A higher categorical approach to Giraud's non-abelian cohomology

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$$
\begin{tikzcd}
x_{fgh} \cdot k \arrow[r, "x_{fg,h} \cdot k"] \arrow[dr, phantom, "\Downarrow x_{fg,h,k}" near start] & x_{fg} \cdot hk \arrow[d, "x_{f,g} \cdot hk"] & & x_{fgh} \cdot k \arrow[r, "x_{fg,h} \cdot k"] \arrow[dr, "x_{f,gh} \cdot k" description] \arrow[dr, phantom, "\Downarrow x_{f,g,h} \cdot k" near end, shift left=3ex] & x_{fg} \cdot hk \arrow[d, "x_{f,g} \cdot hk"] \\
x_{fghk} \arrow[u, "x_{fgh,k}"] \arrow[ur, "x_{fg,hk}" description] \arrow[r, "x_{f,ghk}"'] \arrow[ur, phantom, "\Downarrow x_{f,g,hk}" near end, shift right=3ex] & x_f \cdot ghk & = & x_{fghk} \arrow[u, "x_{fgh,k}"] \arrow[r, "x_{f,ghk}"'] \arrow[ur, phantom, "\Downarrow x_{f,gh,k}" near start] & x_f \cdot ghk
\end{tikzcd}
$$

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Summary

This thesis continues the program of Ross Street and his collaborators to develop a theory of non-abelian cohomology with higher categories as the coefficient objects. The main goal of this thesis is to show how this theory can be extended to recover Giraud's non-abelian cohomology of degree 2, thereby addressing an open problem posed by Street.

The definition of non-abelian cohomology that is adopted in this thesis is one due to Grothendieck, which takes higher stacks as the coefficient objects; the cohomology is the higher category of global sections of the higher stack. The two approaches of Street and Grothendieck are compared and it is shown how they may be reconciled.

The central argument depends on a generalisation of Lawvere's construction for associated sheaves, which yields the 2-stack of gerbes over a site as an associated 2-stack; the stack of liens (or "bands") is the 1-stack truncation thereof.

Much of the work is dedicated to showing how the coherence theory of tricategories, supplemented by results of three-dimensional monad theory and enriched model category theory, provides a practicable model of the tricategory of indexed bicategories over a site, in whose context the theory can be developed. This tricategory is shown to be triequivalent to a full sub-**Gray**-category of the **Gray**-category of indexed 2-categories over the site, thus permitting the tricategorical analogues of limits, colimits, image factorisation systems, and Grothendieck's plus construction to be modelled by strict constructions of **Gray**-enriched category theory.

Statement

I hereby declare that the work in this thesis has not been submitted for a higher degree to any other university or institution.

Alexander Campbell

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Chapter 1

Introduction

> We leave as a future quest the study of the 2-category $\mathcal{H}(\mathcal{E},\mathcal{G})$ versus the categories $\mathcal{H}^2(\mathcal{E},L)$.
>
> Ross Street [Str04]

This thesis continues the program of Ross Street and his collaborators to develop a theory of non-abelian cohomology with higher categories as the coefficient objects (a key reference for which is [Str04]). The main goal of this thesis is to show how this theory can be extended to recover Giraud's non-abelian cohomology of degree 2 [Gir71], thereby answering the call to the quest of Street quoted in the epigraph above.

However, the definition of non-abelian cohomology that we adopt in this thesis is one due to Grothendieck [Gro83], which takes higher stacks as the coefficient objects. In this introductory chapter we compare these two approaches, and show how they may be reconciled. To begin, we reconsider the definitions of abelian (sheaf) cohomology from a perspective more suggestive of the definitions of non-abelian cohomology which follow.

1.1 Abelian cohomology

The definition of the cohomology groups of a topological space with coefficients in an abelian group may be decomposed into the following steps. We define in turn:

(i) a chain complex CX associated to the topological space X,

(ii) the cochain complex $\mathrm{Hom}(C,A)$ associated to a chain complex C and the abelian group A, and

(iii) the cohomology groups H^nE of a cochain complex E.

The nth cohomology group is then defined to be $H^n(X;A) := H^n\,\mathrm{Hom}(CX,A)$.

We recall each of these definitions in reverse order. First, let E be a cochain complex with differentials $d\colon E^n \longrightarrow E^{n+1}$. The abelian group Z^nE of n-cocycles in E is the kernel of the morphism $d\colon E^n \longrightarrow E^{n+1}$ and the abelian group B^nE of n-coboundaries is the image of the morphism $d\colon E^{n-1} \longrightarrow E^n$. For each integer n, we define the nth cohomology group of E to be the quotient abelian group $H^nE := Z^nE/B^nE$.

Next, given a chain complex C with differentials $\partial\colon C_n \longrightarrow C_{n-1}$ and an abelian group A, we can form the cochain complex $\mathrm{Hom}(C, A)$ with $\mathrm{Hom}(C, A)^n = [C_n, A]$ and differentials $[\partial, 1]\colon [C_n, A] \longrightarrow [C_{n+1}, A]$, where $[-, -]$ denotes the usual internal hom of abelian groups.

Finally, there are a variety of methods by which we can assign a chain complex CX to a topological space X. For instance, given a simplicial set K with face maps $d^n_k\colon K_n \longrightarrow K_{n-1}$, we may define the chain complex $\mathbb{Z}K$, with $(\mathbb{Z}K)_n = \mathbb{Z}(K_n)$ and differentials $\partial\colon \mathbb{Z}K_n \longrightarrow \mathbb{Z}K_{n-1}$ given by $\partial = \sum_{k=0}^n (-1)^k \mathbb{Z}d^n_k$, where $\mathbb{Z}\colon \mathbf{Set} \longrightarrow \mathbf{Ab}$ denotes the free abelian group functor. We may then define CX to be the chain complex $\mathbb{Z}(SX)$, where SX is the simplicial set of singular simplices in X, that is, the simplicial set $\mathbf{Top}(J-, X)\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$, where $J\colon \Delta \longrightarrow \mathbf{Top}$ is the functor that sends $[n]$ to the topological n-simplex.

We may recast this definition as follows. Recall that the category **Ch** of chain complexes is a closed category when equipped with the internal hom $[C, D]$ defined by

$$[C, D]_n = \prod_k [C_k, D_{k+n}] \tag{1.1}$$

with differential $\partial\colon [C, D]_n \longrightarrow [C, D]_{n-1}$ such that

$$(\partial f)_k = \partial f_k + (-1)^{n+1} f_{k-1}\partial. \tag{1.2}$$

Furthermore, recall the following description of the 0th homology group of an internal hom chain complex.

Lemma 1.1.1. *Let C and D be chain complexes. The 0th homology group $H_0[C, D]$ of the internal hom $[C, D]$ is the quotient of the abelian group of chain maps $C \longrightarrow D$ by the null-homotopies.*

Proof. A 0-cycle of $[C, D]$ is an element f of $[C, D]_0 = \prod_k [C_k, D_k]$ such that $\partial f = 0$, that is, $\partial f_k - f_{k-1}\partial = 0$ for all k, that is, $f\colon C \longrightarrow D$ is a chain map. Such an f is a 0-boundary if and only if there exists an s in $[C, D]_1 = \prod_k [C_k, D_{k+1}]$ such that $f_k = \partial s = \partial s_k + s_{k-1}\partial$ for all k, which says precisely that f is null-homotopic.

$$\begin{tikzcd}
\cdots & C_{k-1} \arrow[l, "\partial"'] \arrow[d, "f_{k-1}"'] \arrow[dr, "s_{k-1}"] & C_k \arrow[l, "\partial"'] \arrow[d, "f_k"] \arrow[dr, "s_k"] & C_{k+1} \arrow[l, "\partial"'] \arrow[d, "f_{k+1}"] & \cdots \arrow[l, "\partial"'] \\
\cdots & D_{k-1} \arrow[l, "\partial"] & D_k \arrow[l, "\partial"] & D_{k+1} \arrow[l, "\partial"] & \cdots \arrow[l, "\partial"]
\end{tikzcd}$$

$\square$

Now, for an abelian group A, let $\Sigma^n A$ denote the chain complex

$$\cdots \longleftarrow 0 \longleftarrow A \longleftarrow 0 \longleftarrow \cdots$$

with the only non-zero entry A in the nth place.

Proposition 1.1.2. *Let C be a chain complex and A an abelian group. Then for all n there is an isomorphism of abelian groups*

$$H^n(C;A) \cong H_0[C,\Sigma^n A].$$

Proof. We have $[C,\Sigma^n A]_m = \prod_k [C_k,(\Sigma^n A)_{k+m}] \cong [C_{n-m},A]$, with differential $(\partial f)_{n-m} = (-1)^m f\partial$. So the chain complex $[C,\Sigma^n A]$ is equal, possibly up to an immaterial change of sign, to the reverse ordering of the cochain complex $\operatorname{Hom}(C,A)$ shifted n places to the left. Hence $H^n \operatorname{Hom}(C,A) \cong H_0[C,\Sigma^n A]$.

Alternatively, with reference to Lemma 1.1.1, it is evident from the diagram

$$\begin{tikzcd}
\cdots & C_{n-1} \arrow[l, "\partial"'] \arrow[d] \arrow[dr, "s"] & C_n \arrow[l, "\partial"'] \arrow[d, "f"] \arrow[dr] & C_{n+1} \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & 0 \arrow[l] & A \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

that a chain map $C \longrightarrow \Sigma^n A$ amounts to a single morphism $f\colon C_n \longrightarrow A$ such that $f\partial = 0$, i.e. an n-cocycle of the cochain complex $\operatorname{Hom}(C,A)$, and that such a map is null-homotopic if and only if there exists a morphism $s\colon C_{n-1} \longrightarrow A$ such that $f = s\partial$, i.e. f is an n-coboundary of $\operatorname{Hom}(C,A)$. $\square$

Therefore cocycles can be seen as morphisms in a certain category, and their equivalence relation is a notion of homotopy equivalence suitable to that category. We place emphasis on this interpretation in the following definition.

Definition 1.1.3. Let C and A be chain complexes. The *cohomology of C with coefficients in A* is defined to be the internal hom chain complex

$$\mathscr{H}(C;A) := [C,A].$$

Let X be a topological space with associated chain complex SX. The *cohomology of X with coefficients in A* is defined to be the chain complex

$$\mathscr{H}(X;A) := \mathscr{H}(SX,A) = [SX,A].$$

Note that we prefer to define the cohomology to be the chain complex, being content to consider its 0th homology group as something that may be extracted from this more structured object.

Given this definition of the cohomology of a topological space X, by Proposition 1.1.2 we can recover the nth cohomology group of X with coefficients in an abelian group A by

$$H^n(X;A) \cong H_0\mathscr{H}(X;\Sigma^n A).$$

1.2 Abelian sheaf cohomology

Recall that the nth cohomology group of a Grothendieck topos $\mathcal{X}$ (such as the category $\mathrm{Sh}(X)$ of sheaves of sets on a topological space X) with coefficients in an abelian group A in $\mathcal{X}$ (when $\mathcal{X} = \mathrm{Sh}(X)$, a sheaf of abelian groups on X) is defined to be the value of the nth right derived functor of the global sections functor $\mathrm{Hom}(\mathbb{Z}_{\mathcal{X}}, -)\colon \mathbf{Ab}(\mathcal{X}) \longrightarrow \mathbf{Ab}$ at A,

$$H^n(\mathcal{X};A) := \mathbf{R}^n \mathrm{Hom}(\mathbb{Z}_{\mathcal{X}}, -)(A),$$

where $\mathbb{Z}_{\mathcal{X}}$ denotes the free abelian group on the terminal object of $\mathcal{X}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be Grothendieck abelian categories (such as the category of abelian groups in a Grothendieck topos). Recall that the nth right derived functor $\mathbf{R}^n F$ of an additive functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ can be calculated at an object A of $\mathcal{A}$ by the following procedure. Take an injective resolution of A, that is, a long exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

in $\mathcal{A}$, such that each object I^n is injective. Then $\mathbf{R}^n F(A)$ is isomorphic to the nth cohomology object of the cochain complex $FI^0 \longrightarrow FI^1 \longrightarrow \cdots$ in $\mathcal{B}$.

To recast the definition of abelian sheaf cohomology, we recall how the derived functors of homological algebra relate to the derived functors of model category theory. Given a functor $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ between categories with weak equivalences, the right derived functor of F can be characterised with respect to the weak equivalences by a certain universal property. If $\mathcal{M}$ is, moreover, a model category with functorial factorisations (as it will have, for instance, when it is cofibrantly generated), and F preserves weak equivalences between fibrant objects, the right derived functor $\mathbb{R}F$ of F can be calculated by precomposing with a fibrant replacement functor $R\colon \mathcal{M} \longrightarrow \mathcal{M}$, giving $\mathbb{R}F \cong FR\colon \mathcal{M} \longrightarrow \mathcal{N}$ [Rie14].

The category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in a Grothendieck abelian category $\mathcal{A}$ admits a cofibrantly generated model structure, called the injective model structure, whose weak equivalences are the quasi-isomorphisms, cofibrations are the monomorphisms, and is such that the bounded above complexes of injective objects are among the fibrant objects [Hov99b]. Additionally, any additive functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ between Grothendieck abelian categories lifts to a functor $F_*\colon \mathbf{Ch}(\mathcal{A}) \longrightarrow \mathbf{Ch}(\mathcal{B})$, defined pointwise, which preserves chain homotopy equivalences.

Proposition 1.2.1. *Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between Grothendieck abelian categories with lifting $F_*\colon$ $\mathbf{Ch}(\mathcal{A}) \longrightarrow \mathbf{Ch}(\mathcal{B})$. The nth right derived functor $\mathbf{R}^n F$ of F is isomorphic to the composite*

$$\mathcal{A} \xrightarrow{\Sigma^n} \mathbf{Ch}(\mathcal{A}) \xrightarrow{\mathbb{R}F_*} \mathbf{Ch}(\mathcal{B}) \xrightarrow{H_0} \mathcal{B},$$

where $\mathbb{R}F_*$ is the right derived functor of F_* with respect to quasi-isomorphisms.

Proof. Let A be an object of $\mathcal{A}$. Take an injective resolution

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

of A. Let I be the chain complex

$$\cdots \longleftarrow I^1 \longleftarrow I^0 \longleftarrow 0 \longleftarrow \cdots$$

in $\mathcal{A}$, with I^0 in the nth entry. Since I is a bounded above chain complex of injective objects, it is a fibrant object in the injective model structure on $\mathbf{Ch}(\mathcal{A})$. Furthermore, the chain map $\Sigma^n A \longrightarrow I$ given by

$$\begin{tikzcd}
\cdots & 0 \arrow[l] \arrow[d] & A \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & I^1 \arrow[l] & I^0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

is both a monomorphism and a quasi-isomorphism, that is, a trivial cofibration. Hence I is a fibrant replacement of $\Sigma^n A$ in the injective model structure on $\mathbf{Ch}(\mathcal{A})$.

Since quasi-isomorphisms between fibrant objects in the injective model structure on $\mathbf{Ch}(\mathcal{A})$ are chain homotopy equivalences, F_* preserves weak equivalences between fibrant objects, so its right derived functor can be calculated by evaluating at fibrant replacements. Hence $\mathbb{R}F_*(\Sigma^n A) \cong F_* I$.

Finally, it is evident that the 0th homology object of $F_* I$ is equal to the nth cohomology object of the cochain complex $FI^0 \longrightarrow FI^1 \longrightarrow \cdots$. Therefore, we have the isomorphism

$$\mathbf{R}^n F(A) \cong H_0 \mathbb{R}F_*(\Sigma^n A). \qquad \square$$

Note that the formulas (1.1) and (1.2), mutatis mutandis, define a hom-functor $\mathrm{Hom}_{\mathbf{Ch}}\colon \mathbf{Ch}(\mathcal{A})^{\mathrm{op}} \times \mathbf{Ch}(\mathcal{A}) \longrightarrow \mathbf{Ch}$ for any Grothendieck abelian category $\mathcal{A}$. For any object A of $\mathcal{A}$, the lifting $\mathrm{Hom}(A,-)_*\colon \mathbf{Ch}(\mathcal{A}) \longrightarrow \mathbf{Ch}$ is isomorphic to the hom-functor $\mathrm{Hom}_{\mathbf{Ch}}(A,-)$, where by an abuse of notation we let A denote both the object of $\mathcal{A}$ and its 0-suspension chain complex.

A fundamental principle of model category theory is that for objects A and B of a model category $\mathcal{M}$, the "homotopically correct" notion of map $A \longrightarrow B$ is a morphism in $\mathcal{M}$ from a cofibrant replacement QA of A and to a fibrant replacement RB of B. This claim is instantiated by the fact that in the homotopy category $\operatorname{Ho}\mathcal{M}$, the hom-set $\operatorname{Ho}\mathcal{M}(A, B)$ is isomorphic to the quotient of the hom-set $\mathcal{M}(QA, RB)$ by the relation of homotopy equivalence.

Since every object in $\mathbf{Ch}(\mathcal{A})$ is cofibrant for the injective model structure, we may interpret the value of the right derived functor of $\operatorname{Hom}_{\mathbf{Ch}}(C, -)$ at a chain complex D, which is calculated by taking a fibrant replacement of D, as giving the "homotopically correct" chain complex of maps from C to D. Let us denote this chain complex by $\mathbb{R}\operatorname{Hom}(C, D)$.

Definition 1.2.2. Let C and D be chain complexes in a Grothendieck abelian category $\mathcal{A}$. We define the *cohomology of C with coefficients in D* to be the chain complex

$$\mathscr{H}(C; D) := \mathbb{R}\operatorname{Hom}(C, D).$$

Let $\mathcal{X}$ be a Grothendieck topos, and A a chain complex of abelian groups in $\mathcal{X}$. We define the *cohomology of $\mathcal{X}$ with coefficients in A* to be the chain complex

$$\mathscr{H}(\mathcal{X}; A) := \mathscr{H}(\mathbb{Z}_{\mathcal{X}}, A) = \mathbb{R}\operatorname{Hom}(\mathbb{Z}_{\mathcal{X}}, A).$$

By Proposition 1.2.1, we can recover the nth cohomology group of a Grothendieck topos $\mathcal{X}$ with coefficients in an abelian group A in $\mathcal{X}$ by

$$H^n(\mathcal{X}; A) \cong H_0\mathscr{H}(\mathcal{X}; \Sigma^n A).$$

1.3 Grothendieck's definition of non-abelian cohomology

In his unpublished notes *Pursuing Stacks* [Gro83], Grothendieck discusses the theory of higher stacks over topoi as "the natural foundations of non commutative cohomological algebra", i.e. non-abelian cohomology, and higher stacks as "the natural coefficients" for the non-abelian cohomology of topoi. He defines the cohomology of a topos X with coefficients in a higher stack F over X to be the higher category of (global) sections of F. He says, moreover,

> The notion of a stack here appears as the unifying concept for a synthesis of homotopical algebra and non commutative cohomological algebra. This (rahter [sic] than merely furnishing us with still another description of homotopy types, more convenient for expression of the homotopy groups) seems to

> me the real "raison d'être" of the notion of a stack, and the main motivation for pushing ahead a theory of stacks.

Note that by "stack" he means what we are calling higher stack.

This definition of cohomology of a topos with coefficients in a higher stack is the central definition of this thesis. To give it in more detail, we speak in the informal language of higher category theory. Thus we speak naively of n-categories, n-functors, etc., always meaning the "weak" notions, allowing the possibility $n = \omega$, noting that $\omega + 1 = \omega$. We suppose n-categories to form an $(n+1)$-category n-**Cat**, so that for any n-categories A and B, the n-functors $A \longrightarrow B$ form an n-category $\mathrm{Hom}(A, B)$. In this thesis, these definitions will be made rigorous for $n \leq 2$. Note however that we mean "site" in the usual sense, that is, a 1-category equipped with a Grothendieck topology.

Definition 1.3.1. Let $(\mathcal{C}, J)$ be a site. An *n-stack F* over $(\mathcal{C}, J)$ is an $(n+1)$-functor $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow n\text{-}\mathbf{Cat}$, i.e. an object of $\mathcal{F} = \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, n\text{-}\mathbf{Cat})$, such that for every covering sieve $R \in J(C)$, the canonical n-functor

$$FC \longrightarrow \mathcal{F}(R, F)$$

is an equivalence of n-categories.

Let n-Stacks$(\mathcal{C}, J)$ denote the full sub-$(n+1)$-category of $\mathcal{F}$ on the n-stacks.

Much as sheaves of sets (or abelian groups, chain complexes, etc.) can be thought of as such structures varying over a space, we can think of n-stacks as n-categories varying over a space.

Definition 1.3.2. Let $X = (\mathcal{C}, J)$ be a site. The *cohomology of $(\mathcal{C}, J)$ with coefficients in an n-stack F over $(\mathcal{C}, J)$* is the n-category of global sections of F:

$$\mathscr{H}(X; F) := n\text{-Stacks}(\mathcal{C}, J)(1, F),$$

where 1 denotes the terminal sheaf over $\mathcal{C}$.

We record also the natural definition for the case of "constant" coefficients, analogous to that of Section 1.1.

Definition 1.3.3. Let A and B be n-categories. The *cohomology of A with coefficients in B* is the hom-n-category

$$\mathscr{H}(A; B) := \mathrm{Hom}(A, B).$$

The objects of the cohomology n-category $\mathscr{H}(A; B)$, which we may call n-cocycles, are then none other than n-functors $A \longrightarrow B$.

Compare these definitions to Definitions 1.2.2 and 1.1.3 respectively. The cohomology objects are all defined to be certain hom-objects suitable to their type of coefficient object, which is the essential difference between the definitions; the coefficients of abelian cohomology are (sheaves of) chain complexes, while the coefficients of non-abelian cohomology are (higher stacks of) higher categories.

We can therefore motivate this definition of non-abelian cohomology by recognising higher categories as non-abelian versions of chain complexes. One result supporting this idea is that in a finitely complete additive category $\mathcal{A}$, there is an equivalence between the categories of strict ω-categories in $\mathcal{A}$ and of non-negatively graded chain complexes in $\mathcal{A}$ [Bou90, Cra01, BH03]. In particular, given an abelian group A and an integer $n \geq 0$, there is an n-category $K(A, n)$, which corresponds to the chain complex $\Sigma^n A$ under this equivalence, with only one k-cell for $k = 0, \ldots, n-1$, whose n-cells are the elements of A, and all compositions are given by addition.

Recall that the abelian (sheaf) cohomology groups were recovered as the 0-th homology groups of the cohomology chain complexes which we defined. We could have chosen to define non-abelian cohomology to be the set of connected components of the cohomology higher categories, but again we choose to keep our attention on the more structured object.

In an earlier letter to Breen [Gro75], Grothendieck writes of the benefits of the approach to constructing the cohomology of a topos via a potential theory of stacks against the approaches of derived functors and hypercoverings (Verdier's hypercovering theorem [SGA72b]).

> I do not know if a theory of stacks and of operations on them can be written *without* ever using semi-simplicial algebra. If yes, there would be essentially three distinct approaches for constructing the cohomology of a topos:
>
> (a) viewpoint of complexes of sheaves, injective resolutions, derived categories (*commutative homological algebra*)
>
> (b) viewpoint Čechist or semi-simplicial (*homotopical algebra*)
>
> (c) viewpoint of n-stacks (categorical algebra, or *non-commutative homological algebra*).
>
> In (a) one "resolves" the coefficients, in (b) one resolves the base space (or topos), and in (c) it appears one resolves neither the one nor the other.

Remark 1.3.4. A rigorous definition may be made in the context of the theory of ∞-categories [Lur09], whose terminology we adopt for this remark. In this theory, there is an ∞-category $\mathcal{S}$ of ∞-groupoids (modelled by Kan complexes), called the ∞-category of spaces, and for any site $(\mathcal{C}, J)$, there is an ∞-topos $\mathrm{Shv}(\mathcal{C})$ of sheaves of spaces over

$(\mathcal{C}, J)$, which are certain functors $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}$. We can define the cohomology of an ∞-topos $\mathcal{X}$ with coefficients in a sheaf of spaces F to be the hom-∞-groupoid $\mathrm{Map}_{\mathcal{X}}(1, F)$. In fact, [Lur09, Definition 7.2.2.14] defines the nth cohomology group of an ∞-topos $\mathcal{X}$ with coefficients in an abelian group A in $\mathcal{X}$ to be

$$H^n(\mathcal{X}, A) = \pi_0 \mathcal{X}(1, K(A, n)),$$

where $K(A, n)$ denotes a so-called Eilenberg-MacLane sheaf. [Lur09, Remark 7.2.2.17] shows that when $\mathcal{X}$ is the ∞-topos of sheaves of spaces over a site this agrees with the usual sheaf cohomology groups.

1.4 Street's definition of non-abelian cohomology

We now come to Street's definition of non-abelian cohomology. Note that unlike the previous section, in this section when we speak of n-categories and their attendant notions, we mean the strict versions. The origin of this definition is the observation of Roberts [Rob79] that n-categories are the natural coefficients for non-abelian cohomology, as they provide the natural algebraic structure in which the n-cocycle conditions can be expressed. We recall the motivation for this observation given in [Str87].

Recall that for a simplicial set X with face maps $d_k \colon X_{n+1} \longrightarrow X_n$ and an abelian group A, an n-cocycle on X with coefficients in A is a function $f \colon X_n \longrightarrow A$ such that

$$\partial f = \sum_{k=0}^{n+1} (-1)^k f d_k = f d_0 - f d_1 + \cdots + (-1)^{n+1} f d_{n+1} = 0.$$

The 1-cocycle condition is an equation of the form $a_0 - a_1 + a_2 = 0$ in an abelian group. Rewriting this as $a_1 = a_0 + a_2$ removes the need for inverses and so gives an equation which can be expressed in a monoid, or more generally in a category, in which the 1-cocycle condition becomes $a_1 = a_0 \circ a_2$, which is the commutative triangle

$$\begin{tikzcd} \cdot \arrow[rr, "a_1"] \arrow[dr, "a_2"'] & & \cdot \\ & \cdot \arrow[ur, "a_0"'] \arrow[u, phantom, "="] & \end{tikzcd} .$$

Similarly, the 2-cocycle condition $a_0 - a_1 + a_2 - a_3 = 0$ may be generalised first to the equation $a_3 + a_1 = a_0 + a_2$ and then to the equation between composite 2-cells

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr] \arrow[r, phantom, shift right=4ex, "\Downarrow a_1"] & \cdot \\ \cdot \arrow[r] \arrow[u, phantom, shift left=4ex, "\Downarrow a_3"] & \cdot \arrow[u] \end{tikzcd} \quad = \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[d, phantom, shift left=4ex, "\Downarrow a_2"] & \cdot \\ \cdot \arrow[r] \arrow[ur] & \cdot \arrow[u] \arrow[u, phantom, shift right=4ex, "\Downarrow a_0"] \end{tikzcd}$$

which is a commutative tetrahedron in a 2-category.

Generally, the n-cocycle condition should be a commutative $(n+1)$-simplex in an n-category. In [Str87], Street defined a functor $\mathcal{O}\colon \Delta \longrightarrow \omega\text{-}\mathbf{Cat}$, sending $[n]$ to the nth oriental $\mathcal{O}^n$, thought of as the free n-category on the n-simplex. He then defined an $(n+1)$-cocycle in an n-category A to be an $(n+1)$-functor $\mathcal{O}^{n+1} \longrightarrow A$, which necessarily sends the non-trivial $(n+1)$-cell of $\mathcal{O}^{n+1}$ to an identity in A.

Moreover, Roberts [Rob79] suggested that n-cocycles should form an n-category, and he gave definitions for $n = 1, 2$. (Duskin [Dus89] gave a similar definition of a 2-category of 2-cocycles for a simplicial object in a category $\mathcal{C}$ valued in a functor $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$.) Using the theory of parity complexes, Street [Str91] was able to give a general definition for $n = \omega$, whose ingredients we now recall.

The lax Gray tensor product of ω-categories is the unique biclosed monoidal structure on $\omega\text{-}\mathbf{Cat}$ for which $I^n \otimes I^m \cong I^{n+m}$, where I^n denotes the free ω-category on the parity n-cube [Str10]. The left internal hom $[A, B]$ consists of ω-functors $A \longrightarrow B$, oplax natural transformations between these, and higher oplax transfors between these [Cra95].

Recall the universal co-ω-category $\mathbf{2}_*$ in $\omega\text{-}\mathbf{Cat}$, with underlying co-globular ω-category

$$\mathbf{2}_0 \rightrightarrows \mathbf{2}_1 \rightrightarrows \mathbf{2}_2 \rightrightarrows \cdots$$

where $\mathbf{2}_n$ is the free n-cell, that is, $\mathbf{2}_n$ represents the functor $\omega\text{-}\mathbf{Cat} \longrightarrow \mathbf{Set}$ which sends an ω-category to its set of n-cells. For any ω-category A, the globular set $\omega\text{-}\mathbf{Cat}(\mathbf{2}_*, A)$ inherits an ω-category structure which makes it isomorphic to A.

Since ω-categories are the algebras for a finite limit sketch, and the lax Gray tensor product preserves colimits in each variable, the pointwise tensor product $\mathbf{2}_* \otimes \mathcal{O}$ is a co-ω-category in the functor category $[\Delta, \omega\text{-}\mathbf{Cat}]$. Hence for any functor $E\colon \Delta \longrightarrow \omega\text{-}\mathbf{Cat}$, the hom-set $[\Delta, \omega\text{-}\mathbf{Cat}](\mathbf{2}_* \otimes \mathcal{O}, E)$ inherits an ω-category structure.

Definition 1.4.1. Let $E\colon \Delta \longrightarrow \omega\text{-}\mathbf{Cat}$ be a cosimplicial ω-category. The *descent ω-category* of E is the ω-category

$$\mathrm{Desc}\, E = [\Delta, \omega\text{-}\mathbf{Cat}](\mathbf{2}_* \otimes \mathcal{O}, E).$$

Definition 1.4.2. Let $\mathcal{C}$ be a finitely complete category and let X be a simplicial object and A an ω-category in $\mathcal{C}$. The *cohomology ω-category of X with coefficients in A* is the descent ω-category of the cosimplicial ω-category $\mathcal{C}(X, A)\colon \Delta \longrightarrow \omega\text{-}\mathbf{Cat}$,

$$\mathscr{H}(X; A) := \mathrm{Desc}\, \mathcal{C}(X, A).$$

Restricting to the case $\mathcal{C} = \mathbf{Set}$, we can recast this definition as follows.

Lemma 1.4.3. *Let $E\colon \Delta \longrightarrow \omega\text{-}\mathbf{Cat}$ be a cosimplicial ω-category. There is an isomor-*

phism of ω-categories

$$\operatorname{Desc} E \cong \int_n [\mathcal{O}^n, E^n],$$

*where the end is taken in ω-**Cat**.*

Proof. We have the isomorphisms

$$\begin{aligned}
\operatorname{Desc} E &= [\Delta, \omega\text{-}\mathbf{Cat}](\mathbf{2}_* \otimes \mathcal{O}, E) \\
&\cong \int_n \omega\text{-}\mathbf{Cat}(\mathbf{2}_* \otimes \mathcal{O}^n, E^n) \\
&\cong \int_n \omega\text{-}\mathbf{Cat}(\mathbf{2}_*, [\mathcal{O}^n, E^n]) \\
&\cong \omega\text{-}\mathbf{Cat}\left(\mathbf{2}_*, \int_n [\mathcal{O}^n, E^n]\right) \\
&\cong \int_n [\mathcal{O}^n, E^n].
\end{aligned}$$

□

Recall that the nerve functor $N\colon \omega\text{-}\mathbf{Cat} \longrightarrow [\Delta^{\mathrm{op}}, \mathbf{Set}]$ induced by the orientals has a left adjoint $L\colon [\Delta^{\mathrm{op}}, \mathbf{Set}] \longrightarrow \omega\text{-}\mathbf{Cat}$ given by left Kan extension, or equivalently, by the coend formula

$$LX \cong \int^n X_n \times \mathcal{O}^n.$$

Proposition 1.4.4. *Let X be a simplicial set and A an ω-category. Then there is an isomorphism of ω-categories*

$$\mathscr{H}(X; A) \cong [LX, A].$$

Proof. Note that since each X_n is discrete, we have the natural isomorphisms $X_n \otimes - \cong X_n \times -$ and $[X_n, -] \cong \operatorname{Hom}(X_n, -)$, where Hom denotes the cartesian internal hom ω-category. Hence we have the isomorphisms

$$\begin{aligned}
\mathscr{H}(X; A) &= \operatorname{Desc} \operatorname{Hom}(X, A) \\
&\cong \int_n [\mathcal{O}^n, \operatorname{Hom}(X_n, A)] \\
&\cong \int_n [\mathcal{O}^n, [X_n, A]] \\
&\cong \int_n [\mathcal{O}^n \otimes X_n, A] \\
&\cong \int_n [\mathcal{O}^n \times X_n, A] \\
&\cong \left[\int^n \mathcal{O}^n \times X_n, A\right] \\
&\cong [LX, A].
\end{aligned}$$

□

We are thus led to the following definition, which supplements Street's account.

Definition 1.4.5. Let A and B be ω-categories. The *cohomology ω-category of A with coefficients in B* is defined to be the left internal hom for the lax Gray tensor product

$$\mathscr{H}(A;B) := [A,B].$$

By Proposition 1.4.4, we may recover the cohomology ω-category of a simplicial set X with coefficients in an ω-category A as the cohomology ω-category $\mathscr{H}(LX;A) = [LX,A]$ of Definition 1.4.5.

We have separated Street's definition into two parts: (i) one which assigns an ω-category to a simplicial set, and (ii) one which defines the cohomology ω-category of an ω-category with coefficients in another ω-category. Compare these to steps (i) and (ii) in the definition of abelian cohomology in Section 1.1.

The motivation we gave for Street's definition was devoted to the n-categorical expression of the n-cocycle condition. However this relates purely to part (i). Hence we can consider Proposition 1.4.4 as having isolated the essential categorical aspect of Street's definition of non-abelian cohomology. We can more abstractly define a cocycle on an ω-category A valued in an ω-category B to be merely an ω-functor $A \longrightarrow B$, as we did for weak n-categories in Definition 1.3.3.

Now, this definition, dealing as it does with strict ω-categories, is merely a template for a more important definition in the context of weak ω-categories. In [Str04], after giving the above definition, Street goes on to outline how a similar definition of cohomology may be made for weak ω-categories. Nevertheless, the following comments apply to both versions.

Despite their formal similarity, Street's and Grothendieck's definitions are somewhat at odds. The Gray hom $[A,B]$ consists of strict ω-functors, oplax transformations and oplax higher transfors, whereas the hom $\mathrm{Hom}(A,B)$ of Definition 1.3.3 consists of weak ω-functors, weak (or pseudo) natural transformations and weak higher transfors.

Furthermore, LX is not necessarily the ω-category that one might want to associate to a simplicial set X. For if X is the nerve of an ω-category A, then LX is the normal oplax functor classifier of A, which is not equivalent to A. So we can expect that for a simplicial set X which models an ∞-groupoid, the ω-category LX will not be equivalent to that ∞-groupoid.

For these reasons, we reiterate that it is the definitions of Section 1.3 that we adopt in this thesis.

Finally, note that the last section of [Str04] implies a definition of the non-abelian cohomology of a topos $\mathcal{E}$, with coefficient object a functor $\mathcal{E}^{\mathrm{op}} \longrightarrow \omega\text{-}\mathbf{Cat}$. Let $\mathcal{C} = [\mathcal{E}^{\mathrm{op}}, \mathbf{Set}]$, so that A is an ω-category in $\mathcal{C}$. The cohomology of $\mathcal{E}$ with coefficients in A is taken to be the colimit of the cohomology ω-categories $\mathscr{H}(R;A)$, where the colimit is taken over the hypercovers of $\mathcal{E}$, which are certain simplicial objects in $\mathcal{E}$ and hence in

$\mathcal{C} = [\mathcal{E}^{\mathrm{op}}, \mathbf{Set}]$.

We choose to interpret this implicit definition, in light of the results of [DHI04], as an attempt to construct the ω-category of global sections of the associated ω-stack of A over $\mathcal{E}$, in line with Definition 1.3.2, though due in part to the caveats mentioned above we do not expect this to be precisely the result of this construction.

Chapter 2

The central argument

In the previous chapter we gave definitions of abelian cohomology that allowed (sheaves of) chain complexes as the coefficient objects, from which we recovered the usual cohomology groups by taking suitable coefficient chain complexes. Similarly, to recover Giraud's non-abelian cohomology of degree 2 from Grothendieck's definition, we must find suitable coefficient objects, which in this case are 2-stacks. These 2-stacks will be produced from indexed bicategories by the following principle.

A fundamental desideratum of any theory of higher stacks is that n-stacks over a site $(\mathcal{C}, J)$ should form a left exact localisation of indexed (weak) n-categories over $\mathcal{C}$. That is, the inclusion of $(n+1)$-categories

$$\begin{tikzcd} n\text{-Stacks}(\mathcal{C}, J) \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, n\text{-}\mathbf{Cat}) \arrow[l, shift right, "L"'] \end{tikzcd}$$

should have a finite limit preserving left adjoint L (in a higher categorical sense). We say that L sends an indexed n-category to its associated n-stack. The existence and exactness of such an L has been proved for the cases $n = 0$ [SGA72a] and $n = 1$ [Str82b, Str82a]; our Chapter 3 is dedicated to the proof of the case $n = 2$.

In this chapter we present a general method for calculating non-abelian cohomology with coefficients in associated n-stacks, inspired by Lawvere's construction for associated sheaves, which we now recall.

2.1 Lawvere's construction

Grothendieck's proof [SGA72a] of the existence and exactness of the associated sheaf functor uses the "plus construction", which sends a presheaf F on a site $(\mathcal{C}, J)$ to the presheaf F^+ given by

$$F^+C := \operatorname*{colim}_{R \in J(C)^{\mathrm{op}}} [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](R, F).$$

There is a natural comparison map $F \longrightarrow F^+$, and two applications yield the associated sheaf of F.

Lawvere [Law71] gave an alternative construction for associated sheaves in the context of elementary topos theory, where the external infinite colimits of Grothendieck's construction are not available. We recall this construction following [Joh77].

Let $\mathscr{E}$ be an elementary topos with a topology $j\colon \Omega \longrightarrow \Omega$. The image of the morphism j, denoted by Ω_j, classifies j-closed subobjects. Hence, for any object A of $\mathscr{E}$, there is a canonical morphism $A \longrightarrow \Omega_j^A$ corresponding to the j-closure of the diagonal $A \rightarrowtail A \times A$. Take the image MA of this morphism, i.e. its (epi, mono)-factorisation, and let LA be the j-closure thereof.

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, two heads] \arrow[dr, phantom, shift left=4, "||"] & & \Omega_j^A \\
& MA \arrow[ur, tail] \arrow[r, tail] \arrow[r, phantom, shift left=4, "="] & LA \arrow[u, tail]
\end{tikzcd}$$

The composite $A \longrightarrow LA$ witnesses LA as the associated j-sheaf of A.

For our purposes, we suppose the existence and exactness of the associated sheaf functor L for a site $(\mathcal{C}, J)$ to have been established via Grothendieck's construction. Lawvere's construction then provides an additional method for analysing the associated sheaf of a presheaf.

The essential ingredients of Lawvere's construction are the (epi, mono)-factorisation of a morphism and the closure of a subobject. Once we have the associated sheaf functor L, the closure of a subobject $m\colon A \rightarrowtail B$ can be constructed as the pullback

$$\begin{tikzcd}
A \arrow[drr, tail, bend left, "m"] \arrow[dr, tail] \arrow[ddr, bend right, "\eta_A"'] & & \\
& \overline{A} \arrow[r, tail, "\overline{m}"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "\eta_B"] \\
& LA \arrow[r, tail, "Lm"'] & LB
\end{tikzcd}$$

where $\eta\colon 1 \longrightarrow L$ is the unit for the reflection.

This suggests the possibility of generalising Lawvere's construction from an elementary topos with a topology and its (epi, mono) factorisation system to an arbitrary (finitely complete) category with a left exact localisation and a factorisation system. We will see in Theorem 2.1.11 that such a generalisation can be made, assuming a property relating the localisation and the factorisation system.

We first recall the definition and elementary properties of factorisation systems [FK72]. (We spell out these elementary arguments here both for reference and for the benefit of Section 4.2, where they will be generalised to the tricategorical setting.) We make repeated use of the famous "pasting lemma" for pullbacks.

Lemma 2.1.1. *Consider a commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

in a category such that the right-hand square is a pullback. The left-hand square is a pullback if and only if the outer rectangle is a pullback.

Definition 2.1.2. Let $f\colon A \longrightarrow B$ and $g\colon X \longrightarrow Y$ be morphisms in a category $\mathcal{C}$. We say that f and g are *orthogonal*, written $f \perp g$, if the square

$$\begin{tikzcd} \mathcal{C}(B,X) \arrow[r, "{(1,g)}"] \arrow[d, "{(f,1)}"'] \arrow[dr, phantom, "="] & \mathcal{C}(B,Y) \arrow[d, "{(f,1)}"] \\ \mathcal{C}(A,X) \arrow[r, "{(1,g)}"'] & \mathcal{C}(A,Y) \end{tikzcd}$$

is a pullback. That is, for every commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, dotted, "\exists!" description] & Y \end{tikzcd}$$

there exists a unique diagonal filler $B \longrightarrow X$.

Lemma 2.1.3. *A morphism f such that $f \perp f$ is necessarily an isomorphism.*

Proof. The diagonal filler g in the commutative square

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "1"'] \arrow[ur, dotted, "g" description] & B \end{tikzcd}$$

satisfies $gf = 1$ and $fg = 1$, and is therefore the inverse of f. □

Lemma 2.1.4. *Let $\mathcal{J}$ be a class of morphisms in a category $\mathcal{C}$, and let $\mathcal{R}$ be the class of morphisms f in $\mathcal{C}$ such that $j \perp f$ for all $j \in \mathcal{J}$. Then $\mathcal{R}$ has the following properties.*

(i) $\mathcal{R}$ is closed under composition and contains the isomorphisms,

(ii) $\mathcal{R}$ is stable under pullback, and

(iii) if gf and g both belong to $\mathcal{R}$, then so does f.

Dually, the class $\mathcal{L}$ of morphisms f such that $f \perp j$ for all $j \in \mathcal{J}$ has the following properties.

(i) $\mathcal{L}$ is closed under composition and contains the isomorphisms,

(ii) $\mathcal{L}$ is stable under pushout, and

(iii) if gf and f both belong to $\mathcal{L}$, then so does g.

Proof. It suffices to prove these properties with respect to a single morphism $j\colon A \longrightarrow B$ in $\mathcal{J}$.

Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be a composable pair of morphisms, with $g \in \mathcal{R}$. Then, by the pasting lemma for pullbacks, in the diagram

$$\begin{tikzcd}
\mathcal{C}(B,X) \arrow[r, "{(1,f)}"] \arrow[d, "{(j,1)}"'] \arrow[rr, bend left, "{(1,gf)}"] \arrow[dr, phantom, "="] & \mathcal{C}(B,Y) \arrow[r, "{(1,g)}"] \arrow[d, "{(j,1)}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{C}(B,Z) \arrow[d, "{(j,1)}"] \\
\mathcal{C}(A,X) \arrow[r, "{(1,f)}"'] \arrow[rr, bend right, "{(1,gf)}"'] & \mathcal{C}(A,Y) \arrow[r, "{(1,g)}"'] & \mathcal{C}(A,Z)
\end{tikzcd}$$

the left-hand square is a pullback if and only if the outer rectangle is. Hence $f \in \mathcal{R}$ if and only if $gf \in \mathcal{R}$. This proves both the cancellation property (iii) and that $\mathcal{R}$ is closed under composition.

An isomorphism is orthogonal to any morphism since any commutative square of the form

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "\cong"'] \arrow[dr, phantom, "="] & X \arrow[d, "\cong"] \\
Y \arrow[r] & Z
\end{tikzcd}$$

with two opposite sides isomorphisms is a pullback.

