Teachers' Understanding and Use of Mathematical Structure

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November 2015

Submitted in fulfilment of the requirements

for the degree of Master of Research

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Contents

Summary

This research focuses on how junior secondary mathematics teachers can be more effective in teaching mathematics. It is based on the concern that too few secondary school students now study advanced mathematics subjects, possibly because they believe they do not have the ability to do so or because they dislike mathematics. The concept of mathematical structure is researched as a pedagogical approach to teaching mathematics that could not only convey the content and concepts of mathematics, but also engage students, more successfully.

In this small-scale study, I investigated teachers' understanding and promotion of mathematical structure. In doing so, I examined the nature of mathematical structure while acknowledging the previous theorising and research about procedural and conceptual understandings of mathematics teaching. I attempted to identify teachers' awareness of mathematical structure as well as their promotion of structural thinking when teaching.

Five mathematics teachers were surveyed. Three of these were then interviewed and subsequently observed teaching junior secondary mathematics classes. The survey and interviews were concerned with what teachers said about mathematical structure, and the observations identified whether they promoted structural thinking in their teaching. The results, while not conclusive, provide an interesting comparison between what teachers thought was mathematical structure and how they conveyed this understanding when actually teaching mathematics.

Authorship Statement

I hereby certify that this work titled *Teachers' Understanding and Use of Mathematical Structure* has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree to any other university or institution than Macquarie University.

I also certify that the thesis is an original piece of research and it has been written by me. In addition, I certify that all information sources and literature used are indicated in the thesis.

The research presented in this thesis was approved by the Macquarie University Faculty of Human Sciences Human Research Ethics Sub-Committee, No. 5201401133 Con/Met (see Appendix A), and permission to conduct the research was provided by the principal of the participating high school (see Appendix B).

Mark Gronow

27 November 2015

Acknowledgements

The completion of this thesis has been possible with the assistance of many colleagues, friends, and family. Importantly, I am grateful to the principals and staff of the schools where the instruments were piloted and the main study conducted and to The Catholic Education Commission for the award of the Br John Taylor fellowship that gave me the encouragement and confidence to build on my ideas for researching mathematics education. I am also grateful to Tony McArthur as my lead at the Catholic Education Commission; Dr Michael Bezzina, Director of Sydney Catholic Schools Office, as my mentor for the fellowship; and Jim Hanna for his help and support on media relations.

Professor Joanne Mulligan acted as primary supervisor for the thesis and the research project. Her never-ending support and encouragement was always appreciated and respected. It was Joanne's ability to take the idea of mathematical structure and help me build on this so this project could take shape as a suitable and possible research topic. Joanne was able to direct my learning as an independent researcher to achieve the goal of completing this task. Her ability to see how things needed to develop and grow to make this project possible was a the greatest part of the learned experience and I appreciate how she was able to guide me on this path despite my moments of doubt and what seemed to me to be impossibilities. I also thank Dr Michael Cavanagh for his support and assistance as assistant supervisor.

Thank you, also, to my professional writing guide, Dr Robert Trevethan, who helped me develop my academic writing style. His never-ending questioning and excitement about the thesis and its development improved the quality of the final product. Robert's ability to bring out the best of my writing is greatly appreciated.

Thank you to my family, Michele, Emma, and Thomas, for allowing me the indulgence of giving up family time to work on this project. Finally, thanks also to my good friend Adam Lackey for encouraging me to produce the best work possible and for solving the technical issues.

Researcher Background

I am a secondary mathematics teacher. From very early in my career, I realised that to engage students in my classes I needed to understand how students think when being taught mathematics. My beliefs meant challenging the traditional teacher-centred manner in which mathematics is taught. In order to explore these beliefs I began investigating alternative pedagogical approaches. Enrolling in a postgraduate degree at Sydney University in 1988 and studying educational psychology further developed my interest in how to teach mathematics with a focus on how students perceived their learning of mathematics. I was able to apply what I had learnt about students' understanding of mathematics to the classroom and to develop an alternative mathematics curriculum to support this.

Over the next 26 years I have held a variety of positions in different secondary schools, including head of mathematics; assistant principal, curriculum; and deputy principal, administration. During this time, I taught in both single-sex and coeducational schools that were in both metropolitan and rural areas. Additionally, I taught mathematics at all levels from Year 7 to Extension 2 HSC level from the NSW mathematics curriculum. This diverse range of experiences added to my knowledge and understanding of school systems, curriculum, and pedagogy, especially in mathematics.

In my leadership and coordinating roles I was able to implement some change to teaching and learning from a pedagogical and curriculum perspective. The area of mathematics, as a specialty, was my focus. I was particularly interested in why so many students had an aversion to mathematics. As a possible solution to this, I felt teachers needed to teach with attention to students' understanding of mathematics, rather than the delivery of the content

In 2014, I decided to commence further studies in mathematics education. After applying to Macquarie University for doctoral studies in this area, I enrolled in a Master of Research as the beginning of a journey to investigate teaching and learning of mathematics. I was fortunate enough to receive the 2015 Br John Taylor Fellowship in educational research from the Catholic Education Commission of NSW as encouragement to develop my research into mathematics education.

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CHAPTER

1

INTRODUCTION

> A teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions he may give them a taste for, and some means of, independent thinking. Pòlya (1957)

1.1 Context of this study

Pòlya's articulate exhortation in 1957 that teachers' responsibility is to engage and enthuse students about mathematics still has relevance today. With fewer graduates of mathematics from tertiary institutions, declining numbers of mathematics teachers, and a decreasing number of students attempting higher mathematics in secondary schools, it has become incumbent on teachers of mathematics to change the way that mathematics is taught. Students, when given opportunities to engage in mathematical activities that stimulate their interest, are likely to enjoy learning mathematics not only for extrinsic rewards but also opportunities to learn about how the beauty of mathematics shapes our world.

In this research project I explore mathematical structure, a construct that supports learners' deep understanding of mathematics. An attention to mathematical structure that promotes structural thinking can encourage students to become curious about learning mathematics and able to solve mathematical problems confidently.

The focus of this research centres on the construct of mathematical structure or structural thinking in mathematics. Although Australian mathematics teachers may not be familiar with the term mathematical structure, they may be familiar with the principles of mathematical structure and apply those principles in their teaching. My aim for this research project is to explore whether teachers of mathematics understand mathematical structure and apply it when teaching even if they do not always realise they are doing so.

1.2 Background to this study

There has been a highly publicised concern with regard to the preparation of mathematics and science teacher graduates in Australia (O'Kane, 2015). This concern stems from government, industry, and educational institutions' realisation that the declining number of mathematics graduates from universities is now a national problem. In their report *Mathematics, Engineering & Science in the National Interest,* Chubb, Findlay, Du, Burmester, and Kusa (2012) identified this concern and the flow-on effect of fewer qualified mathematics and science teachers in Australian schools.

Chubb et al. (2012) stated that nonqualified mathematics teachers taught 40% of Years 7–10 mathematics classes in Australia. In the same report it was revealed that 12% of Years 11–12 mathematics teachers in metropolitan schools had no university training in mathematics, and in rural areas this increased to 16%. The shortage of qualified mathematics teachers is a critical issue for Australia.

In the report *Science, Technology, Engineering and Mathematics in the National Interest: A Strategic Approach,* delivered by the Office of the Chief Scientist (2013), the authors emphasised the need for highly qualified, respected, and supported mathematics and science teachers.

A report conducted by the Mathematical Association of NSW (MANSW) revealed that the number of students participating in the higher levels of mathematics in NSW secondary schools was declining (MANSW, 2014). Similarly, Barrington and Evans (2014) found that the number of students attempting elementary mathematics in Year 12 across Australia in the previous 10 years had increased while the number of students studying mathematics at higher levels had decreased, thus impacting on the number of students entering mathematics subjects at university.

The low levels of participation in mathematics have been of particular concern for mathematics educators and stakeholders in Australia for over a decade. Plenty and Heubeck (2013) noted that government, community, industry, and higher education groups have raised concerns about the lack of preparation among young people for a technological world. The problem has been acknowledged for some time, Thomas (2000) reported that the consequence of fewer students studying mathematics at university would be even fewer choosing to undertake mathematics teacher education programs, further exacerbating the critical shortage of mathematics teachers with strong mathematical backgrounds. Forgasz (2006) studied patterns and trends of student enrolments in mathematics subjects at university and found that fewer students completed higher mathematics subjects for entry into tertiary studies because these subjects were no longer required as prerequisites for entry into courses like engineering, and students were motivated to choose subjects that maximised their university entrance score. Pitt (2015) identified the same decline of students studying the calculus-based courses in New South Wales because they preferred the general mathematics non-calculus courses. The result has been that students are inadequately prepared for university study. Pitt provided evidence that the scaling algorithm used to derive the Australian Tertiary Admission Rank is a reason why students are choosing the lowest level of mathematics in Year 12.

As participation in mathematics has declined over the past 20 years and smaller proportions of university students are obtaining mathematics qualifications, there has been a concomitant decline in qualified teachers of mathematics (MANSW, 2014). The 2014 MANSW report also noted the increasing number of nonqualified teachers of mathematics. Nonqualified teachers in junior secondary mathematics classes are not engaging students in the discipline. The teachers' understanding of the mathematical content, how to teach it, and how students learn it impacts on student engagement and learning. The Australian *National Numeracy Report* (Commonwealth of Australia, 2008) linked the declining number of students attempting the higher levels of mathematics with the shortage of qualified mathematics teachers. The authors of this report stated that qualified teachers made mathematics more meaningful and produced learners who acquire higher aspirations for future mathematics study. The authors of the report from the Office of the Chief Scientist (2013) singled out inspirational teaching as crucial for nurturing student interest in mathematics and science.

What is happening in the mathematics classroom?

The problem of students losing interest in mathematics begins before senior school calculus courses are considered. Students can begin disengaging from mathematics from early years in primary school and the disengagement can continue into junior secondary school years. Attard (2013) reported on a longitudinal case study that investigated the problem of students' lowered engagement in mathematics and what influenced their engagement during the middle years of schooling. The data collected through interviews, focus group studies, and classroom observations showed that positive pedagogical relationships between the teachers and students were important as the foundation for students maintaining engagement in mathematical learning. Attard (2010) found that teachers without a pedagogical background in mathematics had difficulty explaining mathematical concepts to the students. With the decline in mathematics teacher graduates, the number of teachers without mathematical pedagogical experiences has increased, which adds to student disengagement. Plenty and Heubeck (2011) identified lower motivational attitudes toward mathematics as beginning in early secondary school when students can make decisions about their perceived mathematical ability that then directly influence their engagement with the subject.

Mason, Stephens, and Watson (2009) identified an awareness of mathematical structure as being crucial for teachers to maintain student interest and engagement. They believed that students stop learning when mathematical structure is not appreciated in the classroom. Teachers need to initiate students into mathematical structure, and cultivate it in order to mature this appreciation. By presenting research that supports this view, Mason et al. (2009) argue that students who are not encouraged to observe mathematical structure in their mathematics learning or are not engaged in structural thinking processes become blocked from thinking deeply about mathematics. Their point is that teachers need to support students in developing structural thinking skills, and to do this they must understand what is mathematical structure.

1.3 The research problem

The broad research problem is drawn out of the drought of qualified mathematics teachers. Nonqualified teachers are now required to teach mathematics, and due to their lack of pedagogical training in mathematics education they may tend to teach toward procedural understanding. Skemp (1976) called this instrumental understanding and argued against it nearly 40 years ago. However, it is not only nonqualified mathematics teachers who teach in a procedural manner. Sullivan, Clarke, and Clarke (2009) found that mathematics teachers were not able to create meaningful learning tasks and tended to teach using more procedural methods to solve mathematical problems. Teaching procedural methods does not develop students' ability to think deeply about the mathematics they are learning, yet it dominates the teaching and learning of mathematics. Mathematics teachers can use procedures in their teaching but have a responsibility to develop deeper conceptual understanding in their students. This can be achieved through an awareness of mathematical structure and encouraging students to use structural thinking in solving mathematical problems.

The concern that teachers are more inclined to teach procedurally inspired me to research teachers' understanding of mathematical structure. Following this, I was interested in finding out whether teachers actually demonstrated mathematical structure though their classroom teaching. Mason et al. (2009) identified the benefits of an appreciation of mathematical structure, but there is insufficient evidence that teachers understand mathematical structure, either through their personal knowledge or in their teaching.

1.3.1 Mathematical structure and procedural and conceptual understanding

Skemp (1976) produced his seminal paper about instrumental versus relational understanding about the learning of mathematics. Skemp emphasised the need to change mathematics teaching from an instrumental to a relational focus. His ideas about instrumental and relational understanding in mathematical learning remain central to new theories relating to procedural and conceptual understanding. Sullivan (2011) stated that Skemp's theory of relational understanding is aligned to conceptual understanding as an appreciation of ideas and relationships. Kilpatrick, Swafford, and Findell (2001) and Watson and Sullivan (2008) described procedures as being the ability to use flexible, accurate, efficient, and appropriate methods to solve mathematical problems, and along with these procedures they included the ability to recall mathematical facts readily. This describes what most people remember of their mathematical experiences: rote learning facts and procedures to be reproduced in timed tests.

Teaching procedures to support the development of mathematical concepts are inherent in the teaching and learning of mathematics. The procedure represents the memorised method used to solve a problem, the concept being the mathematical theory, model, or idea the student needs to understand. The teacher's awareness of mathematical structure supports any method of delivery provided a deeper understanding of the concept is achieved. Memorising procedures without understanding the concept or why the procedures are used is a central problem.

Australian mathematics teachers have been identified as teaching predominantly toward a procedural understanding. In the Third International Mathematics and Science Study (TIMSS) 1999 video study, Australia was shown to have a higher proportion of nonqualified mathematics teachers and that teaching methods were dominated by a procedural approach (Lokan, McRae, & Hollingsworth, 2003). There was no identified correlation between nonqualified mathematics teachers and teaching procedurally, although all mathematics teachers need to be aware of the negative effect that a purely procedural approach has on the learning of mathematics. The TIMSS video study identified teachers in countries with the highest scores on TIMSS as teaching toward a higher involvement of conceptual understanding.

Procedural understanding, as previously mentioned, can be characterised by simply memorising a method to obtain a solution to a mathematical question. For example, to find the area of a rectangle, students are taught to multiply length by breadth, where the length and breadth are given as two separate numbers. Students, in a procedural manner, simply learn to multiply the two numbers given without understanding what area is and why the two numbers are multiplied. This approach will yield the correct answer for the area of a rectangle, but is meaningless if the student does not recognise the length and breadth as being the adjacent sides of the rectangle. The words become pointless as many other words or symbols can be used. The multiplication process works for a rectangle but the formula cannot be applied to other plane shapes, for example a triangle. A procedural understanding requires completion of a number of similar examples, explaining the steps to get the answer, setting the students a number of similar examples to practise by repeating the process, and finally assessing the ability to repeat the process in a timed test. Procedural understanding is important in mathematics, but not as the focus. Memorising a method does not develop a deep understanding of concepts. Procedural understanding is specific to the examples given, but the process is unlikely to transfer to other situations.

Conceptual understanding requires knowledge of the basic principles of mathematics. It encourages the learners to think about the mathematics they are learning rather than recalling facts and processes. It is flexible and can be generalised to new situations.

Mason et al. (2009) argued that when mathematical structure is connected to mastering procedures and understanding concepts, mathematical thinking is promoted. They stated further that the learner would understand the relevance of the mathematics being taught, rather than relying on memorising, when the teacher's focus is on mathematical structure. Effective mathematical thinking involves being able to use, explain, and connect mathematical properties. They use specific examples of how mathematical structure bridges the gap between procedural and conceptual understanding of mathematics in teaching and learning.

Mason et al. (2009) strongly suggested that attention to mathematical structure as an overarching theory of procedural and conceptual understanding of mathematics should be addressed in every mathematics classroom. They provide evidence that students' mathematical understanding is enhanced when mathematical structure is the focus of learning. To achieve this, teachers need to acknowledge mathematical structure in the content taught and pedagogy employed, and they need to avoid relying on procedural understanding in teaching mathematics.

Research by Prescott and Cavanagh (2006) has shown that new graduate teachers focused on procedural understanding in their teaching. They demonstrated that these teachers, once they began teaching, relied on their own experiences as students about how mathematics should be taught. Similarly, Bobis (2000) found that new graduate teachers reverted to a teacher-centred approach that could be described as being similar to procedural understanding.

Mason et al. (2009) pointed out that the over-use of procedural methods for teaching mathematical content blocked students' ability to think mathematically.

1.3.2 Mathematical structure and pedagogical content knowledge

Mason et al. (2009) have shown that the teacher's understanding of mathematical structure is a significant component of pedagogical content knowledge (PCK), described by Shulman (1987) as a requirement for good teaching of mathematics.

The mathematical content taught requires an awareness of mathematical structure by the teacher for effective communication to the learners. Clarke, Clarke, and Sullivan (2012) recognised that understanding mathematical content is important for teaching mathematics effectively. Mathematical structure enables the teacher to explain the content so students can relate to it. To achieve this, the teacher can apply mathematical structure through making connections with other learning, recognising any existing patterns, identifying similarities and differences, and generalising results to different situations. The ability to demonstrate these relationships is essential in the mathematics teacher's pedagogy. Attention has been given to developing teacher pedagogical content knowledge (PCK) as a means of improving student learning by mathematical education researchers (Bobis, Anderson, Martin, & Way, 2011; Hill & Ball, 2004). If regarded as a component of mathematical pedagogy, mathematical structure should be included as an important part of PCK. Vale, McAndrew, and Krishnan (2011) found that nonqualified teachers' understanding of mathematical content and concepts improved with an awareness of mathematical structure.

