Local Reflections between Relations, Spans and Polynomials

By

Charles R. Walker

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University
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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Charles R. Walker

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Abstract

Given a locally cartesian closed regular category $\mathcal{E}$, we may form the bicategories of relations, spans, and polynomials. We show that for each hom-category, relations are a reflective subcategory of spans, and spans are a coreflective subcategory of polynomials (with cartesian 2-cells). We then use these local reflections and coreflections to derive the universal property of relations from that of spans, and construct a right adjoint to the inclusion of spans into polynomials in the 2-category of bicategories, lax functors and icons. Moreover, we show that this right adjoint becomes a pseudofunctor if we restrict ourselves to polynomials for which the middle map is a monomorphism, or alternatively if we restrict ourselves to polynomials for which this map is a regular epimorphism.

Contents

1

Introduction

Hermida showed in [1, Theorem A.2] that given a category $\mathcal{E}$ with pullbacks and a bicategory $\mathscr{C}$, giving a pseudofunctor $\mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ is equivalent to giving a pseudofunctor $\mathcal{E} \to \mathscr{C}$ which maps arrows into left adjoints and satisfies a Beck condition. If $\mathcal{E}$ is also regular, this result may be extended to the bicategory of relations $\mathbf{Rel}(\mathcal{E})$ by noticing that for every pair of objects $X, Y \in \mathcal{E}$ the hom-category $\mathbf{Rel}(\mathcal{E})(X, Y)$ is a reflective subcategory of the hom-category $\mathbf{Span}(\mathcal{E})(X, Y)$. The 2-category of bicategories, lax functors and icons introduced by Lack [2] provides a natural setting in which to consider these local reflections; in particular, these local reflections extend to an adjunction in this 2-category. In chapter 3 we introduce the theory of *locally reflective sub-bicategories* of which this is an example, and use this theory to deduce the universal property of relations from that of spans.

We will then extend this idea of considering local reflections between $\mathbf{Rel}(\mathcal{E})$ and $\mathbf{Span}(\mathcal{E})$ to considering local coreflections between $\mathbf{Span}(\mathcal{E})$ and $\mathbf{Poly}(\mathcal{E})$, where $\mathbf{Poly}(\mathcal{E})$ is the bicategory of polynomials with cartesian 2-cells. In order to do this we will need to first introduce a generalization of the standard pullback for a monomorphism p

$$\begin{tikzcd} E \arrow[r, equal] \arrow[d, equal] & E \arrow[d, tail, "p"] \\ E \arrow[r, tail, "p"'] & B \end{tikzcd} \tag{4.1}$$

by realizing that such a pullback is terminal among pullbacks for which the bottom arrow p pulls back into an identity, i.e. pullbacks of the form

$$\begin{tikzcd} E \arrow[r, equal] \arrow[d, "\ell"'] & E \arrow[d, "p\ell"] \\ E \arrow[r, tail, "p"'] & B \end{tikzcd} \qquad (4.2)$$

This universal property allows us to give a suitable generalization of pullbacks of the form (4.1) to the case when p is not necessarily a monomorphism. Throughout chapter 4 we discover many interesting properties of this construction, including results determining precisely when an arrow pulls back into an isomorphism or a monomorphism in a locally cartesian closed category.

Making use of these terminal pullbacks (which we will call *singleton fiber pullbacks*), we will define a lax functor $L : \mathbf{Poly}\,(\mathcal{E}) \to \mathbf{Span}\,(\mathcal{E})$ which exhibits $\mathbf{Span}\,(\mathcal{E})$ as a locally coreflective sub-bicategory of $\mathbf{Poly}\,(\mathcal{E})$. Moreover, we will show that if we restrict ourselves to polynomials of the form

$$\begin{tikzcd} & E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & \\ I & & & J \end{tikzcd}$$

for which p is a monomorphism, then this lax functor L becomes a pseudofunctor. Similarly, we show that if we restrict ourselves to polynomials for which p is a regular epimorphism (when $\mathcal{E}$ is a regular category) then L also reduces to a pseudofunctor.

The original research in this paper includes the definition and universal property of locally reflective sub-bicategories, the universal property of relations, and all of chapters 4 and 5 (aside from the basic theory of relations, spans and polynomials).

2
Background

Throughout category theory it is often prudent to study mathematical structures by analyzing the behavior of morphisms in or out of such structures, and in particular universal properties satisfied by these morphisms. For example a product $A \times B$ may be characterized by the property that giving a morphism into the product $A \times B$ is the same as giving morphisms into both A and B. In a similar fashion, we may also investigate universal properties satisfied by certain categories.

Given a suitable category $\mathcal{E}$, it is possible to construct new categories $\mathbf{Rel}(\mathcal{E})$, $\mathbf{Span}(\mathcal{E})$ and $\mathbf{Poly}(\mathcal{E})$ (which will be defined later). To better understand these constructions we would like to know how giving a functor out of $\mathbf{Rel}(\mathcal{E})$, $\mathbf{Span}(\mathcal{E})$ or $\mathbf{Poly}(\mathcal{E})$ relates to giving a functor out of $\mathcal{E}$. We should note here that $\mathbf{Rel}(\mathcal{E})$, $\mathbf{Span}(\mathcal{E})$ and $\mathbf{Poly}(\mathcal{E})$ are not actually categories, but a 2-dimensional version of categories known as bicategories; the corresponding 2-dimensional version of functor is a pseudofunctor.

2.1 Bicategories

Consider the category $\mathbf{Cat}$ of small categories and functors. We have the functors mapping between categories, but we could also add in natural transformations mapping between functors. We now have a 2-dimensional structure consisting of small categories, functors and natural transformations. Hence we may view $\mathbf{Cat}$ as a 2-dimensional category with the usual objects and morphisms as well as these "morphisms between the morphisms": the natural transformations. Adding in these natural transformations turns out to be very useful; in particular it allows us define adjoint functors.

$\mathbf{Cat}$ with these "2-morphisms" is an example of a 2-category. More generally, we may ask that composition of morphisms is associative only up to isomorphism; this gives the more general notion of a bicategory as defined by Bénabou [3].

Definition 1. A *bicategory* $\mathscr{C}$ consists of

- A collection of objects $\mathscr{C}_0$;
- For each pair of objects $X, Y \in \mathscr{C}_0$, a category $\mathscr{C}(X, Y)$, of which we call its objects 1-cells and morphisms 2-cells;
- For each object $X \in \mathscr{C}_0$, a functor $I_X : 1 \to \mathscr{C}(X, X)$ which picks out the identity 1-cell at X;
- For each triple of objects $X, Y, Z \in \mathscr{C}_0$, a functor $c_{X,Y,Z} : \mathscr{C}(Y, Z) \times \mathscr{C}(X, Y) \to \mathscr{C}(X, Z)$ called horizontal composition, whose action on objects and morphisms we write as $(g, f) \mapsto g \circ f$ and $(\alpha, \beta) \mapsto \alpha * \beta$ respectively;
- Natural isomorphisms

$$
\begin{aligned}
a &: c_{W,X,Z} \circ \left(c_{X,Y,Z} \times \mathrm{id}_{\mathscr{C}(W,X)}\right) \implies c_{W,Y,Z} \circ \left(\mathrm{id}_{\mathscr{C}(Y,Z)} \times c_{W,X,Y}\right);\\
\ell &: c_{W,X,X} \circ \left(I_X \times \mathrm{id}_{\mathscr{C}(W,X)}\right) \implies \mathrm{id}_{\mathscr{C}(W,X)};\\
r &: c_{W,W,X} \circ \left(\mathrm{id}_{\mathscr{C}(W,X)} \times I_W\right) \implies \mathrm{id}_{\mathscr{C}(W,X)};
\end{aligned}
$$

consisting of component 2-cells for every $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$

$$a_{h,g,f} : (hg)\, f \to h\,(gf)\,, \quad \ell_f : 1_W f \to f, \quad r_f : f 1_X \to f$$

known as the associators and left and right unitors respectively. Moreover, we require that the associators satisfy the pentagon identity expressed by the commutivity of the diagram:

$$
\begin{tikzcd}
 & (kh)\,(gf) \arrow[dl, "a_{k,h,gf}"'] & \\
k\,(h\,(gf)) & & ((kh)\,g)\,f \arrow[ul, "a_{kh,g,f}"'] \arrow[d, "a_{k,h,g}f"] \\
k\,((hg)\,f) \arrow[u, "ka_{h,g,f}"] & & (k\,(hg))\,f \arrow[ll, "a_{k,hg,f}"]
\end{tikzcd}
$$

and that the unitors render commutative the diagram:

$$\begin{tikzcd}
(g \circ 1) \circ f \arrow[r, "a_{g,1,f}"] \arrow[d, "r_g f"'] & g \circ (1 \circ f) \arrow[dl, "g\ell_f"] \\
g \circ f &
\end{tikzcd}$$

If the associators and unitors are identities, then we call $\mathscr{C}$ a 2-category.

Pseudofunctors

We all know that the structure preserving maps between categories are functors, but we will need a 2-dimensional analogue of functor to serve as our maps between bicategories. These are called pseudofunctors and consist of an action on objects, as well a functor on each hom-category. We only require that these pseudofunctors preserve identities and composition up to 2-isomorphism.

Definition 2. A *pseudofunctor* $F : \mathscr{C} \to \mathscr{D}$ consists of

- For each object $X \in \mathscr{C}$, an object $FX \in \mathscr{D}$;
- For each pair of objects $X, Y \in \mathscr{C}_0$, a functor $F_{X,Y} : \mathscr{C}(X, Y) \to \mathscr{D}(FX, FY)$;
- For each object $X \in \mathscr{C}_0$, a 2-isomorphism $\lambda_X : 1_{FX} \to F(1_X)$;
- For each triple of objects $X, Y, Z \in \mathscr{C}_0$ and morphisms $f : X \to Y$ and $g : Y \to Z$, a 2-isomorphism $\varphi_{g,f} : F(g)F(f) \to F(gf)$ natural in g and f.

Moreover, we require that the constraints make the associativity diagram

$$\begin{tikzcd}
F(h)(F(g)F(f)) \arrow[r, "F(h)\varphi_{g,f}"] & F(h)F(gf) \arrow[r, "\varphi_{h,gf}"] & F(h(gf)) \\
(F(h)F(g))F(f) \arrow[u, "a_{Fh,Fg,Ff}"] \arrow[r, "\varphi_{h,g}F(f)"'] & F(hg)F(f) \arrow[r, "\varphi_{hg,f}"'] & F((hg)f) \arrow[u, "F\left(a_{h,g,f}\right)"']
\end{tikzcd}$$

commute for composable morphisms f,g and h. We also ask that the identity constraints make the following diagrams commute:

$$\begin{tikzcd}
F(f)1_{FX} \arrow[r, "F(f)\lambda_X"] \arrow[d, "r_{Ff}"'] & F(f)F(1_X) \arrow[d, "\varphi_{f,1_X}"] \\
F(f) & F(f \circ 1_X) \arrow[l, "F\left(r_f\right)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{FY}F(f) \arrow[r, "\lambda_Y F(f)"] \arrow[d, "\ell_{Ff}"'] & F(1_Y)F(f) \arrow[d, "\varphi_{1_Y,f}"] \\
F(f) & F(1_Y \circ f) \arrow[l, "F\left(\ell_f\right)"]
\end{tikzcd}$$

More generally, if we no longer require that the constraints λ and φ are invertible then we have the definition of a *lax functor*. Furthermore, if we reverse the direction of these constraints we then have the notion of an *oplax functor*.

Icons

We now have our 2-dimensional versions of categories and functors, so what about natural transformations? Most authors use oplax transformations as the maps between pseudofunctors but this has a disadvantage: bicategories, pseudofunctors and oplax transformations don't form a bicategory. Fortunately, there is an alternative and simpler notion of map between pseudofunctor to use, known as icons [2]. These icons (which exist only between pseudofunctors which agree on objects) are simply families of natural transformations between those functors on the hom-categories which pseudofunctors consist of. The main advantage of this is that bicategories, pseudofunctors and icons form a 2-category (as shown in [2]) which will make our work considerably simpler.

Definition 3. Given two pseudofunctors $F, G : \mathscr{A} \to \mathscr{B}$ that agree on objects, an *icon* $\alpha : F \implies G$ is a family of natural transformations

$$\begin{tikzcd} \mathscr{A}(X,Y) \arrow[r, bend left, "F_{X,Y}"] \arrow[r, bend right, "G_{X,Y}"'] \arrow[r, phantom, "\Downarrow \alpha_{X,Y}"] & \mathscr{B}(FX, FY) \end{tikzcd}, \qquad X, Y \in \mathscr{A}_{\mathrm{ob}}$$

with components making the following diagrams commute:

$$\begin{tikzcd} F(g)F(f) \arrow[r, "\varphi_{g,f}"] \arrow[d, "\alpha_g * \alpha_f"'] & F(gf) \arrow[d, "\alpha_{gf}"] \\ G(g)G(f) \arrow[r, "\psi_{g,f}"'] & G(gf) \end{tikzcd} \qquad \begin{tikzcd} 1_{FX} \arrow[d, "\lambda_X"'] \arrow[dr, "\omega_X"] & \\ F1_X \arrow[r, "\alpha_{1_X}"'] & G1_X \end{tikzcd}$$

We will denote the 2-category of bicategories, pseudofunctors and icons as $\mathbf{Icon}$. We will denote the corresponding 2-categories with lax functors and oplax functors as $\mathbf{LaxIcon}$ and $\mathbf{OplaxIcon}$ respectively.[1]

Adjunctions

In the 2-category $\mathbf{Cat}$ we have the notion of an adjunction between two functors; we now generalize this to an arbitrary bicategory.

Definition 4. In a bicategory $\mathscr{C}$, an adjunction between two objects $A, B \in \mathscr{C}_0$ consists of two 1-cells $f : A \to B$ and $u : B \to A$, as well as two 2-cells $\varepsilon : fg \implies 1$ and $\eta : 1 \implies gf$ such that the triangle identities $1_f = \varepsilon f \circ f\eta$ and $1_u = u\varepsilon \circ \eta u$ are satisfied. In this case we say f is left adjoint to g; denoted $f \dashv g$.

[1]Due to length constraints, we will omit any discussion of small and large categories.

Mates

Definition 5. Let $\eta, \varepsilon : f \dashv u : B \to A$ and $\eta', \varepsilon' : f' \dashv u' : B \to A$ be two pairs of adjoint morphisms. We say that two 2-cells

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & A' \arrow[d, "f'"] \\ B \arrow[r, "h"'] & B' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[dr, phantom, "\Downarrow\beta"] & A' \\ B \arrow[r, "h"'] \arrow[u, "u"] & B' \arrow[u, "u'"'] \end{tikzcd}$$

are mates under the adjunctions if β is given by the pasting

$$\begin{tikzcd} & A \arrow[r, "g"] \arrow[d, "f\ \Downarrow\alpha", "\Downarrow\varepsilon"'] & A' \arrow[r, "1_{A'}"] \arrow[d, "f'\ \Downarrow\eta'"] & A' \\ B \arrow[ur, bend left, "u"] \arrow[r, "1_B"'] & B \arrow[r, "h"'] & B' \arrow[ur, bend right, "u'"'] & \end{tikzcd}$$

and consequently α is given by the pasting

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[dr, bend right, "f"'] & A \arrow[r, "g"] & A' \arrow[dr, bend left, "f'"] & \\ & B \arrow[r, "h"'] \arrow[u, "u\ \Downarrow\beta"', "\Downarrow\eta"] & B' \arrow[r, "1_{B'}"'] \arrow[u, "u'\ \Downarrow\varepsilon'"'] & B' \end{tikzcd}$$

It follows from the triangle identities that taking mates in this fashion defines a bijection between 2-cells $f'g \to hf$ and 2-cells $gu \to u'h$, which is functorial in a suitable sense: see [4].

2.2 Bicategory of spans

Before defining the bicategory $\mathbf{Poly}\,(\mathcal{E})$ we will study the simpler and more well known construction $\mathbf{Span}\,(\mathcal{E})$, which was introduced by Bénabou in 1967 [3].

Definition 6. Given a category $\mathcal{E}$ with pullbacks (equipped with a choice of pullback for each diagram of the form $X \to Y \leftarrow Z$), we may form a bicategory called $\mathbf{Span}\,(\mathcal{E})$ with objects those of $\mathcal{E}$, 1-cells $A \twoheadrightarrow B$ given by diagrams in $\mathcal{E}$ of the form

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A & & B \end{tikzcd}$$

composition of 1-cells given by taking the chosen pullback

$$\begin{tikzcd} & & X \times_B Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\ & X \arrow[dl, "p"'] \arrow[dr, "q"] & & Y \arrow[dl, "r"'] \arrow[dr, "s"] & \\ A & & B & & C \end{tikzcd}$$

and 2-cells α given by those morphisms between the vertices of two spans which yield commuting diagrams of the form

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "\alpha"] & \\ A & & B \\ & Y \arrow[ul, "p'"] \arrow[ur, "q'"'] & \end{tikzcd}$$

Identities are given by identity spans $X \xleftarrow{1_X} X \xrightarrow{1_X} X$, and composition extends to 2-cells by using the universal property of pullbacks. The uniqueness of the limit of a diagram

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "q"] & & Y \arrow[dl, "r"'] \arrow[dr, "s"] & & Z \arrow[dl, "t"'] \arrow[dr, "u"] & \\ A & & B & & C & & D \end{tikzcd}$$

up to isomorphism yields the appropriate associators, making $\mathbf{Span}\left(\mathcal{E}\right)$ into a bicategory.

Spans as linear maps

A morphism in $\mathbf{Span}\left(\mathbf{Set}\right)$

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ I & & J \end{tikzcd}$$

may be represented by the matrix of sets defined componentwise by

$$\mathbf{B}_{i,j} = \{x \in X : p\left(x\right) = i,\ q\left(x\right) = j\}$$

which induces the "linear mapping"

$$\mathbf{B} : \mathbf{Set}^{I} \to \mathbf{Set}^{J} : \mathbf{v} = \left(v_{1,i} : i \in I\right) \mapsto \mathbf{vB} = \left(\sum_{i \in I} v_{1,i} \times \mathbf{B}_{i,j} : j \in J\right).$$

Composition of spans then corresponds to "matrix multiplication".

Embedding into spans

A category $\mathcal{E}$ with pullbacks (viewed as a locally discrete bicategory, i.e. a bicategory with all 2-cells being identities) may be embedded into the bicategory $\mathbf{Span}\left(\mathcal{E}\right)$ via the pseudofunctor $\left(-\right)_{*} : \mathcal{E} \to \mathbf{Span}\left(\mathcal{E}\right)$ which sends an object of $\mathcal{E}$ to itself, and is defined on morphisms by the assignment

$$A \xrightarrow{f} B \qquad \mapsto \qquad \begin{tikzcd} & A \arrow[dl, "1_A"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

It is not hard to show that this span has right adjoint $B \xleftarrow{f} A \xrightarrow{1_A} A$. We will often denote these left and right adjoints by f_{*} and f^{*} respectively.

