Linear Programming Approach to Discrete Time Optimal Control Problems With Time Discounting

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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any other university or institution.

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Abstract

The linear programming (LP) approach to control systems is based on the fact that the occupational measures generated by admissible controls and the corresponding solutions of a nonlinear system satisfy certain linear equations representing the system's dynamics in an integral form. The idea of such linearization was explored extensively in relation to various deterministic and stochastic problems of optimal control of systems that evolve in continuous time. However, no results based on this idea for deterministic discrete time control systems is available in the literature. The thesis aims at the development of LP based techniques for analysis and solution of a deterministic discrete time optimal control problem with time discounting criteria. To this end, we reformulate the optimal control problem as that of optimization problem on the set of discounted occupational measures and we show that the optimal value of the latter is equal to the optimal value of a certain infinite dimensional (ID) LP problem. We then show that this IDLP problem can be approximated by semi-infinite linear programming problems and subsequently by finite-dimensional ("standard") LP problems. We also indicate a way how a near optimal control of the underlying nonlinear optimal control problem can be constructed on the basis of the solution of an approximating finite-dimensional LP problem.

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Contents

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Introduction

The linear programming (LP) approach to control systems is based on the fact that the occupational measures generated by admissible controls and the corresponding solutions of a nonlinear system satisfy certain linear equations representing the system's dynamics in an integral form. Using this fact, one can reformulate an optimal control problem as an infinite-dimensional linear programming (IDLP) problem. The idea of such linearization was explored extensively in relation to various deterministic and stochastic problems of optimal control of systems that evolve in continuous time (see, e.g., [3–6, 9, 11–15, 18, 20–22] and references therein). However, to the best of our knowledge, no results based on this idea for control systems evolving in discrete time is available in the literature.

The thesis aims at the development of LP based techniques for the analysis and solution of a deterministic discrete time optimal control problem with time discounting criteria. To this end, we reformulate the optimal control problem evolving in discrete time as that of an optimization problem on the set of discounted occupational measures and we show that the optimal value of the latter is equal to the optimal value of a certain IDLP problem. We then show that this IDLP problem can be approximated by semi-infinite linear programming (SILP) problems and subsequently by finite-dimensional ("standard") LP problems. We indicate how a near optimal control of the underlying nonlinear optimal control problem can be constructed on the basis of the solution of an approximating finite-dimensional LP problem. Note that continuous time counterparts of some of our results can be found in [8], [9] and [10]. Some of the results obtained in the thesis are announced in [7].

The thesis is organized as follows. Section 1 is this introduction. In Section 2, we introduce the optimal control problem that is the subject of our consideration and we introduce the IDLP problem that, as established in the thesis, is closely related to the former. Also in this section, we introduce the problem dual with respect to this IDLP problem. The main results of the section are Proposition 2.3 and Theorem 2.5. Proposition 2.3 establishes that the set of discounted occupational measures generated by the admissible controls and the corresponding solutions of the discrete time system is contained in the feasible set of the IDLP problem, while Theorem 2.5 establishes relationships between the IDLP problem and its dual.

In Section 3, we use the dynamic programming principle and one of the duality relationships to establish that the optimal value of the optimal control problem is equal to the optimal value of the IDLP problem (Theorem 3.2), and we state necessary and sufficient optimality conditions in terms of a solution of the problem dual to the IDLP problem (Proposition 3.3). In Section 4 we show that the IDLP problem is approximated by a sequence of SILP problems (Proposition 4.1) and that the SILP problems are, in turn, approximated by finite-dimensional LP problems (Proposition 4.3). We then indicate a way of constructing a near optimal control on the basis of an optimal solution of an approximating finite-dimensional LP problem and we demonstrate the construction with a numerical example.

We conclude this introductory section with some comments and notations. Let Y stand for a compact subset of R^m and U be a compact metric space. Denote by $\mathcal{P}(Y \times U)$ the space of probability measures defined on Borel subsets of $Y \times U$.

Let us endow the space $\mathcal{P}(Y \times U)$ with a metric ρ,

$$\rho(\gamma', \gamma'') \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} \frac{1}{2^j} \left| \int_{Y \times U} q_j(y,u)\gamma'(dy,du) - \int_{Y \times U} q_j(y,u)\gamma''(dy,du) \right|, \tag{1.1}$$

$\forall \gamma', \gamma'' \in \mathcal{P}(Y \times U)$ where $q_j(\cdot), j = 1, 2, \ldots$ is a sequence of Lipschitz continuous functions that is dense in the unit ball of $C(Y \times U)$. This metric is consistent with the weak convergence topology of $\mathcal{P}(Y \times U)$. Namely, a sequence $\gamma^k \in \mathcal{P}(Y \times U)$ converges to $\gamma \in \mathcal{P}(Y \times U)$, that is, $\lim_{k\to\infty} \rho(\gamma^k, \gamma) = 0$, if and only if

$$\lim_{k\to\infty} \int_{Y \times U} q(y,u)\gamma^k(dy,du) = \int_{Y \times U} q(y,u)\gamma(dy,du) \tag{1.2}$$

for any continuous $q(y,u) : Y \times U \to \mathbb{R}$. Note that the space $\mathcal{P}(Y \times U)$ is known to be compact in weak convergence (weak*) topology (see, e.g., [2] or [17]). Using the metric ρ we can consider the "distance" $\rho(\gamma, \Gamma)$ between $\gamma \in \mathcal{P}(Y \times U)$ and $\Gamma \subset \mathcal{P}(Y \times U)$ and define the Hausdorff metric, $\rho_H(\Gamma_1, \Gamma_2)$, between $\Gamma_1 \subset \mathcal{P}(Y \times U)$ and $\Gamma_2 \subset \mathcal{P}(Y \times U)$ as

$$\rho(\gamma, \Gamma) \stackrel{\text{def}}{=} \inf_{\gamma' \in \Gamma} \rho(\gamma, \gamma') \quad \rho_H(\Gamma_1, \Gamma_2) \stackrel{\text{def}}{=} \max \left\{ \sup_{\gamma \in \Gamma_1} \rho(\gamma, \Gamma_2), \sup_{\gamma \in \Gamma_2} \rho(\gamma, \Gamma_2) \right\}. \tag{1.3}$$

Note that, although, by some abuse of terminology, we refer to $\rho_H(\cdot, \cdot)$ as a metric on the set of subsets of $\mathcal{P}(Y \times U)$, it is, in fact, a semimetric on this set (since $\rho_H(\Gamma_1, \Gamma_2) = 0$ implies $\Gamma_1 = \Gamma_2$ if and only if Γ_1 and Γ_2 are closed).

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Occupational Measure Formulation and Duality Results

In this section we introduce the optimal control problem that is the subject of our consideration and we introduce the IDLP problem that, as established in the thesis, is closely related to the former. Also in this section, we introduce the problem dual with respect to this IDLP problem. The main results of the section are Proposition 2.3 and Theorem 2.5. Proposition 2.3 establishes that the set of discounted occupational measures generated by admissible controls and the corresponding solutions of the discrete time system is contained in the feasible set of the IDLP problem, while Theorem 2.5 establishes relationships between the IDLP problem and its dual.

Consider a discrete time control system

$$y(t+1) = f\left(y(t), u(t)\right), \ t = 0, 1, \dots, \quad y(0) = y_0 \ \text{(given)}, \tag{2.1}$$

where $f : \mathbb{R}^m \times U \to \mathbb{R}^m$ is a continuous vector function and $u(t) \in U$, $t = 0, 1, \dots,$ are controls (U being a given compact metric space). Denote by $y(t, y_0, u(\cdot))$ the solution of (2.1) obtained with a control $u(\cdot)$ and initial condition $y(0) = y_0$. We will call $(y(\cdot), u(\cdot))$ an *admissible pair* if $u(t) \in U$ and $y(t) \in Y$ for all $t = 0, 1, \dots,$ where Y is a given non-empty compact subset of $\mathbb{R}^m$. Note that, if $(y(\cdot), u(\cdot))$ is an admissible pair, then $f(y(t), u(t)) \in Y$ for $t = 0, 1, \dots$.

Consider the optimal control problem

$$\inf_{(y(\cdot),u(\cdot))} \sum_{t=0}^{\infty} \alpha^t g\left(y\left(t\right), u\left(t\right)\right) \stackrel{\text{def}}{=} G\left(\alpha, y_0\right), \tag{2.2}$$

where inf is taken over all admissible pairs, $g : \mathbb{R}^m \times U \to \mathbb{R}$ is continuous and α is a discount factor ($0 < \alpha < 1$). We say a pair $(y(\cdot), u(\cdot))$ is optimal in (2.2) if the optimal value is attained along the trajectory generated by the pair, that is $\sum_{t=0}^{\infty} \alpha^t g(y(t), u(t)) = G(\alpha, y_0)$. We are interested in establishing connections of the problem (2.2) with the problem

$$\inf_{\gamma \in W(\alpha, y_0)} \int_{Y \times U} g\left(y, u\right) \gamma\left(dy, du\right) \stackrel{\text{def}}{=} g^*\left(\alpha, y_0\right) \tag{2.3}$$

where

$$W(\alpha, y_0) \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \gamma \in \mathcal{P}(Y \times U) : \int_{Y \times U} \alpha\,(\phi\,(f\,(y,u)) - \phi\,(y)) + \\ (1-\alpha)\,(\phi\,(y_0) - \phi\,(y))\,\gamma\,(dy, du) = 0 \\ \forall \phi \colon Y \times U \to \mathbb{R}\,\text{continuous} \end{array} \right\} \tag{2.4}$$

Note that (2.3) is an IDLP problem since its objective function and its constraints are linear in γ.

Given an admissible pair $(y(\cdot), u(\cdot))$, a probability measure $\gamma^{\alpha}_{y(\cdot),u(\cdot)} \in \mathcal{P}(Y \times U)$ is called the *discounted occupational measure* generated by the pair $(y(\cdot), u(\cdot))$ if, for any Borel set $Q \subset Y \times U$,

$$\gamma^{\alpha}_{y(\cdot),u(\cdot)}(Q) = (1-\alpha) \sum_{t=0}^{\infty} \alpha^t \chi_Q(y(t), u(t)), \tag{2.5}$$

where χ_Q is the indicator function of Q. Note that from this definition it follows that

$$\int_{Y \times U} q(y,u) \gamma^{\alpha}_{y(\cdot),u(\cdot)}(dy, du) = (1-\alpha) \sum_{t=0}^{\infty} \alpha^t q(y(t), u(t)) \tag{2.6}$$

for any $q(\cdot) \in C(Y \times U)$.

Due to (2.6), we may restate the optimal control problem (2.2) as the problem of optimization over the set of discounted occupational measures $\Gamma(\alpha, y_0) = \bigcup \gamma^{\alpha}_{y(\cdot),u(\cdot)}$ (the union is over the admissible pairs $(y(\cdot), u(\cdot))$). Namely,

$$\inf_{\gamma \in \Gamma(\alpha, y_0)} \int_{Y \times U} g\,(y,u)\,\gamma\,(dy, du) = (1-\alpha) G(\alpha, y_0). \tag{2.7}$$

Let us first establish some basic properties of the set $W(\alpha, y_0)$.