Now, let g be a pullback of a morphism $f \in \mathcal{R}$.

$$\begin{tikzcd}
U \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, very near start, "\lrcorner"] & V \arrow[d, "v"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

It follows from the equation

$$\begin{tikzcd}
\mathcal{C}(B,U) \arrow[r, "{(1,g)}"] \arrow[d, "{(j,1)}"'] \arrow[dr, phantom, "="] & \mathcal{C}(B,V) \arrow[d, "{(j,1)}"] \\
\mathcal{C}(A,U) \arrow[r, "{(1,g)}"'] \arrow[d, "{(1,u)}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{C}(A,V) \arrow[d, "{(1,v)}"] \\
\mathcal{C}(A,X) \arrow[r, "{(1,f)}"'] & \mathcal{C}(A,Y)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C}(B,U) \arrow[r, "{(1,g)}"] \arrow[d, "{(1,u)}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{C}(B,V) \arrow[d, "{(1,v)}"] \\
\mathcal{C}(B,X) \arrow[r, "{(1,f)}"'] \arrow[d, "{(j,1)}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{C}(B,Y) \arrow[d, "{(j,1)}"] \\
\mathcal{C}(A,X) \arrow[r, "{(1,f)}"'] & \mathcal{C}(A,Y)
\end{tikzcd}$$

and from the pasting lemma for pullbacks that $j \perp g$. Hence $g \in \mathcal{R}$. □

Lemma 2.1.5. *Let $L \dashv R \colon \mathcal{A} \longrightarrow \mathcal{C}$ be an adjunction. For morphisms f in $\mathcal{C}$ and g in $\mathcal{A}$, $Lf \perp g$ in $\mathcal{A}$ if and only if $f \perp Rg$ in $\mathcal{C}$.*

Proof. It is evident from the natural isomorphisms $\mathcal{A}(LX, Y) \cong \mathcal{C}(X, RY)$ and from the pasting lemma for pullbacks that one square is a pullback if and only if the other square is so.

$$\begin{tikzcd} \mathcal{A}(LD,A) \arrow[r, "{(1,g)}"] \arrow[d, "{(Lf,1)}"'] \arrow[dr, phantom, "="] & \mathcal{A}(LD,B) \arrow[d, "{(Lf,1)}"] \\ \mathcal{A}(LC,A) \arrow[r, "{(1,g)}"'] & \mathcal{A}(LC,B) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(D,RA) \arrow[r, "{(1,Rg)}"] \arrow[d, "{(f,1)}"'] \arrow[dr, phantom, "="] & \mathcal{C}(D,RB) \arrow[d, "{(f,1)}"] \\ \mathcal{C}(C,RA) \arrow[r, "{(1,Rg)}"'] & \mathcal{C}(C,RB) \end{tikzcd}$$

□

Definition 2.1.6. Let $\mathcal{C}$ be a category. A *factorisation system* on $\mathcal{C}$ consists of two classes $(\mathcal{E}, \mathcal{M})$ of morphisms in $\mathcal{C}$ such that

(i) $\mathcal{E}$ and $\mathcal{M}$ are both closed under composition and contain the isomorphisms,

(ii) every morphism f in $\mathcal{C}$ factors as $f = me$ for some $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and

(iii) $e \perp m$ for every $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

Moreover, a factorisation system is *stable* if the class $\mathcal{E}$ is stable under pullbacks.

Lemma 2.1.7. *Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on a category $\mathcal{C}$. Then for any morphism f in $\mathcal{C}$, $f \perp \mathcal{M}$ if and only if $f \in \mathcal{E}$; dually, $\mathcal{E} \perp f$ if and only if $f \in \mathcal{M}$.*

Proof. Let $f \perp \mathcal{M}$ have factorisation $f = me$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$. Then there exists a diagonal filler s in the left-hand square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & C \arrow[d, "m"] \\ B \arrow[r, "1"'] \arrow[ur, dotted, "s"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "e"] \arrow[d, "e"'] \arrow[dr, phantom, "="] & C \arrow[d, "m"] \\ C \arrow[r, "m"'] & B \end{tikzcd}$$

with $sf = e$ and $ms = 1$. Then since $sme = sf = e$ and $msm = 1m = m$, both 1 and sm are diagonal fillers for the right-hand square. Hence $sm = 1$. Therefore m is an isomorphism, and so $f = me$ belongs to $\mathcal{E}$. □

Hence the classes $(\mathcal{E}, \mathcal{M})$ of a factorisation system enjoy the properties of the classes $(\mathcal{L}, \mathcal{R})$ from Lemma 2.1.4. Furthermore, the factorisations of any given morphism are unique up to a unique isomorphism.

Lemma 2.1.8. *Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on a category $\mathcal{C}$. Suppose a morphism f in $\mathcal{C}$ has factorisations $f = me = m'e'$, with $e, e' \in \mathcal{E}$ and $m, m' \in \mathcal{M}$. Then there* *exists a unique comparison morphism making the diagram*

$$\begin{tikzcd} & C \arrow[dr, "m"] \arrow[dd, dotted, "\exists!"] & \\ A \arrow[ur, "e"] \arrow[dr, "e'"'] \arrow[r, phantom, "="] & & B \arrow[l, phantom, "="] \\ & D \arrow[ur, "m'"'] & \end{tikzcd}$$

commute, which is, moreover, an isomorphism.

Proof. Since $e \perp m'$, there exists a diagonal filler s in the square

$$\begin{tikzcd} A \arrow[r, "e'"] \arrow[d, "e"'] & D \arrow[d, "m'"] \\ C \arrow[r, "m"'] \arrow[ur, dotted, "s"] & B \end{tikzcd}$$

which, by cancellation, belongs to $\mathcal{E} \cap \mathcal{M}$. Hence $s \perp s$, and so s is an isomorphism. □

For the following sequence of results, up to and including Theorem 2.1.14, let $\mathcal{C}$ be a category with pullbacks and with a reflective subcategory $\mathcal{A}$ whose reflector $L: \mathcal{C} \longrightarrow \mathcal{A}$ preserves pullbacks. We denote the unit by $\eta: 1 \longrightarrow L$. Without loss of generality, we may assume that the counit is an identity.

Definition 2.1.9. A morphism $f: A \longrightarrow B$ in $\mathcal{C}$ is

(i) a *local isomorphism* if Lf is an isomorphism,

(ii) *cartesian* if the naturality square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\eta_B"] \\ LA \arrow[r, "Lf"'] & LB \end{tikzcd}$$

is a pullback.

Note that a morphism is cartesian in the sense of the above definition if and only if it is cartesian for the functor L in the sense of the theory of fibrations.

Recall the following construction from [CHK85].

Proposition 2.1.10. *The classes (local isomorphism, cartesian) form a stable factorisation system on $\mathcal{C}$.*

Proof. Since L preserves composition and isomorphisms, the class of local isomorphisms is closed under composition and contains the isomorphisms. Any isomorphism is cartesian, and for a composable pair of cartesian morphisms f and g, the outer rectangle in the following diagram is a pullback.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "g"] \arrow[d, "\eta_B"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\eta_C"] \\
LA \arrow[r, "Lf"'] & LB \arrow[r, "Lg"'] & LC
\end{tikzcd}$$

Note that a morphism is cartesian if and only if it is a pullback of a morphism in the subcategory $\mathcal{A}$. For, by definition, a cartesian morphism f is a pullback of Lf. Conversely, suppose f is the pullback of a morphism g in $\mathcal{A}$. Since L preserves this pullback,

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\
B \arrow[r, "v"'] & Y
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & LA \arrow[r, "Lu"] \arrow[d, "Lf"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\
B \arrow[r, "\eta_B"'] & LB \arrow[r, "Lv"'] & Y
\end{tikzcd}$$

by the pasting lemma, the naturality square for f is a pullback, and so f is cartesian.

The factorisation of a morphism $f\colon A \longrightarrow B$ can be constructed by taking the pullback of Lf along η_B, as in the diagram.

$$\begin{tikzcd}
A \arrow[dr, "g"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\eta_A"'] & & \\
& C \arrow[r, "h"] \arrow[d, "k"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\eta_B"] \\
& LA \arrow[r, "Lf"'] & LB
\end{tikzcd}$$

Applying L to this diagram,

$$\begin{tikzcd}
LA \arrow[r, "Lg"] \arrow[dr, "1"'] & LC \arrow[r, "Lh"] \arrow[d, "Lk"] \arrow[dr, phantom, "\lrcorner", very near start] & LB \arrow[d, "1"] \\
& LA \arrow[r, "Lf"'] & LB
\end{tikzcd}$$

we find that Lk is an isomorphism, since it is the pullback of an isomorphism. Hence Lg is an isomorphism. Moreover, since h is a pullback of Lf, it is cartesian. Therefore $f = hg$ is a (local isomorphism, cartesian) factorisation of f.

Let $f\colon A \longrightarrow B$ be a local isomorphism and let $g\colon X \longrightarrow Y$ be cartesian. Then Lf is an isomorphism, so $Lf \perp Lg$. But Lg is a morphism in the subcategory $\mathcal{A}$, so by adjointness we have $f \perp Lg$. Then, since g is a pullback of Lg, we have that $f \perp g$.

Finally, the pullback of a local isomorphism is a local isomorphism, since L preserves pullbacks and the pullback of an isomorphism is an isomorphism. □

Theorem 2.1.11. *Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on $\mathcal{C}$ such that $L(\mathcal{M}) \subseteq \mathcal{M}$. Then there exists a factorisation system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ such that, for a morphism f with*

$(\mathcal{E},\mathcal{M})$-factorisation $f = me$,

(i) $f \in \mathcal{L}$ if and only if m is a local isomorphism,

(ii) $f \in \mathcal{R}$ if and only if $f \in \mathcal{M}$ and f is cartesian.

Moreover, if $(\mathcal{E},\mathcal{M})$ is stable then so is $(\mathcal{L},\mathcal{R})$.

Proof. Since both $\mathcal{M}$ and the class of cartesian morphisms are closed under composition and contain the isomorphisms, $\mathcal{R}$ also enjoys these properties. For an isomorphism f, $f = f1$ is an $(\mathcal{E},\mathcal{M})$-factorisation, so $f \in \mathcal{L}$, since L preserves isomorphisms.

To show that $\mathcal{L}$ is closed under composition, let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ be a composable pair of morphisms in $\mathcal{L}$ with $(\mathcal{E},\mathcal{M})$-factorisations as in the diagram.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \arrow[rr, "g"] \arrow[dr, "e'"'] \arrow[dd, phantom, "="] & & C \\
& D \arrow[ur, "m"'] \arrow[dr, "e''"'] \arrow[u, phantom, "="] & & E \arrow[ur, "m'"'] \arrow[u, phantom, "="] & \\
& & F \arrow[ur, "m''"'] & &
\end{tikzcd}$$

Take the $(\mathcal{E},\mathcal{M})$-factorisation of $e'm$ as indicated. Then, since L preserves the class $\mathcal{M}$, we have $e' \perp Lm''$ and $e'' \perp Lm''$, which by adjointness implies $Le' \perp Lm''$ and $Le'' \perp Lm''$. Since f belongs to $\mathcal{L}$, Lm is an isomorphism, so we have $Lm \perp Lm''$. Hence by composition, $Lm''Le'' = Le'Lm \perp Lm''$. So by cancellation, we have $Lm'' \perp Lm''$. Hence Lm'' is an isomorphism. Since $(m'm'')(e''e)$ is the $(\mathcal{E},\mathcal{M})$-factorisation of gf, we have therefore that $gf \in \mathcal{L}$.

The $(\mathcal{L},\mathcal{R})$-factorisation of a morphism $f\colon A \longrightarrow B$ can be constructed by first taking its $(\mathcal{E},\mathcal{M})$-factorisation $f = me$, and then the (local isomorphism, cartesian)-factorisation of m. The latter is constructed by the pullback

$$\begin{tikzcd}
C \arrow[dr, "g"'] \arrow[drr, bend left, "m"] \arrow[ddr, bend right, "\eta_C"'] & & \\
& D \arrow[r, "h"] \arrow[d, "k"] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "\eta_B"] \\
& LC \arrow[r, "Lm"'] & LB
\end{tikzcd}$$

We have that h is cartesian, and since it is the pullback of $Lm \in \mathcal{M}$, it belongs to $\mathcal{M}$. Then since both hg and h belong to $\mathcal{M}$, by cancellation we have $g \in \mathcal{M}$. Hence the $\mathcal{M}$-part of the composite ge is a local isomorphism, i.e. $ge \in \mathcal{L}$. Therefore $f = h(ge)$ is the $(\mathcal{L},\mathcal{R})$-factorisation of f.

Now let $l\colon A \longrightarrow B$ and $r\colon X \longrightarrow Y$ belong to $\mathcal{L}$ and $\mathcal{R}$ respectively, and let l have $(\mathcal{E},\mathcal{M})$-factorisation $l = me$. Since m is a local isomorphism and r is cartesian, we have that $m \perp r$, and since $e \in \mathcal{E}$ and $r \in \mathcal{M}$, we have that $e \perp r$. Hence, by composition, $l \perp r$.

Suppose $\mathcal{E}$ is stable under pullbacks. Let $l \in \mathcal{L}$ have $(\mathcal{E}, \mathcal{M})$-factorisation $l = me$. Then the pullback l' of l along some morphism f can be given as in the following diagram.

$$\begin{tikzcd}
E \arrow[r, "e'"] \arrow[d] \arrow[rr, bend left, "l'"] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[r, "m'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
A \arrow[r, "e"'] \arrow[rr, bend right, "l"'] & C \arrow[r, "m"'] & B
\end{tikzcd}$$

We have that $e' \in \mathcal{E}$, $m' \in \mathcal{M}$, and m' is a local isomorphism, since all three classes are stable under pullbacks. Hence the $\mathcal{M}$-part of the pullback of l is a local isomorphism, and so the pullback of l is in $\mathcal{L}$. $\square$

Remark 2.1.12. We say that a morphism belonging to $\mathcal{L}$ is a "local $\mathcal{E}$", or "locally $\mathcal{E}$". For example, if $\mathcal{E}$ is the class of "essentially surjective" morphisms, then $\mathcal{L}$ is the class of "locally essentially surjective" morphisms.

Example 2.1.13. For $\mathcal{C}$ an elementary topos with a topology j, L the associated j-sheaf functor, and $(\mathcal{E}, \mathcal{M})$ the (epi, mono) factorisation system, Lawvere's construction is the $(\mathcal{L}, \mathcal{R})$-factorisation of the canonical morphism $A \longrightarrow \Omega_j^A$.

As in this motivating example, we can use the $(\mathcal{L}, \mathcal{R})$ factorisation system to give an alternative construction of the reflection LA of an object A.

Theorem 2.1.14. *Let $f\colon A \longrightarrow B$ be a morphism in $\mathcal{C}$ such that $B \in \mathcal{A}$ and $Lf \in \mathcal{M}$. Then the $(\mathcal{L}, \mathcal{R})$-image of f is the reflection of A.*

Proof. Note that since every morphism in $\mathcal{R}$ is cartesian, we have by orthogonality that every local isomorphism is in $\mathcal{L}$. Therefore in the naturality square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "="] & B \arrow[d, "1"] \\
LA \arrow[r, "Lf"'] & B
\end{tikzcd}$$

we have $\eta_A \in \mathcal{L}$. By assumption, Lf is in $\mathcal{M}$ and moreover it is cartesian, since it is in the subcategory $\mathcal{A}$. Hence $f = Lf \circ \eta_A$ is an $(\mathcal{L}, \mathcal{R})$-factorisation of f. By Lemma 2.1.8, for any $(\mathcal{L},\mathcal{R})$-factorisation $f = rl$, there is a unique comparison morphism

$$\begin{tikzcd}
 & LA \arrow[dr, "Lf"] \arrow[dd, dotted, "\exists!"] & \\
A \arrow[ur, "\eta_A"] \arrow[dr, "l"'] \arrow[r, phantom, "="] & & B \arrow[l, phantom, "="] \\
 & C \arrow[ur, "r"'] &
\end{tikzcd}$$

which is moreover an isomorphism. Hence $l\colon A \longrightarrow C$ witnesses C as the reflection of A. □

Therefore we can recognise that the essential properties of the morphism $A \longrightarrow \Omega_j^A$ for the purpose of Lawvere's construction are that its codomain Ω_j^A is a j-sheaf, and that it is a "local monomorphism".

Recall from elementary topos theory that the generalised elements of the closure $\overline{A}$ of a subobject $m\colon A \rightarrowtail B$ admit the following characterisation. For a morphism $x\colon X \longrightarrow B$, $x \in \overline{A}$ if and only if there exists a dense monomorphism $r\colon R \rightarrowtail X$ such that $xr \in A$, i.e. such that there exists a commutative square

$$\begin{tikzcd}
R \arrow[r] \arrow[d, tail, "r"'] \arrow[dr, phantom, "="] & A \arrow[d, tail, "m"] \\
X \arrow[r, "x"'] & B
\end{tikzcd}$$

There is a similar characterisation for the generalisation of Lawvere's construction.

Theorem 2.1.15. *Let $\mathcal{C}$ be a category with pullbacks and let $(\mathcal{L},\mathcal{R})$ be a stable factorisation system on $\mathcal{C}$ such that every morphism in $\mathcal{R}$ is a monomorphism. Let $f\colon A \longrightarrow B$ be a morphism in $\mathcal{C}$ with $(\mathcal{L},\mathcal{R})$-factorisation $A \xrightarrow{l} C \xrightarrow{r} B$. Then for any object X of $\mathcal{C}$, the hom-set $\mathcal{C}(X,C)$ is isomorphic to the subset of $\mathcal{C}(X,B)$ consisting of those morphisms $x\colon X \longrightarrow B$ for which there exists a commutative square*

$$\begin{tikzcd}
R \arrow[r, "a"] \arrow[d, "p"'] \arrow[dr, phantom, "="] & A \arrow[d, "f"] \\
X \arrow[r, "x"'] & B
\end{tikzcd}$$

with $p \in \mathcal{L}$.

Proof. By assumption, r is a monomorphism, so $(1,r)\colon \mathcal{C}(X,C) \longrightarrow \mathcal{C}(X,B)$ is injective, and $\mathcal{C}(X,C)$ is therefore isomorphic to its image.

Let $x\colon X \longrightarrow B$ belong to the image of $(1,r)$. Then there exists a morphism $y\colon X \longrightarrow$

C such that $x = ry$. The pullback p of l along y, as in the diagram

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "l"] \\ X \arrow[r, "y"] \arrow[d, "1"'] \arrow[dr, phantom, "="] & C \arrow[d, "r"] \\ X \arrow[r, "x"'] & B \end{tikzcd}$$

belongs to $\mathcal{L}$, since $\mathcal{L}$ is stable under pullback. Hence x belongs to the subset of $\mathcal{C}(X, B)$ defined in the statement of the theorem.

Conversely, let $x \in \mathcal{C}(X, B)$ be such that there exists a commutative square as in the statement of the theorem. Then, since $p \perp r$, there is a diagonal filler y

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "p"'] & A \arrow[r, "l"] & C \arrow[d, "r"] \\ X \arrow[urr, dotted, "y"] \arrow[rr, "x"'] & & B \end{tikzcd}$$

such that $x = ry$. Hence x belongs to the image of $(1, r)$. □

Remark 2.1.16. Note that if $(\mathcal{E}, \mathcal{M})$ is another factorisation system on $\mathcal{C}$, such that $\mathcal{E} \subseteq \mathcal{L}$, and if $f \in \mathcal{M}$, then we may take $p \in \mathcal{L} \cap \mathcal{M}$ in the statement of the theorem. Take the $(\mathcal{E}, \mathcal{M})$-factorisation $p = me$. Then, since $f \in \mathcal{M}$, we have $e \perp f$, so there is a diagonal filler h as in the diagram

$$\begin{tikzcd} R \arrow[rr, "a"] \arrow[d, "e"'] & & A \arrow[d, "f"] \\ Y \arrow[urr, dotted, "h"] \arrow[r, "m"'] & X \arrow[r, "x"'] & B \end{tikzcd}$$

Hence there exists a commutative square

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "m"'] \arrow[dr, phantom, "="] & A \arrow[d, "f"] \\ X \arrow[r, "x"'] & B \end{tikzcd}$$

with $m \in \mathcal{L} \cap \mathcal{M}$, since by cancellation, $me = p \in \mathcal{L}$ and $e \in \mathcal{E} \subseteq \mathcal{L}$ imply $m \in \mathcal{L}$.

Corollary 2.1.17. *With the hypotheses of Theorems 2.1.11 and 2.1.15, let $f : A \longrightarrow B$ be a morphism in $\mathcal{C}$ such that $B \in \mathcal{A}$ and $Lf \in \mathcal{M}$. Then for any object $X \in \mathcal{C}$, the hom-set $\mathcal{C}(X, LA)$ is isomorphic to the subset of $\mathcal{C}(X, B)$ consisting of those morphisms*

$x\colon X \longrightarrow B$ for which there exists a commutative square

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "p"'] \arrow[dr, phantom, "="] & A \arrow[d, "f"] \\ X \arrow[r, "x"'] & B \end{tikzcd}$$

with $p \in \mathcal{L}$.

Proof. Combine Theorems 2.1.14 and 2.1.15. □

Regarding $(\mathcal{E}, \mathcal{M})$ as the "image" factorisation system on $\mathcal{C}$, we may think of $(\mathcal{L}, \mathcal{R})$ as the "local image" factorisation system. We may then interpret this corollary as saying that the (generalised) elements of LA are the (generalised) elements of B which are "locally in the image of f".

For interest, we end this section with a generalisation of Theorem 2.1.15 which removes the assumption that $\mathcal{R}$ is contained in the class of monomorphisms.

Lemma 2.1.18. *Let $\mathcal{E}$ and $\mathcal{B}$ be categories and $P\colon \mathcal{E} \longrightarrow \mathcal{B}$ an opfibration. Then for any objects B of $\mathcal{B}$ and E of $\mathcal{E}$, there is an isomorphism*

$$\mathcal{B}(PE, B) \cong \operatorname{colim}\ \left(\mathcal{E}^B \longrightarrow \mathcal{E} \xrightarrow{\mathcal{E}(E,-)} \mathbf{Set}\right).$$

Proof. The Yoneda lemma says that for any object C of a category $\mathcal{C}$, the canonical natural transformation

$$\begin{tikzcd} 1 \arrow[rr, "C"] \arrow[dr, "*"'] & \arrow[d, phantom, "\Rightarrow"] & \mathcal{C} \arrow[dl, "\mathcal{C}(C,-)"] \\ & \mathbf{Set} & \end{tikzcd}$$

is a left Kan extension. Then by the pasting lemma for left Kan extensions [SW78], we have that the rightmost triangle in the equation

$$\begin{tikzcd} 1 \arrow[rr, "PE"] \arrow[dr, "*"'] & \arrow[d, phantom, "\Rightarrow"] & \mathcal{B} \arrow[dl, "\mathcal{B}(PE,-)"] \\ & \mathbf{Set} & \end{tikzcd} \quad = \quad \begin{tikzcd} 1 \arrow[r, "E"] \arrow[dr, "*"'] & \mathcal{E} \arrow[r, "P"] \arrow[d, "\mathcal{E}(E,-)"] & \mathcal{B} \arrow[dl, "\mathcal{B}(PE,-)"] \\ & \mathbf{Set} & \end{tikzcd}$$

is a left Kan extension.

Hence by Lawvere's colimit formula for left Kan extensions, we have that $\mathcal{B}(PE, B)$ is isomorphic to the colimit of the composite

$$P/B \longrightarrow \mathcal{E} \xrightarrow{\mathcal{E}(E,-)} \mathbf{Set}$$

Finally, the functor $\mathcal{E}^B \longrightarrow P/B$ that sends E to $(E, 1_B)$ has a left adjoint, which sends an object (E, f) to its opcartesian lift, and is therefore final. Hence we have the isomorphisms

$$\begin{aligned}\mathcal{B}(PE, B) &\cong \operatorname{colim}\ (P/B \longrightarrow \mathcal{E} \xrightarrow{\mathcal{E}(E,-)} \mathbf{Set}) \\ &\cong \operatorname{colim}\ (\mathcal{E}^B \longrightarrow P/B \longrightarrow \mathcal{E} \xrightarrow{\mathcal{E}(E,-)} \mathbf{Set})\end{aligned}$$

□

Proposition 2.1.19. *Let $\mathcal{C}$ be a category with pullbacks and let $(\mathcal{L}, \mathcal{R})$ be a stable factorisation system on $\mathcal{C}$. Let $f\colon A \longrightarrow B$ be a morphism in $\mathcal{C}$ with $(\mathcal{L}, \mathcal{R})$-factorisation $A \xrightarrow{l} C \xrightarrow{r} B$. Then for any object X of $\mathcal{C}$, the hom-set $\mathcal{C}(X, C)$ is isomorphic to the colimit of the composite*

$$(\mathcal{L}/X)^{op} \longrightarrow [\mathbf{2}, \mathcal{C}]^{op} \xrightarrow{[\mathbf{2},\mathcal{C}](-,f)} \mathbf{Set}$$

Proof. For any $p\colon Y \longrightarrow X$ in $\mathcal{L}$, the pullback that defines the hom-set $[\mathbf{2}, \mathcal{C}](p, f)$ is equal to the pasted pullback on the right-hand side of the equation

$$\begin{tikzcd}
{[\mathbf{2},\mathcal{C}](p,f)} \arrow[r, "\mathrm{cod}"] \arrow[d, "\mathrm{dom}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(X,B) \arrow[d, "{(p,1)}"] \\
\mathcal{C}(Y,A) \arrow[r, "{(1,f)}"'] & \mathcal{C}(Y,B)
\end{tikzcd}
=
\begin{tikzcd}
{[\mathbf{2},\mathcal{C}](p,l)} \arrow[r, "\mathrm{cod}"] \arrow[d, "\mathrm{dom}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(X,C) \arrow[r, "{(1,r)}"] \arrow[d, "{(p,1)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(X,B) \arrow[d, "{(p,1)}"] \\
\mathcal{C}(Y,A) \arrow[r, "{(1,l)}"'] & \mathcal{C}(Y,C) \arrow[r, "{(1,r)}"'] & \mathcal{C}(Y,B)
\end{tikzcd}$$

Hence $[\mathbf{2}, \mathcal{C}](p, f) \cong [\mathbf{2}, \mathcal{C}](p, l)$, and it follows that the functor in the statement of the proposition is naturally isomorphic to the composite functor

$$(\mathcal{L}/X)^{\mathrm{op}} \longrightarrow \mathcal{L}^{\mathrm{op}} \xrightarrow{\mathcal{L}(-,l)} \mathbf{Set}$$

where $\mathcal{L}$ is seen as a full subcategory of $[\mathbf{2}, \mathcal{C}]$. Since $\mathcal{L}$ is stable under pullback, the codomain functor $\mathcal{L} \longrightarrow \mathcal{C}$ is a fibration. Hence by Lemma 2.1.18, the colimit of this composite functor is isomorphic to the hom-set $\mathcal{C}(X, C)$. □

2.2 Factorisation systems in higher categories

In this section we outline how Lawvere's construction may be generalised to calculate non-abelian cohomology with coefficients in associated n-stacks. We resume the informal manner of speaking about higher categories from Section 1.3, in which all notions are intended in the "weak" sense.

First, note that Lawvere's construction, applied to the category of presheaves over a site $(\mathcal{C}, J)$, makes use of the (epi, mono) factorisation system on $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, which is inherited pointwise from the (surjective, injective) factorisation system on $\mathbf{Set}$. We therefore wish to make use of similar factorisation systems on n-$\mathbf{Cat}$.

Definition 2.2.1. An n-functor $F\colon A \longrightarrow B$ between n-categories is

- *essentially surjective* if for every object $b \in B$, there exists an object $a \in A$ and an equivalence $Fa \simeq b$, and
- *an equivalence on homs* if for all objects $a, b \in A$, $A(a,b) \longrightarrow B(Fa, Fb)$ is an equivalence of $(n-1)$-categories.

Note that a function $f\colon A \longrightarrow B$ between sets is essentially surjective if and only if it is surjective, and is an equivalence on homs if and only if for all $a, b \in A$, $fa = fb$ implies $a = b$, i.e. if f is injective. A functor between categories is an equivalence on homs if and only if it is fully faithful.

Recall that the classes (surjective, injective) form a factorisation system on **Set**, and that the classes (essentially surjective, fully faithful) form a (bicategorical) factorisation system on **Cat**. Generally, one expects that for each n, the classes (essentially surjective, equivalence on homs) should form a factorisation system on n-**Cat**, and define factorisation systems on $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, n\text{-}\mathbf{Cat})$ pointwise.

Recall that a morphism $f\colon A \longrightarrow B$ in a category is a monomorphism if and only if the commutative square

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback. Similarly, one expects that for each n, the equivalence on homs n-functors should have a finite limit characterisation, and therefore be preserved by the left exact reflection $L\colon \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, n\text{-}\mathbf{Cat}) \longrightarrow n\text{-}\mathrm{Stacks}(\mathcal{C}, J)$.

Therefore one can expect a higher categorical analogue of Theorem 2.1.11 to give "local" versions of these factorisation systems on $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, n\text{-}\mathbf{Cat})$, to which a higher categorical analogue of Corollary 2.1.17 would apply.

Combining these observations with the definition of the non-abelian cohomology of a site with coefficients in an n-stack, we arrive at our general method.

Method 2.2.2. *Let $X = (\mathcal{C}, J)$ be a site, and let $F \in \mathcal{F} = \mathrm{Hom}(\mathcal{C}^{op}, n\text{-}\mathbf{Cat})$ be an indexed n-category over $\mathcal{C}$. Find a morphism $f\colon F \longrightarrow G$ in $\mathcal{F}$ such that G is an n-stack and Lf is an equivalence on homs. Then the cohomology n-category $\mathscr{H}(X; LF)$ of the site with coefficients in the associated n-stack LF of F is equivalent to the full sub-n-category of the hom-n-category $\mathcal{F}(1, G)$ on those objects $x\colon 1 \longrightarrow G$ for which there exists a square*

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "p"'] \arrow[dr, phantom, "\simeq"] & F \arrow[d, "f"] \\ 1 \arrow[r, "x"'] & G \end{tikzcd}$$

commuting up to equivalence, with p locally essentially surjective.

Proof. By Definition 1.3.2, the cohomology n-category $\mathscr{H}(X; LF)$ is equal to the n-category $\mathcal{F}(1, LF)$. The expected higher categorical analogue of Corollary 2.1.17 yields the equivalence. □

In the following chapters we will make this method rigorous for $n = 2$ and show how it may be used to recover Giraud's definition of non-abelian cohomology of degree 2 in terms of gerbes.

To further illustrate the use of this method, we close this chapter with an outline of how it may applied in the case $n = 1$ to recover the classical definition of non-abelian cohomology of degree 1 in terms of torsors. For details and similar approaches, see [Gir71, Bun79, Str82a, Str04].

Example 2.2.3. Let $X = (\mathcal{C}, J)$ be a site. The inclusion of the 2-category of stacks over $(\mathcal{C}, J)$ into the 2-category of indexed categories over $\mathcal{C}$ has a left biadjoint

$$\begin{tikzcd} \mathrm{Stacks}(\mathcal{C}, J) \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}) \arrow[l, shift right, "L"'] \end{tikzcd}$$

which preserves finite bilimits. The bicategorical factorisation system (essentially surjective, fully faithful) on $\mathbf{Cat}$ lifts pointwise to $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$.

The indexed category $\mathcal{S}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ that sends $C \in \mathcal{C}$ to the category of sheaves $\mathrm{Sh}(\mathcal{C}/C)$ over the site $\mathcal{C}/C$ with the induced topology is a stack.

Let G be a sheaf of groups over $(\mathcal{C}, J)$. As a functor $G\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Gp}$, this may be seen as an object of $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$. This indexed category G is a "pre-stack", and consequently there is a indexed functor $G^{\mathrm{op}} \times G \longrightarrow \mathcal{S}$, whose exponential transpose $y\colon G \longrightarrow \mathcal{S}^{G^{\mathrm{op}}}$ is (pointwise) fully faithful. The codomain of y is the stack $\mathcal{S}^{G^{\mathrm{op}}}$ which sends $C \in \mathcal{C}$ to the category of sheaves of sets over $\mathcal{C}/C$ equipped with a right action of the sheaf of groups $\mathrm{pr}_C\colon G \times C \longrightarrow C$. Therefore we may apply Method 2.2.2 to the indexed functor $y\colon G \longrightarrow \mathcal{S}^{G^{\mathrm{op}}}$ to yield the following description of the cohomology category $\mathscr{H}(X; LG)$.

The category $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})(1, \mathcal{S}^{G^{\mathrm{op}}})$ is equivalent to the category of G-sheaves, i.e., the category of sheaves of sets over $\mathcal{C}$ equipped with a right action of G. Those G-sheaves x that are "locally in the image of y", i.e. those for which there exists an indexed functor $R \longrightarrow G$ with $R \longrightarrow 1$ locally essentially surjective and an isomorphism

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & G \arrow[d, "y"] \\ 1 \arrow[r, "x"'] & \mathcal{S}^{G^{\mathrm{op}}} \end{tikzcd}$$

are precisely those that are locally isomorphic to the right action of G on itself, i.e. the G-torsors. Moreover, the associated stack of G is the stack of G-torsors.

Hence the cohomology category $\mathscr{H}(X; LG)$ is equivalent to the category (in fact a groupoid) of G-torsors over $X = (\mathcal{C}, J)$. The classical cohomology set $H^1(X; G)$ is defined to be the (pointed) set of equivalence classes of G-torsors, and is therefore isomorphic to the set of equivalence classes of the cohomology category $\mathscr{H}(X; LG)$, i.e.,

$$H^1(X; G) \cong \pi_0 \mathscr{H}(X; LG).$$

Chapter 3

A coherent approach to 2-stacks

The purpose of this chapter is to prove that the inclusion of the tricategory of 2-stacks over a site $(\mathcal{C}, J)$ into the tricategory of indexed bicategories over $\mathcal{C}$ has a left triadjoint

$$\begin{tikzcd} \text{2-Stacks}(\mathcal{C}, J) \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat}) \arrow[l, shift right] \end{tikzcd}$$

that preserves finite trilimits. Having established this result, we can apply the method of Chapter 2 to recover Giraud's non-abelian cohomology of degree 2; this is the purpose of Chapters 4 and 5.

Our proof, which uses a tricategorical version of Grothendieck's plus construction, is aided by results of three-dimensional coherence theory. We first show that there is a triadjunction

$$\begin{tikzcd} {[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]} \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat}) \arrow[l, shift right] \end{tikzcd} \tag{3.1}$$

whose right triadjoint is the inclusion, and which induces a triequivalence between $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$ and a full sub-**Gray**-category of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$. This enables us to construct trilimits in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$ as **Gray**-limits in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$, and to define the plus construction as an endo-**Gray**-functor L of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$. Finally, we show that there exists an ordinal α (indeed, one can take $\alpha = 4$), such that the composite

$$\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat}) \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}] \xrightarrow{L^\alpha} [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}] \longrightarrow \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$$

preserves finite trilimits and gives the reflection into 2-stacks.

For the definitions and basic results of the theory of tricategories, we refer the reader to [GPS95, Gur13, Buh15]. Note that we take a common sense approach to working with tricategories; we do not stop to prove all those general, elementary results (such as the trinaturality of the canonical maps, à la [Kel05, Section 1.8]) whose straightforward, but typically lengthy, proofs merely consist of unwinding definitions. Instead, we focus on those results and constructions that specially pertain to our applications.

3.1 Strictification of bicategories

Recall that the (pseudo) Gray tensor product endows the category 2-**Cat** of 2-categories and 2-functors with the structure of a symmetric monoidal closed category. For 2-categories A and B, the internal hom $[A, B]$ is the 2-category of 2-functors, pseudonatural transformations, and modifications. Categories enriched over this monoidal category are called **Gray**-categories, and can be seen as a certain type of strict tricategory. We let **Gray** denote the self-enrichment of 2-**Cat**, which is therefore the **Gray**-category (and hence the tricategory) of 2-categories, 2-functors, pseudonatural transformations, and modifications.

The fundamental result of the coherence theory of tricategories is that every tricategory is triequivalent to a **Gray**-category. For example, let **Bicat** denote the tricategory of bicategories, pseudofunctors, pseudonatural transformations, and modifications. There is a trihomomorphism $\mathrm{st}\colon \mathbf{Bicat} \longrightarrow \mathbf{Gray}$, whose underlying functor $\mathrm{st}\colon \mathbf{Bicat}_0 \longrightarrow$ 2-**Cat** is left adjoint to the inclusion

$$\begin{tikzcd} 2\text{-}\mathbf{Cat} \arrow[r, shift right, hook] \arrow[r, phantom, "\perp"] & \mathbf{Bicat}_0 \arrow[l, shift right, "\mathrm{st}"'] \end{tikzcd}, \tag{3.2}$$

and which is a biequivalence on hom-bicategories. It follows that **Bicat** is triequivalent to the full sub-**Gray**-category of **Gray** on the cofibrant 2-categories, which can be characterised as those 2-categories whose underlying category is free on a graph (for instance, any set seen as a discrete 2-category is cofibrant).

In this section we show that the adjunction (3.2) extends to a triadjunction between tricategories

$$\begin{tikzcd} \mathbf{Gray} \arrow[r, shift right, hook] \arrow[r, phantom, "\perp"] & \mathbf{Bicat} \arrow[l, shift right, "\mathrm{st}"'] \end{tikzcd}. \tag{3.3}$$

For this purpose, we recall how st acts when restricted to a trihomomorphism $\mathrm{st}\colon \mathbf{Gray} \longrightarrow \mathbf{Gray}$.

Let A be a 2-category. The 2-category $\mathrm{st}\,A$ has the same objects as A. A morphism $(f_1, \ldots, f_n)\colon a \longrightarrow b$ consists of a path of morphisms $a \xrightarrow{f_1} \cdots \xrightarrow{f_n} b$ in A, for some $n \geq 0$. A 2-cell $\alpha\colon (f_1, \ldots, f_n) \longrightarrow (g_1, \ldots, g_m)$ consists of a 2-cell $\alpha\colon f_n \circ \cdots \circ f_1 \longrightarrow g_m \circ \cdots \circ g_1$ in A. Horizontal composition is by concatenation of paths, and vertical composition is as in A. Evaluating the composites of paths yields a 2-functor $\varepsilon_A\colon \mathrm{st}\,A \longrightarrow A$, which is the component of the counit ε of the adjunction (3.2) at A.

Let $F\colon A \longrightarrow B$ be a 2-functor. The 2-functor $\mathrm{st}\,F\colon \mathrm{st}\,A \longrightarrow \mathrm{st}\,B$ agrees with F on objects, sends a path $(f_1, \ldots, f_n)\colon a \longrightarrow b$ in A to the path $(Ff_1, \ldots, Ff_n)\colon Fa \longrightarrow Fb$ in B, and sends a 2-cell $\alpha\colon (f_1, \ldots, f_n) \longrightarrow (g_1, \ldots, g_m)$ in $\mathrm{st}\,A$ to the 2-cell $F\alpha\colon (Ff_1, \ldots, Ff_n) \longrightarrow (Fg_1, \ldots, Fg_m)$ in $\mathrm{st}\,B$.

Let $\theta\colon F \longrightarrow G$ be a pseudonatural transformation. The component of the pseudonatural transformation $\mathrm{st}\,\theta\colon \mathrm{st}\,F \longrightarrow \mathrm{st}\,G$ at an object a of A is the unary path $(\mathrm{st}\,\theta)_a = (\theta_a)\colon Fa \longrightarrow Ga$, and its component at a path $(f_1, \dots, f_n)\colon a \longrightarrow b$ in A is the 2-cell $(Ff_1, \dots, Ff_n, \theta_b) \longrightarrow (\theta_a, Gf_1, \dots, Gf_n)$ given by the pasting composite

$$\begin{tikzcd} Fa \arrow[r, "Ff_1"] \arrow[d, "\theta_a"'] \arrow[dr, phantom, "\Downarrow\theta_{f_1}"] & Fa_2 \arrow[d, "\theta_{a_2}"] & \cdots & Fa_n \arrow[r, "Ff_n"] \arrow[d, "\theta_{a_n}"'] \arrow[dr, phantom, "\Downarrow\theta_{f_n}"] & Fb \arrow[d, "\theta_b"] \\ Ga \arrow[r, "Gf_1"'] & Ga_2 & & Ga_n \arrow[r, "Gf_n"'] & Gb \end{tikzcd} \tag{3.4}$$

Let $m\colon \theta \longrightarrow \varphi$ be a modification. The component of the modification $\mathrm{st}\,m\colon \mathrm{st}\,\theta \longrightarrow \mathrm{st}\,\varphi$ at an object a of A is the 2-cell $m_a\colon (\theta_a) \longrightarrow (\varphi_a)$ in $\mathrm{st}\,B$.

The action of st on **Bicat** may be described similarly, although more care must be taken, due to the coherent associativity of composition in a bicategory. Recall that for a bicategory A, the unit $\eta_A\colon A \longrightarrow \mathrm{st}\,A$ is a biequivalence pseudofunctor that is an identity on objects, sends a morphism f in A to the unary path (f), and is an identity on 2-cells.

Note that the trihomomorphism $\mathrm{st}\colon$ **Gray** $\longrightarrow$ **Gray** fails to be a **Gray**-functor: the pseudofunctors $\mathrm{st}\colon [A, B] \longrightarrow [\mathrm{st}\,A, \mathrm{st}\,B]$ are in general not strict, and the composition constraints

$$\begin{tikzcd} {[B,C] \times [A,B]} \arrow[r, "\mathrm{st}\times\mathrm{st}"] \arrow[d, "\circ"'] \arrow[dr, phantom, "\Downarrow\chi"] & {[\mathrm{st}\,B, \mathrm{st}\,C] \times [\mathrm{st}\,A, \mathrm{st}\,B]} \arrow[d, "\circ"] \\ {[A,C]} \arrow[r, "\mathrm{st}"'] & {[\mathrm{st}\,A, \mathrm{st}\,C]} \end{tikzcd}$$

are in general non-identity invertible icons. All the other trihomomorphism coherence cells are identities.

By contrast, the unit and counit of the adjunction (3.2) extend to tritransformations that are as strict as possible.

Lemma 3.1.1. *The natural transformations*

$$\begin{tikzcd} \mathbf{Bicat}_0 \arrow[rr, "1"] \arrow[dr, "\mathrm{st}"'] & \Downarrow\eta & \mathbf{Bicat}_0 \\ & 2\text{-}\mathbf{Cat} \arrow[ur] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & \mathbf{Bicat}_0 \arrow[dr, "\mathrm{st}"] & \\ 2\text{-}\mathbf{Cat} \arrow[ur] \arrow[rr, "1"'] & \Downarrow\varepsilon & 2\text{-}\mathbf{Cat} \end{tikzcd}$$

extend to strict tritransformations

$$\begin{tikzcd} \mathbf{Bicat} \arrow[rr, "1"] \arrow[dr, "\mathrm{st}"'] & \Downarrow\eta & \mathbf{Bicat} \\ & \mathbf{Gray} \arrow[ur] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & \mathbf{Bicat} \arrow[dr, "\mathrm{st}"] & \\ \mathbf{Gray} \arrow[ur] \arrow[rr, "1"'] & \Downarrow\varepsilon & \mathbf{Gray} \end{tikzcd}$$

respectively.

Proof. To show that ε is a strict tritransformation, it suffices to show that for all 2-categories A and B, the square

$$\begin{tikzcd}
{[A,B]} \arrow[r, "\mathrm{st}"] \arrow[d, "1"'] \arrow[dr, phantom, "="] & {[\mathrm{st}\,A, \mathrm{st}\,B]} \arrow[d, "{[1,\varepsilon_B]}"] \\
{[A,B]} \arrow[r, "{[\varepsilon_A,1]}"'] & {[\mathrm{st}\,A, B]}
\end{tikzcd}$$

commutes. For then, since ε sends the components of the icons χ to identities, all higher coherence cells can be chosen to be identities.

By ordinary naturality, we know that this square commutes at the level of objects. Let $\theta\colon F \longrightarrow G\colon A \longrightarrow B$ be a pseudonatural transformation. The pseudonatural transformations $\varepsilon_B \circ \mathrm{st}\,\theta$ and $\theta \circ \varepsilon_A$ are equal, since for each object a of A, they both have the component $\theta_a\colon Fa \longrightarrow Ga$, and for each path $(f_1, \ldots, f_n)\colon a \longrightarrow b$ in A, they both have the component $\theta_b \circ (Ff_n \circ \cdots \circ Ff_1) \longrightarrow (Gf_n \circ \cdots \circ Gf_1) \circ \theta_a$ given by the pasting composite (3.4).

Let $m\colon \theta \longrightarrow \varphi$ be a modification. For each object a of A, both $\varepsilon_B \circ \mathrm{st}\,\theta$ and $\theta \circ \varepsilon_A$ have the component $m_a\colon \theta_a \longrightarrow \varphi_a$. Therefore the square commutes.

Similarly, for bicategories A and B, we know by ordinary naturality that the square

$$\begin{tikzcd}
\mathbf{Bicat}(A,B) \arrow[r, "1"] \arrow[d, "\mathrm{st}"'] \arrow[dr, phantom, "="] & \mathbf{Bicat}(A,B) \arrow[d, "{(1,\eta_B)}"] \\
\mathbf{Bicat}(\mathrm{st}\,A, \mathrm{st}\,B) \arrow[r, "{(\eta_A,1)}"'] & \mathbf{Bicat}(A, \mathrm{st}\,B)
\end{tikzcd}$$

commutes at the level of objects.

