Neumann Boundary Value Problem for the Helmholtz Equation: 2D Arbitrary Boundary

By

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Abstract

This thesis is aimed at presenting the mathematically rigorous analytical-numerical method for solving the Neumann boundary-value problem for Helmholtz equation. It is consistently realised that the idea of analytical regularisation of ill-conditioned integral, integral-differential and series equations of the first kind resulted in the efficient technique and numerical algorithm which allows accurate numerical solution. the presented regularisation technique is successfully used studies of two-dimensional wave scattering by closed and unclosed screens. The thesis concentrates on the screens in the form of infinitely long cylinders with circular and arbitrary cross sections. When the boundary of the cross section is opened (or unclosed) we get a slotted cylinder.

Contents

1 Introduction

1.1 Boundary Value Problem for scattering by an Open Cylinder

The study of wave scattering has a significantly long history across physics, mathematics and engineering; its practical application produced various and many impacts in technology such as telecommunications, radar, lasers, ultrasonic imagining, etc. With such enormous application across numerous fields, accurate mathematical simulation of wave scattering problems has been developed by many researchers for many decades. From an analytical and numerical point of view, it is fair to say that the studies of wave scattering problems on a closed body with smooth surface is well developed. So, it is natural for the study to move on, by considering (open) scatterers with cavities and edges.

The focus of this thesis is to provide an approach to find the solution of the Neumann mixed boundary value problem (BVP) of scattering from an infinitely long (open) cylinder with arbitrary cross section. Due to the phenomena of wave action on the cylinder being identical on each cross section, with an appropriate formulation, the wave scattering problem is then reduced to a 2-dimensional problem where we only need to consider a cross section.

Since this is ultimately a model for a physical problem, we shall discuss some context on the physical side. The total wave, incident wave and scattered wave need to be distinguished. The incident wave is that sent out from some radiating source and it is assumed that the functional form and behaviour is completely known if it were allowed to travel unimpeded. In two dimensions, the wave will take the form of a circular outward travelling wave with a point source or of a plane wave. Note that the plane wave can be thought of a circular wave where the source point is very far away from the observation point, and locally behaves as a plane wave. The total wave is viewed as the resultant wave which describes the behaviour of the incident wave in the presence of the scatterer. Then the scattered wave can be defined as the difference of the total wave and the incident wave.

Our main task of this thesis is to use a mathematically rigorous method to calculate the scattered wave. We assume our waves to have time harmonic dependence $e^{-i\omega t}$, so the waves' spatial behaviour is captured by the solution of Helmholtz equation. Since edges are being introduced in our scatterer, the Meixner integrability condition (or simply, the edge condition) also need to be enforced. Then the solution is bounded, and it is valid to be used to model physical scattering problems which involve finite sources. Moreover, we are expecting the wave to decay at infinity, so the solution also need to obey Sommerfeld radiation conditions. Finally, the solution will also need to satisfy the Neumann boundary condition. A proof in [2] shows that if a solution satisfies all these conditions, then it will be unique.

Geometrically, a cross section of cylinder looks like a closed non-self-crossing contour of finite length. For the cross section of an open (or singly slotted) cylinder, we think of it as a connected subset of the closed contour. The cylinder with circular cross section would be the first special case to examine. For the closed circle case, it allows us to parametrise the surface by the classical cylindrical polar coordinates; it can be solved by the technique of separation of variables. As a result, the wave scattering problem can be solved analytically, and an analytical solution can be obtained. When we move on to the case of an open circle, it turns out the BVP is in the form of dual series equation which can be reduced to a coupled dual series equation which involving sine and cosine function. From there, we can make use of the Jacobi representation of the trigonometric functions, and we are able to apply the Method of Regularisation (MoR) which is described in [3]. This technique was developed alongside that for potential theory, and it can be applied when we have dual series equation in a particular form. The technique leads us to a well-conditioned Fredholm equation of the second kind. Hence, numerical solution can be found by the application of relatively straight forward numerical techniques for the solution of such systems.

For the arbitrary case, since there is no canonical or separable coordinate system in which the Laplace or Helmholtz equation is separable, we shall consider the solution in its integral representation, which involves a Green's function. However, the integral representation of solution results in the form of a first kind Fredholm integral equation, which is inherently an ill-posed problem [6]. When we solve with direct numerical methods, by using a Riemannian sum to convert the integral equation into an infinite system of linear algebraic equations, and computing the solution by a truncation method, the truncation method replaces the infinite system by a finite number N_{tr} of linear algebraic equations, in which all infinite sums are truncated to retain only the variables $x_0, x_1, \dots, x_{tr}$. This approximate numerical solution of the integral equation is shown to be unreliable (see e.g. [5] [7]), and errors will be eventually amplified as more and more terms are taken into consideration.

Clearly another approach (a rigorous one) is required to obtain the numerical solution, and the one we going to discuss in this thesis is the MoR which we have done for the open circle case. The key idea of the MoR is to convert the ill-conditioned first kind Fredholm equation into a well-conditioned second kind Fredholm equation, analytically. The unknown function of first kind Fredholm equation is still the same (in the sense of possessing the same Fourier expansion) after the analytical transformation to second kind Fredholm equation. The conversion is only done to the operator itself but does not actually change any information of the unknown function. Furthermore, the second kind Fredholm equation is guaranteed to have an unique solution, and the solution which is computed from the truncated well-conditioned matrix equation is reliable, stable, numerically accurate and efficient. The precise treatments of the behaviour of second-kind system under truncation can be found in [10], [11].

Moreover, the method and mathematical justification for the Dirichlet case has been studied in [8] and [9] for infinitely long cylinder with arbitrary shaped cross section; several classes of canonical scatterers with edge-cavity structures are examined with the same approach in [4]. However, in comparison, the regularisation of the Neumann case is more difficult than the Dirichlet case since the kernel of the first kind Fredholm equation which we obtained is a strongly singular double normal derivative of the Green function of the Helmholtz equation. We use the results from [13] to transform the integral equation to the equivalent dual series equation by the Fourier expansion of each term. The exponential function can be converted into a coupled dual series equation which involves sine and cosine function. Then, by using the same approach as for the open circle case, apply the MoR to convert the dual series equation to a well-conditioned second kind Fredholm matrix equation. Hence, numerical method can be applied to this appropriately transformed system.

1.2 Outline of the thesis

In chapter 2, we begin by stating the tools which we are going to use for the regularisation. In particular, the Abel transform and the properties of Jacobi polynomials which are mentioned in this chapter play the important roles for the regularisation in the later chapters. Most of the steps during the regularisation will required reader to refer back to this chapter.

In chapter 3, we consider the BVP for the circle in both the closed case and the open case. For the closed case, we shall proceed with the technique of separation of variables and obtain an analytical solution. However, for the open case, the solution is in the form of dual series equation. To solve the dual series equation, we perform the MoR as developed in [3], and convert the dual series equation into Fredholm equations of the second kind to obtain the well conditioned equation that is suitable to compute the solution, numerically.

In chapter 4, we consider the formulation of the problem and its solution in integral representation. First, we formulate the problem so that the reduced 2D problem with different boundary conditions on the screen and aperture result in a Neumann mixed BVP. After enforcing the boundary conditions, the integral representation of the solution results in a Fredholm integral equation of the first kind, an ill-posed problem where the solution cannot reliable be obtained by numerical methods. In addition, the kernel is a strongly singular double normal derivative of the Green's function. To deal with this kernel, we use the results from [13] to convert them into their Fourier series. Hence, we obtain the equivalent dual series equation of the integral equation.

In chapter 5, we start by converting the dual series equation with exponential kernel which we obtained in the previous chapter into a coupled dual series equation which involving sine and cosine function. Since the trigonometric functions can be represented in Jacobi polynomial, we convert the series into a Fourier-Jacobi series, and perform the same regualrisation as for the open circle case to obtained the desired form of Fredholm matrix equation of the second kind.

2
Preliminaries

We shall begin with stating some definitions and results which we are going to use throughout this thesis. They are needed to carry out the regularisation process in later chapter. The topics concerns Abel's integral equation and special function, particularly Jacobi's polynomial. For full details, please consult [3].

2.1 Abel's integral equation

Abel's integral equation is a specific type of integral equation, which is in the form of

$$\int_a^x \frac{U(t)}{(x-t)^\lambda} dt = F(x), \quad x \in [a, b] \tag{2.1}$$

where the function $U(t)$ is the only unknown, $0 < \lambda < 1$ and its unique inversion is as the following

$$U(t) = \frac{\sin \lambda\pi}{\pi} \frac{d}{dt} \int_a^t \frac{F(x)}{(t-x)^\lambda} dx, \quad t \in [a, b].$$

Also, there is a companion form of this integral equation, namely

$$\int_x^b \frac{U(t)}{(t-x)^{1-\lambda} dt} = F(x), \quad x \in [a, b] \tag{2.2}$$

where the function $U(t)$ is the only unknown, $0 < \lambda < 1$ and has the unique inversion in the form of

$$U(t) = -\frac{\sin \lambda\pi}{\pi} \frac{d}{dt} \int_t^b \frac{F(x)}{(x-t)^{1-\lambda}} dx, \quad t \in [a, b].$$

Integral equations which are in the form of (2.1) or (2.2) are called *integral equation of Abel type*.

2.2 Gamma function

The Gamma function Γ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad Re(z) > 0.$$

It is a generalisation of the factorial, widely used across many fields.

The relevant properties which we are going to use in later section is the *recurrence formula*

$$\Gamma(\nu + 1) = \nu\Gamma(\nu)$$

which can be shown by integration by parts.

Further, for positive integer $n \in \mathbb{N}$,

$$\Gamma(n + 1) = n!.$$

Also, a well known identity is $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

2.3 Jacobi polynomials

Jacobi polynomial form a special class of polynomials. For fixed (α, β) with $\alpha > -1$, $\beta > -1$, each Jacobi polynomial $P_n^{(\alpha,\beta)}$ is a polynomial of degree n (where $n = 0, 1, 2, 3, \dots$), and is the solution to the differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} + [\beta - \alpha - (\alpha + \beta + 2)x]\frac{dy}{dx} + n(n + \alpha + \beta + 1)y = 0.$$

They form a complete orthogonal basis for the functional space $L^2[-1, 1]$ with respect to the weight function $w_{(\alpha,\beta)}(x) = (1 - x)^\alpha(1 + x)^\beta$, i.e.

$$\begin{aligned}
\langle P_n^{(\alpha,\beta)}, P_m^{(\alpha,\beta)}\rangle &= \int_{-1}^1 (1 - x)^\alpha(1 + x)^\beta P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx \\
&= h_n^{(\alpha,\beta)}\delta(n, m)
\end{aligned}$$

where

$$\begin{aligned}
h_n^{(\alpha,\beta)} &= \|P_n^{(\alpha,\beta)}\|^2 \\
&= \int_{-1}^1 (1 - x)^\alpha(1 + x)^\beta[P_n^{(\alpha,\beta)}(x)]^2dx \\
&= \frac{2^{\alpha+\beta+1}}{2n + \alpha + \beta + 1}\frac{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{n!\Gamma(n + \alpha + \beta + 1)}
\end{aligned}$$

2.4 Normalised Jacobi polynomials

The normalised Jacobi polynomials

$$\widehat{P}_n^{(\alpha,\beta)}(x) = \frac{1}{\|P_n^{(\alpha,\beta)}\|} P_n^{(\alpha,\beta)}(x)$$

are commonly used, and they form a complete orthonormal basis for the functional space $L^2[-1,1]$.

Further more, when $\alpha = 0$ and $\beta = 1$, the normalised Jacobi polynomials $\widehat{P}_n^{(\alpha,\beta)}(x)$ satisfy the following for $x \in [-1,1]$:

$$\widehat{P}_n^{(0,1)}(x) = -\frac{1}{n+1}\frac{d}{dx}\left[(1-x)\widehat{P}_n^{(0,1)}(x)\right], \tag{2.3}$$

$$\int_x^1 \widehat{P}_n^{(0,1)}(t)dt = \frac{1-x}{n+1}\widehat{P}_n^{(1,0)}(x), \quad x \in [-1,1] \tag{2.4}$$

$$\widehat{P}_n^{(1,0)}(x) = \frac{1+x}{n+1}\frac{d}{dx}\{\widehat{P}_n^{(0,1)}(x)\}. \tag{2.5}$$

2.5 Incomplete scalar product

Another tool which we required is the *incomplete scalar product*, we denote

$$Q_{n,m}^{(\alpha,\beta)}(x) = \int_x^1 (1-t)^\alpha (1+t)^\beta P_n^{(\alpha,\beta)(t)} P_m^{(\alpha,\beta)}(t)dt$$

and the normalised version

$$\widehat{Q}_{n,m}^{(\alpha,\beta)}(x) = \int_x^1 (1-t)^\alpha (1+t)^\beta \widehat{P}_n^{(\alpha,\beta)(t)} \widehat{P}_m^{(\alpha,\beta)}(t)dt.$$

Note that the "incompleteness" is referring to the integration is performed over the subinterval $[x,1]$, and $Q_{m,n}^{(\alpha,\beta)}(-1) = \langle P_n^{(\alpha,\beta)}, P_m^{(\alpha,\beta)}\rangle$

Also, a notable relation which we need in the later section is

$$\widehat{Q}_{n,m}^{(0,1)}(x) = \frac{1-x^2}{m+1}\widehat{P}_n^{(0,1)}(x)\widehat{P}_m^{(1,0)}(x) + \frac{n+1}{m+1}\widehat{Q}_{n,m}^{(1,0)}(x) \tag{2.6}$$

2.6 Properties of Jacobi polynomials

Next, we shall discuss the properties of Jacobi polynomials. Of their many properties, we are only stating those which are relevant to our problem.

For a fixed parameter $\eta \in [0,1)$,

$$P_n^{(\alpha,\beta)}(x) = \frac{(1-x)^{-\alpha}\Gamma(n+1+\alpha)}{\Gamma(1-\eta)\Gamma(n+\alpha+\eta)} \int_x^1 \frac{(1-t)^{\alpha+\eta-1} P_n^{(\alpha+\eta-1,\beta-\eta+1)}(t)}{(t-x)^\eta} dt \tag{2.7}$$

and the companion version

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+x)^{-\beta}\Gamma(n+1+\beta)}{\Gamma(1-\eta)\Gamma(n+\beta+\eta)} \int_{-1}^{x} \frac{(1+t)^{\beta+\eta-1}P_n^{(\alpha-\eta+1,\beta+\eta-1)}(t)}{(x-t)^{\eta}} dt. \tag{2.8}$$

Equation (2.7) express as $(1-x)^{\alpha}P_n^{(\alpha,\beta)}(x)$ as the product of the *Abel integral transform* of $(1-t)^{\alpha+\eta-1}P_n^{(\alpha+\eta-1,\beta-\eta+1)}(t)$ and a constant factor; (2.8) has a similar interpretation.