Lemma 2.1. *The set $W(\alpha, y_0)$ is convex.*

Proof. The proof follows from the fact that the constraints defining $W(\alpha, y_0)$ are linear in γ. In fact, let $\mu, \gamma \in W(\alpha, y_0)$ and let $\omega_\lambda = (1-\lambda)\mu + \lambda\gamma$ for $0 < \lambda < 1$. Then

$$\begin{aligned} &\int_{Y \times U} \alpha\,(\phi\,(f\,(y,u)) - \phi\,(y)) + (1-\alpha)\,(\phi\,(y_0) - \phi\,(y))\,\omega_\lambda\,(dy, du) \\ &= (1-\lambda) \int_{Y \times U} \alpha\,(\phi\,(f\,(y,u)) - \phi\,(y)) + (1-\alpha)\,(\phi\,(y_0) - \phi\,(y))\,\mu\,(dy, du) \\ &+ \lambda \int_{Y \times U} \alpha\,(\phi\,(f\,(y,u)) - \phi\,(y)) + (1-\alpha)\,(\phi\,(y_0) - \phi\,(y))\,\gamma\,(dy, du) \\ &= 0 \end{aligned}$$

since $\mu, \gamma \in W(\alpha, y_0)$. So $(1-\lambda)\mu + \lambda\gamma \in W(\alpha, y_0)$ and thus $W(\alpha, y_0)$ is convex. □

Lemma 2.2. *The set $W(\alpha, y_0)$ is closed.*

Proof. Let $\gamma_k \in W(\alpha, y_0)$ be a sequence that converges to a limit γ. Then, for any $q \in C(Y \times U)$,

$$\lim_{k \to \infty} \int_{Y \times U} q(y,u) \gamma^k(dy, du) = \int_{Y \times U} q(y,u) \gamma(dy, du),$$

and so

$$\begin{aligned}0 &= \lim_{k\to\infty}\int_{Y\times U}\alpha\left(\phi\left(f\left(y,u\right)\right)-\phi\left(y\right)\right)+\left(1-\alpha\right)\left(\phi\left(y_0\right)-\phi\left(y\right)\right)\gamma^k\left(dy,du\right)\\ &= \int_{Y\times U}\alpha\left(\phi\left(f\left(y,u\right)\right)-\phi\left(y\right)\right)+\left(1-\alpha\right)\left(\phi\left(y_0\right)-\phi\left(y\right)\right)\gamma\left(dy,du\right).\end{aligned}$$

Thus $\gamma \in W(\alpha, y_0)$, and so $W(\alpha, y_0)$ is closed. □

Proposition 2.3. $\bar{co}\Gamma(\alpha, y_0) \subset W(\alpha, y_0)$.

Proof. In light of Lemma 2.1 and 2.2 it is sufficient to show that $\Gamma(\alpha, y_0) \subset W(\alpha, y_0)$. Thus, it is sufficient to show that the discounted occupational measure $\gamma^{\alpha}_{y(\cdot),u(\cdot)}$ generated by an arbitrary admissible pair satisfies the inclusion $\gamma^{\alpha}_{y(\cdot),u(\cdot)} \in W(\alpha, y_0)$. In fact, one can write down the following equalities

$$\begin{aligned}\sum_{t=0}^{\infty}\alpha^t\phi\left(y\left(t\right)\right) &= \phi(y_0)+\sum_{t=1}^{\infty}\alpha^t\phi\left(y\left(t\right)\right)\\ &= \phi(y_0)+\sum_{t=0}^{\infty}\alpha^{t+1}\phi\left(y\left(t+1\right)\right)\\ &= \phi(y_0)+\sum_{t=0}^{\infty}\alpha^{t+1}\phi\left(f\left(y\left(t\right),u\left(t\right)\right)\right)\\ &= \phi(y_0)+\alpha\sum_{t=0}^{\infty}\alpha^{t}\phi\left(f\left(y\left(t\right),u\left(t\right)\right)\right).\end{aligned}$$

Multiplying the latter by $(1-\alpha)$, we have

$$(1-\alpha)\sum_{t=0}^{\infty}\alpha^t\phi\left(y\left(t\right)\right) = (1-\alpha)\left[\phi(y_0)+\alpha\sum_{t=0}^{\infty}\alpha^t\phi\left(f\left(y\left(t\right),u\left(t\right)\right)\right)\right]$$

and so, by (2.6), for any admissible pair $(y(\cdot), u(\cdot))$,

$$\begin{aligned}\int_{Y\times U}\phi\left(y\right)\gamma^{\alpha}_{y(\cdot),u(\cdot)}\left(dy,du\right) &= \int_{Y\times U}\left(1-\alpha\right)\phi\left(y_0\right)\gamma^{\alpha}_{y(\cdot),u(\cdot)}\left(dy,du\right)\\ &\quad+\alpha\int_{Y\times U}\phi\left(f\left(y,u\right)\right)\gamma^{\alpha}_{y(\cdot),u(\cdot)}\left(dy,du\right),\end{aligned}$$

the rearrangement of which yields

$$\int_{Y\times U}\alpha\left[\phi\left(f\left(y,u\right)\right)-\phi\left(y\right)\right]+\left(1-\alpha\right)\left[\phi\left(y_0\right)-\phi\left(y\right)\right]\gamma^{\alpha}_{y(\cdot),u(\cdot)}\left(dy,du\right)=0.$$

Thus, $\gamma^{\alpha}_{y(\cdot),u(\cdot)} \in W(\alpha, y_0)$ and so $\Gamma(\alpha, y_0) \subset W(\alpha, y_0)$ as required. □

From (2.7) and Proposition 2.3 it follows that

$$(1-\alpha)G(\alpha, y_0) \geq g^*(\alpha, y_0). \tag{2.8}$$

We now consider the problem

$$\sup_{(\theta,\psi(\cdot))\in\mathbb{R}\times C}\left\{\begin{aligned}\theta \leq g\left(y,u\right)+\alpha\left[\psi\left(f\left(y,u\right)\right)-\psi\left(y\right)\right]\\ +\left(1-\alpha\right)\left[\psi\left(y_0\right)-\psi\left(y\right)\right]\forall\left(y,u\right)\end{aligned}\right\} \stackrel{\text{def}}{=} d^*\left(\alpha, y_0\right), \tag{2.9}$$

which is connected with the IDLP problem (2.3) by duality relations (see statements below). Thus, we call it the problem dual to (2.3). Note that the supremum in (2.9) is taken over pairs $(\theta, \psi(\cdot))$ where $\theta \in \mathbb{R}$ and $\psi(\cdot) \in C$ (the space of continuous functions).

Lemma 2.4. *If $W(\alpha, y_0) \neq \emptyset$ then $d^*(\alpha, y_0) \leq g^*(\alpha, y_0)$.*

Proof. Suppose $W(\alpha, y_0) \neq \emptyset$. Then, for any $\gamma \in W(\alpha, y_0)$, and any pair $(\theta, \psi(\cdot))$ such that

$$\theta \leq g(\alpha, y_0) + \alpha\,[\psi\,(f\,(y,u)) - \psi\,(y)] + (1-\alpha)\,[\psi\,(y_0) - \psi\,(y)] \quad \forall (y,u) \in Y \times U$$

one can obtain (via integration of the latter over γ)

$$\theta \leq \int_{Y\times U} g(y,u)\gamma(dy,du).$$

Hence, by taking the minimum over $\gamma \in W(\alpha, y_0)$ in the right hand side,

$$\theta \leq g^*(\alpha, y_0)$$

and, consequently, the optimal value of (2.3) and it's dual (2.9) satisfy

$$d^*(\alpha, y_0) \leq g^*(\alpha, y_0).$$

□

More elaborated duality relationships between the IDLP problem (2.3) and it's dual (2.9) are established by the following theorem.

Theorem 2.5. *Let $d^*(\alpha, y_0)$ be the optimal value of the dual problem (2.9), then*

(i) The optimal value of the dual problem is bounded (that is, $d^*(\alpha, y_0) < \infty$) if and only if the set $W(\alpha, y_0)$ is not empty.

(ii) If the optimal value $d^*(\alpha, y_0)$ of the dual problem is bounded then

$$d^*(\alpha, y_0) = g^*(\alpha, y_0) \tag{2.10}$$

(iii) The optimal value of the dual problem is unbounded (that is, $d^*(\alpha, y_0) = \infty$) if and only if there exists a function $\psi(\cdot) \in C$ such that

$$\max_{(y,u)\in Y\times U} \{\alpha\,[\psi\,(f\,(y,u)) - \psi\,(y)] + (1-\alpha)\,[\psi\,(y_0) - \psi\,(y)]\} < 0. \tag{2.11}$$

Proof of (iii). If such a function $\psi(\cdot)$ exists, then

$$\min_{(y,u)\in Y\times U} \{-\alpha\,[\psi\,(f\,(y,u)) - \psi\,(y)] - (1-\alpha)\,[\psi\,(y_0) - \psi\,(y)]\} > 0$$

and so

$$\lim_{\beta\to\infty} \min_{(y,u)\in Y\times U} \{g\,(y,u) + \beta\,[-\alpha\,[\psi\,(f\,(y,u)) - \psi\,(y)] - (1-\alpha)\,[\psi\,(y_0) - \psi\,(y)]]\} = \infty.$$

Thus, the optimal value of the dual problem is unbounded.

Conversely, suppose $d^*(\alpha, y_0)$ is unbounded. Then, there exists a sequence $(\theta_k, \psi_k(\cdot))$ such that $\lim_{k\to\infty} \theta_k = \infty$. Now, for all $(y,u) \in Y \times U$

$$\theta_k \le g(y,u) + \alpha \left[\psi_k\left(f(y,u)\right) - \psi_k(y)\right] + (1-\alpha)\left[\psi_k(y_0) - \psi_k(y)\right]$$
$$\Rightarrow 1 \le \frac{1}{\theta_k} g(y,u) + \frac{1}{\theta_k}\left[\alpha\left[\psi_k\left(f(y,u)\right) - \psi_k(y)\right] + (1-\alpha)\left[\psi_k(y_0) - \psi_k(y)\right]\right].$$

For k large enough, $\frac{1}{\theta_k}|g(y,u)| \le \frac{1}{2}$ so

$$\frac{1}{2} \le \frac{1}{\theta_k}\left[\alpha\left[\psi_k\left(f(y,u)\right) - \psi_k(y)\right] + (1-\alpha)\left[\psi_k(y_0) - \psi_k(y)\right]\right]$$

for all $(y,u) \in Y \times U$. Define $\psi(y) = -\frac{1}{\theta_k}\psi_k(y)$ then

$$-\frac{1}{2} \ge \left[\alpha\left(\psi(f(y,u)) - \psi(y)\right) + (1-\alpha)\left(\psi(y_0) - \psi(y)\right)\right]$$

and so $\psi(\cdot)$ satisfies (2.11). □

Before proving parts (i) and (ii) of Theorem 2.5 we must first reformulate the constraint set $W(\alpha, y_0)$ in terms of a countable system of equations. Let $\phi_i(\cdot) \in C$, $i = 1, 2, \ldots$ be a sequence of functions such that any $\psi(\cdot) \in C$ can be approximated by a linear combination of ϕ_i. That is, for any $\psi(\cdot) \in C$ and any $\delta > 0$, there exists $\beta_1, \ldots, \beta_k$ (real numbers) such that

$$\max_{y\in\hat{Y}}\left\{\left|\psi(y) - \sum_{i=1}^{k}\beta_i\phi_i(y)\right|\right\} \le \delta, \tag{2.12}$$

where $\hat{Y}$ is a sufficiently large compact set that contains Y as well as $f(Y,U) \stackrel{\text{def}}{=} \{f(y,u) \mid (y,u) \in Y \times U\}$. An example of such an approximating sequence is the sequence of monomials $y_1^{i_1} \ldots y_m^{i_m}$, $i_1, \ldots, i_m = 0, 1, \ldots$, where y_j $(j = 1, \ldots, m)$ stands for the j-th component of y (see, e.g. [16]).

