CHAPTER 1

INTRODUCTION

The transition from arithmetic to algebra can be a difficult barrier to cross for many students. Students learn about numbers and numerical techniques to solve mathematical problems throughout primary school. Then, as soon as they begin secondary school, they are required to operate on variables, most often represented by letters.

Letters are used in different ways. They can represent the name of a quantity (such as *m* for metre), the attribute of a person (*h* for Sam's height), or a generalised rule ($2n$, where n belongs to the set of integers). When students do not understand all these different meanings of letters and what letters represent in a particular context, they may have difficulties in representing word problems in algebraic form, simplifying algebraic expressions, and solving equations.

The traditional teaching approach used in schools provides limited opportunities for students to experience the multifaceted character of variables. Usually, after only a brief introduction to the use of a variable as an unknown quantity, students are taught algebraic techniques of simplification and representation. Students are thus expected to represent and solve word problems, reason about variables, simplify algebraic expressions and solve equations, despite often possessing only a superficial understanding of the multifaceted character of variables.

For example, in the school where the researcher worked, variables are introduced as an unknown quantity in the beginner algebra course in Year 7 (12 to 13 year olds). All mathematics teachers in the school primarily use an empty box for an unknown number and then gradually replace that box with the letter x. Many students in Year 6 and Year 7 find it difficult to represent an unknown quantity by the letter x. The difficulty in understanding the meaning of variables represented by the letter x makes it difficult for students to appreciate the algebraic activity. Many students think that algebra is difficult and meaningless. Thus, it is necessary to find a teaching approach which could facilitate students' understanding of the meaning of variables. Such an understanding would make algebra meaningful and interesting and students might appreciate algebra as a powerful problem solving tool.

1.1 TEACHING ALGEBRA

There are three different categories of teaching approaches which are commonly used to learn and teach algebra in the junior secondary school. One approach is based on problem solving or modelling, another on generalisations, and another on functions.

Problem-solving approaches are the most commonly used in the junior secondary years. Here, equations are considered as the main objects of algebra and hence the main algebraic activity is to solve equations (Dougherty, 2001; Katz, 2001; Sutherland, 2004). Students learn to reason about a given problem, identify the unknowns, form an equation relating the unknown variables, and then solve the problem. The modelling approach evolved from the problem-solving approach. In the modelling approach, students develop a model of a real world problem and gradually refine it by a process of representation and verification until an accurate algebraic model is obtained (Goos, Stillman, & Vale, 2007).

Generalisation approaches are based on the core idea that variables are generalised numbers and that in learning about the generalisation of number properties students develop skills in algebraic reasoning (Blanton & Kaput, 2005; Carraher, Schliemann, & Schwartz, 2007; Fujii & Stephens, 2001).

Carraher et al. (2007) proposed that algebra be introduced by using different representations of functions. Using functions instead of variables or equations promotes a multiple-value image of variables, as graphs are used to display calculations and algebraic expressions are considered as representations of functions (Chazan & Yerushalmy, 2003).

Each of these three teaching approaches has its merits and disadvantages. For example, the use of restricted arithmetic word problems in the problem-solving approach encourages the conception of letters as unknown quantities. When the purpose of this algebraic activity is just to find the unknown value of variables, students may not think about the relationships between the quantities (Goos, Dole, & Makar, 2007). On the other hand, the use of patterns in the generalisation approach can promote the conception of a variable as a generalised number. However, difficulties with generalisation have also been widely reported; in generalising number patterns students do not necessarily look at the functional relationships between variables (M. MacGregor & Stacey, 1993; Stacey & MacGregor, 1997).

To appreciate and learn algebra with understanding, it is essential to move from thinking about relations between particular numbers to thinking about relations between sets of numbers, and from computing numerical answers to describing relations among variables (Carraher & Schliemann, 2007). The multi-representational technological environments used in functional approaches can facilitate students in building links between symbolic, tabular and graphic representations (Balacheff & Kaput, 1996; Friedlander & Tabach, 2001). However, some students find it difficult to link functional representations, such as tables of values and graphs, to their algebraic representations. Thus it may be necessary to reform the curricular sequence based on the functional approach to bridge the gap between functions and algebraic representations (Yerushalmy, 2000). The concept of a variable promoted by the functional approach is also unclear (Kieran, 2007).