Sinister and Beck functors

A pseudofunctor between bicategories which sends arrows to left adjoints is called *sinister*. We will write Ff^* for the chosen right adjoint of each Ff. Given a sinister pseudofunctor $F : \mathcal{E} \to \mathscr{C}$ (where $\mathcal{E}$ is a locally discrete 2-category) and a pullback square in $\mathcal{E}$

$$\begin{tikzcd} X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X_1 \arrow[d, "x_1"] \\ X_2 \arrow[r, "x_2"'] & X_0 \end{tikzcd} \qquad (\wr)$$

applying F and composing with pseudofunctoriality constraints yields an invertible 2-cell as on the left, and then taking mates gives a 2-cell as on the right:

$$\begin{tikzcd} FX \arrow[r, "Fp_1"] \arrow[d, "Fp_2"'] \arrow[dr, phantom, "\Downarrow\xi"] & FX_1 \arrow[d, "Fx_1"] \\ FX_2 \arrow[r, "Fx_2"'] & FX_0 \end{tikzcd} \qquad \begin{tikzcd} FX \arrow[r, "Fp_1"] \arrow[dr, phantom, "\Downarrow\xi^*"] & FX_1 \\ FX_2 \arrow[u, "Fp_2^*"] \arrow[r, "Fx_2"'] & FX_0 \arrow[u, "Fx_1^*"'] \end{tikzcd}$$

We say that a sinister pseudofunctor F satisfies the *Beck condition* if every such ξ^* is invertible.

An example of a pseudofunctor which is sinister and satisfies the Beck condition is the embedding $(-)_* : \mathcal{E} \to \mathbf{Span}\,(\mathcal{E})$; we have already seen that the embedding maps arrows into left adjoints, and we will now check that the embedding $(-)_*$ satisfies the Beck condition.

Beck-Chevalley isomorphisms

Given a pullback square of the form $(\wr)$, the composites in $\mathbf{Span}\,(\mathcal{E})$

$$\begin{tikzcd}[column sep=small] & X_2 \arrow[dl, "1_{X_2}"'] \arrow[dr, "x_2"] & & X_1 \arrow[dl, "x_1"'] \arrow[dr, "1_{X_1}"] & \\ X_2 & & X_0 & & X_1 \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & X \arrow[dl, "p_2"'] \arrow[dr, "1_X"] & & X \arrow[dl, "1_X"'] \arrow[dr, "p_1"] & \\ X_2 & & X & & X_1 \end{tikzcd}$$

coincide up to an invertible 2-cell (due to the uniqueness of pullbacks up to isomorphism). This shows that the embedding $(-)_*$ satisfies the Beck condition.

There is another Beck functor to consider here, namely the pseudofunctor $\Sigma : \mathcal{E} \to \mathbf{Cat}$ which sends an object $X \in \mathcal{E}$ to its slice category $\mathcal{E}/X$ and sends a morphism $f : A \to B$ to the "compose with f" functor Σ_f. Note that since $\mathcal{E}$ has pullbacks, each Σ_f has a right adjoint: the "pull back along f" functor denoted by Δ_f. To see that the pseudofunctor Σ satisfies the Beck condition, note that for every pullback square $(\wr)$ and map $h : A \to X_2$, we see via the diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\Delta_{p_2}(h)"'] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r, "p_2"] \arrow[d, "p_1"'] & \bullet \arrow[d, "x_2"] \\ \bullet \arrow[r, "x_1"'] & \bullet \end{tikzcd}$$

that we have $\Delta_{x_1}\Sigma_{x_2}(h) \cong \Sigma_{p_1}\Delta_{p_2}(h)$ due to the pullback pasting lemma and uniqueness of pullbacks up to isomorphism. We call such a natural isomorphism $\Delta_{x_1}\Sigma_{x_2} \cong \Sigma_{p_1}\Delta_{p_2}$ a *Beck-Chevalley isomorphism.*

Universal property of spans

Given a bicategory $\mathscr{C}$ and a pseudofunctor $F : \mathbf{Span}(\mathcal{E}) \to \mathscr{C}$

$$\begin{tikzcd} \mathbf{Span}(\mathcal{E}) \arrow[dr, "F"] & \\ & \mathscr{C} \\ \mathcal{E} \arrow[uu, "(-)_*"] \arrow[ur, "\overline{F}"'] & \end{tikzcd}$$

we get a pseudofunctor $F(-)_* : \mathcal{E} \to \mathscr{C}$ by composing with the earlier mentioned embedding.

We may ask what pseudofunctors $\mathcal{E} \to \mathscr{C}$ arise in this way, and if this describes a bijective correspondence between pseudofunctors $\mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ and those pseudofunctors $\mathcal{E} \to \mathscr{C}$ which arise in this way. In practice, we will want to regard certain pseudofunctors as being the same if we are to define a bijection, specifically pseudofunctors which are isomorphic with respect to the maps between pseudofunctors we are using: i.e. pseudofunctors which differ by an invertible icon. It is convenient to state such a "bijection up to isomorphism" as an equivalence of hom-categories of the form

$$\mathbf{Hom}(\mathbf{Span}(\mathcal{E}), \mathscr{C}) \simeq \mathbf{Hom}(\mathcal{E}, \mathscr{C})_{\text{subject to conditions}}$$

The universal property of spans as stated by Hermida was in a tricategorical context making use of modifications [1, Theorem A.2]. Here we describe a simpler 2-categorical version of this result, and briefly discuss why it holds. The universal property of spans is given by the equivalence of categories

$$\mathbf{Icon}(\mathbf{Span}(\mathcal{E}), \mathscr{C}) \simeq \mathbf{Beck}(\mathcal{E}, \mathscr{C})$$

where $\mathbf{Icon}(\mathbf{Span}(\mathcal{E}), \mathscr{C})$ is the category of pseudofunctors $\mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ and icons, and $\mathbf{Beck}(\mathcal{E}, \mathscr{C})$ is the category of sinister Beck morphisms and invertible icons.

Why is this true? Well consider the "composition with the embedding" functor (which may be alternatively viewed as just restricting a pseudofunctor to spans of the form $X \xleftarrow{1_X} X \xrightarrow{f} Y$) defined by the assignment

$$\mathbf{Icon}\left(\mathbf{Span}\left(\mathcal{E}\right),\mathscr{C}\right) \to \mathbf{Beck}\left(\mathcal{E},\mathscr{C}\right)$$

$$\begin{tikzcd} \mathbf{Span}\left(\mathcal{E}\right) \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{C} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathcal{E} \arrow[r, bend left, "F\left(-\right)_*"] \arrow[r, bend right, "G\left(-\right)_*"'] \arrow[r, phantom, "\Downarrow\alpha\left(-\right)_*"] & \mathscr{C} \end{tikzcd}$$

Well defined: Icons map to invertible icons since for any $f \in \mathcal{E}$, the 2-cell $\alpha_{f_*} : Ff_* \to Gf_*$ has an inverse given by the mate of $\alpha_{f^*} : Ff^* \to Gf^*$ under the adjunctions $Ff_* \dashv Ff^*$ and $Gf_* \dashv Gf^*$.

Fully faithful: Why is the assignment $\alpha \mapsto \alpha\left(-\right)_*$ bijective? Well given any invertible icon $\alpha_* : F\left(-\right)_* \to G\left(-\right)_*$, i.e. collection of 2-cells of the form $\alpha_{f_*} : Ff_* \to Gf_*$, we can define the 2-cells $\alpha_{f^*} : Ff^* \to Gf^*$ as $\left(\alpha_{f_*}^{-1}\right)^*$. We can then define the 2-cell for a general span $A \xleftarrow{f} X \xrightarrow{g} B$ as the horizontal composite $\alpha_{g_*} * \alpha_{f^*}$. This tells us the assignment is surjective; also since $\left(\alpha_{f_*}^{-1}\right)^*$ is the only choice for α_{f^*} the assignment is also injective.

Essentially surjective: Given a sinister and Beck pseudofunctor $H : \mathcal{E} \to \mathscr{C}$ we can define the action on a span $A \xleftarrow{f} X \xrightarrow{g} B$ as $H\left(g\right)H\left(f\right)^*$ where $H\left(f\right)^*$ denotes a chosen right adjoint of $H\left(f\right)$. The isomorphisms due to the Beck condition yield the appropriate pseudofunctoriality constraints; furthermore the coherences on our new functor follow from the coherences on H and the functoriality of mates.

Remark 7. (1) This universal property was incorrectly stated in [5] with the maps between pseudofunctors being oplax transformations, which is problematic as pseudofunctors and oplax transformations do not form a category. (2) Again restating the results of [5] in terms of icons, we see that invertible icons correspond to sinister icons (icons $\alpha\left(-\right)_*$ for which each $\left(\alpha_{f_*}^{-1}\right)^*$ is invertible) under this equivalence.

The canonical pseudofunctor $\Phi : \mathbf{Span}\left(\mathcal{E}\right) \to \mathbf{Cat}$

Recall that for any morphism $f : A \to B$, we denoted by $\Delta_f : \mathcal{E}/B \to \mathcal{E}/A$ the "pull back along f" functor between the slice categories, and denoted by $\Sigma_f : \mathcal{E}/A \to \mathcal{E}/B$ its left adjoint: the "compose with f" functor.

Proposition 8. *There is a canonical pseudofunctor* $\Phi : \mathbf{Span}\left(\mathcal{E}\right) \to \mathbf{Cat}$ *defined by taking an object* A *to its slice category* $\mathcal{E}/A$*, with the functors on the hom-categories* $\mathbf{Span}\left(\mathcal{E}\right)\left(A,B\right)$ *being given by the assignment*

$$\begin{tikzcd}[column sep=small]
 & X \arrow[dl, "f'"'] \arrow[dr, "g'"] \arrow[dd, "w"] & \\
A & & B \\
 & Y \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}[column sep=huge]
\mathcal{E}/A \arrow[r, bend left=60, "\Sigma_{g'}\Delta_{f'}"{name=U}] \arrow[r, "\Sigma_g\Sigma_w\Delta_w\Delta_f"'{name=M}] \arrow[r, bend right=60, "\Sigma_g\Delta_f"'{name=D}] & \mathcal{E}/B
\arrow[from=U, to=M, phantom, "\Downarrow"] \arrow[from=M, to=D, phantom, "\Downarrow"]
\end{tikzcd}$$

where the first upper 2-cell on the right comes from the isomorphisms $\Sigma_{gw} \cong \Sigma_g\Sigma_w$ and $\Delta_{fw} \cong \Delta_w\Delta_f$, and the lower one results from the the counit of the adjunction $\varepsilon : \Sigma_w\Delta_w \implies 1$.

Proof. Apply the universal property of spans with the Beck pseudofunctor $\Sigma : \mathcal{E} \to \mathbf{Cat}$ defined by mapping an object X to its slice $\mathcal{E}/X$, and mapping a morphism f to the functor Σ_f. $\square$

The pseudofunctor Φ motivates us to view a span $A \xleftarrow{f} X \xrightarrow{g} B$ as (by abuse of notation) $\Sigma_g\Delta_f$. So how do we compose spans if we view them this way? The earlier mentioned Beck-Chevalley isomorphisms allow us to reduce a composite of spans, i.e. something of the form $\Sigma_{g_1}\Delta_{f_1}\Sigma_{g_2}\Delta_{f_2}$ to $\Sigma_{g_1}\Sigma_{g_2'}\Delta_{f_1'}\Delta_{f_2} \cong \Sigma_{g_1g_2}\Delta_{f_2f_1'}$ which is our resulting span (where $f_1' = \Delta_{g_2}f_1$ and $g_2' = \Delta_{f_1}g_2$).

2.3 Bicategory of polynomials

We have seen that given a category $\mathcal{E}$ with pullbacks, one may construct a new bicategory $\mathbf{Span}(\mathcal{E})$ with morphisms corresponding to "multivariate linear maps", i.e. "matrices". In this section we will construct a bicategory $\mathbf{Poly}(\mathcal{E})$ with morphisms corresponding to "multivariate polynomials". As noted in [6], such structures turn out to be a rich area for study, with applications in areas ranging from computer science [7] and topology [9] to mathematical logic [8, 10].

Definition 9. A *polynomial* in a category $\mathcal{E}$ is a diagram of the form

$$\begin{tikzcd}[column sep=small]
 & E \arrow[dl, "s"'] \arrow[r, "p"] & B \arrow[dr, "t"] & \\
I & & & J
\end{tikzcd}$$

We now consider polynomials in the case $\mathcal{E} = \mathbf{Set}$, and their correspondence to multivariate polynomial maps.

Polynomial maps

We saw earlier that a morphism in $\mathbf{Span}(\mathbf{Set})$ could be viewed as linear map. Similarly (as noted in [11]) a polynomial in $\mathbf{Set}$ (which will be later defined as a morphism in $\mathbf{Poly}(\mathbf{Set})$)

$$\begin{tikzcd} & E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & \\ I & & & J \end{tikzcd}$$

induces the "multivariate polynomial mapping"

$$\mathbf{P} : \mathbf{Set}^I \to \mathbf{Set}^J : (v_{1,i} : i \in I) \mapsto \left(\sum_{b \in B_j} \prod_{e \in E_b} v_{1,s(e)} : j \in J \right)$$

where $B_j := t^{-1}(j)$ and $E_b := p^{-1}(b)$. In the case in which I and J are singleton, this reduces to the polynomial in one variable,

$$\mathbf{P} : \mathbf{Set} \to \mathbf{Set} : v \mapsto \sum_{b \in B} \prod_{e \in E_b} v \cong \sum_{b \in B} v^{E_b}$$

Alternatively, if p is an identity map this reduces to the "linear mapping"

$$\mathbf{P} : \mathbf{Set}^I \to \mathbf{Set}^J : (v_{1,i} : i \in I) \mapsto \left(\sum_{b \in B_j} v_{1,s(b)} : j \in J \right) \cong \left(\sum_{i \in I} v_{1,i} \times \mathbf{B}_{i,j} : j \in J \right)$$

as seen in spans.

In order to define our bicategory $\mathbf{Poly}(\mathcal{E})$, we will assume that $\mathcal{E}$ not only has pullbacks, but also satisfies an additional property:

Definition 10. We say a finitely complete category $\mathcal{E}$ is *locally cartesian closed* if for any morphism $f : A \to B$ the pullback functor $\Delta_f : \mathcal{E}/B \to \mathcal{E}/A$ has a right adjoint Π_f.

Distributivity pullbacks

We now define a special type of pullback introduced by Weber [12], which is related to the distributivity of multiplication over addition, and will be useful for composing polynomials.

Definition 11. Given two composable morphisms $u : X \to A$ and $f : A \to B$, a *pullback around* (f, u) is a diagram

$$\begin{tikzcd} T \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[r, "u"] & A \arrow[d, "f"] \\ Y \arrow[rr, "r"'] & & B \end{tikzcd}$$

such that the outer rectangle is a pullback. A morphism of pullbacks around (f, u) is a pair of morphisms $s : T \to T'$ and $t : Y \to Y'$ such that $p's = p$, $q's = tq$ and $r = r't$. A *distributivity pullback* around (f, u) is a terminal object in the category of pullbacks around (f, u).

As the large square is a pullback, we have a Beck-Chevalley isomorphism $\Delta_f \Sigma_r \implies \Sigma_{up}\Delta_q$. Under the adjunctions $\Delta_f \dashv \Pi_f$ and $\Delta_q \dashv \Pi_q$ this has a mate $\Sigma_r \Pi_q \implies \Pi_f \Sigma_{up}$. Whiskering with Δ_p and composing with the counit $\varepsilon : \Sigma_p \Delta_p \implies 1$ yields a morphism

$$\Sigma_r \Pi_q \Delta_p \implies \Pi_f \Sigma_{up} \Delta_p \cong \Pi_f \Sigma_u \Sigma_p \Delta_p \implies \Pi_f \Sigma_u$$

In fact, this is an isomorphism if and only if this pullback around (p, q, r) is a distributivity pullback [12, Prop 2.2.3].

In a locally cartesian closed category $\mathcal{E}$, given the maps $u : X \to A$ and $f : A \to B$ we can construct a distributivity pullback around (f, u) as the diagram

$$\begin{tikzcd} & A \times_C B \arrow[r, "f'"] \arrow[dl, "e"'] \arrow[d, "\Delta_f(g)"] & B \arrow[d, "g = \Pi_f(u)"] \\ X \arrow[r, "u"'] & A \arrow[r, "f"'] & C \end{tikzcd}$$

where e is the counit of the adjunction $\Delta_f \dashv \Pi_f$ at u; i.e. $\varepsilon_u : \Delta_f \Pi_f (u) \longrightarrow u$. We take this diagram as the chosen distributivity pullback. Given such a diagram, we have a distributivity isomorphism $\Sigma_g \Pi_{f'} \Delta_e \implies \Pi_f \Sigma_u$ as just shown.

Bicategory of polynomials

Definition 12. Given a locally cartesian closed category $\mathcal{E}$, we may form a bicategory $\mathbf{Poly}(\mathcal{E})$ with objects taken as those of $\mathcal{E}$, and morphisms $I \twoheadrightarrow J$ given by diagrams of the form

$$\begin{tikzcd} & E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & \\ I & & & J \end{tikzcd}$$

where to compose a polynomial $I \twoheadrightarrow J$ with a polynomial $J \twoheadrightarrow K$ we form the diagram:

$$\begin{tikzcd} & & & \bullet \arrow[r, "p_2''"] \arrow[dl, "e'"'] \arrow[d, phantom, "(2)"] & \bullet \arrow[r, "p_1'"] \arrow[dl, "e"] \arrow[d, phantom, "(4)"] & \bullet \arrow[dd, "g"] \\ & & \bullet \arrow[r, "p_2'"] \arrow[dl, "s_1''"'] \arrow[d, phantom, "(1)"] & \bullet \arrow[dl, "s_1'"] \arrow[dr, "t_2'"] \arrow[d, phantom, "(3)"] & & \\ & E \arrow[r, "p_2"] \arrow[dl, "s_2"'] & B \arrow[dr, "t_2"] & & E' \arrow[r, "p_1"] \arrow[dl, "s_1"] & B' \arrow[dr, "t_1"] & \\ I & & & J & & & K \end{tikzcd} \qquad (\ddagger)$$

where (1), (2) and (3) are chosen pullbacks, and (4) is a chosen distributivity pullback. A cartesian 2-cell between two polynomials $I \twoheadrightarrow J$ is a pair of arrows f and g yielding the commuting diagram

$$
\begin{tikzcd}
& E \arrow[r, "p"] \arrow[dl, "s"'] \arrow[dd, "f"'] \arrow[ddr, phantom, "\mathrm{pb}"] & B \arrow[dr, "t"] \arrow[dd, "g"] & \\
I & & & J \\
& E' \arrow[ul, "s'"] \arrow[r, "p'"'] & B' \arrow[ur, "t'"'] &
\end{tikzcd}
$$

It is worth noting that there is a more general notion of 2-cell given by diagrams of the form

$$
\begin{tikzcd}
& E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] \arrow[dd, "g"] & \\
I & M \arrow[u, "e"] \arrow[ur, "h"'] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{pb}"] & & J \\
& E' \arrow[ul, "s'"] \arrow[r, "p'"'] & B' \arrow[ur, "t'"'] &
\end{tikzcd}
$$

With these more general 2-cells, for any $f : A \to B$ we have adjunctions (by abuse of notation) $\Sigma_f \dashv \Delta_f \dashv \Pi_f$ in $\mathbf{Poly}\,(\mathcal{E})$ where Σ_f, Δ_f and Π_f are given by the polynomials

$$
A \xleftarrow{1} A \xrightarrow{1} A \xrightarrow{f} B \qquad B \xleftarrow{f} A \xrightarrow{1} A \xrightarrow{1} A \qquad A \xleftarrow{1} A \xrightarrow{f} B \xrightarrow{1} B
$$

respectively. However, we will not use these general 2-cells. Throughout the remainder of this thesis $\mathbf{Poly}\,(\mathcal{E})$ is to be understood as the bicategory of polynomials with *cartesian 2-cells*, unless otherwise stated.

