrable with respect to y for every $y \in \Omega$. Substitute in the bounds for $p_t^{B^c}(x,y)$ stated in chapter 7 as a result of Zhang [74].

$$k(x,y) \sim \frac{1}{\rho(x)} \int_0^\infty \left(1 \wedge \frac{\rho(x)}{1 \wedge \sqrt{t}}\right) \left(1 \wedge \frac{\rho(y)}{1 \wedge \sqrt{t}}\right) \frac{e^{-|x-y|^2/ct}}{t^{n/2}} \frac{\mathrm{d}t}{\sqrt{t}}$$

Evaluate this integral for the upper bound. □

Next is the proof of Theorem 9.1.

Proof of Theorem 9.1. Let T be an integral operator with the following kernel.

$$k(x,y) = \frac{\rho(x)^{k/p}}{\rho(y)^{k/p}} \left(\frac{|x-y|}{1 \wedge |x-y|} \wedge \frac{|x-y|}{\rho(x)}\right) \left(1 \wedge \frac{\rho(y)}{1 \wedge |x-y|}\right) \frac{1}{|x-y|^n}$$

There are multiple cases to consider. Schur's test is used in each case. Each integral uses an appropriate coordinate transfer before evaluation.

Case 1a: suppose $|x-y| < \rho(x) \wedge \rho(y)$. Let $I_1 = \{y \in \Omega : |x-y| < \rho(x) \wedge \rho(y)\}$ and

$I_2 = \{x \in \Omega : |x-y| < \rho(x) \wedge \rho(y)\}$. The triangle inequality implies in this case that $\rho(x) < |x-y| + \rho(y) < 2\rho(y)$ and $\rho(y) < |x-y| + \rho(x) < 2\rho(x)$. Hence $\rho(x) \sim \rho(y)$ so that with $a = p/k$, $v(x,y) = 1$ and $\lambda = |x-y|$ the following two equations hold.

$$1) \qquad \int_{I_1} k(x,y)v(x,y)^{1/p}\,\mathrm{d}y \lesssim \int_{I_1} \frac{\rho(y)^{-a}\rho(x)^{a-1}}{|x-y|^{n-1}}\,\mathrm{d}y \lesssim \int_0^{\rho(x)} \frac{1}{\rho(x)}\,\mathrm{d}\lambda \lesssim 1$$

$$2) \qquad \int_{I_2} k(x,y)v(x,y)^{1/p'}\,\mathrm{d}x \lesssim \int_{I_2} \frac{\rho(y)^{-a}\rho(x)^{a-1}}{|x-y|^{n-1}}\,\mathrm{d}x \lesssim \int_0^{\rho(y)} \frac{1}{\rho(y)}\,\mathrm{d}\lambda \lesssim 1$$

Thus this case satisfies Schur's test with no restriction for k.

Case 1b: suppose $|x-y| \le 3$ and $\rho(x) \wedge \rho(y) \le |x-y|$. If $\rho(x) \le |x-y| \le \rho(y)$ then $\rho(y) \le 2|x-y|$ by the triangle inequality. Similarly if $\rho(y) \le |x-y| \le \rho(x)$ then $\rho(x) \le 2|x-y|$ by the triangle inequality. Hence the conditions of this case imply $\max(\rho(x), \rho(y)) \le 2|x-y| \le 6$. This means x and y are close to each other and to the boundary $\delta\Omega$. Consider first the subcase $\rho(x) \le \rho(y)$. For this case take $v(x,y) = \frac{\rho(x)^m}{\rho(y)^m}$, $a = k/p + m/p$, $b = k/p - m/p'$, $I_3 = \{y \in \Omega : \rho(x) \le \rho(y) \le 2|x-y|\}$, $I_4 = \{x \in \Omega : \rho(x) \le \rho(y) \le 2|x-y|\}$ and $\lambda = |x-y|$. First consider the dy integral.

$$1) \qquad \int_{I_3} k(x,y)v(x,y)^{1/p}\,\mathrm{d}y \lesssim \int_{I_3} \frac{\rho(y)^{1-a}\rho(x)^a}{|x-y|^{n+1}}\,\mathrm{d}y \lesssim \begin{cases} \int_{\rho(x)}^\infty \frac{\rho(x)}{\lambda^2}\,\mathrm{d}\lambda \lesssim 1 & \text{if } 1 \le a \\ \int_{\rho(x)}^\infty \frac{\rho(x)^a}{\lambda^{1+a}}\,\mathrm{d}\lambda \lesssim 1 & \text{if } 0 < a < 1 \end{cases}$$

For the dx integral use local coordinates $x = (x_1, \ldots, x_n)$ chosen so that $x_1 = \psi(x')$ with $x' = (x_2, \ldots, x_n)$, locally describes $\delta\Omega$ and $\rho(x) \sim x_1 - \psi(x')$ for local x. This requires $\delta\Omega$ to be locally Lipschitz. The area $\{x \in \Omega : \rho(x) < 6\}$, around the $C^{1,1}$ compact convex object can be split into a finite number of local coordinates each satisfying the outlined condition. As $\rho(y) < 2|x-y|$ in this case, if $|x'-y'| < \rho(y)/4$ then $|x_1 - y_1| > \rho(y)/4$. Let $s = \rho(x)$, $t = |x_1 - y_1|$ and $z = |x'-y'|$. Then on appropriate local regions,

$$\begin{aligned} 2) \qquad &\int_{I_4} k(x,y)v(x,y)^{1/p'}\,\mathrm{d}x \lesssim \int_{I_4} \frac{\rho(y)^{1-b}\rho(x)^b}{|x-y|^{n+1}}\,\mathrm{d}x \\ &\lesssim \begin{cases} \int_{\rho(y)}^\infty \frac{\rho(y)}{\lambda^2}\,\mathrm{d}\lambda \lesssim 1 & \text{if } 0 \le b \\ \int_0^{\frac{\rho(y)}{4}} \int_{\frac{\rho(y)}{4}}^\infty \frac{\rho(y)^{1-b}}{z^{1-b}t^2}\,\mathrm{d}t\,\mathrm{d}z + \int_{\frac{\rho(y)}{4}}^\infty \int_0^{\rho(y)} \frac{\rho(y)^{1-b}}{s^{-b}z^{1+b}}\,\mathrm{d}s\,\mathrm{d}z \lesssim 1 & \text{if } -1 < b < 0 \end{cases} \end{aligned}$$

holds. The conditions $a > 0$ and $b > -1$ in this case imply $-k < m < kp' - k + p'$ so that m exists only if $k > -1$.

Continue with case 1b, now with subcase $\rho(y) < \rho(x)$. Let $v(x,y)$ and a be as above and $I_5 = \{y \in \Omega : \rho(y) < \rho(x) \leq 2|x-y|\}$. Again use local coordinates, this time so that $\rho(y) \sim y_1 - \psi(y')$ for local y. As $\rho(x) < 2|x-y|$ in this case, if $|x'-y'| < \rho(x)/4$ then $|x_1 - y_1| > \rho(x)/4$. Let $s = \rho(y)$, $t = |x_1 - y_1|$ and $z = |x'-y'|$.

$$\text{1)} \qquad \int_{I_5} k(x,y)v(x,y)^{1/p}\,\mathrm{d}y \lesssim \int_{I_5} \frac{\rho(y)^{1-a}\rho(x)^a}{|x-y|^{n+1}}\,\mathrm{d}y$$
$$\lesssim \begin{cases} \int_{\rho(x)}^{\infty} \frac{\rho(x)}{\lambda^2}\,\mathrm{d}\lambda \lesssim 1 & \text{if } a \leq 1 \\ \int_0^{\rho(x)/4}\int_{\rho(x)/4}^{\infty} \frac{\rho(x)^a}{s^{1+a}}\,\mathrm{d}t\,\mathrm{d}z + \int_{\rho(x)/4}^{\infty}\int_0^{\rho(x)} \frac{s^{1-a}\rho(x)^a}{z^3}\,\mathrm{d}s\,\mathrm{d}z \lesssim 1 & \text{if } 1 < a < 2 \end{cases}$$

For the dx case let $I_6 = \{x \in \Omega : \rho(y) < \rho(x) \leq 2|x-y|\}$, $b = k/p - m/p'$ and $\lambda = |x-y|$.

$$\text{2)} \qquad \int_{I_6} k(x,y)v(x,y)^{1/p'}\,\mathrm{d}x \lesssim \int_{I_6} \frac{\rho(x)^b\rho(y)^{1-b}}{|x-y|^{n+1}}\,\mathrm{d}x$$
$$\lesssim \begin{cases} \int_{\rho(y)}^{\infty} \frac{\rho(y)}{\lambda^2}\,\mathrm{d}\lambda \lesssim 1 & \text{if } b \leq 0 \\ \int_{\rho(y)}^{\infty} \frac{\rho(y)^{1-b}}{\lambda^{2-b}}\,\mathrm{d}\lambda \lesssim 1 & \text{if } 0 < b < 1 \end{cases}$$

The conditions $a < 2$ and $b < 1$ combine for $kp' - k - p' < m < 2p - k$. So m only exists in this case if $k < 2p - 1$. If Ω was a global Lipschitz domain, then case 1b would not need the restriction $|x-y| < 2$ and the kernel would be the same as considered here in cases 1a and 1b. This is referred to in the proof of Theorem 9.2 in the next section.

Case 2a: suppose $3 \leq |x-y|$ and $1 \leq \rho(x) \leq |x-y| \wedge \rho(y)$. The triangle inequality and the geometry of the case imply $|x-y| \leq 2\rho(y) \leq 4|x-y|$. Take $v(x,y) = \frac{\rho(x)^m}{\rho(y)^m}$, $a = k/p + m/p > 1$, $b = k/p - m/p' < 1-n$, $I_7 = \{y \in \Omega : 1 < \rho(x) \leq \rho(y) \sim |x-y|\}$, $I_8 = \{x \in \Omega : 1 < \rho(x) \leq \rho(y) \sim |x-y|\}$, $s = \rho(y)$ and $t = \rho(x)$. Note $\rho(x)$ and $\rho(y)$ are radial in this case in the sense that $\rho(x) \sim |x-c|$ for some $c \in \mathbb{R}^n$, and similar for $\rho(y)$.

$$\text{1)} \qquad \int_{I_7} k(x,y)v(x,y)^{1/p}\,\mathrm{d}y \lesssim \int_{I_7} \frac{\rho(y)^{-a}\rho(x)^{a-1}}{|x-y|^{n-1}}\,\mathrm{d}y \lesssim \int_{\rho(x)}^{\infty} \frac{\rho(x)^{a-1}}{s^a}\,\mathrm{d}s \lesssim 1$$
$$\text{2)} \qquad \int_{I_8} k(x,y)v(x,y)^{1/p'}\,\mathrm{d}x \lesssim \int_{I_8} \frac{\rho(y)^{-b}\rho(x)^{b-1}}{|x-y|^{n-1}}\,\mathrm{d}x \lesssim \int_0^{\rho(y)} \frac{t^{n+b-2}}{\rho(y)^{b+n-1}}\,\mathrm{d}t \lesssim 1$$

As $a > 1$ and $b > 1-n$ in this case, then $p - k < m < kp' - k - p' + np'$ holds which gives the restriction $p - n < k$.

Case 2b: suppose $3 \leq |x-y|$ and $1 \leq \rho(y) \leq |x-y| \wedge \rho(x)$. The triangle inequality and the geometry of the case imply $|x-y| \leq 2\rho(y) \leq 4|x-y|$. Let $v(x,y) = \frac{\rho(x)^m}{\rho(y)^m}$, $a = k/p + m/p < n$, $b = k/p - m/p' < 0$, $I_9 = \{y \in \Omega : 1 < \rho(y) \leq \rho(x) \sim |x-y|\}$ and $I_{10} = \{x \in \Omega : 1 < \rho(y) \leq \rho(x) \sim |x-y|\}$. Further $s = \rho(x)$ and $t = \rho(y)$ can again be described radially here.

$$1) \qquad \int_{I_9} k(x,y)v(x,y)^{1/p}\,\mathrm{d}y \lesssim \int_{I_9} \frac{\rho(y)^{-a}\rho(x)^{a-1}}{|x-y|^{n-1}}\,\mathrm{d}y \lesssim \int_0^{\rho(x)} s^{-a+n-1}\rho(x)^{a-n}\,\mathrm{d}s \lesssim 1$$
$$2) \qquad \int_{I_{10}} k(x,y)v(x,y)^{1/p'}\,\mathrm{d}x \lesssim \int_{I_{10}} \frac{\rho(y)^{-b}\rho(x)^{b-1}}{|x-y|^{n-1}}\,\mathrm{d}x \lesssim \int_{\rho(y)}^{\infty} \frac{\rho(y)^{-b}}{t^{1-b}}\,\mathrm{d}t \lesssim 1$$

As $a < n$ and $b < 0$ in this case, then $kp' - k < m < np - k$ holds which gives a restriction $k < np - n$.

