A STUDY OF MUSIELAK-ORLICZ HARDY SPACES, WEIGHTED MORREY SPACES AND BOUNDEDNESS OF OPERATORS

By

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This thesis entitled:

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I also certify that the thesis is an original piece of research and it has been written by me. Any help and assistance that I have received in my research work and the preparation of the thesis itself has been appropriately acknowledged.

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Tri Dung Tran

Abstract

The main aim of this thesis is to study Musielak-Orlicz Hardy spaces associated with certain operators and to investigate generalized weighted Hardy-Cesàro operators and their commutators on weighted Morrey spaces. The main results of the thesis are presented in the last three chapters.

In Chapter 3, let L be a divergence form elliptic operator with complex bounded measurable coefficients, let ω be a positive Musielak-Orlicz function on $(0,\infty)$ of uniformly strictly critical lower type $p_\omega \in (0,1]$, and let $\rho(x,t) = t^{-1}/\omega^{-1}(x,t^{-1})$ for $x \in \mathbb{R}^n, t \in (0,\infty)$. We first study the Musielak-Orlicz Hardy space $H_{\omega,L}(\mathbb{R}^n)$ and then its dual space $\mathrm{BMO}_{\rho,L^*}(\mathbb{R}^n)$, where L^* denotes the adjoint operator of L in $L^2(\mathbb{R}^n)$. The ρ-Carleson measure characterization and the John-Nirenberg inequality for the space $\mathrm{BMO}_{\rho,L}(\mathbb{R}^n)$ are also established. Then, as applications, we show that the Riesz transform $\nabla L^{-1/2}$ and the Littlewood-Paley g-function g_L map $H_{\omega,L}(\mathbb{R}^n)$ continuously into $L^1(\omega)$.

In Chapter 4, assume that L is an operator which satisfies Davies-Gaffney heat kernel estimates and has a bounded H_∞ functional calculus on $L^2(X)$, where X is a metric space with doubling measure, then we develop a theory of Musielak-Orlicz Hardy spaces associated to L, including a molecular decomposition, square function characterization and duality of Musielak-Orlicz Hardy spaces $H_{L,\omega}(X)$. As applications, we show that L has a bounded holomorphic functional calculus on $H_{L,\omega}(X)$ and the Riesz transform is bounded from $H_{L,\omega}(X)$ to $L^1(\omega)$.

In the last chapter, let $\psi : [0,1] \to [0,\infty)$ and $s : [0,1] \to \mathbb{R}$ be measurable functions, and let Γ be a parameter curve in $\mathbb{R}^n$ given by $(t,x) \in [0,1] \times \mathbb{R}^n \to s(t,x) = s(t)x$. Then we study a new weighted Hardy-Cesàro operator defined by $U_{\psi,s}f(x) = \int\limits_0^1 f\left(s(t)x\right)\psi(t)dt$, for measurable complex-valued functions f on $\mathbb{R}^n$. Under certain conditions on $s(t)$ and on an absolutely homogeneous weight function ω, we characterize the weight function ψ such that $U_{\psi,s}$ is bounded on weighted Morrey spaces $L^{p,\lambda}(\omega)$ and then compute the corresponding operator norm of $U_{\psi,s}$. We also give a necessary and sufficient condition on the function ψ, which ensures the boundedness of the commutator of the operator $U_{\psi,s}$ from weighted central Morrey spaces $\dot{L}^{q,\lambda}(\omega)$ to weighted central Morrey spaces $\dot{L}^{p,\lambda}(\omega)$ $(1 < p < q < \infty)$ for all symbols b in the space $\mathrm{BMO}(\omega)$.

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Contents

1
Introduction

The introduction and study of classical real-variable Hardy and BMO spaces on the Euclidean space $\mathbb{R}^n$ began in the 1960s with the initial paper of Stein and Weiss [74]. Later, this theory was developed systematically by Fefferman and Stein [39] and studied extensively in [24] and [73] as well by many others. Since then these classes of functions have played an important role in a number of analyses, such as modern harmonic analysis and partial differential equations. It is now well known that there are various equivalent characterizations of functions in the classical Hardy space. For instance, the Hardy space $H^1(\mathbb{R}^n)$ can be viewed as the set of functions $f \in L^1(\mathbb{R}^n)$ such that the Riesz transform $\nabla\Delta^{-1/2}f$ belongs to $L^1(\mathbb{R}^n)$. We also have alternative characterizations of $H^1(\mathbb{R}^n)$ via the atomic decomposition or by the square function and the nontangential maximal function associated to the Poisson semigroup generated by the Laplacian. Basically, this standard theory of Hardy spaces is intimately connected with properties of harmonic functions and of the Laplacian.

Nevertheless, it is a fact that there are some situations in which the classical Hardy spaces are not applicable. For example, one considers a general elliptic operator in divergence form with complex bounded coefficients. Let A be an $n \times n$ matrix with entries $\{a_{j,k}\}_{j,\,k=1}^n \subset L^\infty(\mathbb{R}^n, \mathbb{C})$ satisfying the ellipticity conditions; namely, there exist constants $0 < \lambda_A \le \Lambda_A < \infty$ such that for all ξ, $\zeta \in \mathbb{C}^n$,

$$\lambda_A|\xi|^2 \le \mathcal{R}e\langle A\xi, \xi\rangle \quad \text{and} \quad |\langle A\xi, \zeta\rangle| \le \Lambda_A|\xi||\zeta|. \tag{1.0.1}$$

Then the second-order divergence form operator is given by

$$Lf \equiv -\operatorname{div}(A\nabla f), \tag{1.0.2}$$

interpreted in the weak sense via a sesquilinear form.

It is shown that the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^2(\mathbb{R}^n)$ but not bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ (for more details, see [46]). The need for research of new Hardy spaces other than the Hardy space $H^1(\mathbb{R}^n)$ thus naturally arises.

In recent years, function spaces, especially Hardy spaces and BMO spaces associated with different operators have inspired great interest (see, e.g., [5, 8, 9, 33, 35, 36, 45, 46, 84] and references therein). In particular, Auscher, Duong, and McIntosh [5] first introduced the Hardy space $H^1_L(\mathbb{R}^n)$ associated with an operator L whose heat kernel satisfies a pointwise Poisson type upper bound by means of a corresponding variant of the Lusin area function, and established its molecular characterization. Later, Duong and Yan [35, 36] introduced its dual space $\mathrm{BMO}_L(\mathbb{R}^n)$ and established the dual relation between $H^1_L(\mathbb{R}^n)$ and $\mathrm{BMO}_L(\mathbb{R}^n)$. Yan [84] further generalized these results to the Hardy spaces $H^p_L(\mathbb{R}^n)$ with certain $p \le 1$ and their dual spaces. Also, Auscher and Russ [9] studied the Hardy space H^1_L on strongly Lipschitz domains associated with a divergence form elliptic operator L whose heat kernels have the Gaussian upper bounds and regularity. Very recently, Auscher, McIntosh, and Russ [8] treated the Hardy space H^p with $p \in [1, \infty]$ associated to the Hodge Laplacian on a Riemannian manifold with doubling measure. Meanwhile, Hofmann and Mayboroda [46] further studied the Hardy space $H^1_L(\mathbb{R}^n)$ and its dual space adapted to a second-order divergence form elliptic operator L on $\mathbb{R}^n$ with bounded complex coefficients, and these operators may not have the pointwise heat kernel bounds. Then Hofmann, Lu, D. Mitrea, M. Mitrea, and Yan [45] introduced the new Hardy spaces H^p_L, $1 \le p < \infty$, on a metric space X associated to a nonnegative self-adjoint operator L satisfying Davies–Gaffney estimates. The motivation for investigating Hardy spaces, for example, is that boundedness in Hardy spaces can be interpolated with an L^2-boundedness to obtain other L^p-boundednesses. For this particular application, the atomic decomposition of Hardy spaces is very convenient and, as pointed out in some recent works, the set of atoms is even sufficient, so we do not have to study boundedness on the whole Hardy space (which may be difficult to prove; see, e.g., [12] and [15]).

On the other hand, as generalizations of Hardy spaces $H^p(\mathbb{R}^n)$, the Orlicz–Hardy spaces on $\mathbb{R}^n$ and their dual spaces have received considerable attention as well. In particular, Strömberg [76] and Janson [48] introduced generalized Hardy spaces $H_\omega(\mathbb{R}^n)$, via replacing the norm $\|\cdot\|_{L^p(\mathbb{R}^n)}$ by the Orlicz-norm $\|\cdot\|_{L^1(\omega)}$ in the definition of $H^p(\mathbb{R}^n)$, where ω is an Orlicz function on $[0, \infty)$ satisfying some control conditions. Viviani [82] further characterized these spaces H_ω on spaces of homogeneous type via atoms. The dual spaces of these spaces were also studied in [76], [48],[82], and [43]. Very recently, Orlicz–Hardy spaces associated with certain operators have been investigated by a number of mathematicians (see, e.g., [51], [49], [50], and [58] and references therein). In particular, Jiang and Yang [49, 50] introduced the new Orlicz–Hardy spaces associated to divergence form elliptic operators and to nonnegative self-adjoint operators holding Davies–Gaffney estimates. Meanwhile, Liang, D. Yang, and S. Yang [58] presented some applications of Orlicz–Hardy spaces associated with operators satisfying Poisson estimates.

Motivated by all of the above-mentioned facts, we will study generalized Orlicz–Hardy spaces related to generalized Orlicz functions. In this setting, the Orlicz function $\varphi(t)$ is replaced by a function $\omega(x, t)$, called *the Musielak–Orlicz function* (see [28, 66], and Chapter 2, Section 2.5 below), that may vary in the spatial variables and possesses some control conditions. We then introduce a new class of Hardy spaces $H_{\omega,L}$, called *Hardy spaces of Musielak–Orlicz type* associated to certain operators L, and their dual spaces.

In Chapter 2, we will provide some preliminaries and background from harmonic analysis and operator theory. The main parts of the thesis are the next three chapters, which we now give a summary.

Chapter 3 is devoted to the study of Hardy spaces of Musielak–Orlicz type associated to the second-order divergence form elliptic operator L on $\mathbb{R}^n$ defined by (1.0.1) and (1.0.2), and this chapter is organized as follows.

In Section 3.1, we recall some basic notation and known results on second-order divergence form elliptic operators L. Under some basic assumptions on the Musielak–Orlicz function ω considered in Chapter 2, we show that the Musielak–Orlicz Hardy space $H_{\omega,L}(\mathbb{R}^n)$ behaves, in some sense, more closely like the classical Hardy space.

In Section 3.2, we introduce the tent spaces $T_\omega(\mathbb{R}^{n+1}_+)$ associated to ω and establish its atomic characterization (see Theorem 3.2.3 below). By the proof of Theorem 3.2.3, we observe that if a function $f \in T_\omega(\mathbb{R}^{n+1}_+) \cap T^p_2(\mathbb{R}^{n+1}_+)$, $p \in (0,\infty)$, then there exists an atomic decomposition of F which converges in both $T_\omega(\mathbb{R}^{n+1}_+)$ and $T^p_2(\mathbb{R}^{n+1}_+)$ (see Proposition 3.2.5 below). As a consequence, we show that if $f \in T_\omega(\mathbb{R}^{n+1}_+) \cap T^2_2(\mathbb{R}^{n+1}_+)$, then there exists an atomic decomposition of f which converges in both $T_\omega(\mathbb{R}^{n+1}_+)$ and $T^p_2(\mathbb{R}^{n+1}_+)$ for all $p \in [1,2]$ (see Corollary 3.2.6 below). These convergences play a significant role in the whole chapter.

In Section 3.3, we first introduce the Musielak–Orlicz Hardy space $H_{\omega,\,L}(\mathbb{R}^n)$, and then prove that the operator $\pi_{L,M}$, which is introduced in [36] and initially defined on $f \in L^2(\mathbb{R}^{n+1}_+)$ with compact support by

$$\pi_{L,M} f \equiv C_M \int_0^\infty (t^2L)^{M+1} e^{-t^2L} f(\cdot,t)\,\frac{dt}{t}, \tag{1.0.3}$$

where $M \in \mathbb{N}$ and

$$C_M \int_0^\infty t^{2(M+2)} e^{-2t^2}\,\frac{dt}{t} = 1,$$

maps the tent space $T^p_2(\mathbb{R}^{n+1}_+)$ continuously into $L^p(\mathbb{R}^n)$ for $p \in (p_L, \widetilde{p}_L)$ and $T_\omega(\mathbb{R}^{n+1}_+)$ continuously into $H_{\omega,L}(\mathbb{R}^n)$ (see Proposition 3.3.6 below). As a result, we obtain a molecular decomposition for elements in $H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ which converges in $L^p(\mathbb{R}^n)$ for $p \in (p_L, 2]$ (see Proposition 3.3.7 below), and then characterize the Musielak–Orlicz Hardy space $H_{\omega,L}(\mathbb{R}^n)$ via the Lusin-area operator $\mathcal{S}_P$ associated to the Poisson semigroup (see Theorem 3.3.14 below). It should be pointed out here that the divergence form structure of the operator L plays an important role in obtaining this equivalent characterization. Next, due to the molecular decomposition of $H_{\omega,\,L}(\mathbb{R}^n)$, we further obtain the duality between $H_{\omega,\,L}(\mathbb{R}^n)$ and $\mathrm{BMO}_{\rho,\,L^*}(\mathbb{R}^n)$ (see Theorem 3.3.20 below). The rest of Section 3.3 is devoted to establishing the ρ–Carleson measure characterization (see Theorem 3.3.24 below) and the John–Nirenberg inequality (see Theorem 3.3.25 below) for the space $\mathrm{BMO}_{\rho,L}(\mathbb{R}^n)$.

In Section 3.4, as an application, we give some sufficient conditions which guarantee the boundedness of linear or nonnegative sublinear operators from $H_{\omega,L}(\mathbb{R}^n)$ to $L^1(\omega)$. In particular, we show that the Riesz transform $\nabla L^{-1/2}$ and the Littlewood–Paley g-function g_L map $H_{\omega,L}(\mathbb{R}^n)$ continuously into $L^1(\omega)$ (see Theorem 3.4.1 below).

Chapter 3 is essentially based on a revision of the material as it is going to appear in *Nagoya Mathematical Journal* under the title "T. D. Tran, *Musielak–Orlicz Hardy spaces associated with divergence form elliptic operators without weight assumptions*". It should be pointed here that our approach in Chapter 3 and the article above is strongly motivated by Hofmann and Mayboroda [46] and Jiang and Yang [49], and that the majority of our results are closely following those of [49].

However, it should be emphasized that *the theory of Orlicz-Hardy spaces developed in [49] is not true in the setting of Musielak-Orlicz Hardy spaces in general.* So our contribution in Chapter 3 is introducing an appropriate way of generalising the concepts from the Orlicz setting in [49] to the Musielak–Orlicz setting, as well as introducing Assumption (C) (see page 27 and 28) on the Musielak-Orlicz function $\omega(x,t)$ under which one can obtain the results similar to but more general than those proved in [49]. Certainly, this extension to the Musielak–Orlicz setting is non-trivial, even though it naturally and strongly follows from the previous works of [49], [46] and others. Indeed, there are some new technical results that we need to add to the work in Chapter 3. For instance, we illustrate here two of such results.

- Lemma 2.5.3, page 28. In order to obtain this key lemma in the Musielak–Orlicz setting, the functions $\omega^{-1}(x,\cdot)$ and $\rho(x,\cdot)$ on $\mathbb{R}_+$ (in Definition 2.5.2, page 27) need to be defined in a different way, compared to the Orlicz case, see [49, page 1175, line 2-4 from below], which enables us to give a proof to Lemma 2.5.3. In addition, this proof is new even if the Musielak–Orlicz function $\omega(x,t)$ is independent of the space variable x and reduces to the Orlicz function $\omega(t)$.

- Lemma 2.5.1, page 27, which claims the following important properties: under Assumption (C), the function ω is uniformly σ-quasi-subadditive on $\mathbb{R}^n \times [0,\infty)$; namely, there exists a positive constant C such that for all $(x,t_j) \in \mathbb{R}^n \times [0,\infty)$ with $j \in \mathbb{Z}_+$, $\omega(x, \sum_{j=1}^{\infty} t_j) \le C \sum_{j=1}^{\infty} \omega(x,t_j)$. Moreover, if we let $\widetilde{\omega}(x,t) := \int_0^t \frac{\omega(x,s)}{s}\, ds$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$, then $\widetilde{\omega}$ is equivalent to ω.

Note that both Lemma 2.5.1 and Lemma 2.5.3 play a significant role in generalising the well-known theory in the Orlicz setting from [49] to the more general theory in the Musielak–Orlicz setting in Chapter 3.

While our article [81] was submitted, we learned that the theory of Musielak–Orlicz Hardy spaces associated with operators has also been studied independently by the authors in [85]. Later, provided that $\mathcal{X}$ is a metric space with doubling measure then the Musielak–Orlicz Hardy space, associated with a one-to-one operator of type ω having a bounded H_∞ functional calculus in $L^2(\mathcal{X})$ and satisfying the reinforced off-diagonal estimates, was investigated in [17]. As a special case of this setting, the Musielak–Orlicz Hardy space associated with divergence form elliptic operators with complex bounded measurable coefficients on $\mathbb{R}^n$ was also treated. Basically, the Musielak–Orlicz functions ω considered in [85] and [17] are *growth functions*, that is, the functions $\omega(\cdot,t)$ belong to the *uniformly weight class* $\mathbb{A}_\infty$ and satisfy the *uniformly reverse Hölder condition*. However, whether the assumptions on a Musielak–Orlicz function in [85] and [17] and Assumption (C) in Chapter 3 can be comparable is still unclear to us.

As a continuation to the previous work, in Chapter 4 we study a more generalized form of Musielak–Orlicz Hardy spaces $H_{L,\omega}(X)$ associated to operators, and their dual spaces, under the assumptions that the operators have bounded H_∞ functional calculi and satisfy Davies–Gaffney estimates. We remark that our assumption that the operator having a bounded H_∞ functional calculus is much weaker than the usual assumption that L being nonnegative self-adjoint which played an important role in a number of the previous works, see for example [45, 49, 50]. For example, a nonnegative self-adjoint operator L with Davies–Gaffney estimates would satisfy the finite speed propagation property for solutions of the corresponding wave equation, see [71] (see also [45]), and allows us to construct the Hardy space via (ω, M)-atoms, see [45, 49, 50].

There are two key generalizations in this work.

First, we replace the assumption that L is a nonnegative self-adjoint operator by the weaker assumption that L has a bounded H_∞ functional calculus on $L^2(X)$. This would allow a much larger class of applicable operators L. For example, it is well known that in general the second-order divergence form operator L defined by (1.0.1) and (1.0.2) is not a self-adjoint operator, but L has a bounded H_∞ functional calculus on $L^2(X)$; see for example [10]. For another example, one considers the operator $L = b(x)\Delta$, a special case of a second-order elliptic operator in non-divergence form with bounded measurable complex coefficients, where Δ denotes the Laplacian in $\mathbb{R}^n$ and b denotes an ω-accretive function on $\mathbb{R}^n$, $\omega \in [0, \frac{\pi}{2})$, with bounded reciprocal, meaning that b and $\frac{1}{b}$ belong to $L^\infty(\mathbb{R}^n, \mathbb{C})$ and $|\arg b(x)| \le \omega$ for almost all $x \in \mathbb{R}^n$. The operator $L = b(x)\Delta$ is clearly not self-adjoint in general and it has a bounded H_∞ functional calculus on $L^2(X)$, see [62, Proposition 1.1]. Furthermore, if $\Re b(x) \ge \delta > 0$ for almost all $x \in \mathbb{R}^n$, then the semigroup $\{e^{-tL}\}_{t>0}$ satisfies the Davies–Gaffney estimate (4.1.1), see [31].

Second, the Orlicz functions $\varphi(t)$ appearing in many of previous works are replaced by more general Musielak–Orlicz functions $\omega(x,t)$. In the particular case when $\omega = t^p, p \in (0,1]$, our results are in line with those in [30]. In another special case, if ω is an Orlicz function on $\mathbb{R}_+$ with $p_\omega \in (0,1]$, which is continuous, strictly increasing and concave then by Jensen's inequality it can be verified that Assumption (C) on the function ω holds (see Section 2.5 below). In this sense, our work is an extension to [42].

Chapter 4 is organized as follows. In Section 4.1, we present some assumptions on the operator L. In Sections 4.2, we shall introduce Musielak–Orlicz Hardy spaces $H_{L,\omega}(X)$. Then we show that each function in $H_{L,\omega}(X)$ can be represented as a decomposition of (ω, ϵ, M)-molecules and more importantly, the space of all finite linear combinations of (ω, ϵ, M)-molecule is dense in $H_{L,\omega}(X)$. Then the dual spaces of $H_{L,\omega}(X)$ are investigated. In the last section, we consider applications of the holomorphic functional calculus of the operator L and certain Riesz transforms associated to L. By using the molecular decomposition associated to the operator L and the Musielak–Orlicz function, we shall show that L has a bounded holomorphic functional calculus on the Musielak–Orlicz Hardy spaces $H_{L,\omega}(X)$ and the Riesz transforms are bounded from $H_{L,\omega}(X)$ to $L^1(\omega)$.

Chapter 4, in full, is a revision of the material as it is going to appear in *Journal of the Mathematical Society of Japan* under the title "*Musielak–Orlicz Hardy spaces associated to operators satisfying Davies–Gaffney estimates and bounded holomorphic functional calculus*", a joint work of X. T. Duong and T. D. Tran.

In Chapter 5, we investigate generalized weighted Hardy–Cesàro operators and their commutators on weighted central Morrey spaces.

Let us consider the classical Hardy operator U defined by

$$Uf(x) = \frac{1}{x}\int_0^x f(t)dt, x \neq 0$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. A celebrated Hardy integral inequality, see [44], can be formulated as

$$\|Uf\|_{L^p(\mathbb{R})} \leq \frac{p}{p-1}\|f\|_{L^p(\mathbb{R})},$$

where $1 < p < \infty$, in which the constant $\frac{p}{p-1}$ is known as the best constant.

The Hardy integral inequality and its variants have played an important role in various branches of analysis such as approximation theory, differential equations, theory of function spaces, etc. Therefore, the Hardy integral inequalities for operator U and their generalizations have been studied extensively.

Up to now, there are two types of Hardy operator in $n-$dimension case. The first type was introduced by Faris [38] and studied further by M. Christ and L. Grafakos [21] in the equivalent representation

$$\mathcal{H}f(x) = \frac{1}{\Omega_n|x|^n}\int_{|y|<|x|} f(y)dy, \tag{1.0.4}$$

where $\Omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$. Various approaches are described and some extensions are given for the Hardy operators $U, \mathcal{H}$ and their modifications. Particularly, there appeared a lot of papers which have discussed the problems of characterizing the weights (u, v), for which U and its generalizations are of weak and strong type (p, q), or are bounded between Lorentz spaces, BMO spaces, ...(see [1, 16, 37, 60, 64, 68, 70, 72, 73, 75]).

The second type of Hardy operator was introduced by Carton–Lebrun and Fosset [19], in which the authors defined the weighted Hardy operator U_ψ as follows. Let $\psi : [0, 1] \to [0, \infty)$ be a measurable function, and let f be a measurable complex-valued function on $\mathbb{R}^n$. Then the weighted Hardy operator U_ψ is defined by

$$U_\psi f(x) = \int_0^1 f(tx)\psi(t)dt, \quad x \in \mathbb{R}^n. \tag{1.0.5}$$

Under certain conditions on ψ, [19] showed that U_ψ is bounded from $BMO(\mathbb{R}^n)$ into itself. Moreover, U_ψ commutes with the Hilbert transform in the case $n = 1$ and with certain Calderón–Zygmund singular integral operators (and thus with the Riesz transforms) in the case $n \geq 2$.

In [83], J. Xiao obtained that U_ψ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\mathcal{A} := \int_0^1 t^{-n/p}\psi(t)dt < \infty$$

and showed the interesting estimate that the corresponding operator norm is exactly $\mathcal{A}$. J. Xiao [83] also obtained the $BMO(\mathbb{R}^n)-$bounds of U_ψ, which sharpened the main result in [19].

There is a connection between the above two types of Hardy operator. If $\psi(t) = 1$, and $n = 1$, then U_ψ is just reduced to the classical Hardy operator U. If $n \geq 2$, then the restriction of $\mathcal{H}$ on the class of radial functions is U_ψ with $\psi(t) = nt^{n-1}$ (see [21, 88]). Thus, a number of results on the Hardy operators of the second type would be helpful in obtaining certain estimates for the Hardy operators of the first type.

Observe that the value of $U_\psi f$ at a point x just depends on the weight average value of f along a parameter $s(t,x) = tx$. We consider the generalized weighted Hardy–Cesàro operator as follows.

Definition 1.0.1 *Let $\psi : [0,1] \to [0,\infty)$, $s : [0,1] \to \mathbb{R}$ be measurable functions. Then the generalized weighted Hardy–Cesàro operator $U_{\psi,s}$, associated to the parameter curve $s(x,t) := s(t)x$, is defined by*

$$U_{\psi,s}f(x) = \int_0^1 f\left(s(t)x\right)\psi(t)dt, \tag{1.0.6}$$

for measurable complex-valued functions f on $\mathbb{R}^n$. See [22].

Note that the class of operators $U_{\psi,s}$ contains both types of classical Hardy operator and Cesàro operator. If $s(t) = t$, $U_{\psi,s}$ is reduced to U_ψ and if $s(t) = 1/t$, we replace $\psi(t)$ by $t^{-n}\psi(t)$, then $U_{\psi,s}$ is reduced to weighted Cesàro operator

$$V_\psi f(x) = \int_0^1 f\left(x/t\right)t^{-n}\psi(t)dt.$$

The operator V_ψ can be generalized to

$$V_{\psi,s}f(x) = \int_0^1 f\left(s(t)x\right)|s(t)|^n\psi(t)dt. \tag{1.0.7}$$

Definition 1.0.2 *Let b be a locally integrable function on $\mathbb{R}^n$. The commutators of b and operators $U_{\psi,s}, V_{\psi,s}$ are respectively defined by*

$$U^b_{\psi,s}f = bU_{\psi,s}(f) - U_{\psi,s}(bf), \tag{1.0.8}$$

and

$$V^b_{\psi,s}f(x) = bV_{\psi,s}(f) - V_{\psi,s}(bf). \tag{1.0.9}$$

Recently, Z. W. Fu, Z. G. Liu, and S. Z. Lu [41] gave a necessary and sufficient condition on the weight function ψ, which ensures the boundedness of the commutators of weighted Hardy operators U_ψ on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with symbols in $BMO(\mathbb{R}^n)$. In addition, several authors have considered the boundedness and bounds of U_ψ on Morrey spaces, Campanato spaces, $Q^\alpha_{p,q}$–type spaces, Triebel–Lizorkin–type spaces (see [55, 78, 79, 88]).

Our aim in Chapter 5 is to study weighted norm inequalities for the generalized weighted Hardy–Cesàro operator $U_{\psi,s}$ and its commutator $U^b_{\psi,s}$ on weighted Morrey spaces (see Section 5.1 below). More precisely, for weights ω in the class $\mathcal{W}_\alpha$ (see Section 5.2 below), we obtain a necessary and sufficient condition on the function ψ such that $U_{\psi,s}$ is bounded on $L^{p,\lambda}(\omega)$ (Theorem 5.2.4). We also find a characterization on $\psi(t)$ (the condition (ii)) so that for $1 < p < q < \infty$, the commutator $U^b_{\psi,s}$ of the generalized weighted Hardy–Cesàro operator $U_{\psi,s}$ is bounded from weighted central Morrey spaces $\dot{L}^{q,\lambda}(\omega)$ to weighted central Morrey spaces $\dot{L}^{p,\lambda}(\omega)$ for all symbols b in BMO(ω) (see Theorem 5.3.2). For the case $1 < p = q < \infty$, whether the above condition (ii) on $\psi(t)$ in Theorem 5.3.2 is sufficient for the boundedness of $U^b_{\psi,s}$ on $\dot{L}^{p,\lambda}(\omega)$ for all $b \in BMO(\omega)$ is still an interesting open question to us. Instead, we only proved that the condition (ii) is sufficient for the operator $M_{\omega,c}\left(U^b_{\psi,s}(\cdot)\right)$ to be bounded on $L^{p,\lambda}(\omega)$ and on $\dot{L}^{p,\lambda}(\omega)$ for all $b \in BMO(\omega)$, where $M_{\omega,c}$ is the Hardy-Littlewood central maximal operator with respect to the measure $\omega(x)dx$, that is

$$M_{\omega,c}f(x) = \sup_B \frac{1}{\omega(B)} \int_B |f(y)|\omega(y)dy,$$

where the supremum is taken over all balls B centered at the origin and containing x (see Theorem 5.3.7). These results are in line with those obtained in [83], [41], [22] and extend the results in [42].

The majority of the material in Chapter 5 is a revision of the material as it appeared in "T. D. Tran, *Generalized weighted Hardy–Cesàro operators and their commutators on weighted Morrey spaces*, J. Math. Anal. Appl. 412 (2014), no. 2, 1025-1035". It should be pointed out here that recently we were informed that there was a mistake in the proof of Theorem 3.1 in the article mentioned above, and so one direction of this theorem might not be true. In the current thesis, we, taking this notification into account, have revised all the work in [80] and fixed that mistake while writing Chapter 5. We will also submit a corrigendum of the article [80] to Journal of Mathematical Analysis and Applications as soon as we can.

2
Preliminaries

2.1 Spaces of homogeneous type and Muckenhoupt weights

2.1.1 Spaces of homogeneous type

In order to extend the Calderón-Zygmund theory from the Euclidean spaces to more general settings, in [25], the authors introduced certain metric spaces with the doubling measures. These spaces are called spaces of homogeneous type. In this section, we recall definition of spaces of homogeneous type and the Hardy spaces on spaces of homogeneous type, see for example [25].

Let (X, d, μ) be a metric space, with quasi-distance d and μ is a nonnegative, Borel, doubling measure on X.

Denote by $B(x, r)$ the open ball of radius $r > 0$ and center $x \in X$, and by $V(x, r)$ its measure $\mu(B(x, r))$. The doubling property of μ provides that there exists a constant $C > 0$ so that

$$V(x, 2r) \leq CV(x, r) < \infty \tag{2.1.1}$$

for all $x \in X$ and $r > 0$.
Notice that the doubling property (2.1.1) implies that following property that

$$V(x, \lambda r) \leq C\lambda^n V(x, r), \tag{2.1.2}$$

for some positive constant n uniformly for all $\lambda \geq 1, x \in X$ and $r > 0$. There also exists a constant $0 \leq N \leq n$ such that

$$V(x, r) \leq C\Big(1 + \frac{d(x, y)}{r}\Big)^N V(y, r), \tag{2.1.3}$$

uniformly for all $x, y \in X$ and $r > 0$.

To simplify notation, we will often just use B for $B(x_B, r_B)$. Also given $\lambda > 0$, we will write λB for the λ-dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$. For each ball $B \subset X$, we set

$$S_0(B) = B \text{ and } S_j(B) = 2^j B \backslash 2^{j-1} B \text{ for } j \in \mathbb{N}.$$

We now consider some examples on the spaces of homogeneous type in [25]:

(i) $X = \mathbb{R}^n$, $d(x, y) = |x - y|$ and μ is the Lebesgue measure;

(ii) $X = \mathbb{R}^n$, $d(x, y) = \sum_{j=1}^n |x_j - y_j|^{\alpha_j}$ where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $\alpha_1, \ldots, \alpha_n$ are positive numbers, not all equal, and μ is the Lebesgue measure;

(iii) $X = \mathbb{R}_+ = \{r \in \mathbb{R} : r \geq 0\}$, $d\mu(r) = r^{n-1} dr$, d is the usual distance;

(iv) X is a Lipschitz domain in $\mathbb{R}^n$, $d(x, y) = |x - y|$ and μ is the Lebesgue measure.

2.1.2 Calderón-Zygmund operators

Calderón-Zygmund theory plays an important role in studying the boundedness of singular integrals on the setting of spaces of homogenous type introduced by Coifman and Weiss in [25]. Let us recall here the concept of Calderón-Zygmund operators on a space of homogenous type (X, d, μ).

By a kernel on X we shall mean a function $K : \{(x, y) \in X \times X \backslash : x \neq y\} \to \mathbb{C}$ which is locally integrable away from the diagonal.

Definition 2.1.1 *A kernel $K(\cdot, \cdot) \in L^1_{\text{loc}}(X \times X \backslash \{(x, y) : x = y\})$ is called a Calderón-Zygmund kernel, or a C-Z kernel, if there exist two constants $\delta > 0$ and $C > 0$ such that*

(i)

$$|K(x, y)| \leq C \frac{1}{d(x, y)^n} \text{ for all } x \neq y; \tag{2.1.4}$$

(ii)

$$|K(x, y) - K(z, y)| + |K(y, x) - K(y, z)| \leq C\delta \frac{d(x, z)^\delta}{d(x, y)^{n+\delta}} \tag{2.1.5}$$

whenever $d(x, z) < d(x, y)/2$.

Basic examples of the kernels which satisfy (2.1.4) and (2.1.5) are $K(x, y) = (x - y)^{-1}$ and $K(x, y) = |x - y|^{-1}$ on the real line. In many circumstances, condition (2.1.5) can be replaced by the stronger condition

$$|\nabla K(x, y)| \leq \frac{C}{d(x, y)^{n+1}}.$$

A linear operator T is called *a Calderón-Zygmund operator with kernel* $K(\cdot,\cdot)$ *satisfying (2.1.4) and (2.1.5)* if for all $f \in L^\infty(\mu)$ with bounded support and $x \notin suppf$,

$$Tf(x) = \int_X K(x,y)f(y)d\mu(y).$$

The maximal operator T_* *associated with a Calderón-Zygmund operator* T is defined by

$$T_*f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|,$$

where $T_\epsilon f(x) = \int_{d(x,y)\geq\epsilon} K(x,y)f(y)d\mu(y)$.

A Calderón-Zygmund singular integral is a Calderón-Zygmund operator that satisfies

$$Tf(x) = \lim_{\varepsilon\to 0} T_\varepsilon f(x).$$

The following result implies that if a Calderón-Zygmund operator is bounded on L^2 then this operator and its maximal operator both are bounded on L^p for $1 < p < \infty$, and are of weak type $(1, 1)$. Hence, given an operator with a Calderón-Zygmund kernel, the problem reduces to showing that it is bounded on L^2.

Proposition 2.1.2 *If a Calderón-Zygmund operator* T *is bounded on* $L^2(X,\mu)$*, then* T *and* T_* *are bounded on* $L^p(X,\mu)$ *for* $1 < p < \infty$*, and are of weak type* $(1,1)$.

When one says T is bounded on L^p, we mean T can be extended to an L^p-bounded operator. More precisely, we have

$$\|Tf\|_{L^p} \leq C\|f\|_{L^p} \text{ for all } f \in L^p(X).$$

It is important to note that the weak type $(1,1)$ property of an L^2 boundedness singular integral operator still holds if we replace the condition (2.1.5) in Definition 2.1.1 by the Hömander integral condition. Recall that the kernel $K(x,y)$ satisfies the Hömander integral condition if there exist constants $C > 0$ and $\delta > 1$ such that

$$\int_{d(x,y)\geq\delta d(y_1,y)} |K(x,y) - K(x,y_1)|dx \leq C, \tag{2.1.6}$$

for all $y, y_1 \in X$.

Very recently, in connection with the study of the Cauchy integral, the question has arisen of extending the theory of singular integrals to nonhomogeneous spaces like topological spaces X with a pseudo-metric ρ and a measure μ which is not doubling. For example, F. Nazarov, S. Treil and A. Volberg [67], and T. A. Bui and X. T. Duong [18] have shown that the above result remains true when ρ is a metric and there exists a constant $n > 0$ such that for all $x \in X$ and $r > 0$, $\mu(B(x,r)) \leq r^n$.

2.1.3 Muckenhoupt weights

The theory of weighted inequalities for the maximal function and singular integrals is a natural development of Calderón-Zygmund theory. We can describe one of principal problems as follows.

Let M be the Hardy-Littlewood maximal function on X,

$$Mf(x) = \sup_{B \ni x} \frac{1}{V(B)} \int_B |f(y)| d\mu(y)$$

and we expect to characterize the nonnegative measure $d\nu$ so that

$$\int_X |Mf(x)|^p d\nu(x) \leq C \int_X |f(x)|^p d\nu(x) \tag{2.1.7}$$

for some $p, 1 < p < \infty$.