1.2 A PROPOSED NEW APPROACH

It is necessary that students learn about and differentiate between different aspects of variables (unknown quantity, generalised number, function[1]) as all these aspects of variables are used in algebra. Hence it may be more fruitful to use some elements of these different teaching approaches in conjunction with each other. In doing so, students might develop a broader concept of variables.

Trigueros and Ursini (2001) proposed a teaching model called the Three Uses of Variables or 3UV model which approached the learning and teaching of algebra through the concept of a variable (see p.29). However, the results of the 3UV teaching model have not been reported so it is not possible to be sure about the effectiveness of the approach and whether or not students taught via the 3UV model can differentiate and understand the different facets of variables.

In the research reported in this thesis, a Multifaceted Variable Approach (MVA) has been developed and trialled. The MVA builds upon the idea of learning the three aspects of variables (unknown, generalised number and function) in parallel with each other to promote a deep understanding of variables. The MVA is a mixed approach which integrates positive elements of the generalisation approach, the functional approach and the problem-solving approach. In the MVA, students work mathematically to formulate algebraic expressions; they prove, justify and solve word problems; and they learn to translate between numerical, tabular and graphic representations of linear functions. Only then do they move on to algebraic simplifications and the solution of linear equations. In the MVA, students solve problems based on real and familiar contexts to make algebra meaningful and interesting.

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[1] More precisely, this third aspect of a variable should be described as “used to specify a functional relationship”. In this thesis, I use the term “function” as an abbreviation for this description.

The MVA is therefore a combination of teaching resources, a revised learning sequence and novel teaching techniques. The teaching resources contain problems and activities which require students to work mathematically in algebra. Rather than the traditional approach which begins with learning the procedures for substitution and simplifying algebraic expressions, the MVA learning sequence is designed to study all three aspects of variables together in a variety of contexts first. Finally, the teaching techniques incorporated into the MVA emphasise active learning (Anthony, 1996) rather than rote learning.

1.3 AIM OF THE STUDY

The aim of this study is to investigate the effect of the MVA on students' conceptions of variables and their general algebraic competence. The specific research questions are as follows:

1. Does the MVA lead to a deeper conception of variables by students than the traditional approach to teaching algebra in Years 7-8?

2. Does the MVA result in students having superior algebraic competence (in terms of representation of word problems in algebraic form, simplification of algebraic expressions, and solution of linear equations) by the end of Year 8, when compared to the results of traditional algebra teaching?

3 What aspects of the MVA tend to promote or hinder students' conceptual understanding of variables and algebraic competence?

The study was carried out as a longitudinal teaching experiment in a single school. It was completed in two phases: Phase I with students and the teachers of four graded Year 7 classes and Phase II with the same cohort of students and teachers in Year 8. Two classes (Sets 2 and 4) were taught algebra using the MVA and two classes (Sets 1 and 3) were taught using a traditional approach. Data were collected from students through six written assessments and two rounds of student interviews over the duration of the two years of the study. The differences between the students taught by the MVA and the traditional teaching approaches were analysed to assess the effectiveness of the MVA in promoting a deeper conceptual understanding of variables and greater algebraic competence.

1.4 SIGNIFICANCE OF THE STUDY

Student difficulties in learning algebra are well documented. Different teaching experiments indicate that instructional reform has the potential to improve the skills of algebraic reasoning among students (Watson, 2010). It is also possible to minimise student misconceptions by a suitable teaching practice. However, little is known about the effect of different teaching approaches on students' algebraic difficulties (Watson, 2010). This study investigates the effect of an innovative teaching approach (MVA) on student understanding of variables and their general algebraic competence.

The MVA emphasises the understanding of relationships between variables using real contexts and their tabular, algebraic and graphic representations. Algebraic symbolism is central to school algebra and research is needed to investigate the ways in which students can be facilitated in seeing relations between graphical and algebraic representations and making connections between word problems and the generation of equations (Kieran, 2007).