Bobis (2000) reported that effective mathematics teachers understand the interconnectedness of ideas; have an ability to select and use efficient and effective strategies; challenge students to think; and encourage them to explain, listen, and solve problems. Bobis identified with mathematical structure through the strategies she recognised. These are strategies that develop structural thinking in the students. An understanding of mathematical structure will encourage the mathematics teacher to use these strategies in the classroom. Teachers' understanding of mathematical structure will be identified through teaching statements of instruction and questioning.

1.3.3 Mathematical structure and student engagement in mathematics

Student engagement in mathematics is included as a background problem to this research project because teachers can address this issue by paying greater attention to mathematical structure in their classroom. Teachers who have an awareness of mathematical structure can develop structural thinking that engages students. (Mason et al., 2009; Mulligan & Mitchelmore, 2009; Taylor & Wade, 1965).

In their study about improving participation rates in mathematics, Brown, Brown, and Bibby (2008) surveyed over 1,500 students in 17 schools. Results from a questionnaire found that the level of difficulty of the mathematics and personal lack of confidence were reasons for students not continuing with mathematics. These factors, along with a dislike and boredom, as well as a perceived lack of relevance, were also related to students' decision not to study mathematics at senior secondary school level.

Mathematical structure aims to increase student engagement. Mason et al (2009) made it clear that students who are able to think structurally receive intrinsic reward from their enjoyment in mathematics. It is not about the mark on a test or being the fastest to answer the question. Mason et al. (2009) concluded that a teacher's awareness of structural relationships would transform students' mathematical thinking and their disposition to engage.

1.4 What is mathematical structure?

Mathematical structure can be found in connecting mathematical relationships, recognising patterns, identifying similarities and differences, and generalising results. Mason et al. (2009) defined mathematical structure clearly as "the identification of general properties which are instantiated in particular situations as relationships between elements or subsets of elements of a set" (p. 10). They believed that appreciating structure is powerful in developing students' understanding of mathematics and that attention to structure should be an essential part of mathematical teaching and learning. Mathematical structure is a precursor to structural thinking, which can be associated to cognitive structures, producing schemas that are essential in mathematical thinking and successful learning. Mason et al. (2009) stated that mathematical structure is not taught. Rather, it is an understanding of how the procedures and concepts are connected to support student learning.

In light of a significant amount of research into mathematics structure, Taylor and Wade (1965) proposed a theoretical definition as the formation and arrangement of a mathematical system within mathematical properties. The seminal work of Skemp (1976) introduced relational thinking, which has been associated with structure. Others have also referenced mathematical structure. Jones and Bush (1996) use a “building blocks” metaphor to describe mathematical structure, stating that mathematical structure is like the foundation of a building on which the content is built. They identified structural thinking in mathematics as a vehicle for helping students understand and answer the "why" questions in mathematics. Schmidt, Houang, and Cogan (2002) took a different approach to mathematical structure. They were concerned with the deeper sense of mathematical structure as it connects content and its particulars into deeper mathematical understanding. More recently, Mulligan and Mitchelmore (2009) identified structural thinking in preschool patterning strategies, and Vale, McAndrew, and Krishnan (2011) looked at developing structural understandings in out-of-field mathematics teachers.

1.5 Purpose

Various mathematics education researchers have proposed individual definitions of mathematical structure that have similarities to a broad concept, but display individual distinctions (Barnard, 1996; Jones & Bush, 1996; Mason et al., 2009). Others have attempted to identify how mathematical structure and structural thinking impact on students’ mathematical understanding (Jones & Bush, 1996; Mason et al., 2009; Mulligan & Mitchelmore, 2009; Vale et al., 2009). Despite this large body of research about mathematical structure, there is a lack of research about teachers’ understanding of mathematical structure and how they teach with reference to mathematical structure in junior secondary schools. Mason et al. (2009) said that teachers’ awareness of mathematical structure could improve students’ engagement in mathematics. If this is the case, there is a need to identify how the classroom teacher does this.

In this research project I intended first to explore mathematics teachers’ knowledge about mathematical structure and how they perceive its importance in the teaching and learning of mathematics. Following this I intended to identify how the classroom teacher demonstrates this understanding by observing them when teaching mathematics.

In this project, I contribute to existing research about how mathematical structure is perceived and used by mathematics teachers. By means of a survey and interviews I identify teachers' knowledge of mathematical structure, and by observing teachers, I gain evidence of how they apply mathematical structure when teaching mathematics.

To achieve the above aims, I surveyed a group of five mathematics teachers. Following the survey, I interviewed three teachers from this group of five. I then observed the same three teachers as they each taught three junior secondary mathematics classes.

1.6 Research questions

Three main questions, two of which had subquestions, were addressed in this study:

1. Do mathematics teachers demonstrate an awareness of the nature and value of mathematical structure?
    - Do teachers understand mathematical structure?
    - Do teachers recognise the presence of mathematical structure within the NSW mathematics syllabus?
    - Are teachers aware of structural thinking in their students?

2. Do mathematics teachers promote structural thinking when teaching mathematics?
    - Do teachers' explanations about mathematical procedures and concepts in the classroom support structural thinking?
    - Do teachers focus on procedures or concepts when teaching mathematics?

3. Is there a discrepancy between what teachers say and do concerning mathematical structure?

1.7 Significance of this study

The benefit of any advancement in this area of research will be in how teachers' awareness of mathematical structure helps them to support positive experiences in students' mathematics learning. Identifying that teachers do need to have an awareness of mathematical structure could help them to teach students the importance of applying structural thinking when solving mathematical problems. Empowering mathematics teachers with the confidence to think about mathematical structure when teaching will help them to build students' engagement, confidence, understanding, and success in mathematics.

1.8 Thesis structure

This thesis has five chapters. Following this first chapter, the second chapter comprises a literature review and theoretical framework in which I will detail and critique the limited research surrounding mathematical structure and the theory developed for this study. In the third chapter I deal with the design and methodological considerations that are pertinent to my research. Chapter 4 comprises the results I obtained when surveying, interviewing, and observing a small number of mathematics teachers. When presenting the results I also provide some discussion that is specific to those results. In the final chapter, Chapter 5, I discuss the results of this research more fully, particularly in relation to other research, and draw some conclusions based on the findings. I also acknowledge the limitations in this study and identify avenues for future research.

CHAPTER

2

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

2.1 Chapter outline

In this chapter I review the literature related to the concept of mathematical structure and structural thinking in its relevance to teaching and learning. As well as identifying how mathematical structure exists as a focus of mathematics pedagogy, I explore and critically assess how mathematics education researchers have identified mathematical structure and structural thinking and how the concept has been acknowledged as a component of mathematics teachers' pedagogical content knowledge. The focus is on mathematics teachers and their awareness of mathematical structure and promotion of structural thinking. Through this focus I attempt to analyse how authors have described the role of teachers in promoting structural thinking. I develop the theoretical framework behind the development of the components of mathematical structure as connections, recognising patterns, identifying similarities and differences, and generalising (CRIG) based on the literature.

In Chapter 1, I dealt with definitions and the development of the construct of mathematical structure and structural thinking. There are various interpretations by different authors. Mathematical structure, or structure as referred to by Mulligan, Vale, and Stephens (2009), was the focus of a special issue of *Mathematics Education Research Journal* that provided a range of exemplars of the application of mathematical structure in teaching and learning. They viewed structure as a focus in mathematics education, particularly in the area of algebraic thinking and arithmetic processes, and in the development of mathematical representation, symbolising, proofs, generalising, and abstraction.

Mason, Stephens, and Watson, (2009) and Mulligan and Mitchelmore (2009) provided complementary approaches to describing mathematical structure as central to mathematics teaching and learning. Mason et al. (2009) provided examples of teaching methods and tasks that secondary teachers would use to develop their own structural awareness and structural thinking in their students. The teachers' awareness of mathematical structure is said to benefit students as mathematical learners. This is evident from the work of Mulligan and Mitchelmore (2009) about pattern and structure with young children. Their research showed how a structural approach was advantageous for teachers in developing mathematical understanding in their students.

2.2 Mathematics teaching and learning

Mathematical structure is not a term commonly used by teachers in a mathematics classroom environment, but it does have a long history of use by mathematics education researchers. Mason et al. (2009) identified the notion of structure as existing as far back as Euclid, but it might be assumed that mathematical structure has been a part of mathematics since mankind started to think mathematically. Fifty years ago, Taylor and Wade (1965) found that the term structure was starting to occur frequently in the mathematics education research literature, and they questioned what structure in mathematics actually meant. They decided it was unclear, meaning different things to different people, and that an explicit definition would be beneficial. Essentially, they felt it was reasonable to assume that mathematical structure is the "formation, arrangement, or result of putting together of parts".

While not using the term mathematical structure, Skemp (1976) made a distinction between what he referred to as instrumental and relational understandings of mathematical learning. He explained that instrumental understanding was learning a number of fixed plans with starting points and finishing points and explanations of what to do along the way, whereas relational understanding involved building up a conceptual structure or a schema that offered an unlimited number of starting points toward any finishing point, with multiple paths to get there. These terms are identified within the psychology of mathematics education literature through the work of Fischbein (2002, p. 248), and they represent different approaches about how mathematics is taught. The delineation between procedural and conceptual learning and understanding is bridged by mathematical structure, which connects procedures and concepts so that deeper thinking about the mathematics being taught is achieved.

Fischbein (2002) and Stephens (2008) aligned Skemp's relational understanding to structure, and instrumental to procedure. Mason et al. (2009) concluded that while authors have identified learners' ability to use relational or structural thinking, they maintained that it is not a procedure that is to be taught like some mathematical content, and it is the awareness of structural relationships that promotes a similar awareness in students.

Mason et al. (2009) started from the premise that mastering of procedures is important when taking advantage of opportunities to make mathematical sense, but it is of little value to the learner if it remains as a procedure. Procedural learning simply places a burden on the learner to remember, but when procedures are associated with some appreciation of mathematical structure the learning shifts from memorising to understanding the concepts. Richland, Sigler, and Holyoak (2012) were more emphatic in saying that procedures simply left the learner ineffective at any mathematical reasoning. It is structural thinking that allows learners to have confidence in manipulating the procedures taught and apply the concepts to mathematical problems. Using the expression "conceptual structure" Richland et al. (2012) acknowledged that this process allowed learners to make predictions regarding how procedures relate to the solutions and develop new understandings about the concepts. In their study, Richland et al. (2012) examined mathematics knowledge of students who had completed the K–12 mathematics sequence and found these students were unlikely to have flexible reasoning in mathematics. Students in this study saw mathematics as a collection of procedures, rules, and facts to be remembered, and found that this became increasingly difficult as they progressed through the curriculum.

Mathematical structure has its foundations in the connections between procedural understanding, which is the doing of a problem, and conceptual understanding, which is described as the knowing or understanding of why a particular procedure is used. Richland et al. (2012) proposed that students' long-term ability to transfer and engage in mathematical knowledge is achieved through mathematics instruction that focuses on making connections between the mathematics learnt as procedure or concept.

Richland et al. (2012) noticed that, after completing their schooling, students were not flexible in their mathematical reasoning. Instead of being able to recognise relationships between problems or to make inferences about the representations given, they relied on using, often incorrectly, previously memorised procedures, which can be associated to a strictly procedural mode of understanding. By identifying mathematical structure and structural thinking they suggest that the students' ability to engage in these practices is associated with high performance on international mathematics tests.

Richland et al. (2012) reviewed the 1999 Third International Mathematics Science Study (TIMSS) video study. This study involved seven countries, including Australia, in which 100 teachers were randomly selected and a single mathematics lesson was videotaped. A team of international researchers collaborated in developing a reliable coding procedure to gather data about common teaching practices between the countries. Countries in which student performances were the highest had mathematics teachers who were more likely to present problems that were categorised as *Making connections* as opposed to the lesser achieving countries that relied on what was identified as *Using procedures.* Recently, Boaler (2015) also identified the inadequacies of the procedural manner of instruction that has dominated current teaching of mathematics. She has been active in discouraging this approach in favour of deeper understanding through development of conceptual understanding.

2.3 Teachers' awareness of mathematical structure

Research about mathematics teachers' awareness of structure is related to a wide range of studies about pedagogical approaches, mathematics teacher pedagogical content knowledge, and other affective factors such as teacher confidence and self-efficacy. Other studies relate to teacher professional learning. This review focuses on a number of pertinent studies related to the focus of enquiry in this thesis.

Vale, McAndrew, and Krishnan (2011) examined out-of-field (i.e., nonqualified) mathematics teachers after they completed a professional development course for junior secondary mathematics teachers. The course focused on mathematics syllabus content and pedagogy, and the researchers explored teachers' understanding of mathematical connections and their appreciation of mathematical structure. An appreciation and awareness of structure was identified through teachers' recognition of mathematical relationships and properties, resulting in a deepening of their structural understanding of mathematics. These teachers were also able to make the connections between these relationships and properties, which they were able to implement in the classroom to promote student structural thinking.

Vale et al. (2011) point out that this study demonstrates that by taking the position of the learner, practising teachers of secondary mathematics can appreciate structure and make connections with more complex concepts. In the role of learner, teachers can understand student thinking and learn to develop their teaching practice that enables learners to make connections between the mathematical relationships that support mathematical structure. The benefits identified by Vale et al. that teachers, when aware of mathematical structure, do promote structural thinking in their teaching are related to the research questions of this project. The teacher who is aware of mathematical structure makes mathematical connections between current learning, previous learning, and future learning. The expression "knowledge at the mathematical horizon" (Vale et al., 2011, p. 169) is used to describe how the teachers' own mathematical knowledge is required to enhance students' future mathematical learning. A definition of mathematical structure as "building blocks" (Jones & Bush, 1996) would adequately describe this idea. All mathematics learnt forms the foundation for future mathematics to be learnt.

An important outcome from the Vale et al. (2011) research was the impact that the teachers involved experienced. They felt that their deepened awareness of mathematical structure and their ability to explore structure had increased their desire to develop their pedagogical knowledge for classroom practice. In addition to this, the teachers broadened their appreciation that an emphasis in the classroom on procedural understanding limits students in their understanding of mathematics.

Davis and Renert (2013) introduce emergent mathematics as "a sophisticated and largely enactive mix of familiarity with various realisations of mathematical concepts and awareness of the complex process through which mathematics is produced" (p. 247). This complex nature of emergent mathematics can be used to understand the structure of mathematics in that it highlights a teacher's content knowledge. Davis and Renert's study with 22 practising teachers identified how concepts can be represented in different instances and reflects the Mason et al. (2009) definition of structure as "general properties which are instantiated in particular situations" (p. 10). They then go on to introduce the process of substructing which operationalises the structure of emergent mathematical knowledge.

Teachers' own ability to recognise their understanding of mathematical structure is not clear. It is predicted that they can adopt structural behaviours when teaching mathematics through collegial awareness and open discussion. Davis and Renert (2013) acknowledged this was possible. They stated that, working collaboratively, teachers have an ability to adopt alternative approaches to teaching mathematics identified by colleagues without being able to articulate that they are.

In a survey of 39 mathematics teachers, Cavanagh (2006) conducted interviews to examine the extent of the implementation of the working mathematically strand in their teaching. Although this study was concerned with the NSW mathematics syllabus before the introduction of the Australian curriculum, the results are still relevant. A small number of teachers interviewed were able to describe what working mathematically involved and applied it to their teaching. The majority had a very limited understanding of what working mathematically meant. As working mathematically can be closely aligned to mathematical structure, it appears that teachers' lack of acknowledgement of working mathematically can transfer to a similar lack of awareness about mathematical structure.

In this paper on mathematics teachers' responses to working mathematically, Cavanagh (2006) explained how the inclusion of the working mathematically in the NSW mathematics syllabus was a result of a reform to develop students' conceptual understanding, that is: to reason, communicate, and reflect on their learning experiences. This reform describes mathematical structure as it is not bound to content, but related to how mathematical knowledge is applied and communicated.

The teachers Cavanagh (2006) interviewed identified time pressure as one reason for not applying working mathematically in their classroom. According to them, the content-driven curriculum did not encourage teachers to focus on these components of mathematical learning, and the need to prepare students for examinations was a barrier to incorporating activities that encouraged working mathematically. These are realistic and conspicuous reasons for mathematics teachers not to practise working mathematically in their day-to-day teaching. Similarly, teachers might not practise pedagogical approaches espousing mathematical structure because of these conditions.

2.4 Mathematical structure and mathematical pedagogy

Effective mathematical pedagogy should include mathematical structure and guiding and promoting students toward structural thinking.

Vale et al. (2011), in their previously mentioned study, focussed on nonqualified mathematics teachers and demonstrated how mathematical structure improved the teachers' pedagogical content knowledge (PCK) and mathematical content knowledge (MCK). The development of PCK by Shulman (1987) and MCK by Ball, Thames, and Phelps (2008) identified knowledge as being essential for teachers to be effective in the classroom. Shulman's focus was across all disciplines, whereas Ball et al. were more concerned that the teacher had a competent level of understanding of mathematical knowledge or the mathematical content that was required to be taught.