Let $\theta\colon F \longrightarrow G\colon A \longrightarrow B$ be a pseudonatural transformation. The pseudonatural transformations $\eta_B \circ \theta$ and $\mathrm{st}\,\theta \circ \eta_A$ both have the components $(\theta_a)\colon Fa \longrightarrow Ga$ for an object a of A, and $\theta_f\colon (Ff, \theta_b) \longrightarrow (\theta_a, Gf)$ for a morphism $f\colon a \longrightarrow b$ in A.

Let $m\colon \theta \longrightarrow \varphi$ be a modification. The modifications $\eta_B \circ m$ and $\mathrm{st}\,m \circ \eta_A$ both have the component $m_a\colon (\theta_a) \longrightarrow (\varphi_a)$ for an object a of A.

Therefore the square commutes. It is evident from inspection that all higher tritransformation coherence cells can be chosen to be identities. □

Remark 3.1.2. The category 2-**Cat** with the Gray tensor product admits a monoidal model structure in which the weak equivalences are those 2-functors that are biequivalences, the fibrations are the equifibrations, all objects are fibrant, and the cofibrant objects are the cofibrant 2-categories as recalled above [Lac02b, Lac04]. For each 2-category A, the component of the counit $\varepsilon_A\colon \mathrm{st}\,A \longrightarrow A$ is a cofibrant replacement of A.

For any monoidal model category $\mathcal{V}$ with cofibrant unit, [LR15, Proposition A.1] states that if there exists a $\mathcal{V}$-natural transformation $q\colon Q \longrightarrow 1$ which exhibits $Q\colon \mathcal{V} \longrightarrow V$ as a cofibrant replacement $\mathcal{V}$-functor, then all objects of $\mathcal{V}$ must be cofibrant. Therefore, since not every 2-category is cofibrant, $\mathrm{st}\colon \mathbf{Gray} \longrightarrow \mathbf{Gray}$ must fail to be a **Gray**-functor, which we saw above indeed to be the case.

We say that trihomomorphisms $L\colon \mathcal{A} \longrightarrow \mathcal{B}$ and $R\colon \mathcal{B} \longrightarrow \mathcal{A}$ form a *triadjunction*, written $L \dashv R$ with L the left triadjoint and R the right triadjoint, if there exist biequivalences $\mathcal{B}(LA, B) \simeq \mathcal{A}(A, RB)$ trinatural in $A \in \mathcal{A}$ and $B \in \mathcal{B}$. That is, if there exists a tritransformation $\mathcal{B}(L, 1) \longrightarrow \mathcal{A}(1, R)\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \longrightarrow \mathbf{Bicat}$ with all components biequivalences.

We can now establish the fundamental strictification triadjunction (3.3).

Theorem 3.1.3. *The trihomomorphism* $\mathrm{st}\colon \mathbf{Bicat} \longrightarrow \mathbf{Gray}$ *is left triadjoint to the inclusion* $\mathbf{Gray} \longrightarrow \mathbf{Bicat}$*. Each component* $\eta_A\colon A \longrightarrow \mathrm{st}\,A$ *of the unit is a biequivalence in* **Bicat***.*

Proof. It follows from Lemma 3.1.1 that the pseudofunctors

$$\mathbf{Bicat}(A, B) \xrightarrow{\mathrm{st}} [\mathrm{st}\,A, \mathrm{st}\,B] \xrightarrow{[1,\varepsilon_B]} [\mathrm{st}\,A, B] \tag{3.5}$$

are trinatural in $A \in \mathbf{Bicat}$ and $B \in \mathbf{Gray}$. For a bicategory A and a 2-category B, the pasting composites

$$\begin{tikzcd}
\mathbf{Bicat}(A,B) \arrow[r, "\mathrm{st}"] \arrow[dd, "1"'] \arrow[ddr, phantom, "="] & {[\mathrm{st}\,A, \mathrm{st}\,B]} \arrow[r, "{[1,\varepsilon_B]}"] \arrow[d] \arrow[dr, phantom, "="] & {[\mathrm{st}\,A, B]} \arrow[d] \\
 & \mathbf{Bicat}(\mathrm{st}\,A, \mathrm{st}\,B) \arrow[r, "{(1,\varepsilon_B)}"'] \arrow[d, "{(\eta_A,1)}"'] \arrow[dr, phantom, "="] & \mathbf{Bicat}(\mathrm{st}\,A, B) \arrow[d, "{(\eta_A,1)}"] \\
\mathbf{Bicat}(A,B) \arrow[r, "{(1,\eta_B)}"'] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, bend right=15, "\parallel"] & \mathbf{Bicat}(A, \mathrm{st}\,B) \arrow[r, "{(1,\varepsilon_B)}"'] & \mathbf{Bicat}(A,B)
\end{tikzcd}$$

and

$$\begin{tikzcd}
{[\mathrm{st}\,A, B]} \arrow[r] \arrow[dd, "1"'] \arrow[ddr, phantom, "="] & \mathbf{Bicat}(\mathrm{st}\,A, B) \arrow[r, "{(\eta_A,1)}"] \arrow[d, "\mathrm{st}"'] \arrow[dr, phantom, "\simeq"] & \mathbf{Bicat}(A,B) \arrow[d, "\mathrm{st}"] \\
 & {[\mathrm{st}\,\mathrm{st}\,A, \mathrm{st}\,B]} \arrow[r, "{[\mathrm{st}\,\eta_A,1]}"'] \arrow[d, "{[1,\varepsilon_B]}"'] \arrow[dr, phantom, "="] & {[\mathrm{st}\,A, \mathrm{st}\,B]} \arrow[d, "{[1,\varepsilon_B]}"] \\
{[\mathrm{st}\,A, B]} \arrow[r, "{[\varepsilon_{\mathrm{st}\,A},1]}"'] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, bend right=15, "\parallel"] & {[\mathrm{st}\,\mathrm{st}\,A, B]} \arrow[r, "{[\mathrm{st}\,\eta_A,1]}"'] & {[\mathrm{st}\,A, B]}
\end{tikzcd}$$

witness the 2-functor

$$[\mathrm{st}\,A, B] \longrightarrow \mathbf{Bicat}(\mathrm{st}\,A, B) \xrightarrow{(\eta_A,1)} \mathbf{Bicat}(A, B)$$

as a weak inverse of the pseudofunctor (3.5). □

3.2 Strictification of indexed bicategories

In this section we construct the triadjunction (3.1) as a composite of three triadjunctions.

$$
\begin{tikzcd}
{[\mathcal{A}, \mathbf{Gray}]} \arrow[r, shift right, ""'] \arrow[r, phantom, "\perp"] & \mathrm{Hom}_{\mathrm{ls}}(\mathcal{A}, \mathbf{Gray}) \arrow[l, shift right, "Q"'] \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Hom}(\mathcal{A}, \mathbf{Gray}) \arrow[l, shift right, "\simeq"'] \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Hom}(\mathcal{A}, \mathbf{Bicat}) \arrow[l, shift right, "(1,\mathrm{st})"']
\end{tikzcd}
$$

In fact, the left triadjunction is a **Gray**-adjunction and the middle triadjunction is an adjoint equivalence of **Gray**-categories. Here $\mathcal{A}$ can be taken as a (small) "locally cofibrant" **Gray**-category, i.e., a **Gray**-category for which each hom-2-category $\mathcal{A}(A, B)$ is cofibrant.

For tricategories $\mathcal{A}$ and $\mathcal{B}$, we denote the globular set of trihomomorphisms, tritransformations, trimodifications, and perturbations by $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$. When $\mathcal{B}$ is a **Gray**-category or the tricategory **Bicat**, it is known [Gur13, Buh15] that $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ forms a **Gray**-category and a tricategory respectively.

At one end, we can lift the triadjunction of Theorem 3.1.3 to a triadjunction between tricategories of trihomomorphisms. To aid the proof of this result, we extract the method of the proof of Theorem 3.1.3.

Lemma 3.2.1. *Let $L\colon \mathcal{A} \longrightarrow \mathcal{B}$ and $R\colon \mathcal{B} \longrightarrow \mathcal{A}$ be trihomomorphisms between tricategories, and let $\varepsilon\colon LR \longrightarrow 1_{\mathcal{B}}$ and $\eta\colon 1_{\mathcal{A}} \longrightarrow RL$ be tritransformations. Suppose that there exist equivalences $\varepsilon_{LA} \circ L\eta_A \simeq 1_{LA}$ and $R\varepsilon_B \circ \eta_{RB} \simeq 1_{RB}$. Then there exists a triadjunction $L \dashv R$.*

Proof. The two families of pseudofunctors

$$\mathcal{A}(A, RB) \xrightarrow{L} \mathcal{B}(LA, LRB) \xrightarrow{(1,\varepsilon_B)} \mathcal{B}(LA, B)$$

$$\mathcal{B}(LA, B) \xrightarrow{R} \mathcal{A}(RLA, RB) \xrightarrow{(\eta_A,1)} \mathcal{A}(A, RB)$$

inherit trinaturality from ε and η, and are mutually weakly inverse, as witnessed by the pasting composites of pseudonatural equivalences

$$
\begin{tikzcd}
\mathcal{A}(A,RB) \arrow[r, "L"] \arrow[dd, "1"'] & \mathcal{B}(LA,LRB) \arrow[r, "{(1,\varepsilon_B)}"] \arrow[d, "R"'] \arrow[dr, phantom, "\simeq"] & \mathcal{B}(LA,B) \arrow[d, "R"] \\
{} \arrow[r, phantom, "\simeq"] & \mathcal{A}(RLA,RLRB) \arrow[r, "{(1,R\varepsilon_B)}"'] \arrow[d, "{(\eta_A,1)}"'] \arrow[dr, phantom, "\simeq"] & \mathcal{A}(RLA,RB) \arrow[d, "{(\eta_A,1)}"] \\
\mathcal{A}(A,RB) \arrow[r, "{(1,\eta_{RB})}"'] \arrow[rr, bend right, "1"'] & \mathcal{A}(A,RLRB) \arrow[r, "{(1,R\varepsilon_B)}"'] \arrow[d, phantom, "\simeq"] & \mathcal{A}(A,RB) \\
& {} &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\mathcal{B}(LA,B) \arrow[r, "R"] \arrow[dd, "1"'] & \mathcal{A}(RLA,RB) \arrow[r, "{(\eta_A,1)}"] \arrow[d, "L"'] \arrow[dr, phantom, "\simeq"] & \mathcal{A}(A,RB) \arrow[d, "L"] \\
{} \arrow[r, phantom, "\simeq"] & \mathcal{B}(LRLA,LRB) \arrow[r, "{(L\eta_A,1)}"'] \arrow[d, "{(1,\varepsilon_B)}"'] \arrow[dr, phantom, "\simeq"] & \mathcal{B}(LA,LRB) \arrow[d, "{(1,\varepsilon_B)}"] \\
\mathcal{B}(LA,B) \arrow[r, "{(\varepsilon_{LA},1)}"'] \arrow[rr, bend right, "1"'] & \mathcal{B}(LRLA,B) \arrow[r, "{(L\eta_A,1)}"'] \arrow[d, phantom, "\simeq"] & \mathcal{B}(LA,B) \\
& {} &
\end{tikzcd}
$$

□

Proposition 3.2.2. *Let $\mathcal{A}$ be a tricategory. The inclusion of tricategories*

$$
\begin{tikzcd}
\mathrm{Hom}(\mathcal{A},\mathbf{Gray}) \arrow[r, shift right, ""'] \arrow[r, phantom, "\perp"] & \mathrm{Hom}(\mathcal{A},\mathbf{Bicat}) \arrow[l, shift right, "{(1,\mathrm{st})}"']
\end{tikzcd}
$$

has a left triadjoint given by post-composition with the trihomomorphism $\mathrm{st}\colon \mathbf{Bicat} \longrightarrow \mathbf{Gray}$*. Each component $\eta \circ F\colon F \longrightarrow \mathrm{st}\circ F$ of the unit of this triadjunction is a biequivalence in* $\mathrm{Hom}(\mathcal{A},\mathbf{Bicat})$.

Proof. By [Buh15, Theorem A.4], we have that

$$(1,\mathrm{st})\colon\ \mathrm{Hom}(\mathcal{A},\mathbf{Bicat}) \longrightarrow \mathrm{Hom}(\mathcal{A},\mathbf{Gray})$$

is a trihomomorphism between tricategories. The tritransformations η and ε define pointwise tritransformations

$$(1,\eta)\colon 1 \longrightarrow (1,\mathrm{st})\colon\ \mathrm{Hom}(\mathcal{A},\mathbf{Bicat}) \longrightarrow \mathrm{Hom}(\mathcal{A},\mathbf{Bicat})$$

and

$$(1,\varepsilon)\colon (1,\mathrm{st}) \longrightarrow 1\colon\ \mathrm{Hom}(\mathcal{A},\mathbf{Gray}) \longrightarrow \mathrm{Hom}(\mathcal{A},\mathbf{Gray}).$$

Since the triangle identities hold pointwise, Lemma 3.2.1 yields the desired triadjunction.

By [Buh15, Proposition A.16], a tritransformation is a biequivalence in $\mathrm{Hom}(\mathcal{A}, \mathbf{Bicat})$ if and only if each of its components is a biequivalence in **Bicat**. Hence each component $\eta \circ F\colon F \longrightarrow \mathrm{st} \circ F$ of the unit of this triadjunction is a biequivalence, since each of their components $\eta_{FA}\colon FA \longrightarrow \mathrm{st}\, FA$ is a biequivalence. □

At the other end, we can recall the following result of [Buh15], which gives this adjunction as an instance of the coherence theorem for the pseudoalgebras for a **Gray**-monad [Gur13]. For a **Gray**-category $\mathcal{A}$, let $\mathrm{Hom}_{\mathrm{ls}}(\mathcal{A}, \mathbf{Gray})$ denote the full sub-**Gray**-category of $\mathrm{Hom}(\mathcal{A}, \mathbf{Gray})$ on the "locally strict" trihomomorphisms, i.e., those trihomomorphisms $F\colon \mathcal{A} \longrightarrow \mathbf{Gray}$ for which all of the pseudofunctors $F\colon \mathcal{A}(A, B) \longrightarrow [FA, FB]$ are strict 2-functors.

Proposition 3.2.3. *Let $\mathcal{A}$ be a small* **Gray***-category. Then there is a* **Gray***-adjunction*

$$\begin{tikzcd} {[\mathcal{A}, \mathbf{Gray}]} \arrow[r, shift right, ""] \arrow[r, phantom, "\perp"] & \mathrm{Hom}_{\mathrm{ls}}(\mathcal{A}, \mathbf{Gray}) \arrow[l, shift right, "Q"'] \end{tikzcd}$$

whose right adjoint is the inclusion. Each component of the unit of this adjunction is a biequivalence in $\mathrm{Hom}_{\mathrm{ls}}(\mathcal{A}, \mathbf{Gray})$*.*

Proof. This is [Buh15, Theorem 1.8], where it is proved by showing that $\mathrm{Hom}_{\mathrm{ls}}(\mathcal{A}, \mathbf{Gray})$ is isomorphic to the **Gray**-category Ps-T-Alg of pseudoalgebras for the **Gray**-monad T on $[\mathrm{ob}\, \mathcal{A}, \mathbf{Gray}]$ whose **Gray**-category of strict algebras is $[\mathcal{A}, \mathbf{Gray}]$, and which is defined by left Kan extension and restriction along $\mathrm{ob}\, \mathcal{A} \longrightarrow \mathcal{A}$ (and therefore preserves **Gray**-colimits). The result then follows from the coherence theorem [Gur13, Corollary 15.14]. □

We can bridge the gap between the above **Gray**-adjunction and triadjunction by observing that if a **Gray**-category $\mathcal{A}$ is locally cofibrant, then every trihomomorphism in $\mathrm{Hom}(\mathcal{A}, \mathbf{Gray})$ is isomorphic to a locally strict trihomomorphism.

Proposition 3.2.4. *Let $\mathcal{A}$ be a locally cofibrant* **Gray***-category. Then the inclusion* $\mathrm{Hom}_{\mathrm{ls}}(\mathcal{A}, \mathbf{Gray}) \longrightarrow \mathrm{Hom}(\mathcal{A}, \mathbf{Gray})$ *is essentially surjective, and is therefore an equivalence of* **Gray***-categories, which may be promoted to an adjoint equivalence.*

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{ls}}(\mathcal{A}, \mathbf{Gray}) \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Hom}(\mathcal{A}, \mathbf{Gray}) \arrow[l, shift right, "\simeq"'] \end{tikzcd}$$

Proof. Let F be a trihomomorphism $\mathcal{A} \longrightarrow \mathbf{Gray}$. We can construct a locally strict trihomomorphism $\widehat{F}\colon \mathcal{A} \longrightarrow \mathbf{Gray}$ and an invertible tritransformation $\theta\colon F \longrightarrow \widehat{F}$ as follows. First, define $\widehat{F}$ to be equal to F on objects. Then, since each hom-2-category of $\mathcal{A}$ is cofibrant, for each pair of objects $A, B \in \mathcal{A}$ there exists [Lac07] a 2-functor

$\widehat{F}\colon \mathcal{A}(A,B) \longrightarrow [FA,FB]$ and an invertible icon

$$\begin{tikzcd} \mathcal{A}(A,B) \arrow[r, bend left, "F"] \arrow[r, bend right, "\widehat{F}"'] \arrow[r, phantom, "\Downarrow\theta"] & {[FA,FB]}. \end{tikzcd}$$

By taking all the higher coherence cells of the tritransformation θ to be identities, we can derive the trihomomorphism coherence cells for $\widehat{F}$ from those of F. For instance, to assert that the modification Π in the definition of tritransformation is an identity is to assert the equation of the following two pasting composites of pseudonatural transformations

$$\begin{tikzcd} \mathcal{A}(B,C)\times\mathcal{A}(A,B) \arrow[rr, bend left, "F\times F"] \arrow[rr, bend right, "\widehat{F}\times\widehat{F}"'] \arrow[rr, phantom, "\Downarrow\theta\times\theta"] \arrow[d, "\circ"'] \arrow[drr, phantom, "\Downarrow\widehat{\chi}"] & & {[FB,FC]\times[FA,FB]} \arrow[d, "\circ"] \\ \mathcal{A}(A,C) \arrow[rr, "\widehat{F}"'] & & {[FA,FC]} \end{tikzcd}$$

$$\|$$

$$\begin{tikzcd} \mathcal{A}(B,C)\times\mathcal{A}(A,B) \arrow[rr, "F\times F"] \arrow[d, "\circ"'] \arrow[drr, phantom, "\Downarrow\chi"] & & {[FB,FC]\times[FA,FB]} \arrow[d, "\circ"] \\ \mathcal{A}(A,C) \arrow[rr, bend left, "F"] \arrow[rr, bend right, "\widehat{F}"'] \arrow[rr, phantom, "\Downarrow\theta"] & & {[FA,FC]} \end{tikzcd}$$

from which we can define the multiplication constraint $\widehat{\chi}$ for $\widehat{F}$, since the icon θ is invertible. By taking the modification M in the definition of tritransformation to be an identity, we get a definition for the unit constraint $\widehat{\iota}$ for $\widehat{F}$.

$$\begin{tikzcd} 1 \arrow[rr, "1_{FA}"] \arrow[dr, "1_A"'] & \arrow[d, phantom, "\Downarrow\widehat{\iota}"] & {[FA,FA]} \\ & \mathcal{A}(A,A) \arrow[ur, "\widehat{F}"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} 1 \arrow[rr, "1_{FA}"] \arrow[dr, "1_A"'] & \arrow[d, phantom, "\Downarrow\iota"] & {[FA,FA]} \\ & \mathcal{A}(A,A) \arrow[ur, bend left, "F"] \arrow[ur, bend right, "\widehat{F}"'] \arrow[ur, phantom, "\Downarrow\theta"] & \end{tikzcd}$$

Similarly, by considering the axioms for a tritransformation, we can define the higher coherence cells for $\widehat{F}$ in terms of those of F, giving a locally strict trihomomorphism $\widehat{F}$ and a tritransformation $\theta\colon F \longrightarrow \widehat{F}$, which has an inverse in $\mathrm{Hom}(\mathcal{A},\mathbf{Gray})$ whose components are the inverses of the components of θ. $\square$

Composing these three triadjunctions yields the desired strictification triadjunction.

Theorem 3.2.5. *Let $\mathcal{A}$ be a locally cofibrant* **Gray***-category. Then there is a triadjunction*

$$\begin{tikzcd}[\mathcal{A}, \mathbf{Gray}] \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Hom}(\mathcal{A}, \mathbf{Bicat}) \arrow[l, shift right]\end{tikzcd}$$

whose right triadjoint is the inclusion. Each component of the unit of this triadjunction is a biequivalence in $\mathrm{Hom}(\mathcal{A}, \mathbf{Bicat})$.

Proof. For the nonce, let the left adjoint of the adjoint equivalence of Proposition 3.2.4 be denoted by L. By composing the triadjunctions of Propositions 3.2.2, 3.2.3, and 3.2.4, we have the biequivalences

$$\begin{aligned}\mathrm{Hom}(\mathcal{A}, \mathbf{Bicat})(F, G) &\simeq \mathrm{Hom}(\mathcal{A}, \mathbf{Gray})(\mathrm{st} \circ F, G) \\ &\cong \mathrm{Hom}_{\mathrm{ls}}(\mathcal{A}, \mathbf{Gray})(L(\mathrm{st} \circ F), G) \\ &\cong [\mathcal{A}, \mathbf{Gray}](QL(\mathrm{st} \circ F), G)\end{aligned}$$

trinatural in $F \in \mathrm{Hom}(\mathcal{A}, \mathbf{Bicat})$ and $G \in [\mathcal{A}, \mathbf{Gray}]$.

The component of the unit of this triadjunction at an object F of $\mathrm{Hom}(\mathcal{A}, \mathbf{Bicat})$ is given by the composite of biequivalences

$$F \longrightarrow \mathrm{st} \circ F \longrightarrow L(\mathrm{st} \circ F) \longrightarrow QL(\mathrm{st} \circ F)$$

and is therefore a biequivalence. □

We will denote the value of this left triadjoint at an object F of $\mathrm{Hom}(\mathcal{A}, \mathbf{Bicat})$ by F'.

Remark 3.2.6. The left triadjoint of this triadjunction is a biequivalence on hom-bicategories, and hence gives a triequivalence between $\mathrm{Hom}(\mathcal{A}, \mathbf{Bicat})$ and a full sub-**Gray**-category of $[\mathcal{A}, \mathbf{Gray}]$. For if $L \dashv R \colon \mathcal{B} \longrightarrow \mathcal{A}$ is a triadjunction whose unit has every component $\eta_B \colon \longrightarrow RLB$ a biequivalence in $\mathcal{A}$, then for all objects A and B of $\mathcal{A}$, the pseudofunctor $L \colon \mathcal{A}(A, B) \longrightarrow \mathcal{B}(LA, LB)$ is equivalent to a composite of biequivalences

$$\begin{tikzcd}\mathcal{A}(A,B) \arrow[r, "L"] \arrow[d, "{(1,\eta_B)}"'] \arrow[dr, phantom, "\simeq"] & \mathcal{B}(LA,LB) \arrow[r, "1"] \arrow[d, "{(1,L\eta_B)}"'] \arrow[dr, phantom, "\simeq"] & \mathcal{B}(LA,LB) \\ \mathcal{A}(A,RLB) \arrow[r, "L"'] & \mathcal{B}(LA,LRLB) \arrow[ur, "{(1,\varepsilon_{LB})}"'] & \end{tikzcd}$$

and is therefore a biequivalence.

Remark 3.2.7. Since 2-**Cat** is a combinatorial monoidal model category, the **Gray**-category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ admits a **Gray**-enriched model structure for any category $\mathcal{C}$, called the projective model structure, for which the weak equivalences and fibrations are defined pointwise [GM13]. For each object F of $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$, F' is projectively cofibrant, since for any pair of morphisms in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ as in the left-hand side of (3.6),

$$
\begin{tikzcd}
 & X \arrow[d, "\theta"] \\
F' \arrow[r] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & X \arrow[d, "\theta"] \\
\mathrm{st} \circ F \arrow[ur, dotted] \arrow[r, "\varphi"'] & Y
\end{tikzcd}
\tag{3.6}
$$

such that θ is a pointwise trivial fibration, it suffices by adjointness to give a lifting in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})$ as in the right-hand side of (3.6). The components of such a lifting φ' can be constructed pointwise in **Gray**, and since θ is pointwise an equifibration, the tritransformation coherence cells for φ can be lifted to give the coherence cells for φ', and since θ is pointwise fully faithful on homs, the tritransformation axioms for φ imply the axioms for φ'. Therefore, as in the previous remark, the tricategory $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$ is triequivalent to the full sub-**Gray**-category of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ on the projectively cofibrant objects.

Remark 3.2.8. The arguments of Propositions 3.2.3 and 3.2.4 are equally valid for any cocomplete **Gray**-category $\mathcal{B}$ in place of **Gray**. Hence for any small locally cofibrant **Gray**-category $\mathcal{A}$, there is a **Gray**-adjunction

$$
\begin{tikzcd}
{[\mathcal{A}, \mathcal{B}]} \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Hom}(\mathcal{A}, \mathcal{B}) \arrow[l, shift right, "Q"']
\end{tikzcd}
$$

whose right adjoint is the inclusion and such that all the components of the unit are biequivalences in $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$. In particular this is true for $\mathcal{B} = [\mathcal{C}, \mathbf{Gray}]$, for any small **Gray**-category $\mathcal{C}$.

3.3 Finite trilimits

In this section we apply the results of the previous section to show that in a **Gray**-category, (finite) trilimits can be calculated as (finite) **Gray**-limits. It follows that for any small category $\mathcal{C}$, (finite) trilimits in the tricategories **Bicat** and $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$ can be calculated as (finite) **Gray**-limits in the **Gray**-categories **Gray** and $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ respectively.

Let $F \colon \mathcal{A} \longrightarrow \mathbf{Bicat}$ and $G \colon \mathcal{A} \longrightarrow \mathcal{B}$ be trihomomorphisms between tricategories. The *trilimit of G weighted by F* is an object $\{F, G\}_{\mathrm{tri}}$ of $\mathcal{B}$ together with a family of biequivalences

$$\mathcal{B}(B, \{F, G\}_{\mathrm{tri}}) \simeq \mathrm{Hom}(\mathcal{A}, \mathbf{Bicat})(F, \mathcal{B}(B, G-))$$

trinatural in $B \in \mathcal{B}$. By the Yoneda lemma for tricategories [Buh15], such a trinatural biequivalence is induced by a tritransformation $F \longrightarrow \mathcal{B}(\{F, G\}_{\mathrm{tri}}, G-)$, unique up to biequivalence.

We can use the triadjunctions of Theorem 3.2.5 to calculate trilimits in a **Gray**-category as weighted **Gray**-limits (in the sense of enriched category theory).

Proposition 3.3.1. *Let $\mathcal{A}$ be a small locally cofibrant* **Gray***-category, and $\mathcal{B}$ a complete* **Gray***-category. For any trihomomorphism $F\colon \mathcal{A} \longrightarrow$* **Bicat** *and* **Gray***-functor $G\colon \mathcal{A} \longrightarrow \mathcal{B}$, the trilimit $\{F, G\}_{\text{tri}}$ is biequivalent in $\mathcal{B}$ to the* **Gray***-limit $\{F', G\}$.*

If $\mathcal{B}$ is moreover a cocomplete **Gray***-category, then for any trihomomorphism $H\colon \mathcal{A} \longrightarrow \mathcal{B}$, the trilimit $\{F, H\}_{\text{tri}}$ is biequivalent to the* **Gray***-limit $\{F', QH\}$.*

Proof. By Theorem 3.2.5, we have the trinatural biequivalences

$$\begin{aligned}\mathcal{B}(B, \{F, G\}_{\text{tri}}) &\simeq \mathrm{Hom}(\mathcal{A}, \mathbf{Bicat})(F, \mathcal{B}(B, G-)) \\ &\simeq [\mathcal{A}, \mathbf{Gray}](F', \mathcal{B}(B, G-)) \\ &\cong \mathcal{B}(B, \{F', G\})\end{aligned}$$

and so by the Yoneda lemma for tricategories, there is a biequivalence $\{F, G\}_{\text{tri}} \simeq \{F', G\}$.

Moreover, if $\mathcal{B}$ is cocomplete, then by Remark 3.2.8, there is a **Gray**-functor $QH \in [\mathcal{A}, \mathcal{B}]$ together with a biequivalence $H \simeq QH$ in $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$, and so we have the trinatural biequivalences

$$\begin{aligned}\mathcal{B}(B, \{F, H\}_{\text{tri}}) &\simeq \mathrm{Hom}(\mathcal{A}, \mathbf{Bicat})(F, \mathcal{B}(B, H-)) \\ &\simeq \mathrm{Hom}(\mathcal{A}, \mathbf{Bicat})(F, \mathcal{B}(B, QH-)) \\ &\simeq \mathcal{B}(B, \{F, QH\}_{\text{tri}}).\end{aligned}$$

Therefore we have the biequivalence $\{F, H\}_{\text{tri}} \simeq \{F, QH\}_{\text{tri}} \simeq \{F', QH\}$. □

Note that since any small tricategory is triequivalent to a small locally cofibrant **Gray**-category, there is no real loss of generality from the assumptions of this lemma.

We record the fact that right triadjoints preserve trilimits.

Lemma 3.3.2. *Let $L \dashv R\colon \mathcal{C} \longrightarrow \mathcal{B}$ be a triadjunction. Then for any tricategory $\mathcal{A}$ and trihomomorphisms $F\colon \mathcal{A} \longrightarrow$* **Bicat** *and $G\colon \mathcal{A} \longrightarrow \mathcal{C}$, there is a biequivalence $R\{F, G\}_{\text{tri}} \simeq \{F, RG\}_{\text{tri}}$.*

Proof. We have the trinatural biequivalences

$$\begin{aligned}\mathcal{B}(B, R\{F, G\}_{\text{tri}}) &\simeq \mathcal{C}(LB, \{F, G\}_{\text{tri}}) \\ &\simeq \mathrm{Hom}(\mathcal{A}, \mathbf{Bicat})(F, \mathcal{C}(LB, G-)) \\ &\simeq \mathrm{Hom}(\mathcal{A}, \mathbf{Bicat})(F, \mathcal{C}(B, RG-)) \\ &\simeq \mathcal{B}(B, \{F, RG\}_{\text{tri}})\end{aligned}$$

and so by the Yoneda lemma for tricategories there is a biequivalence $R\{F,G\}_{\mathrm{tri}} \simeq \{F,RG\}_{\mathrm{tri}}$. □

Remark 3.3.3. It follows that we can calculate trilimits in **Bicat** and $\mathrm{Hom}(\mathcal{C},\mathbf{Bicat})$ as **Gray**-limits in **Gray** and $[\mathcal{C},\mathbf{Gray}]$ respectively, for $\mathcal{C}$ a small locally cofibrant **Gray**-category. Let $\mathcal{A}$ be a small locally cofibrant **Gray**-category. By post-composition with the left triadjoints of the previous section, any trihomomorphism $G\colon \mathcal{A} \longrightarrow \mathbf{Bicat}$ or $H\colon \mathcal{A} \longrightarrow \mathrm{Hom}(\mathcal{C},\mathbf{Bicat})$ is biequivalent to a trihomomorphism which factors as $\widehat{G}\colon \mathcal{A} \longrightarrow \mathbf{Gray}$ or $\widehat{H}\colon \mathcal{A} \longrightarrow [\mathcal{C},\mathbf{Gray}]$ followed by the inclusion. Since these inclusions are right triadjoints, they preserve the weighted trilimits of $\widehat{G}$ and $\widehat{H}$, which will therefore be biequivalent to the respective weighted trilimits of G and H. Since the weighted trilimits of $\widehat{G}$ and $\widehat{H}$ can be computed as weighted **Gray**-limits, we conclude that any trilimit in **Bicat** or $\mathrm{Hom}(\mathcal{C},\mathbf{Bicat})$ is biequivalent to the image under inclusion of a **Gray**-limit in **Gray** or $[\mathcal{C},\mathbf{Gray}]$ respectively.

We define a trilimit weighted by a trihomomorphism $F\colon \mathcal{A} \longrightarrow \mathbf{Bicat}$ to be a *finite trilimit* if $\mathcal{A}$ has finitely many objects, and if each hom-bicategory $\mathcal{A}(A,B)$ and each bicategory FA is finite (in the sense that they have finitely many objects, morphisms, and 2-cells).

Recall from [Kel82] that a **Gray**-limit weighted by a **Gray**-functor $F\colon \mathcal{A} \longrightarrow \mathbf{Gray}$ is a finite **Gray**-limit if $\mathcal{A}$ has finitely many isomorphism classes of objects, and each hom-2-category and each 2-category FA is finitely presentable (in the sense that they are finitely presentable objects of the category 2-**Cat**).

Theorem 3.3.4. *Let $\mathcal{A}$ be a locally cofibrant* **Gray***-category and let the trihomomorphism $F\colon \mathcal{A} \longrightarrow$* **Bicat** *be a weight for a finite trilimit. Then the* **Gray***-functor $F'\colon \mathcal{A} \longrightarrow$* **Gray** *is a weight for a finite* **Gray***-limit.*

Proof. It suffices to show that if every bicategory FA is finite, then every 2-category $F'A$ is finitely presentable. Recall that by definition, $F'A = Q(\mathrm{st} \circ F)$.

Firstly, if a bicategory B is finite, then $\mathrm{st}\, B$ is a finitely presentable 2-category. This can be seen as follows. Recall from [KL01, Lei04] that the functor $\mathrm{st}\colon \mathbf{Bicat} \longrightarrow 2\text{-}\mathbf{Cat}$ is equivalent to the underlying left adjoint of the inclusion $T\text{-}\mathrm{Alg}_{\mathrm{s}} \longrightarrow \mathrm{Ps}\text{-}T\text{-}\mathrm{Alg}$ for a finitary 2-monad T on the 2-category of **Cat**-enriched graphs. If B is a finite bicategory, then its underlying **Cat**-graph UB is finitely presentable. It follows that $\mathrm{st}\, B$ can be constructed as the codescent object of a codescent diagram of finitely presentable free T-algebras TUB, T^2UB, and T^3UB, which is a certain 2-colimit in $T\text{-}\mathrm{Alg}_s$ [Lac02a]. Moreover, it is a finite 2-colimit, since it can be constructed out of co-iso-inserters and coequifiers. Hence $\mathrm{st}\, B$ is a finite colimit of finitely presentable objects, and is therefore finitely presentable [Kel82].

Similarly, we can show that Q sends pointwise finitely presentable objects of $\mathrm{Hom}(\mathcal{A},\mathbf{Gray})$ to pointwise finitely presentable objects of $[\mathcal{A},\mathbf{Gray}]$. Recall that Q is equivalent to the left **Gray**-adjoint of the inclusion $T\text{-Alg}_{\text{s}} \longrightarrow \text{Ps-}T\text{-Alg}$ for a finitary **Gray**-monad T on $[\text{ob}\,\mathcal{A}, \mathbf{Gray}]$. For a pseudoalgebra X, QX is constructed as the codescent object of a codescent diagram, whose values are the free algebras TX, T^2X, T^3X, and T^4X, which is a certain **Gray**-colimit in $T\text{-Alg}_s$ [Gur13]. Moreover, it is a finite **Gray**-colimit, since it can be constructed out of co-2-inserters, co-3-inserters, and coequifiers.

Since the **Gray**-monad T preserves filtered colimits, the free T-algebra TX on a finitely presentable object X is finitely presentable. Therefore, since **Gray**-colimits in $[\mathcal{A}, \mathbf{Gray}]$ are computed pointwise, we have that for any pointwise finitely presentable $X \in \text{Hom}(\mathcal{A}, \mathbf{Gray})$ and any $A \in \mathcal{A}$, $(QX)A$ is a finite **Gray**-colimit of finitely presentable objects in the **Gray**-category **Gray**, and is therefore finitely presentable. Hence QX is pointwise finitely presentable. □

Corollary 3.3.5. *A* **Gray***-functor between finitely complete* **Gray***-categories that preserves finite* **Gray***-limits also preserves finite trilimits.*

Remark 3.3.6. We observed in Remark 3.3.3 that trilimits in **Bicat** can be computed as **Gray**-limits in **Gray**. It now follows from Theorem 3.3.4 that finite trilimits in **Bicat** can be computed as finite **Gray**-limits in **Gray**.

We close this section with some remarks on filtered colimits, relevant to the discussion of the following section. We first recall a useful result from the theory of model categories concerning filtered colimits of weak equivalences.

Lemma 3.3.7. *Let $\mathcal{A}$ be a filtered category. Then the colimit* **Gray***-functor* $[\mathcal{A}, \mathbf{Gray}] \longrightarrow$ **Gray** *sends pointwise weak equivalences to weak equivalences.*

Proof. Since the model structure on the category 2-**Cat** is finitely combinatorial [Lac02b], this is a special case of the result of [Dug01] that states that that for a finitely combinatorial model category, weak equivalences are closed under filtered colimits. □

We define the *tricolimit* of a trihomomorphism $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ to be an object $\text{tricolim}\, F$ of $\mathcal{B}$ together with biequivalences

$$\mathcal{B}(\text{tricolim}\, F, B) \simeq \text{Hom}(\mathcal{A}, \mathcal{B})(F, \Delta B)$$

trinatural in $B \in \mathcal{B}$, where $\Delta B\colon \mathcal{A} \longrightarrow \mathcal{B}$ denotes the constant trihomomorphism with value B. A *filtered tricolimit* is a tricolimit of a trihomomorphism $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ whose domain $\mathcal{A}$ is a filtered category.

Lemma 3.3.8. *Let $\mathcal{A}$ be a filtered category and let $F\colon \mathcal{A} \longrightarrow$* **Gray** *be a* **Gray***-functor. Then the tricolimit of the composite $F\colon \mathcal{A} \longrightarrow$* **Gray** $\longrightarrow$ **Bicat** *is biequivalent to the* **Gray***-colimit of F.*

Proof. We have the biequivalences

$$\begin{aligned}
[(\operatorname{tricolim} F)', B] &\simeq \mathbf{Bicat}(\operatorname{tricolim} F, B) \\
&\simeq \operatorname{Hom}(\mathcal{A}, \mathbf{Bicat})(F, \Delta B) \\
&\simeq [\mathcal{A}, \mathbf{Gray}](F', \Delta B) \\
&\cong [\operatorname{colim} F', B]
\end{aligned}$$

trinatural in $B \in \mathbf{Gray}$, and hence by the Yoneda lemma for tricategories, we have a biequivalence $\operatorname{tricolim} F \simeq (\operatorname{tricolim} F)' \simeq \operatorname{colim} F'$.

Since the components of the unit of the triadjunction of Theorem 3.2.5 are biequivalences in $\operatorname{Hom}(\mathcal{A}, \mathbf{Bicat})$, it follows from the triangle identity $\varepsilon_F \circ \eta_F \simeq 1_F$ that the component of the counit $F' \longrightarrow F$ is a biequivalence in $\operatorname{Hom}(\mathcal{A}, \mathbf{Bicat})$, and therefore a pointwise weak equivalence in $[\mathcal{A}, \mathbf{Gray}]$. Hence by Lemma 3.3.7, $\operatorname{colim} F' \longrightarrow \operatorname{colim} F$ is a weak equivalence in $\mathbf{Gray}$. Therefore we have the biequivalences in $\mathbf{Bicat}$ $\operatorname{tricolim} F \simeq (\operatorname{tricolim} F)' \simeq \operatorname{colim} F' \simeq \operatorname{colim} F$. □

3.4 The plus construction for associated 2-stacks

As we recalled at the beginning of Section 2.1, Grothendieck's plus construction sends a presheaf F on a site $(\mathcal{C}, J)$ to the presheaf F^+ given by

$$F^+C = \operatorname*{colim}_{R \in J(C)^{\mathrm{op}}} [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](R, F).$$

The natural tricategorical analogue of this construction is to send a trihomomorphism $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Bicat}$ to the trihomomorphism $F^+\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Bicat}$ with

$$F^+C = \operatorname*{tricolim}_{R \in J(C)^{\mathrm{op}}} \operatorname{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(R, F). \tag{3.7}$$

However, by the results of the previous sections of this chapter, we can simplify this definition. First, we can take F to be a $\mathbf{Gray}$-functor $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Gray}$. Next, note that since R is pointwise cofibrant, it is biequivalent in $\operatorname{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})$ to $\operatorname{st} \circ R$, since for any cofibrant 2-category B, the component $\eta_B\colon B \longrightarrow \operatorname{st} B$ of the adjunction (3.2) is isomorphic to a 2-functor, and hence it follows from the triangle identity $\varepsilon_B \circ \eta_B = 1_B$ that $\varepsilon_B\colon \operatorname{st} B \longrightarrow B$ is a biequivalence in the tricategory $\mathbf{Gray}$. Then since pointwise biequivalences are biequivalences in $\operatorname{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})$, there is a biequivalence

$$\operatorname{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(R, F) \simeq \operatorname{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})(QR, F) \simeq \operatorname{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})(R, F).$$

Finally, since $J(C)^{\mathrm{op}}$ is filtered, Lemma 3.3.8 gives that the filtered tricolimit in (3.7) is biequivalent to a filtered colimit in **Gray**.

Therefore we may define the plus construction as follows.

Definition 3.4.1. The *plus construction* is that operation that sends a **Gray**-functor $F \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ to the **Gray**-functor $F^+ \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ given by

$$F^+C := \operatorname*{colim}_{R \in J(C)^{\mathrm{op}}} \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})(R, F).$$

For convenience, in this section we will often denote the **Gray**-category $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})$ by $\widehat{\mathcal{C}}$, and the image $\mathcal{C}(-, f)\colon \mathcal{C}(-, B) \longrightarrow \mathcal{C}(-, C)$ of a morphism $f\colon B \longrightarrow C$ in $\mathcal{C}$ under the Yoneda embedding $\mathcal{C} \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ by $f\colon B \longrightarrow C$.

We verify that F^+ indeed is a **Gray**-functor. We define F^+ on morphisms of $\mathcal{C}$ as follows. Recall that each morphism $f\colon B \longrightarrow C$ in $\mathcal{C}$ induces a functor $f^*\colon J(C) \longrightarrow J(B)$ which sends a covering sieve $R \in J(C)$ to its pullback $f^*R \in J(B)$, and sends an inclusion $u\colon S \longrightarrow R$ in $J(C)$ to the inclusion $f^*u\colon f^*S \longrightarrow f^*R$ given by the canonical morphism as displayed.

$$\begin{tikzcd}
f^*S \arrow[r, "f'"] \arrow[d, dotted, "f^*u"'] \arrow[dd, bend right] \arrow[dr, phantom, "="] & S \arrow[d, "u"] \\
f^*R \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d] \\
B \arrow[r, "f"'] & C
\end{tikzcd}$$

We can then define F^+f to be the 2-functor induced from the colimit by the displayed cone.

$$\begin{tikzcd}
\widehat{\mathcal{C}}(R, F) \arrow[r, "{(f',1)}"] \arrow[d, "{(u,1)}"'] \arrow[dr, phantom, "="] & \widehat{\mathcal{C}}(f^*R, F) \arrow[r] \arrow[d, "{(f^*u,1)}"'] & F^+B \arrow[dl, phantom, "="] \\
\widehat{\mathcal{C}}(S, F) \arrow[r, "{(f',1)}"'] & \widehat{\mathcal{C}}(f^*S, F) \arrow[ur] &
\end{tikzcd}$$

One can easily check that F^+ defines a functor $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Gray}$. Moreover we have the following.

Lemma 3.4.2. *The construction $F \mapsto F^+$ extends to a* **Gray***-functor*

$$L = (-)^+\colon [\mathcal{C}^{op}, \mathbf{Gray}] \longrightarrow [\mathcal{C}^{op}, \mathbf{Gray}].$$

Proof. We can define L on the 1-cells, 2-cells, and 3-cells of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ simultaneously; to recover the definition on a 2-cell, just take the definition on its identity 3-cell, and so on.