The new Australian Mathematics Curriculum (Australian Curriculum Assessment and Reporting Authority, 2011) integrates number and algebra in one strand and stresses the representation of relationships, patterns and structure, functions and logical reasoning. The MVA also emphasises pattern and structure along with functions and the representation of relationships between variables. Therefore, this study will provide some early evidence of the feasibility and effectiveness of integrating these ideas in algebra learning. The results of the study may therefore have implications for local mathematics teachers as they implement the Number and Algebra strand of the Australian Curriculum over the coming years.

1.5 OUTLINE OF THE THESIS

The next chapter summarises the literature on student difficulties associated with algebra learning. Chapter 3 discusses research into some elements of the different teaching approaches, such as the problem solving and modelling approaches, the generalisation approaches, and functional approaches which are integrated into the MVA. Chapter 4 outlines the methodology for the study.

Chapter 5 presents some background information about the participating teachers and summarises the pedagogical approaches of teachers of the experimental and comparison groups. Chapter 6 details an analysis of the student assessments and interviews administered during Phase I, while Chapter 7 presents the results and analysis of student assessments and interviews administered during Phase II of the study.

Chapter 8 discusses the results of the study in terms of the conceptions of students about variables and their algebraic competence resulting from the MVA and the traditional approach. The role of teachers and the teaching resources on student learning is also discussed. Chapter 9 presents some answers to the three research questions and considers some implications for teaching and further research.

CHAPTER 2

STUDENT DIFFICULTIES IN ALGEBRA

2.1 INTRODUCTION

This chapter is a review of research concerning the causes of student difficulties in algebra. The literature suggests that despite consistent efforts to improve the learning and teaching of algebra, students still find the concept of a variable problematic and have difficulty in understanding the structure of algebra and the solution of equations (Booth, 1984; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005). Various reasons for student difficulties in algebra have been suggested. According to researchers, the origin of students' difficulties lies in the transition from arithmetic to algebra, and in their incorrect or incomplete schemas about objects of algebra such as variables, expressions and equations. Moreover, students do not understand the relations between these algebraic objects and the system of axioms which governs these relations and makes algebra a powerful problem-solving tool. Students mostly rely on memorised rules to simplify expressions and solve linear equations procedurally. Therefore, they do not appreciate the relevance and importance of algebra in real life, which also results in a lack of motivation to study this subject.

In mathematics, letters or symbols are used in different ways: for example, to represent vertices of geometrical figures such as quadrilaterals, to denote fixed constants such as π, and to represent variable quantities. In a typical junior secondary school algebra curriculum, letters are introduced as generalised numbers, for example in sequences like 2, 4, 6, 8,…, 2*n*. However, letters can also represent variable quantities (e.g., in finding the area of a rectangle, $A = l \times b$, where l represents length, b represents breadth and A represents area) or specific unknowns (e.g. in equations such as $2x + 3 = 5$). If students are not made explicitly aware of the meaning of variables, these different meanings may later become a source of confusion and misconception for students (Booth, 1995).

The following sections elaborate on the reasons for student difficulties in algebra.

2.2 DIFFICULTIES IN TRANSITION FROM ARITHMETIC TO ALGEBRA

Researchers such as Blanton and Kaput (2005), Carraher and Schleimann (2007), Kieran (1992), and Warren (2003) believe that the traditional separation of arithmetic and algebra is a major reason for student difficulties in algebra. When students start learning algebra in secondary school, they come across new concepts such as variables, new objects such as equations, and later on, new ideas such as functions and parameters (Vergnaud, 1997). During algebra lessons, students assimilate the information provided by their teachers about these newly defined objects and build their own schemas in light of their previous experiences with arithmetic in primary school. Incorrect adaptations from arithmetic such as $3\frac{1}{2} = 3 + \frac{1}{2}$ are also responsible for many student errors in algebra (Stacey & MacGregor, 1994). Difficulties in the transition from arithmetic to algebra also stem from the different nature of problems presented in arithmetic and algebra and the different arithmetic and algebraic procedures used to solve these problems (Bednarz & Janvier, 1996). Generally, in arithmetic the given problems are such that the student proceeds from a known quantity towards an unknown quantity. Students create links between known quantities to find an unknown quantity. For example, consider the problem,

> Sonia has 3 times as many books as Julia and 5 times as many books as Bela. If Sonia has 270 books, how many books do the three children have altogether?