In much the same way as we have the canonical pseudofunctor $\mathbf{Span}\,(\mathcal{E}) \to \mathbf{Cat}$, we also have a canonical pseudofunctor $\mathbf{Poly}\,(\mathcal{E}) \to \mathbf{Cat}$ which sends a general 1-cell (which by abuse of notation, is a composite of the form $\Sigma_t \Pi_p \Delta_s$) to its associated polynomial functor $\Sigma_t \Pi_p \Delta_s : \mathcal{E}/I \to \mathcal{E}/J$ [12, Theorem 3.2.6]. From this viewpoint, we can look at what is going on in our definition of composition of two polynomials, and see why it is defined this way. Starting from an expression of the form

$$\Sigma_{t_1} \Pi_{p_1} \Delta_{s_1} \Sigma_{t_2} \Pi_{p_2} \Delta_{s_2},$$

we first evaluate the pullback (3) and apply the Beck isomorphism to get

$$\Sigma_{t_1} \Pi_{p_1} \Sigma_{t_2'} \Delta_{s_1'} \Pi_{p_2} \Delta_{s_2};$$

we then apply the Beck isomorphism for the pullback (1) to get

$$\Sigma_{t_1} \Pi_{p_1} \Sigma_{t_2'} \Pi_{p_2'} \Delta_{s_1''} \Delta_{s_2};$$

now applying the distributivity law for (4) gives

$$\Sigma_{t_1} \Sigma_g \Pi_{p_1'} \Delta_e \Pi_{p_2'} \Delta_{s_1''} \Delta_{s_2};$$

then applying the Beck isomorphism for (3) gives

$$\Sigma_{t_1}\Sigma_g\Pi_{p_1'}\Pi_{p_2''}\Delta_{e'}\Delta_{s_1''}\Delta_{s_2};$$

giving the polynomial expression

$$\Sigma_{t_1 g}\Pi_{p_1' p_2''}\Delta_{s_2 s_1'' e'}.$$

Remark 13. When we say Beck isomorphism here we mean the isomorphism which exists due to the uniqueness of pullbacks up to isomorphism, as discussed in "Beck-Chevalley isomorphisms" (not the isomorphisms between the functors). Similarly, the distributivity isomorphisms here are due to the uniqueness of distributivity pullbacks up to isomorphism.

Universal property of polynomial composition

Proposition 14. *[12, Prop. 3.1.6]. Suppose we are given two fixed polynomials $P : I \nrightarrow J$ and $Q : J \nrightarrow K$. Consider a category $\mathcal{K}$ with objects given by commuting diagrams of the form*

$$\begin{tikzcd}
 & & A_1 \arrow[r] \arrow[dl] & A_2 \arrow[r] \arrow[dl] \arrow[dr] & A_3 \arrow[dr] & & \\
 & E \arrow[r, "p_2"] \arrow[dl, "s_2"'] & B \arrow[dr, "t_2"] & & E' \arrow[r, "p_1"] \arrow[dl, "s_1"'] & B' \arrow[dr, "t_1"] & \\
I & & & J & & & K
\end{tikzcd}$$

for which the left and right squares are pullbacks, and morphisms given by diagrams

$$\begin{tikzcd}
 & & C_1 \arrow[r] \arrow[d] \arrow[ddl] & C_2 \arrow[r] \arrow[d] \arrow[ddl, dashed] \arrow[ddr, dashed] & C_3 \arrow[d] \arrow[ddr] & & \\
 & & A_1 \arrow[r] \arrow[dl] & A_2 \arrow[r] \arrow[dl] \arrow[dr] & A_3 \arrow[dr] & & \\
 & E \arrow[r, "p_2"] \arrow[dl, "s_2"'] & B \arrow[dr, "t_2"] & & E' \arrow[r, "p_1"] \arrow[dl, "s_1"'] & B' \arrow[dr, "t_1"] & \\
I & & & J & & & K
\end{tikzcd}$$

which commute. Then in this category $\mathcal{K}$, the diagram

$$\begin{tikzcd}
 & & \bullet \arrow[r, "p_2''"] \arrow[dl, "s_1''e'"'] & \bullet \arrow[r, "p_1'"] \arrow[dl, "s_1'e"'] \arrow[dr, "t_2'e"] & \bullet \arrow[dr, "g"] & & \\
 & E \arrow[r, "p_2"] \arrow[dl, "s_2"'] & B \arrow[dr, "t_2"] & & E' \arrow[r, "p_1"] \arrow[dl, "s_1"'] & B' \arrow[dr, "t_1"] & \\
I & & & J & & & K
\end{tikzcd}$$

as defined in (‡) *(which yields the composite $QP : I \nrightarrow K$) is a terminal object.*

This property allows one to verify that $\mathbf{Poly}\,(\mathcal{E})$ satisfies the coherence axioms, and to define horizontal composition of 2-cells. Moreover, this property may to used to check that $\mathbf{Poly}\,(\mathcal{E})$ is indeed a bicategory.

3
Local reflections

Given an adjunction $L \dashv I : \mathcal{B}' \to \mathcal{B}$ in $\mathbf{Cat}$ for which I is a full inclusion, the functor L is known as a reflector, and we say that $\mathcal{B}'$ is a reflective subcategory of $\mathcal{B}$. In this chapter we are interested in the situation in which we have a family of adjunctions between hom-categories

$$L_{X,Y} \dashv I_{X,Y} : \mathscr{B}'(X,Y) \to \mathscr{B}(X,Y), \qquad X, Y \in \mathscr{B}_{\mathrm{ob}}$$

for which $I_{X,Y}$ is a full inclusion (or more generally a fully faithful functor) for every pair of objects X and Y. In particular we would like to know how maps out of $\mathscr{B}'$ correspond to maps out of $\mathscr{B}$ in such a situation.

3.1 The universal property of an adjunction

Given an equivalence $F \dashv G : \mathcal{A} \simeq \mathcal{B}$ in a bicategory $\mathscr{C}$ and an object $\mathcal{C} \in \mathscr{C}$ it's clear that giving a map $H : \mathcal{A} \to \mathcal{C}$ is equivalent to giving a map $\overline{H} : \mathcal{B} \to \mathcal{C}$, as we may compose with F and G.

$$\begin{tikzcd}
 & \mathcal{C} & \\
\mathcal{A} \arrow[ur, dotted, "H"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\perp"] & & \mathcal{B} \arrow[ul, dotted, "\overline{H}"'] \arrow[ll, bend right, "F"']
\end{tikzcd}$$

Now given an adjunction $F \dashv G$ which is not necessarily an equivalence, we should expect to still have some correspondence between maps $\mathcal{A} \to \mathcal{C}$ and maps $\mathcal{B} \to \mathcal{C}$ as an adjunction is a weakened version of equivalence; the only difference being that we do not ask the counit ε and unit η be invertible. We now prove this is indeed the case.

Proposition 15. *An adjunction in a bicategory $\mathscr{C}$,*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, phantom, "\perp"] \arrow[rr, bend right, "G"'] & & \mathcal{B} \arrow[ll, bend right, "F"'] \end{tikzcd}$$

with counit $\varepsilon : FG \to 1_{\mathcal{A}}$ and unit $\eta : 1_{\mathcal{B}} \to GF$ induces an equivalence between 1-cells $H : \mathcal{A} \to \mathcal{C}$ for which $H\varepsilon$ is invertible and 1-cells $\overline{H} : \mathcal{B} \to \mathcal{C}$ for which $\overline{H}\eta$ is invertible, for every object $\mathcal{C} \in \mathscr{C}$. We denote this equivalence by $\mathscr{C}(\mathcal{A}, \mathcal{C})_{\text{inv }\varepsilon} \simeq \mathscr{C}(\mathcal{B}, \mathcal{C})_{\text{inv }\eta}$.

Proof. We consider the composition functors

$$\begin{tikzcd} \mathscr{C}(\mathcal{A}, \mathcal{C})_{\text{inv }\varepsilon} \arrow[rr, bend right, "(-) \circ F"'] & & \mathscr{C}(\mathcal{B}, \mathcal{C})_{\text{inv }\eta} \arrow[ll, bend right, "(-) \circ G"'] \end{tikzcd}$$

and first check that these are well defined. To see the mapping

$$(-) \circ F : \mathscr{C}(\mathcal{A}, \mathcal{C})_{\text{inv }\varepsilon} \to \mathscr{C}(\mathcal{B}, \mathcal{C})_{\text{inv }\eta}$$

is well defined, note that by the triangle identities $\varepsilon F \circ F\eta = 1_F$, and so given a 1-cell $H : \mathcal{A} \to \mathcal{C}$ such that $H\varepsilon$ is invertible we have $H\varepsilon F \circ HF\eta = 1_{HF}$. Hence $HF\eta = (H\varepsilon F)^{-1}$ is invertible. Similarly, one can show $(-) \circ G$ is well defined. We also have natural transformations

$$\begin{aligned} (-) \circ \varepsilon : (-) \circ FG \implies (-) : \quad & \mathscr{C}(\mathcal{A}, \mathcal{C})_{\text{inv }\varepsilon} \to \mathscr{C}(\mathcal{A}, \mathcal{C})_{\text{inv }\varepsilon} \\ (-) \circ \eta : (-) \implies (-) \circ GF : \quad & \mathscr{C}(\mathcal{B}, \mathcal{C})_{\text{inv }\eta} \to \mathscr{C}(\mathcal{B}, \mathcal{C})_{\text{inv }\eta} \end{aligned}$$

which are invertible by definition, and so yield our equivalence of categories. □

Remark 16. (1) This is the version concerning postcomposition. Dually, the equivalence $\mathscr{C}(\mathcal{C}, \mathcal{A})_{\text{inv }\varepsilon} \simeq \mathscr{C}(\mathcal{C}, \mathcal{B})_{\text{inv }\eta}$ holds true if we compose with the counit ε and unit η on the other side. (2) Alternatively, we may see that this universal property follows from the fact that the adjunction $(-) \circ G \dashv (-) \circ F$ restricts to an equivalence on the subcategories of objects for which the unit and counit are invertible.

3.2 Local reflections

In practice we would like to deduce universal properties concerning an entire hom-category. One way do this is to consider adjunctions for which the counit ε is invertible. Let us now restrict our attention to the case $\mathscr{C} = \mathbf{LaxIcon}$.

Definition 17. Let $\mathscr{B}$ and $\mathscr{B}'$ be two bicategories with the same class of objects. We say $\mathscr{B}'$ is a *locally reflective sub-bicategory* of $\mathscr{B}$ if there exists an adjunction in $\mathbf{LaxIcon}$

$$\begin{tikzcd} \mathscr{B}' \arrow[r, bend right, "I"'] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend right, "L"'] \end{tikzcd}$$

for which I is locally fully faithful. If the adjunction is in the other direction, i.e. if $I \dashv L$, we say $\mathscr{B}'$ is a locally coreflective sub-bicategory of $\mathscr{B}$.

The following proposition gives sufficient conditions by which we may recognize one bicategory as a locally reflective sub-bicategory of another. The proof is based on the theory of doctrinal adjunctions, described in [13].

Proposition 18. *Let $\mathscr{B}$ and $\mathscr{B}'$ be two bicategories with the same class of objects. Suppose there exists a $\mathscr{B}^2_{\mathrm{ob}}$-indexed family of adjunctions*

$$\begin{tikzcd} \mathscr{B}'(X,Y) \arrow[r, bend right, "I_{X,Y}"'] \arrow[r, phantom, "\perp"] & \mathscr{B}(X,Y) \arrow[l, bend right, "L_{X,Y}"'] \end{tikzcd}, \qquad (X,Y) \in \mathscr{B}^2_{\mathrm{ob}}$$

such that

(L1) *Every $I_{X,Y}$ is fully faithful;*

(L2) *The family $(I_{X,Y} : X, Y \in \mathscr{B})$ extends to a lax functor $I : \mathscr{B}' \to \mathscr{B}$;*

(L3) *L maps η_f and its whiskers into isomorphisms*[1]*;*

(L4) *L maps the constraints of I into isomorphisms;*

Then the family $(L_{X,Y} : X, Y \in \mathscr{B})$ extends to a pseudofunctor $L : \mathscr{B} \to \mathscr{B}'$ which is left adjoint to I in $\mathbf{LaxIcon}$*, and so $\mathscr{B}'$ is a locally reflective sub-bicategory of $\mathscr{B}$.*

Proof. To give the pseudofunctoriality constraints for L, we first note the mate correspondence between natural transformations

[1]The 2-cell η_f is an arbitrary component of a unit $\eta : 1 \to I_{X,Y} L_{X,Y}$. We mean L with the appropriate indicies X, Y such that applying $L_{X,Y}$ to the 2-cell is well defined, and so we are not assuming the family $\left(L_{X,Y} : X, Y \in \mathscr{B}\right)$ extends to a lax/oplax functor; similarly for axiom L4. Note that this axiom may be replaced by asking that expressions of the form $L_{X,Z}\left(\eta_{Y,Z} \times \eta_{X,Y}\right)$ are invertible.

$$\begin{tikzcd}
\mathscr{B}(Y,Z)\times\mathscr{B}(X,Y) \arrow[r, "\circ"] \arrow[d, "L_{Y,Z}\times L_{X,Y}"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathscr{B}(X,Z) \arrow[d, "L_{X,Z}"] \\
\mathscr{B}'(Y,Z)\times\mathscr{B}'(X,Y) \arrow[r, "\bullet"'] & \mathscr{B}'(X,Z)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{B}(Y,Z)\times\mathscr{B}(X,Y) \arrow[r, "\circ"] \arrow[dr, phantom, "\Downarrow\beta"] & \mathscr{B}(X,Z) \\
\mathscr{B}'(Y,Z)\times\mathscr{B}'(X,Y) \arrow[r, "\bullet"'] \arrow[u, "I_{Y,Z}\times I_{X,Y}"] & \mathscr{B}'(X,Z) \arrow[u, "I_{X,Z}"']
\end{tikzcd}$$

and so taking β to be the lax functoriality constraints of I yields oplax functoriality constraints for L. More explicitly, our α is given at a pair of composable maps f and g by the commuting diagram

$$\begin{tikzcd}
L(gf) \arrow[r, "\alpha_{g,f}"] \arrow[d, "L\left(\eta_g*\eta_f\right)"'] & L(g)\,L(f) \\
L(ILgILf) \arrow[r, "L\left(\beta_{Lg,Lf}\right)"'] & LI(LgLf) \arrow[u, "\varepsilon_{Lg,Lf}"']
\end{tikzcd}$$

and we know $\varepsilon_{Lg,Lf}$ is invertible since each $I_{X,Y}$ is fully faithful, $L\left(\beta_{Lg,Lf}\right)$ is invertible by axiom $(L4)$, and $L\left(\eta_g*\eta_f\right)$ is invertible by axiom $(L3)$, as the horizontal composite $\eta_g*\eta_f$ may be defined in terms of whiskering. Similarly, the nullary constraints for L are given by the mates of those of I:

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "\sigma"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathscr{B}(X,X) \arrow[d, "L_{X,X}"] \\
\mathbf{1} \arrow[r, "\omega"'] & \mathscr{B}'(X,X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \arrow[r, "\sigma"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & \mathscr{B}(X,X) \\
\mathbf{1} \arrow[r, "\omega"'] & \mathscr{B}'(X,X) \arrow[u, "I_{X,X}"']
\end{tikzcd}$$

Thus our constraint $\alpha_X : L(1_X) \to 1_X$ is given by the composite

$$L(1_X) \xrightarrow{L(\beta_X)} LI(1_X) \xrightarrow{\varepsilon_{1_X}} 1_X$$

which is again invertible by axioms $(L1)$ and $(L4)$. To check that L is a pseudofunctor it remains to check coherences of the constraints. Note that the coherences on I tell us the two sides of the cube

$$\begin{tikzcd}[column sep=small]
& \mathscr{B}'(Y,Z)\,\mathscr{B}'(X,Y)\,\mathscr{B}'(W,X) \arrow[rrr, "I_{Y,Z}\times I_{X,Y}\times I_{W,X}"] \arrow[dl, "\mathrm{id}\times\bullet"'] \arrow[dr, "\bullet\times\mathrm{id}"] & & & \mathscr{B}(Y,Z)\,\mathscr{B}(X,Y)\,\mathscr{B}(W,X) \arrow[dr, "\circ\times\mathrm{id}"] \arrow[dll, phantom, "\Downarrow\beta\times I_{W,X}"] & \\
\mathscr{B}'(Y,Z)\,\mathscr{B}'(W,Y) \arrow[dr, "\bullet"'] & \Downarrow a & \mathscr{B}'(X,Z)\,\mathscr{B}'(W,X) \arrow[rrr, "I_{X,Z}\times I_{W,X}"] \arrow[dl, "\bullet"] & \arrow[d, phantom, "\Downarrow\beta"] & & \mathscr{B}(X,Z)\,\mathscr{B}(W,X) \arrow[dl, "\circ"] \\
& \mathscr{B}'(W,Z) \arrow[rrr, "I_{W,Z}"'] & & {} & \mathscr{B}(W,Z) &
\end{tikzcd}$$

$$
\begin{tikzcd}[column sep=small]
& \mathscr{B}'(Y,Z)\,\mathscr{B}'(X,Y)\,\mathscr{B}'(W,X) \arrow[dl, "\mathrm{id}\times\bullet"'] \arrow[rrr, "I_{Y,Z}\times I_{X,Y}\times I_{W,X}"] \arrow[drr, phantom, "\Downarrow I_{Y,Z}\times\beta"] & & & \mathscr{B}(Y,Z)\,\mathscr{B}(X,Y)\,\mathscr{B}(W,X) \arrow[dl, "\mathrm{id}\times\circ"'] \arrow[dr, "\circ\times\mathrm{id}"] & \\
\mathscr{B}'(Y,Z)\,\mathscr{B}'(W,Y) \arrow[rrr, "I_{Y,Z}\times I_{W,Y}"'] \arrow[dr, "\bullet"'] \arrow[drrr, phantom, "\Downarrow\beta"] & & & \mathscr{B}(Y,Z)\,\mathscr{B}(W,Y) \arrow[dr, "\circ"'] \arrow[rr, phantom, "\Downarrow a"] & & \mathscr{B}(X,Z)\,\mathscr{B}(W,X) \arrow[dl, "\circ"] \\
& \mathscr{B}'(W,Z) \arrow[rrr, "I_{W,Z}"'] & & & \mathscr{B}(W,Z) &
\end{tikzcd}
$$

are equal, and so taking mates (which is a bijection compatible with composition) yields the coherence conditions for L. We omit the identity coherences, which may be shown similarly. Finally, it remains to check that we actually get an adjunction in $\mathbf{LaxIcon}$, i.e. we need to show the families of units and counits

$$\varepsilon = (\varepsilon_{X,Y} : X, Y \in \mathscr{B}) : LI \to 1_{\mathscr{B}'}, \qquad \eta = (\eta_{X,Y} : X, Y \in \mathscr{B}) : 1_{\mathscr{B}} \to IL$$

satisfy the conditions for being an icon. (Note that here L is pseudo and I is lax, so that both LI and IL are lax functors. In the last chapter we will consider an example in which I is pseudo and L is oplax). We will just check this for ε, as the proof for η is similar.