Let

$$\begin{tikzcd} F \arrow[r, bend left, "\theta"] \arrow[r, bend right, "\varphi"'] \arrow[r, phantom, "m\Downarrow \overset{\sigma}{\longrightarrow} \Downarrow n"] & G \end{tikzcd}$$

be a 3-cell in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$. Define the 3-cell σ^+ to have the component σ^+_C at an object C of $\mathcal{C}$ given by the colimit of the family of 3-cells $\widehat{\mathcal{C}}(R,\sigma)$ as in

$$\begin{tikzcd} \widehat{\mathcal{C}}(R,F) \arrow[r, bend left, "(1,\theta)"] \arrow[r, bend right, "(1,\varphi)"'] \arrow[r, phantom, "(1{,}m)\Downarrow \overset{(1,\sigma)}{\longrightarrow} \Downarrow(1{,}n)"] & \widehat{\mathcal{C}}(R,G) \end{tikzcd}$$

which is natural in $R \in J(C)$, since for each inclusion $u\colon S \longrightarrow R$ in $J(C)$, we have

$$\widehat{\mathcal{C}}(1,\sigma) \circ \widehat{\mathcal{C}}(u,1) = \widehat{\mathcal{C}}(u,1) \circ \widehat{\mathcal{C}}(1,\sigma).$$

To show that σ^+ is a perturbation (and hence a natural transformation or a modification in the degenerate cases), it remains to show that for each morphism $f\colon B \longrightarrow C$ in $\mathcal{C}$, we have the equation

$$\begin{tikzcd} F^+C \arrow[r, "F^+f"] & F^+B \arrow[r, bend left, "\theta^+_B"] \arrow[r, bend right, "\varphi^+_B"'] \arrow[r, phantom, "m^+_B\Downarrow \overset{\sigma^+_B}{\longrightarrow} \Downarrow n^+_B"] & G^+B \end{tikzcd} = \begin{tikzcd} F^+C \arrow[r, bend left, "\theta^+_C"] \arrow[r, bend right, "\varphi^+_C"'] \arrow[r, phantom, "m^+_C\Downarrow \overset{\sigma^+_C}{\longrightarrow} \Downarrow n^+_C"] & G^+C \arrow[r, "G^+f"] & G^+B \end{tikzcd}$$

This equation holds, since both sides are induced by the cones

$$(1,\sigma) \circ (f',1) = (f',1) \circ (1,\sigma)$$

whose equality is displayed in the following equation.

$$\begin{tikzcd} \widehat{\mathcal{C}}(R,F) \arrow[r, "(f'{,}1)"] \arrow[d] \arrow[dr, phantom, "="] & \widehat{\mathcal{C}}(f^*R,F) \arrow[r, bend left, "(1,\theta)"] \arrow[r, bend right, "(1,\varphi)"'] \arrow[r, phantom, "(1{,}m)\Downarrow \overset{(1,\sigma)}{\longrightarrow} \Downarrow(1{,}n)"] \arrow[d] \arrow[dr, phantom, "="] & \widehat{\mathcal{C}}(f^*R,G) \arrow[d] \\ F^+C \arrow[r, "F^+f"'] & F^+B \arrow[r, bend left, "\theta^+_B"] \arrow[r, bend right, "\varphi^+_B"'] \arrow[r, phantom, "m^+_B\Downarrow \overset{\sigma^+_B}{\longrightarrow} \Downarrow n^+_B"] & G^+B \end{tikzcd}$$

$$\|$$

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\widehat{\mathcal{C}}(R,F) \arrow[rr, bend left, "{(1,\theta)}"] \arrow[rr, bend right, "{(1,\varphi)}"'] \arrow[rr, phantom, "{(1,m)\Downarrow \xrightarrow{(1,\sigma)} \Downarrow(1,n)}"] \arrow[d] \arrow[drr, phantom, "="] & & \widehat{\mathcal{C}}(R,G) \arrow[r, "{(f',1)}"] \arrow[d] \arrow[dr, phantom, "="] & \widehat{\mathcal{C}}(f^*R,G) \arrow[d] \\
F^+C \arrow[rr, bend left, "\theta^+_C"] \arrow[rr, bend right, "\varphi^+_C"'] \arrow[rr, phantom, "{m^+_C\Downarrow \xrightarrow{\sigma^+_C} \Downarrow n^+_C}"] & & G^+C \arrow[r, "G^+f"'] & G^+B
\end{tikzcd}
$$

The **Gray**-functoriality of L follows immediately from the **Gray**-functoriality of each $\widehat{\mathcal{C}}(R,-)\colon [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}] \longrightarrow \mathbf{Gray}$, since all compositions in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ are defined pointwise. $\square$

For each $F \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ and each $C \in \mathcal{C}$, there exists a canonical 2-functor $\eta_{F,C}\colon FC \longrightarrow F^+C$ given by the composite

$$FC \xrightarrow{\cong} [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](C,F) \longrightarrow \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})(C,F) \longrightarrow F^+C,$$

where we have used the Yoneda lemma for **Gray**-categories, and that the identity on $\mathcal{C}(-,C)$ gives a covering sieve of C.

Lemma 3.4.3. *The 2-functors $\eta_{F,C}\colon FC \longrightarrow F^+C$ are the components of a* **Gray***-natural transformation $\eta_F\colon F \longrightarrow F^+$. Moreover, the natural transformations η_F are the components of a* **Gray***-natural transformation $\eta\colon 1 \longrightarrow L$.*

Proof. For each morphism $f\colon B \longrightarrow C$ in $\mathcal{C}$, the naturality square for η_F is given by the pasting composite

$$
\begin{tikzcd}
FC \arrow[r] \arrow[d, "Ff"'] \arrow[dr, phantom, "="] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](C,F)} \arrow[r] \arrow[d, "{(f,1)}"'] \arrow[dr, phantom, "="] & \widehat{\mathcal{C}}(C,F) \arrow[r] \arrow[d, "{(f,1)}"'] \arrow[dr, phantom, "="] & F^+C \arrow[d, "F^+f"] \\
FB \arrow[r] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](B,F)} \arrow[r] & \widehat{\mathcal{C}}(B,F) \arrow[r] & F^+B
\end{tikzcd}
$$

To show that η is a **Gray**-natural transformation is to show that for all $F, G \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$, the square

$$
\begin{tikzcd}
{[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](F,G)} \arrow[r, "1"] \arrow[d, "L"'] \arrow[dr, phantom, "="] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](F,G)} \arrow[d, "{(1,\eta_G)}"] \\
{[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](F^+,G^+)} \arrow[r, "{(\eta_F,1)}"'] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](F,G^+).}
\end{tikzcd}
$$

commutes. It suffices to show that for all perturbations σ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](F,G)$, we have the equation $\sigma^+ \circ \eta_F = \eta_G \circ \sigma$, that is, for each $C \in \mathcal{C}$ we have $\sigma^+_C \circ \eta_{F,C} = \eta_{G,C} \circ \sigma_C$. This follows from the following equation.

$$
\begin{tikzcd}[column sep=large, row sep=huge]
FC \arrow[r, "\eta_{F,C}"] \arrow[d] \arrow[dr, phantom, "="] & F^+C \arrow[r, bend left, "\theta^+_C"] \arrow[r, bend right, "\varphi^+_C"'] \arrow[r, phantom, "m^+_C\Downarrow \overset{\sigma^+_C}{\longrightarrow} \Downarrow n^+_C"] \arrow[dr, phantom, "="] & G^+C \\
{[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](C,F)} \arrow[r] & \widehat{\mathcal{C}}(C,F) \arrow[u] \arrow[r, bend left, "{(1,\theta)}"] \arrow[r, bend right, "{(1,\varphi)}"'] \arrow[r, phantom, "{(1,m)}\Downarrow \overset{(1,\sigma)}{\longrightarrow} \Downarrow{(1,n)}"] & \widehat{\mathcal{C}}(C,G) \arrow[u]
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}[column sep=large, row sep=huge]
FC \arrow[r, bend left, "\theta_C"] \arrow[r, bend right, "\varphi_C"'] \arrow[r, phantom, "m_C\Downarrow \overset{\sigma_C}{\longrightarrow} \Downarrow n_C"] \arrow[d] \arrow[dr, phantom, "="] & GC \arrow[r, "\eta_G"] \arrow[d] \arrow[dr, phantom, "="] & G^+C \\
{[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](C,F)} \arrow[r, bend left, "{(1,\theta)}"] \arrow[r, bend right, "{(1,\varphi)}"'] \arrow[r, phantom, "{(1,m)}\Downarrow \overset{(1,\sigma)}{\longrightarrow} \Downarrow{(1,n)}"] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](C,G)} \arrow[r] & \widehat{\mathcal{C}}(C,G) \arrow[u]
\end{tikzcd}
$$

□

Proposition 3.4.4. *The* **Gray***-functor* $L\colon [\mathcal{C}^{op}, \mathbf{Gray}] \longrightarrow [\mathcal{C}^{op}, \mathbf{Gray}]$ *preserves finite* **Gray***-limits and pointwise weak equivalences, and is accessible.*

Proof. Recall that 2-**Cat** with the Gray tensor product is locally finitely presentable as a closed category, in the sense of [Kel82]. Hence finite **Gray**-limits commute with filtered colimits in **Gray**.

Each **Gray**-functor $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](QR, -)$ preserves pointwise weak equivalences, since it is trinaturally biequivalent to $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(R, -)$, which preserves pointwise biequivalences.

Since $\mathcal{C}$ is small, there exists a cardinal κ such that all covering sieves R are κ-presentable objects of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$, and as in the proof of Theorem 3.3.4, we have that all the objects QR of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ are also κ-presentable.

Hence we have that

$$
\begin{aligned}
L(-)C &= \operatorname*{colim}_{R \in J(C)^{\mathrm{op}}} \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})(R, -) \\
&\cong \operatorname*{colim}_{R \in J(C)^{\mathrm{op}}} [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](QR, -)
\end{aligned}
$$

is a filtered colimit of **Gray**-functors that preserve finite **Gray**-limits, weak equivalences, and κ-filtered colimits. Since limits and colimits in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ are computed pointwise, it follows that L preserves finite **Gray**-limits, pointwise weak equivalences, and κ-filtered colimits. $\square$

We now spell out the definitions of the cells of the 2-categories $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})(R, F)$ for a covering sieve $R \in J(C)$ and a **Gray**-functor $F \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$. We call the objects of these 2-categories "matching families". These definitions are further special cases of the definitions of tritransformation, trimodification, and perturbation given in Appendix A, and we make the same simplifying assumption of normality as discussed therein.

To further simplify notation, we will denote $(Ff)x$ by $x \cdot f$ for each $f \colon A \longrightarrow B$ in $\mathcal{C}$ and x a cell of FB.

Definition 3.4.5. Let R be a sieve on an object C of $\mathcal{C}$, and let $F \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ be a **Gray**-functor. A (normal) *matching family* $x \colon R \longrightarrow F$ consists of the following data:

- for each $f \colon B \longrightarrow C$ in R, an object x_f of FB
- for each $A \xrightarrow{g} B \xrightarrow{f} C$ in R, an adjoint equivalence in FA

$$x_{f,g} \colon x_{fg} \longrightarrow x_f \cdot g$$

- for each $Z \xrightarrow{h} A \xrightarrow{g} B \xrightarrow{f} C$ in R, an invertible 2-cell in FZ

$$\begin{tikzcd} & x_{fg} \cdot h \arrow[dr, "x_{f,g} \cdot h"] \arrow[d, phantom, "\Downarrow x_{f,g,h}"] & \\ x_{fgh} \arrow[ur, "x_{fg,h}"] \arrow[rr, "x_{f,gh}"'] & {} & x_f \cdot gh \end{tikzcd}$$

subject to the following axioms:

- for each $f \colon B \longrightarrow C$ in R, $x_{f,1_B} = 1_{x_f}$
- for each $A \xrightarrow{g} B \xrightarrow{f} C$ in R, $x_{f,1_B,g} = 1_{x_{f,g}} = x_{f,g,1_A}$
- for each $Y \xrightarrow{k} Z \xrightarrow{h} A \xrightarrow{g} B \xrightarrow{f} C$ in R,

$$\begin{tikzcd} x_{fgh} \cdot k \arrow[r, "x_{fg,h} \cdot k"] \arrow[dr, phantom, "\Downarrow x_{fg,h,k}"] & x_{fg} \cdot hk \arrow[d, "x_{f,g} \cdot hk"] \arrow[dl, phantom, "\Downarrow x_{f,g,hk}"] \\ x_{fghk} \arrow[u, "x_{fgh,k}"] \arrow[ur, "x_{fg,hk}"] \arrow[r, "x_{f,ghk}"'] & x_f \cdot ghk \end{tikzcd} = \begin{tikzcd} x_{fgh} \cdot k \arrow[r, "x_{fg,h} \cdot k"] \arrow[dr, "x_{f,gh} \cdot k"] \arrow[dr, phantom, bend left, "\Downarrow x_{f,g,h} \cdot k"] & x_{fg} \cdot hk \arrow[d, "x_{f,g} \cdot hk"] \\ x_{fghk} \arrow[u, "x_{fgh,k}"] \arrow[r, "x_{f,ghk}"'] \arrow[ur, phantom, "\Downarrow x_{f,gh,k}"] & x_f \cdot ghk \end{tikzcd}$$

Definition 3.4.6. A *morphism of matching families* $\varphi\colon x \longrightarrow y$ consists of the following data:

- for each $f\colon B \longrightarrow C$ in R, a morphism $\varphi_f\colon x_f \longrightarrow y_f$ in FB
- for each $A \xrightarrow{g} B \xrightarrow{f} C$ in R, an invertible 2-cell in FA

$$\begin{tikzcd}
x_{fg} \arrow[r, "x_{f,g}"] \arrow[d, "\varphi_{fg}"'] \arrow[dr, phantom, "\Downarrow\varphi_{f,g}"] & x_f \cdot g \arrow[d, "\varphi_f \cdot g"] \\
y_{fg} \arrow[r, "y_{f,g}"'] & y_f \cdot g
\end{tikzcd}$$

subject to the following axioms:

- for each $f\colon B \longrightarrow C$ in R, $\varphi_{f,1_B} = 1_{\varphi_f}$
- for each $Z \xrightarrow{h} A \xrightarrow{g} B \xrightarrow{f} C$ in R,

$$\begin{tikzcd}
& x_{fg} \cdot h \arrow[dr, "x_{f,g} \cdot h"] \arrow[dd, "\varphi_{fg} \cdot h"] & \\
x_{fgh} \arrow[ur, "x_{fg,h}"] \arrow[dd, "\varphi_{fgh}"'] \arrow[r, phantom, "\Downarrow\varphi_{fg,h}"] & \arrow[r, phantom, "\Downarrow\varphi_{f,g} \cdot h"] & x_f \cdot gh \arrow[dd, "\varphi_f \cdot gh"] \\
& y_{fg} \cdot h \arrow[dr, "y_{f,g} \cdot h"] \arrow[d, phantom, "\Downarrow y_{f,g,h}"] & \\
y_{fgh} \arrow[ur, "y_{fg,h}"] \arrow[rr, "y_{f,gh}"'] & \, & y_f \cdot gh
\end{tikzcd}
=
\begin{tikzcd}
& x_{fg} \cdot h \arrow[dr, "x_{f,g} \cdot h"] \arrow[d, phantom, "\Downarrow x_{f,g,h}"] & \\
x_{fgh} \arrow[ur, "x_{fg,h}"] \arrow[rr, "x_{f,gh}"'] \arrow[d, "\varphi_{fgh}"'] & \arrow[d, phantom, "\Downarrow\varphi_{f,gh}"] & x_f \cdot gh \arrow[d, "\varphi_f \cdot gh"] \\
y_{fgh} \arrow[rr, "y_{f,gh}"'] & \, & y_f \cdot gh
\end{tikzcd}$$

Definition 3.4.7. A *transformation of morphisms of matching families* $\alpha\colon \varphi \longrightarrow \psi\colon x \longrightarrow y$ consists of, for each $f\colon B \longrightarrow C$ in R, a 2-cell in FB

$$\begin{tikzcd}
x_f \arrow[r, bend left, "\varphi_f"] \arrow[r, bend right, "\psi_f"'] \arrow[r, phantom, "\Downarrow\alpha_f"] & y_f
\end{tikzcd}$$

such that for each $A \xrightarrow{g} B \xrightarrow{f} C$ in R,

$$\begin{tikzcd}
x_{fg} \arrow[r, "x_{f,g}"] \arrow[d, bend right, "\psi_{fg}"'] \arrow[d, bend left, "\varphi_{fg}"] \arrow[d, phantom, "\overset{\alpha_{fg}}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow\varphi_{f,g}"] & x_f \cdot g \arrow[d, "\varphi_f \cdot g"] \\
y_{fg} \arrow[r, "\varphi_f"'] & y_f \cdot g
\end{tikzcd}
=
\begin{tikzcd}
x_{fg} \arrow[r, "x_{f,g}"] \arrow[d, "\psi_{fg}"'] \arrow[dr, phantom, "\psi_{f,g}\Downarrow"] & x_f \cdot g \arrow[d, bend right, "\psi_f \cdot g"'] \arrow[d, bend left, "\varphi_f \cdot g"] \arrow[d, phantom, "\overset{\alpha_f \cdot g}{\Leftarrow}"] \\
y_{fg} \arrow[r, "\varphi_f"'] & y_f \cdot g
\end{tikzcd}$$

The compositions in the 2-category $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})(R, F)$ of matching families are inherited from the compositions in the 2-categories FB, for $B \in \mathcal{C}$.

Recall that an object F of $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$ is called a 2-*stack* if for every covering sieve $R \in J(C)$, the canonical homomorphism

$$FC \longrightarrow \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(R, F)$$

is a biequivalence. Note that for an object F of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$, it is equivalent that each 2-functor

$$FC \longrightarrow \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})(R, F) \qquad \text{or} \qquad FC \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](QR, F)$$

is a biequivalence.

Remark 3.4.8. We say that a pseudofunctor $F\colon A \longrightarrow B$ is

(i) *biessentially surjective* if for every object b of B, there exists an object a of A and an equivalence $Fa \longrightarrow b$ in B, and

(ii) *an equivalence on homs* if for all pairs of objects a, a' of A, the functor $F_{a,a'}\colon A(a, a') \longrightarrow B(Fa, Fa')$ is an equivalence of categories.

By definition, a pseudofunctor is a biequivalence if and only if it is both biessentially surjective and an equivalence on homs.

Normal terminology would be to call (i) "biessentially surjective on objects", and (ii) "locally an equivalence". For this thesis we have adopted this alternative terminology firstly to avoid such circumlocutions as "locally pointwise esentially surjective on objects on homs", and secondly because the use of "locally (P)" to mean that each functor $F_{a,a'}$ has the property (P) conflicts with our use of the term "locally" as in Remark 2.1.12. We instead say "(P) on homs", for example, "essentially surjective on homs", "faithful on homs", etc.

Lemma 3.4.9. *Let $F \in [\mathcal{C}^{op}, \mathbf{Gray}]$. Then F is a 2-stack if and only if $\eta_F\colon F \longrightarrow F^+$ is a pointwise weak equivalence in $[\mathcal{C}^{op}, \mathbf{Gray}]$, that is, if and only if $\eta_{F,C}\colon FC \longrightarrow F^+C$ is a biequivalence for all $C \in \mathcal{C}$.*

Proof. Suppose F is a 2-stack. Then the component of the unit

$$FC \cong \operatorname*{colim}_{R \in J(C)^{\mathrm{op}}} FC \longrightarrow \operatorname*{colim}_{R \in J(C)^{\mathrm{op}}} \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})(R, F) = F^+C$$

is a filtered colimit of weak equivalences in **Gray**, and is therefore a weak equivalence.

Conversely, suppose that $\eta_F \colon F \longrightarrow F^+$ is a pointwise weak equivalence. Let $R \in J(C)$ be a covering sieve. It is required to show that the 2-functor

$$FC \longrightarrow \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})(R, F) =: \widehat{\mathcal{C}}(R, F) \tag{3.8}$$

is a biequivalence.

We first show that it is faithful on homs. Let $\alpha, \beta \colon \varphi \longrightarrow \psi \colon x \longrightarrow y$ be a parallel pair of 2-cells in FC, and suppose $\alpha|_R = \beta|_R$ in $\widehat{\mathcal{C}}(R, F)$. This implies that $\eta_{F,C}(\alpha) = \eta_{F,C}(\beta)$ in F^+C, and since $\eta_{F,C}$ is faithful on homs, we have that $\alpha = \beta$.

Next, we show that (3.8) is full on homs. Let $\varphi, \psi \colon x \longrightarrow y$ be a parallel pair of morphisms in FC, and let $\beta \colon \varphi|_R \longrightarrow \psi|_R$ be a transformation. Since $\eta_{F,C}$ is full on homs, there exists a 2-cell $\alpha \colon \varphi \longrightarrow \psi$ in FC and a covering sieve $S \subseteq R$ of C such that $\alpha|_S = \beta|_S$.

Let $f \colon B \longrightarrow C$ be a morphism in R. Then f^*S is a covering sieve of B and $\alpha_f|_{f^*S} = \beta_f|_{f^*S}$. Since $FB \longrightarrow \widehat{\mathcal{C}}(f^*S, F)$ is faithful on homs, we have that $\alpha_f = \beta_f$ in FB.

Next, we show that (3.8) is essentially surjective on homs. Let x and y be objects of FC, and let $\psi \colon x|_R \longrightarrow y|_R$ be a morphism of matching families. Since $\eta_{F,C}$ is essentially surjective on homs, there exists a morphism $\varphi \colon x \longrightarrow y$ in FC, a covering sieve $S \subseteq R$ of C, and an isomorphism $\beta \colon \varphi|_S \longrightarrow \psi|_S$ in $\widehat{\mathcal{C}}(S, F)$.

Let $f \colon B \longrightarrow C$ be a morphism in R. Then f^*S is a covering sieve of B, and $\beta_f|_{f^*S} \colon \varphi_f|_{f^*S} \longrightarrow \psi_f|_{f^*S}$ is an isomorphism in $\widehat{\mathcal{C}}(f^*S, F)$. Since $FB \longrightarrow \widehat{\mathcal{C}}(f^*S, F)$ is fully faithful on homs, there exists an isomorphism $\alpha_f \colon \varphi_f \longrightarrow \psi_f$ in FB such that $\alpha_f|_{f^*S} = \beta_f|_{f^*S}$.

Let $A \xrightarrow{g} B \xrightarrow{f} C$ be in R. Then $(fg)^*S$ is a covering sieve of A, and since $FA \longrightarrow \widehat{\mathcal{C}}((fg)^*S, F)$ is faithful on homs, the transformation axiom for β implies the transformation axiom for the 2-cells α_f.

Finally, we show that (3.8) is biessentially surjective. Let $y \colon R \longrightarrow F$ be a matching family. Since $\eta_{F,C}$ is biessentially surjective, there exists an object x of FC, a covering sieve $S \subseteq R$ of C, and an equivalence $\psi \colon x|_S \longrightarrow y_S$ in $\widehat{\mathcal{C}}(S, F)$.

Let $f \colon B \longrightarrow C$ be in R. Then f^*S is a covering sieve of B and $\psi_f|_{f^*S} \colon x_f|_{f^*S} \longrightarrow y_f|_{f^*S}$ is an equivalence in $\widehat{\mathcal{C}}(f^*S, F)$. Since $FB \longrightarrow \widehat{\mathcal{C}}(f^*S, F)$ is an equivalence on homs, there exists an equivalence $\varphi_f \colon x_f \longrightarrow y_f$ in FB and an isomorphism transformation $\alpha_f \colon \varphi_f|_{f^*S} \longrightarrow \psi_f|_{f^*S}$ in $\widehat{\mathcal{C}}(f^*S, F)$.

Let $A \xrightarrow{g} B \xrightarrow{f} C$ be in R. Then $(fg)^*S$ is a covering sieve of A, and there is an isomorphism transformation

$$\begin{tikzcd}
x_{fg}|_{(fg)^*S} \arrow[rr, bend left=20, "(\varphi_f\cdot g)|_{(fg)^*S}"] \arrow[rr, "(\psi_f\cdot g)|_{(fg)^*S}"'] \arrow[rr, phantom, bend left=10, "\Downarrow(\alpha_f\cdot g)|_{(fg)^*S}"] \arrow[dr, "\psi_{fg}|_{(fg)^*S}"] \arrow[dr, bend right=40, "\varphi_{fg}|_{(fg)^*S}"'] \arrow[dr, phantom, bend right=20, "\Swarrow\alpha_{fg}^{-1}|_{(fg)^*S}"] & \arrow[d, phantom, "\Downarrow\psi_{f,g}|_{(fg)^*S}"] & (y_f\cdot g)|_{(fg)^*S} \\
& y_{fg}|_{(fg)^*S} \arrow[ur, bend right=30, "y_{f,g}|_{(fg)^*S}"'] &
\end{tikzcd}$$

in $\widehat{\mathcal{C}}((fg)^*S, F)$. Since $FA \longrightarrow \widehat{\mathcal{C}}((fg)^*S, F)$ is fully faithful on homs, there exists a unique isomorphism 2-cell

$$\begin{tikzcd}
x_{fg} \arrow[rr, "\varphi_f\cdot g"] \arrow[dr, "\varphi_{fg}"'] \arrow[rr, phantom, shift right=12, "\Downarrow\varphi_{f,g}"] & & y_f\cdot g \\
& y_{fg} \arrow[ur, "y_{f,g}"'] &
\end{tikzcd}$$

in FA which restricts on $(fg)^*S$ to the above pasting composite. If $g = 1_B$, then $\varphi_{f,1_B}$ is the extension of an identity, and hence by uniqueness is an identity.

Let $Z \xrightarrow{h} A \xrightarrow{g} B \xrightarrow{f} C$ be in R. Then $(fgh)^*S$ is a covering sieve of Z, and since $FZ \longrightarrow \widehat{\mathcal{C}}((fgh)^*S, F)$ is faithful on homs, the morphism axiom for ψ implies the morphism axiom for φ. $\square$

For each $G \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ and $C \in \mathcal{C}$, let $\ulcorner x \urcorner \colon C \longrightarrow G$ denote the object corresponding to $x \in GC$ under the isomorphism $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](\mathcal{C}(-, C), G) \cong GC$ of the Yoneda lemma.

Lemma 3.4.10. *Let $F \in [\mathcal{C}^{op}, \mathbf{Gray}]$. Then there exists an equivalence* **Gray***-modification $\eta_{F^+} \simeq \eta_F^+$ in $[\mathcal{C}^{op}, \mathbf{Gray}]$.*

Proof. For each matching family $x \colon R \longrightarrow F$, there is a canonical morphism of matching families

$$\begin{tikzcd}
R \arrow[r, "x"] \arrow[d] \arrow[dr, phantom, "\overset{\boldsymbol{x}}{\Rightarrow}"] & F \arrow[d, "\eta_F"] \\
C \arrow[r, "\ulcorner x \urcorner"'] & F^+
\end{tikzcd}$$

in $\widehat{\mathcal{C}}(R, F^+)$.

Its component at a morphism $f \colon B \longrightarrow C$ in R is given by the morphism of matching families

$$\begin{tikzcd}
& B \arrow[dl, "\ulcorner f \urcorner"'] \arrow[dr, "x_f"] \arrow[d, phantom, "\overset{\boldsymbol{x}_f}{\Rightarrow}"] & \\
R \arrow[rr, "x"'] & {} & F
\end{tikzcd}$$

in $\widehat{\mathcal{C}}(B, F)$ whose component at a morphism $g \colon A \longrightarrow B$ in $\mathcal{C}$ is $x_{f,g} \colon x_{fg} \longrightarrow x_f \cdot g$ and whose component at a pair of arrows $Z \xrightarrow{h} A \xrightarrow{g} B$ in $\mathcal{C}$ is

$$\begin{tikzcd}
x_{fgh} \arrow[r, "x_{fg}\cdot h"] \arrow[d, "x_{f,gh}"'] \arrow[dr, phantom, "\Downarrow x_{f,g,h}"] & x_{fg}\cdot h \arrow[d, "x_{f,g}\cdot h"] \\
x_f \cdot gh \arrow[r, "1"'] & x_f \cdot gh
\end{tikzcd}$$

Its component at $A \xrightarrow{g} B \xrightarrow{f} C$ in R is given by the transformation

$$\begin{tikzcd}
& A \arrow[d, "g"] & \\
& B \arrow[dl, "\ulcorner f \urcorner"'] \arrow[dr, "x_f"] & \\
R \arrow[rr, "x"'] \arrow[rr, phantom, bend left=20, "\overset{\boldsymbol{x}_f}{\Rightarrow}"] & & F
\end{tikzcd}
\quad \overset{\boldsymbol{x}_{f,g}}{\Rrightarrow} \quad
\begin{tikzcd}
& A \arrow[dl, "\ulcorner fg \urcorner"'] \arrow[dr, bend right=15, "x_{fg}"'] \arrow[dr, bend left=40, "x_f \cdot g"] \arrow[dr, phantom, "\overset{x_{f,g}}{\Rightarrow}"] & \\
R \arrow[rr, "x"'] \arrow[rr, phantom, bend left=20, "\underset{\boldsymbol{x}_{fg}}{\Rightarrow}"] & & F
\end{tikzcd}$$

whose component at $h\colon Z \longrightarrow A$ in $\mathcal{C}$ is

$$\begin{tikzcd}
x_{fgh} \arrow[rr, "x_{f,gh}"] \arrow[dr, "x_{fg,h}"'] & \Downarrow x^{-1}_{f,g,h} & x_f \cdot gh \\
& x_{fg}\cdot h \arrow[ur, "x_{f,g}\cdot h"'] &
\end{tikzcd}$$

Similarly, for each morphism of matching families $\varphi\colon x \longrightarrow y$, there is a canonical transformation

$$\begin{tikzcd}
R \arrow[r, "y"] \arrow[d] \arrow[dr, phantom, "\overset{\boldsymbol{y}}{\Rightarrow}"] & F \arrow[d, "\eta_F"] \\
C \arrow[r, bend left, "\ulcorner y \urcorner"] \arrow[r, bend right, "\ulcorner x \urcorner"'] \arrow[r, phantom, "\Uparrow \ulcorner \varphi \urcorner"] & F^+
\end{tikzcd}
\quad \overset{\boldsymbol{\varphi}}{\Rrightarrow} \quad
\begin{tikzcd}
R \arrow[r, bend left, "y"] \arrow[r, bend right, "x"'] \arrow[r, phantom, "\Uparrow \varphi"] \arrow[d] \arrow[dr, phantom, "\underset{\boldsymbol{x}}{\Rightarrow}"] & F \arrow[d, "\eta_F"] \\
C \arrow[r, "\ulcorner x \urcorner"'] & F^+
\end{tikzcd}$$

in $\widehat{\mathcal{C}}(R, F^+)$. Its component at a morphism $f\colon B \longrightarrow C$ in R is given by the transformation

$$\begin{tikzcd}
& B \arrow[dl, "\ulcorner f \urcorner"'] \arrow[dr, "y_f"] \arrow[d, phantom, "\overset{\boldsymbol{y}}{\Rightarrow}"] & \\
R \arrow[rr, bend left=10, "y"] \arrow[rr, bend right=10, "x"'] \arrow[rr, phantom, "\Uparrow \varphi"] & & F
\end{tikzcd}
\quad \overset{\boldsymbol{\varphi}_f}{\Rrightarrow} \quad
\begin{tikzcd}
& B \arrow[dl, "\ulcorner f \urcorner"'] \arrow[dr, bend right=15, "x_f"'] \arrow[dr, bend left=40, "y_f"] \arrow[dr, phantom, "\overset{\varphi_f}{\Rightarrow}"] & \\
R \arrow[rr, "x"'] \arrow[rr, phantom, bend left=20, "\underset{\mathbf{x}_f}{\Rightarrow}"] & & F
\end{tikzcd}$$

in $\widehat{\mathcal{C}}(B, F)$ whose component at a morphism $g\colon A \longrightarrow B$ in $\mathcal{C}$ is

$$\begin{tikzcd}
x_{fg} \arrow[r, "\varphi_{fg}"] \arrow[d, "x_{f,g}"'] \arrow[dr, phantom, "\Downarrow \varphi^{-1}_{f,g}"] & y_{fg} \arrow[d, "y_{f,g}"] \\
x_f \cdot g \arrow[r, "\varphi_f \cdot g"'] & y_f \cdot g
\end{tikzcd}$$

Each such morphism $\boldsymbol{x}$ and transformation $\boldsymbol{\varphi}\colon \boldsymbol{x} \longrightarrow \boldsymbol{y}$ gives a morphism $w_{C,x}\colon \eta_{F^+,C}(x) \longrightarrow (\eta_F^+)_C(x)$ and a 2-cell

$$\begin{tikzcd}
\eta_{F^+,C}(x) \arrow[r, "\eta_{F^+,C}(\varphi)"] \arrow[d, "w_{C,x}"'] \arrow[dr, phantom, "\Downarrow w_{C,\varphi}"] & \eta_{F^+,C}(y) \arrow[d, "w_{C,y}"] \\
(\eta_F^+)_C(x) \arrow[r, "(\eta_F^+)_C(\varphi)"'] & (\eta_F^+)_C(y)
\end{tikzcd}$$

in $F^{++}C$. This defines a pseudonatural transformation $w_C\colon \eta_{F^+,C} \longrightarrow (\eta_F^+)_C\colon F^+C \longrightarrow F^{++}C$, which is the component at C of a **Gray**-modification $w\colon \eta_{F^+} \longrightarrow \eta_F^+\colon F^+ \longrightarrow F^{++}$. The (pseudo) inverses of the data defining w assemble to define a pseudo inverse **Gray**-modification for w. $\square$

Proposition 3.4.11. *Let $F, G \in [\mathcal{C}^{op}, \mathbf{Gray}]$ be* **Gray***-functors, and suppose that G is a 2-stack. Then the pseudofunctor*

$$(\eta_F, 1)\colon \operatorname{Hom}(\mathcal{C}^{op}, \mathbf{Bicat})(F^+, G) \longrightarrow \operatorname{Hom}(\mathcal{C}^{op}, \mathbf{Bicat})(F, G)$$

is a biequivalence.

Proof. For simplicity, we will denote the hom-bicategories $\operatorname{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(F, G)$ simply by (F, G). Similar to the above, for $F, G \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$, we can extend the plus construction to give pseudofunctors $+\colon (F, G) \longrightarrow (F^+, G^+)$ such that η remains trinatural. Furthermore, since G is a 2-stack, the **Gray**-natural transformation $\eta_G\colon G \longrightarrow G^+$ has a trinatural weak inverse $\eta_G^\bullet$.

Then the pseudofunctor $(\eta_F, 1)\colon (F^+, G) \longrightarrow (F, G)$ has a weak inverse given by the composite

$$(F, G) \xrightarrow{\ +\ } (F^+, G^+) \xrightarrow{(1,\eta_G^\bullet)} (F^+, G). \tag{3.9}$$

The diagram

$$\begin{tikzcd}
(F,G) \arrow[r, "+"] \arrow[d, "1"'] \arrow[dr, phantom, "\simeq"] & (F^+,G^+) \arrow[r, "(1,\eta_G^\bullet)"] \arrow[d, "(\eta_F,1)"] \arrow[dr, phantom, "\simeq"] & (F^+,G) \arrow[d, "(\eta_F,1)"] \\
(F,G) \arrow[r, "(1,\eta_G)"'] \arrow[rr, bend right, "1"'] & (F,G^+) \arrow[r, "(1,\eta_G^\bullet)"'] \arrow[d, phantom, "\simeq"] & (F,G)
\end{tikzcd}$$

witnesses (3.9) as a weak right inverse of $(\eta_F, 1)$. By Lemma 3.4.10, there exists an equivalence $\eta_{F^+} \simeq \eta_F^+$, and so the diagram

$$\begin{tikzcd}
& (F^+,G) \arrow[rr, "1"] \arrow[dr, "{(1,\eta_G)}"] \arrow[d, phantom, "\simeq"] \arrow[r, phantom, "\simeq"] & & (F^+,G) \\
(F^+,G) \arrow[ur, "1"] \arrow[r, "+"'] \arrow[dr, "{(\eta_F,1)}"'] & (F^{++},G^+) \arrow[r, shift left, "{(\eta_{F^+},1)}"] \arrow[r, shift right, "{(\eta_F^+,1)}"'] \arrow[r, phantom, "\simeq"] & (F^+,G^+) \arrow[ur, "{(1,\eta_G^\bullet)}"'] & \\
& (F,G) \arrow[ur, "+"'] \arrow[u, phantom, "\simeq"] & &
\end{tikzcd}$$

witnesses (3.9) as a weak left inverse of $(\eta_F, 1)$. $\square$

At this point we could show that each iteration of L sends $F \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ to an object $G \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ for which the 2-functor

$$GC \longrightarrow \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})(R, G)$$

is successively faithful on homs, fully faithful on homs, an equivalence on homs, and finally a biequivalence, for all covering sieves $R \in J(C)$, therefore showing that L^4F is the associated 2-stack of F. However, for variety, we instead take the opportunity to show how the transfinite methods of [Kel80] may be adapted to higher categorical contexts.

We define the *pseudocolimit* of a $\mathbf{Gray}$-functor $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ to be an object $\operatorname{pscolim} G$ of $\mathcal{B}$ together with a $\mathbf{Gray}$-natural isomorphism

$$\mathcal{B}(\operatorname{pscolim} G, X) \cong \mathrm{Hom}(\mathcal{A}, \mathcal{B})(G, \Delta X).$$

Note that if $\mathcal{A}$ is a category, then by the $\mathbf{Gray}$-adjunction of Remark 3.2.8, there is an isomorphism $\operatorname{pscolim} G \cong \operatorname{colim} QG$. Hence pseudocolimits indexed by a category $\mathcal{A}$ enjoy the same properties as colimits indexed by $\mathcal{A}$. For instance, if $\mathcal{A}$ is filtered, then finite $\mathbf{Gray}$-limits and pointwise weak equivalences in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ commute with filtered pseudocolimits; also, if $\mathcal{A}$ is κ-filtered for some cardinal κ, then a $\mathbf{Gray}$-functor that preserves κ-filtered colimits will also preserve κ-filtered pseudocolimits.

Let α be an ordinal. We define the transfinite iterate $L^\alpha\colon [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}] \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ of the plus construction L by transfinite recursion as follows. Define $L^0 = 1$. If $\alpha = \beta + 1$, then define $L^{\beta+1} = LL^\beta$, and we have the map $\eta_\beta^{\beta+1} = \eta L^\beta \colon L^\beta \longrightarrow L^{\beta+1}$. If α is a limit ordinal, then define $L^\alpha = \operatorname{pscolim}_{\beta<\alpha} L^\beta$, where the pseudocolimit is taken pointwise, and we have the maps $\eta_\beta^\alpha \colon L^\beta \longrightarrow L^\alpha$ arising from the pseudocolimit cone.

Lemma 3.4.12. *Suppose $P\colon [\mathcal{C}^{op}, \mathbf{Gray}] \longrightarrow [\mathcal{C}^{op}, \mathbf{Gray}]$ is a $\mathbf{Gray}$-functor that preserves finite $\mathbf{Gray}$-limits and pointwise weak equivalences. Then the composite trihomomorphism*

$$\mathrm{Hom}(\mathcal{C}^{op}, \mathbf{Bicat}) \longrightarrow [\mathcal{C}^{op}, \mathbf{Gray}] \xrightarrow{P} [\mathcal{C}^{op}, \mathbf{Gray}] \longrightarrow \mathrm{Hom}(\mathcal{C}^{op}, \mathbf{Bicat})$$

preserves finite trilimits.

Proof. The right-hand inclusion is a right triadjoint, and so preserves trilimits. Since P preserves finite **Gray**-limits, it also preserves finite trilimits. The left triadjoint preserves trilimits up to a pointwise weak equivalence $\{F,G\}'_{\mathrm{tri}} \simeq \{F',G\}' \longrightarrow \{F',G\}$, since the components of the counit of the triadjunction of Theorem 3.2.5 are pointwise weak equivalences, as shown in the proof of Lemma 3.3.8. Finally, P preserves pointwise weak equivalences, and the inclusion sends these to biequivalences. $\square$

Theorem 3.4.13. *For any ordinal α, the composite trihomomorphism*

$$\mathrm{Hom}(\mathcal{C}^{op}, \mathbf{Bicat}) \longrightarrow [\mathcal{C}^{op}, \mathbf{Gray}] \xrightarrow{L^\alpha} [\mathcal{C}^{op}, \mathbf{Gray}] \longrightarrow \mathrm{Hom}(\mathcal{C}^{op}, \mathbf{Bicat})$$

preserves finite trilimits.

Proof. By Lemma 3.4.12, it suffices to show that L^α preserves finite **Gray**-limits and pointwise weak equivalences. We prove this by transfinite induction.

If $\alpha = \beta + 1$ and L^β preserves finite **Gray**-limits and pointwise weak equivalences, then by Proposition 3.4.4, $L^\alpha = LL^\beta$ is the composite of two endo-**Gray**-functors that preserve finite **Gray**-limits and pointwise weak equivalences, and hence also preserves the same.

If α is a limit ordinal, and L^β preserves finite **Gray**-limits and pointwise weak equivalences for all $\beta < \alpha$, then L^α is a filtered pseudocolimit of endo-**Gray**-functors that preserve finite **Gray**-limits and pointwise weak equivalences, and hence also preserves the same. $\square$

Theorem 3.4.14. *Let $F \in [\mathcal{C}^{op}, \mathbf{Gray}]$. Then there exists an ordinal α such that $L^\alpha F$ is a 2-stack. Therefore the composite trihomomorphism*

$$\mathrm{Hom}(\mathcal{C}^{op}, \mathbf{Bicat}) \longrightarrow [\mathcal{C}^{op}, \mathbf{Gray}] \xrightarrow{L^\alpha} \text{2-Stacks}(\mathcal{C}, J)$$

is left triadjoint to the inclusion.

Proof. By Proposition 3.4.4, there exists a cardinal κ such that L preserves κ-filtered pseudocolimits. Let α be a κ-filtered limit ordinal. By Lemma 3.4.9, it suffices to prove that $\eta_{L^\alpha F} \colon L^\alpha F \longrightarrow LL^\alpha F$ is a pointwise biequivalence.

Let

$$F \xrightarrow{\eta_F} LF \xrightarrow{\eta_{LF}} L^2F \xrightarrow{\eta_{L^2F}} L^3F \xrightarrow{\eta_{L^3F}} \cdots \longrightarrow L^\alpha F$$

be the pseudocolimit diagram defining $L^\alpha F$. Since L preserves pseudocolimits indexed by α, applying L to this diagram induces an isomorphism

$$LF \xrightarrow{L\eta_F} L^2F \xrightarrow{L\eta_{LF}} L^3F \xrightarrow{L\eta_{L^2F}} \cdots \longrightarrow P \xrightarrow{\cong} LL^\alpha F$$

where P denotes the pseudocolimit of the segment of the diagram to its left. But by Lemma 3.4.10, there exists a trinatural biequivalence between the diagrams, and hence a biequivalence between their pseudocolimits, as displayed.

$$\begin{tikzcd}
LF \arrow[r, "\eta_{LF}"] \arrow[d, "1"'] \arrow[dr, phantom, "\simeq"] & L^2F \arrow[r, "\eta_{L^2F}"] \arrow[d, "1"] \arrow[dr, phantom, "\simeq"] & L^3F \arrow[r, "\eta_{L^3F}"] \arrow[d, "1"] & \dots \arrow[r] & L^\alpha F \arrow[d, dotted, "\simeq"] & \\
LF \arrow[r, "L\eta_F"'] & L^2F \arrow[r, "L\eta_{LF}"'] & L^3F \arrow[r, "L\eta_{L^2F}"'] & \cdots \arrow[r] & P \arrow[r, "\cong"'] & LL^\alpha F
\end{tikzcd}$$

To show that $\eta_{L^\alpha F}$ is equivalent to the composite $L^\alpha F \longrightarrow P \longrightarrow LL^\alpha F$, and hence a biequivalence, it suffices to show that they arise from equivalent cones. The two respective cones have components

$$L^{\beta+1}F \xrightarrow{\eta^\alpha_{\beta+1}F} L^\alpha F \xrightarrow{\eta_{L^\alpha F}} LL^\alpha F$$

and

$$L^{\beta+1}F \xrightarrow{L(\eta^\alpha_\beta F)} LL^\alpha F.$$

From the pseudocolimit cone defining $L^\alpha F$, we have equivalences

$$\begin{tikzcd}
L^\beta F \arrow[rr, "\eta_{L^\beta F}"] \arrow[dr, "\eta^\alpha_\beta F"'] & \simeq & L^{\beta+1}F \arrow[dl, "\eta^\alpha_{\beta+1}F"] \\
& L^\alpha F &
\end{tikzcd}$$

and therefore we have the equivalences

$$\begin{aligned}
L(\eta^\alpha_\beta F) &\simeq L(\eta^\alpha_{\beta+1}F \circ \eta_{L^\beta F}) \\
&= L\eta^\alpha_{\beta+1}F \circ L\eta_{L^\beta F} \\
&\simeq L\eta^\alpha_{\beta+1}F \circ \eta_{L^{\beta+1}F}.
\end{aligned}$$

Hence $\eta_{L^\alpha F}$ is a biequivalence, and so $L^\alpha F$ is a 2-stack.