Case 2c: suppose $3 \leq |x-y|$ and $\rho(y) \leq 1$. The triangle inequality and the geometry of the case imply $\rho(x) \sim |x-y|$. Let $v(x,y) = \frac{\rho(x)^{m+p'}}{\rho(y)^m}$, $a = k/p + m/p < 2 \wedge (n+1-p')$, $b = k/p - m/p' < 1$, $s = \rho(y)$, $t = \rho(x)$, $I_{11} = \{y \in \Omega : 3\rho(y) \leq 3 \leq |x-y|\}$ and $I_{12} = \{x \in \Omega : 3\rho(y) \leq 3 \leq |x-y|\}$. As $\rho(y) \leq 1$, and as the compact object has diameter 1, the $\mathrm{d}y$ integral is over a volume of size $\lesssim 3^n$. Local coordinates are used in the $\mathrm{d}y$ integral as in earlier cases of this proof.

$$1) \qquad \int_{I_{11}} k(x,y)v(x,y)^{1/p}\,\mathrm{d}y \lesssim \int_{I_{11}} \frac{\rho(y)^{1-a}\rho(x)^{a+p'-2}}{|x-y|^{n-1}}\,\mathrm{d}y \lesssim \int_0^1 s^{1-a}\,\mathrm{d}s \lesssim 1$$
$$2) \qquad \int_{I_{12}} k(x,y)v(x,y)^{1/p'}\,\mathrm{d}x \lesssim \int_{I_{12}} \frac{\rho(y)^{1-b}\rho(x)^{b-2}}{|x-y|^{n-1}}\,\mathrm{d}x \lesssim \int_1^{\infty} \frac{\rho(y)^{1-b}}{t^{2-b}}\,\mathrm{d}t \lesssim 1$$

So as $a < 2 \wedge (n+1-p')$ and $b < 1$ then $kp' - k - p' < m < (2p-k) \wedge (np - p' - k)$ which implies $k < (2p-1) \wedge (np-n)$.

Case 2d: suppose $3 \leq |x-y|$ and $\rho(x) \leq 1$. The triangle inequality and the geometry of the case ensures $\rho(y) \sim |x-y|$ holds. Let $v(x,y) = \frac{\rho(x)^{m-p}}{\rho(y)^m}$, $a = k/p + m/p > 1$, $b = k/p - m/p' > \max(-p, -n+1)$, $s = \rho(y)$ and $t = \rho(x)$. Also consider the sets $I_{13} = \{y \in \Omega : 3\rho(x) \leq 3 \leq |x-y|\}$ and $I_{14} = \{x \in \Omega : 3\rho(x) \leq 3 \leq |x-y|\}$. The $\rho(y)$ term can be described radially. As $\rho(x) \leq 1$ the $\mathrm{d}x$ integral is over an area of size $\lesssim 3^n$.

Local coordinates are used in the dx integral as in earlier cases of this proof.

$$1) \qquad \int_{I_{13}} k(x,y)v(x,y)^{1/p}\,\mathrm{d}y \lesssim \int_{I_{13}} \frac{\rho(y)^{-a}\rho(x)^{a-1}}{|x-y|^{n-1}}\,\mathrm{d}y \lesssim \int_1^\infty s^{-a}\,\mathrm{d}s \lesssim 1$$

$$2) \qquad \int_{I_{14}} k(x,y)v(x,y)^{1/p'}\,\mathrm{d}x \lesssim \int_{I_{14}} \frac{\rho(y)^{-b}\rho(x)^{b+p-1}}{|x-y|^{n-1}}\,\mathrm{d}x \lesssim \int_0^1 t^{b+p-1}\,\mathrm{d}t \lesssim 1$$

So as $a > 1$ and $b > \max(-p, -n+1)$ then $p-k < m < \min(p+p'+kp'-k, kp'-k+np'-p')$ so the restriction here is $\max(-1, p-n) < k$.

Hence in total by Schur's test there is an unweighted bound $\|Tg\|_p \leq \|g\|_p$ for all $g \in \mathcal{L}^p$ when $\max(-1, p-n) < k < 2p-1$. Let $f = g/\rho^{k/p}$ and observe due to the relative kernels that $Tg = \frac{\rho^{k/p}}{\rho}\Delta_\Omega^{-1/2} f$. This implies $\|\frac{1}{\rho}\Delta_\Omega^{-1/2} f\|_{\mathcal{L}^p(\rho^k)} \leq \|f\|_{\mathcal{L}^p(\rho^k)}$ for all $\max(-1, p-n) < k < 2p-1$. □

The remainder of this section regards proving Theorem 9.3 in the exterior domain case. The proof involves enforcing Theorem 6.5 from chapter 6. From the results of chapter 7 the kernel for the operator $\frac{\sqrt{t}}{\rho}e^{-t\Delta_\Omega}$ can be approximated by the following.

$$k_t(x,y) \sim \frac{\sqrt{t}}{\rho(x)}\left(1 \wedge \frac{\rho(x)}{1 \wedge \sqrt{t}}\right)\left(1 \wedge \frac{\rho(y)}{1 \wedge \sqrt{t}}\right)\frac{e^{-|x-y|^2/ct}}{t^{n/2}}$$

First functions need to be chosen to take the place of α and β from the theorems of chapter 6.

Remarks 9.9. The obvious choice for α and β would be straight from the kernel found above.

$$\alpha_{r^2}(x) = \frac{r}{\rho(x)}\left(1 \wedge \frac{\rho(x)}{1 \wedge r}\right) \text{ and } \beta_{r^2}(x) = \left(1 \wedge \frac{\rho(x)}{1 \wedge r}\right)$$

But these do not satisfy the weight conditions of chapter 6. This can be seen by checking condition (6.1a), on a small ball B ($r(B) < 1$) far from the boundary ($\rho(B) > r(B)$).

$$\begin{aligned}&\left(\fint_B \alpha_{r^2}(x)^{-A\gamma}\,\mathrm{d}x\right)^{1/\gamma}\left(\fint_B \beta_{r^2}(x)^{-A\gamma'}\,\mathrm{d}x\right)^{1/\gamma'}\\ &\qquad = \left(\fint_B \rho(x)^{A\gamma}r^{-A\gamma}\,\mathrm{d}x\right)^{1/\gamma}\left(\fint_B 1\,\mathrm{d}x\right)^{1/\gamma'} \geq \rho(B)^{A\gamma}r^{-A\gamma}\end{aligned}$$

which grows as $\rho(B)$ does, so importantly is not bounded by 1.

A larger pair of α and β are needed. The choice below is motivated by the fact the resulting weight class is contained in the weight class A_p^{Ext1} discussed in chapter 8.

Lemma 9.10. *The following α and β do satisfy the weight conditions of chapter 6.*

$$\alpha_{r^2}(x) = \left(\frac{r}{r \wedge \rho(x)} \wedge \frac{r}{1 \wedge r}\right) \text{ and } \beta_{r^2}(x) = \left(1 \wedge \frac{\rho(x)}{1 \wedge r}\right) \tag{9.5}$$

Proof. After observing the trivial conditions of continuity and strict positivity there are four conditions to prove. For condition (6.1a) observe α and β here are larger then those considered in Proposition 8.5 so (6.1a) holds by comparison.

For condition (6.1b) consider first balls B of radius r that satisfy either $r < 1$, or $r < \rho(B)$. These mean $\alpha_{r^2}(x) \lesssim 1$ and $\beta_{r^2}(x) \lesssim 1$ so that (6.1b) holds trivially. The only remaining possibility is B large ($r \geq 1$) and near the boundary ($\rho(B) \lesssim r$). This means $\alpha_{r^2}(x) = r \wedge \frac{r}{\rho(x)}$ and $\beta_{r^2}(x) = \rho(x) \wedge 1$.

$$\begin{aligned}\left(\fint_B \alpha_{r^2}(x)\,\mathrm{d}x\right)\left(\fint_B \beta_{r^2}(x)\,\mathrm{d}x\right) &= \left(\fint_B r \wedge \frac{r}{\rho(x)}\,\mathrm{d}x\right)\left(\fint_B \rho(x) \wedge 1\,\mathrm{d}x\right) \\ &\lesssim \left(r^{1-n} + 1\right)\left(r^{-n} + 1\right) \lesssim 1\end{aligned}$$

It is required here for there to be local coordinate systems near the boundary (for which the boundary is Lipschitz) and roughly spherical coordinates away from the boundary.

Next check condition (6.1c). If $\rho(B) > 2^M r$ then $\alpha_{r^2}(x)$ and $\alpha_{4^M r^2}(x)$ are the same as considered in Proposition 8.5 and so satisfy (6.1c) by that case. Similarly if $2^M r < 1$ then $\alpha_{r^2}(x) = 1 = \alpha_{4^M r^2}(x)$ and (6.1c) holds trivially. For the remaining cases:

- If $\max(1, r) \leq \rho(B) \leq 2^M r$; then $\alpha_{r^2}(x) \sim 1 \sim \dfrac{\rho(B)}{2^M r}\alpha_{4^M r^2}(x)$ and $\frac{1}{2^M} \leq \frac{\rho(B)}{2^M r} \leq 1$.
- If $\max(r, \rho(B)) \leq 1 \leq 2^M r$; then $\alpha_{r^2}(x) \sim 1 \sim \dfrac{1}{2^M r}\alpha_{4^M r^2}(x)$ and $\frac{1}{2^M} \leq \frac{1}{2^M r} \leq 1$.
- If $1 \leq \rho(B) \leq r \leq 2^M r$; then $\alpha_{r^2}(x) \sim \dfrac{r}{\rho(x)} \sim \dfrac{1}{2^M}\alpha_{4^M r^2}(x)$.
- If $\rho(B) \leq 1 \leq r \leq 2^M r$; then $\alpha_{r^2}(x) \sim r \sim \dfrac{1}{2^M}\alpha_{4^M r^2}(x)$.

That $\beta_{r^2}(x)$ also satisfies condition (6.1c) is in the proof of Proposition 8.5.

It remains to prove condition (6.1d). Let I_1 and I_2 be a decomposition of the set of all balls $B \subset \Omega$:

- I_1, contains balls B where either $\rho(B) < r < 1$ or $r < \rho(B) \wedge 1$ or $1 < r < \rho(B)$ so $\alpha_{r^2}(x) \sim 1$ for all $x \in 5B$ when $B \in I_1$; and,
- I_2, are the remaining balls and $\alpha_{r^2}(x) \sim r(1 \wedge \rho(x)^{-1})$ for all $x \in 5B$ when $B \in I_2$.

This satisfies condition (6.1d) and concludes the proof. □

The ingredients to prove the exterior part of Theorem 9.3 are now ready. This also proves the remaining part of Theorem 1.5 via Theorem 8.3.

Proof of Theorem 9.3. It was proven in Lemma 9.10 that the α_{r^2} and β_{r^2} of equation (9.5) satisfy all of Conditions 6.1. Further the kernel $\alpha_t(x)\beta_t(y)\frac{e^{-|x-y|^2/ct}}{t^{n/2}}$ is a pointwise upper bound for the kernel of $\frac{\sqrt{t}}{\rho(x)}e^{-t\Delta_\Omega}$. Hence by Theorem 6.5 from chapter 6 the operator $\frac{\sqrt{t}}{\rho(x)}e^{-t\Delta_\Omega}$ is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for all w in the class $A_p^{\alpha,\beta}$ (specified as A_p^{Ext2} in the case of Theorem 9.3). □

Remarks 9.11. The class A_p^{Ext2} is contained in the class of weights A_p^{Ext1} from Theorem 8.1 for which the heat semigroup was shown to be bounded. This is easy to see as β_{r^2} is the same in both cases, and α_{r^2} in this case is pointwise larger. In the example below it is shown that the class A_p^{Ext2} contains weights outside the Muckenhoupt class A_p. Yet there are also weights in A_p not in A_p^{Ext2} (for example for any fixed p, $w(x) = \rho(x)^p$ is in A_p^{Ext2} but is not in A_p, and yet $w(x) = 1$ is in A_p but is not in A_p^{Ext2} for all $p \geq n$).