Due to the above property of the sequence of functions $\phi_i(\cdot)$, $i = 1, 2, \ldots$, the set $W(\alpha, y_0)$ can be represented in the form

$$W(\alpha, y_0) = \left\{\begin{array}{l} \gamma \in \mathcal{P}(Y \times U) : \displaystyle\int_{Y\times U} \alpha\left(\phi_i\left(f(y,u)\right) - \phi_i(y)\right) + \\ (1-\alpha)\left(\phi_i(y_0) - \phi_i(y)\right)\gamma(dy,du) = 0, \\ i = 1, 2, \ldots \end{array}\right\}, \tag{2.13}$$

where, without loss of generality, one may assume that the functions $\phi_i(\cdot)$ satisfy the following normalization condition

$$\max_{y\in\hat{Y}}\{|\phi_i(y)|\} \le \frac{1}{2^i}, \qquad i = 1, 2, \ldots. \tag{2.14}$$

Let l^1 and l^∞ stand for the Banach spaces of infinite sequences such that, $x = (x_1, x_2, \ldots) \in l^1$ if and only if $\|x\|_{l^1} \stackrel{\text{def}}{=} \sum_i |x_i| < \infty$ and, $\lambda = (\lambda_1, \lambda_2, \ldots) \in l^\infty$ if and only if $\|\lambda\|_{l^\infty} \stackrel{\text{def}}{=} \sup_i |\lambda_i| < \infty$. It is easy to see that given an element $\lambda \in l^\infty$, one can define a linear continuous functional $\lambda(\cdot) : l^1 \to \mathbb{R}$ by the equation

$$\lambda(x) = \sum_{i=1}^{\infty}\lambda_i x_i \quad \forall x \in l^1, \qquad \|\lambda(\cdot)\| = \|\lambda\|_{l^\infty}. \tag{2.15}$$

It is also known (see, e.g. [19, p. 86]) that any continuous linear functional $\lambda(\cdot) : l^1 \to \mathbb{R}$ can be presented in the form (2.15) for some $\lambda \in l^\infty$. Note that from (2.14) it follows that $(\phi_1(y), \phi_2(y), \dots) \in l^1$ for any $y \in Y$, and hence, for any $\lambda = (\lambda_1, \lambda_2, \dots) \in l^\infty$, the function $\phi_\lambda(y)$,

$$\phi_\lambda(y) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} \lambda_i \phi_i(y), \tag{2.16}$$

is continuous.

Proof of Theorem (2.5) (i). We have $d^*(\alpha, y_0) \le g^*(\alpha, y_0)$, so if $W(\alpha, y_0) \ne \emptyset$, then $d^*(\alpha, y_0)$ is bounded. Suppose $d^*(\alpha, y_0) < \infty$, and show that $W(\alpha, y_0)$ is not empty. Assume, on the contrary, that $W(\alpha, y_0)$ is empty and define

$$Q \stackrel{\text{def}}{=} \left\{ \begin{array}{l} x = (x_1, x_2, \dots) \mid \\ x_i = \displaystyle\int_{Y\times U} \alpha \left[\phi_i\left(f\left(y,u\right)\right) - \phi_i\left(y\right)\right] + (1-\alpha)\left[\phi_i\left(y_0\right) - \phi_i\left(y\right)\right] \gamma\left(dy, du\right) \\ \forall \gamma \in \mathcal{P}\left(Y \times U\right) \end{array} \right\}$$

The assumption that $W(\alpha, y_0)$ is empty is equivalent to the assumption that Q does not contain the zero element. By a separation theorem (see, e.g. [19, p. 59]) there exists $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \dots) \in l^\infty$ such that

$$\begin{aligned} 0 = \bar{\lambda}(0) &> \max_{x\in Q} \sum_i \bar{\lambda}_i x_i \\ &= \max_{\gamma\in\mathcal{P}(Y\times U)} \left\{ \int_{Y\times U} \alpha\left[\phi_{\bar{\lambda}}\left(f\left(y,u\right)\right) - \phi_{\bar{\lambda}}\left(y\right)\right] + (1-\alpha)\left[\phi_{\bar{\lambda}}\left(y_0\right) - \phi_{\bar{\lambda}}\left(y\right)\right] \gamma\left(dy, du\right) \right\} \\ &= \max_{(y,u)\in Y\times U} \left\{ \alpha\left[\phi_{\bar{\lambda}}\left(f\left(y,u\right)\right) - \phi_{\bar{\lambda}}\left(y\right)\right] + (1-\alpha)\left[\phi_{\bar{\lambda}}\left(y_0\right) - \phi_{\bar{\lambda}}\left(y\right)\right] \right\}, \end{aligned}$$

where $\phi_{\bar{\lambda}}(y) = \sum_i \bar{\lambda}_i \phi_i(y)$. Thus,

$$\max_{(y,u)\in Y\times U} \left\{ \alpha\left[\phi_{\bar{\lambda}}\left(f\left(y,u\right)\right) - \phi_{\bar{\lambda}}\left(y\right)\right] + (1-\alpha)\left[\phi_{\bar{\lambda}}\left(y_0\right) - \phi_{\bar{\lambda}}\left(y\right)\right] \right\} < 0$$

and so, by part (iii), $d^*(\alpha, y_0)$ is unbounded which is a contradiction. So $W(\alpha, y_0)$ is not empty. □

Proof of Theorem (2.5) (ii). Suppose $d^*(\alpha, y_0)$ is bounded then by part (i) $W(\alpha, y_0) \ne \emptyset$ and, hence, a solution of the problem

$$\inf_{\gamma\in W(\alpha,y_0)} \int_{Y\times U} g(y,u)\gamma(dy,du) \stackrel{\text{def}}{=} g^*(\alpha, y_0)$$

exists. Define $\hat{Q} \subset \mathbb{R} \times l^1$ by

$$\hat{Q} \stackrel{\text{def}}{=} \left\{ \begin{array}{l} (\theta, x) \mid \theta \ge \displaystyle\int_{Y\times U} g(y,u)\gamma(dy,du),\ x = (x_1, x_2, \dots) \\ x_i = \displaystyle\int_{Y\times U} \alpha \left[\phi_i\left(f\left(y,u\right)\right) - \phi_i\left(y\right)\right] + (1-\alpha)\left[\phi_i\left(y_0\right) - \phi_i\left(y\right)\right] \gamma\left(dy, du\right) \\ \forall \gamma \in \mathcal{P}\left(Y \times U\right) \end{array} \right\}$$

For any $i = 1, 2, \ldots$ the point $(\theta_j, 0) \notin \hat{Q}$ where $\theta_j \stackrel{\text{def}}{=} g^*(\alpha, y_0) - \frac{1}{j}$. So, by a separation theorem (see, e.g. [19, p. 59]), there exists a sequence $(k^j, \lambda^j) \in \mathbb{R} \times l^\infty$ (where $\lambda^j = (\lambda_1^j, \lambda_2^j, \ldots)$) such that

$$
\begin{aligned}
k^j \left(g^*(\alpha, y_0) - \frac{1}{j} \right) + \delta^j &\leq \inf_{(\theta, x) \in \hat{Q}} \left\{ k^j \theta + \sum_i \lambda_i^j x_i \right\} \\
&= \inf_{\gamma \in \mathcal{P}(Y \times U)} \left\{ \begin{array}{l} k^j \theta + \displaystyle\int_{Y \times U} \alpha \left[\psi_{\lambda^j} \left(f \left(y, u \right) \right) - \psi_{\lambda^j} \left(y \right) \right] \\ \quad + (1 - \alpha) \left[\psi_{\lambda^j} \left(y_0 \right) - \psi_{\lambda^j} \left(y \right) \right] \gamma \left(dy, du \right) \\ s.t.\, \theta \geq \displaystyle\int_{Y \times U} g(y, u) \gamma(dy, du) \end{array} \right\}
\end{aligned}
\tag{2.17}
$$

where $\delta^j > 0$ for all j and $\psi_{\lambda^j} = \sum_i \lambda_i^j \phi_i(y)$. Firstly note that the above can be true only if $k_j \geq 0$. Let us now show that $k^j > 0$. In fact, if it wasn't, then

$$
\begin{aligned}
0 < \delta^j &\leq \min_{\gamma \in \mathcal{P}(Y \times U)} \left\{ \int_{Y \times U} \alpha \left[\psi_{\lambda^j} \left(f \left(y, u \right) \right) - \psi_{\lambda^j} \left(y \right) \right] + (1 - \alpha) \left[\psi_{\lambda^j} \left(y_0 \right) - \psi_{\lambda^j} \left(y \right) \right] \gamma \left(dy, du \right) \right\} \\
&= \min_{(y,u) \in Y \times U} \left\{ \alpha \left[\psi_{\lambda^j} \left(y + f \left(y, u \right) \right) - \psi_{\lambda^j} \left(y \right) \right] + (1 - \alpha) \left[\psi_{\lambda^j} \left(y_0 \right) - \psi_{\lambda^j} \left(y \right) \right] \right\}
\end{aligned}
$$

and so

$$
\max_{(y,u) \in Y \times U} \left\{ -\alpha \left[\psi_{\lambda^j} \left(f \left(y, u \right) \right) - \psi_{\lambda^j} \left(y \right) \right] - (1 - \alpha) \left[\psi_{\lambda^j} \left(y_0 \right) - \psi_{\lambda^j} \left(y \right) \right] \right\} \leq \delta^j < 0.
$$

That is, $\psi(y) = -\psi_{\lambda^j}(y)$ satisfies (2.11), and so by part (iii), $d^*(\alpha, y_0)$ is unbounded. Thus, $k^j > 0$. Now, dividing (2.17) by k^j, we obtain

$$
\begin{aligned}
g^*(\alpha, y_0) - \frac{1}{j} &< \left(g^*(\alpha, y_0) - \frac{1}{j} \right) + \frac{\delta^j}{k^j} \\
&\leq \min_{\gamma \in \mathcal{P}(Y \times U)} \left\{ \begin{array}{l} \displaystyle\int_{Y \times U} g(y, u) + \frac{1}{k^j} \alpha \left[\psi_{\lambda^j} \left(f \left(y, u \right) \right) - \psi_{\lambda^j} \left(y \right) \right] \\ \qquad\qquad + \dfrac{1}{k^j} \left(1 - \alpha \right) \left[\psi_{\lambda^j} \left(y_0 \right) - \psi_{\lambda^j} \left(y \right) \right] \gamma \left(dy, du \right) \end{array} \right\} \\
&= \min_{(y,u) \in Y \times U} \left\{ \begin{array}{l} g(y, u) + \dfrac{1}{k^j} \alpha \left[\psi_{\lambda^j} \left(f \left(y, u \right) \right) - \psi_{\lambda^j} \left(y \right) \right] \\ \qquad + \dfrac{1}{k^j} \left(1 - \alpha \right) \left[\psi_{\lambda^j} \left(y_0 \right) - \psi_{\lambda^j} \left(y \right) \right] \end{array} \right\} \leq d^*(\alpha, y_0)
\end{aligned}
$$

and so $g^*(\alpha, y_0) \leq d^*(\alpha, y_0)$. Hence, in light of Lemma 2.4 we have $g^*(\alpha, y_0) = d^*(\alpha, y_0)$. □

3

Equality of the Optimal Values of the Optimal Control and IDLP Problems

In this section we establish equality of the optimal values of the discounted optimal control problem and the IDLP problem and establish some necessary and sufficient conditions of optimality that are implied by this equality. We begin by recalling the well-known Dynamic Programming Principle (Proposition 3.1) which underpins the argument used in this section. We then prove the main result of this section (Theorem 3.2) and we conclude with the establishment of necessary and sufficient conditions of optimality (Proposition 3.3).

Everywhere in this section it is assumed that the set Y is forward invariant with respect to the system (2.1). That is, $f(y,u) \in Y$ if $y \in Y$ and $u \in U$. Note that from this assumption it follows that $G(\alpha,\cdot)$ is continuous in Y. This can be established similarly to the way it is established in the continuous time setting (see [1]).