In Euclidean setting $X = \mathbb{R}^n$ and μ is the Lebesgue measure, Muckenhoupt [65] proved that inequality (2.1.7) holds if and only if $d\nu$ is in the class A_p. It is now called a Muckenhoupt class. The extension of Muckenhoupt class of weights to the spaces of homogeneous type was investigated in [77]. In this section, we give a brief definition of the Muckenhoupt class of weights and some their basic properties as well, see for example [77].

For the weight w, we shall mean that w is a non-negative local integrable function on X. We shall denote $w(E) := \int_E w(x) d\mu(x)$ and $V(E) = \mu(E)$ for any measurable set $E \subset X$. For $1 \leq p \leq \infty$ let p' be the conjugate exponent of p, i.e. $1/p + 1/p' = 1$.

We first introduce some notation. We use the notation

$$⨍_B h(x) d\mu(x) = \frac{1}{V(B)} \int_B h(x) d\mu(x).$$

A weight w is a non-negative measurable and locally integrable function on X. We say that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset X$,

$$\Big(⨍_B w(x) d\mu(x) \Big) \Big(⨍_B w^{-1/(p-1)}(x) d\mu(x) \Big)^{p-1} \leq C.$$

For $p = 1$, we say that $w \in A_1$ if there is a constant C such that for every ball $B \subset X$,

$$⨍_B w(y) d\mu(y) \leq C w(x) \text{ for a.e. } x \in B.$$

We set $A_\infty = \cup_{p \geq 1} A_p$.

The reverse Hölder classes are defined in the following way: $w \in RH_q, 1 < q < \infty$, if there is a constant C such that for any ball $B \subset X$,

$$\Big(⨍_B w^q(y) d\mu(y) \Big)^{1/q} \leq C ⨍_B w(x) d\mu(x).$$

The endpoint $q = \infty$ is given by the condition: $w \in RH_\infty$ whenever, there is a constant C such that for any ball $B \subset X$,

$$w(x) \le C \fint_B w(y) d\mu(y) \text{ for a.e. } x \in B.$$

Let $w \in A_\infty$, for $1 \le p < \infty$, the weighted spaces $L^p(X, w)$ can be defined by

$$\Big\{ f : \int_X |f(x)|^p w(x) d\mu(x) < \infty \Big\}$$

with the norm

$$\|f\|_{L^p(X,w)} = \Big(\int_X |f(x)|^p w(x) d\mu(x) \Big)^{1/p}.$$

We sum up some of the standard properties of classes of weights in the following lemma. For the proofs, see for example [7, 29, 77].

Lemma 2.1.3 *The following properties hold:*

(i) $A_1 \subset A_p \subset A_q$ *for* $1 < p \le q < \infty$.

(ii) $RH_\infty \subset RH_q \subset RH_p$ *for* $1 < p \le q < \infty$.

(iii) If $w \in A_p, 1 < p < \infty$, *then there exists* $1 < r < p < \infty$ *such that* $w \in A_r$.

(iv) If $w \in RH_q, 1 < q < \infty$, *then there exists* $q < p < \infty$ *such that* $w \in RH_p$.

(v) $A_\infty = \cup_{1 \le p < \infty} A_p \subset \cup_{1 < q \le \infty} RH_q$

(vi) Let $1 < p_0 < p < q_0 < \infty$. *Then we have*

$$w \in A_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{p})'} \Longleftrightarrow w^{1-p'} \in A_{\frac{p'}{q_0'}} \cap RH_{(\frac{p_0'}{p'})'}.$$

It is important to note that in Euclidean setting, we have $A_\infty = \cup_{1<q\le\infty} RH_q$. However, in general, A_∞ is a proper subset of $\cup_{1<q\le\infty} RH_q$, see for example [77, p. 9].

Lemma 2.1.4 *Let* B *be a ball and* E *be any measurable subset* B. *Let* $w \in A_p, p \ge 1$. *Then, there exists a constant* $C_1 > 0$ *such that*

$$C_1 \Big(\frac{V(E)}{V(B)} \Big)^p \le \frac{w(E)}{w(B)}.$$

If $w \in RH_r, r > 1$. *Then, there exists a constant* $C_2 > 0$ *such that*

$$\frac{w(E)}{w(B)} \le C_2 \Big(\frac{V(E)}{V(B)} \Big)^{\frac{r-1}{r}}.$$

From the first inequality of Lemma 2.1.4, if $w \in A_p, 1 \le p < \infty$ then for any ball $B \subset X$ and $\lambda > 1$, we have

$$w(\lambda B) \le C \lambda^{pn} w(B).$$

2.2 Hardy spaces

2.2.1 Characterizations of classical Hardy spaces

There exists an abundance of equivalent characterizations for Hardy spaces, of which only a few representative ones are discussed in this section. For historical reasons, however, we choose to define Hardy spaces using a more classical approach.

Definition 2.2.1 *Let f be a tempered distribution on $\mathbb{R}^n$, let $0 < p < \infty$, and let P be the Poisson kernel, that is*

$$P(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}\left(1+|x|^2\right)^{\frac{n+1}{2}}}.$$

If the Poisson nontangential maximal function $P^*_\nabla(f)(x) := \sup_{|y-x|<t} |(f * P_t)(y)| \in L^p(\mathbb{R}^n)$, then we say f belongs to the Hardy space $H^p(\mathbb{R}^n)$, where $\{(y,t) : |y-x| < t\}$ is a cone in $\mathbb{R}^{n+1}_+ := \{(y,t) : y \in \mathbb{R}^n, t > 0\}$, and $P_t(x) = t^{-n}P(t^{-1}x)$.

In view of Definition (2.2.1), one can easily see that this definition does not completely get rid of the dependence on harmonic functions because of the Poisson kernel appearing in the definition. Then a natural question arises: can the Poisson kernel in Definition (2.2.1) be replaced by any other kernel of approximation to the identity? Fefferman and Stein gave out an affirmative answer to this question in [39]. They introduced the maximal functions associated with smooth kernel as follows.

Definition 2.2.2 *Suppose $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\int \varphi(x)dx = 1$. If we write*

$$\varphi^*_\nabla(f)(x) = \sup_{|y-x|<t} |(f * \varphi_t)(y)|,$$

then $\varphi^*_\nabla(f)$ is called the φ–nontangential maximal function of f.

Definition 2.2.3 *Let $m \in \mathbb{N}$, and let*

$$K_m = \left\{ \Phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{u \in \mathbb{R}^n, |\alpha| \leq m} (1+|u|)^{m+n} |D^\alpha \Phi(u)| \leq 1 \right\}.$$

If we write

$$f^*_m(x) := \sup_{\Phi \in K_m} \Phi^*_\nabla(f)(x),$$

then f^*_m is called the Grand maximal function of f.

The following basic result is proved by Fefferman and Stein.

Theorem 2.2.4 *Suppose $f \in \mathcal{S}'(\mathbb{R}^n)$, $0 < p < \infty$, and $m > 1 + \frac{n}{p}$. Then the following statements are equivalent.*

(i) $f \in H^p(\mathbb{R}^n)$.

(ii) *There exists a function* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *with* $\int \varphi(x)dx = 1$, *such that* $\varphi^*_\nabla(f) \in L^p(\mathbb{R}^n)$.

(iii) $f^*_m \in L^p(\mathbb{R}^n)$

Let us now recall the notion on the type of operators.

Definition 2.2.5 *Suppose* X *is a quasi-normed linear space and* T *is a sublinear operator. If* T *maps* X *into* L^p *and satisfies*

$$\|Tf\|_p \leq C\|f\|_X,$$

where C *is independent of* f, *then* T *is called to be of type* (X, L^p). *If the above inequality is replaced by*

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leq C\left(\frac{\|f\|_X}{\lambda}\right)^p,$$

where C *is independent of* f *and* λ, *then* T *is called to be of weak type* (X, L^p).

It follows from Theorem (2.2.4) that the maximal operator $T : f \mapsto f^*_m$ is of type (H^p, L^p). In the case $p \geq 1$, we shall obtain further results on the type of this maximal operator in the following, see for example [59] for the proof of the proposition below.

Proposition 2.2.6 *Let* $m \geq 1$ *and* $p > 1$. *Then we have*

(i) $T : f \mapsto f^*_m$ *is of weak type* (L^1, L^1).

(ii) $T : f \mapsto f^*_m$ *is of type* (L^p, L^p).

As an immediate consequence of Theorem (2.2.4) and the above proposition, we have

Proposition 2.2.7 *If* $1 < p < \infty$ *then* $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

Therefore, we are interested in the real H^p spaces only for $0 < p \leq 1$.

In addtion, $H^p(\mathbb{R}^n)$ spaces can be characterized in terms of the Lusin area integral in [39]. This can be precisely formulated as follows.

Proposition 2.2.8 *Suppose* $f \in \mathcal{S}'(\mathbb{R}^n)$ *and* $0 < p \leq 1$. *Then the following statements are equivalent.*

(i) $f \in H^p(\mathbb{R}^n)$.

(ii) $\mathcal{S}(f) \in L^p(\mathbb{R}^n)$ *and* $\lim\limits_{t\to\infty} (f * P_t)(x) = 0$, *where*

$$\mathcal{S}(f)(x) = \left(\iint\limits_{\Gamma(x)} |\nabla(f * P_t)(y)|^2 t^{1-n} dydt\right)^{1/2}$$

and

$$\Gamma(x) = \left\{(y,t) \in \mathbb{R}^{n+1}_+ : |y - x| < t\right\}.$$

2.2.2 Atoms and atomic decompositions

The atomic decomposition theory of $H^p(\mathbb{R}^n)$ spaces marks an important step of further developments on its real variable theory. Using the grand maximal function, Coifman [23] first showed that an element in $H^p(\mathbb{R}^n)$ can be decomposed into a sum of a series of basic elements. Then the study on some analytic problems on $H^p(\mathbb{R}^n)$ is reduced to investigating some properties of these basic elements, and so the problems become quite simple. These basic elements are called atoms. Let us now recall the definition of an atom.

Definition 2.2.9 *Let $0 < p \leq 1 \leq q \leq \infty, p \neq q$, and the nonnegative integer $s \geq [n(1/p-1)]$ ($[x]$ indicates the integer part of x). A function $a(x) \in L^q(\mathbb{R}^n)$ is called a (p, q, s) atom with the center at x_0 if it satisfies the following conditions:*

(i) Supp $a \subset B(x_0, r)$;

(ii) $\|a\|_q \leq |B(x_0, r)|^{1/q-1/p}$;

(iii) $\int a(x)x^\alpha dx = 0,\ 0 \leq |\alpha| \leq s$.

Here, (i) means that an atom must be a function with compact support, (ii) is the size condition of atoms, and (iii) is called the cancellation moment condition. Clearly, a (p, ∞, s) atom must be a (p, q, s) atom, if $p < q < \infty$. Let us now define a class of function space generated by atoms.

Definition 2.2.10 *The atomic Hardy space $H_a^{p,q,s}(\mathbb{R}^n)$ is defined by*

$$H_a^{p,q,s}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' : f(x) = \sum_k \lambda_k a_k(x), \sum_k |\lambda_k|^p < \infty \right\},$$

where each a_k is a (p, q, s) atom.

Setting $H_a^{p,q,s}$ norm of f by

$$\|f\|_{H_a^{p,q,s}} = \inf \left(\sum_k |\lambda_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of $f = \sum_k \lambda_k a_k$. It is easy to verify that $H_a^{p,q,s}(\mathbb{R}^n)$ is a complete metric space with the distance

$$\rho(f, g) = \|f - g\|_{H_a^{p,q,s}}.$$

Particularly, $H_a^{1,q,s}(\mathbb{R}^n)$ is a Banach space. Coifman [23] proved that $H_a^{p,q,s}(\mathbb{R}) = H^p(\mathbb{R})$. This indicates that each element in $H^p(\mathbb{R})$ can be decomposed into a sum of atoms in certain way.

Theorem 2.2.11 *Let $0 < p \leq 1 \leq q \leq \infty, p \neq q$, and the nonnegative integer $s \geq [(1/p-1)]$. Then $H_a^{p,q,s}(\mathbb{R}) = H^p(\mathbb{R})$, and*

$$\|f\|_{H_a^{p,q,s}} \sim \|f\|_{H^p}.$$

Later, R. H. Latter [57] generalized Theorem 2.2.11 to high dimension and obtained the following result.

Theorem 2.2.12 *Let $0 < p \leq 1 \leq q \leq \infty, p \neq q$, and the nonnegative integer $s \geq [n(1/p-1)]$. Then $H_a^{p,q,s}(\mathbb{R}^n) = H^p(\mathbb{R}^n)$, and*

$$\|f\|_{H_a^{p,q,s}} \sim \|f\|_{H^p}.$$

Using atoms is helpful in studying the H^p boundedness of operators.

Proposition 2.2.13 *If for any (p,q,s) atom a, $Ta \in L^r(\mathbb{R}^n)$, and*

$$\|Ta\| \leq C,$$

where C is independent of a, (p,q,s) as in Definition 2.2.9, and $1 \geq r \geq p$, then T is of type (H^p, L^r).

2.2.3 Molecules and molecular decompositions

The basic idea of the atomic decomposition theory of H^p spaces is just to decompose the elements of H^p into a sum of series of atoms in some form. However, we need the compact support condition when working with atoms. Thus, a natural problem that arises is that: Can we choose functions without compact supports as basic elements in the decompositional structure of H^p? We call these functions *molecules.*

Definition 2.2.14 *Let $0 < p \leq 1 \leq q \leq \infty$, and the nonnegative integer $s \geq [n(1/p - 1)]$, $\epsilon > max\left\{\frac{s}{n}, \frac{1}{p}-1\right\}$, $a_0 = 1 - \frac{1}{p} + \epsilon$, and $b_0 = 1 - \frac{1}{q} + \epsilon$. A function $M \in L^q(\mathbb{R}^n)$ is said to be a (p,q,s,ϵ) molecule centered at x_0, if it satisfies the following conditions.*

(i) $|x|^{nb_0} M(x) \in L^q(\mathbb{R}^n)$;

(ii) $N_q(M) := \|M\|_q^{a_0/b_0} \left\| M(\cdot)|\cdot - x_0|^{nb_0} \right\|_q^{1-(a_0/b_0)} < \infty$;

(iii) $\int\limits_{\mathbb{R}^n} M(x) x^\alpha dx = 0, \ |\alpha| \leq s.$

It is well know that an element in H^p can be represented as a sum of a series of molecules in some form.

Proposition 2.2.15 *Let p, q, s and ϵ as in Definition 2.2.14. Then $f \in H_a^{p,q,s}(\mathbb{R}^n)$ if and only if*

$$f(x) = \sum_k \lambda_k M_k(x),$$

where each M_k is a (p,q,s,ϵ) molecule, $\sum_k |\lambda_k|^p < \infty$, and there exists a constant C independent of k such that $N_q(M_k) \leq C$. Moreover,

$$\inf \left\{ \left(\sum_k |\lambda_k|^p \right)^{1/p} : f = \sum_k \lambda_k M_k \right\} \sim \|f\|_{H^p}.$$

For the boundedness of operators on H^p spaces, one of the most interesting questions is that what kind of operators is of type (H^p, H^p). By means of atoms and molecules, this problem is reduced to whether the images of atoms under the action of operators are molecules. More precisely, one has the following result, see [59] for the proof.

Proposition 2.2.16 *Suppose $p_1 \leq p_2 \leq 1$. If for any (p_1, q_1, s_1) atom a, Ta is a $(p_2, q_2, s_2, \epsilon)$ molecule, and satisfies*

$$N_{q_2}(Ta) \leq C,$$

where C is independent of a, then T is of type (H^{p_1}, H^{p_2}), where (p_1, q_1, s_1) is as in Definition 2.2.9 and $(p_2, q_2, s_2, \epsilon)$ is as in Definition 2.2.14.

2.2.4 Interpolation of operators

The classic Marcinkiewicz interpolation theorem of operators shows that if $1 \leq p_1 < p < p_2 \leq \infty$, and the sublinear operator T is of weak type (L^{p_1}, L^{p_1}) and weak type (L^{p_2}, L^{p_2}), then T is of type (L^p, L^p). A natural question that arises is that if $p < 1$, what would happen to the interpolation of operators. The following theorem gives an answer to the above question, see [59]. It is also regarded as a generalization of the Marcinkiewicz interpolation theorem mentioned above.

Theorem 2.2.17 *Assume that $0 < p_1 \leq 1 < p_2 \leq \infty$, and a sublinear operator T is of weak type (H^{p_1}, L^{p_1}) and weak type (L^{p_2}, L^{p_2}).*

(i) If $1 < p < p_2$ then T is of type .

(ii) If $p_1 < p \leq 1$ then T is of type (H^p, L^p).

It should be pointed out that when $p_1 = 1$ and $p_2 = \infty$, there also is an interpolation result that differs from either the Marcinkiewicz theorem or Theorem 2.2.17. This result is formulated as follows, see [59].

Proposition 2.2.18 *Suppose a sublinear T is of type (H^1, L^1) and of type (L^∞_c, BMO), where L^∞_c is the space of bounded measurable functions with compact support and BMO is the space of bounded mean oscillation functions (see Section 2.3 below). Then T is of type (L^p, L^p) for $1 < p < \infty$.*

2.2.5 Coifman and Weiss's Hardy spaces

Let us now recall the definition of the Coifman and Weiss's Hardy spaces $H^p_{CW}(X)$ as in [25]. Let $0 < p \leq 1$. We say that a function a is a $(2, p)$ atom if there exists a ball B such that

(i) supp $a \subset B$;
(ii) $\|a\|_{L^2} \leq V(B)^{1/2-1/p}$;
(iii) $\int a(x)\mu(x) = 0$.

The atomic Hardy space $H^1_{CW}(X)$ is defined as follows. We say that a function $f \in H^1_{CW}(X)$, if $f \in L^1$ and there exist a sequence $(\lambda_j)_{j\in\mathbb{N}} \in l^1$ and a sequence of $(2,1)$-atoms $(a_j)_{j\in\mathbb{N}}$ such that $f = \sum_j \lambda_j a_j$ in $L^1(X)$. We set

$$\|f\|_{H^1_{CW}(X)} = \inf\Big\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j\Big\}.$$

To define the Hardy space $H^p_{CW}(X)$ for $0 < p < 1$, we need to introduce the Lipschitz space $\mathfrak{L}_\alpha$. We say that the function $f \in \mathfrak{L}_\alpha$ if there exists a constant $c > 0$, such that

$$|f(x) - f(y)| \leq c|B|^\alpha$$

for every ball B and $x, y \in B$, and the best constant c can be taken to be the norm of f and is denoted by $\|f\|_{\mathfrak{L}_\alpha}$.

Let $0 < p < 1$ and $\alpha = 1/p - 1$. We say that a function $f \in H^p_{CW}(X)$, if $f \in (\mathfrak{L}_\alpha)^*$, the conjugate space of $\mathfrak{L}_\alpha$, and there are a sequence $(\lambda_j)_{j\in\mathbb{N}} \in l^p$ and a sequence of $(2,p)$-atoms $(a_j)_{j\in\mathbb{N}}$ such that $f = \sum_j \lambda_j a_j$ in $(\mathfrak{L}_\alpha)^*$. We set

$$\|f\|_{H^p_{CW}(X)} = \inf\Big\{(\sum_j |\lambda_j|^p)^{1/p} : f = \sum_j \lambda_j a_j\Big\}.$$

Note that when $0 < p < 1$, $\|\cdot\|_{H^p_{CW}}$ is not a norm but $d(f,g) = \|f - g\|_{H^p_{CW}}$ is a metric.

2.3 BMO spaces

This section is written mainly based on Chapter 6 of [29].

2.3.1 Classical BMO spaces on $\mathbb{R}^n$

Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and a ball B in $\mathbb{R}^n$, let f_B denote the average of f on B:

$$f_B = \frac{1}{|B|}\int_B f.$$

Define the sharp maximal function by

$$M^\sharp f(x) = \sup_B \frac{1}{|B|}\int_B |f - f_B| \tag{2.3.1}$$

where the supremum is taken over all balls B containing x. Each of these integrals measures the mean oscillation of f on the ball B. We say that f has bounded mean oscillation if the function M^f is bounded. The space of functions with this property is denoted by BMO, that is:

$$BMO = \left\{ f \in L^1_{\text{loc}} : M^\sharp f \in L^\infty \right\}.$$

We define a norm on BMO spaces by

$$\|f\|_{BMO} = \left\| M^\sharp f \right\|_\infty .$$

This is not properly a norm since any function which is constant almost everywhere has zero oscillation. However, these are the only functions with this property, and it is customary to think of BMO as the quotient of the above space by the space of constant functions. In other words, two functions which differ by a constant coincide as functions in BMO. On this space $\|f\|_{BMO}$ is a norm and the space is a Banach space.

Proposition 2.3.1

$$\frac{1}{2}\|f\|_{BMO} \leq \sup_B \inf_{a \in \mathbb{C}} \frac{1}{|B|} \int_B |f(x) - a| \, dx \leq \|f\|_{BMO} \tag{2.3.2}$$

and

$$M^\sharp(|f|)(x) \leq 2M^\sharp f(x). \tag{2.3.3}$$

Inequality (2.3.2) defines a norm equivalent to the BMO norm, and provides a way to show that $f \in BMO$ without using its average on B: it suffices to find a constant a (that can depend on B) such that

$$\frac{1}{|B|} \int_B |f(x) - a| \, dx \leq C$$

with C independent of B.

It follows from (2.3.3) that if $f \in BMO$ then $|f|$ is also in BMO. However, the converse is not true. Clearly, $L^\infty \subset BMO$, but there are also unbounded BMO functions. The typical example on $\mathbb{R}$ is

$$f(x) = \begin{cases} \log\left(\frac{1}{|x|}\right) & , |x| < 1 \\ 0 & , |x| \geq 1 \end{cases} .$$

It is well-known that the function f above is a BMO function but the function $\operatorname{sgn}(x) f(x)$ is not in BMO.

The following result shows the connection between BMO and singular integrals.

Proposition 2.3.2 *Let T be a Calderón-Zygmund operator. Then if f is a bounded function of compact support then $Tf \in BMO$, and*

$$\|Tf\|_{BMO} \leq C\|f\|_\infty .$$

In applying interpolation theorems (such as the Marcinkiewicz interpolation theorem) we often use the fact that an operator is bounded from L^∞ to L^∞. It is well-known that we can replace this with the weaker condition of boundedness from L^∞ to BMO.

Proposition 2.3.3 *Let T be a linear operator which is bounded on L^q for some q, $1 < q < \infty$, and is bounded from L^∞ to BMO. Then T is bounded on L^p for all p, $q < p < \infty$.*

2.3.2 The John-Nirenberg inequality

In this section we examine the rate of growth of BMO functions. Let us first consider $\log(1/|x|)$. On the interval $(-a, a)$, its average is $1 - \log(a)$, and given $\lambda > 1$ the set where

$$|\log(1/|x|) - (1 - \log a)| > \lambda$$

has measure $2ae^{-\lambda-1}$. The following result, which is well-known as John-Nirenberg Inequality, shows that in some sense the logarithmic growth is the maximum possible for BMO functions.

Proposition 2.3.4 *Let $f \in BMO$. Then there exist two constants c and C, depending only on the dimension, such that for any ball B in $\mathbb{R}^n$ and any $\lambda > 0$,*

$$|\{x \in B : |f(x) - f_B| > \lambda\}| \leq Ce^{-c\lambda/\|f\|_{BMO}} |B| . \tag{2.3.4}$$

As a consequence of Proposition (2.3.4), one gets the following interesting statements

Proposition 2.3.5 *For all p, $1 < p < \infty$,*

$$\|f\|_{BMO,p} = \sup_B \left(\frac{1}{|B|} \int_B |f(x) - f_B|^p dx \right)^{1/p}$$

is a norm on BMO equivalent to the BMO norm.

Proposition 2.3.6 *Given a function f and suppose that there exist constants C_1, C_2, and K such that for any ball B and any $\lambda > 0$,*

$$|\{x \in B : |f(x) - f_B| > \lambda\}| \leq C_1 e^{-C_2\lambda/K} |B| .$$

Then $f \in BMO$.

2.3.3 Carleson measures

Definition 2.3.7 *A positive measure ν on $\mathbb{R}^{n+1}_+$ is called a Carleson measure if for every cube Q in $\mathbb{R}^n$,*

$$\nu(Q \times (0, l(Q))) \leq C\,|Q|\,,$$

where $l(Q)$ is the side length of Q. The infimum of the possible values of the constant C is called the Carleson constant of ν and is denoted by $\|\nu\|$.

An example of a Carleson measure in the upper half-plane $\mathbb{R}^2_+$ is $drd\theta$ (polar coordinates).

The property underlying the definition of Carleson measures is not limited to cubes but also extends to more general sets. Given an open set $E \subset \mathbb{R}^n$, let

$$\hat{E} = \left\{(x,t) \in \mathbb{R}^{n+1}_+ : B(x,t) \subset E\right\}.$$

then the following is true.

Proposition 2.3.8 *If ν is a Carleson measure on $\mathbb{R}^{n+1}_+$ and $E \subset \mathbb{R}^n$ is open, then*

$$\nu(\hat{E}) \leq C\,\|\nu\|\,|E|\,.$$

The converse of the above proposition is also true, that is, if for any ball B in $\mathbb{R}^n$, $\nu(\hat{B}) \leq C\,|B|$ then ν is a Carleson measure on $\mathbb{R}^{n+1}_+$ and $\|\nu\|$ is comparable with the constant C appearing in the above inequality.

A Carleson measure can be characterized as a measure ν for which the Poisson integral defines a bounded operator from $L^p(\mathbb{R}^n, dx)$ to $L^p(\mathbb{R}^{n+1}_+, \nu)$. This is a consequence of a more general result.

Proposition 2.3.9 *Let ϕ be a bounded, integrable function which is positive, radial and decreasing. For $t > 0$, let $\phi_t(x) = t^{-n}\phi(t^{-1}x)$. Then a measure ν is a Carleson measure if and only if for every p, $1 < p < \infty$,*

$$\int\limits_{\mathbb{R}^{n+1}_+} |\phi_t * f(x)|^p d\nu(x,t) \leq C \int\limits_{\mathbb{R}^n} |f(x)|^p dx. \qquad (2.3.5)$$

The constant C is comparable with $\|\nu\|$.

Carleson measures can also be characterized in terms of BMO functions.

Proposition 2.3.10 *Let $b \in BMO$ and let ψ be a Schwartz function such that $\int\limits_{\mathbb{R}^n} \psi = 0$. Then the measure ν defined by*

$$d\nu = |b * \psi_t(x)|^2 \frac{dxdt}{t}$$

is a Carleson measure such that $\|\nu\|$ is dominated by $\|b\|^2_{BMO}$.

2.3.4 Dual space of BMO space

In [39], Fefferman and Stein proved that the dual space of the bounded mean oscillation function space is just the Hardy space $H^1(\mathbb{R}^n)$. Their result can be stated as follows.

Theorem 2.3.11 *If $g \in BMO(\mathbb{R}^n)$ then*

$$Lf = \int_{\mathbb{R}^n} f(x)g(x)dx$$

is a bounded linear functional on $H^1(\mathbb{R}^n)$. Conversely, for any bounded linear functional L on $H^1(\mathbb{R}^n)$, there exists a function $g \in BMO(\mathbb{R}^n)$ such that

$$Lf = \int_{\mathbb{R}^n} f(x)g(x)dx, \forall f \in H^1(\mathbb{R}^n).$$

2.4 Holomorphic functional calculus

Given an operator T, one can define $T^k = T(T \dots (T))$ by taking the composition of T with itself k times. Thus for a polynomial $p(z) = a_k z^k + \cdots a_1 z + a_0$, the operator $p(T)$ can be defined naturally as $a_k T^k + \cdots a_1 T + a_0 I$. Hence for a rational function $r(z) = \frac{p(z)}{q(z)}$ where $q(z) = b_m z^m + \cdots b_1 z + b_0$, the operator $r(T)$ can be defined as

$$r(T) = \frac{a_k T^k + \cdots a_1 T a_0 I}{b_m T^m + \cdots b_1 T + b_0 I} = (a_k T^k + \cdots a_1 T a_0 I)(b_m T^m + \cdots b_1 T + b_0 I)^{-1}$$

assuming that the inverse operator $(b_m T^m + \cdots b_1 T + b_0 I)^{-1}$ exists. The aim of defining holomorphic functional calculus of an operator T is to extend the rational function of operator $r(T)$ to $f(T)$ where f is a suitable holomorphic function. In this section, we now present some preliminary definitions of holomorphic functional calculi which was introduced by A. McIntosh [61].

Let $0 \leq \theta < \nu < \pi$. We define the closed sector in the complex plane $\mathbb{C}$

$$S_\theta := \{z \in \mathbb{C} : |\arg z| \leq \theta\}$$

and denote the interior of S_θ by S_θ^0.
We present the following subspaces of the space $H(S_\nu^0)$ of all holomorphic functions on S_ν^0:

$$H_\infty(S_\nu^0) := \{b \in H(S_\nu^0) : ||b||_\infty < \infty\},$$

where $||b||_\infty := \sup\{|b(z)| : z \in S_\nu^0\}$, and

$$\Psi(S_\nu^0) := \{\psi \in H(S_\nu^0) : \exists s > 0, |\psi(z)| \leq c|z|^s(1 + |z|^{2s})^{-1}\}.$$

Recall that a closed operator L in $L^2(X)$ is said to be of type θ if $\sigma(L) \subset S_\theta$, and for each $\nu > \theta$ there exists a constant c_ν such that

$$||(L - \lambda I)^{-1}|| \leq c_\nu |\lambda|^{-1}, \lambda \notin S_\nu.$$

If L is of type θ and $\psi \in \Psi(S^0_\nu)$, for $f \in L^2(X)$, we define $\psi(L) \in \mathcal{L}(L^2(X), L^2(X))$ by putting

$$\psi(L)f = \frac{1}{2\pi i}\int_\Gamma (L - \lambda I)^{-1} f\psi(\lambda)d\lambda,$$

where Γ is the contour $\{z = re^{\pm i\xi} : r > 0\}$ parametrized clockwise around S_θ, and $\theta < \xi < \nu$. Since

$$\left\|\int\limits_\Gamma (L - \lambda I)^{-1} f\psi(\lambda)d\lambda\right\|_{L^2(X)} \leq \int\limits_0^\infty \left\|(L - \lambda I)^{-1} f\right\|_{L^2(X)} |\psi(\lambda)|\, d\,|\lambda|$$

$$\leq \|f\|_{L^2(X)} \int\limits_0^\infty \frac{c_1 c_2 |\lambda|^s}{|\lambda|\left(1 + |\lambda|^{2s}\right)} d\,|\lambda| < \infty,$$

the integral above is absolutely convergent and defines $\psi(L)$ as a bounded operator from $L^2(X)$ into $L^2(X)$. It is straightforward to show, using Cauchy's theorem, that the definition is independent of the choice of $\xi \in (\theta, \nu)$. If, in addition, L is one-one and has dense range and if $b \in H_\infty(S^0_\nu)$, then $b(L)$ can be defined by

$$b(L) = [\psi(L)]^{-1}(b\psi)(L),$$

where $\psi(z) = z(1 + z)^{-2}$. It can be shown that $b(L)$ is a well-defined linear operator in $L^2(X)$, see [61]. We say that L has a bounded H_∞ functional calculus in $L^2(X)$ if there exists $c_{\nu,2} > 0$ such that $b(L) \in \mathcal{L}(L^2(X), L^2(X))$, and for $b \in H_\infty(S^0_\nu)$,

$$||b(L)|| \leq c_{\nu,2}||b||_\infty.$$

In [61] it was proved that L has a bounded H_∞ functional calculus in $L^2(X)$ if and only if for any non-zero function $\psi \in \Psi(S^0_\nu)$, L satisfies the square function estimate and its reverse

$$c_1||g||_2 \leq \Big(\int_0^\infty ||\psi_t(L)g||_2^2 \frac{dt}{t}\Big)^{1/2} \leq c_2||g||_2 \tag{2.4.1}$$

for some $0 < c_1 \leq c_2 < \infty$, where $\psi_t(x) = \psi(tx)$. More precisely, the authors in [61] proved the following result.

Proposition 2.4.1 *Let L be a one-to-one operator of type ω in a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(i) for every μ in (ω, π), L has a bounded $H_\infty(S^0_\mu)$ functional calculus, that is, for all $f \in H_\infty(S^0_\mu)$, $f(L) \in \mathcal{L}(X)$ and

$$\|f(L)\| \leq C\|f\|_\infty \qquad \forall f \in H_\infty(S^0_\mu);$$

(ii) there exists μ in (ω, π) such that L has a bounded $H_\infty(S^0_\mu)$ functional calculus;

(iii) $\{L^{is} : s \in \mathbb{R}\}$ is a C^0 group, and for every μ in (ω, π),

$$\left\|L^{is}\right\| \leq Ce^{\mu|s|} \qquad \forall s \in \mathbb{R};$$

(iv) if A and B are non-negative self-adjoint operators, and U and V are isometries such that $L = UA$ and $L^* = VB$, then for all α in (0,1), $\mathcal{D}(L^\alpha) = \mathcal{D}(A^\alpha)$, $\mathcal{D}(L^{*\alpha}) = \mathcal{D}(B^\alpha)$, and

$$C^{-1}\|A^\alpha u\| \le \|L^\alpha u\| \le C\|A^\alpha u\| \qquad \forall u \in \mathcal{D}(L^\alpha)$$

and

$$C^{-1}\|B^\alpha v\| \le \|L^{*\alpha} v\| \le C\|B^\alpha v\| \qquad \forall v \in \mathcal{D}(L^{*\alpha});$$

(v) for every μ in (ω,π), and every $\psi \in \Psi(S^0_\mu)$,

$$C^{-1}\|u\| \le \left[\int_0^\infty \|\psi(tL)u\|^2 \frac{dt}{t}\right]^{1/2} \le C\|u\| \qquad \forall u \in \mathcal{H}$$

and

$$C^{-1}\|u\| \le \left[\int_0^\infty \|\psi(tL^*)u\|^2 \frac{dt}{t}\right]^{1/2} \le C\|u\| \qquad \forall u \in \mathcal{H}.$$

As noted in [61], positive self-adjoint operators satisfy the quadratic estimate (2.4.1). So do normal operators with spectra in a sector, and maximal accretive operators. We refer the reader to [87] for precise definitions of these classes of operators. For detailed study on operators which have holomorphic functional calculi, see the work of [61].

It is interesting to compare H_∞ functional calculus with more classical theories. Hörmander's multiplier theorem, applied to radial functions, tells us that if $m : \mathbb{R}^+ \to \mathbb{C}$ satisfies the conditions

$$\left|\frac{d^k}{dx^k} m(x)\right| \le C|x|^{-k} \qquad \forall x \in \mathbb{R}^+,$$

whenever $0 \le k \le [n/2]+1$ ($[x]$ denoting the integer part of x), then $m(\Delta)$ is a bounded map on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. With a view towards proving Hörmander type theorems using H_∞ functional calculus, the authors in [27] established a connection between the two types of calculus as below.

Given a function m on $\mathbb{R}^+$, one writes m_e for the function on $\mathbb{R}$ obtained by composing with the exponential, that is, $m_e = m \circ \exp$.

For any positive number α, let $\Lambda^\alpha_{\infty,1}(\mathbb{R}^+)$ be the set of all bounded continuous functions m on $\mathbb{R}^+$ such that $\|m\|_{\Lambda^\alpha_{\infty,1}} < \infty$, where

$$\|m\|_{\Lambda^\alpha_{\infty,1}} = \sum_{n\in\mathbb{Z}} 2^{|n|\alpha} \left\| m_e * \check{\phi}_n \right\|_\infty .$$

Here, for all ξ in $\mathbb{R}$,

$$\phi_0(\xi) = (2 - 2|\xi|)_+ - (1 - 2|\xi|)_+$$
$$\phi_1(\xi) = (1 - 2|\xi - 1|)_+ + (1/2 - |\xi - 3/2|)_+$$

and

$$\phi_{n\varepsilon}(\xi) = \phi_1(2^{1-n}\varepsilon\xi) \qquad \forall n \in \mathbb{Z}^+, \forall \varepsilon \in \{\pm 1\}.$$

This space is sometimes called a Lipschitz space, and sometimes a Besov space.

Proposition 2.4.2 *Suppose that L is a one-to-one operator of type 0. Then the following conditions are equivalent:*

(i) L admits a bounded $H_\infty(S^0_\mu)$ functional calculus for all positive μ, and there exist α and A in $\mathbb{R}^+$ such that

$$\|m(L)\| \le A\mu^{-\alpha}\|m\|_\infty \qquad \forall m \in H_\infty(S^0_\mu), \forall \mu \in \mathbb{R}+;$$

(ii) L admits a bounded $\Lambda^\alpha_{\infty,1}(\mathbb{R}^+)$ functional calculus.

