UNITS of SKEW MONOIDAL CATEGORIES and SKEW MONOIDALES in SPAN

by

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Declaration

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of the requirements for a degree to any other university or institution other then Macquarie University.

I also certify that this thesis is an original piece of research and has been written by me. Any help and assistance that I have recieved in my research work and the preparation of the thesis itself has been appropriately acknowledged.

In addition, I certify that all information sources and literature used are indicated in the thesis.

date:

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Abstract

This thesis is about skew monoidal categories and consists of two relatively independent chapters, the first of which shows that the units of a skew monoidal category are unique up to a unique isomorphism, and internalises this fact to skew monoidales. Some benefits of certain extra structure on the unit maps are also discussed. We include some remarks on the unit conditions for a monoidal functor between skew monoidal categories that generalises the earlier uniqueness result. In the second, an interesting characterisation of a skew monoidale in the monoidal bicategory Span is given, generalising the case where the unit of the skew monoidale is of a certain restricted form, along with an example. Finally in an appendix, we show that the five axioms of a skew monoidal category are independent.

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Chapter 1

Introduction

Generalisations of the notion of monoidal category have been studied almost as long as the notion itself; several of these involve relaxing the invertibility of the maps expressing the associativity and unit conditions. Once invertibility is dropped the directions of these constraints must be specified; one such choice leads to the notion of skew monoidal category.

For a *left skew monoidal category* $\mathcal{C}$ with tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and unit object I, the natural families of *lax constraints* have the following orientations

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$$

$$\lambda_X : I \otimes X \longrightarrow X$$

$$\rho_X : X \longrightarrow X \otimes I.$$

Given the orientation of these lax constraints for a skew monoidal category, we now need to ask what particular (coherence) equations they satisfy. Mac Lane in [18] shows that a list of five axioms is sufficient for the coherence for monoidal categories. Kelly in [14] found that there were redundancies in that list and reduced it down to two. However this reduction relied on the invertibility of the associativity and unit maps.

In the context of skew monoidal categories no such invertibility is assumed and so we require all of the five axioms of Mac Lane. These five conditions are the pentagon for α, one relating $\alpha_{X,I,Y}$, λ_Y and ρ_X, one relating $\alpha_{I,X,Y}$ and λ, one relating $\alpha_{X,Y,I}$ and ρ, and one relating λ_I and ρ_I.

In [6], Burroni defines a pseudocategory, where a pseudocategory with one object amounts to a category with the same orientations for the α and ρ as above but with the direction of λ reversed. Grandis in [11], defines a biased d-lax 2-category, where a one object version amounts to a category where the direction of α is reversed but the orientations of λ and ρ are the same as a left skew monoidal category. However guided by directed homotopy, the biased d-lax 2-categories of Grandis require six axioms, with instead two equations relating λ_I and ρ_I. On the other hand, the lax monoidal categories described in [8] provides an unbiased generalisation of a monoidal category where now for each $n \in \mathbb{N}$ there is a functor $\otimes_n \colon \mathcal{C}^n \longrightarrow \mathcal{C}$ and two other structural maps satisfying three axioms.

It should be noted that there is an analogous notion of a *right skew monoidal category* where the constraints have their directions reversed. A left skew monoidal structure on $\mathcal{C}$ yields a right one both on $\mathcal{C}^{\mathrm{op}}$ (in which morphisms are reversed) and on $\mathcal{C}^{\mathrm{rev}}$ (in which the tensor is switched) and so a left one on $(\mathcal{C}^{\mathrm{op}})^{\mathrm{rev}}$. In [1], Altenkirch, Chapman and Uustalu, while studying relative monads, show a certain functor category is (left) skew monoidal, they call it lax monoidal. Independently, and motivated by bialgebroids, Szlachányi in [21], first names and studies (both left and right) skew monoidal categories as such. In this text what we call a skew monoidal category is usually referred to as a left skew monoidal category, and what could have been referred to as a skew pseudomonoid we call a skew monoidale. (However, Uustalu in [22], calls what we call a left skew monoidal category, a right skew monoidal category.)

One of the first observations about a monoid is that its unit (if it exists) is unique, as shown by the equality $i = i.j = j$. In a monoidal category these equalities become isomorphisms $I \cong I \otimes J \cong J$; where now in this context there is also a uniqueness result. We show, in Chapter 2, an analogous result for the units of a skew monoidal category. In this context we no longer have isomorphisms $I \cong I \otimes J$ or $I \otimes J \cong J$ but only the maps $I \longrightarrow I \otimes J$ and $I \otimes J \longrightarrow J$. Thus it might seem that uniqueness up to isomorphism is lost, but surprisingly, it turns out that the composite $I \longrightarrow I \otimes J \longrightarrow J$ *is* invertible, and we do still have a uniqueness result for this isomorphism.

In Section 2.2 we establish that the units of a skew monoidal category are unique up to a unique isomorphism; this is the analogue for skew monoidal categories of Proposition 1.7 in [15]. This was shown for monoidal categories by Kock in [15], where earlier references are given to this result by Saavedra Rivano in [20]. The coherence results for monoidal categories with units, by Mac Lane in [18], would imply that the isomorphisms between the units are unique. The proofs here follow the same methods employed in [15] where in our context, we define the category of units for a skew monoidal category and show that it is terminal. We then impose some extra structure on the unit maps of a skew monoidal category, such as requiring that λ be invertible, and remark on some consequences of this extra structure. Section 2.3 consists of some remarks on the unit conditions of a monoidal functor between skew monoidal categories which allows us to generalise the uniqueness result of the previous section. In Section 2.4 we internalise the main result of Section 2.2 to skew monoidales; that is, out of the cartesian monoidal 2-category **Cat** and into a monoidal bicategory, although by the coherence results of [10] it sufficies to work in a Gray monoid.

A general classification of skew monoidales in a monoidal bicategory in terms of simplicial maps from the Catalan simplicial set into the nerve of the monoidal bicategory is shown in [5]. In Chapter 3 of this thesis we consider a skew monoidale in the monoidal bicategory Span. Since their introduction by Bénabou in [4], Span and the Span construction are ubiquitous in higher category theory. This is mainly due to the fact that a category can be regarded as a monad in the bicategory of spans Span, and various generalisations. However, what interests us is Span not just as a bicategory but as a monoidal bicategory made monoidal using the cartestian product of sets. Skew monoidales (= skew pseudomonoids) were defined by Lack and Street in [17], where they also show that quantum categories are skew monoidal objects, with a certain unit, in an appropriate monoidal bicategory. This contains as a special case the fact that categories are equivalently skew monoidales C in the monoidal bicategory Span with tensor product given by

$$C \times C \xleftarrow{(s,t)} E \xrightarrow{t} C$$

for some set E, and where the unit is assumed to be of the form

$$I \xleftarrow{!} C \xrightarrow{1} C \ ;$$

where I is a terminal object in **Set**.

In Chapter 3 we characterise skew monoidales in Span without any restrictions on the unit of the skew monoidale. This means that the tensor product for the skew monoidale C is given by

$$C \times C \xleftarrow{(s,r)} E \xrightarrow{t} C$$

for some set E, and where the unit has the form

$$I \xleftarrow{!} U \xrightarrow{j} C \ .$$

This characterisation follows some lengthy but not difficult calculations in Section 3.2, which are made easier using the concrete form a pullback takes in **Set**. We recover the fact in [17], that categories are equivalently skew monoidales in Span with a unit of a certain restricted type. Section 3.3 collects the extra structure obtained from a skew monoidale in the form of a functor R with some interesting properties.

We finish the chapter with a simple example of a skew monoidale (actually just a monoidale) in Span whose unit is not of the restricted type previously considered.

In the first appendix we show the independence of the five axioms for a skew monoidal category. The second appendix consists of the definition of a Gray monoid.

Chapter 2

Skew Monoidal Categories

2.1 Skew Semimonoidal Categories

A *skew semimonoidal category* is a triple $(\mathcal{C}, \otimes, \alpha)$ where $\mathcal{C}$ is a category equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called *tensor product*), and a natural family of *lax constraints* α whose components have the form

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$$

subject to the condition that the following diagram commutes

$$\begin{tikzcd}
 & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W,X,Y \otimes Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha_{W \otimes X,Y,Z}"] \arrow[d, "\alpha_{W,X,Y} \otimes 1_Z"'] & & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha_{W,X \otimes Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "1_W \otimes \alpha_{X,Y,Z}"']
\end{tikzcd} \tag{2.1}$$

for all objects W, X, Y and Z.

2.2 The Category of Units

A skew monoidal category is a skew semimonoidal category equipped with a chosen unit, in a sense to be defined below. We shall see that if such a unit exists, it is unique up to isomorphism. More precisely, there is a unique compatible isomorphism, in the sense that it is a morphism in the *category of units*, which we now define.

Given a skew semimonoidal category $(\mathcal{C}, \otimes, \alpha)$, we form a category $\mathcal{U}(\mathcal{C})$ as follows. The *objects* are triples (I, λ, ρ) where I is an object of $\mathcal{C}$ and where λ and ρ are natural families of *lax constraints* whose components have the form

$$\lambda_X \colon I \otimes X \longrightarrow X$$

$$\rho_X \colon X \longrightarrow X \otimes I$$

subject to four conditions asserting that the following diagrams commute:

$$\begin{tikzcd} (I \otimes X) \otimes Y \arrow[rr, "\alpha_{I,X,Y}"] \arrow[dr, "\lambda_X \otimes 1_Y"'] & & I \otimes (X \otimes Y) \arrow[dl, "\lambda_{X \otimes Y}"] \\ & X \otimes Y & \end{tikzcd} \tag{2.2}$$

$$\begin{tikzcd} (X \otimes I) \otimes Y \arrow[r, "\alpha_{X,I,Y}"] & X \otimes (I \otimes Y) \arrow[d, "1_X \otimes \lambda_Y"] \\ X \otimes Y \arrow[u, "\rho_X \otimes 1_Y"] \arrow[r, "1_{X \otimes Y}"'] & X \otimes Y \end{tikzcd} \tag{2.3}$$

$$\begin{tikzcd} (X \otimes Y) \otimes I \arrow[rr, "\alpha_{X,Y,I}"] & & X \otimes (Y \otimes I) \\ & X \otimes Y \arrow[ul, "\rho_{X \otimes Y}"] \arrow[ur, "1_X \otimes \rho_Y"'] & \end{tikzcd} \tag{2.4}$$

$$
\begin{tikzcd}
I \arrow[rr, "1_I"] \arrow[dr, "\rho_I"'] & & I \\
& I \otimes I \arrow[ur, "\lambda_I"'] &
\end{tikzcd}
\quad . \tag{2.5}
$$

An *arrow* of $\mathcal{U}(\mathcal{C})$ from (I, λ, ρ) to (J, λ', ρ') is given by an arrow $\varphi \colon I \longrightarrow J$ in $\mathcal{C}$ such that the following two triangles commute

$$
\begin{tikzcd}
I \otimes X \arrow[rr, "\varphi \otimes 1_X"] \arrow[dr, "\lambda_X"'] & & J \otimes X \arrow[dl, "\lambda'_X"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes I \arrow[rr, "1_X \otimes \varphi"] & & X \otimes J \\
& X \arrow[ul, "\rho_X"] \arrow[ur, "\rho'_X"'] &
\end{tikzcd}
\tag{2.6}
$$

The composition of arrows in $\mathcal{U}(\mathcal{C})$ is then given by the composition in $\mathcal{C}$.

Given two objects (I, λ, ρ) and (J, λ', ρ') of $\mathcal{U}(\mathcal{C})$ we *define* $\varphi_{I,J} \colon I \longrightarrow J$ to be the following composite

$$
I \xrightarrow{\rho'_I} I \otimes J \xrightarrow{\lambda_J} J \; ; \tag{2.7}
$$

so with this notation $\varphi_{J,I} \colon J \longrightarrow I$ is the following composite

$$
J \xrightarrow{\rho_J} J \otimes I \xrightarrow{\lambda'_I} I \; .
$$

When no confusion arises we will call these maps just φ.

Lemma 2.2.1. *The map $\varphi_{I,J}$ defined by (2.7) is an arrow in $\mathcal{U}(\mathcal{C})$ from (I, λ, ρ) to (J, λ', ρ').*

Proof. We show that the first diagram of (2.6) commutes by considering the following diagram

$$
\begin{tikzcd}
I \otimes X \arrow[r, "\rho'_I \otimes 1_X"] \arrow[ddr, "1_{X \otimes Y}"'] & (I \otimes J) \otimes X \arrow[r, "\lambda_J \otimes 1_X"] \arrow[d, "\alpha_{I,J,X}"'] & J \otimes X \arrow[d, "\lambda'_X"] \\
& I \otimes (J \otimes X) \arrow[ur, "\lambda_{J \otimes X}"'] \arrow[d, "1_I \otimes \lambda'_X"] & X \\
& I \otimes X \arrow[ur, "\lambda_X"'] &
\end{tikzcd}
$$

in which the left-hand triangle commutes by equation (2.3) for (J, λ', ρ'), the right-hand triangle commutes by equation (2.2) for (I, λ, ρ), and the rectangle commutes by the naturality of λ. The right-hand side of (2.6) is analogous. $\square$

Proposition 2.2.2. *There is exactly one morphism from (I, λ, ρ) to (J, λ', ρ') in $\mathcal{U}(\mathcal{C})$.*

Proof. Suppose we have another morphism τ from I to J in $\mathcal{U}(\mathcal{C})$, and consider the following diagram

$$\begin{tikzcd}
I \arrow[r, "\rho'_I"] \arrow[d, "\tau"'] & I \otimes J \arrow[d, "\tau \otimes 1_J"'] \arrow[dr, "\lambda_J"] & \\
J \arrow[r, "\rho'_J"'] \arrow[rr, bend right, "1_J"'] & J \otimes J \arrow[r, "\lambda'_J"'] & J
\end{tikzcd}$$

The square commutes by the naturality of ρ', the triangle commutes by the assumption that τ satisfies the left-hand side of equation (2.6), and the semi-circle commutes by (2.5) for (J, λ', ρ'). This shows that $\tau = \varphi$. $\square$

Corollary 2.2.3. *Any two objects (I, λ, ρ) and (J, λ', ρ') in $\mathcal{U}(\mathcal{C})$ are isomorphic.*

Proof. Both $\varphi_{J,I} \circ \varphi_{I,J}$ and 1_I are arrows from (I, λ, ρ) to (I, λ, ρ) in $\mathcal{U}(\mathcal{C})$ so by uniqueness they are equal. That $\varphi_{I,J} \circ \varphi_{J,I} = 1_J$ is analogous. $\square$

We may combine the previous two results into:

Theorem 2.2.4. *For a skew semimonoidal category $\mathcal{C}$, if $\mathcal{U}(\mathcal{C})$ is non-empty then it is equivalent to the terminal category.*

Thus a skew semimonoidal category is a *skew monoidal category* if $\mathcal{U}(\mathcal{C})$ is non-empty. Proposition 2.2.2 and Corollary 2.2.3 then imply that the units for a skew monoidal category are unique up to a unique isomorphism (if they exist).