In this problem, the question can be solved by starting with the known quantity (270), dividing it by 3 and 5 respectively to find the number of books that Julia and Bela have, and finally adding all three numbers together to find the total number of books.

In comparison, algebraic problems proceed from an unknown to a known quantity and they are designed so that students need to use the relationships between the variables. For example, consider the problem

> Sonia, Julia and Bela had 414 books altogether. Find the number of books owned by each girl if Sonia had 3 times as many books as Julia and 5 times as many books as Bela,

Here, finding the solution starts with unknown quantities which are the number of books owned by Sonia, Julia and Bela. The solution proceeds by representing the relationships between the number of books owned by Sonia and the number of books owned by Julia and Bela.

Stacey (1999) refers to this transition from thinking about a known quantity to thinking about an unknown quantity as the transition from arithmetic thinking to algebraic thinking, and claims that this transition is difficult for students. While working with over 1000 secondary school students, Stacey (1999) found that students preferred to use arithmetic methods like “guess and check” and logical arithmetic reasoning for solving word problems. For example, to solve the problem, Mark and Jan share \$47, but Mark gets \$5 more than Jan. How much do they each get?

students used “guess and check” by finding different pairs of whole numbers to stand for the shares of Mark and Jan and checking to see which pair totals to 47 and differs by 5. Those students who solved the problem by logical arithmetic reasoning subtracted 5 from 47, divided the answer thus obtained by 2 to find Jan’s share, then added 5 to Jan’s share to find Mark’s share. Very few students represented this problem algebraically and solved the equation to find the solution. Moreover, the success rate of students using algebraic methods was very low.

Students prefer to use arithmetic methods and logical reasoning to solve word problems because representing a problem in algebraic form is difficult for students (Koedinger & Nathan, 2004). A reason for this difficulty is that students often cannot link the meaning of variables with the symbols used to represent them (Watson, 2010).

2.3 MISCONCEPTIONS REGARDING THE CONCEPT OF A VARIABLE

Many students think of an algebraic symbol such as *x* as an unknown quantity and few students consider the possibility that an unknown symbol can be a variable having multiple values (Küchemann, 1981). One possible reason for this is the previous experience of students in arithmetic, where they usually calculate a single numerical value as an answer to a problem. Students also tend to look for whole numbers as the possible solutions of equations and do not even consider the possibility of an answer in the form of a fraction or a decimal (Warren, 2003). This is because arithmetic questions in primary school mostly involve working with whole numbers and less time is spent on solving complex questions involving decimals and fractions.

A large number of students do not think of letters as representing numbers at all: They ignore their presence, assign no particular meaning to them, or treat them as objects (e.g., 4*y* to stand for 4 lots of *y,* where *y* can represent a yacht) (Booth, 1995). Similar results were obtained by MacGregor and Stacey (1997) in studies involving 2000 Australian students in 22 secondary schools.

Küchemann (1981) and MacGregor and Stacey (1997) both developed a categorisation of students’ interpretation of literal symbols. A summary extracted from their interpretation of letters is described here.

Students interpret letters which are used to represent variables as:

- Letter evaluated (e.g., in questions of the type $x + 3 = 6$ where the value of x can be found intuitively without any calculation) (Küchemann, 1981),

- Letters as a specific unknown (e.g., the n in an "n-sided figure" where n cannot be evaluated until the number of sides is known (Küchemann, 1981),

- Letters as generalised numbers (e.g., in questions of the type "$p + q = 20$, $p < q$, $p = ?$" where p and q can have an infinite number of values) (Küchemann, 1981),

- Letters as variables (e.g., in questions of the type: "Which is larger $3n$ or $n + 4$?") (Küchemann, 1981),

- Letter ignored (Students ignore letters while solving equations of the type "$a + b = 12$, $a + b + 6 =?$" where students can ignore $a + b$ and still find the answer. Or in the problem, "If Con is 10 cm taller than Cassy and Cassy's height is h, what would be Con's height?" where students ignore h and give Con's height as 20 cm) (Küchemann, 1981; M. MacGregor & Stacey, 1997).