2.5 Musielak–Orlicz-type functions

Let us first present some notions on Musielak–Orlicz-type functions.

A function $\omega : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is nondecreasing, $\omega(0) = 0$, $\omega(t) > 0$ for $t > 0$ and $\lim_{t\to\infty}\omega(t) = \infty$.

A function $\omega : X \times [0,\infty) \to [0,\infty)$ is called a *Musielak–Orlicz function* if the function $\omega(x,\cdot) : [0,\infty) \to [0,\infty)$ is an Orlicz function for each $x \in X$ and the function $\omega(\cdot,t)$ is a measurable function for each $t \in [0,\infty)$.

Let ω be a Musielak–Orlicz function. The function ω is said to be of uniformly upper type p (resp., uniformly lower type p) for certain $p \in [0, \infty)$ if there exists a positive constant C such that for all $x \in X$, $t \ge 1$ (resp., $t \in (0,1]$) and $s \in (0,\infty)$,

$$\omega(x,st) \le Ct^p\omega(x,s). \tag{2.5.1}$$

If ω is of both uniformly upper type p_1 and lower type p_0, then ω is said to be of type $(p_0,\, p_1)$. A typical example of such ω is

$$\omega(x,t) := f(x)g(t)$$

for $x \in X$ and $t \in [0,\infty)$, where f is a positive measurable function on X, and g is an Orlicz function on $[0,\infty)$ of upper type p_1 and lower type p_0. Another example of Musielak–Orlicz function ω of uniformly upper type $p \in (0,1]$ is, for instance,

$$\omega(x,t) = \frac{t^p}{f(x) + [\log(e+t)]^\alpha},$$

where $\alpha \in [0,1]$ and f is a positive measurable function on X. It is also interesting to observe that if

$$\omega(x,t) = \frac{t^p}{f(x) + g(t)},$$

where f is a positive measurable function on X, and g is a decreasing positive function on $[0,\infty)$, then ω is a Musielak–Orlicz function of uniformly lower type p.

Let

$$p^+_\omega \equiv \inf\{p > 0 :\ \exists C > 0 \text{ such that (2.5.1) holds for all } x \in X,\ t \in [1,\infty),\ s \in (0,\infty)\},$$

and let

$$p_\omega^- \equiv \sup\{p>0:\ \exists C>0 \text{ such that (2.5.1) holds for all } x\in X,\ t\in(0,1],\ s\in(0,\infty)\}.$$

The function ω is said to be of strictly uniformly lower type p if for all $x\in X$, $t\in(0,1)$, and $s\in(0,\infty)$, $\omega(x,st)\le t^p\omega(x,s)$. One then defines

$$p_\omega \equiv \sup\{p>0:\omega(x,st)\le t^p\omega(x,s) \text{ holds for all } x\in X,\ s\in(0,\infty) \text{ and } t\in(0,1)\}.$$

It is easy to see that $p_\omega\le p_\omega^-\le p_\omega^+$ for all ω. In what follows, p_ω, p_ω^-, and p_ω^+ are called the *strictly critical lower type index*, the *critical lower type index*, and the *critical upper type index* of ω, respectively.

In the sequel, we assume that ω satisfies the following assumptions.

Assumption (A) Suppose that ω is a Musielak–Orlicz function which is of uniformly upper type 1 and with $p_\omega\in(0,1]$. In addition, for every $x\in X$, $\omega(x,\cdot)$ is continuous, strictly increasing on $\mathbb{R}_+$.

Note that if ω satisfies Assumption (A), then it has the following properties (see [56, Lemma 4.1]).

Lemma 2.5.1 (i) *The function ω is uniformly σ-quasi-subadditive on $X\times[0,\infty)$; namely, there exists a positive constant C such that for all $(x,t_j)\in X\times[0,\infty)$ with $j\in\mathbb{Z}_+$, $\omega(x,\sum_{j=1}^\infty t_j)\le C\sum_{j=1}^\infty\omega(x,t_j)$.*

(ii) *Let $\widetilde{\omega}(x,t):=\int_0^t\frac{\omega(x,s)}{s}\,ds$ for all $(x,t)\in X\times[0,\infty)$. Then $\widetilde{\omega}$ is equivalent to ω; moreover, $\widetilde{\omega}$ also satisfies Assumption (A).*

Convention (B) From Assumption (A), it follows that $0<p_\omega\le p_\omega^-\le p_\omega^+\le 1$. In what follows, if (2.5.1) holds for p_ω^+ with $t\in[1,\infty)$, then we choose $\widetilde{p}_\omega\equiv p_\omega^+$; otherwise $p_\omega^+<1$ and we choose $\widetilde{p}_\omega\in(p_\omega^+,1)$ to be close enough to p_ω^+.

Let ω satisfy Assumption (A). A measurable function f on X is said to be in the Lebesgue-type space $L(\omega)$ if

$$\int_X \omega(x,|f(x)|)\,d\mu(x)<\infty.$$

Moreover, for any $f\in L(\omega)$, define

$$||f||_{L(\omega)}=\inf\Big\{\lambda>0:\int_X\omega\Big(x,\frac{|f(x)|}{\lambda}\Big)d\mu(x)\le 1\Big\}.$$

The function ρ defined below plays an important role in the sequel.

Definition 2.5.2 *For each $x\in X$, we define the function $\omega^{-1}(x,\cdot)$ and $\rho(x,\cdot)$ on $\mathbb{R}_+$ as follows:*

$$\omega^{-1}(x,t)\equiv\sup\{s\ge 0:\omega(x,s)\le t\} \tag{2.5.2}$$

and

$$\rho(x,t)\equiv\frac{t^{-1}}{\omega^{-1}(x,t^{-1})}. \tag{2.5.3}$$

Then it is easy to see that $\omega^{-1}(x, \cdot)$ is continuous, strictly increasing and that for every $x \in X$,

$$\omega^{-1}(x, \omega(x, t)) = t$$

and

$$\omega(x, \omega^{-1}(x, t)) = t.$$

Moreover, the types of ω and ω^{-1} have the following relation.

Lemma 2.5.3 *Let $0 < p \le q \le 1$. If ω is of type (p, q), then ω^{-1} is of type (q^{-1}, p^{-1}).*

Proof. By symmetry, it suffices to show that if ω is of uniformly lower type p, then ω^{-1} is of uniformly upper type p^{-1}. Suppose that there exists a constant $C \ge 1$ such that for all $x \in X$, $s \ge 0$, $t \le 1$,

$$\omega(x, st) \le Ct^p \omega(x, s). \tag{2.5.4}$$

Then for any $x \in X$, $s \ge 0$, $t \ge 1$, and $u \ge 0$ such that $\omega(x, u) \le st$, it follows from (2.5.4) that

$$\omega\left(x, \frac{u}{t^{\frac{1}{p}}}\right) \le \frac{C\omega(x, u)}{t} \le Cs.$$

This implies that

$$u \le t^{\frac{1}{p}} \omega^{-1}(x, Cs),$$

and hence

$$\omega^{-1}(x, st) \le t^{\frac{1}{p}} \omega^{-1}(x, Cs). \tag{2.5.5}$$

On the other hand, observe that

$$\omega^{-1}(x, Cs) = \sup\left\{\lambda \ge 0 : \frac{\omega(x, \lambda)}{C} \le s\right\}$$

and, by (2.5.4), for any $\lambda \ge 0$,

$$\omega\left(x, \frac{\lambda}{C^{\frac{2}{p}}}\right) \le \frac{\omega(x, \lambda)}{C},$$

then we deduce that

$$\omega^{-1}(x, Cs) \le C^{\frac{2}{p}} \omega^{-1}(x, s), \tag{2.5.6}$$

which together with (2.5.5) completes the proof of Lemma 2.5.3.

□

Hereafter, we shall assume the following condition on the function ω.

Assumption (C) Let ω satisfy Assumption (A) and the following conditions:

(i) there exist positive constants C_1, C_2 such that for any $x \in X$, $C_1 \le \omega(x, 1) \le C_2$;

(ii) there exists a positive constant C such that for any locally integrable positive function f on X, for any ball B in X,

$$\frac{1}{|B|}\int_B \omega\,(x, f(x))d\mu(x) \le C \inf_{x \in X} \omega\left(x, \frac{1}{|B|}\int_B f(y)d\mu(y)\right).$$

Remark 2.5.4 A typical example of Musielak–Orlicz function ω that satisfies Assumption (C) is $\omega(x,t) = h(x)\varphi(t)$ for all $x \in X$ and $t \in [0,\infty)$, where h is a measurable function on X so that there exist positive constants C_1, C_2 such that, for any $x \in X$, $C_1 \le h(x) \le C_2$ and φ is an increasing, continuous, and concave Orlicz function on $[0,\infty)$ with $p_\varphi \in (0,1]$.

3

Musielak–Orlicz Hardy spaces associated with divergence form elliptic operators

3.1 Basic notation and known facts on the divergence form elliptic operator L

We first make some conventions. Throughout the whole chapter, L always denotes the second-order divergence form operator as defined in (1.0.1) and (1.0.2). We denote by C a positive constant which is independent of the main parameters, but which may vary from line to line. The symbol $X \lesssim Y$ means that there exists a positive constant C such that $X \le CY$.
Following [46], let us recall the following indices.

$$p_L \equiv \inf \left\{ p \ge 1 : \sup_{t>0} \|e^{-tL}\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} < \infty \right\}$$

and

$$\widetilde{p}_L \equiv \sup \left\{ p \le \infty : \sup_{t>0} \|e^{-tL}\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} < \infty \right\}.$$

It was proved by Auscher [3] that if $n = 1, 2$, then $p_L = 1$ and $\widetilde{p}_L = \infty$, and that if $n \ge 3$, then $p_L < 2n/(n+2)$ and $\widetilde{p}_L > 2n/(n-2)$. One could detail other situations: for example, if the matrix A is real-valued, then the heat kernel has Gaussian bounds and so $p_L = 1$ and $\widetilde{p}_L = \infty$, due to Theorem 4 in [10] or the case of a higher-order operator (see Section 7.2 of [5]).

Recall that for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the Lusin area function is defined by

$$\mathcal{S}_L f(x) \equiv \left(\iint_{\Gamma(x)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2}. \tag{3.1.1}$$

where, $\Gamma(x) \equiv \{(y,t) \in \mathbb{R}^n \times (0,\infty) : |x-y| < t\}$.

A family $\{S_t\}_{t>0}$ of operators is said to satisfy the L^2 off-diagonal estimates, which are also called the *Gaffney estimates* (see [46]), if there exist positive constants c, C, and β such that for arbitrary closed sets E, $F \subset \mathbb{R}^n$,

$$\|S_t f\|_{L^2(F)} \le C e^{-\left(\frac{\mathrm{dist}\,(E,F)^2}{ct}\right)^\beta} \|f\|_{L^2(E)}$$

for every $t > 0$ and every $f \in L^2(\mathbb{R}^n)$ supported in E. In the sequel, we will need the following important results which were obtained in [3], [6], [46], and [47].

Lemma 3.1.1 ([47]) *If two families of operators, $\{S_t\}_{t>0}$ and $\{T_t\}_{t>0}$, satisfy Gaffney estimates, then so does $\{S_tT_t\}_{t>0}$. Moreover, there exist positive constants c, C, and β such that for arbitrary closed sets E, $F \subset \mathbb{R}^n$,*

$$\|S_sT_t f\|_{L^2(F)} \le C e^{-\left(\frac{\mathrm{dist}\,(E,F)^2}{c\max\{s,t\}}\right)^\beta} \|f\|_{L^2(E)}$$

for every s, $t > 0$ and every $f \in L^2(\mathbb{R}^n)$ supported in E.

Lemma 3.1.2 ([6, 47]) *The families*

$$\{e^{-tL}\}_{t>0}, \quad \{tLe^{-tL}\}_{t>0}, \quad \{t^{1/2}\nabla e^{-tL}\}_{t>0}, \tag{3.1.2}$$

as well as

$$\{(I+tL)^{-1}\}_{t>0}, \quad \{t^{1/2}\nabla(I+tL)^{-1}\}_{t>0}, \tag{3.1.3}$$

are bounded on $L^2(\mathbb{R}^n)$ uniformly in t and satisfy the Gaffney estimates with positive constants c, C depending on n, λ_A, Λ_A as in (1.0.1) only. For the operators in (3.1.2), $\beta = 1$, while in (3.1.3), $\beta = 1/2$.

Lemma 3.1.3 ([3, 46]) *There exist $c, C \in (0,\infty)$ such that*

(i) *for every p and q with $p_L < p \le q < \widetilde{p}_L$, the families $\{e^{-tL}\}_{t>0}$ and $\{tLe^{-tL}\}_{t>0}$ satisfy $L^p - L^q$ off-diagonal estimates; that is, for arbitrary closed sets E, $F \subset \mathbb{R}^n$,*

$$\|e^{-tL}f\|_{L^q(F)} + \|tLe^{-tL}f\|_{L^q(F)} \le Ct^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} e^{-\frac{\mathrm{dist}\,(E,F)^2}{ct}} \|f\|_{L^p(E)}$$

for every $t > 0$ and every $f \in L^p(\mathbb{R}^n)$ supported in E, and thus the operators $\{e^{-tL}\}_{t>0}$ and $\{tLe^{-tL}\}_{t>0}$ are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with the norm $Ct^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}$;

(ii) *for every $p \in (p_L, \widetilde{p}_L)$, the family $\{(I+tL)^{-1}\}_{t>0}$ satisfies $L^p - L^p$ off-diagonal estimates; that is, for arbitrary closed sets E, $F \subset \mathbb{R}^n$,*

$$\|(I+tL)^{-1}f\|_{L^q(F)} \le Ct^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} e^{-\frac{\mathrm{dist}\,(E,F)}{ct^{1/2}}} \|f\|_{L^p(E)}$$

for every $t > 0$ and every $f \in L^p(\mathbb{R}^n)$ supported in E.

Lemma 3.1.4 ([46]) *Let $k \in \mathbb{N}$ and $p \in (p_L, \widetilde{p}_L)$. Then the operator given by for any $f \in L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$\mathcal{S}_L^k f(x) \equiv \left(\iint_{\Gamma(x)} |(t^2L)^k e^{-t^2L} f(y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2},$$

is bounded on $L^p(\mathbb{R}^n)$.

3.2 Tent spaces associated to Musielak–Orlicz functions

In this section, we will deal with the tent spaces associated to Musielak–Orlicz functions. Let us first recall some notions.

For any $\nu > 0$ and $x \in \mathbb{R}^n$, denote by

$$\Gamma_\nu(x) \equiv \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \nu t\}$$

the cone of aperture ν with vertex $x \in \mathbb{R}^n$, where $\mathbb{R}^{n+1}_+ \equiv \mathbb{R}^n \times (0,\infty)$. For any closed set F of $\mathbb{R}^n$, denote by $\mathcal{R}_\nu F$ the union of all cones with vertices in F, that is, $\mathcal{R}_\nu F \equiv \cup_{x\in F}\Gamma_\nu(x)$; and for any open set O in $\mathbb{R}^n$, denote the tent over O by $T_\nu(O) \equiv [\mathcal{R}_\nu(O^\complement)]^\complement$. Apparently,

$$T_\nu(O) = \{(x,t) \in \mathbb{R}^n \times (0,\infty) : \ \mathrm{dist}\,(x, O^\complement) \geq \nu t\}.$$

In addition, we denote $\Gamma_1(x)$, $\mathcal{R}_1(F)$, and $T_1(O)$ simply by $\Gamma(x)$, $\mathcal{R}(F)$, and $\widehat{O}$, respectively.

Let F be a closed subset of $\mathbb{R}^n$, and let $O \equiv F^\complement$. Assume that $|O| < \infty$. For a fixed $\gamma \in (0,1)$, we say that $x \in \mathbb{R}^n$ has the *global γ-density* with respect to F if

$$\frac{|B(x,r)\cap F|}{|B(x,r)|} \geq \gamma$$

for all $r > 0$. Denote by F^* the set of all such x and $O^* \equiv (F^*)^\complement$. Clearly, F^* is a closed subset of F, and $O \subset O^*$. In fact, we have

$$O^* = \{x \in \mathbb{R}^n : \mathcal{M}(\chi_O)(x) > 1-\gamma\},$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal function on $\mathbb{R}^n$. As a result, it follows from the weak type $(1,1)$ of $\mathcal{M}$ that $|O^*| \leq C(\gamma)|O|$, where $C(\gamma)$ is a constant depending on γ.

We recall here the following lemma whose proof is similar to that of [24, Lemma 2] and so we omit the details.

Lemma 3.2.1 *Let $\nu,\ \eta \in (0,\infty)$. Then there exist positive constants $\gamma \in (0,1)$ and $C(\gamma,\nu,\eta)$ such that for any closed subset F of $\mathbb{R}^n$ whose complement has finite measure and any nonnegative measurable function H on $\mathbb{R}^{n+1}_+$,*

$$\iint_{\mathcal{R}_\nu(F^*)} H(y,t)t^n\,dy\,dt \leq C(\gamma,\nu,\eta)\int_F \left\{\iint_{\Gamma_\eta(x)} H(y,t)\,dy\,dt\right\} dx,$$

where F^* denotes the set of points in $\mathbb{R}^n$ with global γ-density with respect to F.

For $\nu \in (0,\infty)$, for all measurable functions g on $\mathbb{R}^{n+1}_+$ and all $x \in \mathbb{R}^n$, let us denote

$$\mathcal{A}_\nu(g)(x) \equiv \left(\iint_{\Gamma_\nu(x)} |g(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2},$$

and denote $\mathcal{A}_1(g)$ simply by $\mathcal{A}(g)$.

Following [24], the tent space $T_2^p(\mathbb{R}_+^{n+1})$ for $p \in (0,\infty)$ is defined as the space of all measurable functions g such that $\|g\|_{T_2^p(\mathbb{R}_+^{n+1})} \equiv \|\mathcal{A}(g)\|_{L^p(\mathbb{R}^n)} < \infty$.

Now let ω satisfy Assumption (A) (see page 27). Then we define the tent space $T_\omega(\mathbb{R}_+^{n+1})$ associated to the function ω as the space of measurable functions g on $\mathbb{R}_+^{n+1}$ such that $\mathcal{A}(g) \in L(\omega)$ with the norm defined by

$$\|g\|_{T_\omega(\mathbb{R}_+^{n+1})} \equiv \|\mathcal{A}(g)\|_{L(\omega)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \omega\left(x, \frac{\mathcal{A}(g)(x)}{\lambda}\right) dx \le 1\right\}.$$

Note that the definition of the tent space $T_\omega(\mathbb{R}_+^{n+1})$ above is similar to that of [43].

Let $p \in (1,\infty)$, let ω satisfy Assumption (C) (see page 27 and 28), and let ρ be the function defined by (2.5.3). A function a on $\mathbb{R}_+^{n+1}$ is called an (ω,p)-*atom* if

(i) there exists a ball $B \subset \mathbb{R}^n$ such that $\operatorname{supp} a \subset \widehat{B}$;

(ii) $\|a\|_{T_2^p(\mathbb{R}_+^{n+1})} \le |B|^{1/p-1} \inf_{x\in B} [\rho(x,|B|)]^{-1}$.

Furthermore, if a is an (ω,p)-atom for all $p \in (1,\infty)$, we then call a an (ω,∞)-atom.

Remark 3.2.2 (i) It is not difficult to verify that for a function ω satisfying Assumption (C), there exist positive constants K_1, K_2 such that for any $x \in \mathbb{R}^n$, $K_1 \le \omega^{-1}(x,1) \le K_2$ and hence $\inf_{x\in B} [\rho(x,|B|)]^{-1}$ is strictly positive.

(ii) In addition, for all (ω,p)-atoms a, we have $\|a\|_{T_\omega(\mathbb{R}_+^{n+1})} \lesssim 1$.

We are now ready to obtain the atomic characterization of the tent space $T_\omega(\mathbb{R}_+^{n+1})$.

Theorem 3.2.3 *Let ω satisfy Assumption (C). Then for any $f \in T_\omega(\mathbb{R}_+^{n+1})$, there exist (ω,∞)-atoms $\{a_j\}_{j=1}^\infty$ and numbers $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that for almost every $(x,t) \in \mathbb{R}_+^{n+1}$,*

$$f(x,t) = \sum_{j=1}^\infty \lambda_j a_j(x,t). \tag{3.2.1}$$

Moreover, there exists a positive constant C such that for all $f \in T_\omega(\mathbb{R}_+^{n+1})$,

$$\Lambda(\{\lambda_j a_j\}_j) \equiv \inf\left\{\lambda > 0 : \sum_{j=1}^\infty |B_j| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda_j|}{\lambda\,|B_j| \sup_{x\in B_j} \rho\,(x,|B_j|)}\right) \le 1\right\} \le C\|f\|_{T_\omega(\mathbb{R}_+^{n+1})}, \tag{3.2.2}$$

where $\widehat{B_j}$ appears as the support of a_j.

Proof. We will exploite ideas found in [24, proof of Theorem 1] and [49, Theorem 3.1], and prove this theorem by using a similar proof of [49, Theorem 3.1] with certain modifications as our setting of Musielak-Orlicz function is more general.

Let us first show the decomposition (3.2.1) holds. Assume that $f \in T_\omega(\mathbb{R}_+^{n+1})$. If for any $k \in \mathbb{Z}$, we set $O_k \equiv \{x \in \mathbb{R}^n : \mathcal{A}(f)(x) > 2^k\}$ then O_k is an open set and $|O_k| < \infty$. The complement of O_k is denoted by F_k.

For any non-positive integer k, it follows from Lemma 3.2.1 that

$$\iint_{\mathcal{R}(F_k^*)} |f(y,t)|^2 \frac{dy\,dt}{t} \quad \lesssim \int_{F_k} \iint_{\Gamma(x)} |f(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\,dx \lesssim \int_{F_k} \mathcal{A}(f)(x)^2\,dx.$$

Note that ω is of uniformly upper type 1 and $1 \lesssim \omega(x,1)$. So we obtain the following estimate

$$\int_{F_k} \mathcal{A}(f)(x)^2\,dx \lesssim \int_{F_k} \omega(x, \mathcal{A}(f)(x))dx,$$

which together with the previous inequalities implies

$$\iint_{\mathcal{R}(F_k^*)} |f(y,t)|^2 \frac{dy\,dt}{t} \lesssim \int_{F_k} \omega(x, \mathcal{A}(f)(x))dx. \tag{3.2.3}$$

Observe that if let $k \to -\infty$ then by the definition of F_k, $\int_{F_k} \omega(x, \mathcal{A}(f)(x))dx \to 0$. Therefore, the estimate (3.2.3) implies that $f = 0$ almost everywhere in $\cap_{k\in\mathbb{Z}} \mathcal{R}(F_k^*)$, and $\operatorname{supp} f \subset \{\cup_{k\in\mathbb{Z}} \widehat{O_k^*} \cup E\}$, where $E \subset \mathbb{R}_+^{n+1}$ and $\iint_E \frac{dx\,dt}{t} = 0$.

At this stage, we will borrow the same idea in the proof of [49, Theorem 3.1] about employing the Whitney decomposition to the sets O_k^*, so that we can obtain a set I_k of indices and a family $\{Q_{k,j}\}_{j\in I_k}$ of cubes satisfying the following properties:

(i) $\cup_{j\in I_k} Q_{k,j} = O_k^*$, and $Q_{k,j} \cap Q_{k,i} = \emptyset$ if $i \neq j$.

(ii) $\sqrt{n}\ell(Q_{k,j}) \leq \operatorname{dist}(Q_{k,j}, (O_k^*)^{\complement}) \leq 4\sqrt{n}\ell(Q_{k,j})$, where $\ell(Q_{k,j})$ denotes the side length of $Q_{k,j}$.

Then, for each $j \in I_k$, we choose a ball $B_{k,j}$ with the same center as $Q_{k,j}$ and with radius $\frac{11}{2}\sqrt{n}$-times $\ell(Q_{k,j})$. Then we set $\lambda_{k,j} \equiv 2^k |B_{k,j}| \sup_{x\in B_{k,j}} [\rho(x, |B_{k,j}|)]$ and

$$a_{k,j} \equiv 2^{-k} |B_{k,j}|^{-1} \inf_{x\in B_{k,j}} [\rho(x, |B_{k,j}|)]^{-1} f \chi_{A_{k,j}},$$

where $A_{k,j} \equiv \widehat{B_{k,j}} \cap (Q_{k,j} \times (0,\infty)) \cap (\widehat{O_k^*} \setminus \widehat{O_{k+1}^*})$. Then it follows from the inclusion $\{(Q_{k,j} \times (0,\infty)) \cap (\widehat{O_k^*} \setminus \widehat{O_{k+1}^*})\} \subset \widehat{B_{k,j}}$ that $f = \sum_{k\in\mathbb{Z}} \sum_{j\in I_k} \lambda_{k,j} a_{k,j}$ almost everywhere.

Our next step is to show that for each $k \in \mathbb{Z}$ and $j \in I_k$, $a_{k,j}$ is an (ω,∞)-atom supported in $\widehat{B}_{k,j}$. In fact, taking any $p \in (1,\infty)$ and any $h \in T_2^q(\mathbb{R}_+^{n+1})$ with $\|h\|_{T_2^q(\mathbb{R}_+^{n+1})} \leq 1$, where q is the conjugate index of p. Then it follows from Lemma 3.2.1, from the Cauchy-Schwarz inequality and from the fact $A_{k,j} \subset (\widehat{O_{k+1}^*})^{\complement} = \mathcal{R}(F_{k+1}^*)$ that

$$\begin{aligned} |\langle a_{k,j}, h\rangle| \quad &\leq \iint_{\mathbb{R}_+^{n+1}} |(a_{k,j}\chi_{A_{k,j}})(y,t) h(y,t)| \frac{dy\,dt}{t} \\ &\lesssim \int_{F_{k+1}} \iint_{\Gamma(x)} |a_{k,j}(y,t) h(y,t)| \frac{dy\,dt}{t^{n+1}}\,dx \lesssim \int_{(O_{k+1})^{\complement}} \mathcal{A}(a_{k,j})(x) \mathcal{A}(h)(x) d\mu(x). \end{aligned}$$

Therefore, applying the Hölder inequality with the pair (p,q) to the rightmost side of the above estimates gives

$$|\langle a_{k,j}, h\rangle| \lesssim 2^{-k} |B_{k,j}|^{-1} \inf_{x\in B_{k,j}} [\rho(x, |B_{k,j}|)]^{-1} \left(\int_{B_{k,j}\cap O_{k+1}^{\complement}} [\mathcal{A}(f)(x)]^p\,dx \right)^{1/p} \|h\|_{T_2^q(\mathbb{R}_+^{n+1})}.$$

Note that

$$\left(\int_{B_{k,j}\cap O_{k+1}^{\complement}} [\mathcal{A}(f)(x)]^p\, dx\right)^{1/p} \leq 2^{k+1}|B_{k,j}|^{1/p}.$$

Now it is clear to see that

$$|\langle a_{k,j}, h\rangle| \lesssim |B_{k,j}|^{1/p-1} \inf_{x\in B_{k,j}} [\rho\,(x, |B_{k,j}|)]^{-1},$$

which implies that $a_{k,j}$ is an (ω, p)-atom supported in $\widehat{B}_{k,j}$ for all $p \in (1, \infty)$. In other words, $a_{k,j}$ is an (ω, ∞)-atom.

Finally, for any $\lambda > 0$, put

$$Z_\lambda = \sum_{k\in\mathbb{Z}} \sum_{j\in I_k} |B_{k,j}| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda_{k,j}|}{\lambda\,|B_{k,j}| \sup\limits_{x\in B_{k,j}} \rho\,(x, |B_{k,j}|)}\right).$$

Then by the choice of $B_{k,j}$, we have

$$Z_\lambda \lesssim \sum_{k\in\mathbb{Z}} \sum_{j\in I_k} |Q_{k,j}| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{2^k}{\lambda}\right).$$

But from the property (i) of the Whitney decomposition to the set O_k^*, it follows that

$$\sum_{k\in\mathbb{Z}} \sum_{j\in I_k} |Q_{k,j}| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{2^k}{\lambda}\right) \lesssim \sum_{k\in\mathbb{Z}} |O_k^*| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{2^k}{\lambda}\right) \lesssim \sum_{k\in\mathbb{Z}} |O_k| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{2^k}{\lambda}\right),$$

which implies that

$$Z_\lambda \lesssim \sum_{k\in\mathbb{Z}} |O_k| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{2^k}{\lambda}\right) := W_\lambda. \tag{3.2.4}$$

Let us now estimate W_λ as follows:

$$W_\lambda \lesssim \sum_{k\in\mathbb{Z}} \int_{O_k} \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{2^k}{\lambda}\right) dx \lesssim \int_{\mathbb{R}^n} \sum_{k<\log_2[\mathcal{A}(f)(x)]} \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{2^k}{\lambda}\right) dx.$$

Since ω is of upper type 1, we have

$$\int_{\mathbb{R}^n} \sum_{k<\log_2[\mathcal{A}(f)(x)]} \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{2^k}{\lambda}\right) dx \lesssim \int_{\mathbb{R}^n} \sum_{k<\log_2[\mathcal{A}(f)(x)]} \int_{2^k}^{2^{k+1}} \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{t}{\lambda}\right) \frac{dt}{t}\, dx.$$

Therefore, we deduce

$$W_\lambda \lesssim \int_{\mathbb{R}^n} \sum_{k<\log_2[\mathcal{A}(f)(x)]} \int_{2^k}^{2^{k+1}} \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{t}{\lambda}\right) \frac{dt}{t}\, dx \lesssim \int_{\mathbb{R}^n} \int_0^{\frac{2\mathcal{A}(f)(x)}{\lambda}} \omega(x,t) \frac{dt}{t}\, dx.$$

Then in view of Lemma 2.5.1, we further obtain

$$W_\lambda \lesssim \int_{\mathbb{R}^n} \omega\left(x, \frac{\mathcal{A}(f)(x)}{\lambda}\right) dx. \tag{3.2.5}$$

Lastly, combining (3.2.4) and (3.2.5) yields

$$Z_\lambda \lesssim \int_{\mathbb{R}^n} \omega\left(x, \frac{\mathcal{A}(f)(x)}{\lambda}\right) dx, \tag{3.2.6}$$

which implies that (3.2.2) holds, and hence, completes the proof of Theorem 3.2.3.

□

Remark 3.2.4 (i) The main ideas from [49] employed in the proof of Theorem 3.2.3 are that the set $A_{k,j}$ are independent of ω, and that atoms can be rescaled.

(ii) Let $\{\lambda_j^i\}_{i,j} \subset \mathbb{C}$ and $\{a_j^i\}_{i,j}$ be (ω, p)-atoms for certain $p \in (1, \infty)$, where $i = 1, 2$. If $\sum_j \lambda_j^1 a_j^1$, $\sum_j \lambda_j^2 a_j^2 \in T_\omega(\mathbb{R}_+^{n+1})$, then since ω is subadditive and of strictly uniformly lower type p_ω, one has

$$[\Lambda(\{\lambda_j^i a_j^i\}_{i,j})]^{p_\omega} \le \sum_{i=1}^{2} [\Lambda(\{\lambda_j^i a_j^i\}_j)]^{p_\omega}.$$

Proposition 3.2.5 *Let ω satisfy Assumption (C), and let $p \in (0, \infty)$. If $f \in (T_\omega(\mathbb{R}_+^{n+1}) \cap T_2^p(\mathbb{R}_+^{n+1}))$, then there exists a decomposition into (ω, ∞)-atoms for f as in (3.2.1) that converges in both spaces $T_\omega(\mathbb{R}_+^{n+1})$ and $T_2^p(\mathbb{R}_+^{n+1})$.*

Proof. The proof that there exists a decomposition into (ω, ∞)-atoms for f as in (3.2.1) that converges in $T_2^p(\mathbb{R}_+^{n+1})$ is analogous to that of [49, Proposition 3.1] and we omit the details. We only need to show that the decomposition (3.2.1) converges in $T_\omega(\mathbb{R}_+^{n+1})$. To this end, let us first show that

$$E_N := \int_{\mathbb{R}^n} \omega\left(x, \mathcal{A}\left(f - \sum_{|k|+|j| \le N} \lambda_{k,j} a_{k,j}\right)(x)\right) dx \to 0$$

as $N \to 0$. In fact, it follows from the Assumption (C) of ω and the subadditive property of $\mathcal{A}$ that

$$\begin{aligned} E_N &\le \sum_{|k|+|j|>N} \int_{\mathbb{R}^n} \omega\left(x, \mathcal{A}(\lambda_{k,j} a_{k,j})(x)\right) dx \\ &\lesssim \sum_{|k|+|j|>N} |B_{k,j}| \inf_{x \in \mathbb{R}^n} \omega\left(x, \frac{1}{|B_{k,j}|} \int_{\mathbb{R}^n} \mathcal{A}(\lambda_{k,j} a_{k,j})(x)\, dx\right). \end{aligned}$$

Note that for each $k \in \mathbb{Z}$ and $j \in I_k$, applying the Hölder inequality gives

$$\frac{1}{|B_{k,j}|} \int_{\mathbb{R}^n} \mathcal{A}(\lambda_{k,j} a_{k,j})(x)\, dx \le \frac{|\lambda_{k,j}|}{|B_{k,j}|^{1/2}} \|a_{k,j}\|_{T_2^2(\mathbb{R}_+^{n+1})} \le \frac{|\lambda_{k,j}|}{|B_{k,j}| \sup_{x \in B_{k,j}} \rho(x, |B_{k,j}|)},$$

which implies that

$$E_N \lesssim \sum_{|k|+|j|>N} |B_{k,j}| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda_{k,j}|}{|B_{k,j}| \sup\limits_{x\in B_{k,j}} \rho\left(x, |B_{k,j}|\right)}\right). \tag{3.2.7}$$

With this inequality at hand, in the light of the estimate (3.2.6), it is clear to see that $E_N \to 0$ as $N \to 0$.

As a result, for any $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that as $N > N_0$,

$$\int_{\mathbb{R}^n} \omega\left(x, \frac{1}{\epsilon}\mathcal{A}\left[f - \sum_{|k|+|j|\le N} \lambda_{k,j}a_{k,j}\right](x)\right) dx \le 1,$$

that is, as $N > N_0$,

$$\|f - \sum_{|k|+|j|\le N} \lambda_{k,j}a_{k,j}\|_{T_\omega(\mathbb{R}^{n+1}_+)} \le \epsilon.$$

Therefore, (3.2.1) holds in $T_\omega(\mathbb{R}^{n+1}_+)$ and we complete the proof of Proposition 3.2.5.

□

One consequence of Proposition 3.2.5 is the following result which plays a significant role in this chapter.

Corollary 3.2.6 *Let ω satisfy Assumption (C). If $f \in T_\omega(\mathbb{R}^{n+1}_+) \cap T^2_2(\mathbb{R}^{n+1}_+)$, then $f \in T^p_2(\mathbb{R}^{n+1}_+)$ for all $p \in [1, 2]$, and hence, there exists a decomposition into (ω, ∞)-atoms for f as in (3.2.1) that converges in both spaces $T_\omega(\mathbb{R}^{n+1}_+)$ and $T^p_2(\mathbb{R}^{n+1}_+)$.*

Proof. Assume that $f \in T_\omega(\mathbb{R}^{n+1}_+) \cap T^2_2(\mathbb{R}^{n+1}_+)$. In view of Proposition 3.2.5, it suffices to show that $f \in T^p_2(\mathbb{R}^{n+1}_+)$ for all $p \in [1, 2]$. To this end, we write

$$\int_{\mathbb{R}^n} [\mathcal{A}(f)(x)]^p\, dx = \int_{\{x\in\mathbb{R}^n:\, \mathcal{A}(f)(x)<1\}} [\mathcal{A}(f)(x)]^p\, dx + \int_{\{x\in\mathbb{R}^n:\, \mathcal{A}(f)(x)\ge1\}} [\mathcal{A}(f)(x)]^p\, dx.$$

But then

$$\int_{\{x\in\mathbb{R}^n:\, \mathcal{A}(f)(x)<1\}} [\mathcal{A}(f)(x)]^p\, dx \le \int_{\{x\in\mathbb{R}^n:\, \mathcal{A}(f)(x)<1\}} \mathcal{A}(f)(x)\, dx.$$

Since ω is of uniformly upper type 1, we have

$$\int_{\{x\in\mathbb{R}^n:\, \mathcal{A}(f)(x)<1\}} \mathcal{A}(f)(x)\, dx \lesssim \int_{\{x\in\mathbb{R}^n:\, \mathcal{A}(f)(x)<1\}} \omega(x, \mathcal{A}(f)(x))\, dx \lesssim \|f\|_{T_\omega(\mathbb{R}^{n+1}_+)}.$$

So we deduce

$$\int_{\{x\in\mathbb{R}^n:\, \mathcal{A}(f)(x)<1\}} [\mathcal{A}(f)(x)]^p\, dx \lesssim \|f\|_{T_\omega(\mathbb{R}^{n+1}_+)}. \tag{3.2.8}$$

On the other hand, observe that

$$\int_{\{x\in\mathbb{R}^n:\,\mathcal{A}(f)(x)\ge 1\}} [\mathcal{A}(f)(x)]^p\,dx \le \int_{\{x\in\mathbb{R}^n:\,\mathcal{A}(f)(x)\ge 1\}} [\mathcal{A}(f)(x)]^2\,dx \le \|f\|^2_{T_2^2(\mathbb{R}^{n+1}_+)}. \quad (3.2.9)$$

Now it follows from (3.2.8) and (3.2.9) that $f \in T_2^p(\mathbb{R}^{n+1}_+)$.