When $\eta = 0$, we have the following identities

$$(1-x)^{\alpha+1}P_n^{(\alpha+1,\beta-1)}(x) = (n+\alpha+1)\int_x^1 (1-t)^{\alpha}P_n^{(\alpha,\beta)}(t)dt \tag{2.9}$$

and

$$(1+x)^{\beta+1}P_n^{(\alpha-1,\beta+1)}(x) = (n+\beta+1)\int_{-1}^{x} (1+t)^{\beta}P_n^{(\alpha,\beta)}(t)dt. \tag{2.10}$$

Also, the *Rodrigues's formula* for the Jacobi's polynomial is

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{1}{(1-x)^{\alpha}(1+x)^{\beta}}\left(\frac{d}{dx}\right)^n\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right] \tag{2.11}$$

from which can be deduced

$$-2n(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x) = \frac{d}{dx}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}P_{n-1}^{(\alpha+1,\beta+1)}(x)\right] \tag{2.12}$$

Note that the relation (2.7)-(2.10) are all integral representations of Abel's type; they play the key role to solve the dual series equation in our problem.

2.7 Integrals involving arcsine function

The definite integral

$$\int_{-1}^{z} \frac{\frac{\pi}{2} + \arcsin x}{\sqrt{1-x}\sqrt{z-x}} dx = -\pi \ln\left(\frac{1-z}{2}\right) \tag{2.13}$$

will occur in the process of later calculation, which may be evaluated from the transform

$$-\sqrt{x}\pi \ln\cos\frac{\phi}{2} = \int_0^{\phi} \frac{\theta \sin\frac{1}{2}\theta}{\sqrt{\cos\theta - \cos\phi}} d\theta.$$

The companion version is

$$\int_{z}^{1} \frac{\frac{\pi}{2} - \arcsin x}{\sqrt{1+x}\sqrt{x-z}} dx = -\pi \ln\left(\frac{1+z}{2}\right), \tag{2.14}$$

which can be obtained by a change of variables.

These two definite integrals will occur specifically during the regularisation of the cosine system in the later sections.

2.8 The $l^2(\mu)$ space

The $l^2(\mu)$ is denoted by the space of sequences $\{x_n\}_{n=0}^{\infty}$ satisfying

$$\sum_{n=0}^{\infty} n^{\mu}|x_n|^2 < \infty. \tag{2.15}$$

Some steps among the regularisation are required to interchange the summation and integration, more precisely, the convergent of the term by term integration. And we want the coefficients of the series to fall into this space in order to claim the series to be *uniformly Abel summable*, in which term by term integration is permissable.

3

The Neumann BVP for a Circular Cylinder

3.1 Closed Circle

The Neumann BVP for a closed circle is a classical problem, which we examine this classical problem before we proceed to an open cylinder.

An infinitely long closed circular cylinder is placed parallel to the z-axis, so each of the cross section is perpendicular to the z-axis, and also the origin is the centre of the circular cross section on the xy-plane. So, the wave fronts strike the cylinder in the same way as z varies, and of the form

$$v^{sc} = u^{sc}(x, y)e^{-i\omega t}$$

where the time harmonic part $e^{-i\omega t}$ is what we have previously assumed; thus we only need to consider a 2D problem.

The scatterd field u^{sc} satisfies the Helmholtz equation

$$(\triangle + k^2)u = 0,$$

where $\triangle$ refers to the Laplacian and k is a constant (wave number). And in the Neumann problem, u^{sc} satisfies the boundary condition

$$\frac{\partial u^{sc}}{\partial n} = -\frac{\partial u^{inc}}{\partial n} \tag{3.1}$$

on the boundary, where u^{inc} refers to the incident field, and n is the outward pointing normal. In addition, the scattered field must obey the Sommerfeld radiation condition.

Since the cross section is a circle, we work with polar coordinates (ρ, θ), and we have the polar form Helmholtz equation as the following

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u^{sc}}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u^{sc}}{\partial \theta^2} + k^2 u^{sc} = 0. \tag{3.2}$$

By the technique of separation of variable with time harmonic dependence $e^{-i\omega t}$, we then obtain the outwardly travelling wave solution of (3.2) in the form of

$$u^{sc}(\rho,\theta) = \sum_{n=-\infty}^{\infty} E_n e^{in\theta} H_n^{(1)}(k\rho) \tag{3.3}$$

where $H_n^{(1)}$ is the Hankel function of the first kind, E_n are constants to be found. It can be shown that (3.3) satisfies the Sommerfeld radiation condition, which captures the behaviour of wave decaying at a far distance.

In polar coordinates, the Neumann boundary condition (3.1) is in the form of

$$\frac{\partial u^{sc}}{\partial \rho} = -\frac{\partial u^{inc}}{\partial \rho}$$

on the surface $\rho = a$, where $\rho = \sqrt{x^2 + y^2}$. Note that the normal derivative in boundary condition for a circle is simply the derivative with respect to the radius ρ.

After enforcing the boundary condition (where incident field $u^{inc}(\rho,\theta) = e^{-ik\rho\cos\theta}$), we obtain the solution

$$u^{sc}(\rho,\theta) = \sum_{n=-\infty}^{\infty} -i^n \frac{J_n'(ka)}{H_n^{(1)'}(ka)} H_n^{(1)}(k\rho) e^{in\theta} \tag{3.4}$$

where J_n' is the first derivative of Bessel function of the first kind.

And so, we solve the BVP analytically, and we have the solution in a Fourier series, where convergence can be guaranteed. However, it may not the case when we have the BVP for an open cylinder.

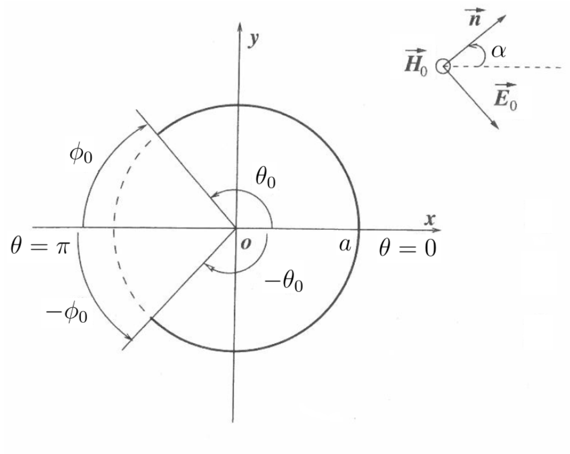
3.2 Open Circle

Now we are going to examine the same BVP for an infinitely long open circular cylinder. We shall follow the same formulation which was described in the closed circle case. The *screen* $\theta \in [-\theta_0, \theta_0]$ and the *aperture* $\theta \in (\theta_0, \pi) \cup (-\pi, -\theta_0)$ at a fixed radius $\rho = a$. The boundary conditions are

$$\frac{\partial u^{sc}}{\partial \rho}(\theta, a) = -\frac{\partial u^{inc}}{\partial \rho}(\theta, a) = 0 \quad \theta \in [-\theta_0, \theta_0], \tag{3.5}$$

$$u^{sc}(\theta, a) = u^{inc}(\theta, a) \quad \theta \in (\theta_0, \pi) \cup (-\pi, -\theta_0). \tag{3.6}$$

In this setting, we then have a *Neumann mixed boundary value problem.* Note that the incident wave is completely known.



Thus, we seek a scattered field solution in the form

$$u^{sc}(\theta, \rho) \;=\; \sum_{m=0}^{\infty} (-i)^m R_m(k\rho) \left[(2-\delta_{0,m})\, x_m \cos m\theta + 2y_m \sin m\theta\right] \tag{3.7}$$

where

$$R_m(k\rho) = \left\{ \begin{array}{cc} J_m(k\rho), & \rho < a \\ \frac{J'_m(ka)}{H_m^{(1)\prime}(ka)} H_m^{(1)}(k\rho), & \rho > a \end{array} \right\} \tag{3.8}$$

and $\{x_m\}_{m=0}^{\infty}$ and $\{y_m\}_{m=0}^{\infty}$ are unknowns to be found.

After enforcing the boundary conditions (3.5) and (3.6), the following (notice $\{x_m\}$ and $\{y_m\}$ are decoupled):

$$\sum_{m=1}^{\infty} \frac{i^m}{H_m^{(1)\prime}(ka)} y_m \sin m\phi = 0, \quad \phi \in (0, \phi_0), \tag{3.9}$$

$$\sum_{m=1}^{\infty} i^m J'_m(ka)\,(y_m + \sin m\alpha) \sin m\phi = 0, \quad \phi \in (\phi_0, \pi), \tag{3.10}$$

and

$$-\frac{x_0}{2H_1^{(1)}(ka)} + \sum_{m=1}^{\infty} \frac{i^m}{H_m^{(1)\prime}(ka)} x_m \cos m\phi = 0, \quad \phi \in (0, \phi_0), \tag{3.11}$$

$$-\frac{1}{2} J_1(ka)\,(1 + x_0) + \sum_{m=1}^{\infty} i^m J'_m(ka)\,(x_m + \cos m\alpha) \cos m\phi = 0, \quad \phi \in (\phi_0, \pi), \tag{3.12}$$

where $\phi = \pi - \theta$, $\phi_0 = \pi - \theta_0$ and α is the angle of incidence measured from the plane $\theta = 0$. (so that the incident wave $u^{inc} = e^{-ik(x\cos\alpha + y\sin\alpha)}$).

Introduce the rescaled unknown coefficients

$$\{X_m, Y_m\} = \frac{i^m}{H_m^{(1)\prime}(ka)} \{x_m, y_m\}. \tag{3.13}$$

From the asymptotic expansion

$$J_m(z) = \frac{z^m}{2^m\Gamma(m+1)}\left\{1 - \frac{z^2}{4(m+1)} + \frac{z^4}{32(m+1)(m+2)} + O\left(\frac{z^6}{m^3}\right)\right\} \tag{3.14}$$

$$H_m^{(1)}(z) = -\frac{i}{\pi}\frac{2^m\Gamma(m)}{z^m}\left\{1 + \frac{z^2}{4(m-1)} + \frac{z^4}{32(m-1)(m-2)} + O\left(\frac{z^6}{m^3}\right)\right\} \tag{3.15}$$

as $m \to \infty$, we deduce that

$$J_m'(ka)H_m^{(1)\prime}(ka) = \frac{i}{\pi}\frac{m}{(ka)^2}\left[1 + O\left(\frac{ka}{m^2}\right)\right]$$

as $m \to \infty$, so that the parameter

$$\varepsilon_m = 1 + i\pi\frac{(ka)^2}{m}J_m'(ka)H_m^{(1)\prime}(ka) \tag{3.16}$$

is asymptotically small as $m \to \infty$, i.e. $\varepsilon_m = O(\frac{ka}{m^2})$.

Hence, we convert (3.9)-(3.12) to

$$\sum_{m=1}^{\infty} Y_m \sin m\phi = 0, \quad \phi \in (0, \phi_0), \tag{3.17}$$

$$\sum_{m=1}^{\infty} m\left[X_m(1-\varepsilon_m) - p_m^s\right]\sin m\phi = 0, \quad \phi \in (\phi_0, \pi), \tag{3.18}$$

and

$$-\frac{x_0}{2H_1^{(1)}(ka)} + \sum_{m=1}^{\infty} X_m \cos m\phi = 0, \quad \phi \in (0, \phi_0), \tag{3.19}$$

$$\frac{i}{2}\pi(ka)^2 J_1(ka)(1+x_0) + \sum_{m=1}^{\infty} m\left[X_m(1-\varepsilon_m) - p_m^c\right]\cos m\phi = 0, \quad \phi \in (\phi_0, \pi), \tag{3.20}$$

where

$$\left\{\begin{array}{c} p_m^c \\ p_m^s \end{array}\right\} = \frac{i}{m}\pi(ka)^2 J_m'(ka)\left\{\begin{array}{c} \cos m\alpha \\ \sin m\alpha \end{array}\right\}.$$

From now on, we shall call (3.17) and (3.18) the sine system, (3.19) and (3.20) the cosine system.

3.3 Electromagnetic Field and The Edge Condition

The BVP comprising the Helmholtz equation the boundary conditions on the scatterer and the Sommerfeld radiation condition has a unique solution for a closed body (such as a circle). However an additional condition must be imposed in order to guarantee uniqueness for an open body (such as an open circular arc). The scattered field u^{sc} must satisfy the so-called edge condition (or the finite energy condition), for any arbitrary finite area V,

$$\int_V |\vec{E}|^2+|\vec{H}|^2 dV < \infty, \tag{3.21}$$

where $\vec{E}$ is the electric field, $\vec{H}$ is the magnetic field.

We consider the electromagnetic field of transverse electric type (H-polarisation, the only non-zero components are H_z, E_ρ, E_ϕ); the magnetic component of the electromagnetic field is oriented along the z-axis. We wish to find the total electromagnetic field

$$H_z\vec{i}_z = u^{tot}(x,y)\vec{i}_z$$

or equivalently, to find the scattered field

$$\begin{aligned} H_z^{sc}\vec{i}_z &= u^{sc}(x,y)\vec{i}_z \\ &= (u^{tot}(x,y) - u^{inc}(x,y)) \cdot \vec{i}_z \end{aligned}$$

resulting from the scattering of the incident wave by the screen.

This physical problem, thus defined, is described by the boundary value problem formulated above in terms of Helmholtz equation, with Neumann boundary conditions for the longitudinal component of the magnetic field $H_z^{sc} = u^{sc}(x,y)$. The two other non-zero field components are found from the relations:

$$\begin{aligned} E_\rho^{sc} &= -\frac{i}{k\rho}\frac{\partial H_z^{sc}}{\partial \phi}; \\ E_\phi^{sc} &= \frac{i}{k}\frac{\partial H_z^{sc}}{\partial \rho}. \end{aligned}$$

It can be shown that (3.21) can be reduced to

$$\int\limits_V |\bigtriangledown u^{sc}|^2 + k^2|u^{sc}|^2 dV < \infty$$

for any arbitrary finite area V.