Next, we shall see that either λ or ρ determines the other.

Corollary 2.2.5. *If* (I, λ, ρ') *and* (I, λ, ρ) *are in* $\mathcal{U}(\mathcal{C})$ *then* $\rho' = \rho$.

Proof. Consider the unique morphism $\varphi_{J,I} \colon (I, \lambda, \rho') \longrightarrow (I, \lambda, \rho)$ where $J = (I, \lambda, \rho')$. By (2.5), this must be 1_I; then by (2.6) we deduce that $\rho = \rho'$. □

Corollary 2.2.6. *If* (I, λ, ρ) *and* (I, λ', ρ) *are in* $\mathcal{U}(\mathcal{C})$ *then* $\lambda = \lambda'$.

Proof. Dually, by reversing the tensor and the direction of arrows, we can instead repeat the above argument instead using $\varphi_{I,J}$. □

Remark 2.2.7. Equation (2.1) or the pentagon equation was not used in Proposition 2.2.2 or its Corollaries. This leads to the possibility of similar results about the units of skew versions of categories not satisfing (2.1) such as in [13].

Remark 2.2.8. The proof of Lemma 2.2.1 uses equations (2.2), (2.3) and (2.4) but not (2.1) or (2.5), while the proof of Proposition 2.2.2 uses the equations (2.5) and (2.6). Now suppose that λ and ρ satisfy only (2.2), (2.3) and (2.4). Then the composite $I \xrightarrow{\rho_I} I \otimes I \xrightarrow{\lambda_I} I$ satisfies (2.6) and so (2.5) becomes a special case of the uniqueness result in Proposition 2.2.2.

We denote a skew monoidal category by the 6-tuple $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$. For the following proposition, we use the fact that, as $\otimes$ is a bifunctor, the interchange law holds, in particular,

$$(f \otimes 1) \circ (1 \otimes g) = (1 \otimes g) \circ (f \otimes 1).$$

Proposition 2.2.9. *Let* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ *be a skew monoidal category . If there exists an object* J *and an isomorphism* $\varphi \colon J \longrightarrow I$ *in* $\mathcal{C}$ *then* (J, λ', ρ') *is also a unit of* $\mathcal{C}$*, where* $\lambda'_X \colon J \otimes X \longrightarrow X$ *and* $\rho'_X \colon X \longrightarrow X \otimes J$ *are given by the following composites:*

$$J \otimes X \xrightarrow{\varphi \otimes 1_X} I \otimes X \xrightarrow{\lambda_X} X \qquad\qquad X \xrightarrow{\rho_X} X \otimes I \xrightarrow{1_X \otimes \varphi^{-1}} X \otimes J$$

Proof. We need to show that these composites satisfy the four conditions in the definition. Consider the following diagrams.

For the one on the left, the square commutes by the naturality of α and the triangle commutes by (2.2). For the one on the right, the square commutes by the naturality of α and the triangle commutes by (2.4).

$$\begin{tikzcd}
(J \otimes X) \otimes Y \arrow[rr, "\alpha"] \arrow[d, "(\varphi\otimes 1)\otimes 1"] & & J \otimes (X \otimes Y) \arrow[d, "\varphi\otimes 1"'] \\
(I \otimes X) \otimes Y \arrow[rr, "\alpha"] \arrow[dr, "\lambda\otimes 1"'] & & I \otimes (X \otimes Y) \arrow[dl, "\lambda"] \\
& X \otimes Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \otimes Y) \otimes J \arrow[rr, "\alpha"] & & X \otimes (Y \otimes J) \\
(X \otimes Y) \otimes I \arrow[rr, "\alpha"] \arrow[u, "1\otimes\varphi^{-1}"] & & X \otimes (Y \otimes I) \arrow[u, "1\otimes(1\otimes\varphi^{-1})"'] \\
& X \otimes Y \arrow[ul, "\rho"] \arrow[ur, "1\otimes\rho"'] &
\end{tikzcd}$$

For the following diagram, the square commutes by the naturality of α and the outside commutes by (2.3). The semicircle on the left commutes as φ is an isomorphism. Thus the irregular lower region commutes as required.

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "\alpha"] & & X \otimes (I \otimes Y) \arrow[dd, bend left, "1\otimes\lambda"] \\
(X \otimes J) \otimes Y \arrow[rr, "\alpha"] \arrow[u, "(1\otimes\varphi)\otimes 1"'] & & X \otimes (J \otimes Y) \arrow[u, "1\otimes(\varphi\otimes 1)"] \\
(X \otimes I) \otimes Y \arrow[u, "(1\otimes\varphi^{-1})\otimes 1"'] \arrow[uu, bend left, "(1\otimes 1)\otimes 1"] & X \otimes Y \arrow[l, "\rho\otimes 1"'] \arrow[r, "1"'] & X \otimes Y
\end{tikzcd}$$

For the final diagram, the top right square commutes by the interchange law, the bottom right square commutes by the naturality of λ, the top left square commutes by the naturality of ρ and below this, the upper triangle commutes by (2.5).

$$\begin{tikzcd}
J \arrow[r, "\rho_J"] \arrow[d, "\varphi"'] \arrow[dd, bend right=60, "1"'] & J \otimes I \arrow[r, "1\otimes\varphi^{-1}"] \arrow[d, "\varphi\otimes 1"] & J \otimes J \arrow[d, "\varphi\otimes 1"] \\
I \arrow[r, "\rho_I"] \arrow[d, "\varphi^{-1}"'] \arrow[dr, "1"'] & I \otimes I \arrow[r, "1\otimes\varphi^{-1}"] \arrow[d, "\lambda_I"] & I \otimes J \arrow[d, "\lambda_J"] \\
J \arrow[r, "\varphi"'] \arrow[rr, bend right=40, "1"'] & I \arrow[r, "\varphi^{-1}"'] & J
\end{tikzcd}$$

$\square$

Proposition 2.2.10. *If* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ *is a skew monoidal category then* $(I \otimes I, \lambda_X \circ (\lambda_I \otimes 1_X), (1_X \otimes \rho_I) \circ \rho_X)$ *is a unit of* $\mathcal{C}$ *if and only if* $\lambda_I \colon I \otimes I \longrightarrow I$ *is invertible (or equivalently,* ρ_I *is invertible).*

Proof. Assuming that $(I \otimes I, \lambda_X \circ (\lambda_I \otimes 1_X), (1_X \otimes \rho_I) \circ \rho_X)$ is a unit we can use Lemma 2.2.1 and Proposition 2.2.2 and just show that $\varphi_{I\otimes I, I} = \lambda_I$ by considering the following diagram.

$$\begin{tikzcd}
I \otimes I \arrow[rr, "\rho_{I\otimes I}"] \arrow[d, "\lambda_I"'] & & (I \otimes I) \otimes I \arrow[d, "\lambda_I \otimes 1_I"] \\
I \arrow[rr, "\rho_I"] \arrow[dr, "1_I"'] & & I \otimes I \arrow[dl, "\lambda_I"] \\
& I &
\end{tikzcd}$$

where the square commutes by the naturality of ρ and the triangle commutes by (2.5). Conversely, if $\lambda_I \colon I \otimes I \longrightarrow I$ is invertible then by (2.5) $\rho_I \circ \lambda_I = 1_{I\otimes I}$, so then $\lambda_I^{-1} = \rho_I$ and we can apply Proposition 2.2.9. $\square$

A skew monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ is *weakly normal* if it also satifies the condition that $\rho_I \circ \lambda_I = 1_{I \otimes I}$; equivalently, if λ_I or ρ_I (and so both) is invertible.

Proposition 2.2.11. *If $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ is a weakly normal skew monoidal category then the monoid* $\mathrm{End}(I)$ *of endomorphisms of the unit object I is commutative.*

Proof. As $\lambda_I \colon I \otimes I \longrightarrow I$ is invertible, it induces an isomorphism $\psi \colon \mathrm{End}(I \otimes I) \longrightarrow \mathrm{End}(I)$ defined by $\psi(\gamma) = \lambda_I \circ \gamma \circ \lambda_I^{-1}$. For $f \in \mathrm{End}(I)$ we deduce, by the naturality of λ, that

$$\begin{aligned} f &= f \circ \lambda_I \circ \lambda_I^{-1} \\ &= \lambda_I \circ (1_I \otimes f) \circ \lambda_I^{-1} \\ &= \psi(1_I \otimes f) \end{aligned}$$

Similarly, using the naturality of λ^{-1} we get $f = \psi(f \otimes 1_I)$.

So for $f, g \in \mathrm{End}(I)$ we have, by the interchange law, that

$$\begin{aligned} f \circ g &= \psi(f \otimes 1) \circ \psi(1 \otimes g) \\ &= \psi((f \otimes 1) \circ (1 \otimes g)) \\ &= \psi((1 \otimes g) \circ (f \otimes 1)) \\ &= \psi(1 \otimes g) \circ \psi(f \otimes 1) \\ &= g \circ f \end{aligned}$$

□

Remark 2.2.12. Let R-**Mod** denote the category of left R-modules over some ring R. Regarding R as a left module over itself using its product, and noticing that $\mathrm{End}(R)$ is the monoid R if we regard R as a monoid under multiplication, we can use Lemma 2.2.11 to conclude that if R is a non-commutative ring then R is not the unit object for a weakly normal skew monoidal structure on R-**Mod**.

A skew monoidal category is *left normal* if λ is invertible. This implies that tensoring on the left by I is an equivalence. Using the naturality of λ and the invertibilty of λ_X we deduce that

$$\lambda_{I\otimes X} = 1_I \otimes \lambda_X$$

A skew monoidal category is *right normal* if ρ is invertible and *normal* if both λ and ρ are invertible.

Remark 2.2.13. If $\mathcal{C}$ is a left normal skew monoidal category then for any units I and J in $\mathcal{C}$ we have $I \otimes J \cong J$ and so $I \otimes J$ is also a unit by Proposition 2.2.10. Thus, if $\mathcal{C}$ is a left normal skew monoidal category then the $\otimes$ from $\mathcal{C}$ applied to $\mathcal{U}(\mathcal{C})$ gives $\mathcal{U}(\mathcal{C})$ the structure of a skew semimonoidal category.

Lemma 2.2.14. *If* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ *only satisfies* (2.2) *and* (2.3) *with both* λ *and* ρ *being invertible then* (2.5) *holds.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
(I\otimes I)\otimes X \arrow[r, "\alpha"] \arrow[dr, "\lambda_I\otimes 1_X"] & I\otimes(I\otimes X) \arrow[d, "1_I\otimes\lambda_X=\lambda_{I\otimes X}"] \\
I\otimes X \arrow[u, "\rho_I\otimes 1_X"] \arrow[r, "1"'] & I\otimes X
\end{tikzcd}$$

The outside commutes by (2.3), the upper triangle commutes by (2.2) where we used the assumption of λ being invertible and the resulting identity that $1_I \otimes \lambda_X = \lambda_{I\otimes X}$, so then the lower triangle commutes. Now taking $X = I$ and using the assumption that ρ is a natural isomorphism we get (2.5). $\square$

2.3 Monoidal Functors

Let $(\mathcal{C}, \otimes', I, \alpha', \lambda', \rho')$ and $(\mathcal{D}, \otimes, J, \alpha, \lambda, \rho)$ be skew monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, φ, F_0) where $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor of the underlying categories, F_0 is a morphism $J \longrightarrow F(I)$ in $\mathcal{D}$, and φ is a natural transformation with components $\varphi_{X,Y}\colon F(X) \otimes F(Y) \longrightarrow F(X \otimes' Y)$ such that the following diagrams commute.

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "\alpha_{F(X),F(Y),F(Z)}"] \arrow[d, "\varphi_{X,Y} \otimes 1_{F(Z)}"'] & F(X) \otimes (F(Y) \otimes F(Z)) \arrow[d, "1_{F(X)} \otimes \varphi_{Y,Z}"] \\
F(X \otimes' Y) \otimes F(Z) \arrow[d, "\varphi_{X \otimes' Y, Z}"'] & F(X) \otimes F(Y \otimes' Z) \arrow[d, "\varphi_{X, Y \otimes' Z}"] \\
F((X \otimes' Y) \otimes' Z) \arrow[r, "F(\alpha'_{X,Y,Z})"'] & F(X \otimes' (Y \otimes' Z))
\end{tikzcd} \tag{2.8}$$

$$\begin{tikzcd}
J \otimes F(X) \arrow[r, "\lambda_{F(X)}"] \arrow[d, "F_0 \otimes 1_{F(X)}"'] & F(X) \\
F(I) \otimes F(X) \arrow[r, "\varphi_{I,X}"'] & F(I \otimes' X) \arrow[u, "F(\lambda'_X)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[r, "\rho_{F(X)}"] \arrow[d, "F(\rho'_X)"'] & F(X) \otimes J \arrow[d, "1_{F(X)} \otimes F_0"] \\
F(X \otimes' I) & F(X) \otimes F(I) \arrow[l, "\varphi_{X,I}"]
\end{tikzcd} \tag{2.9}$$

A monoidal functor between skew monoidal categories is *normal* if F_0 is an isomorphism, and is *strong* if both φ and F_0 are isomorphisms. If the skew monoidal categories were monoidal then these are the usual notions of lax, normal, and strong monoidal functors.