Vale et al.'s, (2011) introduced a professional development program that focused on both mathematical content and pedagogy. At the end of the professional development program the teachers in their study reflected on their learnt experiences of understanding the mathematical connections and their appreciation of pedagogical knowledge. Their reflections indicated that they had been able to both deepen and broaden their knowledge of teaching junior secondary mathematics and had developed their capacity to support students' learning of mathematics. The researchers indicated the need for further research in the area of teachers' awareness of mathematical structure and how teachers could be encouraged to embed structure in their teaching practice. The development of professional development programs such as that implemented by Vale et al. (2011) can cultivate teachers' awareness of mathematical structure so they can implement strategies that will deepen their students' mathematical understanding and structural thinking.

Identifying whether teachers do attend to mathematical structure or aspects of structure is important in considering whether mathematical structure is a determinant of students' mathematical understanding. Mathematical problems used by teachers during classroom instruction need to reflect an awareness and appreciation of mathematical structure, and the directions given to students should allow for the development of structural thinking. Mason et al. (2009) highlighted teachers' ability to transfer their structural awareness to the students through an awareness of structural relationships, and the situations and properties where these relationships.

Hill et al. (2004) measured teachers' mathematical knowledge by developing a survey. They set out to identify the requirements for mathematics teachers' understanding of the multidimensional requirements for teaching elementary mathematics. This included the teachers' content knowledge and student knowledge of content. Drawing on notions of pedagogical content knowledge from the work of Shulman (1987), the authors argued that the mathematical content that teachers must know to teach needs to be mapped precisely.

The need to identify how teachers can embed structure into to their lessons then becomes a focus of mathematical pedagogy. The teacher's awareness of mathematical structure becomes a critical focal point before it can be persuasively applied into the teacher's utterances within the classroom. Leinhardt and Smith (1985) found in their study of experienced and beginning teachers that as teachers began to connect the knowledge to their lessons their students' competence improved.

The research questions for this project are concerned with whether mathematics teachers are aware of the nature and value of mathematical structure and whether they promote structural thinking when actually teaching mathematics. Is it possible that a teacher can promote structural thinking, but not have an understanding of mathematical structure? Cavanagh (2006) showed that teachers did not identify with working mathematically when interviewed, but in their classroom behaviours working mathematically was identified. In the situation of a mathematics teacher, they may be promoting structural thinking without being aware of the underlying principles of mathematical structure.

This section indicates that mathematics teachers require content knowledge and an understanding of how students learn this content. Identifying mathematical structure as a requirement for knowledge and understanding when teaching is not easily recognised. Mathematical structure is a part of mathematical knowledge that reinforces content knowledge, but as a multidimensional consideration it is difficult for a teacher to be aware of it in its totality.

2.5 Mathematical structure in mathematics curricula

The notion of structure can be traced through the development of curricula. In 1963, the United States of America Department of Education listed as one goal "to help each child understand the structure of mathematics, its laws and its principles, its sequence and order and the ways in which mathematics as a system expands to meet these needs" (Taylor &Wade, 1965).

Mathematical structure can be found in current international mathematics curriculum documents. For example, it appears as a component of the American mathematics curriculum document *Common Core State Standards for Mathematics Initiative* "*Common Core*" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The inclusion of "look for and make use of structure" as a component of the common core is one of the eight standards of mathematical practices used to demonstrate what students are doing when they learn mathematics. It recognises mathematical structure as a component of student learning and understanding of mathematics.

Mathematical structure can be identified in the Australian Curriculum—Mathematics (Australian Curriculum, Assessment and Reporting Authority, 2015) through the four proficiency strands of understanding, fluency, problem-solving, and reasoning. These proficiency strands reflect the multidimensional aspects of mathematical structure and they support how the content is taught and the development of the thinking and doing of mathematics. Essentially, these proficiency strands could be identified as aligning with the development of structural thinking skills. In the Australian curriculum, the lack of use of the term does not mean that the concept of mathematical structure is not important to mathematics teaching and learning. The use of the term structure is more aligned with the *Number and algebra* strand of the curriculum, but there are many examples of its application throughout other strands.

In the NSW mathematics syllabus for the Australian curriculum (NSW Board of Studies, 2012) the proficiency strands of the Australian curriculum are re-worked as working mathematically. Mathematical structure represents the teachers' appreciation of the complexities of the learners' mathematical awareness. Stephens (2008, October) highlighted structural awareness as related to the range of possibilities that a learner may attend to when doing mathematics. The teachers' operational definition for mathematical structure therefore includes: awareness of the different ways learners' attend to mathematics from the particular to the general, and the identifying of mathematical relationships through connections, patterns, similarities and differences. To develop learners' structural thinking, the teacher is aware that affective understanding impacts on how the learners will attend and focus their attention. An effective pedagogical approach recognised in the NSW mathematics syllabus (NSW Board of Studies, 2012) as working mathematically, will attend to this.

Mathematical structure can be identified in working mathematically through the communicating, problem solving, reasoning, understanding, and fluency components. Working mathematically can be associated with students' behaviours that are closely aligned to mathematical structure and structural thinking. To identify teachers' awareness of mathematical structure, the components of mathematical structure were developed. The next section introduces components of mathematical structure that aim to help identify teachers' awareness of mathematical structure.

2.6 Theoretical framework: Components of mathematical structure

This study relies on the theoretical framework developed from the work of Mason, et al. (2009) to identify the awareness of structure through the different forms. The following section uses Mason et al.'s forms to extract identifiable components of mathematical structure in order for these to be observed and described. In doing so, these components can be recognised in the teachers' knowledge and teaching practices.

Entries in Table 2.1 represent the forms of mathematical structure as explicated by Mason et al. (2009). I used these as a foundation for components that I developed for the present study because they describe mathematical structure, they would be recognised by mathematics teachers and be observable when mathematics was being taught. Unlike the working mathematically components that are identified in student behaviours, these components of mathematical structure are distinguishable in what teachers say and what they do.

Table 2.1

Comparison of Mason et al. Forms of Mathematical Structure and Components of Mathematical Structure

	Mason et al. forms of mathematical structure	Components of mathematical structure
1.	Holding wholes (gazing)	Connecting prior and future learning
2.	Recognising relationships	Recognising and producing patterns
3.	Discerning details	Identifying similarities and difference
4.	Perceiving properties	Generalising across properties
5.	Reasoning	Generalising a specific situation.

Mason et al.'s (2009) first form of holding wholes (gazing) is interpreted as connection because the reference here is between viewing or gazing by making connections that represent the whole. The second form of recognising relationships I associated with patterning because mathematics relationships are often recognised within patterns. The third form, discerning details, relates directly to equivalences or similarities and differences as essential components within mathematical knowledge. Combining the two forms: perceiving properties and reasoning, creates the single component of generalising.

In the following section I will outline where these components of mathematical structure appear in the reviewed literature.

The importance of these components for this research project is that they will become the focus of how I identify a teacher's awareness of mathematical structure and how the teachers are able to promote structural thinking through their utterances in the classroom. I established the following components for the purpose of my main study as aspects of mathematical knowledge and processes that support mathematical structure.

- **Connections** with prior and future mathematical learning and with other mathematical concepts (Albert, Corea, & Macadino, 2012; Barnard, 1996; Jones & Bush, 1996; Richland et al., 2012; Vale et al., 2011).

- **Recognition** of patterns and relationships through identifying and reproducing content and concepts (Mulligan & Mitchelmore, 2009; Papic, Mulligan, & Mitchelmore, 2011; Stephens, 2008).
- **Identification** of similarities, difference and equivalences in all content and concepts (Barnard, 1996; Jones & Bush, 1996; Stephens, 2008, October; Vale et al., 2011).
- **Generalisation** of expressions to a situation, relationships between properties, explaining, and justifying conclusions (Albert et al., 2012; Stephens, 2008; Vale et al., 2011; Watson & Mason, 2005).

Vale et al. (2011) introduced the relationships of these components, which I will refer to as CRIG components, as awareness and knowledge of mathematical structure. Vale et al. felt that structural thinking is evident when learners are able to see relationships and make generalisations in the mathematics. Fundamental to this was the interrelated nature of the components of mathematical structure. These represent mathematical knowledge that can be identified in mathematics teaching and learning, but they do not exist in isolation and can appear as combinations of each.

The next section focuses on the research about the components of mathematical structure and where these components appear in the NSW mathematics syllabus (NSW Board of Studies, 2012). Although The NSW syllabus does not recognise the term mathematical structure, the components of mathematical structure do appear in the content strands and in the syllabus outcomes from early Stage 1 (preschool) to Stage 4 (Years 7 & 8).

2.6.1 Connections

The National Council of Teachers of Mathematics (2009, p. 19) identified that connecting a context or concept to previous knowledge allowed for an informal process of understanding a situation. Making connections with prior learning of mathematics represents a component of mathematical structure that supports students' reasoning.

The connections component dominates mathematical structure and structural thinking. Authors investigating how we think about mathematics constantly make the connections between present, prior, and future learning. Connections require the recalling a piece of mathematical knowledge and then reapplying it or adapting it to a new piece of knowledge. The NSW mathematics K–10 syllabus recognises that "students develop understanding and fluency in mathematics through inquiry and exploring and connecting mathematical concepts" (NSW Board of Studies, 2012). Connections in mathematical structure are found in recalling and reapplying a fact, procedure, or method that is to be used in a new context. It is not only connecting of content knowledge, but also the procedures and concepts behind the content. In the case of a procedure, the student needs to connect each step involved when solving a mathematical problem in a coherent manner. The concepts taught might appear with one piece of content knowledge, but students need to see how it can connect to new content. When the teacher is able to associate an example with prior experience, that teacher is making connections. It is expected that this would reinforce the students' understanding. The NSW mathematics syllabus has as one of its outcomes for working mathematically that a student "communicates and connects mathematical ideas using appropriate terminology, diagrams and symbols" (NSW Board of Studies, 2012—working mathematical outcome MA4-1WM).

Albert et al. (2012) recognised the importance of making connections between past, present, and future learning experiences and knowledge developed. They encouraged teachers to ask questions of the students that were beyond their level of actual development, while asking them to apply their prior knowledge. Albert et al. found that when the teachers facilitated a connection with the students' prior knowledge in this way, the students were able to apply it to new problems.

Structural thinking allows for flexible thinking. This encourages the development of connections between the different representations—a development that results in understanding. The different representations have connecting elements. These connections give fluidity between different situations so that knowledge learnt can be applied to other contexts. An example is a mathematical concept given in a visual form that can be moved toward a physical form through symbols and language. Albert et al. (2012) give the example of the numerals 1 to 10. They are represented as quantity in a visual or concrete form but can also be connected to symbols, written words, and sounds.

Jones and Bush (1996) noted that there is limited understanding of how connections help students, but once made, there is a deepening of mathematical understanding. Vale et al. (2011) recognised that connection is fundamental to structural thinking and that connections between various representations of mathematics are central to learning and that teachers of mathematics should be familiar with methods that can connect the different representations.

The National Governors Association Center for Best Practices & Council of Chief State School Officers (2010) recognised that the use and connection of mathematical representations was a powerful engager of student learning: "Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving" (p. 24). The various forms of connections between the different mathematical representations can be found in visual, symbolic, verbal, contextual, and physical arrangements. These forms provide different representations of how the content may be taught and the many ways that the learner may understand the required knowledge.

Effective teaching would include knowledge of how procedures and concepts relate to past, present, and future learning. Vale et al.'s (2011) reference to the horizon indicates that the teacher needs to know how pedagogical and content connections in mathematics are effective in developing structural thinking in students.

2.6.2 Recognising patterns

Recognising patterns occurs as an innate observation of the natural world. Children are able to recognise and observe patterns before reaching school. Once introduced to mathematics at school, children are exposed to patterns through formalised learning processes that follow the content strands of the syllabus. Patterning is identified extensively throughout primary and junior secondary education (early Stage 1 to Stage 4) to support student mathematics learning. In the NSW mathematics syllabus (NSW Board of Studies, 2012), patterns are associated with the *Number and algebra* strand with its stated aim to "develop efficient strategies for numerical calculation, recognise patterns, describe relationships and apply algebraic techniques and generalisation". Early Stage 1–Stage 3 has the content strand of *Patterns and algebra*, that has recognising patterns as an outcome for students' mathematical learning, and in Stage 4, the *Number and algebra* content strand has "create and displays number patterns" (MA4-11NA) as an outcome. The overarching statement states: "Students develop efficient strategies for numerical calculation, recognise patterns, and describe relationships" (NSW Board of Studies, 2012, p. 18).

The importance of patterning, awareness of patterns, and reproducing patterns has been widely accepted as essential for mathemical development, which can be associated to mathematical structure as it generates mathematical knowledge and understanding. Papic, Mulligan & Mitchelmore, (2011) noted that awareness of patterning and structural relationships is essential in mathematical learning. They found that young children's identification of pattern structure was crucial in forming and designing future learning models.

Mulligan and Mitchelmore (2009) identified structural thinking in primary students as a requirement for developing mathematical competence. They proposed a construct, namely awareness of mathematical pattern and structure (AMPS). To be able reproduce a pattern requires the ability to generalise the structure of the pattern. Patterning is essential in developing mathematical understanding of other concepts. Generalising here relates to one aspect of patterning.

Stephens (2008) noted that structural thinking is much more than simply seeing a pattern. Merely recounting a pattern without the ability to replicate it is not demonstrating awareness of the property. The ability to generate a pattern to other examples illustrates a feature of structural thinking.

Teaching with a focus on patterning creates relevance to the real world, which becomes important for students' understanding. When students identify, and use patterning, structural thinking competence develops, as shown by Mulligan and Mitchelmore (2009).

2.6.3 Identifying similarities and differences

Early mathematics includes making decisions about differences, whether things are equal or unequal, bigger or smaller, and how to recognise these differences. Identifying similarities and differences is essentially built on sorting and classifying objects into like or unlike categories, equivalence as a different notion of sameness can also develop through experiencing and extends into more subtle differences in mathematical representations. Warren and Cooper (2009) identified primary school children as often misrepresenting the equal sign by not identifying the symbol as representing sameness, but as an operator, meaning to do something, just as an addition sign means to sum. They noted that this confusion then carries through to secondary and tertiary studies, which affects overall mathematics learning.

Identifying similarities and differences as a component of mathematical structure is essential in developing students' deeper structural awareness. It helps reveal that essential features of mathematical ideas persist despite their various forms (Zimba, 2011). This empowers one to consider similarities and difference when regarding other concepts. Jones and Bush (1996) recommended visual forms of diagrams, charts, tables, mind maps, and flowcharts as sources that can be used to help develop mathematical structure. Students can see the similarities and differences as concepts that are developed through the hierarchies that will allow additional and more complicated concepts to be added.

Barnard (1996) argued that mathematical learning that is rich in structure engages the learner in thinking deeply about the mathematics. He developed structurally rich tasks as examples of precision or exactness. His examples of recognising equivalences demonstrate how similarities and differences support mathematical structure.

Identifying similarities and differences are necessary for all mathematics teachers. Encouraging students to identify and understand equivalence is important, as is the ability to see differences when identifying quantities, and values. Teachers need to continually encourage students to reflect on what is similar and different to develop structural thinking competence.

2.6.4 Generalising

Of all the CRIG components, generalising is the most universal. It has application across all the previous components, yet can still be identified individually. Mason et al. (2009) wrote that appreciation of structure has to do with the experience of generality. The NSW mathematics syllabus K–10 (NSW Board of Studies, 2102) *Number and algebra* content stipulates that "students develop efficient strategies for numerical calculation, recognise patterns, describe relationships and apply algebraic techniques and generalisation". Warren (2008), in conducting her Early Algebraic Thinking Project (EATP), which followed the development of algebraic thinking in 6- to -11-year-olds, found that in equivalences and equations the ability to find a solution and then generalise to a real world problem was a key aspect of identifying structural thinking. She found that both young and older students were able to generalise basic equations.

Watson and Mason (2005) focused on learners generating their own mathematical examples from given situations. They identified these as examples of anything the learner might be able to take from a given situation to generalise into a new idea. They stress that while each mathematical problem has its own particularity, the solution process evolves into the generalisation.

Stephens (2008), in applying structural thinking to designing arithmetic questions, explained that structural thinking involves being able to go from several instances of the same thing and then being able to generalise the property. He noted that young children could articulate a generalised structural principle underlying the whole problem.

Concept formation is a process that involves generalising as a result of an interaction between the concrete toward the abstract that is associated with mathematical structure. Albert et al. (2012) indicated that conceptual mastery is an ability to generalise what is learned from one situation and extrapolate it to a different situation, or the "transition from one structure of generalisation to another" (p. 21). Unlike connections, generalising is developing a *what happens next* scenario.

Generalising has a broad application for all mathematics teachers. It is applied to all mathematical learning situations. Teachers who encourage students to generalise are promoting structural thinking, because they are asking students to recognise similarities and differences between contexts, identify relationships and properties, and ultimately express these generalisations algebraically.

2.7 CRIG and this project

A generalised notion of mathematical structure, based on the varying descriptions given by the authors, and its association to the working mathematically component of the NSW mathematics syllabus (NSW Board of Studies, 2012) is given. Mathematical structure represents the building blocks of mathematical learning, and is identified in the CRIG components of connecting mathematical concepts, recognising and reproducing patterns, identifying similarities and differences, and generalising results. Structural thinking requires using these components, yet they are not always considered when solving problems. Students need to learn structural components as an aid to working mathematically.