This imply that $\{x_m\}_{m=0}^\infty$ and $\{y_m\}_{m=0}^\infty$ belong to the functional class $l^2(1)$, i.e.

$$\sum_{m=0}^\infty m|x_m|^2 < \infty, \quad \sum_{m=0}^\infty m|y_m|^2 < \infty.$$

Further, by the asymptotic formulae (3.15), $\{X_m\}_{m=0}^\infty$ and $\{Y_m\}_{m=0}^\infty$ also belong to the functional class $l^2(1)$.

3.4 Scheme of Solution

As for the solution of dual series equation, we are going to use the theory which described in chapter 2 of [3]. First, we replace the trigonometric functions of (3.17)-(3.20) by their Jacobi representations

$$\sin m\phi = \frac{\sqrt{\pi}}{2}\frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(\cos\phi) \tag{3.22}$$

and

$$\cos m\phi = \sqrt{\pi}\frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}P_m^{(-\frac{1}{2},-\frac{1}{2})}(\cos\phi). \tag{3.23}$$

The representation of Jacobi polynomials as Abel integral transforms (see Section 2.6) allows each equations to be expressed as Abel's integral equations with known inversion formula. This is perhaps the key step in our regularisation procedure. So, by applying Abel's integral transform, each original coupled system of dual series equation come to be represented in the form of a piecewise continuous function $F(\phi)$ defined over the full interval $[0,\pi]$, where

$$F(\phi) = \begin{cases} F_1(\phi), & \phi \in [0,\phi_0] \\ F_2(\phi), & \phi \in [\phi_0,\pi] \end{cases}$$

and the function $F(\phi)$, $F_1(\phi)$ and $F_2(\phi)$ are represented in their Fourier-Jacobi series

$$F(\phi) = \sum_{m=1}^{\infty} Z_m P_m^{(\alpha,\beta)}(\cos\phi),$$
$$F_1(\phi) = \sum_{m=1}^{\infty} A_m Z_m P_m^{(\alpha,\beta)}(\cos\phi),$$
$$F_2(\phi) = \sum_{m=1}^{\infty} B_m Z_m P_m^{(\alpha,\beta)}(\cos\phi).$$

Then, by using the orthogonality of the Jacobi's polynomial on the interval $\phi \in [0,\pi]$, we attain the second kind Fredholm infinite system of linear algebraic equations, i.e. an infinite system of the form

$$Z_n + \sum_{m=1}^{\infty} Z_m H_{n,m} = C_n,$$

where Fredholm alternative works, and the equations can be solved by the truncation method (see [1], [3]).

We explain the details of this strategy in the remaining sections.

3.5 Regularisation of the Sine System

By setting $t = \cos\phi$, $t_0 = \cos\phi_0$, and making use of (3.22), the sine system (3.17)-(3.18) can be rewritten as

$$\sum_{m=1}^{\infty}\frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}m\left[Y_m(1-\varepsilon_m)-p_m^s\right]P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(t) = 0, \quad t \in (-1,t_0), \tag{3.24}$$

$$\sum_{m=1}^{\infty}\frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}Y_m P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(t) = 0, \quad t \in (t_0,1). \tag{3.25}$$

Our first task is to combine (3.24) and (3.26) into a piecewise continuous function, but due to the asymptotic behaviour of Gamma functions and Jacobi polynomial, the two equations have different convergence rates. So, we shall first unify their convergence rate.

3.5.1 The interval $(-1, t_0)$ of the sine system

Mutiply both sides of (3.24) by $(1+t)^{\frac{1}{2}}$ and integrate over the interval $(-1, x)$, to obtain

$$\sum_{m=1}^{\infty} \frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})} m\left[Y_m(1-\varepsilon_m) - p_m^s\right] \int_{-1}^{x} (1+t)^{\frac{1}{2}} P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(t)dt = 0.$$

Note that the interchanging order between integral and summation is justified by the edge condition. Then, by applying the relation (2.10) with $\alpha = \beta = \frac{1}{2}$ and $n = m-1$:

$$\int_{-1}^{x} (1+t)^{\frac{1}{2}} P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(t)dt = \frac{(1+x)^{\frac{3}{2}}}{(m+\frac{1}{2})} P_{m-1}^{(-\frac{1}{2},\frac{3}{2})}(x),$$

and we obtain

$$\sum_{m=1}^{\infty} m\frac{\Gamma(m+1)}{\Gamma(m+\frac{3}{2})} \left[Y_m(1-\varepsilon_m) - p_m^s\right] P_{m-1}^{(-\frac{1}{2},\frac{3}{2})}(x) = 0, \quad x \in (-1, t_0). \tag{3.26}$$

Next, setting $n = m-1$, $\alpha = -\frac{1}{2}$, $\beta = \frac{3}{2}$ and $\eta = \frac{1}{2}$, we have the formula (2.8) as

$$P_{m-1}^{(-\frac{1}{2},\frac{3}{2})}(x) = \frac{(1+x)^{-\frac{3}{2}}}{\sqrt{\pi}} \frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+1)} \int_{-1}^{x} \frac{(1+t)P_{m-1}^{(0,1)}(t)}{(x-t)^{\frac{1}{2}}} dt.$$

Substitute this into (3.26) and interchange the summation and integral, we have

$$\int_{-1}^{x} \frac{(1+t)\sum\limits_{m=1}^{\infty} m\left[Y_m(1-\varepsilon_m) - p_m^s)\right] P_{m-1}^{(0,1)}(t)}{(x-t)^{\frac{1}{2}}} dt = 0.$$

This is in the form of homogeneous Abel integral equation, so

$$\sum_{m=1}^{\infty} m\left[Y_m(1-\varepsilon_m) - p_m^s)\right] P_{m-1}^{(0,1)}(t) = 0, \quad t \in (-1, t_0). \tag{3.27}$$

3.5.2 The interval $(t_0, 1)$ of the sine system

On the other hand, by setting $\alpha = \beta = \eta = \frac{1}{2}$, $n = m-1$, and interchange the role of x and t, (2.7) then becomes

$$P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(t) = \frac{(1-t)^{-\frac{1}{2}}}{\sqrt{\pi}} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m)} \int_{t}^{1} \frac{P_{m-1}^{(0,1)}(x)}{(x-t)^{\frac{1}{2}}} dx, \quad t \in (t_0, 1).$$

Substitute into (3.26) and interchange the integral and summation, we obtain

$$\int_t^1 \frac{\sum_{m=1}^{\infty} mY_m P_{m-1}^{(0,1)}(x)}{(x-t)^{\frac{1}{2}}} dx = 0.$$

This is again a homogeneous Abel integral equation, so we have

$$\sum_{m=1}^{\infty} mY_m P_{m-1}^{(0,1)}(x) = 0, \quad x \in (t_0, 1). \tag{3.28}$$

3.5.3 The combined sine system

It is clear that (3.27) and (3.28) have the same convergence rate, so we combine them and get

$$\sum_{m=1}^{\infty} mY_m P_{m-1}^{(0,1)}(t) = \begin{cases} \sum_{m=1}^{\infty} m\left(Y_m \varepsilon_m - p_m^s\right) P_{m-1}^{(0,1)}(t) & t \in (-1, t_0) \\ 0 & t \in (t_0, 1) \end{cases}$$

Replace the Jacobi polynomials with the normalised Jacobi polynomials by using the relation $P_{m-1}^{(0,1)}(t) = \sqrt{\frac{2}{m}}\widehat{P}_{m-1}^{(0,1)}(t)$, so we have

$$\sum_{m=1}^{\infty} \hat{Y}_m \widehat{P}_{m-1}^{(0,1)}(t) = \begin{cases} \sum_{m=1}^{\infty} \left(\hat{Y}_m \varepsilon_m - \hat{p}_m^s\right) \widehat{P}_{m-1}^{(0,1)}(t), & t \in (-1, t_0), \\ 0, & t \in (t_0, 1), \end{cases} \tag{3.29}$$

where

$$\left\{\sqrt{2m}Y_m, \sqrt{2m}p_m^s\right\} = \left\{\hat{Y}_m, \hat{p}_m^s\right\}.$$

Finally, we shall multiply both sides of (3.29) by $(1+t)\widehat{P}_{n-1}^{(0,1)}(t)$ and integrate over $(-1, 1)$, which lead us to

$$\sum_{m=1}^{\infty} \int_{-1}^{1} \hat{Y}_m (1+t)\widehat{P}_{m-1}^{(0,1)}(t)\widehat{P}_{n-1}^{(0,1)}(t)dt = \sum_{m=1}^{\infty} \int_{-1}^{t_0} (\hat{Y}_m \varepsilon_m - \hat{p}_m^s)(1+t)\widehat{P}_{m-1}^{(0,1)}(t)\widehat{P}_{n-1}^{(0,1)}(t)dt.$$

The right hand side can be rewritten as

$$\begin{aligned} &\sum_{m=1}^{\infty} \int_{-1}^{t_0} (\hat{Y}_m \varepsilon_m - \hat{p}_m^s)(1+t)\widehat{P}_{m-1}^{(0,1)}(t)\widehat{P}_{n-1}^{(0,1)}(t)dt \\ &= \sum_{m=1}^{\infty} \int_{-1}^{1} (\hat{Y}_m \varepsilon_m - \hat{p}_m^s)(1+t)\widehat{P}_{m-1}^{(0,1)}(t)\widehat{P}_{n-1}^{(0,1)}(t)dt \\ &\quad - \sum_{m=1}^{\infty} \int_{t_0}^{1} (\hat{Y}_m \varepsilon_m - \hat{p}_m^s)(1+t)\widehat{P}_{m-1}^{(0,1)}(t)\widehat{P}_{n-1}^{(0,1)}(t)dt. \end{aligned} \tag{3.30}$$

Recall that the set of Jacobi polynomials are orthogonal in the interval $(-1, 1)$, and make use of the notation of the incomplete scalar product (Section 2.5). Thus, we have our desired Fredholm equation of the second kind.

$$(1-\varepsilon_n)\hat{Y}_n + \sum_{m=1}^{\infty} \hat{Y}_m \varepsilon_m \widehat{Q}^{(0,1)}_{m-1,n-1}(t_0) = \hat{p}^s_n - \sum_{m=1}^{\infty} \hat{p}^s_m \widehat{Q}^{(0,1)}_{m-1,n-1}(t_0) \tag{3.31}$$

for $n = 1, 2, 3 \ldots$

It can be checked taht, the coefficient matrix associated with (3.31) has the form $I + A$ where I is the identity matrix and

$$A_{mn} = \begin{cases} \varepsilon_m \widehat{Q}^{(0,1)}_{m-1.n-1}(t_0), & m \neq n, \\ \varepsilon_m (\widehat{Q}^{(0,1)}_{m-1.n-1}(t_0) - 1), & m = n. \end{cases}$$

The matrix A represent a compact operator (on l^2).

3.6 Regularisation of the Cosine System

First, using (3.23), we rewrite the cosine system (3.19)-(3.20) as

$$\sum_{m=1}^{\infty} m \frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})} \left[X_m(1-\varepsilon_m) - p^c_m\right] P_m^{(-\frac{1}{2},-\frac{1}{2})}(t) = -\frac{i}{2}\sqrt{\pi}(ka)^2 J_1(ka)(1+x_0); t \in (-1, t_0) \tag{3.32}$$

$$\sum_{m=1}^{\infty} \frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})} X_m P_m^{(-\frac{1}{2},-\frac{1}{2})}(t) = \frac{x_0}{2\sqrt{\pi} H_1^{(1)}(ka)}; t \in (t_0, 1). \tag{3.33}$$

According to the theory developed in [3], as we are taking $\eta = \frac{1}{2}$ invariably, the same approach in Section 3.5 fails at the index where $\alpha = \beta = -\frac{1}{2}$. For this reason, we shall first increase the indices by using the Rodrigues's formula.

Setting $\alpha = \beta = -\frac{1}{2}$, integrating both sides of (2.12) over $(-1, t)$ and $(t, 1)$, we obtain

$$\begin{aligned} \int_{-1}^{t} (1-x^2)^{-\frac{1}{2}} P_m^{(-\frac{1}{2},-\frac{1}{2})}(x)dx &= -\frac{1}{2m}(1-t^2)^{\frac{1}{2}} P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(t) \\ \int_{t}^{1} (1-x^2)^{-\frac{1}{2}} P_m^{(-\frac{1}{2},-\frac{1}{2})}(x)dx &= \frac{1}{2m}(1-t^2)^{\frac{1}{2}} P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(t). \end{aligned} \tag{3.34}$$

Replacing the role of x and t, then we integrate both sides of (3.32)-(3.33) with $(1-x^2)^{-\frac{1}{2}}$. Then, by using the two formulas of (3.34), we have the dual series equation

$$\sum_{m=1}^{\infty} \frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})} \left[X_m(1-\varepsilon_m) - p^c_m\right] P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(t)$$
$$= i\sqrt{\pi}(ka)^2 J_1(ka)(1+x_0) \frac{\left[\frac{\pi}{2} + \arcsin t\right]}{(1-t^2)^{\frac{1}{2}}}, \qquad t \in (-1, t_0), \tag{3.35}$$

$$\sum_{m=1}^{\infty}\frac{\Gamma(m)}{\Gamma(m+\frac{1}{2})}X_m P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(t)=\frac{x_0}{\sqrt{\pi}H_1^{(1)}(ka)}\frac{\left[\frac{\pi}{2}-\arcsin t\right]}{(1-t^2)^{\frac{1}{2}}},\quad t\in(t_0,1). \tag{3.36}$$

Again, as before the aim is to unify the convergence rate of each equation, so we can combine them as a piecewise continuous function. The remaining procedure is now very similar to the regularisation of the sine system.