Proposition 2.3.1. *Let $(\mathcal{C}, \otimes', I, \alpha', \lambda', \rho')$ and $(\mathcal{D}, \otimes, J, \alpha, \lambda, \rho)$ be skew monoidal categories and let F be a functor and φ a natural transformation such that (2.8) holds. Then there is at most one F_0 such that (2.9) holds.*

Proof. Let F_0' be another such morphism in $\mathcal{D}$, so in particular $F_0'\colon J \longrightarrow F(I)$ satisfies the equations in (2.9). Consider the following diagram.

$$
\begin{tikzcd}
J \arrow[r, "\rho_J"] \arrow[d, "F_0'"'] \arrow[rr, bend left, "1_J"] & J \otimes J \arrow[r, "\lambda_J"] \arrow[d, "F_0' \otimes 1_J"'] \arrow[ddr, "1_J \otimes F_0"] & J \arrow[ddd, bend left, "F_0"] \\
F(I) \arrow[r, "\rho_{F(I)}"] \arrow[d, "F(\rho_I)"'] \arrow[dd, bend right, "1_{F(I)}"'] & F(I) \otimes J \arrow[d, "1_{F(I)} \otimes F_0"'] & \\
F(I \otimes' I) \arrow[drr, "F(\lambda_I)"] & F(I) \otimes F(I) \arrow[l, "\varphi_{I,I}"'] & J \otimes F(I) \arrow[l, "F_0' \otimes 1_{F(I)}"'] \arrow[d, "\lambda_{F(I)}"'] \\
F(I) \arrow[rr, "1_{F(I)}"'] & & F(I)
\end{tikzcd}
$$

The part involving the semicircles on the top and left-hand side commute by (2.5). The top square commutes by the naturality of ρ, and the square below it commutes by the right-hand equation in (2.9). The triangle next to the squares commutes by the interchange law. The bottom triangle commutes by the left-hand equation in (2.9) and the remaining part of the diagram (on the right) commutes by the naturality of λ. The commutativity of the exterior gives the required uniqueness. □

Remark 2.3.2. If $F_0 \colon J \longrightarrow F(I)$ is an isomorphism in $\mathcal{D}$ then by Proposition 2.2.9, $F(I)$ is also a unit in $\mathcal{D}$. Now, by Proposition 2.2.2, there is a unique morphism between these units, namely $\varphi_{J,F(I)}$ and using the naturality of λ it can be shown that $\varphi_{J,F(I)} = F_0$.

Remark 2.3.3. This lemma generalises the uniqueness result of Proposition 2.2.2, which we may recover on taking the two skew monoidal categories to be the same, F to be the identity functor, and φ the identity natural transformation. It also implies the uniqueness of units for monoids in a skew monoidal category by taking $\mathcal{C} = 1$.

Remark 2.3.4. We denote by **SkMon** the category with *objects* skew monoidal categories and *1-cells* monoidal functors; and **SkSemiMon** the category with *objects* skew semimonoidal categories and *1-cells* semimonoidal functors (drop the F_0 conditions for the unit).

We denote the obvious forgetful functor where we drop all reference to units and any associated conditions by $V\colon \mathbf{SkMon} \longrightarrow \mathbf{SkSemiMon}$. For an object $\mathcal{C}$ of **SkSemiMon**, that is, any skew semimonoidal category, the fibre of V at $\mathcal{C}$ is the category $\mathcal{U}(\mathcal{C})$ of units of $\mathcal{C}$.

The uniqueness of F_0 in Proposition 2.3.1 implies that the forgetful functor V is faithful. Moreover, the uniqueness and existence results from Section 2.2 imply that V is full on isomorphisms in **SkSemiMon**, and by Proposition 2.2.9 V is also an isofibration.

2.4 Skew Monoidales

The results of Section 2.2 can be lifted to skew monoidales, these were first defined in [17] as an internal version of a skew monoidal category. So in this section we internalise the main result of Section 2.2. By the coherence results of [10], however, it will suffice to work in a Gray monoid; see Appendix B.

Let $\mathcal{B}$ be a Gray monoid; see Appendix B for an explicit definition. Note that for 1-cells $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ in a Gray monoid, the only structural 2-cells are the invertible 2-cells of the form

$$\begin{tikzcd} A \otimes B \arrow[r, "1\otimes g"] \arrow[d, "f\otimes 1"'] \arrow[dr, phantom, "\cong"] & A \otimes B' \arrow[d, "f\otimes 1"] \\ A' \otimes B \arrow[r, "1\otimes g"'] & A' \otimes B' \end{tikzcd}$$

or tensors and composites thereof. In this section we denote them with the symbol $\cong$ as above. These 2-cells satisfy some axioms which we do not list here but will appeal to throughout the rest of this section; see Appendix B once again. We write I for the unit object of the Gray monoid.

2.4.1 Skew Semimonoidales

A *skew semimonoidal* structure on an object A in $\mathcal{B}$ consists of a morphism $p\colon A \otimes A \longrightarrow A$ called the *tensor product*, and a 2-cell

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "1\otimes p"] \arrow[d, "p\otimes 1"'] \arrow[dr, phantom, "\alpha\Uparrow"] & A \otimes A \arrow[d, "p"] \\ A \otimes A \arrow[r, "p"'] & A \end{tikzcd}$$

subject to the "pentagon" axiom

$$
\begin{tikzcd}
 & & A \otimes A \otimes A \arrow[drr, "p\otimes 1"] \arrow[dd, "1\otimes p"'] \arrow[ddrr, phantom, "\Leftarrow \alpha"] & & \\
A \otimes A \otimes A \otimes A \arrow[urr, "p\otimes 1\otimes 1"] \arrow[dd, "1\otimes 1\otimes p"'] \arrow[drr, phantom, "\cong"] & & & & A \otimes A \arrow[dd, "p"] \\
 & & A \otimes A \arrow[drr, "p"] & & \\
A \otimes A \otimes A \arrow[urr, "p\otimes 1"] \arrow[drr, "1\otimes p"'] \arrow[rrrr, phantom, "\Leftarrow \alpha"] & & & & A \\
 & & A \otimes A \arrow[urr, "p"'] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & A \otimes A \otimes A \arrow[drr, "p\otimes 1"] \arrow[d, phantom, "\Leftarrow \alpha\otimes 1"] & & \\
A \otimes A \otimes A \otimes A \arrow[urr, "p\otimes 1\otimes 1"] \arrow[drr, "1\otimes p\otimes 1"] \arrow[dd, "1\otimes 1\otimes p"'] & & {} & & A \otimes A \arrow[dd, "p"] \\
 & & A \otimes A \otimes A \arrow[urr, "p\otimes 1"'] \arrow[dd, "1\otimes p"] \arrow[dl, phantom, "\Leftarrow 1\otimes\alpha"] \arrow[dr, phantom, "\Leftarrow \alpha"] & & \\
A \otimes A \otimes A \arrow[drr, "1\otimes p"'] & {} & & {} & A \\
 & & A \otimes A \arrow[urr, "p"'] & &
\end{tikzcd}
\tag{2.10}
$$

An object A of $\mathcal{B}$ equipped with such a skew semimonoidal structure is called a *skew semimonoidale* in $\mathcal{B}$; we denote it by (A, p, α).

A skew semimonoidale in the cartesian monoidal 2-category **Cat** of categories, functors and natural transformations is a skew semimonoidal category.

2.4.2 The Category of Units

If (A, p, α) is a skew semimonoidale in $\mathcal{B}$, we form its *category of units* $\mathcal{U}(A, p, \alpha)$ as follows. The *objects* are triples (j, λ, ρ), called *units*, where j is a morphism $j\colon I \longrightarrow A$ in $\mathcal{B}$ equipped with 2-cells, denoted by λ and ρ, called the *left unit* and *right unit constraints*. These have the form

$$
\begin{tikzcd}
A \otimes A \arrow[dr, "p"'] \arrow[r, phantom, shift right=4ex, "\Rightarrow\lambda"] & A \arrow[l, "j\otimes 1"'] \arrow[d, "1"] \\
 & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[d, "1\otimes j"] \\
A \arrow[ur, "1"] \arrow[r, phantom, shift left=4ex, "\Rightarrow\rho"] & A \otimes A \arrow[l, "p"]
\end{tikzcd}
$$

and are required to satisfy the following four equations

$$
\begin{tikzcd}
A\otimes A \arrow[r, "j\otimes 1\otimes 1"] \arrow[d, "1\otimes p"'] \arrow[dr, phantom, "\cong"] & A\otimes A\otimes A \arrow[r, "p\otimes 1"] \arrow[d, "1\otimes p"'] \arrow[dr, phantom, "\overset{\alpha}{\Longleftarrow}"] & A\otimes A \arrow[d, "p"] \\
A \arrow[r, "j\otimes 1"] \arrow[rr, bend right=40, "1"'] & A\otimes A \arrow[r, "p"] \arrow[d, phantom, "\lambda\Downarrow"] & A \\
& {} &
\end{tikzcd}
=
\begin{tikzcd}
A\otimes A \arrow[r, "j\otimes 1\otimes 1"] \arrow[d, "1\otimes p"'] \arrow[rr, bend right=20, "1"] & A\otimes A\otimes A \arrow[r, "p\otimes 1"] \arrow[d, phantom, "\lambda\otimes 1\Downarrow"] & A\otimes A \arrow[d, "p"] \\
A \arrow[rr, bend right=40, "1"'] & \cong & A
\end{tikzcd}
\qquad (2.11)
$$

$$
\begin{tikzcd}
A\otimes A \arrow[r, "1\otimes 1\otimes j"] \arrow[d, "p\otimes 1"'] \arrow[dr, phantom, "\cong"] & A\otimes A\otimes A \arrow[r, "1\otimes p"] \arrow[d, "p\otimes 1"'] \arrow[dr, phantom, "\Uparrow\alpha"] & A\otimes A \arrow[d, "p"] \\
A \arrow[r, "1\otimes j"'] \arrow[rr, bend right=40, "1"'] & A\otimes A \arrow[r, "p"'] \arrow[d, phantom, "\rho\Uparrow"] & A \\
& {} &
\end{tikzcd}
=
\begin{tikzcd}
A\otimes A \arrow[r, "1\otimes 1\otimes j"] \arrow[d, "p\otimes 1"'] \arrow[rr, bend right=20, "1"] & A\otimes A\otimes A \arrow[r, "1\otimes p"] \arrow[d, phantom, "1\otimes\rho\Uparrow"] & A\otimes A \arrow[d, "p"] \\
A \arrow[rr, bend right=40, "1"'] & \cong & A
\end{tikzcd}
\qquad (2.12)
$$

$$
\begin{tikzcd}
A\otimes A \arrow[rr, "1"] \arrow[dr, "1\otimes j\otimes 1"'] \arrow[ddr, bend right, "1"'] \arrow[r, phantom, "\Downarrow\rho\otimes 1", shift right=4ex, start anchor=east, end anchor=west] & {} & A\otimes A \arrow[dd, "p"] \arrow[dl, phantom, "\overset{\alpha}{\Longleftarrow}"] \\
\arrow[r, phantom, "\Swarrow 1\otimes\lambda"] & A\otimes A\otimes A \arrow[ur, "p\otimes 1"'] \arrow[d, "1\otimes p"'] & \\
& A\otimes A \arrow[r, "p"'] & A
\end{tikzcd}
=
\begin{tikzcd}
A\otimes A \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\equiv"] & A\otimes A \arrow[d, "p"] \\
A\otimes A \arrow[r, "p"'] & A
\end{tikzcd}
\qquad (2.13)
$$

$$
\begin{tikzcd}
I \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, phantom, "\cong"] & C \arrow[d, "1\otimes j"] \arrow[dr, bend left, "1"] \arrow[dr, phantom, "\Swarrow\rho"] & \\
C \arrow[r, "j\otimes 1"'] \arrow[rr, bend right=40, "1"'] & C\otimes C \arrow[r, "p"'] \arrow[d, phantom, "\lambda\Downarrow"] & C \\
& {} &
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, phantom, "\equiv"] & C \arrow[d, "1"] \\
C \arrow[r, "1"'] & C
\end{tikzcd}
\qquad (2.14)
$$

The *arrows* of $\mathcal{U}(A,p,\alpha)$ from (j',λ',ρ') to (j,λ,ρ) are given by the 2-cells $j' \xrightarrow{\varphi} j$ in $\mathcal{B}$ satisfying the following equations

$$
\begin{tikzcd}
p(j'\otimes 1) \arrow[d, "p(\varphi\otimes 1)"'] \arrow[dr, "\lambda'"] & \\
p(j\otimes 1) \arrow[r, "\lambda"'] & 1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& p(1\otimes j') \arrow[d, "p(1\otimes\varphi)"] \\
1 \arrow[ur, "\rho'"] \arrow[r, "\rho"'] & p(1\otimes j)
\end{tikzcd}
\qquad (2.15)
$$

In the case of a skew semimonoidal category seen as a skew semimonoidale (A, p, j) for **Cat**, this agrees with the previous definition.

Given (j', λ', ρ') and (j, λ, ρ) in $\mathcal{U}(A, p, \alpha)$ we denote the following 2-cell by $\varphi_{j',j}$.

$$
\begin{tikzcd}
& I \arrow[dl, "j'"'] \arrow[dr, "j"] & \\
A \arrow[dr, "1\otimes j"] \arrow[ddr, "1"'] & \cong & A \arrow[dl, "j'\otimes 1"'] \arrow[ddl, "1"] \\
{} \arrow[r, phantom, "\Rightarrow \rho"] & A\otimes A \arrow[d, "p"] \arrow[r, phantom, "\lambda' \Rightarrow"] & {} \\
& A &
\end{tikzcd}
$$

When no confusion will arise we drop the subscripts and simply write φ.

Lemma 2.4.1. *The 2-cell φ is an arrow in $\mathcal{U}(A, p, \alpha)$.*

Proof. We need to show that φ satisfies (2.15). We shall only verify the equation involving ρ; the one involving λ is similar.

