# Restriction categories and their free cocompletion

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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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# Abstract

The universal property of the Yoneda embedding which exhibits the presheaf category $\widehat{\mathbf{C}}$ as the free cocompletion of $\mathbf{C}$ is well-known to category theorists. On the other hand, restriction categories are less well-studied (having only been introduced since the early 1990's). In this thesis, we describe free cocompletion within the restriction setting by introducing the notion of restriction presheaf. We also motivate and give a definition of cocomplete $\mathcal{M}$-category and cocomplete restriction category.

# Contents

# 1
# Introduction

The notion of restriction category was first introduced by Marco Grandis [1], and studied extensively by Cockett and Lack [2] [3] [4] as a continuation of work done on categories of partial maps. Informally, a restriction category is a category $\mathbf{C}$ where every arrow is assigned an idempotent on its domain object which measures its degree of partiality. In the case where $\mathbf{C}$ is the category of sets and partial functions, this idempotent gives precisely the domain of definition of the partial function.

In the world of ordinary categories, we have a good notion of cocompleteness, and it is well-known that for any small category $\mathbf{C}$, the Yoneda embedding $\mathrm{y}\colon \mathbf{C} \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ has the universal property of exhibiting the presheaf category $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ as the free cocompletion of $\mathbf{C}$. We would like to be able to define a notion of cocompleteness with respect to restriction categories, as well as an analogue of free cocompletion in the restriction world. To address these two questions, we begin by revising necessary background material from Cockett and Lack [2] in Chapter 2.

Chapters 3 and 4 are original work. In Chapter 3, we introduce the notion of restriction presheaf on any restriction category $\mathbf{X}$. We define the restriction presheaf category $\widehat{\mathbf{X}}_r$, and give an embedding $\mathrm{y}_r\colon \mathbf{X} \to \widehat{\mathbf{X}}_r$. We show that this embedding $\mathrm{y}_r$ is the same embedding described in Cockett and Lack [2] (up to an equivalence).

In Chapter 4, we define cocomplete $\mathcal{M}$-category and cocomplete restriction category, and show that the embedding $\mathrm{y}_r$ exhibits the restriction presheaf category $\widehat{\mathbf{X}}_r$ as the free cocompletion of any small restriction category $\mathbf{X}$. We conclude by discussing possible continuation of this work in relation to the notion of join restriction category [5].

# 2

# Restriction categories

## 2.1 Restriction structures and total maps

Everything in this chapter is revision of material from Cockett and Lack [2]. A *restriction category* is a category $\mathbf{X}$ together with a family of assignations $\{F_{A,B}\}$ (one for each pair of objects $A, B \in \mathbf{X}$), where each $F_{A,B}$ is a map

$$F_{A,B} \colon \mathbf{X}(A, B) \to \mathbf{X}(A, A), \quad f \mapsto \overline{f}$$

and $\overline{f}$ satisfies the following conditions:

(R1) $f \circ \overline{f} = f$

(R2) $\overline{g} \circ \overline{f} = \overline{f} \circ \overline{g}$ for $f \colon A \to B, g \colon A \to C$

(R3) $\overline{g \circ \overline{f}} = \overline{g} \circ \overline{f}$ for $f \colon A \to B, g \colon A \to C$

(R4) $\overline{h} \circ f = f \circ \overline{h \circ f}$ for $f \colon A \to B, h \colon B \to C$

We call the family of assignations $\{F_{A,B}\}_{A,B \in \mathbf{X}}$ the *restriction structure* on $\mathbf{X}$, and call $\overline{f}$ the *restriction* of $f$.

**Example 1.** *Denote by* **Pfn** *the category of sets and partial functions. We can make* **Pfn** *into a restriction category by defining the restriction of each partial function* $f \colon A \rightharpoonup B$ *to be an another partial function* $\overline{f} \colon A \rightharpoonup A$ *as follows:*

$$\overline{f}(a) = \begin{cases} a; & \text{if } f(a) \text{ is defined at } a \in A \\ \text{undefined;} & \text{otherwise} \end{cases}$$

**Example 2.** *Let* $\mathbf{M}$ *denote the one-object category whose arrows are the natural numbers (including* $0$*), and where composition is defined by* $m \circ n = \max(m, n)$*. Then* $\mathbf{M}$ *may be given a restriction structure by defining* $\overline{n} = n$*.*

The restriction of any arrow in a restriction category has certain properties, and we list some of them below.

**Lemma 3.** *Suppose* $\mathbf{X}$ *is a restriction category, and* $f \colon A \to B$ *and* $g \colon B \to C$ *are arrows in* $\mathbf{X}$*. Then*

(1) $\overline{f}$ *is idempotent.*

(2) $\overline{f} \circ \overline{g \circ f} = \overline{g \circ f}$

(3) $\overline{\overline{g} \circ f} = \overline{g \circ f}$

(4) $\overline{f} = \overline{\overline{f}}$

(5) *If* $f$ *is a monomorphism, then* $\overline{f} = 1_A$*.*

*Proof.* (1) Applying (R3) and (R1), we get $\overline{f} \circ \overline{f} = \overline{f \circ \overline{f}} = \overline{f}$.

(2) Applying (R2) then (R3) and finally (R1) gives

$$\overline{f} \circ \overline{g \circ f} = \overline{g \circ f} \circ \overline{f} = \overline{g \circ f \circ \overline{f}} = \overline{g \circ f}$$

(3) Applying (R4) then (R3), and using the previous result gives

$$\overline{\overline{g} \circ f} = \overline{f \circ \overline{g \circ f}} = \overline{f} \circ \overline{g \circ f} = \overline{g \circ f}$$

(4) By Lemma 3 (3),

$$\overline{f} = \overline{f \circ 1_A} = \overline{\overline{f} \circ 1_A} = \overline{\overline{f}}$$

(5) By (R1), $f \circ \overline{f} = f = f \circ 1_A$. Therefore, if $f$ is monic, then $\overline{f} = 1_A$. □

We say that a map $f$ in $\mathbf{X}$ is a *restriction idempotent* if $f = \overline{f}$. So what maps in $\mathbf{X}$ are of this form? By the previous result, maps of the form $f = \overline{g}$ (for some $g$) are restriction idempotents. Hence, $f$ is a restriction idempotent if and only if $f = \overline{g}$ (for some $g$).

Now we can also define a partial order on the set of restriction idempotents on a fixed object in a restriction category. If $e \colon A \to A$ and $e' \colon A \to A$ are two such restriction idempotents, then define $e \leq e'$ if and only if $e = e'e$. For example, in the restriction category of sets and partial functions, the restriction idempotents $e$ and $e'$ correspond with subsets of $A$, and the ordering described correspond with the usual ordering of subsets. More generally, if $\mathbf{X}$ is a restriction category and $A, B$ are objects in $\mathbf{X}$, we can define a partial order on $\mathbf{X}(A, B)$ by $f \leq g$ if and only if $f = g \circ \overline{f}$.

**Definition 4.** *A map* $f \colon A \to B$ *in a restriction category* $\mathbf{X}$ *is called* total *if* $\overline{f} = 1_A$*.*

So by Lemma 3 (5), all monomorphisms in a restriction category are total (and in particular, the identity maps are total).

**Proposition 5.** *Let* $\mathbf{X}$ *be a restriction category. Denote by* $\mathsf{Total}(\mathbf{X})$ *the structure containing all objects of* $\mathbf{X}$ *and total maps in* $\mathbf{X}$*. Then* $\mathsf{Total}(\mathbf{X})$ *is a subcategory of* $\mathbf{X}$*.*

*Proof.* We just need to check that $\mathsf{Total}(\mathbf{X})$ contains the identity maps and that the composition of two total maps is total. Certainly $\mathsf{Total}(\mathbf{X})$ contains the identities (as identities are total). Now suppose $f: A \to B$ and $g: B \to C$ are total. Then by Lemma 3 (3),

$$\overline{g \circ f} = \overline{\overline{g} \circ f} = \overline{f} = 1_A$$

and so $g \circ f$ is total. Hence $\mathsf{Total}(\mathbf{X})$ is a subcategory of $\mathbf{X}$. □

## 2.2 The 2-category of restriction categories

If $\mathbf{X}$ and $\mathbf{Y}$ are restriction categories, then a *restriction functor* $F: \mathbf{X} \to \mathbf{Y}$ is a functor between the underlying categories such that $F(\overline{f}) = \overline{F(f)}$ for all arrows $f \in \mathbf{X}$. Also, we call a natural transformation between two restriction functors $\alpha: F \Rightarrow G$ a *restriction transformation* if $\overline{\alpha_A} = 1_{FA}$ for all $A \in \mathbf{X}$ (components are total).

Restriction categories (objects), restriction functors (1-cells) and restriction transformations (2-cells) form a 2-category which we denote by **rCat**. There is an obvious forgetful 2-functor $U: \mathbf{rCat} \to \mathbf{Cat}$.

## 2.3 Split restriction categories

Recall that an idempotent $e: A \to A$ in an ordinary category is said to be *split* if there exist two maps $m: B \to A$ and $r: A \to B$ such that $mr = e$ and $rm = 1_B$. Now let $\overline{f}: A \to A$ be the restriction of some map $f: A \to B$ in a restriction category $\mathbf{X}$, and suppose $\overline{f}$ splits. Then there exist maps $m: C \to A$ and $r: A \to C$ such that $mr = \overline{f}$ and $rm = 1_C$ (for some $C \in \mathbf{X}$). But this implies

$$\overline{r} = \overline{\overline{m}r} = \overline{mr} = \overline{\overline{f}} = \overline{f}$$

by Lemma 3 (3) and the fact $m$ is monic. In other words, split restriction idempotents must be of the form $\overline{r} = mr$ for some $m$ satisfying the condition $rm = 1$. We call $m$ a *restriction monic* if there exists an $r$ such that $mr = \overline{r}$ and $rm = 1$.

If all restriction idempotents in a restriction category $\mathbf{X}$ split, we call $\mathbf{X}$ a *split restriction category*. This gives a new 2-category $\mathbf{rCat}_s$ whose objects are split restriction categories, 1-cells are restriction functors and 2-cells are restriction transformations. Clearly, $\mathbf{rCat}_s$ is a full sub-2-category of **rCat**.

Now there exists a 2-functor $K_r: \mathbf{rCat} \to \mathbf{rCat}_s$ which takes any restriction category $\mathbf{X}$ to a split restriction category $K_r(\mathbf{X})$ with the following data:

- **Objects**: An object of $K_r(\mathbf{X})$ is a pair $(A, e_A)$, where $A$ is an object in $\mathbf{X}$ and $e_A: A \to A$ is some restriction idempotent in $\mathbf{X}$.

- **Arrows**: An arrow $(A, e_A)$ to $(B, e_B)$ is a triple $(e_A, f, e_B)$, where $f\colon A \to B$ is an arrow in $\mathbf{X}$ such that $e_B f e_A = f$.

- **Identity**: The identity $1_{(A,e_A)}$ is given by $1_{(A,e_A)} = (e_A, e_A, e_A)$.

- **Composition**: The composite of $(e_A, f, e_B)\colon (A, e_A) \to (B, e_B)$ with $(e_B, g, e_C)\colon (B, e_B) \to (C, e_C)$ is given by $(e_A, gf, e_C)\colon (A, e_A) \to (C, e_C)$.

- **Restriction**: The restriction on $(e_A, f, e_B)$ is given by $\overline{(e_A, f, e_B)} = (e_A, \overline{f} e_A, e_A)$.

It is clear that a structure with the above data is a category (as composition of arrows in $\mathbf{X}$ is associative). Also, the definition of the restriction structure makes sense since

$$e_A \overline{f} e_A e_A = (e_A \overline{f}) e_A e_A = (\overline{f} e_A) e_A e_A = \overline{f} e_A$$

(both $e_A$ and $\overline{f}$ are restriction idempotents). All that remains is to check that this definition satisfies the restriction axioms; see [2, p. 242] for details. Now to see that all restriction idempotents split in $K_r(\mathbf{X})$, consider an arrow $(e_A, f, e_B)\colon (A, e_A) \to (B, e_B)$, with restriction given by $\overline{(e_A, f, e_B)} = (e_A, \overline{f} e_A, e_A)$. Then $(e_A, \overline{f} e_A, e_A) = (\overline{f} e_A, \overline{f} e_A, e_A) \circ (e_A, \overline{f} e_A, \overline{f} e_A)$

$$\begin{tikzcd}
(A, e_A) \arrow[rr, "{(e_A, \overline{f} e_A, e_A)}"] \arrow[dr, "{(e_A, \overline{f} e_A, \overline{f} e_A)}"'] & & (A, e_A) \\
& (A, \overline{f} e_A) \arrow[ur, "{(\overline{f} e_A, \overline{f} e_A, e_A)}"'] &
\end{tikzcd}$$

and

$$(e_A, \overline{f} e_A, \overline{f} e_A) \circ (\overline{f} e_A, \overline{f} e_A, e_A) = (\overline{f} e_A, \overline{f} e_A, \overline{f} e_A) = 1_{(A, \overline{f} e_A)}$$

So every restriction idempotent splits in $K_r(\mathbf{X})$.

There is a canonical embedding $J\colon \mathbf{X} \to K_r(\mathbf{X})$ which has the following data:

- **Objects**: Let $A$ be an object in $\mathbf{X}$. Then $JA = (A, 1_A)$.

- **Arrows**: Let $f\colon A \to B$ be an arrow in $\mathbf{X}$. Then $Jf\colon (A, 1_A) \to (B, 1_B)$ is given by $Jf = (1_A, f, 1_B)$.

Clearly $J$ is a restriction functor since

$$J(\overline{f}) = (1_A, \overline{f}, 1_A) = (1_A, \overline{f} 1_A, 1_A) = \overline{(1_A, f, 1_B)} = \overline{Jf}$$

**Proposition 6.** *Let $\mathbf{X}$ be a restriction category and $\mathcal{E}$ a split restriction category. Then the functor $(-) \circ J\colon \mathbf{rCat}_s(K_r(\mathbf{X}), \mathcal{E}) \to \mathbf{rCat}(\mathbf{X}, \mathcal{E})$ is an equivalence of categories.*

*Proof.* We begin by showing that $(-) \circ J$ is essentially surjective. So given $H\colon \mathbf{X} \to \mathcal{E}$, we must find some $H_!\colon K_r(\mathbf{X}) \to \mathcal{E}$ with $H_! \circ J \cong H$. To define $H_!$, let $(A, e_A) \in K_r(\mathbf{X})$ be given. By assumption, the restriction idempotent $He_A\colon HA \to HA$ splits in $\mathcal{E}$; so choose some splitting (below):

$$\begin{tikzcd} HA \arrow[rr, "He_A"] \arrow[dr, "r_A"'] & & HA \\ & S_{e_A} \arrow[ur, "m_A"'] & \end{tikzcd}$$

And now define $H_!\colon K_r(\mathbf{X}) \to \mathcal{E}$ on objects by $H_!(A, e_A) = S_{e_A}$. Note that since splittings of idempotents are unique up to isomorphism, we have $H_!JA = H_!(A, 1_A) \cong HA$.

Now let $(e_A, f, e_B)\colon (A, e_A) \to (B, e_B)$ be an arrow in $K_r(\mathbf{X})$. Since both $S_{e_A}$ and $S_{e_B}$ are part of the splittings of $He_A$ and $He_B$ respectively, we get the commutative diagram below:

$$\begin{tikzcd} & HA \arrow[r, "Hf"] & HB \arrow[d, "r_B"'] \arrow[dr, "He_B"] & \\ HA \arrow[ur, "He_A"] \arrow[dr, "r_A"'] & & & HB \\ & S_{e_A} \arrow[uu, "m_A"'] \arrow[r, dashed, "H_!(e_A, f, e_B)"'] & S_{e_B} \arrow[ur, "m_B"'] & \end{tikzcd}$$

So define $H_!$ on arrows by $H_!(e_A, f, e_B) = r_B \circ Hf \circ m_A$. This may be checked to make $H_!$ a functor, and in particular, $H_! \circ J \cong H$. This shows that $F$ is essentially surjective; it remains to show that $F$ is fully faithful.

Suppose $G, H\colon K_r(\mathbf{X}) \to \mathcal{E}$ are two restriction functors and $\alpha\colon GJ \Rightarrow HJ$ is a restriction transformation. We require a unique restriction transformation $\widetilde{\alpha}\colon G \Rightarrow H$ such that $\widetilde{\alpha} \circ J = \alpha$. Given $(A, e) \in K_r(\mathbf{X})$ (where $e\colon A \to A$ is a restriction idempotent), we define the component of $\widetilde{\alpha}$ at $(A, e)$ by the arrow which makes the following diagram commute:

$$\begin{tikzcd} F(A,e) \arrow[r, dashed, "\widetilde{\alpha}_{(A,e)}"] \arrow[d, "F(e,e,1)"'] & G(A,e) \\ F(A,1) \arrow[r, "\alpha_A"'] & G(A,1) \arrow[u, "G(1,e,e)"'] \end{tikzcd}$$

It is now easy to check that $\widetilde{\alpha}$ is natural. However, it is also a restriction transformation since

$$\begin{aligned}
\overline{\widetilde{\alpha}_{(A,e)}} &= \overline{G(1,e,e) \circ \alpha_A \circ F(e,e,1)} \\
&= \overline{\overline{\overline{G(1,e,e)} \circ \alpha_A} \circ F(e,e,1)} && \text{(repeated use of fact } \overline{gf} = \overline{\overline{g}f}\text{)} \\
&= \overline{\overline{G(1,e,1) \circ \alpha_A} \circ F(e,e,1)} && \text{(}G\text{ is a restriction functor)} \\
&= \overline{\overline{\alpha_A \circ F(1,e,1)} \circ F(e,e,1)} && (\alpha \text{ is natural}) \\
&= \overline{F(1,e,1) \circ F(e,e,1)} && (\alpha_A \text{ is total; } F(1,e,1) \text{ is restriction idempotent}) \\
&= \overline{F(e,e,1)} = F(e,e,e) = 1_{F(A,e)}
\end{aligned}$$

This shows the existence of $\widetilde{\alpha}$; uniqueness follows by comparison with previous diagram. Therefore $(-) \circ J\colon \mathbf{rCat}_s(K_r(\mathbf{X}), \mathcal{E}) \to \mathbf{rCat}(\mathbf{X}, \mathcal{E})$ is an equivalence. □

## 2.4 Stable system of monics and $\mathcal{M}$-categories

In any category $\mathbf{C}$, a *system of monics* $\mathcal{M}$ is defined to be a collection of monics in $\mathbf{C}$ such that

- $\mathcal{M}$ contains the isomorphisms, and
- If $m\colon A \to B$ and $n\colon B \to C$ are in $\mathcal{M}$, then so is their composite $n \circ m$.

Such a system of monics $\mathcal{M}$ is said to be *stable* if for all $m \in \mathcal{M}$ and $f \in \mathbf{C}$, the pullback of $m$ along $f$ exists and is in $\mathcal{M}$. That is, there is a pullback square of the following form:

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "m"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

We call the above pullback $(A', m', f')$ of $m$ along $f$ an $\mathcal{M}$-pullback. We define an $\mathcal{M}$-category to be a pair $(\mathbf{C}, \mathcal{M})$ of a category $\mathbf{C}$ and a stable system of monics $\mathcal{M}$ in $\mathbf{C}$. If $(\mathbf{D}, \mathcal{N})$ is also an $\mathcal{M}$-category (with $\mathcal{N}$ being a stable system of monics in $\mathbf{D}$), then an $\mathcal{M}$-functor $(\mathbf{C}, \mathcal{M}) \to (\mathbf{D}, \mathcal{N})$ is a functor $F\colon \mathbf{C} \to \mathbf{D}$ with the following properties:

- If $m \in \mathcal{M}$, then $Fm \in \mathcal{N}$, and
- If $\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "m"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$ is an $\mathcal{M}$-pullback in $\mathbf{C}$, then $\begin{tikzcd} FA' \arrow[r, "Ff'"] \arrow[d, "Fm'"'] \arrow[dr, phantom, "\lrcorner", very near start] & FA \arrow[d, "Fm"] \\ FB' \arrow[r, "Ff"'] & FB \end{tikzcd}$ is a pullback in $\mathbf{D}$.

A natural transformation $\alpha\colon F \Rightarrow G$ between two $\mathcal{M}$-functors is called an $\mathcal{M}$-*cartesian natural transformation* if for all $m\colon X \to Y$ in $\mathcal{M}$, the following naturality square is a pullback:

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Fm"'] \arrow[dr, phantom, "\lrcorner", very near start] & GX \arrow[d, "Gm"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

$\mathcal{M}$-categories (objects), $\mathcal{M}$-functors (1-cells) and $\mathcal{M}$-cartesian natural transformations (2-cells) form a 2-category, which we denote by $\mathcal{M}\mathbf{Cat}$.