3.6.1 The interval $(-1, t_0)$ of the cosine system

First, we integrate both sides of (3.35) with $(1+t)^{\frac{1}{2}}$ over the interval $(-1, x)$, so

$$\sum_{m=1}^{\infty}\frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}\left[X_m(1-\varepsilon_m)-p_m^c\right]\int_{-1}^{x}(1+t)^{\frac{1}{2}}P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(t)dt$$
$$=i\sqrt{\pi}(ka)^2J_1(ka)(1+x_0)\int_{-1}^{x}\frac{(1+t)^{\frac{1}{2}}\left[\frac{\pi}{2}+\arcsin t\right]}{(1-x^2)^{\frac{1}{2}}}dt.$$

Note that the interchanging order of integration and summation is again justified by the edge conditions. Then, making use of (2.10) with $\alpha=\beta=\frac{1}{2}$ and $n=m-1$,

$$\int_{-1}^{x}(1+t)^{\frac{1}{2}}P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(t)dt=\frac{(1+x)^{\frac{3}{2}}}{\left(m+\frac{1}{2}\right)}P_{m-1}^{(-\frac{1}{2},\frac{3}{2})}(x),$$

and evaluating the integral, we get

$$(1+x)^{\frac{3}{2}}\sum_{m=1}^{\infty}\frac{\Gamma(m+1)}{\Gamma(m+\frac{3}{2})}\left[X_m(1-\varepsilon_m)-p_m^c\right]P_{m-1}^{(-\frac{1}{2},\frac{3}{2})}(x)$$
$$=2i\sqrt{\pi}(ka)^2J_1(ka)(1+x_0)\left[2(1+x)^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}\left(\arcsin x+\frac{\pi}{2}\right)\right].$$

Next, setting $n=m-1$, $\alpha=-\frac{1}{2}$, $\beta=\frac{3}{2}$ and $\eta=\frac{1}{2}$, we use the formula (2.8) as

$$P_{m-1}^{(-\frac{1}{2},\frac{3}{2})}(x)=\frac{(1+x)^{-\frac{3}{2}}}{\sqrt{\pi}}\frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+1)}\int_{-1}^{x}\frac{(1+t)P_{m-1}^{(0,1)}(t)}{(x-t)^{\frac{1}{2}}}dt,$$

and we obtain an equation in the form of integral equation of Abel type

$$\int_{-1}^{x}\frac{U(t)}{(x-t)^{\frac{1}{2}}}dt=F(x),$$

where

$$U(t)=(1+t)\sum_{m=1}^{\infty}\left[X_m(1-\varepsilon_m)-p_m^c\right]P_{m-1}^{(0,1)}(t)$$

and

$$F(x)=2i\pi(ka)^2J_1(ka)(1+x_0)\left[2(1+x)^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}\left(\arcsin x+\frac{\pi}{2}\right)\right].$$

The unique solution obtained by inversion of the Abel integral equation is

$$\begin{aligned} U(t) &= \frac{1}{\pi}\frac{d}{dt}\int_{-1}^{t}\frac{F(x)}{(t-x)^{\frac{1}{2}}}dx \\ &= 2i(ka)^2 J_1(ka)(1+x_0)\frac{d}{dt}\int_{-1}^{t}\frac{\left[2(1+x)^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}\left(\arcsin x+\frac{\pi}{2}\right)\right]}{(t-x)^{\frac{1}{2}}}dx. \end{aligned}$$

By integration by parts, we have

$$U(t) = 2i(ka)^2 J_1(ka)(1+x_0)\frac{d}{dt}\int_{-1}^{t}\frac{(t-x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}\left(\arcsin x+\frac{\pi}{2}\right)dx.$$

Interchanging the differential and integral,

$$U(t) = i(ka)^2 J_1(ka)(1+x_0)\int_{-1}^{t}\frac{\left(\arcsin x+\frac{\pi}{2}\right)}{(1-x)^{\frac{1}{2}}(t-x)^{\frac{1}{2}}}dx.$$

The integral is in the form of (2.13), and so we have

$$\begin{aligned} (1+t)\sum_{m=1}^{\infty}&\left[X_m(1-\varepsilon_m)-p_m^c\right]P_{m-1}^{(0,1)}(t) \\ &= -\pi i(ka)^2 J_1(ka)(1+x_0)\ln\left(\frac{1-t}{2}\right); t\in(-1,t_0) \end{aligned} \quad (3.37)$$

3.6.2 The interval $(t_0, 1)$ of the cosine system

For convenient purpose, we restate (3.36), replacing t with x

$$\sum_{m=1}^{\infty}\frac{\Gamma(m)}{\Gamma(m+\frac{1}{2})}X_m P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(x) = \frac{x_0}{\sqrt{\pi}H_1^{(1)}(ka)}\frac{\left[\frac{\pi}{2}-\arcsin x\right]}{(1-x^2)^{\frac{1}{2}}}, \quad x\in(t_0,1).$$

By setting $\alpha=\beta=\eta=\frac{1}{2}$, $n=m-1$ in (2.7), we have

$$P_{m-1}^{(\frac{1}{2},\frac{1}{2})}(x) = \frac{(1-x)^{-\frac{1}{2}}}{\sqrt{\pi}}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m)}\int_{x}^{1}\frac{P_{m-1}^{(0,1)}(t)}{(t-x)^{\frac{1}{2}}}dt \quad x\in(t_0,1).$$

Substitute into (3.36), and interchange the integral and summation, we have

$$\int_{x}^{1}\frac{U(t)}{(t-x)^{\frac{1}{2}}}dt = F(x),$$

where

$$U(t) = \sum_{m=1}^{\infty}X_m P_{m-1}^{(0,1)}(t)$$

and

$$F(x) = \frac{x_0}{H_1^{(1)}(ka)} \frac{\left[\frac{\pi}{2} - \arcsin x\right]}{(1+x)^{\frac{1}{2}}}.$$

The unique solution obtained by inversion is

$$\begin{aligned} U(t) &= -\frac{1}{\pi}\frac{d}{dt}\int_t^1 \frac{F(x)}{(x-t)^{\frac{1}{2}}}dx \\ &= -\frac{x_0}{\pi H_1^{(1)}(ka)}\frac{d}{dt}\int_t^1 \frac{\left[\frac{\pi}{2} - \arcsin x\right]}{(1+x)^{\frac{1}{2}}(x-t)^{\frac{1}{2}}}dx, \end{aligned}$$

but the integral may be evaluated using (2.14), so we have

$$U(t) = \frac{x_0}{H_1^{(1)}(ka)}\frac{d}{dt}\left(\ln\left(\frac{1+t}{2}\right)\right).$$

Thus,

$$(1+t)\sum_{m=1}^{\infty} X_m P_{m-1}^{(0,1)}(t) = \frac{x_0}{H_1^{(1)}(ka)}, \quad t \in (t_0, 1). \tag{3.38}$$

3.6.3 The combined cosine system

Since (3.37) and (3.38) clearly have the same convergence rate, so we combine them as a piecewise continuous function, as follows:

$$(1+t)\sum_{m=1}^{\infty} X_m P_{m-1}^{(0,1)}(t) = \begin{cases} F(t), & t \in (-1, t_0), \\ \frac{x_0}{H_1^{(1)}(ka)}, & t \in (t_0, 1) \end{cases}$$

where

$$F(t) = (1+t)\sum_{m=1}^{\infty}\left[X_m\varepsilon_m + p_m^c\right]P_{m-1}^{(0,1)}(t) - \pi i(ka)^2 J_1(ka)(1+x_0)\ln\left(\frac{1-t}{2}\right).$$

Replace the Jacobi polynomials with the normalised Jacobi polynomials by using the relation $P_{m-1}^{(0,1)}(t) = \sqrt{\frac{2}{m}}\widehat{P}_{m-1}^{(0,1)}(t)$ to obtain

$$(1+t)\sum_{m=1}^{\infty} \hat{X}_m \widehat{P}_{m-1}^{(0,1)}(t) = \begin{cases} \widehat{F}(t), & t \in (-1, t_0) \\ \frac{x_0}{H_1^{(1)}(ka)}, & t \in (t_0, 1) \end{cases} \tag{3.39}$$

where

$$\widehat{F}(t) = (1+t)\sum_{m=1}^{\infty}\left[\hat{X}_m\varepsilon_m + \hat{p}_m^c\right]\widehat{P}_{m-1}^{(0,1)}(t) - \pi i(ka)^2 J_1(ka)(1+x_0)\ln\left(\frac{1-t}{2}\right),$$

$$\left\{\hat{X}_m, \hat{p}_c^m\right\} = \sqrt{\frac{2}{m}}\left\{X_m, p_c^m\right\}.$$

The requirement on the solution class forces the function to be continuous at the point $t = t_0$. This condition uniquely defines the constant x_0:

$$\frac{x_0}{H_1^{(1)}(ka)} = (1+t_0)\sum_{m=1}^{\infty}\left[\hat{X}_m\varepsilon_m + \hat{p}_m^c\right]\widehat{P}_{m-1}^{(0,1)}(t_0) - \pi i(ka)^2 J_1(ka)(1+x_0)\ln\left(\frac{1-t_0}{2}\right),$$

which can be reduced to

$$x_0 = -\frac{i\pi(ka)^2 J_1(ka)H_1^{(1)}(ka)\ln\left(\frac{1-t_0}{2}\right)}{\varkappa(ka,t_0)} + (1+t_0)\frac{H_1^{(1)}(ka)}{\varkappa(ka,t_0)}\sum_{m=1}^{\infty}\left[\hat{X}_m\varepsilon_m + \hat{p}_m^c\right]\widehat{P}_{m-1}^{(0,1)}(t_0) \tag{3.40}$$

where

$$\varkappa(ka,t_0) = 1 + i\pi(ka)^2 J_1(ka)H_1^{(1)}(ka)\ln\left(\frac{1-t_0}{2}\right).$$

To deduce the second kind matrix equation for the coefficients $\left\{\hat{X}_m\right\}_{m=1}^{\infty}$, we use the orthogonal features of the Jacobi polynomials, i.e. integrate both sides of (3.39) by $\widehat{P}_{n-1}^{(0,1)}(t)$ over $(-1,1)$, and we have

$$\begin{aligned}\hat{X}_n = &\sum_{m=1}^{\infty}\left[\hat{X}_m\varepsilon_m + \hat{p}_m^c\right]\int_{-1}^{t_0}(1+t)\widehat{P}_{m-1}^{(0,1)}(t)\widehat{P}_{n-1}^{(0,1)}(t)dt \\ &- \pi i(ka)^2 J_1(ka)(1+x_0)\int_{-1}^{t_0}\ln\left(\frac{1-t}{2}\right)\widehat{P}_{n-1}^{(0,1)}(t)dt \\ &+ \frac{x_0}{H_1^{(1)}(ka)}\int_{t_0}^{1}\widehat{P}_{n-1}^{(0,1)}(t)dt.\end{aligned} \tag{3.41}$$

For the first integral, we could use the same trick as (3.30) in the regularisation of sine system. So, we have the first integral as

$$\hat{X}_n\varepsilon_n + \hat{p}_n^c + \sum_{m=1}^{\infty}\left[\hat{X}_m\varepsilon_m + \hat{p}_m^c\right]\widehat{Q}_{m-1,n-1}^{(0,1)}(t_0). \tag{3.42}$$

For the second integral, we use integration by parts and obtain

$$\int_{-1}^{t_0}\ln\left(\frac{1-t}{2}\right)\widehat{P}_{n-1}^{(0,1)}(t)dt = -\frac{1-t_0}{n}\ln\left(\frac{1-t_0}{2}\right)\widehat{P}_{n-1}^{(1,0)}(t_0) - \frac{1}{n}\int_{-1}^{t_0}\widehat{P}_{n-1}^{(1,0)}(t)dt.$$

By using (2.5), we have the identity

$$\int_{-1}^{t_0}\widehat{P}_{n-1}^{(1,0)}(t)dt = \frac{1+t_0}{m}\widehat{P}_{n-1}^{(0,1)}(t_0)$$

and so we can deduce the integral to

$$\int_{-1}^{t_0}\ln\left(\frac{1-t}{2}\right)\widehat{P}_{n-1}^{(0,1)}(t)dt = -\frac{1-t_0}{n}\ln\left(\frac{1-t_0}{2}\right)\widehat{P}_{n-1}^{(1,0)}(t_0) - \frac{1+t_0}{n^2}\widehat{P}_{n-1}^{(0,1)}(t_0). \tag{3.43}$$

For the third integral, we simply apply (2.4)

$$\int_{t_0}^{1} \widehat{P}_{n-1}^{(0,1)}(t)dt = \frac{1-t_0}{n}\widehat{P}_{n-1}^{(1,0)}(t_0). \tag{3.44}$$

Substitute (3.40), (3.42), (3.43) and (3.44) into (3.41); we then obtain our desired Fredholm equation of the second kind

$$(1-\varepsilon_n)\hat{X}_n + \sum_{m=1}^{\infty} \hat{X}_m \varepsilon_m S_{mn}(ka,t_0) = \frac{\xi(ka)}{\varkappa(ka,t_0)H_1^{(1)}(ka)}\Phi_m(t_0) + \hat{p}_n^c - \sum_{m=1}^{\infty} \hat{p}_m^c S_{mn}(ka,t_0) \tag{3.45}$$

for $n = 1, 2, 3, \ldots$ and

$$\begin{aligned} S_{mn}(ka,t_0) &= \widehat{Q}_{m-1,n-1}^{(1,0)}(t_0) - \frac{\xi(ka)}{\varkappa(ka,t_0)}\Phi_m(t_0)\Phi_n(t_0), \\ \xi(ka) &= i\pi(ka)^2 J_1(ka)H_1^{(1)}(ka), \\ \Phi_s(t_0) &= (1+t_0)\frac{\hat{P}_{s-1}^{(0,1)}(t_0)}{s} \quad (s \geq 1). \end{aligned}$$

Evaluating the coefficient matrix associated with (3.45), it can be confirmed that it is a compact perturbation of the identity.

3.7 Summary of the Regularisation

Let us summarise what we have done for the open case.

By (3.40), we have

$$x_0 = -\frac{i\pi(ka)^2 J_1(ka)H_1^{(1)}(ka)\ln\left(\frac{1-t_0}{2}\right)}{\varkappa(ka,t_0)} + (1+t_0)\frac{H_1^{(1)}(ka)}{\varkappa(ka,t_0)}\sum_{m=1}^{\infty}\left[\hat{X}_m\varepsilon_m + \hat{p}_m^c\right]\widehat{P}_{m-1}^{(0,1)}(t_0).$$

By (3.45), (3.31), for $n = 1, 2, 3, \ldots$, we have

$$(1-\varepsilon_n)\hat{Y}_n + \sum_{m=1}^{\infty} \hat{Y}_m \varepsilon_m \widehat{Q}_{m-1,n-1}^{(0,1)}(t_0) = \hat{p}_n^s - \sum_{m=1}^{\infty} \hat{p}_m^s \widehat{Q}_{m-1,n-1}^{(0,1)}(t_0),$$

and

$$(1-\varepsilon_n)\hat{X}_n + \sum_{m=1}^{\infty} \hat{X}_m \varepsilon_m S_{mn}(ka,t_0) = \frac{\xi(ka)}{\varkappa(ka,t_0)H_1^{(1)}(ka)}\Phi_m(t_0) + \hat{p}_n^c - \sum_{m=1}^{\infty} \hat{p}_m^c S_{mn}(ka,t_0)$$

where

$$\begin{aligned}
S_{mn}(ka, t_0) =& \widehat{Q}^{(1,0)}_{m-1,n-1}(t_0) - \frac{\xi(ka)}{\varkappa(ka, t_0)}\Phi_m(t_0)\Phi_n(t_0), \\
\xi(ka) =& i\pi(ka)^2 J_1(ka) H_1^{(1)}(ka), \\
\varkappa(ka, t_0) =& 1 + i\pi(ka)^2 J_1(ka) H_1^{(1)}(ka) \ln\left(\frac{1-t_0}{2}\right), \\
\Phi_s(t_0) =& (1+t_0)\frac{\hat{P}^{(0,1)}_{s-1}(t_0)}{s} \quad (s \geq 1).
\end{aligned}$$

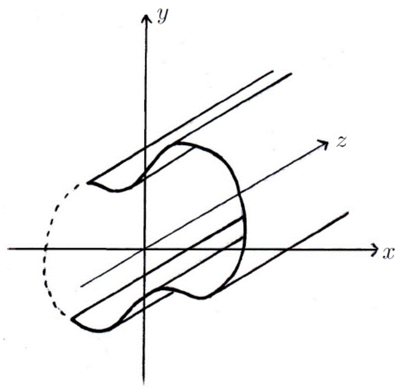
Hence, we have now obtained a well conditioned set of equations, where the solution now can be reliably computed by standard numerical methods.