Compatibility with identities: Under the adjunction $L_{X,X} \dashv I_{X,X}$ the map $\beta_X : 1_X \to I_{X,X}(1_X)$ corresponds to the map $\alpha_X : L_{X,X}(1_X) \to 1_X$. The identity condition on being an icon asks that $\varepsilon_{1_X} \circ \left(L\beta_X \circ \alpha_X^{-1}\right)$ is the identity, which is the case as α_X maps to β_X and then $\varepsilon_{1_X} \circ L\beta_X$ (which must agree with α_X) under the adjunction.

Compatibility with composition: The condition for ε being compatible with the constraints of LI asks that

$$
\begin{tikzcd}
L(Ig \circ If) \arrow[r, "\alpha_{Ig,If}"] \arrow[d, "L\beta_{gf}"'] & LIg \bullet LIf \arrow[d, "\varepsilon_g * \varepsilon_f"] \\
LI(g \circ f) \arrow[r, "\varepsilon_{gf}"'] & gf
\end{tikzcd}
$$

commutes, which is true since given the pasting diagram

$$
\begin{tikzcd}
{} \arrow[r, "I\times I"] \arrow[dr, phantom, "\Downarrow\varepsilon\times\varepsilon"] & {} \arrow[r, "\circ"] \arrow[d, "L\times L"'] \arrow[dr, phantom, "\Downarrow\alpha"] & {} \arrow[r, "L"] \arrow[d, "L"'] \arrow[dr, phantom, "\Downarrow L\eta"] & {} \arrow[r, "\mathrm{id}"] \arrow[dr, phantom, "\Downarrow\varepsilon"] & {} \\
{} \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & {} \arrow[r, "\bullet"'] & {} \arrow[r, "\mathrm{id}"'] & {} \arrow[u, "LI"'] \arrow[r, "\mathrm{id}"'] & {} \arrow[u, "\mathrm{id}"']
\end{tikzcd}
$$

we may cancel the right 2 squares by the triangle identities, or alternatively replace the left 3 squares by $L\beta$ due to the definition of mate. □

The following proposition gives means by which the later mentioned bicategory of relations may be constructed from the bicategory of spans.

Proposition 19. *Let $\mathscr{B}$ be a bicategory. Suppose that we are given a $\mathscr{B}^2_{\text{ob}}$-indexed family of categories and adjunctions*

$$
\begin{tikzcd}
\mathcal{C}_{X,Y} \arrow[rr, bend right, "I_{X,Y}"'] \arrow[rr, phantom, "\perp"] & & \mathscr{B}(X,Y) \arrow[ll, bend right, "L_{X,Y}"']
\end{tikzcd}, \qquad (X,Y) \in \mathscr{B}^2_{\text{ob}}
$$

such that

(L1) *Every $I_{X,Y}$ is fully faithful;*

(L3) *L maps η_f and its whiskers into isomorphisms.*

Then we may form a bicategory $\mathscr{B}'$ where

- *$\mathscr{B}'$ has the same objects as $\mathscr{B}$;*
- *For every $X, Y \in \mathscr{B}$, $\mathscr{B}'(X,Y) := \mathcal{C}_{X,Y}$;*
- *For every $X \in \mathscr{B}$, $1_X := L_{X,X}(1_X) \in \mathscr{B}'$;*
- *The composition functor $\bullet$ in $\mathscr{B}'$ renders commutative the diagram*

$$
\begin{tikzcd}
\mathscr{B}'(Y,Z) \times \mathscr{B}'(X,Y) \arrow[r, "\bullet"] \arrow[d, "I_{Y,Z} \times I_{X,Y}"'] & \mathscr{B}'(X,Z) \\
\mathscr{B}(Y,Z) \times \mathscr{B}(X,Y) \arrow[r, "\circ"'] & \mathscr{B}(X,Z) \arrow[u, "L_{X,Z}"']
\end{tikzcd}
$$

Moreover, the family $(I_{X,Y} : X, Y \in \mathscr{B})$ extends to a lax functor $I : \mathscr{B}' \to \mathscr{B}$ with constraints

$$
\begin{aligned}
I(g) \circ I(f) &\xrightarrow{\eta_{I(g)\circ I(f)}} IL(Ig \circ If) = I(g \bullet f) \\
1_X &\xrightarrow{\quad\eta_{1_X}\quad} IL(1_X) = I(1_X)
\end{aligned}
$$

Note that since we have all axioms L1-L4 of Proposition 18 satisfied, it then follows that $\mathscr{B}'$ is a locally reflective sub-bicategory of $\mathscr{B}$, and moreover the family $(L_{X,Y} : X, Y \in \mathscr{B})$ extends to a pseudofunctor $L : \mathscr{B} \to \mathscr{B}'$.

Proof. For a verification that the coherence axioms for a bicategory are satisfied we refer the reader to the work of Day [14], who did this calculation in the case of one-object bicategories (which may be identified with monoidal categories). Alternatively, this proposition may be seen as a special case of [13, Theorem 3.3]. $\square$

Lemma 20. *Suppose that the adjunction $L \dashv I$ in* $\mathbf{LaxIcon}$ *exhibits $\mathscr{B}'$ as a locally reflective sub-bicategory of $\mathscr{B}$. Then the following are equivalent for any pseudofunctor $F : \mathscr{B} \to \mathscr{C}$: (1) $F \cong FIL$; (2) $F\eta$ is an invertible icon; (3) F inverts all elements of* $\Sigma = \{$2-cells $\alpha \in \mathscr{B} : L\alpha$ is invertible in $\mathscr{B}'\}$.

Proof.

(1 $\implies$ 2): Suppose that $\alpha : FIL \implies F$ is an invertible icon. Then we have the commuting square for the composite $\alpha * \eta$;

$$\begin{tikzcd} FIL \arrow[r, "\alpha"] \arrow[d, "FIL\eta"'] & F \arrow[d, "F\eta"] \\ FILIL \arrow[r, "\alpha IL"'] & FIL \end{tikzcd}$$

As $\alpha, \alpha IL$ and $FIL\eta$ are invertible we conclude that $F\eta$ is also invertible.

(2 $\implies$ 3): Given an $\alpha : f \to g$ such that $L\alpha$ is invertible we consider the naturality square

$$\begin{tikzcd} Ff \arrow[r, "F\alpha"] \arrow[d, "F\eta_f"'] & Fg \arrow[d, "F\eta_g"] \\ FILf \arrow[r, "FIL\alpha"'] & FILg \end{tikzcd}$$

to see that $F\alpha$ must also be invertible.

(3 $\implies$ 1): Since $L\eta$ is invertible we have $\eta \in \Sigma$, hence $F\eta : F \to FIL$ is invertible and so $F \cong FIL$. $\square$

Theorem 21 (Universal properties of local reflections)**.** *Suppose that we are given an adjunction $L \dashv I$ in* $\mathbf{LaxIcon}$ *which exhibits $\mathscr{B}'$ as a locally reflective sub-bicategory of $\mathscr{B}$. Composition with L and I then yields the equivalences*

$$\mathbf{LaxIcon}\left(\mathscr{B}', \mathscr{C}\right) \simeq \mathbf{LaxIcon}\left(\mathscr{B}, \mathscr{C}\right)_{\text{inv } \eta}$$
$$\mathbf{Icon}\left(\mathscr{B}', \mathscr{C}\right) \simeq \mathbf{Icon}\left(\mathscr{B}, \mathscr{C}\right)_{\text{inv } \eta}$$

where $\mathscr{C}$ is an arbitrary bicategory, and a functor $F : \mathscr{B} \to \mathscr{C}$ "inverts η" when it maps every 2-cell

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "I_{X,Y} L_{X,Y} f"'] \arrow[r, phantom, "\Downarrow \eta_{X,Y}(f)"] & Y \end{tikzcd}$$

into a 2-isomorphism; that is, when $F\eta$ is an invertible icon.

Proof. The first equivalence is merely the universal property of the adjunction $L \dashv I$ due to Proposition 15. As $(-) \circ L : \mathbf{Icon}\left(\mathscr{B}', \mathscr{C}\right) \to \mathbf{Icon}\left(\mathscr{B}, \mathscr{C}\right)_{\text{inv } \eta}$ is a restriction of this equivalence, it's still fully faithful. Well definedness is clear as L is a pseudofunctor by [13, Theorem 1.5], and for essential surjectivity note that for a given pseudofunctor $F : \mathscr{B} \to \mathscr{C}$ which inverts η, we may compose with I to obtain the lax functor $FI : \mathscr{B}' \to \mathscr{C}$. Now since L inverts the constraints of I (this follows from the diagram for compatibility of ε with the constraints in Proposition 18 and noting the $\varepsilon's$ and α are invertible and so $L\beta$ be also), any constraint σ of I lies in Σ and so since $F\eta$ is invertible, it follows that F inverts the constraints of I. Hence FI is in fact a pseudofunctor. Finally, $FIL \cong F$ via $F\eta$ and so we have essential surjectivity. □

Remark 22. Using (1) $\iff$ (2) of Lemma 20 it is not hard to show that the equivalence $\mathbf{LaxIcon}(\mathscr{B}', \mathscr{C}) \simeq \mathbf{LaxIcon}(\mathscr{B}, \mathscr{C})_{\text{inv}\ \eta}$ may be realized as the universal property of the bi-coequalizer

$$\begin{tikzcd} \mathscr{B} \arrow[r, shift left, "1"] \arrow[r, shift right, "IL"'] & \mathscr{B} \arrow[r, "L"] & \mathscr{B}' \end{tikzcd}$$

The 2-category $\mathbf{Icon}$ is a 2-category of bicategories (families of hom-categories), pseudofunctors (families of functors between the hom-categories) and icons (families of natural transformations between these functors on the hom-categories) and so we should expect $\mathbf{Icon}$ to be somewhat analogous to $\mathbf{Cat}$. For example the property $[\mathcal{A}^{\text{op}}, \mathcal{B}^{\text{op}}] \cong [\mathcal{A}, \mathcal{B}]^{\text{op}}$ in $\mathbf{Cat}$ will extend to $\mathbf{Icon}$ if we reverse the arrows in the hom-categories (i.e. reverse the 2-cells).

Corollary 23. *Suppose that we are given an adjunction $\varepsilon, \eta : I \dashv L : \mathscr{B} \to \mathscr{B}'$ in $\mathbf{OplaxIcon}$ where I is locally fully faithful; then L is a pseudofunctor and we have the equivalences*

$$\mathbf{OplaxIcon}\left(\mathscr{B}', \mathscr{C}\right) \simeq \mathbf{OplaxIcon}(\mathscr{B}, \mathscr{C})_{\text{inv}\ \varepsilon}$$
$$\mathbf{Icon}\left(\mathscr{B}', \mathscr{C}\right) \simeq \mathbf{Icon}(\mathscr{B}, \mathscr{C})_{\text{inv}\ \varepsilon}$$

Proof. This is true since the adjunction $I \dashv L : \varepsilon, \eta$ in $\mathbf{OplaxIcon}$ becomes an adjunction $L' \dashv I' : \varepsilon', \eta'$ in $\mathbf{LaxIcon}$ upon reversing the 2-cells. This yields the equivalence

$$\mathbf{Icon}\left(\mathscr{B}', \mathscr{C}\right) \cong \mathbf{Icon}\left(\left(\mathscr{B}'\right)^{\text{co}}, \mathscr{C}^{\text{co}}\right)^{\text{op}} \simeq \left[\mathbf{Icon}\left(\left(\mathscr{B}'\right)^{\text{co}}, \mathscr{C}^{\text{co}}\right)_{\text{inv}\ \eta'}\right]^{\text{op}} \cong \mathbf{Icon}(\mathscr{B}, \mathscr{C})_{\text{inv}\ \varepsilon}.$$

where $\varepsilon = (\eta')^{\text{op}}$. As earlier, L is a pseudofunctor by [13, Theorem 1.5]. □

Remark 24. In chapter 5 we are interested in the situation where we are given an adjunction $I \dashv L : \mathscr{B} \to \mathscr{B}'$ in $\mathbf{LaxIcon}$ wherein I is locally fully faithful; i.e. the situation of locally coreflective sub-bicategories (note that in such a situation I is pseudo [13]). We might expect to again have the equivalence $\mathbf{Icon}(\mathscr{B}', \mathscr{C}) \simeq \mathbf{Icon}(\mathscr{B}, \mathscr{C})_{\text{inv}\ \varepsilon}$; however, this is not the case since there is no reason for composition with the lax functor L to map pseudofunctors to pseudofunctors. In fact, the correct "universal property of local reflections" in this case concerns composition on the other side, and is given by $\mathbf{Icon}\,(\mathscr{C}, \mathscr{B}') \simeq \mathbf{Icon}\,(\mathscr{C}, \mathscr{B})_{\text{inv}\,\varepsilon}$; though we will not use this equivalence.

3.3 Relations

3.3.1 Regular categories

In $\mathbf{Set}$, given a function $f : A \to B$ we may form the image $\{f(a) : a \in A\}$, and then write the function f as a surjection followed by an inclusion

$$A \overset{\overline{f}}{\twoheadrightarrow} \{f(a) : a \in A\} \rightarrowtail B$$

We would like to perform these so called image factorizations in categories $\mathcal{E}$ aside from $\mathbf{Set}$. We now describe a type of category in which we can always perform these factorizations.

Definition 25. A category $\mathcal{E}$ is said to be *regular* if

- $\mathcal{E}$ is finitely complete;
- The kernel pair of any morphism $f : A \to B$

$$\begin{tikzcd} A \times_B A \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

admits a coequalizer

$$\begin{tikzcd} A \times_B A \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & A \arrow[r, two heads, "q"] & R; \end{tikzcd}$$

- Regular epimorphisms (epimorphisms which arise as a coequalizer) pull back into regular epimorphisms.

Remark 26. In a regular category $\mathcal{E}$, any morphism $f : A \to B$ can be written as a regular epimorphism followed by a monomorphism [15, Theorem 2.1.3], by factoring through the coequalizer of the kernel pair. This is known as an image factorization.

Definition 27. A *strong epimorphism* is an epimorphism p such that given any commuting diagram of the form

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "p"', two heads] & B \arrow[d, "i", tail] \\ C \arrow[r, "s"'] \arrow[ur, dashed, "h"] & D \end{tikzcd}$$

for which i is a monomorphism, there exists a unique arrow $h : C \to B$ making the diagram commute.

Remark 28. In a regular category, an epimorphism is strong if and only if it is regular [15, Prop. 2.1.4].

3.3.2 The bicategory of relations

Let us now modify the bicategory $\mathbf{Span}(\mathcal{E})$ slightly so as to construct a new bicategory $\mathbf{Rel}(\mathcal{E})$ known as the bicategory of relations.

Definition 29. Let $\mathcal{E}$ be a regular category (equipped with a choice of pullbacks and image factorizations). We define the bicategory $\mathbf{Rel}(\mathcal{E})$ with objects being those of $\mathcal{E}$, 1-cells $A \nrightarrow B$ given by those spans

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A & & B \end{tikzcd}$$

which are *jointly mono,* i.e. spans that satisfy the following property: if $pm = pn$ and $qm = qn$ then $m = n$. Note that we may view such a span as a monomorphism into the product

$$(p, q) : X \rightarrowtail A \times B$$

Composition of 1-cells is given by taking the chosen pullback

$$\begin{tikzcd} & & X \times_B Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\ & X \arrow[dl, "p"'] \arrow[dr, "q"] & & Y \arrow[dl, "r"'] \arrow[dr, "s"] & \\ A & & B & & C \end{tikzcd}$$

as before, and then viewing this a map into the product

$$(p\pi_1, s\pi_2) : X \times_B Y \to A \times C$$

and finally taking the resulting monomorphism

$$\mathrm{im}\,(p\pi_1, s\pi_2) \rightarrowtail A \times C$$

from the image factorization (which is unique up to isomorphism by [15, Theorem 2.1.3]) as the result. The 2-cells α are given by those morphisms between the vertices of two jointly mono spans which yield commuting diagrams of the form

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "\alpha"] & \\ A & & B \\ & Y \arrow[ul, "p'"] \arrow[ur, "q'"'] & \end{tikzcd}$$

We will soon check well definedness of $\mathbf{Rel}\,(\mathcal{E})$ as a bicategory, via an application of Proposition 19.

Reflecting spans onto relations

As our first example of local reflections we consider the case with $\mathscr{B}' = \mathbf{Rel}\,(\mathcal{E})$ and $\mathscr{B} = \mathbf{Span}\,(\mathcal{E})$. Now, our family of adjunctions

$$\begin{tikzcd} \mathbf{Rel}\,(\mathcal{E})\,(X,Y) \arrow[r, bend right, "I_{X,Y}"'] \arrow[r, phantom, "\perp"] & \mathbf{Span}\,(\mathcal{E})\,(X,Y) \arrow[l, bend right, "L_{X,Y}"'] \end{tikzcd}, \qquad (X,Y) \in \mathcal{E}_{\text{ob}}^2$$

are given by first noting that

$$\mathbf{Rel}\,(\mathcal{E})\,(X,Y) \cong \mathcal{E}_{\text{mono}}/X \times Y, \qquad \mathbf{Span}\,(\mathcal{E})\,(X,Y) \cong \mathcal{E}/X \times Y$$

and defining the functor $L_{X,Y} : \mathcal{E}/X \times Y \to \mathcal{E}_{\text{mono}}/X \times Y$ on objects by taking the resulting monomorphism from the chosen image factorization, and on morphisms by

$$\begin{tikzcd} R \arrow[rr, "\alpha"] \arrow[dr, "p"'] & & R' \arrow[dl, "q"] \\ & X \times Y & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, two heads, "\overline{p}"'] & R' \arrow[d, two heads, "\overline{q}"] \\ \mathrm{im}p \arrow[r, dashed, "L\alpha"] \arrow[d, tail, "p'"'] & \mathrm{im}q \arrow[d, tail, "q'"] \\ X \times Y \arrow[r, "1_{X\times Y}"'] & X \times Y \end{tikzcd} \qquad (\dagger)$$

where $L\alpha$ is the arrow which exists due to $\overline{p}$ being a strong epimorphism, and q' a monomorphism. We simply define $I_{X,Y}$ to be the inclusion.