Example 9.12. The weight $w(x) = \rho(x)^k$ is in the weight class A_p^{Ext2} if and only if k satisfies $\max(-1, p-n) < k < 2p-1$. For the *if* part use that $\frac{1}{\rho(x)}\Delta_\Omega^{-1/2}f(x)$ is a pointwise upper bound for $\frac{\sqrt{t}}{\rho(x)}e^{-t\Delta_\Omega}f(x)$, and it was shown that $\frac{1}{\rho(x)}\Delta_\Omega^{-1/2}f(x)$ is bounded for all weights $\rho(x)^k$ for $\max(-1, p-n) < k < 2p-1$ in the proof of Theorem 9.1.

For the *only if* part consider first $r < 1$ and B close to the boundary $\rho(B) \lesssim r$. In such a case $\alpha \sim 1$ and $\beta \sim \rho/r$ so that the following holds.

$$A_p^{Ext2}(\rho^k) = A_p^{\alpha,\beta}(\rho^k) = \sup_B \left(\fint_B \rho(x)^k \,\mathrm{d}x \right) \left(\fint_B \frac{\rho(x)^{p'}}{r^{p'}} \rho(x)^{-kp'/p} \,\mathrm{d}x \right)^{p/p'}$$

If $k \leq -1$ then $\fint_B \rho(x)^k \,\mathrm{d}x$ is not integrable near the boundary. Further if $k \geq 2p-1$ then $\fint_B \frac{\rho(x)^{p'}}{r^{p'}} \rho(x)^{-kp'/p} \,\mathrm{d}x$ is not integrable near the boundary.

Secondly consider $r > 1$ and $\rho(B) = 1$. Without loss of generality suppose the compact object Ω is exterior to is centred at the origin. Then $\rho(x) \sim |x|$ and $\rho(y) \sim |y|$.

$$\begin{aligned}
&\left(\fint_B \alpha_{r^2}(x)^p w(x)\,\mathrm{d}x\right)\left(\fint_B \beta_{r^2}(x)^{p'} w(x)^{-p'/p}\,\mathrm{d}x\right)^{p/p'} \\
&\qquad = \frac{1}{r^n}\int_B r^p \rho(x)^{k-p}\,\mathrm{d}x\Big(\frac{1}{r^n}\int_B \rho(x)^{-kp'/p}\,\mathrm{d}x\Big)^{p/p'} \\
&\qquad = \frac{1}{r^n}\int_1^r r^p \lambda^{k-p+n-1}\,\mathrm{d}\lambda\Big(\frac{1}{r^n}\int_1^r \lambda^{-kp'/p+n-1}\,\mathrm{d}\lambda\Big)^{p/p'} \\
&\qquad = \frac{1}{r^{np}}\begin{cases} r^{k+n} & \text{if } np-n<k \\ r^{k+n}\left(r^{-kp'/p+n}\right)^{p/p'} & \text{if } p-n<k<np-n \\ \left(r^{-kp'/p+n}\wedge 1\right)^{p/p'} & \text{if } k<p-n \end{cases}
\end{aligned}$$

Only in the case where $p-n<k<np-n$ do these integrals remain finite as r tends to infinity. This concludes the example.

9.2 Case of a Global Lipschitz Domain

In this section the argument of the previous section is followed and adapted to the case of the area above a Lipschitz curve. The result is proofs of Theorem 9.2 and the global Lipschitz part of Theorem 9.3. Where possible the proofs will be shortened by direct comparisons to the previous section.

Lemma 9.13. *The Hardy operator in the case of the area above a $C^{1,1}$ global Lipschitz and bounded curve can be written as an integral operator with associated kernel $k(x,y)$ bounded above by the following.*

$$k(x,y) \lesssim \left(1 \wedge \frac{|x-y|}{\rho(x)}\right)\left(1 \wedge \frac{\rho(y)}{|x-y|}\right)\frac{1}{|x-y|^n}$$

Proof. As in the proof of Lemma 9.8, use that the Hardy inequality is known to be given in terms of an integral over the heat semigroup. So the kernel of the Hardy inequality is given by an integral over the heat kernel,

$$k(x,y) = \frac{c}{\rho(x)}\int_0^\infty p_t(x,y)\frac{\mathrm{d}t}{\sqrt{t}}$$

which is a well defined value, integrable with respect to y for every $x \in \Omega$. Substitute in the upper bound for $p_t(x,y)$ found in chapter 7 based on the result of Song in [62].

$$k(x,y) \sim \frac{1}{\rho(x)} \int_0^\infty \left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) \frac{e^{-|x-y|^2/ct}}{t^{n/2}} \frac{\mathrm{d}t}{\sqrt{t}}$$

The integral with respect to t can be evaluated to have appropriate upper bound. □

Proof of Theorem 9.2. Let T be an integral operator with the following associated kernel.

$$k(x,y) = \frac{\rho(x)^{k/p}}{\rho(y)^{k/p}} \left(1 \wedge \frac{|x-y|}{\rho(x)}\right) \left(1 \wedge \frac{\rho(y)}{|x-y|}\right) \frac{1}{|x-y|^n}$$

In cases 1a and 1b of the proof of Theorem 9.1 a kernel of the precise form above was dealt with, and in case 1b the condition $|x-y| \leq 3$ was not used (apart from determining the kernel shape). These two cases allow Schur's test to be used with the kernel given above to get a bound $\|Tg\|_p \lesssim \|g\|_p$ for all $-1 < k < 2p-1$ and $g \in \mathcal{L}^p(\Omega)$. Substitute $f = \rho^{-k/p} g$ and $Tg = \frac{\rho^{k/p}}{\rho} \Delta_\Omega^{-1/2} f$ to get $\|\frac{1}{\rho} \Delta_\Omega^{-1/2} f\|_{\mathcal{L}^p(\rho^k)} \lesssim \|f\|_{\mathcal{L}^p(\rho^k)}$ for all $-1 < k < 2p-1$. □

The remainder of this section finds a weight class for $\frac{1}{\rho} e^{-t\Delta_\Omega}$ in the global Lipschitz domain case. As in the previous section the obvious choices for α and β here do not satisfy the required conditions from chapter 6.

Remarks 9.14. Consider the functions α and β given by,

$$\alpha_{r^2}(x) = \frac{r}{\rho(x)} \left(1 \wedge \frac{\rho(x)}{r}\right) \text{ and } \beta_{r^2}(x) = \left(1 \wedge \frac{\rho(x)}{r}\right)$$

for all $x \in \Omega$ and $t > 0$. These do not satisfy condition (6.1a) from chapter 6. The proof involves observing the following for any ball B where $\rho(B) > r(B)$:

$$\begin{aligned} &\left(\fint_B \alpha_{r^2}(x)^{-A\gamma} \,\mathrm{d}x\right)^{1/\gamma} \left(\fint_B \beta_{r^2}(x)^{-A\gamma'} \,\mathrm{d}x\right)^{1/\gamma'} \\ &\qquad = \left(\fint_B \rho(x)^{A\gamma} r^{-A\gamma} \,\mathrm{d}x\right)^{1/\gamma} \left(\fint_B 1 \,\mathrm{d}x\right)^{1/\gamma'} \geq \rho(B)^{A\gamma} r^{-A\gamma} \end{aligned}$$

which grows as $\rho(B)$ does.

So again the results of chapter 6 cannot be used directly. Instead consider once more an α, β pair larger then those proposed above, chosen so the final weight class will be contained in the weight class for the heat semigroup.

Lemma 9.15. *The following α and β do satisfy the weight conditions of chapter 6.*

$$\alpha_{r^2}(x) = 1 \text{ and } \beta_{r^2}(x) = \left(1 \wedge \frac{\rho(x)}{r}\right) \tag{9.6}$$

Proof. The continuity and strict positivity conditions are trivial leaving four conditions to prove. Firstly the α and β considered here are pointwise larger than their counterparts in the proof of Proposition 8.4. Hence condition (6.1a) holds by comparison.

$$\left(\fint_B \alpha_{r^2}(x)^{-A\gamma}\, \mathrm{d}x\right)^{1/\gamma} \left(\fint_B \beta_{r^2}(x)^{-A\gamma'}\, \mathrm{d}x\right)^{1/\gamma'} \lesssim 1$$

For condition (6.1b) use $\alpha_{r^2}(x) \leq 1$ and $\beta_{r^2}(x) \leq 1$ so that condition (6.1b) will hold trivially.

$$\left(\fint_B \alpha_{r^2}(x)\, \mathrm{d}x\right) \left(\fint_B \beta_{r^2}(x)\, \mathrm{d}x\right) \lesssim 1$$

Condition (6.1c) similarly holds trivially for α, as $\alpha_{r^2}(x) = 1$ for all x and r. For β condition (6.1c) is proven to hold in the proof of Proposition 8.4. Condition (6.1d) is trivial as $\alpha_{r^2}(x) = 1$ on all balls $B \subset \Omega$. □

All the ingredients are now ready to prove the global Lipschitz part of Theorem 9.3.

Proof of Theorem 9.3. It was proven in Lemma 9.15 that the α_{r^2} and β_{r^2} of equation (9.6) satisfy all of Conditions 6.1. Further the kernel $\alpha_t(x)\beta_t(y)\frac{e^{-|x-y|^2/ct}}{t^{n/2}}$ is a pointwise upper bound for the kernel of $\frac{\sqrt{t}}{\rho(x)}e^{-t\Delta_\Omega}$. Hence by Theorem 6.5 from chapter 6 the operator $\frac{\sqrt{t}}{\rho(x)}e^{-t\Delta_\Omega}$ is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for all w in the class $A_p^{\alpha,\beta}$ (Specified as A_p^{Lip2} in Theorem 9.3). □

Remarks 9.16. Given the weight classes A_p^{Lip1} (8.1) and A_p^{Lip2} (9.3) for the operators $e^{-t\Delta_\Omega}$ and $\frac{\sqrt{t}}{\rho}e^{-t\Delta_\Omega}$ respectively, observe that $A_p \subset A_p^{Lip2} \subset A_p^{Lip1}$ holds. This is evident by looking at pointwise upper bounds for α and β in each case. Further the class A_p^{Lip2} contains weights outside the Muckenhoupt class A_p (in the example below it is shown that $\rho(x)^p$ is in $A_p^{\alpha,\beta}$ yet such a weight is not in A_p).

It remains to check the form of typical members of the weight class.

Example 9.17. The weight $\rho(x)^k$ is in the weight class A_p^{Lip2} if and only if k satisfies $-1 < k < 2p-1$. It is near the boundary that the powers are restricted. For the *if* part use that $\frac{1}{\rho(x)}\Delta_\Omega^{-1/2} f(x)$ is a pointwise upper bound for $\frac{\sqrt{t}}{\rho(x)} e^{-t\Delta_\Omega} f(x)$, and it was shown that $\frac{1}{\rho(x)}\Delta_\Omega^{-1/2} f(x)$ is bounded for all weights $\rho(x)^k$ for $-1 < k < 2p-1$ in the proof of Theorem 9.2.

For the *only if* part consider $r < 1$ and B radius r close to the boundary, $\rho(B) \lesssim r$. In such a case $\alpha \sim 1$ and $\beta \sim \rho/r$.

$$A_p^{Lip2}(\rho^k) = A_p^{\alpha,\beta}(\rho^k) = \sup_B \left(\fint_B \rho(x)^k \, \mathrm{d}x \right) \left(\fint_B \frac{\rho(x)^{p'}}{r^{p'}} \rho(x)^{-kp'/p} \, \mathrm{d}x \right)^{p/p'}$$

If $k \le -1$ then $\fint_B \rho(x)^k \, \mathrm{d}x$ is not integrable near the boundary. Further if $k \ge 2p-1$ then $\fint_B \frac{\rho(x)^{p'}}{r^{p'}} \rho(x)^{-kp'/p} \, \mathrm{d}x$ is not integrable near the boundary. This verifies the example and matches the range of weights found for the Hardy operator $\frac{1}{\rho}\Delta_\Omega^{-1/2}$ in this global Lipschitz case.

9.3 Further Examples

In this section there are remarks on a few further examples. To begin consider the area below a parabola case.

9.3.1 Exterior of a parabola

For the exterior of the parabola $x_2 = x_1^2$ in $\mathbb{R}^2$ the kernel of the Hardy inequality is,

$$k(x,y) \sim \frac{\rho(x)^{-1}}{|x-y|^{n-1}} \left[\Big(1 \wedge \frac{\rho(x)}{|x-y|}\Big) \Big(1 \wedge \frac{\rho(x)}{R \wedge |x-y|}\Big) \Big(1 \wedge \frac{\rho(y)}{|x-y|}\Big) \Big(1 \wedge \frac{\rho(y)}{R \wedge |x-y|}\Big) \right]^{1/2}$$

where $R = \max(1, x_2)$. The Hardy operator can be shown in this case to be bounded for weights $w(x) \sim \rho(x)^k$ whenever $-1 < k < 2p-1$ and $p/2 - n \le k \le np - n - p/2$. To find a weight class for the operator $\frac{\sqrt{t}}{\rho} e^{-t\Delta_\Omega}$ in this case use the following α and β.

$$\alpha_{r^2}(x) = \Big(\frac{r}{\rho(x) \wedge r} \wedge \frac{r}{1 \wedge r} \Big)^{1/2} \qquad \beta_{r^2}(x) = \left[\Big(1 \wedge \frac{\rho(y)}{r}\Big) \Big(1 \wedge \frac{\rho(y)}{1 \wedge r}\Big) \right]^{1/2}$$

This pair will satisfy the required conditions of chapter 6.