## 2.5 The category of partial maps

Given any $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, we may construct a restriction category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, called the *category of* $\mathcal{M}$-*partial maps in* $\mathbf{C}$. Its objects are the same as those in $\mathbf{C}$, and an arrow from $A$ to $B$ is an equivalence class of triples $[(A', m, f)]$, where $m\colon A' \to A$ is in $\mathcal{M}$ and $f\colon A' \to B$ is a map in $\mathbf{C}$:

$$\begin{tikzcd} & A' \arrow[dl, "m"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

We say $(A', m, f) \sim (A'', m', f')$ if there exists an isomorphism $\varphi\colon A'' \to A'$ making the following diagram commute:

$$\begin{tikzcd} & A'' \arrow[dl, "m'"'] \arrow[d, "\varphi"] \arrow[dr, "f'"] & \\ A & A' \arrow[l, "m"] \arrow[r, "f"'] & B \end{tikzcd}$$

(From now on, we shall dispense with the bracket notation if the meaning is clear from the context).

The identity arrow on $A \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$ is given by $(A, 1_A, 1_A)$, and the composite of an arrow $(A', m, f)\colon A \to B$ with $(B', m', f')\colon B \to C$ is defined to be $(B', m', f') \circ (A', m, f) = (A'', mm'', f'f'')$, where $(A'', m'', f'')$ is a pullback of $m'$ along $f$:

$$\begin{tikzcd} & & A'' \arrow[dl, "m''"'] \arrow[dr, "f''"] & & \\ & A' \arrow[dl, "m"'] \arrow[dr, "f"] & & B' \arrow[dl, "m'"'] \arrow[dr, "f'"] & \\ A & & B & & C \end{tikzcd}$$

It is easy to see that composition is well-defined on equivalence classes, and the equivalence relation ensures that this composition is strictly associative and unital.

**Proposition 7.** *Suppose $(\mathbf{C}, \mathcal{M})$ is an $\mathcal{M}$-category. Then $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a split restriction category, with restriction given by $\overline{(A', m, f)} = (A', m, m)$ for all arrows $(A', m, f)\colon A \to B$, and the total maps in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ are of the form $(A, 1_A, g)\colon A \to B$.*

*Proof.* The first step simply involves checking that the given restriction structure satisfies axioms (R1)-(R4) (see [2, p. 246-247]). Also, note that each restriction idempotent $\overline{(A', m, f)}$ is split since $(A', m, 1_{A'}) \circ (A', 1_{A'}, m) = (A', 1_{A'}, 1_{A'})$ and $(A', 1_{A'}, m) \circ (A', m, 1_{A'}) = (A', m, m)$. Finally, suppose $(A', m, f)$ is total. Then $(A', m, m) \sim (A, 1_A, 1_A)$, implying $m\colon A' \to A$ is an isomorphism. Therefore, $(A', m, f) \sim (A, 1_A, fm^{-1})$.

$$\begin{tikzcd} & A \arrow[dl, "1_A"'] \arrow[d, "m^{-1}"] \arrow[dr, "fm^{-1}"] & \\ A & A' \arrow[l, "m"] \arrow[r, "f"'] & B \end{tikzcd}$$

Hence, a map is total in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ if and only if it is of the form $(A, 1_A, g)$. $\square$

**Proposition 8.** *There exists a 2-functor* $\mathsf{Par}\colon \mathcal{M}\mathbf{Cat} \to \mathbf{rCat}_s$ *which, on objects, takes* $(\mathbf{C}, \mathcal{M})$ *to* $\mathsf{Par}(\mathbf{C}, \mathcal{M})$.

*Proof.* We need to define a functor

$$\mathsf{Par}\colon \mathcal{M}\mathbf{Cat}\big((\mathbf{C}, \mathcal{M}), (\mathbf{D}, \mathcal{N})\big) \to \mathbf{rCat}_s\big(\mathsf{Par}(\mathbf{C}, \mathcal{M}), \mathsf{Par}(\mathbf{D}, \mathcal{N})\big)$$

So let $F\colon (\mathbf{C}, \mathcal{M}) \to (\mathbf{D}, \mathcal{N})$ be an $\mathcal{M}$-functor and define $\mathsf{Par}(F)$ as follows:

- **Objects**: $\mathsf{Par}(F)(A) = FA$
- **Arrows**: $\mathsf{Par}(F)(A', m, f) = (FA', Fm, Ff)$

Then $\mathsf{Par}(F)$ is a functor by the fact $\mathcal{M}$-functors preserve $\mathcal{M}$-pullbacks. But $\mathsf{Par}(F)$ is also a restriction functor since

$$\mathsf{Par}(F)(A', m, m) = (FA', Fm, Fm) = \overline{(FA', Fm, Ff)} = \overline{\mathsf{Par}(F)(A', m, f)}$$

Now let $\alpha\colon F \Rightarrow G$ be an $\mathcal{M}$-cartesian natural transformation and define $\mathsf{Par}(\alpha)\colon \mathsf{Par}(F) \Rightarrow \mathsf{Par}(G)$ componentwise by

$$\mathsf{Par}(\alpha)_A = (FA, 1_{FA}, \alpha_A)\colon FA \to GA$$

It is straightforward to show that $\mathsf{Par}(\alpha)$ is natural, and hence $\mathsf{Par}\colon \mathcal{M}\mathbf{Cat} \to \mathbf{rCat}_s$ is a 2-functor. $\square$

## 2.6 The 2-functor $\mathcal{M}\mathsf{Total}$

In fact, the 2-functor $\mathsf{Par}\colon \mathcal{M}\mathbf{Cat} \to \mathbf{rCat}_s$ just defined is an equivalence of 2-categories. The 2-functor in the other direction, $\mathcal{M}\mathsf{Total}\colon \mathbf{rCat}_s \to \mathcal{M}\mathbf{Cat}$, is defined as follows:

If $\mathbf{X}$ is a split restriction category, then

$$\mathcal{M}\mathsf{Total}(\mathbf{X}) = (\mathsf{Total}(\mathbf{X}), \mathcal{M}_{\mathbf{X}})$$

where $\mathcal{M}_{\mathbf{X}}$ are the restriction monics in $\mathbf{X}$.

**Proposition 9.** $\mathcal{M}\mathsf{Total}(\mathbf{X})$ *is an* $\mathcal{M}$*-category.*

*Proof.* See Cockett, Lack [2, p. 249] for full details. $\square$

If $F\colon \mathbf{X} \to \mathbf{Y}$ is a restriction functor, define $\mathsf{Total}(F)\colon \mathsf{Total}(\mathbf{X}) \to \mathsf{Total}(\mathbf{Y})$ to be the restriction of $F$ to $\mathsf{Total}(\mathbf{X})$. Then we may show that $\mathsf{Total}(F)$ is a restriction functor, and moreover, that it preserves $\mathcal{M}$-pullbacks. Therefore, defining $\mathcal{M}\mathsf{Total}(F) = \mathsf{Total}(F)$ makes $\mathcal{M}\mathsf{Total}(F)\colon (\mathsf{Total}(\mathbf{X}), \mathcal{M}_{\mathbf{X}}) \to (\mathsf{Total}(\mathbf{Y}), \mathcal{M}_{\mathbf{Y}})$ an $\mathcal{M}$-functor. Finally, if $\alpha\colon F \Rightarrow G$ is a restriction transformation, then we may show that $\mathsf{Total}(\alpha)$ is $\mathcal{M}_{\mathbf{X}}$-cartesian. Hence, defining $\mathcal{M}\mathsf{Total}(\alpha) = \mathsf{Total}(\alpha)$ makes $\mathcal{M}\mathsf{Total}$ a 2-functor.

**Theorem 10.** *The* 2*-categories* $\mathbf{rCat}_s$ *and* $\mathcal{M}\mathbf{Cat}$ *are equivalent.*

*Proof.* To show equivalence, we need to define natural isomorphisms $\Phi\colon 1_{\mathbf{rCat}_s} \Rightarrow \mathsf{Par} \circ \mathcal{M}\mathsf{Total}$ and $\Psi\colon \mathcal{M}\mathsf{Total} \circ \mathsf{Par} \Rightarrow 1_{\mathcal{M}\mathbf{Cat}}$. First consider $\Phi$, and let us define $\Phi_{\mathbf{X}}\colon \mathbf{X} \to \mathsf{Par}(\mathsf{Total}(\mathbf{X}), \mathcal{M}_{\mathbf{X}})$ on objects by $\Phi_{\mathbf{X}}(A) = A$, and on arrows by $\Phi_{\mathbf{X}}(f) = (A', m, fm)$ (where $mr = \overline{f}$ and $rm = 1_{A'}$ for some $r\colon A \to A'$). Then $\Phi_{\mathbf{X}}$ is a functor, and is in fact, a restriction functor since

$$\Phi_{\mathbf{X}}(\overline{f}) = (A', m, \overline{f}m) = (A', m, (mr)m) = (A', m, m) = \overline{\Phi_{\mathbf{X}}(f)}$$

In addition, $\Phi_{\mathbf{X}}$ is an isomorphism and $\Phi$ is natural, making $\Phi$ a natural isomorphism (see [2, p. 250-251] for details).

On the other hand, define $\Psi_{(\mathbf{C},\mathcal{M})}\colon \big(\mathsf{Total}\big(\mathsf{Par}(\mathbf{C}, \mathcal{M})\big), \mathcal{M}_{\mathsf{Par}(\mathbf{C},\mathcal{M})}\big) \to (\mathbf{C}, \mathcal{M})$ on objects by $\Psi_{(\mathbf{C},\mathcal{M})}(A) = A$ and on arrows by $\Psi_{(\mathbf{C},\mathcal{M})}(A', 1_{A'}, f) = f$. Then clearly $\Psi_{(\mathbf{C},\mathcal{M})}$ is an isomorphism. Further, $\Psi$ is natural and so $\Psi$ is a natural isomorphism. Hence, $\mathcal{M}\mathsf{Total}$ and $\mathsf{Par}$ are part of an equivalence of 2-categories. □

## 2.7 New $\mathcal{M}$-categories from existing ones

In this final section, we recall a way of creating a new $\mathcal{M}$-category from an existing $\mathcal{M}$-category.

Let $(\mathbf{C}, \mathcal{M})$ be an $\mathcal{M}$-category, and consider the presheaf category $\widehat{\mathbf{C}} = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ on $\mathbf{C}$. We would like to give a stable system of monics in $\widehat{\mathbf{C}}$, say $\widehat{\mathcal{M}}$, so that $(\widehat{\mathbf{C}}, \widehat{\mathcal{M}})$ is an $\mathcal{M}$-category. We say that $\mu\colon P \Rightarrow Q$ is in $\widehat{\mathcal{M}}$ (or $\mu$ is an $\widehat{\mathcal{M}}$-map) if for all $\delta\colon \mathrm{y}C \Rightarrow Q$ (where y is the Yoneda embedding), there exists a $D \in \mathbf{C}$ and a monic $n\colon D \to C$ in $\mathcal{M}$ such that the following is a pullback:

$$\begin{tikzcd} \mathrm{y}D \arrow[r, "\gamma"] \arrow[d, "\mathrm{y}n"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\mu"] \\ \mathrm{y}C \arrow[r, "\delta"'] & Q \end{tikzcd}$$

To check that $\widehat{\mathcal{M}}$ is a system of monics in $\widehat{\mathbf{C}}$, we first must check it contains all the isomorphisms in $\widehat{\mathbf{C}}$. But if $\mu$ is any isomorphism, then for each $\delta\colon \mathrm{y}C \Rightarrow Q$, the following is a pullback:

$$\begin{tikzcd} \mathrm{y}C \arrow[r, "\mu^{-1}\circ\delta"] \arrow[d, "\mathrm{y}(1_C)"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\mu"] \\ \mathrm{y}C \arrow[r, "\delta"'] & Q \end{tikzcd}$$

So $\mu \in \widehat{\mathcal{M}}$. It is also easy to see that $\widehat{\mathcal{M}}$ is closed under composition, since if $\sigma\colon T \Rightarrow P$ and $\mu\colon P \Rightarrow Q$ are in $\widehat{\mathcal{M}}$ (with $\delta\colon \mathrm{y}C \Rightarrow Q$ given), then the following outer square is a pullback (by the standard pasting properties of pullbacks):

$$\begin{tikzcd}
\mathrm{y}E \arrow[r, "\beta"] \arrow[d, "\mathrm{y}m"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "\sigma"] \\
\mathrm{y}D \arrow[r, "\gamma"] \arrow[d, "\mathrm{y}n"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\mu"] \\
\mathrm{y}C \arrow[r, "\delta"'] & Q
\end{tikzcd}$$

Therefore it remains to show that $\widehat{\mathcal{M}}$ is stable. So let $\tau \colon R \Rightarrow Q$ be an arrow in $\widehat{\mathbf{C}}$. Then because $\widehat{\mathbf{C}}$ is complete, there exists a pullback square:

$$\begin{tikzcd}
S \arrow[r, "\gamma"] \arrow[d, "\mu'"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\mu"] \\
R \arrow[r, "\tau"'] & Q
\end{tikzcd}$$

We need to show $\mu'$ is an $\widehat{\mathcal{M}}$-map. So let $\delta \colon \mathrm{y}C \Rightarrow R$ be given. Now because $\mu$ is an $\widehat{\mathcal{M}}$-map, this means given the composite $\tau \circ \delta \colon \mathrm{y}C \Rightarrow Q$, there is a map $n \colon D \to C$ making the outer square a pullback:

$$\begin{tikzcd}
\mathrm{y}D \arrow[rr, bend left] \arrow[d, "\mathrm{y}n"'] & S \arrow[r, "\gamma"] \arrow[d, "\mu'"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\mu"] \\
\mathrm{y}C \arrow[r, "\delta"'] & R \arrow[r, "\tau"'] & Q
\end{tikzcd}$$

But by the standard pasting properties of pullbacks, this implies there is a unique map $\mathrm{y}D \Rightarrow S$ making the left square a pullback. Hence, $\mu'$ is an $\widehat{\mathcal{M}}$-map and $(\widehat{\mathbf{C}}, \widehat{\mathcal{M}})$ is an $\mathcal{M}$-category.

Note that $\widehat{\mathcal{M}}$ is *not* the smallest stable system of monics generated by the collection $\{\mathrm{y}m \mid m \in \mathcal{M}\}$ (see Lemma 2.4 [3]).

3

# Restriction presheaves

## 3.1 Restriction presheaves

In the previous chapter, we recalled the notion of restriction category and saw there was an equivalence between $\mathbf{rCat}_s$ (the 2-category of split restriction categories) and $\mathcal{M}\mathbf{Cat}$ (the 2-category of $\mathcal{M}$-categories). Cockett and Lack [2, p. 252] uses this equivalence to describe an embedding of any restriction category $\mathbf{X}$ into $\mathsf{Par}\Big(\mathsf{Total}\widehat{(K_r(\mathbf{X}))}, \widehat{\mathcal{M}}_{K_r(\mathbf{X})}\Big)$. The goal of this chapter will be to reformulate this embedding in a more straightforward manner, in terms of the following notion of restriction presheaf.

**Definition 11.** *Let $\mathbf{X}$ be a restriction category. A restriction presheaf on $\mathbf{X}$ is a presheaf $P\colon (U\mathbf{X})^{\mathrm{op}} \to \mathbf{Set}$ together with a family of assignations $\{F_A\}$ (one for each object $A \in \mathbf{X}$), where each $F_A$ is a map*

$$F_A\colon PA \to \mathbf{X}(A, A), \quad x \mapsto \overline{x}$$

*and $\overline{x}$ is a restriction idempotent satisfying the following three axioms:*

*(A1)* $x \cdot \overline{x} = x$

*(A2)* $\overline{x \cdot \overline{f}} = \overline{x} \circ \overline{f}$, *where* $f\colon A \to B$ *in* $\mathbf{X}$

*(A3)* $\overline{x} \circ g = g \circ \overline{x \cdot g}$, *where* $g\colon B \to A$ *in* $\mathbf{X}$

Here, the notation $x \cdot \overline{x}$ denotes the element $P(\overline{x})(x)$ (see Mac Lane and Moerdijk [6, p. 25]). We call the family of assignations $\{F_A\}_{A\in\mathbf{X}}$ the *restriction structure* on $P$, and $\overline{x}$ the restriction of $x$.

**Lemma 12.** *Suppose $P$ is a restriction presheaf on $\mathbf{X}$, and $x \in PA$ (where $A \in \mathbf{X}$). Let $g\colon B \to A$ be an arrow in $\mathbf{X}$. Then*

1. $\overline{g} \circ \overline{x \cdot g} = \overline{x \cdot g}$

2. $\overline{\overline{x} \circ g} = \overline{x \cdot g}$

*Proof.* 1. By (R2), (A2) and (R1),

$$\overline{g} \circ \overline{x \cdot g} = \overline{x \cdot g} \circ \overline{g} = \overline{(x \cdot g) \cdot \overline{g}} = \overline{x \cdot (g \circ \overline{g})} = \overline{x \cdot g}$$

2. By (A3), (R3) and the previous result,

$$\overline{\overline{x} \circ g} = \overline{g \circ \overline{x \cdot g}} = \overline{g} \circ \overline{x \cdot g} = \overline{x \cdot g} \qquad \square$$

**Proposition 13.** *The restriction structure on any presheaf is unique, if it exists.*

*Proof.* Let $P$ be a presheaf on $\mathbf{X}$, and let $\{F_A\}_{A \in \mathbf{X}}$ and $\{G_A\}_{A \in \mathbf{X}}$ be restriction structures on $P$. Let $A \in \mathbf{X}$, $x \in PA$ be arbitrary and let $F_A(x) = \overline{x}$, $G_A(x) = \widetilde{x}$. Then using (A1), (A2) and noting that $\overline{x}$ and $\widetilde{x}$ are restriction idempotents, we get

$$\widetilde{x} = \widetilde{x \cdot \overline{x}} = \widetilde{x} \circ \overline{x} = \overline{x} \circ \widetilde{x} = \overline{x \cdot \widetilde{x}} = \overline{x} \qquad \square$$

We make one more observation regarding the restriction structure of any restriction presheaf.

Let $\mathbf{X}$ be any restriction category and $P$ a presheaf on $\mathbf{X}$. Let $A$ be an object in $\mathbf{X}$ and $x \in PA$. Now denote by $\mathcal{P}$ the partially ordered set of restriction idempotents on $A$ with ordering given by $e \leq e'$ if and only if $e = e'e$. Suppose the restriction idempotent $e \colon A \to A$ satisfies the condition $x \cdot e = x$. Then this implies

$$\overline{x \cdot e} = \overline{x} \circ e = e \circ \overline{x} = \overline{x}$$

or, $\overline{x} \leq e$. Therefore, if $\mathcal{P}$ has a least element, then the restriction of $x$ must be that least element.

## 3.2 The category of restriction presheaves

**Definition 14.** *Let* $\mathbf{X}$ *be a restriction category. We define a new restriction category* $\widehat{\mathbf{X}}_r$ *with the following data:*

- ***Objects****: Restriction presheaves on* $\mathbf{X}$
- ***Arrows****: If* $P, Q$ *are restriction presheaves on* $\mathbf{X}$*, then an arrow is a natural transformation* $\alpha \colon P \Rightarrow Q$.
- ***Restriction****: The restriction on* $\alpha \colon P \Rightarrow Q$, $\overline{\alpha} \colon P \Rightarrow P$*, is given componentwise by*

  $$\overline{\alpha}_A(x) = x \cdot \overline{\alpha_A(x)}$$

  *for every* $A \in \mathbf{X}$ *and* $x \in PA$*. Note that* $\overline{\alpha}$ *is natural since for every* $f \colon B \to A$,

  $$\overline{\alpha}_B(x \cdot f) = x \cdot \left(f \circ \overline{\alpha_B(x \cdot f)}\right) = x \cdot \left(f \circ \overline{\alpha_A(x) \cdot f}\right) = x \cdot \left(\overline{\alpha_A(x)} \circ f\right) = \overline{\alpha}_A(x) \cdot f$$

  *(by axioms and naturality of* $\alpha$*).*

*The identity arrow on $P$ is the identity natural transformation $1_P\colon P \Rightarrow P$, and composition is the usual composition of natural transformations.*

Clearly $\widehat{\mathbf{X}}_r$ is a category, so all that remains is to check it is a restriction category.

(R1) Let $\alpha\colon P \Rightarrow Q$. Then $\alpha\overline{\alpha} = \alpha$ if and only if $\alpha_A(\overline{\alpha}_A(x)) = \alpha_A(x)$ for all $A \in \mathbf{X}$ and $x \in PA$. But

$$\alpha_A(\overline{\alpha}_A(x)) = \alpha_A\left(x \cdot \overline{\alpha_A(x)}\right) = \alpha_A(x) \cdot \overline{\alpha_A(x)} = \alpha_A(x)$$

by (A1) and naturality of $\alpha$. Hence $\alpha\overline{\alpha} = \alpha$.