The composite $1 \xrightarrow{\rho'} p(1\otimes j') \xrightarrow{p(1\otimes\varphi)} p(1\otimes j)$ appearing in (2.15) may be constructed as the following pasting composite.

$$
\begin{tikzcd}
& & A \arrow[dll, bend right, "1"'] \arrow[dl, "1\otimes j'"'] \arrow[dr, "1\otimes j"] \arrow[dl, phantom, bend right, "\rho' \Downarrow"] & \\
A \arrow[ddddrr, bend right, "1"'] \arrow[dddrr, phantom, "\cong"] & A\otimes A \arrow[l, "p"] \arrow[dr, "1\otimes 1\otimes j"] \arrow[ddr, "1"'] & \cong & A\otimes A \arrow[dl, "1\otimes j'\otimes 1"'] \arrow[ddl, "1"] \\
& {} \arrow[r, phantom, "\Rightarrow 1\otimes\rho"] & A\otimes A\otimes A \arrow[d, "1\otimes p"] \arrow[r, phantom, "1\otimes\lambda' \Rightarrow"] & {} \\
& & A\otimes A \arrow[d, "p"] & \\
& & A &
\end{tikzcd}
$$

Using equation (2.12) this is equal to

$$
\begin{tikzcd}
 & & & A \arrow[dlll, bend right, "1"'] \arrow[dll, "1\otimes j'"] \arrow[dr, "1\otimes j"] \arrow[dll, phantom, bend right=20, "\Downarrow\rho'"] \arrow[d, phantom, "\cong"] & \\
A \arrow[dr, "1\otimes j"'] \arrow[dddrrr, bend right, "1"'] & A\otimes A \arrow[l, "p"'] \arrow[drr, "1\otimes 1\otimes j"] \arrow[d, phantom, "\cong"] & & & A\otimes A \arrow[dl, "1\otimes j'\otimes 1"'] \arrow[ddl, "1"] \\
 & A\otimes A \arrow[ddrr, "p"'] \arrow[ddl, phantom, "\Uparrow\rho"] & & A\otimes A\otimes A \arrow[ll, "p\otimes 1"'] \arrow[d, "1\otimes p"] \arrow[dl, phantom, "\Uparrow\alpha"] \arrow[r, phantom, "\Rightarrow 1\otimes\lambda'"] & \\
 & & & A\otimes A \arrow[d, "p"] & \\
 & & & A &
\end{tikzcd}
$$

which, by equations (B.2) and (B.1), is equal to

$$
\begin{tikzcd}
 & & & A \arrow[dlll, bend right, "1"'] \arrow[dr, "1\otimes j"] & \\
A \arrow[dr, "1\otimes j"'] \arrow[dddrrr, bend right, "1"'] \arrow[rr, phantom, "\cong"] & & & & A\otimes A \arrow[dlll, bend right, "1"'] \arrow[dl, "1\otimes j'\otimes 1"'] \arrow[ddl, "1"] \arrow[dl, phantom, bend right=30, "\Downarrow\rho'\otimes 1"] \\
 & A\otimes A \arrow[ddrr, "p"'] \arrow[ddl, phantom, "\Uparrow\rho"] & & A\otimes A\otimes A \arrow[ll, "p\otimes 1"'] \arrow[d, "1\otimes p"] \arrow[dl, phantom, "\Uparrow\alpha"] \arrow[r, phantom, "\Rightarrow 1\otimes\lambda'"] & \\
 & & & A\otimes A \arrow[d, "p"] & \\
 & & & A &
\end{tikzcd}
$$

which finally, by equation (2.13), is equal to

$$
\begin{tikzcd}
 & & A \arrow[dll, bend right, "1"'] \arrow[dr, "1\otimes j"] & \\
A \arrow[dr, "1\otimes j"] \arrow[dddrr, bend right, "1"'] \arrow[rr, phantom, "\cong"] & & & A\otimes A \arrow[dll, bend right, "1"'] \arrow[ddl, "1"] \\
 & A\otimes A \arrow[ddr, "p"] \arrow[rr, phantom, "\equiv"] \arrow[dl, phantom, "\Nearrow\rho"] & & \\
 & & A\otimes A \arrow[d, "p"] & \\
 & & A &
\end{tikzcd}
$$

□

Proposition 2.4.2. *There is exactly one morphism from (j, λ, ρ) to (j', λ', ρ') in $\mathcal{U}(A, p, \alpha)$.*

Proof. Let τ be another such 2-cell $\begin{tikzcd} I \arrow[r, bend left, "j"] \arrow[r, bend right, "j'"'] \arrow[r, phantom, "\tau\Downarrow"] & A \end{tikzcd}$ in $\mathcal{U}(A, p, \alpha)$, so, in particular it satisfies

$$
\begin{tikzcd}
A \arrow[d, "j\otimes 1"'] \arrow[dr, "1"] \arrow[dr, phantom, "\Rightarrow\lambda", shift right=3] & \\
A\otimes A \arrow[r, "p"'] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[d, bend right, "j\otimes 1"'] \arrow[d, bend left, "j'\otimes 1"] \arrow[d, phantom, "\Rightarrow \tau\otimes 1"] \arrow[dr, bend left, "1"] & \\
A\otimes A \arrow[r, "p"'] \arrow[ur, phantom, "\Uparrow\lambda'"] & A
\end{tikzcd}
\tag{2.16}
$$

By assumption (2.16), the 2-cell ϕ

$$
\begin{tikzcd}
 & I \arrow[dl, "j"'] \arrow[dr, "j'"] & \\
A \arrow[dr, "1\otimes j'"] \arrow[ddr, "1"'] \arrow[rr, phantom, "\cong"] & & A \arrow[dl, "j\otimes 1"'] \arrow[ddl, "1"] \\
 & A\otimes A \arrow[d, "p"] \arrow[dl, phantom, "\Nearrow\rho'"] \arrow[dr, phantom, "\lambda\Searrow"] & \\
 & A &
\end{tikzcd}
$$

is equal to

$$
\begin{tikzcd}
& I \arrow[dl, "j"'] \arrow[dr, "j'"] & \\
A \arrow[dr, "1\otimes j'"] \arrow[ddr, "1"'] & \cong & A \arrow[dl, bend right, "j\otimes 1"'] \arrow[dl, bend left, "j'\otimes 1"] \arrow[dl, phantom, "\tau\otimes 1\,\Downarrow"] \arrow[ddl, bend left=60, "1"] \\
\arrow[r, phantom, "\Rightarrow\rho'"] & A\otimes A \arrow[d, "p"] \arrow[r, phantom, "\lambda'\Rightarrow"] & \\
& A &
\end{tikzcd}
$$

which, by equation (B.2), is equal to

$$
\begin{tikzcd}
& I \arrow[dl, bend right, "j"'] \arrow[dl, bend left, "j'"] \arrow[dl, phantom, "\tau\Downarrow"] \arrow[dr, "j'"] & \\
A \arrow[dr, "1\otimes j'"] \arrow[ddr, "1"'] & \cong & A \arrow[dl, "j'\otimes 1"'] \arrow[ddl, "1"] \\
\arrow[r, phantom, "\Rightarrow\rho'"] & A\otimes A \arrow[d, "p"] \arrow[r, phantom, "\lambda'\Rightarrow"] & \\
& A &
\end{tikzcd}
$$

which finally, by equation (2.14), is equal to

$$
\begin{tikzcd}
I \arrow[r, "j'"] \arrow[d, bend right, "j"'] \arrow[d, bend left, "j'"] \arrow[d, phantom, "\overset{\Rightarrow}{\tau}"] & A \arrow[d, "1"] \arrow[dl, phantom, "\equiv"] \\
A \arrow[r, "1"'] & A
\end{tikzcd}
$$

□

A skew semimonoidale in a Gray monoid $\mathcal{B}$ is a *skew monoidale* in $\mathcal{B}$ if $\mathcal{U}(A, p, \alpha)$ is non-empty. We denote such a skew monoidale by $(A, p, j, \alpha, \lambda, \rho)$.

The results proven above now imply, as in the case of the previous section, the following:

Theorem 2.4.3. *The units of a skew monoidale $(A, p, j, \alpha, \lambda, \rho)$ are unique up to a unique isomorphism (if they exist).*

Remark 2.4.4. (Theory of Skew Monoidales) It should not be a surprise that the elementary nature of the proofs in Section 2.2 carry over to this setting, especially if we were to write the axioms for a skew monoidale not as pasting diagrams in a Gray monoid but as equations between the 1-cells p and j. Consider equations (2.14) and (2.13) rewritten as

$$\begin{tikzcd} p(1\otimes j)j \arrow[r, "\cong"] & p(j\otimes 1)j \arrow[d, "\lambda j"] \\ j \arrow[u, "\rho j"] \arrow[r, "1"'] & j \end{tikzcd} \qquad \begin{tikzcd} p(p\otimes 1)(1\otimes j\otimes 1) \arrow[r, "\alpha(1\otimes j\otimes 1)"] & p(1\otimes p)(1\otimes j\otimes 1) \arrow[d, "p(1\otimes\lambda)"] \\ p \arrow[u, "p(\rho\otimes 1)"] \arrow[r, "1"'] & p \end{tikzcd}$$

These equations "look" like the corresponding equations (2.5) and (2.3) from the previous section. In Houston's 2007 thesis [12] there is defined a formal language for a collection of objects, 1-cells, 2-cells, and equations between the 2-cells, that admits an interpretation, or model, in a monoidal bicategory which he has called a "calculus of components". The calculus of components was used in [12] to show that some results for pseudomonoids(= monoidales) follow formally using the formal language from the corresponding result in the cartesian monoidal 2-category **Cat**. As noted in [12], the formal language is not completely general, it applies to a collection of 1-cells of the form $A_1\otimes \cdots\cdots \otimes A_n \longrightarrow B$ where these can then create, by tensoring and composition, composite 1-cells into a single target object such that the composite 1-cells are of the same form of the original collection. It was also noted in [12] that the calculus of components should be regarded as a higher dimensional analogue of the typed languages for monoidal categories defined by C. Barry Jay in [2].

Our only remark is that, we can use the calculus of components, from [12], to form a theory of skew monoidales since the only real difference to a theory of pseudomonoids(= monoidales), as presented by Houston in [12], is that we would need three basic 2-cells as opposed to six, and five equations between the 2-cells as opposed to two (not counting the invertibilty equations). With this in mind we then recognise that the results and proofs of this section are formally identical to the proofs in Section 2.2. That is, a formal proof in the language or theory of skew monoidales is the "same" as the proof of the corresponding result for skew moniodal categories from Section 2.2. For example, the use of one of the derivation rules for the equations between the 2-cells called the naturality axiom in [12] corresponds to our use of naturality in the proof of Propostion 2.2.2. So if we were to define a theory of skew monoidales using the calculus of components we could deduce the results of this section from Section 2.2.

Chapter 3

Skew Monoidales in Span

3.1 Span as a Monoidal Bicategory

Recall from Section 2.4 the definition of and notation for a skew monoidale in a monoidal bicategory $\mathcal{B}$. In this chapter we are interested in the case where $\mathcal{B}$ is Span. We first remind the reader of some details of Span.

The objects of Span are those of **Set** ; so A, B, C ..... are sets. We denote the terminal object in **Set** by 1 and the unique arrow into it by !.

An arrow $r\colon A \longrightarrow\!\!\!\!\!\!\!\mapsto B$ is a span $r = (f, R, g)$ in **Set**, as in (a), where composition of these arrows is by pullback (pullback along g and h), as in (b), and the identity arrow is the span (c) below.

$$(a)\quad \begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd} \qquad (b)\quad \begin{tikzcd} & R \arrow[dl] \arrow[dr, "g"] & & T \arrow[dl, "h"'] \arrow[dr] & \\ A & & B & & C \end{tikzcd} \qquad (c)\quad \begin{tikzcd} & C \arrow[dl, "1"'] \arrow[dr, "1"] & \\ C & & C \end{tikzcd}$$

A 2-cell from (w, R, x) to (f, S, g) is a map $\tau\colon R \longrightarrow S$ in **Set** such that the following commutes

$$
\begin{tikzcd}
& R \arrow[d, "\tau"] \arrow[ddl, "w"'] \arrow[ddr, "x"] & \\
& S \arrow[dl, "f"] \arrow[dr, "g"'] & \\
A & & B
\end{tikzcd}
$$

As **Set** is a category with finite products as well as pullbacks (in the presence of a terminal object, finite products can be obtained as a special case of pullbacks) then the bicategory Span has a monoidal product on it induced by the cartesian product of sets. To calculate a left whiskering, such as in the following diagram (on the left), we first form the pullbacks of f and w along v, then we use the fact that $f\tau = w$ and $v1 = v$ to construct the dotted arrow in the diagram on the right.

$$
\begin{tikzcd}
& & & R \arrow[d, "\tau"] \arrow[ddl, "w"'] \arrow[ddr, "x"] & \\
& E \arrow[dl, "u"'] \arrow[dr, "v"] & & S \arrow[dl, "f"] \arrow[dr, "g"'] & \\
C & & A & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& & E \times_A R \arrow[dll] \arrow[d, dotted] \arrow[drr] & & \\
E \arrow[ddd, bend right, "u"'] \arrow[dr, equal, "1"'] & & E \times_A S \arrow[dl] \arrow[dr] & & R \arrow[dl, "\tau"] \arrow[ddd, bend left, "x"] \\
& E \arrow[dl, "u"] \arrow[dr, "v"'] & & S \arrow[dl, "f"] \arrow[dr, "g"'] & \\
C & & A & & B
\end{tikzcd}
$$

3.2 Notation and Calculations

The motivation for this section is from [17] where skew monoidales in Span with a unit of the form $(!, C, 1)\colon 1 \longrightarrow\!\!\!\!\!\!\mapsto C$ are shown to be equivalent to categories. Here we give a characterisation of a general skew monoidale in Span.

Consider a skew monoidale in Span with underlying object the set C.