Indeed consider any case where the operator involved has a kernel that is an appropriate weighted average of the exterior domain and global Lipschitz domain case kernels. That is consider for some $q > 1$ the operator with kernel,

$$k(x,y) \sim \frac{\rho(x)^{-1}}{|x-y|^{n-1}}\Big(1 \wedge \frac{\rho(x)}{|x-y|}\Big)^{\frac{1}{q}}\Big(1 \wedge \frac{\rho(x)}{R \wedge |x-y|}\Big)^{\frac{1}{q'}}\Big(1 \wedge \frac{\rho(y)}{|x-y|}\Big)^{\frac{1}{q}}\Big(1 \wedge \frac{\rho(y)}{R \wedge |x-y|}\Big)^{\frac{1}{q'}}$$

for appropriate R. It would be expected that such a weight class contains $\rho(x)^k$ for all $-1 < k < 2p-1$ and $p/q' - n \leq k \leq pn - n - p/q$. This also means that the unweighted bound ($w(x) = 1$) holds only if $p/q' < n$ (q' dual to q). The α and β pair to use here is,

$$\alpha_{r^2}(x) = \Big(\frac{r}{\rho(x) \wedge r} \wedge \frac{r}{R \wedge r}\Big)^{1/q'} \qquad \beta_{r^2}(x) = \Big(1 \wedge \frac{\rho(y)}{r}\Big)^{1/q}\Big(1 \wedge \frac{\rho(y)}{R \wedge r}\Big)^{1/q'}$$

which will satisfy the required conditions of chapter 6. The resulting weight class satisfies $A_p^{\alpha,\beta}(w) \leq A_p^{Lip2}(w)^{1/q} A_p^{Ext2}(w)^{1/q'}$ (taking balls over the appropriate domain, and with appropriate inclusion of an R term).

9.3.2 Non-doubling Example

Consider the region Ω given by $\{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_2 < e^{x_1}\}$. Consider further the regions $S_1 = \{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_2\}$ and $S_2 = \{(x_1,x_2) \in \mathbb{R}^2 : -e^{x_1} < x_2 < e^{x_1}\}$ so that $\Omega = S_1 \cap S_2$. Using the maximal principle, the Dirichlet heat kernel on Ω is bounded by the Dirichlet heat kernels of S_1 and S_2. Hence for $x \in \Omega$ closer to the boundary $x_2 = 0$,

$$\frac{1}{\rho_\Omega(x)}\Delta_\Omega^{-1/2} f(x) \leq \frac{1}{\rho_{S_1}(x)}\Delta_{S_1}^{-1/2} f(x)$$

whereas for $x \in \Omega$ closer to the boundary $x_2 = e^{x_1}$,

$$\frac{1}{\rho_\Omega(x)}\Delta_\Omega^{-1/2} f(x) \leq \frac{1}{\rho_{S_2}(x)}\Delta_{S_2}^{-1/2} f(x)$$

holds. Here $\rho_\Omega(x)$ is the minimal distance to the boundary $\delta\Omega$, whereas $\rho_{S_1}(x)$ and $\rho_{S_2}(x)$ are the minimal distances to δS_1 and δS_2 respectively. Define the sets Ω_1 and Ω_2 by $\Omega_1 = \{x \in \Omega : \rho_{S_1}(x) \leq \rho_{S_2}(x)\}$ and $\Omega_2 = \{x \in \Omega : \rho_{S_2}(x) \leq \rho_{S_1}(x)\}$.

$$\|\frac{1}{\rho_\Omega}\Delta_\Omega^{-1/2} f\|_{\mathcal{L}^p(\Omega,w)} \leq \|\frac{1}{\rho_{S_1}}\Delta_{S_1}^{-1/2} f\|_{\mathcal{L}^p(\Omega_1,w)} + \|\frac{1}{\rho_{S_2}}\Delta_{S_2}^{-1/2} f\|_{\mathcal{L}^p(\Omega_2,w)}$$

Use that S_1 is the area above a $C^{1,1}$ Lipschitz and bounded curve, and S_2 can be found to have heat kernel upper bound comparable to that of above a $C^{1,1}$ Lipschitz and bounded curve. This means that first term on the right is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for weights $w(x) \sim \rho_{S_1}(x)^k$ with $-1 < k < 2p-1$, and the second term on the right is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for weights $w(x) \sim \rho_{S_2}(x)^j$ with $-1 < j < 2p-1$. In total a combination,

$$w(x) \sim \begin{cases} \rho(x)^k & x \in \Omega_1 \\ \rho(x)^j & x \in \Omega_2 \end{cases} \qquad -1 \leq j, k \leq 2p-1$$

is a weight for this case. Weight classes for the operator $\frac{\sqrt{t}}{\rho} e^{-t\Delta_\Omega}$ can be determined by similarly comparing to the global Lipschitz case.

Chapter 10:

Riesz Transform for the Dirichlet Laplacian

In this chapter Theorem 1.6 is proven. This is the culmination of the application part of this thesis. Theorem 1.6 classifies a set of weights w for which,

$$|||\nabla\Delta_\Omega^{-1/2} f|||_{\mathcal{L}^p(w)} \lesssim \|f\|_{\mathcal{L}^p(w)} \tag{10.1}$$

holds with Ω an exterior domain. There are two parts to this classification. The first part uses Theorem 1.3 (proven in chapter 5) to find an initial class of weights. This initial class is non-optimal due to the $\mathcal{L}^2$ conditions of Theorem 1.3. The second part extends this initial class to an optimal range.

The following elaborations of Theorem 1.6 are the main results of this chapter.

THEOREM 10.1. *Suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is the exterior of a $C^{1,1}$ compact convex object*[1]*. Then the Riesz transform based on the Dirichlet Laplacian associated to Ω is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$, $p \geq 2$, for all weights $w(x) \sim \rho(x)^k$ where $-1 < k < 2p-1$ and $p - n < k < np - n$.*

THEOREM 10.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is a $C^{1,1}$ global Lipschitz domain*[2]*. Then the Riesz transform based on the Dirichlet Laplacian on Ω is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$, $p \geq 2$, for all weights $w(x) \sim \rho(x)^k$ where $-1 < k < 2p - 1$.*

Results from the previous four chapters have proven all conditions necessary to invoke Theorem 1.3 bar one: it remains to prove weighted $\mathcal{L}^2$ Riesz transform bounds away from the boundary. The weighted $\mathcal{L}^2$ bounds are considered in section 10.1 below. Section 10.2

[1] Same remark on convexity as in Theorems 8.1 and 9.1.

[2] Recall this as the area above or below a $C^{1,1}$ Lipschitz and bounded curve in $\mathbb{R}^n$.

then contains proofs of Theorems 10.1 and 10.2. The chapter finishes with section 10.3 where examples are given regarding which parts of the results can be considered optimal.

10.1 A Weighted $\mathcal{L}^2$ Riesz Transform

The Dirichlet Laplacian is the unique self-adjoint operator on Ω satisfying,

$$\int_\Omega |\nabla f(x)|^2 \,\mathrm{d}x = \int_\Omega |\Delta_\Omega^{1/2} f(x)|^2 \,\mathrm{d}x \tag{10.2}$$

for all $f \in W_0^{1,2}(\Omega)$. Unweighted $\mathcal{L}^2$ Riesz transform bounds hold automatically on Ω by such a definition. This is discussed in the introduction chapter. In this section the interest is in when weighted $\mathcal{L}^2$ bounds will hold. The self-adjoint property of Δ_Ω is not enough for a weighted $\mathcal{L}^2$ Riesz transform bound. Properties of the weights are also needed.

Proposition 10.3. *Let $\Omega \subset \mathbb{R}^n$ be either the exterior of a $C^{1,1}$ compact object with $n \geq 3$, or a $C^{1,1}$ global Lipschitz domain. Suppose that w is a weight and that there exists a constant $k \in \mathbb{R}$ such that $w(x) \sim \rho(x)^k$. Then the Riesz transform $\nabla\Delta_\Omega^{-1/2}$ is bounded $\mathcal{L}^2(w) \to \mathcal{L}^2(w)$ for all $-1 < k < 3$.*

Proof. The proof starts with the $k \geq 0$ case. The $k < 0$ case is very similar. Consider a covering of Ω of disjoint balls $\{B_i\}_{i\in I}$ where each ball B_i satisfies $r(B_i) \lesssim \rho(B_i)$ (this can be constructed by a disjoint Whitney covering where the gaps between balls are then covered ensuring both the disjoint, and the $r(B) \lesssim \rho(B)$ conditions hold). For each B_i consider the set $E_i = \{x \in \Omega : \rho(x) < 3\rho(B_i)\}$. Then for all $x \in E_i$ and $y \in B_i$ the weight w satisfies $w(x) \sim \rho(x)^k \leq 3^k\rho(B_i)^k \leq 3^k\rho(y)^k$. Take f supported on B_i and use that $\nabla\Delta_\Omega^{-1/2}$ is bounded $\mathcal{L}^2 \to \mathcal{L}^2$ without weight by definition.

$$\begin{aligned}
\||\nabla\Delta_\Omega^{-1/2} f|\|_{\mathcal{L}^2(E_i,w)} &\lesssim \Big(\int_{E_i} |\nabla\Delta_\Omega^{-1/2} f(x)|^2 w(x)\,\mathrm{d}x\Big)^{1/2} \\
&\lesssim 3^{k/2}\rho(B_i)^{k/2}\Big(\int_{E_i} |\nabla\Delta_\Omega^{-1/2} f(x)|^2\,\mathrm{d}x\Big)^{1/2} \\
&\lesssim 3^{k/2}\Big(\int_{B_i} |f(x)|^2\rho(y)^k\,\mathrm{d}x\Big)^{1/2} \lesssim 3^{k/2}\Big(\int_{B_i} |f(x)|^2 w(x)\,\mathrm{d}x\Big)^{1/2}
\end{aligned}$$

It remains to consider when $x \in E_i^c$. For this part consider the operator $e^{-t\Delta_\Omega}$. Continue with f supported on B_i and use the result of Li and Yau [50]. In section 8.3 Li and Yau's result was shown to imply in both the exterior and global Lipschitz cases that: $\sqrt{t}|\nabla p_t(x,y)| \leq p_t(x,y) + \frac{\sqrt{t}}{\rho(x)} p_t(x,y)$.

$$
\begin{aligned}
&\|\nabla \Delta_\Omega^{-1/2} f\|_{\mathcal{L}^2(E_i^c,w)} \\
&\lesssim \Big(\int_{E_i^c} |\nabla \Delta_\Omega^{-1/2} f(x)|^2 w(x)\,\mathrm{d}x\Big)^{1/2} \\
&\lesssim \Big(\int_{E_i^c} \Big| \int_0^\infty \int_{B_i} \nabla p_t(x,y) f(y)\,\mathrm{d}y \frac{\mathrm{d}t}{\sqrt{t}} \Big|^2 w(x)\,\mathrm{d}x\Big)^{1/2} \\
&\lesssim \Big(\int_{E_i^c} \Big| \int_{B_i} \int_0^\infty p_t(x,y) f(y) \frac{\mathrm{d}t}{\sqrt{t}}\,\mathrm{d}y \Big|^2 w(x)\,\mathrm{d}x\Big)^{1/2} \\
&\qquad\qquad + \Big(\int_{E_i^c} \Big| \frac{1}{\rho(x)} \int_0^\infty p_t(x,y) f(y) \frac{\mathrm{d}t}{\rho(x)} \Big|^2 w(x)\,\mathrm{d}x\Big)^{1/2} \\
&\lesssim \Big(\int_{E_i^c} \Big| \int_{B_i} \frac{f(y)}{|x-y|^n}\,\mathrm{d}y \Big|^2 w(x)\,\mathrm{d}x\Big)^{1/2} + \Big(\int_{E_i^c} \Big| \frac{1}{\rho(x)} \Delta_\Omega^{-1/2} f(x) \Big|^2 w(x)\,\mathrm{d}x\Big)^{1/2} \qquad (10.3)
\end{aligned}
$$