$\square$

For $p\in(0,\infty)$, let us denote by $T^c_\omega(\mathbb{R}^{n+1}_+)$ and $T_2^{p,c}(\mathbb{R}^{n+1}_+)$ the set of all functions in $T_\omega(\mathbb{R}^{n+1}_+)$ and $T_2^p(\mathbb{R}^{n+1}_+)$ with compact supports, respectively. Then we have the following result.

Lemma 3.2.7 *Let ω satisfy Assumption (C). Then $T^c_\omega(\mathbb{R}^{n+1}_+)$ coincides with $T_2^{2,c}(\mathbb{R}^{n+1}_+)$.*

Proof. Let us first prove $T^c_\omega(\mathbb{R}^{n+1}_+)\subset T_2^{2,c}(\mathbb{R}^{n+1}_+)$. It suffices to show that $T^c_\omega(\mathbb{R}^{n+1}_+)\subset T_2^{p,c}(\mathbb{R}^{n+1}_+)$ for certain $p\in(0,\infty)$, in view of [49, Lemma 3.3, (i)]. In fact, given any $f\in T^c_\omega(\mathbb{R}^{n+1}_+)$ and a compact set K in $\mathbb{R}^{n+1}_+$ such that $\operatorname{supp} f\subset K$. If we choose a ball B in $\mathbb{R}^n$ such that $K\subset \widehat{B}$, then $\operatorname{supp}\mathcal{A}(f)\subset B$. Now we write

$$\int_{\mathbb{R}^n} [\mathcal{A}(f)(x)]^{p_\omega}\,dx = \int_{\{x\in\mathbb{R}^n:\,\mathcal{A}(f)(x)<1\}} [\mathcal{A}(f)(x)]^{p_\omega}\,dx + \int_{\{x\in\mathbb{R}^n:\,\mathcal{A}(f)(x)\ge 1\}} [\mathcal{A}(f)(x)]^{p_\omega}\,dx,$$

where p_ω is the uniformly lower type index of ω. Observe that the first term of the right hand side above is dominated by $C|B|$ while the second is bounded by $C\int_{\mathbb{R}^n}\omega(x,\mathcal{A}(f)(x))\,dx$. Thus, $f\in T_2^{p_\omega,c}(\mathbb{R}^{n+1}_+)$.

Conversely, taking any $f\in T_2^{1,c}(\mathbb{R}^{n+1}_+)\equiv T_2^{2,c}(\mathbb{R}^{n+1}_+)$ and a compact set K in $\mathbb{R}^{n+1}_+$ such that $\operatorname{supp} f\subset K$. Then there exists a ball B such that $K\subset\widehat{B}$ and $\operatorname{supp}\mathcal{A}(f)\subset B$. Now, using the uniformly upper type 1 property and the condition (i) in Assumption (C) of ω, we obtain

$$\int_{\mathbb{R}^n} \omega(x,\mathcal{A}(f)(x))\,dx \lesssim \int_{\{x\in\mathbb{R}^n:\,\mathcal{A}(f)(x)<1\}} \omega(x,\mathcal{A}(f)(x))\,dx + \int_{\{x\in\mathbb{R}^n:\,\mathcal{A}(f)(x)\ge 1\}} \mathcal{A}(f)(x)\,dx.$$

Note that the entire right hand side of the above estimate is dominated by $C(|B| + \|f\|_{T_2^1(\mathbb{R}^{n+1}_+)})$. So we have $f\in T^c_\omega(\mathbb{R}^{n+1}_+)$, which completes the proof of Lemma 3.2.7.

$\square$

3.3 Musielak–Orlicz spaces and their dual spaces

Employing some ideas from [36, 46, 49], our aim in this section is to introduce the Musielak–Orlicz Hardy spaces associated to L via the Lusin area function $\mathcal{S}_L$ defined as in (3.1.1). Then we will study their dual spaces. In the entire section, Assumption (C) is always posed on the Musielak–Orlicz function ω. In addition, for a ball $B\equiv B(x_B,r_B)$, we set $U_0(B)\equiv B$, and for $j\in\mathbb{N}$, $U_j(B)\equiv B(x_B,2^jr_B)\setminus B(x_B,2^{j-1}r_B)$.

Definition 3.3.1 *Let ω satisfy Assumption (C). A function $f \in L^2(\mathbb{R}^n)$ is said to be in $\widetilde{H}_{\omega,L}(\mathbb{R}^n)$ if $\mathcal{S}_L f \in L(\omega)$; moreover, define*

$$\|f\|_{H_{\omega,L}(\mathbb{R}^n)} \equiv \|\mathcal{S}_L f\|_{L(\omega)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \omega \left(x, \frac{\mathcal{S}_L f(x)}{\lambda} \right) dx \le 1 \right\}.$$

The Musielak–Orlicz Hardy space $H_{\omega,L}(\mathbb{R}^n)$ is defined to be the completion of $\widetilde{H}_{\omega,L}(\mathbb{R}^n)$ in the norm $\|\cdot\|_{H_{\omega,L}(\mathbb{R}^n)}$.

Remark 3.3.2 (i) It is well-known that $\widetilde{H}_{\omega,\,L}(\mathbb{R}^n)$ is dense in $H_{\omega,\,L}(\mathbb{R}^n)$, see for example [87, page 56]. Moreover, if $\{f_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $\widetilde{H}_{\omega,\,L}(\mathbb{R}^n)$, then there exists a unique $f \in H_{\omega,\,L}(\mathbb{R}^n)$ such that $\lim_{k\to\infty} \|f_k - f\|_{H_{\omega,\,L}(\mathbb{R}^n)} = 0$.

(ii) For any $f_1,\ f_2 \in H_{\omega,L}(\mathbb{R}^n)$, we have the following:

$$\|f_1 + f_2\|^{p_\omega}_{H_{\omega,L}(\mathbb{R}^n)} \le \|f_1\|^{p_\omega}_{H_{\omega,L}(\mathbb{R}^n)} + \|f_2\|^{p_\omega}_{H_{\omega,L}(\mathbb{R}^n)}.$$

In fact, let $\lambda_1 \equiv \|f_1\|^{p_\omega}_{H_{\omega,L}(\mathbb{R}^n)}$ and $\lambda_2 \equiv \|f_2\|^{p_\omega}_{H_{\omega,L}(\mathbb{R}^n)}$, from the subadditivity, the continuity, and the uniformly lower type p_ω of ω, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} \omega \left(x, \frac{\mathcal{S}_L(f_1+f_2)(x)}{(\lambda_1+\lambda_2)^{1/p_\omega}} \right) dx \quad &\le \sum_{i=1}^{2} \int_{\mathbb{R}^n} \omega \left(x, \frac{\mathcal{S}_L(f_i)(x)}{(\lambda_1+\lambda_2)^{1/p_\omega}} \right) dx \\
&\le \sum_{i=1}^{2} \frac{\lambda_i}{\lambda_1+\lambda_2} \int_{\mathbb{R}^n} \omega \left(x, \frac{\mathcal{S}_L(f_i)(x)}{\lambda_i^{1/p_\omega}} \right) dx \le 1,
\end{aligned}$$

which implies that $\|f_1+f_2\|_{H_{\omega,L}(\mathbb{R}^n)} \le (\|f_1\|^{p_\omega}_{H_{\omega,L}(\mathbb{R}^n)} + \|f_2\|^{p_\omega}_{H_{\omega,L}(\mathbb{R}^n)})^{1/p_\omega}$.

Definition 3.3.3 *Let $q \in (p_L, \widetilde{p}_L)$, let $M \in \mathbb{N}$, and let $\epsilon \in (0,\infty)$. A function $\alpha \in L^q(\mathbb{R}^n)$ is called an (ω, q, M, ϵ)-molecule adapted to B if there exists a ball B such that*

(i) $\|\alpha\|_{L^q(U_j(B))} \lesssim 2^{-j\epsilon}|2^jB|^{1/q-1} \inf_{x\in B} [\rho(x, |2^jB|)]^{-1}$, *$j \in \mathbb{Z}_+$;*

(ii) *for every $k = 1,\ldots,M$ and $j \in \mathbb{Z}_+$, there holds*

$$\|(r_B^{-2}L^{-1})^k\alpha\|_{L^q(U_j(B))} \lesssim 2^{-j\epsilon}|2^jB|^{1/q-1} \inf_{x\in B} \left[\rho\left(x, \left|2^jB\right|\right)\right]^{-1}.$$

Finally, if α is an (ω, q, M, ϵ)-molecule for all $q \in (p_L, \widetilde{p}_L)$, then α is called an $(\omega, \infty, M, \epsilon)$-molecule.

3.3.1 Molecular decompositions of $H_{\omega,L}(\mathbb{R}^n)$

Let us begin this section with the following lemmas.

Lemma 3.3.4 *Let $\pi_{L,M}$ be as in (1.0.3). Then the operator $\pi_{L,M}$, initially defined on $T_2^{p,c}(\mathbb{R}^{n+1}_+)$, extends to a bounded linear operator from $T_2^p(\mathbb{R}^{n+1}_+)$ to $L^p(\mathbb{R}^n)$, where $p \in (p_L, \widetilde{p}_L)$.*

Proof. We refer the reader to [49, Proposition 4.1, (i)] for the proof of Lemma 3.3.4.

$\square$

Lemma 3.3.5 *Let $\pi_{L,M}$ be as in (1.0.3). Assume that a is an (ω,∞)-atom supported in a ball B. Then for any fixed $\epsilon\in(0,\infty)$, $\alpha=\pi_{L,M}(a)$ is a multiple of an $(\omega,\infty,M,\epsilon)$-molecule adapted to the ball B.*

Proof. The proof of this lemma is closely similar to the work from line 3, page 1187 to line 2, page 1188 of [49]. Hence we omit the details here.

$\square$

Our first result in this section is now formulated as follows.

Proposition 3.3.6 *Let ω satisfy Assumption (C), let $M\in\mathbb{N}$ and $M>\frac{n}{2}(\frac{1}{p_\omega}-\frac{1}{2})$, and let $\pi_{L,M}$ be as in (1.0.3). Then the operator $\pi_{L,M}$, initially defined on $T_\omega^c(\mathbb{R}_+^{n+1})$, extends to a bounded linear operator from $T_\omega(\mathbb{R}_+^{n+1})$ to $H_{\omega,L}(\mathbb{R}^n)$.*

Proof. First assume that $f\in T_\omega^c(\mathbb{R}_+^{n+1})$. Then it follows from Theorem 3.2.3 that there exist (ω,∞)-atoms $\{a_j\}_{j=1}^\infty$ and numbers $\{\lambda_j\}_{j=1}^\infty\subset\mathbb{C}$ such that $f=\sum_{j=1}^\infty\lambda_j a_j$ pointwise and $\Lambda(\{\lambda_j a_j\}_j)\lesssim\|f\|_{T_\omega(\mathbb{R}_+^{n+1})}$.

In addition, in view of Lemma 3.2.7, we deduce $f\in T_2^{2,c}(\mathbb{R}_+^{n+1})$, which if combined with Corollary 3.2.6 and Lemma 3.3.4 implies that

$$\pi_{L,M}(f)=\sum_{j=1}^\infty\lambda_j\pi_{L,M}(a_j)\equiv\sum_{j=1}^\infty\lambda_j\alpha_j$$

in $L^p(\mathbb{R}^n)$, for all $p\in(p_L,2]$. As a result, using the subadditivity and the continuity of ω, and the fact that the operator $\mathcal{S}_L$ is bounded on $L^p(\mathbb{R}^n)$, we obtain

$$\int_{\mathbb{R}^n}\omega(x,\mathcal{S}_L(\pi_{L,M}(f))(x))\,dx\le\sum_{j=1}^\infty\int_{\mathbb{R}^n}\omega(x,|\lambda_j|\mathcal{S}_L(\alpha_j)(x))\,dx. \tag{3.3.1}$$

Observe further that by Lemma 3.3.5 for any fixed $\epsilon\in(0,\infty)$, $\alpha_j=\pi_{L,M}(a_j)$ is a multiple of an $(\omega,\infty,M,\epsilon)$-molecule adapted to B_j for each j.

Now let $q\in(p_L,\widetilde{p}_L)$, and let $\epsilon>n(\frac{1}{p_\omega}-\frac{1}{\widetilde{p}_\omega})$, where $\widetilde{p}_\omega$ is as in Convention (B). Our next step is to prove that for any (ω,q,M,ϵ)-molecule α adapted to the ball $B\equiv B(x_B,r_B)$ and $\lambda\in\mathbb{C}$,

$$\int_{\mathbb{R}^n}\omega(x,|\lambda|\mathcal{S}_L(\alpha)(x))\,dx\lesssim|B|\inf_{x\in\mathbb{R}^n}\omega\left(x,\frac{|\lambda|}{|B|\sup_{x\in B}\rho(x,|B|)}\right). \tag{3.3.2}$$

Once (3.3.2) is proved, it follows that $\|\alpha\|_{H_{\omega,L}(\mathbb{R}^n)}\lesssim 1$, and (together with (3.3.1)) that for all $f\in T_\omega^c(\mathbb{R}_+^{n+1})$,

$$\int_{\mathbb{R}^n}\omega(x,\mathcal{S}_L(\pi_{L,M}(f))(x))\,dx\lesssim\sum_{j=1}^\infty|B_j|\inf_{x\in\mathbb{R}^n}\omega\left(x,\frac{|\lambda_j|}{|B_j|\sup_{x\in B_j}\rho(x,|B_j|)}\right).$$

In other words, for all $f \in T_\omega^c(\mathbb{R}_+^{n+1})$, we deduce

$$\|\pi_{L,M}(f)\|_{H_{\omega,L}(\mathbb{R}^n)} \lesssim \Lambda(\{\lambda_j a_j\}_j) \lesssim \|f\|_{T_\omega(\mathbb{R}_+^{n+1})},$$

which implies (ii) via a density argument.

So, it remains to prove (3.3.2). Observe that if $q > 2$, then an (ω, q, M, ϵ)-molecule is also an $(\omega, 2, M, \epsilon)$-molecule. As a result, it now suffices to show (3.3.2) for $q \in (p_L, 2]$. In fact, we will follow the similar idea as in the proof of [49, Proposition 4.2] to estimate

$$\int_{\mathbb{R}^n} \omega(x, |\lambda| \mathcal{S}_L(\alpha)(x))\, dx \lesssim \sum_{j=0}^{\infty} \mathrm{H}_j + \sum_{j=0}^{\infty} \mathrm{I}_j, \tag{3.3.3}$$

where

$$\mathrm{H}_j = \int_{\mathbb{R}^n} \omega(x, |\lambda| \mathcal{S}_L([I - e^{-r_B^2 L}]^M (\alpha \chi_{U_j(B)}))(x))\, dx \tag{3.3.4}$$

and

$$\mathrm{I}_j = \sup_{1 \le k \le M} \int_{\mathbb{R}^n} \omega\left(x, |\lambda| \mathcal{S}_L \left\{ \left[\frac{k}{M} r_B^2 L e^{-\frac{k}{M} r_B^2 L}\right]^M (\chi_{U_j(B)} (r_B^{-2} L^{-1})^M \alpha) \right\}(x)\right) dx. \tag{3.3.5}$$

Let us first estimate the terms H_j as in (3.3.4). For each $j \ge 0$, set $B_j \equiv 2^j B$, then it follows from Assumption (C) that

$$\begin{aligned}
\mathrm{H}_j &\lesssim \sum_{k=0}^{\infty} \int_{U_k(B_j)} \omega(x, |\lambda| \mathcal{S}_L([I - e^{-r_B^2 L}]^M (\alpha \chi_{U_j(B)}))(x))\, dx \\
&\lesssim \sum_{k=0}^{\infty} |2^k B_j| \inf_{x \in \mathbb{R}^n} \omega\left(x, \frac{|\lambda|}{|2^k B_j|} \int_{U_k(B_j)} \mathcal{S}_L([I - e^{-r_B^2 L}]^M (\alpha \chi_{U_j(B)}))(x)\, dx\right).
\end{aligned}$$

Next, applying the Hölder inequality implies that

$$\mathrm{H}_j \lesssim \sum_{k=0}^{\infty} |2^k B_j| \inf_{x \in \mathbb{R}^n} \omega\left(x, \frac{|\lambda|}{|2^k B_j|^{1/q}} \|\mathcal{S}_L([I - e^{-r_B^2 L}]^M (\alpha \chi_{U_j(B)}))\|_{L^q(U_k(B_j))}\right). \tag{3.3.6}$$

Let us recall here, in view of the proof of [46, Lemma 4.2, (4.22) and (4.27)], that for $k = 0,\ 1,\ 2,$

$$\|\mathcal{S}_L([I - e^{-r_B^2 L}]^M (\alpha \chi_{U_j(B)}))\|_{L^q(U_k(B_j))} \lesssim \|\alpha\|_{L^q(U_j(B))},$$

and for $k \ge 3$,

$$\|\mathcal{S}_L([I - e^{-r_B^2 L}]^M (\alpha \chi_{U_j(B)}))\|_{L^q(U_k(B_j))}^2 \lesssim k \left(\frac{1}{2^{k+j}}\right)^{4M + 2(n/2 - n/q)} \|\alpha\|_{L^q(U_j(B))}^2.$$

Bringing these estimates to (3.3.6), together with Definition 3.3.3, yields

$$\begin{aligned}
\mathrm{H}_j &\lesssim |B_j| \inf_{x \in \mathbb{R}^n} \omega\left(x, \frac{|\lambda| 2^{-j\epsilon}}{|B_j| \sup_{x \in B_j} \rho(x, |B_j|)}\right) \\
&\quad + \sum_{k=3}^{\infty} |2^k B_j| \inf_{x \in \mathbb{R}^n} \omega\left(x, \frac{|\lambda| \sqrt{k} 2^{-(2M + n/2 - n/q)(j+k) - j\epsilon}}{|2^k B_j|^{1/q} |B_j|^{1 - 1/q} \sup_{x \in B_j} \rho(x, |B_j|)}\right).
\end{aligned}$$

After some simple calculations, using the uniformly lower type p_ω of ω, we obtain

$$\begin{aligned}\mathrm{H}_j \quad &\lesssim 2^{-jp_\omega\epsilon}\left\{1+\sum_{k=3}^{\infty}\sqrt{k}2^{kn(1-p_\omega/q)}2^{-p_\omega(2M+n/2-n/q)(j+k)}\right\}\times\\ &|B_j|\inf_{x\in\mathbb{R}^n}\omega\left(x,\frac{|\lambda|}{|B_j|\sup\limits_{x\in B_j}\rho(x,|B_j|)}\right).\end{aligned}$$

Note that $2Mp_\omega > n(1-p_\omega/2)$, so the above estimate implies that

$$\mathrm{H}_j \lesssim 2^{-jp_\omega\epsilon}|B_j|\inf_{x\in\mathbb{R}^n}\omega\left(x,\frac{|\lambda|}{|B_j|\sup\limits_{x\in B_j}\rho(x,|B_j|)}\right). \tag{3.3.7}$$

With the estimates (3.3.7) on H_j, exploiting again the fact that ω is of uniformly lower type p_ω gives

$$\sum_{j=0}^{\infty}\mathrm{H}_j \lesssim \sum_{j=0}^{\infty}2^{-jp_\omega\epsilon}|B_j|\left\{\frac{|B|\sup\limits_{x\in B}\rho(x,|B|)}{|B_j|\sup\limits_{x\in B_j}\rho(x,|B_j|)}\right\}^{p_\omega}\inf_{x\in\mathbb{R}^n}\omega\left(x,\frac{|\lambda|}{|B|\sup\limits_{x\in B}\rho(x,|B|)}\right).$$

Since ω^{-1} is of uniformly lower type $1/\widetilde{p}_\omega$ by Lemma 2.5.3, the previous inequality further implies that

$$\begin{aligned}\sum_{j=0}^{\infty}\mathrm{H}_j \quad &\lesssim \sum_{j=0}^{\infty}2^{-jp_\omega\epsilon}|B_j|\left\{\frac{|B|}{|B_j|}\right\}^{p_\omega/\widetilde{p}_\omega}\inf_{x\in\mathbb{R}^n}\omega\left(x,\frac{|\lambda|}{|B|\sup\limits_{x\in B}\rho(x,|B|)}\right)\\ &\lesssim \sum_{j=0}^{\infty}2^{-jp_\omega\epsilon}2^{jn(1-p_\omega/\widetilde{p}_\omega)}|B|\inf_{x\in\mathbb{R}^n}\omega\left(x,\frac{|\lambda|}{|B|\sup\limits_{x\in B}\rho(x,|B|)}\right).\end{aligned}$$

Eventually, note that $\epsilon > n(1/p_\omega - 1/\widetilde{p}_\omega)$, we then obtain the desired estimate

$$\sum_{j=0}^{\infty}\mathrm{H}_j \lesssim |B|\inf_{x\in\mathbb{R}^n}\omega\left(x,\frac{|\lambda|}{|B|\sup\limits_{x\in B}\rho(x,|B|)}\right). \tag{3.3.8}$$

Similarly, we also obtain

$$\sum_{j=0}^{\infty}\mathrm{I}_j \lesssim |B|\inf_{x\in\mathbb{R}^n}\omega\left(x,\frac{|\lambda|}{|B|\sup\limits_{x\in B}\rho(x,|B|)}\right),$$

which if combined with (3.3.8) completes the proof of (3.3.2), and hence, the proof of Proposition 3.3.6.

$\square$

Proposition 3.3.7 *Let ω satisfy Assumption (C), let $\epsilon > n(1/p_\omega - 1/p_\omega^+)$, and let $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$. If $f \in H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $f \in L^p(\mathbb{R}^n)$ for all $p \in (p_L, 2]$ and there exist $(\omega, \infty, M, \epsilon)$-molecules $\{\alpha_j\}_{j=1}^\infty$ and numbers $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that*

$$f = \sum_{j=1}^{\infty} \lambda_j \alpha_j \tag{3.3.9}$$

in both $H_{\omega,L}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ for all $p \in (p_L, 2]$. Moreover, there exists a positive constant C independent of f such that for all $f \in H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$\Lambda(\{\lambda_j \alpha_j\}_j) \equiv \inf \left\{ \lambda > 0 : \sum_{j=1}^{\infty} |B_j| \inf_{x \in \mathbb{R}^n} \omega \left(x, \frac{|\lambda_j|}{\lambda |B_j| \sup\limits_{x \in B_j} \rho(x, |B_j|)} \right) \le 1 \right\} \le C \|f\|_{H_{\omega,L}(\mathbb{R}^n)}, \tag{3.3.10}$$

where for each j, α_j is adapted to the ball B_j.

Proof. Assume that $f \in H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then it was pointed out in the proof of [49, Proposition 4.2], using the $L^2(\mathbb{R}^n)$ functional calculi for L, that

$$f = C_M \int_0^\infty (t^2 L)^{M+2} e^{-2t^2 L} f \frac{dt}{t} = \lim_{N \to \infty} \pi_{L,M}(\chi_{O_N}(t^2 L e^{-t^2 L} f))$$

in $L^2(\mathbb{R}^n)$, where $\pi_{L,M}$, C_M are as in (1.0.3), and for each $N \in \mathbb{N}$, $O_N := \{(x,t) \in \mathbb{R}_+^{n+1} : |x| < N, 1/N < t < N\}$.

Since $f \in H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, it is easy to verify that $t^2 L e^{-t^2 L} f \in T_2^2(\mathbb{R}_+^{n+1}) \cap T_\omega(\mathbb{R}_+^{n+1})$, using Definition 3.3.1 and Lemma 3.1.4. It then follows from Corollary 3.2.6 that $t^2 L e^{-t^2 L} f \in T_2^p(\mathbb{R}_+^{n+1})$, which if combined with Lemma 3.3.4 implies that $\{\pi_{L,M}(\chi_{O_N}(t^2 L e^{-t^2 L} f))\}_N$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$. At this stage, by taking subsequence together with the fact that $f = \lim_{N \to \infty} \pi_{L,M}(\chi_{O_N}(t^2 L e^{-t^2 L} f))$ in $L^2(\mathbb{R}^n)$, we obtain

$$f = \lim_{N \to \infty} \pi_{L,M}(\chi_{O_N}(t^2 L e^{-t^2 L} f))$$

in $L^p(\mathbb{R}^n)$.

On the other hand, it follows from Proposition 3.2.5 that there exist (ω, ∞)-atoms $\{a_j\}_{j=1}^\infty$ and numbers $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $t^2 L e^{-t^2 L} f = \sum_{j=1}^\infty \lambda_j a_j$ in $T_2^p(\mathbb{R}_+^{n+1})$ and $\Lambda(\{\lambda_j a_j\}_j) \lesssim \|t^2 L e^{-t^2 L} f\|_{T_\omega(\mathbb{R}_+^{n+1})}$. From here, we again apply Lemma 3.3.4 to get

$$f = \pi_{L,M}(t^2 L e^{-t^2 L} f) = \sum_{j=1}^{\infty} \lambda_j \pi_{L,M}(a_j) \equiv \sum_{j=1}^{\infty} \lambda_j \alpha_j \tag{3.3.11}$$

in $L^p(\mathbb{R}^n)$ for $p \in (p_L, 2]$.
Note that for each j, α_j is a multiple of an $(\omega, \infty, M, \epsilon)$-molecule for any $\epsilon > 0$, $M \in \mathbb{N}$ and $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$ (by Lemma 3.3.5). As a result, the decomposition (3.3.9) holds in $L^p(\mathbb{R}^n)$ for $p \in (p_L, 2]$.

In addition, observe that $\Lambda(\{\lambda_j\alpha_j\}_j) = \Lambda(\{\lambda_j a_j\}_j) \lesssim \|t^2Le^{-t^2L}f\|_{T_\omega(\mathbb{R}^{n+1}_+)}$, which implies (3.3.10).

To finish the proof of Proposition 3.3.7, it remains to show that (3.3.9) holds in $H_{\omega,L}(\mathbb{R}^n)$. In fact, it follows from Lemma 3.1.4, the decomposition (3.3.11), and together with the continuity and the subadditivity of ω that

$$\int_{\mathbb{R}^n} \omega\left(x, \mathcal{S}_L\left(f - \sum_{j=1}^N \lambda_j\alpha_j\right)(x)\right) dx \le \sum_{j=N+1}^\infty \int_{\mathbb{R}^n} \omega\left(x, \mathcal{S}_L(\lambda_j\alpha_j)(x)\right) dx.$$

Let us now choose $\widetilde{p}_\omega$ as in Convention (C) such that $\epsilon > n(1/p_\omega - 1/\widetilde{p}_\omega)$, which is guaranteed by the assumption $\epsilon > n(1/p_\omega - 1/p_\omega^+)$. Then by (3.3.2), we have

$$\sum_{j=N+1}^\infty \int_{\mathbb{R}^n} \omega\left(x, \mathcal{S}_L(\lambda_j\alpha_j)(x)\right) dx \lesssim \sum_{j=N+1}^\infty |B_j| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda_j|}{|B_j| \sup_{x\in B_j}|\rho(x,|B_j|)}\right).$$

Finally, we will follow a similar argument as in the proof of Proposition 3.2.5 to obtain $f = \sum_{j=1}^\infty \lambda_j\alpha_j$ in $H_{\omega,L}(\mathbb{R}^n)$.

□

As a consequence of Proposition 3.3.7, we have the following result.

Corollary 3.3.8 *Let ω satisfy Assumption (C), let $\epsilon > n(1/p_\omega - 1/p_\omega^+)$, and let $q \in (p_L, \widetilde{p}_L)$ and $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$. Then for every $f \in H_{\omega,L}(\mathbb{R}^n)$, there exist (ω, q, M, ϵ)-molecules $\{\alpha_j\}_{j=1}^\infty$ and numbers $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $f = \sum_{j=1}^\infty \lambda_j\alpha_j$ in $H_{\omega,L}(\mathbb{R}^n)$. Moreover, if letting $\Lambda(\{\lambda_j\alpha_j\}_j)$ be as in (3.3.10), then there exists a positive constant C independent of f such that $\Lambda(\{\lambda_j\alpha_j\}_j) \le C\|f\|_{H_{\omega,L}(\mathbb{R}^n)}$.*

Proof. We prove this result by using a density argument. For any $f \in H_{\omega,L}(\mathbb{R}^n)$, there exist $\{f_k\}_{k=1}^\infty \subset (H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n))$ such that for all $k \in \mathbb{N}$,

$$\|f - f_k\|_{H_{\omega,L}(\mathbb{R}^n)} \le 2^{-k}\|f\|_{H_{\omega,L}(\mathbb{R}^n)}.$$

If we set $f_0 \equiv 0$, then $f = \sum_{k=1}^\infty (f_k - f_{k-1})$ in $H_{\omega,L}(\mathbb{R}^n)$. On the other hand, it follows from Proposition 3.3.7 that for all $k \in \mathbb{N}$, there exist (ω, q, M, ϵ)-molecules $\{\alpha_j^k\}_{j=1}^\infty$ and numbers $\{\lambda_j^k\}_{j=1}^\infty \subset \mathbb{C}$ such that $f_k - f_{k-1} = \sum_{j=1}^\infty \lambda_j^k\alpha_j^k$ in $H_{\omega,L}(\mathbb{R}^n)$ and $\Lambda(\{\lambda_j^k a_j^k\}_j) \lesssim \|f_k - f_{k-1}\|_{H_{\omega,L}(\mathbb{R}^n)}$.
Therefore, we have $f = \sum_{k,j=1}^\infty \lambda_j^k\alpha_j^k$ in $H_{\omega,L}(\mathbb{R}^n)$. In addition, by Remark 3.2.4, we deduce

$$[\Lambda(\{\lambda_j^k\alpha_j^k\}_{k,j})]^{p_\omega} \le \sum_{k=1}^\infty [\Lambda(\{\lambda_j^k a_j^k\}_j)]^{p_\omega} \lesssim \sum_{k=1}^\infty \|f_k - f_{k-1}\|_{H_{\omega,L}(\mathbb{R}^n)}^{p_\omega} \lesssim \|f\|_{H_{\omega,L}(\mathbb{R}^n)}^{p_\omega},$$

which completes the proof of Corollary 3.3.8.

□

Let us denote by $H^{q,M,\epsilon}_{\omega,\text{fin}}(\mathbb{R}^n)$ the set of all finite combinations of (ω, q, M, ϵ)-molecules then we immediately obtain the following density result.

Corollary 3.3.9 *Let ω satisfy Assumption (C), let $\epsilon > n(1/p_\omega - 1/p_\omega^+)$, and let $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$. Then the space $H^{q,M,\epsilon}_{\omega,\text{fin}}(\mathbb{R}^n)$ is dense in the space $H_{\omega,L}(\mathbb{R}^n)$.*

3.3.2 An equivalent characterization of $H_{\omega,L}(\mathbb{R}^n)$

In this subsection, we will characterize the Musielak–Orlicz Hardy space $H_{\omega,L}(\mathbb{R}^n)$ via the Lusin-area operator $\mathcal{S}_P$ associated to the Poisson semigroup. The divergence form structure of the operator L plays an important role in obtaining this characterization.

Let us first recall here that for any $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the Lusin-area operator $\mathcal{S}_P$ associated to the Poisson semigroup is defined by

$$\mathcal{S}_P f(x) \equiv \left(\iint_{\Gamma(x)} |t\nabla e^{-t\sqrt{L}} f(y)|^2 \frac{dy\,dt}{t} \right)^{1/2}. \tag{3.3.12}$$

Then we define the space $H_{\omega,\mathcal{S}_P}(\mathbb{R}^n)$ as follows.

Definition 3.3.10 *Let ω satisfy Assumption (A). A function $f \in L^2(\mathbb{R}^n)$ is said to be in $\widetilde{H}_{\omega,\mathcal{S}_P}(\mathbb{R}^n)$ if $\mathcal{S}_P(f) \in L(\omega)$. In addition, let us define*

$$\|f\|_{H_{\omega,\mathcal{S}_P}(\mathbb{R}^n)} \equiv \|\mathcal{S}_P(f)\|_{L(\omega)} = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \omega\left(x, \frac{\mathcal{S}_P(f)(x)}{\lambda}\right) dx \le 1 \right\}.$$

The space $H_{\omega,\mathcal{S}_P}(\mathbb{R}^n)$ is defined to be the completion of $\widetilde{H}_{\omega,\mathcal{S}_P}(\mathbb{R}^n)$ in the norm $\|\cdot\|_{H_{\omega,\mathcal{S}_P}(\mathbb{R}^n)}$.

Below are the well-known results on the Lusin-area operator $\mathcal{S}_P$ associated to the Poisson semigroup.

Lemma 3.3.11 *The Lusin-area operator $\mathcal{S}_P$ associated to the Poisson semigroup is bounded on $L^2(\mathbb{R}^n)$.*

Proof. We refer the reader to the proof of (5.15) in [46].

□

Lemma 3.3.12 *There exists a positive constant C such that for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$\widetilde{\mathcal{S}}_P f(x) \le C\mathcal{S}_P f(x), \tag{3.3.13}$$

where

$$\widetilde{\mathcal{S}}_P f(x) \equiv \left(\iint_{\Gamma(x)} |t^2 L e^{-t\sqrt{L}} f(y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2}.$$

Proof. See the proof of [46, Lemma 5.4].

□

In what follows, we also need the following useful result, which is a general variant of [49, Lemma 5.1] on the boundedness of linear or nonnegative sublinear operators from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$.

Lemma 3.3.13 *Let $q \in (p_L, 2]$, let ω satisfy Assumption (C), let $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$, and let $\epsilon > n(1/p_\omega - 1/p_\omega^+)$. Suppose that T is a nonnegative sublinear (resp., linear) operator which maps $L^q(\mathbb{R}^n)$ continuously into weak-$L^q(\mathbb{R}^n)$. If there exists a positive constant C such that for all $(\omega, \infty, M, \epsilon)$-molecules α adapted to balls B and $\lambda \in \mathbb{C}$,*

$$\int_{\mathbb{R}^n} \omega\left(x, T(\lambda\alpha)(x)\right) dx \leq C|B| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda|}{|B| \sup\limits_{x\in B} \rho(x, |B|)}\right), \tag{3.3.14}$$

then T extends to a bounded sublinear (resp., linear) operator from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$. Moreover, there exists a positive constant $\widetilde{C}$ such that for all $f \in H_{\omega,L}(\mathbb{R}^n)$, $\|Tf\|_{L(\omega)} \leq \widetilde{C}\|f\|_{H_{\omega,L}(\mathbb{R}^n)}$.

Proof. Because of the density of $H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ in $H_{\omega,L}(\mathbb{R}^n)$, it suffices to prove Lemma 3.3.13 for $H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

In fact, for any $f \in H_{\omega,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $f \in L^q(\mathbb{R}^n)$ with $q \in (p_L, 2]$, in the light of Proposition 3.3.7, there exist $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ and $(\omega, \infty, M, \epsilon)$-molecules $\{\alpha_j\}_{j=1}^\infty$ such that $f = \sum_{j=1}^\infty \lambda_j \alpha_j$ in both $H_{\omega,L}(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$, and $\Lambda(\{\lambda_j \alpha_j\}_j) \lesssim \|f\|_{H_{\omega,L}(\mathbb{R}^n)}$.

Now, assume that T is linear, then since T is of weak type (q, q) so $T(f) = \sum_{j=1}^\infty T(\lambda_j \alpha_j)$ almost everywhere.