The 1-cells of a Skew Monoidale:

The tensor $p\colon C \times C \longrightarrow\!\!\!\!\!\!\mapsto C$ has the form

$$\begin{tikzcd} & E \arrow[dl, "(s,r)"'] \arrow[dr, "t"] & \\ C \times C & & C \end{tikzcd}$$

So for $f \in E$ we will record this data as $s(f) \overset{f}{\dashrightarrow} t(f)$ and $r(f) \in C$.

The unit $j\colon 1 \longrightarrow\!\!\!\!\!\!\mapsto C$ has the form

$$\begin{tikzcd} & U \arrow[dl, "!"'] \arrow[dr, "j"] & \\ 1 & & C \end{tikzcd}$$

So for $u \in U$ we will record this data as $j(u) \in C$.

Given a skew monoidale, with its unit having the restricted form $(!, C, 1)\colon 1 \longrightarrow\!\!\!\!\!\!\mapsto C$, it will become evident when dealing with the general case below, that this forces the first span to be of the form $C \times C \xleftarrow{(s,t)} E \xrightarrow{t} C$, and that it defines a category with E as its set of arrows. Conversely, given a category $C = (C_1, C_0, 1, s, t, \circ)$, we construct the following two spans: $C_0 \times C_0 \xleftarrow{(s,t)} C_1 \xrightarrow{t} C_0$, and $1 \xleftarrow{!} C_0 \xrightarrow{1} C_0$. The 2-cell structure making this category into a skew monoidale comes from the composition and identity arrows of the category, with the skewness arising from the non-symmetric nature of the first span.

The 2-cells of a Skew Monoidale:

What is now required is a long and often repetitive calculation with, when we include the equations between the 2-cells, sixteen pullback constructions in **Set**; so we will present enough of it to introduce and justify the supporting notation that will form our input for a further characterisation.

For the 2-cell λ: $p(j \times 1) \Longrightarrow 1$ we need to consider the following composite

$$\begin{tikzcd}
& U \times C \arrow[dl, "!\times 1"'] \arrow[dr, "j\times 1"] & & E \arrow[dl, "(s,r)"'] \arrow[dr, "t"] & \\
1 \times C & & C \times C & & C
\end{tikzcd}$$

First we need to form the following pullback

$$\begin{tikzcd}
P \arrow[r, "q"] \arrow[d, "p"'] & E \arrow[d, "s"] \\
U \arrow[r, "j"'] & C
\end{tikzcd} \tag{3.1}$$

then the required composite is

$$\begin{tikzcd}
& & P \arrow[dl, "(p,rq)"'] \arrow[dr, "q"] & & \\
& U \times C \arrow[dl, "!\times 1"'] \arrow[dr, "j\times 1"] & & E \arrow[dl, "(s,r)"'] \arrow[dr, "t"] & \\
1 \times C & & C \times C & & C
\end{tikzcd}$$

so we finally have for the 2-cell λ, a function which we also denote by λ, such that the following diagram commutes,

$$\begin{tikzcd}
 & P \arrow[d, "\lambda"] \arrow[ddl, "rq"'] \arrow[ddr, "tq"] & \\
 & C \arrow[dl, "1"] \arrow[dr, "1"'] & \\
C & & C
\end{tikzcd}$$

it can only exist if $rq = tq$ and is then given as a morphism in **Set** by the common value

$$rq = tq. \tag{3.2}$$

As we are in **Set** we can write P as $P = \big\{(u, f)|u \in U, f \in E,\ \ j(u) = s(f)\big\}$ with $p(u, f) = u$ and $q(u, f) = f$ as the projections. With our notation, the elements in P look like $j(u) \overset{f}{\dashrightarrow} y$. We can now record the effect of λ as : $j(u) \overset{f}{\dashrightarrow} y \xmapsto{\ \ \lambda\ \ } y = r(f)$. Thus the existence of λ implies that if $j(u) \overset{f}{\dashrightarrow} y$ then $y = r(f)$, and the map itself sends $(u,\ j(u) \overset{f}{\dashrightarrow} y\ )$ to y.

In the case of a category (that is, the case where U is C and the unit is of the form $1 \xleftarrow{!} C \xrightarrow{1} C$) then $P = E = C_1$ and $j = 1$ forces $r = t$, so λ is just t .

For the 2-cell $\rho\colon 1 \Longrightarrow p(1 \times j)$ we first need to construct the following pullback

$$\begin{tikzcd}
B \arrow[r, "k"] \arrow[d, "m"'] & E \arrow[d, "r"] \\
U \arrow[r, "j"'] & C
\end{tikzcd}$$

In the diagram below

$$
\begin{tikzcd}
& & C \arrow[d, "\rho"] \arrow[dd, bend right, "{(1,\psi)}"'] \arrow[ddrr, bend left, "\phi"] \arrow[dddl, bend right, "1"'] \arrow[dddrrr, bend left, "1"] & & & \\
& & B \arrow[dl, "{(sk,m)}"'] \arrow[dr, "k"] & & & \\
& C \times U \arrow[dl, "proj_1"'] \arrow[dr, "1\times j"] & & E \arrow[dl, "{(s,r)}"'] \arrow[dr, "t"] & & \\
C & & C \times C & & C &
\end{tikzcd}
\tag{3.3}
$$

the square is the pullback involved in the composite $p(1\times j)$, so to give $\rho\colon 1 \Longrightarrow p(1\times j)$ is equivalently to give $\phi\colon C \longrightarrow E$ and $\psi\colon C \longrightarrow U$ satisfying $t\phi = 1$, $s\phi = 1$, and $r\phi = j\psi$. That these equations hold can be seen by the following diagrams extracted from (3.3).

$$
\begin{tikzcd}
& C \arrow[d, "{(\psi,\phi)}"] \arrow[ddl, bend right, "1"'] \arrow[ddr, bend left, "1"] & \\
& B \arrow[dl, "sk"'] \arrow[dr, "tk"] & \\
C & & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[dr, "{(\psi,\phi)}"] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "\psi"'] & & \\
& B \arrow[r, "k"] \arrow[d, "m"'] & E \arrow[d, "r"] \\
& U \arrow[r, "j"'] & C
\end{tikzcd}
$$

We record for later use that

$$r\phi = j\psi.$$

As we are in **Set**, $B = \big\{(u,f) | u \in U, f \in E,\ j(u) = r(f)\big\}$ with $m(u,f) = u$ and $k(u,f) = f$ as the projections. With respect to our notation, the elements in B look like $(j(u) = r(f),\ x \overset{f}{\dashrightarrow} y\ )$ so we record the effect of ϕ as

$$x \in C \xmapsto{\ \phi\ } (\ x \overset{\phi_x}{\dashrightarrow} x\ )$$

then $\psi_x \in U$ satisfies $j(\psi_x) = r(\phi_x)$.

Note that in the case of a category then $B = E = C_1$ and so ρ is just the identity.

For the 2-cell $\alpha\colon p(p\times 1)\Longrightarrow p(1\times p)$ we need the following two pullbacks

$$
\begin{tikzcd}
X \arrow[r, "l"] \arrow[d, "h"'] & E \arrow[d, "s"] \\
E \arrow[r, "t"'] & C
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Y \arrow[r, "y"] \arrow[d, "e"'] & E \arrow[d, "r"] \\
E \arrow[r, "t"'] & C
\end{tikzcd}
$$

The objects X and Y will appear as the vertex of the spans $p(p\times 1)$ and $p(1\times p)$, respectively.

In the diagram below

$$
\begin{tikzcd}
 & & X \arrow[d, "\alpha"] \arrow[dddll, bend right, "(sh,rh,rl)"'] \arrow[ddl, bend right, "(sh,\tau)"'] \arrow[ddr, bend left, "\delta"] \arrow[dddrr, bend left, "tl"] & & \\
 & & Y \arrow[dl, "(sy,e)"] \arrow[dr, "y"'] & & \\
 & C\times E \arrow[dl, "1\times(s,r)"] \arrow[dr, "1\times t"'] & & E \arrow[dl, "(s,r)"] \arrow[dr, "t"'] & \\
C\times C\times C & & C\times C & & C
\end{tikzcd}
\tag{3.4}
$$

the square is the pullback involved in the composite $p(1\times p)$, so to give $\alpha\colon p(p\times 1)\Longrightarrow p(1\times p)$ is equivalently to give $\tau\colon X\longrightarrow E$ and $\delta\colon X\longrightarrow E$ satisfying $t\delta = tl$, $s\delta = sh$,$s\tau = rh$, $r\tau = rl$, and $r\delta = t\tau$. That these equations hold can be seen by the following diagrams extracted from (3.4).

$$
\begin{tikzcd}
 & X \arrow[d, "{(\tau,\delta)}"] \arrow[ddl, bend right, "(sh,rh,rl)"'] \arrow[ddr, bend left, "tl"] & \\
 & Y \arrow[dl, "(sy,se,re)"] \arrow[dr, "ty"'] & \\
C\times C\times C & & C
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[dr, "{(\tau,\delta)}"] \arrow[drr, bend left, "\delta"] \arrow[ddr, bend right, "\tau"'] & & \\
 & Y \arrow[r, "y"] \arrow[d, "e"'] & E \arrow[d, "r"] \\
 & E \arrow[r, "t"'] & C
\end{tikzcd}
$$

As we are in **Set**, $X = \big\{(f,g)|f,g\in E; t(f)=s(g)\big\}$ with $l(f,g)=g$ and $h(f,g)=f$ as the projections. Similarly, $Y = \big\{(f,g)|f,g\in E; t(f)=r(g)\big\}$ with $y(f,g)=g$ and

$e(f,g) = f$ as its projections. So with respect to our notation, elements of X look like $x \overset{f}{\dashrightarrow} y \overset{g}{\dashrightarrow} z$ and elements of Y look like ($x \overset{f}{\dashrightarrow} r(g)$, $y \overset{g}{\dashrightarrow} z$) with $r(f)$, $r(g)$ $\in C$ and we record the effect of δ as

$$x \overset{f}{\dashrightarrow} y \overset{g}{\dashrightarrow} z \xmapsto{\ \delta\ } x \overset{gf}{\dashrightarrow} z$$

and τ as

$$x \overset{f}{\dashrightarrow} y \overset{g}{\dashrightarrow} z \xmapsto{\ \tau\ } r(f) \overset{g^f}{\dashrightarrow} r(gf)$$

with $r(g^f) = r(g)$ in C.

Note that δ gives us a map from x to z which we have called gf. We want to interpret the set E as a set of arrows and gf as a composite (with ϕ_x as an identity), indeed, that this is the composite in a category will be shown below. The map τ gives us a map from $r(f)$ to $r(gf)$ which we have called g^f. This map will form the basis of our characterisation for the resulting "extra" structure given on the category.

We now consider the **equations between the 2-cells** and just do one calculation to give the reader an indication of how the final relations are obtained. Consider the left-hand side of equation (2.10) and the whiskering

$$\begin{tikzcd} C \times C \times C \times C \arrow[r, "p\times 1\times 1"] & C \times C \times C \arrow[r, "p\times 1"] \arrow[d, "1\times p"'] \arrow[dr, phantom, "\Leftarrow\alpha"] & C \times C \arrow[d, "p"] \\ & C \times C \arrow[r, "p"'] & C \end{tikzcd}$$

For this we need to compose

$$\begin{tikzcd}
 & & & X \arrow[d, "(\tau{,}\delta)"] \arrow[ddl, bend right, "(sh{,}rh{,}rl)"'] \arrow[ddr, bend left, "tl"] & \\
 & E\times C\times C \arrow[dl, "(s{,}r)\times1\times1"'] \arrow[dr, "t\times1\times1"] & & Y \arrow[dl, "(sy{,}se{,}re)"] \arrow[dr, "ty"] & \\
C\times C\times C\times C & & C\times C\times C & & C
\end{tikzcd}$$

First form the pullbacks

$$\begin{tikzcd}
Q \arrow[r, "l'"] \arrow[d, "y'"'] & Y \arrow[d, "y"] \\
X \arrow[r, "l"'] & E
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
W \arrow[r, "w"] \arrow[d, "h'"'] & X \arrow[d, "h"] \\
X \arrow[r, "l"'] & E
\end{tikzcd}$$

then we have the following pullbacks

$$\begin{tikzcd}
Q \arrow[r, "l'"] \arrow[d, "(hy'{,}sel'{,}rel')"'] & Y \arrow[d, "(sy{,}se{,}re)"] \\
E\times C\times C \arrow[r, "t\times1\times1"'] & C\times C\times C
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "w"] \arrow[d, "(hh'{,}rhw{,}rlw)"'] & X \arrow[d, "(sh{,}rh{,}rl)"] \\
E\times C\times C \arrow[r, "t\times1\times1"'] & C\times C\times C
\end{tikzcd}$$

and so form

$$\begin{tikzcd}
 & & W \arrow[dll, "(hh'{,}rhw{,}rlw)"'] \arrow[d, "\mu"] \arrow[drr, "w"] & & \\
E\times C\times C \arrow[dr, equal, "1"'] \arrow[dd, bend right, "(s{,}r)\times1\times1"'] & & Q \arrow[dl, "(hy'{,}sel'{,}rel')"'] \arrow[dr, "l'"] & & X \arrow[dl, Rightarrow, "(\tau{,}\delta)"'] \arrow[dd, bend left, "tl"] \\
 & E\times C\times C \arrow[dl, "(s{,}r)\times1\times1"] \arrow[dr, "t\times1\times1"] & & Y \arrow[dl, "(sy{,}se{,}re)"] \arrow[dr, "ty"] & \\
C\times C\times C\times C & & C\times C\times C & & C
\end{tikzcd}$$

As before, to give the map $p(p\times 1)(p\times 1\times 1) \xrightarrow{\alpha(p\times 1\times 1)} p(1\times p)(p\times 1\times 1)$ is equivalently to give $\gamma\colon W \longrightarrow E$ and $\epsilon\colon W \longrightarrow Y$ as in the diagram below.