Proposition 3.1 (Dynamic Programming Principle)**.**

$$G(\alpha, y) = \min_{u\in U} \{g(y,u) + \alpha G(\alpha, f(y,u))\} \tag{3.1}$$

Proof.

$$\begin{aligned}
G(\alpha, y) &= \inf_{y(\cdot),u(\cdot)} \sum_{t=0}^{\infty} \alpha^t g(y(t),u(t)) \\
&= \inf_{y(\cdot),u(\cdot)} \left\{ g(y(0),u(0)) + \sum_{t=1}^{\infty} \alpha^t g(y(t),u(t)) \right\} \\
&= \inf_{u(0)} \left\{ g(y(0),u(0)) + \inf_{u(1),u(2),\dots} \left\{ \sum_{t=1}^{\infty} \alpha^t g(y(t),u(t)) \right\} \right\} \\
&= \inf_{u(0)} \left\{ g(y(0),u(0)) + \alpha \inf_{u(1),u(2),\dots} \left\{ \sum_{t=0}^{\infty} \alpha^t g(y(t+1),u(t+1)) \right\} \right\} \\
&= \min_{u} \{g(y,u) + \alpha G(\alpha, f(y,u))\}.
\end{aligned}$$

The minimizer in the above expression exists due to the continuity of $G(\alpha,\cdot)$. □

The main result of this section is the following theorem.

Theorem 3.2. *The equality*

$$g^*(\alpha, y_0) = (1-\alpha)G(\alpha, y_0) \tag{3.2}$$

is valid.

Proof. As a direct consequence of Proposition 3.1, we have

$$G(\alpha, y) \leq g(y,u) + \alpha G(\alpha, f(y,u))$$

for all $(y,u) \in Y \times U$. Thus,

$$(1-\alpha)G(\alpha, y_0) \leq g(y,u) + \alpha\left[G(\alpha, f(y,u)) - G(\alpha, y)\right] + (1-\alpha)\left[G(\alpha, y_0) - G(\alpha, y)\right]$$

for all $(y,u) \in Y \times U$ and so, by the very definition of $d^*(\alpha, y_0)$ (see (2.9)),

$$d^*(\alpha, y_0) \geq (1-\alpha)G(\alpha, y_0) \tag{3.3}$$

since $G(\alpha, \cdot)$ is continuous.

This inequality, together with (2.8) and Theorem 2.5 (ii) prove (3.2). □

Note that the Dynamic Programming Principle equation (3.1) can be rewritten in the form

$$\min_{u \in U}\{g(y,u) + \alpha G(\alpha, f(y,u))\} - G(\alpha, y) = 0. \tag{3.4}$$

It is well known that a feedback control $u(y)$ is optimal if and only if

$$u(y) = \arg\min_{u \in U}\{g(y,u) + \alpha G(\alpha, f(y,u))\}.$$

Below, we establish another version of this result by considering the inequality version of equation (3.4). We shall say that $\psi(\cdot) \in C$ is a solution of the inequality form of (3.4) if

$$g(y,u) + \alpha\psi(f(y,u)) - \psi(y) \geq 0 \qquad \forall (y,u) \in Y \times U. \tag{3.5}$$

This will be referred to as the Bellman inequality. Note that this is equivalent to

$$g(y,u) + \alpha\left[\psi(f(y,u)) - \psi(y)\right] - (1-\alpha)\psi(y) \geq 0 \qquad \forall (y,u) \in Y \times U. \tag{3.6}$$

We will consider solutions of this inequality that satisfy the additional condition

$$\psi(y_0) = G(\alpha, y_0). \tag{3.7}$$

Observe that, by (3.4), the optimal value function $G(\alpha, y)$ is a solution of the Bellman inequality satisfying this condition.

Proposition 3.3. *Let $\psi(\cdot)$ be a solution of (3.5) satisfying (3.7). An admissible pair $(y(\cdot), u(\cdot))$ is optimal in the optimal control problem (2.2) if and only if*

$$g(y(t), u(t)) + \alpha\psi\left(f(y(t), u(t))\right) - \psi(y(t)) = 0 \qquad \forall t = 0, 1, \ldots, \tag{3.8}$$

$$u(t) = \arg\min_{u \in U}\{g(y(t), u) + \alpha\psi(f(y(t), u))\} \qquad \forall t = 0, 1, \ldots, \tag{3.9}$$

$$y(t) = \arg\min_{y \in Y}\{g(y, u(t)) + \alpha\psi(f(y, u(t))) - \psi(y)\} \qquad \forall t = 0, 1, \ldots. \tag{3.10}$$

In addition, if $(y(t), u(t))$ is optimal in (2.2), then

$$\psi(y(t)) = G(\alpha, y(t)) \qquad \forall t = 0, 1, \ldots. \tag{3.11}$$

Proof. Suppose $(y(\cdot), u(\cdot))$ is optimal in (2.2). That is,

$$\sum_{t=0}^{\infty} \alpha^t g(y(t), u(t)) = G(\alpha, y_0). \tag{3.12}$$

We will show that (3.8), (3.9) and (3.10) are valid. Consider the partial sum

$$\sum_{t=0}^{T} \alpha^{t+1}\psi(y(t+1)) - \alpha^t \psi(y(t)) = \alpha^{T+1}\psi(y(T+1)) - \psi(y_0). \tag{3.13}$$

Since $(y(\cdot), u(\cdot))$ is an admissible pair we have $y(t) \in Y$ for all $t = 0, 1, \dots$. Furthermore, $\psi(\cdot)$ is continuous on the compact set Y and thus, $|\psi(y(t))| \leq const$ for all $t = 0, 1, \dots$. Thus, by taking $T \to \infty$ we obtain

$$\begin{aligned} -\psi(y_0) &= \sum_{t=0}^{\infty} \alpha^{t+1}\psi(f(y(t), u(t))) - \alpha^t \psi(y(t)) \\ &= \sum_{t=0}^{\infty} \alpha^t \left[\alpha\psi(f(y(t), u(t))) - \psi(y(t))\right] \\ &= \sum_{t=0}^{\infty} \alpha^t \left[\alpha \left(\psi(f(y(t), u(t))) - \psi(y(t))\right) - (1-\alpha)\psi(y(t))\right]. \end{aligned}$$

Thus,

$$\sum_{t=0}^{\infty} \alpha^t \left[\alpha \left(\psi(f(y(t), u(t))) - \psi(y(t))\right) + (1-\alpha)\left(\psi(y_0) - \psi(y(t))\right)\right] = 0. \tag{3.14}$$

Hence, (3.12) can be rewritten as

$$\sum_{t=0}^{\infty} \alpha^t \begin{bmatrix} g(y(t), u(t)) + \alpha\left(\psi(f(y(t), u(t))) - \psi(y(t))\right) \\ + (1-\alpha)\left(\psi(y_0) - \psi(y(t))\right) \end{bmatrix} = G(\alpha, y_0) \tag{3.15}$$

and so

$$\sum_{t=0}^{\infty} \alpha^t \begin{bmatrix} g(y(t), u(t)) + \alpha\left(\psi(f(y(t), u(t))) - \psi(y(t))\right) \\ + (1-\alpha)\left(\psi(y_0) - G(\alpha, y_0) - \psi(y(t))\right) \end{bmatrix} = 0. \tag{3.16}$$

Since $\psi(y_0) = G(\alpha, y_0)$, we have

$$\sum_{t=0}^{\infty} \alpha^t \left[g(y(t), u(t)) + \alpha\left(\psi(f(y(t), u(t))) - \psi(y(t))\right) - (1-\alpha)\psi(y(t))\right] = 0, \tag{3.17}$$

which is equivalent to

$$\sum_{t=0}^{\infty} \alpha^t \left[g(y(t), u(t)) + \alpha\psi(f(y(t), u(t))) - \psi(y(t))\right] = 0. \tag{3.18}$$

From (3.5) and (3.18) it follows that (3.8) is valid. Furthermore,

$$(y(t), u(t)) = \underset{(y,u)\in Y\times U}{\arg\min} \{g(y,u) + \alpha\psi(f(y,u)) - \psi(y)\} \tag{3.19}$$

which is equivalent to (3.9) and (3.10).

Conversely, suppose (3.8), (3.9) and (3.10) are satisfied. Then (3.18) is valid. Since $\psi(y_0) = G(\alpha, y_0)$, (3.16) is satisfied which together with (3.14) implies (3.12). Hence, $(y(t), u(t))$ is optimal in (2.2).

Finally, we will show that (3.11) is true if $(y(\cdot), u(\cdot))$ is optimal in (2.2). Fix $T = 0, 1, \ldots,$ and consider again the partial sum (3.13). After a simple rearrangement, we have

$$\alpha^{T+1}\psi(y(T+1)) = \sum_{t=0}^{T} \alpha^{t+1}\psi(y(t+1)) - \alpha^t\psi(y(t)) + \psi(y_0).$$

Since $\psi(y_0) = G(\alpha, y_0) = \sum_{t=0}^{\infty} \alpha^t g(y(t), u(t))$,

$$\alpha^{T+1}\psi(y(T+1)) = \sum_{t=0}^{T} \alpha^{t+1}\psi(y(t+1)) - \alpha^t\psi(y(t)) + \sum_{t=0}^{\infty} \alpha^t g(y(t), u(t)),$$

which after rearranging yields

$$\begin{aligned}\alpha^{T+1}\psi(y(T+1)) &= \sum_{t=0}^{T} \alpha^t \left[g(y(t), u(t)) + \alpha\psi(y(t+1)) - \psi(y(t))\right] \\ &\quad + \sum_{t=T+1}^{\infty} \alpha^t g(y(t), u(t)) \\ &= \sum_{t=T+1}^{\infty} \alpha^t g(y(t), u(t)).\end{aligned}$$

The latter equality is valid due to the fact that $g(y(t), u(t)) + \alpha\psi(y(t+1)) - \psi(y(t)) = 0$ for all t. So,

$$\begin{aligned}\alpha^{T+1}\psi(y(T+1)) &= \sum_{t=0}^{\infty} \alpha^{t+T+1} g(y(t+T+1)) \\ &= \alpha^{T+1} \sum_{t=0}^{\infty} \alpha^t g(y(t+T+1))\end{aligned}$$

that is $\psi(y(T+1)) = G(\alpha, y(T+1))$. By re-indexing and recalling that $G(\alpha, y(0)) = \psi(y(0))$ we have $G(\alpha, y(t)) = \psi(y(t))$ for all $t = 0, 1, \ldots$ and so (3.11) is valid. □

Let us now establish a connection between the solutions of the Bellman inequality (3.5) and solutions of the dual problem (2.9). We will call $\psi(\cdot)$ a solution of the dual problem (2.9) if

$$g(y, u) + \alpha\left[\psi(f(y, u)) - \psi(y)\right] + (1-\alpha)\left[\psi(y_0) - \psi(y)\right] \geq d^*(\alpha, y_0) \qquad \forall (y, u) \in Y \times U \tag{3.20}$$

Observe that if $\psi(\cdot)$ is a solution of (2.9) then $\tilde{\psi}(\cdot) = \psi(\cdot) + const$ is as well.

Lemma 3.4. *If $\psi(\cdot)$ is a solution of the Bellman inequality (3.5) that satisfies (3.7) then $\psi(\cdot)$ is also a solution of the dual problem (2.9). Conversely if $\psi(\cdot)$ is a solution of (2.9) then $\tilde{\psi}(\cdot) = \psi(\cdot) - \psi(y_0) + G(\alpha, y_0)$ is a solution of (3.5) that satisfies (3.7).*

Proof. Let $\psi(\cdot)$ be a solution of (3.5) that satisfies (3.7). We have

$$g(y,u) + \alpha\psi(f(y,u)) - \psi(y) \geq 0$$

$$\Rightarrow g(y,u) + \alpha\left[\psi(f(y,u)) - \psi(y)\right] + (1-\alpha)\left[-\psi(y)\right] \geq 0$$

$$\Rightarrow g(y,u) + \alpha\left[\psi(f(y,u)) - \psi(y)\right] + (1-\alpha)\left[\psi(y_0) - \psi(y)\right] \geq (1-\alpha)G(\alpha, y_0) \quad (3.21)$$

where the last inequality follows from (3.7). Since $d^*(\alpha, y_0) = (1-\alpha)G(\alpha, y_0)$ it follows that $\psi(\cdot)$ is a solution of (2.9).