(R2) Let $\alpha\colon P \Rightarrow Q$ and $\beta\colon P \Rightarrow R$. Then for all $A \in \mathbf{X}$ and $x \in PA$,

$$\begin{aligned}
\overline{\alpha}_A\left(\overline{\beta}_A(x)\right) &= \overline{\alpha}_A\left(x \cdot \overline{\beta_A(x)}\right) = \overline{\alpha}_A(x) \cdot \overline{\beta_A(x)} \\
&= \left(x \cdot \overline{\alpha_A(x)}\right) \cdot \overline{\beta_A(x)} = x \cdot \left(\overline{\alpha_A(x)} \circ \overline{\beta_A(x)}\right) \\
&= x \cdot \left(\overline{\beta_A(x)} \circ \overline{\alpha_A(x)}\right) = \overline{\beta}_A(\overline{\alpha}_A(x))
\end{aligned}$$

So $\overline{\alpha}\overline{\beta} = \overline{\beta}\overline{\alpha}$.

(R3) Let $\alpha, \beta$ be as before. We need to show $\overline{\alpha\overline{\beta}} = \overline{\alpha}\overline{\beta}$. Now for all $A \in \mathbf{X}$ and $x \in PA$,

$$\begin{aligned}
\overline{\alpha\overline{\beta}}_A(x) &= x \cdot \overline{(\alpha\overline{\beta})_A(x)} = x \cdot \overline{\alpha_A\left(\overline{\beta}_A(x)\right)} \\
&= x \cdot \overline{\alpha_A\left(x \cdot \overline{\beta_A(x)}\right)} = x \cdot \overline{\alpha_A(x) \cdot \overline{\beta_A(x)}} \\
&= x \cdot \left(\overline{\alpha_A(x)} \circ \overline{\beta_A(x)}\right) = \overline{\alpha}_A\left(\overline{\beta}_A(x)\right)
\end{aligned}$$

So $\overline{\alpha\overline{\beta}} = \overline{\alpha}\overline{\beta}$.

(R4) Let $\gamma\colon Q \Rightarrow R$. To show $\overline{\gamma}\alpha = \alpha\overline{\gamma\alpha}$, let $A \in \mathbf{X}$ as before, and $x \in PA$. Then

$$\begin{aligned}
(\overline{\gamma}\alpha)_A(x) &= \overline{\gamma}_A(\alpha_A(x)) = \alpha_A(x) \cdot \overline{\gamma_A(\alpha_A(x))} = \alpha_A\left(x \cdot \overline{(\gamma\alpha)_A(x)}\right) \\
&= \alpha_A\left(\overline{\gamma\alpha}_A(x)\right) = \left(\alpha\overline{\gamma\alpha}\right)_A(x)
\end{aligned}$$

Therefore $\overline{\gamma}\alpha = \alpha\overline{\gamma\alpha}$ as required.

Hence, $\overline{\alpha}_A(x) = x \cdot \overline{\alpha_A(x)}$ gives a restriction structure on $\widehat{\mathbf{X}}_r$, making $\widehat{\mathbf{X}}_r$ a restriction category which we call the *category of restriction presheaves* on $\mathbf{X}$.

## 3.3 Total maps in the category of restriction presheaves

Let $\alpha\colon P \Rightarrow Q$ be total in $\widehat{\mathbf{X}}_r$, where $\mathbf{X}$ is any restriction category. Then $\overline{\alpha} = 1_P$, or $\overline{\alpha}_A(x) = 1_{PA}(x) = x$ for all $A \in \mathbf{X}$ and $x \in PA$. That is, $x \cdot \overline{\alpha_A(x)} = x$. But this implies $\overline{x} \leq \overline{\alpha_A(x)}$ since

$$\overline{x} = \overline{x \cdot \overline{\alpha_A(x)}} = \overline{x} \circ \overline{\alpha_A(x)} = \overline{\alpha_A(x)} \circ \overline{x}$$

On the other hand, we have $\overline{\alpha_A(x)} \leq \overline{x}$ as

$$\overline{\alpha_A(x)} = \overline{\alpha_A(x \cdot \overline{x})} = \overline{\alpha_A(x) \cdot \overline{x}} = \overline{\alpha_A(x)} \circ \overline{x} = \overline{x} \circ \overline{\alpha_A(x)}$$

(by naturality of $\alpha$). Therefore, a map $\alpha$ in $\widehat{\mathbf{X}}_r$ is total if and only if $\overline{\alpha_A(x)} = \overline{x}$ for all $A \in \mathbf{X}$ and $x \in PA$. In other words, if and only if $\alpha$ preserves restrictions.

## 3.4 The restriction Yoneda embedding

We now describe a restriction functor $\mathrm{y}_r\colon \mathbf{X} \to \widehat{\mathbf{X}}_r$ which is an analogue of the Yoneda embedding within the restriction setting. On objects, we define $\mathrm{y}_r(A)$ to be the representable $\mathbf{X}(-, A)$ equipped with the restriction operations $\mathbf{X}(B, A) \to \mathbf{X}(B, B)$ from those of $\mathbf{X}$. It is easy to see that the restriction presheaf axioms are satisfied.

To define $\mathrm{y}_r$ on arrows, let $f\colon A \to B$ be in $\mathbf{X}$. Then define $\mathrm{y}_r(f)\colon \mathbf{X}(-, A) \Rightarrow \mathbf{X}(-, B)$ componentwise by $(\mathrm{y}_r f)_X = f \circ (-)$ for each $X \in \mathbf{X}$. All that remains is to show $\mathrm{y}_r$ is a restriction functor. Now for $X \in \mathbf{X}$ and $x \in \mathbf{X}(X, A)$, we have

$$\begin{aligned}(\overline{\mathrm{y}_r f})_X(x) &= x \cdot \overline{(\mathrm{y}_r f)_X(x)} = x \cdot \overline{f \circ x} = (\mathrm{y}_r A)(\overline{f \circ x})(x) \\ &= x \circ \overline{f \circ x} = \overline{f} \circ x = (\mathrm{y}_r \overline{f})_X(x)\end{aligned}$$

by (R4). Therefore, $\overline{\mathrm{y}_r f} = \mathrm{y}_r(\overline{f})$, making $\mathrm{y}_r$ a restriction functor. We call the restriction functor $\mathrm{y}_r$ the *restriction Yoneda embedding*.

We list one further property of $\widehat{\mathbf{X}}_r$.

## 3.5 The category of restriction presheaves is split

**Proposition 15.** *Let* $\mathbf{X}$ *be a restriction category. Then* $\widehat{\mathbf{X}}_r$ *is a split restriction category.*

*Proof.* Suppose $\overline{\alpha}\colon P \Rightarrow P$ is a restriction idempotent in $\widehat{\mathbf{X}}_r$. Since $\widehat{U\mathbf{X}}$ is complete, the equaliser of $1_P$ and $\overline{\alpha}\colon P \Rightarrow P$ exists (which we denote by $\mu\colon Q \Rightarrow P$). Componentwise, for every $A \in \mathbf{X}$, $\mu_A$ is the inclusion $\mu_A\colon QA \to PA$ and $QA = \{x \in PA \mid \overline{\alpha}_A(x) = x\}$.

Now let $\rho\colon P \Rightarrow Q$ be the unique map which makes the following diagram commute:

$$\begin{tikzcd} Q \arrow[r, "\mu"] & P \arrow[r, shift left, "1_P"] \arrow[r, shift right, "\overline{\alpha}"'] & P \\ P \arrow[u, "\rho"] \arrow[ur, "\overline{\alpha}"'] & & \end{tikzcd}$$

By definition, $\mu\rho = \overline{\alpha}$, and precomposing both sides by $\mu$ gives $\mu\rho\mu = \overline{\alpha}\mu = \mu$. Therefore, $\rho\mu = 1_P$ (as $\mu$ is monic) and $\overline{\alpha}\colon P \Rightarrow P$ is split. So all that remains is to show $\overline{\alpha}$ is a split restriction idempotent (by giving $Q$ a restriction structure). But because $QA \subset PA$ for every $A \in \mathbf{X}$, giving $Q$ the same restriction structure as $P$ does the job. (That is, every $x \in QA \subset PA$ will satisfy axioms (A1)-(A3)). Therefore, $\widehat{\mathbf{X}}_r$ is a split restriction category. □

Before moving on to the main results of this chapter, let us recall the restriction category $K_r(\mathbf{X})$ whose objects are pairs $(A, e_A)$ (with $e_A$ a restriction idempotent), and whose arrows are of the form $(e_A, f, e_B)\colon (A, e_A) \to (B, e_B)$ (with $f$ satisfying the condition $e_B f e_A = f$), and the restriction functor $J\colon \mathbf{X} \to K_r(\mathbf{X})$, given on objects by $JA = (A, 1_A)$ and on arrows by $Jf = (1_A, f, 1_B)$ (with $f\colon A \to B \in \mathbf{X}$).

**Proposition 16.** *The categories* $\widehat{\mathbf{X}}_r$ *and* $\widehat{K_r(\mathbf{X})}_r$ *are equivalent.*

*Proof.* To show these categories are equivalent, we need a restriction functor $F\colon \widehat{K_r(\mathbf{X})}_r \to \widehat{\mathbf{X}}_r$ which is fully faithful and essentially surjective. So let us define $F$ in the following way:

$$\begin{tikzcd}
\widehat{K_r(\mathbf{X})}_r \arrow[r, "F"] \arrow[d, hook] & \widehat{\mathbf{X}}_r \arrow[d, hook] \\
\widehat{K_r(\mathbf{X})} \arrow[r, "(-)\circ J^{\mathrm{op}}"'] & \widehat{\mathbf{X}}
\end{tikzcd}$$

It is well-known that the bottom arrow is an equivalence of categories. We aim to show that it restricts back to an equivalence $F$ (as displayed) between the corresponding subcategories of restriction presheaves.

On objects, we must have $FP = P \circ J^{\mathrm{op}}$. To see that this is a restriction presheaf on $\mathbf{X}$, we need to show for every $A \in \mathbf{X}$ and $x \in (PJ^{\mathrm{op}})(A) = P(A, 1_A)$, there is a restriction idempotent $\overline{x}\colon A \to A$ in $\mathbf{X}$ satisfying axioms (A1)-(A3). But we know that $P$ is a restriction presheaf on $K_r(\mathbf{X})$. That is, for $(A, 1_A) \in K_r(\mathbf{X})$ and $y \in P(A, 1_A)$, there is a restriction idempotent $\overline{y}\colon (A, 1_A) \to (A, 1_A)$ of the form $\overline{y} = (1_A, f, 1_A)$ satisfying the relevant axioms (where $f \in \mathbf{X}$). Note that $f$ is a restriction idempotent in $\mathbf{X}$ since $\overline{\overline{y}} = \overline{y}$ implies $(1_A, \overline{f}, 1_A) = (1_A, f, 1_A)$. Therefore, taking $\overline{x} = f$ ensures that $P \circ J^{\mathrm{op}}$ is a restriction presheaf on $\mathbf{X}$.

Now on arrows $\alpha\colon P \Rightarrow Q$ in $\widehat{K_r(\mathbf{X})}_r$, we must have $F(\alpha) = \alpha \circ J^{\mathrm{op}}$. To see that this makes $F$ a restriction functor, let $A \in \mathbf{X}$ and $x \in P(A, 1_A)$ be arbitrary. Then

$$(\overline{\alpha} \circ J^{\mathrm{op}})_A(x) = \overline{\alpha}_{(A,1_A)}(x) = x \cdot \overline{\alpha_{(A,1_A)}(x)} = x \cdot \overline{(\alpha \circ J^{\mathrm{op}})_A(x)} = \left(\overline{\alpha \circ J^{\mathrm{op}}}\right)_A(x)$$

implies $F(\overline{\alpha}) = \overline{F(\alpha)}$, and so $F$ is a restriction functor.

Since $\widehat{K_r(\mathbf{X})} \simeq \widehat{\mathbf{X}}$ in **Cat**, such an $F$ must be fully faithful. Hence, it remains to show $F$ is essentially surjective. That is, given a restriction presheaf $P\colon \mathbf{X}^{\mathrm{op}} \to \mathbf{Set}$, there is a $\widetilde{P}\colon K_r(\mathbf{X})^{\mathrm{op}} \to \mathbf{Set}$ such that $\widetilde{P} \circ J^{\mathrm{op}} \cong P$. So define $\widetilde{P}$ on objects $(A, e_A) \in K_r(\mathbf{X})$ by $\widetilde{P}(A, e_A) = \{x \in PA \mid x \cdot e_A = x\}$. [Note this implies $\widetilde{P}(A, 1_A) = PA$]. For an arrow $(e_A, f, e_B)\colon (A, e_A) \to (B, e_B)$, we need $\widetilde{P}(e_A, f, e_B)$ to be a function $\widetilde{P}(B, e_B) \to \widetilde{P}(A, e_A)$. Knowing that $Pf\colon PB \to PA$ and $\widetilde{P}(A, e_A) \subset PA$, define $\widetilde{P}(e_A, f, e_B) = Pf$. We just need to check for all $x \in \widetilde{P}(B, e_B)$, $(Pf)(x) \in \widetilde{P}(A, e_A)$. But $f \circ e_A = (e_B f e_A)e_A = f$ implies

$$(x \cdot f) \cdot e_A = x \cdot (f \circ e_A) = x \cdot f$$

Therefore $(Pf)(x) \in \widetilde{P}(A, e_A)$, making $\widetilde{P}$ a presheaf on $K_r(\mathbf{X})$, with $\widetilde{P} \circ J^{\mathrm{op}} = P$. If we can define a restriction structure on $\widetilde{P}$, then $F$ would be essentially surjective.

To do this, let $(A, e) \in K_r(\mathbf{X})$ and $x \in \widetilde{P}(A, e)$ be arbitrary. Now because $x \in \widetilde{P}(A, e) \subset PA$ and $P$ is a restriction presheaf, there exists a restriction idempotent $\overline{x}\colon A \to A$ associated to $x$. So define the restriction of $x \in \widetilde{P}(A, e)$ to be $\widetilde{x} = (e, \overline{x}, e)$. First, observe that $\widetilde{x}$ is an idempotent. Second, $\widetilde{x}$ is an arrow in $K_r(\mathbf{X})$ since

$$e\overline{x}e = ee\overline{x} = e\overline{x} = \overline{x}e = \overline{x \cdot e} = \overline{x}$$

Therefore, it remains to show that $\widetilde{x} = (e, \overline{x}, e)$ satisfies the restriction presheaf axioms.

Now $x \cdot \widetilde{x} = x \cdot (e, \overline{x}, e) = \widetilde{P}(e, \overline{x}, e)(x) = P(\overline{x})(x) = x \cdot \overline{x} = x$. Also, given an arrow $(e_A, f, e_B)\colon (A, e_A) \to (B, e_B)$ in $K_r(\mathbf{X})$, we have

$$\begin{aligned}
x \cdot \overline{\widetilde{(e_A, f, e_B)}} &= x \cdot \widetilde{(e_A, \overline{f e_A}, e_A)} = \widetilde{x \cdot \overline{f e_A}} = (e_A, \overline{x \cdot \overline{f e_A}}, e_A) = (e_A, \overline{(x \cdot e_A)\overline{f}}, e_A) \\
&= (e_A, \overline{x \cdot \overline{f}}, e_A) = (e_A, \overline{x} \circ \overline{f}, e_A) = (e_A, \overline{x}, e_A) \circ (e_A, \overline{f}, e_A) \\
&= \widetilde{x} \circ (e_A, \overline{f e_A}, e_A) = \widetilde{x} \circ \overline{(e_A, f, e_B)}
\end{aligned}$$

Finally, suppose $(e_B, g, e_A)\colon (B, e_B) \to (A, e_A)$ is another arrow in $K_r(\mathbf{X})$. Then

$$\begin{aligned}
\widetilde{x} \circ (e_B, g, e_A) &= (e_A, \overline{x}, e_A) \circ (e_B, g, e_A) = (e_B, \overline{x} \circ g, e_A) = (e_B, g \circ \overline{x \cdot g}, e_A) \\
&= (e_B, g, e_A) \circ (e_B, \overline{x \cdot g}, e_B) = (e_B, g, e_A) \circ \widetilde{x \cdot g} \\
&= (e_B, g, e_A) \circ x \cdot \widetilde{(e_B, g, e_A)}
\end{aligned}$$

Therefore, $\widetilde{x}$ defines a restriction structure on $\widetilde{P}$, making $\widetilde{P}$ a restriction presheaf. Hence, $\widehat{\mathbf{X}}_r \simeq \widehat{K_r(\mathbf{X})}_r$. □

## 3.6 An equivalence in Cat

Cockett and Lack [2, p. 252] describes the following chain of embeddings of any restriction category $\mathbf{X}$. First we take $\mathbf{X}$ to $K_r(\mathbf{X})$ via the functor $J$ (taking objects $A$ to $(A, 1_A)$). Next, by the fact that $\mathcal{M}\mathsf{Total}$ and $\mathsf{Par}$ are part of an equivalence of 2-categories, there is an isomorphism $\Phi_{K_r(\mathbf{X})}$ taking $K_r(\mathbf{X})$ to the restriction category $\mathsf{Par}\big(\mathsf{Total}(K_r(\mathbf{X})), \mathcal{M}_{K_r(\mathbf{X})}\big)$. We then compose with the restriction functor $\mathsf{Par}(\mathrm{y})$ into $\mathsf{Par}\big(\widehat{\mathsf{Total}(K_r(\mathbf{X}))}, \widehat{\mathcal{M}}_{K_r(\mathbf{X})}\big)$. This yields the following composite embedding of $\mathbf{X}$:

$$\mathbf{X} \xrightarrow{J} K_r(\mathbf{X}) \xrightarrow{\Phi_{K_r(\mathbf{X})}} \mathsf{Par}\big(\mathsf{Total}(K_r(\mathbf{X})), \mathcal{M}_{K_r(\mathbf{X})}\big) \xrightarrow{\mathsf{Par}(\mathrm{y})} \mathsf{Par}\big(\widehat{\mathsf{Total}(K_r(\mathbf{X}))}, \widehat{\mathcal{M}}_{K_r(\mathbf{X})}\big)$$

However, we have already seen that the restriction Yoneda embedding $\mathrm{y}_r$ gives an embedding of any restriction category $\mathbf{X}$ into the restriction presheaf category $\widehat{\mathbf{X}}_r$. The remainder of this chapter will therefore be devoted to showing that $\mathrm{y}_r$ is the same embedding described by Cockett and Lack (up to an equivalence). To do this, we prove two important results; the first is an equivalence in **Cat**, and the second is an equivalence in $\mathcal{M}$**Cat**.

**Theorem 17.** *Let* $\mathbf{C}$ *be any category and* $\mathcal{M}$ *a stable system of monics in* $\mathbf{C}$. *Then*

$$\widehat{\mathbf{C}} \simeq \mathsf{Total}\left(\widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})}_r\right)$$

*Proof.* We begin by defining a functor $F\colon \widehat{\mathbf{C}} \to \mathsf{Total}\left(\widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})}_r\right)$ on objects $P \in \widehat{\mathbf{C}}$. Let $F(P) = \widetilde{P}$, where $\widetilde{P}\colon \mathsf{Par}(\mathbf{C}, \mathcal{M})^{\mathrm{op}} \to \mathbf{Set}$ has the following data:

- **Objects**: If $X$ is an object of $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, then $\widetilde{P}(X)$ is the set of equivalence classes

  $$\widetilde{P}(X) = \Big\{[(Y, m, f)] \mid Y \in \mathsf{Par}(\mathbf{C}, \mathcal{M}), m \in \mathcal{M}, f \in PY\Big\}$$

  where $(Y, m, f) \sim (Z, n, g)$ if there exists an isomorphism $\varphi\colon Z \to Y$ such that $n = m\varphi$ and $g = f \cdot \varphi$. Alternatively, we may think of an element in $\widetilde{P}(X)$ as a span from $X$ to $\star$

$$\begin{tikzcd} & Y \arrow[dl, "m"'] \arrow[dr, "f"] & \\ X & & \star \end{tikzcd}$$

where $\star$ is some formal object outside of $\mathbf{C}$. Again, we will write $(Y, m, f)$ instead of $[(Y, m, f)]$ where the context is clear.

- **Arrows**: If $(B, n, g)\colon Z \to X$ is an arrow in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ and $(Y, m, f) \in \widetilde{P}(X)$, then define
$$\left(\widetilde{P}(B, n, g)\right)(Y, m, f) = (B \times_X Y, nm', f \cdot g')$$
where $(B \times_X Y, m', g')$ is a pullback of $m$ along $g$ and $f \cdot g' = (Pg')(f)$.

$$\begin{tikzcd} & & B \times_X Y \arrow[dl, "m'"'] \arrow[dr, "g'"] & & \\ & B \arrow[dl, "n"'] \arrow[dr, "g"] & & Y \arrow[dl, "m"'] \arrow[dr, "f"] & \\ Z & & X & & \star \end{tikzcd}$$

Informally, we will sometimes denote $\left(\widetilde{P}(B, n, g)\right)(Y, m, f)$ by $(Y, m, f) \cdot (B, n, g)$ for notational purposes.

The restriction on each $(Y, m, f) \in \widetilde{P}(X)$ is to be given by $\overline{(Y, m, f)} = (Y, m, m)$. This makes $F(P) = \widetilde{P}$ a restriction presheaf.