Let $G \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ be a 2-stack. By transfinite induction, the pseudofunctor

$$(\eta_\alpha, 1)\colon \operatorname{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(L^\alpha F, G) \longrightarrow \operatorname{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(F, G)$$

is a trilimit of biequivalences in **Bicat**, and so is a biequivalence. Since $L^\alpha F$ is a 2-stack, this establishes the triadjunction. $\square$

Chapter 4

Non-abelian cohomology of degree 2

The purpose of this chapter is to prove the case $n = 2$ of Method 2.2.2. In Section 4.1 we prove the basic properties of tripullbacks, thus enabling us to give in Section 4.2 a tricategorical version of Section 2.1. In Section 4.3 we establish a factorisation system on the tricategory $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$, to which we can apply the tricategorical analogue of Corollary 2.1.17 to prove the case $n = 2$ of Method 2.2.2, which we state in the final section of this chapter.

4.1 Tripullbacks

The development of Section 2.1 makes repeated use of pullbacks and their elementary properties. In this section we prove the analogous results for tripullbacks. Throughout this section, let $\mathcal{A}$ denote the free category on the graph

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \tag{4.1}$$

A strict trihomomorphism $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ into a tricategory $\mathcal{B}$ amounts to a cospan

$$A \xrightarrow{f} C \xleftarrow{g} B$$

in $\mathcal{B}$. The trilimit of such a trihomomorphism weighted by the constant trihomomorphism $\Delta 1\colon \mathcal{A} \longrightarrow \mathbf{Bicat}$ is called the *tripullback* of the pair of morphisms f and g.

To facilitate the manipulation of tripullbacks, we show that they can be calculated as a certain type of weighted **Gray**-limit. The *equi-comma object* of a pair of arrows f and g as above in a **Gray**-category $\mathcal{B}$ is the **Gray**-limit of $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ weighted by the functor $J\colon \mathcal{A} \longrightarrow \mathbf{Gray}$ that picks out the cospan

$$1 \xrightarrow{0} \mathbb{E} \xleftarrow{1} 1,$$

where $\mathbb{E}$ denotes the free 2-category containing an adjoint equivalence $0 \longrightarrow 1$.

First, without loss of generality, we may take G to be a diagram in a **Gray**-category $\mathcal{B}$; in particular we have, as in Remark 3.3.3, that tripullbacks in **Bicat** and $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$ are biequivalent to tripullbacks in the **Gray**-categories **Gray** and $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ respectively.

Next, since the weight $\Delta 1$ is pointwise cofibrant, the universal property of the tripullback is equivalent to

$$\mathcal{B}(B, \{\Delta 1, G\}_{\mathrm{tri}}) \simeq \mathrm{Hom}(\mathcal{A}, \mathbf{Gray})(\Delta 1, \mathcal{B}(B, G-)).$$

Finally, the diagram

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr, phantom, "= \ 0"] & 1 \arrow[d] \arrow[dr, phantom, "\simeq"] & 1 \arrow[l] \arrow[d] \\ 1 \arrow[r, "0"'] & \mathbb{E} & 1 \arrow[l, "1"] \end{tikzcd}$$

defines a tritransformation $0\colon \Delta 1 \longrightarrow J$ in $\mathrm{Hom}(\mathcal{A}, \mathbf{Gray})$. We will show that for any **Gray**-functor $K\colon \mathcal{A} \longrightarrow \mathbf{Gray}$, the 2-functor

$$[\mathcal{A}, \mathbf{Gray}](J, K) \longrightarrow \mathrm{Hom}(\mathcal{A}, \mathbf{Gray})(\Delta 1, K) \tag{4.2}$$

given by pre-composition with the tritransformation $0\colon \Delta 1 \longrightarrow J$ is a biequivalence, and therefore that equi-comma objects satisfy the universal property of tripullbacks.

Let K pick out the pair of 2-functors $F\colon A \longrightarrow C \longleftarrow B\colon G$. As usual for weighted limits in the base category of enrichment, $[\mathcal{A}, \mathbf{Gray}](J, K)$ is the equi-comma 2-category of F over G, and $\mathrm{Hom}(\mathcal{A}, \mathbf{Gray})(\Delta 1, K)$ is the tripullback of F and G. We now spell out the definitions of equi-comma 2-category and tripullback in **Gray**, with reference to Appendix A, and once again making the same simplifying assumption of normality as therein.

Definition 4.1.1. The *equi-comma 2-category* of a pair of 2-functors $F\colon A \longrightarrow C \longleftarrow B\colon G$ is the 2-category such that:

- an object (a, s, b) consists of an object a of A, an object b of B, and an adjoint equivalence $s\colon Fa \longrightarrow Gb$ in C,
- a morphism $(f, \sigma, g)\colon (a, s, b) \longrightarrow (a', s', b')$ consists of a morphism $f\colon a \longrightarrow a'$ in A, a morphism $g\colon b \longrightarrow b'$ in B, and an invertible 2-cell

  $$\begin{tikzcd} Fa \arrow[r, "s"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow\sigma"] & Gb \arrow[d, "Gg"] \\ Fa' \arrow[r, "s'"'] & Gb' \end{tikzcd}$$

  in C,

- a 2-cell $(\alpha, \beta)\colon (f, \sigma, g) \longrightarrow (f', \sigma', g')$ consists of a 2-cell $\alpha\colon f \longrightarrow f'$ in A and a 2-cell $\beta\colon g \longrightarrow g'$ in B such that

$$\begin{tikzcd}
Fa \arrow[r, "s"] \arrow[d, bend right=40, "Ff'"'] \arrow[d, bend left=40, "Ff"] \arrow[d, phantom, "\overset{F\alpha}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow\sigma"] & Gb \arrow[d, "Gg"] \\
Fa' \arrow[r, "s'"'] & Gb'
\end{tikzcd}
=
\begin{tikzcd}
Fa \arrow[r, "s"] \arrow[d, "Ff'"'] \arrow[dr, phantom, "\sigma'\Downarrow"] & Gb \arrow[d, bend right=40, "Gg'"'] \arrow[d, bend left=40, "Gg"] \arrow[d, phantom, "\overset{G\beta}{\Leftarrow}"] \\
Fa' \arrow[r, "s'"'] & Gb'
\end{tikzcd}$$

and all compositions are inherited from the 2-categories A, B, and C.

Definition 4.1.2. The *tripullback* of a pair of 2-functors $F\colon A \longrightarrow C \longleftarrow B\colon G$ is the 2-category such that:

- an object (a, b, c, r, t) consists of objects $a \in A$, $b \in B$, and $c \in C$, and adjoint equivalences $r\colon c \longrightarrow Fa$ and $t\colon c \longrightarrow Gb$ in C,
- a morphism $(f, g, h, \rho, \tau)\colon (a, b, c, r, t) \longrightarrow (a', b', c', r', t')$ consists of morphisms $f\colon a \longrightarrow a'$ in A, $g\colon b \longrightarrow b'$ in B, and $h\colon c \longrightarrow c'$ in C, and invertible 2-cells

$$\begin{tikzcd}
c \arrow[r, "r"] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\rho"] & Fa \arrow[d, "Ff"] \\
c' \arrow[r, "r'"'] & Fa'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
c \arrow[r, "t"] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\tau"] & Gb \arrow[d, "Gg"] \\
c' \arrow[r, "t'"'] & Gb'
\end{tikzcd}$$

  in C,
- a 2-cell $(\alpha, \beta, \gamma)\colon (f, g, h, \rho, \tau) \longrightarrow (f', g', h', \rho', \tau')$ consists of 2-cells $\alpha\colon f \longrightarrow f'$ in A, $\beta\colon g \longrightarrow g'$ in B, and $\gamma\colon c \longrightarrow c'$ in C, such that

$$\begin{tikzcd}
c \arrow[r, "r"] \arrow[d, bend right=40, "h'"'] \arrow[d, bend left=40, "h"] \arrow[d, phantom, "\overset{\gamma}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow\rho"] & Fa \arrow[d, "Ff"] \\
c' \arrow[r, "r'"'] & Fa'
\end{tikzcd}
=
\begin{tikzcd}
c \arrow[r, "r"] \arrow[d, "h'"'] \arrow[dr, phantom, "\rho'\Downarrow"] & Fa \arrow[d, bend right=40, "Ff'"'] \arrow[d, bend left=40, "Ff"] \arrow[d, phantom, "\overset{F\alpha}{\Leftarrow}"] \\
c' \arrow[r, "r'"'] & Fa'
\end{tikzcd}$$

  and

$$\begin{tikzcd}
c \arrow[r, "t"] \arrow[d, bend right=40, "h'"'] \arrow[d, bend left=40, "h"] \arrow[d, phantom, "\overset{\gamma}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow\tau"] & Gb \arrow[d, "Gg"] \\
c' \arrow[r, "t'"'] & Gb'
\end{tikzcd}
=
\begin{tikzcd}
c \arrow[r, "t"] \arrow[d, "h'"'] \arrow[dr, phantom, "\tau'\Downarrow"] & Gb \arrow[d, bend right=40, "Gg'"'] \arrow[d, bend left=40, "Gg"] \arrow[d, phantom, "\overset{G\beta}{\Leftarrow}"] \\
c' \arrow[r, "t'"'] & Gb'
\end{tikzcd}$$

and all compositions are inherited from the 2-categories A, B, and C.

The 2-functor (4.2) then corresponds to the 2-functor from the equi-comma 2-category to the tripullback of F and G which acts as follows:

- on objects, $(a, s, b) \mapsto (a, b, Fa, 1, s)$,
- on morphisms, $(f, \sigma, g) \mapsto (f, g, Ff, 1, \sigma)$,
- on 2-cells, $(\alpha, \beta) \mapsto (\alpha, \beta, F\alpha)$.

Lemma 4.1.3. *Let $F\colon A \longrightarrow C \longleftarrow B\colon G$ be a cospan in* **Gray**. *Then the equi-comma 2-category of F over G is biequivalent to the tripullback of F and G. Therefore tripullbacks in* **Bicat** *can be calculated as equi-comma objects in* **Gray**.

Proof. We show that the 2-functor (4.2) is a biequivalence. For convenience, let E and T denote the equi-comma 2-category and the tripullback respectively.

Let $(f, \sigma, g), (f', \sigma', g')\colon (a, s, b) \longrightarrow (a', s', b')$ be a parallel pair of morphisms in E. To give a 2-cell $(f, g, Ff, 1, \sigma) \longrightarrow (f', g', Ff', 1, \sigma')$ in T is to give 2-cells $\alpha\colon f \longrightarrow f'$ in A, $\beta\colon g \longrightarrow g'$ in B, and $\gamma\colon Ff \longrightarrow Ff'$ in C, such that $\gamma = F\alpha$ and

$$\begin{tikzcd} Fa \arrow[r, "s"] \arrow[d, bend right=50, "Ff'"'] \arrow[d, bend left=50, "Ff"] \arrow[d, phantom, "\overset{\gamma}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow\sigma"] & Gb \arrow[d, "Gg"] \\ Fa' \arrow[r, "s'"'] & Gb' \end{tikzcd} = \begin{tikzcd} Fa \arrow[r, "s"] \arrow[d, "Ff'"'] \arrow[dr, phantom, "\sigma'\Downarrow"] & Gb \arrow[d, bend right=50, "Gg'"'] \arrow[d, bend left=50, "Gg"] \arrow[d, phantom, "\overset{G\beta}{\Leftarrow}"] \\ Fa' \arrow[r, "s'"'] & Gb' \end{tikzcd}$$

But this amounts precisely to a 2-cell $(\alpha, \beta)\colon (f, \sigma, g) \longrightarrow (f', \sigma', g')$ in E. Hence (4.2) is fully faithful on homs.

Let (a, s, b) and (a', s', b') be objects in E, and let $(f, g, h, \rho, \tau)\colon (a, b, Fa, 1, s) \longrightarrow (a', b', Fa', 1, s')$ be a morphism in T. Then $(f, \sigma, g)\colon (a, s, b) \longrightarrow (a', s', b')$ is a morphism in E, where σ is defined to be the pasting composite

$$\begin{tikzcd} Fa \arrow[r, "s"] \arrow[d, bend right=50, "Ff"'] \arrow[d, bend left=50, "h"] \arrow[d, phantom, "\overset{\rho^{-1}}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow\tau"] & Gb \arrow[d, "Gg"] \\ Fa' \arrow[r, "s'"'] & Gb' \end{tikzcd}$$

and $(1, 1, \rho)\colon (f, g, Ff, 1, \sigma) \longrightarrow (f, g, h, \rho, \tau)$ is an invertible 2-cell in T. Hence (4.2) is an equivalence on homs.

Let (a, b, c, r, t) be an object of T. Then (a, s, b) is an object of E, where s is defined to be the composite

$$Fa \xrightarrow{r^\bullet} c \xrightarrow{t} Gb,$$

and $(1, 1, r^\bullet, \varepsilon^{-1}, 1)\colon (a, b, Fa, 1, s) \longrightarrow (a, b, c, r, t)$ is an equivalence in T, where $\varepsilon\colon rr^\bullet \longrightarrow 1$ is the counit of the adjoint equivalence $r \dashv r^\bullet$. Therefore (4.2) is a biequivalence.

Since any cospan in **Bicat** is biequivalent to one in **Gray**, and the inclusion **Gray** $\longrightarrow$ **Bicat** preserves trilimits, we have that any tripullback in **Bicat** can be calculated as an equi-comma 2-category in **Gray**. □

Remark 4.1.4. Let E denote the equi-comma 2-category of a cospan $F\colon A \longrightarrow C \longleftarrow B\colon G$. There is a pseudonatural equivalence

$$\begin{tikzcd} E \arrow[r, "Q"] \arrow[d, "P"'] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}"] & B \arrow[d, "G"] \\ A \arrow[r, "F"'] & C \end{tikzcd} \tag{4.3}$$

where P and Q are the obvious projections, and the component of θ at an object (a, s, b) is s, and its component at a morphism $(f, \sigma, g)\colon (a, s, b) \longrightarrow (a', s', b')$ is σ.

By contrast, the cone for a tripullback in a tricategory consists of equivalence 2-cells

$$\begin{tikzcd} T \arrow[r] \arrow[d] \arrow[dr] & B \arrow[d, "g"] \arrow[dl, phantom, near start, "\Rightarrow"] \\ A \arrow[r, "f"'] \arrow[ur, phantom, near start, "\Leftarrow"] & C \end{tikzcd} . \tag{4.4}$$

We will nonetheless speak of squares in tricategories of the form

$$\begin{tikzcd} U \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\simeq"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd} \tag{4.5}$$

as being tripullbacks; we can interpret this to mean that the cone

$$\begin{tikzcd} U \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, "fp" description] & Y \arrow[d, "g"] \arrow[dl, phantom, near start, "\simeq"] \\ X \arrow[r, "f"'] \arrow[ur, phantom, near start, "="] & Z \end{tikzcd}$$

has the universal property of a tripullback. Conversely, for any tripullback (4.4), we can compose one of the two equivalence 2-cells with a weak inverse of the other to give a single equivalence as in (4.5).

Hereafter, when we need to calculate with tripullbacks in **Bicat**, we will reckon with equi-comma 2-categories. Moreover, since tripullbacks are defined representably in a general tricategory, it suffices to prove the basic properties of tripullbacks in **Bicat**, and hence in **Gray**.

Lemma 4.1.5. *The tripullback of a biequivalence is a biequivalence.*

Proof. Let $F\colon A \longrightarrow C$ be a 2-functor, and let its equi-comma 2-category over a 2-functor $G\colon B \longrightarrow C$ be denoted as in (4.3). We prove separately that if F is biessentially surjective, then so is Q, and that if F is an equivalence on homs, then so is Q. Since a 2-functor is a biequivalence if and only if it is both biessentially surjective and an equivalence on homs, it follows that if F is a biequivalence, then so is Q.

Suppose F is biessentially surjective. Let b be an object of B. Then there exists an object a of A and an equivalence $s\colon Fa \longrightarrow Gb$ in C. Hence (a, s, b) is an object of E such that $Q(a, s, b) = b$, and so Q is surjective on objects.

Suppose F is an equivalence on homs. Let $(\alpha, \beta), (\alpha', \beta)\colon (f, \sigma, g) \longrightarrow (f', \sigma', g')$ be a parallel pair of 2-cells in E. Then we have the equation

$$
\begin{tikzcd}
Fa \arrow[r, "s"] \arrow[d, bend right=40, "Ff'"'] \arrow[d, bend left=40, "Ff"] \arrow[d, phantom, "\overset{F\alpha}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow\sigma"] & Gb \arrow[d, "Gg"] \\
Fa' \arrow[r, "s'"'] & Gb'
\end{tikzcd}
=
\begin{tikzcd}
Fa \arrow[r, "s"] \arrow[d, bend right=40, "Ff'"'] \arrow[d, bend left=40, "Ff"] \arrow[d, phantom, "\overset{F\alpha'}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow\sigma"] & Gb \arrow[d, "Gg"] \\
Fa' \arrow[r, "s'"'] & Gb'
\end{tikzcd}.
$$

But σ is invertible and s is an equivalence, so $F\alpha = F\alpha'$. Hence $\alpha = \alpha'$, and so Q is faithful on homs.

Now let $(f, \sigma, g), (f', \sigma', g')\colon (a, s, b) \longrightarrow (a', s', b')$ be a parallel pair of morphisms in E. Let $\beta\colon g \longrightarrow g'$ be a 2-cell in B. Then, since F is full on homs, there exists a 2-cell $\alpha\colon f \longrightarrow f'$ such that $F\alpha$ is equal to the pasting composite

$$
\begin{tikzcd}
 & Fa' \arrow[dr, "s'"] \arrow[drr, bend left, "1"] \arrow[d, phantom, "\Downarrow\sigma^{-1}"] & & \\
Fa \arrow[ur, "Ff"] \arrow[r, "s"'] \arrow[dr, "Ff'"'] & Gb \arrow[r, bend left, "Gg"] \arrow[r, bend right, "Gg'"'] \arrow[r, phantom, "\Downarrow G\beta"] & Gb' \arrow[r, "s'^{\bullet}"'] \arrow[u, phantom, "\Downarrow\eta"] \arrow[d, phantom, "\Downarrow\eta^{-1}"] & Fa' \\
 & Fa' \arrow[ur, "s'"'] \arrow[urr, bend right, "1"'] \arrow[u, phantom, "\Downarrow\sigma'"] & &
\end{tikzcd}.
$$

Hence there is a 2-cell $(\alpha, \beta)\colon (f, \sigma, g) \longrightarrow (f', \sigma', g')$ in E, and so Q is full on homs.

Finally, let (a, s, b) and (a', s', b') be objects of E, and let $b\colon g \longrightarrow g'$ be a morphism in B. Since F is essentially surjective on homs, there exists a morphism $f\colon a \longrightarrow a'$ in A and an invertible 2-cell

$$
\begin{tikzcd}
Fa \arrow[r, "s"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\overset{\rho}{\Leftarrow}"] & Gb \arrow[d, "Gg"] \\
Fa' & Gb' \arrow[l, "s'^{\bullet}"]
\end{tikzcd}
$$

in C. Hence $(f, \sigma, g)\colon (a, s, b) \longrightarrow (a', s', b')$ is a morphism in E, where σ is the pasting composite

$$\begin{tikzcd}
Fa \arrow[rr, "s"] \arrow[d, "Ff"'] & \arrow[d, phantom, "\overset{\rho}{\Leftarrow}"] & Gb \arrow[d, "Gg"] \\
Fa' \arrow[dr, "s'"'] & \arrow[l, phantom, ""] \arrow[d, phantom, "\overset{\varepsilon^{-1}}{\Leftarrow}"] & Gb' \arrow[ll, "s'^{\bullet}"'] \arrow[dl, "1"] \\
& Gb' &
\end{tikzcd}.$$

Hence Q is surjective on homs, and therefore an equivalence of homs. $\square$

Lemma 4.1.6. *Let*

$$\begin{tikzcd}
D \arrow[r, "K"] \arrow[d, "H"'] \arrow[dr, phantom, "\overset{\varphi}{\Rightarrow}"] & B \arrow[d, "G"] \\
A \arrow[r, "F"'] & C
\end{tikzcd}$$

be a diagram in a tricategory, such that φ is an equivalence. If F and K are biequivalences, then this square is a tripullback.

Proof. It suffices to check this in **Gray**. Take the equi-comma 2-category E of F over G. Since F is a biequivalence, so is Q. Then the canonical 2-functor $L\colon D \longrightarrow E$, as in the diagram

$$\begin{tikzcd}
D \arrow[dr, dotted, "L"] \arrow[drr, bend left, "K"] \arrow[ddr, bend right, "H"'] & & \\
& E \arrow[r, "Q"] \arrow[d, "P"] \arrow[dr, phantom, "\simeq"] & B \arrow[d, "G"] \\
& A \arrow[r, "F"'] & C
\end{tikzcd}$$

satisfies $QL = K$, and is therefore a biequivalence. Hence the square is biequivalent to an equi-comma square, and is therefore a tripullback. $\square$

We now prove the pasting lemma for tripullbacks.

Lemma 4.1.7. *Consider a diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq"] & C \arrow[d] \\
D \arrow[r] & E \arrow[r] & F
\end{tikzcd}$$

in a tricategory such that the right-hand square is a tripullback. The left-hand square is a tripullback if and only if the outer rectangle is a tripullback.

Proof. Again, it suffices to check this in **Gray**. Let

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq"] & B \arrow[r] \arrow[d, "P"] \arrow[dr, phantom, "\simeq"] & C \arrow[d, "H"] \\
D \arrow[r, "K"'] & E \arrow[r, "G"'] & F
\end{tikzcd} \tag{4.6}$$

be a diagram in **Gray**. For convenience, let us denote the equi-comma 2-category of G over H by (G, H).

By direct calculation, one can show that the square of canonical 2-functors

$$\begin{tikzcd} (K,P) \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq"] & (GK,H) \arrow[d] \\ B \arrow[r] & (G,H) \end{tikzcd}$$

is a tripullback. But the right-hand square in (4.6) is a tripullback square precisely when the 2-functor $B \longrightarrow (G, H)$ is a biequivalence. Hence its tripullback $(K, P) \longrightarrow (GK, H)$ is also a biequivalence.

Similarly, the left-hand square and the outer rectangle are tripullback squares precisely when the canonical 2-functors $A \longrightarrow (K, P)$ and $A \longrightarrow (GK, H)$ respectively are biequivalences. But this latter 2-functor factors as $A \longrightarrow (K, P) \longrightarrow (GK, H)$, and the second factor is a biequivalence. Hence the left-hand square is a tripullback if and only if the outer rectangle is so. $\square$

4.2 The tricategorical Lawvere construction

Having proved the basic properties of tripullbacks, we can now give the tricategorical analogue of the development of Section 2.1. Both the statements and proofs of that section can be reproduced virtually word for word, once the appropriate tricategorical substitutions have been made, such as tripullback for pullback, biequivalence for isomorphism, and so on.

Definition 4.2.1. Let $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ be morphisms in a tricategory $\mathcal{B}$. We say that f and g are *orthogonal* (in the tricategorical sense), written $f \perp g$, if the square

$$\begin{tikzcd} \mathcal{B}(B,X) \arrow[r, "{(1,g)}"] \arrow[d, "{(f,1)}"'] \arrow[dr, phantom, "\simeq"] & \mathcal{B}(B,Y) \arrow[d, "{(f,1)}"] \\ \mathcal{B}(A,X) \arrow[r, "{(1,g)}"'] & \mathcal{B}(A,Y) \end{tikzcd}$$

is a tripullback.

When $\mathcal{B}$ is a **Gray**-category, $f \perp g$ if and only if the canonical 2-functor from $\mathcal{B}(B, X)$ to the equi-comma 2-category of the cospan $(1, g) : \mathcal{B}(A, X) \longrightarrow \mathcal{B}(A, Y) \longleftarrow \mathcal{B}(B, Y) : (f, 1)$ is a biequivalence. That this 2-functor is biessentially surjective means that for every equivalence 2-cell

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\simeq"] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] & Y \end{tikzcd}$$

there exists a morphism $h\colon B \longrightarrow X$, equivalence 2-cells $u \simeq hf$ and $gh \simeq v$, and an invertible 3-cell

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\simeq"] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] & Y \end{tikzcd} \quad \cong \quad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, "h"] & Y \end{tikzcd} .$$

We call such an $h\colon B \longrightarrow X$ a diagonal filler for the square. Note that since the canonical 2-functor is an equivalence on homs, if $h, h'\colon B \longrightarrow X$ are diagonal fillers for the same square, then there exists an equivalence $h \simeq h'$. By taking triequivalent **Gray**-categories, we can see that the same is true for orthogonality in any tricategory.

Lemma 4.2.2. *A morphism f such that $f \perp f$ is necessarily a biequivalence.*

Proof. There exists a diagonal filler g in the commutative square

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "1"'] \arrow[ur, dotted, "g"] & B \end{tikzcd}$$

and hence equivalences $gf \simeq 1$ and $fg \simeq 1$. Therefore g is a weak inverse of f. □

We say that a class of morphisms $\mathcal{M}$ in a tricategory is *replete* if whenever $f \in \mathcal{M}$ and $f \simeq g$ is an equivalence, then $g \in \mathcal{M}$.

Lemma 4.2.3. *Let $\mathcal{J}$ be a class of morphisms in a tricategory $\mathcal{B}$, and let $\mathcal{R}$ be the class of morphisms f in $\mathcal{B}$ such that $j \perp f$ for all $j \in \mathcal{J}$. Then $\mathcal{R}$ has the following properties.*

(i) $\mathcal{R}$ is closed under composition, contains the biequivalences, and is replete,

(ii) $\mathcal{R}$ is stable under tripullback, and

(iii) if gf and g both belong to $\mathcal{R}$, then so does f.

Dually, the class $\mathcal{L}$ of morphisms f such that $f \perp j$ for all $j \in \mathcal{J}$ has the following properties.

(i) $\mathcal{L}$ is closed under composition, contains the biequivalences, and is replete,

(ii) $\mathcal{L}$ is stable under tripushout, and

(iii) if gf and f both belong to $\mathcal{L}$, then so does g.

Proof. It suffices to prove these properties with respect to a single morphism $j\colon A \longrightarrow B$ in $\mathcal{J}$.

Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be a composable pair of morphisms, with $g \in \mathcal{R}$. Then, by the pasting lemma for tripullbacks, in the diagram

$$\begin{tikzcd}
\mathcal{B}(B,X) \arrow[r, "{(1,f)}"] \arrow[d, "{(j,1)}"'] \arrow[rr, bend left, "{(1,gf)}"] \arrow[dr, phantom, "\simeq"] & \mathcal{B}(B,Y) \arrow[r, "{(1,g)}"] \arrow[d, "{(j,1)}"'] \arrow[dr, phantom, "\simeq"] & \mathcal{B}(B,Z) \arrow[d, "{(j,1)}"] \\
\mathcal{B}(A,X) \arrow[r, "{(1,f)}"'] \arrow[rr, bend right, "{(1,gf)}"'] & \mathcal{B}(A,Y) \arrow[r, "{(1,g)}"'] & \mathcal{B}(A,Z)
\end{tikzcd}$$

the left-hand square is a tripullback if and only if the outer rectangle is. Hence $f \in \mathcal{R}$ if and only if $gf \in \mathcal{R}$. This proves both the cancellation property (iii) and that $\mathcal{R}$ is closed under composition.

By Lemma 4.1.6, a biequivalence is orthogonal to any morphism. Moreover, the class $\mathcal{R}$ is replete, since if a square as on the left-hand side

$$\begin{tikzcd}
W \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\simeq"] & X \arrow[d] \\
Y \arrow[r, "v"'] & Z
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
W \arrow[r, bend left, "u'"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\simeq"] \arrow[d] \arrow[dr, phantom, "\simeq"] & X \arrow[d] \\
Y \arrow[r, bend left, "v"] \arrow[r, bend right, "v'"'] \arrow[r, phantom, "\simeq"] & Z
\end{tikzcd}
\tag{4.7}$$

is a tripullback, then for any equivalences $u \simeq u'$ and $v \simeq v'$, the square on the right-hand side is also a tripullback.

Now, let g be a tripullback of a morphism $f \in \mathcal{R}$.

$$\begin{tikzcd}
U \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\simeq"] & V \arrow[d, "v"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

It follows from the isomorphism

$$\begin{tikzcd}
 & \mathcal{B}(B,V) \arrow[rr, "{(1,v)}"] \arrow[dr, "{(j,1)}"] & & \mathcal{B}(B,Y) \arrow[dr, "{(j,1)}"] & \\
\mathcal{B}(B,U) \arrow[ur, "{(1,g)}"] \arrow[dr, "{(j,1)}"'] \arrow[rr, phantom, "\simeq"] & & \mathcal{B}(A,V) \arrow[rr, "{(1,v)}"] \arrow[ur, phantom, "\simeq"] \arrow[dr, phantom, "\simeq"] & & \mathcal{B}(A,Y) \\
 & \mathcal{B}(A,U) \arrow[ur, "{(1,g)}"'] \arrow[rr, "{(1,u)}"'] & & \mathcal{B}(A,X) \arrow[ur, "{(1,f)}"'] &
\end{tikzcd}$$

$$\cong \quad \begin{tikzcd}[column sep=small]
 & \mathcal{B}(B,V) \arrow[rr, "{(1,v)}"] \arrow[dr, phantom, "\simeq"] & & \mathcal{B}(B,Y) \arrow[dr, "{(j,1)}"] & \\
\mathcal{B}(B,U) \arrow[ur, "{(1,g)}"] \arrow[rr, "{(1,u)}"'] \arrow[dr, "{(j,1)}"'] & & \mathcal{B}(B,X) \arrow[ur, "{(1,f)}"'] \arrow[dr, "{(j,1)}"'] \arrow[rr, phantom, "\simeq"] & & \mathcal{B}(A,Y) \\
 & \mathcal{B}(A,U) \arrow[rr, "{(1,u)}"'] \arrow[ur, phantom, "\simeq"] & & \mathcal{B}(A,X) \arrow[ur, "{(1,f)}"'] &
\end{tikzcd}$$

and from the pasting lemma for tripullbacks that $j \perp g$. Hence $g \in \mathcal{R}$. $\square$

Lemma 4.2.4. *Let $L \dashv R \colon \mathcal{A} \longrightarrow \mathcal{B}$ be a triadjunction. For morphisms f in $\mathcal{B}$ and g in $\mathcal{A}$, $Lf \perp g$ in $\mathcal{A}$ if and only if $f \perp Rg$ in $\mathcal{B}$.*

Proof. It is evident from the trinatural biequivalences $\mathcal{A}(LX,Y) \simeq \mathcal{B}(X,RY)$ and from the pasting lemma for tripullbacks that one square is a tripullback if and only if the other square is so.

$$\begin{tikzcd}
\mathcal{A}(LD,A) \arrow[r, "{(1,g)}"] \arrow[d, "{(Lf,1)}"'] \arrow[dr, phantom, "\simeq"] & \mathcal{A}(LD,B) \arrow[d, "{(Lf,1)}"] \\
\mathcal{A}(LC,A) \arrow[r, "{(1,g)}"'] & \mathcal{A}(LC,B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}(D,RA) \arrow[r, "{(1,Rg)}"] \arrow[d, "{(f,1)}"'] \arrow[dr, phantom, "\simeq"] & \mathcal{B}(D,RB) \arrow[d, "{(f,1)}"] \\
\mathcal{B}(C,RA) \arrow[r, "{(1,Rg)}"'] & \mathcal{B}(C,RB)
\end{tikzcd}$$

$\square$

Definition 4.2.5. Let $\mathcal{B}$ be a tricategory. A (tricategorical) *factorisation system* on $\mathcal{B}$ consists of two classes $(\mathcal{E},\mathcal{M})$ of morphisms in $\mathcal{B}$ such that

(i) $\mathcal{E}$ and $\mathcal{M}$ are both closed under composition, contain the biequivalences, and are replete,

(ii) every morphism f in $\mathcal{B}$ factors up to equivalence as $f \simeq me$ for some $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and

(iii) $e \perp m$ for every $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

Moreover, a factorisation system is *stable* if the class $\mathcal{E}$ is stable under tripullbacks.

Lemma 4.2.6. *Let $(\mathcal{E},\mathcal{M})$ be a factorisation system on a tricategory $\mathcal{B}$. Then for any morphism f in $\mathcal{B}$, $f \perp \mathcal{M}$ if and only if $f \in \mathcal{E}$; dually, $\mathcal{E} \perp f$ if and only if $f \in \mathcal{M}$.*

Proof. Let $f \perp \mathcal{M}$ have factorisation $f \simeq me$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$. Then there exists a diagonal filler s in the left-hand square

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "f"'] \arrow[dr, phantom, "\simeq", near start] & C \arrow[d, "m"] \\
B \arrow[r, "1"'] \arrow[ur, dotted, "s" description] \arrow[ur, phantom, "\simeq", near end, shift right=3] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "e"'] \arrow[dr, phantom, "="] & C \arrow[d, "m"] \\
C \arrow[r, "m"'] & B
\end{tikzcd}$$

with $sf \simeq e$ and $ms \simeq 1$. Then we have equivalences $sme \simeq sf \simeq e$ and $msm \simeq 1m \simeq m$, both 1 and sm are both diagonal fillers for the right-hand square. Hence there is an equivalence $sm \simeq 1$. Therefore m is a biequivalence, and so $f \simeq me$ belongs to $\mathcal{E}$. $\square$

Hence the classes $(\mathcal{E}, \mathcal{M})$ of a factorisation system enjoy the properties of the classes $(\mathcal{L}, \mathcal{R})$ from Lemma 4.2.3. Moreover, any two factorisations of a given morphism are biequivalent in the following sense.

Lemma 4.2.7. *Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on a tricategory $\mathcal{B}$. Suppose a morphism f in $\mathcal{B}$ has factorisations $f \simeq me$ and $f \simeq m'e'$, with $e, e' \in \mathcal{E}$ and $m, m' \in \mathcal{M}$. Then there exists a comparison morphism making the diagram*

$$
\begin{tikzcd}
 & C \arrow[dd, dotted] \arrow[dr, "m"] & \\
A \arrow[ur, "e"] \arrow[dr, "e'"'] \arrow[r, phantom, "\simeq"] & & B \arrow[l, phantom, "\simeq"] \\
 & D \arrow[ur, "m'"'] &
\end{tikzcd}
$$

commute up to equivalence, which is, moreover, a biequivalence.

Proof. Since $e \perp m'$, there exists a diagonal filler s in the square

$$
\begin{tikzcd}
A \arrow[r, "e'"] \arrow[d, "e"'] \arrow[dr, phantom, "\simeq" near start] & D \arrow[d, "m'"] \\
C \arrow[r, "m"'] \arrow[ur, dotted, "s"] \arrow[ur, phantom, "\simeq" near end, shift right=2ex] & B
\end{tikzcd}
$$

which, by repleteness and cancellation, belongs to $\mathcal{E} \cap \mathcal{M}$. Hence $s \perp s$, and so s is a biequivalence. $\square$

For the following sequence of results, up to and including Theorem 4.2.11, let $\mathcal{B}$ be a tricategory with tripullbacks and with a reflective sub-tricategory $\mathcal{A}$ whose reflector $L\colon \mathcal{B} \longrightarrow \mathcal{A}$ preserves tripullbacks. We denote the unit by $\eta\colon 1 \longrightarrow L$, and suppress mention of the biequivalence counit.

Definition 4.2.8. A morphism $f\colon A \longrightarrow B$ in $\mathcal{B}$ is

(i) a *local biequivalence* if Lf is a biequivalence,

(ii) *cartesian* if the naturality square

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\simeq"] & B \arrow[d, "\eta_B"] \\
LA \arrow[r, "Lf"'] & LB
\end{tikzcd}
$$

is a tripullback.

Proposition 4.2.9. *The classes (local biequivalence, cartesian) form a stable factorisation system on $\mathcal{B}$.*

Proof. Since L preserves composition up to equivalence and preserves biequivalences, the class of local isomorphisms is closed under composition, contains the biequivalences, and is replete. By Lemma 4.1.6 every biequivalence is cartesian, and for a composable pair of cartesian morphisms f and g, the outer rectangle in the right-hand of the following diagram is a tripullback.

$$\begin{tikzcd} A \arrow[r, "gf"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\simeq"] & C \arrow[d, "\eta_C"] \\ LA \arrow[r, "L(gf)"'] & LC \end{tikzcd} \quad \cong \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "gf"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\simeq"] & B \arrow[r, "g"] \arrow[d, "\eta_B"] \arrow[dr, phantom, "\simeq"] & C \arrow[d, "\eta_C"] \\ LA \arrow[r, "Lf"'] \arrow[rr, bend right, "L(gf)"'] & LB \arrow[r, "Lg"'] & LC \end{tikzcd}$$

Moreover, the class of cartesian morphisms is replete, by the fact quoted at (4.7).

Note that a morphism is cartesian if and only if it is a tripullback of a morphism in the sub-tricategory $\mathcal{A}$. For, by definition, a cartesian morphism f is a tripullback of Lf. Conversely, suppose f is the tripullback of a morphism g in $\mathcal{A}$. Since L preserves this tripullback,

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\simeq"] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] & Y \end{tikzcd} \quad \cong \quad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\simeq"] & LA \arrow[r, "Lu"] \arrow[d, "Lf"] \arrow[dr, phantom, "\simeq"] & X \arrow[d, "g"] \\ B \arrow[r, "\eta_B"'] & LB \arrow[r, "Lv"'] & Y \end{tikzcd}$$

by the pasting lemma, the naturality square for f is a tripullback, and so f is cartesian.

The factorisation of a morphism $f\colon A \longrightarrow B$ can be constructed by taking the tripullback of Lf along η_B, as in the diagram.

$$\begin{tikzcd} A \arrow[dr, "g"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\eta_A"'] & & \\ & C \arrow[r, "h"] \arrow[d, "k"] \arrow[dr, phantom, "\simeq"] & B \arrow[d, "\eta_B"] \\ & LA \arrow[r, "Lf"'] & LB \end{tikzcd}$$

Applying L to this diagram,

$$\begin{tikzcd} LA \arrow[r, "Lg"] \arrow[dr, "1"'] & LC \arrow[r, "Lh"] \arrow[d, "Lk"] \arrow[dl, phantom, "\simeq"] \arrow[dr, phantom, "\simeq"] & LB \arrow[d, "1"] \\ & LA \arrow[r, "Lf"'] & LB \end{tikzcd}$$

we find that Lk is a biequivalence, since it is the tripullback of a biequivalence. Hence Lg is a biequivalence. Moreover, since h is a tripullback of Lf, it is cartesian. Therefore $f \simeq hg$ is a (local biequivalence, cartesian) factorisation of f.

Let $f\colon A \longrightarrow B$ be a local biequivalence and let $g\colon X \longrightarrow Y$ be cartesian. Then Lf is a biequivalence, so $Lf \perp Lg$. But Lg is a morphism in the sub-tricategory $\mathcal{A}$, so by triadjointness we have $f \perp Lg$. Then, since g is a tripullback of Lg, we have that $f \perp g$.

Finally, the tripullback of a local biequivalence is a local biequivalence, since L preserves tripullbacks and the tripullback of a biequivalence is a biequivalence. □

Theorem 4.2.10. *Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on $\mathcal{B}$ such that $L(\mathcal{M}) \subseteq \mathcal{M}$. Then there exists a factorisation system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{B}$ such that, for a morphism f with $(\mathcal{E}, \mathcal{M})$-factorisation $f \simeq me$,*

(i) $f \in \mathcal{L}$ if and only if m is a local biequivalence,

(ii) $f \in \mathcal{R}$ if and only if $f \in \mathcal{M}$ and f is cartesian.

Moreover, if $(\mathcal{E}, \mathcal{M})$ is stable then so is $(\mathcal{L}, \mathcal{R})$.

Proof. Since both $\mathcal{M}$ and the class of cartesian morphisms are closed under composition, contain the biequivalences, and are replete, $\mathcal{R}$ also enjoys these properties. For a biequivalence f, $f \simeq f1$ is an $(\mathcal{E}, \mathcal{M})$-factorisation, so $f \in \mathcal{L}$, since L preserves biequivalences.

To show that $\mathcal{L}$ is closed under composition, let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ be a composable pair of morphisms in $\mathcal{L}$ with $(\mathcal{E}, \mathcal{M})$-factorisations as in the diagram.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \arrow[rr, "g"] \arrow[dr, "e'"'] \arrow[dd, phantom, "\simeq"] & & C \\
& D \arrow[u, phantom, "\simeq"] \arrow[ur, "m"'] \arrow[dr, "e''"'] & & E \arrow[u, phantom, "\simeq"] \arrow[ur, "m'"'] & \\
& & F \arrow[ur, "m''"'] & &
\end{tikzcd}$$

Take the $(\mathcal{E}, \mathcal{M})$-factorisation of $e'm$ as indicated. Then, since L preserves the class $\mathcal{M}$, we have $e' \perp Lm''$ and $e'' \perp Lm''$, which by triadjointness implies $Le' \perp Lm''$ and $Le'' \perp Lm''$. Since f belongs to $\mathcal{L}$, Lm is a biequivalence, so we have $Lm \perp Lm''$. Hence by composition and repleteness, $Lm''Le'' \simeq Le'Lm \perp Lm''$. So by cancellation, we have $Lm'' \perp Lm''$. Hence Lm'' is a biequivalence. Since $(m'm'')(e''e)$ is a $(\mathcal{E}, \mathcal{M})$-factorisation of gf, we have therefore that $gf \in \mathcal{L}$.

The $(\mathcal{L}, \mathcal{R})$-factorisation of a morphism $f\colon A \longrightarrow B$ can be constructed by first taking its $(\mathcal{E}, \mathcal{M})$-factorisation $f \simeq me$, and then the (local biequivalence, cartesian)-

factorisation of m. The latter is constructed by the tripullback

$$\begin{tikzcd} C \arrow[dr, "g"] \arrow[drr, bend left, "m"] \arrow[ddr, bend right, "\eta_C"'] & & \\ & D \arrow[r, "h"'] \arrow[d, "k"] \arrow[dr, phantom, "\simeq"] & B \arrow[d, "\eta_B"] \\ & LC \arrow[r, "Lm"'] & LB \end{tikzcd}$$

We have that h is cartesian, and since it is the tripullback of $Lm \in \mathcal{M}$, it belongs to $\mathcal{M}$. Then since both hg and h belong to $\mathcal{M}$, by cancellation we have $g \in \mathcal{M}$. Hence the $\mathcal{M}$-part of the composite ge is a local biequivalence, i.e. $ge \in \mathcal{L}$. Therefore $f \simeq h(ge)$ is a $(\mathcal{L}, \mathcal{R})$-factorisation of f.

Now let $l \colon A \longrightarrow B$ and $r \colon X \longrightarrow Y$ belong to $\mathcal{L}$ and $\mathcal{R}$ respectively, and let l have $(\mathcal{E}, \mathcal{M})$-factorisation $l \simeq me$. Since m is a local biequivalence and r is cartesian, we have that $m \perp r$, and since $e \in \mathcal{E}$ and $r \in \mathcal{M}$, we have that $e \perp r$. Hence, by composition, $l \perp r$.

Suppose $\mathcal{E}$ is stable under tripullbacks. Let $l \in \mathcal{L}$ have $(\mathcal{E}, \mathcal{M})$-factorisation $l \simeq me$. Then the tripullback l' of l along some morphism f can be given by taking successive tripullbacks, as in the following diagram.

$$\begin{tikzcd} E \arrow[r, "e'"] \arrow[d] \arrow[rr, bend left, "l'"] \arrow[dr, phantom, "\simeq"] & D \arrow[r, "m'"] \arrow[d] \arrow[dr, phantom, "\simeq"] & X \arrow[d, "f"] \\ A \arrow[r, "e"'] \arrow[rr, bend right, "l"'] & C \arrow[r, "m"'] & B \end{tikzcd}$$

We have that $e' \in \mathcal{E}$, $m' \in \mathcal{M}$, and m' is a local biequivalence, since all three classes are stable under tripullbacks. Hence the $\mathcal{M}$-part of the tripullback of l is a local biequivalence, and so the tripullback of l is in $\mathcal{L}$. $\square$

Theorem 4.2.11. *Let $f \colon A \longrightarrow B$ be a morphism in $\mathcal{B}$ such that $B \in \mathcal{A}$ and $Lf \in \mathcal{M}$. Then the $(\mathcal{L}, \mathcal{R})$-image of f is the reflection of A.*

Proof. Note that since every morphism in $\mathcal{R}$ is cartesian, we have by orthogonality that every local biequivalence is in $\mathcal{L}$. Therefore in the naturality square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\simeq"] & B \arrow[d, "1"] \\ LA \arrow[r, "Lf"'] & B \end{tikzcd}$$

we have $\eta_A \in \mathcal{L}$. By assumption, Lf is in $\mathcal{M}$ and moreover it is cartesian, since it is in the sub-tricategory $\mathcal{A}$. Hence $f \simeq Lf \circ \eta_A$ is an $(\mathcal{L}, \mathcal{R})$-factorisation of f. By Lemma 4.2.7, for any $(\mathcal{L}, \mathcal{R})$-factorisation $f \simeq rl$, there is a comparison morphism

$$\begin{tikzcd}
 & LA \arrow[dr, "Lf"] \arrow[dd, dotted] & \\
A \arrow[ur, "\eta_A"] \arrow[dr, "l"'] \arrow[r, phantom, "\simeq"] & {} \arrow[r, phantom, "\simeq"] & B \\
 & C \arrow[ur, "r"'] &
\end{tikzcd}$$

which is moreover a biequivalence. Hence $l \colon A \longrightarrow C$ witnesses C as the reflection of A. $\square$

Let $F \colon X \longrightarrow Y$ be a pseudofunctor between bicategories, and suppose it is an equivalence on homs. Then F gives a biequivalence between X and the full sub-bicategory of Y on those objects y for which there exists an object x of X and an equivalence $Fx \simeq y$, that is, the "replete full image" of F.