Conversely, suppose $\psi(\cdot)$ is a solution of (2.9). Then (3.21) is satisfied. Let $\tilde{\psi}(\cdot) = \psi(\cdot) - \psi(y_0) + G(\alpha, y_0)$, since $\tilde{\psi}(\cdot) - \psi(\cdot) = const$ from (3.21) if follows that

$$g(y,u) + \alpha\left[\tilde{\psi}(f(y,u)) - \tilde{\psi}(y)\right] + (1-\alpha)\left[\tilde{\psi}(y_0) - \tilde{\psi}(y)\right] \geq (1-\alpha)G(\alpha, y_0). \quad (3.22)$$

Now $\tilde{\psi}(y_0) = G(\alpha, y_0)$, and the substitution into (3.22) gives

$$g(y,u) + \alpha\tilde{\psi}(f(y,u)) - \tilde{\psi}(y) \geq 0$$

thus $\tilde{\psi}(\cdot)$ is a solution of (3.5) satisfying (3.7). $\square$

Note that, due to Theorem 2.5 (ii) and (3.2), $G(\alpha, y_0)$ is a solution of (2.9). Note also that the set of solutions of problem (2.9) can be much broader, as illustrated by the following example (taken from [7]).

Example. Consider the problem

$$\begin{aligned}
&\text{Minimize} \sum_{t=0}^{\infty} \alpha^t g(y(t)),\\
&\text{Subject to:}\\
&y(t+1) = u(t),\ t \in \{0, 1 \ldots\},\\
&y(0) = y_0,\\
&u(t) \in [0,1],\\
&y(t) \in [0,1],
\end{aligned}$$

where function g is increasing on $[0,1]$ and $g(0) = 0$.

It is clear that the optimal control is $u \equiv 0$ with the corresponding trajectory

$$y(t) = \begin{cases} y_0, & t = 0,\\ 0, & t > 0, \end{cases}$$

and the value function is $G(\alpha, y_0) = g(y_0)$.

Let us show that, if $\psi : [0,1] \to \mathbb{R}$ is such that $\psi(y_0) = g(y_0)$, $\psi(0) = g(0) = 0$, and $0 \leq \psi(y) \leq g(y)$ for all $y \in [0,1]$, then ψ is a solution of (2.9). Indeed, for such ψ we have

$$\begin{aligned}
&\min_{(y,u)\in[0,1]\times[0,1]} \{g(y) + \alpha\psi(u) - \psi(y) + (1-\alpha)\psi(y_0)\}\\
&= (1-\alpha)\psi(y_0) = (1-\alpha)G(\alpha, y_0) = g^*(\alpha, y_0),
\end{aligned}$$

where the last equality follows from (3.2). Therefore, ψ is a solution of (2.9) due to Theorem 2.5 (ii). $\square$

4

Finite Dimensional Approximations

This section consists of two subsections. In subsection 4.1 we show that the IDLP problem is approximated by a sequence of SILP problems (Proposition 4.1) and that the SILP problems are, in turn, approximated by finite-dimensional LP problems (Proposition 4.3). In subsection 4.2 we discuss a method of constructing a near optimal feedback control on the basis of a solution of an approximating finite-dimensional LP problem, and we illustrate the construction with a numerical example.

4.1 Approximating Semi-infinite and Finite LP Problems

Let $\phi_i(\cdot) \in C, i = 1, 2, \ldots$ be a sequence of functions such that any $\phi(\cdot) \in C$ is approximated by a linear combination of ϕ_i. Note that it is everywhere assumed that $\phi_i(\cdot) \equiv 1$ for exactly one i. As has been mentioned in Section 2, we can use such a sequence to represent the set $W(\alpha, y_0)$ in the form of a countable system of equations:

$$W(\alpha, y_0) = \left\{ \begin{array}{l} \gamma \in \mathcal{P}(Y \times U) : \int_{Y \times U} \alpha\, (\phi_i\,(f\,(y, u)) - \phi_i\,(y)) + \\ (1 - \alpha)\, (\phi_i\,(y_0) - \phi_i\,(y))\, \gamma\,(dy, du) = 0, \\ i = 1, 2, \ldots \end{array} \right\}. \tag{4.1}$$

We now define the set $W_N(\alpha, y_0)$ as a truncation of the system of equations in (4.1):

$$W_N(\alpha, y_0) = \left\{ \begin{array}{l} \gamma \in \mathcal{P}(Y \times U) : \int_{Y \times U} \alpha\, (\phi_i\,(f\,(y, u)) - \phi_i\,(y)) + \\ (1 - \alpha)\, (\phi_i\,(y_0) - \phi_i\,(y))\, \gamma\,(dy, du) = 0, \\ i = 1, 2, \ldots, N. \end{array} \right\}. \tag{4.2}$$

Everywhere in what follows we will be assuming that, for any N, the functions $\phi_i(\cdot)$ are linearly independent in the sense that from the fact that

$$\sum_{i=1}^{N} v_i \phi_i(y) = const \qquad \forall y \in Y',$$

where Y' is a subset of Y with nonempty interior, it follows that $v_i = 0, \forall i = 1, \ldots, N$. Note that this property is satisfied if monomials are used as $\phi_i(y)$.

Consider the following SILP problem

$$\min_{\gamma \in W_N} \int_{Y \times U} g(y,u)\gamma(dy,du) \stackrel{\text{def}}{=} G_N(\alpha, y_0). \tag{4.3}$$

Observe that $W_N(\alpha, y_0)$ is a convex and compact subset of $\mathcal{P}(Y \times U)$ and that

$$W_1(\alpha, y_0) \supset W_2(\alpha, y_0) \supset \cdots \supset W_N(\alpha, y_0) \supset \cdots \supset W(\alpha, y_0), \tag{4.4}$$

which implies

$$G_1(\alpha, y_0) \leq G_2(\alpha, y_0) \leq \cdots \leq G_N(\alpha, y_0) \leq \cdots \leq g^*(\alpha, y_0). \tag{4.5}$$

Note also that the set $W_N(\alpha, y_0)$ is empty if (2.11) is true with $\psi(y) = \sum_{i=1}^N v_i \phi_i(y)$, where v_i are real numbers.

Proposition 4.1. *The set $W(\alpha, y_0)$ is not empty if and only if $W_N(\alpha, y_0)$ is not empty for all $N \geq 1$. If $W(\alpha, y_0)$ is not empty then*

$$\lim_{N \to \infty} \rho_H(W_N(\alpha, y_0), W(\alpha, y_0)) = 0 \tag{4.6}$$

and

$$\lim_{N \to \infty} G_N(\alpha, y_0) = g^*(\alpha, y_0). \tag{4.7}$$

Furthermore, if γ_N is a solution of (4.3) and $\lim_{N' \to \infty} \rho(\gamma_{N'}, \gamma) = 0$ *for some subsequence of integers N' tending to infinity, then γ is a solution of (2.3). If the solution γ^* of problem (2.3) is unique, then, for any solution γ_N of (4.3),*

$$\lim_{N \to \infty} \rho(\gamma_N, \gamma^*) = 0. \tag{4.8}$$

Proof. The validity of (4.7) follows from the validity of (4.6). The other statements in the proposition follow from (4.7) and (4.6). Since $W \subset W_N$, to prove that (4.6) is valid, it is enough to show that

$$\lim_{N \to \infty} \sup_{\gamma \in W_N(\alpha, y_0)} \rho\left(\gamma, W(\alpha, y_0)\right) = 0. \tag{4.9}$$

Assume it is not true. There exists a positive number δ, a subsequence of positive integers $N' \to \infty$ and a sequence of probability measures $\gamma_{N'} \in W_{N'}(\alpha, y_0)$ such that $\rho(\gamma_{N'}, W(\alpha, y_0)) \geq \delta$. Due to the fact that $\mathcal{P}(Y \times U)$ is weakly compact, one may assume (without loss of generality) that there exists $\bar{\gamma} \in \mathcal{P}(Y \times U)$ such that

$$\lim_{N' \to \infty} \rho(\gamma_{N'}, \bar{\gamma}) = 0 \quad \Rightarrow \quad \rho(\bar{\gamma}, W(\alpha, y_0)) \geq \delta. \tag{4.10}$$

From the fact that $\gamma_{N'} \in W_{N'}$ it follows that, for any integer i and $N' \geq i$,

$$\int_{Y \times U} \psi_i(y,u)\gamma_{N'}(dy,du) = 0 \quad \Rightarrow \quad \int_{Y \times U} \psi_i(y,u)\bar{\gamma}(dy,du) = 0, \tag{4.11}$$

where $\psi_i(y,u) = \alpha[\phi_i(f(y,u)) - \phi_i(y)] + (1-\alpha)[\phi_i(y_0) - \phi_i(y)]$. Since this is valid for any $i = 1, 2, \ldots$, it follows that $\bar{\gamma} \in W(\alpha, y_0)$, which contradicts (4.10). This proves (4.6). □

Assumption 1. *For any* $\psi(y)$ *that is presentable in the form*

$$\psi(y) = \sum_{i=1}^{N} v_i \phi_i(y), \tag{4.12}$$

the inequality

$$\alpha\,[\psi(f(y,u)) - \psi(y)] + (1-\alpha)\,[\psi(y_0) - \psi(y)] \geq 0 \qquad \forall\,(y,u) \in Y \times U \tag{4.13}$$

is valid only if

$$v_i = 0,\ \ \forall i = 1, 2, \ldots, N. \tag{4.14}$$

Let

$$\mathcal{R}_{y_0} \stackrel{\text{def}}{=} \left\{ y \,\middle|\, \begin{array}{l} y = y(t),\ \text{for some } t = 0, 1, \ldots \text{ where,} \\ (y(\cdot), u(\cdot)) \text{ is admissible with } y(0) = y_0 \end{array} \right\}. \tag{4.15}$$

That is, $\mathcal{R}_{y_0}$ is the set of points reachable (in finite time) along trajectories of (2.1). The following proposition gives some sufficient conditions for Assumption 1 to be true.