The second part above is the Hardy operator dealt with in the previous chapter. For the first part use Hölder's inequality.

$$
\int_{E_i^c} \Big| \int_{B_i} \frac{f(y)}{|x-y|^n}\,\mathrm{d}y \Big|^2 w(x)\,\mathrm{d}x \lesssim \Big(\int_{B_i} |f(y)|^2 w(y)\,\mathrm{d}y\Big) \Big(\int_{E_i^c} \int_{B_i} \frac{\mathrm{d}y}{w(y)|x-y|^{2n}}\, w(x)\,\mathrm{d}x\Big)
$$

Next use that $\rho(x) \lesssim |x-y|$ as $x \in E_i^c$ and $y \in B_i$, and that $w(x) \sim \rho(x)^k$. The geometry of Ω implies E_i^c can be described radially around y_0 where y_0 is the centre of B_i. Further the similarity $|x-y| \sim |x-y_0|$ holds.

$$
\begin{aligned}
\int_{E_i^c} \Big| \int_{B_i} \frac{f(y)}{|x-y|^n}\,\mathrm{d}y \Big|^2 w(x)\,\mathrm{d}x &\lesssim \int_{B_i} |f(y)|^2 w(y)\,\mathrm{d}y \int_{E_i^c} \int_{B_i} \frac{1}{\rho(B_i)^k |x-y_0|^{2n-k}}\,\mathrm{d}y\,\mathrm{d}x \\
&\lesssim \int_{B_i} |f(y)|^2 w(y)\,\mathrm{d}y \int_{\rho(B_i)}^\infty \frac{r^n}{\rho(B_i)^k \lambda^{n-k+1}}\,\mathrm{d}\lambda \\
&\lesssim \int_{B_i} |f(x)|^2 w(x)\,\mathrm{d}x
\end{aligned}
$$

The integration required $k < n$. This can be improved to $k < n+2$ in the global Lipschitz case by using in equation (10.3) the heat kernel upper bound for the global Lipschitz case found in chapter 7. In total equation (10.3) gives a bound of $\int_{B_i} f^2 w\,\mathrm{d}x$ whenever the Hardy inequality is bounded for that same weight. The full result of the proposition now follows by considering all the balls B_i.

$$\begin{aligned}
\||\nabla\Delta_\Omega^{-1/2}f|\|_{\mathcal{L}^2(\Omega,w)} &\leq \sum_{i\in I}\||\nabla\Delta_\Omega^{-1/2}f1_{B_i}|\|_{\mathcal{L}^2(\Omega,w)} \\
&\leq \sum_{i\in I}\left(\||\nabla\Delta_\Omega^{-1/2}f1_{B_i}|\|_{\mathcal{L}^2(E_i,w)} + \||\nabla\Delta_\Omega^{-1/2}f1_{B_i}|\|_{\mathcal{L}^2(E_i^c,w)}\right) \\
&\lesssim \sum_{i\in I}\|f\|_{\mathcal{L}^2(B_i,w)} \lesssim \|f\|_{\mathcal{L}^2(\Omega,w)}
\end{aligned}$$

The final line used that the balls involved were constructed disjoint. The proof is similar for $k < 0$ case, though E_i will be $\{x \in \Omega : \rho(x) > \rho(B_i)/2\}$. The $\mathcal{L}^2$ Hardy inequality boundedness in both the exterior domain and above a Lipschitz curve cases is proven in the previous chapter to hold when $w(x) = \rho(x)^k$ and $-1 < k < 3$. This is the biggest restriction on k when $n \geq 3$. □

A similar proof can be used to show a range of weights for an $\mathcal{L}^p$ bound given one $\mathcal{L}^p$ weighted or unweighted result. This is considered towards the end of the next section. Proposition 10.3 implies the local $\mathcal{L}^2$ weighted Riesz transform condition (5.6) needed for Theorem 1.3.

Remarks 10.4. The proof of the weighted $\mathcal{L}^2$ bound relied heavily on the knowledge of unweighted $\mathcal{L}^2$ bounds. The establishment of general conditions to ensure an unweighted equivalence $\|\sqrt{L}f\|_2 \sim \||\nabla f|\|_2$ is known as the *Kato square root problem*, originally conjectured by Kato in [45]. The equivalence is trivial when L is the Laplacian on $\mathbb{R}^n$, the Dirichlet Laplacian on $\Omega \subset \mathbb{R}^n$, or the Laplace-Beltrami operator on a general manifold with an *integration by parts* type structure. Similar results hold for Schrödinger operators. Other cases are more difficult. The paper [6] includes the following summary on the known results for elliptic operators.

Proposition 10.5 (From [6])**.** *Let $L = -div(A\nabla f)$ be a differential operator on $\mathbb{R}^n$ where $A = A(x)$ is a matrix of L^∞ complex coefficients and satisfies ellipticity conditions,*

$$\lambda|\xi|^2 \leq \Re[A\xi\cdot\xi^\perp] \qquad \textit{and} \qquad |A\xi\cdot\zeta^\perp| \leq \Lambda|\xi||\zeta|$$

for all $\zeta, \xi \in \mathbb{C}^n$ and some $0 < \lambda \leq \Lambda < \infty$. Then $\|\sqrt{L}f\|_2 \sim \||\nabla f|\|_2$ holds with constants depending only on n, λ and Λ.

Cases where such operators are considered on subsets on $\mathbb{R}^n$ has also been researched, refer to [6] and references therein for details.

10.2 A Weighted $\mathcal{L}^p$ Riesz Transform

To get to the full result of Theorems 10.1 and 10.2, first presented is an inferior result.

Proposition 10.6. *Let $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ be the exterior of a compact convex object. Suppose that w is a weight and that there exists a constant $k \in \mathbb{R}$ such that $w(x) \sim \rho(x)^k$. Then the associated Riesz transform satisfies a weighted $\mathcal{L}^p$ bound,*

$$\||\nabla \Delta_\Omega^{-1/2} f|\|_{\mathcal{L}^p(w)} \lesssim \|f\|_{\mathcal{L}^p(w)}$$

for all $p > 2$ and $\max(-1, p-n) < k < p+1$.

Proof. For this Theorem to hold, the conditions of Theorem 1.3 need to hold in this case. These conditions are:

- A heat kernel $p_t(x,y)$ of the form $\alpha_t(x)\beta_t(x)t^{-n/2}e^{-d(x,y)^2/ct}$ for some $\alpha \lesssim 1$ and $\beta \lesssim 1$ that also satisfy $\alpha_t(x) \leq 2\alpha_{2t}(x)$ and $\beta_t(x) \leq 2\beta_{2t}(x)$ (condition 5.1). This holds by Theorem 7.1.

- That with α and β the same as above, both $w^{2/p} \in A_2^{\alpha,\beta}$ and $w^{q/p} \in A_R^{\alpha,\beta}$ hold, with some $q > p$ and $R = 1 + \frac{q}{2}$ (see equation 5.2 for a reminder of the $A_p^{\alpha,\beta}$ class). Weights $w(x)^{q/p} \sim \rho(x)^{kq/p}$ are in $A_R^{\alpha,\beta}$ for all $-R-1 < kq/p < 2R-1$ (from Remarks 8.7) which rearranges for $-p/2 - p/q < k < (q+1)p/q$. This allows any $-1 < k < p+1$ choosing q close enough to p. This is one of the biggest restrictions on k. For the $w^{2/p} \in A_2^{\alpha,\beta}$ part, this holds for $w(x)^{2/p} \sim \rho(x)^{2k/p}$ whenever $-3 < 2k/p < 3$ (again from Remarks 8.7). Hence $-3p/2 < k < 3p/2$ from this part.

- That $w \in A_\infty$ (condition 5.3). This is true for weights $w(x) \sim \rho(x)^k$ whenever $k > -1$. This can be observed from $\rho(x)^k \in A_s$ whenever $-1 < k < s-1$ proven in Example 9.5.

- A weighted local Poincaré estimate for balls away from the boundary ($c_0 r < \rho(B)$) with weight $w^{2/p}$ (condition 5.4). This is known to hold if $w^{2/p}$ satisfies an A_2 condition away from the boundary. This is true for $w(x)^{2/p} \sim \rho(x)^{2k/p}$ for all k (it is important that this is only away from the boundary).

- A weighted local semigroup unit condition on balls away from the boundary (condition 5.5). This is proven in Theorem 8.2 to hold whenever $w(x) \sim \rho(x)^k$ for all k.

- A weighted $\mathcal{L}^2(w^{2/p})$ Riesz transform bound on balls away from the boundary (condition 5.6). This is proven to hold in Proposition 10.3 for all weights of the form $w(x)^{2/p} \sim \rho(x)^k$ with k in the range $-1 < 2k/p < 3$ which implies the range $-p/2 < k < 3p/2$.

- A weighted local $\mathcal{L}^2(w^{2/p})$ Davies-Gaffney estimate (condition 5.7). This is proven in Proposition 8.12 to be true whenever $w(x) \sim \rho(x)^k$ and the heat semigroup $e^{-t\Delta_\Omega}$ and the Hardy inequality $\frac{1}{\rho}\Delta_\Omega^{-1/2}$ satisfy weighted $\mathcal{L}^2(w)$ bounds for that same weight. The biggest restriction on k comes from the Hardy inequality and that restriction on k is given by $-1 < 2k/p < 3$ which implies $-p/2 < k < 3p/2$. This restriction is from Theorem 9.1.

- A weighted $\mathcal{L}^q(w^{q/p})$ Gaffney estimate for some $q > p$ (condition 5.8). This is stated in Theorem 8.3 to hold whenever the heat semigroup $e^{-t\Delta_\Omega}$ and a variation on the heat semigroup $\frac{\sqrt{t}}{\rho}e^{-t\Delta_\Omega}$ are $\mathcal{L}^q(w^{q/p})$ bounded. In chapter 9 a class of weights for $\frac{\sqrt{t}}{\rho}e^{-t\Delta_\Omega}$ is found that is a subset of the class of weights for $e^{-t\Delta_\Omega}$. When $w(x) \sim \rho(x)^k$ an $\mathcal{L}^q(w^{q/p})$ bound holds for $\frac{\sqrt{t}}{\rho}e^{-t\Delta_\Omega}$ if $-1 < kq/p < 2q-1$ implying $-p/q < k < 2p - p/q$. For q close enough to p this allows any $-1 < k < 2p - 1$. See Example 9.12.

- A weighted $\mathcal{L}^p(w)$ Hardy inequality (condition 5.9). Theorem 9.1 states that this is true for weights $w(x) \sim \rho(x)^k$ whenever $\max(-1, p - n) < k < 2p - 1$.

Thus by Theorem 1.3 (or equivalently by Theorem 5.1 and associated corollaries), this application theorem holds so that,

$$\||\nabla\Delta^{-1/2}f|\|_{\mathcal{L}^p(w)} \leq c\|f\|_{\mathcal{L}^p(w)}$$

for some constant c whenever $w(x) \sim \rho(x)^k$ and $\max(-1, p-n) < k < p+1$. □

Proposition 10.7. *Let $\Omega \subset \mathbb{R}^n$ be a global Lipschitz domain. Suppose that w is a weight and that there exists a constant $k \in \mathbb{R}$ such that $w(x) \sim \rho(x)^k$. Then the associated Riesz transform satisfies a weighted $\mathcal{L}^p$ bound,*

$$\||\nabla\Delta_\Omega^{-1/2}f|\|_{\mathcal{L}^p(w)} \lesssim \|f\|_{\mathcal{L}^p(w)}$$

for all $p \geq 2$ and $-1 < k < p+1$.

Proof. Similar to the previous proof, for this theorem to hold the conditions of Theorem 1.3 need to hold in this case. These results are all proven in similar locations to those given in the proof above. The end result again has $w(x) \sim \rho(x)^k$ where k depends most upon the $w^{q/p} \in A_R^{\alpha,\beta}$ and Hardy conditions which leads to the restriction $-1 < k < p+1$. □

The above two propositions above are inferior versions of Theorems 10.1 and 10.2. The problem is the condition $w^{q/p} \in A_R^{\alpha,\beta}$ with $R = 1 + \frac{q}{2}$ and α and β from the heat semigroup condition (5.1). This condition could be weakened by use of a higher power Poincaré condition within Theorem 1.3. Instead however, a proof similar to that used to prove the $\mathcal{L}^2$ weighted Riesz transform bounds in section 10.1, is now used to extend the $\mathcal{L}^p$ case.

Propositions 10.6 and 10.7 are used in this proof to establish the existence of a least one $\mathcal{L}^p$ bound. For the global Lipschitz domain case it would have been good enough above to use Theorem 1.1 to establish an unweighted $\mathcal{L}^p$ bound to use in the proof of Theorem 10.2. However in the exterior case Theorem 1.3 had to be used because unweighted bounds do not exist in that case for $p > n$.