We call a morphism $f \colon A \longrightarrow B$ in a tricategory $\mathcal{B}$ *representably an equivalence on homs* if for every object X of $\mathcal{B}$, the pseudofunctor $\mathcal{B}(X, f) \colon \mathcal{B}(X, A) \longrightarrow \mathcal{B}(X, B)$ is an equivalence on homs.

Theorem 4.2.12. *Let $\mathcal{B}$ be a tricategory with tripullbacks and let $(\mathcal{L}, \mathcal{R})$ be a stable factorisation system on $\mathcal{B}$ such that every morphism in $\mathcal{R}$ is representably an equivalence on homs. Let $f \colon A \longrightarrow B$ be a morphism in $\mathcal{B}$ with $(\mathcal{L}, \mathcal{R})$-factorisation $A \xrightarrow{l} C \xrightarrow{r} B$. Then for each object X of $\mathcal{B}$, the hom-bicategory $\mathcal{B}(X, C)$ is biequivalent to the full sub-bicategory of $\mathcal{B}(X, B)$ on those morphisms $x \colon X \longrightarrow B$ for which there exists a square*

$$\begin{tikzcd}
R \arrow[r, "a"] \arrow[d, "p"'] \arrow[dr, phantom, "\simeq"] & A \arrow[d, "f"] \\
X \arrow[r, "x"'] & B
\end{tikzcd}$$

with $p \in \mathcal{L}$.

Proof. By assumption, r is representably an equivalence on homs, so $(1, r) \colon \mathcal{B}(X, C) \longrightarrow \mathcal{B}(X, B)$ is an equivalence on homs, and $\mathcal{B}(X, C)$ is therefore biequivalent to its replete full image.

Let $x \colon X \longrightarrow B$ belong to the replete full image of $(1, r)$. Then there exists a morphism

$y\colon X \longrightarrow C$ such that $x \simeq ry$. The tripullback p of l along y, as in the diagram

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "p"'] \arrow[dr, phantom, "\simeq"] & A \arrow[d, "l"] \\ X \arrow[r, "y"] \arrow[d, "1"'] \arrow[dr, phantom, "\simeq"] & C \arrow[d, "r"] \\ X \arrow[r, "x"'] & B \end{tikzcd}$$

belongs to $\mathcal{L}$, since $\mathcal{L}$ is stable under tripullback. Hence x belongs to the full sub-bicategory of $\mathcal{B}(X, B)$ defined in the statement of the theorem.

Conversely, let $x \in \mathcal{B}(X, B)$ be such that there exists a square as in the statement of the theorem. Then, since $p \perp r$, there is a diagonal filler y

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "p"'] \arrow[dr, phantom, "\simeq"] & A \arrow[r, "l"] & C \arrow[d, "r"] \\ X \arrow[urr, dotted, "y"] \arrow[rr, "x"'] \arrow[urr, phantom, shift right=3ex, "\simeq"] & & B \end{tikzcd}$$

such that $x \simeq ry$. Hence x belongs to the replete full image of $(1, r)$. $\square$

Corollary 4.2.13. *With the hypotheses of Theorems 4.2.10 and 4.2.12, let $f\colon A \longrightarrow B$ be a morphism in $\mathcal{B}$ such that $B \in \mathcal{A}$ and $Lf \in \mathcal{M}$. Then for any object $X \in \mathcal{B}$, the hom-bicategory $\mathcal{B}(X, LA)$ is biequivalent to the full sub-bicategory of $\mathcal{B}(X, B)$ on those morphisms $x\colon X \longrightarrow B$ for which there exists a square*

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "p"'] \arrow[dr, phantom, "\simeq"] & A \arrow[d, "f"] \\ X \arrow[r, "x"'] & B \end{tikzcd}$$

with $p \in \mathcal{L}$.

Proof. Combine Theorems 4.2.11 and 4.2.12. $\square$

4.3 A factorisation system for indexed pseudofunctors

To establish the tricategorical case of Method 2.2.2, it remains to show that the classes (pointwise biessentially surjective, pointwise an equivalence on homs) form a (tricategorical) factorisation system on $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$. We do this via a factorisation system (in the **Gray**-enriched sense) on $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$.

A (**Gray**-enriched) factorisation system on a **Gray**-category $\mathcal{B}$ consists of two classes $(\mathcal{E}, \mathcal{M})$ of morphisms in $\mathcal{B}$ such that

(i) $\mathcal{E}$ and $\mathcal{M}$ are both closed under composition and contain the isomorphisms,

(ii) every morphism f in $\mathcal{B}$ factors as $f = me$ for some $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and

(iii) for every $e \in \mathcal{E}$ and $m \in \mathcal{M}$, the square

$$\begin{tikzcd} \mathcal{B}(B,X) \arrow[r, "{(1,m)}"] \arrow[d, "{(e,1)}"'] \arrow[dr, phantom, "="] & \mathcal{B}(B,Y) \arrow[d, "{(e,1)}"] \\ \mathcal{B}(A,X) \arrow[r, "{(1,m)}"'] & \mathcal{B}(A,Y) \end{tikzcd}$$

is a pullback in **Gray**.

Proposition 4.3.1. *The classes (bijective on objects, isomorphism on homs) form a* **Gray***-enriched factorisation system on* **Gray***.*

Proof. Both classes are clearly closed under composition and contain the isomorphisms. Given a 2-functor $F\colon A \longrightarrow B$, let C be the 2-category with the same objects as A, and with hom-categories $C(a,b) = B(Fa, Fb)$. Then F factors as a bijective on objects 2-functor $E\colon A \longrightarrow C$ followed by an isomorphism on homs 2-functor $M\colon C \longrightarrow B$.

Now, let $E\colon A \longrightarrow B$ be bijective on objects and let $M\colon X \longrightarrow Y$ be an isomorphism on homs. To show that the square

$$\begin{tikzcd} {[B,X]} \arrow[r, "{[1,M]}"] \arrow[d, "{[E,1]}"'] \arrow[dr, phantom, "="] & {[B,Y]} \arrow[d, "{[E,1]}"] \\ {[A,X]} \arrow[r, "{[1,M]}"'] & {[A,Y]} \end{tikzcd}$$

is a pullback in **Gray**, i.e. a pullback in 2-**Cat**, it suffices to show that for every 2-category U, the square

$$\begin{tikzcd} 2\text{-}\mathbf{Cat}(U,[B,X]) \arrow[r, "{(1,[1,M])}"] \arrow[d, "{(1,[E,1])}"'] \arrow[dr, phantom, "="] & 2\text{-}\mathbf{Cat}(U,[B,Y]) \arrow[d, "{(1,[E,1])}"] \\ 2\text{-}\mathbf{Cat}(U,[A,X]) \arrow[r, "{(1,[1,M])}"'] & 2\text{-}\mathbf{Cat}(U,[A,Y]) \end{tikzcd}$$

is a pullback in **Set**. By the tensor-hom adjunction for the **Gray** tensor product $\otimes$, this is equivalent to each square

$$\begin{tikzcd} 2\text{-}\mathbf{Cat}(U\otimes B,X) \arrow[r, "{(1,M)}"] \arrow[d, "{(1\otimes E,1)}"'] \arrow[dr, phantom, "="] & 2\text{-}\mathbf{Cat}(U\otimes B,Y) \arrow[d, "{(1\otimes E,1)}"] \\ 2\text{-}\mathbf{Cat}(U\otimes A,X) \arrow[r, "{(1,M)}"'] & 2\text{-}\mathbf{Cat}(U\otimes A,Y) \end{tikzcd}$$

being a pullback in **Set**. Since the **Gray** tensor product is just the cartesian product of sets at the level of objects, the 2-functor $1 \otimes E \colon U \otimes A \longrightarrow U \otimes B$ is bijective on objects.

Hence it suffices to show that $E \perp M$ in the ordinary sense, for every bijective on objects 2-functor E and isomorphism on homs 2-functor M. Let

$$\begin{tikzcd} A \arrow[r, "U"] \arrow[d, "E"'] \arrow[dr, phantom, "="] & X \arrow[d, "M"] \\ B \arrow[r, "V"'] & Y \end{tikzcd}$$

be a commutative square in 2-**Cat**. A diagonal filler $G \colon B \longrightarrow X$ is uniquely determined by the function $G_0 \colon B_0 \longrightarrow X_0$ on objects, and the functors $G \colon B(b,c) \longrightarrow X(Gb, Gc)$ on hom-categories. But these must make the diagrams

$$\begin{tikzcd} A_0 \arrow[r, "U_0"] \arrow[d, "E_0"'] \arrow[dr, phantom, "="] & X_0 \\ B_0 \arrow[ur, "G_0"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X(Gb,Gc) \arrow[d, "M"] \\ B(b,c) \arrow[ur, "G"] \arrow[r, "MG"'] \arrow[ur, phantom, "=", shift right=3] & Y(MGb, MGc) \end{tikzcd}$$

commute, and by assumption, the function E_0 and the functors $M \colon X(Gb, Gc) \longrightarrow Y(MGb, MGc)$ are isomorphisms, hence the 2-functor G is uniquely determined. One can easily check that this defines a 2-functor G such that $GE = U$ and $MG = V$. □

As usual, we can lift this factorisation system to a pointwise defined factorisation system on $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$. To prevent overuse of the word "pointwise", given a class of maps $\mathcal{J}$ in **Gray** or **Bicat**, we will say that a tritransformation $\theta \colon F \longrightarrow G$ belongs to $\mathcal{J}$ if for every object C of $\mathcal{C}$, the component $\theta_C \colon FC \longrightarrow GC$ belongs to $\mathcal{J}$. So for instance we say that a morphism $\theta \colon F \longrightarrow G$ in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})$ is "fully faithful on homs" if for every object C of $\mathcal{C}$, the 2-functor $\theta_C \colon FC \longrightarrow GC$ is fully faithful on homs.

Proposition 4.3.2. *The classes (bijective on objects, isomorphism on homs) form a* **Gray***-enriched factorisation system on* $[\mathcal{C}^{op}, \mathbf{Gray}]$.

Proof. Both classes are closed under composition and contain the isomorphisms, since these are defined pointwise. We can construct factorisations as follows. Let $\theta \colon F \longrightarrow G$ be a **Gray**-natural transformation. Factorise each component $\theta_C \colon FC \longrightarrow GC$ as a bijective on objects 2-functor $\varphi_C \colon FC \longrightarrow HC$ and an isomorphism on homs 2-functor $\psi_C \colon HC \longrightarrow GC$. For each morphism $f \colon B \longrightarrow C$ in C, define Hf to be the unique 2-functor making the diagram

$$\begin{tikzcd} FC \arrow[r, "\varphi_C"] \arrow[d, "Ff"'] \arrow[dr, phantom, "="] & HC \arrow[r, "\psi_C"] \arrow[d, dotted, "Hf"] \arrow[dr, phantom, "="] & GC \arrow[d, "Gf"] \\ FB \arrow[r, "\varphi_B"'] & HB \arrow[r, "\psi_B"'] & GB \end{tikzcd}$$

commute, which exists and is unique since $\varphi_C \perp \psi_B$. This makes H into a **Gray**-functor $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Gray}$, and φ and ψ into **Gray**-natural transformations. Hence $\theta = \psi\varphi$ is a (bijective on objects, isomorphism on homs)-factorisation of θ.

It remains to show that for every bijective on objects **Gray**-natural transformation $\varphi\colon F \longrightarrow G$, and every isomorphism on homs **Gray**-natural transformation $\psi\colon H \longrightarrow K$, the square

$$\begin{tikzcd}
{[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](G,H)} \arrow[r, "{(1,\psi)}"] \arrow[d, "{(\varphi,1)}"'] \arrow[dr, phantom, "="] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](G,K)} \arrow[d, "{(\varphi,1)}"] \\
{[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](F,H)} \arrow[r, "{(1,\psi)}"'] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](F,K)}
\end{tikzcd}$$

is a pullback in **Gray**, or equivalently by the end-formula for the hom-2-categories in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$, that the square

$$\begin{tikzcd}
{\int_C [GC,HC]} \arrow[r, "{\int_C (1,\psi_C)}"] \arrow[d, "{\int_C (\varphi_C,1)}"'] \arrow[dr, phantom, "="] & {\int_C [GC,KC]} \arrow[d, "{\int_C (\varphi_C,1)}"] \\
{\int_C [FC,HC]} \arrow[r, "{\int_C (1,\psi_C)}"'] & {\int_C [FC,KC]}
\end{tikzcd}$$

is a pullback in **Gray**. This square is a pullback since ends commute with pullbacks, and each square

$$\begin{tikzcd}
{[GC,HC]} \arrow[r, "{[1,\psi_B]}"] \arrow[d, "{[\varphi_C,1]}"'] \arrow[dr, phantom, "="] & {[GC,KC]} \arrow[d, "{[\varphi_C,1]}"] \\
{[FC,HC]} \arrow[r, "{[1,\psi_B]}"'] & {[FC,KC]}
\end{tikzcd}$$

is a pullback. $\square$

To derive tricategorical orthogonality from the **Gray**-enriched orthogonality of Proposition 4.3.2, we use a 2-categorical generalisation of a result of [JS93], which gives a condition under which a pullback in **Gray** is also a tripullback.

We say that a 2-functor $F\colon A \longrightarrow B$ is an *equifibration* if

(i) for every object a of A and equivalence $g\colon Fa \longrightarrow b$ in B, there exists an equivalence $f\colon a \longrightarrow a'$ in A such that $Ff = g$, and

(ii) for every morphism $f\colon a \longrightarrow a'$ in A and invertible 2-cell $\beta\colon Ff \longrightarrow g$ in B, there exists an invertible 2-cell $\alpha\colon f \longrightarrow f'$ in A such that $F\alpha = \beta$.

Recall that the equifibrations are the fibrations in the model structure on 2-**Cat**, and that we can take "adjoint equivalence" in place of "equivalence" in (i) [Lac04].

Lemma 4.3.3. *Let $F\colon A \longrightarrow C \longleftarrow B\colon G$ be a cospan of 2-functors, and suppose F is an equifibration. Then the canonical 2-functor from the pullback P to the equi-comma 2-category E is a biequivalence.*

Proof. The canonical 2-functor $P \longrightarrow E$ is always fully faithful on homs, regardless of the equifibration assumption. So it remains to show that this 2-functor is biessentially surjective and essentially surjective on homs.

Let (a, s, b) be an object of E, where $s\colon Fa \longrightarrow Gb$ is an adjoint equivalence in C. Since F is an equifibration, there exists an adjoint equivalence $r\colon a \longrightarrow a'$ in A such that $Fr = s$. Then (a', b) is an object of P, and $(r, 1, 1)\colon (a, s, b) \longrightarrow (a', 1, b)$ is an equivalence is E. Hence the canonical 2-functor $P \longrightarrow E$ is biessentially surjective.

Now, let (a, b) and (a', b') be objects of P, and let $(f, \sigma, g)\colon (a, 1, b) \longrightarrow (a', 1, b')$ be a morphism in E, so that $f\colon a \longrightarrow a'$ and $g\colon b \longrightarrow b'$ are morphisms in A and B respectively, and $\sigma\colon Gg \longrightarrow Ff$ is an invertible 2-cell in C. Since F is an equifibration, there exists an invertible 2-cell $\rho\colon f' \longrightarrow f$ in A such that $F\rho = \sigma$. Then $(f', g)\colon (a, b) \longrightarrow (a', b')$ is a morphism in P, and $(\rho, 1)\colon (f', 1, g) \longrightarrow (f, \sigma, g)$ is an invertible 2-cell in E. Hence the canonical 2-functor $P \longrightarrow E$ is essentially surjective on homs. □

The fact that **Gray** and $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ are model **Gray**-categories ensures an ample supply of equifibrations.

Lemma 4.3.4. *Let $\mathcal{K}$ be a model* **Gray***-category. Then for any cofibrant object A and fibration $f\colon X \longrightarrow Y$ in $\mathcal{K}$, the 2-functor $(1, f)\colon \mathcal{K}(A, X) \longrightarrow \mathcal{K}(A, Y)$ is an equifibration.*

Proof. By definition [Hov99a], if $\mathcal{K}$ is a model **Gray**-category, then $\mathcal{K}(-,-)\colon \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \longrightarrow$ **Gray** is a right Quillen bifunctor, which means in part that for any cofibration $i\colon A \longrightarrow B$ and any fibration $f\colon X \longrightarrow Y$, the canonical 2-functor into the pullback

$$\begin{tikzcd}
\mathcal{K}(B,X) \arrow[dr, dotted] \arrow[drr, bend left, "{(1,f)}"] \arrow[ddr, bend right, "{(i,1)}"'] & & \\
& \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{K}(B,Y) \arrow[d, "{(i,1)}"] \\
& \mathcal{K}(A,X) \arrow[r, "{(1,f)}"'] & \mathcal{K}(A,Y)
\end{tikzcd}$$

is a fibration in **Gray**, i.e., an equifibration. But for a cofibrant object A in $\mathcal{K}$, the unique morphism $0 \longrightarrow A$ is a cofibration, and therefore for any fibration $f\colon X \longrightarrow Y$, this gives that the 2-functor $(1, f)\colon \mathcal{K}(A, X) \longrightarrow \mathcal{K}(A, Y)$ is an equifibration. □

Recall the F' notation for the values of the left triadjoint of the triadjunction of Theorem 3.2.5.

Theorem 4.3.5. *The classes (biessentially surjective, equivalence on homs) form a (tricategorical) stable factorisation system on* $\mathrm{Hom}(\mathcal{C}^{op}, \mathbf{Bicat})$.

Proof. Both classes are closed under composition, contain the biequivalences, and are replete, since these are defined pointwise, and the corresponding properties hold for the classes in **Bicat**.

To construct the factorisation of a morphism $\varphi\colon F \longrightarrow G$ in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$, it suffices to factorise the morphism $\varphi'\colon F' \longrightarrow G'$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$, since φ is equivalent to a composite of φ' with biequivalences. But we can factorise φ' as a bijective on objects morphism followed by an isomorphism on homs morphism, as in Proposition 4.3.2.

We can reduce the verification of tricategorical orthogonality to cases of **Gray**-enriched orthogonality as follows. Let $\theta\colon F \longrightarrow G$ be a biessentially surjective morphism in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$. Then $\theta'\colon F' \longrightarrow G'$ is a biessentially surjective morphism in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$. Factorise θ' as $\theta' = \psi\varphi$, where $\varphi\colon F' \longrightarrow H$ is bijective on objects, and ψ is an isomorphism on homs, and hence a biequivalence in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$. Then $\mathrm{st}\circ\varphi\colon \mathrm{st}\circ F' \longrightarrow \mathrm{st}\circ H$ is a bijective on objects morphism in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})$. Since the left **Gray**-adjoint $Q\colon \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}) \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ is constructed by **Gray**-colimits (as in the proof of Theorem 3.3.4), and the class of bijective on objects 2-functors is closed under **Gray**-colimits, we therefore have that $\varphi'\colon (F')' \longrightarrow H'$ is a bijective on objects morphism in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$. Therefore θ is equivalent to a composite of φ' with biequivalences, and so we conclude that we can replace any biessentially surjective morphism in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$ by a bijective on objects morphism between objects of the form F' in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$.

Similarly, we can replace any morphism in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$ that is an equivalence on homs by a morphism in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ that is (pointwise) both an isomorphism on homs and an equifibration. First, we can replace any such morphism $\theta\colon F \longrightarrow G$ by the morphism $\theta'\colon F' \longrightarrow G'$, which is also an equivalence on homs. Next, take a (trivial cofibration, fibration) factorisation $\theta' = \pi\iota$ for the projective model structure on $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$; note that since ι is a weak equivalence, we have that π is an equivalence on homs. Next, take the (bijective on objects, isomorphism on homs)-factorisation $\pi = \psi\varphi$; note that φ is a pointwise biequivalence. One can check directly by construction that since π is a pointwise equifibration, then so is ψ. Hence θ is equivalent to a composite of ψ and biequivalences in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$.

Therefore, to show that any biessentially surjective morphism is orthogonal to any equivalence on homs morphism in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$, it suffices to show that for any bijective on objects morphism $\varphi\colon F' \longrightarrow G'$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ and any isomorphism on homs and equifibration morphism $\psi\colon H \longrightarrow K$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$, the square

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(G', H) \arrow[r, "{(1,\psi)}"] \arrow[d, "{(\varphi,1)}"'] \arrow[dr, phantom, "\simeq"] & \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(G', K) \arrow[d, "{(\varphi,1)}"] \\
\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(F', H) \arrow[r, "{(1,\psi)}"'] & \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(F', K)
\end{tikzcd}$$

is a tripullback. But since F' and G' are strictifications, this square is biequivalent to the square

$$\begin{tikzcd} {[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](G', H)} \arrow[r, "{(1,\psi)}"] \arrow[d, "{(\varphi,1)}"'] \arrow[dr, phantom, "="] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](G', K)} \arrow[d, "{(\varphi,1)}"] \\ {[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](F', H)} \arrow[r, "{(1,\psi)}"'] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](F', K)} \end{tikzcd} \tag{4.8}$$

which, by Proposition 4.3.2, is a pullback in **Gray**. Moreover, since F' is a cofibrant object and ψ is a fibration in the **Gray**-enriched model category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$, equipped with the projective model structure, the 2-functor $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](F', \psi)$ is an equifibration. Therefore, by Lemma 4.3.3, the pullback (4.8) is also a tripullback. Therefore $\varphi \perp \psi$.

Finally, this factorisation system is stable, since tripullbacks are computed pointwise in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$, and as in the proof of Lemma 4.1.5, biessentially surjective 2-functors are stable under tripullback in **Gray**, and hence in **Bicat**, and every biessentially surjective pseudofunctor can be replaced by a biessentially surjective 2-functor. □

To show that the factorisation system of Theorem 4.3.5 satisfies the hypotheses of Theorem 4.2.10, it remains to show that there is a finite trilimit characterisation of the class of pointwise equivalence on homs morphisms in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$. For this would imply that this class is preserved by any finite trilimit perserving trihomomorphism $L\colon \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat}) \longrightarrow \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$. Since both this class of morphisms and trilimits are defined pointwise, it suffices to show that equivalence on homs pseudofunctors have a finite trilimit characterisation in **Bicat**. Moreover, it suffices to check this for 2-functors in **Gray**.

Furthermore, since trilimits in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$ are calculated pointwise, and for each object X of $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$, the trihomomorphism

$$\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(X, -)\colon \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat}) \longrightarrow \mathbf{Bicat}$$

preserves trilimits, such a (finite) trilimit characterisation will ensure that a morphism in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$ is pointwise an equivalence on homs if and only if it is representably an equivalence on homs.

Once again, let $\mathcal{A}$ denote the free category on the graph (4.1). Let $\mathbf{2}_2$ denote the free 2-category containing a 2-cell, and let $K\colon \mathcal{A} \longrightarrow \mathbf{Gray}$ be the **Gray**-functor that picks out the cospan

$$1 \xrightarrow{0} \mathbf{2}_2 \xleftarrow{1} 1.$$

We call the **Gray**-limit of a cospan in **Gray** weighted by K its 2-comma 2-category. We spell out the definitions of the cells of this 2-category.

Definition 4.3.6. The *2-comma 2-category* of a pair of 2-functors $F\colon A \longrightarrow C \longleftarrow B\colon G$ is the 2-category $F//G$ such that:

- an object (a, s, σ, t, b) consists of an object a of A, an object b of B, and a 2-cell

$$\begin{tikzcd} Fa \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\sigma"] & Gb \end{tikzcd}$$

  in C,

- a morphism $(f, \lambda, \rho, g)\colon (a, s, \sigma, t, b) \longrightarrow (a', s', \sigma', t', b')$ consists of a morphism $f\colon a \longrightarrow a'$ in A, a morphism $g\colon b \longrightarrow b'$ in B, and invertible 2-cells

$$\begin{tikzcd} Fa \arrow[r, "s"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow\lambda"] & Gb \arrow[d, "Gg"] \\ Fa' \arrow[r, "s'"'] & Gb' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Fa \arrow[r, "t"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow\rho"] & Gb \arrow[d, "Gg"] \\ Fa' \arrow[r, "t'"'] & Gb' \end{tikzcd}$$

  in C, such that

$$\begin{tikzcd} Fa \arrow[r, "s"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow\lambda"] & Gb \arrow[d, "Gg"] \\ Fa' \arrow[r, bend left, "s'"] \arrow[r, bend right, "t'"'] \arrow[r, phantom, "\Downarrow\sigma'"] & Gb' \end{tikzcd} = \begin{tikzcd} Fa \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\sigma"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow\rho"] & Gb \arrow[d, "Gg"] \\ Fa' \arrow[r, "t'"'] & Gb' \end{tikzcd}$$

- a 2-cell $(\alpha, \beta)\colon (f, \lambda, \rho, g) \longrightarrow (f', \lambda', \rho', g')$ consists of a 2-cell $\alpha\colon f \longrightarrow f'$ in A and a 2-cell $\beta\colon g \longrightarrow g'$ in B, such that

$$\begin{tikzcd} Fa \arrow[r, "s"] \arrow[d, bend right, "Ff'"'] \arrow[d, bend left, "Ff"] \arrow[d, phantom, "\overset{F\alpha}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow\lambda"] & Gb \arrow[d, "Gg"] \\ Fa' \arrow[r, "s'"'] & Gb' \end{tikzcd} = \begin{tikzcd} Fa \arrow[r, "s"] \arrow[d, "Ff'"'] \arrow[dr, phantom, "\lambda'\Downarrow"] & Gb \arrow[d, bend right, "Gg'"'] \arrow[d, bend left, "Gg"] \arrow[d, phantom, "\overset{G\beta}{\Leftarrow}"] \\ Fa' \arrow[r, "s'"'] & Gb' \end{tikzcd}$$

  and

$$\begin{tikzcd} Fa \arrow[r, "t"] \arrow[d, bend right, "Ff'"'] \arrow[d, bend left, "Ff"] \arrow[d, phantom, "\overset{F\alpha}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow\rho"] & Gb \arrow[d, "Gg"] \\ Fa' \arrow[r, "t'"'] & Gb' \end{tikzcd} = \begin{tikzcd} Fa \arrow[r, "t"] \arrow[d, "Ff'"'] \arrow[dr, phantom, "\rho'\Downarrow"] & Gb \arrow[d, bend right, "Gg'"'] \arrow[d, bend left, "Gg"] \arrow[d, phantom, "\overset{G\beta}{\Leftarrow}"] \\ Fa' \arrow[r, "t'"'] & Gb' \end{tikzcd},$$

and all compositions are inherited from the 2-categories A, B, and C.

Lemma 4.3.7. *A 2-functor $F\colon A \longrightarrow B$ is an equivalence on homs if and only if the canonical 2-functor from $[\mathbf{2}_2, A]$ to the 2-comma 2-category $F//F$ is a biequivalence.*

Proof. Note that $[\mathbf{2}_2, A]$ is the 2-comma 2-category $1_A//1_A$. The canonical 2-functor $U\colon [\mathbf{2}_2, A] \longrightarrow F//F$ is always faithful on homs, regardless of hypothesis.

Suppose F is an equivalence on homs. Let $(f, \lambda, \rho, g), (f', \lambda', \rho', g')\colon (a, s, \sigma, t, b) \longrightarrow (a', s', \sigma', t', b')$ be a parallel pair of morphisms in $[\mathbf{2}_2, A]$, and let $(\alpha, \beta)\colon (f, F\lambda, F\rho, g) \longrightarrow (f, F\lambda', F\rho', g')$ be a 2-cell in $F//F$. Then since F is faithful on homs, $(\alpha, \beta)\colon (f, \lambda, \rho, g) \longrightarrow (f, \lambda', \rho', g')$ is a 2-cell in $[\mathbf{2}_2, A]$. Hence U is fully faithful on homs.

Now, let (a, s, σ, t, b) and $(a', s', \sigma', t', b')$ be a pair of objects in $[\mathbf{2}_2, A]$, and let (f, λ, ρ, g): $(a, Fs, F\sigma, Ft, b) \longrightarrow (a', Fs', F\sigma', Ft', b')$ be a morphism in $F//F$. Since F is fully faithful on homs, there exist invertible 2-cells

$$\begin{tikzcd} a \arrow[r, "s"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\kappa"] & b \arrow[d, "g"] \\ a' \arrow[r, "s'"'] & b' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} a \arrow[r, "t"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\pi"] & b \arrow[d, "g"] \\ a' \arrow[r, "t'"'] & b' \end{tikzcd}$$

in A such that $F\kappa = \lambda$ and $F\pi = \rho$. Then, since F is faithful on homs, (f, κ, π, g): $(a, s, \sigma, t, b) \longrightarrow (a', s', \sigma', t', b')$ is a morphism in $[\mathbf{2}_2, A]$, such that $U(f, \kappa, \pi, g) = (f, \lambda, \rho, g)$. Hence U is a (surjective) equivalence on homs.

Now, let (a, s, σ, t, b) be an object of $F//F$. Since F is an equivalence on homs, there exists a 2-cell $\rho\colon p \longrightarrow q$ in A, and invertible 2-cells $\gamma\colon Fp \longrightarrow s$ and $\delta\colon Fq \longrightarrow t$ in B such that $\sigma\gamma = \delta(F\rho)$. Then $(1, \gamma, \delta, 1)\colon (a, Fp, F\rho, Fq, b) \longrightarrow (a, s, \sigma, t, b)$ is an equivalence in $F//F$. Hence U is a biequivalence.

Conversely, suppose U is a biequivalence. Let $\alpha, \alpha'\colon f \longrightarrow f'\colon a \longrightarrow a'$ be a parallel pair of 2-cells in A such that $F\alpha = F\alpha'$. Then $(f, 1, 1, f), (f', 1, 1, f')\colon (a, 1, 1, 1, a) \longrightarrow (a', 1, 1, 1, a')$ is a parallel pair of morphisms in $[\mathbf{2}_2, A]$, and $(\alpha, \alpha')\colon (f, 1, 1, f) \longrightarrow (f', 1, 1, f')$ is a 2-cell in $F//F$. Then, since U is full on homs, $(\alpha, \alpha')\colon (f, 1, 1, f) \longrightarrow (f', 1, 1, f')$ is a 2-cell in $[\mathbf{2}_2, A]$, and so $\alpha = \alpha'$. Hence F is faithful on homs.

Now, let $f, g\colon a \longrightarrow b$ be a parallel pair of morphisms in A, and let $\beta\colon Ff \longrightarrow Fg$ be an invertible 2-cell in B. Then $(g, \beta, \beta, f)\colon (a, 1, 1, 1, a) \longrightarrow (b, 1, 1, 1, b)$ is a morphism in $F//F$. Then, since U is essentially surjective on homs, there exists a morphism $(k, \alpha, \gamma, h)\colon (a, 1, 1, 1, a) \longrightarrow (b, 1, 1, 1, b)$ in $[\mathbf{2}_2, A]$ and an invertible 2-cell $(\eta, \varepsilon)\colon (g, \beta, \beta, f) \longrightarrow (k, F\alpha, F\gamma, h)$ in $F//F$. Then $\eta^{-1}\alpha\varepsilon\colon f \longrightarrow g$ is a 2-cell in A, and $F(\eta^{-1}\alpha\varepsilon) = \beta$.

Now, let $f, g\colon a \longrightarrow b$ be a parallel pair of morphisms in A, and let $\beta\colon Ff \longrightarrow Fg$ be an arbitrary 2-cell in B. Then (a, Ff, β, Fg, b) is an object of $F//F$. Since U is biessentially surjective, there exists an object (a', f', α, g', b') of $[\mathbf{2}_2, A]$, and an equivalence $(u, \lambda, \rho, v)\colon (a, Ff, \beta, Fg, b) \longrightarrow (A', Ff', F\alpha, Fg', b')$ in $F//F$. Then since λ and ρ are invertible 2-cells, there exist, by the previous paragraph, invertible 2-cells

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\kappa"] & b \arrow[d, "v"] \\ a' \arrow[r, "f'"'] & b' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} a \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\pi"] & b \arrow[d, "v"] \\ a' \arrow[r, "g'"'] & b' \end{tikzcd}$$

in A, such that $F\kappa = \lambda$ and $F\pi = \rho$. Then

$$\begin{tikzcd} & b \arrow[dr, "v"] \arrow[drr, bend left, "1"] \arrow[d, phantom, "\Downarrow\kappa"] & \arrow[d, phantom, "\Downarrow\eta"] & \\ a \arrow[ur, "f"] \arrow[r, "u"'] \arrow[dr, "g"'] & a' \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[d, phantom, "\Downarrow\pi^{-1}"] & b' \arrow[r, "v^\bullet"'] \arrow[d, phantom, "\Downarrow\eta^{-1}"] & b \\ & b \arrow[ur, "v"'] \arrow[urr, bend right, "1"'] & & \end{tikzcd}$$

is a 2-cell $f \longrightarrow g$ in A which is sent by F to β. Hence F is full on homs.

Finally, let a and b be a pair of objects of A, and let $g\colon Fa \longrightarrow Fb$ be a morphism in B. Then $(a, g, 1, g, b)$ is an object of $F//F$. Since U is biessentially surjective, there exists an object (c, u, γ, v, d) in $[\mathbf{2}_2, A]$ and an equivalence $(f, \mu, \nu, h)\colon (c, Fu, F\gamma, Fv, d) \longrightarrow (a, g, 1, g, b)$ in $F//F$. Then $huf^\bullet\colon a \longrightarrow b$ is a morphism in A and

$$\begin{tikzcd} & Fc \arrow[r, "Fu"] \arrow[d, "Ff"] \arrow[dr, phantom, "\Downarrow\mu"] & Fd \arrow[d, "Fh"] \\ Fa \arrow[ur, "Ff^\bullet"] \arrow[r, "1"'] \arrow[r, phantom, bend left, "\Downarrow\varepsilon"] & Fa \arrow[r, "g"'] & Fb \end{tikzcd}$$

is an invertible 2-cell in B. Hence F is essentially surjective on homs. □

Although the 2-comma 2-category $F//F$ was defined as a **Gray**-limit, this is indeed a finite trilimit characterisation of equivalence on homs 2-functors. Firstly, since $\mathbf{2}_2$ is a cofibrant 2-category, we have the equivalences

$$\begin{aligned} \mathbf{Bicat}(\mathbf{2}_2, [X, A]) &\simeq [\mathbf{2}_2, [X, A]] \\ &\cong [X, [\mathbf{2}_2, A]] \end{aligned}$$

for all 2-categories X, and therefore $[\mathbf{2}_2, A]$ satisfies the universal property of the trilimit of $A\colon 1 \longrightarrow \mathbf{Gray}$ weighted by $\mathbf{2}_2\colon 1 \longrightarrow \mathbf{Bicat}$. Similarly, letting 2 denote the discrete 2-category with two objects, $[2, A]$ is a trilimit for all 2-categories A. Finally, for any

2-functor $F\colon A \longrightarrow B$, the 2-comma 2-category is a pullback

$$\begin{tikzcd} F//F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathbf{2}_2, B]} \arrow[d] \\ {[2, A]} \arrow[r, "{[1,F]}"'] & {[2, B]} \end{tikzcd}$$

in **Gray**. The 2-functor $2 \longrightarrow \mathbf{2}_2$ is equal to the composite of the exterior of the diagram

$$\begin{tikzcd} 2 \arrow[r] \arrow[d] & \mathbf{2} \arrow[d] \\ \mathbf{2} \arrow[r] & \mathbf{2} +_2 \mathbf{2} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r] & \mathbf{2}_2 \end{tikzcd}$$

where $\mathbf{2}$ denotes the free category containing a morphism, and is therefore a cofibration, since $2 \longrightarrow \mathbf{2}$ and $\mathbf{2}+_2\mathbf{2} \longrightarrow \mathbf{2}_2$ are both generating cofibrations for the model structure on 2-**Cat** [Lac02b], and the class of cofibrations is closed under composition and stable under pushout. Therefore, by Lemma 4.3.4, we have that $[\mathbf{2}_2, B] \longrightarrow [2, B]$ is an equifibration, and hence by Lemma 4.3.3 this pullback has the universal property of a tripullback. Hence $F//F$ can be calculated as a finite trilimit.

We end this section with a lemma which we will see implies a refinement of Corollary 4.2.13 when applied to the factorisation system of this section.

Lemma 4.3.8. *For every object G of* $\mathrm{Hom}(\mathcal{C}^{op}, \mathbf{Bicat})$*, there exists an object F of* $[\mathcal{C}^{op}, \mathbf{Set}]$ *and a biessentially surjective morphism $F \longrightarrow G$ in* $\mathrm{Hom}(\mathcal{C}^{op}, \mathbf{Bicat})$*.*

Proof. First suppose that G is an object of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$, i.e. a **Gray**-functor $G\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **Gray**, or equivalently a functor $G\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ 2-**Cat**. By composing with the functor $(-)_0\colon$ 2-**Cat** $\longrightarrow$ **Set** that sends a 2-category A to its set A_0 of objects, we get a functor $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$. Then the inclusions $FC = (GC)_0 \longrightarrow GC$ form the components of a (**Gray**-)natural transformation $F \longrightarrow G$, which is pointwise bijective on objects.

Now for an arbitrary object G of $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$, the unit $G \longrightarrow G'$ of the triadjunction of Theorem 3.2.5 is a biequivalence in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$, so has a weak inverse biequivalence $G' \longrightarrow G$. Since G' is an object of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$, there exists an F as above and a pointwise bijective on objects morphism $F \longrightarrow G'$, and hence a pointwise biessentially surjective morphism $F \longrightarrow G' \longrightarrow G$. $\square$

4.4 Calculating cohomology

We have now assembled all the prerequisites to prove the case $n = 2$ of Method 2.2.2. Recall the case $n = 2$ of Definition 1.3.2.

Definition 4.4.1. Let $X = (\mathcal{C}, J)$ be a site. The *cohomology bicategory* of $(\mathcal{C}, J)$ with coefficients in a 2-stack F over $(\mathcal{C}, J)$ is the bicategory of global sections of F:

$$\mathscr{H}(X; F) := \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(1, F),$$

where 1 denotes the constant trihomomorphism with value 1.

Theorem 4.4.2. *Let $X = (\mathcal{C}, J)$ be a site, and let $F \in \mathcal{F} = \mathrm{Hom}(\mathcal{C}^{op}, \mathbf{Bicat})$ be an indexed bicategory over $\mathcal{C}$. Let $f\colon F \longrightarrow G$ be a morphism in $\mathcal{F}$ such that G is a 2-stack and Lf is (pointwise) an equivalence on homs. Then the cohomology bicategory $\mathscr{H}(X; LF)$ of the site with coefficients in the associated 2-stack LF of F is biequivalent to the full sub-bicategory of the hom-bicategory $\mathcal{F}(1, G)$ on those objects $x\colon 1 \longrightarrow G$ for which there exists a locally surjective morphism $r\colon R \longrightarrow 1$ in $[\mathcal{C}^{op}, \mathbf{Set}]$, a morphism $a\colon R \longrightarrow F$, and an equivalence*

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "r"'] \arrow[dr, phantom, "\simeq"] & F \arrow[d, "f"] \\ 1 \arrow[r, "x"'] & G \end{tikzcd}$$

Proof. By Definition 4.4.1, the cohomology bicategory $\mathscr{H}(X; LF)$ is the bicategory $\mathcal{F}(1, LF)$. The results of Section 4.3 show that the factorisation system (biessentially surjective, equivalence on homs) on $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$ satisfy the hypotheses of Theorems 4.2.10 and 4.2.12. Hence Corollary 4.2.13 yields a biequivalence between $\mathscr{H}(X; LF)$ and the full sub-bicategory of $\mathcal{F}(1, G)$ on those objects $x\colon 1 \longrightarrow G$ for which there exists a square of the form

$$\begin{tikzcd} P \arrow[r, "a"] \arrow[d, "p"'] \arrow[dr, phantom, "\simeq"] & F \arrow[d, "f"] \\ 1 \arrow[r, "x"'] & G \end{tikzcd}$$

where p is locally biessentially surjective. But by Lemma 4.3.8, there exists an object R of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and a biessentially surjective morphism $R \longrightarrow P$. Then the unique morphism $R \longrightarrow P \longrightarrow 1$ is locally surjective in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and so for every such x there exists a square as in the statement of the theorem. $\square$

Chapter 5

Torsors, gerbes, and Giraud's H^2

In this final chapter we apply Theorem 4.4.2 to the indexed 2-functor

$$\mathrm{Tors}\colon \mathbb{G} \longrightarrow \mathbb{S} \tag{5.1}$$

over a site $X = (\mathcal{C}, J)$ (where $\mathbb{G}$ is the indexed 2-category of sheaves of groups and $\mathbb{S}$ is the 2-stack of stacks over X, and Tors sends a sheaf of groups G to the stack of G-torsors) to show that the associated 2-stack $L\mathbb{G}$ of $\mathbb{G}$ is the 2-stack of gerbes over the site; the stack $\mathbb{L}$ of liens (sometimes translated as "bands" or "ties") is (equivalent to) the 1-stack truncation $\tau_1 L\mathbb{G}$ thereof. Therefore we can recover Giraud's non-abelian $H^2(X; M)$, for M a lien, as the set of connected components of the fibre of the 2-functor $\mathscr{H}(X; L\mathbb{G}) \longrightarrow \mathscr{H}(X; \mathbb{L})$ over M.

Throughout this chapter, let $(\mathcal{C}, J)$ be a fixed site. We refer the reader to [Gir71] for the definitions and basic properties of torsors and gerbes.

We first define the indexed 2-functor (5.1) in more detail. Recall that for any presheaf $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, its category of elements $\mathrm{el}\, F$ inherits a canonical topology J_F from $(\mathcal{C}, J)$; a sieve on an object (x, C) of $\mathrm{el}\, F$ amounts to a sieve on C in $\mathcal{C}$, and the set of covering sieves of (x, C) is defined to be $J_F(x, C) = J(C)$. In particular, for each object C of $\mathcal{C}$, the slice category $\mathcal{C}/C$ inherits a canonical topology J_C. Moreover, for each morphism $\varphi\colon F \longrightarrow G$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, pre-composition with the functor $\varphi_*\colon \mathrm{el}\, F \longrightarrow \mathrm{el}\, G$ yields a 2-functor

$$\varphi^*\colon \mathrm{Hom}((\mathrm{el}\, G)^{\mathrm{op}}, \mathbf{Cat}) \longrightarrow \mathrm{Hom}((\mathrm{el}\, F)^{\mathrm{op}}, \mathbf{Cat}) \tag{5.2}$$

which sends sheaves of groups and stacks over $(\mathrm{el}\, G, J_G)$ to sheaves of groups and stacks over $(\mathrm{el}\, F, J_F)$ respectively.

We define $\mathbb{G}$ and $\mathbb{S}$ to be the **Gray**-functors $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Gray}$ that send an object C of $\mathcal{C}$ to the full sub-2-categories of $\mathrm{Hom}((\mathcal{C}/C)^{\mathrm{op}}, \mathbf{Cat})$ on the sheaves of groups and the stacks over $(\mathcal{C}/C, J_C)$ respectively, and are such that for each morphism $f\colon B \longrightarrow C$ in $\mathcal{C}$, $\mathbb{G}f\colon \mathbb{G}C \longrightarrow \mathbb{G}B$ and $\mathbb{S}f\colon \mathbb{S}C \longrightarrow \mathbb{S}B$ are restrictions of the 2-functor f^* in (5.2).

For each object C of $\mathcal{C}$, there is a 2-functor $\mathrm{Tors}_C\colon \mathbb{G}C \longrightarrow \mathbb{S}C$ that sends a sheaf of groups G over $(\mathcal{C}/C, J_C)$ to the stack of G-torsors $\mathrm{Tors}\, G$ over $(\mathcal{C}/C, J_C)$. The stack $\mathrm{Tors}\, G$ sends an object $f\colon B \longrightarrow C$ of $\mathcal{C}/C$ to the category of (f^*G)-torsors over $(\mathcal{C}/B, J_B)$. Hence these 2-functors Tors_C are the components of a **Gray**-natural transformation $\mathrm{Tors}\colon \mathbb{G} \longrightarrow \mathbb{S}$.