Proposition 4.2. *Suppose* $int(cl\mathcal{R}_{y_0}) \neq \emptyset$*. Then, for* $\psi(\cdot)$ *of the form (4.12), the inequality (4.13) is valid only if (4.14) is valid (that is, Assumption 1 is valid).*

Proof. Let $\psi(\cdot)$ be of the form (4.12) satisfying (4.13). For any admissible pair (y,u) that satisfies $y(0) = y_0$ we have shown already in (3.14) that

$$\sum_{t=0}^{\infty} \alpha\,[\psi(f(y(t),u(t))) - \psi(y(t))] + (1-\alpha)\,[\psi(y_0) - \psi(y(t))] = 0,$$

which together with (4.13) imply

$$\alpha\,[\psi(f(y(t),u(t))) - \psi(y(t))] + (1-\alpha)\,[\psi(y_0) - \psi(y(t))] = 0 \quad \forall t \in \{0, 1, \ldots\}.$$

Rearranging the latter and recalling that $f(y(t),u(t)) = y(t+1)$ yields

$$\alpha\,[\psi(y(t+1)) - \psi(y(t))] = (1-\alpha)\,[\psi(y(t)) - \psi(y_0)] \quad \forall t \in \{0, 1, \ldots\} \tag{4.16}$$

which gives

$$\psi(y(t)) = \psi(y_0) \qquad \forall t \in \{0, 1, \ldots\}. \tag{4.17}$$

To see this, first take $t = 0$. Then (4.16) gives $\psi(y(1)) - \psi(y_0) = 0$. Substitution of this into (4.16) for $t = 1$ yields $\psi(y(2)) - \psi(y(1)) = 0$. Continuing inductively, we see that $\psi(y(t+1)) = \psi(y(t))\ \forall t = 0, 1, \ldots,$ thus proving (4.17). Consequently, by definition of $\mathcal{R}_{y_0}$ (see (4.15)),

$$\psi(y) = \psi(y_0)\ \ \forall y \in \mathcal{R}_{y_0} \quad \Rightarrow \quad \psi(y) = \psi(y_0)\ \ \forall y \in cl\mathcal{R}_{y_0}.$$

Thus,

$$\sum_{i=1}^{N} v_i \phi_i(y) = \sum_{i=1}^{N} v_i \phi_i(y_0) = const \qquad \forall y \in cl\mathcal{R}_{y_0}$$

and so $v_i = 0$ for $i = 1, \ldots, N$ due to linear independence of $\{\phi_i\}_{i=1}^{N}$. This proves the proposition. □

Let $\Delta > 0$, and let Borel sets $Q_{k,l}^{\Delta} \subset Y \times U$ ($k = 1, \ldots, K^{\Delta}$ and $l = 1, \ldots, L^{\Delta}$) be defined in such a way that $Q_{k,l}^{\Delta} \cap Q_{k',l'}^{\Delta} = \emptyset$ (if $k \neq k'$ and/or $l \neq l'$), $\cup_{k,l} Q_{k,l}^{\Delta} = Y \times U$ and

$$\sup_{(y,u)\in Q_{k,l}^{\Delta}} \|(y,u) - (y_k, u_l)\| \leq c\Delta, \qquad c = \text{const}, \tag{4.18}$$

for some point $(y_k, u_l) \in Q_{k,l}^{\Delta}$. It is assumed (from now on) that U is a compact subset of $\mathbb{R}^n$ and $\|\cdot\|$ stands for a norm in $\mathbb{R}^{n+m}$. For convenience, the sets $Q_{k,l}^{\Delta}$ will be referred to as cells. Fix the points (y_k, u_l) for $k = 1, \ldots, K^{\Delta}$, $l = 1, \ldots, L^{\Delta}$ and define a polyhedral set $W_N^{\Delta} \subset \mathbb{R}^{L^{\Delta}+K^{\Delta}}$ by the equation

$$W_N^{\Delta} \stackrel{\text{def}}{=} \left\{ \gamma = \{\gamma_{k,l}\} \geq 0 \;\middle|\; \begin{array}{l} \sum_{k,l} \gamma_{k,l} = 1, \\ \sum_{k,l} \psi_i(y_k, u_l)\gamma_{k,l} = 0, \quad i = 1, 2, \ldots, N \end{array} \right\} \tag{4.19}$$

where $\psi_i(y,u) = \alpha\,[\phi_i(f(y,u)) - \phi_i(y)] + (1-\alpha)\,[\phi_i(y_0) - \phi_i(y)]$ and $\sum_{k,l} = \sum_{k=1}^{K^{\Delta}} \sum_{l=1}^{L^{\Delta}}$. Consider now the finite dimensional LP problem

$$\min_{\gamma \in W_N^{\Delta}} \sum_{k,l} g(y_k, u_l)\gamma_{k,l} \stackrel{\text{def}}{=} G_N^{\Delta}(\alpha, y_0). \tag{4.20}$$

Note that the set W_N^{Δ} is a set of probability measures on $Y \times U$ which assign nonzero probabilities only to the points (y_k, u_l) and, as such

$$W_N^{\Delta} \subset W_N \qquad \Rightarrow \qquad G_N^{\Delta}(\alpha, y_0) \geq G_N(\alpha, y_0). \tag{4.21}$$

Proposition 4.3. *Let Assumption 1 be satisfied. The set W_N is not empty if and only if there exists $\Delta_0 > 0$ such that W_N^{Δ} is not empty for all $\Delta \leq \Delta_0$. If W_N is not empty, then*

$$\lim_{\Delta \to 0} \rho_H\left(W_N^{\Delta}, W_N\right) = 0 \tag{4.22}$$

and

$$\lim_{\Delta \to 0} G_N^{\Delta} = G_N. \tag{4.23}$$

Also, if γ_N^{Δ} is a solution of problem (4.20) and $\lim_{\Delta' \to 0} \rho(\gamma_N^{\Delta'}, \gamma_N) = 0$ for some sequence Δ' tending to zero, then γ_N is a solution of (4.3). If the solution γ_N of problem (4.3) is unique, then, for any solution γ_N^{Δ} of (4.20),

$$\lim_{\Delta \to 0} \rho(\gamma_N^{\Delta}, \gamma_N) = 0. \tag{4.24}$$

Proof. Observe that by (4.21) W_N is not empty if W_N^{Δ} is not empty. Suppose W_N is not empty, we will show that W_N^{Δ} is not empty for Δ small enough and that (4.22) is valid (the validity of (4.23) follows from (4.22)); the other statements in the proposition are immediate consequences of (4.22) and (4.23).

For brevity, let us denote $\psi_i(y,u) = \alpha[\phi_i(f(y,u)) - \phi_i(y)] + (1-\alpha)[\phi_i(y_0) - \phi_i(y)]$. From (4.18) and the fact that $\psi_i(y,u)$ are continuous it follows that

$$\sup_{(y,u)\in Q_{k,l}^{\Delta}} |\psi_i(y,u) - \psi_i(y_k, u_l)| \leq \kappa(\Delta), \quad i = 1, \ldots, N \tag{4.25}$$

for some $\kappa(\Delta)$ such that $\lim_{\Delta\to 0}\kappa(\Delta) = 0$. Define the set $Z_N^\Delta \subset \mathbb{R}^{L^\Delta + K^\Delta}$ by the equation

$$Z_N^\Delta = \left\{ \gamma = \{\gamma_{k,l}\} \geq 0 \;\middle|\; \begin{array}{l} \sum_{k,l} \gamma_{k,l} = 1 \\ \left| \sum_{k,l} \psi_i(y_k, u_l)\gamma_{k,l} \right| \leq \kappa(\Delta), \quad i = 1, \ldots, N \end{array} \right\}. \tag{4.26}$$

For any Δ, let $\gamma^\Delta \in W_N$ be such that $\rho(\gamma^\Delta, Z_N^\Delta) = \max_{\gamma\in W_N} \rho(\gamma, Z_N^\Delta)$ (γ^Δ exists since W_N is compact) we want to show that

$$\lim_{\Delta\to 0} \max_{\gamma\in W_N} \rho(\gamma, Z_N^\Delta) = \lim_{\Delta\to 0} \rho(\gamma^\Delta, Z_N^\Delta) = 0. \tag{4.27}$$

Let $\gamma_{k,l}^\Delta \overset{\text{def}}{=} \int_{Q_{k,l}^\Delta} \gamma^\Delta(dy, du)$. By (4.25),

$$\begin{aligned} \left| \sum_{k,l} \psi_i(y_k, u_l)\gamma_{k,l}^\Delta \right| &= \left| \sum_{k,l} \psi_i(y_k, u_l)\gamma_{k,l}^\Delta - \int_{Y\times U} \psi_i(y, u)\gamma^\Delta(dy, du) \right| \\ &\leq \sum_{k,l} \int_{Q_{k,l}^\Delta} |\psi_i(y_k, u_l) - \psi_i(y, u)|\, \gamma^\Delta(dy, du) \\ &\leq \kappa(\Delta), \qquad i = 1, 2, \ldots, N. \end{aligned} \tag{4.28}$$

Hence, denoting $\tilde{\gamma}^\Delta = (\gamma_{k,l}^\Delta)$ we obtain $\tilde{\gamma}^\Delta \in Z_N^\Delta$ and consequently that

$$\rho\left(\tilde{\gamma}^\Delta, Z_N^\Delta\right) = 0. \tag{4.29}$$

Let $q(y, u) : Y \times U \to \mathbb{R}^1$ be an arbitrary continuous function and let $\kappa_q(\Delta)$ be such that

$$\sup_{(y,u)\in Q_{k,l}^\Delta} |q(y, u) - q(y_k, u_l)| \leq \kappa_q(\Delta), \qquad \lim_{\Delta\to 0} \kappa_q(\Delta) = 0. \tag{4.30}$$

Then

$$\begin{aligned} &\left| \int_{Y\times U} q(y, u)\gamma^\Delta(dy, du) - \sum_{k,l} q(y_k, u_l)\gamma_{k,l}^\Delta \right| \\ =&\left| \sum_{k,l} \int_{Q_{k,l}} q(y, u)\gamma^\Delta(dy, du) - \sum_{k,l} \int_{Q_{k,l}} q(y_k, u_l)\gamma^\Delta(dy, du) \right| \\ \leq& \kappa_q(\Delta). \end{aligned} \tag{4.31}$$

Since the last inequality is valid for an arbitrary continuous function $q(y, u)$ it follows that $\lim_{\Delta\to 0} \rho(\gamma^\Delta, \tilde{\gamma}^\Delta) = 0$ which, together with (4.29) implies (4.27).

By (4.21), $\max_{\gamma\in W_N^\Delta} \rho(\gamma, W_N) = 0$. Hence, to prove (4.22), it is enough to establish that

$$\lim_{\Delta\to 0} \max_{\gamma\in W_N} \rho(\gamma, W_N^\Delta) = 0. \tag{4.32}$$

Since (by the triangle inequality),

$$\max_{\gamma\in W_N} \rho(\gamma, W_N^\Delta) \leq \max_{\gamma\in W_N} \rho(\gamma, Z_N^\Delta) + \max_{\gamma\in Z_N^\Delta} \rho(\gamma, W_N^\Delta) \tag{4.33}$$

and since (4.27) has already been verified, equality (4.32) will be established if we show that

$$\lim_{\Delta\to 0} \max_{\gamma\in Z_N^\Delta} \rho(\gamma, W_N^\Delta) = \lim_{\Delta\to 0} \rho(\tilde{\gamma}^\Delta, W_N^\Delta) = 0, \tag{4.34}$$

where $\tilde{\gamma}^\Delta = \{\tilde{\gamma}_{k,l}^\Delta\} \in Z_N^\Delta$ is such that $\rho(\tilde{\gamma}^\Delta, W_N^\Delta) = \max_{\gamma\in Z_N^\Delta} \rho(\gamma, W_N^\Delta)$.

Let $q_j(\cdot)$, $j = 1, 2, \ldots$ be a sequence of Lipschitz continuous functions which is dense in the unit ball of $C(Y \times U)$. Consider the finite-dimensional linear program:

$$F_J(\Delta) \stackrel{\text{def}}{=} \min_{\gamma=\{\gamma_{k,l}\}\in W_N^\Delta} \sum_{j=1}^{J} \frac{1}{2^j} \left| \sum_{k,l} q_j(y_k, u_l)\gamma_{k,l} - \sum_{k,l} q_j(y_k, u_l)\tilde{\gamma}_{k,l}^\Delta \right|. \tag{4.35}$$

To prove that (4.34) is valid, it is enough to show that

$$\lim_{\Delta\to 0} F_J(\Delta) = 0, \qquad J = 1, 2, \ldots. \tag{4.36}$$

Below it is shown that the optimal value of the problem dual to (4.35) tends to zero as Δ tends to zero. Since the latter coincides with $F_J(\Delta)$, this will prove (4.36). Also, from (4.36) is follows that $F_J(\Delta)$ is bounded and, hence, W_N^Δ is not empty for Δ small enough.