4

The Neumann BVP For An Open Cylinder of Arbitrary Profile

In this chapter, we turn our attention to cylindrical structures of constant cross-section which have a single aperture. The Neumann BVP is reformulated as an integral equation, which in turn is converted to an equivalent dual series equations.

4.1 Formulation of The Boundary Value Problem

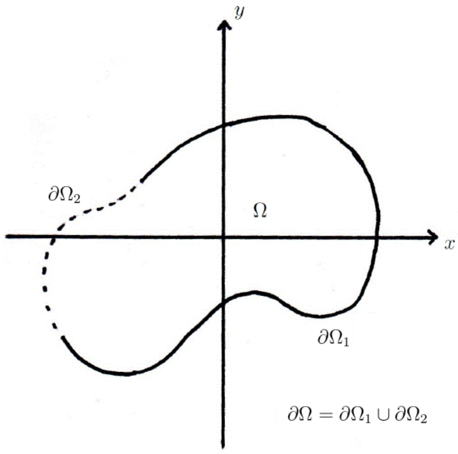


Consider an infinitely long cylinder of arbitrary shape in which a single aperture is opened. We refer to the portion of the cross-section that remains of the opening the aperture as the screen.

The physical phenomena of wave action on a cylinder is identical on every cross section, so we shall focus the problem only on a cross section, and consider a two dimensional BVP.

The cylinder is assumed to be an infinitely thin, perfectly conducting cavity and each cross section is identical along the length of the cylinder. We place the origin inside the cylinder, and set the cylinder parallel along the z-axis such that every cross section is perpendicular to the z-axis. In this way, the problem is independent of z-axis, and we can put our focus on the cross section which is on the xy-plane (the plane where $z = 0$).

Let Ω be a finite bounded region of $\mathbb{R}^2$ to represent the cross section of our cylinder, and $\partial\Omega$ denoted as the boundary of Ω. Geometrically, $\partial\Omega$ looks like a simple non-self-crossing closed contour around the origin.



We let $\partial\Omega_1$ be a connected closed subset of $\partial\Omega$ to represent the *screen*, and $\partial\Omega_2$ as the compliment of $\partial\Omega_1$ in $\partial\Omega$ to represent the *aperture*, so

$$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2,$$

the two points where the screen meets the aperture are referred to as the *edges*.

We seek the scattered field u which satisfies the Helmholtz equation

$$(\triangle + k^2)u = 0, \tag{4.1}$$

where $\triangle$ refers to the Laplacian, k is a constant, such that the solution u also satisfies

(i) $\frac{\partial u}{\partial n} = 0$ on the screen, $u = 0$ on the aperture,

(ii) the Sommerfield radiation condition at infinity,

(iii) the Edge condition,

for an infinitely long open cylinder with arbitrary cross section.

4.2 The Parametrisation

The contour $\partial\Omega$ is parametrised by a smooth 2π-periodic vector function

$$\eta(\theta) = (x(\theta), y(\theta))$$

for $\theta \in [-\pi, \pi]$.

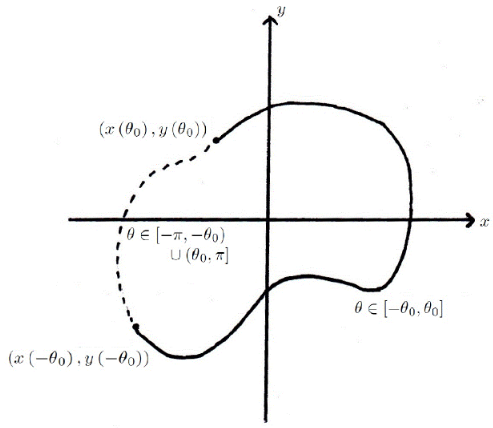
We assume that the condition $l(\theta) = \sqrt{[x'(\theta)]^2 + [y'(\theta)]^2} > 0$ holds to provide one-to-one mapping for $\eta = \eta(\theta)$; the point $\eta(\theta)$ is moving in an anti-clockwise direction along the contour $\partial\Omega$.

We may choose $\theta_0 \in (0, \pi)$ to parametrise the screen by

$$\partial\Omega_1 = \{\eta(\theta); \theta \in [-\theta_0, \theta_0]\},$$

and the aperture by

$$\partial\Omega_2 = \{\eta(\theta); \theta \in [-\pi, -\theta_0) \cup (\theta_0, \pi]\}.$$



For any two points on $\partial\Omega$ corresponding to $\eta(\theta)$ and $\eta(\tau)$, the function $R(\theta, \tau)$: $[-\pi, \pi] \times [-\pi, \pi] \to \mathbb{R}$ which measures the distance between the two points is defined by

$$R(\theta, \tau) = \sqrt{[x(\theta) - x(\tau)]^2 + [y(\theta) - y(\tau)]^2}.$$

4.3 The Boundary Conditions

Recalling previous terminology, the *incident*, *scattered* and *total field*, which will be denoted u^{inc}, u^{sc} and u^{tot}, respectively. So, we have the total field as

$$u^{tot} = u^{inc} + u^{sc}. \tag{4.2}$$

Recall that the incident field is completely known, and all the fields above satisfy the Helmholtz equation.

The boundary condition on the screen $\frac{\partial u^{tot}}{\partial \mathbf{n}}(\mathbf{x}) = 0$, implies for $h > 0$

$$\lim_{h\to 0^+} \frac{\partial u^{sc}}{\partial \mathbf{n}}(\mathbf{x} \pm h\mathbf{n}_{\mathbf{x}}) = -\frac{\partial u^{inc}}{\partial \mathbf{n}}(\mathbf{x}) \tag{4.3}$$

for $\mathbf{x} \in \partial\Omega_1$, where $\mathbf{n}_{\mathbf{x}}$ is the outward normal to $\partial\Omega_1$ at the point $\mathbf{x}$.

Similarly, for the aperture, we have

$$u^{sc}(\mathbf{x}) = -u^{inc}(\mathbf{x}) \tag{4.4}$$

for $\mathbf{x} \in \partial\Omega_2$. With such formulation, we have a Neumann mixed boundary value problem.

As mentioned in the introduction, we want our solution u^{sc} to behave as an outgoing wave at infinity and vanish at infinity, so the solution needs to satisfy the Sommerfeld radiation conditions

$$|\sqrt{r}u^{sc}| < K$$

for some constant K, and

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^{sc}}{\partial r} - iku^{sc}\right) = 0$$

uniformly with respect to the direction of r, where $r = |\mathbf{x}|$ is the distance of point $\mathbf{x}$ from origin. In some contexts, the conditions are also referred as the boundary condition at infinity.

Additionally, we want our field to be bounded. Thus in an electromagnetic context, an edge condition namely the Meixner Integrability Condition

$$\int_V \left(|\vec{E}|^2 + |\vec{H}|^2\right) dV < \infty \tag{4.5}$$

for any arbitrary finite area V also need to be enforced, where $\vec{E}$ is the eletric field, $\vec{H}$ is the magnetic field. This condition is relating to the finiteness of the energy of the scattered field within any arbitrarily chosen finite area that may contain the edges, and we already have a brief discussion on Section (3.3), which can reduced to

$$\int\limits_V \left(|\nabla u^{sc}|^2 + k^2|u^{sc}|^2\right) dV < \infty$$

for any arbitrary finite area V.

As mentioned in Section 1.1, a proof in [2] shows that if a solution satisfies all these conditions, then it will be unique.

4.4 Solution in Integral representation

We refer to the points $(x, y) \in \Omega$ as the *interior*, and for $(x, y) \in \mathbb{R}^2\backslash\Omega$ as *exterior*. Note that the boundary $\partial\Omega$ is not included in both interior and exterior. We denote the interior field as u_i and the exterior field as u_e. In combination, for example u_e^{sc} means the exterior scattered field.

Since there is no classical or separable coordinate system for arbitrary cross section, we shall consider the integral representation of the solution.

Let $\mathbf{x_0} \in \Omega$ be a point, then we shall have the interior solution in the form of

$$u_i^{sc}(\mathbf{x_0}) = \int_{\partial\Omega} u_i^{sc}(\mathbf{x})\frac{\partial G}{\partial \mathbf{n_x}}(\mathbf{x},\mathbf{x_0}) - G(\mathbf{x},\mathbf{x_0})\frac{\partial u_i^{sc}}{\partial \mathbf{n}}(\mathbf{x})dl_{\mathbf{x}}. \tag{4.6}$$

And for a point $\mathbf{x_0} \in \mathbb{R}^2\backslash\Omega$, the exterior solution is

$$u_e^{sc}(\mathbf{x_0}) = -\int_{\partial\Omega} u_e^{sc}(\mathbf{x})\frac{\partial G}{\partial \mathbf{n_x}}(\mathbf{x},\mathbf{x_0}) - G(\mathbf{x},\mathbf{x_0})\frac{\partial u_e^{sc}}{\partial \mathbf{n}}(\mathbf{x})dl_{\mathbf{x}}. \tag{4.7}$$

In both solutions, G is the *free space Green's function*, and $\mathbf{n_x}$ is the normal vector pointing outward of Ω. Note that (4.6) will be zero when $\mathbf{x_0} \in \mathbb{R}^2\backslash\Omega$; likewise, (4.7) is zero when $\mathbf{x_0} \in \Omega$.

So, for $\mathbf{x_0} \notin \partial\Omega$, we can express the whole solution in the form of

$$u^{sc}(\mathbf{x_0}) = u_i^{sc}(\mathbf{x_0}) + u_e^{sc}(\mathbf{x_0}). \tag{4.8}$$

Therefore, we have

$$\begin{aligned} u^{sc}(\mathbf{x_0}) &= u_i^{sc}(\mathbf{x_0}) + u_e^{sc}(\mathbf{x_0}) \\ &= \int_{\partial\Omega} u_i^{sc}(\mathbf{x})\frac{\partial G}{\partial \mathbf{n_x}}(\mathbf{x},\mathbf{x_0}) - G(\mathbf{x},\mathbf{x_0})\frac{\partial u_i^{sc}}{\partial \mathbf{n}}(\mathbf{x})dl_{\mathbf{x}} \\ &\quad - \int_{\partial\Omega} u_e^{sc}(\mathbf{x})\frac{\partial G}{\partial \mathbf{n_x}}(\mathbf{x},\mathbf{x_0}) - G(\mathbf{x},\mathbf{x_0})\frac{\partial u_e^{sc}}{\partial \mathbf{n}}(\mathbf{x})dl_{\mathbf{x}}. \end{aligned}$$

With some rearrangement, we have

$$\begin{aligned} u^{sc}(\mathbf{x_0}) = \int_{\partial\Omega} & [u_i^{sc}(\mathbf{x}) - u_e^{sc}(\mathbf{x})]\frac{\partial G}{\partial \mathbf{n_x}}(\mathbf{x},\mathbf{x_0}) \\ & - G(\mathbf{x},\mathbf{x_0})\left[\frac{\partial u_i^{sc}}{\partial \mathbf{n}}(\mathbf{x}) - \frac{\partial u_e^{sc}}{\partial \mathbf{n}}(\mathbf{x})\right]dl_{\mathbf{x}} \end{aligned} \tag{4.9}$$

By enforcing (4.3) and (4.4), we shall have

$$\begin{aligned} u^{sc}(\mathbf{x_0}) = \int_{\partial\Omega_1} & [u_i^{sc}(\mathbf{x}) - u_e^{sc}(\mathbf{x})]\frac{\partial G}{\partial \mathbf{n_x}}(\mathbf{x},\mathbf{x_0}) \\ - \int_{\partial\Omega_2} & G(\mathbf{x},\mathbf{x_0})\left[\frac{\partial u_i^{sc}}{\partial \mathbf{n}}(\mathbf{x}) - \frac{\partial u_e^{sc}}{\partial \mathbf{n}}(\mathbf{x})\right]dl_{\mathbf{x}}. \end{aligned} \tag{4.10}$$

Since no scattering is happening on the aperture $\partial\Omega_2$, we can reduce (4.10) to an equation

$$u^{sc}(\mathbf{x_0}) = \int\limits_{\partial\Omega_1} [u_i^{sc}(\mathbf{x}) - u_e^{sc}(\mathbf{x})] \frac{\partial G}{\partial \mathbf{n_x}}(\mathbf{x}, \mathbf{x_0}) dl_{\mathbf{x}}. \tag{4.11}$$

For convenient purpose, we shall denote

$$z(\mathbf{x}) = u_i^{sc}(\mathbf{x}) - u_e^{sc}(\mathbf{x}).$$

Hence, for $\mathbf{x_0} \notin \partial\Omega_1$, the integral representation of the scattered field is

$$u^{sc}(\mathbf{x_0}) = \int\limits_{\partial\Omega_1} z(\mathbf{x}) \frac{\partial G}{\partial \mathbf{n_x}}(\mathbf{x}, \mathbf{x_0}) dl_{\mathbf{x}}. \tag{4.12}$$

Enforcing the boundary condition (4.3), we obtain

$$\lim_{h\to 0^+} \frac{\partial}{\partial \mathbf{n_{x_0}}} \int\limits_{\partial\Omega_1} z(\mathbf{x}) \frac{\partial G}{\partial \mathbf{n_x}}(\mathbf{x}, \mathbf{x_0} \pm h\mathbf{n_{x_1}}) dl_{\mathbf{x}} = -\frac{\partial u^{inc}}{\partial \mathbf{n_{x_0}}}(\mathbf{x_0}) \tag{4.13}$$

where $n_{\mathbf{x_0}}$ and $n_{\mathbf{x}}$ are the normal to $\mathbf{x_0}$ and $\mathbf{x}$, respectively, $z(\mathbf{x})$ is unknown function.