To see this is an adjunction, note that we may take our $\eta_{X,Y} : 1 \implies I_{X,Y}L_{X,Y}$ to be defined on a general component (a morphism p into the product $X \times Y$) by

$$\begin{tikzcd} R \arrow[r, two heads, "\eta_p"] \arrow[d, "p"'] & \mathrm{im}p \arrow[dl, tail, "I_{X,Y}L_{X,Y}p"] \\ X \times Y & \end{tikzcd}$$

and take $\varepsilon : L_{X,Y}I_{X,Y} \implies 1$ to be the identity (by defining L to do nothing on mono 1-cells, i.e. we choose the image factorization of a mono to be an identity followed by itself). It is then easily checked that $L\eta$ and ηI are identities, so that the triangle identities are satisfied.

Whiskering

We now check that L maps η and its whiskers into isomorphisms. Note first that L maps all regular epimorphisms into identities by our definition of L, and so as η consists of regular epimorphisms $L\eta$ is an identity. We now consider the diagram where we whisker a regular epi 2-cell (in particular a component of η) by a 1-cell on the right:

$$\begin{tikzcd}
& & T\times_B Y \arrow[dl, "a"'] \arrow[d, dotted, "\gamma"] \arrow[ddr, bend left, "b"] & & \\
& T \arrow[d, "\eta"'] \arrow[ddl, bend right, "m"'] \arrow[ddr, bend left, "n"] & X\times_B Y \arrow[dl, "r'"'] \arrow[dr, "s'"] & & \\
& X \arrow[dl, "p"'] \arrow[dr, "q"] & & Y \arrow[dl, "r"'] \arrow[dr, "s"] & \\
A & & B & & C
\end{tikzcd}$$

Now $q\eta a = na = rb$ so there is an induced map γ into the pullback $X \times_B Y$ which is the whiskering of η by the span $(r, s) : B \nrightarrow C$. Also both r, s', r', q and $r, s'\gamma, a, q\eta$ are pullback squares. By the pullback pasting lemma, we conclude that a, η, r', γ is also a pullback, and so since we're in a regular category we know that γ is a regular epi, and hence is sent to an isomorphism under L. The argument for whiskering on the left is similar. Hence by Proposition 19, $\mathbf{Rel}(\mathcal{E})$ is a locally reflective sub-bicategory of $\mathbf{Span}(\mathcal{E})$, and moreover by Theorem 21 we have the equivalence

$$\mathbf{Icon}(\mathbf{Rel}(\mathcal{E}), \mathscr{C}) \simeq \mathbf{Icon}(\mathbf{Span}(\mathcal{E}), \mathscr{C})_{\text{inv } \eta}$$

where a functor $F : \mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ inverts η when $F\eta$ is an invertible icon. We will now use this equivalence to derive the universal property of relations from that of spans.

Definition 30. Let $\mathcal{E}$ be a regular category, and let $\mathscr{C}$ be a bicategory. We define $\mathbf{Beck}_{\text{reg epi}}(\mathcal{E}, \mathscr{C})$ to be the category of Beck functors $H : \mathcal{E} \to \mathscr{C}$ for which $\varepsilon : H(p) H(p)^* \to 1$ is invertible for every regular epimorphism p (Beck functors which map regular epimorphisms to reflections), and invertible icons between them.

Theorem 31 (Universal property of relations)**.** *Let $\mathcal{E}$ be a regular category, and let $\mathscr{C}$ be a bicategory. We then have the equivalence*

$$\mathbf{Icon}(\mathbf{Rel}(\mathcal{E}), \mathscr{C}) \simeq \mathbf{Icon}(\mathbf{Span}(\mathcal{E}), \mathscr{C})_{\text{inv } \eta} \simeq \mathbf{Beck}_{\text{reg epi}}(\mathcal{E}, \mathscr{C})$$

Proof. Given a Beck functor $H : \mathcal{E} \to \mathscr{C}$ we get a corresponding functor $\overline{H} : \mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ under the equivalence $\mathbf{Icon}(\mathbf{Span}(\mathcal{E}), \mathscr{C}) \simeq \mathbf{Beck}(\mathcal{E}, \mathscr{C})$. Now by Lemma 20 the functor $\overline{H}$:

$\mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ inverts η, i.e. $\overline{H}\eta$ is invertible, if and only if $\overline{H}$ inverts all elements of

$$\Sigma = \{\text{2-cells } \alpha \in \mathbf{Span}(\mathcal{E}) : L\alpha \text{ is invertible in } \mathbf{Rel}(\mathcal{E})\}$$

It is clear from our definition of $L\alpha$ given by the diagram (†), that if α is a regular epimorphism then $L\alpha$ is as well (by the commutivity of the top right square). Moreover, the commutivity of the bottom right square means $L\alpha$ must also be mono and hence $L\alpha$ is invertible if α is a regular epi 2-cell. Hence Σ contains all regular epimorphisms, and so if $\overline{H}\eta$ is invertible, then $\overline{H}$ maps all regular epi 2-cells into isomorphisms. The converse, that if $\overline{H}$ maps regular epis into isomorphisms then $\overline{H}\eta$ is invertible, is trivial as the components of η are regular epimorphisms. Now, to ask that this pseudofunctor $\overline{H}$ maps any regular epi 2-cell α

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, two heads, "\alpha"] & \\ A & & B \\ & Y \arrow[ul, "p'"] \arrow[ur, "q'"'] & \end{tikzcd}$$

into an isomorphism, is to ask that

$$H(q)\,H(p)^* = H(q'\alpha)\,H(p'\alpha)^* \cong H(q')\,H(\alpha)\,H(\alpha)^*\,H(p')^* \xrightarrow{H(q')\,\varepsilon_\alpha H(p')^*} H(q')\,H(p')^*$$

is invertible. It is thus sufficient to ask that every $\varepsilon_\alpha : H(\alpha)\,H(\alpha)^* \to 1$ is invertible, and moreover necessary as we may consider 2-cells

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\alpha"] \arrow[dd, two heads, "\alpha"] & \\ Y & & Y \\ & Y \arrow[ul, "1_Y"] \arrow[ur, "1_Y"'] & \end{tikzcd}$$

Hence to give a pseudofunctor $\mathbf{Rel}(\mathcal{E}) \to \mathscr{C}$ is to give a Beck functor $H : \mathcal{E} \to \mathscr{C}$ for which $\varepsilon : H(p)\,H(p)^* \to 1$ is invertible for every regular epimorphism p. $\square$

Remark 32. Noting that (1) $\iff$ (2) of Lemma 20 holds in a general bicategory $\mathscr{C}$ (not just $\mathbf{LaxIcon}$), it is easily seen that if $H(p)$ is a reflection, then $H(p)$ may be realized as the bicoequalizer

$$\begin{tikzcd} \bullet \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "H(p)^*H(p)"'] & \bullet \arrow[r, "H(p)"] & \bullet \end{tikzcd}$$

4

Pulling back into isomorphisms

We all know that if we take a pullback square

$$\begin{tikzcd} X \arrow[r, "p'"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

then to say that both f' and p' are isomorphisms is to say that f and p are the same modulo an isomorphism, and are both monomorphisms. But what does it mean if we only ask that p' is an isomorphism? In this chapter we will answer this question in the context of locally cartesian closed categories.

To find out what it means for p' to be invertible, a natural question to ask is how the case in which we only ask that p' is invertible relates to the case where both f' and p' are invertible. Simplifying our question by only considering those those isomorphisms which are identities, this question reduces to asking what is special about the standard pullback of a monomorphism p along itself

$$\begin{tikzcd} E \arrow[r, equal] \arrow[d, equal] & E \arrow[d, tail, "p"] \\ E \arrow[r, tail, "p"'] & B \end{tikzcd} \tag{4.1}$$

relative to pullbacks for which we only ask that the bottom arrow p pulls back into an identity, i.e. those pullbacks of the form

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\ell"'] & A \arrow[d, "p\ell"] \\ E \arrow[r, tail, "p"'] & B \end{tikzcd} \tag{4.2}$$

We notice here that pullbacks of the form (4.2) admit a unique factorization through pullbacks of the form (4.1)

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[dd, "\ell"'] & A \arrow[dd, "p\ell"] \\
& \\
E \arrow[r, tail, "p"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, dotted, "\ell"'] & A \arrow[d, dotted, "\ell"] \\
E \arrow[r, equal] \arrow[d, equal] & E \arrow[d, tail, "p"] \\
E \arrow[r, tail, "p"'] & B
\end{tikzcd}
$$

In categorical language, we say that the pullback (4.1) is terminal among pullbacks of the form (4.2). Notice that this is only valid when p is a monomorphism; if p is not a monomorphism then (4.1) is not a pullback, and it is unclear if there is still a terminal pullback among pullbacks of the form (4.2). This is a question we will explore in this chapter.

4.1 Extracting the singleton fibers

Suppose we are given a function $p : E \to B$ in $\mathbf{Set}$. A common exercise is to restrict the domain of our function to a subset $\overline{E}_p \subseteq E$ so that the function restricted to our new domain $p : \overline{E}_p \to B$ is injective. This can be done in a canonical way; namely by taking $\overline{E}_p$ to be union of the singleton fibers of p. More precisely,

$$\overline{E}_p = \{e \in E \mid pe \text{ has singleton fibre}\} = \left\{e \in E \mid \forall e' \in E,\ pe = pe' \implies e = e'\right\}.$$

Denoting the subset inclusion $\overline{E}_p \subseteq E$ as the function $k_p : \overline{E}_p \hookrightarrow E$, we have in $\mathbf{Set}$ the property that for any given function p, there exists a canonical choice of injection k_p such that pk_p is injective. Note that this is the best *canonical* choice; we could for example include the empty set into E but then we are forgetting more elements of E than necessary. Conversely, there are often better choices than $\overline{E}_p$, for example restricting $x^2 : \mathbb{R} \to \mathbb{R}$ to non-negative reals instead of the only singleton fiber $\{0\}$, however such a choice is not canonical. Since k_p is the best canonical choice, we will not be surprised to find that certain constructions involving the inclusion k_p satisfy universal properties. We will now generalize this notion to any locally cartesian closed category $\mathcal{E}$.

Definition 33. Given any morphism $p : E \to B$ in a locally cartesian closed category $\mathcal{E}$, we define $k_p := \Pi_{\pi_1}(\delta_p)$ where δ_p and π_1 are defined by the pullback diagram

$$\begin{tikzcd}
E \arrow[drr, equal, bend left, "1_E"] \arrow[dr, dashed, "\delta_p"] \arrow[ddr, equal, bend right, "1_E"'] & & \\
& E \times_B E \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & E \arrow[d, "p"] \\
& E \arrow[r, "p"'] & B
\end{tikzcd}$$

Throughout the rest of this thesis, we will assume $p : E \to B$ is a fixed map in a locally cartesian closed category $\mathcal{E}$, with δ_p, π_1, π_2 and k_p defined as above, unless otherwise stated.

Remark 34. As δ_p is a monomorphism, it follows that $k_p = \Pi_{\pi_1}(\delta_p)$ is also a monomorphism by the following property of dependent products.

Lemma 35. *Dependent products preserve isomorphisms and monomorphisms*

Proof. As the dependent product functor $\Pi_p : \mathcal{E}/E \to \mathcal{E}/B$ is a right adjoint, it preserves limits, and in particular terminal objects; hence isomorphisms into E map into isomorphisms into B. For convenience, we will choose Π_p such that identities are mapped into identities.

We also note that since Π_p preserves limits, it will preserve the standard pullback of a monomorphism along itself, and so Π_p preserves monomorphisms. Hence for any monomorphism $m : X \hookrightarrow E$ in $\mathcal{E}$, (which gives the monomorphism $m : m \hookrightarrow 1_E$ in the slice $\mathcal{E}/E$) we get the monomorphism in the slice category $\mathcal{E}/B$

$$\begin{tikzcd}
Y \arrow[r, tail, "\Pi_p(m)"] \arrow[dr, "\Pi_p(m)"'] & B \arrow[d, equal] \\
& B
\end{tikzcd}$$

and so taking $\Pi_p(m)\, a = \Pi_p(m)\, b$ we have the commuting diagram

$$\begin{tikzcd}
T \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[dr, "\Pi_p(m)\, a = \Pi_p(m)\, b"'] & Y \arrow[r, tail, "\Pi_p(m)"] \arrow[d, "\Pi_p(m)"] & B \arrow[dl, equal] \\
& B &
\end{tikzcd}$$

forcing $a = b$ since $\Pi_p(m)$ is mono in $\mathcal{E}/B$. Thus $\Pi_p(m)$ is a monomorphism in $\mathcal{E}$. □

4.2 Pulling back into isomorphisms

In this section we will give necessary and sufficient conditions by which a fixed morphism $p : E \to B$ in a locally cartesian closed category $\mathcal{E}$ pulls back along another morphism $f : A \to B$ into an isomorphism. We then demonstrate that for such a morphism p, there is indeed a terminal pullback among pullbacks of the form (4.2).

Lemma 36. *The monomorphism δ_p is the equalizer of π_1 and π_2.*

Proof. Suppose we are given a morphism u such that $\pi_1 u = \pi_2 u$. Then both u and $\delta_p \pi_1 u$ render commutative the following diagram

$$\begin{tikzcd}
E \arrow[dr, shift left, "\delta_p\pi_1 u"] \arrow[dr, shift right, "u"'] \arrow[drr, bend left, "\pi_1 u = \pi_2 u"] \arrow[ddr, bend right, "\pi_1 u = \pi_2 u"'] & & \\
& E \times_B E \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & E \arrow[d, "p"] \\
& E \arrow[r, "p"'] & B
\end{tikzcd}$$

since $\pi_1 \delta_p \pi_1 u = \pi_1 u$ and $\pi_2 \delta_p \pi_1 u = \pi_1 u = \pi_2 u$, and so $u = \delta_p (\pi_1 u)$. The fact that δ_p is a monomorphism ensures the uniqueness property of the equalizer. □

Proposition 37. *Let a morphism $h : A \to E$ be given in the locally cartesian closed category $\mathcal{E}$, define $(A \times_B E, \zeta_1, h \times 1)$ to be the pullback*

$$\begin{tikzcd}
A \times_B E \arrow[r, "\zeta_1"] \arrow[d, "h \times 1"'] & A \arrow[d, "h"] \\
E \times_B E \arrow[r, "\pi_1"'] & E
\end{tikzcd}$$

Then the following are equivalent: (1) h admits a (unique) factorization through $k_p : \overline{E}_p \hookrightarrow E$; (2) $h \times 1$ admits a (unique) factorization through $\delta_p : E \hookrightarrow E \times_B E$; (3) $h\zeta_1 = \pi_2 (h \times 1)$; (4) ζ_1 is invertible; (5) $\eta_h : h \to \Delta_p \Sigma_p h$ is invertible.

Proof. (1) $\iff$ (2): Under the adjunction $\Delta_{\pi_1} \dashv \Pi_{\pi_1} : \mathcal{E}/E \times_B E \to \mathcal{E}/E$, we have a factorization $h = k_p f$, i.e. a morphism $f : h \to \Pi_{\pi_1} \delta_p = k_p$ in the slice $\mathcal{E}/E$, if and only if there is a morphism $g : h \times 1 = \Delta_{\pi_1} h \to \delta_p$ in the slice $\mathcal{E}/E \times_B E$, i.e. a morphism $g : A \times_B E \to E$ such that $h \times 1 = \delta_p g$.

(2) $\iff$ (3): Since δ_p equalizes π_1 and π_2, we have a factorization $h \times 1 = \delta_p g$ if and only if $\pi_1 (h \times 1) = \pi_2 (h \times 1)$, i.e. $h\zeta_1 = \pi_2 (h \times 1)$.

(2) $\implies$ (4): Assuming $h \times 1 = \delta_p g$ gives the composite of pullbacks

$$\begin{tikzcd}
A \times_B E \arrow[r, equal] \arrow[d, "g"'] & A \times_B E \arrow[r, "\zeta_1"] \arrow[d, "g"'] & A \arrow[dd, "h"] \\
E \arrow[r, equal] \arrow[d, equal] & E \arrow[d, "\delta_p"'] & \\
E \arrow[r, "\delta_p"'] & E \times_B E \arrow[r, "\pi_1"'] & E
\end{tikzcd}$$

and since the identity $\pi_1\delta_p$ must pullback into an isomorphism, we know that ζ_1 is invertible.

(4) $\implies$ (2): By assuming that ζ_1 is invertible, and considering the pullback composite

$$\begin{tikzcd}
\bullet \arrow[r, "r"] \arrow[d, "u"'] & A \times_B E \arrow[r, "\zeta_1"] \arrow[d, "h \times 1"] & A \arrow[d, "h"] \\
E \arrow[r, "\delta_p"'] & E \times_B E \arrow[r, "\pi_1"'] & E
\end{tikzcd}$$

we see that $\zeta_1 r$ (a pullback of an identity), and hence r, is invertible. Hence $h \times 1 = \delta_p u r^{-1}$.

(4) $\iff$ (5): Recall that η_h is defined as the unique map making the following diagram commute (wherein the bottom right square is a pullback)

$$\begin{tikzcd}
A \arrow[dr, dashed, "\eta_h"] \arrow[drr, equal, bend left] \arrow[ddr, bend right, "h"'] & & \\
& A \times_B E \arrow[r, "\gamma_1"] \arrow[d, "\Delta_p\Sigma_p h"] & A \arrow[d, "ph"] \\
& E \arrow[r, "p"'] & B
\end{tikzcd}$$

and this pullback coincides with the composite

$$\begin{tikzcd}
A \times_B E \arrow[r, "\zeta_1"] \arrow[d, "h \times 1"'] & A \arrow[d, "h"] \\
E \times_B E \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"'] & E \arrow[d, "p"] \\
E \arrow[r, "p"'] & B
\end{tikzcd}$$

modulo an isomorphism. It is then clear that ζ_1 is invertible iff γ_1 is invertible iff η_h is invertible. $\square$

We will now give our description of precisely when a morphism $p : E \to B$ pulls back into an isomorphism, which we will use often throughout the remainder of this thesis.