Remark 5.1. For $\mathbb{G}$ and $\mathbb{S}$ to be objects of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ requires that the relevant full sub-2-categories of the 2-categories $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$ are objects of **Gray**. To ensure this, we assume the existence of inaccessible cardinals κ_0, κ_1, and κ_2 (or equivalently Grothendieck universes $\mathcal{U}_0$, $\mathcal{U}_1$, and $\mathcal{U}_2$) such that the category **Set** of κ_0-small sets is κ_1-small, the 2-category **Cat** of κ_1-small categories is κ_2-small, and the objects of **Gray** and **Bicat** are the κ_2-small 2-categories and bicategories respectively.

To show that the indexed 2-functor (5.1) satisfies the hypotheses of Theorem 4.4.2, we must show that its codomain $\mathbb{S}$ is a 2-stack, and that the associated 2-stack trihomomorphism L sends it to a pointwise equivalence on homs morphism.

Our reasoning on the indexed 2-categories $\mathbb{G}$ and $\mathbb{S}$ will be aided by the following lemmas.

Lemma 5.2. *For every presheaf F on $\mathcal{C}$, the 2-categories* $\mathrm{Hom}(\mathcal{C}^{op}, \mathbf{Gray})(F, \mathbb{G})$ *and* $\mathrm{Hom}(\mathcal{C}^{op}, \mathbf{Gray})(F, \mathbb{S})$ *are biequivalent to the full sub-2-categories of* $\mathrm{Hom}((\mathrm{el}\, F)^{op}, \mathbf{Cat})$ *on the sheaves of groups and stacks respectively. Moreover, for every morphism of presheaves $\varphi\colon F \longrightarrow G$, there is an equivalence*

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{C}^{op}, \mathbf{Gray})(G, \mathbb{S}) & \xrightarrow{\varphi^*} & \mathrm{Hom}(\mathcal{C}^{op}, \mathbf{Gray})(F, \mathbb{S}) \\ \simeq\big\downarrow & \simeq & \big\downarrow\simeq \\ \mathrm{Stacks}(\mathrm{el}\, G, J_G) & \xrightarrow[\varphi^*]{} & \mathrm{Stacks}(\mathrm{el}\, F, J_F) \end{array}$$

and similarly for $\mathbb{G}$.

Proof. We describe the biequivalence for $\mathbb{S}$ at the level of objects; the rest is straightforward, and the proof for $\mathbb{G}$ is analogous. Let $S\colon (\mathrm{el}\, F)^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ be a stack. We define a **Gray**-natural transformation $F \longrightarrow \mathbb{S}$ whose component at an object C of $\mathcal{C}$ sends an element $x \in FC$ to the stack

$$(\mathcal{C}/C)^{\mathrm{op}} \xrightarrow{x_*} (\mathrm{el}\, F)^{\mathrm{op}} \xrightarrow{S} \mathbf{Cat}\,. \tag{5.3}$$

Conversely, let $\sigma\colon F \longrightarrow \mathbb{S}$ be a tritransformation in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Gray})$. We define the stack $S\colon (\mathrm{el}\, F)^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ to send the object (x, C) of $\mathrm{el}\, F$ to the value of the stack $\sigma_C(x)\colon (\mathcal{C}/C)^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ at 1_C, and to send a morphism $f\colon ((Ff)x, B) \longrightarrow (x, C)$ in $\mathrm{el}\, F$

to the composite functor

$$(\sigma_C(x))(1_C) \xrightarrow{\quad f^* \quad} (\sigma_C(x))(f) \xrightarrow[\simeq]{((\sigma_f(x))(1_B))^\bullet} (\sigma_B((Ff)x))(1_B).$$

Applying the construction of the first paragraph to this stack S gives the **Gray**-natural transformation $\tau\colon F \longrightarrow \mathbb{S}$ whose component at C sends an element $x \in FC$ to the stack (5.3). Then there is an equivalence of stacks $S \simeq \sigma_C(x)$ over $(\mathcal{C}/C, J_C)$ whose component at $f\colon B \longrightarrow C$ is

$$S((Ff)x, B) := (\sigma_B((Ff)x))(1_B) \xrightarrow[\simeq]{(\sigma_f(x))(1_B)} (\sigma_C(x))(f).$$

These form the components of an equivalence tritransformation $\tau \simeq \sigma$. □

We spell out the definition of bicategorical orthogonality for morphisms in a 2-category.

Definition 5.3. Let $f\colon A \longrightarrow B$ and $g\colon X \longrightarrow Y$ be morphisms in a 2-category. We say that f and g are *orthogonal* (in the bicategorical sense), written $f \perp g$, if the following hold:

- for every square as on the left-hand side of (5.4) there exists a morphism $h\colon B \longrightarrow X$ and isomorphisms as on the right-hand side of (5.4), satisfying the equation

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] & Y \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong", near start] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, "h"] \arrow[ur, phantom, "\cong", near end, shift right=3] & Y \end{tikzcd} \tag{5.4}$$

- for all parallel pairs of morphisms $h, k\colon B \longrightarrow X$, and 2-cells

$$\begin{tikzcd} & B \arrow[dr, "h"] & \\ A \arrow[ur, "f"] \arrow[dr, "f"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & X \\ & B \arrow[ur, "k"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X \arrow[dr, "g"] & \\ B \arrow[ur, "h"] \arrow[dr, "k"'] \arrow[rr, phantom, "\Downarrow\beta"] & & Y \\ & X \arrow[ur, "g"'] & \end{tikzcd}$$

  such that $g\alpha = \beta f$, there exists a unique 2-cell $\gamma\colon h \longrightarrow k$ such that $\alpha = \gamma f$ and $\beta = g\gamma$.

As recalled in the introduction to Chapter 2, the inclusion of the 2-category of stacks over $(\mathcal{C}, J)$ into the 2-category of indexed categories over $\mathcal{C}$ has a left biadjoint

$$\begin{tikzcd} \mathrm{Stacks}(\mathcal{C}, J) \arrow[r, hook, shift right=1.5] \arrow[r, phantom, "\perp"] & \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}). \arrow[l, "L"', shift right=1.5] \end{tikzcd}$$

By the bicategorical analogue of Propositions 2.1.10 and 4.2.9, the classes (local equivalence, cartesian) form a stable factorisation system (in the bicategorical sense) on

$\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$. Here a morphism $f\colon X \longrightarrow Y$ is a local equivalence if Lf is an equivalence, and is cartesian if the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "\cong"] & Y \arrow[d, "\eta_Y"] \\ LX \arrow[r, "Lf"'] & LY \end{tikzcd}$$

is a bipullback.

For any object B of a 2-category $\mathcal{K}$, let $\mathcal{K}/B$ denote the (strict) slice 2-category, whose objects (X, f) are morphisms $f\colon X \longrightarrow B$ in $\mathcal{K}$, whose morphisms $h\colon (X, f) \longrightarrow (Y, g)$ are morphisms $h\colon X \longrightarrow Y$ in $\mathcal{K}$ such that $gh = f$, and whose 2-cells $\alpha\colon h \longrightarrow k$ are 2-cells $\alpha\colon h \longrightarrow k$ in $\mathcal{K}$ such that $g\alpha = f$. For a presheaf F on $\mathcal{C}$, let $\mathrm{Cart}\,/F$ denote the full sub-2-category of the slice 2-category $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})/F$ on the cartesian morphisms $X \longrightarrow F$.

Note that while the 2-category $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$ does not have all pullbacks, it does have pullbacks of cospans whose common codomain is **Set**-valued. For bipullbacks, and indeed iso-comma objects, do exist in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$, and can be computed pointwise as iso-comma categories. But the iso-comma category of a pair of functors with discrete codomain is isomorphic to their pullback. Hence for any morphism of presheaves $\varphi\colon F \longrightarrow G$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, there is a 2-functor $\varphi^*\colon \mathrm{Cart}\,/G \longrightarrow \mathrm{Cart}\,/F$ defined by 2-pullback.

Lemma 5.4. *For every presheaf F on $\mathcal{C}$, there is an equivalence of 2-categories*

$$\mathrm{Stacks}(\mathrm{el}\,F, J_F) \simeq \mathrm{Cart}\,/F,$$

and for every morphism of presheaves $\varphi\colon F \longrightarrow G$, there is an equivalence

$$\begin{tikzcd} \mathrm{Stacks}(\mathrm{el}\,G, J_G) \arrow[r, "\varphi^*"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\simeq"] & \mathrm{Stacks}(\mathrm{el}\,F, J_F) \arrow[d, "\simeq"] \\ \mathrm{Cart}\,/G \arrow[r, "\varphi^*"'] & \mathrm{Cart}\,/F \end{tikzcd}$$

Proof. For each presheaf F there is an equivalence

$$\mathrm{Hom}((\mathrm{el}\,F)^{\mathrm{op}}, \mathbf{Cat}) \simeq \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})/F$$

that respects the change of base morphisms φ^*; see [BS03] for a proof in terms of the equivalent 2-categories of fibrations. Under these equivalences, the stacks over $(\mathrm{el}\,F, J_F)$ correspond to the morphisms $X \longrightarrow F$ which are right orthogonal (in the bicategorical sense) to every covering sieve $R \longrightarrow C$ in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$. Hence it remains to show that a morphism $X \longrightarrow Y$ in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$ is right orthogonal to every covering sieve if and only if it is cartesian.

If $X \longrightarrow Y$ is cartesian, then it is right orthogonal to all local equivalences in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$, and so is in particular right orthogonal to every covering sieve. Conversely, suppose $g\colon X \longrightarrow Y$ is right orthogonal to every covering sieve, that is, for every covering sieve $R \longrightarrow C$, the square

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})(C, X) \arrow[r, "{(1,g)}"] \arrow[d] \arrow[dr, phantom, "="] & \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})(C, Y) \arrow[d] \\
\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})(R, X) \arrow[r, "{(1,g)}"'] & \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})(R, Y)
\end{tikzcd}$$

is a bipullback in $\mathbf{Cat}$. By the bicategorical Yoneda lemma, and by the fact that filtered colimits in $\mathbf{Cat}$ commute with bipullbacks, this means that the square

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "="] & Y \arrow[d, "\eta_Y"] \\
X^+ \arrow[r, "g^+"'] & Y^+
\end{tikzcd}$$

is a bipullback in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$, where X^+ denotes the bicategorical plus construction [Str82a]. But since the bicategorical plus construction preserves bipullbacks, and by the bicategorical analogue of Lemma 3.4.10, we have that both of the squares

$$\begin{tikzcd}
X^+ \arrow[r, "g^+"] \arrow[d, "\eta_{X^+}"'] \arrow[dr, phantom, "="] & Y^+ \arrow[d, "\eta_{Y^+}"] \\
X^{++} \arrow[r, "g^{++}"'] & Y^{++}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X^{++} \arrow[r, "g^{++}"] \arrow[d, "\eta_{X^{++}}"'] \arrow[dr, phantom, "="] & Y^{++} \arrow[d, "\eta_{Y^{++}}"] \\
X^{+++} \arrow[r, "g^{+++}"'] & Y^{+++}
\end{tikzcd}$$

are bipullbacks. Then, since the associated stack 2-functor L is given by three applications of the plus construction, we have that the square

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "="] & Y \arrow[d, "\eta_Y"] \\
LX \arrow[r, "Lg"'] & LY
\end{tikzcd}$$

is a bipullback. Hence g is cartesian. $\square$

Proposition 5.5. *The indexed 2-category $\mathbb{S}$ of stacks is a 2-stack.*

Proof. By Lemmas 5.2 and 5.4, it suffices to show that for every covering sieve $R \in J(C)$, the 2-functor $\mathrm{Cart}/C \longrightarrow \mathrm{Cart}/R$ is a biequivalence. Indeed, we will show that for any local isomorphism $f\colon F \longrightarrow G$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, the 2-functor $f^*\colon \mathrm{Cart}/G \longrightarrow \mathrm{Cart}/F$ is a biequivalence.

Let $g\colon X \longrightarrow F$ be a cartesian morphism in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$. Take the (local equivalence, cartesian)-factorisation $fg = me$, and take the pullback as in the diagram.

$$\begin{tikzcd}
X \arrow[dr, "h"] \arrow[drr, bend left, "e"] \arrow[ddr, bend right, "g"'] & & \\
& Z \arrow[r, "q"] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "m"] \\
& F \arrow[r, "f"'] & G
\end{tikzcd}$$

Then q is a local equivalence and p is cartesian, since they are bipullbacks of a local isomorphism and a cartesian morphism respectively. But e is a local equivalence and g is cartesian, so by cancellation, and since the class of local equivalences satisfies the 2-out-of-3 property, h is both a local equivalence and cartesian, and is therefore an equivalence. Hence we have found an object m of $\mathrm{Cart}\,/G$ and an equivalence $g \simeq p = f^*m$ in $\mathrm{Cart}\,/F$. Hence the 2-functor f^* is biessentially surjective.

Now let $g\colon X \longrightarrow G$ and $h\colon Y \longrightarrow G$ be cartesian morphisms, and let

$$\begin{tikzcd}
f^*X \arrow[rr, "r"] \arrow[dr, "f^*g"'] & \arrow[d, phantom, "="] & f^*Y \arrow[dl, "f^*h"] \\
& F &
\end{tikzcd}$$

be a morphism in $\mathrm{Cart}\,/F$. Then the pullback $p\colon f^*X \longrightarrow X$ of f along g is a local equivalence, since it is the bipullback of a local isomorphism, and h is cartesian, so there exists a diagonal filler $s\colon X \longrightarrow Y$ as in the following diagram.

$$\begin{tikzcd}
f^*X \arrow[r, "r"] \arrow[d, "p"'] & f^*Y \arrow[r, "q"] \arrow[d, phantom, "\cong"] & Y \arrow[d, "h"] \\
X \arrow[urr, "s"] \arrow[rr, "g"'] & \arrow[ur, phantom, "="] & G
\end{tikzcd}$$

Hence we have the diagram as on the left-hand of (5.5).

$$\begin{tikzcd}
f^*X \arrow[r, "p"] \arrow[d, "r"] \arrow[dd, bend right=60, "f^*g"'] \arrow[dr, phantom, "\cong"] & X \arrow[d, "s"] \\
f^*Y \arrow[r, "q"'] \arrow[d, "f^*h"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "h"] \\
F \arrow[r, "f"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*X \arrow[r, "p"] \arrow[d, "f^*s"] \arrow[dd, bend right=60, "f^*g"'] \arrow[dr, phantom, "="] & X \arrow[d, "s"] \\
f^*Y \arrow[r, "q"'] \arrow[d, "f^*h"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "h"] \\
F \arrow[r, "f"'] & G
\end{tikzcd}
\tag{5.5}$$

But f^*s was defined to be the unique morphism making the diagram on the right-hand side of (5.5) commute. So since the bottom square is also a bipullback, there exists an isomorphism $r \cong f^*s$ in $\mathrm{Cart}\,/F$. Hence the 2-functor f^* is essentially surjective on homs.

Now, let

$$\begin{tikzcd} X \arrow[rr, "r"] \arrow[dr, "g"'] & \arrow[d, phantom, "="] & Y \arrow[dl, "h"] \\ & G & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[rr, "s"] \arrow[dr, "g"'] & \arrow[d, phantom, "="] & Y \arrow[dl, "h"] \\ & G & \end{tikzcd}$$

be a parallel pair of morphisms in Cart $/G$, and let

$$\begin{tikzcd} f^*X \arrow[rr, bend left, "f^*r"] \arrow[rr, bend right, "f^*s"'] \arrow[rr, phantom, "\Downarrow\sigma"] \arrow[dr, "f^*g"'] & \arrow[d, phantom, "="] & f^*Y \arrow[dl, "f^*h"] \\ & F & \end{tikzcd}$$

be a 2-cell in Cart $/F$. Then $p\colon F^*X \longrightarrow X$ is a local equivalence and $h\colon Y \longrightarrow B$ is cartesian, and so by the bicategorical orthogonality $p \perp h$, there exists a unique 2-cell $\rho\colon r \longrightarrow s$ in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$ such that $q\sigma = \rho p$ and $g = h\rho$. But the 2-cell $f^*\rho$ was defined to be the unique 2-cell $f^*r \longrightarrow f^*s$ such that $q(f^*\rho) = \rho p$ and $(f^*h)(f^*\rho) = f^*g$. Hence we have that there exists a unique 2-cell $\sigma\colon r \longrightarrow s$ in Cart $/G$ such that $f^*\rho = \sigma$. Therefore the 2-functor f^* is fully faithful on homs, and hence a biequivalence. $\square$

Therefore we have proved that the indexed 2-functor (5.1) satisfies the first hypothesis for Theorem 4.4.2. We prove the second hypothesis in a few steps as follows.

Lemma 5.6. *The indexed* 2*-functor* $\mathrm{Tors}\colon \mathbb{G} \longrightarrow \mathbb{S}$ *is pointwise fully faithful on homs.*

Proof. It suffices to show that for every $C \in \mathcal{C}$ and for all sheaves of groups G and H over $(\mathcal{C}/C, J_C)$, the functor

$$\mathrm{Hom}((\mathcal{C}/C)^{\mathrm{op}}, \mathbf{Cat})(G, H) \longrightarrow \mathrm{Hom}((\mathcal{C}/C)^{\mathrm{op}}, \mathbf{Cat})(\mathrm{Tors}\, G, \mathrm{Tors}\, H) \tag{5.6}$$

is fully faithful. But the morphism $\eta_H\colon H \longrightarrow \mathrm{Tors}\, H$ that picks out the trivial H-torsor is pointwise (and hence representably) fully faithful, so by the equation

$$\begin{tikzcd} \mathrm{Hom}((\mathcal{C}/C)^{\mathrm{op}}, \mathbf{Cat})(G, H) \arrow[r, "1"] \arrow[d] \arrow[dr, phantom, "="] & \mathrm{Hom}((\mathcal{C}/C)^{\mathrm{op}}, \mathbf{Cat})(G, H) \arrow[d, "{(1,\eta_H)}"] \\ \mathrm{Hom}((\mathcal{C}/C)^{\mathrm{op}}, \mathbf{Cat})(\mathrm{Tors}\, G, \mathrm{Tors}\, H) \arrow[r, "\simeq", "{(\eta_G,1)}"'] & \mathrm{Hom}((\mathcal{C}/C)^{\mathrm{op}}, \mathbf{Cat})(G, \mathrm{Tors}\, H) \end{tikzcd}$$

in which the bottom functor is an equivalence since $\eta_G\colon G \longrightarrow \mathrm{Tors}\, G$ is a local equivalence and $\mathrm{Tors}\, H$ is a stack, we have that the functor (5.6) is fully faithful. $\square$

It remains to show that the indexed 2-functor (5.1) is locally essentially surjective on homs. To prove this, we use an alternative special characterisation of equivalence on homs 2-functors to Lemma 4.3.7.

Let $f\colon A \longrightarrow C \longleftarrow B\colon g$ be a cospan in a **Gray**-category. We call its **Gray**-limit weighted by the cospan

$$1 \xrightarrow{0} \mathbf{2} \xleftarrow{1} 1$$

the *pseudo-comma object* f/g of f over g. For a cospan in **Gray**, the explicit definition of the pseudo-comma 2-category F/G is identical to the definition of the equi-comma 2-category of F over G in Definition 4.1.1, except that for an object (a, s, b), the morphism $s\colon Fa \longrightarrow Gb$ is not required to be an equivalence.

Lemma 5.7. *Let $F\colon A \longrightarrow B$ be a 2-functor that is fully faithful on homs. Then the canonical 2-functor $[\mathbf{2}, A] \longrightarrow F/F$ is an equivalence on homs, and is a biequivalence if and only if F is an equivalence on homs.*

Proof. The proof is straightforward and much the same as the proof of Lemma 4.3.7. □

As in the discussion around Lemma 4.3.7, Lemma 5.7 provides a finite trilimit characterisation of when a pointwise fully faithful on homs morphism in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$ is pointwise an equivalence on homs. Note in particular that the pseudo-comma 2-category F/F can be calculated as a pullback

$$\begin{tikzcd} F/F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathbf{2}, B]} \arrow[d] \\ {[2, A]} \arrow[r, "{[1,F]}"'] & {[2, B]} \end{tikzcd}$$

which is also a tripullback, since $2 \longrightarrow \mathbf{2}$ is a cofibration.

Therefore, since the associated 2-stack trihomomorphism $L\colon \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat}) \longrightarrow \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})$ preserves finite trilimits, to show that a fully faithful on homs morphism $f\colon X \longrightarrow Y$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ is such that $Lf\colon LX \longrightarrow LY$ is an equivalence on homs, it suffices to show that the canonical morphism $\{\mathbf{2}, X\} \longrightarrow f/f$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ is a local biequivalence. (Here $\{\mathbf{2}, X\}$ denotes the **Gray**-cotensor product.) But since this canonical morphism is an equivalence on homs, it suffices to show that it is locally biessentially surjective.

Lemma 5.8. *A morphism $f\colon X \longrightarrow Y$ in $\mathrm{Hom}(\mathcal{C}^{op}, \mathbf{Bicat})$ is locally biessentially surjective if and only if for every $y \in YC$, there exists a locally surjective morphism $r\colon R \longrightarrow C$ in $[\mathcal{C}^{op}, \mathbf{Set}]$, a morphism $x\colon R \longrightarrow X$, and an equivalence*

$$\begin{tikzcd} R \arrow[r, "x"] \arrow[d, "r"'] \arrow[dr, phantom, "\simeq"] & X \arrow[d, "f"] \\ C \arrow[r, "y"'] & Y \end{tikzcd}.$$

Proof. Let $X \longrightarrow Z \longrightarrow Y$ be a (locally biessentially surjective, cartesian equivalence on homs)-factorisation of f, which exists by Theorems 4.2.10 and 4.3.5. Then f is locally biessentially surjective if and only if the right factor $Z \longrightarrow Y$ is a biequivalence. But this right factor is an equivalence on homs, and so is a biequivalence if and only if it is biessentially surjective.

By Theorem 4.2.12, for each object C of $\mathcal{C}$, the replete full image of $ZC \longrightarrow YC$ is the full sub-bicategory of YC on those objects y for which there exists a square of the form

$$\begin{tikzcd} R \arrow[r, "x"] \arrow[d, "r"'] \arrow[dr, phantom, "\simeq"] & X \arrow[d, "f"] \\ C \arrow[r, "y"'] & Y \end{tikzcd}$$

in which r is locally biessentially surjective. But by Lemma 4.3.8 we can replace any such square by one for which r is a locally surjective morphism in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. Hence $Z \longrightarrow Y$ is biessentially surjective if and only if there exists a square as in the statement of the lemma for all $y \in YC$. $\square$

Proposition 5.9. *The indexed* 2*-functor* $\mathrm{Tors}\colon \mathbb{G} \longrightarrow \mathbb{S}$ *is sent by the associated* 2*-stack trihomomorphism to a pointwise equivalence on homs morphism.*

Proof. By the above discussion, it suffices to show that the canonical morphism $\{\mathbf{2}, \mathbb{G}\} \longrightarrow \mathrm{Tors} / \mathrm{Tors}$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ is locally biessentially surjective. Let C be an object of $\mathcal{C}$. An object of $(\mathrm{Tors} / \mathrm{Tors})C$ consists of a pair of sheaves of groups G and H over $(\mathcal{C}/C, J_C)$ together with a morphism $\varphi\colon \mathrm{Tors}\, G \longrightarrow \mathrm{Tors}\, H$ in $\mathrm{Hom}((\mathcal{C}/C)^{\mathrm{op}}, \mathbf{Cat})$. The composite morphism $G \longrightarrow \mathrm{Tors}\, G \longrightarrow \mathrm{Tors}\, H$ amounts to picking out an H-torsor X and a morphism of sheaves of groups $f\colon G \longrightarrow \mathrm{Aut}_H(X)$, where $\mathrm{Aut}_H(X)$ denotes the sheaf of H-object automorphisms of X [Gir71, III.1.2.4].

Any H-torsor X is locally isomorphic to the trivial H-torsor H_d, that is, there exists a locally surjective morphism $r\colon R \longrightarrow C$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and an isomorphism $r^*X \cong r^*H_d$ of r^*H-torsors in $\mathrm{Hom}((\mathrm{el}\, R)^{\mathrm{op}}, \mathbf{Cat})$ [Gir71, Lemme III.1.4.3]. But the sheaf of automorphisms $\mathrm{Aut}_H(H_d)$ of the trivial H-torsor is isomorphic to H [Gir71, Proposition III.1.2.7], so the morphism r^*f is isomorphic to a morphism of sheaves of groups $g\colon r^*G \longrightarrow r^*H$ over $(\mathrm{el}\, R, J_R)$. This corresponds to a morphism in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}](R, \mathbb{G})$ and hence a morphism $R \longrightarrow \{\mathbf{2}, \mathbb{G}\}$.

All told, this assembles to an equivalence

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[d, "r"'] \arrow[dr, phantom, "\simeq"] & \{\mathbf{2}, \mathbb{G}\} \arrow[d] \\ C \arrow[r, "(G,\varphi,H)"'] & \mathrm{Tors} / \mathrm{Tors} \end{tikzcd}$$

and hence by Lemma 5.8, the canonical morphism $\{\mathbf{2}, \mathbb{G}\} \longrightarrow \mathrm{Tors}\,/\,\mathrm{Tors}$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Gray}]$ is locally biessentially surjective. $\square$

Therefore the indexed 2-functor (5.1) satisfies the hypotheses of Theorem 4.4.2. We therefore have that the cohomology bicategory $\mathscr{H}(X; L\mathbb{G})$ is biequivalent to the global section of $\mathbb{S}$, i.e. the stacks over X, that are in the "local full image" of $\mathrm{Tors}\colon \mathbb{G} \longrightarrow \mathbb{S}$. We now show that these are precisely the gerbes over X.

Recall that a stack S over a site $(\mathcal{C}, J)$ is a *gerbe* if for all C in $\mathcal{C}$ the category SC is a groupoid, and if the indexed functors $S \longrightarrow 1$ and $S^{\mathbf{2}} \longrightarrow S \times S$ are locally essentially surjective. (Here $S^{\mathbf{2}}$ denotes the cotensor product in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$.) This indeed agrees with Giraud's definition [Gir71, Définition III.2.1.1], since $S^{\mathbf{2}} \longrightarrow S \times S$ is locally essentially surjective if and only if for all $x, y \in SC$, its bipullback along $(x, y)\colon C \longrightarrow S \times S$ is locally essentially surjective. Since this bipullback is equivalent to

$$\begin{tikzcd} \mathrm{Hom}_S(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & S^{\mathbf{2}} \arrow[d] \\ C \arrow[r, "(x,y)"'] & S \times S \end{tikzcd}$$

this definition is equivalent to Giraud's definition.

Lemma 5.10. *Let $f\colon F \longrightarrow G$ be a locally surjective morphism in $[\mathcal{C}^{op}, \mathbf{Set}]$. Then the 2-functor*

$$f^*\colon \mathrm{Hom}((\mathrm{el}\, G)^{op}, \mathbf{Cat}) \longrightarrow \mathrm{Hom}((\mathrm{el}\, F)^{op}, \mathbf{Cat}) \tag{5.7}$$

preserves bilimits, and reflects locally essentially surjective morphisms. Moreover if S is a stack over $(\mathrm{el}\, G, J_F)$ such that f^*G is (pointwise) groupoidal, then S is groupoidal. Hence if S is a stack over $(\mathrm{el}\, G, J_F)$ such that f^*S is a gerbe over $(\mathrm{el}\, F, J_F)$, then S is a gerbe.

Proof. For convenience, let $\mathcal{K} = \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$. As in the proof of Lemma 5.4, the 2-functor $f^*\colon \mathcal{K}/G \longrightarrow \mathcal{K}/F$ shares whatever bicategorical properties the 2-functor (5.7) possesses. Firstly, f^* has a left 2-adjoint $f_*\colon \mathcal{K}/F \longrightarrow \mathcal{K}/G$ given by post-composition with f, and therefore preserves bilimits.

Next, note the locally essentially surjective morphisms in $\mathrm{Hom}((\mathrm{el}\, G)^{\mathrm{op}}, \mathbf{Cat})$ correspond under the equivalence $\mathrm{Hom}((\mathrm{el}\, G)^{\mathrm{op}}, \mathbf{Cat}) \simeq \mathcal{K}/G$ to the morphisms

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & \arrow[d, phantom, "="] & Y \arrow[dl, "q"] \\ & G & \end{tikzcd} \tag{5.8}$$

in $\mathcal{K}/G$ such that g is locally essentially surjective in $\mathcal{K}$.

Let g be a morphism in $\mathcal{K}/G$ as in (5.8), and suppose f^*g is a locally essentially surjective morphism in $\mathcal{K}/F$. Recall that f^*g is defined to be the unique morphism making the following diagram commute.

$$\begin{tikzcd} P \arrow[r, "f^*g"] \arrow[d, "r"] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[r, "q'"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \arrow[r, "q"'] & G \end{tikzcd}$$

Then r and s are locally essentially surjective in $\mathcal{K}$, since they are both bipullbacks of f, and f^*g is locally essentially surjective by assumption. Hence by composition and cancellation, g is locally essentially surjective. Hence f^* reflects locally essentially surjective morphisms.

Finally, let $f = me$ be the (epi,mono)-factorisation of f in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, and note that since f is locally surjective, m is a dense mono. Since $e\colon F \longrightarrow H$ is pointwise surjective, we have that the corresponding functor $e_*\colon \operatorname{el} F \longrightarrow \operatorname{el} H$ is surjective on objects. But for any surjective on objects functor $T\colon \mathcal{A} \longrightarrow \mathcal{B}$, the 2-functor $T^*\colon \operatorname{Hom}(\mathcal{B}, \mathbf{Cat}) \longrightarrow \operatorname{Hom}(\mathcal{A}, \mathbf{Cat})$ reflects pointwise groupoidal objects.

Since $f^* \cong m^*e^*$, it remains to show that if S is a stack over $(\operatorname{el} G, J_G)$ such that m^*S is groupoidal, then S is groupoidal. Let S correspond to the cartesian morphism $s\colon X \longrightarrow G$, and let $(x, C) \in \operatorname{el} G$. It suffices to show that the category $(\mathcal{K}/G)(x, s)$ is a groupoid. Then since m is a dense mono, the pullback $r\colon R \longrightarrow C$ of m along x is a covering sieve of C in $\mathcal{C}$.

$$\begin{tikzcd} R \arrow[r, "u"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[d, "m"] \\ C \arrow[r, "x"'] & G \end{tikzcd}$$

Since s is cartesian, there is an equivalence

$$(\mathcal{K}/G)(x, s) \simeq (\mathcal{K}/G)(xr, s) = (\mathcal{K}/G)(mu, s).$$

But by the 2-adjunction $m_* \dashv m^*$, there is an isomorphism

$$(\mathcal{K}/G)(mu, s) \cong (\mathcal{K}/H)(u, m^*s).$$

This last category is a groupoid, since by assumption, m^*s corresponds to a pointwise (and hence representably) groupoidal object of $\operatorname{Hom}((\operatorname{el} H)^{\mathrm{op}}, \mathbf{Cat})$. Hence S is groupoidal. □

Theorem 5.11. *Let $X = (\mathcal{C}, J)$ be a site. Then the cohomology bicategory $\mathscr{H}(X; L\mathbb{G})$ with coefficients in the associated 2-stack of the indexed 2-category $\mathbb{G}$ is biequivalent to the 2-category of gerbes over X, that is, the full sub-2-category of $\operatorname{Hom}(\mathcal{C}^{op}, \mathbf{Cat})$ on the gerbes.*

Proof. By Propositions 5.5 and 5.9, the indexed 2-functor $\operatorname{Tors}\colon \mathbb{G} \longrightarrow \mathbb{S}$ satisfies the hypotheses of Theorem 4.4.2. Moreover, by Lemma 5.2, the bicategory $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(1, \mathbb{S})$ is biequivalent to the 2-category of stacks over $(\mathcal{C}, J)$. Hence it remains to show that a stack S is a gerbe if and only if it is locally equivalent to the stack of torsors for a "locally given group", that is, there exists a locally surjective morphism $r\colon R \longrightarrow 1$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, a morphism $A\colon R \longrightarrow \mathbb{G}$, and an equivalence as displayed.

$$\begin{tikzcd} R \arrow[r, "A"] \arrow[d, "r"'] \arrow[dr, phantom, "\simeq"] & \mathbb{G} \arrow[d, "\mathrm{Tors}"] \\ 1 \arrow[r, "S"'] & \mathbb{S} \end{tikzcd} \tag{5.9}$$

First, suppose $S \in \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$ is a gerbe. Then $S \longrightarrow 1$ locally admits a section, that is, there exists a locally surjective morphism $r\colon R \longrightarrow 1$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ such that r^*S admits a global section $s\colon 1 \longrightarrow r^*S$ in $\mathrm{Hom}((\mathrm{el}\, R)^{\mathrm{op}}, \mathbf{Cat})$. Hence r^*S is a trivialised gerbe, and so there exists a sheaf of groups A over $(\mathrm{el}\, R, J_R)$ (the sheaf of automorphisms of s) and an equivalence $r^*S \simeq \mathrm{Tors}\, A$ in $\mathrm{Hom}((\mathrm{el}\, R)^{\mathrm{op}}, \mathbf{Cat})$ [Gir71, Corollaire III.2.2.6]. But this gives a morphism $A\colon R \longrightarrow \mathbb{G}$ and an equivalence as in (5.9) as desired.

For the converse, suppose there exists a square as in (5.9). For any sheaf of groups over a site, its associated stack of torsors is a gerbe over that site [Gir71, Théorème III.1.4.5]. Hence it suffices to show that a stack that is locally equivalent to a gerbe is a gerbe. So let S be a stack over $(\mathcal{C}, J)$, and let $r\colon R \longrightarrow 1$ be a locally surjective morphism in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, and suppose r^*S is (equivalent to) a gerbe in $\mathrm{Hom}((\mathrm{el}\, R)^{\mathrm{op}}, \mathbf{Cat})$. Then by Lemma 5.10, S is a gerbe over $(\mathcal{C}, J)$. □

Similarly, we have that for every object C of $\mathcal{C}$, $(L\mathbb{G})C$ is biequivalent to the full sub-2-category of $\mathrm{Hom}((\mathcal{C}/C)^{\mathrm{op}}, \mathbf{Cat})$ on the gerbes over $(\mathcal{C}/C, J_C)$. Thus we say that the associated 2-stack of the indexed 2-category of groups is the 2-stack of gerbes.

Giraud's non-abelian cohomology sets $H^2(X; M)$ are defined to be certain sets of equivalence classes of gerbes. The coefficient objects M are *liens*, which are defined to be the global sections of a certain stack $\mathbb{L}$. As can be seen either from the definition [Gir71, Définition IV.1.1.6], or by [Gir71, Corollaire IV.1.1.8], this stack is the associated stack of the indexed category

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathbb{G}} 2\text{-}\mathbf{Cat} \xrightarrow{(\pi_0)_*} \mathbf{Cat}$$

where $(\pi_0)_*\colon 2\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}$ is the left adjoint to the inclusion, and sends a 2-category A to the category $(\pi_0)_*A$ with the same objects as A and whose hom-sets are the sets of connected components of the hom-categories of A, i.e., $(\pi_0)_*A(a, b) = \pi_0 A(a, b)$.

Note that the functor $(\pi_0)_*$ extends to a triadjunction

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, shift right, ""'] \arrow[r, phantom, "\perp"] & \mathbf{Bicat}, \arrow[l, shift right, "(\pi_0)_*"'] \end{tikzcd}$$

which extends to a pointwise defined triadjunction

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}) \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat}). \arrow[l, shift right, "(1,(\pi_0)_*)"'] \end{tikzcd}$$

We can also characterise the stack $\mathbb{L}$ of liens as the 1-stack truncation of the 2-stack $L\mathbb{G}$ of gerbes. The 1-*stack truncation* of a 2-stack F over a site $(\mathcal{C}, J)$ is a stack $\tau_1 F$ over $(\mathcal{C}, J)$ such that for every stack G, there exists a trinatural equivalences

$$\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})(\tau_1 F, G) \simeq \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(F, G).$$

Hence τ_1 is left triadjoint to the inclusion

$$\begin{tikzcd} \mathrm{Stacks}(\mathcal{C}, J) \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \text{2-}\mathrm{Stacks}(\mathcal{C}, J). \arrow[l, shift right, "\tau_1"'] \end{tikzcd} \tag{5.10}$$

In general, we can calculate $\tau_1 F$ as the associated stack of the indexed category $(\pi_0)_* \circ F$. If $F = LE$ is the associated 2-stack of an indexed 2-category E, then for any stack G we have the equivalences

$$\begin{aligned} \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})(\tau_1(LE), G) &\simeq \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(LE, G) \\ &\simeq \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Bicat})(E, G) \\ &\simeq \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})((\pi_0)_* \circ E, G) \\ &\simeq \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})(L((\pi_0)_* \circ E), G). \end{aligned}$$

Hence there is an equivalence of stacks $\tau_1(LE) \simeq L((\pi_0)_* \circ E)$. Taking $E = \mathbb{G}$, this shows that the 1-stack truncation of the 2-stack of gerbes is equivalent to the stack $\mathbb{L}$ of liens.

Hence there is a canonical morphism of stacks $L\mathbb{G} \longrightarrow \mathbb{L}$, which is the unit of the triadjunction (5.10). This induces a (2-)functor

$$\mathscr{H}(X; L\mathbb{G}) \longrightarrow \mathscr{H}(X; \mathbb{L}) \tag{5.11}$$

which is equivalent to the (2-)functor

$$\mathbf{lien}\colon \mathrm{Gerbes}(X) \longrightarrow \mathrm{Liens}(X)$$

from the 2-category of gerbes to the category of liens over the site, as described in [Gir71, IV.2.2], recalling that the category $\mathscr{H}(X; \mathbb{L})$ is defined to be the category of global sections of $\mathbb{L}$, that is, the category of liens. For if G is a gerbe over X, then there exists a locally surjective morphism $r\colon R \longrightarrow 1$, a sheaf of groups A over $(\mathrm{el}\, R, J_R)$, and an equivalence $r^* G \simeq A$. Hence there exists an isomorphism $r^* \mathbf{lien}(G) \cong \mathbf{lien}(\mathrm{Tors}\, A)$ of liens over

$(\operatorname{el} R, J_R)$ [Gir71, Corollaire IV.2.2.4]. By the isomorphism

$$\begin{tikzcd} \mathbb{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & (\pi_0)_* \circ \mathbb{G} \arrow[d] \\ L\mathbb{G} \arrow[r] & \mathbb{L} \end{tikzcd}$$

we have that $\mathbf{lien}(\operatorname{Tors} A)$ is isomorphic to the lien associated to A [Gir71, Définition IV.1.1.6]. Hence the value of the functor (5.11) at G and the lien $\mathbf{lien}(G)$ associated to G [Gir71, IV.2.2.2.2] define isomorphic objects of the descent category of the truncated cosimplicial category

$$\operatorname{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})(R, \mathbb{L}) \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \operatorname{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})(R \times R, \mathbb{L}) \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \operatorname{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})(R \times R \times R, \mathbb{L})$$

which is equivalent to the category $\operatorname{Liens}(X)$, since $R \longrightarrow 1$ is locally surjective and $\mathbb{L}$ is a stack.

Let M be a lien over a site $X = (\mathcal{C}, J)$. Giraud defines the set $H^2(X; M)$ to be the set of connected components of the fibre of the 2-functor $\mathbf{lien}\colon \operatorname{Gerbes}(X) \longrightarrow \operatorname{Liens}(X)$ over M, that is, the set of connected components of the pullback

$$\begin{tikzcd} \mathbf{lien}^{-1} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Gerbes}(X) \arrow[d, "\mathbf{lien}"] \\ 1 \arrow[r, "M"'] & \operatorname{Liens}(X) \end{tikzcd}$$

in 2-$\mathbf{Cat}$ [Gir71, Définition IV.3.1.1, Corollaire IV.2.2.7]. However, by [Gir71, Propositions IV.2.2.6, IV.2.3.18], the 2-functor $\mathbf{lien}$ is an equifibration, so the pullback 2-category is equivalent to the tripullback 2-category (which we call the "pseudo-fibre" over M), whose set of connected components is therefore in bijection with the set $H^2(X; M)$.

Hence we have proved the following result.

Theorem 5.12. *The set $H^2(X; M)$ is in bijection with the set of connected components of the pseudo-fibre of the canonical functor $\mathscr{H}(X; L\mathbb{G}) \longrightarrow \mathscr{H}(X; \mathbb{L})$ over the lien M.*

Chapter 6

Conclusion

To conclude, I would like to reflect on the relation of this thesis to earlier works on the higher categorical approach to non-abelian cohomology. This thesis is primarily a response to Ross Street's paper [Str04]. As recalled in Section 1.4, that paper gives a definition of non-abelian cohomology with (strict) ω-categories as the coefficient objects (and outlines a definition for weak ω-categories), and relates it in low dimensions (i.e. non-abelian cohomology of degree 1) to descent theory, or equivalently, the theory of stacks [Str03].

The principal struggle in my reading of that paper was to identify a context in which its various constructions and definitions could be united and seen as natural consequences of some central guiding principle. The suggested connection to the theory of stacks led me to Grothendieck's *Pursuing stacks* [Gro83], in which I found much discussion of the theory of stacks as "the natural foundations" of non-abelian cohomology and of (higher) stacks as "the natural coefficients" for non-abelian cohomology, and most importantly the definition of the non-abelian cohomology of a topos with coefficients in a higher stack F as the higher category of global sections of F, as recalled in Section 1.3.

This definition suggested that the constructions and definitions in Street's paper could be seen as methods for constructing such higher categories of global sections of higher stacks. With this in mind, a further reading of Section 7 of Street's paper suggested that the important construction was that of the associated (higher) stack of an indexed (higher) category over a topos or a site. Indeed, this was corroborated by the result of Bunge [Bun79], which I learned from [Str82a] and is implicit in [Str04, Section 7], that the associated stack of an internal category A in a topos is the stack of A-torsors.

In Street's account, the A-torsors are defined to be the locally representable discrete fibrations on A, and the stack of A-torsors is shown to be the image of a certain "yoneda morphism" $A \longrightarrow \mathcal{P}A$, where $\mathcal{P}A$ is the "presheaf object of A", for the (locally surjective on objects, cover cartesian fully faithful) factorisation system for indexed (or "parametrized") functors, defined in [Str04, Section 5]. Trying to understand how this factorisation system could construct associated stacks, I naturally sought an analogous construction within ordinary topos theory. This search culminated in finding Lawvere's construction for associated sheaves, in the form of [Joh77, Exercise 3.4], as recalled in Section 2.1 of this thesis.

By this point the essential ingredients for the central argument of Chapter 2 had been identified, and I then went on to develop this approach to tackle the open problem with which [Str04] ends, which we quoted as the epigraph to Chapter 1. This thesis presents my solution to this problem, which involves crucially both a conceptual understanding (and consequent reframing) of the question within its natural theoretical context, and the necessary development of the relevant mathematical theory to provide an answer.

Finally, I would like to relate Theorems 5.11 and 5.12 to other results in the literature. Duskin's approach [Dus13] to studying Giraud's non-abelian cohomology of degree 2 in terms of structures internal to a topos is naturally related to the results of this thesis. For a topos E (seen as a site with its canonical topology) and a lien L over E, Duskin shows that Giraud's $H^2(E; L)$ is in bijection with a certain set of equivalence classes of "bouquets" in E, which are the non-empty connected groupoids in E, and a certain set of equivalence classes of "2-cocycles" over hypercovers in E "with coefficients in a locally group". The latter is clearly related to the indexed 2-category $\mathbb{G}$, and we claim that the bijection between the set of equivalence classes of gerbes and the set of equivalence classes of 2-cocycles over hypercovers with coefficients in a locally given group is somehow a shadow of the fact that the associated 2-stack of $\mathbb{G}$ is the 2-stack of gerbes.

Similarly, Jardine [Jar10] shows that Giraud's $H^2(\mathcal{C}; L)$ for a site $\mathcal{C}$ and a lien L over $\mathcal{C}$ and other sets of equivalence classes of gerbes over $\mathcal{C}$ are in bijection with sets of equivalence classes of 2-cocycles, defined in a different fashion to Duskin, with values in certain subobjects of a certain presheaf of 2-groupoids, again related to $\mathbb{G}$.

Appendix A

Some tricategorical definitions

For reference, we record the definitions of tritransformations, trimodifications, and perturbations between **Gray**-functors $\mathcal{A} \longrightarrow \mathcal{B}$, supposing that $\mathcal{A}$ is a category. The simplified expression of the coherence cells and axioms in this special case reveal a formal similarity between the definition of tritransformation and the definition of pseudofunctor. One can exploit this formal similarity by copying the proof that any pseudofunctor is isomorphic to a normal pseudofunctor, to prove that any tritransformation is isomorphic to a normal tritransformation (i.e. one for which the coherence modification M is an identity). Hence we take all tritransformations in these definitions to be normal.