Clearly, (4.13) is a Fredhom integral equation of the first kind. Further more, the kernel

$$\frac{\partial^2 G}{\partial \mathbf{n_{x_0}} \partial \mathbf{n_x}}(\mathbf{x}, \mathbf{x_0})$$

is highly singular when $\mathbf{x} \to \mathbf{x_0}$.

4.5 Transformation of Integral Equation to Its Equivalent Series Equations

It can be shown by using polar coordinates, that the well known free space Green's function for the 2D Helmholtz equation is

$$G(r) = -\frac{i}{4} H_0^{(1)}(kr) \tag{4.14}$$

where $r = |\mathbf{x} - \mathbf{x_0}|$, $H_0^{(1)}$ is the Hankel function of the first kind at order 0, or in terms of Bessel and Neumann functions,

$$G(r) = -\frac{i}{4} \left[J_0(kr) + iY_0(kr) \right].$$

Although the Bessel function $J_0(x)$ is bounded (in particular at $x = 0$), the Neumann function $Y_0(x)$ has a logarithmic order singularity at $x = 0$. That Y_0 has a logarithmic order singularity is most easily seen in its expansion

$$Y_0(x) = \frac{2}{\pi} \left[J_0(x) \left(\ln \left(\frac{x}{2} - \gamma \right) \right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m} (m!)^2} x^{2m} \right]$$

where γ is the Euler's constant and $h_m = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m}$. We shall note here that

$$\lim_{x \to 0^+} J_0(x) = 1$$

and

$$\lim_{x \to 0^+} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m} = 0$$

since the sum is absolutely convergent for all values of x. It is very clear from this that we have

$$Y_0(x) \sim \ln\left(\frac{x}{2}\right)$$

as $x \to 0$. Hence, a highly singular function (or singularity of order 2) results when we consider the second derivatives of the Neumann function $Y_0(x)$, and by extension, the Green's function.

Let $\mathbf{x_0} = \eta(\theta)$ and $\mathbf{x} = \eta(\tau)$, and we denote

$$D_0(\theta, \tau) = \left[\frac{\partial^2 G}{\partial \mathbf{n_{x_0}} \partial \mathbf{n_x}}(\mathbf{x}, \mathbf{x_0})\right]_{\mathbf{x_0}=\eta(\theta), \mathbf{x}=\eta(\tau)}.$$

At this point, let us define the function (recall $l(\theta)$ was defined in Section 4.2)

$$K(\theta, \tau) = 2\pi l(\theta) l(\tau) D_0(\theta, \tau) + \left[4\sin^2\left(\frac{\theta - \tau}{2}\right)\right]^{-1} \tag{4.15}$$

and the function

$$K_s(\theta, \tau) = K(\theta, \tau) - \frac{kl(\theta)kl(\tau)}{2} \ln\left|2\sin\frac{\theta - \tau}{2}\right|. \tag{4.16}$$

Hence,

$$2\pi l(\theta) l(\tau) D_0(\theta, \tau) = K_s(\theta, \tau) + \frac{kl(\theta)kl(\tau)}{2} \ln\left|2\sin\frac{\theta - \tau}{2}\right| - \left[4\sin^2\left(\frac{\theta - \tau}{2}\right)\right]^{-1} \tag{4.17}$$

It was shown in [13] that the function (4.15) has only a logarithmic singularity, and the function (4.16) is smooth with all its first order and second order derivatives have only logarithmic singularities. The decomposition (4.17) is the basis for obtaining the dual series equation that are equivalent to the integral equation (4.13).

It is proved in [13] that (4.13) with the described parameterisation $\eta(\theta)$ of contour $\partial\Omega_1$ may be equivalently reduced to the equation

$$\frac{d^2}{d\theta^2} \int_{-\pi}^{\pi} \hat{z}(\tau) \ln\left|2\sin\left(\frac{\theta - \tau}{2}\right)\right| d\tau + \int_{-\pi}^{\pi} \hat{z}(\tau) K(\theta, \tau) d\tau = F(\theta) \tag{4.18}$$

where

$$\hat{z}(\tau) = \begin{cases} z(\tau); & \tau \in [-\theta_0, \theta_0] \\ 0; & \tau \in [-\pi, -\theta_0) \cup (\theta_0, \pi] \end{cases} \tag{4.19}$$

and

$$F(\theta) = -2\pi l(\theta) \left[\frac{\partial u^{inc}}{\partial \mathbf{n_{x_0}}}(\mathbf{x_0}) \right]_{\mathbf{x_0}=\eta(\theta)}.$$

Next, we shall consider the Fourier expansion of each function in (4.18).

4.5.1 The Unknown Function

Let

$$\hat{z}(\theta) = \sum_{n=-\infty}^{\infty} \xi_n e^{in\theta}, \quad \theta \in [-\pi, \pi]. \tag{4.20}$$

Since $\hat{z}(\tau)$ is the unknown function in (4.19), then obviously the coefficients ξ_n are also unknown. From the form of $\hat{z}(\tau)$ in (4.19), it is clear that $\hat{z}(\tau)$ is piecewise continous.

4.5.2 The Function $F(\theta)$

Let

$$\frac{1}{\pi} F(\theta) = \sum_{n=\infty}^{\infty} f_n e^{in\theta} \quad \theta \in [-\pi, \pi] \tag{4.21}$$

be its Fourier expansion; all the coefficients f_n are known.

4.5.3 The Log Function

The expansion of the log function of the particular form involved is a fairly famous result and common in many tables for Fourier expansions. Nonetheless, the expansion

$$\ln \left| 2 \sin \left(\frac{\theta - \tau}{2} \right) \right| = -\frac{1}{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e^{in(\theta-\tau)}}{|n|} \quad \theta, \tau \in [-\pi, \pi] \tag{4.22}$$

will be derived here.

Using the standard Taylor series

$$\ln(1-z) = -\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \dots \right)$$

and noting that it converges for $|z| \leq 1$ except when $z = 1$, we have

$$\ln\left(1 - e^{i\theta}\right) = -\left(e^{i\theta} + \frac{e^{2i\theta}}{2} + \frac{e^{3i\theta}}{3} + \dots \right) \quad \text{for real } \theta.$$

Taking real parts, we have

$$\ln\left|1 - e^{i\theta}\right| = -\frac{1}{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{|n|} e^{in\theta}$$

and noting that

$$\begin{aligned}|1-e^{i\theta}| &= \sqrt{(1-\cos\theta)^2+\sin^2\theta}\\ &= \sqrt{2-2\cos\theta}\\ &= \sqrt{4\sin^2\frac{\theta}{2}}\\ &= 2\left|\sin\frac{\theta}{2}\right|\end{aligned}$$

we have the desired form.

4.5.4 The Function $K(\theta,\tau)$

We consider the double Fourier series expansion of $K(\theta,\tau)$:

$$K(\theta,\tau)=\sum_{p=\infty}^{\infty}\sum_{n=-\infty}^{\infty}k_{p,n}e^{i(p\theta+n\tau)} \quad \theta,\tau\in[-\pi,\pi]. \tag{4.23}$$

Of the four terms being expanded, the function $K(\theta,\tau)$ is undoubtedly the most interesting in terms of Fourier expansions.

Recall that in Section 4.2, we have

$$\begin{aligned}\eta(\theta) &= (x(\theta),y(\theta)),\\ l(\theta) &= \sqrt{[x'(\theta)]^2+[y'(\theta)]^2},\\ R(\theta,\tau) &= \sqrt{[x(\theta)-x(\tau)]^2+[y(\theta)-y(\tau)]^2}.\end{aligned}$$

We need to calculate the mixed second directional derivative of the Green's function (4.14), namely

$$D_0(\theta,\tau)=\left[\frac{\partial^2 G}{\partial\mathbf{n}_{\mathbf{x_0}}\partial\mathbf{n}_{\mathbf{x}}}(\mathbf{x},\mathbf{x_0})\right]_{\mathbf{x_0}=\eta(\theta),\mathbf{x}=\eta(\tau)}.$$

After a simple, but bulky calculation, it was obtained (see [13]):

$$D_0(\theta,\tau)=-\frac{i}{4}\frac{1}{l(\theta)l(\tau)}\left\{k^2C(\theta,\tau)U(\theta,\tau)-\frac{1}{(kR(\theta,\tau))^2}V(\theta,\tau)(k^2A(\theta,\tau))(k^2B(\theta,\tau))\right\}$$

where

$$\begin{aligned}A(\theta,\tau) &= (x(\theta)-x(\tau))y'(\tau)-(y(\theta)-y(\tau))x'(\tau),\\ B(\theta,\tau) &= (x(\theta)-x(\tau))y'(\theta)-(y(\theta)-y(\tau))x'(\theta),\\ C(\theta,\tau) &= x'(\theta)x'(\tau)+y'(\theta)y'(\tau),\\ U(\theta,\tau) &= \frac{1}{kR(\theta,\tau)}H_1^{(1)}(kR(\theta,\tau)),\\ V(\theta,\tau) &= \frac{2}{kR(\theta,\tau)}H_1^{(1)}(kR(\theta,\tau))-H_0^{(1)}(kR(\theta,\tau)).\end{aligned}$$

To find the Fourier coefficients of $K(\theta,\tau)$, we need to calculate $K_s(\theta,\tau)$. When $\theta \neq \tau$, we calculate $K_s(\theta,\tau)$ by the formula (4.17); when $\theta = \tau$, it was proved that in [13], the function K_s has the following form:

$$\begin{aligned} K_s(\theta,\theta) = &\frac{(kl)^2}{2}\ln\frac{kl}{2} - \frac{1}{(kl)^2}\left[\frac{k^2\Upsilon^{(1,3)}(\theta)}{6} - \frac{k^2\Upsilon^{(2,2)}(\theta)}{4}\right] \\ &-\frac{1}{2}\frac{(k^2\Psi^{(1,2)}(\theta))^2}{(kl)^4} - (kl)^2\left[i\frac{\pi}{4} - \frac{\gamma}{2} + \frac{1}{4}\right] + \frac{1}{12} \end{aligned}$$

where γ is the Euler's constant, and

$$\begin{aligned} l &= l(\theta), \\ \Upsilon^{(i,j)}(\theta) &= x^{(i)}(\theta)x^{(j)}(\theta) + y^{(i)}(\theta)y^{(j)}(\theta), \\ \Psi^{(i,j)}(\theta) &= x^{(i)}(\theta)y^{(j)}(\theta) - y^{(i)}(\theta)x^{(j)}(\theta), \end{aligned}$$

the indices i,j are referring to the orders of derivatives. For numerical purpose K_s is sufficiently smooth that we may calculate the Fourier coefficients of $K_s(\theta,\tau)$ by the fast Fourier transform algorithm.

Recalling that

$$K(\theta,\tau) = \frac{kl(\theta)kl(\tau)}{2}\ln\left|2\sin\frac{\theta-\tau}{2}\right| + K_s(\theta,\tau),$$

it follows that the Fourier coefficients $\{k_{s,n}\}_{s,n\in\mathbb{Z}}$ of $K(\theta,\tau)$ are equal to the sum of the Fourier coefficients of the functions on the right hand side. So if

$$\begin{aligned} K_s(\theta,\tau) &= \sum_{p=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} k^s_{p,n}e^{i(p\theta+n\tau)}, \\ L(\theta,\tau) &= \frac{kl(\theta)kl(\tau)}{2}\ln\left|2\sin\frac{\theta-\tau}{2}\right| \\ &= \sum_{p=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} L_{p,n}e^{i(p\theta+n\tau)}, \end{aligned}$$

then the Fourier coefficient of $K(\theta,\tau)$ is

$$k_{p,n} = k^s_{p,n} + L_{p,n}. \tag{4.24}$$

Furthermore, it was shown that with the expansions

$$kl(\theta) = \sum_{m=-\infty}^{\infty} l_m e^{im\theta}$$

and

$$\ln\left|2\sin\frac{\theta-\tau}{2}\right| = \sum_{n=-\infty}^{\infty}\Lambda_n e^{int}$$

$$\text{where } \Lambda_n = \begin{cases} 0, & n=0 \\ -\frac{1}{2}\frac{1}{|n|} & n\neq 0 \end{cases}$$

then

$$L_{p,n} = \frac{1}{2} \sum_{r=\infty}^{\infty} l_{p-r} l_{n+r} \Lambda_r.$$

4.5.5 The Equivalent dual series equation

Thus, we substitute (4.20) and (4.22) into the first integral of (4.18), then we integrate and differentiate to obtain

$$\frac{d^2}{d\theta^2} \int_{-\pi}^{\pi} \hat{z}(\tau) \ln \left| 2 \sin \left(\frac{\theta - \tau}{2} \right) \right| d\tau = \pi \sum_{n=-\infty}^{\infty} |n| \xi_n e^{in\theta}. \tag{4.25}$$

As for the second integral of (4.18), we substitute (4.20), (4.23) into it, then perform integration by parts and we have

$$\int_{-\pi}^{\pi} \hat{z}(\tau) K(\theta, \tau) d\tau = 2\pi \sum_{n=-\infty}^{\infty} e^{in\theta} \sum_{p=-\infty}^{\infty} k_{n,-p} \xi_p \tag{4.26}$$

where $k_{n,p}$ are calculated by (4.24).

Finally, substitute (4.21), (4.25) and (4.26) into (4.18), and we obtain the desired equivalent dual series equation

$$\sum_{n=-\infty}^{\infty} \left\{ |n| \xi_n + 2 \sum_{p=-\infty}^{\infty} (k_{n,-p}) \xi_p - f_n \right\} e^{in\theta} = 0 \quad \text{for} \quad \theta \in [-\theta_0, \theta_0],$$

$$\sum_{n=-\infty}^{\infty} \xi_n e^{in\theta} = 0 \quad \text{for} \quad \theta \in [-\pi, -\theta_0) \cup (\theta_0, \pi].$$

Once we solve the dual series equation above, we have then solved our BVP.