Next, let T be a nonnegative sublinear operator, then it follows from the assumption that T maps $L^q(\mathbb{R}^n)$ continuously into weak-$L^q(\mathbb{R}^n)$ that

$$\sup_{t>0} t^{1/q} \left|\left\{x \in \mathbb{R}^n : \left|T(f)(x) - T\left(\sum_{j=1}^N \lambda_j \alpha_j\right)(x)\right| > t\right\}\right| \lesssim \left\|f - \sum_{j=1}^N \lambda_j \alpha_j\right\|_{L^q(\mathbb{R}^n)}.$$

Since $\left\|f - \sum_{j=1}^N \lambda_j \alpha_j\right\|_{L^q(\mathbb{R}^n)} \to 0$ as $N \to 0$, the above estimate implies that there exists a subsequence $\{N_k\}_k \subset \mathbb{N}$ such that

$$T\left(\sum_{j=1}^{N_k} \lambda_j \alpha_j\right) \to T(f)$$

almost everywhere, as $k \to \infty$. At this stage, using the nonnegativity and the sublinearity of T further gives that

$$\begin{aligned} T(f) - \sum_{j=1}^\infty T(\lambda_j \alpha_j) \quad &= T(f) - T\left(\sum_{j=1}^{N_k} \lambda_j \alpha_j\right) + T\left(\sum_{j=1}^{N_k} \lambda_j \alpha_j\right) - \sum_{j=1}^\infty T(\lambda_j \alpha_j) \\ &\leq T(f) - T\left(\sum_{j=1}^{N_k} \lambda_j \alpha_j\right). \end{aligned}$$

Then letting $k \to \infty$, it is easy to see that

$$T(f) \leq \sum_{j=1}^{\infty} T(\lambda_j \alpha_j) \tag{3.3.15}$$

almost everywhere. Finally, it follows from (3.3.15), the subadditivity and the continuity of ω and from the condition (3.3.14) that

$$\begin{aligned}\int_{\mathbb{R}^n} \omega\left(x, T(f)(x)\right) dx &\lesssim \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \omega\left(x, T(\lambda_j \alpha_j)(x)\right) dx \\ &\lesssim \sum_{j=1}^{\infty} |B_j| \inf_{x \in \mathbb{R}^n} \omega\left(x, \frac{|\lambda_j|}{|B_j| \sup\limits_{x \in B_j} \rho(x, |B_j|)}\right),\end{aligned}$$

which implies that

$$\|T(f)\|_{L(\omega)} \lesssim \Lambda(\{\lambda_j \alpha_j\}_j) \lesssim \|f\|_{H_{\omega,L}(\mathbb{R}^n)},$$

and hence completes the proof of Lemma 3.3.13.

$\square$

Our main result is now formulated as follows.

Theorem 3.3.14 *Let ω satisfy Assumption (C). Then the spaces $H_{\omega,L}(\mathbb{R}^n)$ and $H_{\omega,\mathcal{S}_P}(\mathbb{R}^n)$ coincide with equivalent norms.*

Proof. Let us first prove that $H_{\omega,\mathcal{S}_P}(\mathbb{R}^n) \subset H_{\omega,L}(\mathbb{R}^n)$. In fact, let $\epsilon > n(\frac{1}{p_\omega} - \frac{1}{p_\omega^+})$, let $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$, and let $f \in H_{\omega,\mathcal{S}_P}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Clearly, it follows from Lemma 3.3.11 and Lemma 3.3.12 that

$$\|\widetilde{\mathcal{S}}_P f\|_{L^2(\mathbb{R}^n)} \lesssim \|\mathcal{S}_P f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$$

and

$$\|\widetilde{\mathcal{S}}_P f\|_{L(\omega)} \lesssim \|f\|_{H_{\omega,\mathcal{S}_P}(\mathbb{R}^n)}.$$

As a result of that, it implies that $t^2 L e^{-t\sqrt{L}} f \in (T_\omega(\mathbb{R}^{n+1}_+) \cap T_2^2(\mathbb{R}^{n+1}_+))$. Then, by the $L^2(\mathbb{R}^n)$-functional calculi for f, we deduce

$$f = \frac{\widetilde{C}}{C_M} \pi_{L,M}(t^2 L e^{-t\sqrt{L}} f)$$

in $L^2(\mathbb{R}^n)$, where $\pi_{L,M}$ and C_M are as in (1.0.3) and $\widetilde{C}$ is the positive constant such that

$$\widetilde{C} \int_0^\infty t^{2(M+1)} e^{-t^2} t^2 e^{-t} \frac{dt}{t} = 1.$$

At this stage, it follows from the fact that $t^2Le^{-t\sqrt{L}}f \in T_\omega(\mathbb{R}^{n+1}_+)$ and from Proposition 3.3.6 that $f \in H_{\omega,L}(\mathbb{R}^n)$, and that

$$\|f\|_{H_{\omega,L}(\mathbb{R}^n)} \lesssim \|t^2Le^{-t\sqrt{L}}f\|_{T_\omega(\mathbb{R}^{n+1}_+)} \sim \|\widetilde{\mathcal{S}}_P f\|_{L(\omega)} \lesssim \|f\|_{H_{\omega,\mathcal{S}_P}(\mathbb{R}^n)},$$

which implies that $H_{\omega,\mathcal{S}_P}(\mathbb{R}^n) \subset H_{\omega,L}(\mathbb{R}^n)$ via a density argument.

Conversely, by using the estimates from (5.16) to (5.22) in the proof of [46, Theorem 5.3] and by following the same arguments to the proof of (3.3.2) in Proposition 3.3.6, we obtain the following similar estimate

$$\int_{\mathbb{R}^n} \omega(x, |\lambda|\mathcal{S}_P(\alpha)(x))\,dx \lesssim |B| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda|}{|B| \sup_{x\in B} \rho(x,|B|)}\right) \tag{3.3.16}$$

for any $(\omega, 2, M, \epsilon)$-molecule α adapted to the ball B and any $\lambda \in \mathbb{C}$.
Finally, it follows from (3.3.16), from Lemma 3.3.11 and Lemma 3.3.13 that

$$\|f\|_{H_{\omega,\mathcal{S}_P}(\mathbb{R}^n)} = \|\mathcal{S}_P f\|_{L(\omega)} \lesssim \|f\|_{H_{\omega,L}(\mathbb{R}^n)},$$

which implies that $H_{\omega,L}(\mathbb{R}^n) \subset H_{\omega,\mathcal{S}_P}(\mathbb{R}^n)$.

$\square$

3.3.3 Dual spaces of $H_{\omega,L}(\mathbb{R}^n)$

Our purpose in this section is to study the dual space of the Musielak–Orlicz Hardy space $H_{\omega,L}(\mathbb{R}^n)$. First we need to introduce some notation and notions.

Following [46], for $\epsilon > 0$ and $M \in \mathbb{N}$, we define the space

$$\mathcal{M}^{M,\epsilon}_\omega(L) \equiv \{\mu \in L^2(\mathbb{R}^n) : \|\mu\|_{\mathcal{M}^{M,\epsilon}_\omega(L)} < \infty\},$$

where

$$\|\mu\|_{\mathcal{M}^{M,\epsilon}_\omega(L)} \equiv \sup_{j\geq 0}\left\{2^{j\epsilon}|B(0,2^j)|^{1/2} \sup_{x\in B(0,2^j)} \rho(x,|B(0,2^j)|) \sum_{k=0}^{M} \|L^{-k}\mu\|_{L^2(U_j(B(0,1)))}\right\}.$$

It follows directly from the definition above and the definition of an $(\omega, 2, M, \epsilon)$-molecule that if $\phi \in \mathcal{M}^{M,\epsilon}_\omega(L)$ with norm 1, then ϕ is an $(\omega, 2, M, \epsilon)$-molecule adapted to $B(0,1)$. Conversely, if α is an $(\omega, 2, M, \epsilon)$-molecule adapted to certain ball, then $\alpha \in \mathcal{M}^{M,\epsilon}_\omega(L)$.

In addition, it is easy to see that $(I - A_t^*)^M f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ in the sense of distributions, where A_t being either $(I + t^2L)^{-1}$ or e^{-t^2L}, and $f \in (\mathcal{M}^{M,\epsilon}_\omega(L))^*$, the dual space of $\mathcal{M}^{M,\epsilon}_\omega(L)$. Indeed, for any ball B, any $\psi \in L^2(B)$, the Gaffney estimates from Lemmas 3.1.1 and 3.1.2 imply that $(I - A_t)^M\psi \in \mathcal{M}^{M,\epsilon}_\omega(L)$ for all $\epsilon > 0$ and any fixed $t \in (0,\infty)$. Therefore, we obtain

$$|\langle (I - A_t^*)^M f, \psi\rangle| \equiv |\langle f, (I - A_t)^M\psi\rangle| \leq C(t, r_B, \mathrm{dist}\,(B,0))\|f\|_{(\mathcal{M}^{M,\epsilon}_\omega(L))^*}\|\psi\|_{L^2(B)},$$

which results in the fact that $(I - A_t^*)^M f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ in the sense of distributions.

Next, for any $M \in \mathbb{N}$, let us define

$$\mathcal{M}^M_{\omega,L^*}(\mathbb{R}^n) \equiv \bigcap_{\epsilon > n(1/p_\omega - 1/p_\omega^+)} (\mathcal{M}^{M,\epsilon}_\omega(L))^*.$$

Definition 3.3.15 *Let $q \in (p_L, \widetilde{p}_L)$, let ω satisfy Assumption (C), and let $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$. A functional $f \in \mathcal{M}^M_{\omega,L}(\mathbb{R}^n)$ is said to be in $\mathrm{BMO}^{q,M}_{\rho,L}(\mathbb{R}^n)$ if*

$$\|f\|_{\mathrm{BMO}^{q,M}_{\rho,L}(\mathbb{R}^n)} \equiv \sup_{B\subset\mathbb{R}^n} \frac{1}{\sup\limits_{x\in B}\rho(x,|B|)} \left[\frac{1}{|B|}\int_B |(I - e^{-r_B^2 L})^M f(x)|^q\,dx\right]^{1/q} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

In particular, we denote $BMO^{2,M}_{\rho,L}(\mathbb{R}^n)$ simply by $\mathrm{BMO}^M_{\rho,L}(\mathbb{R}^n)$. The proofs of Lemmas 3.3.16, 3.3.17, and 3.3.18 below are similar to those of Lemmas 8.1, 8.3 of [46] and Lemma 4.3 of [49], respectively, and hence we skip them here.

Lemma 3.3.16 *Let ω, q, and M be as in Definition 3.3.15. Then a functional $f \in \mathrm{BMO}^{q,M}_{\rho,L}(\mathbb{R}^n)$ if and only if $f \in \mathcal{M}^M_{\omega,L}(\mathbb{R}^n)$ and*

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{\sup\limits_{x\in B}\rho(x,|B|)} \left[\frac{1}{|B|}\int_B |(I - (I + r_B^2 L)^{-1})^M f(x)|^q\,dx\right]^{1/q} < \infty.$$

Moreover, the term appearing on the left-hand side of the above formula is equivalent to $\|f\|_{\mathrm{BMO}^{q,M}_{\rho,L}(\mathbb{R}^n)}$.

Lemma 3.3.17 *Let ω and M be as in Definition 3.3.15. Then there exists a positive constant C such that for all $f \in \mathrm{BMO}^M_{\rho,L}(\mathbb{R}^n)$,*

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{\sup\limits_{x\in B}\rho(x,|B|)} \left[\frac{1}{|B|}\iint_{\widehat{B}} |(t^2 L)^M e^{-t^2 L} f(x)|^2 \frac{dx\,dt}{t}\right]^{1/2} \le C\|f\|_{\mathrm{BMO}^M_{\rho,L}(\mathbb{R}^n)}.$$

Lemma 3.3.18 *Let ω, ρ and M be as in Definition 3.3.15, let $q \in (p_{L^*}, 2]$, let ϵ, $\epsilon_1 > 0$, and let $\widetilde{M} > M + \epsilon_1 + \frac{n}{4}$. Suppose that $f \in \mathcal{M}^M_{\omega,L^*}(\mathbb{R}^n)$ satisfies*

$$\int_{\mathbb{R}^n} \frac{|(I - (I + L^*)^{-1})^M f(x)|^q}{1 + |x|^{n+\epsilon_1}}\,dx < \infty. \tag{3.3.17}$$

Then for every $(\omega, q', \widetilde{M}, \epsilon)$-molecule α,

$$\langle f, \alpha\rangle = \widetilde{C}_M \iint_{\mathbb{R}^{n+1}_+} (t^2 L^*)^M e^{-t^2 L^*} f(x)\overline{t^2 L e^{-t^2 L}\alpha(x)}\,\frac{dx\,dt}{t},$$

where q' is the conjugate index of q and $\widetilde{C}_M$ is the positive constant satisfying

$$\widetilde{C}_M \int_0^\infty t^{2(M+1)} e^{-2t^2}\,\frac{dt}{t} = 1.$$

Remark 3.3.19 *It follows from Lemma 3.3.16 that if $f \in \mathrm{BMO}^{q,M}_{\rho,L}(\mathbb{R}^n)$ then f satisfies (3.3.17) for all $\epsilon_1 > 0$. Thus, Lemma 3.3.18 holds for all $f \in \mathrm{BMO}^{q,M}_{\rho,L}(\mathbb{R}^n)$.*

We are now ready to state the following main results.

Theorem 3.3.20 *Let ω satisfy Assumption (C), let $\epsilon > n(1/p_\omega - 1/p_\omega^+)$, and let $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$ and $\widetilde{M} > M + \frac{n}{4}$. In addition, assume that there exists a positive constant K such that for all balls B in $\mathbb{R}^n$, for all $x \in \mathbb{R}^n$,*

$$|B|\,\omega\left[x, K\inf_{x\in B}\omega^{-1}\left(x, |B|^{-1}\right)\right] \geq 1. \tag{3.3.18}$$

Then $(H_{\omega,L}(\mathbb{R}^n))^*$, the dual space of $H_{\omega,L}(\mathbb{R}^n)$, coincides with $\mathrm{BMO}^{M}_{\rho,L^*}(\mathbb{R}^n)$ in the following sense.

(i) Let $g \in \mathrm{BMO}^{M}_{\rho,L^*}(\mathbb{R}^n)$. Then the linear functional ℓ, which is initially defined on $H^{2,\widetilde{M},\epsilon}_{\omega,\mathrm{fin}}(\mathbb{R}^n)$ by

$$\ell(f) \equiv \langle g, f\rangle, \tag{3.3.19}$$

has a unique extension to $H_{\omega,L}(\mathbb{R}^n)$ with $\|\ell\|_{(H_{\omega,L}(\mathbb{R}^n))^*} \leq C\|g\|_{\mathrm{BMO}^{M}_{\rho,L^*}(\mathbb{R}^n)}$, where C is a positive constant independent of g.

(ii) Conversely, for any $\ell \in (H_{\omega,L}(\mathbb{R}^n))^*$, then $\ell \in \mathrm{BMO}^{M}_{\rho,L^*}(\mathbb{R}^n)$, (3.3.19) holds for all $f \in H^{2,M,\epsilon}_{\omega,\mathrm{fin}}(\mathbb{R}^n)$, and $\|\ell\|_{\mathrm{BMO}^{M}_{\rho,L^*}(\mathbb{R}^n)} \leq C\|\ell\|_{(H_{\omega,L}(\mathbb{R}^n))^*}$, where C is a positive constant independent of ℓ.

Proof. Before coming to the proof of Theorem 3.3.20, we need the following lemma.

Lemma 3.3.21 *Let ω satisfy the assumptions of Theorem 3.3.20, and let $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ and $\Lambda(\{\lambda_j a_j\}_j)$ be as in Theorem 3.2.3. Then we have*

$$\sum_{j=1}^\infty |\lambda_j| \leq C\Lambda(\{\lambda_j a_j\}_j) \leq C\|f\|_{T_\omega(\mathbb{R}^{n+1}_+)}.$$

Proof of Lemma 3.3.21. Take any $\lambda > 0$ such that

$$\sum_{j=1}^\infty |B_j| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda_j|}{\lambda|B_j|\sup_{x\in B_j}\rho(x, |B_j|)}\right) \leq 1. \tag{3.3.20}$$

If there is some λ_j such that $K\lambda < |\lambda_j|$, then by (3.3.18), we see that, for any $x \in \mathbb{R}^n$,

$$|B_j|\,\omega\left(x, \frac{|\lambda_j|}{\lambda|B_j|\sup_{x\in B_j}\rho(x, |B_j|)}\right) > |B_j|\,\omega\left[x, K\inf_{x\in B_j}\ \omega^{-1}\left(x, |B_j|^{-1}\right)\right] \geq 1,$$

which contradicts with (3.3.20). Hence $K\lambda \geq |\lambda_j|$ for all λ_j. Since ω is of uniformly upper type 1, we deduce that

$$|B_j|\,\omega\left(x, \frac{|\lambda_j|}{\lambda\,|B_j|\sup\limits_{x\in B_j}\rho\left(x,|B_j|\right)}\right) \geq \frac{|\lambda_i|}{K\lambda}\,|B_j|\,\omega\left[x, K\inf_{x\in B_j}\,\omega^{-1}\left(x,|B_j|^{-1}\right)\right] \geq \frac{|\lambda_i|}{K\lambda},$$

which, together with (3.3.20) and the definition of $\Lambda(\{\lambda_j a_j\}_j)$ in Theorem 3.2.3, completes the proof of Lemma 3.3.21.

□

Proof of Theorem 3.3.20. Assume $g \in \mathrm{BMO}^M_{\rho,L^*}(\mathbb{R}^n)$.

For any $f \in H^{2,\widetilde{M},\epsilon}_{\omega,\mathrm{fin}}(\mathbb{R}^n) \subset H_{\omega,L}(\mathbb{R}^n)$, we have $f \in L^2(\mathbb{R}^n)$. Setting $h = t^2Le^{-t^2L}f$, then it follows from Lemma 3.1.4 that $h \in (T_\omega(\mathbb{R}^{n+1}_+) \cap T^2_2(\mathbb{R}^{n+1}_+))$. Therefore, in the light of Theorem 3.2.3, there exist $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ and (ω,∞)-atoms $\{a_j\}_{j=1}^\infty$ supported in $\{\widehat{B}_j\}_{j=1}^\infty$ such that $h = \sum_{j=1}^\infty \lambda_j a_j$ and (3.2.2) holds.

On the other hand, since $g \in \mathrm{BMO}^M_{\rho,L^*}(\mathbb{R}^n)$ so Lemma 3.3.18 holds for g (see Remark 3.3.19). As a result, we can write

$$\begin{aligned}
|\langle g, f\rangle| \quad &= \left|C_{\widetilde{M}}\iint_{\mathbb{R}^{n+1}_+}(t^2L^*)^M e^{-t^2L^*}g(x)\overline{t^2Le^{-t^2L}f(x)}\frac{dx\,dt}{t}\right| \\
&\lesssim \sum_{j=1}^\infty|\lambda_j|\iint_{\mathbb{R}^{n+1}_+}|(t^2L^*)^M e^{-t^2L^*}g(x)\overline{a_j(x,t)}|\frac{dx\,dt}{t} \\
&\lesssim \sum_{j=1}^\infty|\lambda_j|\|a_j\|_{T^2_2(\mathbb{R}^{n+1}_+)}\left(\iint_{\widehat{B}_j}|(t^2L^*)^M e^{-t^2L^*}g(x)|^2\frac{dx\,dt}{t}\right)^{1/2},
\end{aligned}$$

where the last estimate follows from applying the Cauchy-Schwarz inequality. At this stage, we observe that for each j,

$$\|a_j\|_{T^2_2(\mathbb{R}^{n+1}_+)} \leq \frac{|B_j|^{-1/2}}{\sup\limits_{x\in B_j}\rho(x,B_j)},$$

which, together with Lemma 3.3.17, implies

$$|\langle g, f\rangle| \lesssim \sum_{j=1}^\infty|\lambda_j|\|g\|_{\mathrm{BMO}^M_{\rho,L^*}(\mathbb{R}^n)}.$$

Thus, we reach, in view of Lemma 3.3.21, the following estimates

$$|\langle g, f\rangle| \lesssim \|t^2Le^{-t^2L}f\|_{T_\omega(\mathbb{R}^{n+1}_+)}\|g\|_{\mathrm{BMO}^M_{\rho,L^*}(\mathbb{R}^n)} \lesssim \|f\|_{H_{\omega,L}(\mathbb{R}^n)}\|g\|_{\mathrm{BMO}^M_{\rho,L^*}(\mathbb{R}^n)}. \tag{3.3.21}$$

Finally, due to Corollary 3.3.9, we obtain (i) by a standard density argument.

Conversely, for any $\ell \in (H_{\omega,L}(\mathbb{R}^n))^*$, we need to show that $\ell \in \mathrm{BMO}^M_{\rho,L^*}(\mathbb{R}^n)$. First, for any $(\omega, 2, M, \epsilon)$-molecule α, since $\|\alpha\|_{H_{\omega,L}(\mathbb{R}^n)} \lesssim 1$, so $|\ell(\alpha)| \lesssim \|\ell\|_{(H_{\omega,L}(\mathbb{R}^n))^*}$, which implies that $\ell \in \mathcal{M}^M_{\omega,L^*}(\mathbb{R}^n)$. It remains to show, by Lemma 3.3.16, that

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{\sup\limits_{x\in B}\rho(x,|B|)} \left(\frac{1}{|B|}\int_B |(I-[(I+r_B^2L)^{-1}]^*)^M\ell(x)|^2\,dx\right)^{1/2} \lesssim \|\ell\|_{(H_{\omega,L}(\mathbb{R}^n))^*}.$$

To this end, for any ball B, let $\phi \in L^2(B)$ such that

$$\|\phi\|_{L^2(B)} \le \frac{1}{|B|^{1/2}\sup\limits_{x\in B}\rho(x,|B|)}$$

and set $\widetilde{\alpha} \equiv (I-[I+r_B^2L]^{-1})^M\phi$, where $2M > n(1/p_\omega - 1/2)$. Let us next show that $\widetilde{\alpha}$ is a multiple of an $(\omega, 2, M, \epsilon)$-molecule. In fact, for each $j \in \mathbb{Z}_+$ and $k = 0, 1, \ldots, M$, it follows from Lemma 3.1.3 that

$$\begin{aligned}
\|(r_B^2L)^{-k}\widetilde{\alpha}\|_{L^2(U_j(B))} &= \|(I-[I+r_B^2L]^{-1})^{M-k}(I+r_B^2L)^{-k}\phi\|_{L^2(U_j(B))}\\
&\lesssim \exp\left\{-\frac{\mathrm{dist}\,(B,U_j(B))}{cr_B}\right\}\|\phi\|_{L^2(B)}\\
&\lesssim 2^{-2j(M+\epsilon)}2^{jn(1/p_\omega-1/2)}\frac{1}{|2^jB|^{1/2}\sup\limits_{x\in B}\rho(x,|2^jB|)}\\
&\lesssim 2^{-2j\epsilon}\frac{1}{|2^jB|^{1/2}\sup\limits_{x\in B}\rho(x,|2^jB|)}.
\end{aligned}$$

On the other hand, observe that for any fixed $t > 0$, $(I-([I+t^2L]^{-1})^*)^M\ell$ is well defined and belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. So we obtain

$$|\langle (I-[(I+r_B^2L)^{-1}]^*)^M\ell,\phi\rangle| = |\langle \ell,(I-[I+r_B^2L]^{-1})^M\phi\rangle| = |\langle \ell,\widetilde{\alpha}\rangle| \lesssim \|\ell\|_{(H_{\omega,L}(\mathbb{R}^n))^*}. \tag{3.3.22}$$

Eventually, we now write

$$\begin{aligned}
&\frac{1}{\sup\limits_{x\in B}\rho(x,|B|)}\left(\frac{1}{|B|}\int_B |(I-[(I+r_B^2L)^{-1}]^*)^M\ell(x)|^2\,dx\right)^{1/2}\\
&\quad= \sup_{\|\phi\|_{L^2(B)}\le 1}\left|\left\langle \ell,(I-[I+r_B^2L]^{-1})^M\frac{\phi}{|B|^{1/2}\sup\limits_{x\in B}\rho(x,|B|)}\right\rangle\right| \lesssim \|\ell\|_{(H_{\omega,L}(\mathbb{R}^n))^*},
\end{aligned}$$

where the last estimate comes from (3.3.22). We complete the proof of Theorem 3.3.20.

□

Remark 3.3.22 It follows from the proof of Theorem 3.3.20 that the spaces $\mathrm{BMO}^M_{\rho,L}(\mathbb{R}^n)$ for all $M > \frac{n}{2}(\frac{1}{p_\omega}-\frac{1}{2})$ coincide with equivalent norms. Thus, we will denote $\mathrm{BMO}^M_{\rho,L}(\mathbb{R}^n)$ simply by $\mathrm{BMO}_{\rho,L}(\mathbb{R}^n)$.

3.3.4 The Carleson measure and the John–Nirenberg inequality

In this section, we aim to characterize the space $\mathrm{BMO}_{\rho,L^*}(\mathbb{R}^n)$ via the ρ-Carleson measure and then establish the John–Nirenberg inequality for elements in $\mathrm{BMO}_{\rho,L^*}(\mathbb{R}^n)$, where L^* stands for the conjugate operator of L in $L^2(\mathbb{R}^n)$.

Definition 3.3.23 *A measure $d\mu$ on $\mathbb{R}^{n+1}_+$ is called a ρ-Carleson measure if*

$$\|d\mu\|_\rho \equiv \sup_{B\subset\mathbb{R}^n}\left\{\frac{1}{|B|[\sup_{x\in B}\rho(x,|B|)]^2}\iint_{\widehat{B}}|\,d\mu|\right\}<\infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$.

Theorem 3.3.24 *Let ω satisfy Assumption (C), and let $M>\frac{n}{2}(\frac{1}{p_\omega}-\frac{1}{2})$.*

(i) *If $f\in\mathrm{BMO}_{\rho,L^*}(\mathbb{R}^n)$, then $d\mu_f$ is a ρ-Carleson measure and there exists a positive constant C independent of f such that $\|d\mu_f\|_\rho\le C\|f\|^2_{\mathrm{BMO}_{\rho,L^*}(\mathbb{R}^n)}$, where*

$$d\mu_f \equiv \left|(t^2L^*)^M e^{-t^2L^*}f(x)\right|^2\frac{dx\,dt}{t}. \tag{3.3.23}$$

(ii) *Conversely, if $f\in\mathcal{M}^M_{\omega,L^*}(\mathbb{R}^n)$ satisfies (3.3.17) with certain $q\in(p_{L^*},2]$ and $\epsilon_1>0$, (3.3.18) holds and $d\mu_f$ is a ρ-Carleson measure, then $f\in\mathrm{BMO}_{\rho,L^*}(\mathbb{R}^n)$. Moreover, there exists a positive constant C independent of f such that $\|f\|^2_{\mathrm{BMO}_{\rho,L^*}(\mathbb{R}^n)}\le C\|d\mu_f\|_\rho$, where $d\mu_f$ is as in (3.3.23).*

Proof. Applying Lemma 3.3.17 yields (i).

Conversely, assume that $f\in\mathcal{M}^M_{\omega,L^*}(\mathbb{R}^n)$ satisfies (3.3.17) with certain $q\in(p_{L^*},2]$ and $\epsilon_1>0$. Then for any $\widetilde{M}>M+\epsilon_1+\frac{n}{4}$ and any $\epsilon>n(\frac{1}{p_\omega}-\frac{1}{p_\omega^+})$, it follows from Lemma 3.3.18 that

$$\langle f,g\rangle=\widetilde{C}_M\iint_{\mathbb{R}^{n+1}_+}(t^2L^*)^M e^{-t^2L^*}f(x)\overline{t^2Le^{-t^2L}g(x)}\frac{dx\,dt}{t},$$

if g is a finite combination of $(\omega,q',\widetilde{M},\epsilon)$-molecules and q' is the conjugate of q. In addition, according to (3.3.21), one can observe that

$$|\langle f,g\rangle|\lesssim\|d\mu_f\|^{1/2}_\rho\|g\|_{H_{\omega,L}(\mathbb{R}^n)}.$$

Finally, note that $H^{q',\widetilde{M},\epsilon}_{\omega,\mathrm{fin}}$ is dense in $H_{\omega,L}(\mathbb{R}^n)$, so we can use a standard density argument to obtain that $f\in(H_{\omega,L}(\mathbb{R}^n))^*$. This if combined with Theorem 3.3.20 implies that $f\in\mathrm{BMO}_{\rho,L^*}(\mathbb{R}^n)$ and $\|f\|^2_{\mathrm{BMO}_{\rho,L^*}(\mathbb{R}^n)}\lesssim\|\,du_f\|_\rho$.

$\square$

Using the same arguments as in the proof of [49, Theorem 6.2], we obtain the following result.

Theorem 3.3.25 *Let ω satisfy Assumption (C), let (3.3.18) hold, and let $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$. Then the spaces $\mathrm{BMO}^{q,M}_{\rho,L^*}(\mathbb{R}^n)$ for all $q \in (p_{L^*}, \widetilde{p}_{L^*})$ coincide with equivalent norms.*

It could be of interest to put forward the following comment: these kind of results are a well-known consequence of John–Nirenberg inequalities, as explained in this current section. Recently, such self-improving properties have been studied in a very abstract setting (see [11, 13, 14, 52]). Moreover, in [13], applications for functional spaces (Hardy spaces and Sobolev spaces) associated to the same (than here) second-order divergence operator are obtained. In [13],[52],[11], the main assumption to get this self-improving property (the John–Nirenberg inequality) is related to the behavior of the "weight" ρ, if it is doubling or increasing (with respect to the ball). They only consider weights, which are "x"-independent. So the results obtained in this current paper are interesting since they deal with an "x"-dependent weight ρ. However, whether it is possible to compare Assumption (C) and (3.3.18) (required in Theorem 3.3.25) with the doubling property required for an "x"-independent weight is still an interesting question. We believe that they are in general incomparable.

3.4 Some applications

In this section, as an application of the theory developed in the previous sections, we will show the boundedness on Musielak–Orlicz Hardy spaces of the Riesz transform and of the Littlewood–Paley g-function associated with the operator L.

Let us recall here that for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the Littlewood–Paley g-function g_L is defined by

$$g_L f(x) \equiv \left(\int_0^\infty |t^2 L e^{-t^2 L} f(x)|^2 \frac{dt}{t}\right)^{1/2}.$$

It is well-known that g_L is bounded on $L^2(\mathbb{R}^n)$, see [46, Proof of Theorem 3.4]. We are now ready to claim the following result which is similar but more general than [46, Theorems 3.2 and 3.4] and [49, Theorem 7.1].

Theorem 3.4.1 *Let ω satisfy Assumption (C), and let $p \in (p_L, 2]$. Suppose that the nonnegative sublinear operator or linear operator T is bounded on $L^p(\mathbb{R}^n)$ and there exist $C > 0$, $M \in \mathbb{N}$ and $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$ such that for all closed sets E, F in $\mathbb{R}^n$ with $\operatorname{dist}(E, F) > 0$ and all $f \in L^p(\mathbb{R}^n)$ supported in E,*

$$\|T(I - e^{-tL})^M f\|_{L^p(F)} \le C \left(\frac{t}{\operatorname{dist}(E,F)^2}\right)^M \|f\|_{L^p(E)} \tag{3.4.1}$$

and

$$\|T(tLe^{-tL})^M f\|_{L^p(F)} \le C \left(\frac{t}{\operatorname{dist}(E,F)^2}\right)^M \|f\|_{L^p(E)} \tag{3.4.2}$$

for all $t > 0$. Then T extends to a bounded sublinear or linear operator from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$. In particular, the Riesz transform $\nabla L^{-1/2}$ and the Littlewood–Paley g-function g_L are bounded from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$.

Proof. Take any $\epsilon > n(1/p_\omega - 1/\widetilde{p}_\omega)$, where $\widetilde{p}_\omega$ is as in Convention (B). In view of Lemma 3.3.13, if we wish to show that T extends to a bounded sublinear or linear operator from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$, then it suffices to show that for all $\lambda \in \mathbb{C}$ and $(\omega, \infty, M, \epsilon)$-molecules α adapted to balls B,

$$\int_{\mathbb{R}^n} \omega\left(x, T(\lambda\alpha)(x)\right) dx \lesssim |B| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda|}{|B| \sup_{x\in B} \rho(x, |B|)}\right). \tag{3.4.3}$$

To this end, we will follow the idea in [49, Theorem 7.1] to estimate

$$\int_{\mathbb{R}^n} \omega(x, T(\lambda\alpha)(x))\, dx \lesssim \sum_{j=0}^{\infty} \mathrm{P}_j + \sum_{j=0}^{\infty} \mathrm{Q}_j, \tag{3.4.4}$$

where

$$\mathrm{P}_j = \int_{\mathbb{R}^n} \omega(x, |\lambda| T([I - e^{-r_B^2 L}]^M (\alpha\chi_{U_j(B)}))(x))\, dx \tag{3.4.5}$$

and

$$\mathrm{Q}_j = \sup_{1\le k\le M} \int_{\mathbb{R}^n} \omega\left(x, |\lambda| T\left\{\left[\frac{k}{M} r_B^2 L e^{-\frac{k}{M} r_B^2 L}\right]^M (\chi_{U_j(B)} (r_B^{-2} L^{-1})^M \alpha)\right\}(x)\right) dx. \tag{3.4.6}$$

Let us now estimate the terms P_j in (3.4.5). Denote $B_j = 2^j B$, for each $j \ge 0$, then it follows from Assumption (C) that

$$\begin{aligned}
\mathrm{P}_j &\lesssim \sum_{k=0}^{\infty} \int_{U_k(B_j)} \omega(x, |\lambda| T([I - e^{-r_B^2 L}]^M (\alpha\chi_{U_j(B)}))(x))\, dx \\
&\lesssim \sum_{k=0}^{\infty} |2^k B_j| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda|}{|2^k B_j|} \int_{U_k(B_j)} T([I - e^{-r_B^2 L}]^M (\alpha\chi_{U_j(B)}))(x)\, dx\right).
\end{aligned}$$

Then applying the Hölder inequality implies that

$$\mathrm{P}_j \lesssim \sum_{k=0}^{\infty} |2^k B_j| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda|}{|2^k B_j|^{1/p}} \|T([I - e^{-r_B^2 L}]^M (\alpha\chi_{U_j(B)}))\|_{L^p(U_k(B_j))}\right). \tag{3.4.7}$$

At this stage, it follows from Lemma 3.1.3, the condition (3.4.1) and the $L^p(\mathbb{R}^n)$-boundedness of T that for $k = 0,\ 1,\ 2$,

$$\|T([I - e^{-r_B^2 L}]^M (\alpha\chi_{U_j(B)}))\|_{L^p(U_k(B_j))} \lesssim \|\alpha\|_{L^p(U_j(B))},$$

and that for $k \ge 3$,

$$\|T([I - e^{-r_B^2 L}]^M (\alpha\chi_{U_j(B)}))\|_{L^p(U_k(B_j))} \lesssim \left(\frac{1}{2^{k+j}}\right)^{2M} \|\alpha\|_{L^p(U_j(B))}^2.$$

Plugging these properties into the right hand quantity of (3.4.7), together with Definition 3.3.3, implies that

$$\begin{aligned}
\mathrm{P}_j &\lesssim |B_j| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda| 2^{-j\epsilon}}{|B_j| \sup\limits_{x\in B_j} \rho(x,|B_j|)}\right) \\
&\quad + \sum_{k=3}^{\infty} |2^k B_j| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda| 2^{-(2M)(j+k)-j\epsilon}}{|2^k B_j|^{1/p} |B_j|^{1-1/p} \sup\limits_{x\in B_j} \rho(x,|B_j|)}\right).
\end{aligned}$$

After some simple calculations, using the uniformly lower type index p_ω of ω, we deduce

$$\mathrm{P}_j \lesssim 2^{-jp_\omega\epsilon}\left\{1+\sum_{k=3}^{\infty} 2^{kn(1-p_\omega/p)} 2^{-2Mp_\omega(j+k)}\right\} |B_j| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda|}{|B_j| \sup\limits_{x\in B_j} \rho(x,|B_j|)}\right).$$

Observe that $2Mp_\omega > n(1-p_\omega/2)$, so it follows from the above estimate that

$$\mathrm{P}_j \lesssim 2^{-jp_\omega\epsilon} |B_j| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda|}{|B_j| \sup\limits_{x\in B_j} \rho(x,|B_j|)}\right). \tag{3.4.8}$$

With the estimates (3.4.8) on P_j in hand, again note that ω is of uniformly lower type p_ω, we obtain

$$\sum_{j=0}^{\infty} \mathrm{P}_j \lesssim \sum_{j=0}^{\infty} 2^{-jp_\omega\epsilon} |B_j| \left\{\frac{|B| \sup\limits_{x\in B} \rho(x,|B|)}{|B_j| \sup\limits_{x\in B_j} \rho(x,|B_j|)}\right\}^{p_\omega} \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda|}{|B| \sup\limits_{x\in B} \rho(x,|B|)}\right).$$

Since ω^{-1} is of uniformly lower type $1/\widetilde{p}_\omega$ (see Lemma 2.5.3), the above inequality implies that

$$\begin{aligned}
\sum_{j=0}^{\infty} \mathrm{P}_j &\lesssim \sum_{j=0}^{\infty} 2^{-jp_\omega\epsilon} |B_j| \left\{\frac{|B|}{|B_j|}\right\}^{p_\omega/\widetilde{p}_\omega} \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda|}{|B| \sup\limits_{x\in B} \rho(x,|B|)}\right) \\
&\lesssim \sum_{j=0}^{\infty} 2^{-jp_\omega\epsilon} 2^{jn(1-p_\omega/\widetilde{p}_\omega)} |B| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda|}{|B| \sup\limits_{x\in B} \rho(x,|B|)}\right).
\end{aligned}$$

Finally, note that $\epsilon > n(1/p_\omega - 1/\widetilde{p}_\omega)$, we then obtain the following desired estimate.

$$\sum_{j=0}^{\infty} \mathrm{P}_j \lesssim |B| \inf_{x\in\mathbb{R}^n} \omega\left(x, \frac{|\lambda|}{|B| \sup\limits_{x\in B} \rho(x,|B|)}\right). \tag{3.4.9}$$

Using the similar arguments above, we also obtain

$$\sum_{j=0}^{\infty} \mathrm{Q}_j \lesssim |B| \inf_{x \in \mathbb{R}^n} \omega\left(x, \frac{|\lambda|}{|B| \sup\limits_{x \in B} \rho(x, |B|)}\right),$$

which, together with (3.4.9) and (3.4.4), yields the condition (3.4.3). That is, T is bounded from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$.