Mason et al., (2009) pointed out that mathematical structure could not be taught. This made identifying teachers' awareness of structural relationships difficult. Yet, the teachers' awareness of structure is considered in their pedagogical practices that include mathematical structure. The identification of the CRIG components is important in this study because they become a basis for recording teachers' awareness of mathematical structure. Teachers can acknowledge the CRIG components in what they say when in professional discussion and in what they do when teaching mathematics. Acknowledging these components may help monitor teachers' awareness of mathematical structure.

2.8 Context of this project

This research project is based around the question of what teachers know about mathematical structure and whether they promote structural thinking when teaching mathematics. Mathematical structure is recognised as being a bridge between what Mason et al. (2009) described as the "mythical chasm" between procedural and conceptual understanding. Teacher awareness of mathematical structure could build this bridge.

CHAPTER

3

DESIGN AND METHODS

3.1 Introduction

This chapter commences with a description of the design and method of the study. The context of the research is introduced, followed by the development of the instruments and how these were trialled and refined in preparation for the main study. The second part of the chapter addresses the methodology of the main study, including the participants, procedures, and methods of data analysis.

The main study involved a small-scale exploratory and descriptive investigation of junior secondary mathematics teachers' awareness and pedagogical practices in relation to mathematical structure and structural thinking.

3.2 Methodological bases

The purpose of this study is to identify whether teachers are aware of mathematical structure and to determine whether or not they promote structural thinking when teaching mathematics. In particular, I focus on the teachers' promotion of structural thinking through the CRIG (connections, recognising patterns, identifying similarities and differences, and generalising) components of mathematical structure. Survey, interview, and observation instruments were developed to enable interpretation of multiple sources of data to answer the research questions. Anderson, Sullivan, and White (2004), when researching mathematics teachers' beliefs about problem solving, also used surveys, interviews, and observations to gather data about teacher beliefs and practices that were considered complex to measure. From this present study, the data obtained will form the development of a larger doctoral study. The scope and depth of the study is thus limited as an exploratory phase given the parameters of the thesis. The three instruments were aligned with the research questions. The first research question deals with teachers' awareness of mathematical structure and the second focuses on teachers' promotion of structural thinking in their pedagogical practice. The third research question is concerned with the differences between what teachers say about mathematical structure and what they do to promote structure when teaching mathematics. The survey and interview data will contribute to answering the first research question. The observation instrument is closely aligned to the second question. The third question is answered by noting differences between data obtained from the interviews and observations. The instruments are:

1. A survey with Likert scale response options.
2. An interview schedule comprising structured questions.
3. An observation template used to record pedagogical practices that focus on teacher utterances during mathematics teaching.

These instruments were refined through a pilot phase in preparation for the main study. The following describes the processes involved in the development of these instruments.

3.3 Development of the methods

This is an exploratory study of mathematics teachers' knowledge and pedagogical practices. The assumption that underlies this study is that teachers lack an understanding of mathematical structure. The instruments were designed to allow the teachers to articulate their perceptions and ideas (survey and interview) and to observe their acknowledgement of the CRIG components through the language used when teaching mathematics.

To identify teachers' awareness of mathematical structure and how structural thinking is promoted in the classroom, I drew upon the work of Anderson et al. (2004), who used a questionnaire and interviews to measure teacher problem-solving beliefs. In recognising the limitation of a survey, they suggested that data from the survey could be complemented by the interviews. They studied the extremes of traditional to contemporary teaching and then introduced classroom observations to triangulate the data collected from the survey and interviews.

In a similar manner, I planned to use instruments that could assess teachers' awareness of mathematical structure and the promotion of structural thinking in their pedagogical practices. The consolidation of Mason, Stephens and Watsons' (2011) forms in Chapter 2 into the CRIG components is my attempt to identify mathematical structure in teachers' use of language when teaching mathematics.

The consolidation of Mason et al.'s (2011) frames into the CRIG components was intended to identify mathematical structure in what teachers say and what they do. Mason et al. asserted that structural thinking lies on a continuum, so it is expected that the teachers' awareness of mathematical structure might also be on a continuum. I also envisaged that awareness of mathematical structure would transform students' mathematical thinking only when the teacher is aware of structural relationships. It would be expected that this would be achieved at varying degrees depending on the teachers' level of awareness.

Chick, Baker, Pham, and Cheng's (2006) study was a point of reference for the development of the method for this study. In this Australian study, 14 Grade 5/6 teachers completed a questionnaire regarding teaching decimals and then participated in follow-up interviews. The authors developed a framework of teacher pedagogical content knowledge (PCK) grouped into three categories comprising knowledge of teaching strategies for mathematics, knowledge of students' thinking, and pedagogical knowledge in a content context. Their coding and sorting approach was a guide to the process implemented for analysing the interview and observation data collected in this study.

3.3.1 Survey

The first instrument to be developed was the survey. It was intended to identify any understanding of structure the teachers may have through the terms mathematical structure, structural thinking, and associated CRIG components. A question on the appearance of the term structure in the NSW mathematics syllabus was also included.

It was considered that teachers may have an underlying understanding of the principles of mathematical structure but that they might not be familiar with formal expressions connected to mathematical structure. This survey may be the first time that some of these teachers had heard of mathematical structure or structural thinking.

The first survey questions developed were modelled on the framework of Mason et al. (2009), explained in Chapter 2. The first question introduces the term mathematical structure and asks teachers to consider this in terms of mathematical thinking. The second question asks teachers to acknowledge mathematical structure in the NSW mathematics syllabus. This question reflects the role of structure included in the *Common Core State Standards for Mathematics Initiative* "*Common Core*" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Subsequent questions were based on research that identified aspects of mathematical structure and structural thinking: problem solving (Anderson et al., 2004), making connections (Vale, McAndrew, & Krishnan, 2011), patterning (Mulligan & Mitchelmore, 2009; Papic, 2007), similarities and differences (Barnard, 1996), and generalising (Stephens, 2008: Warren, 2008). Additional questions regarding teacher mathematical content knowledge (MCK; Hill, Schilling, & Ball, 2004) and pedagogical content knowledge (PCK; Shulman, 1987) were included to uncover teachers' pedagogical practices that may promote structural thinking.

Using this research and other relevant studies presented in Chapter 2, I constructed a bank of 50 questions that addressed the first research question and my assumption of teachers' lack of understanding of structure. These questions were reviewed, following the pilot, before a final set of questions was retained. The questions were to be answered according to five Likert-type response options of disagree, partially disagree, neither agree or disagree, partially agree, and agree.

I intended the survey to be easy for teachers to complete without losing interest. The questions were written with the intention of allowing the teacher to become familiar with the term mathematical structure. When generating these questions, I focused on the guidelines from Crawford (1997) to create an effective survey, namely that:

- The questions should meet the objectives of the research.
- Participants must understand the questions to avoid misleading responses.

- Questions should be worded and organised so participants give an accurate and unbiased response.
- The questions should be easy for the participants to answer.
- Questions should be brief and precise for the participants to complete.

Prior to piloting the survey, some rewriting and refinements were made so that repetition among questions was eliminated. As excessive time would be required to complete the original survey, the number of questions was reduced from 50 to 22. This revised survey was submitted to the Macquarie University ethics committee, and approved. This revised survey is provided in Appendix C.

The 22 questions for the survey fell within four groups, shown in Table 3.1, as indicators of mathematical structure, CRIG components, structural thinking, and pedagogy and content. Mathematical structure cannot be isolated to a single aspect of mathematics teaching so the survey questions within each group addressed mathematical structure from different perspectives, allowing participants to familiarise themselves with the overall concept. Included in Group 1 was a question concerned with whether mathematical structure was identifiable in the NSW mathematics syllabus.

Table 3.1
Survey Questions by Group

Group	Questions	Topic
1	1–6, 20	Mathematical structure
2	7–11	CRIG components of mathematical structure
3	12–19	Structural thinking
4	21–22	Mathematical pedagogy and content

Specifically, questions in Groups 1 and 2 focused on mathematics teachers' awareness of mathematical structure, either as the term mathematical structure or as a CRIG component. Group 3 focused on a teacher's ability to recognise structural thinking, and Group 4 would provide data relevant to teachers' opinions about pedagogy and content.

3.3.2 Interview schedule

The second instrument was designed as an interview for teachers. This explored teachers' perceptions of mathematical structure and applications of structural thinking. The questions in the interview schedule were intended to obtain information that expanded on the survey questions, particularly focused on the terms mathematical structure and structural thinking.

These interview questions:

- Were specific to a discussion point.
- Were open-ended to allow for personal opinions.
- Were clear with precise wording.
- Required the teachers to draw upon their knowledge and beliefs.

They allowed for the interviewees to respond openly without prompting. For this study this was important in determining how teachers viewed mathematical structure within their own pedagogical understandings. Raymond (1997) also used interviews when investigating the differences between teachers' beliefs about mathematics and teaching practices. Raymond's interview questions were useful as exemplars for the design of the questions.

As there are few instruments that associate teacher awareness and pedagogical practices with CRIG components of mathematical structure I relied on similar studies of teachers' knowledge of mathematical structure. The study by Vale et al. (2011) was relevant as it introduced teachers to mathematical structure and then analysed their reflections about understanding mathematical connections and appreciation of mathematical structure. These aspects then informed the development of the structured interview in my study. The interview questions were open-ended questions that allowed for the interviewees to respond without restriction.

I decided in advance not use any prompts. However, I decided that the teachers interviewed needed a clear definition of mathematical structure and structural thinking to avoid any misunderstandings, so at the beginning of the interview I provided them with a brief description of mathematical structure. Anderson et al. (2004) gave definitions to illustrate meanings of terms such as open-ended to overcome any misinterpretations. The following is the description that I read to the interviewees:

> *Some authors describe mathematical structure as the building blocks of mathematical learning. Mathematical structure can be found in connecting mathematical concepts, recognising and reproducing patterns, identifying similarities and differences, and generalising results. Students who perform structural thinking use these skills without always considering them when solving problems. Many students need to be taught these skills when introduced to concepts as a reminder of how to think mathematically.*

It was expected that teachers might be able to articulate the processes that underlie mathematical structure and structural thinking through their knowledge of the working mathematically strand of the NSW mathematics syllabus (NSW Board of Studies, 2012). It was not expected that they would necessarily be familiar with the theoretical concept of mathematical structure that has been described in the previous chapters.

The first draft of the interview schedule comprised 11 questions that were grouped into the following categories:

- The nature of mathematical structure and how it is used in the classroom (Questions 1–3).
- Mathematical structure in the NSW mathematics syllabus (Question 4).
- Students and structural thinking (Questions 5–11).

The initial set of questions was intended to further explore what had been introduced in the survey. The survey questions required teachers' immediate response. Teachers did not have to think deeply about their response to the questions. The interview questions were designed to examine teachers' awareness of structure further and probe their knowledge of structure at a deeper level. The questions were intended for the teachers to verbalise their knowledge and awareness of mathematical structure. Mason et al. (2009) acknowledged the benefits of students' ability to think structurally, so questions included in the interview also investigated how the teachers identified structural thinking in their students.

3.3.3 Observation template

The third instrument was a template to be used when observing teachers' mathematical pedagogical practices to identify whether structural thinking was promoted in their teaching. Merriam (2009) claimed that observation as a tool for data collection has benefits in that it allows the research to be undertaken in a natural environment. The observation instrument was regarded as an essential component of this study as it allowed for the "what they do" aspect of the research question. Raymond (1997) used observation in her study in which she identified differences between a teacher's mathematics beliefs and practices. This was useful as a model for this study as the research question here may identify a difference between what teachers say about structure compared with what they do to promote it.

The initial observation template was designed so that I could record every teacher utterance that related to a CRIG component. As I was to be both scribe and observer, I was conscious of difficulties in accurately recording all the essential utterances. The first draft of this template included separate sections for the CRIG components. Table 3.2 contains two categories each utterance was allocated to, as a CRIG, and a statement: explanation, instruction, question, or response.

Table 3.2

Teacher Utterances to be Identified During Classroom Observation

Number	Category	Utterances
1	CRIG	Connection to prior or future learning
2	CRIG	Recognising patterns
3	CRIG	Identifying similarities and differences
4	CRIG	Generalising
5	Statement	Explanation
6	Statement	Instruction
7	Statement	Question asked
8	Statement	Response to student

My intention was to take a seat at the back of the classroom, making no contact with the teacher or the students. Video and/or audio recording were not used as they did not have ethical approval. As the teachers were the focus of this project, students' participation was not included.

3.4 Pilot testing and refining of instruments

3.4.1 Recruitment of participants

Pilot testing was conducted with two separate groups of mathematics teachers. First, I was able to recruit a convenience sample of 10 heads of mathematics departments or assistant heads of mathematics from Catholic secondary schools in the Broken Bay Catholic diocese while they were attending a network meeting. They participated in the trial of the survey only.

The second group were three experienced mathematics teachers from a Catholic secondary boys school located on Sydney's northern beaches. They participated in pilot testing all three instruments. To secure their participation, I approached the principal of their school for permission to conduct the pilot. After receiving the principal's approval, I met with the head teacher of mathematics who identified these three teachers as prospective participants. I approached the teachers via email outlining my research project. Their role in the project was made clear as piloting the instruments, not provision of data. All three agreed to be involved, and we arranged an initial meeting time to discuss the process to be undertaken. The email also included a link to the survey.

A subsequent meeting was arranged to discuss the survey and interview questions. A time to observe one mathematics lesson of each teacher was also arranged. They were asked that the observed lesson not deviate from any regular lesson, and to introduce me as a visiting teacher. They were also told that my role during the lesson was to observe their pedagogical strategies that would help to develop the best method to record and code teachers' utterances on an observation template.

3.4.2 Administration of instruments and subsequent adjustments

3.4.2.1 Survey

The participants in the first group completed the survey at the meeting and a focus group discussion regarding the survey took place subsequently. This group gave positive feedback about the survey.

Completion of the survey by the second group was minimal. Only one of the three teachers had completed the survey at the time of the meeting. The feedback from that person was similar to that of the first group.

Given the feedback that I obtained about the survey, I proceeded in using this version for the main study. All 22 questions from the survey appear in Appendix C.

3.4.2.2 Interview schedule

The interview schedule as described in Section 3.2.2 had 11 questions. The second group examined these questions, noting that some were repetitive and could be consolidated and that other questions were too broad and did not allow for in-depth responses. In light of this feedback, the interview schedule was reduced to six questions. It is provided in Appendix D. The first two questions were concerned with the teachers' understanding of mathematical structure. Question 3 was intended to gain insight into mathematical structure and the NSW mathematical syllabus. Three final questions were concerned with structural thinking in students.

3.4.2.3 Observation template

Piloting of the observation template took place across three lessons with the three teachers involved. During the first of these teaching sessions, the original observation template (refer to Section 3.3.3) was used. Following the second observation lesson, I recorded the utterances and categorised them. By reformatting the observation template I created a simpler template in the form of a table to record teacher utterances, allowing for categorising post-observation. This observation template was used in the final pilot observation lesson. It is provided in Appendix E.

3.5 The main study

3.5.1 Participants

For the main study, I approached the principal of a secondary Catholic boys school on the northern beaches of Sydney[1]. This school was not the same school that had been involved in the pilot study. The principal gave permission to conduct the research at this school and for mathematics teachers to be involved in this research project (see Appendix B).

Initial conversations with the head teacher of mathematics confirmed that all eight mathematics teachers at the school would be asked to complete the survey and that three of that group would be invited to participate in the interview and observation components of the project. The head of mathematics approached these three participants and they agreed to be involved. Each teacher completed the ethics-approved consent form (see Appendix B).

3.5.2 Participants in interviews and observation sessions

The three teachers interviewed and observed were asked to complete an information questionnaire at the beginning of the interview session. Table 3.3 contains the demographic details of the two female and one male teachers involved in this part of the study. The two teachers with most experience had mathematics as their second teaching subject and the teacher with the mathematics qualification was the least experienced of the three.

3.5.3 Procedures for data collection

Data collection occurred during Term 2, 2015. The teachers were asked to complete the survey within a week commencing on 11 May, the interviews were conducted on 18 May, and the lesson observations took place during the week beginning 25 May.

[1] This school was chosen as I live and work in this area and am familiar with this school and acquainted with the principal.

Table 3.3

Teacher Demographic Information

Question	Teacher A	Teacher B	Teacher C
For how many years have you been teaching?	12	3	17
What is your teaching qualification?	BSc Grad Dip Ed	BA Dip Ed	Dip PE & Maths
What university did you graduate from?	USyd, CSU	ACU	Johannesburg College of Ed
What other subjects are you teaching?	None	None	PDHPE
Is maths your first subject?	No	Yes	No
What regions have you taught in?	Metro	Metro	Metro
What recent professional development have you done?	MANSW New Head Teachers	MANSW Ext 1	CSO maths focus day
What professional association are you a member of?	MANSW	MANSW	MANSW
Do you use in technology in the classroom?	Yes	No	Yes
Have you participated in other research?	Yes	No	Yes

3.5.3.1 Survey

The survey data were collected online via the SurveyMonkey software. A URL provided immediate access to the survey for all participants.

The link to the survey was emailed to all eight mathematics teachers with a request that they complete the survey online within a week so that the survey would have been completed before the interviews were conducted. There was some delay in teachers completing this survey, so follow-up emails as reminders to complete it were sent. Five of the eight mathematics teachers at the school completed the survey and did so by responding to every question.