Now suppose $\alpha\colon P \Rightarrow Q$ is an arrow in $\widehat{\mathbf{C}}$. Let $F(\alpha) = \widetilde{\alpha}$, whose component at $X \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$ is defined as follows:

$$\widetilde{\alpha}_X(Y, m, f) = \left(Y, m, \alpha_Y(f)\right)$$

(Note that $\alpha_Y(f) \in QY$). To show that $\widetilde{\alpha}$ is natural, we need to show the following diagram commutes:

$$\begin{tikzcd} \widetilde{P}X \arrow[r, "\widetilde{\alpha}_X"] \arrow[d, "\widetilde{P}(B,n,g)"'] & \widetilde{Q}X \arrow[d, "\widetilde{Q}(B,n,g)"] \\ \widetilde{P}Z \arrow[r, "\widetilde{\alpha}_Z"'] & \widetilde{Q}Z \end{tikzcd}$$

$$\begin{array}{ccc} (Y, m, f) & \longmapsto & \left(Y, m, \alpha_Y(f)\right) \\ \downarrow & & \downarrow \\ (B \times_X Y, nm', f \cdot g') \longmapsto \left(B \times_X Y, nm', \alpha_{B \times_X Y}(f \cdot g')\right) & \overset{?}{=} & \left(B \times_X Y, nm', \alpha_Y(f) \cdot g'\right) \end{array}$$

But $\alpha_{B \times_X Y}(f \cdot g') = \alpha_Y(f) \cdot g'$ by naturality of $\alpha$, and so $\widetilde{\alpha}$ is natural. It remains to check that $\widetilde{\alpha}$ is total. That is, $\overline{\widetilde{\alpha}_X(Y, m, f)} = \overline{(Y, m, f)}$ for every $X \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$ and $(Y, m, f) \in \widetilde{P}X$. But

$$\overline{\widetilde{\alpha}_X(Y, m, f)} = \overline{\left(Y, m, \alpha_Y(f)\right)} = (Y, m, m) = \overline{(Y, m, f)}$$

Hence, $F\colon \widehat{\mathbf{C}} \to \mathsf{Total}\left(\widehat{\mathsf{Par}(\mathbf{C},\mathcal{M})}_r\right)$ is a functor. (Checking that $F$ maps identities to identities and preserves composition is trivial).

The next step is to define a functor $G\colon \mathsf{Total}\left(\widehat{\mathsf{Par}(\mathbf{C},\mathcal{M})}_r\right) \to \widehat{\mathbf{C}}$ in the other direction. So let $P$ be an object in $\widehat{\mathsf{Par}(\mathbf{C},\mathcal{M})}_r$ and let $G(P) = \dot{P}$, where $\dot{P}$ has the following data:

- **Objects**: If $X$ is an object of $\mathbf{C}$ (and hence an object in $\mathsf{Par}(\mathbf{C},\mathcal{M})$), then

$$\dot{P}(X) = \left\{x \mid x \in PX, \overline{x} = 1_X \in \mathsf{Par}(\mathbf{C},\mathcal{M})\right\}$$

  Note that $\dot{P}X \subset PX$.

- **Arrows**: If $f\colon Z \to X$ is an arrow in $\mathbf{C}$, then

$$\dot{P}(f) = P(Z, 1_Z, f)$$

  To see that $\dot{P}(f)$ is well-defined, first observe that $P(Z, 1_Z, f)$ is a function from $PX$ to $PZ$. Now let $x \in \dot{P}X$. Then

$$\overline{P(Z, 1_Z, f)(x)} = \overline{x \cdot (Z, 1_Z, f)} = \overline{\overline{x} \circ (Z, 1_Z, f)} = 1_Z$$

  and so $\dot{P}(f)$ is a function from $\dot{P}X$ to $\dot{P}Z$.

Clearly $\dot{P}$ is a presheaf on $\mathbf{C}$. To define $G$ on arrows $\alpha\colon P \Rightarrow Q$, let $G(\alpha) = \dot{\alpha}$ and define $\dot{\alpha}$ componentwise by

$$\dot{\alpha}_X(x) = \alpha_X(x)$$

Note that $\alpha_X(x) \in \dot{Q}X$ since $\alpha$ total implies $\overline{\alpha_X(x)} = \overline{x} = 1_X$ (as $x \in \dot{P}X$). It is then easy to check that $G$ is a functor. We now show that $F$ and $G$ are part of an equivalence of categories. That is, there exist natural isomorphisms $\eta\colon 1_{\widehat{\mathbf{C}}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow 1_{\mathsf{Total}\left(\widehat{\mathsf{Par}(\mathbf{C},\mathcal{M})}_r\right)}$.

First, to define $\eta$, we need to give its components $\eta_P\colon P \to GF(P)$ for every presheaf $P$. This means we need to give natural isomorphisms $(\eta_P)_X\colon PX \to \dot{\tilde{P}}X$ for every $X \in \mathbf{C}$. But

$$\dot{\tilde{P}}X = \left\{(Y, m, f) \mid \overline{(Y, m, f)} = (X, 1_X, 1_X), f \in PX\right\} = \left\{(X, 1_X, f) \mid f \in PX\right\}$$

So clearly $(\eta_P)_X$ defined by $(\eta_P)_X(f) = (X, 1_X, f)$ is an isomorphism (for every $f \in PX$).

To show naturality of $\eta_P$, let $P \in \widehat{\mathbf{C}}$ be given and suppose $g\colon Z \to X$ is an arrow in $\mathbf{C}$. Then because $(X, 1_X, f) \cdot (Z, 1_Z, g) = (Z, 1_Z, f \cdot g)$ in $\mathsf{Par}(\mathbf{C},\mathcal{M})$ (for any $f \in PX$), the following diagram commutes:

$$\begin{tikzcd}
PX \arrow[r, "(\eta_P)_X"] \arrow[d, "Pg"'] & \dot{\tilde{P}}X \arrow[d, "\dot{\tilde{P}}g = (-)\cdot(Z,1_Z,g)"] \\
PZ \arrow[r, "(\eta_P)_Z"'] & \dot{\tilde{P}}Z
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & (X, 1_X, f) \arrow[d, mapsto] \\
f \cdot g \arrow[r, mapsto] & (Z, 1_Z, f \cdot g)
\end{tikzcd}$$

Therefore, $\eta_P$ is natural in $X$ for every presheaf $P$. To show that $\eta$ is natural in $P$, we need to show the following diagram commutes for all arrows $\alpha\colon P \Rightarrow Q$ in $\widehat{\mathbf{C}}$:

$$
\begin{tikzcd}
P \arrow[r, "\eta_P"] \arrow[d, "\alpha"'] & \dot{\widetilde{P}} \arrow[d, "\dot{\widetilde{\alpha}}"] \\
Q \arrow[r, "\eta_Q"'] & \dot{\widetilde{Q}}
\end{tikzcd}
$$

That is, for every $X \in \mathbf{C}$, we have $(\eta_Q \circ \alpha)_X = (\dot{\widetilde{\alpha}} \circ \eta_P)_X$, or

$$(\eta_Q)_X\big(\alpha_X(f)\big) = \dot{\widetilde{\alpha}}_X\big((\eta_P)_X(f)\big)$$

for all $f \in PX$. But

$$\dot{\widetilde{\alpha}}_X\big((\eta_P)_X(f)\big) = \widetilde{\alpha}_X\big((\eta_P)_X(f)\big) = \widetilde{\alpha}_X(X, 1_X, f) = \big(X, 1_X, \alpha_X(f)\big) = (\eta_Q)_X\big(\alpha_X(f)\big)$$

Therefore, $\eta$ is natural in $P$, and hence $\eta\colon 1_{\widehat{\mathbf{C}}} \Rightarrow GF$ is a natural isomorphism.

Likewise, to define $\varepsilon$, we need to give components $\varepsilon_P\colon FG(P) \Rightarrow P$ for every restriction presheaf $P\colon \mathsf{Par}(\mathbf{C}, \mathcal{M})^{\mathrm{op}} \to \mathbf{Set}$. That is, we need to give natural isomorphisms $(\varepsilon_P)_X\colon \widetilde{\dot{P}}X \to PX$ for every $X \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$. By definition,

$$\widetilde{\dot{P}}X = \Big\{(Y, m, f) \mid Y \in \mathsf{Par}(\mathbf{C}, \mathcal{M}), m \in \mathcal{M}, f \in PY, \overline{f} = 1_Y\Big\}$$

Now for every $(Y, m, f) \in \widetilde{\dot{P}}X$, there is an arrow $(Y, m, 1_Y)\colon X \to Y$ in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ and hence a function $P(Y, m, 1_Y)\colon PY \to PX$. So define $(\varepsilon_P)_X$ by

$$(\varepsilon_P)_X(Y, m, f) = f \cdot (Y, m, 1_Y)$$

To define its inverse $(\varepsilon_P)_X^{-1}$, let $x \in PX$ so that $\overline{x} = (Z, n, n)$ for some $Z \in \mathbf{C}$ and $n \in \mathcal{M}$. This gives an arrow $(Z, 1_Z, n)\colon Z \to X$ in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ and a function $P(Z, 1_Z, n)\colon PX \to PZ$. Since

$$\overline{x \cdot (Z, 1_Z, n)} = \overline{\overline{x} \circ (Z, 1_Z, n)} = \overline{(Z, n, n) \circ (Z, 1_Z, n)} = \overline{(Z, 1_Z, n)} = 1_Z$$

define $(\varepsilon_P)_X^{-1}\colon PX \to \widetilde{\dot{P}}(X)$ by

$$(\varepsilon_P)_X^{-1}(x) = \big(Z, n, x \cdot (Z, 1_Z, n)\big)$$

To see that $(\varepsilon_P)_X^{-1}$ really is the inverse of $(\varepsilon_P)_X$, first note that for all $(Y, m, f) \in \widetilde{\dot{P}}X$,

$$\overline{f \cdot (Y, m, 1_Y)} = \overline{\overline{f} \circ (Y, m, 1_Y)} = \overline{(Y, m, 1_Y)} = (Y, m, m)$$

as $\overline{f} = 1_Y$, and so

$$
\begin{aligned}
(\varepsilon_P)_X^{-1}\big((\varepsilon_P)_X(Y, m, f)\big) &= (\varepsilon_P)_X^{-1}\big(f \cdot (Y, m, 1_Y)\big) = \Big(Y, m, \big(f \cdot (Y, m, 1_Y)\big) \cdot (Y, 1_Y, m)\Big) \\
&= \Big(Y, m, f \cdot \big((Y, m, 1_Y) \circ (Y, 1_Y, m)\big)\Big) = \big(Y, m, f \cdot (Y, 1_Y, 1_Y)\big) \\
&= (Y, m, f)
\end{aligned}
$$

Similarly, for $x \in PX$ and $\overline{x} = (Z, n, n)$,

$$\begin{aligned}(\varepsilon_P)_X\Big((\varepsilon_P)_X^{-1}(x)\Big) &= (\varepsilon_P)_X\Big(Z, n, x \cdot (Z, 1_Z, n)\Big) = \Big(x \cdot (Z, 1_Z, n)\Big) \cdot (Z, n, 1_Z)\\ &= x \cdot \Big((Z, 1_Z, n) \circ (Z, n, 1_Z)\Big) = x \cdot (Z, n, n) = x \cdot \overline{x}\\ &= x\end{aligned}$$

Therefore, $(\varepsilon_P)_X \colon \widetilde{\dot{P}}X \to PX$ is an isomorphism. To show naturality of $\varepsilon_P$ in $X$, we need to show the following diagram commutes for all arrows $(B, n, g) \colon Z \to X$ in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$:

$$\begin{tikzcd} \widetilde{\dot{P}}X \arrow[r, "(\varepsilon_P)_X"] \arrow[d, "\widetilde{\dot{P}}(B,n,g) = (-)\cdot(B,n,g)"'] & PX \arrow[d, "P(B,n,g)"] \\ \widetilde{\dot{P}}Z \arrow[r, "(\varepsilon_P)_Z"'] & PZ \end{tikzcd}$$

Figure 3.1: Naturality of $\varepsilon_P$

So let $(Y, m, f) \in \widetilde{\dot{P}}X$, and let $(B \times_X Y, m', g')$ be a pullback of $m$ along $g$:

$$\begin{tikzcd} B \times_X Y \arrow[r, "g'"] \arrow[d, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "m"] \\ B \arrow[r, "g"'] & X \end{tikzcd}$$

Then the top composite from Figure 3.1 is given by

$$f \cdot \Big((Y, m, 1_Y) \circ (B, n, g)\Big) = f \cdot (B \times_X Y, nm', g')$$

On the other hand, the bottom composite is given by

$$\begin{aligned}(\varepsilon_P)_Z\Big(B \times_X Y, nm', (\dot{P}g')f\Big) &= (\varepsilon_P)_Z\Big(B \times_X Y, nm', f \cdot (B \times_X Y, 1, g')\Big)\\ &= f \cdot \Big((B \times_X Y, 1, g') \circ (B \times_X Y, nm', 1)\Big)\\ &= f \cdot (B \times_X Y, nm', g')\end{aligned}$$

Therefore, the square in Figure 3.1 commutes and so $\varepsilon_P$ is natural in $X$.

Finally, to show $\varepsilon$ is natural in $P$, let $\alpha \colon P \Rightarrow Q$ be an arrow in $\mathsf{Total}\Big(\widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})_r}\Big)$. We need to show the following diagram commutes:

$$\begin{tikzcd} \widetilde{\dot{P}} \arrow[r, "\varepsilon_P"] \arrow[d, "\widetilde{\dot{\alpha}}"'] & P \arrow[d, "\alpha"] \\ \widetilde{\dot{Q}} \arrow[r, "\varepsilon_Q"'] & Q \end{tikzcd}$$

That is, for all $X \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$, $(\alpha \circ \varepsilon_P)_X = (\varepsilon_Q \circ \widetilde{\dot{\alpha}})_X$ or

$$\alpha_X\Big((\varepsilon_P)_X(Y, m, f)\Big) = (\varepsilon_Q)_X\left(\widetilde{\dot{\alpha}}_X(Y, m, f)\right)$$

for all $(Y, m, f) \in \tilde{\dot{P}}X$. But

$$\begin{aligned}(\varepsilon_Q)_X\Big(\widetilde{\alpha}_X(Y, m, f)\Big) &= (\varepsilon_Q)_X\Big(Y, m, \dot{\alpha}_Y(f)\Big) = (\varepsilon_Q)_X\Big(Y, m, \alpha_Y(f)\Big) \\ &= \alpha_Y(f) \cdot (Y, m, 1_Y) = \alpha_X\Big(f \cdot (Y, m, 1_Y)\Big) = \alpha_X\Big((\varepsilon_P)_X(Y, m, f)\Big)\end{aligned}$$

by naturality of $\alpha$. Therefore, $\varepsilon\colon FG \Rightarrow 1_{\mathsf{Total}\left(\mathsf{Par}\widehat{(\mathbf{C},\mathcal{M})}_r\right)}$ is a natural isomorphism and the following categories are equivalent:

$$\widehat{\mathbf{C}} \simeq \mathsf{Total}\left(\mathsf{Par}\widehat{(\mathbf{C}, \mathcal{M})}_r\right)$$ □

## 3.7 An equivalence in $\mathcal{M}$Cat

The following lemma will be useful in proving our second equivalence.

**Lemma 18.** *Let $m\colon A \to B$ be a monomorphism in some category $\mathbf{C}$, and suppose the pullback of $m$ along $f\colon C \to B$ exists.*

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "n"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "m"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

*Then $n$ is an isomorphism if and only if $f = mg$ (for some unique $g\colon C \to A$).*

*Proof.* ($\Rightarrow$) If $n$ is an isomorphism, define $g = hn^{-1}$. Such a $g$ is automatically unique since $m$ is monic.

($\Leftarrow$) Suppose $f = mg$ and consider the following diagram.

$$\begin{tikzcd} C \arrow[dr, dashed, "n'"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "1_C"'] & & \\ & D \arrow[r, "h"] \arrow[d, "n"'] & A \arrow[d, "m"] \\ & C \arrow[r, "f"'] & B \end{tikzcd}$$

By definition, $nn' = 1_C$, and precomposing by $n$ gives $nn'n = n$. But $n$ monic implies $n'n = 1_D$, and so $n$ is an isomorphism. □

**Theorem 19.** *Let $\mathbf{C}$ be a category and $\mathcal{M}$ a stable system of monics in $\mathbf{C}$. Then*

$$\left(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}\right) \simeq \mathcal{M}\mathsf{Total}\left(\mathsf{Par}\widehat{(\mathbf{C}, \mathcal{M})}_r\right)$$

*Proof.* To show equivalence, we need to give two $\mathcal{M}$-functors (one in each direction). First consider the functor $F\colon \widehat{\mathbf{C}} \to \mathsf{Total}\left(\mathsf{Par}\widehat{(\mathbf{C}, \mathcal{M})}_r\right)$ defined in Theorem 17. If we can show

$F$ takes $\widehat{\mathcal{M}}$-maps to restriction monics in $\widehat{\mathsf{Par}(\mathbf{C},\mathcal{M})}_r$, then $F$ is an $\mathcal{M}$-functor. (Note that $F$ automatically preserves $\widehat{\mathcal{M}}$-pullbacks since $F$ is right adjoint to $G$).

So let $\mu\colon P \Rightarrow Q$ be an $\widehat{\mathcal{M}}$-map. To show $F(\mu) = \widetilde{\mu}\colon \widetilde{P} \Rightarrow \widetilde{Q}$ is a restriction monic, we need to show $\widetilde{\mu}$ is an equaliser of $1_{\widetilde{Q}}$ and $\alpha\colon \widetilde{Q} \Rightarrow \widetilde{Q}$ for some restriction idempotent $\alpha$. To define $\alpha$, let $X \in \mathsf{Par}(\mathbf{C},\mathcal{M})$ and $(Z,n,g) \in \widetilde{Q}X$. Now $g \in QZ$, and so by Yoneda, there exists a corresponding natural transformation $\langle g\rangle\colon yZ \Rightarrow Q$. By definition of an $\widehat{\mathcal{M}}$-map, there exists an arrow $m_g\colon B \to Z$ in $\mathcal{M}$ making the following square a pullback:

$$\begin{tikzcd} \mathrm{y}B \arrow[r] \arrow[d, "\mathrm{y}m_g"'] \arrow[dr, phantom, very near start, "\lrcorner"] & P \arrow[d, "\mu"] \\ \mathrm{y}Z \arrow[r, "\langle g\rangle"'] & Q \end{tikzcd}$$

So define $\alpha$ by its components $\alpha_X\colon \widetilde{Q}X \to \widetilde{Q}X$, where

$$\alpha_X(Z,n,g) = (B, nm_g, g\cdot m_g)$$

(Observe that $nm_g \in \mathcal{M}$ as both $n, m_g \in \mathcal{M}$. Also, $g\cdot m_g \in QB$ since $Q(m_g)\colon QZ \to QB$).

To see that $\alpha_X$ is well-defined (and hence $\alpha$), we need to show that if $m'_g\colon B' \to Z$ satisfies the same condition as $m_g$, then $(B, nm_g, g\cdot m_g) = (B', nm'_g, g\cdot m'_g)$. This will be true if there exists an isomorphism $\varphi\colon B \to B'$ such that $m_g = m'_g \circ \varphi$. By definition, $\mathrm{y}m_g$ and $\mathrm{y}m'_g$ are both pullbacks of $\mu$ along $\langle g\rangle$. Therefore, there exists an isomorphism $\psi\colon \mathrm{y}B \to \mathrm{y}B'$ such that $\mathrm{y}m_g = \mathrm{y}m'_g \circ \psi$. As y is full and faithful, this implies the existence of an isomomorphism $\varphi\colon B \to B'$ such that $m_g = m'_g \circ \varphi$. Therefore $\alpha$ is well-defined.

We now show that $\alpha$ is natural and is a restriction idempotent. To complete the proof that $\widetilde{\mu}$ is a restriction monic, we show $\widetilde{\mu}$ equalises $1_{\widetilde{Q}}$ and $\alpha$.