Let us rewrite the problem (4.35) in the equivalent form:

$$F_J(\Delta) = \min_{\gamma=\{\gamma_{k,l}\in W_N^\Delta\}} \sum_{j=1}^{J} \frac{1}{2^j}\theta_j, \tag{4.37}$$

where

$$-\sum_{k,l} q_j(y_k, u_l)\gamma_{k,l} + \theta_j \geq -\sum_{k,l} q_j(y_k, u_l)\tilde{\gamma}_{k,l}^\Delta, \tag{4.38}$$

$$\sum_{k,l} q_j(y_k, u_l)\gamma_{k,l} + \theta_j \geq \sum_{k,l} q_j(y_k, u_l)\tilde{\gamma}_{k,l}^\Delta. \tag{4.39}$$

The problem dual to (4.37)-(4.39) is

$$F_J(\Delta) = \max_{\lambda_i,\mu_j,\eta_j,\zeta} \sum_{j=1}^{J} (-\mu_j + \eta_j) \left(\sum_{k,l} q_j(y_k, u_l)\tilde{\gamma}_{k,l}^\Delta \right) + \zeta, \tag{4.40}$$

where λ_i, $i = 1, \ldots, N$; μ_j, η_j, $j = 1, \ldots, J$, and ζ satisfy the following relationships:

$$\sum_{i=1}^{N} \lambda_i \psi_i(y_k, u_l) + \sum_{j=1}^{J} (-\mu_j + \eta_j) q_j(y_k, u_l) + \zeta \leq 0, \tag{4.41}$$

$l = 1, \ldots, L^\Delta$, $k = 1, \ldots, K^\Delta$, and

$$\mu_j + \eta_j = \frac{1}{2^j}, \quad \mu_j \geq 0, \quad \eta_j \geq 0, \quad j = 1, \ldots, J. \tag{4.42}$$

Before proving (4.36) let us verify that $F_J(\Delta)$ is bounded for Δ small enough (which, by (4.35), is equivalent to $W_N^\Delta \neq \emptyset$). Assume that it is not. Then there exists a sequence Δ^r, $r = 1, 2, \ldots$, $\lim_{r\to\infty} \Delta^r = 0$, and sequences $\lambda_i^r, \mu_j^r, \eta_j^r, \zeta^r$, satisfying (4.41) and (4.42) with $\Delta = \Delta^r$, $r = 1, 2, \ldots$, such that $\lim_{r\to\infty}(|\zeta^r| + \sum_{i=1}^N |\lambda_i^r|) = \infty$ and

$$\lim_{r\to\infty} \frac{\zeta^r}{|\zeta^r| + \sum_{i=1}^N |\lambda_i^r|} \stackrel{\text{def}}{=} a \geq 0, \qquad \lim_{r\to\infty} \frac{\lambda_i^r}{|\zeta^r| + \sum_{i=1}^N |\lambda_i^r|} \stackrel{\text{def}}{=} v_i, \tag{4.43}$$

where

$$a + \sum_{i=1}^{N} |v_i| = 1. \tag{4.44}$$

Dividing (4.41) by $|\zeta^r| + \sum_{i=1}^{N} |\lambda_i^r|$ and passing to the limit as $r \to \infty$, one can obtain

$$\sum_{i=1}^{N} v_i \psi_i(y,u) + a \leq 0 \quad \forall (y,u) \in Y \times U, \tag{4.45}$$

where it is taken into account that every point $(y,u) \in Y \times U$ can be presented as the limit of (y_k, u_l) belonging to the sequence of cells $Q_{k,l}^{\Delta^r}$ such that $(y_k, u_l) \in Q_{k,l}^{\Delta^r}$.

Two cases are possible: $a > 0$ and $a = 0$. If $a > 0$, then the validity of (4.45) implies that the function $\phi(y) \stackrel{\text{def}}{=} \sum_{i=1}^{N} v_i \phi_i(y)$ satisfies (2.11) which would lead to W_N being empty.

The set W_N however, is not empty (by our assumption) and, hence, the only case to consider is $a = 0$. In this case, (4.45) becomes

$$\sum_{i=1}^{N} v_i \psi_i(y,u) \leq 0 \qquad \forall (y,u) \in Y \times U. \tag{4.46}$$

By Assumption 1, (4.46) can be valid only with all v_i being equal to zero. This contradicts (4.44) and, thus, proves that $F_J(\Delta)$ is bounded for Δ small enough (and that W_N^Δ is not empty).

From the fact that $F_J(\Delta)$ is bounded it follows that a solution λ_i^Δ, $i = 1, \ldots, N$; $\mu_i^\Delta, \eta_j^\Delta$, $j = 1, \ldots, J$, and ζ^Δ of the problem (4.40)-(4.42) exists. Using this solution, one can obtain the following estimates:

$$\begin{aligned}
0 \leq F_J(\Delta) &= \sum_{j=1}^{J} (-\mu_j^\Delta + \eta_j^\Delta) \left(\sum_{k,l} q_j(y_k, u_l) \tilde{\gamma}_{k,l}^\Delta \right) + \zeta^\Delta \\
&= \sum_{k,l} \tilde{\gamma}_{k,l}^\Delta \left(\sum_{j=1}^{J} (-\mu_j^\Delta + \eta_j^\Delta) q_j(y_k, u_l) \right) + \zeta^\Delta \\
&\leq \sum_{k,l} \tilde{\gamma}_{k,l}^\Delta \left(- \sum_{i=1}^{N} \lambda_i^\Delta \psi_i(y_k, u_l) - \zeta^\Delta \right) + \zeta^\Delta \\
&= - \sum_{i=1}^{N} \lambda_i^\Delta \left(\sum_{k,l} \psi_i(y_k, u_l) \tilde{\gamma}_{k,l}^\Delta \right) \leq \sum_{i=1}^{N} |\lambda_i^\Delta| \kappa(\Delta),
\end{aligned} \tag{4.47}$$

where the last inequality is implied by the fact that $\tilde{\gamma}^\Delta = \{\tilde{\gamma}_{k,l}^\Delta\} \in Z_N^\Delta$ (see (4.26))

To prove (4.36), it is now sufficient to show that $\sum_{i=1}^{N} |\lambda_i^\Delta|$ remains bounded as $\Delta \to 0$. Assume it is not. Then there exists a sequence Δ^r, $r = 1, 2, \ldots$, $\lim_{r \to \infty} \Delta^r = 0$, and sequences $\lambda_i^r, \mu_j^r, \eta_j^r, \zeta^r$, satisfying (4.41) and (4.42) with $\Delta = \Delta^r$, $r = 1, 2, \ldots$, such that

$$\lim_{r \to \infty} \sum_{i=1}^{N} |\lambda_i^r| = \infty, \quad \lim_{r \to \infty} \frac{\zeta^r}{\sum_{i=1}^{N} |\lambda_i^r|} = 0, \quad \lim_{r \to \infty} \frac{\lambda_i^r}{\sum_{i=1}^{N} |\lambda_i^r|} = v_i, \quad \sum_{i=1}^{N} |v_i| = 1. \tag{4.48}$$

Dividing (4.41) by $\sum_{i=1}^{N} |\lambda_i^r|$ and passing to the limit as $r \to \infty$, one obtains that the inequality (4.46) is valid, which, by Assumption 1, implies that $v_i = 0$, $i = 1, \ldots, N$. This contradicts the last equality in (4.48) and, thus, proves (4.36). $\square$

4.2 Construction of Near Optimal Controls

Let γ^* be an optimal solution of the IDLP problem (2.3) and let it be generated by an admissible pair $(y^{\gamma^*}(\cdot), u^{\gamma^*}(\cdot))$. Then, by (2.6),

$$\begin{aligned} g^*(\alpha, y_0) &= \int_{Y\times U} g(y,u)d\gamma^* \\ &= (1-\alpha)\sum_{t=0}^{\infty} \alpha^t g(y^{\gamma^*}(t), u^{\gamma^*}(t)). \end{aligned} \tag{4.49}$$

Consequently, by Theorem 3.2,

$$\begin{aligned} (1-\alpha)\sum_{t=0}^{\infty} \alpha^t g(y^{\gamma^*}(t), u^{\gamma^*}(t)) &= (1-\alpha)G(\alpha, y_0) \\ \Rightarrow \quad \sum_{t=0}^{\infty} \alpha^t g(y^{\gamma^*}(t), u^{\gamma^*}(t)) &= G(\alpha, y_0). \end{aligned} \tag{4.50}$$

That is, $(y^{\gamma^*}(\cdot), u^{\gamma^*}(\cdot))$ is an optimal solution of the optimal control problem (2.2).

Define

$$\Theta \stackrel{\text{def}}{=} \left\{(y,u) \mid (y,u) = (y^{\gamma^*}(t), u^{\gamma^*}(t)) \text{ for some } t \in \{0, 1, \ldots\}\right\}. \tag{4.51}$$

and denote

$$\mathcal{Y} \stackrel{\text{def}}{=} \{y \mid (y,u) \in \Theta\}, \qquad \psi(y) \stackrel{\text{def}}{=} u \text{ for } (y,u) \in \Theta, \tag{4.52}$$

where, it is assumed that if $(y, u') \in \Theta$ and $(y, u'') \in \Theta$ then $u' = u''$. The set Θ can be considered as the graph of the optimal feedback control function $\psi(y)$, which is defined on the optimal state trajectory $\mathcal{Y}$.

Let $\gamma_N^\Delta \stackrel{\text{def}}{=} \{\gamma_{k,l}^\Delta\}$ be a basic optimal solution of the finite dimensional LP problem (4.20). Assume that the optimal solution γ^* of the IDLP problem (2.3) is unique. From (4.8) and (4.24), it follows that

$$\lim_{N\to\infty} \limsup_{\Delta\to 0} \rho(\gamma_N^\Delta, \gamma^*) = 0. \tag{4.53}$$

That is, γ_N^Δ can be interpreted as an approximation of γ^* if N is large and Δ is small enough.

Let

$$\Theta_N^\Delta \stackrel{\text{def}}{=} \left\{(y_k, u_l) \mid \gamma_{k,l}^\Delta > 0\right\}, \tag{4.54}$$

and denote

$$\mathcal{Y}_N^\Delta \stackrel{\text{def}}{=} \left\{y \mid (y,u) \in \Theta_N^\Delta\right\}, \qquad \psi_N^\Delta(y) \stackrel{\text{def}}{=} u \text{ for } (y,u) \in \Theta_N^\Delta, \tag{4.55}$$

where again it is assumed that if $(y,u') \in \Theta_N^\Delta$ and $(y,u'') \in \Theta_N^\Delta$ then $u' = u''$. Note that the function $\psi_N^\Delta(y)$ is defined on the set $\mathcal{Y}_N^\Delta \subset Y$. Let us extend the definition of $\psi_N^\Delta(\cdot)$ to the whole of Y by using one of the available interpolating schemes.

Conjecture 1. *Under certain conditions, the control $\psi_N^\Delta(y)$ is near optimal in the optimal control problem* (2.2) *(in the sense that the value of the objective function obtained with this control is close to the optimal one) if N is large enough and Δ is small enough.*

We do not give the proof of this conjecture in the thesis. It will be a topic of our future research. We believe that it can be established on the basis of the fact that the sets Θ_N^Δ and Θ are, in a certain sense, close if N is large and Δ is small enough (the latter, in turn, can be proved on the basis of (4.53) similarly to the way it is proved in the continuous time setting; see Propositions 10 and 11 in [9]).