- Letter as a label or an abbreviated word (e.g., the letters that are used to label the vertices of a triangle or the use of *Dh* to represent David's height) (M. MacGregor & Stacey, 1997).

Steinle, Gvozdenko, Price, Stacey, and Pierce (2009) classified student misconceptions regarding variables as numerical and non numerical. Non numerical misconceptions include letter as a label and letters ignored, and numerical misconceptions are the misconceptions related to the numerical values attained by variables. Some numerical misconceptions identified by MacGregor and Stacey (1997) and Perso (1991) are listed here. According to MacGregor and Stacey (1997)

- students use alphabetical values for letters (i.e. they substitute a numerical value in place of a variable based on the alphabetical position of that variable, such as c corresponds to 3),

- students use different letters to represent different numbers,

- students substitute arbitrary numerical values for letters,

- students think that a letter standing alone is equal to 1 (M. MacGregor & Stacey, 1997; Perso, 1991).

Perso (1991) identified further misconceptions about the possible values assigned to letters. For example, students think that

- letters are sequential (a = 3 implies that b = 4),
- each letter has a unique value (for example x and y cannot have the same value simultaneously),
- letters have a place value (Given that 2st = 250, if s = 5 then t must be 0).

Küchemann (1981) noticed that only 40% of students aged 13-15 years reached a stage of cognitive development where they were able to interpret symbols as unknown quantities or as variables. This was reflected in their symbolic interpretation of relationships in problem-solving activities. For example, while solving $p + q = 20$, students were often content to find only one set of numbers for which this relationship is true.

MacGregor and Stacey (1997) attributed the cause of student errors regarding variables to factors such as misleading teaching resources (e.g., “fruit-salad algebra” where a is for apples and b is for bananas, 4y means 4 lots of y’s, etc), making inferences from new schemas (e.g.,, assuming that x means 1 because $x + 2x = 3x$), reasoning about an unfamiliar new symbol system (e.g., if height is h then Con’s height can be represented by *Ch*) and relating algebraic symbols with symbolic notations used in other subjects (such as symbolic representation of elements in the periodic table in Chemistry).

Küchemann (1981) developed a framework of cognitive levels which corresponds to Piaget’s stages of cognitive development and incorporates the six types of interpretations of letters by students mentioned earlier. Table 2.1 describes the algebraic level, Piaget’s stage of cognitive development, the type of problems used to assess each level, and students’ concept of a variable at that level. As shown in Table 2.1, students who consider variables as an object, or ignore them, or consider them as a place holder are at the first stage of algebraic understanding. In contrast, students who consider variables as specific unknowns, generalised numbers or variables according to the context are at the final stage of algebraic understanding.

Students who interpret letters as specific unknowns and not as generalised numbers or as variable quantities, learn the procedures of manipulation and substitution without assigning any meaning to the symbols involved (Booth, 1995). Misconceptions about the concept of a variable are also responsible for the difficulties which students face in equation solving

(Perso, 1992). Moreover, if students do not understand what the letters are referring to, they often become disinterested in studying algebra (Stacey & Chick, 2004).