Proof of Theorems 10.1 and 10.2. Take $w(x) \sim \rho(x)^k$. The proof starts with the $k \geq p$ case. The $k < p$ case is identical. Consider a covering of Ω of disjoint balls $\{B_i\}_{i \in I}$

where each ball B satisfies $r(B) < \rho(B)$. Take f supported on B_i and for each B_i consider the set $E_i = \{x \in \Omega : \rho(x) < 3\rho(B_i)\}$. For all $x \in E_i$ and $y \in B_i$ the inequalities $\rho(x)^{k-p} \leq 3^{k-p}\rho(B_i)^{k-p} \leq 3^{k-p}\rho(y)^{k-p}$ hold. Use that $\nabla\Delta_\Omega^{-1/2}$ is bounded $\mathcal{L}^p(w) \to \mathcal{L}^p(w)$ for the weight $w(x) = \rho(x)^p$ by the previous proposition.

$$\begin{aligned}
|||\nabla\Delta_\Omega^{-1/2} f|||^p_{\mathcal{L}^p(E_i,w)} &\lesssim \int_{E_i} |\nabla\Delta_\Omega^{-1/2} f(x)|^p w(x)\,\mathrm{d}x \\
&\lesssim 3^{k-p}\rho(B_i)^{k-p} \int_{E_i} |\nabla\Delta_\Omega^{-1/2} f(x)|^p \rho(x)^p\,\mathrm{d}x \\
&\lesssim 3^{k-p} \int_{B_i} |f(x)|^p \rho(x)^k\,\mathrm{d}x \lesssim 3^{k-p} \int_{B_i} |f(x)|^p w(x)\,\mathrm{d}x
\end{aligned}$$

It remains to consider when $x \in E_i^c$. Continue with f supported on B_i and for this part consider the operator $e^{-t\Delta_\Omega}$. Use Li-Yau result [50] to claim the inequality given by $\sqrt{t}|\nabla p_t(x,y)| \lesssim p_t(x,y) + \frac{\sqrt{t}}{\rho(x)} p_t(x,y)$ (already dealt with in section 8.3) to be used momentarily.

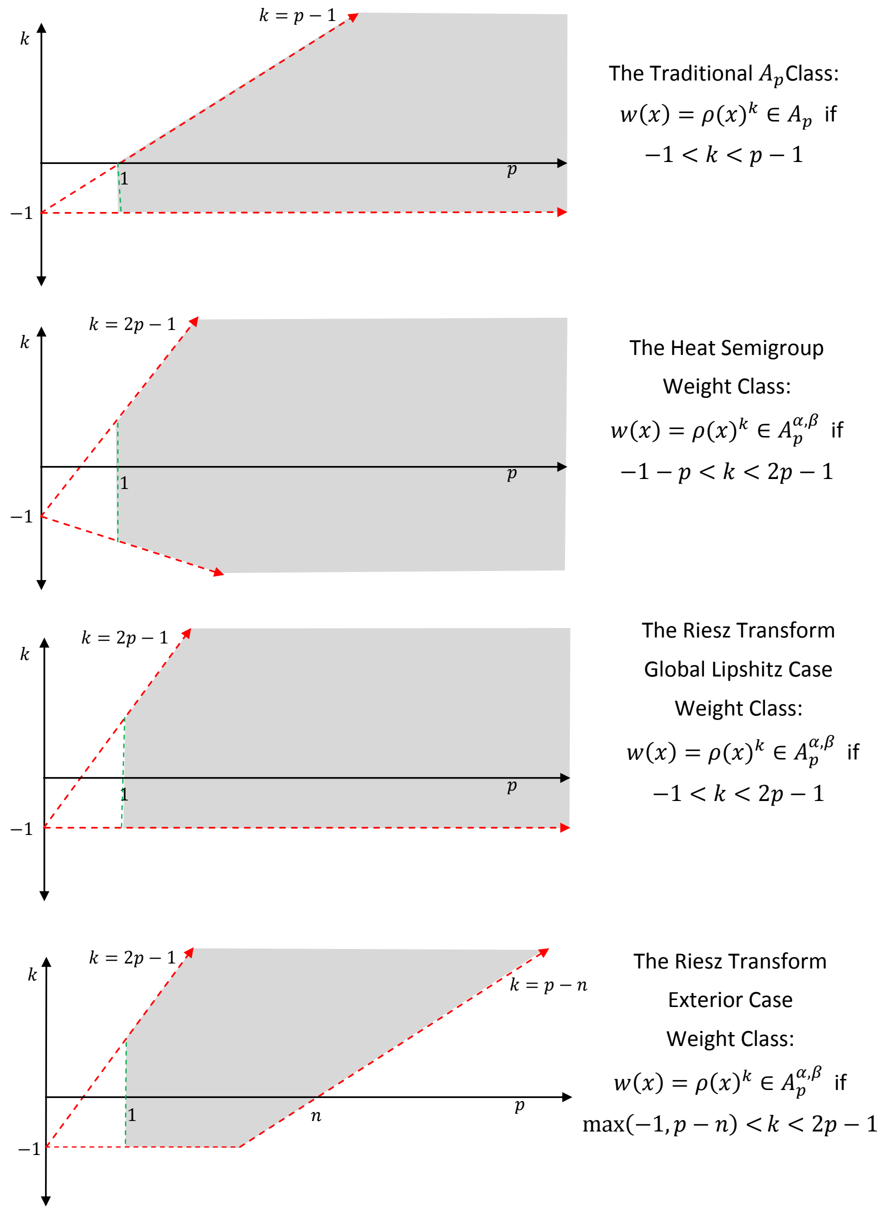
$$\begin{aligned}
&|||\nabla\Delta_\Omega^{-1/2} f|||_{\mathcal{L}^p(E_i^c,w)} \\
&\lesssim \left(\int_{E_i^c} |\nabla\Delta_\Omega^{-1/2} f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p} \\
&\lesssim \left(\int_{E_i^c} \Big|\int_0^\infty \int_{B_i} \nabla p_t(x,y) f(y)\,\mathrm{d}y \frac{\mathrm{d}t}{\sqrt{t}}\Big|^p w(x)\,\mathrm{d}x\right)^{1/p} \\
&\lesssim \left(\int_{E_i^c} \Big|\int_{B_i} \frac{f(y)}{|x-y|^n}\,\mathrm{d}y\Big|^p w(x)\,\mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_{E_i^c} \Big|\frac{1}{\rho(x)} \Delta_\Omega^{-1/2} f(x)\Big|^p w(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \qquad (10.4)
\end{aligned}$$

The second part above is the Hardy operator dealt with in the previous chapter. For the first part use Hölder's inequality.

$$\begin{aligned}
&\int_{E_i^c} \Big|\int_{B_i} \frac{f(y)}{|x-y|^n}\,\mathrm{d}y\Big|^p w(x)\,\mathrm{d}x \\
&\qquad \lesssim \left(\int_{B_i} |f(y)|^p w(y)\,\mathrm{d}y\right) \left(\int_{E_i^c} \left[\int_{B_i} \frac{1}{w(y)^{p'/p}|x-y|^{np'}}\,\mathrm{d}y\right]^{p/p'} w(x)\,\mathrm{d}x\right)
\end{aligned}$$

Also use that on such domains $\rho(x) \lesssim |x-y|$, and $|x-y| \sim |x-y_0|$ where y_0 is the

Figure 10.1: Range of weights of the form $\rho(x)^k$.



centre of B_i. Together with $w(x) \sim \rho(x)^k$ and $r(B_i) \lesssim \rho(B_i)$ this gives,

$$\begin{aligned}
\int_{E_i^c} \Big| \int_{B_i} \frac{f(y)}{|x-y|^n} \, \mathrm{d}y \Big|^p w(x) \, \mathrm{d}x &\lesssim \left(\int_{B_i} |f(y)|^p w(y) \, \mathrm{d}y \right) \left(\int_{E_i^c} \frac{|B_i|^{p-1}}{\rho(B_i)^k |x-y_0|^{np-k}} \, \mathrm{d}x \right) \\
&\lesssim \left(\int_{B_i} |f(y)|^p w(y) \, \mathrm{d}y \right) \left(\int_{\rho(B_i)}^{\infty} \frac{r^{np-n}}{\rho(B_i)^k \lambda^{np-n-k+1}} \, \mathrm{d}\lambda \right) \\
&\lesssim \int_{B_i} |f(x)|^p w(x) \, \mathrm{d}x
\end{aligned}$$

where the integration requires $k < np - n$. This restriction on k can be improved in the global Lipschitz case to $k < np + p - n$. This works by using the heat kernel upper bound from chapter 7 in equation (10.4) to get an additional $\frac{\rho(y)}{|x-y|}$ term. These k restrictions are weaker then the $k < 2p - 1$ required when considering the Hardy operator. In total equation (10.4) gives a bound of $\int_{B_i} |f|^p w \, \mathrm{d}x$ whenever the Hardy inequality is bounded for the same weight. The full result of the theorem now follows by considering all balls B_i.

$$\begin{aligned}
\| |\nabla \Delta_\Omega^{-1/2} f| \|_{\mathcal{L}^p(\Omega,w)} &\leq \sum_{i \in I} \| |\nabla \Delta_\Omega^{-1/2} f 1_{B_i}| \|_{\mathcal{L}^p(\Omega,w)} \\
&\leq \sum_{i \in I} \left(\| |\nabla \Delta_\Omega^{-1/2} f 1_{B_i}| \|_{\mathcal{L}^p(E_i,w)} + \| |\nabla \Delta_\Omega^{-1/2} f 1_{B_i}| \|_{\mathcal{L}^p(E_i^c,w)} \right) \\
&\lesssim \sum_{i \in I} \|f\|_{\mathcal{L}^p(B_i,w)} \lesssim \|f\|_{\mathcal{L}^p(\Omega,w)}
\end{aligned}$$

The last step holds as the covering was chosen disjoint. The proof is similar for $k < p$ case, but involves integrating over $\rho(x)^k$ near the boundary so $k > -1$ is required along with the usual $\mathcal{L}^p(w)$ Hardy operator bounds. The $-1 < k < p$ range has already been considered in Propositions 10.6 and 10.7. The Hardy inequality $\mathcal{L}^p(w)$ boundedness part is proven in the previous chapter to hold when $-1 < k < 2p - 1$ and $p - n < k < np - n$ whenever $n \geq 3$ for the exterior case, and for $-1 < k < 2p - 1$ in the above a Lipschitz function case. This is what most restricts the range of k. □

Similar results can be found for the area below a parabola case and the non-doubling example $\{(x_1, x_2) \in \mathbb{R}^n : 0 < x_2 < e_1^x\}$ discussed throughout this thesis. The ranges of k for weights $w(x) \sim \rho(x)^k$ would continue to be based on the ranges of k found for the Hardy inequality in the previous chapter. The remainder of this chapter is dedicated to discussion and examples regarding the weight classes.

10.3 Optimisation of Results

Four examples are considered in this section. The first is an example showing general weighted $\mathcal{L}^p$ bounds for the Riesz transform will not hold with weights of the form $w(x) \sim \rho(x)^k$ with $k \leq -1$ or $k \geq 2p-1$, proving that part of the results optimal. The second consideration is an example from [48] proving the Riesz transform is not $\mathcal{L}^p(w)$ bounded with $w(x) = 1$ and $p > n$ in the exterior of a compact object case, proving that part of the results optimal. The third consideration is from [43] and contains remarks as to why the theorems include smooth boundary among the conditions, and the final part looks at the preservation case as a limit of non-preservation cases.

10.3.1 The Halfspace Example

Proposition 10.8. *The Hilbert Transform (the 1-dimensional Riesz transform) on the halfspace $\mathbb{R}^+$ is not bounded with weight $w(x) = \rho(x)^k$ if $k \leq -1$ or $k \geq 2p-1$.*

Proof. The Hilbert Transform on the halfspace $\mathbb{R}^+$ has known associated kernel.

$$k(x, y) = \frac{1}{x-y} - \frac{1}{x+y}$$

Let $f(x) = x1_{[0,1]}$, then $f \in \mathcal{L}^p(\frac{1}{x})$ for every $p \in [1, \infty]$. Apply the Hilbert transform to f to get,

$$Hf(x) = x \ln \left| \frac{x-1}{x+1} \right| - 2$$

for every $x > 0$. The part -2 means that as $x \to 0$, $Hf(x) \to -2$. This means $Hf \notin \mathcal{L}^p(\frac{1}{x})$ for any $p \in [1, \infty]$ due to the $\frac{1}{x}$ weight not being integrable at the boundary $x = 0$.