**1. Naturality of $\alpha$**

To see that $\alpha$ is natural, let $(A,m,h)\colon Y \to X$ be an arrow in $\mathsf{Par}(\mathbf{C},\mathcal{M})$. We need to show the following diagram commutes:

$$\begin{tikzcd} \widetilde{Q}X \arrow[r, "\alpha_X"] \arrow[d, "\widetilde{Q}(A,m,h) = (-)\cdot(A,m,h)"'] & \widetilde{Q}X \arrow[d, "\widetilde{Q}(A,m,h)"] \\ \widetilde{Q}Y \arrow[r, "\alpha_Y"'] & \widetilde{Q}Y \end{tikzcd}$$

Figure 3.2: Naturality square of $\alpha$

Let $(Z,n,g) \in \widetilde{Q}X$, and consider $\widetilde{Q}(A,m,h)(Z,n,g) = (Z,n,g)\cdot(A,m,h)$. Suppose $(A \times_X Z, m', g')$ is a pullback of $n$ along $h$:

$$\begin{tikzcd}
& & A \times_X Z \arrow[dl, "m'"'] \arrow[dr, "g'"] & & \\
& A \arrow[dl, "m"'] \arrow[dr, "h"] & & Z \arrow[dl, "n"'] \arrow[dr, "g"] & \\
Y & & X & & \star
\end{tikzcd}$$

Then $\widetilde{Q}(A, m, h)(Z, n, g) = (A \times_X Z, mm', g \cdot g')$. Applying $\alpha_Y$ gives

$$\alpha_Y(A \times_X Z, mm', g \cdot g') = \big(B', mm'm_{g \cdot g'}, g \cdot (g'm_{g \cdot g'})\big)$$

for some $B'$ and $m_{g \cdot g'} \colon B' \to A \times_X Z$. Now since $m_g \in \mathcal{M}$, we may consider a pullback of $m_g$ along $g'$:

$$\begin{tikzcd}
(A \times_X Z) \times_Z B \arrow[r, "g''"] \arrow[d, "m'_g"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "m_g"] \\
A \times_X Z \arrow[r, "g'"'] & Z
\end{tikzcd}$$

By the special pasting properties of pullbacks and the fact that the Yoneda embedding preserves all pullbacks, the outer square below is a pullback:

$$\begin{tikzcd}
\mathrm{y}(A \times_X B) \arrow[r, "\mathrm{y}g''"] \arrow[d, "\mathrm{y}m'_g"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{y}B \arrow[r] \arrow[d, "\mathrm{y}m_g"] \arrow[dr, phantom, very near start, "\lrcorner"] & P \arrow[d, "\mu"] \\
\mathrm{y}(A \times_X Z) \arrow[r, "\mathrm{y}g'"'] & \mathrm{y}Z \arrow[r, "\langle g \rangle"'] & Q
\end{tikzcd}$$

But since $\langle g \rangle \circ \mathrm{y}g' = \langle g \cdot g' \rangle$ by naturality of the bijection $QZ \cong \widehat{\mathbf{C}}(\mathrm{y}Z, Q)$ in $Z$, there is an isomorphism $\delta$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{y}B' \arrow[dr, "\delta"] \arrow[drr, bend left] \arrow[ddr, bend right, "\mathrm{y}m_{g \cdot g'}"'] & & \\
& \mathrm{y}(A \times_X B) \arrow[r] \arrow[d, "\mathrm{y}m'_g"'] & P \arrow[d, "\mu"] \\
& \mathrm{y}(A \times_X Z) \arrow[r, "\langle g \cdot g' \rangle"'] & Q
\end{tikzcd}$$

And because y is full and faithful, there is a unique isomorphism $d \colon B' \to A \times_X B$ (with $\mathrm{y}d = \delta$) making the diagram below commute:

$$\begin{tikzcd}
B' \arrow[r, "d"] \arrow[dr, "m_{g \cdot g'}"'] & A \times_X B \arrow[d, "m'_g"] \\
& A \times_X Z
\end{tikzcd}$$

Now consider the top composite in Figure 3.2. By definition, $\alpha_X(Z, n, g) = (B, nm_g, g \cdot m_g)$ and so applying $\widetilde{Q}(A, m, h)$ gives

$$\left(A \times_X B, mm'm'_g, g \cdot (m_g g'')\right)$$

due to the existence of a pullback square below:

$$\begin{tikzcd}
& & A \times_X B \arrow[dl, "m'm'_g"'] \arrow[dr, "g''"] & & \\
& A \arrow[dl, "m"'] \arrow[dr, "h"] & & B \arrow[dl, "nm_g"'] \arrow[dr, "g\cdot m_g"] & \\
Y & & X & & \star
\end{tikzcd}$$

Therefore, to show that $\alpha$ is natural is to show there is an isomorphism $B' \to A \times_X B$ making the following diagram commute:

$$\begin{tikzcd}
& B' \arrow[dl, "mm'm_{g\cdot g'}"'] \arrow[d] \arrow[dr, "g\cdot(g'm_{g\cdot g'})"] & \\
Y & A \times_X B \arrow[l, "mm'm'_g"] \arrow[r, "g\cdot(m_g g'')"'] & \star
\end{tikzcd}$$

Now $d$ certainly makes the left triangle commute since $m_{g\cdot g'} = m'_g d$ (by definition). But it also makes the right triangle commute since

$$g \cdot (g'm_{g\cdot g'}) = g \cdot \left(g'(m'_g d)\right) = g \cdot \left((m_g g'')d\right) = \left(g \cdot (m_g g'')\right) \cdot d$$

as $m'_g$ is the pullback of $m_g$ along $g'$. Therefore, as there is an isomorphism making both triangles commute, $\alpha$ is natural.

### 2. $\alpha$ is a restriction idempotent

To show $\alpha$ is a restriction idempotent, let $X \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$ and $(Z, n, g) \in \widetilde{Q}X$. Then

$$\begin{aligned}
\overline{\alpha}_X(Z, n, g) &= (Z, n, g) \cdot \overline{\alpha_X(Z, n, g)} = (Z, n, g) \cdot \overline{(B, nm_g, g \cdot m_g)} \\
&= (Z, n, g) \cdot (B, nm_g, nm_g) = (B, nm_g, g \cdot m_g) \\
&= \alpha_X(Z, n, g)
\end{aligned}$$

by observing that $(B, 1_B, m_g)$ is a pullback of $n$ along $nm_g$

$$\begin{tikzcd}
& & B \arrow[dl, "1_B"'] \arrow[dr, "m_g"] & & \\
& B \arrow[dl, "nm_g"'] \arrow[dr, "nm_g"] & & Z \arrow[dl, "n"'] \arrow[dr, "g\in QZ"] & \\
X & & X & & \star
\end{tikzcd}$$

Therefore, $\alpha$ is a restriction idempotent.

**3. $\widetilde{\mu}\colon \widetilde{P} \Rightarrow \widetilde{Q}$ is an equaliser of $\alpha$ and $1_{\widetilde{Q}}$**

To show $\widetilde{\mu}$ is an equaliser of $\alpha$ and $1_{\widetilde{Q}}$, we need to show $\widetilde{\mu}_X$ is an equaliser of $\alpha_X$ and $1_{\widetilde{Q}X}$ in **Set** (for all $X \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$). This amounts to showing that:

- $\widetilde{\mu}_X$ is injective, and
- $(Z, n, g) \in \widetilde{Q}X$ satisfies $(Z, n, g) = \widetilde{\mu}_X(Y, m, f) = \big(Y, m, \mu_Y(f)\big)$ for some $(Y, m, f) \in \widetilde{P}X$ if and only if $\alpha_X(Z, n, g) = (Z, n, g)$

To show $\widetilde{\mu}_X$ is injective, suppose $\widetilde{\mu}_X(Y, m, f) = \widetilde{\mu}_X(Y', m', f')$. That is, there exists an isomorphism $\varphi\colon Y' \to Y$ making the following diagram commute:

$$\begin{tikzcd}
 & Y' \arrow[dl, "m'"'] \arrow[d, "\varphi"] \arrow[dr, "\mu_{Y'}(f')"] & \\
X & Y \arrow[l, "m"] \arrow[r, "\mu_Y(f)"'] & \star
\end{tikzcd}$$

If we can show that this implies $f \cdot \varphi = f'$, then $(Y, m, f) = (Y', m', f')$. Now by naturality of $\mu$, we have $\mu_{Y'}(f \cdot \varphi) = \mu_Y(f) \cdot \varphi$. But by assumption, $\mu_Y(f) \cdot \varphi = \mu_{Y'}(f')$, and so $\mu_{Y'}(f \cdot \varphi) = \mu_{Y'}(f')$. Since $\mu$ is monic, $f \cdot \varphi = f'$, and therefore $\widetilde{\mu}_X$ is injective.

To prove the second claim, let $(Z, n, g) \in \widetilde{Q}X$, and suppose $\alpha_X(Z, n, g) = (Z, n, g)$. That is, $(B, nm_g, g \cdot m_g) = (Z, n, g)$, or equivalently, $m_g$ is an isomorphism making the following diagram commute:

$$\begin{tikzcd}
 & B \arrow[dl, "nm_g"'] \arrow[d, "m_g"] \arrow[dr, "g \cdot m_g"] & \\
X & Z \arrow[l, "m"] \arrow[r, "g"'] & \star
\end{tikzcd}$$

Since functors preserve isomorphisms and the Yoneda embedding is full and faithful, $m_g$ is an isomorphism if and only if $\mathrm{y}m_g$ is an isomorphism. By Lemma 18, $\mathrm{y}m_g$ is an isomorphism if and only if $\langle g \rangle = \mu \circ \langle h \rangle$, for some $\langle h \rangle\colon \mathrm{y}Z \Rightarrow P$.

$$\begin{tikzcd}
\mathrm{y}B \arrow[r] \arrow[d, "\mathrm{y}m_g"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\mu"] \\
\mathrm{y}Z \arrow[ur, dashed, "\langle h \rangle"] \arrow[r, "\langle g \rangle"'] & Q
\end{tikzcd}$$

And by naturality of the Yoneda bijection $\widehat{\mathbf{C}}(\mathrm{y}Z, P) \cong PZ$ in $P$, the condition $\langle g \rangle = \mu \circ \langle h \rangle$ is equivalent to the statement that $g = \mu_Z(h)$ for some $h \in PZ$. But this is the same as saying that the following diagram commutes:

$$\begin{tikzcd} & Z \arrow[dl, "n"'] \arrow[d, "1_Z"] \arrow[dr, "\mu_Z(h)"] & \\ X & Z \arrow[l, "n"] \arrow[r, "g"'] & \star \end{tikzcd}$$

for some $h \in PZ$, or $(Z, n, g) = (Z, n, \mu_Z(h)) = \widetilde{\mu}_X(Z, n, h)$ (where $(Z, n, h) \in \widetilde{P}X$). Therefore, $\widetilde{\mu}_X$ is an equaliser of $1_{\widetilde{Q}X}$ and $\alpha_X$ in **Set**, and hence $\widetilde{\mu}$ is an equaliser of $1_{\widetilde{Q}}$ and $\alpha$. This makes $\widetilde{\mu}$ a restriction monic, and so $F\colon \widehat{\mathbf{C}} \to \mathsf{Total}\left(\widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})_r}\right)$ is an $\mathcal{M}$-functor.

Now for a functor in the other direction, consider the functor $G\colon \mathsf{Total}\left(\widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})_r}\right) \to \widehat{\mathbf{C}}$ defined in Theorem 17. Again, we will show that $G$ is also an $\mathcal{M}$-functor.

So let $\mu\colon P \Rightarrow Q$ be a restriction monic in $\widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})_r}$. To show that $\dot{\mu}\colon \dot{P} \Rightarrow \dot{Q}$ is an $\widehat{\mathcal{M}}$-map, we need to show given any $\langle\theta\rangle\colon \mathrm{y}C \Rightarrow \dot{Q}$, there is an object $D$, an arrow $m\colon D \to C$ in $\mathcal{M}$ and a $\langle\delta\rangle\colon \mathrm{y}D \Rightarrow \dot{P}$ making the following square a pullback:

$$\begin{tikzcd} \mathrm{y}D \arrow[r, "\langle\delta\rangle"] \arrow[d, "\mathrm{y}m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{P} \arrow[d, "\dot{\mu}"] \\ \mathrm{y}C \arrow[r, "\langle\theta\rangle"'] & \dot{Q} \end{tikzcd}$$

We make two observations. First, $D, m$ and $\langle\delta\rangle$ must satisfy the condition $\dot{\mu} \circ \langle\delta\rangle = \langle\theta\rangle \circ \mathrm{y}m$. But naturality of the bijection $\dot{Q}C \cong \widehat{\mathbf{C}}(\mathrm{y}C, \dot{Q})$ in $C$ implies $\langle\theta\rangle \circ \mathrm{y}m = \langle\theta \cdot m\rangle$ (where $\theta \in \dot{Q}C$), and naturality of $\widehat{\mathbf{C}}(\mathrm{y}D, \dot{P}) \cong \dot{P}D$ in $\dot{P}$ implies $\dot{\mu} \circ \langle\delta\rangle = \langle\dot{\mu}_D(\delta)\rangle$ (where $\delta \in \dot{P}D$). So $D, m$ and $\langle\delta\rangle$ must satisfy the following condition (in $\dot{Q}D$):

$$\dot{\mu}_D(\delta) = \theta \cdot m = (\dot{Q}m)(\theta) \tag{3.1}$$

Second, $D, m$ and $\langle\delta\rangle$ must make the following a pullback in **Set** (for all $X \in \mathbf{C}$):

$$\begin{tikzcd} \mathbf{C}(X, D) = \mathrm{y}DX \arrow[r, "\langle\delta\rangle_X = \delta\cdot(-)"] \arrow[d, "(\mathrm{y}m)_X = m\circ(-)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{P}X \arrow[d, "\dot{\mu}_X"] \\ \mathbf{C}(X, C) = \mathrm{y}CX \arrow[r, "\langle\theta\rangle_X = \theta\cdot(-)"'] & \dot{Q}X \end{tikzcd}$$

Note this amounts to showing that given any $f \in \mathbf{C}(X, C)$ and $x \in \dot{P}X$ with $(\dot{Q}f)(\theta) = \dot{\mu}_X(x)$, there is a unique $g \in \mathbf{C}(X, D)$ such that:

$$(\dot{P}g)(\delta) = \delta \cdot (X, 1_X, g) = x, \quad \text{and} \quad m \circ g = f \tag{3.2}$$

Now to define $D$ and $m$, we note that there is a $\rho\colon Q \Rightarrow P$ such that $\mu\rho = \bar{\rho}$ and $\rho\mu = 1_P$ (as $\mu$ is a restriction monic). Since $\theta \in \dot{Q}C \subset QC$, applying $\rho_C\colon QC \to PC$ to $\theta$ gives an element $\rho_C(\theta) \in PC$, and taking its restriction gives

$$\overline{\rho_C(\theta)} = (D, m, m) \in \mathsf{Par}(\mathbf{C}, \mathcal{M})(C, C)$$

for some $D \in \mathbf{C}$ and $m\colon D \to C$ in $\mathcal{M}$. This gives us our object $D$ and arrow $m$.

To define $\delta \in \dot{P}D$, consider the arrow $(D, 1_D, m) \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$. As $P$ is a restriction presheaf on $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, applying $P$ to $(D, 1_D, m)$ gives a function $P(D, 1_D, m)\colon PC \to PD$. So define

$$\delta = P(D, 1_D, m)\big(\rho_C(\theta)\big) = \rho_C(\theta) \cdot (D, 1_D, m)$$

We just need to show $\delta \in \dot{P}D$, or alternatively, $\overline{\delta} = (D, 1_D, 1_D)$. But

$$\begin{aligned}\overline{\delta} &= \overline{\rho_C(\theta) \cdot (D, 1_D, m)} = \overline{\overline{\rho_C(\theta)} \circ (D, 1_D, m)} = \overline{(D, m, m) \circ (D, 1_D, m)} \\ &= \overline{(D, 1_D, m)} = (D, 1_D, 1_D)\end{aligned}$$

Hence $\delta \in \dot{P}D$. All that remains is to show that $D, m$ and $\delta$ satisfy (3.1) and (3.2).

To see that $D, m$ and $\delta$ satisfy (3.1), we substitute and get

$$\begin{aligned}\dot{\mu}_D(\delta) &= \mu_D\big(\rho_C(\theta) \cdot (D, 1_D, m)\big) = \mu_C\big(\rho_C(\theta)\big) \cdot (D, 1_D, m) = \overline{\rho}_C(\theta) \cdot (D, 1_D, m) \\ &= \big(\theta \cdot \overline{\rho_C(\theta)}\big) \cdot (D, 1_D, m) = \theta \cdot \big(\overline{\rho_C(\theta)} \circ (D, 1_D, m)\big) = \theta \cdot (D, 1_D, m) \\ &= Q(D, 1_D, m)(\theta) = (\dot{Q}m)(\theta)\end{aligned}$$

using the fact that $\mu\rho = \bar{\rho}$.

To see that $D, m$ and $\delta$ also satisfy (3.2), suppose $\theta \cdot (X, 1_X, f) = (\dot{Q}f)(\theta) = \dot{\mu}_X(x) = \mu_X(x)$ for some $f \in \mathbf{C}(X, C)$ and $x \in \dot{P}X$. Then applying $\rho_X$ to both sides gives

$$\rho_C(\theta) \cdot (X, 1_X, f) = \rho_X\big(\theta \cdot (X, 1_X, f)\big) = \rho_X(\mu_X(x)) = x$$

since $\rho\mu = 1_P$. Now the condition $\delta \cdot (X, 1_X, g) = x$ is equivalent to

$$\rho_C(\theta) \cdot \big((D, 1_D, m) \circ (X, 1_X, g)\big) = \rho_C(\theta) \cdot (X, 1, mg) = x$$

(after substitution). Therefore, to show that $D, m$ and $\delta$ satisfy (3.2), it suffices to find a $g\colon X \to D$ such that $mg = f$ (as $g$ will automatically be unique by the fact $m$ is monic).

So consider the composite $(D, m, m) \circ (X, 1_X, f) = (E, m', mf')$.

$$\begin{tikzcd} & & E \arrow[dl, "m'"'] \arrow[dr, "f'"] & & \\ & X \arrow[dl, "1_X"'] \arrow[dr, "f"] & & D \arrow[dl, "m"'] \arrow[dr, "m"] & \\ X & & C & & C \end{tikzcd}$$

If $m'$ is an isomorphism (with inverse given by $(m')^{-1}$), then $f' \circ (m')^{-1}$ will give a map $X$

to $D$. Now

$$
\begin{aligned}
\theta\cdot(E,m',mf') &= \theta\cdot\Big((D,m,m)\circ(X,1_X,f)\Big) && \\
&= (\theta\cdot\overline{\rho_C(\theta)})\cdot(X,1_X,f) && \text{(definition of } \overline{\rho_C(\theta)}\text{)} \\
&= \overline{\rho}_C(\theta)\cdot(X,1_X,f) && \text{(definition of } \overline{\rho}\text{)} \\
&= \overline{\rho}_X\Big(\theta\cdot(X,1_X,f)\Big) && \text{(naturality of } \overline{\rho}\text{)} \\
&= \overline{\rho}_X(\mu_X(x)) && \text{(by assumption)} \\
&= \mu_X(\rho_X(\mu_X(x))) && \\
&= \mu_X(x) && \\
&= \theta\cdot(X,1_X,f) && \text{(by assumption)}
\end{aligned}
$$

and so $\overline{\theta\cdot(E,m',mf')} = \overline{\theta\cdot(X,1_X,f)}$. But since $\theta\in\dot{Q}C\subset QC$,

$$\overline{\theta\cdot(E,m',mf')} = \overline{\overline{\theta}\circ(E,m',mf')} = \overline{(E,m',mf')} = (E,m',m')$$

and

$$\overline{\theta\cdot(X,1_X,f)} = \overline{\overline{\theta}\circ(X,1_X,f)} = \overline{(X,1_X,f)} = (X,1_X,1_X)$$

as $\overline{\theta} = (C,1_C,1_C)$. That is, $(E,m',m') = (X,1_X,1_X)$ and so by definition, $m'$ must be an isomorphism. This gives a unique $g = f'\circ(m')^{-1}$ in $\mathbf{C}(X,D)$ satisfying the condition

$$mg = m\circ f'\circ(m')^{-1} = (f\circ(m'))\circ(m')^{-1} = f$$

Therefore, $D, m$ and $\delta$ satisfy (3.1) and (3.2). Hence, $\dot{\mu}\colon \dot{P}\Rightarrow\dot{Q}$ is an $\widehat{\mathcal{M}}$-map and so $G$ is an $\mathcal{M}$-functor.

Finally, by the previous theorem, there exist isomorphisms $\eta\colon 1_{\widehat{\mathbf{C}}}\Rightarrow GF$ and $\varepsilon\colon FG\Rightarrow 1_{\mathsf{Total}\left(\mathsf{Par}\widehat{(\mathbf{C},\mathcal{M})}_r\right)}$. Therefore, all naturality squares must be pullback squares and so both $\eta$ and $\varepsilon$ must be $\mathcal{M}$-cartesian. Hence,

$$\left(\widehat{\mathbf{C}},\widehat{\mathcal{M}}\right)\simeq\mathcal{M}\mathsf{Total}\left(\mathsf{Par}\widehat{(\mathbf{C},\mathcal{M})}_r\right) \qquad \square$$

We now use the above result to derive the following fact.

**Lemma 20.** *There is an equivalence of restriction categories*

$$L\colon \mathsf{Par}\left(\widehat{\mathbf{C}},\widehat{\mathcal{M}}\right)\to\mathsf{Par}\widehat{(\mathbf{C},\mathcal{M})}_r$$

*In addition, this equivalence L makes the following diagram commute:*

$$
\begin{tikzcd}
\mathsf{Par}(\mathbf{C},\mathcal{M}) \arrow[rr, "\mathsf{Par}(\mathrm{y})"] \arrow[dr, "\mathrm{y}_r"'] & & \mathsf{Par}\left(\widehat{\mathbf{C}},\widehat{\mathcal{M}}\right) \arrow[dl, "L"] \\
& \mathsf{Par}\widehat{(\mathbf{C},\mathcal{M})}_r &
\end{tikzcd}
$$

*That is,* $\mathrm{y}_r = L\circ\mathsf{Par}(\mathrm{y})$.