Theorem 38. *Given a pullback of morphisms $p : E \to B$ and $f : A \to B$ in a locally cartesian closed category $\mathcal{E}$,*

$$\begin{tikzcd}
A \times_B E \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & A \arrow[d, "f"] \\
E \arrow[r, "p"'] & B
\end{tikzcd}$$

The following are equivalent: (1) f admits a factorization through $pk_p : \overline{E}_p \to B$; (2) γ_1 is invertible; (3) $\varepsilon_f : \Sigma_p\Delta_p f \to f$ is invertible.

Proof. (1 $\implies$ 2): Suppose that $f : A \to B$ may be written as $A \xrightarrow{\overline{f}} \overline{E}_p \xrightarrow{k_p} E \xrightarrow{p} B$. We may then form the pullbacks

$$\begin{tikzcd} A \times_B E \arrow[r, "\zeta_1"] \arrow[d, "k_p\overline{f} \times 1"'] & A \arrow[d, "k_p\overline{f}"] \\ E \times_B E \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

As $k_p\overline{f}$ admits a factorization through k_p, we conclude that ζ_1 is invertible by Proposition 37. By the uniqueness of pullback up to isomorphism, we can write $\gamma_1 = \alpha\zeta_1$ for some isomorphism α, and so γ_1 is invertible.

($2 \implies 1$): Suppose now that γ_1 is invertible, so that $f = p\gamma_2\gamma_1^{-1}$. We may then form the composite of pullbacks (where we know $p\gamma_2\gamma_1^{-1} = f$)

$$\begin{tikzcd} A \times_B E \arrow[r, "\zeta_1"] \arrow[d, "\gamma_2\gamma_1^{-1} \times 1"'] & A \arrow[d, "\gamma_2\gamma_1^{-1}"] \\ E \times_B E \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

As before, the uniqueness of pullback up to isomorphism means $\alpha^{-1}\gamma_1 = \zeta_1$ for some invertible α, and so ζ_1 is invertible. Hence $\gamma_2\gamma_1^{-1}$ admits a factorization through k_p by Proposition 37, i.e. we may write $\gamma_2\gamma_1^{-1} = k_p\overline{f}$, so that $f = p\gamma_2\gamma_1^{-1} = pk_p\overline{f}$ as required.

($2 \iff 3$): This is since $\varepsilon_f = \gamma_1$ in this case. □

Lemma 39. *For all maps $f : A \to E$ and $u : A \to \overline{E}_p$ in the locally cartesian closed category $\mathcal{E}$, if $pf = pk_pu$ then $f = k_pu$.*

Proof. Suppose $pf = pk_pu$, and then consider the pullback diagram

$$\begin{tikzcd} A \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, equal, "1_A"] \arrow[ddr, bend right, equal, "f"] \arrow[ddr, bend right=50, equal, "k_pu"'] & & \\ & A \times_B E \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & A \arrow[d, "pk_pu = pf"] \\ & E \arrow[r, "p"'] & B \end{tikzcd}$$

Now since pk_pu factors through pk_p we know that γ_1 is invertible by Theorem 38, and so $\alpha = \gamma_1^{-1}$ regardless of whether we place k_pu or f on the far left. We can thus conclude that $k_pu = f = \gamma_2\alpha$ as required. □

Corollary 40. *For all maps $p : E \to B$ in a locally cartesian closed category $\mathcal{E}$, pk_p is a monomorphism.*

Proof. If $pk_pm = pk_pn$ for morphisms $m, n : X \to \overline{E}_p$ we have by Lemma 39 that $k_pm = k_pn$, and so since k_p is mono $m = n$. Hence pk_p is also a monomorphism. □

Corollary 41. *For all maps $p : E \to B$ in a locally cartesian closed category $\mathcal{E}$, p is a monomorphism if and only if k_p is an isomorphism.*

Proof. ($\Longrightarrow$): Assuming that p is a monomorphism we have the pullback square

$$\begin{tikzcd} E \arrow[r, equal] \arrow[d, equal] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

As the top morphism is invertible we can apply Theorem 38 and conclude that $p = pk_pu$ for some $u : E \to \overline{E}_p$. By Lemma 39 this means $k_pu = 1_E$. Hence $k_puk_p = k_p$ and so since k_p is mono we have $uk_p = 1_{\overline{E}_p}$. Hence k_p is an isomorphism with inverse u.

($\Longleftarrow$): Suppose k_p is invertible. Then since pk_p is a monomorphism so is $p = pk_pk_p^{-1}$. □

4.2.1 Universal property

Given a morphism $p : E \to B$ in a category $\mathcal{E}$ with pullbacks, consider the category in which objects are pullbacks of the form

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\ell"'] & A \arrow[d, "p\ell"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

and in which a morphism of pullbacks $h : (p, p\ell_1) \to (p, p\ell_2)$ is a factorization

$$\begin{tikzcd} X \arrow[r, equal] \arrow[dd, "\ell_1"'] & X \arrow[dd, "p\ell_1"] \\ \\ E \arrow[r, "p"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, equal] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, equal] \arrow[d, "\ell_2"'] & Y \arrow[d, "p\ell_2"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

Definition 42. We define the *singleton fiber pullback* about a morphism $p : E \to B$ to be the terminal object in this category, if it exists. We have shown that in a locally cartesian closed category $\mathcal{E}$, this *singleton fiber pullback* about p is given by

$$\begin{tikzcd} \overline{E}_p \arrow[r, equal] \arrow[d, tail, "k_p"'] \arrow[dr, phantom, "\text{sfpb}"] & \overline{E}_p \arrow[d, tail, "pk_p"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

where k_p is defined as in Definition 33.

4.3 Pulling back into monomorphisms

In much the same way as we can describe when an arrow pulls back into an isomorphism, we can describe when an arrow pulls back into a monomorphism. This description of when an arrow pulls back into a monomorphism easily follows from the previous section, as we can pull back such a monomorphism along itself to get the standard pullback (4.1), and apply Theorem 38.

Theorem 43. *Given a pullback of morphisms $p : E \to B$ and $f : A \to B$ in a locally cartesian closed category $\mathcal{E}$,*

$$\begin{tikzcd} A \times_B E \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

The following are equivalent: (1) γ_1 is a monomorphism; (2) γ_2 admits a (unique) factorization through k_p; (3) f admits a (unique) factorization through $\Pi_p(k_p)$.

Proof. ($1 \implies 2$): Suppose that γ_1 is a monomorphism. We then have the composite of pullbacks

$$\begin{tikzcd} A \times_B E \arrow[r, equal] \arrow[d, equal] & A \times_B E \arrow[d, tail, "\gamma_1"] \\ A \times_B E \arrow[r, tail, "\gamma_1"] \arrow[d, "\gamma_2"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

and so we may write $f\gamma_1 = pk_pu$ for some morphism u by Theorem 38. As $f\gamma_1 = p\gamma_2$ we have $pk_pu = p\gamma_2$ and so $k_pu = \gamma_2$ by Lemma 39.

($2 \implies 1$): Suppose that $\gamma_2 = k_pu$ for some morphism u, and then consider the diagram wherein all squares are pullbacks

$$\begin{tikzcd} A \times_B E \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \times_B E \arrow[r, "k_pu"] \arrow[d, "\gamma_1"] & E \arrow[d, "p"] \\ A \times_B E \arrow[r, "\gamma_1"] \arrow[d, "k_pu"'] & A \arrow[r, "f"'] \arrow[d, "f"] & B \\ E \arrow[r, "p"'] & B & \end{tikzcd}$$

As $f\gamma_1 = pk_pu$ factors through pk_p, we have that β, and similarly α, must be invertible as a consequence of Theorem 38. Hence γ_1 is a monomorphism.

($2 \iff 3$): Under the adjunction $\Delta_p \dashv \Pi_p$, to give a morphism $u : \Delta_p(f) \to k_p$ in the slice category $\mathcal{E}/E$ is to give a morphism $\widetilde{u} : f \to \Pi_p(k_p)$ in the slice category $\mathcal{E}/B$. In other words, to give a morphism u such that $\gamma_2 = k_p u$ is to give a morphism $\widetilde{u}$ such that $f = \Pi_p(k_p)\,\widetilde{u}$. $\square$

Remark 44. (1) In the case $\mathcal{E} = \mathbf{Set}$, the map $\Pi_p(k_p)$ is given by the inclusion

$$\Pi_p(k_p) : \left\{ b \in B : \left| p^{-1}(b) \right| \leq 1 \right\} \hookrightarrow B$$

(2) We have discussed when arrows pull back into isomorphisms and monomorphisms, but have not said anything about epimorphisms. It is an easy exercise to prove that an arrow p pulls back along an arrow f into a split epimorphism if and only if f factors through p, in any category $\mathcal{E}$ with pullbacks.

4.3.1 Distributivity pullbacks with invertible counit

Given morphisms $p : E \to B$ and $h : A \to E$ in a locally cartesian closed category $\mathcal{E}$, it is sometimes the case that the distributivity pullback around p and h has the form

$$\begin{tikzcd}
A \arrow[r, "p'"] \arrow[d, "h"'] & Y \arrow[d, "\Pi_p(h)"] \\
E \arrow[r, "p"'] & B
\end{tikzcd}$$

This happens precisely when the counit component $\varepsilon_h : \Delta_p \Pi_p(h) \to h$ is invertible. We now give sufficient (but not necessary) conditions for this component to be invertible.

Proposition 45. *Given a morphism $u : A \to \overline{E}_p$, if we take a distributivity pullback around p and $k_p u$, then the counit component $\varepsilon : \Delta_p \Pi_p(k_p u) \to k_p u$ is invertible.*

Proof. To see that the counit component ε given below is a split epimorphism, form the pullback around p and $k_p u$ given by Theorem 38

$$\begin{tikzcd}
& A \arrow[r, equal] \arrow[d, dotted, "\alpha"] \arrow[dl, equal, bend right] & A \arrow[d, dotted, "\beta"'] \arrow[dd, bend left, "pk_p u"] \\
& X \arrow[r, "p'"] \arrow[dl, "\varepsilon"'] & Y \arrow[d, "\Pi_p(k_p u)"] \\
A \arrow[r, "k_p u"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

and apply the universal property of the distributivity pullback to construct the morphisms α and β. Furthermore, to see that ε is a monomorphism assume that $\varepsilon a = \varepsilon b$ for two morphisms $a, b : Z \to X$ and consider the pullback around p and $k_p u$

$$\begin{tikzcd}
& Z \arrow[r, equal] \arrow[d, dotted, shift right, "a"'] \arrow[d, dotted, shift left, "b"] \arrow[ddl, bend right, "\varepsilon a = \varepsilon b"'] & Z \arrow[d, dotted, shift right, "p'b"'] \arrow[d, dotted, shift left, "p'b"] \arrow[ddd, bend left=60, "pk_pu\varepsilon a = pk_pu\varepsilon b"] \\
& X \arrow[r, "p'"] \arrow[dl, "\varepsilon"'] & Y \arrow[dd, "\Pi_p\left(k_pu\right)"'] \\
A \arrow[dr, "k_pu"'] & & \\
& E \arrow[r, "p"'] & B
\end{tikzcd}$$

which forces $a = b$ by the uniqueness of the induced arrow into the distributivity pullback. Hence ε is invertible as required. □

Remark 46. It is tempting to say that the counit of $\Delta_p \dashv \Pi_p$ at an object h is invertible if and only if the unit of $\Sigma_p \dashv \Delta_p$ is invertible at an object h. This would mean that $\varepsilon_h : \Delta_p\Pi_p(h) \to h$ is invertible if and only if h has the form $h = k_pu$; however this is clearly false. For example consider the distributivity pullback around a morphism $p : E \to B$ (which is not mono) and $h = 1_E$. In this case ε_h is invertible, but $h = 1_E$ does not have the required form.

Corollary 47. *If p is a monomorphism, then any distributivity pullback around $p : E \to B$ and another morphism $h : A \to E$ has invertible counit component.*

Proof. This is since k_p is invertible when p is a monomorphism as shown in Corollary 41, and so we may write $h = k_pu$ where $u = k_p^{-1}h$. □

Another corollary of Proposition 45 is that if an arrow pulls back into an isomorphism, then this isomorphism necessarily arises from pulling a monomorphism back along itself. We will state this result with only those isomorphisms which are identities for simplicity.

Corollary 48. *In a locally cartesian closed category $\mathcal{E}$, any pullback square in which p pulls back into an identity admits a factorization*

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[dd, "\ell"'] & A \arrow[dd, "p\ell"] \\
& \\
E \arrow[r, "p"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, tail, "p'"] \\
A \arrow[r, tail, "p'"] \arrow[d, "\ell"'] & X \arrow[d, "\Pi_p\left(\ell\right)"] \\
E \arrow[r, "p"'] & B
\end{tikzcd}$$

Proof. Applying Theorem 43 we see that ℓ is of the form k_pu since p pulls back into an identity, which is a monomorphism. Hence the distributivity pullback around p and ℓ has invertible counit component. That the arrow p' above is a monomorphism is equivalent to ℓ factoring through k_p by the same theorem. □

4.3.2 Universal property

Given a morphism $p : E \to B$ in a locally cartesian closed category $\mathcal{E}$, we may consider the distributivity pullback around p and k_p given by the diagram

$$\begin{tikzcd} & \overline{E}_p \arrow[r, tail, "p'"] & Y \arrow[dd, tail, "\Pi_p\,(k_p)"] \\ \overline{E}_p \arrow[ur, equal] \arrow[dr, "k_p"'] & & \\ & E \arrow[r, "p"'] & B \end{tikzcd}$$

The universal property of this distributivity pullback shows that it is the terminal object in the category of pullbacks of the form

$$\begin{tikzcd} A \times_B E \arrow[r, tail, "\gamma_1"] \arrow[d, "\gamma_2"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

for which γ_1 is a monomorphism, since by Theorem 43 γ_1 is a monomorphism if and only if this pullback is a pullback around p and k_p.

4.4 Extending pullbacks along singleton fibers

In the next chapter we will often want to pull a morphism $\ell : C \to E$ back along the inclusion of the singleton fibers of p, namely $k_p : \overline{E}_p \hookrightarrow E$. We now demonstrate that such a pullback has a particularly simple form if ℓ arises from pulling back a morphism along p.

Proposition 49. *If the right square below is a pullback, then there exists a pullback as on the left (where u is uniquely determined since k_p is mono)*

$$\begin{tikzcd} \overline{C}_q \arrow[r, "k_q"] \arrow[d, "u"'] & C \arrow[r, "q"] \arrow[d, "\ell"'] & D \arrow[d, "t"] \\ \overline{E}_p \arrow[r, "k_p"'] & E \arrow[r, "p"'] & B \end{tikzcd}$$

Proof. Given the pullback square on the right, we can form the pullback (X, a, b) of ℓ and k_p as below, and then since $tqb = pk_pa$ we know that if we extend the diagram with pullback squares to

$$\begin{tikzcd}
& C \times_D X \arrow[r, "\zeta_1"] \arrow[d, "\zeta_2"'] & X \arrow[d, "b"] \\
& C \times_D C \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & C \arrow[d, "q"] \\
X \arrow[r, "b"] \arrow[d, "a"'] & C \arrow[r, "q"] \arrow[d, "\ell"'] & D \arrow[d, "t"] \\
\overline{E}_p \arrow[r, "k_p"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

then ζ_1 is invertible by Theorem 38. Hence we may write $qb = qk_q m$ for some morphism m again by Theorem 38, and so $b = k_q m$ by Lemma 39. Note that since k_p is mono (and pullbacks preserve monos), we know that $\Delta_\ell(k_p) = b = k_q m$ is mono, and so m is also a monomorphism.

Now observe that in the top pullback of b and γ_1, we could have just taken the pullback of k_q and γ_1 instead. Again this would give an invertible top arrow by Theorem 38, and so we can conclude that $tqk_q = pk_p u$ for some u by the same theorem. Since $tq = p\ell$ this says $p\ell k_q = pk_p u$, which implies that $\ell k_q = k_p u$ by Lemma 39. But considering the diagram

$$\begin{tikzcd}
\overline{C}_q \arrow[rrd, tail, bend left, "k_q"] \arrow[rd, dashed, "h"] \arrow[rdd, bend right, "u"'] & & \\
& X \arrow[r, tail, "b = k_q m"] \arrow[d, "a"'] & A \arrow[d, "\ell"] \\
& \overline{E}_p \arrow[r, tail, "k_p"'] & E
\end{tikzcd}$$

where h is the induced map into the pullback, we see that $k_q mh = k_q$ and since k_q is mono this means $mh = 1_{\overline{C}_q}$. We also know that m is mono and so since $mhm = m$ we have $hm = 1_X$ as well. Thus m defines an isomorphism between X and $\overline{C}_q$, and so we have the composite

$$\begin{tikzcd}
\overline{C}_q \arrow[r, "m^{-1}"] \arrow[d, "u = am^{-1}"'] & X \arrow[r, "b"] \arrow[d, "a"'] & C \arrow[r, "q"] \arrow[d, "\ell"'] & D \arrow[d, "t"] \\
\overline{E}_p \arrow[r, equal] & \overline{E}_p \arrow[r, "k_p"'] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

with $bm^{-1} = k_q$ as required. □

5
Polynomial reflections

In order to motivate this chapter let us look at the relationship between $\mathbf{Span}\,(\mathbf{Set})$ and $\mathbf{Poly}\,(\mathbf{Set})$. Recall that the 1-cells in $\mathbf{Span}\,(\mathbf{Set})$ and $\mathbf{Poly}\,(\mathbf{Set})$ are analogous to linear and polynomial maps in many variables. Let us consider one variable maps for simplicity.

Trivially, a polynomial of the form ax (which we may think of as a 1-cell in $\mathbf{Span}\,(\mathbf{Set})$) may be written as the polynomial $0 + ax + 0x^2 + \cdots$ (which we may think of as a 1-cell in $\mathbf{Poly}\,(\mathbf{Set})$). We denote this inclusion of "spans" into "polynomials" by the function R. Conversely, given a polynomial $a_0 + a_1x + a_2x^2 + \cdots$ we may extract the term a_1x. We denote this reflection from "polynomials" to "spans" as L. It is clear that the inclusion R of "spans" into "polynomials" is functorial, i.e. that

$$R\,(ax) \circ R\,(bx) = R\,(ax \circ bx) = 0 + abx + 0x^2 + \cdots$$

But this is not so for our reflector L mapping "polynomials" into "spans"; for example

$$L\left(x^2\right) \circ L\,(x+1) = 0 \circ (x+1) = 0$$

differs from

$$L\left(x^2 \circ (x+1)\right) = L\left(x^2 + 2x + 1\right) = 2x$$

However, if we restrict ourselves to polynomials of the form $a_0 + a_1x$, which we will call "mono polynomials", then we indeed have functoriality since

$$L\,((a_0 + a_1x) \circ (b_0 + b_1x)) = L\,(a_0 + a_1\,(b_0 + b_1x)) = a_1b_1x$$

and

$$L\left(a_0+a_1x\right)\circ L\left(b_0+b_1x\right)=a_1x\circ b_1x=a_1b_1x$$

Similarly, if we restrict ourselves to polynomials for which the constant term vanishes, i.e. those of the form $\sum_{n\geq 1}a_nx^n$ (which one often does in the context of composing formal power series), then L is functorial. We will call these "epic polynomials". In fact the mapping L is also functorial on epi-mono factorizations, i.e.

$$L\left(\left(a_0+a_1x\right)\circ\sum_{n\geq 1}b_nx^n\right)=a_1b_1x=a_1x\circ b_1x=L\left(a_0+a_1x\right)\circ L\left(\sum_{n\geq 1}b_nx^n\right)$$

Now considering this from a more categorical viewpoint, we ask if this may be categorified to give an adjunction between L and $R:\mathbf{Span}\left(\mathcal{E}\right)\hookrightarrow\mathbf{Poly}\left(\mathcal{E}\right)$, and in particular is this a reflection or a coreflection. We may also ask if L is functorial if we restrict ourselves to polynomials of the form

$$\begin{tikzcd} & E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & \\ I & & & J \end{tikzcd}$$

for which p is a monomorphism, or alternatively if we restrict ourselves to polynomials for which p is a regular epimorphism in a regular category $\mathcal{E}$.