3.5.3.2 Interviews

The three teachers involved in the interview component of the study were among the five who completed the survey. I met with each teacher in a meeting room at the school at mutually convenient times. Each interview lasted approximately 10 minutes. Consent had been given by the teachers for these interviews to be recorded. I used a personal audio recording device (apple iPhone 6 plus) for all interviews. These recordings were then transcribed into a Word document. Notes were not taken during the interviews. The interview statements were then entered into NVivo for coding.

3.5.3.3 Observations

Three mathematics lessons conducted by each of the three teachers for Year 7 and 8 classes were observed. The observations followed consecutive classes across a week. During the week I visited the school only for the time of the observations. There was no communication between me and the teachers outside this time. No information or feedback about the observation was given to the teachers at the beginning or end of the lesson. I was not aware of any corroboration between the teachers.

Table 3.4 contains information about each teacher's class and class ability level. The classes were being taught from the number and algebra strand of the NSW mathematics syllabus. All classes were finalising the current topic and were involved in revision as preparation for the end-of-semester examination that was to take place the following week. The topic taught to the classes was ideal for this study as it could expose a higher level of identification to the CRIG components, particularly with respect to recognising patterns, as the outcomes from the NSW mathematics syllabus places patterning within the number and algebra strand.

I observed each lesson from a position that was unobtrusive to the classroom setting (i.e., seated at the back of room). I had no interaction with the teacher or the students. My presence in the classroom appeared to have little impact on the students' involvement in the lesson. I cannot make any assumptions about whether or not the teachers' lesson strategies were influenced by, or changed, as a result of my presence during the lesson.

Table 3.4

Teacher, Class, and Student Ability Level for Classroom Observations

Teacher	Class	Ability level of students
A	Year 7	Mixed
A	Year 7	Mixed
A	Year 7	Mixed
B	Year 8	Low
B	Year 8	Low
B	Year 8	Low
C	Year 7	Mixed
C	Year 7	Mixed
C	Year 7	Mixed

3.5.4 Analyses

Within this section I outline how data from the three instruments were analysed. I used information from the literature on mathematical structure and structural thinking to assist in the analysis of the data.

Individual data for each teacher for lessons taught were recorded and coded. However, an analysis of the individual teachers' responses was beyond the scope of this project and was not considered in addressing the research questions. All data collected were therefore analysed as a group, except for any singular anomalies.

3.5.4.1 Survey

SurveyMonkey produced survey results in terms of averages and frequencies.

3.5.4.2 Interviews

I transcribed the audio recordings from the interviews directly to a Word document and then checked for accuracy by reading over the transcripts while listening to the recording. The transcriptions were then copied into NVivo, where common themes were coded for analysis. The coding process initially allocated responses from the mathematical structure questions to a CRIG component. Questions related to structural thinking were categorised as follows: recognising structural thinking in students, student engagement when involved in structural thinking, and the benefits of structural thinking.

3.5.4.3 Observations

The teachers' utterances recorded in the Word document were transferred to an Excel spreadsheet for subsequent coding and analysis. These utterances were categorised as belonging to one of the CRIG components and were then allocated to subcategories of question/instruction, superficial/analytical, and content/concept. The level of attention to structure through Mason et al.'s mathematical structure continuum was noted by allocation between the subcategories. The utterances identified as rich in structure were allocated to the analytical category, and if they were weaker in structure they were allocated to the superficial category. Subsequently, the utterances were placed into the subcategories of *Conceptual understanding* if they related to concepts and *Procedural understanding* if they related to content.

3.6 Summary

The contents of this chapter explain the design and refining of the three instruments used in the main study. The main study is described with regard to its implementation, data collection, and analysis. The following chapter contains the analysis and discussion of results obtained from surveying, interviewing, and observing this small group of teachers.

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CHAPTER

4

RESULTS

4.1 Introduction

This chapter contains the results of the data from each of the three instruments, an explanation of the approach taken to analyse the results, and a short description of relevant aspects of the results. In addressing the research questions, the literature reviewed in Chapter 2 provided foundations to help interpret the data. The results are presented in tabular form and interpreted from varying perspectives. The survey provides an indication of the teachers' understanding of structure, the interviews detail what teachers know about structure and identifying it in their students, and the observation data indicate the extent to which teachers' utterances did or did not reflect structure.

4.2 Survey

Table 4.1 contains a summary of the data from the survey indicating average scores from the five-point Likert scales to the items within the four question groups. A score of 5 would mean that there was absolute agreement with all the questions asked in that particular group, and a score of 1 would indicate that all teachers disagreed with the statements. A score of 3 would indicate that the teachers, on average, neither agreed nor disagreed with the statements. Averages of the responses to each of the 22 individual survey questions are provided in Appendix C.

From the entries in Table 4.1 it appears that the teachers have a high level of awareness of mathematical structure as the averages are close to the maximum score.

The slightly higher response on Group 4 acknowledges teachers' awareness of pedagogy and content knowledge as being very important. The initial assumption, made in Section 3.3 that teachers lack an understanding of mathematical structure, is not reflected in these results.

Table 4.1

Survey Question Group Averages from Likert Scale Responses

Group	Question	Survey classification	Average
1	1–6, 20	Mathematical structure	4.56
2	7–11	CRIG components of mathematical structure	4.56
3	12–19	Structural thinking	4.18
4	21–22	Mathematical pedagogy and content	4.60

4.3 Interviews

From the interviews, my aim was to identify any aspects related to mathematical structure. While reflecting on the research from the literature review, I used content analysis to categorise concepts by isolating words and phrases within the written texts. The references to mathematical structure in those words and phrases connected similar relationships and meanings. These coding categories were determined after the interview data were collected and recorded.

The teachers' responses to the first question reflected their interpretations concerning the meaning of structure in terms of pedagogical or organisational aspects. This was reflected in their comments, which fell into three main groups.

a) First, some comments were based on the building block analogy, identified as a definition of mathematical structure by Jones and Bush (1996), that refers to building on previously learnt content:

- Building on previous knowledge, taking it to the next step. (Teacher A)
- I think I feel like structure in maths is all based on—it's sort of like a building blocks: A lesson would depend on the lesson before that. It's sort of like everything is getting added on each lesson. (Teacher B)
- I am always going over what we did the previous lesson and I know especially with the seniors I say, "Remember using this in Year 8?" (Teacher C)
- I suppose with that thing is relying on previous knowledge. Are they able to draw on that in the most efficient ways and then same sort of question? If you throw fraction, do they know their fraction work in algebraic fractions? Is there structural thinking? Do they know their operations adding, subtracting,

multiplying, and dividing? Can they go back to when they learnt that in Year 7? Are they able to draw on that? (Teacher C)

b) Second, structure was referred to in terms of organisational features such as lesson and curriculum structure:

- So I think the structure that teachers use is different to the structure of the curriculum, as it has to be more generic. (Teacher A)
- I think putting, like in a Stage 5 putting (Stage 4) at the beginning of it. Putting Stage 4 like things you need to do it as a review before that. (Teacher B)

c) Third, structure in mathematics was identified with regard to how a mathematical problem is presented in the written process:

- If they were able to structure the [written] solutions. (Teacher A)
- I am very big on believing in understanding the content, then be able to show that [written] process—which a lot of students cannot do. (Teacher A)

The above statements demonstrate a variety of interpretations of the term structure, whether as mathematical structure or structure as a process of completing a problem that has an introduction, body, and conclusion—a process to gain a solution that both teachers and students can develop. These examples demonstrate that individuals do have different definitions of mathematical structure.

After reviewing the interview responses for all questions, I reanalysed how those responses might have been associated with mathematical structure, in particular whether some statements showed a greater awareness than did others. To do this, I categorised selected words and phrases made by the teachers during the interview as being either specific or nonspecific. I defined a specific statement as one that related directly to mathematics, for example, “Doing series and sequences, and I took them back to tables of values”. I defined a nonspecific statement as one that had no reference to an example of mathematics. An example of a nonspecific statement was “Recognising similarities and differences, I do that”. Table 4.2 contains the frequencies of specific and nonspecific responses.

The higher frequency of nonspecific responses was regarded as demonstrating teachers' lack of awareness of mathematical structure. These statements were vague, and lacked any perception of support for the development of students' structural thinking. These statements, related to mathematical structure through key words that linked them to the CRIG components of connection, recognising patterns, identifying similarities and differences, and generalising, but were meaningless in the way that teachers' used the key words ie. "students' really struggle with generalisation", and it was not observed in the classroom observation.

Table 4.2

Frequencies of Teachers' Specific and Nonspecific Responses

Type of statement	Frequency
Specific	4
Nonspecific	28

Table 4.3 contains the frequencies of nonspecific and specific responses made in reference to each of the four CRIG components from all the questions in the interview.

Table 4.3

Frequency of Teachers' CRIG Specific/Nonspecific Responses to Interview Questions

CRIG component	Specific/nonspecific example	Frequency
Connections	Specific	0
	Nonspecific	13
Recognising patterns	Specific	1
	Nonspecific	8
Identifying similarities and differences	Specific	1
	Nonspecific	2
Generalising	Specific	2
	Nonspecific	5

There was no assumption that an interview response allocated within the CRIG components would be automatically regarded as structurally rich. Within each of the CRIG components there were responses that displayed varying levels of awareness about mathematical structure. The specific responses are more inclined toward structural awareness, but there were fewer of these in all four CRIG categories. The higher frequency of nonspecific responses represents a lack of awareness of mathematical structure.

Table 4.4 contains the only examples of the specific statements, all of which were made by the same teacher, Teacher C, that were related to a CRIG component. The entries in this table provide valuable exemplars of this teacher's understanding and use of mathematical structure. The small number of specific statements suggests that overall Teachers A and B were either not aware of mathematical structure or were unable to describe or exemplify it.

Table 4.4

Teacher C's Statements About CRIG Components

CRIG component	Examples of teacher's statement
Recognising patterns	Doing series and sequences and I took them back to the table of values and linear [expressions] and getting them to look for the patterns.
Identifying similarities and differences	Recognising, for example, let's say difference of two squares where you don't have a perfect square.
Generalising	Looking for the generalising and coming up with the formula
	Can they recognise the pattern generalisation that it doesn't have to have a perfect square

Table 4.5 contains examples of nonspecific statements made by the teachers about the CRIG components.

Table 4.5

Examples of Nonspecific Statements Made by Teachers About CRIG Components

CRIG component	Examples of teachers' statement	Teacher
Connection	Building on knowledge.	A
	Remember from last lesson.	B
Recognising patterns	I tell them their brain recognises the pattern.	A
Identifying similarities and differences	Similarities and difference. I do that as well.	B
	Able to recognise the different things	C
Generalising	They sort of generalise.	B
	Students really struggle with generalising.	C

As can be seen from the entries in Table 4.5, each teacher made nonspecific statements that were attached to the CRIG components. That they did so indicates some understanding of the CRIG components, but that alone does not indicate that these statements are rich in mathematical structure.

The interview questions 4, 5, and 6 (see Appendix D) focused specifically on how teachers promoted structural thinking. I connected relevant words or phrases from their responses to an aspect of structural thinking that would identify teachers' awareness of mathematical structure. These aspects were:

- *Recognising* structural thinking in students
- Student *engagement* when involved in structural thinking in mathematics
- *Benefits* of structural thinking in mathematics.

Table 4.6 contains examples of statements that relate to these three aspects.

Table 4.6

Examples of Teacher Statements That Identify Aspects of Structural Thinking

Aspects of structural thinking	Examples of teachers' statement	Teacher
Recognising structural thinking in students	I'd recognise structural thinking when [students] asking questions.	C
Student engagement when involved in structural thinking in mathematics	We need to identify the structure that they need to learn and then once they can see the achievement they are going to be engaged.	A
Benefits of structural thinking in mathematics	Doing structural thinking it sets them for deeper understanding.	B

Entries in Table 4.7 indicate the frequency of references made about the above three themes. It is evident that there were more statements that were related to recognising structural thinking in students, but overall there was not a big difference between the aspects of structural thinking.

Table 4.7

Students' Structural Thinking in Mathematics

Aspects of structural thinking	Frequency
Recognising structural thinking in students.	20
Student engagement when involved in structural thinking in mathematics.	14
Benefits of structural thinking in mathematics.	14
Total	48

The frequency of these statements suggests that teachers can recognise and express the use and benefits of structural thinking in their students.

Table 4.8 displays the frequency of statements made that acknowledge student behaviours that are linked to structural thinking.

Table 4.8

Frequency of Statements Identifying Structural Thinking Behaviours in Students

Classification	Student behaviour		Frequency
Recognising structural thinking in students	Questions asked by students		9
	Visual observations of students		3
	Students' working out	Processing	4
		Written setting	4
Student engagement when involved in structural thinking in mathematics	Impact on student achievement		3
	Students' confidence in doing mathematics		3
	Students' ability to stay on task		6
	Ability levels of students		2
Benefits of structural thinking in mathematics	Demonstrated knowledge by students		6
	Ability to take ownership		2
	Problem solving		6

The entries in Table 4.8 demonstrate that teachers identify with structural thinking through students' behaviours. The variety of behaviours recognised within the aspects of structural thinking give some indication that teachers are aware of mathematical structure when they are actually teaching mathematics and observing students in the classroom.

4.4 Observations

Classroom lesson observations comprised three lessons for each teacher and one observation template per lesson. Use of an Excel spreadsheet allowed for allocating and filtering of utterances into the CRIG components, then I allocated the subcategories. The subcategories were identified from the literature, with the intent that they would help during the process of data analysis. This also allowed for tallying the subcategories that would give further data for analysis.

In total, 227 recorded teacher utterances were documented. These are shown in Appendix H. Table 4.9 contains a breakdown of the frequency and percentages of utterances associated with each of the CRIG components. I initially felt that the allocation of utterances to the CRIG components would be central in measuring teachers' structural awareness when teaching mathematics.

Table 4.9

Teacher CRIG Statements Made during Observations

CRIG component	Frequency	Percentage
Connection	38	16.7
Recognising patterns	33	14.5
Identifying similarities and differences	52	22.9
Generalising	104	44.8
Total	227	100

The expectation was made in Section 3.5.3.3 that there would be strong evidence of utterances related to recognising patterns because patterns are emphasised in the number and algebra strand of the NSW mathematics syllabus. This was not the case, as this category had the fewest number recorded utterances.

After the allocation of utterances to a CRIG component, I became aware that this association does not necessarily indicate an effective pedagogical strategy reflecting mathematical structure. As this study is attempting to identify teachers' awareness of structure, I needed to consider how an utterance can be identified as an awareness of structure or no structural awareness, or somewhere inbetween. Further breakdown of these utterances into subcategories may identify the teachers' level of structural awareness.

I was interested in categorising an utterance according to whether it indicated a high or low grading of structural thinking. Any utterance that I felt promoted structural thinking at a higher level, I coded as *analytical*. I coded as *superficial* any utterances suggesting that the promotion of structural thinking was not as strong. The aim was to identify whether or not teachers' awareness of mathematical structure could be differentiated in their utterances. Table 4.10 contains examples of the type of utterance categorised as either analytical or superficial.

Table 4.10

Examples of Utterances Identified as Analytical or Superficial Statements

Statement	Examples of teachers' utterances	Teacher
Analytical	You know what you are doing. Now you have to understand why you are doing it.	A
	It could go to the biggest number in the world.	B
	What does the denominator tell me about the fraction?	C
Superficial	I am going to show you another way of doing this.	A
	We never add the denominators.	B
	How do I identify an improper fraction?	C

Table 4.11 contains the frequencies and percentages of analytical and superficial statements. The number of superficial statements is almost double that of analytical statements made during the lessons. This does not deny the presence of structural understanding, but does indicate that teachers were not promoting structural thinking at a higher level.

Table 4.11

Frequency and Percentage of Analytical and Superficial Statements Made by Teachers

Type of statement		Frequency	Percentage
Analytical		79	34.8
Superficial		148	64.2
	Total	227	100

I chose to introduce a further categorisation of each utterance to further identify whether teachers focused on procedures or concepts when attempting to promote structural thinking. Two new domains were introduced. The first of these was the *concept* domain, where utterances were used in an attempt to explain or question why something was done in a particular way, thus characterising conceptual understanding.

The second was the *content* domain in which teachers' utterances were topic-oriented and stated or questioned what to do to solve a problem. I regarded these utterances as being associated to a procedural understanding. Table 4.12 contains examples of utterances that were categorised as either concept or content.

Table 4.12

Examples of Utterances Made by Teachers Identifying Concept or Content

Domain	Teachers' utterances	Teacher
Concept	Dividing by a quarter is the same as multiplying by ….?	A
	You do multiplication before you do addition or subtraction	B
	Associative law we can swap the numbers around to make it easier to solve.	C
Content	The biggest mistake made is they flip the first	A
	What you do to the bottom you do to the top.	B
	Addition of two negative numbers will always give you a negative.	C

The division of concept and content statements is displayed in Table 4.13 as frequencies and percentages. This table shows that there was an approximately equal distribution of concept and content statements, although there was a slightly stronger trend was toward content utterances.

Table 4.13

Teacher Utterances Made during Observations as Concept or Content Statements

Domain	Frequency	Percentage
Concept	104	44.8
Content	123	54.2
Total number of statements recorded	227	100

After categorising all the teachers' statements into the subcategories, as shown in Tables 4.9, 4.11, and 4.13, I was interested in the outcome when utterances were allocated as a combination of the three subcategories. My intention for doing this was to answer the second research question associated with whether teachers promote structural thinking. I felt that by drawing the utterances toward a narrower perspective I could determine whether certain kinds of utterances promoted structural thinking. For example, utterances promoting high-level structural thinking would be those coded as both analytical and conceptual; those at the other extreme would superficial and content related. Figure 4.1 identifies the tree diagram that I used in the breakdown of each statement made by a teacher into a specific classification. These classifications follow the breakdown from the CRIG component to an *analytical* or *superficial* statement and then to a *concept* or *content* domain. Each statement was therefore categorised into one of 16 areas from the CRIG components and their subcategories.