*Proof.* Since $\mathsf{Par}$ and $\mathcal{M}\mathsf{Total}$ are 2-equivalences, the following is an isomorphism of categories:

$$\mathcal{M}\mathbf{Cat}\Big((\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), \mathcal{M}\mathsf{Total}\big(\widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})}_r\big)\Big) \cong \mathbf{rCat}\Big(\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), \widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})}_r\Big)$$

Define $L$ to be the unique transpose of $F\colon (\widehat{\mathbf{C}}, \widehat{\mathcal{M}}) \to \mathcal{M}\mathsf{Total}\big(\widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})}_r\big)$, where $F$ is the $\mathcal{M}$-functor from Theorem 19. Likewise, define $\widetilde{\mathrm{y}_r}\colon (\mathbf{C}, \mathcal{M}) \to \mathcal{M}\mathsf{Total}\big(\widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})}_r\big)$ to be the unique transpose of $\mathrm{y}_r$. Explicitly, $\widetilde{\mathrm{y}_r}$ is the unique map making the diagram below commute:

$$\begin{tikzcd}
\mathbf{C} \arrow[r, dashed, "\widetilde{\mathrm{y}_r}"] \arrow[d, hook] & \mathsf{Total}\big(\widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})}_r\big) \arrow[d, hook] \\
\mathsf{Par}(\mathbf{C}, \mathcal{M}) \arrow[r, "\mathrm{y}_r"'] & \widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})}_r
\end{tikzcd}$$

Now if the following diagram commutes in $\mathcal{M}\mathbf{Cat}$, then $\mathrm{y}_r = L \circ \mathsf{Par}(\mathrm{y})$.

$$\begin{tikzcd}
(\mathbf{C}, \mathcal{M}) \arrow[rr, "\mathrm{y}"] \arrow[dr, "\widetilde{\mathrm{y}_r}"'] & & (\widehat{\mathbf{C}}, \widehat{\mathcal{M}}) \arrow[dl, "F"] \\
& \mathcal{M}\mathsf{Total}\big(\widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})}_r\big) &
\end{tikzcd}$$

But the above diagram will commute if the following diagram commutes in $\mathbf{Cat}$:

$$\begin{tikzcd}
\mathbf{C} \arrow[rr, "\mathrm{y}"] \arrow[dr, "\widetilde{\mathrm{y}_r}"'] & & \widehat{\mathbf{C}} \arrow[dl, "F"] \\
& \mathsf{Total}\big(\widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})}_r\big) &
\end{tikzcd}$$

So let $A \in \mathbf{C}$. Then by definition, $\widetilde{\mathrm{y}_r}(A) = \mathsf{Par}(\mathbf{C}, \mathcal{M})(-, A)$. On the other hand, $F\mathrm{y}A = F(\mathbf{C}(-, A))$, which is defined on objects $B \in \mathsf{Par}(\mathbf{C}, \mathcal{M})$ by

$$F(\mathbf{C}(-, A))(B) = \Big\{[(Y, m, f)] \mid Y \in \mathsf{Par}(\mathbf{C}, \mathcal{M}), m : Y \to B \in \mathcal{M}, f \in \mathbf{C}(Y, A)\Big\}$$

Alternatively, the elements of $F(\mathbf{C}(-, A))(B)$ are equivalence classes of the form

$$\begin{tikzcd}
& Y \arrow[dl, "m \in \mathcal{M}"'] \arrow[dr, "f"] & \\
B & & A
\end{tikzcd}$$

Clearly $F(\mathbf{C}(-, A))(B) = \mathsf{Par}(\mathbf{C}, \mathcal{M})(B, A) = (\widetilde{\mathrm{y}_r}A)(B)$. Likewise, if $(Z, n, g)\colon C \to B$ is an arrow in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, then $F(\mathbf{C}(-, A))(Z, n, g) = (-) \circ (Z, n, g) = (\widetilde{\mathrm{y}_r}A)(Z, n, g)$. Therefore, $\widetilde{\mathrm{y}_r}(A) = F\mathrm{y}A$.

Now let $h\colon B \to C$ be an arrow in $\mathbf{C}$, so that $\mathrm{y}h\colon \mathbf{C}(-,B) \Rightarrow \mathbf{C}(-,C)$ is defined componentwise by $(\mathrm{y}h)_D = h \circ (-)$. Applying $F$ to $\mathrm{y}h$ gives $F\mathrm{y}h\colon \mathsf{Par}(\mathbf{C},\mathcal{M})(-,B) \Rightarrow \mathsf{Par}(\mathbf{C},\mathcal{M})(-,C)$, whose component at $D \in \mathsf{Par}(\mathbf{C},\mathcal{M})$ is given by

$$(F\mathrm{y}h)_D(Z,n,g) = \Big(Z,n,(\mathrm{y}h)_Z(g)\Big) = (Z,n,hg) = (B,1_B,h) \circ (Z,n,g)$$

(for some $(Z,n,g) \in \mathsf{Par}(\mathbf{C},\mathcal{M})(D,B)$). But $\widetilde{\mathrm{y}_r}(h) = \mathrm{y}_r(B,1_B,h)$, whose component at $D \in \mathsf{Par}(\mathbf{C},\mathcal{M})$ is also given by $\Big(\mathrm{y}_r(B,1_B,h)\Big)_D = (B,1_B,h) \circ (-)$. Therefore, $(F\mathrm{y})(h) = \widetilde{\mathrm{y}_r}(h)$, and so $F\mathrm{y} = \widetilde{\mathrm{y}_r}$ in **Cat**. Hence, $\mathrm{y}_r = L \circ \mathsf{Par}(\mathrm{y})$. □

## 3.8 An embedding of restriction categories

Recall the following Cockett and Lack embedding of any restriction category $\mathbf{X}$:

$$\mathbf{X} \xrightarrow{J} K_r(\mathbf{X}) \xrightarrow{\Phi_{K_r(\mathbf{X})}} \mathsf{Par}\Big(\mathsf{Total}(K_r(\mathbf{X})), \mathcal{M}_{K_r(\mathbf{X})}\Big) \xrightarrow{\mathsf{Par}(\mathrm{y})} \mathsf{Par}\Big(\widehat{\mathsf{Total}(K_r(\mathbf{X}))}, \widehat{\mathcal{M}_{K_r(\mathbf{X})}}\Big)$$

However, we have seen that the restriction functor $\mathrm{y}_r\colon \mathbf{X} \to \widehat{\mathbf{X}}_r$ is also an embedding of $\mathbf{X}$. We now show that the Cockett and Lack embedding and the restriction Yoneda embedding are equal up to an equivalence (by using the fact $L\colon \mathsf{Par}\Big(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}\Big) \to \widehat{\mathsf{Par}(\mathbf{C},\mathcal{M})}_r$ is an equivalence). But first, we need the following lemma.

**Lemma 21.** *Suppose* $\mathbf{X}$ *and* $\mathbf{Z}$ *are restriction categories, and* $H\colon \mathbf{X} \to \mathbf{Z}$ *is a fully faithful restriction functor. Then the following diagram commutes up to isomorphism:*

$$\begin{tikzcd} \mathbf{X} \arrow[r, "H"] \arrow[d, "\mathrm{y}_r"'] \arrow[dr, phantom, "\cong"] & \mathbf{Z} \arrow[d, "\mathrm{y}_r"] \\ \widehat{\mathbf{X}}_r & \widehat{\mathbf{Z}}_r \arrow[l, "(-)\circ H^{\mathrm{op}}"] \end{tikzcd}$$

*That is,* $\mathrm{y}_r \cong \Big((-) \circ H^{\mathrm{op}}\Big) \circ \mathrm{y}_r \circ H$.

*Proof.* Let $A \in \mathbf{X}$, so that $\mathrm{y}_r(A) = \mathbf{X}(-,A)$ and $\Big((-)\circ H^{\mathrm{op}}\Big)\circ \mathrm{y}_r \circ H(A) = \mathbf{Z}\Big(H(-),HA\Big)$. But since $H$ is fully faithful, $\theta_A\colon \mathbf{X}(B,A) \cong \mathbf{Z}\Big(HB,HA\Big)$ for all $B \in \mathbf{X}$, and this isomorphism is clearly natural. As the following square commutes (for all $f\colon A \to B$ in $\mathbf{X}$):

$$\begin{tikzcd} \mathbf{X}(-,A) \arrow[r, "\theta_A"] \arrow[d, "f\circ(-)"'] & \mathbf{Z}\Big(H(-),HA\Big) \arrow[d, "Hf\circ(-)"] \\ \mathbf{X}(-,B) \arrow[r, "\theta_B"'] & \mathbf{Z}\Big(H(-),HB\Big) \end{tikzcd} \qquad \begin{tikzcd} h : D \to A \arrow[r, mapsto] \arrow[d, mapsto] & Hh \arrow[d, mapsto] \\ fh \arrow[r, mapsto] & Hfh = Hf \circ Hh \end{tikzcd}$$

it follows that $\mathrm{y}_r \cong \Big((-) \circ H^{\mathrm{op}}\Big) \circ \mathrm{y}_r \circ H$. □

**Theorem 22.** *The restriction Yoneda embedding* $\mathrm{y}_r$ *is the same embedding given by Cockett and Lack (up to an equivalence). That is, the following diagram commutes up to isomorphism:*

$$\begin{tikzcd}
 & \mathbf{X} \arrow[dl, "\mathsf{Par}(\mathrm{y}) \circ \Phi_{K_r(\mathbf{X})} \circ J"'] \arrow[dr, "\mathrm{y}_r"] \arrow[d, phantom, "\cong"] & \\
\mathsf{Par}\left(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}\right) \arrow[r, "L"'] & \mathsf{Par}\widehat{(\mathbf{C}, \mathcal{M})}_r \arrow[r, "(-) \circ \left(\Phi_{K_r(\mathbf{X})} \circ J\right)^{\mathrm{op}}"'] & \widehat{\mathbf{X}}_r
\end{tikzcd}$$

*where* $\mathbf{C} = \mathsf{Total}(K_r(\mathbf{X}))$ *and* $\mathcal{M} = \mathcal{M}_{K_r(\mathbf{X})}$, *and the composite* $\left((-) \circ \left(\Phi_{K_r(\mathbf{X})} \circ J\right)^{\mathrm{op}}\right) \circ L$ *is an equivalence.*

*Proof.* Consider the following diagram:

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "\Phi_{K_r(\mathbf{X})} \circ J"] \arrow[d, "\mathrm{y}_r"'] \arrow[dr, phantom, "\cong"] & \mathsf{Par}(\mathbf{C}, \mathcal{M}) \arrow[r, "\mathsf{Par}(\mathrm{y})"] \arrow[d, "\mathrm{y}_r"] & \mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}) \arrow[d, "L"] \\
\widehat{\mathbf{X}}_r & \mathsf{Par}\widehat{(\mathbf{C}, \mathcal{M})}_r \arrow[l, "(-) \circ \left(\Phi_{K_r(\mathbf{X})} \circ J\right)^{\mathrm{op}}"] \arrow[r, equal] & \mathsf{Par}\widehat{(\mathbf{C}, \mathcal{M})}_r
\end{tikzcd}$$

The left square commutes up to isomorphism by Lemma 21, and the right square commutes by Lemma 20. Also, $\left((-) \circ \left(\Phi_{K_r(\mathbf{X})} \circ J\right)^{\mathrm{op}}\right) \circ L$ is an equivalence since $L$ and $(-) \circ J$ are both equivalences, and $\Phi_{K_r(\mathbf{X})}$ is an isomorphism of categories. $\square$

# 4

# Free cocompletion

It is a well-known fact that the Yoneda embedding exhibits the presheaf category $\widehat{\mathbf{C}}$ as the free cocompletion of $\mathbf{C}$ (for $\mathbf{C}$ small). That is, for any cocomplete category $\mathcal{E}$, the functor $(-) \circ \mathrm{y} \colon \mathbf{Cocts}(\widehat{\mathbf{C}}, \mathcal{E}) \to \mathbf{Cat}(\mathbf{C}, \mathcal{E})$ is an equivalence of categories, where $\mathbf{Cocts}(\widehat{\mathbf{C}}, \mathcal{E})$ denotes the category of cocontinuous functors from $\widehat{\mathbf{C}}$ to $\mathcal{E}$ and their natural transformations. The aim of this chapter will be to present an analogue of the above phenomenon for restriction categories. To do this, we require a notion of cocompleteness with respect to $\mathcal{M}$-categories and restriction categories.

## 4.1 $\mathcal{M}$-categories and their free cocompletion

To motivate our definition of cocomplete $\mathcal{M}$-category, recall that the presheaf category $\widehat{\mathbf{C}} = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is cocomplete for any small $\mathbf{C}$. It therefore makes sense that for any $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$ (with $\mathbf{C}$ small), the $\mathcal{M}$-category $(\widehat{\mathbf{C}}, \widehat{\mathcal{M}})$ ought to be cocomplete. However, $(\widehat{\mathbf{C}}, \widehat{\mathcal{M}})$ has the additional property of being a *classified* $\mathcal{M}$-category (see Proposition 2.5 from [3]). That is, the inclusion $\widehat{\mathbf{C}} \hookrightarrow \mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}})$ has a right adjoint. This leads to the following definition.

**Definition 23** (Cocomplete $\mathcal{M}$-category)**.** *An $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$ is* cocomplete *if*

- $\mathbf{C}$ *is cocomplete, and*
- *the inclusion $E \colon \mathbf{C} \hookrightarrow \mathsf{Par}(\mathbf{C}, \mathcal{M})$ defined on objects by $E(A) = A$ and on arrows by $E(f) = (1_A, f)$ preserves colimits.*

Since $\mathcal{M}$-functors are functors of their underlying categories, it makes sense to define a cocontinuous $\mathcal{M}$-functor as follows.

**Definition 24** (Cocontinuous $\mathcal{M}$-functor)**.** *An $\mathcal{M}$-functor $G\colon (\mathbf{C},\mathcal{M}) \to (\mathbf{D},\mathcal{N})$ is* cocontinuous *if the underlying functor $UG\colon \mathbf{C} \to \mathbf{D}$ is cocontinuous.*

We denote by $\mathbf{Cocts}\mathcal{M}\mathbf{Cat}$, the 2-category of cocomplete $\mathcal{M}$-categories, cocontinuous $\mathcal{M}$-functors and $\mathcal{M}$-cartesian natural transformations. Now given $\mathcal{M}$-categories $(\mathbf{C},\mathcal{M})$ and $(\mathbf{D},\mathcal{N})$ (with $(\mathbf{D},\mathcal{N})$ cocomplete), we would like to show that the functor

$$(-)\circ \mathrm{y}\colon \mathbf{Cocts}\mathcal{M}\mathbf{Cat}\Big((\widehat{\mathbf{C}},\widehat{\mathcal{M}}),(\mathbf{D},\mathcal{N})\Big) \to \mathcal{M}\mathbf{Cat}\Big((\mathbf{C},\mathcal{M}),(\mathbf{D},\mathcal{N})\Big)$$

is an equivalence of categories. The proof will require the following three lemmas.

**Lemma 25.** *Let $\mathbf{X}$ be a restriction category and $L\colon \mathbf{I} \to U\mathbf{X}$ a functor whose colimit exists and the injections $(p_I\colon LI \to \operatorname{colim} L)_{I\in\mathbf{I}}$ are total. Suppose $\varepsilon\colon L \Rightarrow L$ is a natural transformation such that $\varepsilon_I$ is a restriction idempotent for every $I \in \mathbf{I}$. Then the unique map $\theta\colon \operatorname{colim} L \to \operatorname{colim} L$ making the following diagram commute (for each $I \in \mathbf{I}$):*

$$\begin{tikzcd} LI \arrow[r, "p_I"] \arrow[d, "\varepsilon_I"'] & \operatorname{colim} L \arrow[d, "\theta"] \\ LI \arrow[r, "p_I"'] & \operatorname{colim} L \end{tikzcd}$$

*is a restriction idempotent.*

*Proof.* To show that $\theta$ is a restriction idempotent, we just need to show $\overline{\theta}$ satisfies the same property as $\theta$ (by uniqueness). That is, $\overline{\theta}\circ p_I = p_I \circ \varepsilon_I$. But

$$\overline{\theta}\circ p_I = p_I \circ \overline{\theta\circ p_I} = p_I \circ \overline{p_I \circ \varepsilon_I} = p_I \circ \overline{\overline{p_I}\circ \varepsilon_I} = p_I \circ \overline{\varepsilon} = p_I \circ \varepsilon$$

since $p_I$ is total for every $I \in \mathbf{I}$ and $\varepsilon$ is a restriction idempotent. Therefore, $\overline{\theta}$ is a restriction idempotent. □

**Lemma 26.** *Suppose $(\mathbf{D},\mathcal{N})$ is a cocomplete $\mathcal{M}$-category, $H, K\colon \mathbf{I} \to \mathbf{D}$ are functors, and $\alpha\colon H \Rightarrow K$ is a natural transformation such that for each $I \in \mathbf{I}$ and $f\colon I \to J$, we have*

- *$\alpha_I\colon HI \to KI \in \mathcal{N}$, and*
- $\begin{tikzcd} HI \arrow[r, "Hf"] \arrow[d, "\alpha_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & HJ \arrow[d, "\alpha_J"] \\ KI \arrow[r, "Kf"'] & KJ \end{tikzcd}$ *is a pullback square.*

*Let $(\operatorname{colim} H, p_I)_{I\in\mathbf{I}}$ and $(\operatorname{colim} K, q_I)_{I\in\mathbf{I}}$ be colimiting cocones, and let $\theta\colon \operatorname{colim} H \to \operatorname{colim} K$ be the unique map making each square of the following form commute:*

$$\begin{tikzcd} HI \arrow[r, "p_I"] \arrow[d, "\alpha_I"'] & \operatorname{colim} H \arrow[d, "\theta"] \\ KI \arrow[r, "q_I"'] & \operatorname{colim} K \end{tikzcd}$$

*In this situation, the map $\theta$ is in $\mathcal{N}$ and each of the above squares is a pullback.*

*Proof.* Applying the inclusion $E$ to the following commuting square

$$\begin{tikzcd} HI \arrow[r, "p_I"] \arrow[d, "\alpha_I"'] & \operatorname{colim} H \arrow[d, "\theta"] \\ KI \arrow[r, "q_I"'] & \operatorname{colim} K \end{tikzcd}$$

we get the following commutative diagram in $\mathsf{Par}(\mathbf{D}, \mathcal{N})$ (for every $I \in \mathbf{I}$):

$$\begin{tikzcd} HI \arrow[r, "{(1,p_I)}"] \arrow[d, "{(1,\alpha_I)}"'] & \operatorname{colim} H \arrow[d, "{(1,\theta)}"] \\ KI \arrow[r, "{(1,q_I)}"'] & \operatorname{colim} K \end{tikzcd}$$

Now there is a natural transformation $\beta\colon EK \Rightarrow EH$ whose component at $I \in \mathbf{I}$ is $(\alpha_I, 1)$. To see this defines a natural transformation, let $f\colon I \to J$ be an arrow in $\mathbf{I}$. We require the following square to commute:

$$\begin{tikzcd} KI \arrow[r, "{(1,Kf)}"] \arrow[d, "{\beta_I = (\alpha_I,1)}"'] & KJ \arrow[d, "{(\alpha_J,1)}"] \\ HI \arrow[r, "{(1,Hf)}"'] & HJ \end{tikzcd}$$

But as $(HI, \alpha_I, Hf)$ is a pullback of $\alpha_J$ along $Kf$ (by assumption), we have

$$(1, Hf) \circ (\alpha_I, 1) = (\alpha_I, Hf) = (\alpha_J, 1) \circ (1, Kf)$$

Hence, $\beta\colon EK \Rightarrow EH$ is a natural transformation with $\beta_I = (\alpha_I, 1)$. Since $E(\operatorname{colim} K) \cong \operatorname{colim} EK$ and $E(\operatorname{colim} H) \cong \operatorname{colim} EH$, the universal property of colimit induces a unique map $(n, g)\colon \operatorname{colim} K \to \operatorname{colim} H$ in $\mathsf{Par}(\mathbf{D}, \mathcal{N})$ making the top square below commute for every $I \in \mathbf{I}$:

$$\begin{tikzcd} KI \arrow[r, "{(1,q_I)}"] \arrow[d, "{(\alpha_I,1)}"'] & \operatorname{colim} K \arrow[d, "{(n,g)}"] \\ HI \arrow[r, "{(1,p_I)}"] \arrow[d, "{(1,\alpha_I)}"'] & \operatorname{colim} H \arrow[d, "{(1,\theta)}"] \\ KI \arrow[r, "{(1,q_I)}"'] & \operatorname{colim} K \end{tikzcd}$$

Figure 4.1: Commuting squares in $\mathsf{Par}(\mathbf{D}, \mathcal{N})$

The composite $(1, \alpha_I) \circ (\alpha_I, 1) = (\alpha_I, \alpha_I)$ is the component of a natural transformation $\varepsilon\colon EK \Rightarrow EK$ at $I$. But since $(\alpha_I, \alpha_I) = \overline{(\alpha_I, \alpha_I)}$ and each of the maps $(1, q_I)$ is total, the composite $(1, \theta) \circ (n, g) = (n, \theta g)$ must be a restriction idempotent (by LEMMA 25). In particular, this implies $n = \theta g$ since $(n, \theta g) = \overline{(n, \theta g)} = (n, n)$.