5

The Analytical Regularisation of The Solution

5.1 The Coupled dual series equation

In the last section of the previous chapter, we obtained the equivalent series representation of the integral equation, and our task now is to find the set of unknown Fourier coefficients $\{\xi_n\}_{n=-\infty}^{\infty}$ for the unknown function $\hat{z}(\tau)$. This is advantageous, because when we convert the dual series equation into another form, and obtained the set of values $\{\xi_n\}_{n=-\infty}^{\infty}$, we do not change any information of $\hat{z}(\tau)$. Our approach uses the MoR which has been used in Chapter 3 for the open circle case: it converted the dual series equation into a matrix equation which is in the form of a well conditioned second kind of Fredholm equation.

Now, let us consider the dual series equation

$$\sum_{n=-\infty}^{\infty} \left\{ |n|\xi_n + 2\sum_{p=-\infty}^{\infty} (k_{n,-p})\xi_p - f_n \right\} e^{in\theta} = 0 \quad \text{for} \quad \theta \in [-\theta_0, \theta_0], \tag{5.1}$$

$$\sum_{n=-\infty}^{\infty} \xi_n e^{in\theta} = 0 \quad \text{for} \quad \theta \in [-\pi, -\theta_0) \cup (\theta_0, \pi]. \tag{5.2}$$

First, we separate the sums in the way of

$$\sum_{n=-\infty}^{\infty} b_n = b_0 + \sum_{n=1}^{\infty} (b_n + b_{-n}),$$

and employing the identities

$$e^{in\theta} = \cos n\theta + i \sin n\theta,$$
$$e^{-in\theta} = \cos n\theta - i \sin n\theta.$$

Then, by taking the real part and imaginary part separately, we decouple (5.1) and (5.2) into a coupled dual series equation as the following

$$\sum_{n=1}^{\infty} (y_n - c_n) \sin n\theta = 0, \quad \theta \in [0, \theta_0], \tag{5.3}$$

$$\sum_{n=1}^{\infty} n y_n \sin n\theta = 0, \quad \theta \in [\theta_0, \pi] \tag{5.4}$$

and

$$\sum_{n=1}^{\infty} (x_n - a_n) \cos n\theta = a_0, \quad \theta \in [0, \theta_0], \tag{5.5}$$

$$\sum_{n=1}^{\infty} n x_n \cos n\theta = -\xi_0, \quad \theta \in [\theta_0, \pi], \tag{5.6}$$

where

$$x_n = \frac{\xi_n + \xi_{-n}}{n}, \qquad f_n^+ = f_n + f_{-n},$$
$$y_n = \frac{\xi_n - \xi_{-n}}{n}, \qquad f_n^- = f_n - f_{-n}$$

and

$$a_0 = f_0 - 2\xi_0[k_{0,0}] - 2\sum_{p=1}^{\infty} [k_{0,-p} + k_{0,p}] p x_p,$$

$$a_n = f_n^+ - 2\xi_0[k_{n,0} + k_{-n,0}] - 2\sum_{p=1}^{\infty} ([k_{n,-p} + k_{-n,-p}] + [k_{-n,p} + k_{n,p}])\, p x_p,$$

$$c_n = f_n^- - 2\xi_0[k_{n,0} + k_{-n,0}] - 2\sum_{p=1}^{\infty} ([k_{n,-p} - k_{-n,-p}] + [k_{-n,p} - k_{n,p}])\, p y_p.$$

Recall the edge condition (4.5), which was imposed on the coefficients $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ and restricts them to a solution class $l^2(1)$ (following the notation of (2.15)):

$$\begin{aligned} \{x_n\}_{n\in\mathbb{N}} \in l^2(1) &\Rightarrow \sum_{n=0}^{\infty} n|x_n|^2 < \infty, \\ \{y_n\}_{n\in\mathbb{N}} \in l^2(1) &\Rightarrow \sum_{n=0}^{\infty} n|y_n|^2 < \infty. \end{aligned} \tag{5.7}$$

From (5.7), we obtain asymptotic estimates for $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$:

$$\begin{aligned} x_n &= O\left(n^{-\frac{3}{2}}\right), \\ y_n &= O\left(n^{-\frac{3}{2}}\right) \end{aligned} \tag{5.8}$$

as $n \to \infty$. These two conditions allow us to interchange the order of integration and summation.

Following the same tradition, we shall refer (5.3)-(5.4) as the sine system, (5.5)-(5.6) as the cosine system.

In the remaining section of the chapter, we shall focus on the regularisation of (5.3)-(5.6) by following the same scheme which described in Chapter 3 (or in [3]). In fact, most of the integrals turn out to be the same as Section 3.6.

5.2 Regularisation of the sine system

Apply the Jacobi representation of the sine function (3.22) in Chapter 3, and denote $t_0 = \cos\theta_0$ and $t = \cos\theta$ (so $t \in [-1,1]$). Hence, we have the sine system in the form of

$$\sum_{n=1}^{\infty} n y_n \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(t) = 0, \quad t \in [-1, t_0], \tag{5.9}$$

$$\sum_{n=1}^{\infty} (y_n - c_n) \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(t) = 0, \quad t \in [t_0, 1]. \tag{5.10}$$

It can be shown using (5.8) and asymptotic behaviour of Jacobi polynomials and Gamma function that the sum (5.9) has convergence rate $O\left(n^{-\frac{1}{2}}\right)$ and (5.10) has the faster convergence rate $O\left(n^{-\frac{3}{2}}\right)$ as $n \to \infty$. Our task is again unifying their convergence rate, so we can combine them as a piecewise function.

First, multiply both sides of (5.9) by $(1+t)^{\frac{1}{2}}$, then by applying (2.10),

$$\int_{-1}^{x} (1+t)^{\frac{1}{2}} P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(t)dt = \frac{(1+x)^{\frac{3}{2}}}{n+\frac{1}{2}} P_{n-1}^{(-\frac{1}{2},\frac{3}{2})}(x),$$

to integrate term by term over the interval $[-1, x]$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \int_{-1}^{x} n y_n \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} (1+t)^{\frac{1}{2}} P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(t) dt \\
&\quad = \sum_{n=1}^{\infty} n y_n \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \int_{-1}^{x} (1+t)^{\frac{1}{2}} P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(t) dt \\
&\quad = \sum_{n=1}^{\infty} n y_n \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \frac{(1+x)^{\frac{3}{2}}}{n+\frac{1}{2}} P_{n-1}^{(-\frac{1}{2},\frac{3}{2})}(x) \\
&\quad = (1+x)^{\frac{3}{2}} \sum_{n=1}^{\infty} n y_n \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} P_{n-1}^{(-\frac{1}{2},\frac{3}{2})}(x) \\
&\quad = 0, \quad x \in [-1, t_0].
\end{aligned}$$

Since the product is zero on the interval $[-1, t_0]$, hence

$$\sum_{n=1}^{\infty} n y_n \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} P_{n-1}^{(-\frac{1}{2},\frac{3}{2})}(x) = 0, \quad x \in [-1, t_0]. \tag{5.11}$$

And now we apply the Abel's transform technique. Taking $\eta = \frac{1}{2}$, we have (2.8) in the form of

$$P_{n-1}^{(-\frac{1}{2},\frac{3}{2})}(x) = \frac{(1+x)^{-\frac{3}{2}}}{\sqrt{\pi}} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \int_{-1}^{x} \frac{(1+t) P_{n-1}^{(0,1)}(t)}{(x-t)^{\frac{1}{2}}} dt. \tag{5.12}$$

Substituting (5.12) to (5.11), we have

$$\frac{(1+t)^{-\frac{3}{2}}}{\sqrt{\pi}} \int_{-1}^{t} \frac{\sum_{n=1}^{\infty} n y_n (1+t) P_{n-1}^{(0,1)}(t)}{(x-t)^{\frac{1}{2}}} dx = 0, \quad t \in [-1, t_0]. \tag{5.13}$$

This is in the form of an homogeneous Abel's integral equation, and the unique inversion is

$$\sum_{n=1}^{\infty} n y_n P_{n-1}^{(0,1)}(t) = 0, \quad t \in [-1, t_0]. \tag{5.14}$$

Next, we shall consider equation (5.10). By taking $\eta = \frac{1}{2}$, we have (2.7) as

$$P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(x) = \frac{(1-x)^{-\frac{1}{2}}}{\sqrt{\pi}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)} \int_{x}^{1} \frac{P_{n-1}^{(0,1)}(t)}{(t-x)^{\frac{1}{2}}} dt, \quad x \in [t_0, 1]. \tag{5.15}$$

By replacing the role of t by x in (5.10), and then substitute into (5.15) to obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty}&(y_n-c_n)\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\left[\frac{(1-x)^{-\frac{1}{2}}}{\sqrt{\pi}}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}\int_x^1\frac{P_{n-1}^{(0,1)}(t)}{(t-x)^{\frac{1}{2}}}dt\right]\\
&=\sum_{n=1}^{\infty}(y_n-c_n)\frac{n}{\sqrt{\pi}}(1-x)^{-\frac{1}{2}}\left[\int_x^1\frac{P_{n-1}^{(0,1)}(t)}{(t-x)^{\frac{1}{2}}}dt\right]\\
&=\frac{(1-x)^{-\frac{1}{2}}}{\sqrt{\pi}}\int_t^1\frac{\sum\limits_{n=1}^{\infty}n(y_n-c_n)P_{n-1}^{(0,1)}(t)}{(t-x)^{\frac{1}{2}}}dt\\
&=0,\quad t\in[t_0,1].
\end{aligned}
$$

Since the product is identically zero in the interval $[t_0,1]$, we have

$$\int_x^1\frac{\sum\limits_{n=1}^{\infty}n(y_n-c_n)P_{n-1}^{(0,1)}(t)}{(t-x)^{\frac{1}{2}}}dt=0,$$

which is in the form of homogeneous integral equation of Abel type, the unique inversion is

$$\sum_{n=1}^{\infty}n(y_n-c_n)P_{n-1}^{(0,1)}(t)=0,\quad t\in[t_0,1]. \tag{5.16}$$

It is clear that (5.14) and (5.16) have the same convergence rate, so we combine them and obtain the piecewise continuous function

$$\sum_{n=1}^{\infty}ny_nP_{n-1}^{(0,1)}(t)=\begin{cases}0, & x\in[-1,t_0],\\ \sum\limits_{n=1}^{\infty}nc_nP_{n-1}^{(0,1)}(t), & x\in[t_0,1].\end{cases}$$

By using the relation $P_{n-1}^{(0,1)}(t)=\sqrt{\frac{2}{n}}\widehat{P}_{n-1}^{(0,1)}(t)$, we replace the Jacobi polynomial by the normalised Jacobi polynomial and obtain

$$\sum_{n=1}^{\infty}\sqrt{2n}y_n\widehat{P}_{n-1}^{(0,1)}(t)=\begin{cases}0, & x\in[-1,t_0],\\ \sum\limits_{n=1}^{\infty}\sqrt{2n}c_n\widehat{P}_{n-1}^{(0,1)}(t), & x\in[t_0,1].\end{cases} \tag{5.17}$$

Recall that the set $\left\{\widehat{P}_{n-1}^{(0,1)}(t)\right\}_{n\in\mathbb{N}}$ is orthonormal in $[-1,1]$, so

$$\int_{-1}^{1}(1+t)\widehat{P}_{n-1}^{(0,1)}(t)\widehat{P}_{m-1}^{(0,1)}(t)dt=\delta_{m,n}.$$

Thus, by multiplying both sides of (5.17) with $(1+t)\widehat{P}_{m-1}^{(0,1)}(t)$ and integrating over $[-1,1]$, we obtain the infinite system of equations

$$\sqrt{2m}y_m=\sum_{n=1}^{\infty}\sqrt{2n}c_n\widehat{Q}_{n-1,m-1}^{(0,1)}(t_0) \tag{5.18}$$

for $m=1,2,3,\ldots$

5.3 Regularisation of the cosine system

By employing (3.23) and denoting $t_0 = \cos\theta_0$ and $t = \cos\theta$, the cosine system is hence in the form of

$$\sum_{n=1}^{\infty} n x_n \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} P_n^{(-\frac{1}{2},-\frac{1}{2})}(t) = -\frac{\xi_0}{\sqrt{\pi}}, \quad t \in [-1, t_0], \tag{5.19}$$

$$\sum_{n=1}^{\infty} (x_n - a_n) \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} P_n^{(-\frac{1}{2},-\frac{1}{2})}(t) = \frac{a_0}{\sqrt{\pi}}, \quad t \in [t_0, 1]. \tag{5.20}$$

Since $\alpha = \beta = -\frac{1}{2}$, a preliminary step is needed: we shall increase the value of α and β. Recall the two formulas which deduced from the Rodrigues's formula in Section 3.6:

$$\begin{aligned} \int_{-1}^{x} (1-t^2)^{-\frac{1}{2}} P_n^{(-\frac{1}{2},-\frac{1}{2})}(t)dt &= -\frac{1}{2n}(1-x^2)^{\frac{1}{2}} P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(x), \\ \int_{x}^{1} (1-t^2)^{-\frac{1}{2}} P_n^{(-\frac{1}{2},-\frac{1}{2})}(t)dt &= \frac{1}{2n}(1-x^2)^{\frac{1}{2}} P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(x). \end{aligned} \tag{5.21}$$

We perform term by term integration on (5.19) and (5.20) with weight $(1-t^2)^{-\frac{1}{2}}$ over the interval $[-1, x]$ and $[x, 1]$, respectively. Then, by using (5.21), we obtain the following

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} x_n P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(x) = \frac{2\xi_0}{\sqrt{\pi}} \frac{\frac{\pi}{2} + \arcsin x}{(1-x^2)^{\frac{1}{2}}}, \quad x \in [-1, t_0], \tag{5.22}$$

$$\sum_{n=1}^{\infty} \frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})} (x_n - a_n) P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(x) = \frac{2a_0}{\sqrt{\pi}} \frac{\frac{\pi}{2} - \arcsin x}{(1-x^2)^{\frac{1}{2}}}, \quad x \in [t_0, 1]. \tag{5.23}$$

The dual series equation are now having the indices $\alpha = \beta = \frac{1}{2}$, and we can proceed the similar steps as the regularisation of the sine system.

By using (2.10), we have (5.22) as

$$\begin{aligned} (1+x)^{\frac{3}{2}} \sum_{n=1}^{\infty} & \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} x_n P_{n-1}^{(-\frac{1}{2},\frac{3}{2})}(x) \\ &= \frac{4\xi_0}{\sqrt{\pi}} \left[2(1+x)^{\frac{1}{2}} - (1-x)^{\frac{1}{2}} \left(\frac{\pi}{2} + \arcsin x \right) \right], \quad x \in [-1, t_0]. \end{aligned} \tag{5.24}$$

(Note that for convenient purpose, we have retained the parameter of (5.24) as x.)