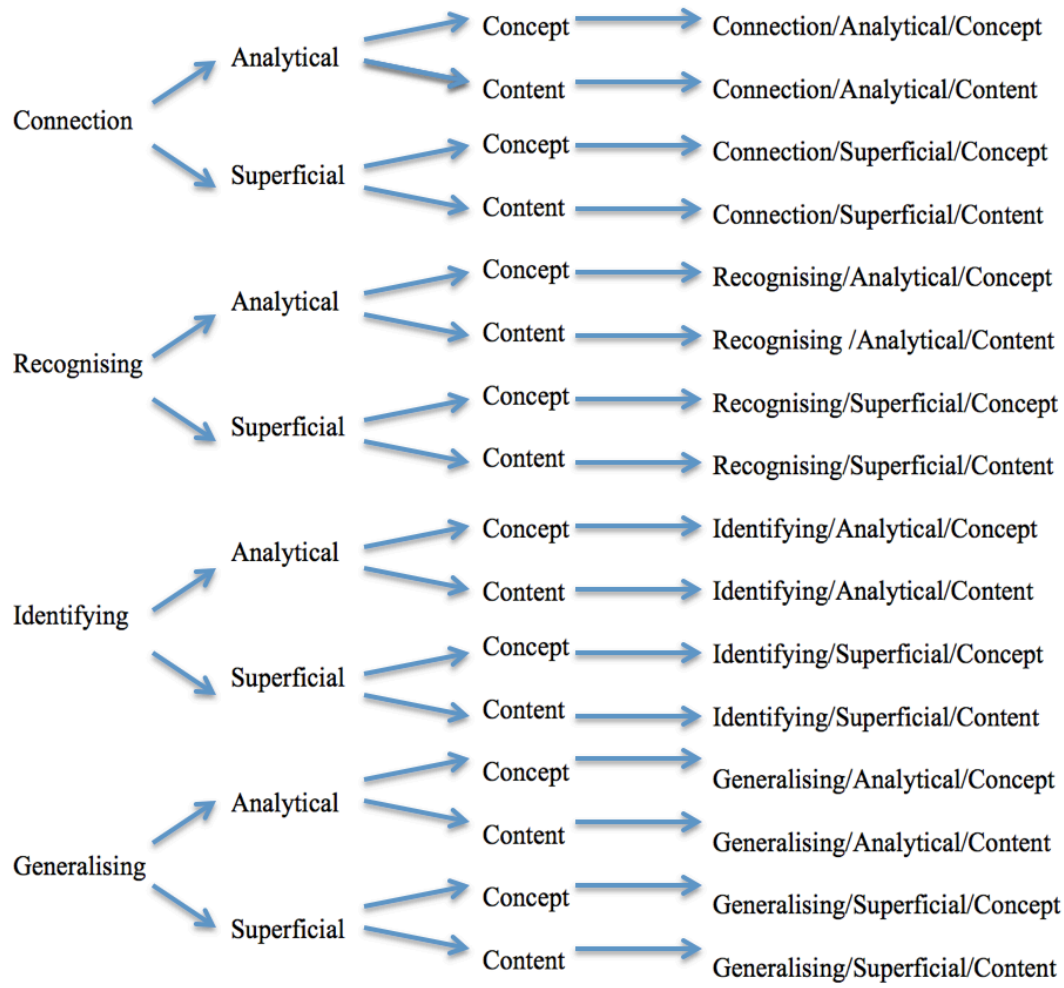

Figure 4.1. Tree diagram of breakdown of teacher utterances into subcategories.

Table 4.14 contains the frequency of the utterances generated by the category combinations from Figure 4.1. The generalising component subcategories of analytical/concept and superficial/ content had the highest frequencies (37 and 34 utterances respectively). However, generalising utterances that are analytical/concept in nature indicate a focus on mathematical structure, whereas the superficial/content utterances do not. Generalising utterances occur at the extremes of the two domains, analytical/concept and superficial/content, which causes confusion in identifying the pedagogical practice the teachers, are applying. A parallel set of results is evident within the connection component where the subcategories of analytical/concept and superficial/ content had the highest frequencies (14 and 17 utterances respectively). These opposing results indicate there are strong indications of attention to structure as well as a strong absence of attention to structure.

Table 4.14

Teacher Utterances as Combined Categories Frequency

CRIG component	Name of utterance	Domain	Frequency
Connection	Analytical	Concept	14
		Content	3
	Superficial	Concept	4
		Content	17
Recognising	Analytical	Concept	6
		Content	2
	Superficial	Concept	4
		Content	21
Identifying	Analytical	Concept	3
		Content	3
	Superficial	Concept	14
		Content	32
Generalising	Analytical	Concept	37
		Content	11
	Superficial	Concept	22
		Content	34
		Total	227

Table 4.15 provides a different perspective of teachers' utterances. As the CRIG components are not included, only the concept or content domain and analytical or superficial groupings are considered. This table presents new information about how structural thinking can be identified as conceptual or procedural (content) understanding.

Table 4.15

Frequency of Concept/Content Statement as Analytical/Superficial

Concept/content	Analytical/superficial	Frequency
Concept	Analytical	60
	Superficial	44
Content	Analytical	19
	Superficial	104
	Total	227

When the data are categorised in this way they produce four separate categories. The highest frequency occurs when utterances reflected teaching toward procedural understanding (content/superficial) statements. These types of utterances do not promote structural thinking. At the other extreme is the lowest frequency associated with procedural (content/analytical) statements. This table also provides an indication that teachers' pedagogical practices are almost evenly divided between conceptual (concept) and procedural (content) understanding (104 to 123 utterances respectively).

In an attempt to further identify the teachers' awareness of mathematical structure, I considered the frequency of concept/analytical utterances. The concept/analytical statement combination lies at the higher end of the mathematical structure continuum, and is therefore most likely to promote structural thinking. The proportion of these utterances is approximately a quarter of the total number of utterances (60/227). This indicates that all teachers made an attempt to promote structural thinking. Comparing this with teachers' interview responses, where only one teacher identified with mathematical structure awareness, as shown in Table 4.2, there is little evidence that teachers are aware of the value of mathematical structure and the minimal attempts to promote structural thinking may not be sufficient to make a difference to students' engagement. In fact, the lack of utterances that reflect structural awareness is an indication of the overall problem that there is not sufficient attention paid to mathematical structure when teaching mathematics. There is a small amount of evidence indicating an attempt toward using mathematical structure, but if there were an understanding of the value of mathematical structure in teaching mathematics these utterances would appear at a much higher frequency.

It is evident that the endorsement that the teachers gave to mathematical structure in the survey does not correspond with what they do in the classroom. The fewer number of structural related utterances confirms that teachers need to develop a greater awareness of mathematical structure to promote structural thinking when teaching mathematics. Despite some occasional attempts to promote structure, there is a noticeable discrepancy between what teachers indicated they knew about structure in the survey about mathematical structure and what they do to promote it when actually teaching mathematics.

4.5 Summary

In this chapter I considered the results from each instrument independently but also made some comparisons between them. From these results, an insight into the differences between what teachers think is mathematical structure, what they say about mathematical structure, and what they do to promote mathematical structure when teaching mathematical structure has been uncovered. In Chapter 5, I will discuss these results and draw some conclusions in response to the research questions.

CHAPTER

5

DISCUSSION AND IMPLICATIONS

5.1 Introduction

This chapter begins with a summary of the main findings of the study, followed by a consideration of the limitations of the study. This leads into a discussion of the implications for future research, particularly in teaching, learning, and curriculum, with potential for influencing professional development programs for teachers. The chapter concludes with insights gained from this research.

5.2 Summary of main findings

This thesis documents a small study exploring junior secondary teachers' understanding of mathematical structure. Data were collected from three sources: a survey, interviews, and observations. The importance of teacher awareness of structure as a pedagogical tool for promoting structural thinking was the main focus of this study.

The research questions focused on two main aspects, namely what teachers say they know about mathematical structure and how they use language in the classroom to promote structural thinking. The inconsistency between the results in the survey, interviews, and observations provide disparate answers to these questions. This is demonstrated by the teachers rating their understanding of mathematical structure as being toward the highest score of "agree" on the items in the survey, but being unable to provide adequate definitions of mathematical structure in the interviews. Observing teachers' pedagogical practices identified that the language used to promote structural thinking was inconsistent with the survey and interview data.

The survey results alone answer the first research question in the positive, that teachers are well aware of the nature and value of mathematical structure. Yet, the descriptions of mathematical structure given by the teachers during the interviews were conflicting without any definitive evidence of them understanding the nature and value of mathematical structure. The exception was one teacher who made four responses that demonstrated an awareness of structure. Four responses, from one teacher, are not enough to state that teachers are aware of mathematical structure.

Observations of the teachers in their actual teaching of mathematics revealed limited reference of mathematical structure in the language used during pedagogical practices. Their utterances occurred predominantly in a procedural understanding approach, which limited the promotion of structural thinking at a deeper level. There were, nevertheless, some examples of statements of conceptual understanding that attempted to promote a deeper structural thinking.

An interesting component of the research is what the teachers said during the interviews that identified their awareness of mathematical structure and what they did when teaching mathematics that promoted structural thinking. In essence, the interview results show that the teachers predominantly made statements that were weak in structural awareness and, as indicated above, only one teacher showed any structural awareness. This was not reflected in what the teachers did when teaching mathematics, where a quarter of all teachers' utterances were identified as analytical/concept, which gives some indication of structural understanding.

In comparing the interview and observational data, the teachers' emphasis on individual CRIG components shifted. In the interviews, teachers tended to articulate more about connection and recognising patterns, but in the classroom their attention to mathematical structure was aligned to generalising. When asked to identify mathematical structure verbally, identifying similarities and differences was the weakest. With regard to the CRIG categories, therefore, there appears to be a discrepancy in teachers' awareness of mathematical structure in what they say as compared with how they promote structural thinking when actually teaching mathematics.

Data obtained from the interviews and the observations indicate that teachers do not have a deep understanding of the term mathematical structure. The benefits of structural thinking in students' learning are acknowledged, but are not verbalised clearly when talking about the nature and value of mathematical structure to teaching and learning.

5.3 Limitations of the study

In this section I discuss aspects of the study that can be regarded as limitations for the effectiveness of the study to be able to address the research questions.

5.3.1 Scope

As this is a small-scale study, any conclusions drawn from the data may be relevant only within the context of this research.

Some demographic variables related to the participants were obtained, but were not analysed. As an isolated group of teachers, they may have different standards to mathematics teachers in other schools. This could have caused some bias in the results. Additional information about teacher demographics, and a more detailed analysis, could have improved the results.

Further limitations to the scope of the study include the failure to acknowledge the levels of awareness of mathematical structure. The majority of the data collected in the interviews and observations were allocated to one of two categories. This is a dichotomous grading between the categories, rather than a continuum or a progressive scale, and therefore, it is limited in identifying teachers' awareness of mathematical structural.

5.3.2 CRIG components

The CRIG components were identified in the literature and related to the forms presented by Mason, Stephens, and Watson (2009). I identified two problems regarding categorising teacher utterances to a CRIG component. First, an assumption was made that if an utterance was allocated to a CRIG component then it provided an indication of structural awareness. However, it became obvious that not all utterances allocated to a CRIG component were able to indicate an awareness of mathematical structure. The second problem was that I believed that all utterances would fit into a single CRIG component. While analysing the observation data I realised that utterances could be allocated to more than one of the CRIG components.

To overcome this, an all-encompassing selection of CRIG options should exist. The utterances that overlap can be allocated to a new combined CRIG component category, and all utterance recorded should be identified at a progressive stage of structural awareness.

5.3.3 Syllabus

Only one question on the survey and another in the interviews addressed mathematical structure in the NSW mathematics syllabus. This was considered within the literature review as being relevant to the background of the study in that mathematical structure is related to the working mathematically component of the NSW mathematics syllabus. These results were expected to show that mathematical structure was like working mathematically which is not readily acknowledged by teachers (Cavanagh, 2006). The responses from the questions did not support Cavanagh's findings. This could have been explored further through more directed and probing questions in the survey and interviews.

5.3.4 Instruments

In reviewing the data, I was able to identify aspects of the instruments that had created misleading results or made the results obtained difficult to interpret. Most of these problems are associated with the design of the instruments.

5.3.4.1 Survey

In retrospect, the survey did not yield worthwhile information regarding teachers' understanding of mathematical structure. The responses could not answer the question regarding teachers' awareness of the nature and value of mathematical structure.

Reasons why the teachers responded at the top level of "partially agree" and "agree" may be attributed to the design and scope of the questions and inadequate piloting. With a larger sample, validation of the survey items could have been obtained. The apparent high endorsement of structure could be a result of certain words in the questions such as “identify”, “include”, “recognise”, and “consider” which tended to encourage an “agree” response. Although this problem was not revealed in the piloting process, it became obvious during the data analysis. At the survey development phase, the focus was on the questions, not the possible responses. If this had been noticed during the piloting, changes to the survey questions could have been made.

5.3.4.2 Interviews

The interview questions could have been further developed to invite more depth in the teachers’ responses. The interview sessions lasted on average only 10 minutes, indicating that the teachers’ responses were lacking in depth.

5.3.4.3 Observations

The observation component of this study proved to be the most challenging because of the range of data collected and the coding used.

The collection and recording of utterances started with deciding whether the coding process was accurate and whether it included all appropriate utterances. The context of each utterance was not considered. This study could have been improved by using an alternative model, such as TIMSS (Lokan, McRae, & Hollingsworth, 2003). This study was not going to be of that scale, so within its parameters it did provide a trial of a method and instrumentation that can be built on for any further study.

The use of the CRIG components of mathematical structure was vital during the development of the instruments. I identified the CRIG components as different categories of mathematical structure. This was an assumption made early in the development of this study, but it fails to recognise that not all utterances allocated to a CRIG category are indicative of mathematical structure.

The additional coding between two categories, which was intended to refine the analysis of utterances further, did not allow for identification within the structural thinking continuum identified by Mason et al. (2009).

5.4 Implications

Mason et al. (2009) espoused the importance of students engaging in structural thinking before they are able think deeply about mathematics, and they argued that this will happen only when the teacher is structurally aware. Teachers, both qualified and nonqualified, need to be made aware of mathematical structure not only in their mathematical knowledge but also in their pedagogical practices. The role of the teacher in delivering the mathematical content, by procedural or conceptual understandings, is essential for creating engaged students. The teacher's awareness of mathematical structure helps develop teaching practices that support the higher range of the structural thinking continuum and helps overcome what Mason et al. (2009) called the "mythical chasm" between procedural and conceptual understandings of mathematics learning.

5.5 Future research

There is a pressing need to investigate the teacher's role in creating a learning environment rich in the appreciation of mathematical structure through what they say and what they do. Further research needs to explore how mathematical structure can be developed in teachers' pedagogical practices. Vale, McAndrew, and Krishnan (2011) showed that when out-of-field teachers were introduced to mathematical structure, their understanding of the syllabus content and their teaching practices improved. This was evident from the report by Vale et al (2011) that the teachers' reflections as learners, helped them develop an understanding of student thinking, clarified their learning goals, and developed their teaching practices that enabled students to connect with their learning. The evidence of teacher practice improving was analysed from teacher reflective records. However, there is no data to support their improved practice over time. Certain components of teacher backgrounds such as teaching qualification and years of experience would need to be considered in further research.

Using the Mason et al. (2009) structural thinking continuum, future research could be directed to investigate how professional development programs could be designed to demonstrate how teacher utterances and behaviours can enhance mathematical structure.

5.5.1 Teaching, learning, and the curriculum

Due to many factors, the current focus in teaching mathematics is on the delivery of syllabus content. This detracts from its delivery and the teachers' ability to promote structural thinking effectively. The syllabus documents need to embed mathematical structure through the working mathematically component of the syllabus content.

Cavanagh (2006) discussed a range of reasons for the lack of attention to working mathematically. School requirements such as formal assessment and reporting procedures do not support teachers to commit to developing structural thinking because of the need to meet administrative deadlines. The support for mathematics teachers to promote structural thinking must be developed by executive staff members.

5.5.2 Mathematical structure and teacher professional development

There is a need for all teachers of mathematics to develop a greater awareness of mathematical structure. The focus of Australian teachers toward a procedural approach to mathematics teaching has seen Australia's decline on international test score rankings (TIMSS) (Lokan et al., 2003). Lokan et al. (2003) argued that Australian students would benefit from higher-level problems, discussion of solutions, and opportunities to explore their thinking. This would be a mathematical pedagogy rich in mathematical structure.

Sullivan, Clarke, & Clarke, (2009) provided an example of how teachers, when given opportunities to create alternative learning opportunities, chose a traditional approach that involved following a single procedure toward the answer. Mathematical structure can still exist in a procedural approach, but teachers need to be made aware of this. Stephens (2008) demonstrated how mathematical structure supports procedures when describing how students made generalisations after completing several instances of the same problem. A professional development program that supports mathematical structure across mathematical pedagogical practices would show this. Such programs should involve professional dialogue. Davis and Renert (2013) discovered that collaborative discussion promoted greater awareness of mathematical structure.