Remark 50. It turns out that the categorification of the mapping $L:\sum_{n\geq 0}a_nx^n\mapsto a_1x$ corresponds to extracting the singleton fibers, and so we will make considerable use of the previous chapter.

5.1 The coreflector

We intend to define a local coreflection in $\mathbf{LaxIcon}$

$$\begin{tikzcd} \mathbf{Span}\left(\mathcal{E}\right) \arrow[r, bend right, "R"'] \arrow[r, phantom, "\top"] & \mathbf{Poly}\left(\mathcal{E}\right) \arrow[l, bend right, "L"'] \end{tikzcd}$$

where R is the inclusion and L is the coreflector. (Note that L is the *right* adjoint to R here; however, we will later flip the adjunction by reversing the 2-cells.) It is clear how to define the inclusion R, but it remains to define our categorification of the mapping $L:\sum_{n\geq 0}a_nx^n\mapsto a_1x$.

5.1.1 Defining the coreflector locally

We will define our coreflector L by reversing the 2-cells in $\mathbf{Span}\left(\mathcal{E}\right)$ and $\mathbf{Poly}\left(\mathcal{E}\right)$ and applying Proposition 18. The first step is to define our family of adjunctions

$$\begin{tikzcd} \mathbf{Span}(\mathcal{E})^{\mathrm{co}}(I,J) \arrow[rr, bend right, "R_{I,J}"'] \arrow[rr, phantom, "\perp"] & & \mathbf{Poly}(\mathcal{E})^{\mathrm{co}}(I,J) \arrow[ll, bend right, "L_{I,J}"'] \end{tikzcd}, \qquad (I,J) \in \mathcal{E}_{\mathrm{ob}}$$

To do this, we first define the family of functors

$$L_{I,J} : \mathbf{Poly}(\mathcal{E})^{\mathrm{co}}(I,J) \to \mathbf{Span}(\mathcal{E})^{\mathrm{co}}(I,J)$$

by the assignment

$$\begin{tikzcd} & E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & \\ I & & & J \\ & C \arrow[r, "q"'] \arrow[ul, "s'"] \arrow[uu, "\ell"] & D \arrow[ur, "t'"'] \arrow[uu, "t"'] & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & \overline{E}_p \arrow[dl, "sk_p"'] \arrow[dr, "tpk_p"] & \\ I & & J \\ & \overline{C}_q \arrow[ul, "s'k_q"] \arrow[ur, "t'qk_q"'] \arrow[uu, "\Delta_{k_p}(\ell)"'] & \end{tikzcd}$$

Remark 51. Note that Proposition 49 shows that $\Delta_{k_p}(\ell)$ may be realized as a morphism $\overline{C}_q \to \overline{E}_p$ uniquely, so that this is indeed well defined.

We then define the family of functors

$$R_{I,J} : \mathbf{Span}(\mathcal{E})^{\mathrm{co}}(I,J) \to \mathbf{Poly}(\mathcal{E})^{\mathrm{co}}(I,J)$$

by the inclusion

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[dr, "t"] & \\ I & & J \\ & C \arrow[ul, "u"] \arrow[ur, "v"'] \arrow[uu, "f"'] & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & E \arrow[r, equal] \arrow[dl, "s"'] & E \arrow[dr, "t"] & \\ I & & & J \\ & C \arrow[r, equal] \arrow[ul, "u"] \arrow[uu, "f"] & C \arrow[ur, "v"'] \arrow[uu, "f"'] & \end{tikzcd}$$

To see that this indeed gives a family of adjunctions $L_{I,J} \dashv R_{I,J}$, note that we may take $\varepsilon : L_{I,J}R_{I,J} \to 1$ simply as the identity, and take our $\eta : 1 \to R_{I,J}L_{I,J}$ to be given at a polynomial $P = (t,p,s)$ by

$$\begin{tikzcd} & E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & \\ I & & & J \\ & \overline{E}_p \arrow[r, equal] \arrow[ul, "sk_p"] \arrow[uu, "k_p"] \arrow[ur, phantom, "\mathrm{sfpb}"] & \overline{E}_p \arrow[ur, "tpk_p"'] \arrow[uu, "pk_p"'] & \end{tikzcd}$$

In other words, our η consists of the singleton fiber pullbacks. For the triangle identities, note that if p is an identity above then this 2-cell η_P may be taken to be an identity. Moreover, if we apply L to the component of η at (t,p,s) we get an identity 2-cell since $\Delta_{k_p}(k_p) = 1_{\overline{E}_p}$ as k_p is a monomorphism (with a suitable choice of the pullback functor Δ_{k_p}).

5.1.2 Extending the coreflector

Clearly the family of inclusions $R_{I,J}$ extend to a pseudofunctor (and in particular a lax functor) $R : \mathbf{Span}(\mathcal{E})^{\mathrm{co}} \to \mathbf{Poly}(\mathcal{E})^{\mathrm{co}}$, and so we may apply Proposition 18 without axiom $L3$. The loss of axiom $L3$ means the induced oplax left adjoint $L : \mathbf{Poly}(\mathcal{E})^{\mathrm{co}} \to \mathbf{Span}(\mathcal{E})^{\mathrm{co}}$ might not have invertible constraints.

Reversing the 2-cells to dispose of the "co" on the bicategories flips the adjunction, leaves R as a pseudofunctor and turns the oplax L into a lax L. We thus have the following result.

Theorem 52. *Given a locally cartesian closed category $\mathcal{E}$, the lax functor $L : \mathbf{Poly}(\mathcal{E}) \to \mathbf{Span}(\mathcal{E})$ defined by the assignment*

$$\begin{tikzcd} & E \arrow[r, "p"] \arrow[dl, "s"'] \arrow[d, "a"'] & B \arrow[dr, "t"] \arrow[d, "b"] & \\ I & X \arrow[l, "u"] \arrow[r, "q"'] & Y \arrow[r, "v"'] & J \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & \overline{E}_p \arrow[dl, "sk_p"'] \arrow[dr, "tpk_p"] \arrow[d, "\Delta_{k_q}(a)"] & \\ I & \overline{X}_q \arrow[l, "uk_q"] \arrow[r, "vqk_q"'] & J \end{tikzcd}$$

with k_p and k_q defined as in Definition 33 is right adjoint to the inclusion $R : \mathbf{Span}(\mathcal{E}) \hookrightarrow \mathbf{Poly}(\mathcal{E})$ in $\mathbf{LaxIcon}$*. Moreover, composition with the maps L and R induce an equivalence*

$$\mathbf{LaxIcon}(\mathbf{Span}(\mathcal{E}), \mathscr{C}) \simeq \mathbf{LaxIcon}(\mathbf{Poly}(\mathcal{E}), \mathscr{C})_{\text{inv sfpb}}$$

where the RHS consists only of those functors $F : \mathbf{Poly}(\mathcal{E}) \to \mathscr{C}$ which map all 2-cells of the form

$$\begin{tikzcd} & \overline{E}_p \arrow[r, equal] \arrow[dl, "k_p"'] \arrow[d, "k_p"'] \arrow[dr, phantom, "\text{sfpb}"] & \overline{E}_p \arrow[dr, "pk_p"] \arrow[d, "pk_p"] & \\ E \arrow[r, equal] & E \arrow[r, "p"'] & B & B \arrow[l, equal] \end{tikzcd}$$

into isomorphisms. Note that it suffices to consider only these 2-cells since a general η_P comes from whiskering this with Σ_t and Δ_s.

5.2 Functoriality on monic and epic polynomials

In this section we show that if we restrict ourselves to polynomials $P = (t, p, s)$ for which p is a monomorphism, or alternatively if we restrict ourselves to polynomials for which p is a regular epimorphism in a regular category, then this lax functor $L : \mathbf{Poly}(\mathcal{E}) \to \mathbf{Span}(\mathcal{E})$ becomes a pseudofunctor (or equivalently the oplax $L : \mathbf{Poly}(\mathcal{E})^{\mathrm{co}} \to \mathbf{Span}(\mathcal{E})^{\mathrm{co}}$ reduces to a pseudofunctor).

5.2.1 A distributivity pullback lemma

Since pullbacks are symmetric, the pullback pasting lemma applies whether we are composing pullbacks horizontally or vertically. However, this is not the case for distributivity pullbacks. The horizontal "distributivity pullback lemma" is described in [12, Lemma 2.2.4]. Here we prove the version concerning composing distributivity pullbacks vertically.

Lemma 53 (Vertical distributivity pullback lemma)**.** *Suppose we are given a diagram of the form*

$$
\begin{tikzcd}
& & \bullet \arrow[r, "r"] \arrow[dl, "\varepsilon"'] \arrow[ddr, phantom, "\text{pb}"] & \bullet \arrow[dd, "v"] \\
& \bullet \arrow[dl, "d"'] \arrow[dr, "c"] \arrow[dd, phantom, "\text{pb}"] & & \\
\bullet \arrow[dr, "b"'] & & \bullet \arrow[r, "q"] \arrow[dl, "e"] \arrow[ddr, phantom, "\text{dpb}"] & \bullet \arrow[dd, "u"] \\
& \bullet \arrow[dr, "a"'] & & \\
& & \bullet \arrow[r, "p"'] & \bullet
\end{tikzcd}
$$

Then if the outside diagram is a distributivity pullback around (p, ab)*, the top pullback is a distributivity pullback around* (q, c).

Proof. Suppose we are given a pullback around (q, c) which consists of morphisms $\widetilde{v}, \widetilde{r}$ and $\widetilde{\varepsilon}$ such that $qc\widetilde{\varepsilon} = \widetilde{v}\widetilde{r}$. Since the outside is a distributivity pullback around (p, ab), there are induced arrows α and β such that $d\varepsilon\alpha = d\widetilde{\varepsilon}$, $r\alpha = \beta\widetilde{r}$ and $uv\beta = u\widetilde{v}$.

$$
\begin{tikzcd}
& & \bullet \arrow[r, "\widetilde{r}"] \arrow[d, dotted, "\alpha"] \arrow[ddl, bend right, "\widetilde{\varepsilon}"'] \arrow[dddll, bend right, "d\widetilde{\varepsilon}"'] & \bullet \arrow[d, dotted, "\beta"'] \arrow[ddd, bend left, "\widetilde{v}"] \\
& & \bullet \arrow[r, "r"] \arrow[dl, "\varepsilon"'] \arrow[ddr, phantom, "\text{pb}"] & \bullet \arrow[dd, "v"] \\
& \bullet \arrow[dl, "d"'] \arrow[dr, "c"] \arrow[dd, phantom, "\text{pb}"] & & \\
\bullet \arrow[dr, "b"'] & & \bullet \arrow[r, "q"] \arrow[dl, "e"] \arrow[ddr, phantom, "\text{dpb}"] & \bullet \arrow[dd, "u"] \\
& \bullet \arrow[dr, "a"'] & & \\
& & \bullet \arrow[r, "p"'] & \bullet
\end{tikzcd}
$$

We now check that $\varepsilon\alpha = \widetilde{\varepsilon}$ and $v\beta = \widetilde{v}$. Note that in order to prove that $\varepsilon\alpha = \widetilde{\varepsilon}$ it suffices to show that $c\varepsilon\alpha = c\widetilde{\varepsilon}$. This is because if $c\varepsilon\alpha = c\widetilde{\varepsilon}$ it then follows that $\varepsilon\alpha = \widetilde{\varepsilon}$ by uniqueness in the following diagram wherein the bottom right square is a pullback

$$
\begin{tikzcd}
\bullet \arrow[dr, shift left, "\widetilde{\varepsilon}"] \arrow[dr, shift right, "\varepsilon\alpha"'] \arrow[drr, bend left, "d\varepsilon\alpha = d\widetilde{\varepsilon}"] \arrow[ddr, bend right, "c\varepsilon\alpha = c\widetilde{\varepsilon}"'] & & \\
& \bullet \arrow[r, "d"] \arrow[d, "c"'] & \bullet \arrow[d, "b"] \\
& \bullet \arrow[r, "e"'] & \bullet
\end{tikzcd}
$$

The fact that $c\varepsilon\alpha = c\widetilde{\varepsilon}$ and $v\beta = \widetilde{v}$ follows from the uniqueness of the induced arrows into the bottom distributivity pullback via the diagram

$$
\begin{tikzcd}
& \bullet \arrow[r, "\widetilde{r}"] \arrow[d, shift right, "c\widetilde{\varepsilon}"'] \arrow[d, shift left, "c\varepsilon\alpha"] \arrow[ddl, bend right, "ec\varepsilon\alpha = bd\varepsilon\alpha = bd\widetilde{\varepsilon} = ec\widetilde{\varepsilon}"'] & \bullet \arrow[d, shift right, "\widetilde{v}"'] \arrow[d, shift left, "v\beta"] \arrow[ddd, bend left=60, "uv\beta = u\widetilde{v}"] \\
& \bullet \arrow[r, "q"] \arrow[dl, "e"'] & \bullet \arrow[dd, "u"] \\
\bullet \arrow[dr, "a"'] & \arrow[r, phantom, "\text{dpb}"] & \\
& \bullet \arrow[r, "p"'] & \bullet
\end{tikzcd}
$$

For uniqueness, note that the given pullback $(\widetilde{v}, \widetilde{r}, \widetilde{\varepsilon})$ around (q, c) may be extended to the pullback $(u\widetilde{v}, \widetilde{r}, d\widetilde{\varepsilon})$ around (p, ab). Supposing we are given two morphisms (α, β) and (γ, δ) of pullbacks around (q, c) from $(\widetilde{v}, \widetilde{r}, \widetilde{\varepsilon})$ to (v, r, ε), we get two morphisms with the same data, (α, β) and (γ, δ), from $(u\widetilde{v}, \widetilde{r}, d\widetilde{\varepsilon})$ to $(uv, r, d\varepsilon)$. By the fact that the outside diagram is a distributivity pullback we conclude $(\alpha, \beta) = (\gamma, \delta)$. $\square$

5.2.2 Whiskering on the right

We now consider whiskering a 2-cell η_P by a polynomial Q on the right. Our goal here is to show that if P is a regular epi polynomial, or if Q is a mono polynomial, then $L\,(Q\eta_P)$ is invertible. We first calculate the resulting 2-cell $Q\eta_P$ by applying the universal property of polynomial composition given in Proposition 14; this resulting 2-cell is given by the diagram

$$
\begin{tikzcd}
 & & & & & \bullet \arrow[r, equal] \arrow[dll, "\varepsilon"'] \arrow[ddd, dotted, "\alpha"] & \bullet \arrow[r, "w"] \arrow[dll, "\varepsilon"] \arrow[ddd, dotted, "\beta"] & \bullet \arrow[ddd, dotted, "\gamma"] \arrow[ddddd, bend left, "\Pi_q\left(\Delta_u\left(tpk_p\right)\right)"] & \\
 & & & \bullet \arrow[r, equal] \arrow[dll, "\widetilde{u}"'] & \bullet \arrow[dll, "\widetilde{u}"] \arrow[ddddrr, "\Delta_u\left(tpk_p\right)"] & & & & \\
 & \overline{E}_p \arrow[r, equal] \arrow[ddd, tail, "k_p"] & \overline{E}_p \arrow[ddd, tail, "pk_p"] & & & & & & \\
 & & & & & \bullet \arrow[r, "p''"] \arrow[dll, "e'"'] & \bullet \arrow[r, "q''"] \arrow[dll, "e"] & \bullet \arrow[dd, "\Pi_q\left(t'\right)"] & \\
 & & & \bullet \arrow[r, "p'"] \arrow[dll, "u''"'] & \bullet \arrow[dll, "u'"] \arrow[drr, "t'"] & & & & \\
 & E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[drr, "t"] & & & & X \arrow[r, "q"] \arrow[dll, "u"'] & Y \arrow[dr, "v"] & \\
I & & & & J & & & & K
\end{tikzcd}
$$

where α, β and γ are the induced arrows into the polynomial composite. Now as p'' pulls back into an isomorphism, we must have that $\alpha = k_{p''}h$ and $\beta = p''k_{p''}h$ for some h. We hence have a diagram

$$
\begin{tikzcd}
 & & \bullet \arrow[r, "w"] \arrow[dl, "h"'] \arrow[dr, phantom, "\text{pb}"] & \bullet \arrow[dd, "\gamma"] \\
 & \bullet \arrow[dl, "\Delta_{k_{p'}}\left(e'\right)"'] \arrow[dr, "p''k_{p''}"] \arrow[dd, phantom, "\text{pb}"] & & \\
\bullet \arrow[dr, "p'k_{p'}"'] & & \bullet \arrow[r, "q''"] \arrow[dl, "e"] \arrow[dr, phantom, "\text{dpb}"] & \bullet \arrow[dd, "\Pi_q\left(t'\right)"] \\
 & \bullet \arrow[dr, "t'"'] & & \\
 & & \bullet \arrow[r, "q"'] & \bullet
\end{tikzcd}
$$

where the square on the LHS is the pullback composite

$$
\begin{tikzcd}
\bullet \arrow[r, "k_{p''}"] \arrow[d, "\Delta_{k_{p'}}\left(e'\right)"'] & \bullet \arrow[r, "p''"] \arrow[d, "e'"'] & \bullet \arrow[d, "e"] \\
\bullet \arrow[r, "k_{p'}"'] & \bullet \arrow[r, "p'"'] & \bullet
\end{tikzcd}
$$

given by Proposition 49. Now by the same proposition we know that $t'p'k_{p'}$ is the same as $\Delta_u\left(tpk_p\right)$ modulo an isomorphism, since we have the pullback composite

$$
\begin{tikzcd}
\bullet \arrow[r, "k_{p'}"] \arrow[d, "\Delta_{k_p}\left(u''\right)"'] & \bullet \arrow[r, "p'"] \arrow[d, "u''"'] & \bullet \arrow[r, "t'"] \arrow[d, "u'"'] & X \arrow[d, "u"] \\
\overline{E}_p \arrow[r, "k_p"'] & E \arrow[r, "p"'] & B \arrow[r, "t"'] & J
\end{tikzcd}
$$