$$\begin{tikzcd}
 & & W \arrow[dll, "{(hh',rhw,rlw)}"'] \arrow[ddl, bend right, "\gamma"'] \arrow[d, "\mu"] \arrow[ddr, bend left, "\epsilon"] \arrow[drr, "w"] & & \\
E\times C\times C \arrow[dd, bend right, "{(s,r)\times 1\times 1}"'] \arrow[dr, equal, "1"'] & & Q \arrow[dl, "{(hy',sel',rel')}"'] \arrow[dr, "l'"] & & X \arrow[dl, Rightarrow, "{(\tau,\delta)}"'] \arrow[dd, bend left, "tl"] \\
 & E\times C\times C \arrow[dl, "{(s,r)\times 1\times 1}"] \arrow[dr, "t\times 1\times 1"] & & Y \arrow[dl, "{(sy,se,re)}"'] \arrow[dr, "ty"] & \\
C\times C\times C\times C & & C\times C\times C & & C
\end{tikzcd}$$

From this diagram we now establish some relationship between (γ,ϵ) and (τ,δ). We get $\gamma = hh'$ and $\epsilon = (\tau,\delta)w = (\tau w,\delta w)$. Now writing these as functions into just the set E we recall the previous pullbacks we had constructed and consider the following diagram

$$\begin{tikzcd}
 & & W \arrow[d, "\mu"] \arrow[dddll, bend right, "\gamma=hh'"'] \arrow[ddd, bend left, "y\epsilon=\delta w"'] \arrow[ddr, bend left, "\epsilon"] \arrow[dddrr, bend left, "\tau w=e\epsilon"] & & \\
 & & Q \arrow[dl, "y'"'] \arrow[dr, "l'"] & & \\
 & X \arrow[dl, "h"'] \arrow[dr, "l"] & & Y \arrow[dl, "y"] \arrow[dr, "e"] & \\
E \arrow[dr, "t"'] & & E \arrow[dl, "s"] \arrow[dr, "r"'] & & E \arrow[dl, "t"] \\
 & C & & C &
\end{tikzcd}$$

From this diagram we have $\gamma = hh'$, $y\epsilon = y(\tau w,\delta w) = \delta w$ and $e\epsilon = e(\tau w,\delta w) = \tau w$. As we are in **Set**, $W = \{(x_1,x_2)|x_1,x_2 \in X; l(x_1) = h(x_2)\}$ with projections $h'(x_1,x_2) = x_1$ and $w(x_1,x_2) = x_2$. Similarly, $Q = \{(x,z)|x \in X, z \in Y; l(x) = y(z)\}$ with projections $y'(x,z) = x$ and $l'(x,z) = z$. So with respect to our notation, the elements of the set W look like $a \overset{f}{\dashrightarrow} b \overset{g}{\dashrightarrow} c \overset{k}{\dashrightarrow} d$ with $r(f)$, $r(g)$, $r(k) \in C$ and the elements of Q look like ($a \overset{f}{\dashrightarrow} b \overset{g}{\dashrightarrow} c$, $v \overset{k}{\dashrightarrow} r(g)$) with $r(f)$, $r(g)$ and $r(k) \in C$.

So the 2-cell under consideration gives for $W \longrightarrow Q$ that

$$a \overset{f}{\dashrightarrow} b \overset{g}{\dashrightarrow} c \overset{k}{\dashrightarrow} d \quad \Longrightarrow \quad (\, a \overset{f}{\dashrightarrow} b \,,\; r(g) \overset{k^g}{\dashrightarrow} r(kg) \,,\; b \overset{kg}{\dashrightarrow} d \,)$$

Now observe that the two sides of the cube (2.10) act as in the diagram below,

$$\begin{tikzcd}
 & (gf, g^f, h) \arrow[r, mapsto, "\alpha(1\otimes p\otimes 1)"] & (gf, h^{gf}, h(gf)) \arrow[d, mapsto, "p(1\otimes\alpha)"] \\
(f, g, h) \arrow[ur, mapsto, "p(\alpha\otimes 1)"] \arrow[d, mapsto, "\alpha(p\otimes 1\otimes 1)"'] & & ((h^{gf})^{g^f}, h^{gf}g^f, h(gf)) \\
(f, h^g, hg) \arrow[dr, mapsto, "\cong"'] & & (h^g, (hg)^f, (hg)f) \\
 & (h^g, f, hg) \arrow[ur, mapsto, "\alpha(1\otimes 1\otimes p)"'] &
\end{tikzcd}$$

so the cube commutes if and only if the two expressions in the lower right corner agree; in other words, if the following equations, $h^g = (h^{gf})^{g^f}$, $(hg)^f = h^{gf}g^f$, and $(hg)f = h(gf)$ hold, for a composable triple of arrows. The remaining four equations are analyzed similarly, and the results summarized below.

Summary: We summarize all the calculations with respect to a skew monoidale into the notation introduced earlier to get:

For the 1-cell $p\colon C \times C \longrightarrow\!\!\!\!\!\!\mid\;\; C$ with vertex E : For $f \in E$, $x \overset{f}{\dashrightarrow} y$ for $x, y \in C$ and $r(f) \in C$.

For the 1-cell $j\colon 1 \longrightarrow\!\!\!\!\!\!\mid\;\; C$ with vertex U : For $u \in U$ that $j(u) \in C$.

For the 2-cell λ : if $j(u) \overset{f}{\dashrightarrow} y$ then $y = r(f)$ in C.

For the 2-cell ρ : for $x \in C$ we have $x \overset{\phi_x}{\dashrightarrow} x$ in E and $\psi_x \in U$ with $j(\psi_x) = r(\phi_x)$.

For the 2-cell α : if $x \overset{f}{\dashrightarrow} y \overset{g}{\dashrightarrow} z$ then $r(f) \overset{g^f}{\dashrightarrow} r(gf)$ and $x \overset{gf}{\dashrightarrow} z$ are both in E with $r(g^f) = r(g)$.

For the equation between the 2-cells involving (λ, ρ) : For $j(u) \in C$ we have $\psi_{j(u)} = u$, that is, $\psi j = 1$.

For the equation between the 2-cells involving (ρ, α) : For $x \overset{f}{\dashrightarrow} y$ we have $\psi_y = \psi_{r(f)}$, $\phi_y f = f$, and $\phi_y^f = \phi_{r(g)}$.

For the equation between the 2-cells involving (λ, α) : For $j(u) \overset{f}{\dashrightarrow} y \overset{g}{\dashrightarrow} z$ we have $g^f = g$.

For the equation between the 2-cells involving (ρ, α, λ) : For $x \overset{f}{\dashrightarrow} y$ we have $f\phi_x = f$.

For the equation between the 2-cells involving (α, α) (the pentagon) :
For $x \overset{f}{\dashrightarrow} y \overset{g}{\dashrightarrow} z \overset{h}{\dashrightarrow} a$ we have $(hg)f = h(gf)$, $(hg)^f = h^{gf} g^f$, and $h^g = (h^{gf})^{g^f}$.

We conclude that we can now safely rename ϕ_x as 1_x and change our notation for $x \dashrightarrow^{f} y$ to an arrow $x \xrightarrow{f} y$ and with the condition that $(hg)f = h(gf)$ obtain a category with some extra structure consisting of :

(a) for each morphism f an object $r(f)$.

(b) a set U with a function j from U to the set of objects.

(c) for each composable pair $x \xrightarrow{f} y \xrightarrow{g} z$ a map $r(f) \xrightarrow{g^f} r(gf)$ with $r(g^f) = r(g)$.

(d) for each object c an element $\psi_c \in U$.

satisfying the following

$$\text{For } u \in U \quad \text{that} \quad \psi j(u) = u. \tag{3.5}$$

$$\text{For } x \in C \quad \text{that} \quad r(1_x) = j\psi. \tag{3.6}$$

$$\text{For } j(u) \xrightarrow{f} y \xrightarrow{g} z \quad \text{that} \quad g^f = g. \tag{3.7}$$

$$\text{For } x \xrightarrow{f} y\,, r(f) \in C \quad \text{that} \quad 1_y^f = 1_{r(f)}. \tag{3.8}$$

$$\text{For } x \xrightarrow{f} y\,, r(f) \in C \quad \text{that} \quad \psi_y = \psi_{r(f)}. \tag{3.9}$$

$$\text{For } x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} a \quad \text{that} \quad (hg)^f = h^{gf} g^f. \tag{3.10}$$

$$\text{For } x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} a \quad \text{that} \quad h^g = (h^{gf})^{g^f}. \tag{3.11}$$

Before we consider these equations again, we notice that from (3.5) j is already injective.

Lemma 3.2.1. *If j is surjective then $r = t$.*

Proof. If j is surjective then by (3.1) q is also surjective. Since $rq = tq$ by (3.2), we can conclude that $r = t$. □

So with the assumption that j is surjective we see that a skew monoidale in Span is precisely a category. The extra structure given by τ and the map g^f reduces to $g^f = g$ for all $f, g \in E$ by (3.7). This recovers the result in [17] where the skew monoidale in Span assumed the unit was of the form

$$\begin{tikzcd} & C \arrow[dl, "!"'] \arrow[dr, "1"] & \\ 1 & & C \end{tikzcd}$$

3.3 A Characterisation

3.3.1 Coslice Category

In this subsection we use the notation of [19] to denote the coslice category or under-category of a category, which we now define.

Let C be a category and x an object of C, then the *coslice* category denoted by $(x \downarrow C)$ has *objects* the arrows of C with source x, that is, $x \xrightarrow{f} y$ which we sometimes denote by the pairs (f, y); and *arrows* those $g\colon (f, y) \longrightarrow (f', z)$ where $y \xrightarrow{g} z$ is an arrow of C such that $f' = gf$, which we usually denote as $(f, y) \xrightarrow{g} (gf, z)$. It is useful sometimes to write these arrows as the following triangles

$$\begin{tikzcd} x \arrow[d, "f"'] \arrow[dr, "gf"] & \\ y \arrow[r, "g"'] & z \end{tikzcd}$$

There is an evident functor $\mathrm{Cod}_x\colon (x \downarrow C) \longrightarrow C$ defined on objects by $x \xrightarrow{f} y \longmapsto y$ and on arrows by $(f, y) \xrightarrow{g} (gf, z) \longmapsto y \xrightarrow{g} z$.

Note: Let A and B be categories and x an object of A. For a functor $T\colon A \longrightarrow B$ there is an induced functor $(x \downarrow A) \xrightarrow{(x \downarrow T)} (Tx \downarrow B)$ sending an object $x \xrightarrow{f} y$ to $Tx \xrightarrow{Tf} Ty$ and an arrow

$$\begin{tikzcd} x \arrow[d, "f"'] \arrow[dr, "gf"] & \\ y \arrow[r, "g"'] & z \end{tikzcd} \qquad \longmapsto \qquad \begin{tikzcd} Tx \arrow[d, "Tf"'] \arrow[dr, "T(gf)=T(g)T(f)"] & \\ Ty \arrow[r, "Tg"'] & Tz \end{tikzcd}$$

Let C be a category and $f\colon x \longrightarrow y$ be an object of $(x \downarrow C)$; we remind the reader of the coslice category $(f \downarrow (x \downarrow C))$. This category has as its objects the morphisms in $(x \downarrow C)$ starting at f denoted by $f \xrightarrow{g} gf$ and as its morphisms the commuting triangles between its objects which we denote by

$$
\begin{tikzcd}
f \arrow[d, "g"'] \arrow[dr, "hg"] & \\
gf \arrow[r, "h"'] & hgf
\end{tikzcd}
$$

we sometimes denote them by $g \xrightarrow{h} hg$.

The functor $(f \downarrow \mathrm{Cod}_x) : \ (f \downarrow (x \downarrow C)) \longrightarrow (y \downarrow C)$ is invertible; it sends an object $f \xrightarrow{g} gf$ to g and a morphism

$$
\begin{tikzcd}
f \arrow[d, "g"'] \arrow[dr, "hg"] & \\
gf \arrow[r, "h"'] & hgf
\end{tikzcd}
\qquad \longmapsto \qquad
\begin{tikzcd}
\mathrm{Cod}(f) \arrow[d, "g"'] \arrow[dr, "hg"] & \\
\mathrm{Cod}(gf) \arrow[r, "h"'] & \mathrm{Cod}(hgf)
\end{tikzcd}
$$

3.3.2 The Functor R_x

From the previous sections we have seen that a skew monoidale C in Span gives rise to a category $\mathbb{C}$ with some extra structure via the function g^f and equations (3.5) - (3.11). In this section we use some of these equations to obtain a functor from a coslice category of $\mathbb{C}$ to $\mathbb{C}$ and relate the remaining equations to this functor.

For $x \in \mathbb{C}$ we use equations (3.8) and (3.10) to *define* a functor $R_x \colon (x \downarrow \mathbb{C}) \longrightarrow \mathbb{C}$ sending an object $x \xrightarrow{f} y$ to $r(f)$ and an arrow $(f, y) \xrightarrow{g} (gf, z)$ to $r(f) \xrightarrow{g^f} r(gf)$. When it is clear in context we write that on the objects $R_x(f) = r(f)$ and on the arrows $R_x(g) = g^f$. We check that we do have a functor.

We have by definition that $R_x(hg) = (hg)^f$ and $R_x(h)R_x(g) = h^{gf}g^f$ and by (3.10) these agree so that R_x preserves composition. Similarly by (3.8), R_x preserves identities and so is a functor.

We now express equation (3.11) in terms of the functor R_x. However for the benefit of the reader we will explicitly describe the functor $(f \downarrow R_x f) : (f \downarrow (x \downarrow \mathbb{C})) \longrightarrow (R_x f \downarrow \mathbb{C})$ which is defined on objects by $f \xrightarrow{g} gf \longmapsto r(f) \xrightarrow{g^f} r(gf)$ and on arrows by

$$\begin{tikzcd} f \arrow[d, "g"'] \arrow[dr, "hg"] & \\ gf \arrow[r, "h"'] & hgf \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} r(f) \arrow[d, "g^f"'] \arrow[dr, "h^{gf}g^f = (hg)^f"] & \\ r(gf) \arrow[r, "h^{gf}"'] & r(hgf) \end{tikzcd}$$

The above remark allows us to conclude that equation (3.11) asserts that the following diagram commutes (it agrees on objects since $r(g^f) = r(g)$).

$$\begin{tikzcd} (y \downarrow \mathbb{C}) \arrow[rr, "R_y"] & & \mathbb{C} \\ (f \downarrow (x \downarrow \mathbb{C})) \arrow[u, "(f \downarrow Cod_x)"] \arrow[rr, "(f \downarrow R_x f)"'] & & (R_x f \downarrow \mathbb{C}) \arrow[u, "R_{R_x f}"'] \end{tikzcd} \tag{3.12}$$

In the following section we consider the remaining structure involving U, j, and ψ.