Applying (5.15) and (5.12),

$$P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(x) = \frac{(1-x)^{-\frac{1}{2}}}{\sqrt{\pi}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)} \int_x^1 \frac{P_{n-1}^{(0,1)}(t)}{(t-x)^{\frac{1}{2}}} dt,$$

$$P_{n-1}^{(-\frac{1}{2},\frac{3}{2})}(x) = \frac{(1+x)^{-\frac{3}{2}}}{\sqrt{\pi}} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \int_{-1}^{x} \frac{(1+t) P_{n-1}^{(0,1)}(t)}{(x-t)^{\frac{1}{2}}} dt,$$

we can convert (5.23) and (5.24) into integral equations of Abel type:

$$\int_x^1 \frac{\sum_{n=1}^{\infty}(x_n - a_n)P_{n-1}^{(0,1)}(t)}{(t-x)^{\frac{1}{2}}}dt = 2a_0\frac{\frac{\pi}{2} - \arcsin x}{(1+x)^{\frac{1}{2}}}, \quad x \in [t_0, 1],$$

$$\int_{-1}^x \frac{(1+t)\sum_{n=1}^{\infty} x_n P_{n-1}^{(0,1)}(t)}{(x-t)^{\frac{1}{2}}}dt = 4\xi_0\left[2(1+x)^{\frac{1}{2}} - (1-x)^{\frac{1}{2}}\left(\frac{\pi}{2} + \arcsin x\right)\right], \quad x \in [-1, t_0].$$

Inversion of the equations gives us

$$\sum_{n=1}^{\infty}(x_n - a_n)P_{n-1}^{(0,1)}(t) = -\frac{2a_0}{\pi}\frac{d}{dt}\int_t^1 \frac{\frac{\pi}{2} - \arcsin x}{(1+x)^{\frac{1}{2}}(x-t)^{\frac{1}{2}}}dx \tag{5.25}$$

for $t \in [t_0, 1]$, and

$$(1+t)\sum_{n=1}^{\infty} x_n P_{n-1}^{(0,1)}(t) = \frac{4\xi_0}{\pi}\frac{d}{dt}\int_{-1}^t \frac{2(1+x)^{\frac{1}{2}} - (1-x)^{\frac{1}{2}}\left(\frac{\pi}{2} + \arcsin x\right)}{(t-x)^{\frac{1}{2}}}dx \tag{5.26}$$

for $t \in [-1, t_0]$.

Note that the integrals on the right hand side of (5.25) and (5.26) have been evaluated in Section 3.6.1 and Section 3.6.2. So, we have

$$\frac{d}{dt}\int_t^1 \frac{\frac{\pi}{2} - \arcsin x}{(1+x)^{\frac{1}{2}}(x-t)^{\frac{1}{2}}}dx = -\frac{\pi}{1+t}, \tag{5.27}$$

$$\frac{d}{dt}\int_{-1}^t \frac{2(1+x)^{\frac{1}{2}} - (1-x)^{\frac{1}{2}}\left(\frac{\pi}{2} + \arcsin x\right)}{(t-x)^{\frac{1}{2}}}dx = -\frac{\pi}{2}\ln\left(\frac{1-t}{2}\right). \tag{5.28}$$

Now, substituting (5.27) and (5.28) into (5.25) and (5.26), respectively, and combine, we obtain the piecewise equation

$$(1+t)\sum_{n=1}^{\infty} x_n P_{n-1}^{(0,1)}(t) = \begin{cases} -2\xi_0 \ln\left(\frac{1-t}{2}\right), & t \in [-1, t_0], \\ 2a_0 + (1+t)\sum_{n=1}^{\infty} a_n P_{n-1}^{(0,1)}(t), & t \in [t_0, 1]. \end{cases} \tag{5.29}$$

By rescaling the Jacobi polynomial using $P_{n-1}^{(0,1)}(t) = \sqrt{\frac{2}{n}}\widehat{P}_{n-1}^{(0,1)}(t)$, (5.29) becomes

$$(1+t)\sum_{n=1}^{\infty}\sqrt{\frac{2}{n}} x_n \widehat{P}_{n-1}^{(0,1)}(t) = \begin{cases} -2\xi_0 \ln\left(\frac{1-t}{2}\right), & t \in [-1, t_0], \\ 2a_0 + (1+t)\sum_{n=1}^{\infty} \sqrt{\frac{2}{n}}\widehat{P}_{n-1}^{(0,1)}(t), & t \in [t_0, 1]. \end{cases} \tag{5.30}$$

The solution belonging to the functional class $l^2(1)$ results in continuity being imposed on the piecewise continuous function in (5.30) at $t = t_0$, and hence we have

$$\xi_0 = -\frac{a_0}{\ln\left(\frac{1-t_0}{2}\right)} - \frac{1+t_0}{2\ln\left(\frac{1-t_0}{2}\right)}\sum_{n=1}^{\infty}\sqrt{\frac{2}{n}} a_n \widehat{P}_{n-1}^{(0,1)}(t_0). \tag{5.31}$$

Now, we employ the orthogonality of the Jacobi polynomials. Multiplying both sides of (5.30) by $\widehat{P}^{(0,1)}_{m-1}(t)$ and integrate over $[-1,1]$, we obtain

$$\sqrt{\frac{2}{m}}x_m = -2\xi_0\int_{-1}^{t_0}\ln\left(\frac{1-t}{2}\right)\widehat{P}^{(0,1)}_{m-1}(t)dt+2a_0\int_{t_0}^{1}\widehat{P}^{(0,1)}_{m-1}(t)dt+\sum_{n=1}^{\infty}\sqrt{\frac{2}{n}}a_n\widehat{Q}^{(0,1)}_{n-1,m-1}(t_0). \tag{5.32}$$

Note that the first integral on the left hand side of (5.32) is already been evaluated in Section 3.6.3. So, we have

$$\int_{-1}^{t_0}\ln\left(\frac{1-t}{2}\right)\widehat{P}^{(0,1)}_{m-1}(t)dt = -\frac{1-t_0}{m}\ln\left(\frac{1-t_0}{2}\right)\widehat{P}^{(1,0)}_{m-1}(t_0)-\frac{1+t_0}{m^2}\widehat{P}^{(0,1)}_{m-1}(t_0). \tag{5.33}$$

From (2.4), the second integral of (5.32) is

$$\int_{t_0}^{1}\widehat{P}^{(0,1)}_{m-1}(t)dt = \frac{1-t_0}{m}\widehat{P}^{(1,0)}_{m-1}(t_0). \tag{5.34}$$

And now, we shall substitute (5.31), (5.33), and (5.34) into (5.32), and obtain

$$\begin{aligned}\sqrt{\frac{2}{m}}x_m =& -2a_0\frac{1+t_0}{m^2}\frac{\hat{P}^{(0,1)}_{m-1}(t_0)}{\ln\left(\frac{1-t_0}{2}\right)}\\ &+\sum_{n=1}^{\infty}\sqrt{\frac{2}{n}}a_n\left[\widehat{Q}^{(0,1)}_{n-1,m-1}(t_0)-\frac{(1-t_0^2)}{m}\widehat{P}^{(0,1)}_{n-1}(t_0)\widehat{P}^{(1,0)}_{m-1}(t_0)\right]\\ &-\sum_{n=1}^{\infty}\sqrt{\frac{2}{n}}a_n\left[\frac{(1+t_0)^2}{m^2}\frac{\widehat{P}^{(0,1)}_{n-1}(t_0)\widehat{P}^{(0,1)}_{m-1}(t_0)}{\ln\left(\frac{1-t_0}{2}\right)}\right].\end{aligned} \tag{5.35}$$

Finally, replace $\widehat{Q}^{(0,1)}_{n-1,m-1}(t_0)$ by using the relation (2.6), we therefore obtain the infinite linear system of equations

$$\begin{aligned}\sqrt{2m}x_m =& -2a_0\frac{1+t_0}{m}\frac{\widehat{P}^{(0,1)}_{m-1}(t_0)}{\ln\left(\frac{1-t_0}{2}\right)}\\ &+\sum_{n=1}^{\infty}\sqrt{2n}a_n\widehat{Q}^{(1,0)}_{n-1,m-1}(t_0)\\ &-\sum_{n=1}^{\infty}\sqrt{2n}a_n\frac{(1+t_0)^2}{nm}\frac{\widehat{P}^{(0,1)}_{n-1}(t_0)\widehat{P}^{(0,1)}_{m-1}(t_0)}{\ln\left(\frac{1-t_0}{2}\right)}\end{aligned} \tag{5.36}$$

for $m = 1, 2, 3, \ldots$

5.4 Summary of the Chapter

At the beginning of the chapter, we considered the Fourier series of the integral representation of the scattered field solution:

$$\sum_{n=-\infty}^{\infty}\left\{|n|\xi_n + 2\sum_{m=-\infty}^{\infty}(k_{n,-m})\xi_m - f_n\right\}e^{in\theta} = 0 \quad \text{for} \quad \theta \in [-\theta_0, \theta_0],$$
$$\sum_{n=-\infty}^{\infty}\xi_n e^{in\theta} = 0 \quad \text{for} \quad \theta \in [-\pi, -\theta_0) \cup (\theta_0, \pi].$$

Using the identities $e^{in\theta} = \cos n\theta + i\sin n\theta$ and $e^{-in\theta} = \cos n\theta - i\sin n\theta$ lead to a coupled dual series equation:

$$\begin{cases} \sum\limits_{n=1}^{\infty}(y_n - c_n)\sin n\theta = 0, & \theta \in [0, \theta_0], \\ \sum\limits_{n=1}^{\infty} n y_n \sin n\theta = 0, & \theta \in [\theta_0, \pi] \end{cases}$$

and

$$\begin{cases} \sum\limits_{n=1}^{\infty}(x_n - a_n)\cos n\theta = a_0, & \theta \in [0, \theta_0], \\ \sum\limits_{n=1}^{\infty} n x_n \cos n\theta = -\xi_0, & \theta \in [\theta_0, \pi]. \end{cases}$$

Then, replacing the sine and cosine functions by their Jacobi polynomial representations, and using the properties of Jacobi polynomial, we reduced the dual series into integral equation of Abel type which have unique inversion. Finally, by the orthogonality of Jacobi polynomials, we have the following:

$$\begin{aligned}
\sqrt{2m}y_m =& \sum_{n=1}^{\infty}\sqrt{2n}c_n\widehat{Q}^{(0,1)}_{n-1,m-1}(t_0), \\
\sqrt{2m}x_m =& -2a_0\frac{1+t_0}{m}\frac{\widehat{P}^{(0,1)}_{m-1}(t_0)}{\ln\left(\frac{1-t_0}{2}\right)} \\
&+ \sum_{n=1}^{\infty}\sqrt{2n}a_n\left[\widehat{Q}^{(1,0)}_{n-1,m-1}(t_0) - \frac{(1+t_0)^2}{nm}\frac{\widehat{P}^{(0,1)}_{n-1}(t_0)\widehat{P}^{(0,1)}_{m-1}(t_0)}{\ln\left(\frac{1-t_0}{2}\right)}\right]
\end{aligned}$$

where

$$
\begin{aligned}
x_n &= \frac{\xi_n + \xi_{-n}}{n},\\
y_n &= \frac{\xi_n - \xi_{-n}}{n},\\
f_n^+ &= f_n + f_{-n},\\
f_n^- &= f_n - f_{-n},\\
a_0 &= f_0 - 2\xi_0[k_{0,0}] - 2\sum_{p=1}^{\infty}[k_{0,-p} + k_{0,p}]px_p,\\
a_n &= f_n^+ - 2\xi_0[k_{n,0} + k_{-n,0}] - 2\sum_{p=1}^{\infty}\left([k_{n,-p} + k_{-n,-p}] + [k_{-n,p} + k_{n,p}]\right)px_p,\\
c_n &= f_n^- - 2\xi_0[k_{n,0} + k_{-n,0}] - 2\sum_{p=1}^{\infty}\left([k_{n,-p} - k_{-n,-p}] + [k_{-n,p} - k_{n,p}]\right)py_p,\\
\xi_0 &= \frac{a_0}{\ln\left(\frac{1-t_0}{2}\right)} - \frac{1+t_0}{2\ln\left(\frac{1-t_0}{2}\right)}\sum_{n=1}^{\infty}\sqrt{\frac{2}{n}}a_n\widehat{P}_{n-1}^{(0,1)}(t_0).
\end{aligned}
$$

Furthermore, by rescaling the unknowns

$$X_0 = 2\xi_0, \quad X_n = \sqrt{2n}x_n, \quad Y_n = \sqrt{2n}y_n,$$

then it can be shown that when we combine these two systems, we obtain the desired Fredholm equation of the second kind. In conjunction, one extra equation will occurs when we substitute both a_0 and a_n into ξ_0.

5.5 Conclusion

In this thesis, we begin by considering the wave scattering problems for the infinite circular cylinder in both open and closed case. By using the known fact that the wave's identical behaviour along the cylinder, we then reduced the problem into a two dimensional BVP. Since circle can be parametrised by using the cylindrical polar coordinate, so for the closed case, we show using the separation of variable technique and obtain an analytical solution. The analytical solution is in the form of Fourier series, a well known series where the convergent can be guaranteed. As for the open case, we have the solution resulted in a coupled dual series equation. To solve such equations, we used MoR to convert the dual series equation into the well-behaved Fredholm equation of the second kind, in which the solution can be computed by truncation method.

Then, we moved to the formulation of the wave scattering problems for an infinite long open cylinder with arbitrary cross section. With the appropriate setting, the problem is independent of the z-axis and thus simplified the problem to two dimensional BVP. Using the standard representation of the scattered field in terms of the surface quantities and a Green's function, the scattered field required us to solve a first kind Fredholm integral equation. Moreover, the kernel is a highly singular second normal derivatives of Green's function. To deal with this, we used the results from [13] to transform the integral equation to its equivalent Fourier series representation and resulted a dual series equation.

To solve dual series equation, we first decoupled the exponential kernel into sine and cosine functions, and that leads us to a coupled dual series equation which is similar to the open circular cylinder case. The similar approach have been apply, we converted the sine and cosine functions into their Jacobi polynomial representation, then apply the Abel's transform technique and orthogonality of Jacobi polynomial to convert the series into an infinite system of linear algebraic function which in the form of a well conditioned second kind Fredholm equation.

Thus, the solution can be obtained by numerical method, and we shall conclude the thesis here.

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