Now stacking the bottom square on top this time, we get the following diagram, where the composite $(n, g) \circ (1, \theta)$ is the unique arrow making the diagram commute:

$$\begin{tikzcd}
HI \arrow[r, "{(1,p_I)}"] \arrow[d, "{(1,\alpha_I)}"'] & \operatorname{colim} H \arrow[d, "{(1,\theta)}"] \\
KI \arrow[r, "{(1,q_I)}"] \arrow[d, "{(\alpha_I,1)}"'] & \operatorname{colim} K \arrow[d, "{(n,g)}"] \\
HI \arrow[r, "{(1,p_I)}"'] & \operatorname{colim} H
\end{tikzcd}$$

The composite $(\alpha_I, 1) \circ (1, \alpha_I) = (1, 1)$ is the component of a natural transformation $\gamma\colon EH \Rightarrow EH$ at $I$, and is clearly a restriction idempotent. Since $(1, 1)\colon \operatorname{colim} H \to \operatorname{colim} H$ makes the outer square commute, we must have $(n, g) \circ (1, \theta) = (1, 1)$ by uniqueness. That is, if $(B, n', \theta')$ is a pullback of $n$ along $\theta$,

$$\begin{tikzcd}
 & & B \arrow[dl, "n'"'] \arrow[dr, "\theta'"] & & \\
 & \operatorname{colim} H \arrow[dl, "1"'] \arrow[dr, "\theta"] & & A \arrow[dl, "n"'] \arrow[dr, "g"] & \\
\operatorname{colim} H & & \operatorname{colim} K & & \operatorname{colim} H
\end{tikzcd}$$

then $n'$ must be an isomorphism making the following diagram commute:

$$\begin{tikzcd}
 & B \arrow[dl, "n'"'] \arrow[d, dashed, "n'"] \arrow[dr, "g\theta'"] & \\
\operatorname{colim} H & \operatorname{colim} H \arrow[l, "1"] \arrow[r, "1"'] & \operatorname{colim} H
\end{tikzcd}$$

A quick summary of the results so far gives

- $\theta = n\theta'(n')^{-1}$ (by pullback), and
- $g\theta'(n')^{-1} = 1$ (by commuting triangle), and
- $\theta g = n$ (since $(n, \theta g)$ is a restriction idempotent)

But $n\theta'(n')^{-1}g = \theta g = n$ implies $\theta'(n')^{-1}g = 1$ (as $n \in \mathcal{N}$). Therefore, $g$ must also be an isomorphism (and so $g^{-1} \in \mathcal{N}$). Hence, as $\theta = ng^{-1}$ and $n, g^{-1} \in \mathcal{N}$, we conclude that $\theta$ is in $\mathcal{N}$. This proves the first claim of the lemma.

To prove the second part of the lemma, observe that the following diagram commutes:

$$\begin{tikzcd}
 & A \arrow[dl, "n"'] \arrow[d, dashed, "g"] \arrow[dr, "g"] & \\
\operatorname{colim} K & \operatorname{colim} H \arrow[l, "\theta"] \arrow[r, "1"'] & \operatorname{colim} H
\end{tikzcd}$$

(since $\theta g = n$ and $g$ is an isomorphism). That is, $(n, g) = (\theta, 1)$. Replacing $(n, g)$ with $(\theta, 1)$ in FIGURE 4.1 gives the following commuting diagram for all $I \in \mathbf{I}$:

$$\begin{tikzcd} KI \arrow[r, "{(1,q_I)}"] \arrow[d, "{(\alpha_I,1)}"'] & \operatorname{colim} K \arrow[d, "{(\theta,1)}"] \\ HI \arrow[r, "{(1,p_I)}"'] & \operatorname{colim} H \end{tikzcd}$$

The bottom composite is $(1, p_I) \circ (\alpha_I, 1) = (\alpha_I, p_I)$, and since this is required to be equal to $(\theta, 1) \circ (1, q_I)$

$$\begin{tikzcd} & & B \arrow[dl, "\theta''"'] \arrow[dr, "q'_I"] & & \\ & KI \arrow[dl, "1"'] \arrow[dr, "q_I"] & & \operatorname{colim} H \arrow[dl, "\theta"'] \arrow[dr, "1"] & \\ KI & & \operatorname{colim} K & & \operatorname{colim} H \end{tikzcd}$$

we must have $B = HI$, $\theta'' = \alpha_I$, $q'_I = p_I$, and so the following square must be a pullback:

$$\begin{tikzcd} HI \arrow[r, "p_I"] \arrow[d, "\alpha_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{colim} H \arrow[d, "\theta"] \\ KI \arrow[r, "q_I"'] & \operatorname{colim} K \end{tikzcd}$$

(for all $I \in \mathbf{I}$). □

**Lemma 27.** *Let $(\mathbf{D}, \mathcal{N})$ be a cocomplete $\mathcal{M}$-category. Suppose functors $H, K \colon \mathbf{I} \to \mathbf{D}$ and natural transformation $\alpha \colon H \Rightarrow K$ satisfy assumptions from* LEMMA *26 so that the following is a pullback for every $I \in \mathbf{I}$*

$$\begin{tikzcd} HI \arrow[r, "p_I"] \arrow[d, "\alpha_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{colim} H \arrow[d, "\theta"] \\ KI \arrow[r, "q_I"'] & \operatorname{colim} K \end{tikzcd}$$

*and $\theta \in \mathcal{N}$. Let $n \colon X \to Y \in \mathcal{N}$ and suppose there exist maps $x \colon \operatorname{colim} H \to X$ and $y \colon \operatorname{colim} K \to Y$ such that the following outer square is a pullback for all $I \in \mathbf{I}$ and the right hand square commutes:*

$$\begin{tikzcd} HI \arrow[r, "p_I"] \arrow[d, "\alpha_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{colim} H \arrow[r, "x"] \arrow[d, "\theta"'] & X \arrow[d, "n"] \\ KI \arrow[r, "q_I"'] & \operatorname{colim} K \arrow[r, "y"'] & Y \end{tikzcd}$$

*Then the right hand square is also a pullback.*

*Proof.* Applying the inclusion $E\colon \mathbf{D} \hookrightarrow \mathsf{Par}(\mathbf{D}, \mathcal{N})$, we get the following diagram:

$$\begin{tikzcd}
KI \arrow[r, "{(1,q_I)}"] \arrow[d, "{(\alpha_I,1)}"'] & \operatorname{colim} K \arrow[r, "{(1,y)}"] \arrow[d, "{(\theta,1)}"] & Y \arrow[d, "{(n,1)}"] \\
HI \arrow[r, "{(1,p_I)}"] \arrow[d, "{(1,\alpha_I)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{colim} H \arrow[r, "{(1,x)}"] \arrow[d, "{(1,\theta)}"'] & X \arrow[d, "{(1,n)}"] \\
KI \arrow[r, "{(1,q_I)}"'] & \operatorname{colim} K \arrow[r, "{(1,y)}"'] & Y
\end{tikzcd}$$

where all squares commute, except with the possible exception of the top right square. Focusing entirely upon this top right square, we see that the bottom composite is given by $(1, x) \circ (\theta, 1) = (\theta, x)$, and the top composite is $(n, 1) \circ (1, y) = (\theta', x')$ (where $(Z, \theta', x')$ is a pullback of $n$ along $y$). So if we can show that this square commutes (that is, $\theta' = \theta$ and $x' = x$), then the result follows.

Now both $(\theta, x)$ and $(n, 1) \circ (1, y)$ are maps out of $\operatorname{colim} K$, and therefore, if $(\theta, x) \circ (1, q_I) = (n, 1) \circ (1, y) \circ (1, q_I)$ for all $I \in \mathbf{I}$, then they must be equal.

$$\begin{tikzcd}
KI \arrow[r, "{(1,q_I)}"] & \operatorname{colim} K \arrow[r, shift left, "{(\theta,x)}"] \arrow[r, shift right, "{(n,1)\circ(1,y)}"'] & X
\end{tikzcd}$$

By commutativity of the top left square, we have

$$(\theta, x) \circ (1, q_I) = (1, x) \circ (1, p_I) \circ (\alpha_I, 1) = (1, x) \circ (\alpha_I, p_I) = (\alpha_I, xp_I)$$

But

$$(n, 1) \circ (1, y) \circ (1, q_I) = (n, 1) \circ (1, yq_I) = (\alpha_I, xp_I)$$

since by assumption, the following square is a pullback square:

$$\begin{tikzcd}
HI \arrow[r, "xp_I"] \arrow[d, "\alpha_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "n"] \\
KI \arrow[r, "yq_I"'] & Y
\end{tikzcd}$$

Therefore, the top right square commutes in $\mathsf{Par}(\mathbf{C}, \mathcal{N})$ and

$$\begin{tikzcd}
\operatorname{colim} H \arrow[r, "x"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "n"] \\
\operatorname{colim} K \arrow[r, "y"'] & Y
\end{tikzcd}$$

is a pullback. □

We are now in a position to prove the following theorem.

**Theorem 28.** *Let* $(\mathbf{C}, \mathcal{M})$ *and* $(\mathbf{D}, \mathcal{N})$ *be* $\mathcal{M}$*-categories, and suppose* $(\mathbf{D}, \mathcal{N})$ *is cocomplete. Then*

$$(-) \circ \mathrm{y} \colon \mathbf{Cocts}\mathcal{M}\mathbf{Cat}\Big((\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), (\mathbf{D}, \mathcal{N})\Big) \to \mathcal{M}\mathbf{Cat}\Big((\mathbf{C}, \mathcal{M}), (\mathbf{D}, \mathcal{N})\Big)$$

*is an equivalence of categories.*

*Proof.* We begin by showing that $(-) \circ \mathrm{y}$ is essentially surjective. Recall that

$$(-) \circ \mathrm{y} \colon \mathbf{Cocts}(\widehat{\mathbf{C}}, \mathbf{D}) \to \mathbf{Cat}(\mathbf{C}, \mathbf{D})$$

is an equivalence of categories, giving us a cocontinuous functor $F_! \colon \widehat{\mathbf{C}} \to \mathbf{D}$ such that $F_! \circ y \cong F$ for every $F \colon \mathbf{C} \to \mathbf{D}$. Therefore, if we can show $F_!$ is an $\mathcal{M}$-functor for every $\mathcal{M}$-functor $F$, then $(-) \circ \mathrm{y}$ is essentially surjective.

### 1. $F_!$ takes $\widehat{\mathcal{M}}$-maps to $\mathcal{N}$-maps

Let $\mu \colon P \to Q$ be an $\widehat{\mathcal{M}}$-map. Since every presheaf may be written as a colimit of representables, let $Q \cong \operatorname{colim}(\mathrm{y}\pi_Q \colon \operatorname{el} Q \to \widehat{\mathbf{C}})$, where $\operatorname{el} Q$ is the category of elements of $Q$ and $\pi_Q \colon \operatorname{el} Q \to \mathbf{C}$ is the obvious projection map. Now let $K = \mathrm{y}\pi_Q$, so that our colimiting cocone becomes $(\operatorname{colim} K, q_I)_{I \in \operatorname{el} Q}$ (where each $q_I$ is an arrow of the form $\mathrm{y}D_I \to Q$ for some $D_I \in \mathbf{C}$). By definition of an $\widehat{\mathcal{M}}$-map, for every $I \in \operatorname{el} Q$, there is some $m_I \colon C_I \to D_I$ in $\mathcal{M}$ making the following square a pullback:

$$\begin{tikzcd}
\mathrm{y}C_I \arrow[r] \arrow[d, "\mathrm{y}m_I"'] \arrow[dr, phantom, very near start, "\lrcorner"] & P \arrow[d, "\mu"] \\
KI = \mathrm{y}D_I \arrow[r, "q_I"'] & Q
\end{tikzcd}$$

This gives a functor $H \colon \operatorname{el} Q \to \widehat{\mathbf{C}}$, which on objects takes $I$ to $HI = \mathrm{y}C_I$, and on arrows takes $f \colon I \to J$ to the unique map $Hf$ making the following diagram commute and the left square a pullback (by the pasting properties of pullbacks):

$$\begin{tikzcd}
HI \arrow[r, dashed, "Hf"] \arrow[rr, bend left, "p_I"] \arrow[d, "\mathrm{y}m_I"'] \arrow[dr, phantom, very near start, "\lrcorner"] & HJ \arrow[r] \arrow[d, "\mathrm{y}m_J"'] \arrow[dr, phantom, very near start, "\lrcorner"] & P \arrow[d, "\mu"] \\
KI \arrow[r, "Kf"'] \arrow[rr, bend right, "q_I"'] & KJ \arrow[r, "q_J"'] & Q
\end{tikzcd}$$

We can also define a natural transformation $\alpha \colon H \Rightarrow K$ whose component at $I$ is $\alpha_I = \mathrm{y}m_I$. Since colimits in presheaf categories are stable under pullback, $(P, p_I)_{I \in \operatorname{el} Q}$ is a colimiting cocone since $(Q, q_I)$ is, and so we may write $P \cong \operatorname{colim} H$.

Recalling that the functor $F_!$ preserves colimits and that $F_! \circ \mathrm{y} \cong F$, applying $F_!$ to the above diagram yields:

$$\begin{tikzcd}
FC_I \arrow[r, "Ff_C"] \arrow[d, "F_!(\alpha_I) \cong Fm_I"'] \arrow[rr, bend left, "F_!(p_I)"] & FC_J \arrow[r] \arrow[d, "Fm_J"'] & F_!P \cong \operatorname{colim} F_!H \arrow[d, "F_!(\mu)"] \\
FD_I \arrow[r, "Ff_D"'] \arrow[rr, bend right, "F_!(q_I)"'] & FD_J \arrow[r, "F_!(q_J)"'] & F_!Q \cong \operatorname{colim} F_!K
\end{tikzcd}$$

Figure 4.2: Result of applying $F_!$

where $f_C\colon C_I \to C_J$ and $f_D\colon D_I \to D_J$ are maps in $\mathbf{C}$ corresponding to $f$ in $\operatorname{el} Q$ (and $(F_!P, F_!(p_I))$ and $(F_!Q, F_!(q_I))$ are colimiting cocones). Since

$$\begin{tikzcd}
HI = \mathrm{y}C_I \arrow[r, "\mathrm{y}f_C"] \arrow[d, "\mathrm{y}m_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{y}C_J \arrow[d, "\mathrm{y}m_J"] \\
KI = \mathrm{y}D_I \arrow[r, "\mathrm{y}f_D"'] & \mathrm{y}D_J
\end{tikzcd}$$

is a pullback, the following must also be a pullback:

$$\begin{tikzcd}
C_I \arrow[r, "f_C"] \arrow[d, "m_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_J \arrow[d, "m_J"] \\
D_I \arrow[r, "f_D"'] & D_J
\end{tikzcd}$$

and is in fact an $\mathcal{M}$-pullback. Therefore, as $F$ is an $\mathcal{M}$-functor, the left square in Figure 4.2 must be a pullback, and $Fm_I \in \mathcal{N}$ for all $I \in \operatorname{el} Q$. Hence, as $F_!(\mu)\colon \operatorname{colim} F_!H \to \operatorname{colim} F_!K$ is the unique map making Figure 4.2 commute, $F_!(\mu) \in \mathcal{N}$ by Lemma 26.

Observe that by the same lemma, the right hand square in Figure 4.2 is also a pullback for all $J \in \operatorname{el} Q$, which means that $F_!$ preserves pullbacks of the form:

$$\begin{tikzcd}
\mathrm{y}C_I \arrow[r] \arrow[d, "\mathrm{y}m_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\mu"] \\
\mathrm{y}D_I \arrow[r, "q_I"'] & Q
\end{tikzcd}$$

for all $I \in \operatorname{el} Q$ and $\mu \in \widehat{\mathcal{M}}$.

**2. $F_!$ preserves $\widehat{\mathcal{M}}$-pullbacks**

Let $P, P', Q, Q'$ be presheaves on $\mathbf{C}$ and $\mu, \mu'$ be $\widehat{\mathcal{M}}$-maps. Suppose the following is an $\widehat{\mathcal{M}}$-pullback:

$$\begin{tikzcd} P \arrow[r, "r_P"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\lrcorner", very near start] & P' \arrow[d, "\mu'"] \\ Q \arrow[r, "r_Q"'] & Q' \end{tikzcd}$$

Then for all $I \in \mathrm{el}\, Q$, all squares below are pullbacks:

$$\begin{tikzcd} \mathrm{y}C_I = HI \arrow[r, "p_I"] \arrow[d, "\alpha_I = \mathrm{y}m_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r, "r_p"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\lrcorner", very near start] & P' \arrow[d, "\mu'"] \\ \mathrm{y}D_I = KI \arrow[r, "q_I"'] & Q \arrow[r, "r_Q"'] & Q' \end{tikzcd}$$

Applying $F_!$ to the above diagram gives

$$\begin{tikzcd} FC_I \arrow[r, "F_!(p_I)"] \arrow[d, "Fm_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & F_!P \arrow[r, "F_!(r_p)"] \arrow[d, "F_!(\mu)"'] & F_!P' \arrow[d, "F_!(\mu') \in \mathcal{N}"] \\ FD_I \arrow[r, "F_!(q_I)"'] & F_!Q \arrow[r, "F_!(r_Q)"'] & F_!Q' \end{tikzcd}$$

where $F_!(\mu) \in \mathcal{N}$. In particular, as $F_!$ preserves pullbacks of the form

$$\begin{tikzcd} \mathrm{y}C_I \arrow[r] \arrow[d, "\mathrm{y}m_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\mu"] \\ \mathrm{y}D_I \arrow[r, "q_I"'] & Q \end{tikzcd}$$

both the left hand square and the outer square are pullbacks. Therefore, by Lemma 27, the right hand square must also be a pullback and so $F_!$ preserves $\widehat{\mathcal{M}}$-pullbacks. Hence, $F_!\colon (\widehat{\mathbf{C}}, \widehat{\mathcal{M}}) \to (\mathbf{D}, \mathcal{N})$ is an $\mathcal{M}$-functor and so $(-) \circ \mathrm{y}$ is essentially surjective.

For the final part of the proof, we need to show $(-) \circ \mathrm{y}$ is full and faithful. That is, suppose $F, G\colon (\widehat{\mathbf{C}}, \widehat{\mathcal{M}}) \to (\mathbf{D}, \mathcal{N})$ are cocontinuous $\mathcal{M}$-functors. Then for all $\mathcal{M}$-cartesian $\alpha\colon F\mathrm{y} \Rightarrow G\mathrm{y}$, we must show there is a unique $\widehat{\mathcal{M}}$-cartesian $\widetilde{\alpha}\colon F \Rightarrow G$ such that $\widetilde{\alpha} \circ \mathrm{y} = \alpha$. But this condition holds if the following statement is true: for all natural transformations $\widetilde{\alpha}\colon F \Rightarrow G$, if $\widetilde{\alpha} \circ \mathrm{y}\colon F\mathrm{y} \Rightarrow G\mathrm{y}$ is $\mathcal{M}$-cartesian, then $\widetilde{\alpha}$ is $\widehat{\mathcal{M}}$-cartesian.

[To see this, observe that $\widetilde{\alpha} \circ \mathrm{y}\colon F\mathrm{y} \Rightarrow G\mathrm{y}$ being $\mathcal{M}$-cartesian implying $\widetilde{\alpha}$ is $\widehat{\mathcal{M}}$-cartesian amounts to the statement that the following is a pullback in **Set**:

$$\begin{tikzcd} \mathcal{M}\mathbf{Cat}(F, G) \arrow[r, "(-)\circ \mathrm{y}"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}\mathbf{Cat}(F\mathrm{y}, G\mathrm{y}) \arrow[d, hook] \\ \mathbf{Nat}(F, G) \arrow[r, "(-)\circ \mathrm{y}"'] & \mathbf{Nat}(F\mathrm{y}, G\mathrm{y}) \end{tikzcd}$$

However, because $(-) \circ \mathrm{y}\colon \mathbf{Nat}(F, G) \to \mathbf{Nat}(F\mathrm{y}, G\mathrm{y})$ is an isomorphism (by the universal property of the Yoneda embedding), this implies the restriction of $(-) \circ \mathrm{y}$ to $\mathcal{M}\mathbf{Cat}(F, G)$ and $\mathcal{M}\mathbf{Cat}(F\mathrm{y}, G\mathrm{y})$ must also be an isomorphism].