Next consider the weight $w(x) = x^{2p-1}$. Notice that functions of the form given by $f(x) = \frac{1}{x^2 \ln(x)} 1_{[0,\frac{1}{2}]}$ are in $\mathcal{L}^p(x^{2p-1})$ for all $p > 1$. With such a function f and $x > 1$

observe that,

$$\begin{aligned}Hf(x) &= \int_0^{1/2} \Big(\frac{1}{x-y} - \frac{1}{x+y}\Big)\frac{1}{y^2\ln(y)}\,\mathrm{d}y \\ &= \int_0^{1/2} \Big(\frac{2}{(x-y)(x+y)}\Big)\frac{1}{y\ln(y)}\,\mathrm{d}y \geq \frac{2}{x(x+1)}\int_0^{1/2}\frac{1}{y\ln(y)}\,\mathrm{d}y\end{aligned}$$

The dy integral is infinite. So it is not feasible to have general bounds for $\rho(x)^k$ if $k \geq 2p-1$. Hence the Hilbert Transform on the halfspace is only bounded for weights $w(x) \sim \rho(x)^k$ if $-1 < k < 2p-1$. Similar examples can be constructed for halfspaces of the form $\mathbb{R}^{n-1} \times \mathbb{R}^+$. □

10.3.2 Counterexample from Killip et al

Proposition 10.9 (Proposition 7.2 in [48])**.** *The Riesz transform $\nabla\Delta_\Omega^{-1/2}$ is not bounded $\mathcal{L}^p(\Omega) \to \mathcal{L}^p(\Omega)$ in the case of Ω as the exterior of a compact convex object whenever $p > n$.*

Proof. Working with a ball radius 1. Choose $\phi \in C^\infty(\mathbb{R})$ where $\phi(r) = 0$ for $r > 2$ and $\phi(r) = 1$ for $r < 1$. Then define $\chi(r) = \phi(r/R) - \phi((r-1)/\epsilon)$ as an approximation of the characteristic function of the annulus $1 + \epsilon < r < R$. Let u be an eigenfunction for the Dirichlet Laplacian, It is known that $|u(r)| \leq 1 \wedge (r-1)$ in this case. Using that the annulus $1+\epsilon < r < 1+2\epsilon$ has area approximately ϵ, and that the annulus $R < r < 2R$ has area approximately R^n the following holds.

$$\||u\nabla\chi|\|_{\mathcal{L}^p(\Omega)} \lesssim \left(\frac{[\sup_{1<r<1+2\epsilon} u]^p\epsilon}{\epsilon^p} + \frac{R^n}{R^p}\right)^{1/p} \lesssim \epsilon^{1/p} + R^{n/p-1} \to 0$$

So if $p > n$ the above tends to 0 as ϵ tends to 0 and R tends to infinity. Use this to show a further result.

$$\begin{aligned}\lim_{\substack{\epsilon\to 0\\ R\to\infty}} \||\nabla(\chi u)|\|_{\mathcal{L}^p(\mathbb{R}^n)} &= \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \|\chi|\nabla u|\|_{\mathcal{L}^p(\mathbb{R}^n)} + \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \|u|\nabla\chi|\|_{\mathcal{L}^p(\mathbb{R}^n)} \\ &\geq \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \|\chi|\nabla u|\|_{\mathcal{L}^p(\mathbb{R}^n)} = \||\nabla u|\|_{\mathcal{L}^p(\mathbb{R}^n)}\end{aligned}$$

Then as the eigenvalue λ of the eigenfunction u tends to 0.

$$\lim_{\lambda\to 0}\lim_{\substack{\epsilon\to 0\\ R\to\infty}}\|\nabla(\chi u)\|_{\mathcal{L}^p(\mathbb{R}^n)} \geq \lim_{\lambda\to 0}\|\nabla u\|_{\mathcal{L}^p(\mathbb{R}^n)} \geq 1$$

It remains to find an estimate on $\Delta_\Omega^{-1/2}$. By complex interpolation due to the multiplier theorem,

$$\|\Delta_\Omega(\chi u - u)\|^2_{\mathcal{L}^p(\Omega)} \lesssim \|(\chi-1)u\|_{\mathcal{L}^p(\Omega)}\|\Delta_\Omega(\chi-1)u\|_{\mathcal{L}^p(\Omega)}$$

and for R large enough this improves.

$$\|\Delta_\Omega(\chi u - u)\|^2_{\mathcal{L}^p(\Omega)} \lesssim \epsilon^{1+1/p}\|\Delta_\Omega(\chi-1)u\|_{\mathcal{L}^p(\Omega)}$$

Dealing with the remaining part,

$$\begin{aligned}\|\Delta_\Omega(\chi u - u)\|^2_{\mathcal{L}^p(\Omega)} &\lesssim \epsilon^{1+1/p}\Big[\|u\Delta_\Omega\chi\|_{\mathcal{L}^p(\Omega)} + \|\nabla\chi\cdot\nabla u\|_{\mathcal{L}^p(\Omega)} + \lambda^2\|(\chi-1)u\|_{\mathcal{L}^p(\Omega)}\Big]\\ &\lesssim \epsilon^{2/p} + R^{n/p-1} + \lambda^2\|u\|_{\mathcal{L}^p}\end{aligned}$$

and take $\epsilon \to 0$ and $R \to \infty$ and $\lambda \to 0$ to get a contradiction which proves the proposition. $\square$

10.3.3 Boundary Smoothness

In all the main theorems of this thesis the boundaries of the domains considered have been smooth, $C^{1,1}$. The smoothness is used mainly in chapter 7 to build heat kernel upper bounds on the domain. In the paper [43] by Jerison and Kenig examples are included that show in a bounded domain case, a non-smooth boundary implies that the Riesz transform may not be bounded even for $p > 4$. In the case of this thesis the domains have been mainly unbounded, however it would be expected that similar problems would occur. In the case of [43] smooth boundary domains had corresponding $\mathcal{L}^p \to \mathcal{L}^p$ Riesz transform boundedness for all $1 < p < \infty$. This is different from the unbounded domain cases, where the exterior of a convex object has corresponding Riesz transform unbounded in $\mathcal{L}^p$ for $p > n$ (when unweighted) even when the boundary is smooth.

10.3.4 The Limit Towards Preservation

As previously mentioned, when a preservation condition holds the Riesz transform is bounded by techniques in [5]. Further if there are Gaussian lower bounds on the heat kernel (as often occurs in preservation cases) the Hardy inequality may not be bounded. In this section consider a limit of non-preservation cases to a preservation one.

Example 10.10. Fix $f \in C_c^\infty$ and let Ω be the space $\Omega = \{x \in \mathbb{R}^n : x_1 > -\alpha\}$. The idea of this example is that f will stay fixed, whilst the boundary $x_1 = -\alpha$ of Ω moves away. When f is far enough from the boundary the non-preservation case approaches the preservation case. The following two heat kernel derivatives are well known;

$$\nabla p_t^{\mathbb{R}^n}(x,y) = c_n \frac{(y-x)e^{-|x-y|^2/4t}}{t^{n/2+1}}$$

and,

$$\nabla p_t^{\Omega}(x,y) = c_n \frac{(y-x)e^{-|x-y|^2/4t}}{t^{n/2+1}} - c_n \frac{(\bar{y}-x)e^{-|x-\bar{y}|^2/4t}}{t^{n/2+1}}$$

where $\bar{y}$ is the reflection of y over the boundary given by $\bar{y} = (-2\alpha - y_1, y_2, \ldots, y_n)$. This second kernel is only valid when $x_1, y_1 > -\alpha$. Observe that,

$$\lim_{\alpha\to\infty} \nabla p_t^{\Omega}(x,y)f(y) \to \nabla p_t^{\mathbb{R}^n}(x,y)f(y)$$

is a pointwise limit. It is not difficult to check that,

$$|\nabla p_t^{\Omega}(x,y)f(y)| \lesssim |\nabla p_t^{\mathbb{R}^n}(x,y)f(y)| \lesssim \frac{e^{-|x-y|^2/4t}}{t^{n/2+1/2}} f(y)$$

which is an integrable function with respect to y. So by Lebesgue dominated convergence,

$$\lim_{\alpha\to\infty} \nabla e^{-t\Delta_\Omega} f(x) \to \nabla e^{-t\Delta_{\mathbb{R}^n}} f(x)$$

holds as another pointwise limit. Further the value $|\nabla e^{-t\Delta_\Omega} f(x)|$ can be bounded above by a constant of the form $ct^{-1/2}\|f\|_2$, using Hölder's inequality and the presented heat kernel gradient. Thus again by Lebesgue dominated convergence,

$$\lim_{\alpha\to\infty}\int_{\epsilon}^{1/\epsilon}\nabla e^{-t\Delta_\Omega}f(x)\frac{\mathrm{d}t}{\sqrt{t}}\to\int_{\epsilon}^{1/\epsilon}\nabla e^{-t\Delta_{\mathbb{R}^n}}f(x)\frac{\mathrm{d}t}{\sqrt{t}}$$

also holds as a pointwise limit. It is known that the $\mathcal{L}^p$ norm of the terms on the left is bounded by a constant $\|f\|_p$.

$$\lim_{\alpha\to\infty}\Big\|\int_{\epsilon}^{1/\epsilon}\nabla e^{-t\Delta_\Omega}f(\cdot)\frac{\mathrm{d}t}{\sqrt{t}}\Big\|_p\to\Big\|\int_{\epsilon}^{1/\epsilon}\nabla e^{-t\Delta_{\mathbb{R}^n}}f(\cdot)\frac{\mathrm{d}t}{\sqrt{t}}\Big\|_p$$

Thus as the non-preservation cases tends to the preservation case the Riesz transforms match. In this way preservation cases can be thought of as the limit of non-preservation cases. It is interesting to note that in the same case,

$$\lim_{\alpha\to\infty}|\nabla e^{-t\Delta_\Omega}1_\Omega(x)|\to 0$$

which comes from observing that condition (1.3) implies,

$$\left(\fint_B\rho(x)^2\,\mathrm{d}x\right)\left(\fint_B|\nabla e^{-t\Delta_\Omega}1_\Omega(x)|^2\,\mathrm{d}x\right)\leq c$$

where the c will not change with α as each α leads to geometrically the same problem. Then $\rho(x)$ is the minimal distance from the point x to the boundary which gets large as $\alpha\to\infty$. So fixing B, as $\alpha\to\infty$ then $\rho(x)\to\infty$ so $\fint_B|\nabla e^{-t\Delta_\Omega}1_\Omega(x)|^2\,\mathrm{d}x\to 0$, which is an approach to preservation.

Bibliography

[1] H. Aikawa, T. Lundh, and T. Mizutani, *Martin boundary of a fractal domain*, Potential Anal. **18** (2003), no. 4, 311–357, MR:1953266, doi:10.1023/A:1021823023212.

[2] D. Albrecht, X. T. Duong, and A. McIntosh, *Operator theory and harmonic analysis*, Instructional Workshop on Analysis and Geometry, Part III (Canberra, 1995), Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 34, Austral. Nat. Univ., Canberra, 1996, MR:1394696, pp. 77–136.

[3] A. Ancona, *Sur la théorie du potentiel dans les domaines de John*, Publ. Mat. **51** (2007), no. 2, 345–396, MR:2334795, doi:10.5565/PUBLMAT_51207_05.

[4] P. Auscher, *On necessary and sufficient conditions for L^p estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates*, Mem. Amer. Math. Soc. **186** (2007), no. 871, xviii+75, MR:2292385, doi:10.1090/memo/0871.

[5] P. Auscher, T. Coulhon, X. T. Duong, and S. Hofmann, *Riesz transforms on manifolds and heat kernel regularity*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 6, 911–957, MR:2119242, doi:10.1016/j.ansens.2004.10.003.

[6] P. Auscher, S. Hofmann, M. Lacey, J. Lewis, A. McIntosh, and P. Tchamitchian, *The solution of Kato's conjectures*, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), no. 7, 601–606, MR:1841892, doi:10.1016/S0764-4442(01)01893-6.

[7] P. Auscher and J. M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. III. Harmonic analysis of elliptic operators*, J. Funct. Anal. **241** (2006), no. 2, 703–746, MR:2271934, doi:10.1016/j.jfa.2006.07.008.

[8] ———, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. I. General operator theory and weights*, Adv. Math. **212** (2007), no. 1, 225–276, MR:2319768, doi:10.1016/j.aim.2006.10.002.

[9] ———, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. II. Off-diagonal estimates on spaces of homogeneous type*, J. Evol. Equ. **7** (2007), no. 2, 265–316, MR:2316480, doi:10.1007/s00028-007-0288-9.