Definition A.1. Let $\mathcal{A}$ be a category, $\mathcal{B}$ a **Gray**-category, and $F, G\colon \mathcal{A} \longrightarrow \mathcal{B}$ **Gray**-functors. A (normal) *tritransformation* $\theta\colon F \longrightarrow G$ consists of the following data:

- for each object A of $\mathcal{A}$, a morphism $\theta_A\colon FA \longrightarrow GA$ in $\mathcal{B}$

- for each morphism $f\colon A \longrightarrow B$ in $\mathcal{A}$, an adjoint equivalence 2-cell in $\mathcal{B}$

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\theta_A"'] \arrow[dr, phantom, "\Downarrow\theta_f"] & FB \arrow[d, "\theta_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

  which we will also write as $\theta_f\colon \theta_B.Ff \longrightarrow Gf.\theta_A$

- for each composable pair of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$, an invertible 3-cell in $\mathcal{B}$

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\theta_A"'] \arrow[dr, phantom, "\Downarrow\theta_f"] & FB \arrow[r, "Fg"] \arrow[d, "\theta_B"] \arrow[dr, phantom, "\Downarrow\theta_g"] & FC \arrow[d, "\theta_C"] \\ GA \arrow[r, "Gf"'] & GB \arrow[r, "Gg"'] & GC \end{tikzcd} \quad \overset{\Pi_{f,g}}{\Rrightarrow} \quad \begin{tikzcd} FA \arrow[r, "F(gf)"] \arrow[d, "\theta_A"'] \arrow[dr, phantom, "\Downarrow\theta_{gf}"] & FC \arrow[d, "\theta_C"] \\ GA \arrow[r, "G(gf)"'] & GC \end{tikzcd}$$

which we will also write as

$$
\begin{tikzcd}
& Gg.\theta_B.Ff \arrow[dr, "1.\theta_f"] & \\
\theta_C.Fg.Ff \arrow[ur, "\theta_g.1"] \arrow[rr, "\theta_{gf}"'] & \arrow[u, phantom, "\Downarrow\Pi_{f,g}"] & Gg.Gf.\theta_A
\end{tikzcd}
$$

subject to the following axioms:

- for each object A of $\mathcal{A}$, $\theta_{1_A} = 1_{FA}$
- for each morphism $f\colon A \longrightarrow B$ in $\mathcal{A}$, $\Pi_{f,1_B} = 1_{\theta_f} = \Pi_{1_A,f}$
- for each $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in $\mathcal{A}$,

$$
\begin{tikzcd}
Gh.\theta_C.Fg.Ff \arrow[r, "1.\theta_g.1"] \arrow[dr, phantom, "\Downarrow\Pi_{g,h}.1"] & Gh.Gg.\theta_B.Ff \arrow[d, "1.\theta_f"] & & Gh.\theta_C.Fg.Ff \arrow[r, "1.\theta_g.1"] \arrow[dr, "1.\theta_{gf}"] & Gh.Gg.\theta_B.Ff \arrow[d, "1.1.\theta_f"] \arrow[dl, phantom, "\Downarrow 1.\Pi_{f,g}"] \\
\theta_D.Fh.Fg.Ff \arrow[u, "\theta_h.1.1"] \arrow[ur, "\theta_{hg}.1"] \arrow[r, "\theta_{hgf}"'] & Gh.Gg.Gf.\theta_A \arrow[ul, phantom, "\Downarrow\Pi_{f,hg}"] & = & \theta_D.Fh.Fg.Ff \arrow[u, "\theta_h.1"] \arrow[r, "\theta_{hgf}"'] \arrow[ur, phantom, "\Downarrow\Pi_{gf,h}"] & Gh.Gg.Gf.\theta_A
\end{tikzcd}
$$

Definition A.2. A *trimodification* $m\colon \theta \longrightarrow \varphi$ consists of the following data:

- for each object A of $\mathcal{A}$, a 2-cell in $\mathcal{B}$

$$
\begin{tikzcd}
FA \arrow[r, bend left, "\theta_A"] \arrow[r, bend right, "\varphi_A"'] \arrow[r, phantom, "\Downarrow m_A"] & GA
\end{tikzcd}
$$

which we will also write as $m_A\colon \theta_A \longrightarrow \varphi_A$

- for each morphism $f\colon A \longrightarrow B$ in $\mathcal{A}$, an invertible 3-cell in $\mathcal{B}$

$$
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, bend right=40, "\varphi_A"'] \arrow[d, bend left=40, "\theta_A"] \arrow[d, phantom, "\overset{m_A}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow\theta_f"] & FB \arrow[d, "\theta_B"] \\
GA \arrow[r, "Gf"'] & GB
\end{tikzcd}
\quad \overset{m_f}{\Rrightarrow} \quad
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "\varphi_f\Downarrow"] & FB \arrow[d, bend right=40, "\varphi_B"'] \arrow[d, bend left=40, "\theta_B"] \arrow[d, phantom, "\overset{m_B}{\Leftarrow}"] \\
GA \arrow[r, "Gf"'] & GB
\end{tikzcd}
$$

which we will also write as

$$
\begin{tikzcd}
\theta_B.Ff \arrow[r, "\theta_f"] \arrow[d, "m_B.1"'] \arrow[dr, phantom, "\Downarrow m_f"] & Gf.\theta_A \arrow[d, "1.m_A"] \\
\varphi_B.Ff \arrow[r, "\varphi_f"'] & Gf.\varphi_A
\end{tikzcd}
$$

subject to the following axioms:

- for each object A of $\mathcal{A}$, $m_{1_A} = 1_{m_A}$
- for each $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$,

$$
\begin{tikzcd}
 & Gg.\theta_B.Ff \arrow[dr, "1.\theta_f"] \arrow[dd, "1.m_B.1"] & \\
\theta_C.Fg.Ff \arrow[ur, "\theta_g.1"] \arrow[dd, "m_C.1.1"'] \arrow[r, phantom, "\Downarrow m_g.1"] & & Gg.Gf.\theta_A \arrow[dd, "1.1.m_A"] \arrow[l, phantom, "\Downarrow 1.m_f"'] \\
 & Gg.\varphi_B.Ff \arrow[dr, "1.\varphi_f"] \arrow[d, phantom, "\Downarrow \Pi_{f,g}"] & \\
\varphi_C.Fg.Ff \arrow[ur, "\varphi_g.1"] \arrow[rr, "\varphi_{gf}"'] & {} & Gg.Gf.\theta_A
\end{tikzcd}
=
\begin{tikzcd}
 & Gg.\theta_B.Ff \arrow[dr, "1.\theta_f"] \arrow[d, phantom, "\Downarrow \Pi_{f,g}"] & \\
\theta_C.Fg.Ff \arrow[ur, "\theta_g.1"] \arrow[rr, "\theta_{gf}"'] \arrow[d, "m_c.1"'] & {} & Gg.Gf.\theta_A \arrow[d, "1.m_A"] \\
\varphi_C.Fg.Ff \arrow[rr, "\varphi_{gf}"'] \arrow[urr, phantom, "\Downarrow m_{gf}"] & & Gg.Gf.\varphi_A
\end{tikzcd}
$$

Definition A.3. A *perturbation* $\sigma\colon m \longrightarrow n$ consists of the following data:

- for each object A of $\mathcal{A}$, a 3-cell in $\mathcal{B}$

$$
\begin{tikzcd}
FA \arrow[r, bend left=50, "\theta_A"] \arrow[r, bend right=50, "\varphi_A"'] \arrow[r, phantom, "m_A\Downarrow \xrightarrow{\sigma_A} \Downarrow n_A"] & GA
\end{tikzcd}
$$

which we will also write as

$$
\begin{tikzcd}
\theta_A \arrow[r, bend left=50, "m_A"] \arrow[r, bend right=50, "n_A"'] \arrow[r, phantom, "\Downarrow \sigma_A"] & \varphi_A
\end{tikzcd}
$$

subject to the following axiom:

- for each morphism $f\colon A \longrightarrow B$ in $\mathcal{A}$,

$$
\begin{tikzcd}
\theta_B.Ff \arrow[r, "\theta_f"] \arrow[d, bend right=30, "n_B.1"'] \arrow[d, bend left=30, "m_B.1"] \arrow[d, phantom, "\overset{\sigma_B.1}{\Leftarrow}"] & Gf.\theta_A \arrow[d, "1.m_A"] \arrow[dl, phantom, "\Downarrow m_f"] \\
\varphi_B.Ff \arrow[r, "\varphi_f"'] & Gf.\varphi_A
\end{tikzcd}
=
\begin{tikzcd}
\theta_B.Ff \arrow[r, "\theta_f"] \arrow[d, "n_B.1"'] \arrow[dr, phantom, "n_f\Downarrow"] & Gf.\theta_A \arrow[d, bend right=30, "1.n_A"'] \arrow[d, bend left=30, "1.m_A"] \arrow[d, phantom, "\overset{1.\sigma_A}{\Leftarrow}"] \\
\varphi_B.Ff \arrow[r, "\varphi_f"'] & Gf.\varphi_A
\end{tikzcd}
$$

Appendix B

A note on parity structures

In this appendix we present some simplifications and clarifications of the basic theory of parity complexes [Str91, Str94] and pasting schemes [Joh89]. These structures were introduced to enable the explicit description of the ω-categories freely generated by certain higher dimensional pasting diagrams. Important diagram shapes are the oriented simplexes, cubes, and globes, and their products; the free ω-categories on these diagrams have played a central role in the constructions of the simplicial nerve of an ω-category [Str87, Ver08], the Gray tensor product of ω-categories [Cra95], and the higher dimensional descent construction [Str04].

In Section 1, we introduce a system of axioms on the common underlying structure of parity complexes and pasting schemes, here called parity structures, which are all satisfied by the above examples. Then using a series of results presented in Section 2, we show in Sections 3 and 4 that these axioms imply all the axioms required for the constructions of [Str91] and [Joh89], with one minor exception. In addition, we take the opportunity to make some expository remarks on the definition of cell in [Str91], which we have found helpful for the understanding of that paper.

In the constructions of [Str91] and [Joh89], the cells of the free ω-category generated by a parity structure C (satisfying suitable axioms) are described as certain subsets of C. This method relies crucially on some property of *loop-freeness* of C, as is already evident in the 1-dimensional situation of a category freely generated by a directed graph. Both of these articles utilise an order they denote by $\lhd$, which we define in Section 3, to enforce loop-freeness, by introducing the axiom that $\lhd$ is antisymmetric. However, as described in Section 5 of [Str91], this order only compares elements of the same dimension, and so to prove that the product of two parity complexes is a parity complex, a stronger order $\blacktriangleleft$ is introduced which compares elements of all dimension. Remarkably, for many examples, including all those mentioned above, this order is in fact *linear*. We take this property as our axiom **(L)**. This property has also been used to characterise those globular sets underlying globular pasting diagrams, see [Str00].

Despite taking a somewhat different perspective to this note and to [Str91] and [Joh89], we have been informed by the results of [Ste93, Ste98]. In particular, [Ste98] gives two necessary conditions that a parity structure must satsify in order to generate a free ω-category. The second is a certain globularity condition, a slightly weaker version of which we take as our axiom **(G)**, using a formula for source and target which is also shown in [Ste98] to be necessary. Our axiom **(C)** can be seen as an adaptation of the first necessary condition.

It should be noted that we do not seek here to capture every instance of pasting diagrams; such a task belongs properly to the theory of computads [Bat98]. Here we are concerned only with a very restrictive class of loop-free diagrams; an example of a simple pasting diagram for which $\blacktriangleleft$ is not antisymmetric is given in [Str91]. Rather, the purpose of this theory is to give simple, accessible constructions of the ω-categories freely generated by those diagrams mentioned in the first paragraph.

Beyond the pragmatic reasons mentioned above, we hope that the perspective of this note will be vindicated by a simplification of the proof that a parity structure satisfying our axioms generates a free ω-category. In particular, the $\blacktriangleleft$ order was not used in the proofs of [Str91] and [Joh89], and so provides a promising prospect for a new proof.

B.1 Parity structures

In this section we introduce the notion of parity structure and the axioms **(L)**, **(G)**, and **(C)**.

A *graded set* is a set C equipped with a function $C \to \mathbb{N}$. As in [Str91], we denote the fibre above $n \in \mathbb{N}$ by C_n, and we let $C^{(n)}$ denote the graded set $\sum_{0\leq k\leq n} C_k$ (where throughout this note we use the summation notation to emphasise that the unions are disjoint). We say that C is n-dimensional if n is the smallest natural number such that $C = C^{(n)}$, or equivalently, the largest natural number such that C_n is nonempty. Note that any finite graded set is n-dimensional for some $n \in \mathbb{N}$.

For a set A, we denote its set of subsets by $\mathcal{P}A$. We use the convention that a function $f\colon A \longrightarrow \mathcal{P}B$ is extended to $\mathcal{P}A$ by $f(X) = \bigcup_{x\in X} f(x)$, for $X \subseteq A$.

Definition B.1.1. A *parity structure* is a finite graded set C equipped with functions

$$-,+\colon C_{n+1} \longrightarrow \mathcal{P}C_n \setminus \{\emptyset\}$$

for each $n \geq 0$.

Following [Str91], we write x^- and x^+ for the value of these functions at $x \in C$, and call their elements the *negative faces* and *positive faces* of x, respectively. For $X \subseteq C$, we write $X^\varepsilon = \bigcup_{x\in X} x^\varepsilon$ for $\varepsilon \in \{-,+\}$ as above, and $X^\mp = X^- \setminus X^+$ and $X^\pm = X^+ \setminus X^-$.

Swapping the roles of $-$ and $+$ in a parity structure C for all $n \in \mathbb{N}$ yields another parity structure C^{op} called the *dual* of C. Note that the three axioms we introduce in this section are all self-dual, meaning that they hold in C if and only if they hold in C^{op}. Hence we will typically only prove one dual of a statement, as necessary.

For the remainder of this section let C denote a parity structure.

B.1.1 Linearity

Our first axiom is our condition of loop-freeness. We define the relation $\blacktriangleleft$ to be the preorder on C generated by (i.e. the reflexive transitive closure of) the relations

$$\begin{cases} a \blacktriangleleft x & \text{if} \quad a \in x^- \\ y \blacktriangleleft b & \text{if} \quad b \in y^+ \end{cases}$$

for $x, y \in C$. For subsets $A, B \subseteq C$, we write $A \blacktriangleleft B$ to indicate that $a \blacktriangleleft b$ for all $a \in A$ and $b \in B$.

Axiom (L). *The relation $\blacktriangleleft$ on C is a linear order.*

B.1.2 Globularity

To give the remaining two axioms, we introduce part of the definition of the ω-category generated by a parity structure. Our approach to this definition is closer to that of [Joh89] than that of [Str91]; in Section 3.1 we will make some comments on comparison. In this section we express the source and target formulas of [Ste93, Ste98] in the style of [Joh89], for general subsets of C. In the next section we give the conditions on a subset of C for it to be a cell, which is essentially the definition of [Joh89]. The only remaining piece of the definition for an ω-category is composition; this is given by union of cells as subsets of C.

A subset $A \subseteq C$ is called a *subcomplex* of C when $x^- \cup x^+ \subseteq A$ for all $x \in A$. For $x \in C$, we define $R(x)$ to be the smallest subcomplex of C containing x, and we define the subsets

$$E(x) = R(x) \setminus R(x^-)$$

and

$$B(x) = R(x) \setminus R(x^+),$$

called the *end* and *beginning* of x. As above, we define these on arbitrary subsets of C by taking unions.

Definition B.1.2. For $A \subseteq C$ and $n \in \mathbb{N}$, we define the sets

$$s_n A = A^{(n+1)} \setminus E(A_{n+1})$$

and

$$t_n A = A^{(n+1)} \setminus B(A_{n+1}).$$

Our second axiom is a minimal condition of globularity for these source and target formulas. We prove in Theorem B.2.8 that in the presence of Axiom **(L)**, this implies all the globularity conditions for an arbitary subset of C.

Axiom (G). *For each $x \in C_n$, $n \geq 2$,*

$$s_{n-2}s_{n-1}R(x) = s_{n-2}t_{n-1}R(x)$$

and

$$t_{n-2}s_{n-1}R(x) = t_{n-2}t_{n-1}R(x).$$

Using the formulas for s_n and t_n, we may restate the axiom as follows.

Axiom (G). *For each $x \in C$,*

$$E(x) \cup E(x^-) = B(x) \cup E(x^+)$$

and

$$E(x) \cup B(x^-) = B(x) \cup B(x^+).$$

Proof. Let $x \in C_n$. If $n = 0, 1$, the equations are trivial, so let $n \geq 2$. Then

$$s_{n-1}R(x) = R(x) \setminus E(x) = R(x^-).$$

So

$$s_{n-2}s_{n-1}R(x) = R(x) \setminus (E(x) \cup E(x^-)).$$

Similarly, we get

$$s_{n-2}t_{n-1}R(x) = R(x) \setminus (B(x) \cup E(x^+)).$$

Hence the equation

$$s_{n-2}s_{n-1}R(x) = s_{n-2}t_{n-2}R(x)$$

is equivalent to the equation

$$E(x) \cup E(x^-) = B(x) \cup E(x^+).$$ □

For many of the proofs of Section 2, it is this form of the axiom that we most readily use.

B.1.3 Cellularity

In both [Str91] and [Joh89], the crucial notion characterising cells is that of well-formedness. We say that a subset $X \subseteq C$ is *well-formed* if

$$x, y \in X, x \neq y \implies x^- \cap y^- = \emptyset = x^+ \cap y^+$$

and if X_0 has at most one element.

Note that by the formulas for s_n and t_n we have that for any $A \subseteq C$,

$$(s_n A)_n = A_n \setminus A^+_{n+1} \qquad \text{and} \qquad (t_n A)_n = A_n \setminus A^-_{n+1}. \tag{*}$$

Hence

$$\sum_{n\in\mathbb{N}} s_n A = A \setminus A^+ \qquad \text{and} \qquad \sum_{n\in\mathbb{N}} t_n A = A \setminus A^-.$$

Definition B.1.3. A subset $A \subseteq C$ is called a *cell* if

- $s_n A$ and $t_n A$ are subcomplexes for all $n \in \mathbb{N}$, and
- $A \setminus A^+$ and $A \setminus A^-$ are well-formed.

Note that since C is finite, we have that A is an n-dimensional subset of C for some minimal $n \in \mathbb{N}$. Hence $s_n A = A = t_n A$ and A is a subcomplex. In this case we call A an n-cell.

Axiom (C). *For each $x \in C$, $R(x)$ is a cell.*

B.2 Basic results

In this section we prove a collection of results, organised by their dependency on the axioms **(L)**, **(G)**, and **(C)**. These will all go toward proving, in Sections 3 and 4, that these axioms imply those of parity complexes and pasting schemes.

Proposition B.2.1. *Let C be a parity structure. Then $B(x) \cap E(x) = \{x\}$ for all $x \in C$.*

Proof. $B(x) \cap E(x) = R(x) \setminus (R(x^-) \cup R(x^+)) = \{x\}$. □

B.2.1 Assuming (L)

For this section, let C denote a parity structure satisfying axiom **(L)**.

Proposition B.2.2. *For all $x \in C$, $x^- \cap x^+ = \emptyset$.*

Proof. By definition, $x^- \blacktriangleleft x \blacktriangleleft x^+$. Hence the result follows from the antisymmetry of $\blacktriangleleft$. □

B.2.2 Assuming (G)

For this section, let C denote a parity structure satisfying axiom **(G)**.

Proposition B.2.3. *For all $x \in C$,*

$$E(x^+) \cup B(x^-) = E(x^-) \cup B(x^+).$$

In particular,

$$x^{++} \cup x^{--} = x^{-+} \cup x^{+-}.$$

Proof. Axiom **(G)** implies $E(x^+) \subseteq E(x) \cup E(x^-)$. But

$$E(x^+) \cap E(x) \subseteq E(x) \setminus \{x\} = E(x) \setminus B(x) \subseteq B(x^+),$$

using axiom **(G)** for the inclusions and Proposition B.2.1 for the equality. Hence $E(x^+) \subseteq E(x^-) \cup B(x^+)$.

Axiom **(G)** implies $E(x^-) \subseteq B(x) \cup E(x^+)$. But as in the first case,

$$E(x^-) \cap B(x) \subseteq B(x) \setminus \{x\} = B(x) \setminus E(x) \subseteq B(x^-).$$

Hence $E(x^-) \subseteq E(x^+) \cup B(x^-)$.

The second statement is the restriction of the first equation to the top-dimensional elements. □

Proposition B.2.4. *For each $x \in C$,*

$$E(x) + E(x^-) = B(x) + E(x^+)$$

and

$$E(x) + B(x^-) = B(x) + B(x^+).$$

Proof. The equality of the unions is axiom **(G)**. Note that $E(X) \subseteq R(X)$ for any $X \subseteq C$. Then since we defined $E(x) = R(x) \setminus R(x^-)$, we have

$$E(x) \cap E(x^-) = \emptyset = E(x) \cap B(x^-).$$ □

B.2.3 Assuming (L) and (G)

For this section, let C denote a parity structure satisfying axioms **(L)** and **(G)**.

Proposition B.2.5. *Let $x, y \in C$. If $x \blacktriangleleft y$ and $x \neq y$, then $B(x) \cap E(y) = \emptyset$.*

Proof. Let $a \in C$. Axiom **(G)** implies $E(x) \setminus \{x\} \subseteq E(x^+)$. Hence each $a \in E(x) \setminus \{x\}$ has dimension less than x and is a positive face of a positive face of ... x, and so $x \blacktriangleleft a$, by antisymmetry of $\blacktriangleleft$. Dually, we have that $b \in B(x)$ implies $b \blacktriangleleft x$. Hence $b \blacktriangleleft x \blacktriangleleft a$, so $b \neq a$. □

Proposition B.2.6. *For all $x \in C$,*

$$E(x) \setminus \{x\} = E(x^+) \cap B(x^+)$$

and

$$B(x) \setminus \{x\} = E(x^-) \cap B(x^-).$$

Proof. Axiom **(G)** implies

$$E(x) \subseteq (B(x) \cup E(x^+)) \cap (B(x) \cup B(x^+)) = B(x) \cup (E(x^+) \cap B(x^+)).$$

Hence

$$E(x) \setminus \{x\} = E(x) \setminus B(x) \subseteq E(x^+) \cap B(x^+),$$

using Proposition B.2.1 for the equality.

Conversely, Axiom **(G)** implies

$$E(x^+) \cap B(x^+) \subseteq (E(x) \cup E(x^-)) \cap (E(x) \cup B(x^-)) = E(x) \cup (E(x^-) \cap B(x^-)).$$

But $x^- \blacktriangleleft x \blacktriangleleft x^+$ implies that $B(x^-) \cap E(x^+) = \emptyset$, by Proposition B.2.5. Hence

$$E(x^+) \cap B(x^+) \subseteq E(x).$$ □

Proposition B.2.7. *For all $x \in C$,*

$$E(x^+) + B(x^-) = E(x^-) + B(x^+).$$

In particular,

$$x^{++} + x^{--} = x^{-+} + x^{+-}.$$

Proof. The equality of the unions is Proposition B.2.3. It remains to prove that the unions are disjoint. Since $x^- \blacktriangleleft x \blacktriangleleft x^+$, we have $B(x^-) \cap E(x^+) = \emptyset$ by Proposition B.2.5. By Proposition B.2.4 we have

$$E(x^-) \cap B(x^+) = ((B(x) \cup E(x^+)) \setminus E(x)) \cap ((E(x) \cup B(x^-)) \setminus B(x)) \subseteq E(x^+) \cap B(x^-) = \emptyset.$$

The second statement is the restriction of the first equation to the top-dimensional elements. □

Note that this result shows that we may construct a chain complex $\hat{C}$ from C, as in [Str91], by taking $\hat{C}_n$ to be the free abelian group on the set C_n, and defining $d\colon \hat{C}_{n+1} \longrightarrow \hat{C}_n$ by $d(x) = x^+ - x^-$. The second assertion of Proposition B.2.7 is then exactly $dd = 0$.

Theorem B.2.8. *For any subset $A \subseteq C$ and any $n \geq 2$, we have*

$$s_{n-2}s_{n-1}A = s_{n-2}A = s_{n-2}t_{n-1}A$$

and

$$t_{n-2}s_{n-1}A = t_{n-2}A = t_{n-2}t_{n-1}A.$$

Proof. We prove the first line, the other follows dually. As for the equivalent form of axiom **(G)**, this is equivalent to

$$E(A_n) \cup E(A_{n-1} \setminus A_n^+) = A_n \cup E(A_{n-1}) = B(A_n) \cup E(A_{n-1} \setminus A_n^-).$$

We demonstrate these equations as (1) = (2) = (3).

(1)$\subseteq$(2). It suffices to show $E(A_n) \subseteq A_n \cup E(A_{n-1})$.

Let $x \in E(A_n)$. Then there exists $y \in A_n$ such that $x \in E(y)$. Then **(G)** implies $x \in B(y) \cup E(y+)$.

If $x \in B(y)$, then $x \in B(y) \cap E(y) = \{y\} \subseteq A_n$, using Proposition B.2.1.

Else, $x \in E(y^+) \subseteq E(A_{n-1})$.

(3)$\subseteq$(2). It suffices to show $B(A_n) \subseteq A_n \cup E(A_{n-1})$.

Let $x \in B(A_n)$. Then there exists $y \in A_n$ such that $x \in B(y)$. Then **(G)** implies $x \in E(y) \cup E(y^-)$.

If $x \in E(y)$, then $x \in B(y) \cap E(y) = \{y\} \subseteq A_n$.

Else, $x \in E(y^-) \subseteq E(A_{n-1})$.

(2)$\subseteq$(1). It suffices to show $E(A_{n-1}) \subseteq E(A_n) \cup E(A_{n-1} \setminus A_n^+)$.

Let $x \in E(A_{n-1})$. Let u be the minimum element of A_{n-1} such that $x \in E(u)$. If $u \notin A_n^+$, we are done.

Hence suppose $u \in A_n^+$. Then there exists $y \in A_n$ such that $u \in y^+$. So $x \in E(y^+)$. Then **(G)** implies $x \in E(y) \cup E(y^-)$.

If $x \in E(y)$, then $x \in E(y) \subseteq E(A_n)$.

Else $x \in E(y^-)$. So there exists $v \in y^-$ such that $x \in E(v)$. But $v \blacktriangleleft y \blacktriangleleft u$, contradicting minimality of u.

(2)$\subseteq$(3). It suffices to show $E(A_{n-1}) \subseteq B(A_n) \cup E(A_{n-1} \setminus A_n^-)$.

Let $x \in E(A_{n-1})$. Let u be the maximum element of A_{n-1} such that $x \in E(u)$. If $u \notin A_n^-$ we are done.

Hence suppose $u \in A_n^-$. Then there exists $y \in A_n$ such that $u \in y^-$. Hence $x \in E(y^-)$. Then **(G)** implies $x \in B(y) \cup E(y^+)$.

If $x \in B(y)$, then $x \in B(y) \subseteq B(A_n)$.

Else $x \in E(y^+)$. So there exists $v \in y^+$ such that $x \in E(y)$. But $u \blacktriangleleft y \blacktriangleleft v$, contradicting maximality of u. □

B.2.4 Assuming (L), (G), and (C)

For this section, let C denote a parity structure satisfying axioms **(L)**, **(G)**, and **(C)**.

For each $x \in C$ we define the sets

$$\mu(x) = R(x) \setminus R(x)^+ \qquad \text{and} \qquad \pi(x) = R(x) \setminus R(x)^-.$$

Then, by (*) in Section 1.3, we have

$$\mu(x)_n = (s_n R(x))_n \qquad \text{and} \qquad \pi(x) = (t_n R(x))_n.$$

Using these sets, we may give a direct formula for the sources and targets of cells of the form $R(x)$. The proof follows that of Proposition 4.11 of [Ste98].

Lemma B.2.9. *For all $x \in C$, $n \in \mathbb{N}$ we have*

$$s_n R(x) = R(\mu(x)_n) \qquad \textit{and} \qquad t_n R(x) = R(\pi(x)_n).$$

Proof. Since $R(x)$ is a cell, we have that $s_n R(x)$ is a subcomplex of C, Then since $\mu(x)_n = (s_n R(x))_n$, we have

$$R(\mu(x)_n) \subseteq s_n R(x).$$

Conversely, note that for any $A \subseteq C$ and $k \in \mathbb{N}$ we have

$$(s_k A)_k \cap (t_k A)_k = A_k \setminus (A_{k+1}^+ \cup A_{k+1}^-).$$

Let $k < n$. Then

$$(s_k R(x))_k \cap (t_k R(x))_k = \emptyset.$$

Let $a \in (s_n R(x))_k$. Then $a \notin s_k R(x) \cap t_k R(x)$. But

$$s_k R(x) \cap t_k R(x) = s_k s_n R(x) \cap t_k s_n R(x),$$

by Theorem B.2.8. Hence

$$a \in (s_n R(x))_{k+1}^+ \cup (s_n R(x))_{k+1}^-.$$

So there exists $b \in (s_n R(x))_{k+1}$ such that $a \in b^- \cup b^+$. By induction, we get that

$$s_n R(x) \subseteq R(s_n R(x))_n) = R(\mu(x)_n). \qquad \square$$

Proposition B.2.10. *For all $x \in C$, we have*

$$\mu(x)_n^{\mp} = \mu(x)_{n-1} = \pi(x)_n^{\mp}$$

and

$$\mu(x)_n^{\pm} = \pi(x)_{n-1} = \pi(x)_n^{\pm}.$$

Proof. By Theorem B.2.8 we have

$$s_{n-1} s_n R(x) = s_{n-1} R(x) = s_{n-1} t_n R(x).$$

Recall that $\mu(x)_{n-1} = (s_{n-1}R(x))_{n-1}$ and for any subset $A \subseteq C$ we have $(s_{n-1}A)_{n-1} = A_{n-1} \setminus A_n^+$. Hence it suffices to show that we have $A_{n-1} \setminus A_n^+ = A_n^{\mp}$ for $A = s_n R(x)$ and $A = t_n R(x)$.

But in both cases we have that A is a subcomplex of C, since $R(x)$ is a cell, so $A_n^- \subseteq A_{n-1}$. So it suffices to show $A_{n-1} \setminus A_n^+ \subseteq A_n^-$, that is $A_{n-1} \subseteq A_n^- \cup A_n^+$. But this follows from Lemma B.2.9. $\square$

Corollary B.2.11. *For all $x \in C$, we have $\mu(x) \blacktriangleleft x \blacktriangleleft \pi(x)$.*

Proof. For each $n \in \mathbb{N}$ we have $\mu(x)_{n-1} = \mu(x)_n^{\mp}$, so by induction we have that every element of $\mu(x)$ is a negative face of a negative face of ... x. $\square$

B.3 Parity complexes

In the following two sections, we directly compare the above development with the basic theory of [Str91] and [Joh89]. We have freely used the notation of these two articles, but in some cases we have given a different definition to the same symbol. We prove that in all cases the notions coincide under our axioms, but for clarity, we denote a symbol H, say, by $\widetilde{H}$ when given its original definition.

In this section we give the definition of parity complex and the axioms required for the constructions of [Str91]. We prove that all these axioms hold for a parity structure satisfying our axioms **(L)**, **(G)**, and **(C)**, with one minor exception that we address after proving the theorem.

First we introduce the original order used to enforce a property of loop-freeness on a parity structure, as described in the introduction of this note. For a subset S of a parity structure C, define the relation $\lhd_S$ to be the preorder generated on S by the relation $x < y$ if $x^+ \cap y^- \neq \emptyset$ for $x, y \in S$. We simply write $\lhd$ for $\lhd_C$.

Definition B.3.1. A *parity complex* is a parity structure C such that for all $x, y \in C$ we have

- **Axiom 0.** $x^- \cap x^+ = \emptyset$
- **Axiom 1.** $x^{--} \cup x^{++} = x^{-+} \cup x^{+-}$
- **Axiom 2.** x^- and x^+ are both well-formed
- **Axiom 3.**
  (a) $x \lhd y \lhd x$ implies $x = y$
  (b) $x \lhd y$, $x \in z^\varepsilon$, $y \in z^\eta$ imply $\varepsilon = \eta$, for $\varepsilon, \eta \in \{-, +\}$.

This is the definition given at the beginning of [Str91], here expressed in terms of parity structures; note that for simplicity we only consider finite parity complexes. However, through the course of that article, further properties on a parity complex are introduced to ensure the validity of certain arguments. For instance, in Section 5, the $\blacktriangleleft$ order is introduced to prove that the product of two parity complexes is a parity complex, using the following axiom.

- **Axiom 4.** $\blacktriangleleft$ is an antisymmetric order.

Also, in Section 4, a "globularity condition" is introduced to ensure that every element of the parity complex determines a cell. This condition is given in terms of the sets $\widetilde{\mu}(x)$ and $\widetilde{\pi}(x)$ for $x \in C_n$, which are defined inductively as follows.

- $\widetilde{\mu}(x)_n = \{x\} = \widetilde{\pi}(x)_n$
- $\widetilde{\mu}(x)_{k-1} = \widetilde{\mu}(x)_k^{\mp}$ and $\widetilde{\pi}(x)_{k-1} = \widetilde{\pi}(x)_k^{\pm}$ for $0 < k < n$.

This is the pair of subsets giving the cell determined by x according to the definitions of [Str91], as we will describe in Section 3.1. The globularity condition then states

- **Axiom 5.** $\widetilde{\mu}(x)_{k-1} = \widetilde{\pi}(x)_k^{\mp}$ and $\widetilde{\pi}(x)_{k-1} = \widetilde{\mu}(x)_k^{\pm}$ for $0 < k < n$.

In [Str94], which corrected some errors in [Str91], it was shown that these axioms suffice to demonstrate that the construction of [Str91] gives the ω-category freely generated by a parity complex. In particular, the following result, which is used in the proof of the "excision of extremals" algorithm of [Str91], is shown. For $R \subseteq S \subseteq C$, say that R is a $\lhd$-interval of S if $u, v \in R$, $w \in S$, $u \lhd_S w \lhd_S v$ imply $w \in R$.

Proposition B.3.2. *Let $x \in C$ and let $S \subseteq C$ be well-formed such that $\mu(x) \subseteq S$. Then $\mu(x)$ is a $\lhd$-interval of S.*

Proof. This is proved in Propositions 1.4 and 1.5 of [Str94]. In fact, an inspection of the proof makes it clear that the results of our Proposition B.2.10 and its Corollary suffice to demonstrate this result. □

Theorem B.3.3. *A parity structure satisfying the axioms* ***(L)****,* ***(G)****, and* ***(C)*** *also satisfies the axioms 0,1,2,4,5.*

Proof. Note that Axioms 0 and 1 are the results of Propositions B.2.2 and B.2.3. Axiom 2 is a part of axiom **(C)**, since for $x \in C_n$ we have

$$(s_{n-1}R(x))_{n-1} = R(x)_{n-1} \setminus R(x)^+_n = (x^- \cup x^+) \setminus x^+ = x^-,$$

since $x^- \cap x^+ = \emptyset$. Axiom 4 is part of our axiom **(L)**, since a linear order is in particular antisymmetric. Note that by Proposition B.2.10, we have $\mu(x) = \widetilde{\mu}(x)$ and $\pi(x) = \widetilde{\pi}(x)$ for all $x \in C$, and Axiom 5 holds. □

Some discussion is required for Axiom 3. First note that $\blacktriangleleft$ is a stronger order than $\lhd$, for $x < y$ means there exists $z \in x^+ \cap y^-$, and so $x \blacktriangleleft z \blacktriangleleft y$. Hence $x \lhd y$ implies $x \blacktriangleleft y$. Hence Axiom 3(a), asserting the antisymmetry of $\lhd$, follows from the antisymmetry of $\blacktriangleleft$. This is addressed in Section 5 of [Str91], where it is further shown that "half" of Axiom 3(b) also follows from antisymmetry of $\blacktriangleleft$. The other half then follows from the property that the "odd dual" of C has $\blacktriangleleft$. Hence we could take this property as a further axiom. However, observe that one important case of this other half, namely $x^{-+} \cap x^{+-} = \emptyset$ follows from our axioms, as shown in Proposition B.2.7. Also, no such condition is required in [Joh89], so it is plausible that our axioms suffice.

B.3.1 Cells

In this section we give some motivation for the notion of cell given in [Str91], which we call (S-)cells. Rather than a subcomplex A, such a cell is a pair of subsets (M, P). The key idea behind the definition is that

$$M = \sum_{n \in \mathbb{N}} (s_n A)_n \qquad \text{and} \qquad P = \sum_{n \in \mathbb{N}} (t_n A)_n.$$

We show that the conditions for (M, P) to be an (S-)cell follow from the conditions for A to be a cell.

The new key notion for this definition is that of *movement*, as defined in Section 2 of [Str91].

Definition B.3.4. Let S, M, P be subsets of a parity structure C. We say that S *moves M to P* (written $S\colon M \longrightarrow P$) if

$$P = (M \cup S^+) \setminus S^- \qquad \text{and} \qquad M = (P \cup S^-) \setminus S^+.$$

Note that we may equivalently write this condition as the single equation

$$M + S^+ = P + S^-.$$

We claim that movement is essentially an artefact of globularity for subcomplexes.

Lemma B.3.5. *Let A be a subcomplex of a parity structure C. Then for all $n \geq 1$ we have*

$$A_n\colon (s_{n-1}A)_{n-1} \longrightarrow (t_{n-1}A)_{n-1}.$$

Proof. Since $(s_{n-1}A)_{n-1} = A_{n-1} \setminus A_n^+$ and dually, this condition is equivalent to the equation

$$(A_{n-1} \setminus A_n^+) + A_n^+ = (A_{n-1} \setminus A_n^-) + A_n^-.$$

Disjointness is immediate, and since A is a subcomplex, both of these unions are equal to A_{n-1}. □

Definition B.3.6. Let C be a parity complex. An *(S-)cell* is a pair (M, P) of non-empty well-formed subsets of C such that M and P both move M to P.

Recall

$$\sum_{n\in\mathbb{N}} s_n A = A \setminus A^+ \qquad \text{and} \qquad \sum_{n\in\mathbb{N}} t_n A = A \setminus A^-.$$

Theorem B.3.7. *Let C be a parity structure satisfying Axioms **(L)** and **(G)**. Then any cell $A \subseteq C$ gives an (S-)cell by $M = A \setminus A^-$ and $P = A \setminus A^+$.*

Proof. By definition of a cell, $A \setminus A^-$ and $A \setminus A^+$ are well formed. Hence it remains only to prove the movement condition. Note that each $s_n A$ and $t_n A$ is a subcomplex of C. It suffices to check movement dimension-wise, so we need to check $M_n, P_n\colon M_{n-1} \longrightarrow P_{n-1}$. But this is just

$$(s_n A)_n, (t_n A)_n\colon (s_{n-1}A)_{n-1} \longrightarrow (t_{n-1}A)_{n-1},$$

since $(s_k A)_k = A_{k-1} \setminus A_k^+$ and dually. And since

$$s_{n-1}s_n A = s_{n-1}A = t_{n-1}s_n A,$$

by Theorem B.2.8, the result follows from Lemma B.3.5. □

Finally, we would like a result showing that these two notions of cell are equivalent. Given the above construction, and since

$$A = (A \setminus A^+) \cup (A \setminus A^-) \cup (A^+ \cap A^-) = (A \setminus A^+) \cap (A \setminus A^-) \cap (A^+ \cup A^-),$$

the reverse construction of a cell from an (S-)cell could be given in any of the following ways.

- $A = R(M \cup P)$
- $A = R(M \cap P)$
- inductively, for (M, P) n-dimensional subsets,
  - $A_n = M_n = P_n$
  - $A_k = M_k \cup P_k \cup (A^+_{k+1} \cap A^-_{k+1})$ for $k < n$.

We suspect that the third definition would give the most direct proof.

B.4 Pasting schemes

In this section we give the definition of loop-free pasting scheme in terms of parity structures, and show that any parity structure satisfying the axioms **(L)**, **(G)**, and **(C)** is a loop-free pasting scheme.

In [Joh89], a pasting scheme is defined to be a graded set C together with relations E^i_j, B^i_j between C_j and C_i for $j \leq i$, satisfying certain axioms. We restate this definition in terms of parity structures by taking, for each $x \in C_k$,

$$x^- = \{y \in C_{k-1} \colon xB^k_{k-1}y\} \qquad \text{and} \qquad x^+ = \{y \in C_{k-1} \colon xE^k_{k-1}y\}.$$

In [Joh89], the sets $\widetilde{E}(x)$ and $\widetilde{B}(x)$ are defined for $x \in C_n$ inductively as follows.

- $\widetilde{E}(x)_n = \{x\}$ and $\widetilde{B}(x)_n = \{x\}$, for $x \in C_n$,
- $\widetilde{E}(x)_k = \widetilde{E}(x^+)_k \cap \widetilde{B}(x^+)_k$ and $\widetilde{B}(x)_k = \widetilde{B}(x^-)_k \cap \widetilde{E}(x^-)_k$ for $k < n$.

Definition B.4.1. A *pasting scheme* is a parity structure C such that for all $x \in C$ we have

$$\widetilde{E}(x^+) \cup \widetilde{B}(x^-) = \widetilde{E}(x^-) \cup \widetilde{B}(x^+).$$

Rather than giving formulas for all sources and targets, [Joh89] proceeds by giving a formula for the top-dimensional source and target, and defining the others by iteration as approriate by globularity. Let A be an n-dimensional subset of C. Then define

$$\mathrm{dom}A = A \setminus \widetilde{E}(A_n) \qquad \text{and} \qquad \mathrm{cod}A = A \setminus \widetilde{B}(A_n),$$

and

- $\widetilde{s}_k A = A = \widetilde{t}_k A$ for $k \geq n$
- $\widetilde{s}_k A = \mathrm{dom}^{n-k} A$ and $\widetilde{t}_k A = \mathrm{cod}^{n-k} A$ for $k < n$.

We now give the definition of cell as given in [Joh89].

Definition B.4.2. A subset $A \subseteq C$ is called *compatible* if it is an n-dimensional subcomplex for some n, and A_n is well-formed.

Definition B.4.3. A subset $A \subseteq C$ is a *(J-)cell* if $\widetilde{s}_k A$ and $\widetilde{t}_k A$ are compatible for all k. Further, if A is n-dimensional, we say that A is an n-(J-)cell.

Proposition B.4.4. *Let C be a parity structure satisfying axioms **(L)** and **(G)**. Then a subset $A \subseteq C$ is an n-cell if and only if it is an n-(J-)cell.*

Proof. By Proposition B.2.6, we have that $\widetilde{E}(x) = E(x)$ and $\widetilde{B}(x) = B(x)$. Hence by Theorem B.2.8, our definitions of source and target coincide. Under both definitions, A is a subcomplex of C. The only thing left to check is that $s_k A$ is a k-dimensional for all $k \leq n$. By the definition, it is at most k-dimensional, and so it suffices to show that $(s_k A)_k = A_k \setminus A^+_{k+1}$ is nonempty. To this end, let x be the minimum element of A_k under $\blacktriangleleft$. If $x \notin A^+_{k+1}$, we are done. Else, $x \in A^+_{k+1}$, and so there exists $y \in A_{k+1}$ such that $x \in y^+$. Then for any $u \in y^-$ we have $u \in A_k$ and $u \blacktriangleleft y \blacktriangleleft x$, contradicting minimality of x. $\square$

Definition B.4.5. A *loop-free pasting scheme* is a pasting scheme satisfying the axioms

(i) $\widetilde{B}(x) \cap \widetilde{E}(x) = \{x\}$ and $x \lhd y$ implies $\widetilde{B}(x) \cap \widetilde{E}(y) = \emptyset$

(ii) $R(x)$ is a (J-)cell

(iii) for any n-(J-)cell A and $x \in C$ such that $\widetilde{s}_n R(x) \subseteq A$, then $\mu(x)_n$ is a $\lhd$-interval of A_n.

Theorem B.4.6. *Any parity structure satisfying the axioms **(L)**, **(G)**, and **(C)** is a loop-free pasting scheme.*

Proof. As in the proof of Proposition B.4.4, we have $\widetilde{E}(x) = E(x)$ and $\widetilde{B}(x) = B(x)$. The condition in the definition of pasting scheme is our Proposition B.2.3. Since $\blacktriangleleft$ is stronger than $\lhd$, the loop-free axiom (i) is our Propositions B.2.1 and B.2.5. By Proposition B.4.4, axiom (ii) is exactly our axiom **(C)**. Finally, Axiom (iii) is Proposition B.3.2. $\square$

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