In particular, the operators g_L and $\nabla L^{-1/2}$ were shown in [46, Theorem 3.4] to satisfy the conditions (3.4.1) and (3.4.2). Therefore, g_L and $\nabla L^{-1/2}$ are bounded from $H_{\omega,L}(\mathbb{R}^n)$ to $L(\omega)$, which completes the proof of Theorem 3.4.1.

$\square$

4

Musielak–Orlicz Hardy spaces associated to operators satisfying Davies–Gaffney estimates and bounded holomorphic functional calculus

4.1 Assumptions on the operator L

Let X be a metric space, with a distance d and μ is a nonnegative, Borel, doubling measure on X. Let L be a linear operator of type θ on $L^2(X)$ with $\theta < \pi/2$, hence L generates a holomorphic semigroup e^{zL}, $|\arg(z)| < \pi/2 - \theta$. Throughout this chapter, we always suppose that the space X is of homogeneous type, and that the operator L satisfies the following assumptions.

(i) The operator L has a bounded H_∞-calculus on $L^2(X)$.

(ii) The operator L generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ which satisfies the Davies–Gaffney estimates; that is, there exist positive constants C_1 and C_2 such that for all closed sets E and F in $X, t \in (0, \infty)$ and $f \in L^2(X)$ supported in E,

$$||e^{-tL}f||_{L^2(F)} \leq C_1 \exp\Big\{ -\frac{d(E,F)^2}{C_2 t}\Big\}||f||_{L^2(E)}, \tag{4.1.1}$$

where $d(E, F)$ is the distance between E and F in X.

Remark 4.1.1 We now give a list of examples of differential operators which satisfy assumptions (i) and (ii):

(α) Second-order elliptic divergence form operators defined by (1.0.2), acting on the Euclidean space $\mathbb{R}^n$. Note that these operators in general are neither self-adjoint nor having Gaussian heat kernel bounds. See [10] and Section 2 of [6].

(β) The operators $L = b(x)\Delta$ as described in Chapter 1 of this thesis.

(γ) Schrödinger operators with nonnegative potentials and magnetic Schrödinger operators. These operators are self-adjoint and possess Gaussian upper bounds on heat kernels. See, for example, Section 1 and 3 of [34].

(δ) Laplace–Beltrami operators on all complete Riemannian manifolds. These operators are self-adjoint and satisfy the Davies–Gaffney estimates (but not Gaussian heat kernel bounds) in general setting. See [4], Section 3.1.

Lemma 4.1.2 ([47]) *Assume that the families of operators $\{S_t\}_{t>0}$ and $\{T_t\}_{t>0}$ satisfy Davies–Gaffney estimates (4.1.1). Then there exist two constants $C \geq 0$ and $c > 0$ such that, for every $t > 0$, every closed subsets E and F of X and every function f supported in E, one has*

$$||S_s T_t f||_{L^2(F)} \leq C \exp\Big\{ -\frac{d(E,F)^2}{c\max\{s,t\}}\Big\}||f||_{L^2(E)}.$$

Lemma 4.1.3 *Let L satisfy assumptions (i) and (ii). Then for any fixed $k \in \mathbb{N}$, the following family of operators $\{(tL)^k e^{-tL}\}_{t>0}$ satisfies Davies–Gaffney estimates (4.1.1).*

Proof. The proof of this lemma is similar to one in [45] and hence we omit the details here.

4.2 Musielak–Orlicz Hardy spaces associated to operators

4.2.1 Tent spaces on spaces of homogeneous type

Given $x \in X$ and $\alpha > 0$, the cone of aperture α and vertex x is the set

$$\Gamma^\alpha(x) := \{(y,t) \in X \times (0,\infty) : d(y,x) < \alpha t\}.$$

For any closed subset $F \subset X$, define a saw-tooth region $R^\alpha(F) = \bigcup_{x\in F} \Gamma^\alpha(x)$. For simplicity, we will often write $R(F)$ instead of $R^1(F)$. If O is an open subset of X, and we denote by E^c the complement of a set E, then the tent over O, denoted by $\widehat{O}$, is defined as

$$\widehat{O} := [R(O^c)]^c := \{(x,t) \in X \times (0,\infty) : d(x,O^c) \geq t\}$$

For each measurable function g on $X \times (0,\infty)$ and $x \in X$, define

$$\mathcal{A}(g)(x) := \Big(\int_{\Gamma(x)} |g(y,t)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t}\Big)^{1/2}.$$

When $X = \mathbb{R}^n$ Coifman, Meyer and Stein [24] introduced the tent spaces $T_2^p(\mathbb{R}^{n+1}_+)$ for $p \in (0,\infty)$. The tent spaces $T_2^p(X)$ on spaces of homogeneous type were studied by Russ [69]. The function g is said to belong to the space $T_2^p(X)$ with $p \in (0,\infty)$ if $||g||_{T_2^p(X)} = ||\mathcal{A}(g)||_{L^p} < \infty$. Then, Harbourne, Salinas and Viviani [43] introduced the tent spaces $T_\omega(\mathbb{R}^{n+1}_+)$ associated to ω. Now let ω satisfy Assumption (A) (see page 27). Then we define $T_\omega(X)$ as the space of all measurable functions g on $X \times (0, \infty)$ such that $\mathcal{A}(g) \in L^1(\omega)$, and for any $g \in T_\omega(X)$, one defines

$$||g||_{T_\omega(X)} = ||\mathcal{A}(g)||_{L^1(\omega)}.$$

Definition 4.2.1 *Let ω satisfy Assumption (C) (see page 27 and 28), and let ρ be the function defined by (2.5.3) in Definition 2.5.2. A function a on $X \times (0, \infty)$ is called a $T_\omega(X)$-atom if*

(i) there exists a ball $B \subset X$ such that supp $a \subset \widehat{B}$;

(ii) $||a||_{T_2^2(X)} \leq [V(B)]^{-1/2} \inf_{x \in B} [\rho(x, V(B))]^{-1}$.

Remark 4.2.2 (i) It is not difficult to verify that for a function ω satisfying Assumption (C), there exist positive constants K_1, K_2 such that for any $x \in X$, $K_1 \leq \omega^{-1}(x, 1) \leq K_2$ and hence $\inf_{x \in B} [\rho(x, V(B))]^{-1}$ is strictly positive.
(ii) In addition, for all $T_\omega(X)$-atoms a, we have $||a||_{T_\omega(X)} \lesssim 1$.

For the functions in the space $T_\omega(X)$, we have the following atomic decomposition.

Proposition 4.2.3 *Let ω satisfy Assumption (C). Then for any $f \in T_\omega(X)$, there exist $T_\omega(X)$-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that for almost every $(x, t) \in X \times (0, \infty)$*

$$f(x, t) = \sum_{j=1}^\infty \lambda_j a_j(x, t). \tag{4.2.1}$$

Moreover, there exists a positive constant C such that for all $f \in T_\omega(X)$,

$$\Lambda(\{\lambda_j\}) = \inf \left\{ \lambda > 0 : \sum_{j=1}^\infty V(B_j) \inf_{x \in X} \omega\Big(x, \frac{|\lambda_j|}{\lambda V(B_j) \sup_{y \in B_j} \rho(y, V(B_j))} \leq 1 \Big) \right\} \leq C ||f||_{T_\omega(X)}, \tag{4.2.2}$$

where $\widehat{B_j}$ appears as the support of a_j.

Proof. The proof of Proposition 4.2.3 is similar to those of [24, Theorem 1], [69, Theorem 1.1], [49, Theorem 3.1] and [50, Theorem 3.1] with minor modifications, thus we omit the details.

The following proposition on the convergence of (4.2.1) plays a significant role in the remaining part of this chapter. The proof of it is analogous to that of [49, Proposition 3.1] and we omit the details.

Proposition 4.2.4 *Let ω satisfy Assumption (C). If $f \in T_\omega(X) \cap T_2^2(X)$, then the decomposition (4.2.1) holds in both $T_\omega(X)$ and $T_2^2(X)$.*

4.2.2 Musielak–Orlicz Hardy spaces associated to L

For all functions $f \in L^2(X)$, the Lusin-area function $S_L(f)$ is defined by setting,

$$S_L f(x) := \Big(\int \int_{\Gamma(x)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \Big)^{1/2}, x \in X.$$

The Musielak–Orlicz Hardy space $H_{L,\omega}(X)$ is defined as the completion of

$$\{f \in L^2(X) : ||S_L f||_{L^1(\omega)} < \infty\}$$

with the norm

$$||f||_{H_{L,\omega}(X)} = ||S_L f||_{L^1(\omega)}.$$

Noting that if $\omega(t) = t, t \in (0,\infty)$ then the space $H_{L,\omega}(X)$ turns out to be the space $H^1_L(X)$ in [45]. Furthermore, if $\omega(t) = t^p, t \in (0,\infty)$ and $p \in (0,1]$, the space $H_{L,\omega}(X)$ is just the space $H^p_L(X)$ considered in [30]. We now introduce the notions of (ω, M, ϵ)-molecule as follows.

Let us denote by $D(T)$ the domain of an unbounded operator T and by $T^k = T \dots T$ the $k-$fold composition of T with itself.

Definition 4.2.5 *A function $m \in L^2(X)$ is called an (ω, M, ϵ)-molecule associated to the operator L if there exist a function $b \in D(L^M)$ and a ball B such that*

(i) $m = L^M b$;

(ii) for every $k = 0, 1, \dots, M.$ and $j \in \mathbb{Z}_+$

$$||(r_B^2 L)^k b||_{L^2(S_j(B))} \lesssim r_B^{2M} 2^{-j\epsilon} [V(2^j B)]^{-1/2} \inf_{x\in B} [\rho(x, V(2^j B))]^{-1}.$$

Theorem 4.2.6 *Let L satisfy assumptions (i) and (ii), ω satisfy Asumption (C), $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$ and $0 < \epsilon < 2M - n(\frac{1}{p_\omega} - \frac{1}{2})$. Then for all $f \in H_{L,\omega}(X) \cap L^2(X)$, there exist (ω, ϵ, M)-molecules $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that*

$$f = \sum_{j=1}^\infty \lambda_j a_j$$

in both $H_{L,\omega}(X)$ and $L^2(X)$. Moreover, there exists a positive constant C such that for all $f \in H_{L,\omega}(X) \cap L^2(X)$,

$$\Lambda(\{\lambda_j a_j\}_j) = \inf \Big\{ \lambda > 0 : \sum_{j=1}^\infty V(B_j) \inf_{x\in X} \omega\Big(x, \frac{|\lambda_j|}{\lambda V(B_j) \sup_{y\in B_j} \rho(y, V(B_j))} \le 1\Big) \Big\} \le C ||f||_{H_{L,\omega}(X)}, \tag{4.2.3}$$

where B_j is the ball associated with (ω, ϵ, M)-molecule a_j.

Before giving a proof of Theorem 4.2.6, we consider the following operator

$$\pi_{L,M}(F)(x) = \int_0^\infty (t^2 L e^{-t^2 L})^M (F(\cdot,t))(x) \frac{dt}{t},$$

for all $F \in L^2(X \times (0,\infty))$ with bounded support. The bound

$$||\pi_{L,M}(F)||_{L^2(X)} \leq C||F||_{T_2^2(X)}, \forall M \geq 1 \tag{4.2.4}$$

follows readily by duality and the L^2 quadratic estimate (2.4.1). Moreover, we have the following proposition.

Proposition 4.2.7 *Let a be a $T_\omega(X)$-atom associated to a ball $B \subset X$ and $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$. Then, $\pi_{L,M} a$ is an (ω, ϵ, M)-molecule (up to a harmless constant); moreover, $\pi_{L,M} a \in H_{L,\omega}(X)$.*

Proof. Setting

$$b = \int_0^\infty t^{2M} (e^{-t^2 L})^M a(\cdot, t) \frac{dt}{t}.$$

Since a is a $T_\omega(X)$-atom associated to a ball $B \subset X$, supp $a \subset \widehat{B} = \{(x,t) \in X \times (0,\infty) : d(x, B^c) \geq t\} \subset B \times [0, r_B]$. Thus, the integral $b = \int_0^\infty t^{2M}(e^{-t^2L})^M a(\cdot,t)\frac{dt}{t}$ is well defined and $\pi_{L,M} a = L^M b$.
For any $h \in L^2(S_j(B))$ with norm 1 and $k \in \{0, 1, \ldots, M\}$, one has

$$\begin{aligned}
&\Big|\int_X (r_B^2 L)^k b(x) \overline{h(x)} d\mu(x)\Big| \\
&= \Big| \int_X \int_0^\infty t^{2M} (r_B^2 L)^k (e^{-t^2 L})^M a(x,t) \overline{h(x)} \frac{dt}{t} d\mu(x)\Big| \\
&\leq \Big| \iint_{\widehat{B}} t^{2(M-k)} (r_B^2)^k a(x,t) (e^{-t^2 L^*})^{M-k} \overline{(t^2 L^* e^{-t^2 L^*})^k h(x)} \frac{dt}{t} d\mu(x)\Big| \\
&\leq r_B^{2M} ||a||_{T_2^2(X)} \Big(\iint_{\widehat{B}} |(e^{-t^2 L^*})^{M-k} \overline{(t^2 L^* e^{-t^2 L^*})^k h(x)}|^2 \frac{dt}{t} d\mu(x)\Big)^{1/2} \\
&\leq r_B^{2M} [V(B)]^{-1/2} \inf_{x \in B} [\rho(x, V(B))]^{-1} \Big(\iint_{\widehat{B}} |(e^{-t^2 L^*})^{M-k} \overline{(t^2 L^* e^{-t^2 L^*})^k h(x)}|^2 \frac{dt}{t} d\mu(x)\Big)^{1/2}.
\end{aligned}$$

If $j \geq 3$ then we have

$$\begin{aligned}
&\Big(\iint_{\widehat{B}} |(e^{-t^2 L^*})^{M-k} \overline{(t^2 L^* e^{-t^2 L^*})^k h(x)}|^2 \frac{dt}{t} d\mu(x)\Big)^{1/2} \\
&\leq C \Big(\int_0^{r_B} e^{\frac{-d(B, S_j(B))^2}{ct^2}} \frac{dt}{t}\Big)^{1/2} \\
&\leq C \Big(\int_0^{r_B} \Big(\frac{t}{2^j r_B}\Big)^{4M} \frac{dt}{t}\Big)^{1/2} \\
&\leq C 2^{-2Mj}.
\end{aligned}$$

If $j = 0, 1, 2$, it is simple to see that

$$\Big(\int\int_{\widehat{B}} |(e^{-t^2L^*})^{M-k}\overline{(t^2L^*e^{-t^2L^*})^k h(x)}|^2 \frac{dt}{t} d\mu(x)\Big)^{1/2} \leq C2^{-2Mj}.$$

All in all, one has

$$\Big|\int_X (r_B^2 L)^k b(x)\overline{h(x)} d\mu(x)\Big| \leq C r_B^{2M} 2^{-2Mj}[V(B)]^{-1/2} \inf_{x\in B}[\rho(x, V(B))]^{-1}$$

which implies

$$||(r_B^2 L)^k b||_{L^2(S_j(B))} \leq C r_B^{2M} 2^{-2Mj}[V(B)]^{-1/2} \inf_{x\in B}[\rho(x, V(B))]^{-1}.$$

Since ω is of uniformly lower type p_ω, ρ is of uniformly upper type $\frac{1}{p_\omega} - 1$ by Lemma 2.5.3. Then we have

$$\begin{aligned}
||(r_B^2 L)^k b||_{L^2(S_j(B))} &\leq C r_B^{2M} 2^{-2Mj}[V(B)]^{-1/2} \inf_{x\in B}[\rho(x, V(B))]^{-1} \\
&\leq C r_B^{2M} 2^{-2Mj} 2^{\frac{n}{2}j}\Big(\frac{V(2^jB)}{V(B)}\Big)^{1/p_\omega - 1}[V(2^jB)]^{-1/2} \inf_{x\in B}[\rho(x, V(2^jB))]^{-1} \\
&\leq C r_B^{2M} 2^{(-2M+\frac{n}{2}+\frac{n}{p_\omega}-n+\epsilon)j} 2^{-j\epsilon}[V(2^jB)]^{-1/2} \inf_{x\in B}[\rho(x, V(2^jB))]^{-1} \\
&\leq C r_B^{2M} 2^{(-2M-\frac{n}{2}+\frac{n}{p_\omega}+\epsilon)j} 2^{-j\epsilon}[V(2^jB)]^{-1/2} \inf_{x\in B}[\rho(x, V(2^jB))]^{-1}
\end{aligned}$$

Due to $(-2M - \frac{n}{2} + \frac{n}{p_\omega} + \epsilon) < 0$, we obtain that

$$||(r_B^2 L)^k b||_{L^2(S_j(B))} \leq C r_B^{2M} 2^{-j\epsilon}[V(2^jB)]^{-1/2} \inf_{x\in B}[\rho(x, V(2^jB))]^{-1}.$$

Therefore, $\pi_{L,M}a$ is an (ω, ϵ, M)-molecule.
It remains to show that $\alpha = \pi_{L,M}a \in H_{L,\omega}(X)$. Write

$$\int_X \omega(x, S_L(\lambda\alpha)(x))d\mu(x) = \sum_{j=0}^{\infty}\int_X \omega(x, S_L(\lambda\alpha\chi_{S_j(B)})(x))d\mu(x) = \sum_{j=0}^{\infty} A_j$$

for all $j \in \mathbb{N}$.
By Assumption (C) and the Hölder inequality, for each $j \in \mathbb{N}$, one obtains

$$\begin{aligned}
A_j &\leq \sum_{k=0}^{\infty}\int_{S_k(2^jB)} \omega(x, S_L(\lambda\alpha\chi_{S_j(B)})(x))d\mu(x) \\
&\leq \sum_{k=0}^{\infty} V(2^{k+j}B)\inf_{x\in X}\omega\Big(x, \frac{|\lambda|\int_{S_k(2^jB)}|S_L(\alpha\chi_{S_j(B)})(y)|d\mu(y)}{V(2^{k+j}B)}\Big) \\
&\leq \sum_{k=0}^{\infty} V(2^{k+j}B)\inf_{x\in X}\omega\Big(x, \frac{|\lambda| ||S_L(\alpha\chi_{S_j(B)})||_{L^2(S_k(2^jB))}}{V(2^{k+j}B)^{1/2}}\Big).
\end{aligned}$$

For $k = 0, 1, 2$, one has

$$\begin{aligned}||S_L(\alpha\chi_{S_j(B)})||_{L^2(S_k(2^jB))} &\le C||\alpha||_{L^2(S_j(B))} \\ &\le C2^{-j\epsilon}[V(2^jB)]^{-1/2}\inf_{x\in B}[\rho(x, V(2^jB))]^{-1}.\end{aligned}$$

For $k \ge 3$, write

$$\begin{aligned}||S_L(\alpha\chi_{S_j(B)})||^2_{L^2(S_k(2^jB))} &= \int\limits_{S_k(2^jB)} \Big(\int\limits_0^{\frac{d(x,x_B)}{4}} + \int\limits_{\frac{d(x,x_B)}{4}}^{\infty} \Big) \int\limits_{d(x,y)<t} |t^2Le^{-t^2L}\alpha|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} d\mu(x) \\ &= I_j + II_j.\end{aligned}$$

To estimate I_j, we set $U_{kj}(B) := \{y \in X : d(x, y) \le \frac{d(x,x_B)}{4}$ for certain $x \in S_k(2^jB)\}$. Then, for each $z \in S_j(B)$ and $y \in U_{kj}(B)$, we have $d(y, z) \ge 2^{k+j-2}r_B$. It follows from the fact that

$$\int\limits_{d(x,y)<t} V(x, t)^{-1}d\mu(x) \le C$$

and $\alpha = L^Mb$ that

$$\begin{aligned}I_j &\le C \int\limits_0^{2^{k+j+1}r_B} \int\limits_{S_j(B)} |(t^2L)^{M+1}e^{-t^2L}b\chi_{S_j(B)}(y)|^2d\mu(y)\frac{dt}{t^{4M+1}} \\ &\le C||b||^2_{L^2(S_j(B))} \int\limits_0^{2^{k+j+1}r_B} \exp\Big(- \frac{d(U_{kj}(B), S_j(B))^2}{ct^2}\Big)\frac{dt}{t^{4M+1}} \\ &\le C||b||^2_{L^2(S_j(B))} \int\limits_0^{2^{k+j+1}r_B} \exp\Big(- \frac{(2^{k+j-2}r_B)^2}{ct^2}\Big)\frac{dt}{t^{4M+1}} \\ &\le C||b||^2_{L^2(S_j(B))} \int\limits_0^{2^{k+j+1}r_B} \Big(\frac{t}{2^{k+j}r_B}\Big)^{4M+4}\frac{dt}{t^{4M+1}} \\ &\le C||b||^2_{L^2(S_j(B))}\frac{2^{-4(k+j)M}}{(r_B)^{4M}} \le C2^{-4(k+j)M}2^{-2\epsilon j}[V(2^jB)]^{-1}\inf_{x\in B}[\rho(x, V(2^jB))]^{-2}.\end{aligned}$$

Finally, for the term II_j we obtain

$$\begin{aligned}II_j &\le C \int\limits_{2^{k+j-1}r_B}^{\infty} \int\limits_{S_j(B)} |(t^2L)^{M+1}e^{-t^2L}b\chi_{S_j(B)}(y)|^2d\mu(y)\frac{dt}{t^{4M+1}} \\ &\le C||b||^2_{L^2(S_j(B))} \int\limits_{2^{k+j-1}r_B}^{\infty} \frac{dt}{t^{4M+1}} \\ &\le C2^{-4(k+j)M}2^{-2\epsilon j}[V(2^jB)]^{-1}\inf_{x\in B}[\rho(x, V(2^jB))]^{-2}.\end{aligned}$$

It therefore, from the estimates for I_j, II_j above, the uniformly lower type p_ω of ω together with the fact that $-2Mp_\omega + n(1 - p_\omega/2) < 0$, implies that

$$\int\limits_{S_j(B)} \omega(x, S_L(\alpha)(x))d\mu(x)$$

$$\leq C \sum_{k=0}^{\infty} 2^{(-2(k+j)M-j\epsilon)p_\omega} V(2^{k+j}B) \inf_{x\in X} \omega\Big(x, \frac{|\lambda|}{[V(2^{k+j}B)]^{1/2}[V(2^jB)]^{1/2} \sup\limits_{y\in B}\rho(y, V(2^jB))}\Big)$$

$$\leq C \sum_{k=0}^{\infty} 2^{(-2(k+j)M-j\epsilon)} 2^{kn(1-p_\omega/2)p_\omega} V(2^jB) \inf_{x\in X} \omega\Big(x, \frac{|\lambda|}{[V(2^jB)] \sup\limits_{y\in B}\rho(y, V(2^jB))}\Big).$$

Note that we can choose $\widetilde{p}_\omega$ as in Convention (B) such that $n\Big(\frac{1}{p_\omega} - \frac{1}{\widetilde{p}_\omega}\Big) < \epsilon$. It therefore, together with the uniformly lower type $1/\widetilde{p}_\omega - 1$ of ρ by Lemma 2.5.3, yields

$$\sum_{j=0}^{\infty} \int\limits_{S_j(B)} \omega(x, S_L(\lambda\alpha)(x))d\mu(x)$$

$$\leq C \sum_{j=0}^{\infty} 2^{-\epsilon p_\omega j} (V(2^jB)\Big(\frac{V(B)}{V(2^jB)}\Big)^{p_\omega/\widetilde{p}_\omega} \inf_{x\in X} \omega\Big(x, \frac{|\lambda|}{V(B) \sup\limits_{y\in B}\rho(y, V(B))}\Big)$$

$$\leq C \sum_{j=0}^{\infty} 2^{-\epsilon p_\omega j} 2^{(1-p_\omega/\widetilde{p}_\omega)nj} V(B) \inf_{x\in X} \omega\Big(x, \frac{|\lambda|}{V(B)) \sup\limits_{y\in B}\rho(y, V(B))}\Big)$$

$$\leq CV(B) \inf_{x\in X} \omega\Big(x, \frac{|\lambda|}{V(B)) \sup\limits_{y\in B}\rho(y, V(B))}\Big),$$

which completes the proof of Proposition 4.2.7.

□

Proof of Theorem 4.2.6. Since $f \in L^2(X)$ and T has a bounded holomorphic functional calculus on $L^2(X)$, there exists a constant $C_{M,L}$ such that

$$f = C_{M,L} \int\limits_0^\infty (t^2 L e^{-t^2 L})^{M+1} f \frac{dt}{t}.$$

By definition of $H_{L,\omega}(X)$ and the quadratic estimate (2.4.1), $t^2 L e^{-t^2 L} f \in T_\omega(X) \cap T_2^2(X)$. Thanks to Proposition 4.2.4 and Proposition 4.2.7, we easily deduce

$$f = C_{M,L}\pi_{L,M}(t^2 L e^{-t^2 L} f) = C_{M,L} \sum_{j=0}^{\infty} \lambda_j \pi_{L,M}(a_j)$$

in both $L^2(X)$ and $H_{L,\omega}(X)$ and $\Lambda(\{\lambda_j a_j\}_j) \leq C||f||_{H_{L,\omega}(X)}$, which completes the proof of Theorem 4.2.6.

$\square$

By the density of $H_{L,\omega}(X)\cap L^2(X)$ in $H_{L,\omega}(X)$, we conclude the following corollaries.

Corollary 4.2.8 *Let the operator L satisfy Assumptions (i) and (ii), ω satisfy Assumption (C) and $M > \frac{n}{2}(1/p_\omega - 1/2)$. Then for all $f \in H_{L,\omega}(X)$, there exist a sequence of (ω, ϵ, M)-molecules $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \in \mathbb{C}$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $H_{L,\omega}(X)$. Moreover, there exists a positive constant C independent of f such that*

$$\Lambda(\{\lambda_j a_j\}_j) \leq C||f||_{H_{L,\omega}(X)},$$

where $\Lambda(\{\lambda_j a_j\}_j)$ was defined by (4.2.3).

Corollary 4.2.9 *Let the operator L satisfy Assumptions (i) and (ii), ω satisfy Assumption (C) and $M > \frac{n}{2}(1/p_\omega - 1/2)$. Then for all $0 < \epsilon < 2M - n(1/p_\omega - 1/2)$, the spaces $H^{mol,\epsilon,M}_{\omega,fin}$ are dense in $H_{L,\omega}(X)$ where $H^{mol,\epsilon,M}_{\omega,fin}$ denote the spaces of finite linear combinations of (ω, ϵ, M).*

4.2.3 Dual spaces of Musielak–Orlicz Hardy spaces

In this subsection, we study the dual space of the Musielak–Orlicz Hardy spaces $H_{L,\omega}(X)$. Let $\phi = L^M v$ be a function in $L^2(X)$, where $v \in D(L^M)$. Following [45, 46] for $\epsilon > 0$, $M \in \mathbb{N}$ and fixed $x_0 \in X$ we introduce the norm

$$||\phi||_{\mathcal{M}^{M,\epsilon}_\omega(L)} = \sup_{j\in\mathbb{Z}_+}\{2^{j\epsilon}[V(x_0, 2^j)]^{1/2} \sup_{x\in B(x_0,2^j)} \rho(x, V(x_0, 2^j)) \sum_{k=0}^{M} ||L^k v||_{L^2(S_j(B(x_0,1)))}\}$$

where

$$\mathcal{M}^{M,\epsilon}_\omega(L) := \{\phi = L^M v \in L^2(X) : ||\phi||_{\mathcal{M}^{M,\epsilon}_\omega(L)} < \infty\}.$$

Let $(\mathcal{M}^{M,\epsilon}_\omega(L))^*$ be the dual of $\mathcal{M}^{M,\epsilon}_\omega(L)$ and denote either $(I + t^2L)^{-1}$ or e^{-t^2L} by A_t. Then for any $f \in (\mathcal{M}^{M,\epsilon}_\omega(L))^*$, $(I - A_t^*)^M f$ belongs to $L^2_{loc}(X)$ in the sense of distributions, see [45, 46].
For any $M \in \mathbb{N}$, one defines

$$\mathcal{M}^M_{\omega,L^*}(X) := \bigcap_{\epsilon>n(1/p_\omega-1/p_\omega^+)} (\mathcal{M}^{M,\epsilon}_\omega(L))^*.$$

Definition 4.2.10 *Let the operator L satisfy Assumptions (i) and (ii), ω satisfy Assumption (C) and $M > \frac{n}{2}(1/p_\omega - 1/2)$. A functional $f \in \mathcal{M}^M_{\omega,L^*}(X)$ is said to be in $BMO^M_{\rho,L}(X)$ if*

$$||f||_{BMO^M_{\rho,L}(X)} = \sup_{B\subset X} \frac{1}{\sup_{x\in B}\rho(x, V(B))}\left[\frac{1}{V(B)}\int_B |(I - e^{-r_B^2 L})^M f(x)|^2 d\mu(x)\right]^{1/2} < \infty,$$

where the supremum is taken over all balls B in X.

We have the following characterizations of the spaces $BMO^M_{\rho,L}(X)$.

Proposition 4.2.11 *Let the operator L satisfy Assumptions (i) and (ii), ω satisfy Assumption (C) and $M > \frac{n}{2}(1/p_\omega - 1/2)$. Then $f \in BMO^M_{\rho,L}(X)$ if and only if $f \in \mathcal{M}^M_{\omega,L^*}(X)$ and*

$$\sup_{B\subset X} \frac{1}{\sup\limits_{x\in B}\rho(x, V(B))}\Big[\frac{1}{V(B)}\int_B |(I-(I+r_B^2L)^{-1})^M f(x)|^2 d\mu(x)\Big]^{1/2} < \infty.$$

Moreover,

$$||f||_{BMO^M_{\rho,L}(X)} \approx \sup_{B\subset X} \frac{1}{\sup\limits_{x\in B}\rho(x, V(B))}\Big[\frac{1}{V(B)}\int_B |(I-(I+r_B^2L)^{-1})^M f(x)|^2 d\mu(x)\Big]^{1/2}.$$

Proposition 4.2.12 *Let the operator L satisfy Assumptions (i) and (ii), ω satisfy Assumption (C) and $\epsilon > n(1/p_\omega - 1/p_\omega^+)$. Then there exists a positive constant C such that for all $f \in BMO^M_{\rho,L}(X)$,*

$$\sup_{B\subset X} \frac{1}{\sup\limits_{x\in B}\rho(x, V(B))}\Big[\frac{1}{V(B)}\int_{\widehat{B}} |(t^2L)^M e^{-t^2L} f(x)|^2 \frac{d\mu(x)dt}{t}\Big]^{1/2} \le C||f||_{BMO^M_{\rho,L}(X)}.$$

The proofs of two above propositions are similar to Lemmas 8.1 and 8.3 in [46] and hence we omit the details.

We are now in position to obtain the main result in this subsection.

Theorem 4.2.13 *Let the operator L satisfy Assumptions (i) and (ii), ω satisfy Assumption (C). Then $(H_{L,\omega}(X))^*$, the dual space of $H_{L,\omega}(X)$, coincides with $\mathrm{BMO}^M_{\rho,L^*}(X)$ in the following sense:*

(i) For any functional $f \in BMO^M_{\rho,L^*}(X)$ and $M > \max\left\{\frac{n}{2}(1/p_\omega - 1) + 1, \frac{n}{4}\right\}$, the linear functional given by

$$\ell(g) := < f, g >,$$

which is initially defined on $H^{mol,\epsilon,2\widetilde{M}}_{\omega,fin}$ with $\widetilde{M} > M + \frac{N}{2}(1/p_\omega - 1/2)$ and $\widetilde{M} - \frac{n}{2}(1/p_\omega - 1) > \epsilon > \frac{N}{2}(1/p_\omega - 1/2)$ (N is a constant appearing in (2.1.3)), has a unique extension to $H_{L,\omega}(X)$ with

$$||\ell||_{(H_{L,\omega}(X))^*} \le C||f||_{BMO^M_{\rho,L^*}(X)},$$

where C is a positive constant independent of f.

(ii) Conversely, for any $\ell \in (H_{L,\omega}(X))^*$ and $M > \frac{n}{2}(1/p_\omega - 1/2)$ there exists a function $f \in BMO^M_{\rho,L^*}(X)$ such that

$$\ell(g) = \langle f, g\rangle$$

for all $g \in H^{mol,M,\epsilon}_{\omega,fin}$ and $||f||_{BMO^M_{\rho,L^*(X)}} \le C||\ell||_{(H_{L,\omega}(X))^*}$, where C is a positive constant independent of ℓ.

Before coming to the proof of Theorem 4.2.13, we need the following results.

Lemma 4.2.14 *There exists a collection of open sets* $\{Q_\alpha^k \subset X : k \in \mathbb{Z}, \alpha \in I_k\}$, *where* I_k *denotes certain (possibly finite) index set depending on* k, *and constants* $\delta \in (0,1), a_0 \in (0,1)$ *and* $C_1 \in (0,\infty)$ *such that*

(i) $\mu(X \setminus \cup_\alpha Q_\alpha^k) = 0$ *for all* $k \in \mathbb{Z}$;

(ii) *if* $i \geq k$, *then either* $Q_\alpha^i \subset Q_\beta^k$ *or* $Q_\alpha^i \cap Q_\beta^k = \emptyset$;

(iii) *for* (k,α) *and each* $i < k$, *there exists a unique* β *such* $Q_\alpha^k \subset Q_\beta^i$;

(iv) *the diameter of* $Q_\alpha^k \leq C_1 \delta^k$;

(v) *each* Q_α^k *contains certain ball* $B(z_\alpha^k, a_0\delta^k)$.

Proof. The proof of this lemma can be found in [20].

Theorem 4.2.15 *Let* $M > \max\left\{\frac{n}{2}(1/p_\omega - 1/2) + 1, \frac{n}{4}\right\}$ *and* $0 < \epsilon < 2M - n(1/p_\omega - 1/2)$. *Suppose that* $f = \sum_{i=1}^{l} \lambda_i a_i$ *where* $\{a_i\}_{i=1}^l$ *is a family of* $(\omega, \epsilon, 2M)$*-molecules and* $\sum_{i=1}^{l} |\lambda_i| < \infty$. *Then there exists a representation* $f = \sum_{i=1}^{K} \mu_i m_i$, *where the* m_i'*s are* (ω, ϵ, M)*-molecules and*

$$\sum_{i=1}^{K} |\mu_i| \leq C||f||_{H_{L,\omega}(X)},$$

with $C = C(\epsilon, M)$.

Proof. Indeed, we can adapt the ideas in the proof of Theorem 5.3 in [45] with minor modifications to obtain the proof of Theorem 4.2.15. Instead of dealing with atoms as in [45], we work on molecules by decomposing the underline space X into annuli according to the balls associated with the molecules. We therefore omit the details.