So suppose $F, G\colon (\widehat{\mathbf{C}}, \widehat{\mathcal{M}}) \to (\mathbf{D}, \mathcal{N})$ are cocontinuous $\mathcal{M}$-functors, and $\widetilde{\alpha}\colon F \Rightarrow G$ is a natural transformation such that $\widetilde{\alpha} \circ \mathrm{y}\colon F\mathrm{y} \Rightarrow G\mathrm{y}$ is $\mathcal{M}$-cartesian. That is, for any $m\colon A \to B \in \mathcal{M}$, the following square is a pullback:

$$\begin{tikzcd} F\mathrm{y}A \arrow[r, "(\widetilde{\alpha}\mathrm{y})_A"] \arrow[d, "F\mathrm{y}m"'] \arrow[dr, phantom, "\lrcorner", very near start] & G\mathrm{y}A \arrow[d, "G\mathrm{y}m"] \\ F\mathrm{y}B \arrow[r, "(\widetilde{\alpha}\mathrm{y})_B"'] & G\mathrm{y}B \end{tikzcd}$$

We would like to show for any $\mu\colon P \Rightarrow Q \in \widehat{\mathcal{M}}$, the following square is also a pullback

$$\begin{tikzcd} FP \arrow[r, "\widetilde{\alpha}_P"] \arrow[d, "F\mu"'] & GP \arrow[d, "G\mu"] \\ FQ \arrow[r, "\widetilde{\alpha}_Q"'] & GQ \end{tikzcd}$$

Figure 4.3: Required to show above square is a pullback

where both $F\mu$ and $G\mu$ are in $\mathcal{N}$ (as $F, G$ are $\mathcal{M}$-functors). We saw previously that writing $Q \cong \operatorname{colim} \mathrm{y}\pi_Q = \operatorname{colim} K$ makes the left square below a pullback

$$\begin{tikzcd} FHI \arrow[r, "Fp_I"] \arrow[d, "Fm_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{colim} FH \cong FP \arrow[r, "\widetilde{\alpha}_P"] \arrow[d, "F\mu"'] & GP \arrow[d, "G\mu"] \\ FKI \arrow[r, "Fq_I"'] & \operatorname{colim} FK \cong FQ \arrow[r, "\widetilde{\alpha}_Q"'] & GQ \end{tikzcd}$$

for every $I \in \operatorname{el} Q$ (as $F$ is cocontinuous). Therefore, by Lemma 27, if we can show that the outer square is also a pullback, then the square on the right will be a pullback. So consider the following diagram:

$$\begin{tikzcd} FHI \arrow[r, "\widetilde{\alpha}_{HI} = (\widetilde{\alpha}\mathrm{y})_{C_I}"] \arrow[d, "Fm_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & GHI \arrow[r, "Gp_I"] \arrow[d, "Gm_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & GP \cong \operatorname{colim} GH \arrow[d, "G\mu"] \\ FKI \arrow[r, "\widetilde{\alpha}_{KI} = (\widetilde{\alpha}\mathrm{y})_{D_I}"'] & GKI \arrow[r, "Gq_I"'] & GQ \cong \operatorname{colim} GK \end{tikzcd}$$

By assumption, the left square is a pullback, and so is the right square; therefore, the outer square must be a pullback. But $\widetilde{\alpha}_P \circ Fp_I = Gp_I \circ \widetilde{\alpha}_{HI}$ and $\widetilde{\alpha}_Q \circ Fq_I = Gq_I \circ \widetilde{\alpha}_{KI}$ as the following square commutes

$$\begin{tikzcd} FHI \arrow[r, "\widetilde{\alpha}_{HI}"] \arrow[d, "Fp_I"'] & GHI \arrow[d, "Gp_I"] \\ FP \arrow[r, "\widetilde{\alpha}_P"'] & GP \end{tikzcd}$$

for each $I \in \mathrm{el}\, Q$ (by naturality of $\widetilde{\alpha}$). This shows the naturality square in FIGURE 4.3 is a pullback for each $\mu\colon P \Rightarrow Q$ in $\widehat{\mathcal{M}}$, and completes the proof that $(-) \circ \mathrm{y}$ is full and faithful. Hence,

$$(-) \circ \mathrm{y}\colon \mathbf{Cocts}\mathcal{M}\mathbf{Cat}\Big((\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), (\mathbf{D}, \mathcal{N})\Big) \to \mathcal{M}\mathbf{Cat}\Big((\mathbf{C}, \mathcal{M}), (\mathbf{D}, \mathcal{N})\Big)$$

is an equivalence of categories. □

## 4.2 Restriction categories and their free cocompletion

To motivate our definition of cocomplete restriction category, consider the split restriction category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$. Applying $\mathrm{y}_r$ gives $\widehat{\mathsf{Par}(\mathbf{C}, \mathcal{M})}_r \simeq \mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}})$, which ought to be cocomplete with respect to restriction categories (in the same way $\widehat{\mathbf{C}}$ is cocomplete).

So what properties does $\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}})$ have? First, it is a split restriction category, with $\mathsf{Total}\big(\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}})\big) \cong \widehat{\mathbf{C}}$. That is, its subcategory of total maps is cocomplete. Also, by PROPOSITION 2.5 of [3], the $\mathcal{M}$-category $(\widehat{\mathbf{C}}, \widehat{\mathcal{M}})$ is classified, which means $\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}})$ is a classified restriction category. In other words, the inclusion $\widehat{\mathbf{C}} \hookrightarrow \mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}})$ has a right adjoint. We therefore give the following definition.

**Definition 29** (Cocomplete restriction category)**.** *A restriction category* $\mathbf{X}$ *is cocomplete if:*

- $\mathbf{X}$ *is split,*
- $\mathsf{Total}(\mathbf{X})$ *is cocomplete, and*
- *the inclusion* $\mathsf{Total}(\mathbf{X}) \hookrightarrow \mathbf{X}$ *preserves all colimits*

**Example 30.** *Consider again the restriction category of sets and partial functions (***Pfn***) where the restriction on each arrow $f\colon A \rightharpoonup B$ is given by the idempotent $\overline{f}\colon A \rightharpoonup A$ defined as follows:*

$$\overline{f}(a) = \begin{cases} a; & \text{if } f(a) \text{ is defined at } a \in A \\ \text{undefined}; & \text{otherwise} \end{cases}$$

*Clearly* **Pfn** *is a split restriction category and* $\mathsf{Total}(\mathbf{Pfn}) = \mathbf{Set}$ *is cocomplete. Also, the inclusion* $\mathbf{Set} \hookrightarrow \mathbf{Pfn}$ *has a right adjoint and so* $\mathsf{Total}(\mathbf{X}) \hookrightarrow \mathbf{X}$ *preserves all colimits. Therefore,* **Pfn** *is a cocomplete restriction category.*

The following is an example of a split restriction category which is *not* cocomplete despite the fact that its subcategory of total maps is cocomplete (as the inclusion fails to preserve all colimits).

**Example 31.** *Consider the split restriction category* $\mathsf{Par}(\mathbf{Ab}, \mathcal{M})$*, where* **Ab** *denotes the category of abelian groups and* $\mathcal{M}$ *is a stable system of monics in* **Ab** *containing all zero maps and the diagonal* $\Delta\colon G \to G \times G, g \mapsto (g, g)$*. Clearly* $\mathsf{Par}(\mathbf{Ab}, \mathcal{M})$ *is split and* $\mathsf{Total}\big(\mathsf{Par}(\mathbf{Ab}, \mathcal{M})\big) \cong \mathbf{Ab}$ *is cocomplete. Let* $\mathbb{Z}$ *denote the group of integers under addition and* $0$ *the trivial group. Now the inclusion* $\mathbf{Ab} \hookrightarrow \mathsf{Par}(\mathbf{Ab}, \mathcal{M})$ *takes the coproduct* $(\mathbb{Z} \oplus \mathbb{Z}, \iota_1, \iota_2)$ *in* **Ab** *to the diagram*

$$\mathbb{Z} \xrightarrow{(1,\iota_1)} \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{(1,\iota_2)} \mathbb{Z}$$

*in* $\mathsf{Par}(\mathbf{Ab}, \mathcal{M})$ *(where* $\iota_1, \iota_2$ *are the coproduct injections). However,* $\big(\mathbb{Z} \oplus \mathbb{Z}, (1, \iota_1), (1, \iota_2)\big)$ *is not a coproduct in* $\mathsf{Par}(\mathbf{Ab}, \mathcal{M})$ *as both the maps* $(0, 0)$ *and* $(\Delta, 0)$ *make the following diagram commute:*

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "(1,\iota_1)"] \arrow[dr, "(0,0)"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[d, shift right, "(0,0)"'] \arrow[d, shift left, "(\Delta,0)"] & \mathbb{Z} \arrow[l, "(1,\iota_2)"'] \arrow[dl, "(0,0)"] \\ & 0 & \end{tikzcd}$$

*Therefore,* $\mathsf{Par}(\mathbf{Ab}, \mathcal{M})$ *is* not *cocomplete as a restriction category.*

**Definition 32** (Cocontinuous restriction functor)**.** *A restriction functor* $F\colon \mathbf{X} \to \mathbf{Y}$ *is* cocontinuous *if* $\mathsf{Total}(F)\colon \mathsf{Total}(\mathbf{X}) \to \mathsf{Total}(\mathbf{Y})$ *is cocontinuous.*

Denote by **CoctsRCat** the 2-category of cocomplete restriction categories, cocontinuous restriction functors and restriction transformations. We would now like to show the restriction Yoneda embedding $\mathrm{y}_r\colon \mathbf{X} \to \widehat{\mathbf{X}}_r$ exhibits the restriction presheaf category $\widehat{\mathbf{X}}_r$ as the restriction free cocompletion of $\mathbf{X}$.

**Theorem 33.** *Let* $\mathcal{E}$ *be a cocomplete restriction category. Then the functor*

$$(-) \circ \mathrm{y}_r\colon \mathbf{CoctsRCat}(\widehat{\mathbf{X}}_r, \mathcal{E}) \to \mathbf{rCat}(\mathbf{X}, \mathcal{E})$$

*is an equivalence of categories.*

*Proof.* We begin by proving the following equivalence

$$\mathbf{CoctsRCat}(\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), \mathcal{E}) \simeq \mathbf{rCat}(\mathbf{X}, \mathcal{E})$$

(where $\mathbf{C} = \mathsf{Total}(K_r(\mathbf{X}))$ and $\mathcal{M} = \mathcal{M}_{K_r(\mathbf{X})}$). We know from Proposition 6 that $(-) \circ J\colon \mathbf{rCat}(K_r(\mathbf{X}), \mathcal{E}) \to \mathbf{rCat}(\mathbf{X}, \mathcal{E})$ is an equivalence of categories, and also that $(-) \circ \Phi_{K_r(\mathbf{X})}\colon \mathbf{rCat}(\mathsf{Par}(\mathbf{C}, \mathcal{M}), \mathcal{E}) \to \mathbf{rCat}(K_r(\mathbf{X}), \mathcal{E})$ is an isomorphism. Therefore, if the functor $(-) \circ \mathsf{Par}(\mathrm{y})\colon \mathbf{CoctsRCat}(\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), \mathcal{E}) \to \mathbf{rCat}(\mathsf{Par}(\mathbf{C}, \mathcal{M}), \mathcal{E})$ is an equivalence, then the following composite will also be an equivalence:

$$\mathbf{CoctsRCat}(\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), \mathcal{E}) \xrightarrow{(-)\circ\mathsf{Par}(\mathrm{y})} \mathbf{rCat}(\mathsf{Par}(\mathbf{C}, \mathcal{M}), \mathcal{E}) \xrightarrow{(-)\circ\Phi_{K_r(\mathbf{X})}} \mathbf{rCat}(K_r(\mathbf{X}), \mathcal{E}) \xrightarrow{(-)\circ J} \mathbf{rCat}(\mathbf{X}, \mathcal{E})$$

Now the fact $\mathcal{E} \cong \mathsf{Par}(\mathbf{D}, \mathcal{N})$ (where $\mathbf{D} = \mathsf{Total}(\mathcal{E})$ and $\mathcal{N}$ contains the restriction monics in $\mathcal{E}$) implies there is an isomorphism

$$\mathbf{CoctsRCat}(\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), \mathcal{E}) \cong \mathbf{CoctsRCat}\big(\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), \mathsf{Par}(\mathbf{D}, \mathcal{N})\big)$$

But since $\mathsf{Par}\colon \mathcal{M}\mathbf{Cat} \to \mathbf{rCat}_s$ and $\mathcal{M}\mathsf{Total}\colon \mathbf{rCat}_s \to \mathcal{M}\mathbf{Cat}$ are 2-equivalences, the following is also an equivalence:

$$\mathbf{CoctsRCat}\big(\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), \mathsf{Par}(\mathbf{D}, \mathcal{N})\big) \simeq \mathbf{Cocts}\mathcal{M}\mathbf{Cat}\big((\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), (\mathbf{D}, \mathcal{N})\big)$$

Now consider the diagram below:

$$\begin{tikzcd}
\mathbf{CoctsRCat}(\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), \mathcal{E}) \arrow[r, "(-)\circ \mathsf{Par}(\mathrm{y})"] \arrow[d, "\cong"'] & \mathbf{rCat}(\mathsf{Par}(\mathbf{C}, \mathcal{M}), \mathcal{E}) \\
\mathbf{CoctsRCat}\big(\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), \mathsf{Par}(\mathbf{D}, \mathcal{N})\big) \arrow[r, "(-)\circ \mathsf{Par}(\mathrm{y})"] \arrow[d, "\simeq"'] & \mathbf{rCat}\big(\mathsf{Par}(\mathbf{C}, \mathcal{M}), \mathsf{Par}(\mathbf{D}, \mathcal{N})\big) \arrow[u, "\cong"'] \\
\mathbf{Cocts}\mathcal{M}\mathbf{Cat}\big((\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), (\mathbf{D}, \mathcal{N})\big) \arrow[r, "(-)\circ \mathrm{y}"'] & \mathcal{M}\mathbf{Cat}\big((\mathbf{C}, \mathcal{M}), (\mathbf{D}, \mathcal{N})\big) \arrow[u, "\simeq"']
\end{tikzcd}$$

Therefore, $(-) \circ \mathsf{Par}(\mathrm{y})\colon \mathbf{CoctsRCat}(\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), \mathcal{E}) \to \mathbf{rCat}(\mathsf{Par}(\mathbf{C}, \mathcal{M}), \mathcal{E})$ will be an equivalence if $(-) \circ \mathrm{y}\colon \mathbf{Cocts}\mathcal{M}\mathbf{Cat}\big((\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), (\mathbf{D}, \mathcal{N})\big) \to \mathcal{M}\mathbf{Cat}\big((\mathbf{C}, \mathcal{M}), (\mathbf{D}, \mathcal{N})\big)$ is an equivalence. But this is true by THEOREM 28, and so

$$\mathbf{CoctsRCat}(\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), \mathcal{E}) \simeq \mathbf{rCat}(\mathbf{X}, \mathcal{E})$$

Recall from the previous chapter that both the functors $L\colon \mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}) \to \mathsf{Par}\widehat{(\mathbf{C}, \mathcal{M})}_r$ and $(-) \circ (\Phi_{K_r(\mathbf{X})} \circ J)^{\mathrm{op}} : \mathsf{Par}\widehat{(\mathbf{C}, \mathcal{M})}_r \to \widehat{\mathbf{X}}_r$ were equivalences. Therefore, the following composite is also an equivalence:

$$\begin{tikzcd}
\mathbf{CoctsRCat}(\widehat{\mathbf{X}}_r, \mathcal{E}) \arrow[d, "(-)\circ\left((-)\circ(\Phi_{K_r(\mathbf{X})}\circ J)^{\mathrm{op}}\right)"] \\
\mathbf{CoctsRCat}(\mathsf{Par}\widehat{(\mathbf{C}, \mathcal{M})}_r, \mathcal{E}) \arrow[d, "(-)\circ L"] \\
\mathbf{CoctsRCat}(\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}), \mathcal{E}) \arrow[d, "(-)\circ \mathsf{Par}(\mathrm{y})"] \\
\mathbf{rCat}(\mathsf{Par}(\mathbf{C}, \mathcal{M}), \mathcal{E}) \arrow[d, "(-)\circ \Phi_{K_r(\mathbf{X})}"] \\
\mathbf{rCat}(K_r(\mathbf{X}), \mathcal{E}) \arrow[d, "(-)\circ J"] \\
\mathbf{rCat}(\mathbf{X}, \mathcal{E})
\end{tikzcd}$$

But by THEOREM 22, there exists an isomorphism between $\mathrm{y}_r$ and the following composite:

$$\mathbf{X} \xrightarrow{J} K_r(\mathbf{X}) \xrightarrow{\Phi_{K_r(\mathbf{X})}} \mathsf{Par}(\mathbf{C}, \mathcal{M}) \xrightarrow{\mathsf{Par}(\mathrm{y})} \mathsf{Par}\left(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}\right) \xrightarrow{L} \mathsf{Par}\widehat{(\mathbf{C}, \mathcal{M})}_r \xrightarrow{(-)\circ\Phi^{\mathrm{op}}_{K_r(\mathbf{X})}} \widehat{K_r(\mathbf{X})}_r \xrightarrow{(-)\circ J^{\mathrm{op}}} \widehat{\mathbf{X}}_r$$

Hence, $(-) \circ \mathrm{y}_r\colon \mathbf{CoctsRCat}(\widehat{\mathbf{X}}_r, \mathcal{E}) \to \mathbf{rCat}(\mathbf{X}, \mathcal{E})$ is an equivalence of categories. □

A consequence of the above result is that for any cocomplete restriction category $\mathcal{E}$ and restriction functor $H\colon \mathbf{X} \to \mathcal{E}$, there is a unique cocontinuous restriction functor $H_!\colon \widehat{\mathbf{X}}_r \to \mathcal{E}$ (up to isomorphism) such that $H_! \circ \mathrm{y}_r \cong H$

$$
\begin{tikzcd}
\widehat{\mathbf{X}}_r \arrow[r, "H_!"] & \mathcal{E} \cong \mathsf{Par}(\mathbf{D}, \mathcal{N}) \\
\widehat{K_r(\mathbf{X})}_r \arrow[u, "(-) \circ J^{\mathrm{op}}"] & \\
\mathsf{Par}\widehat{(\mathbf{C}, \mathcal{M})}_r \arrow[u, "(-) \circ \Phi^{\mathrm{op}}_{K_r(\mathbf{X})}"] & \\
\mathsf{Par}(\widehat{\mathbf{C}}, \widehat{\mathcal{M}}) \arrow[u, "L", "\cong"'] & \\
\mathsf{Par}(\mathbf{C}, \mathcal{M}) \arrow[u, "\mathsf{Par}(\mathrm{y})"] & \\
K_r(\mathbf{X}) \arrow[u, "\Phi_{K_r(\mathbf{X})}"] & \\
\mathbf{X} \arrow[u, "J"] \arrow[uuuuuu, bend left=60, "\mathrm{y}_r"] \arrow[uuuuuur, bend right, "H"'] &
\end{tikzcd}
$$

We say that the restriction Yoneda embedding exhibits the restriction presheaf category $\widehat{\mathbf{X}}_r$ as the *restriction free cocompletion* of $\mathbf{X}$.

# 5
# Conclusion

We introduced the notion of restriction presheaf and saw that the category of restriction presheaves had a canonical split restriction structure. We then defined a notion of cocompleteness in **rCat**, and showed that the restriction Yoneda embedding $y_r$ exhibited the restriction presheaf category $\widehat{\mathbf{X}}_r$ as the free cocompletion of $\mathbf{X}$ (for small $\mathbf{X}$).

A possible continuation of this work would be to extend this to involve *join restriction categories*. Recall that a join restriction category is a restriction category $\mathbf{X}$ such that for any two objects $A, B \in \mathbf{X}$, the join of any *compatible* subset $S \subset \mathbf{X}(A, B)$ exists and satisfies certain axioms [5]. In the same way we defined restriction presheaf, cocompleteness and free cocompletion of restriction categories, we may repeat the same process but for join restriction categories. Understanding colimits in the join restriction setting will give us another way of understanding the meaning of *assembling* local pieces of data.

# References


[1] M. Grandis. *Cohesive categories and manifolds*. Ann. Mat. Pura Appl. (4) **157**, 199 (1990). URL http://dx.doi.org/10.1007/BF01765319.

[2] J. R. B. Cockett and S. Lack. *Restriction categories. I. Categories of partial maps*. Theoret. Comput. Sci. **270**(1-2), 223 (2002). URL http://dx.doi.org/10.1016/S0304-3975(00)00382-0.

[3] J. R. B. Cockett and S. Lack. *Restriction categories. II. Partial map classification*. Theoret. Comput. Sci. **294**(1-2), 61 (2003). Category theory and computer science, URL http://dx.doi.org/10.1016/S0304-3975(01)00245-6.

[4] R. Cockett and S. Lack. *Restriction categories. III. Colimits, partial limits and extensivity*. Math. Structures Comput. Sci. **17**(4), 775 (2007). URL http://dx.doi.org/10.1017/S0960129507006056.

[5] X. Guo. *Products, Joins, Meets, and Ranges in Restriction Categories*. Ph.D. thesis, Department of Computer Science, University of Calgary (2012).

[6] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic*. Universitext (Springer-Verlag, New York, 1994). A first introduction to topos theory, Corrected reprint of the 1992 edition.