[10] ———, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. IV. Riesz transforms on manifolds and weights*, Math. Z. **260** (2008), no. 3, 527–539, MR:2434468, doi:10.1007/s00209-007-0286-1.

[11] R. Bañuelos, R. D. DeBlassie, and R. Smits, *The first exit time of planar Brownian motion from the interior of a parabola*, Ann. Probab. **29** (2001), no. 2, 882–901, MR:1849181, doi:10.1214/aop/1008956696.

[12] K. Bogdan, *Sharp estimates for the Green function in Lipschitz domains*, J. Math. Anal. Appl. **243** (2000), no. 2, 326–337, MR:1741527, doi:10.1006/jmaa.1999.6673.

[13] B. Bongioanni, E. Harboure, and O. Salinas, *Classes of weights related to Schrödinger operators*, J. Math. Anal. Appl. **373** (2011), no. 2, 563–579, MR:2720705, doi:10.1016/j.jmaa.2010.08.008.

[14] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, *Boundary behaviour of nonnegative solutions of elliptic operators in divergence form*, Indiana Univ. Math. J. **30** (1981), no. 4, 622–640, MR:620271, doi:10.1512/iumj.1981.30.30049.

[15] L. Caffarelli and S. Salsa, *A geometric approach to free boundary problems*, Graduate Studies in Mathematics, vol. 68, American Mathematical Society, Providence, RI, 2005, MR:2145284.

[16] G. Carron, T. Coulhon, and A. Hassel, *Riesz transform and L^p-cohomology for manifolds with Euclidean ends*, Duke Math. J. **133** (2006), no. 1, 59–93, MR:2219270, doi:10.1215/S0012-7094-06-13313-6.

[17] T. Coulhon and X. T. Duong, *Riesz transforms for* $1 \leq p \leq 2$, Trans. Amer. Math. Soc. **351** (1999), no. 3, 1151–1169, MR:1458299, doi:10.1090/S0002-9947-99-02090-5.

[18] ———, *Riesz transforms for* $p > 2$, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), no. 11, 975–980, MR:1838122, doi:10.1016/S0764-4442(01)01981-4.

[19] ———, *Riesz transform and related inequalities on non-compact Riemannian manifolds*, Comm. Pure Appl. Math. **56** (2003), no. 12, 1728–1751, MR:2001444, doi:10.1002/cpa.3040.

[20] E. B. Davies, *The equivalence of certain heat kernel and Green function bounds*, J. Funct. Anal. **71** (1987), no. 1, 88–103, MR:879702, doi:10.1016/0022-1236(87)90017-6.

[21] ———, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989, MR:990239, doi:10.1017/CBO9780511566158.

[22] ———, *Pointwise bounds on the space and time derivatives of heat kernels*, J. Operator Theory **21** (1989), no. 2, 367–378, MR:1023321.

[23] E. B. Davies and B. Simon, *Ultracontractivity and heat kernels for Schrödinger operators and Dirichlet Laplacians*, J. Funct. Anal. **59** (1984), 335–395, MR:766493, doi:10.1016/0022-1236(84)90076-4.

[24] J. L. Doob, *Measure theory*, Graduate Texts in Mathematics, vol. 143, Springer-Verlag, New York, 1994, MR:1253752, doi:10.1007/978-1-4612-0877-8.

[25] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7, Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London, 1958, MR:0117523.

[26] X. T. Duong and A. McIntosh, *Singular integral operators with non smooth kernels on irregular domains*, Rev. Mat. Iberoamericana **15** (1999), no. 2, 233–265, MR:1715407, doi:10.1007/s11512-006-0021-x.

[27] X. T. Duong, E. M. Ouhabaz, and A. Sikora, *Plancherel-type estimates and sharp spectral multipliers*, J. Funct. Anal. **196** (2002), no. 2, 443–485, MR:1943098, doi:10.1016/S0022-1236(02)00009-5.

[28] X. T. Duong and D. W. Robinson, *Semigroup kernels, Poisson bounds, and holomorphic functional calculus*, J. Funct. Anal. **142** (1996), no. 1, 89–128, MR:1419418, doi:10.1002/cpa.20080.

[29] E. B. Fabes, N. Garofalo, and S. Salsa, *A backward Harnark inequality and Fatou theorem for nonnegative solutions of parabolic equations*, Illinois J. Math. **30** (1986), no. 4, 536–565, MR:857210, projecteuclid:1256064230.

[30] E. Gagliardo, *On integral transformations with positive kernel*, Proc. Amer. Math. Soc. **16** (1965), 429–434, MR:0177314.

[31] J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, Notas de Matemtica [Mathematical Notes], 104. North-Holland Publishing Co., Amsterdam, 1985, MR:807149.

[32] A. A. Grigor′yan, *The heat equation on noncompact Riemannian manifolds.*, Mat. Sb. **182** (1991), no. 1, 55–87, translation in Math. USSR-Sb. **72** (1992), no. 1, 4777, MR:1098839.

[33] ———, *Heat kernel upper bounds on a complete non-compact manifold*, Rev. Mat. Iberoamericana **10** (1994), no. 2, 395–452, MR:1286481, doi:10.4171/RMI/157.

[34] ———, *Upper bounds for derivatives of the heat kernel on an arbitrary complete manifold*, J. Funct. Anal. **127** (1995), no. 2, 363–389, MR:1317722, doi:10.1006/jfan.1995.1016.

[35] A. A. Grigor′yan and L. Saloff-Coste, *Dirichlet heat kernel in the exterior of a compact set*, Comm. Pure Appl. Math. **55** (2002), no. 1, 93–133, MR:1857881, doi:10.1002/cpa.10014.

[36] M. Grüter and K. Widman, *The Green function for uniformly elliptic equations*, Manuscripta Math. **37** (1982), no. 3, 303–342, MR:657523, doi:10.1007/BF01166225.

[37] P. Gyrya, *Heat kernel estimates for inner uniform subsets of Harnack-type Dirichlet spaces*, Thesis (Ph.D.)–Cornell University. ProQuest LLC, Ann Arbor, MI, 2007, MR:2710640.

[38] P. Gyrya and L. Saloff-Coste, *Neumann and Dirichlet heat kernels in inner uniform domains*, no. 336, Astérisque, 2011, MR:2807275.

[39] M. Haase, *The functional calculus for sectorial operators*, Operator Theory: Advances and Applications, vol. 169, Birkhäuser Verlag, Basel, 2006, MR:2244037 doi:10.1007/3-7643-7698-8.

[40] G. H. Hardy, *Note on a theorem of Hilbert*, Math. Z. **6** (1920), no. 3-4, 314–317, MR:1544414, doi:10.1007/BF01199965.

[41] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs. Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993, MR:1207810.

[42] E. Hopf, *The general temporally discrete Markoff process*, J. Rational Mech. Anal. **3** (1954), 13–45, MR:0060181.

[43] D. Jerison and C. E. Kenig, *The inhomogenous Dirichlet problem in Lipshitz domains*, J. Funct. Anal. **130** (1995), no. 1, 161–219, MR:1331981, doi:10.1006/jfan.1995.1067.

[44] R. Johnson and C. J. Neugebauer, *Change of variable results for A_p - and reverse Hölder* RH_r *- classes*, Trans. Amer. Math. Soc. **328** (1991), no. 2, 639666, MR:1018575, doi:10.2307/2001798.

[45] T. Kato, *Fractional powers of dissipative operators*, J. Math. Soc. Japan **13** (1961), 246–274, MR:0138005.

[46] ———, *Perturbation theory for linear operators*, Reprint of the 1980 edition. Classics in Mathematics, Springer-Verlag, Berlin, 1995, MR:1335452, doi:10.1007/BF02399203.

[47] J. T. Kemper, *A boundary Harnack principle for Lipschitz domains and the principle of positive singularities*, Comm. Pure Appl. Math. **25** (1972), 247–255, MR:0293114.

[48] R. Killip, M. Visan, and X. Zhang, *Harmonic analysis outside a convex obstacle*, arXiv:1205.5784 (2012).

[49] ———, *Quintic NLS in the exterior of a strictly convex obstacle*, arXiv:1208.4904 (2012).

[50] P. Li and S. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), no. 3-4, 153–201, MR:834612, doi:10.1007/BF02399203.

[51] J. Lierl and L. Saloff-Coste, *Scale-invariant boundary Harnack principle in inner uniform domains*, arXiv:1110.2763 (2012).

[52] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226, MR:0293384.

[53] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931–954, MR:0100158.

[54] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer–Verlag, New York, 1983, MR:MR710486, doi:10.1007/978-1-4612-5561-1.

[55] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159 (2002).

[56] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional analysis*, second ed., Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980, MR:751959.

[57] W. Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987, MR:924157.

[58] L. Saloff-Coste, *A note on Poincaré, Sobolev, and Harnack inequalities*, Internat. Math. Res. Notices (1992), no. 2, 27–38, MR:1150597, doi:10.1155/S1073792892000047.

[59] S. Salsa, *Some properties of nonnegative solutions of parabolic differential operators*, Ann. Mat. Pura Appl. (4) **128** (1981), 193–206, MR:640782, doi:10.1007/BF01789473.

[60] Z. Shen, *Bounds of Riesz transforms on L^p spaces for second order elliptic operators*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 173–197, MR:2141694, `http://aif.cedram.org/item?id=AIF_2005__55_1_173_0`.

[61] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 3, 447–526, MR:0670130, doi:10.1090/S0273-0979-1982-15041-8.

[62] R. Song, *Estimates on the Dirichlet heat kernel of domains above the graphs of bounded $C^{1,1}$ functions*, Glas. Mat. Ser. III **39(59)** (2004), no. 2, 275–288, MR:2109269, doi:10.3336/gm.39.2.09.

[63] P. Souplet and Q. S. Zhang, *Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds*, Bull. London Math. Soc. **38** (2006), no. 6, 1045–1053, MR:2285258, doi:10.1112/S0024609306018947, arXiv:math/0502079.

[64] E. M. Stein, *On the maximal ergodic theorem*, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 1894–1897, MR:0131517.

[65] ———, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970, MR:0290095.

[66] ———, *Topics in harmonic analysis related to the Littlewood–Paley theory*, Annals of Mathematics Studies, No. 63, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1970, MR:0252961.

[67] ———, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*, With the assistance of Timothy S. Murphy. Princeton Mathematical Series, vol. 43, Princeton University Press, 1993, MR:1232192.

[68] R. S. Strichartz, *Analysis of the Laplacian on the complete Riemannian manifold*, J. Funct. Anal. **52** (1983), no. 1, 48–79, MR:0705991, doi:10.1016/0022-1236(83)90090-3.

[69] H. Triebel, *The structure of functions*, [2012 reprint of the 2001 original] Modern Birkhäuser Classics, no. 97, Birkhäuser/Springer Basel AG, Basel, 2001, reprint MR:3013187, original MR:1851996.

[70] J. Väisälä, *Relatively and inner uniform domains*, Conform. Geom. Dyn. **2** (1998), 56–88, MR:1637079, doi:10.1090/S1088-4173-98-00022-8.

[71] N. T. Varopoulos, *Gaussian estimates in Lipshitz domains*, Canad. J. Math. **55** (2003), no. 2, 401–431, MR:1969798, doi:10.4153/CJM-2003-018-9.

[72] J. Zhang and J. Zheng, *Scattering theory for nonlinear Schrödinger equations with inverse-square potential*, J. Funct. Anal. **267** (2014), no. 8, 2907–2932, MR:3255478, doi:10.1016/j.jfa.2014.08.012, arXiv:1312.2294.

[73] Q. S. Zhang, *The boundary behavior of heat kernels of Dirichlet Laplacians*, J. Differential Equations **182** (2002), no. 2, 416–430, MR:1900329, doi:10.1006/jdeq.2001.4112.

[74] ———, *The global behavior of heat kernels in exterior domains*, J. Funct. Anal. **200** (2003), no. 1, 160–176, MR:1974093, doi:10.1016/S0022-1236(02)00074-5.

[75] ———, *Some gradient estimates for the heat equation on domains and for an equation by Perelman*, Int. Math. Res. Not. (2006), MR:2250008, doi:10.1155/IMRN/2006/92314, arXiv:math/0605518.

[76] ———, *Sobelov inequalities, heat kernels under ricci flow, and the Poincaré conjecture*, CRC Press, Boca Raton, FL, 2011, MR:2676347.

[77] Z. Zhao, *Green's function for Schrödinger operators and conditioned Feynman-Kac gauge*, J. Math. Anal. Appl. **116** (1986), no. 2, 309–334, MR:842803, doi:10.1016/S0022-247X(86)80001-4.