$\square$

Proof of Theorem 4.2.13. Let m be an $(\omega, \epsilon, \widetilde{M})$-molecule associated with a ball $B \subset X$. Then there exists a function b such that the conditions (i) and (ii) in Definition 4.2.5 hold. Then we have,

$$\begin{aligned} r_B^{2\widetilde{M}} m &= (r_B^2 L)^{\widetilde{M}} b = (I - (I + r_B^2 L)^{-1})^M (I + r_B^2 L)^M (r_B^2 L)^{\widetilde{M} - M} b \\ &= \sum_{k=0}^{M} C_M^k (I - (I + r_B^2 L)^{-1})^M (r_B^2 L)^{\widetilde{M} - k} b. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&|\langle f,m\rangle| = r_B^{-2\widetilde{M}}|\langle f,(r_B^2L)^{\widetilde{M}}b\rangle| \\
&\le Cr_B^{-2\widetilde{M}}\sum_{k=0}^{M}\Big|\int_X (I-(I+r_B^2L^*)^{-1})^M f(x)\overline{(r_B^2L)^{\widetilde{M}-k}b(x)}d\mu(x)\Big| \\
&\le Cr_B^{-2\widetilde{M}}\sum_{k=0}^{M}\sum_{j=0}^{\infty}\Big(\int_{S_j(B)}\Big|(I-(I+r_B^2L^*)^{-1})^M f(x)\Big|^2 d\mu(x)\Big)^{1/2}\times ||(r_B^2L)^{\widetilde{M}-k}b||_{L^2(S_j(B))} \\
&\le C\sum_{j=0}^{\infty}2^{-\epsilon j}[V(2^jB)]^{-1/2}\sup_{x\in B}[\rho(x,V(2^jB))]^{-1}\Big(\int_{S_j(B)}\Big|(I-(I+r_B^2L^*)^{-1})^M f(x)\Big|^2 d\mu(x)\Big)^{1/2}.
\end{aligned}
\tag{4.2.5}
$$

With notations as in Lemma 4.2.14 we choose an integer k_j, for each $j\in\mathbb{Z}$, such that $C_1\delta^{k_j}\le 2^jr_B < C_1\delta^{k_j-1}$. Set

$$M_j = \{\beta\in I_{k_0} : Q_\beta^{k_0}\cap B(x_B, C_1\delta^{k_j-1})\neq\emptyset\}.$$

Then, for each $j\in\mathbb{Z}$,

$$S_j(B)\subset B(x_B,C_1\delta^{k_j-1})\subset \cup_{\beta\in M_j}Q_\beta^{k_0}\subset B(x_B,2C_1\delta^{k_j-1}).$$

From (ii) in Lemma 4.2.14 we can assume that the sets $Q_\beta^{k_0}$ for all $\beta\in M_j$ are pairwise disjoint. Furthermore, it follows from (iv) and (v) that there exists $z_\beta^{k_0}\in Q_\beta^{k_0}$ such that

$$B(z_\beta^{k_0},a_0\delta^{k_0})\subset Q_\beta^{k_0}\subset B(z_\beta^{k_0},C_1\delta^{k_0})\subset B(z_\beta^{k_0},r_B)\subset B(z_\beta^{k_0},C_1\delta^{k_0-1}). \tag{4.2.6}$$

Therefore, from (2.1.3), Proposition 4.2.11 together with the fact that ρ is of type $(\frac{1}{p_\omega^+}-1,\frac{1}{p_\omega^-}-1)$ and $p_\omega\le p_\omega^-\le p_\omega^+$, we have

$$
\begin{aligned}
&\Big(\int_{S_j(B)}\Big|(I-(I+r_B^2L^*)^{-1})^M f(x)\Big|^2 d\mu(x)\Big)^{1/2} \\
&\qquad\le \Big(\sum_{\beta\in M_j}\int_{B(z_\beta^{k_0},r_B)}\Big|(I-(I+r_B^2L^*)^{-1})^M f(x)\Big|^2 d\mu(x)\Big)^{1/2} \\
&\qquad\le C||f||_{BMO^M_{\rho,L^*}(X)}\Big(\sum_{\beta\in M_j}V(B(z_\beta^{k_0},r_B))\sup_{x\in B(z_\beta^{k_0},r_B)}\rho(x,V(B(z_\beta^{k_0},r_B)))^2\Big)^{1/2} \\
&\qquad\le C2^{jN(1/p_\omega-1)}||f||_{BMO^M_{\rho,L^*}(X)}V(B(x_B,2C_1\delta^{k_j-1}))^{1/2}\sup_{x\in B}\rho(x,V(B(x_B,r_B))) \\
&\qquad\le C2^{jN(1/p_\omega-1)}||f||_{BMO^M_{\rho,L^*}(X)}V(2^jB)^{1/2}\sup_{x\in B}\rho(x,V(B)).
\end{aligned}
\tag{4.2.7}
$$

Combination of two estimates (4.2.5) and (4.2.7) gives

$$\begin{aligned}
|\langle f, m\rangle| &\leq C\sum_{j=0}^{\infty} 2^{-\epsilon j}2^{jN(1/p_\omega-1)}\Big(\frac{V(B)}{V(2^jB)}\Big)^{1/p_\omega-1}||f||_{BMO^M_{\rho,L^*}(X)} \\
&\leq C\sum_{j=0}^{\infty} 2^{j(-\epsilon+N(1/p_\omega-1))}||f||_{BMO^M_{\rho,L^*}(X)} \\
&\leq C||f||_{BMO^M_{\rho,L^*}(X)}.
\end{aligned} \tag{4.2.8}$$

Now for any $g \in H^{mol,\epsilon,2\widetilde{M}}_{\omega,fin}$ then by Theorem 4.2.15, there exists a representation $g = \sum_{i=1}^{K} \mu_i m_i$, where the $m_i's$ are $(\omega, \epsilon, \widetilde{M})$-molecules and

$$\sum_{i=1}^{K} |\mu_i| \leq C||g||_{H_{L,\omega}(X)}. \tag{4.2.9}$$

As a result, it follows from (4.2.8) and (4.2.9) that

$$\begin{aligned}
\langle f, g\rangle &= \sum_{i=1}^{K} |\mu_i|\langle f, m_i\rangle \\
&\leq C\sum_{i=1}^{K} |\mu_i|||f||_{BMO^M_{\rho,L^*}(X)} \\
&\leq C||g||_{H_{L,\omega}(X)}||f||_{BMO^M_{\rho,L^*}(X)}.
\end{aligned}$$

The proof of part (i) of Theorem 4.2.13 is complete.

Conversely, we will adapt the ideas in [46] to give the proof of part (ii) of Theorem 4.2.13. Observe that for each (ω, ϵ, M)-molecule m,

$$|\ell(m)| \leq C||\ell||_{(H_{L,\omega}(X))^*}.$$

Since each element in $\mathcal{M}^{M,\epsilon}_\omega(L)$ is also an (ω, ϵ, M)-molecule associated to the ball $B(x_0, 1)$ which implies that ℓ defines a linear function on $\mathcal{M}^{M,\epsilon}_\omega(L)$ for every $\epsilon > 0, M > \frac{n}{2}(1/p_\omega - 1/2)$. Therefore, $(I - (I + t^2L^*))^M\ell$ is well defined and belongs to L^2_{loc} for all $t > 0$. Fix a ball B and let $\phi \in L^2(B)$ such that $||\phi||_{L^2(B)} \leq 1$. Then one can check that

$$\widetilde{m} = (I - (I + r_B^2L)^{-1})^M\phi$$

is an (ω, ϵ, M)-molecule for every $\epsilon > 0$ and hence $||\widetilde{m}||_{H_{L,\omega}(X)} \leq C$. Consequently, we have

$$\begin{aligned}
| < (I - (I + r_B^2L^*)^{-1})^M\ell, \phi > | &= | < \ell, (I - (I + r_B^2L)^{-1})^M\phi > | \\
&= | < \ell, \widetilde{m} > | \leq C||\ell||_{(H_{L,\omega}(X))^*},
\end{aligned}$$

which further implies that

$$\frac{1}{\sup\limits_{x\in B}\rho(x, V(B))}\Big(\frac{1}{V(B)}\int_B |(I - (I + r_B^2L^*)^{-1})^M\ell(x)|^2d\mu(x)\Big)^{1/2} \leq C||\ell||_{(H_{L,\omega}(X))^*}.$$

for all balls B. Thus, $\ell \in BMO^M_{\rho,L^*}(X)$ and $\|\ell\|_{BMO^M_{\rho,L^*}(X)} \leq C\|\ell\|_{(H_{L,\omega}(X))^*}$, which completes the proof of part (ii).

□

4.3 Riesz transform and holomorphic functional calculus

4.3.1 Holomorphic functional calculus

Lemma 4.3.1 *Let the operator L satisfy Assumptions (i) and (ii), ω satisfy Assumption (C) and $M > \frac{n}{2}(\frac{1}{p_\omega} - \frac{1}{2})$. Suppose that T is linear (resp. nonnegative sublinear) operator which maps $L^2(X)$ continuous into weak-$L^2(X)$. If there exists a positive constant C such that for any (ω, ϵ, M)-molecule a,*

$$\int_X \omega(x, T(\lambda a)(x))d\mu(x) \leq CV(B)\inf_{x\in X}\omega\Big(x, \frac{|\lambda|}{V(B)\sup_{y\in B}\rho(y, V(B))}\Big), \tag{4.3.1}$$

then T extends to a bounded linear (resp. sublinear) operator from $H_{L,\omega(X)}$ to $L^1(\omega)$; moreover, there exists a positive constant C' such that

$$||Tf||_{L^1(\omega)} \leq C'||f||_{H_{L,\omega}(X)},$$

for all $f \in H_{L,\omega}(X)$.

Proof. It follows from Theorem 4.2.6 that for every $f \in H_{L,\omega}(X) \cap L^2(X)$, there exist a sequence of (ω, ϵ, M)-molecules $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \in \mathbb{C}$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in both $H_{L,\omega}(X)$ and $L^2(X)$. Moreover, there exists a positive constant C such that

$$\Lambda(\{\lambda_j a_j\}_j) \leq C||f||_{H_{L,\omega}(X)}.$$

Thus, if T is linear, then it follows from the fact that T is of week type (2,2) that $T(f) = \sum_{j=1}^\infty T(\lambda_j a_j)$ almost everywhere. If T is a non-negative sublinear operator, then

$$\sup_{t>0} t^{1/2}\mu\left\{x \in X : \left|T(f)(x) - T\left(\sum_{j=1}^N \lambda_j a_j\right)(x)\right| > t\right\} \leq C\left\|f - \sum_{j=1}^N \lambda_j a_j\right\|_{L^2(X)}.$$

Note that $\left\|f - \sum_{j=1}^N \lambda_j a_j\right\|_{L^2(X)} \to 0$ as $N \to \infty$. Thus, there exists a subsequence $\{N_k\}_k \subset \mathbb{N}$ such that

$$T\left(\sum_{j=1}^{N_k} \lambda_j a_j\right) \to T(f)$$

almost everywhere, as $k \to \infty$. Observe that

$$T(f) - \sum_{j=1}^{\infty} T(\lambda_j a_j) = T(f) - T\left(\sum_{j=1}^{N_k} \lambda_j a_j\right) + T\left(\sum_{j=1}^{N_k} \lambda_j a_j\right) - \sum_{j=1}^{\infty} T(\lambda_j a_j).$$

Hence, it follows from the non-negativity and the sublinearity of T that

$$T(f) - \sum_{j=1}^{\infty} T(\lambda_j a_j) \leq T(f) - T\left(\sum_{j=1}^{N_k} \lambda_j a_j\right).$$

By letting $k \to \infty$, we see that $T(f) \leq \sum_{j=1}^{\infty} T(\lambda_j a_j)$ almost everywhere. Thus, by the subadditivity and the continuity of ω, we deduce that

$$\int_X \omega(x, T(f)(x)) d\mu(x) \leq C \sum_{j=1}^{\infty} \int_X \omega(x, T(\lambda_j a_j)(x)) d\mu(x). \tag{4.3.2}$$

Finally, it follows from (4.3.1) and (4.3.2) that

$$\|T(f)\|_{L^1(\omega)} \leq C\Lambda\left(\{\lambda_j a_j\}_j\right) \leq C\|f\|_{H_{L,\omega}(X)},$$

which, combined with the density of $H_{L,\omega}(X) \cap L^2(X)$ in $H_{L,\omega}(X)$, completes the proof of the above lemma.

□

Theorem 4.3.2 *Let L be of type θ on $L^2(X)$ with $0 \leq \theta < \pi/2$ and satisfy (i) and (ii), ω satisfy (C) and $\theta < \nu < \pi$. Then, for any $f \in H_\infty(S_\nu^0)$, $f(L)$ is bounded on $H_{L,\omega}(X)$, that is, for any $g \in H_{L,\omega}(X)$*

$$||f(L)g||_{H_{L,\omega}(X)} \leq C||f||_\infty ||g||_{H_{L,\omega}(X)}. \tag{4.3.3}$$

Proof. Choose $M > \frac{n}{2}(1/p_\omega - 1/2)$ and $\widetilde{p}_\omega > p_\omega$ close enough to p_ω (as in Convention (B)) so that there exists ϵ satisfying

$$n\Big(\frac{1}{p_\omega} - \frac{1}{\widetilde{p}_\omega}\Big) < \epsilon < 2M + \frac{n}{2} - \frac{n}{p_\omega}.$$

With any (ω, ϵ, M)-molecule m associated to a ball $B \subset X$, we will claim that

$$\int_X \omega(x, S_L(\lambda f(L)m)(x)) d\mu(x) \leq C||f||_\infty V(B) \inf_{x \in X} \omega\Big(x, \frac{|\lambda|}{V(B) \sup_{y \in B} \rho(y, V(B))}\Big). \tag{4.3.4}$$

Once (4.3.4) is proved, (4.3.3) follows by Lemma 4.3.1.
Let us prove (4.3.4). Write

$$\int_X \omega(x, S_L(\lambda f(L)m)(x)) d\mu(x) \leq \sum_{j=0}^{\infty} \int_X \omega(x, S_L(\lambda f(L)m \cdot \chi_{S_j(B)})(x)) d\mu(x) = \sum_{j=0}^{\infty} A_j$$

for all $j \in \mathbb{N}$.
Since ω satisfies Assumption (C), by the Hölder inequality, for each $j \in \mathbb{N}$, one obtains

$$\begin{aligned}
A_j &\le \sum_{k=0}^{\infty} \int_{S_k(2^jB)} \omega(x, S_L(\lambda f(L)m \cdot \chi_{S_j(B)})(x)) d\mu(x) \\
&\le \sum_{k=0}^{\infty} V(2^{k+j}B) \inf_{x\in X} \omega\Big(x, \frac{|\lambda| \int_{S_k(2^jB)} |S_L(f(L)m \cdot \chi_{S_j(B)})(y)| d\mu(y)}{V(2^{k+j}B)}\Big) \\
&\le \sum_{k=0}^{\infty} V(2^{k+j}B) \inf_{x\in X} \omega\Big(x, \frac{|\lambda| ||S_L(f(L)m \cdot \chi_{S_j(B)})||_{L^2(S_k(2^jB))}}{V(2^{k+j}B)^{1/2}}\Big).
\end{aligned}$$

For $k = 0, 1, 2,$

$$\begin{aligned}
||S_L(f(L)m \cdot \chi_{S_j(B)})||_{L^2(S_k(2^jB))} &\le C||f(L)m \cdot \chi_{S_j(B)}||_{L^2(X)} \\
&\le C||f||_\infty ||m||_{S_j(B)} 2^{-j\epsilon} [V(2^jB)]^{-1/2} \inf_{x\in B} [\rho(x, V(2^jB))]^{-1}.
\end{aligned}$$

For $k \ge 3$, write

$$\begin{aligned}
&||S_L(f(L)m \cdot \chi_{S_j(B)})||^2_{L^2(S_k(2^jB))} \\
&= \int_{S_k(2^jB)} \Big(\int_0^{\frac{d(x,x_B)}{4}} + \int_{\frac{d(x,x_B)}{4}}^{\infty} \Big) \int_{d(x,y)<t} |t^2 L e^{-t^2L} f(L)m \cdot \chi_{S_j(B)}|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} d\mu(x) \\
&= I_j + II_j.
\end{aligned}$$

Let us estimate I_j. It can be verified that there exists a positive constant C such that for all closed sets E and F in $X, t \in (0, \infty)$ and $g \in L^2(X)$ supported in E,

$$||(tL)^{M+1} e^{-tL} f(L)g||_{L^2(F)} \le C\Big(\frac{t}{d(E,F)^2}\Big)^{M+1} ||g||_{L^2(E)}. \tag{4.3.5}$$

Setting $U_{kj}(B) := \{y \in X : d(x,y) \le \frac{d(x,x_B)}{4}$ for certain $x \in S_k(2^jB)\}$, then for each $z \in S_j(B)$ and $y \in U_{kj}(B)$, we have $d(y,z) \ge 2^{k+j-2} r_B$. Combining $\int_{d(x,y)<t} V(x,t)^{-1} d\mu(x) < c$, $m = L^M b$ and (4.3.5), one gets

$$\begin{aligned}
I_j &\le C \int_0^{2^{k+j+1}r_B} \int_{S_j(B)} |(t^2L)^{M+1} e^{-t^2L} f(L)b \cdot \chi_{S_j(B)}(y)|^2 d\mu(y) \frac{dt}{t^{4M+1}} \\
&\le C||f||^2_\infty ||b||^2_{L^2(S_j(B))} \int_0^{2^{k+j+1}r_B} \Big(\frac{ct^2}{d(U_{kj}(B), S_j(B))^2}\Big)^{2M+2} \frac{dt}{t^{4M+1}} \\
&\le C||f||^2_\infty 2^{-4(j+k)M} 2^{-2\epsilon j} [V(2^jB)]^{-1} \inf_{x\in B} [\rho(x, V(2^jB))]^{-2}.
\end{aligned}$$

For the term II_j, we have

$$II_j \leq C \int_{2^{k+j-1}r_B}^{\infty} \int_{S_j(B)} |(t^2L)^{M+1}e^{-t^2L}f(L)b\cdot\chi_{S_j(B)}(y)|^2 d\mu(y)\frac{dt}{t^{4M+1}}$$

$$\leq C||f||_\infty^2||b||^2_{L^2(S_j(B))}\int_{2^{k+j-1}r_B}^{\infty}\frac{dt}{t^{4M+1}}$$

$$\leq C||f||_\infty^2 2^{-4(k+j)M}2^{-2\epsilon j}[V(2^jB)]^{-1}\inf_{x\in B}[\rho(x,V(2^jB))]^{-2}.$$

Further going, from the estimates for I_j, II_j, the strictly lower type p_ω of ω together with the fact that $-2Mp_\omega + n(1-p_\omega/2) < 0$, we obtain

$$\int_{S_j(B)} \omega(x, S_L(f(L)m)(x))d\mu(x)$$

$$\leq C||f||_\infty \sum_{k=0}^{\infty} 2^{(-2(k+j)M-j\epsilon)p_\omega}V(2^{k+j}B)\inf_{x\in X}\omega\Big(x, \frac{|\lambda|}{[V(2^{k+j}B)]^{1/2}[V(2^jB)]^{1/2}\sup_{y\in B}\rho(y,V(2^jB))}\Big)$$

$$\leq C||f||_\infty \sum_{k=0}^{\infty} 2^{(-2(k+j)M-j\epsilon)}2^{kn(1-p_\omega/2)p_\omega}V(2^jB)\inf_{x\in X}\omega\Big(x, \frac{|\lambda|}{[V(2^jB)]\sup_{y\in B}\rho(y,V(2^jB))}\Big)$$

Since ρ is of uniformly lower type $1/\widetilde{p}_\omega - 1$, we further have

$$\sum_{j=0}^{\infty}\int_{S_j(B)} \omega(x, S_L(\lambda f(L)m)(x))d\mu(x)$$

$$\leq C||f||_\infty \sum_{j=0}^{\infty} 2^{-\epsilon p_\omega j}(V(2^jB)\Big(\frac{V(B)}{V(2^jB)}\Big)^{p_\omega/\widetilde{p}_\omega}\inf_{x\in X}\omega\Big(x, \frac{|\lambda|}{V(B)\sup_{y\in B}\rho(y,V(B))}\Big)$$

$$\leq C||f||_\infty \sum_{j=0}^{\infty} 2^{-\epsilon p_\omega j}2^{1-p_\omega/\widetilde{p}_\omega)nj}V(B)\inf_{x\in X}\omega\Big(x, \frac{|\lambda|}{V(B)\sup_{y\in B}\rho(y,V(B))}\Big).$$

Noting that since $n\Big(\frac{1}{p_\omega} - \frac{1}{\widetilde{p}_\omega}\Big) < \epsilon$ and $M > \frac{n}{2}\Big(\frac{1}{p_\omega} - \frac{1}{2}\Big)$, we learn that

$$\sum_{j=0}^{\infty}\int_{S_j(B)} \omega(x, S_L(\lambda f(L)m)(x))d\mu(x) \leq C||f||_\infty V(B)\inf_{x\in X}\omega\Big(x, \frac{|\lambda|}{V(B)\sup_{y\in B}\rho(y,V(B))}\Big).$$

□

4.3.2 Riesz transforms

Assume that D is a densely defined linear operator on $L^2(X)$ which possesses the following properties:

(i) $DL^{-1/2}$ is bounded on $L^2(X)$.

(ii) The family of operators $\{\sqrt{t}De^{-tL}\}_{t>0}$ satisfies the Davies–Gaffney estimate (4.1.1).

Examples in which conditions (i) and (ii) are satisfied include the case $X = \mathbb{R}^n$, D being the gradient operator, and L the second-order divergence form operator; also, X any non-compact complete Riemannian manifold, D the Riemannian gradient, and L the Laplace–Beltrami operator. See, for example, [3, 4, 26].

Theorem 4.3.3 *For any $f \in H_{L,\omega}(X)$,*

$$||DL^{-1/2}f||_{L^1(\omega)} \leq C||f||_{H_{L,\omega}(X)}.$$

Before giving the proof of Theorem 4.3.3, we state the following lemma.

Lemma 4.3.4 *For every $M \in \mathbb{N}$, all closed sets E, F in X with $d(E,F) > 0$ and every $f \in L^2(X)$ supported in E, one has*

$$||DL^{-1/2}(I - e^{-tL})^M f||_{L^2(F)} \leq C\Big(\frac{t}{d(E,F)^2}\Big)^M ||f||_{L^2(E)}, \ \forall t > 0, \tag{4.3.6}$$

and

$$||DL^{-1/2}(tLe^{-tL})^M f||_{L^2(F)} \leq C\Big(\frac{t}{d(E,F)^2}\Big)^M ||f||_{L^2(E)}, \ \forall t > 0. \tag{4.3.7}$$

Proof. The proof of Lemma 4.3.4 is completely analogous to one of Lemma 2.2 in [47] and we omit it here.

□

Proof of Theorem 4.3.3. Choose $M > \frac{n}{2}(1/p_\omega - 1/2)$. Let m is an (ω, ϵ, M)-molecule associated to a ball B, $\epsilon < 2M + \frac{n}{2} - \frac{n}{p_\omega}$. Then there exists a function b such that $m = L^M b$. Setting $T = DL^{-1/2}$ and write

$$\begin{aligned}
&\int_X \omega(x, T(\lambda m)(x))d\mu(x) \\
&\leq \int_X \omega(x, |\lambda| T((I - e^{r_B^2 L})^M m(x))d\mu(x) + \int_X \omega(x, |\lambda| T([I - (I - e^{r_B^2 L})^M]m(x))d\mu(x) \\
&\leq \sum_{j=0}^{\infty} \int_X \omega(x, |\lambda| T((I - e^{r_B^2 L})^M (m \cdot \chi_{S_j(B)})(x))d\mu(x) \\
&\quad + \sum_{j=0}^{\infty} \int_X \omega(x, |\lambda| T([I - (I - e^{r_B^2 L})^M](m \cdot \chi_{S_j(B)})(x))d\mu(x) \\
&\leq \sum_{j=0}^{\infty} I_j + \sum_{j=0}^{\infty} II_j.
\end{aligned}$$

We estimate the term I_j first. By the Hölder inequality, we obtain

$$
\begin{aligned}
I_j &\leq \sum_{k=0}^{\infty} \int_{S_k(2^jB)} \omega(x, |\lambda| T((I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})(x)) d\mu(x) \\
&\leq \sum_{k=0}^{\infty} V(2^{k+j}B) \inf_{x\in X} \omega\Big(x, \frac{|\lambda|}{V(2^{k+j}B)} \int_{S_k(2^jB)} T((I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})(y)) d\mu(y)\Big) \\
&\leq \sum_{k=0}^{\infty} V(2^{k+j}B) \inf_{x\in X} \omega\Big(x, \frac{|\lambda|}{[V(2^{k+j}B)]^{1/2}} ||T((I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})||_{L^2(S_k(2^jB))}\Big).
\end{aligned}
$$

For $k = 0, 1, 2$, it follows from Lemma 4.3.4 that

$$
||T((I-e^{r_B^2 L})^M)(m\cdot\chi_{S_j(B)})||_{L^2(S_k(2^jB))} \leq C||m||_{L^2(S_j(B))} \leq C2^{-\epsilon j}[V(2^jB)]^{-1/2} \inf_{x\in B}[\rho(x, V(2^jB))]^{-1}.
$$

and for $k \geq 3$ that

$$
\begin{aligned}
&||T((I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})||_{L^2(S_k(2^jB))} \\
&\qquad \leq C2^{-2M(k+j)}||m||_{L^2(S_j(B))} \\
&\qquad \leq C2^{-2M(k+j)}2^{-\epsilon j}[V(2^jB)]^{-1/2} \inf_{x\in B}[\rho(x, V(2^jB))]^{-1}.
\end{aligned}
$$

At this stage, by the same argument used in the proof of Theorem 4.3.2, we obtain

$$
\sum_{j=0}^{\infty} I_j \leq CV(B) \inf_{x\in X} \omega\Big(x, \frac{|\lambda|}{V(B) \sup_{y\in B} \rho(y, V(B))}\Big).
$$

We now proceed with terms $II_j, j = 0, 1, \ldots$ Also, by the Hölder inequality, we obtain

$$
\begin{aligned}
II_j &\leq \sum_{k=0}^{\infty} \int_{S_k(2^jB)} \omega(x, |\lambda| T(I - (I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})(x)) d\mu(x) \\
&\leq \sum_{k=0}^{\infty} V(2^{k+j}B) \inf_{x\in X} \omega\Big(x, \frac{|\lambda|}{V(2^{k+j}B)} \int_{S_k(2^jB)} T(I - (I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})(y) d\mu(y)\Big) \\
&\leq \sum_{k=0}^{\infty} V(2^{k+j}B) \inf_{x\in X} \omega\Big(x, \frac{|\lambda|}{[V(2^{k+j}B)]^{1/2}} ||T(I - (I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})||_{L^2(S_k(2^jB))}\Big) \\
&\leq \sum_{k=0}^{\infty} II_j^k.
\end{aligned}
$$

Next we have

$$
I - (I - e^{r_B^2 L})^M = \sum_{k=1}^{M} c_k e^{-kr_B^2 L},
$$

where $c_k := (-1)^{k+1}\frac{M!}{(M-k)!k!}$. Therefore,

$$
\begin{aligned}
II_j^k &\leq C \sup_{1\leq k\leq M} ||Te^{-kr_B^2L}m\cdot\chi_{S_j(B)}||_{L^2(S_k(2^jB))} \\
&\leq C \sup_{1\leq k\leq M} \Big|\Big|T\Big(\frac{k}{M}r_B^2Le^{-\frac{k}{M}r_B^2L}\Big)^M(r_B^{-2}L^{-1})^Mm\cdot\chi_{S_j(B)}\Big|\Big|_{L^2(S_k(2^jB))}.
\end{aligned}
$$

At this point, repeating the argument used to estimate I_j, we also obtain that

$$\sum_{j=0}^{\infty} II_j \leq CV(B)\inf_{x\in X}\omega\Big(x, \frac{|\lambda|}{V(B)\sup_{y\in B}\rho(y,V(B))}\Big).$$

Combining obtained estimates gives

$$\int_X \omega(x, T(\lambda m(x)))d\mu(x) \leq CV(B)\inf_{x\in X}\omega\Big(x, \frac{|\lambda|}{V(B)\sup_{y\in B}\rho(y,V(B))}\Big).$$

This, together with Lemma 4.3.1, therefore completes our proof.

$\square$

5

Generalized weighted Hardy-Cesàro operators and their commutators on weighted Morrey spaces

5.1 Basic notation

Let ω be a weight function, namely ω is a measurable function on $\mathbb{R}^n$ and $\omega(x) > 0$ a.e. $x \in \mathbb{R}^n$. We denote by $BMO(\omega)$ the space of all functions f, which are of bounded mean oscillation with weight ω, that is

$$\|f\|_{BMO(\omega)} = \sup_B \frac{1}{\omega(B)} \int_B |f(x) - f_{B,\omega}|\omega(x)dx < \infty, \tag{5.1.1}$$

where the supremum is taken over all $n-$dimensional balls B. Here, $\omega(B) = \int_B \omega(x)dx$, and $f_{B,\omega}$ is the mean value of f on B with weight ω:

$$f_{Q,\omega} = \frac{1}{\omega(Q)} \int_Q f(x)\omega(x)dx.$$

The case $\omega \equiv 1$ of (5.1.1) corresponds to the class of functions of bounded mean oscillation of F. John and L. Nirenberg [53]. It is easy to see that $L^\infty(\mathbb{R}^n) \subset BMO(\omega)$. In the sequel, we need the following well-known result (see [73]).

Lemma 5.1.1 *Assume that ω is a weight function with the doubling property, that is for some positive constant C, we have*

$$\omega\left(B(x, 2r)\right) \leq C\omega\left(B(x, r)\right),$$

for all $x \in \mathbb{R}^n$ and $r > 0$. Then, for any $1 < p < \infty$, there exists some positive contant C_p such that

$$\|f\|_{BMO^p(\omega)} := \sup_B \left(\frac{1}{\omega(B)} \int_B |f(x) - f_{B,\omega}|^p \omega(x) dx \right)^{1/p} \leq C_p \|f\|_{BMO(\omega)}.$$

On the other hand, the classical Morrey spaces, which are natural generalizations of $L^p(\mathbb{R}^n)$, were introduced by Morrey in [63] to investigate the local behavior of solutions to second-order elliptic partial differential equations. After that, K. Yasuo and S. Satoru in [86] defined weighted Morrey spaces to study the boundedness of classical operators in harmonic analysis such as the Hardy-Littlewood maximal operator, Calderón-Zygmund operators and fractional integral operators.

Definition 5.1.2 *Let ω be a weight function and $B(a, R)$ be a ball centered at a with radius R. Let $1 \leq p < \infty$ and $-1/p \leq \lambda \leq 0$. The weighted Morrey space $L^{p,\lambda}(\omega)$ is defined by*

$$L^{p,\lambda}(\omega) := \left\{ f \in L^p_{loc}(\omega) : \|f\|_{L^{p,\lambda}(\omega)} < \infty \right\}, \tag{5.1.2}$$

where

$$\|f\|_{L^{p,\lambda}(\omega)} = \sup_{a \in \mathbb{R}^n, R > 0} \left(\frac{1}{\omega(B(a,R))^{1+p\lambda}} \int_{B(a,R)} |f(x)|^p \omega(x) dx \right)^{1/p}. \tag{5.1.3}$$

Remark 5.1.3 *If ω has the doubling property then in (5.1.3) one can replace balls $B(a, R)$ by balls $B(0, R)$ centered at the origin.*

Remark 5.1.4 *(1) If $\omega \equiv 1$, then $L^{p,\lambda}(\omega) = L^{p,\lambda}(\mathbb{R}^n)$, the classical Morrey spaces.*

(2) Assume that ω has the doubling property. If $\lambda = -1/p$, $L^{p,-1/p}(\omega) = L^p(\omega)$. If $\lambda = 0$, $L^{p,0}(\omega) = L^\infty(\omega)$ by the Lebesgue differentiation theorem with respect to ω.

In what follows, we will use the following useful variant of the maximal theorem for the Hardy-Littlewood maximal operator M_ω with respect to the measure $\omega(x)dx$, that is

$$M_\omega f(x) = \sup_B \frac{1}{\omega(B)} \int_B |f(y)| \omega(y) dy,$$

where the supremum is taken over all balls B containing x. See [86, Theorem 3.1].

Lemma 5.1.5 *Assume that ω has the doubling property, $1 < p < \infty$ and $-1/p < \lambda < 0$. Then the operator M_ω is bounded on $L^{p,\lambda}(\omega)$.*

5.2 Bounds of generalized weighted Hardy-Cesàro operators on weighted Morrey spaces $L^{p,\lambda}(\omega)$

In this section, we will show the boundedness of the generalized Hardy-Cesàro operator $U_{\psi,s}$ on spaces $L^{p,\lambda}(\omega)$ for the class of weights ω below and compute the corresponding operator norm.

Definition 5.2.1 *Let α be a real number. Then $\mathcal{W}_\alpha$ denotes the set of all weight functions ω on $\mathbb{R}^n$, which are absolutely homogeneous of degree α, that is $\omega(tx) = |t|^\alpha \omega(x)$, for all $t \in \mathbb{R} \setminus \{0\}$, $x \in \mathbb{R}^n$ and $0 < \int\limits_{S_n} \omega(y) d\sigma(y) < \infty$, where $S_n = \{x \in \mathbb{R}^n : |x| = 1\}$.*

Recall that if we define the measure ρ on $(0, \infty)$ by $\rho(E) = \int_E r^{n-1} dr$, and the map $\Phi(x) = \left(|x|, \frac{x}{|x|}\right)$, then there exists a unique Borel measure σ on S_n such that $\rho \times \sigma$ is the Borel measure induced by Φ from Lebesgue measure on $\mathbb{R}^n$ $(n > 1)$ (see [40, page 78], [54, page 142] for more details).

Let us describe some typical examples and properties of $\mathcal{W}_\alpha$. Note that a weight $\omega \in \mathcal{W}_\alpha$ may not belong to $L^1_{\text{loc}}(\mathbb{R}^n)$. In fact, we observe that if $\omega \in \mathcal{W}_\alpha$, then $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$ if and only if $\alpha > -n$. If $n = 1$, then $\omega(x) = c|x|^\alpha$, for some positive constant c. For $n \geq 1$, $\omega(x) = |x|^\alpha$ is in $\mathcal{W}_\alpha$ and has the doubling property when $\alpha > -n$. If ω_1, ω_2 are in $\mathcal{W}_\alpha$, so are $\theta\omega_1 + \lambda\omega_2$ for all $\theta, \lambda > 0$.

Lemma 5.2.2 *For any real number $\alpha > -n$, if $\omega \in \mathcal{W}_\alpha$ then there exists a positive constant C such that for all $R > 0$,*

$$\omega(B(0,R)) = \int_{B(0,R)} \omega(x) dx = C R^{n+\alpha}.$$

Proof. It comes from the standard integral calculation that

$$\int_{B(0,R)} \omega(x) dx = \int_0^R dr \int_{S(0,r)} \omega(y) d\sigma(y) = \int_0^R r^{n+\alpha-1} dr \int_{S_n} \omega(y) d\sigma(y).$$

So one can choose $C = \frac{1}{n+\alpha} \int_{S_n} \omega(y) d\sigma(y)$ to complete the proof of Lemma 5.2.2.

□

The next lemma plays an important role in our proofs of theorems in the sequel.

Lemma 5.2.3 *Assume that $\omega \in \mathcal{W}_\alpha (\alpha > -n)$ and has the doubling property, $1 < q < \infty$ and $-1/q < \lambda < 0$. Then $h(x) = |x|^{(n+\alpha)\lambda} \in L^{q,\lambda}(\omega)$ and $\|h\|_{L^{q,\lambda}(\omega)} > 0$.*

Proof. By Remark 5.1.3, one can replace balls $B(a,R)$ by balls $B(0,R)$ in the definition of weighted Morrey spaces. Then we have

$$\begin{aligned}
&\frac{1}{\omega(B(0,R))^{1+q\lambda}} \int_{B(0,R)} |x|^{(n+\alpha)q\lambda} \omega(x) dx \\
&= \frac{1}{\omega(B(0,R))^{1+q\lambda}} \int_0^R dr \int_{S(0,r)} r^{(n+\alpha)q\lambda} \omega(y) d\sigma(y) \\
&= \frac{1}{\omega(B(0,R))^{1+q\lambda}} \int_0^R r^{(n+\alpha)q\lambda+n+\alpha-1} dr \int_{S_n} \omega(y) d\sigma(y) \\
&= \frac{1}{\omega(B(0,R))^{1+q\lambda}} \frac{R^{(n+\alpha)(q\lambda+1)}}{(n+\alpha)(q\lambda+1)} \int_{S_n} \omega(y) d\sigma(y),
\end{aligned}$$

which, together with Lemma 5.2.2, completes the proof of Lemma 5.2.3.