Since the outside is a distributivity pullback around q and $t'p'k_{p'} \cong \Delta_u\left(tpk_p\right)$ we conclude that the top pullback is a distributivity pullback around q'' and $p''k_{p''}$. Hence we may write $\gamma = \Pi_{q''}\left(p''k_{p''}\right)$. It remains to check that in the diagram

$$\begin{tikzcd} \bullet \arrow[r, "k_w"] \arrow[d, "\phi"'] & \bullet \arrow[r, equal] \arrow[d, "k_{p''}h"'] & \bullet \arrow[r, "w"] \arrow[d, "p''k_{p''}h"'] & \bullet \arrow[d, "\gamma = \Pi_{q''}\left(p''k_{p''}\right)"] \\ \bullet \arrow[r, "k_{q''p''}"'] & \bullet \arrow[r, "p''"'] & \bullet \arrow[r, "q''"'] & \bullet \end{tikzcd}$$

ϕ is invertible. But ϕ is invertible if and only if $q''p''k_{q''p''}$ factors through $\Pi_{q''}\left(p''k_{p''}\right)$, using the fact that $\Pi_{q''}\left(p''k_{p''}\right)$ is mono as $p''k_{p''}$ is mono and Theorem 38. That is, ϕ is invertible if and only if there is a morphism $q''p''k_{q''p''} \to \Pi_{q''}\left(p''k_{p''}\right)$ in the slice $\mathcal{E}/B$, which corresponds under the adjunction $\Delta_{q''} \dashv \Pi_{q''}$ to a morphism $\Delta_{q''}\left(q''p''k_{q''p''}\right) \to p''k_{p''}$. But due to Theorem 38, there exists a morphism $\Delta_{q''}\left(q''p''k_{q''p''}\right) \to p''k_{p''}$, i.e. a factorization of $\Delta_{q''}\left(q''p''k_{q''p''}\right)$ through $p''k_{p''}$, if and only if in the pullback composite

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\xi"'] & \bullet \arrow[d, "p''"] \\ \bullet \arrow[r, "\Delta_{q''}\left(q''p''k_{q''p''}\right)"] \arrow[d, "\delta"'] & \bullet \arrow[d, "q''"] \\ \bullet \arrow[r, "q''p''k_{q''p''}"'] & \bullet \end{tikzcd}$$

ξ is invertible. Upon noting that the outside square is actually the singleton fiber pullback about $q''p''$, we see that $\delta\xi$ defines an isomorphism and so ξ is a split monomorphism and δ is a split epimorphism. Now if p'' is a regular epi, then so is ξ making ξ invertible. Alternatively, if q'' is mono then δ must also be mono, making δ invertible, and so $\xi = \delta^{-1}(\delta\xi)$ is also invertible. Hence ϕ defined earlier is indeed invertible, and so we have shown $L(Q\eta_P)$ is an isomorphism provided P is an epic polynomial, or Q a mono polynomial.

5.2.3 Whiskering on the left

Similarly to the previous result, we aim to show that if P is an epic polynomial, or if Q is a mono polynomial, then $L(\eta_Q P)$ is invertible. We again calculate $\eta_Q P$ using the universal property of polynomial composition from Proposition 14, given by the diagram

$$\begin{tikzcd}[column sep=small]
 & & & & & \bullet \arrow[r, "\widetilde{p}"] \arrow[dd, dotted, "\alpha"'] \arrow[ddddllll, "u''"'] & \bullet \arrow[rr, equal] \arrow[dd, dotted, "\beta"'] \arrow[dr, "\widetilde{t}"] \arrow[ddddllll] & & \bullet \arrow[dd, dotted, "\gamma"'] \arrow[dr, "\widetilde{t}"] & & \\
 & & & & & & & \overline{X}_q \arrow[rr, equal] \arrow[dddl, tail, "k_q"'] & & \overline{X}_q \arrow[dddl, tail, "qk_q"] & \\
 & & & & & \bullet \arrow[r, "p''"] \arrow[dll, "e'"'] & \bullet \arrow[rr, "q''"] \arrow[dll, "e"] & & \bullet \arrow[dd, "\Pi_q(t')"'] & & \\
 & & & \bullet \arrow[r, "p'"] \arrow[dll] & \bullet \arrow[dll, "u'"] \arrow[drr, "t'"] & & & & & & \\
 & E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[drr, "t"] & & & & X \arrow[rr, "q"] \arrow[dll, "u"'] & & Y \arrow[drr, "v"] & & \\
I & & & & J & & & & & & K
\end{tikzcd}$$

We will first show that γ is merely $q''k_{p''}$ by making use of the universal property of singleton fiber pullbacks described in Definition 42.

Lemma 54. *The pullback square in the above diagram*

$$\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "\beta"'] & \bullet \arrow[d, "\gamma"] \\
\bullet \arrow[r, "q''"'] & \bullet
\end{tikzcd}$$

is actually a singleton fiber pullback.

Proof. To prove this square is a singleton fiber pullback it suffices to show that this square is terminal among pullbacks of the form

$$\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "\ell"'] & \bullet \arrow[d, "q''\ell"] \\
\bullet \arrow[r, "q''"'] & \bullet
\end{tikzcd}$$

So let ℓ be any morphism giving such a pullback, and consider the commuting diagram

$$\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[dd, bend right=60, "\ell"'] & \bullet \arrow[dd, bend left=60, "q''\ell"] \\
\bullet \arrow[r, equal] \arrow[d, "\beta"'] \arrow[ddd, bend right=60, "k_q\widetilde{t}"'] & \bullet \arrow[d, "\gamma"] \\
\bullet \arrow[r, "q''"'] \arrow[d, "e"'] & \bullet \arrow[dd, "\Pi_q(t')"] \\
\bullet \arrow[d, "t'"'] & \\
\bullet \arrow[r, "q"'] & \bullet
\end{tikzcd}$$

with our goal being to find a unique y such that $\beta y = \ell$. We note that $t'e\ell = k_q x$ for a unique x by Theorem 43, as q pulls back into the identity monomorphism. Now considering the pullback diagram

$$\begin{tikzcd}
\bullet \arrow[dr, dotted, "y"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "e\ell"'] & & \\
& \bullet \arrow[r, "\widetilde{t}"] \arrow[d, tail, "r"'] & \bullet \arrow[d, tail, "k_q"] \\
& \bullet \arrow[r, "t'"] \arrow[d, "u'"'] & \bullet \arrow[d, "u"] \\
& \bullet \arrow[r, "t"'] & \bullet
\end{tikzcd}$$

we see that $x = \widetilde{t}y$ for some y, and so $t'e\ell = k_q\widetilde{t}y$. It remains to show that our y satisfies $\beta y = \ell$. We observe that since the commutivity property of the universal property of polynomial composition forces $u'e\beta = \Delta_t\left(uk_q\right)$ in the diagram for $\eta_Q P$ above, we have $r = e\beta$ by uniqueness of the induced arrow into the pullback (u, u', t, t') above. Forming the pullback around q and t'

$$\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, dotted, shift right, "\ell"'] \arrow[d, dotted, shift left, "\beta y"] \arrow[dd, bend right=60, "e\ell = e\beta y"'] & \bullet \arrow[d, dotted, shift right, "q''\beta y"'] \arrow[d, dotted, shift left, "q''\ell"] \arrow[ddd, bend left=60, "qt'e\ell"] \\
\bullet \arrow[r, "q''"'] \arrow[d, "e"'] & \bullet \arrow[dd, "\Pi_q\left(t'\right)"'] \\
\bullet \arrow[d, "t'"'] & \\
\bullet \arrow[r, "q"'] & \bullet
\end{tikzcd}$$

and noting that both ℓ and βy satisfy the property of the induced maps into the distributivity pullback, we see that $\beta y = \ell$ by uniqueness. Lastly, let us suppose that we have two solutions y_1 and y_2 such that $\beta y_1 = \beta y_2 = \ell$. Then as $t'e\beta = k_q\widetilde{t}$ we have $k_q\widetilde{t}y_1 = k_q\widetilde{t}y_2 = t'e\ell$ and so as k_q is mono $\widetilde{t}y_1 = \widetilde{t}y_2$. Hence both y_1 and y_2 give a commuting diagram

$$\begin{tikzcd}
\bullet \arrow[dr, shift left, "y_1"] \arrow[dr, shift right, "y_2"'] \arrow[drr, bend left, "\widetilde{t}y_1 = \widetilde{t}y_2"] \arrow[dddr, bend right, "u'e\beta y_1 = u'e\beta y_2"'] & & \\
& \bullet \arrow[r, "\widetilde{t}"] \arrow[d, tail, "r = e\beta"'] & \bullet \arrow[d, tail, "k_q"] \\
& \bullet \arrow[r, "t'"] \arrow[d, "u'"'] & \bullet \arrow[d, "u"] \\
& \bullet \arrow[r, "t"'] & \bullet
\end{tikzcd}$$

which forces $y_1 = y_2$ by uniqueness. $\square$

Finally, we consider the composite of pullbacks

$$\begin{tikzcd}
\bullet \arrow[r, tail] \arrow[d, "\psi"'] & \bullet \arrow[d, "p''"] \\
\bullet \arrow[r, tail] \arrow[d, "\varphi"'] & \bullet \arrow[d, "q''"] \\
\bullet \arrow[r, tail, "q''p''k_{q''p''}"'] & \bullet
\end{tikzcd}$$

and note that this is just the singleton fiber pullback about $q''p''$, so the composite $\varphi\psi$ is invertible, and so ψ is split mono and φ is split epi. If p'' is a regular epi, then so is ψ forcing both ψ and φ to be invertible. If q'' is mono then so is φ, forcing both φ and ψ to be invertible. As φ is invertible in either case, we may write $q''p''k_{q''p''} = q''k_{q''}x$ for some x by Theorem 38. It follows by the same theorem that in the pullback

$$\begin{tikzcd}
\bullet \arrow[r, tail] \arrow[d, tail, "\phi"'] & \bullet \arrow[d, tail, "\gamma = q''k_{q''}"] \\
\bullet \arrow[r, tail, "q''p''k_{q''p''}"'] & \bullet
\end{tikzcd}$$

ϕ is invertible, as required. We have now shown that if P is an epic polynomial, or if Q is a monic polynomial, then $L(\eta_Q P)$ is invertible.

5.2.4 Functoriality on epic and monic polynomials

We now give sufficient conditions by which the oplax constraints of L are invertible.

Corollary 55. *If P is a regular epi polynomial, or Q mono polynomial, then the oplax constraint $L(QP) \to L(Q)\,L(P)$ is invertible.*

Proof. It suffices to show that $L(\eta_Q * \eta_P)$ is invertible, as the invertibility of this guarantees the invertibility of the oplax constraints of L by the argument of Proposition 18. Now the horizontal composite $\eta_Q * \eta_P$ may be defined in terms of whiskering as

$$\eta_Q * \eta_P = ILQ\eta_P \circ \eta_Q P$$

We have shown that $L(ILQ\eta_P)$ is invertible in 'whiskering on the right', and that $L(\eta_Q P)$ is invertible by the argument in 'whiskering on the left'. As L is strictly functorial with respect to vertical composition of 2-cells, we see that $L(\eta_Q * \eta_P) = L(ILQ\eta_P) \circ L(\eta_Q P)$ is invertible as required. □

Remark 56. In particular, this means the reflectors $L|_{\text{mono}} : \mathbf{Poly}(\mathcal{E})^{\text{co}}_{\text{mono}} \to \mathbf{Span}(\mathcal{E})^{\text{co}}$ and $L|_{\text{epic}} : \mathbf{Poly}(\mathcal{E})^{\text{co}}_{\text{epic}} \to \mathbf{Span}(\mathcal{E})^{\text{co}}$ are pseudofunctors, where $\mathbf{Poly}(\mathcal{E})_{\text{mono}}$ denotes the full sub-bicategory of $\mathbf{Poly}\left(\mathcal{E}\right)$ containing only monic polynomials, and $\mathbf{Poly}\left(\mathcal{E}\right)_{\text{epic}}$ denotes the sub-bicategory containing only regular epi polynomials, where $\mathcal{E}$ is a regular category.

5.2.5 Universal properties of epic and monic polynomials

Applying Proposition 15 to the adjunction $L \vdash R : \mathbf{Span}\left(\mathcal{E}\right) \hookrightarrow \mathbf{Poly}\left(\mathcal{E}\right)_{\text{mono/epic}}$ in $\mathbf{Icon}$ it follows that for any locally cartesian closed category $\mathcal{E}$,

$$\mathbf{Icon}\left(\mathbf{Span}\left(\mathcal{E}\right), \mathscr{C}\right) \simeq \mathbf{Icon}\left(\mathbf{Poly}\left(\mathcal{E}\right)_{\text{mono}}, \mathscr{C}\right)_{\text{inv sfpb}}$$

and for any locally cartesian closed regular category $\mathcal{E}$,

$$\mathbf{Icon}\left(\mathbf{Span}\left(\mathcal{E}\right), \mathscr{C}\right) \simeq \mathbf{Icon}\left(\mathbf{Poly}\left(\mathcal{E}\right)_{\text{epic}}, \mathscr{C}\right)_{\text{inv sfpb}}$$

Remark 57. (1) Note that L (which is a pseudofunctor when restricted to epic or monic polynomials) is right adjoint to the inclusion pseudofunctor R in $\mathbf{Icon}$, $\mathbf{LaxIcon}$, and $\mathbf{OplaxIcon}$, and so the universal property of the adjunction $\left(R \dashv L\right)|_{\text{mono/epic}}$ given by Proposition 15 we just mentioned in the case of $\mathbf{Icon}$, also holds in $\mathbf{LaxIcon}$ and $\mathbf{OplaxIcon}$. (2) Note that these equivalences may be regarded as restrictions of the universal properties of monic and epic polynomials with cartesian 2-cells.

6
Future directions

We are now familiar with the universal property of spans

$$\mathbf{Icon}\left(\mathbf{Span}\left(\mathcal{E}\right),\mathscr{C}\right)\simeq\mathbf{Beck}\left(\mathcal{E},\mathscr{C}\right)$$

and the universal property of relations

$$\mathbf{Icon}\left(\mathbf{Rel}\left(\mathcal{E}\right),\mathscr{C}\right)\simeq\mathbf{Beck}_{\text{reg epi}}\left(\mathcal{E},\mathscr{C}\right);$$

however, it remains to find a universal property for the bicategory of polynomials $\mathbf{Poly}\left(\mathcal{E}\right)$. We expect the bicategory of polynomials with general 2-cells (in which case we have adjunctions between 1-cells $\Sigma_p \dashv \Delta_p \dashv \Pi_p$) to have a universal property analogous to that of spans, involving a Beck condition and a distributivity condition. With the cartesian 2-cells (for which we don't have an adjunction $\Delta_p \dashv \Pi_p$), we expect the universal property to be more complicated; though in this case we may make use of our adjunction between the inclusion R and coreflector L:

$$\begin{tikzcd} \mathbf{Span}\left(\mathcal{E}\right) \arrow[r, shift left, "R"] \arrow[r, phantom, "\perp"] & \mathbf{Poly}\left(\mathcal{E}\right) \arrow[l, shift left, "L"] \end{tikzcd}.$$

In future work we aim to derive universal properties of the bicategory $\mathbf{Poly}\left(\mathcal{E}\right)$ with both the cartesian and general 2-cells. We then wish to develop a theory of $\mathbf{Poly}\left(\mathcal{E}\right)$-enriched categories (in the context of indexed/fibered categories) as has been developed for $\mathbf{Span}\left(\mathcal{E}\right)$-enriched categories. Furthermore, we aim to describe algebraic structures such as groups, monoids and modules in terms of $\mathbf{Poly}\left(\mathcal{E}\right)$-enriched categories.

References

[1] C. Hermida. *Representable multicategories*. Adv. Math. **151**(2), 164 (2000). URL `http://dx.doi.org/10.1006/aima.1999.1877`.

[2] S. Lack. *Icons*. Appl. Categ. Structures **18**(3), 289 (2010). URL `http://dx.doi.org/10.1007/s10485-008-9136-5`.

[3] J. Bénabou. *Introduction to bicategories*. In *Reports of the Midwest Category Seminar*, pp. 1–77 (Springer, Berlin, 1967). URL `http://dx.doi.org/10.1007/BFb0074299`.

[4] G. M. Kelly and R. Street. *Review of the elements of* 2*-categories*. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pp. 75–103. Lecture Notes in Math., Vol. 420 (Springer, Berlin, 1974). URL `http://dx.doi.org/10.1007/BFb0063101`.

[5] R. J. M. Dawson, R. Paré, and D. A. Pronk. *Universal properties of Span*. Theory Appl. Categ. **13**, No. 4, 61 (2004). URL `http://www.tac.mta.ca/tac/volumes/13/4/13-04.pdf`.

[6] N. Gambino and J. Kock. *Polynomial functors and polynomial monads*. Math. Proc. Cambridge Philos. Soc. **154**(1), 153 (2013). URL `http://dx.doi.org/10.1017/S0305004112000394`.

[7] M. Abbott, T. Altenkirch, and N. Ghani. *Containers: constructing strictly positive types*. Theoret. Comput. Sci. **342**(1), 3 (2005). URL `http://dx.doi.org/10.1016/j.tcs.2005.06.002`.

[8] T. von Glehn. *Polynomials and models of type theory*. Ph.D. thesis, University of Cambridge (2015).

[9] T. Bisson and A. Joyal. *The Dyer-Lashof algebra in bordism (extended abstract)*. C. R. Math. Rep. Acad. Sci. Canada **17**(4), 135 (1995).

[10] J.-Y. Girard. *Normal functors, power series and λ-calculus*. Ann. Pure Appl. Logic **37**(2), 129 (1988). URL `http://dx.doi.org/10.1016/0168-0072(88)90025-5`.

[11] J. Kock. *Notes on polynomial functors* (2009). URL `http://mat.uab.es/~kock/cat/polynomial.pdf`.

[12] M. Weber. *Polynomials in categories with pullbacks*. Theory Appl. Categ. **30**, No. 16, 533 (2015). URL `http://www.tac.mta.ca/tac/volumes/30/16/30-16.pdf`.

[13] G. M. Kelly. *Doctrinal adjunction*. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pp. 257–280. Lecture Notes in Math., Vol. 420 (Springer, Berlin, 1974). URL `http://dx.doi.org/10.1007/BFb0063105`.

[14] B. Day. *Construction of Biclosed Categories*. Ph.D. thesis, University of New South Wales (1970). URL `http://www.math.mq.edu.au/~street/DayPhD.pdf`.

[15] F. Borceux. *Handbook of categorical algebra. 2*, vol. 51 of *Encyclopedia of Mathematics and its Applications* (Cambridge University Press, Cambridge, 1994). Categories and structures.