To illustrate our construction, let us consider the following example

$$\inf_{(y(\cdot),u(\cdot))} \sum_{t=0}^{\infty} \alpha^t \left[-y_1(t)u_2(t) + y_2(t)u_1(t)\right] \stackrel{\text{def}}{=} G(\alpha, y_0) \tag{4.56}$$

where $\alpha = 0.9$, $-1 \le u_i(t) \le 1$ and

$$y_i(t+1) = \frac{1}{2}y_i(t) - \frac{1}{2}u_i(t) \text{ for } i = 1, 2. \tag{4.57}$$

Let $y(t) \in Y = [-1, 1] \times [-1, 1]$ and let

$$\begin{aligned} y_1(0) &= \frac{1}{2}, \\ y_2(0) &= \frac{1}{4}. \end{aligned} \tag{4.58}$$

Fix integer $N > 1$. Let $\Delta_1 > 0$ and $\Delta_2 > 0$ be sufficiently small. Define

$$y_{i,1} = -1 + i\Delta_1, \quad y_{j,2} = -1 + j\Delta_1, \quad u_{k,1} = -1 + k\Delta_2, \quad u_{l,2} = -1 + l\Delta_2 \tag{4.59}$$

where $i, j = 0, 1, \ldots, \frac{2}{\Delta_1}$ and $k, l = 0, 1, \ldots, \frac{2}{\Delta_2}$ (Δ_1, Δ_2 being chosen in such a way that $K^\Delta \stackrel{\text{def}}{=} \frac{2}{\Delta_1}$ and $L^\Delta \stackrel{\text{def}}{=} \frac{2}{\Delta_2}$ are integers). Consider the finite dimensional LP problem

$$\text{minimize} \sum_{i=1}^{K^\Delta}\sum_{j=1}^{K^\Delta}\sum_{k=1}^{L^\Delta}\sum_{l=1}^{L^\Delta} (-y_{i,1}u_{l,2} + y_{j,2}u_{k,1})\gamma_{i,j,k,l} \tag{4.60}$$

subject to

$$\sum_{i=1}^{K^\Delta}\sum_{j=1}^{K^\Delta}\sum_{k=1}^{L^\Delta}\sum_{l=1}^{L^\Delta} \gamma_{i,j,k,l} = 1 \tag{4.61}$$

and

$$\sum_{i=1}^{K^\Delta}\sum_{j=1}^{K^\Delta}\sum_{k=1}^{L^\Delta}\sum_{l=1}^{L^\Delta} \left[\begin{array}{r} \alpha\left(\left(\frac{1}{2}y_{i,1} - \frac{1}{2}u_{k,1}\right)^m \left(\frac{1}{2}y_{j,2} - \frac{1}{2}u_{l,2}\right)^n - y_{i,1}^m y_{j,2}^n\right) \\ + (1-\alpha)\left(\frac{1}{2^m}\frac{1}{4^n} - y_{i,1}^m y_{j,2}^n\right) \end{array}\right] \gamma_{i,j,k,l} = 0. \tag{4.62}$$

where $m = 0, \ldots, N_0$ and $n = 0, \ldots, N_0$ ($m + n \ge 1$) (compare with the LP problem (4.20)).

The LP problem was solved using IBM CPLEX with $N_0 = 7$ (thus giving a total of 49 constraints), $\Delta_1 = 0.0125$ ($K^\Delta = 160$) and $\Delta_2 = \frac{1}{2}$ ($L^\Delta = 4$). In doing so, the values of $\gamma_{i,j,k,l}$ were obtained, and the optimal value was evaluated to be ≈ -1.013. The grid points corresponding to $\gamma_{i,j,k,l} > 0$ are marked with red dots in Figure 4.1 and their size is scaled proportionally to the magnitude of $\gamma_{i,j,k,l}$. Due to the interpretation of $\gamma_{i,j,k,l}$ as an approximation of the discounted occupational measure generated by the optimal trajectory, one may assume that the later passes through neighbourhoods of the points $(y_{i,1}, y_{j,2}, u_{k,1}, u_{l,2})$ corresponding to positive $\gamma_{i,j,k,l}$. We have applied the control $\psi_N^\Delta(y)$ in the system (4.56) with $y_0 = (\frac{1}{2}, \frac{1}{4})$ for 100 time periods and obtained the trajectory represented by the blue line in Figure 4.1. The trajectory becomes close to the square like figure within 4 time periods and moves along this figure exactly starting from the moment $t = 7$. The objective value thus obtained was ≈ -0.986, which is close to the optimal value of the LP problem (-1.013). This indicates that the control obtained is near optimal (due to Propositions 4.1 and 4.3). Note that the interpolation of $\psi_N^\Delta(y)$ to all of Y was not necessary as the trajectory passed exactly through points in $\mathcal{Y}_N^\Delta$.

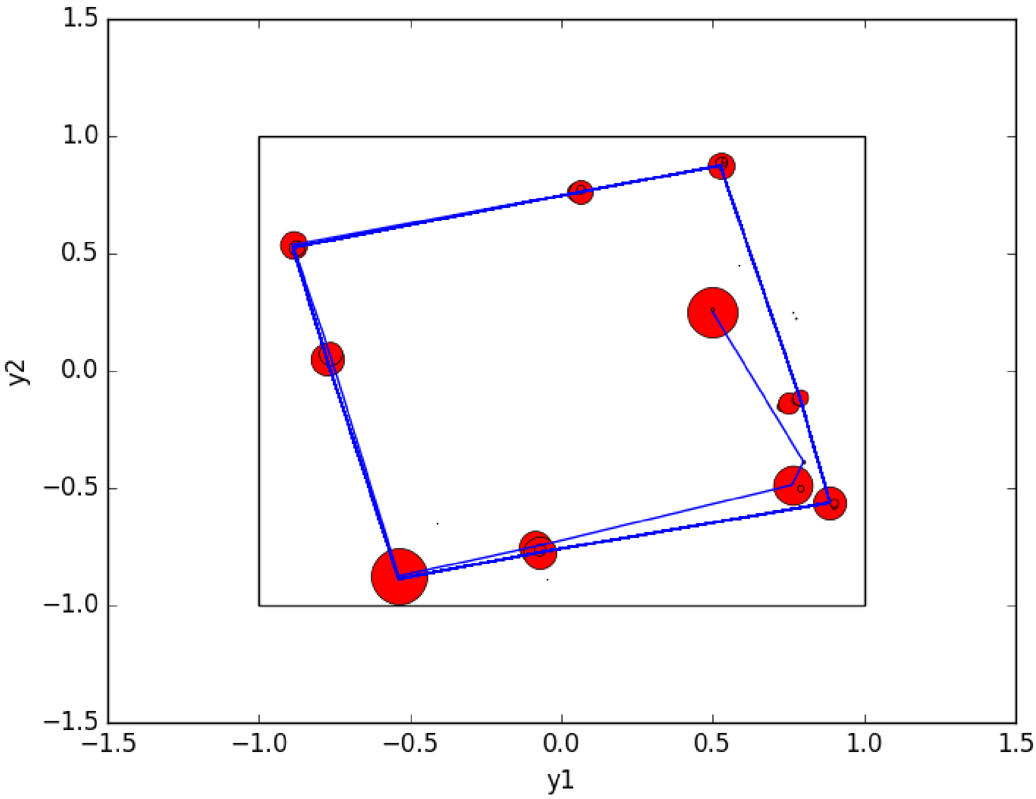

Figure 4.1: The optimal state trajectory of the system.

5

Future Research Directions

We have shown that the optimal value of a deterministic discrete time optimal control problem with time discounting is equal to the optimal value of the associated IDLP problem. The optimal value of the IDLP problem is approximated by the optimal values of SILP problems which are, in turn, approximated by the optimal values of the finite-dimensional LP problems. These finite-dimensional problems are readily solvable by one of the many software packages available. We further conjectured that a near optimal feedback control may be constructed on the basis of an optimal solution of an approximating finite-dimensional LP problem. Future research directions are as follows.

- Prove Conjecture 1.
- Develop a LP approach to long-run average optimal control problems in discrete time.
- Explore possibilities of using LP based techniques for the construction of near optimal controls for discrete time optimal control problems in higher dimensions.

References

[1] Bardi, M. and Capuzzo-Dolcetta, I. (2008). *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Springer Science & Business Media.

[2] Bertsekas, D. P. and Shreve, S. E. (1978). *Stochastic Optimal Control: The Discrete-Time Case*. Academic Press, New York.

[3] Bhatt, A. G. and Borkar, V. S. (1996). Occupation Measures for Controlled Markov Processes: Characterization and Optimality. *The Annals of Probability*, 24(3):1531–1562.

[4] Evans, L. C. and Gomes, D. (2002). Linear Programming Interpretations of Mather's Variational Principle. *ESAIM: Control, Optimisation and Calculus of Variations*, 8:693–702.

[5] Finlay, L., Gaitsgory, V., and Lebedev, I. (2008). Duality in Linear Programming Problems Related to Deterministic Long Run Average Problems of Optimal Control. *Proceedings of the IEEE Conference on Decision and Control*.

[6] Fleming, W. H. and Vermes, D. (1989). Convex Duality Approach to the Optimal Control of Diffusions. *SIAM Journal on Control and Optimization*, 27(5):1136–1155.

[7] Gaitsgory, V., Parkinson, A., and Shvartsman, I. (2016). Linear Programming Formulation of an Infinite Horizon Optimal Control Problem in Discrete Time. In *Proceedings of the 55th IEEE Conference on Decision and Control*, pages 4481–4483.

[8] Gaitsgory, V. and Quincampoix, M. (2009). Linear Programming Approach To Deterministic Infinite Horizon Optimal Control Problems With Discounting. *SIAM Journal on Control and Optimization*, 48(4):2480–2512.

[9] Gaitsgory, V. and Rossomakhine, S. (2006). Linear Programming Approach to Deterministic Long Run Average Problems of Optimal Control. *SIAM Journal on Control and Optimization*, 44(6):2006–2037.

[10] Gaitsgory, V., Rossomakhine, S., and Thatcher, N. (2012). Approximate Solution of the HJB Inequality Related to the Infinite Horizon Optimal Control Problem With Discounting. *Dynamics of Continuous, Discrete and Impulsive Systems*, 19:65–92.

[11] Gomes, D. A. and Oberman, A. M. (2004). Computing the Effective Hamiltonian Using a Variational Approach. *SIAM Journal on Control and Optimization*.

[12] Helmes, K. and Stockbridge, R. H. (2000). Numerical Comparison of Controls and Verification of Optimality for Stochastic Control Problems. *Journal of Optimization Theory and Applications*, 106(1):107–127.

[13] Hernandez-Hernandez, D., Hernandez-Lerma, O., and Taksar, M. (1996). The Linear Programming Approach to Deterministic Optimal Control Problems. *Applicationes Mathematicae*, 24(1):17–33.

[14] Kurtz, T. G. and Stockbridge, R. H. (1998). Existence of Markov Controls and Characterization of Optimal Markov Controls. *SIAM Journal on Control and Optimization*, 36(2):609–653.

[15] Lasserre, J. B., Henrion, D., Prieur, C., and Trélat, E. (2008). Nonlinear Optimal Control via Occupation Measures and LMI-Relaxations. *SIAM Journal on Control and Optimization*.

[16] Llavona, J. G. (1986). Approximation of Continuously Differentiable Functions. In Llavona, J. G., editor, *North-Holland Mathematical Studies*. North-Holland.

[17] Parthasarathy, K. R. (1967). *Probability measures on metric spaces*. Academic Press, New York.

[18] Rubio, J. E. (1986). *Control and Optimization: The Linear Treatment of Nonlinear Problems*. Manchester University Press, Manchester.

[19] Rudin, W. (1987). *Real and complex analysis*. New York, NY: McGraw-Hill, 3rd ed. edition.

[20] Stockbridge, R. H. (1990a). Time-Average Control of Martingale Problems: A Linear Programming Formulation. *The Annals of Probability*, 18(1):206–217.

[21] Stockbridge, R. H. (1990b). Time-Average Control of Martingale Problems: Existence of a Stationary Solution. *The Annals of Probability*, 18(1):190–205.

[22] Vinter, R. (1993). Convex Duality and Nonlinear Optimal Control. *SIAM Journal on Control and Optimization*, 31(2):518–538.