3.3.3 The Function E

We *define* a function E on the set of objects of the category $\mathbb{C}$ by $E(x) = r(1_x)$. Using (3.8) and $r(g^f) = r(g)$ (for a composable pair of morphisms), we note that if $x \xrightarrow{f} y$ then $E(r(f)) = E(y)$. Taking $f = 1_x$ we find that $E(E(x)) = E(r(1_x)) = E(x)$, so E is idempotent.

From equation (3.5), $\psi j = 1$, and equation (3.6), $j\psi_x = r(1_x)$, we can define U, j, and ψ as a splitting of E. So in terms of the functor R_x we have $E(x) = R_x(1_x)$ for each object x in the category $\mathbb{C}$. With this notation, equation (3.9) then asserts that the following diagram commutes on the objects of the respective categories:

$$\begin{tikzcd} Ob(x \downarrow \mathbb{C}) \arrow[r, "R_x"] \arrow[d, "Cod_x"'] & Ob(\mathbb{C}) \arrow[d, "E"] \\ Ob(\mathbb{C}) \arrow[r, "E"'] & Ob(\mathbb{C}) \end{tikzcd} \tag{3.13}$$

Following an object $x \xrightarrow{f} y$ of $(x \downarrow \mathbb{C})$ around (3.13) then asserts in terms of the functor R_x that $R_y(1_y) = R_{R_xf}(1_{R_xf})$ and as R_x is a functor we also have $R_{R_xf}(1_{R_xf}) = R_{R_xf}(R_x(1_y))$.

However if we follow the object $y \xrightarrow{1_y} y$ of $(y \downarrow \mathbb{C})$ around (3.12) (really we follow $f \xrightarrow{1_y} 1_y f$ of $(f \downarrow (x \downarrow \mathbb{C}))$ around (3.12)) we get that $R_y(1_y) = R_{R_xf}(R_x(1_y))$. So we have shown:

Lemma 3.3.1. *If* (3.12) *holds then so does* (3.13).

We now consider the remaining equation (3.7) in terms of the functor R_x. It is the statement that if for $j(u) \xrightarrow{f} y \xrightarrow{g} z$ then $g^f = g$.

As $\psi j = 1$ it can be shown that $x = j\psi x$ if and only if there exist a u such that $x = ju$. So for the u where $x = ju$ then $x = E(x) = R_x(1_x)$ (We could now define U to be those x for which $x = R_x(1_x)$). So we conclude that (3.7) is the statement that if $x = R_x(1_x)$ then $R_x = \mathrm{Cod}_x$.

Conclusion: A skew monoidale C in Span amounts to a category $\mathbb{C}$ with

(a) a functor $R_x \colon (x \downarrow \mathbb{C}) \longrightarrow \mathbb{C}$ for each x in $\mathbb{C}$.

(b) if $x = R_x(1_x)$ then $R_x = \mathrm{Cod}_x$.

(c) R_x satisfies (3.12), that is, for an arrow $x \xrightarrow{f} y$ in $\mathbb{C}$ the following commutes

$$\begin{tikzcd}
(y \downarrow \mathbb{C}) \arrow[r, "R_y"] & \mathbb{C} \\
(f \downarrow (x \downarrow \mathbb{C})) \arrow[u, "(f\downarrow Cod_x)"] \arrow[r, "(f\downarrow R_x f)"'] & (R_x f \downarrow \mathbb{C}) \arrow[u, "R_{R_x f}"']
\end{tikzcd}$$

Note: For each $x \in C$, the case when $j = 1$ (equivalently, j is surjective) corresponds to $R_x = \mathrm{Cod}_x$.

3.3.4 The Simplicial category and the Decalage Functor

We recall some standard facts about the simplex category $\mathbf{\Delta}$, before using it in our characterisation. There are many references for this section we use [19] and [9].

The simplicial category $\mathbf{\Delta}$ has as objects the finite ordinals $\mathbf{n} = \{0, 1, \ldots, n-1\}$ and morphisms the order-preserving functions $\xi : \mathbf{m} \longrightarrow \mathbf{n}$ with composition that of functions; the composite of order preserving functions is again order preserving.

If $0 \leq i \leq n$, we write $\delta_i \colon \mathbf{n} \to \mathbf{n{+}1}$ for the injective order-preserving function where $\delta_i(k)$ is equal to k if $k < i$ and $k+1$ otherwise. Similarly, if $0 \leq i \leq n-1$, we write $\sigma_i \colon \mathbf{n{+}1} \to \mathbf{n}$ for the order-preserving surjective function where $\sigma_i(k)$ is equal to k if $k \leq i$ and $k-1$ otherwise. We call these maps coface and codegeneracy maps respectively and they satisfy the well known simplicial identities which allow for a presentation of $\mathbf{\Delta}$ with the δ_i and σ_i as its generators and the simplicial identities as its relations, see [19]. A simplicial set is a contravariant functor from $\mathbf{\Delta}$ to $\mathbf{Set}$. The category $\mathbf{Simp}$ of simplicial sets and simplicial maps between them is defined to be the functor category $[\mathbf{\Delta}^{\mathbf{op}}, \mathbf{Set}]$. For a functor $S \colon \mathbf{\Delta}^{\mathbf{op}} \longrightarrow \mathbf{Set}$ we write S_n for $S(\mathbf{n})$. It can be shown that the data for a simplicial set can be specified by the sets S_n and maps $d_i \colon S_n \longrightarrow S_{n-1}$ and $s_i \colon S_n \longrightarrow S_{n-1}$ where for $0 \leq i \leq n$ we define d_i as $S\delta_i$ and s_i as $S\sigma_i$. We call these face and degeneracy maps and they satisfy relations dual to those in $\mathbf{\Delta}$, and they allow us to display a simplicial set S by

$$
\begin{tikzcd}
S : & \cdots\cdots S_n \arrow[r, bend left=70, "d_n"] \arrow[r, bend left=20, "d_0"] & S_{n-1} \arrow[l, bend left=20, "s_{n-1}"] \arrow[l, bend left=70, "s_0"] \cdots\cdots & S_2 \arrow[r, bend left=60, "d_2"] \arrow[r, bend left=40, "d_1"] \arrow[r, bend left=20, "d_0"] & S_1 \arrow[l, bend left=20, "s_1"] \arrow[l, bend left=50, "s_0"] \arrow[r, bend left=50, "d_1"] \arrow[r, bend left=20, "d_0"] & S_0 \arrow[l, bend left=20, "s_0"]
\end{tikzcd}
$$

For a simplicial set S we consider the shift or decalage functor $\mathrm{Dec} \colon \mathbf{Simp} \longrightarrow \mathbf{Simp}$ which removes the 0-th face and degeneracy maps, shifts dimension so that

$(\mathrm{Dec}(S))_n = S_{n+1}$ and shifts indices on the remaining face and degeneracy maps down by 1 so that $d_i \colon (\mathrm{Dec}(S))_n \longrightarrow (\mathrm{Dec}(S))_{n-1}$ is $d_{i+1} \colon S_{n+1} \longrightarrow S_n$ and $s_i \colon (\mathrm{Dec}(S))_n \longrightarrow (\mathrm{Dec}(S))_{n+1}$ is $s_{i+1} \colon S_{n+1} \longrightarrow S_{n+2}$. We depict this as

$$\begin{tikzcd}
\mathrm{Dec}(S): & \cdots\cdots S_{n+1} \arrow[r, bend left=20, "d_1"] \arrow[r, bend left=80, "d_{n+1}"] & S_n \arrow[l, bend left=20, "s_n"] \arrow[l, bend left=80, "s_1"] & \cdots\cdots S_3 \arrow[r, bend left=20, "d_1"] \arrow[r, bend left=45, "d_2"] \arrow[r, bend left=70, "d_3"] & S_2 \arrow[l, bend left=20, "s_2"] \arrow[l, bend left=45, "s_1"] \arrow[r, bend left=20, "d_1"] \arrow[r, bend left=50, "d_2"] & S_1 \arrow[l, bend left=20, "s_1"]
\end{tikzcd}$$

Given a category C we can form the nerve $N(C)$ of C, it is the well known simplicial set where the face and degeneracy maps are those given in [19] and where

$$\begin{aligned}
N(C)_0 &= \text{set of objects in } C \\
N(C)_1 &= \text{set of morphisms in } C \\
N(C)_2 &= \text{set of composable pairs of morphisms in } C \\
&\vdots \\
N(C)_n &= \text{set of composable } n\text{-tuples of morphisms in } C.
\end{aligned}$$

With the above discussion in mind we see that if C is a category then so is $\mathrm{Dec}(C)$ where

$$\begin{aligned}
\mathrm{Dec}(C)_0 &= \text{set of morphisms in } C \\
\mathrm{Dec}(C)_1 &= \text{set of composable pairs of morphisms in } C \\
&\vdots \\
\mathrm{Dec}(C)_n &= \text{set of composable } (n+1)\text{-tuples of morphisms in } C.
\end{aligned}$$

Recall that in the category **Cat** of small categories and functors, the coproduct of a family of categories is their disjoint union. For I a set and $(C_i)_{i \in I}$ a family of objects in **Cat** we write $\coprod_{i \in I} C_i$ for the coproduct of the family $(C_i)_{i \in I}$. Now with this notation and from the functors Cod_x we can form a functor from $\coprod_{x \in C} (x \downarrow C)$ to C which we denote by Cod.

Having described above what the functor Dec does on objects of **Cat** we notice for a category C, that $\mathrm{Dec}(C) = \coprod_{x \in C}(x \downarrow C)$. So to complete this (brief) description of Dec as an endofunctor from **Cat** we need to describe what it does on arrows of **Cat**. Let $F\colon X \longrightarrow C$ be a functor where X and C are categories. As we need a functor from a coproduct in **Cat**, it is sufficient, for each $x \in X$, to specify a functor from $(x \downarrow X)$ to $\mathrm{Dec}(C)$ where $\mathrm{Dec}(C) = \coprod_{c \in C}(c \downarrow C)$. We define the functor $\mathrm{Dec}(F)_x\colon (x \downarrow X) \longrightarrow \mathrm{Dec}(C)$ by the following composite

$$(x \downarrow X) \xrightarrow{(x \downarrow F)} (F(x) \downarrow C) \xrightarrow{\textit{inclusion}} \mathrm{Dec}(C)$$

So we have a functor $\mathrm{Dec}(F)\colon \mathrm{Dec}(X) \longrightarrow \mathrm{Dec}(C)$.

Using these constructions we can rewrite the previous description of a skew monoidale in Span as:

Conclusion: A skew monoidale C in Span amounts to a category $\mathbb{C}$ with

(a) a functor $R\colon \mathrm{Dec}(\mathbb{C}) \longrightarrow \mathbb{C}$, where

(b) R makes the following diagram commute

$$\begin{tikzcd}
\mathrm{Dec}(\mathbb{C}) \arrow[r, "R"] & \mathbb{C} \\
\mathrm{Dec}(\mathrm{Dec}(\mathbb{C})) \arrow[u, "\mathrm{Dec}(\mathrm{Cod})"] \arrow[r, "\mathrm{Dec}(R)"'] & \mathrm{Dec}(\mathbb{C}) \arrow[u, "R"']
\end{tikzcd}$$

(c) such that, if $x = R(1_x)$ then $R_x = \mathrm{Cod}_x$.

Note that when starting with just a category then $R = Cod$.

3.4 An Example

In this section we denote by (M, μ, η) or just M a monoid in the monoidal category $(\mathbf{Set}, \times, 1)$ where the tensor product is the cartesian product $\times$ and $1 = \{\star\}$ denotes a one point set as its unit. Here the two arrows μ and η in **Set** satisfy the usual equations (see [19]). For $\mu\colon M \times M \longrightarrow M$ and for $a, b \in M$ we write $\mu(a, b) = a.b$ and write for $\eta(\{\star\}) = 1_M$, we sometimes just write $\eta(\{\star\}) = 1$ where it should be clear in context what 1 represents.

We recall the embedding $(-)_\star\colon \mathbf{Set} \longrightarrow \mathrm{Span}$ which is the identity on objects and assigns to the morphism $f\colon A \longrightarrow B$ the following span

$$f_\star = \begin{tikzcd} & A \arrow[dl, "1_A"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

In fact, this is a strong monoidal pseudofunctor and as a consequence sends monoids in **Set** to monoidales in Span. We can therefore consider a monoid (M, μ, η) in **Set** as a (skew) monoidale in Span.

The 1-cell $p\colon C \times C \longrightarrow\!\!\!\!\!\mapsto C$ for a skew monoidale in Span is given by

$$\begin{tikzcd} & M \times M \arrow[dl, "(\pi_1,\pi_2)"'] \arrow[dr, "\mu"] & \\ M \times M & & M \end{tikzcd}$$

where $\pi_i\colon M \times M \longrightarrow M$ is defined by $\pi_i(m_1, m_2) = m_i$ for i=1,2 and $m_1, m_2 \in M$.