$\square$

In [42], Z. Fu and S. LU proved that for $1 < p < \infty$ and $-1/p < \lambda < 0$, U_ψ is bounded on Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ if and only if

$$\mathcal{A} = \int_0^1 t^{n\lambda}\psi(t)dt < \infty.$$

They also showed that $\mathcal{A}$ is the $L^{p,\lambda}$-operator norm of U_ψ. Our main result in this section is formulated as follows.

Theorem 5.2.4 *Assume that $\omega \in \mathcal{W}_\alpha(\alpha > -n)$ and has the doubling property, $1 < p < \infty$ and $-1/p < \lambda < 0$. Let $s : [0;1] \to \mathbb{R}$ be a measurable function such that $|s(t)| > 0$ a.e. $t \in [0,1]$. Then $U_{\psi,s}$ is bounded on $L^{p,\lambda}(\omega)$ if and only if*

$$\int_0^1 |s(t)|^{(n+\alpha)\lambda}\psi(t)dt < \infty. \tag{5.2.1}$$

Moreover, when (5.2.1) holds, the operator norm of $U_{\psi,s}$ on $L^{p,\lambda}(\omega)$ is given by

$$\|U_{\psi,s}\|_{L^{p,\lambda}(\omega)\to L^{p,\lambda}(\omega)} = \int_0^1 |s(t)|^{(n+\alpha)\lambda}\psi(t)dt. \tag{5.2.2}$$

Proof. Suppose that (5.2.1) holds. For each $f \in L^{p,\lambda}(\omega)$, since $s(t) \neq 0$ almost everywhere, ω is homogeneous of degree α and applying Minkowski's inequality (see [44, page 14]) we obtain

$$\begin{aligned}
&\left(\frac{1}{\omega(B(a,R))^{1+p\lambda}}\int_{B(a,R)} |U_{\psi,s}f(x)|^p\omega(x)dx\right)^{1/p}\\
&\leq \int_0^1 \left(\frac{1}{\omega(B(a,R))^{1+p\lambda}}\int_{B(a,R)} |f(s(t)x)|^p\omega(x)dx\right)^{1/p}\psi(t)dt\\
&= \int_0^1 \left(\frac{1}{\omega\left(s(t)B(a,R)\right)^{1+p\lambda}}\int_{s(t)B(a,R)} |f(x)|^p\omega(x)dx\right)^{1/p}|s(t)|^{(n+\alpha)\lambda}\psi(t)dt\\
&\leq \|f\|_{L^{p,\lambda}(\omega)}\cdot\int_0^1 |s(t)|^{(n+\alpha)\lambda}\psi(t)dt < \infty.
\end{aligned}$$

Thus, $U_{\psi,s}$ is defined as a bounded operator on $L^{p,\lambda}(\omega)$ and

$$\|U_{\psi,s}\|_{L^{p,\lambda}(\omega)\to L^{p,\lambda}(\omega)} \leq \int_0^1 |s(t)|^{(n+\alpha)\lambda}\psi(t)dt. \tag{5.2.3}$$

Conversely, assume that $U_{\psi,s}$ is defined as a bounded operator on $L^{p,\lambda}(\omega)$. Due to Lemma 5.2.3, we take $h(x) = |x|^{(n+\alpha)\lambda} \in L^{p,\lambda}(\omega)$. It is easy to see that

$$U_{\psi,s}h = h\int_0^1 |s(t)|^{(n+\alpha)\lambda}\psi(t)dt. \tag{5.2.4}$$

Combining (5.2.3) and (5.2.4), we get the desired result immediately.

$\square$

Analogous to the proof of Theorem 5.2.4, we could find a sufficient condition on ψ such that the integral operator $\mathcal{U}_{\psi,s}$, which is defined by

$$\mathcal{U}_{\psi,s}f(x) = \int_0^\infty f\left(s(t)x\right)\psi(t)dt,$$

is bounded on $L^{p,\lambda}(\omega)$.

Theorem 5.2.5 *Assume that $\omega \in \mathcal{W}_\alpha(\alpha > -n)$ and has the doubling property, $1 < p < \infty$ and $-1/p < \lambda < 0$. Let $s : [0;1] \to \mathbb{R}$ be a measurable function such that $|s(t)| > 0$ a.e. $t \in [0,1]$. Then $\mathcal{U}_{\psi,s}$ is bounded on $L^{p,\lambda}(\omega)$ if*

$$\int_0^\infty |s(t)|^{(n+\alpha)\lambda}\psi(t)dt < \infty. \tag{5.2.5}$$

Moreover, when (5.2.5) holds, we have

$$\|\mathcal{U}_{\psi,s}\|_{L^{p,\lambda}(\omega)\to L^{p,\lambda}(\omega)} \leq \int_0^\infty |s(t)|^{(n+\alpha)\lambda}\psi(t)dt. \tag{5.2.6}$$

Corollary 5.2.6 *Assume that $\omega \in \mathcal{W}_\alpha(\alpha > -n)$ and has the doubling property, $1 < p < \infty$ and $-1/p < \lambda < 0$. Let $s : [0;1] \to \mathbb{R}$ be a measurable function such that $|s(t)| > 0$ a.e. $t \in [0,1]$. Then $V_{\psi,s}$ is bounded on $L^{p,\lambda}(\omega)$ if and only if*

$$\int_0^1 |s(t)|^{(n+\alpha)\lambda+n}\psi(t)dt < \infty. \tag{5.2.7}$$

Moreover, when (5.2.7) holds, the operator norm of $V_{\psi,s}$ on $L^{p,\lambda}(\omega)$ is given by

$$\|V_{\psi,s}\|_{L^{p,\lambda}(\omega)\to L^{p,\lambda}(\omega)} = \int_0^1 |s(t)|^{(n+\alpha)\lambda+n}\psi(t)dt. \tag{5.2.8}$$

Proof. This is an immediate consequence of Theorem 5.2.4 with the relation $V_{\psi,s}f(x) = U_{|s(\cdot)|^n\psi,s}f(x)$.

$\square$

5.3 Commutators of generalized Hardy-Cesàro operators

Recently, the authors in [41] established a necessary and sufficient condition on the weight function $\psi(t)$, which ensures the boundedness of the commutators (with symbols in $BMO(\mathbb{R}^n)$) of weighted Hardy operators U_ψ and weighted Cesàro operators V_ψ on $L^p(\mathbb{R}^n)$. Then in [42] these results have been extended to classical Morrey spaces. The purpose of this section is to extend the results mentioned above to generalized Hardy-Cesàro operators $U_{\psi,s}$ on weighted central Morrey spaces. Let us first recall here the definition of the weighted central Morrey spaces.

Definition 5.3.1 *Let ω be a weight function. Let $1 \le p < \infty$ and $-1/p \le \lambda \le 0$. The weighted central Morrey space $\dot{L}^{p,\lambda}(\omega)$ is defined by*

$$\dot{L}^{p,\lambda}(\omega) := \left\{ f \in L^p_{loc}(\omega) : \|f\|_{\dot{L}^{p,\lambda}(\omega)} < \infty \right\}, \tag{5.3.1}$$

where

$$\|f\|_{\dot{L}^{p,\lambda}(\omega)} = \sup_{R>0} \left(\frac{1}{\omega(B(0,R))^{1+p\lambda}} \int_{B(0,R)} |f(x)|^p \omega(x) dx \right)^{1/p}. \tag{5.3.2}$$

The first result we obtain in this section is the following.

Theorem 5.3.2 *Let $\omega \in \mathcal{W}_\alpha (\alpha > -n)$ hold the doubling property, $1 < p < q < \infty$ and $-1/q < \lambda < 0$. Assume that $s : [0;1] \to \mathbb{R}$ is a measurable function such that $0 < |s(t)| \le 1$ a.e. $t \in [0,1]$. Then (i) and (ii) are equivalent:*

(i) $U^b_{\psi,s}$ is bounded from $\dot{L}^{q,\lambda}(\omega)$ to $\dot{L}^{p,\lambda}(\omega)$ for all $b \in BMO(\omega)$;

(ii) $\int_0^1 |s(t)|^{(n+\alpha)\lambda} \left|\log \frac{2}{|s(t)|}\right| \psi(t) dt < \infty$.

Before coming to the proof of Theorem 5.3.2, we need the following two lemmas (see [22]). For convenience of the reader, the proofs to these lemmas are presented along with them.

Lemma 5.3.3 *If $\omega \in \mathcal{W}_\alpha$ and has the doubling property, then $\log |x| \in BMO(\omega)$.*

Proof. To prove $\log |x| \in BMO(\omega)$, for any $x_0 \in \mathbb{R}^n$ and $r > 0$, we need to find a constant $c_{x_0,r}$, such that $\frac{1}{\omega(B(x_0,r))} \int_{|x-x_0|\le r} |\log |x| - c_{x_0,r}| \, \omega(x) dx$ is uniformly bounded. Since

$$\frac{1}{\omega(B(x_0,r))} \int\limits_{|x-x_0|\le r} |\log |x| - c_{x_0,r}| \, \omega(x) dx$$

$$= \frac{r^{\alpha+n}}{\omega(B(x_0,r))} \int\limits_{|z-r^{-1}x_0|\le 1} |\log |z| - \log r - c_{x_0,r}| \, \omega(z) dz$$

$$= \frac{1}{\omega(B(r^{-1}x_0,1))} \int\limits_{|z-r^{-1}x_0|\le 1} |\log |z| - \log r - c_{x_0,r}| \, \omega(z) dz,$$

we may take $c_{x_0,r} = c_{r^{-1}x_0,1} + \log r$, and so things reduce to the case that $r = 1$ and x_0 is arbitrary. Let

$$A_{x_0} = \frac{1}{\omega(B(x_0,1))} \int\limits_{|z-x_0|\le 1} |\log |z| - c_{x_0,1}| \, \omega(z) dz.$$

If $|x_0| \le 2$, we take $c_{x_0,1} = 0$, and observe that

$$A_{x_0} \le \frac{1}{\omega\left(B(x_0,1)\right)} \int\limits_{|z|\le 3} \log 3 \cdot \omega(z)dz = \log 3 \cdot \frac{\omega\left(B(0,3)\right)}{\omega\left(B(x_0,1)\right)}$$

$$\le \log 3 \cdot \frac{\omega\left(B(x_0,6)\right)}{\omega\left(B(x_0,1)\right)} \le C < \infty,$$

where the last inequality comes from the assumption that ω has the doubling property.

If $|x_0| \ge 2$, take $c_{x_0,1} = \log|x_0|$. In this case, notice that

$$A_{x_0} = \frac{1}{\omega\left(B(x_0,1)\right)} \int\limits_{B(x_0,1)} \left|\log \frac{|z|}{|x_0|}\right| \omega(z)dz$$

$$\le \frac{1}{\omega\left(B(x_0,1)\right)} \int\limits_{B(x_0,1)} \max\left\{\log \frac{|x_0|+1}{|x_0|}, \log \frac{|x_0|}{|x_0|-1}\right\} \cdot \omega(z)dz$$

$$\le \max\left\{\log \frac{|x_0|+1}{|x_0|}, \log \frac{|x_0|}{|x_0|-1}\right\} \le \log 2.$$

Thus $\log|x|$ is in $BMO(\omega)$.

$\square$

Lemma 5.3.4 *Let ω be a doubling weight function. Then, there exists a positve constant C such that for any balls $B_1 = B(x_1, r_1)$, $B_2 = B(x_2, r_2)$, whose intersection is not empty, and $\frac{1}{2}r_2 \le r_1 \le 2r_2$, then $\omega(B) \le C\omega(B_i)$, $i = 1, 2$. Here, B is the smallest ball which contains both B_1 and B_2. Moreover, for each function $b \in BMO(\omega)$, we have*

$$|b_{B_1,\omega} - b_{B_2,\omega}| \le 2C\|b\|_{BMO(\omega)}.$$

Proof. Since ω has the doubling property, there exists a constant C_1 such that $\omega\left(B(x,2r)\right) \le C_1\omega\left(B(x,r)\right)$, for any $x \in \mathbb{R}^n$ and $r > 0$. Without loss of generality, we assume $r_2 \le r_1 \le 2r_2$. Let $B_1 = B(x_1, r_1)$, $B_2 = B(x_2, r_2)$ be two balls, whose intersection is not empty and $r_2 \le r_1 \le 2r_2$. Take $x \in B_1 \cap B_2$. Then,

$$\omega(B) \le \omega\left(B(x, 2r_1\right) \le C_1\omega\left(B(x, r_1)\right) \le C_1\omega\left(B(x_1, 2r_1)\right) \le C_1^2\omega\left(B_1\right),$$

and

$$\omega(B) \le \omega\left(B(x, 4r_2\right) \le C_1^2\omega\left(B(x, r_2)\right) \le C_1^3\omega\left(B(x_2, r_2)\right).$$

Thus we can choose the constant $C = \max\{C_1^2, C_1^3\}$.

It is clear that

$$|b_{B_1,\omega} - b_{B_2,\omega}| \le |b_{B_1,\omega} - b_{B,\omega}| + |b_{B,\omega} - b_{B_2,\omega}|.$$

Now

$$|b_{B,\omega} - b_{B_1,\omega}| = \left| b_{B,\omega} - \frac{1}{\omega(B_1)} \int_{B_1} b(y)\omega(y)dy \right|$$

$$\leq \frac{1}{\omega(B_1)} \int_{B_1} |b(y) - b_{B,\omega}| \, \omega(y)dy \leq \frac{C}{\omega(B)} \int_{B} |b(y) - b_{B,\omega}| \, \omega(y)dy \leq C\|b\|_{BMO(\omega)}.$$

The left term is estimated in a similar way.

$\square$

Proof of Theorem 5.3.2. Assume that (ii) holds. Let B be any ball centered at the origin, let f be any function in $\dot{L}^{q,\lambda}(\omega)$, and let b be any function in BMO(ω). Then it follows from the Minkowski inequality that

$$\begin{aligned} K &= \left(\frac{1}{\omega(B)^{1+\lambda p}} \int_{B} \left| U^{b}_{\psi,s} f(y) \right|^{p} \omega(y)dy \right)^{1/p} \\ &\leq \int_{0}^{1} \left(\frac{1}{\omega(B)^{1+\lambda p}} \int_{B} |(b(y) - b\,(s(t)y))\, f\,(s(t)y)|^{p} \omega(y)dy \right)^{1/p} \psi(t)dt. \end{aligned}$$

Applying the following elementary inequality

$$3^{p-1} \left(|x|^{p} + |y|^{p} + |z|^{p} \right) \geq |x + y + z|^{p}, x, y, z \in \mathbb{C}$$

to the right-hand side of the above estimate gives

$$\begin{aligned} K &\leq C \int_{0}^{1} \left(\frac{1}{\omega(B)^{1+\lambda p}} \int_{B} \left|(b(y) - b_{B,\omega})\, f\,(s(t)y)\right|^{p} \omega(y)dy \right)^{1/p} \psi(t)dt \\ &+ C \int_{0}^{1} \left(\frac{1}{\omega(B)^{1+\lambda p}} \int_{B} \left|\left(b_{B,\omega} - b_{s(t)B,\omega}\right) f\,(s(t)y)\right|^{p} \omega(y)dy \right)^{1/p} \psi(t)dt \\ &+ C \int_{0}^{1} \left(\frac{1}{\omega(B)^{1+\lambda p}} \int_{B} \left|\left(b(s(t)y) - b_{s(t)B,\omega}\right) f\,(s(t)y)\right|^{p} \omega(y)dy \right)^{1/p} \psi(t)dt \\ &:= K_1 + K_2 + K_3, \end{aligned}$$

where the constant C depends only on p.

Set $r = \frac{pq}{q-p}$. Let us now estimate the term K_1. It is clear to see that applying the Hölder inequality with the pair $\left(l = \frac{q}{q-p}, l' = \frac{q}{p}\right)$ and applying Lemma 5.1.1 yield

$$\begin{aligned}
K_1 &\leq \frac{C}{\omega(B)^\lambda} \int_0^1 \left(\frac{1}{\omega(B)} \int_B |f(s(t)y)|^q \omega(y)dy\right)^{1/q} \left(\frac{1}{\omega(B)} \int_B |b(y) - b_{B,\omega}|^r \omega(y)dy\right)^{1/r} \psi(t)dt \\
&\leq \frac{C\|b\|_{BMO(\omega)}}{\omega(B)^\lambda} \int_0^1 \left(\frac{1}{\omega(B)} \int_B |f(s(t)y)|^q \omega(y)dy\right)^{1/q} \psi(t)dt \\
&\leq C\|b\|_{BMO(\omega)} \int_0^1 \left(\frac{1}{\omega(B)^{1+\lambda q}} \int_B |f(s(t)y)|^q \omega(y)dy\right)^{1/q} \psi(t)dt \\
&\leq C\|b\|_{BMO(\omega)} \|f\|_{\dot{L}^{q,\lambda}(\omega)} \int_0^1 |s(t)|^{(n+\alpha)\lambda} \psi(t)dt.
\end{aligned}$$

Similarly, one can use the same argument above to obtain

$$K_3 \leq C\|b\|_{BMO(\omega)} \|f\|_{\dot{L}^{q,\lambda}(\omega)} \int_0^1 |s(t)|^{(n+\alpha)\lambda} \psi(t)dt.$$

For the last term K_2, let us express this term as

$$K_2 = \frac{C}{\omega(B)^\lambda} \int_0^1 \left(\frac{1}{\omega(B)} \int_B |f(s(t)y)|^p \omega(y)dy\right)^{1/p} \left|b_{B,\omega} - b_{s(t)B,\omega}\right| \psi(t)dt.$$

Then the Hölder inequality with the pair $(q/p, (q/p)')$ implies that

$$\begin{aligned}
K_2 &\leq \frac{C}{\omega(B)^\lambda} \int_0^1 \left(\frac{1}{\omega(B)} \int_B |f(s(t)y)|^q \omega(y)dy\right)^{1/q} \left|b_{B,\omega} - b_{s(t)B,\omega}\right| \psi(t)dt \\
&\leq C\|f\|_{\dot{L}^{q,\lambda}(\omega)} \int_0^1 \left|b_{B,\omega} - b_{s(t)B,\omega}\right| |s(t)|^{(n+\alpha)\lambda} \psi(t)dt \\
&\leq C\|f\|_{\dot{L}^{q,\lambda}(\omega)} \sum_{k=0}^{\infty} \int_{2^{-(k+1)} \leq |s(t)| \leq 2^{-k}} \left|b_{B,\omega} - b_{s(t)B,\omega}\right| |s(t)|^{(n+\alpha)\lambda} \psi(t)dt.
\end{aligned}$$

Observe that for each $k \in \mathbb{N}$,

$$\left|b_{B,\omega} - b_{s(t)B,\omega}\right| \leq \sum_{i=0}^{k} \left|b_{2^{-(i+1)}B,\omega} - b_{2^{-i}B,\omega}\right| + \left|b_{2^{-(k+1)}B,\omega} - b_{s(t)B,\omega}\right|.$$

Then it follows from Lemma 5.3.4 that

$$K_2 \leq C\|f\|_{\dot{L}^{q,\lambda}(\omega)} \|b\|_{BMO(\omega)} \sum_{k=0}^{\infty} \int_{2^{-(k+1)} \leq |s(t)| \leq 2^{-k}} (k+2)|s(t)|^{(n+\alpha)\lambda} \psi(t)dt.$$

So, we obtain

$$K_2 \leq C\|f\|_{\dot{L}^{q,\lambda}(\omega)} \|b\|_{BMO(\omega)} \int\limits_0^1 |s(t)|^{(n+\alpha)\lambda} \psi(t) \left(2 + \log \frac{1}{|s(t)|} \right) dt,$$

which, if combined with the last estimates of K_1 and K_3 above, implies (i).

Conversely, assume that (i) holds. Taking $h(x) = |x|^{(n+\alpha)\lambda} \in \dot{L}^{s,\lambda}(\omega)$ for any $s > 1$, and $b_0 = \log|x| \in BMO(\omega)$ (see Lemma 5.3.3). Then it is easy to see that

$$U_{\psi,s}^{b_0} h = h \int_0^1 |s(t)|^{(n+\alpha)\lambda} \psi(t) \log \frac{1}{|s(t)|} dt.$$

It thus follows from (i) that

$$\int_0^1 |s(t)|^{(n+\alpha)\lambda} \psi(t) \log \frac{1}{|s(t)|} dt < \infty. \tag{5.3.3}$$

On the other hand, we take $g(x) = |x|^{(n+\alpha)\lambda} 1\chi_{B(0,1)}(x)$ and consider the family of functions $b \in BMO(\omega)$ given by $b(x) = 1\chi_{B(0,1)}(x)\mathrm{sgn}(\sin \pi d|x|)$, $d \in \mathbb{N}$. Then one has

$$\begin{aligned} U_{\psi,s}^b g(x) &= b(x)|x|^{(n+\alpha)\lambda} \int_0^1 |s(t)|^{(n+\alpha)\lambda} \psi(t) dt \\ &- |x|^{(n+\alpha)\lambda} \int_0^1 |s(t)|^{(n+\alpha)\lambda} \psi(t) \mathrm{sgn}(\sin \pi d |s(t)||x|) dt, |x| < 1. \end{aligned}$$

By Riemann-Lebesgue Lemma, one can take d, depending on ψ, so large that

$$|U_{\psi,s}^b g(x)| \geq \frac{1}{2} |x|^{(n+\alpha)\lambda} \int_0^1 |s(t)|^{(n+\alpha)\lambda} \psi(t) dt, \frac{1}{2} < |x| < 1.$$

It follows from the above inequality and the boundedness of the operator $U_{\psi,s}^b$ from $\dot{L}^{q,\lambda}(\omega)$ to $\dot{L}^{p,\lambda}(\omega)$ that

$$\int_0^1 |s(t)|^{(n+\alpha)\lambda} \psi(t) dt \leq C \|U_{\psi,s}^b\|_{\dot{L}^{q,\lambda}(\omega) \to \dot{L}^{p,\lambda}(\omega)}, \tag{5.3.4}$$

where the constant C depends only on p, q and λ. Finally, combining (5.3.3) and (5.3.4) implies (ii).

□

Corollary 5.3.5 *Let $\omega \in \mathcal{W}_\alpha (\alpha > -n)$ hold the doubling property, $1 < p < q < \infty$ and $-1/q < \lambda < 0$. Assume that $s : [0;1] \to \mathbb{R}$ is a measurable function such that $0 < |s(t)| \leq 1$ a.e. $t \in [0,1]$. Then (iii) and (iv) are equivalent:*

(iii) $V_{\psi,s}^b$ is bounded from $\dot{L}^{q,\lambda}(\omega)$ to $\dot{L}^{p,\lambda}(\omega)$ for all $b \in BMO(\omega)$;

(iv) $\int_0^1 |s(t)|^{(n+\alpha)\lambda+n} \left|\log\frac{2}{|s(t)|}\right| \psi(t)dt < \infty.$

Remark 5.3.6 *One can observe from the proof of Theorem 5.3.2 that the condition (ii) is also necessary for the commutator $U^b_{\psi,s}$ to be bounded on $\dot{L}^{p,\lambda}(\omega)$ (the case $p = q$) for all $b \in BMO(\omega)$. However, whether the condition (ii) is sufficient for the boundedness of $U^b_{\psi,s}$ on $\dot{L}^{p,\lambda}(\omega)$ for all $b \in BMO(\omega)$ is still an interesting open question. Up to now, what we can prove is the following result.*

Theorem 5.3.7 *Let $\omega \in \mathcal{W}_\alpha(\alpha > -n)$ hold the doubling property, $1 < p < \infty$ and $-1/p < \lambda < 0$. Assume that $s : [0;1] \to \mathbb{R}$ is a measurable function such that $0 < |s(t)| \leq 1$ a.e. $t \in [0,1]$. Then (ii) is sufficient for (i') to be true:*

(i') $M_{\omega,c}\left(U^b_{\psi,s}(\cdot)\right)$ *is bounded on $L^{p,\lambda}(\omega)$ and on $\dot{L}^{p,\lambda}(\omega)$ for all $b \in BMO(\omega)$, where $M_{\omega,c}$ is the Hardy-Littlewood central maximal operator with respect to the measure $\omega(x)dx$, that is*

$$M_{\omega,c}f(x) = \sup_B \frac{1}{\omega(B)} \int_B |f(y)|\omega(y)dy,$$

where the supremum is taken over all balls B centered at the origin and containing x.

(ii) $\int_0^1 |s(t)|^{(n+\alpha)\lambda} \left|\log\frac{2}{|s(t)|}\right| \psi(t)dt < \infty.$

Proof of Theorem 5.3.7. Assume that (ii) holds. Let b be any function in $BMO(\omega)$ and let f be any function in $L^{p,\lambda}(\omega)$. Then for any ball B centered at the origin of $\mathbb{R}^n$, $x \in B$, one has

$$\begin{aligned}
&\frac{1}{\omega(B)}\int_B \left|U^b_{\psi,s}f(y)\right|\omega(y)dy \\
&\leq \frac{1}{\omega(B)}\int_B\int_0^1 |(b(y) - b(s(t)y))\, f(s(t)y)|\,\psi(t)\omega(y)dtdy \\
&\leq \frac{1}{\omega(B)}\int_0^1\int_B |(b(y) - b(s(t)y))\, f(s(t)y)|\,\omega(y)dy\psi(t)dt \\
&\leq \frac{1}{\omega(B)}\int_0^1\int_B |(b(y) - b_{B,\omega})\, f(s(t)y)|\,\omega(y)dy\psi(t)dt \\
&\quad + \frac{1}{\omega(B)}\int_0^1\int_B \left|\left(b_{B,\omega} - b_{s(t)B,\omega}\right) f(s(t)y)\right|\omega(y)dy\psi(t)dt \\
&\quad + \frac{1}{\omega(B)}\int_0^1\int_B \left|\left(b(s(t)y) - b_{s(t)B,\omega}\right) f(s(t)y)\right|\omega(y)dy\psi(t)dt \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

Choose a fixed number $r \in (1,p)$ and denote r' the conjugate of r (that is, $1/r + 1/r' = 1$). Then we will estimate J_1, J_3 and J_2 as follows.

It follows from the Hölder inequality and Lemma 5.1.1 that

$$J_1 \le \int_0^1 \left(\frac{1}{\omega(B)}\int_B |f(s(t)y)|^r \omega(y)dy\right)^{1/r} \left(\frac{1}{\omega(B)}\int_B |b(y)-b_{B,\omega}|^{r'}\omega(y)dy\right)^{1/r'} \psi(t)dt$$

$$\le C\|b\|_{BMO(\omega)} \int_0^1 \left(\frac{1}{\omega(s(t)B)}\int_{s(t)B} |f(y)|^r\omega(y)dy\right)^{1/r} \psi(t)dt$$

$$\le C\|b\|_{BMO(\omega)} \int_0^1 \left(M_\omega f^r(s(t)x)\right)^{1/r}\psi(t)dt,$$

where C is a constant, which depends only on ω and p.

Similarly, applying the Hölder inequality and Lemma 5.1.1 again gives

$$J_3 \le \int_0^1 \left(\frac{1}{\omega(B)}\int_B |f(s(t)y)|^r \omega(y)dy\right)^{1/r} \left(\frac{1}{\omega(B)}\int_B \left|b(s(t)y)-b_{s(t)B,\omega}\right|^{r'}\omega(y)dy\right)^{1/r'} \psi(t)dt.$$

$$= \int_0^1 \left(\frac{1}{\omega(s(t)B)}\int_{s(t)B} |f(y)|^r\omega(y)dy\right)^{1/r} \left(\frac{1}{\omega(s(t)B)}\int_{s(t)B} \left|b(y)-b_{s(t)B,\omega}\right|^{r'}\omega(y)dy\right)^{1/r'} \psi(t)dt$$

$$\le C\|b\|_{BMO(\omega)} \int_0^1 \left(M_\omega f^r(s(t)x)\right)^{1/r}\psi(t)dt.$$

For the remaining term J_2, we will estimate J_2 as follows:

$$J_2 = \frac{1}{\omega(B)}\int_0^1\int_B \left|\left(b_{B,\omega}-b_{s(t)B,\omega}\right) f(s(t)y)\right| \omega(y)dy\psi(t)dt$$

$$= \int_0^1 \left(\frac{1}{\omega(B)}\int_B |f(s(t)y)|\,\omega(y)dy\right) \left|b_{B,\omega}-b_{s(t)B,\omega}\right| \psi(t)dt$$

$$\le \int_0^1 \left(M_\omega f^r(s(t)x)\right)^{1/r} \left|b_{B,\omega}-b_{s(t)B,\omega}\right| \psi(t)dt$$

$$= \sum_{k=0}^{\infty} \int_{2^{-(k+1)}\le s(t)\le 2^{-k}} \left(M_\omega f^r(s(t)x)\right)^{1/r} \left|b_{B,\omega}-b_{s(t)B,\omega}\right| \psi(t)dt$$

$$\le \sum_{k=0}^{\infty} \int_{2^{-(k+1)}\le s(t)\le 2^{-k}} \left(M_\omega f^r(s(t)x)\right)^{1/r} \left(\sum_{i=0}^{k} \left|b_{2^{-(i+1)}B,\omega}-b_{2^{-i}B,\omega}\right| + \left|b_{2^{-(k+1)}B,\omega}-b_{s(t)B,\omega}\right|\right)\psi(t)dt.$$

At this stage, in the light of Lemma 5.3.4, we can observe that $\left|b_{2^{-(i+1)}B,\omega}-b_{2^{-i}B,\omega}\right| \le C\|b\|_{BMO(\omega)}$, and $\left|b_{2^{-(k+1)}B,\omega}-b_{s(t)B,\omega}\right| \le C\|b\|_{BMO(\omega)}$ as $2^{-(k+1)} \le s(t) \le 2^{-k}$. So we obtain

$$J_2 \le C\|b\|_{BMO(\omega)} \sum_{k=0}^{\infty} \int_{2^{-(k+1)}\le s(t)\le 2^{-k}} \left(M_\omega f^r(s(t)x)\right)^{1/r}(k+2)\,\psi(t)dt.$$

Then it follows from the inequality above that

$$J_2 \leq C\|b\|_{BMO(\omega)} \int_0^1 \left(M_\omega f^r(s(t)x)\right)^{1/r} \left(2 + \left|\log \frac{1}{|s(t)|}\right|\right) \psi(t)dt.$$

Combining the estimates of J_1, J_2 and J_3 gives the following estimate:

$$\frac{1}{\omega(B)} \int_B \left|U^b_{\psi,s} f(y)\right| \omega(y)dy \leq C\|b\|_{BMO(\omega)} \int_0^1 \left(M_\omega f^r(s(t)x)\right)^{1/r} \left(2 + \left|\log \frac{1}{|s(t)|}\right|\right) \psi(t)dt.$$

Taking the supremum from this inequality over such all balls B containing x, we gain

$$M_{\omega,c}\left(U^b_{\psi,s} f\right)(x) \leq C\|b\|_{BMO(\omega)} \int_0^1 \left(M_\omega f^r(s(t)x)\right)^{1/r} \left(2 + \left|\log \frac{1}{|s(t)|}\right|\right) \psi(t)dt.$$

Therefore, for any balls B, by Minkowski's inequality and the inequality above, it is clear to see that

$$\begin{aligned}
&\left(\frac{1}{\omega(B)^{1+p\lambda}} \int_B \left(M_{\omega,c}\left(U^b_{\psi,s} f\right)(x)\right)^p \omega(x)dx\right)^{1/p} \\
&\leq C\|b\|_{BMO(\omega)} \left(\frac{1}{\omega(B)^{1+p\lambda}} \int_B \left(\int_0^1 \left(M_\omega f^r(s(t)x)\right)^{1/r} \left(2 + \left|\log \frac{1}{|s(t)|}\right|\right) \psi(t)dt\right)^p \omega(x)dx\right)^{1/p} \\
&\leq C\|b\|_{BMO(\omega)} \int_0^1 \left(\frac{1}{\omega(B)^{1+p\lambda}} \int_B \left(M_\omega f^r(s(t)x)\right)^{p/r} \omega(x)dx\right)^{1/p} \psi(t) \left(2 + \left|\log \frac{1}{|s(t)|}\right|\right) dt \\
&\leq C\|b\|_{BMO(\omega)} \|f\|_{L^{p,\lambda}(\omega)} \int_0^1 |s(t)|^{(n+\alpha)\lambda} \psi(t) \left(2 + \left|\log \frac{1}{|s(t)|}\right|\right) dt,
\end{aligned}$$

where the last inequality is deduced from applying Lemma 5.1.5 and the fact that $f \in L^{p,\lambda}(\omega)$ if and only if $f^r \in L^{p/r,\lambda r}(\omega)$. Thus $M_{\omega,c}\left(U^b_{\psi,s}(\cdot)\right)$ is bounded on $L^{p,\lambda}(\omega)$.

To prove the boundedness of $M_{\omega,c}\left(U^b_{\psi,s}(\cdot)\right)$ on $\dot{L}^{p,\lambda}(\omega)$, we use the same arguments above and taking the following into account: instead of using the Hardy-Littlewood maximal operator M_ω and Lemma 5.1.5, we employ the Hardy-Littlewood central maximal operator $M_{\omega,c}$ and the following lemma whose proof is similar to that of [86, Theorem 3.1] with slight modifications.

Lemma 5.3.8 *Assume that ω has the doubling property, $1 < p < \infty$ and $-1/p < \lambda < 0$. Then the operator $M_{\omega,c}$ is bounded on $\dot{L}^{p,\lambda}(\omega)$.*

□

5.4 Higher order commutators on weighted central Morrey spaces

Given a vector $\boldsymbol{b} = (b_1, \ldots, b_k)$, the higher order commutator of the generalized weighted Hardy-Cesàro operator is defined by

$$U_{\psi,s}^{\boldsymbol{b}} f(x) = \int_0^1 \left(\prod_{j=1}^{k} (b_j(x) - b_j(s(t)x)) \right) f(s(t)x)\psi(t)dt.$$

When $k = 0$, we understand that $U_{\psi,s}^{\boldsymbol{b}} = U_{\psi,s}$. When $k = 1$, $U_{\psi,s}^{\boldsymbol{b}} = U_{\psi,s}^{b}$. Similarly, the higher order commutator of the generalized weighted Cesàro operator is defined by

$$V_{\psi,s}^{\boldsymbol{b}} f(x) = \int_0^1 \left(\prod_{j=1}^{k} (b_j(x) - b_j(s(t)x)) \right) f(s(t)x)|s(t)|^n \psi(t)dt.$$

The notation $\boldsymbol{b} \in BMO(\omega)$ below will mean that all $b_i \in BMO(\omega)$ for $1 \leq i \leq k$.

Using the same method in the proof of Theorem 5.3.2 as well as induction, we can obtain a similar result for the higher order commutator of the generalized weighted Hardy-Cesàro operator.

Theorem 5.4.1 *Let $\omega \in \mathcal{W}_\alpha (\alpha > -n)$ hold the doubling property, $1 < p < q < \infty$ and $-1/q < \lambda < 0$. Assume that $s : [0; 1] \to \mathbb{R}$ is a measurable function such that $0 < |s(t)| \leq 1$ a.e. $t \in [0, 1]$. Then (1) and (2) are equivalent:*

(1) $U_{\psi,s}^{\boldsymbol{b}}$ is bounded from $\dot{L}^{q,\lambda}(\omega)$ to $\dot{L}^{p,\lambda}(\omega)$ for all $\boldsymbol{b} \in BMO(\omega)$;

(2) $\int_0^1 |s(t)|^{(n+\alpha)\lambda} \left|\log \frac{2}{|s(t)|}\right|^k \psi(t)dt < \infty$.

Also, (3) and (4) are equivalent:

(3) $V_{\psi,s}^{\boldsymbol{b}}$ is bounded from $\dot{L}^{q,\lambda}(\omega)$ to $\dot{L}^{p,\lambda}(\omega)$ for all $\boldsymbol{b} \in BMO(\omega)$;

(4) $\int_0^1 |s(t)|^{(n+\alpha)\lambda+n} \left|\log \frac{2}{|s(t)|}\right|^k \psi(t)dt < \infty$.

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