This study could be the basis for further research in this area leading toward a professional development program that could, following from the work of Vale et al. (2011), develop a deeper awareness of mathematical structure. These same authors' demonstrated how teacher professional development could enhance teachers' appreciation of mathematical structure. Further promotion of mathematical structure would benefit student engagement and have the potential to see increased enrolments of students in higher levels of senior secondary mathematics courses.

5.6 Concluding remarks

5.6.1 What they say versus what they do

The original assumption that teachers do not have an understanding of mathematical structure does not preclude the possibility that some of their pedagogical practices do promote structural thinking. This is reflective of the difference between what they say and what they do. Davis and Renert (2013) identified that teachers' beliefs are often different from their practices.

5.6.2 Future directions

Mathematics is in danger of becoming more inaccessible and unattractive in the eyes of young learners in schools. It has a poor image within the general population as a subject that is neither understood nor enjoyed by the majority of people, and there is a pseudo elitist attitude that only some people understand it. There is therefore a strong need to continue investigation into how mathematical structure can put mathematics back in the mainstream as a desirable and important skill—a valuable and interesting subject to study, that is an essential skill for all people.

Given this background to the state of mathematics as a domain in general, and specifically mathematics education, I saw a need to find out why students are turning away from mathematics. Initially, it was an issue of engagement (Attard, 2012) and, while showing the importance of engagement to be a component of student learning, there were no direct findings about what could be done to engage students in learning mathematics. My research project began with the teacher as the protagonist in the student–teacher relationship and as the main influence on student engagement (Attard, 2010). It became clear that through mathematical structure the teacher's pedagogical approach to learning mathematics in the classroom would impact on students' engagement through developing structural thinking skills. These skills are more likely to create an interested learner, not guided by extrinsic rewards, but a learner who enjoys and feels good about understanding this subject.

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APPENDIX A

Ethics Approval

This appendix contains the ethics approval from the Macquarie University Faculty of Human Sciences Human Research Ethics Sub-Committee, No. 5201401133 Con/Met.

From: "Fhs Ethics" <fhs.ethics@mq.edu.au>
Subject: RE: HS Ethics Application - Approved (5201401133)(Con/Met)
Date: 16 February 2015 12:23:19 pm AEDT
To: "A/Prof Joanne Mulligan" <joanne.mulligan@mq.edu.au>
Cc: "Mr Mark Thomas Gronow" <mark.gronow@students.mq.edu.au>

Dear A/Prof Mulligan,

Re: "Mathematics teachers' understanding of structural thinking in mathematics" (5201401133)

Thank you very much for your response. Your response has addressed the issues raised by the Faculty of Human Sciences Human Research Ethics Sub-Committee and approval has been granted, effective 16th February 2015. This email constitutes ethical approval only.

This research meets the requirements of the National Statement on Ethical Conduct in Human Research (2007). The National Statement is available at the following web site:

http://www.nhmrc.gov.au/_files_nhmrc/publications/attachments/e72.pdf.

The following personnel is authorised to conduct this research:

A/Prof Joanne Mulligan
Mr Mark Thomas Gronow

Please note the following standard requirements of approval:

1. The approval of this project is conditional upon your continuing compliance with the National Statement on Ethical Conduct in Human Research (2007).

2. Approval will be for a period of five (5) years subject to the provision of annual reports.

   Progress Report 1 Due: 16th February 2016
   Progress Report 2 Due: 16th February 2017
   Progress Report 3 Due: 16th February 2018
   Progress Report 4 Due: 16th February 2019
   Final Report Due: 16th February 2020

   NB. If you complete the work earlier than you had planned you must submit a Final Report as soon as the work is completed. If the project has been discontinued or not commenced for any reason, you are also required to submit a Final Report for the project.

   Progress reports and Final Reports are available at the following website:

   http://www.research.mq.edu.au/for/researchers/how_to_obtain_ethics_approval
   /human_research_ethics/forms

3. If the project has run for more than five (5) years you cannot renew approval for the project. You will need to complete and submit a Final Report and submit a new application for the project. (The five year limit on renewal of approvals allows the Sub-Committee to fully re-review research in an environment where legislation, guidelines and requirements are continually changing, for example, new child protection and privacy laws).

4. All amendments to the project must be reviewed and approved by the Sub-Committee before implementation. Please complete and submit a Request for Amendment Form available at the following website:

   ttp://www.research.mq.edu.au/for/researchers/how_to_obtain_ethics_approval/human_research_ethics/forms

5. Please notify the Sub-Committee immediately in the event of any adverse effects on participants or of any unforeseen events that affect the continued ethical acceptability of the project.

6. At all times you are responsible for the ethical conduct of your research in accordance with the guidelines established by the University. This information is available at the following websites:

   http://www.mq.edu.au/policy

   http://www.research.mq.edu.au/for/researchers/how_to_obtain_ethics_approval/human_research_ethics/policy

If you will be applying for or have applied for internal or external funding for the above project it is your responsibility to provide the Macquarie University's Research Grants Management Assistant with a copy of this email as soon as possible. Internal and External funding agencies will not be informed that you have approval for your project and funds will not be released until the Research Grants Management Assistant has received a copy of this email.

If you need to provide a hard copy letter of approval to an external organisation as evidence that you have approval, please do not hesitate to contact the Ethics Secretariat at the address below.

Please retain a copy of this email as this is your official notification of ethics approval.

Yours sincerely,

Dr Anthony Miller
Chair
Faculty of Human Sciences
Human Research Ethics Sub-Committee

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APPENDIX B

Information Materials and Consent Forms for School Principal and Participating Teachers

This appendix contains two sets of information materials and consent forms: one for the principal of the school at which the research was conducted, and the other for the three teachers who participated.

Each set was duplicated so that one signed copy was retained by the investigator, and the other signed copy was retained by the signatories at the school. Only one copy from each set is included in this appendix

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Email: joanne.mulligan@mq.edu.au

Chief Investigator/Supervisor: Joanne Mulligan
Chief Investigator/Supervisor title: Associate Professor

Principal information and Consent Form

Name of Project: Mathematics teachers understanding of structural thinking in mathematics

You school is invited to participate in a study of mathematics teachers' understanding of mathematical structure. The purpose of the study is to investigate the how teachers of mathematics use structure in their lessons to engage students.

The study is being conducted by Mr Mark Gronow Department of Education, Faculty of Human Sciences, Macquarie University, +6(0)432232454, mark.gronow@students.mq.edu.au as the Co-Investigator. This research project is being conducted to meet the requirements of Master of Research under the supervision of Assoc. Professor Joanne Mulligan, +61(0)298508621, joanne.mulligan@mq.edu.au of the Faculty of Human Sciences, Department of Education.

If you decide to participate, teacher participants will be asked to complete a Likert scale questionnaire and be interviewed by the Co-Investigator. A digital recording of the interview will be made for the purpose of analysis.

The Co-Investigator may request of the participants to observe two mathematics lessons of Year 7 & 8 classes. During these lessons the Co-Investigator, will observe the lessons and monitor the teaching of structure.

Any information or personal details gathered in the course of the study are confidential, except as required by law. No individual will be identified in any publication of the results. The results of this project will be made available, on request, to you, the mathematics staff and the school executive through a written report.

Only the Co-Investigator and the Chief Investigator will have access to the data collected. All the data will be stored securely at the Chief Investigators office at Macquarie University and kept there for 5 years, after which they will be destroyed. The data may be used to prepare publications for professional, academic journals, web sites and presentations to teachers and other professionals. However, the data used will be treated in such a way that no one will be able to identify any specific participants or the school. All data collected and reports written will be made available to the participants during the research process and at its completion.

Participation in this study is entirely voluntary: you are not obliged to participate and if you decide to participate, you are free to withdraw at any time without having to give a reason and without consequence. Your school is invited to participate in the project and you give your consent for teachers on your staff to participate in these activities by completing the attached consent form.

This information sheet is for you to keep. If you have any questions at all about the project, please contact me at any time.

Yours sincerely,

Co-Investigator
Mark Gronow
Tel. 0432232454
E-mail: mark.gronow@students.mq.edu.au

Chief Investigator
Associate Professor Joanne Mulligan,
Tel. (02) 9850 8621
Fax. (02) 9850 8674
E-mail: joanne.mulligan@mq.edu.au

I, …………………………….. *(participant's name)*, have read and understand the information above and any questions I have asked have been answered to my satisfaction. I agree to participate in this research, knowing that I can withdraw from further participation in the research at any time without consequence. I have been given a copy of this form to keep.

Participant's Name:____
(Block letters)

Participant's Signature: __________________________ Date:___________________

Investigator's Name:____
(Block letters)

Investigator's Signature: _____________________ ___ Date:___________________

The ethical aspects of this study have been approved by the Macquarie University Human Research Ethics Committee. If you have any complaints or reservations about any ethical aspect of your participation in this research, you may contact the Committee through the Director, Research Ethics (telephone (02) 9850 7854; email ethics@mq.edu.au). Any complaint you make will be treated in confidence and investigated, and you will be informed of the outcome.

(INVESTIGATOR'S COPY)

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Phone +61 (0)2 9850 8621
Fax: +61(0)2 9850 8674
Email: joanne.mulligan@mq.edu.au

Chief Investigator/Supervisor: Joanne Mulligan
Chief Investigator/Supervisor title: Associate Professor

Participant information and Consent Form

Name of Project: Mathematics teachers understanding of structural thinking in mathematics.

You are invited to participate in a study of mathematics teachers' understanding of mathematical structure. The purpose of the study is to investigate the how teachers of mathematics use structure in their lessons to engage students.

The study is being conducted by Mr Mark Gronow Department of Education, Faculty of Human Sciences, Macquarie University, +61(0)298508621, mark.gronow@students.mq.edu.au as the Co-Investigator. This research project is being conducted to meet the requirements of Master of Research under the supervision of Assoc. Professor Joanne Mulligan, +61(0)298508621, joanne.mulligan@mq.edu.au of the Faculty of Human Sciences, Department of Education.

Participants will be asked to complete a 30-minute Likert scale questionnaire. Three teachers will be asked be involved in a follow up interview, of approximately 30 minutes, and observation of two mathematics lessons. A digital recording of the interview will be made for the purpose of analysis. The Co-Investigator, will observe the lessons to monitor the teaching of structure.

Any information or personal details gathered in the course of the study are confidential, except as required by law. No individual will be identified in any publication of the results. The results of this project will be made available, on request, the mathematics staff and the school executive through a written report. You are able to discuss the project at anytime with the Co-Investigator.

Only the Co-Investigator and the Chief Investigator will have access to the data collected. All the data will be stored securely at the Chief Investigator's office at Macquarie University and kept there for 5 years, after which they will be destroyed. The data may be used in publications for professional, academic journals, web sites and presentations to teachers and other professionals. However, the data used will be treated in such a way that no one will be able to identify any specific participants or the school. All data collected and reports written will be made available to the participants during the research process and at its completion.

Participation in this study is entirely voluntary: you are not obliged to participate and if you decide to participate, you are free to withdraw at any time without having to give a reason and without consequence. You are invited to participate in the project and give your consent to participate in these activities by completing the attached consent form.

If at anytime you have cause for complaint you can contact the Chief Investigator.

This information sheet is for you to keep. If you have any questions at all about the project, please contact me at any time.

Yours sincerely,

Co-Investigator
Mark Gronow
Tel. 0432232454
E-mail: mark.gronow@students.mq.edu.au

Chief Investigator
Associate Professor Joanne Mulligan,
Tel. (02) 9850 8621
Fax. (02) 9850 8674
E-mail: joanne.mulligan@mq.edu.au

I, …………………………….. *(participant's name*), have read and understand the information above and any questions I have asked have been answered to my satisfaction. I agree to participate in this research, knowing that I can withdraw from further participation in the research at any time without consequence. I have been given a copy of this form to keep.

Participant's Name:____
(Block letters)

Participant's Signature: ______________________________ Date:______________________

Investigator's Name:____
(Block letters)

Investigator's Signature: _________________________ ___ Date:______________________

The ethical aspects of this study have been approved by the Macquarie University Human Research Ethics Committee. If you have any complaints or reservations about any ethical aspect of your participation in this research, you may contact the Committee through the Director, Research Ethics (telephone (02) 9850 7854; email ethics@mq.edu.au). Any complaint you make will be treated in confidence and investigated, and you will be informed of the outcome.

(INVESTIGATOR'S COPY)

APPENDIX C

The Survey

This appendix contains the 22-item survey that teachers at the participating high school were asked to complete.

1. Mathematical structure forms the basis of all mathematical thinking.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

2. Mathematical structure is identified in the Year 7 - 10 NSW Syllabus for the Australian Curriculum.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

3. Mathematical structure is included in mathematics teaching.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

4. Mathematical structure is used in problem solving.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

5. Mathematical structure supports mathematical understanding.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

6. Mathematics teachers consider mathematical structure in their teaching.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

7. Mathematics teachers identify patterning when teaching.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

8. Mathematics teachers identify the differences in problems.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

9. Mathematics teachers identify the similarities in problems.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

10. Mathematics teachers make connections to students' previous learning.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

11. Mathematics teachers make generalisations when solving mathematical problems.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

12. Structural thinking is required to solve mathematical problems.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

13. Students can generalise when problem solving.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

14. Students can recognise patterns in mathematics.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

15. Students can reproduce a pattern in mathematics.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

16. Students make connections to previous learning.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

17. Students recognise differences in mathematical problems.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

18. Students recognise similarities in mathematical problems.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

19. Students identify patterns in mathematics.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

20. Teaching mathematical structure requires an understanding of mathematics.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

21. Teaching mathematics requires an understanding of mathematical pedagogy.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

22. Teaching mathematics requires an understanding of syllabus content from K-12.

Disagree	Partially Disagree	Niether Agree or Disagree	Partially Agree	Agree
○	○	○	○	○

APPENDIX D

The Interview Schedule

This appendix contains the definition of mathematical structure read aloud to the teachers at the beginning of the interview, and the interview questions read aloud to the teachers.

Teachers were given the following brief outline of what is structure and structural thinking in mathematics before the interview questions.

> Some authors describe mathematical structure as the building blocks of mathematical learning. Mathematical structure can be found in connecting mathematical concepts, recognising and reproducing patterns, identifying similarities and differences, and generalising results. Students who perform structural thinking use these skills without always considering them when solving problems. Many students need to be taught these skills when introduced to concepts as a reminder of how to think mathematically.

1. Can you give example(s) of another expression(s) you would use to describe “mathematical structure”?

2. When do you use mathematical structure in your teaching?

3. In what way does the NSW syllabus for the Australian curriculum identify mathematical structure?

4. How do you encourage your students be use structural thinking skills in mathematics?

5. How would you recognise structural thinking in any of your students?

6. What are the benefits of students using structural thinking in mathematics?

APPENDIX E

The Observation Template

This appendix contains the final version of the observation template that was used after piloting of the template had altered the original version in two stages.

Mathematical structure categories to be identified

Connecting (C)	Recognition (R)	Identifying (I)	Generalising/Reasoning (G)
Prior and future mathematical learning, other mathematical concepts, application to real life examples	Noticing, identifying and reproducing patterns and relationships in content and concepts.	Similarities and difference in all content and concepts.	Generalising expressions, relationships, and complexities in content and concepts are generalised through representing, explaining and justifying conclusions.

	Statement	Category
1.		
2.		
3.		
4.		
5.		
6.		
7.		
8.		
9.		
10.		
11.		
12.		
13.		
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21.		
22.		
23.		
24.		
25.		
26.		

APPENDIX F

Participant Background Questionnaire

This appendix contains the questionnaire that was completed by the participants of the interview and observation components of the main study.

For how many years have you been teaching? ________________

What is your teaching qualification? ________________

What university did you graduate from? ________________

What other subjects are you teaching? ________________

Is maths your first subject? ________________

What regions have you taught in? ________________

What recent professional development have you done? ________________

What professional association are you a member of? ________________

Do you use in technology in the classroom? ________________

Have you participated in other research? ________________

APPENDIX G

Average of Responses to Survey Items

This appendix contains the averages for each question from the survey completed by the five mathematics teachers from the 5-point Likert scale with 1 the lowest and 5 the highest.

These questions were designed to identify the teachers' familiarity with mathematical structure and structural thinking—where this concept appears in the mathematical teaching and learning, curriculum documents, and student thinking.

Question	Survey item	Average
1	Mathematical structure forms the basis of all mathematical thinking.	4.2
2	Mathematical structure is identified in the Year 7–10 NSW syllabus for the Australian curriculum.	3.8
3	Mathematical structure is included in mathematics teaching.	4.4
4	Mathematical structure is used in problem solving.	4.6
5	Mathematical structure supports mathematical understanding.	4.6
6	Mathematics teachers consider mathematical structure in their teaching.	4.6
7	Mathematics teachers identify patterning when teaching.	4.6
8	Mathematics teachers identify the differences in problems.	4.6
9	Mathematics teachers identify the similarities in problems.	4.8
10	Mathematics teachers make connections to students' previous learning.	4.8
11	Mathematics teachers make generalisations when solving mathematical problems.	4.0
12	Structural thinking is required to solve mathematical problems.	4.6
13	Students can generalise in problem solving.	3.4
14	Students can recognise patterns in mathematics.	4.2
15	Students can reproduce a pattern in mathematics.	4.2
16	Students make connections to previous learning.	4.2
17	Students recognise differences in mathematical problems.	4.2
18	Students recognise similarities in mathematical problems.	4.2
19	Students identify patterns in mathematics.	4.4
20	Teaching mathematical structure requires an understanding of mathematics.	5.0
21	Teaching mathematics requires an understanding of mathematical pedagogy.	5.0
22	Teaching mathematics requires an understanding of syllabus content from K–12.	4.8

APPENDIX H

Teachers' Utterances

Related to Mathematical Structure

This appendix contains the utterances made by the teachers during the observations conducted during the main study. The categorisations relating to each utterance appear in the four columns to the right of the utterance.