The 1-cell $j\colon 1 \longrightarrow\!\!\!\!\!\mapsto C$ for a skew monoidale in Span is given by

$$\begin{tikzcd} & 1 \arrow[dl, "1"'] \arrow[dr, "\eta"] & \\ 1 & & M \end{tikzcd}$$

With these choices for p and j, the 2-cell $\rho\colon 1 \Longrightarrow p(1 \times j)$ for this skew monoidale is given by the following diagram

$$
\begin{tikzcd}
 & & M \arrow[dddll, bend right, "1"'] \arrow[ddl, bend right, "{(1,!)}"'] \arrow[d, Rightarrow, "\rho"] \arrow[ddr, bend left, "{(1,\eta)}"] \arrow[dddrr, bend left, "1"] & & \\
 & & M \times 1 \arrow[dl, "{(\pi_1,\pi_2)}"'] \arrow[dr, "1\times\eta"] & & \\
 & M \times 1 \arrow[dl, "\pi_1"'] \arrow[dr, "1\times\eta"] & & M \times M \arrow[dl, "{(\pi_1,\pi_2)}"'] \arrow[dr, "\mu"] & \\
M & & M \times M & & M
\end{tikzcd}
$$

and the 2-cell $\alpha\colon p(p \times 1) \Longrightarrow p(1 \times p)$ is given by

$$
\begin{tikzcd}
 & & M \times M \times M \arrow[dddll, bend right, "{(\pi_1,\pi_2,\pi_3)}"'] \arrow[ddl, bend right, "{(\pi_1,\pi_{23})}"'] \arrow[d, Rightarrow, "\alpha=1"] \arrow[ddr, bend left, "1\times\mu"] \arrow[dddrr, bend left, "\mu(\mu\times 1)"] & & \\
 & & M \times M \times M \arrow[dl, "{(\pi_1,\pi_{23})}"'] \arrow[dr, "1\times\mu"] & & \\
 & M \times M \times M \arrow[dl, "{1\times(\pi_1,\pi_2)}"'] \arrow[dr, "1\times\mu"] & & M \times M \arrow[dl, "{(\pi_1,\pi_2)}"'] \arrow[dr, "\mu"] & \\
M \times M \times M & & M \times M & & M
\end{tikzcd}
$$

where $\pi_{23}\colon M \times M \times M \longrightarrow M \times M$ is defined as $\pi_{23}(m_1, m_2, m_3) = (m_2, m_3)$. We will now describe the resulting monoidale in terms of the characterisation of skew monoidales in Span given in the previous sections.

So with these choices for p and j, M is a category whose *objects* are the elements of the set M and whose *arrows* are the pairs $(a, b) \in M \times M$ with source $\pi_1(a, b) = a$ and target $\mu(a, b) = a.b$ which we represent as $a \xrightarrow{b} a.b$. The composition of arrows in M and the functor $R\colon \mathrm{Dec}(M) \longrightarrow M$ are both defined by the 2-cell $\alpha\colon p(p \times 1) \Longrightarrow p(1 \times p)$. The composition of arrows in M is then given by $(a, b, c) \xmapsto{1\times\mu} (a, b.c)$ for $(a, b, c) \in M \times M \times M$ and so the composite $a \xrightarrow{b} a.b \xrightarrow{c} (a.b).c$ is given by $a \xrightarrow{b.c} a.(b.c)$.

For the functor $R\colon \operatorname{Dec}(M) \longrightarrow M$ and the p and j chosen from M we have on the objects of $\operatorname{Dec}(M)$ that $R((a,b)) = \pi_2(a,b) = b$ or $R(\ a \xrightarrow{b} a.b\ ) = b$ and on the arrows of $\operatorname{Dec}(M)$ we have that $R((a,b,c)) = \pi_{23}(a,b,c) = (b,c)$ or $R(\ a \xrightarrow{b} a.b \xrightarrow{c} (a.b).c\ ) = b \xrightarrow{c} b.c$.

The identity arrow for the category M exists via the 2-cell $\rho\colon 1 \Longrightarrow p(1\times j)$ and is represented as $a \xrightarrow{1} a.1 = a$.

Remark 3.4.1. The monoids in **Set** constitute a category **Mon** and the above example defines the object part of a functor $T\colon \mathbf{Mon} \longrightarrow \mathbf{Cat}$. For a morphism of monoids $f\colon (M,\mu,\eta) \longrightarrow (M',\mu',\eta')$ the induced functor $TM \longrightarrow TM'$ sends an object m to fm and a morphism (m,n) to (fm,fn).

Remark 3.4.2. Considering a category as a partial monoid and using the notation of [19]; we can instead start with a (small) category C where O, A and $A\times_O A$ respectively denotes the sets of objects, arrows and composable arrows of C.

The tensor for a monoidale in Span is given by

$$\begin{tikzcd} & A\times_O A \arrow[dl, "(\pi_1,\pi_2)"'] \arrow[dr, "comp"] & \\ A\times A & & A \end{tikzcd}$$

The unit for that monoidale in Span is given by

$$\begin{tikzcd} & O \arrow[dl, "!"'] \arrow[dr, "id"] & \\ 1 & & A \end{tikzcd}$$

Remark 3.4.3. The following is a non-trivial example given by Stephen Lack at a talk to the Australian Category Seminar [16].

Batanin and Markl in [3] define a strict operadic category as a category $\mathbb{C}$ equipped with a cardinality functor into **sFSet**, the skeletal category of finite sets, where each connected component of $\mathbb{C}$ has a chosen terminal object. One of the axioms for a strict operadic category requires the existence of a family of functors from a slice category of $\mathbb{C}$ into $\mathbb{C}$, for the chosen terminal object this is required to be the domain functor. Lack has shown that strict operadic categories are equivalent to left normal skew monoidales in $\mathrm{Span}([\mathbb{N}, \mathbf{Set}])$. Here $\mathbb{N}$ denotes the set of natural numbers, seen as a discrete category, and the functor category $[\mathbb{N}, \mathbf{Set}]$ is given a monoidal structure via Day's convolution.

Appendix A

Independence of the Axioms

In this section we show that the five axioms for a skew monoidal category, given by equations (2.1), (2.2), (2.3), (2.4), and (2.5), are independent. The underlying category we use is **Set** where the cartesian product between two sets is denoted by $\times$; we often identify the cartesian product of a one-point set with a set as the set itself and $X \times Y$ with $Y \times X$ in what follows.

For a set M, define a tensor product on **Set** by $X \otimes Y = M \times X \times Y$; this gives a functor $\mathbf{Set} \times \mathbf{Set} \xrightarrow{\otimes} \mathbf{Set}$. If M has a product $M \times M \xrightarrow{\mu} M$, denoted by $\mu(m,n) = m.n$, then there is a natural transformation $\alpha\colon M \times M \times X \times Y \times Z \longrightarrow M \times X \times M \times Y \times Z$ given by sending (m,n,x,y,z) to $(m.n,x,m,y,z)$. Let I be a one-point set, and $1 \in M$. The map $\lambda\colon I \otimes X (= M \times X) \longrightarrow X$ defined by sending (m,x) to x and the map $\rho\colon X \longrightarrow X \otimes I (= M \times X)$ defined by sending x to $(x,1)$ are both natural transformations. With these maps, equations (2.2) and (2.5) are automatic, while equation (2.1) asks that the product on M is associative, equation (2.3) asks that $1 \in M$ is a right identity on M, and equation (2.4) asks that $1 \in M$ is a left identity on M. (These maps are based on the constructions in the first section of [17].)

We take for M the following three sets. The M defined by the table on the left has a left and right identity but is not associative, so equation (2.1) does not hold but the other four equations do.

.	1	a	b
1	1	a	b
a	a	1	b
b	b	a	1

.	1	a
1	1	a
a	1	a

.	1	a
1	1	1
a	a	a

The M defined by the table in the middle has no right identity, but has a left identity and is associative. In this case, equation (2.3) does not hold but the other four equations do. The M defined by the table on the right has no left identity, but has a right identity and is associative. In this case, the equation (2.4) does not hold but the other four equations do.

Thus each of the equations (2.1), (2.3) and (2.4) is independent of the remaining four equations. By reversing the tensor, direction of arrows and the order of composition we notice that equation (2.2) and equation (2.4) are dual, so statements such as independence holds for one if and only if it holds for the other. Thus independence of equation (2.2) follows from the independence of equation (2.4).

This leaves equation (2.5), for which we take the tensor product to be the cartesian product, so $X \otimes Y = X \times Y$. The map $\alpha\colon (X \times Y) \times Z \longrightarrow X \times (Y \times Z)$ is the usual associativity isomorphism $(X \times Y) \times Z \cong X \times (Y \times Z)$, and I is given by $\{a, b\}$. The map $\lambda\colon I \times X \longrightarrow X$ is defined by sending (i, x) to x and the map $\rho\colon X \longrightarrow X \times I$ defined by sending x to (x, a) are natural transformations. In this case, equation (2.5) asks for the elements of I to be identical which is not the case here, so equation (2.5) is not satisfied but it is easy to see that the other four equations hold.

With these four examples and duality we have shown:

Proposition A.0.4. *The five axioms for a skew monoidal category are independent.*

Appendix B

Gray Monoids

In this section, following [7], we explicitly record the definition of a Gray monoid.

A *Gray monoid* $\mathbb{M}$ is a 2-category equipped with the following:

(1) an object I;

(2) for all objects A, two 2-functors $L_A = A \otimes - \colon \mathbb{M} \longrightarrow \mathbb{M}$ and $R_A = - \otimes A \colon \mathbb{M} \longrightarrow \mathbb{M}$ satisfying the following equations for all objects A and B:

$L_I = R_I = 1_{\mathbb{M}}$, $R_B L_A = L_A R_B$, $L_A(B) = R_B(A)$ which allows us to *define* $A \otimes B$ as $L_A(B)$, $L_{A\otimes B} = L_A L_B$, and $R_{A\otimes B} = R_B R_A$.

(3) for all arrows $f \colon A \longrightarrow A'$, $g \colon B \longrightarrow B'$, an invertible 2-cell $c_{f,g}$

$$\begin{tikzcd}
A \otimes B \arrow[r, "A\otimes g"] \arrow[d, "f\otimes B"'] \arrow[dr, phantom, "\overset{\Longrightarrow}{c_{f,g}}"] & A \otimes B' \arrow[d, "f\otimes B'"] \\
A' \otimes B \arrow[r, "A'\otimes g"'] & A' \otimes B'
\end{tikzcd}$$

satisfying the following axioms:

(a) if both f and g are identities then $c_{f,g}$ is an identity,

(b) for all arrows $f \colon A \longrightarrow A'$, $g \colon B \longrightarrow B'$, $h \colon C \longrightarrow C'$, the following equations hold:

$$\mathrm{A}\otimes(c_{g,h}) = c_{A\otimes g,h},\ c_{f,g} \otimes C = c_{f,g\otimes C}, \text{ and } c_{f,B\otimes h} = c_{f\otimes B,h}. \tag{B.1}$$

(c) for all arrows $f, h\colon A \longrightarrow A'$, $g, k\colon B \longrightarrow B'$, and 2-cells $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\alpha"] & A' \end{tikzcd}$, $\begin{tikzcd} B \arrow[r, bend left, "g"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & B' \end{tikzcd}$

$$
\begin{tikzcd}
A \otimes B \arrow[rr, bend left, "A\otimes k"] \arrow[rr, "A\otimes g"'] \arrow[rr, phantom, bend left=20, "\Uparrow A\otimes\beta"] \arrow[d, "f\otimes B"'] \arrow[drr, phantom, "\overset{\Longrightarrow}{c_{f,g}}"] & & A \otimes B' \arrow[d, bend right, "f\otimes B'"'] \arrow[d, bend left, "h\otimes B'"] \arrow[d, phantom, "\overset{\Longrightarrow}{\alpha\otimes B'}"] \\
A' \otimes B \arrow[rr, "A'\otimes g"'] & & A' \otimes B'
\end{tikzcd}
=
\begin{tikzcd}
A \otimes B \arrow[rr, "A\otimes k"] \arrow[d, bend right, "f\otimes B"'] \arrow[d, bend left, "h\otimes B"] \arrow[d, phantom, "\overset{\alpha\otimes B}{\Longrightarrow}"] \arrow[drr, phantom, "\overset{c_{f,g}}{\Longrightarrow}"] & & A \otimes B' \arrow[d, "h\otimes B'"] \\
A' \otimes B \arrow[rr, bend left, "A'\otimes k"] \arrow[rr, bend right, "A'\otimes g"'] \arrow[rr, phantom, "\Uparrow A'\otimes\beta"] & & A' \otimes B'
\end{tikzcd}
\tag{B.2}
$$

(d) for all arrows $f\colon A \longrightarrow A'$, $g\colon B \longrightarrow B'$, $f'\colon A' \longrightarrow A''$, $g'\colon B' \longrightarrow B''$,

$$
\begin{tikzcd}
A \otimes B \arrow[r, "A\otimes g"] \arrow[d, "f\otimes B"'] \arrow[dr, phantom, "\overset{\Longrightarrow}{c_{f,g}}"] & A \otimes B' \arrow[r, "A\otimes g'"'] \arrow[d, "f\otimes B'"] \arrow[dr, phantom, "\overset{\Longrightarrow}{c_{f,g'}}"] & A \otimes B'' \arrow[d, "f\otimes B''"] \\
A' \otimes B \arrow[r, "A'\otimes g"'] \arrow[d, "f'\otimes B"'] \arrow[dr, phantom, "\overset{\Longrightarrow}{c_{f',g}}"] & A' \otimes B' \arrow[r, "A'\otimes g'"'] \arrow[d, "f'\otimes B'"] \arrow[dr, phantom, "\overset{\Longrightarrow}{c_{f',g'}}"] & A' \otimes B'' \arrow[d, "f'\otimes B''"] \\
A'' \otimes B \arrow[r, "A''\otimes g"'] & A'' \otimes B' \arrow[r, "A''\otimes g'"'] & A'' \otimes B''
\end{tikzcd}
=
\begin{tikzcd}
A \otimes B \arrow[r, "A\otimes g'g"] \arrow[d, "f'f\otimes B"'] \arrow[dr, phantom, "\overset{\Longrightarrow}{c_{f'f,g'g}}"] & A \otimes B'' \arrow[d, "f'f\otimes B'"] \\
A'' \otimes B \arrow[r, "A''\otimes g'g"'] & A'' \otimes B''
\end{tikzcd}
$$

Identifying $f \otimes g$ as $R'_B(f) \circ L_A(g)$ and $\alpha \otimes \beta$ as $R'_B(\alpha) \circ L_A(\beta)$ makes each Gray monoid a monoidal bicategory.

The coherence theorem in [10] implies that

Theorem B.0.5. *Every monoidal bicategory is monoidally biequivalent to a Gray monoid.*

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