Utterance	CRIG Category	Question or instruction	Superficial / Analytical	Content/ Concept
Who can remember the rule for multiplying fractions?	Connection	Question	Analytical	Concept
Remember we went …	Connection	Instruction	Superficial	Concept
Isn't it better to know why?	Connection	Question	Analytical	Concept
You know what to do but you need to understand why	Connection	Instruction	Superficial	Concept
We are explaining why	Connection	Instruction	Analytical	Concept
Let's go back to the rules we used before	Connection	Instruction	Analytical	Concept
Remember what we said about multiples	Connection	Instruction	Superficial	Concept
What does evaluate mean?	Connection	Question	Analytical	Concept
You do multiplication before you do addition or subtraction	Connection	Instruction	Superficial	Concept
What law do you use?	Connection	Question	Analytical	Concept
Who can remember what is meant by the reciprocal?	Connection	Question	Analytical	Concept
What happens if we multiply 2 by a half, what is the result?	Connection	Question	Analytical	Concept
What was the main issue we had when adding or subtracting fractions?	Connection	Question	Analytical	Concept
What does this number tell you?	Connection	Question	Analytical	Concept
What type of number is 2?	Connection	Question	Analytical	Concept
What type of number is 100?	Connection	Question	Analytical	Concept
Please remind me, the factors of a number are getting bigger or smaller.	Connection	Question	Analytical	Concept
Multiplication is repeated addition.	Connection	Instruction	Analytical	Concept
You have to know it next year.	Connection	Instruction	Superficial	Content
Can anyone remember the divisibility rules from last term?	Connection	Question	Analytical	Content

We were talking about how we divide fractions. There is a phrase that we used can you remember it.	Connection	Instruction	Superficial	Content
It's your times tables.	Connection	Instruction	Superficial	Content
Remember, we are going to try and simplify first.	Connection	Instruction	Superficial	Content
Is it because you knew your times tables	Connection	Instruction	Superficial	Content
We have done this before	Connection	Instruction	Superficial	Content
What law did we speak about?	Connection	Question	Analytical	Content
Who knows what the distributive law is?	Connection	Question	Analytical	Content
Who can remind me.	Connection	Instruction	Superficial	Content
Remember what we said about multiples.	Connection	Instruction	Superficial	Content
Who can remember what a factor tree is?	Connection	Question	Superficial	Content
What factor tree are we drawing?	Connection	Question	Superficial	Content
Who can remember the word we use to remember corresponding, cointerior and alternate angles on parallel lines?	Connection	Instruction	Superficial	Content
What are these amounts?	Connection	Question	Superficial	Content
How do we go from a percentage to a fraction?	Connection	Question	Superficial	Content
It was a while ago we were doing this.	Connection	Instruction	Superficial	Content
I asked you that question the other day.	Connection	Question	Superficial	Content
Remember we spoke about reciprocal.	Connection	Instruction	Superficial	Content
How do we go from a fraction to a percentage?	Connection	Question	Superficial	Content
When you simplify a fraction what are you doing?	Generalising	Question	Analytical	Concept

What's the problem here?	Generalising	Question	Analytical	Concept
What does that mean?	Generalising	Question	Analytical	Concept
What can I do here?	Generalising	Question	Analytical	Concept
How do I know that I could have simplified more at the beginning?	Generalising	Question	Analytical	Concept
Write it down as a mathematical problem.	Generalising	Instruction	Analytical	Concept
Tell me what the question means.	Generalising	Instruction	Analytical	Concept
What do you have to do to the first number?	Generalising	Question	Superficial	Concept
What would you do next?	Generalising	Question	Superficial	Concept
Can you see what we got?	Generalising	Question	Superficial	Concept
What can we simplify?	Generalising	Question	Superficial	Concept
What is a fraction? What is it we are doing?	Generalising	Question	Superficial	Content
How much easier is it to go…	Generalising	Instruction	Superficial	Concept
I am going to show you another way of doing this.	Generalising	Instruction	Superficial	Concept
What can we do here?	Generalising	Question	Analytical	Concept
What does that mean?	Generalising	Question	Analytical	Concept
What do I do?	Generalising	Question	Analytical	Concept
What happens when I go?	Generalising	Question	Analytical	Concept
Can you simplify?	Generalising	Question	Analytical	Concept
What did you do here?	Generalising	Question	Analytical	Concept
What did you do in your head?	Generalising	Question	Analytical	Concept
Then what do we do?	Generalising	Question	Analytical	Concept
What will I do?	Generalising	Question	Analytical	Concept
What did you actually do?	Generalising	Question	Analytical	Content
Can you see why you times and tip? (Dividing fractions)	Generalising	Instruction	Analytical	Content
You know what you are doing now you have to understand why you are doing it.	Generalising	Instruction	Analytical	Content

What did we do here	Generalising	Instruction	Analytical	Concept
Put the number sentence into words.	Generalising	Instruction	Analytical	Concept
We have to work out a strategy	Generalising	Instruction	Analytical	Concept
Can you see what we did here?	Generalising	Question	Superficial	Concept
What have we done with the division and multiplication sign?	Generalising	Question	Superficial	Content
How do we multiply our fractions?	Generalising	Question	Superficial	Content
How do we multiply fractions?	Generalising	Question	Superficial	Concept
What does 4 divided by a third mean?	Generalising	Question	Superficial	Content
Tell me sentence in words what tells me to do (10÷5)	Generalising	Instruction	Superficial	Content
Give me a word sentence explaining 4÷ 1/3	Generalising	Instruction	Superficial	Content
If you multiply by two you divide by a half	Generalising	Instruction	Superficial	Content
Adding fractions what do we need to do?	Generalising	Question	Superficial	Content
Whatever we do to the bottom we do the top	Generalising	Instruction	Superficial	Content
We can't add or subtract until we have the same denominator	Generalising	Instruction	Superficial	Content
You recognise the numbers they mean something to you.	Generalising	Instruction	Analytical	Concept
Why do it this way?	Generalising	Question	Analytical	Concept
How do we multiply fractions?	Generalising	Question	Superficial	Content
What did you do?	Generalising	Question	Analytical	Concept
You need to be able to realise that by making 3 a fraction (3/1) then you can use the same rule.	Generalising	Instruction	Superficial	Content
Can I simplify	Generalising	Instruction	Superficial	Concept
What's the next step?	Generalising	Question	Superficial	Concept

The biggest mistake made is they flip the first.	Generalising	Instruction	Superficial	Content
How can you write that as a number sentence	Generalising	Instruction	Analytical	Concept
What is a fraction?	Generalising	Question	Analytical	Concept
What does it actually mean?	Generalising	Question	Analytical	Concept
A fraction is a division	Generalising	Question	Analytical	Content
Everything is linked	Generalising	Instruction	Superficial	Concept
What do we have to do?	Generalising	Question	Superficial	Concept
What is the base?	Generalising	Question	Superficial	Content
What is the index	Generalising	Instruction	Superficial	Content
If I said to you evaluate, what does that mean?	Generalising	Question	Analytical	Concept
These are the words you need to remember	Generalising	Instruction	Superficial	Content
What is the multiple?	Generalising	Question	Analytical	Content
What do I write?	Generalising	Question	Analytical	Concept
How else would you evaluate it	Generalising	Instruction	Analytical	Concept
It could go to the biggest number in the world	Generalising	Instruction	Analytical	Concept
This is the mistake that people do	Generalising	Instruction	Analytical	Concept
What if it was five cubed?	Generalising	Question	Superficial	Content
What is the rule when we multiply two fractions?	Generalising	Question	Superficial	Content
Is that simplified?	Generalising	Question	Superficial	Concept
What has to be the same when you add two fractions?	Generalising	Question	Superficial	Content
Evaluate, calculate or find the answer to	Generalising	Instruction	Superficial	Concept
What's the rule when we divide fractions?	Generalising	Question	Superficial	Content
The reason why I am putting the times is how else will I evaluate it	Generalising	Instruction	Superficial	Content

We never add the denominators	Generalising	Instruction	Superficial	Content
What you do to the bottom you do to the top.	Generalising	Instruction	Superficial	Content
How would I work this out?	Generalising	Question	Analytical	Concept
Tell me how you did it?	Generalising	Question	Superficial	Concept
Tell me how you did question 2?	Generalising	Question	Superficial	Concept
Where did you go wrong?	Generalising	Question	Superficial	Concept
What would we do if we decrease?	Generalising	Question	Superficial	Content
Because I am decreasing, what do I do?	Generalising	Question	Superficial	Content
Did we come up with a general rule?	Generalising	Question	Analytical	Concept
What does the denominator tell me about the fraction?	Generalising	Question	Analytical	Content
What the law states is the numbers can be shared?	Generalising	Question	Analytical	Concept
When you add them together what do you get?	Generalising	Question	Superficial	Concept
What law applies?	Generalising	Question	Superficial	Concept
How did you work out what one-third was?	Generalising	Question	Superficial	Content
There is some symmetry about this.	Generalising	Question	Superficial	Concept
Generalisation for when going in the same direction add them and put the sign of the direction.	Generalising	Instruction	Superficial	Content
Associative law we can swap the numbers around to make it easier to solve.	Generalising	Instruction	Superficial	Concept
Letters in mathematics are called pronumerals and can stand for any number.	Generalising	Instruction	Analytical	Concept

Why is it an equilateral triangle?	Generalising	Question	Analytical	Content
If all the angles are the same what does it tell us about the sides?	Generalising	Question	Analytical	Content
What does the word "regular" mean in geometry?	Generalising	Question	Superficial	Content
How do I know from the diagram that these lines are parallel?	Generalising	Question	Superficial	Content
If we have an isosceles triangle what do we know about the base angles?	Generalising	Question	Analytical	Content
What do you know about the base angle?	Generalising	Question	Analytical	Content
What doesn't make sense?	Generalising	Question	Analytical	Concept
What do we do when ….	Generalising	Instruction	Analytical	Concept
What would that one be?	Generalising	Question	Superficial	Content
What do we replace the "of" with?	Generalising	Question	Superficial	Content
Why is it over 100? (%)	Generalising	Question	Superficial	Content
Who's got another way?	Generalising	Question	Analytical	Concept
Give me an example of…	Generalising	Instruction	Analytical	Content
What happens when you multiply a number by one?	Generalising	Question	Superficial	Content
Who can expand on that definition of whole numbers?	Generalising	Question	Superficial	Concept
General rule if signs are different subtract smallest from biggest then put the sign of the biggest number in front of the answer.	Generalising	Instruction	Superficial	Content
What have we got here?	Identifying	Question	Superficial	Concept

Is that the same thing?	Identifying	Question	Superficial	Concept
What's the difference between these and the previous questions.	Identifying	Instruction	Superficial	Concept
What's the difference between that and that? (points to two expressions on the board)	Identifying	Instruction	Superficial	Concept
Dividing by a quarter is the same as multiplying by ….?	Identifying	Question	Superficial	Concept
What's the opposite of times?	Identifying	Question	Superficial	Concept
It's making you think about numbers.	Identifying	Instruction	Superficial	Concept
You could've done that but I want to show you something else here.	Identifying	Instruction	Superficial	Content
8 ÷ ½ is the same as 8 times 2	Identifying	Instruction	Superficial	Content
Dividing by a number is the same as multiplying by the inverse	Identifying	Instruction	Superficial	Content
Make 3 into an improper fraction 3/1	Identifying	Instruction	Superficial	Content
Why do we put 3 over 1?	Identifying	Question	Analytical	Concept
That would leave it the same	Identifying	Instruction	Superficial	Concept
You must keep the first fraction the same.	Identifying	Instruction	Superficial	Content
What's the problem here?	Identifying	Question	Analytical	Concept
Have a look at this -7 -2 and have a look at this -7 x -2 what is different?	Identifying	Question	Superficial	Content
Are these like terms?	Identifying	Question	Superficial	Content
What's the x?	Identifying	Question	Superficial	Content
Why am I putting times here and the last questions had just x and y?	Identifying	Question	Superficial	Content
What's in between the 2 and the x?	Identifying	Question	Superficial	Content
What is the like term to $8x^2$?	Identifying	Question	Superficial	Content

Is 2x and 4x the same?	Identifying	Question	Superficial	Content
Is there a like term to 7xy?	Identifying	Question	Superficial	Content
Can you see a like term to $2x^2y^2$?	Identifying	Question	Superficial	Content
You can't add non like terms	Identifying	Instruction	Analytical	Content
This is a completely different question	Identifying	Instruction	Superficial	Concept
5 squared does not mean five times two	Identifying	Instruction	Superficial	Content
This means 5 x 5 x 5.	Identifying	Instruction	Superficial	Content
They are exactly the same	Identifying	Instruction	Superficial	Content
Adding and subtracting like terms is different to evaluate	Identifying	Instruction	Superficial	Content
I like that you included the minus	Identifying	Instruction	Superficial	Content
That's the difference between what we were doing before and what we are doing now.	Identifying	Instruction	Superficial	Concept
When you add two fractions what needs to be the same?	Identifying	Question	Superficial	Content
They are equivalent fractions.	Identifying	Instruction	Superficial	Content
Recognise the different numbers.	Identifying	Instruction	Superficial	Content
This fraction stays the same	Identifying	Instruction	Analytical	Content
Subtraction is the same	Identifying	Instruction	Superficial	Concept
That's one way of doing it, What's another way of doing it	Identifying	Instruction	Analytical	Content
Are these equivalent, the same thing?	Identifying	Question	Superficial	Content
There are two types of factor trees.	Identifying	Instruction	Superficial	Content
Which is the biggest?	Identifying	Question	Superficial	Content
Are you going ascending or descending?	Identifying	Question	Superficial	Content
Have a look what we did here.	Identifying	Question	Superficial	Concept
You might get something like.	Identifying	Instruction	Superficial	Concept

What ever your amount is here it has to be the same in your answer	Identifying	Instruction	Superficial	Concept
It's still out of 100	Identifying	Instruction	Superficial	Content
Why are they different?	Identifying	Instruction	Analytical	Concept
How do I identify an improper fraction?	Identifying	Question	Superficial	Content
These are all the same as equivalent fractions.	Identifying	Instruction	Superficial	Content
2/5 is the same as 4/10	Identifying	Instruction	Superficial	Content
Can you look at the numbers? What do you notice about the signs? The same digits but different signs.	Identifying	Instruction	Superficial	Content
They look similar.	Identifying	Instruction	Superficial	Content
Whatever you do to the numerator you must do to the denominator.	Recognising	Question	Superficial	Content
Can you see two numbers that have the same factor?	Recognising	Question	Superficial	Content
Can you see any common factors?	Recognising	Question	Superficial	Content
How many fifths do we have in four wholes?	Recognising	Question	Superficial	Content
The first thing I said to you was how many fifths are in 4, so how many two-fifths are there?	Recognising	Question	Superficial	Content
Anything else I can simplify?	Recognising	Question	Superficial	Content
What's the inverse of two	Recognising	Instruction	Analytical	Content
What is our common denominator?	Recognising	Question	Superficial	Content
What are the multiples of 2?	Recognising	Question	Superficial	Content
What is the common multiple of 3?	Recognising	Question	Superficial	Content

What do I multiply the 2 by to get the 6?	Recognising	Question	Superficial	Concept
What do we have to do to the half?	Recognising	Question	Superficial	Concept
What simplifying can I do?	Recognising	Question	Superficial	Content
You know what your denominator is going to be	Recognising	Instruction	Superficial	Content
What do you notice?	Recognising	Question	Analytical	Concept
We are actually using the distributive law in our heads.	Recognising	Instruction	Analytical	Concept
What do I know about the two lines? How do I know that?	Recognising	Question	Superficial	Concept
What type of triangle is that?	Recognising	Question	Superficial	Content
How can you tell this is an isosceles triangle?	Recognising	Question	Superficial	Content
An isosceles triangle has how many sides equal?	Recognising	Question	Superficial	Content
Easiest way to order fractions is to put them with common denominators.	Recognising	Instruction	Superficial	Content
Can you see ….	Recognising	Instruction	Superficial	Concept
What did we say the pattern was?	Recognising	Question	Analytical	Concept
Look for the pattern.	Recognising	Instruction	Analytical	Concept
Let’s look for a pattern.	Recognising	Instruction	Analytical	Concept
What sign is if I go to the left?	Recognising	Question	Superficial	Content
What is the pattern with the minuses?	Recognising	Question	Analytical	Content
Addition of two negative numbers will always give you a negative.	Recognising	Instruction	Superficial	Content

Think of direction left – negative and right – positive.	Recognising	Instruction	Superficial	Content
'Lots of' is multiplication.	Recognising	Instruction	Superficial	Content
These numbers are symmetrical +8 and -8	Recognising	Instruction	Superficial	Content
What's a plus and a minus give us	Recognising	Instruction	Superficial	Content
How many ways can you represent that number?	Recognising	Question	Analytical	Concept