Table 2.1
Algebraic Levels, Piaget's stage of Cognitive Development, Sample Items, Student Understanding of Variables

Algebraic Levels	Piaget's Stage	Items	Students' understanding
Level 1	Early concrete	$x + 2 = 3$ $2x + 5x = ?$	Demonstrate the ability to solve arithmetic questions involving numbers only. Consider letters as objects, Evaluate letters, or ignore letters, or conjoin terms.
Level 2	Late Concrete	Finding perimeters of pentagons using numbers and letters or both	Evaluate letters, consider letters as objects. Do not realise that letters represent unknowns, generalised numbers and variables. Demonstrate more willingness to accept an algebraic expression as an answer to a problem.
Level 3	Early Formal	$p + q = 20$, $p < q$, $p = ?$	Understand letters to be specific unknowns. Accept $3x + 1$ as a complete answer.
Level 4	Late Formal	$(x + 1)^3 + x = 349$ Which is larger $3n$ or n?	Understand letters to be specific unknowns, or in some cases, to be generalised numbers or variables

2.4 PROBLEMS RELATED TO THE REPRESENTATION AND SOLUTION OF A LINEAR EQUATION

The process of algebraic problem solving is completed in two stages. In the first stage, the relationship between the variables is identified and then represented in the form of an algebraic equation. In the second stage, a solution of the problem is obtained by solving the formulated equation. Students' difficulties in both these stages of problem solving are well documented and some are discussed here.

2.4.1 Errors in representation

Factors such as misinterpreting the meaning of variables and attempting to translate directly from words to algebraic expressions contribute to student errors in representation (Stacey & MacGregor, 1999). Students who do not understand the meaning of letters used to represent variables are unlikely to interpret algebraic expressions correctly (Booth, 1995). In fact some students do not assign any meaning to algebraic expressions such as $x + 3$. These students consider such expressions incomplete in the absence of an equality sign and a right-hand term (Chalouh & Herscovics, 1988; Kieran, 1983).

Another cause for errors in representation is to use a letter as an abbreviation for a word instead of representing the numerical value of the letter (Clement, 1982). For example, when Clement asked tertiary calculus students to write an algebraic statement for the relationship,

> There are six times as many students as professors in this university

he found that one third of the students represented this relationship incorrectly as $P = 6S$ instead of $S = 6P$. Clement (1982) called this error the reversal error. The reversal error was the result of using a letter as an abbreviation for a word instead of representing the numerical value of the letter (as mentioned previously). Another explanation given by Clement was that in algebra $6 \times s$ is written as $6s$, which encourages the tendency to read $6s$ as 6 students.

Herscovics (1989) and Laborde (1990) reasoned that the reversal error is due to a direct translation from words to symbols. They proposed that difficulties in translating statements from direct language to algebraic expressions are not due to incorrect concepts (Krishner, Awtry, McDonald, & Grey, 1991) but are caused by trying to represent what students understand about a situation intuitively (Stacey & MacGregor, 1993).

Sometimes students conjoin algebraic expressions when they translate from words to algebraic expressions. For example, Küchemann (1981) reported that students represent an algebraic sum as a product (e.g., they represent the perimeter of an equilateral triangle with side *e* as *eee*). This error called conjoining error, has also been reported by other researchers such as Falle (2007) and Stacey and MacGregor (1999). Evidence suggests that this error is widespread among students and is hard to remedy (Booth, 1984).

Generally, students who make this error are at a lower level of algebraic competency (Falle, 2007; Küchemann, 1981). However, Falle (2007) observed that even high-ability students who had a procedural understanding of algebraic techniques made this error when presented with an unfamiliar problem.

Researchers have proposed different reasons for conjoining errors. For example, Stacey and MacGregor (1993) argued that conjoining errors were due to an incorrect adaptation from arithmetic to algebra, an incorrect reasoning made by students about an unfamiliar notation system, or a failure to understand the relationship between addition and multiplication. For example, in arithmetic students can write $4 + \frac{1}{2}$ as $4\frac{1}{2}$ so they expect that they can combine 4 and x together as $4 + x = 4x$. Another suggested reason for conjoining is that students do not see an expression such as $3e + 4f$ as an algebraic object in its own right and therefore acceptable as an answer (Booth, 1995; Küchemann, 1981). Students conjoin terms such as $4 + x = 4x$ because they believe that the addition sign is a direction to calculate (Warren, 2003; Wong, 1997). Also students sometimes process expressions from left to right without considering the sequence of operations in which they must be performed (Tall & Thomas, 1991). For example, students may read 2 + 3x from left to right and add 2 + 3 first, obtaining the answer 5x. Tall and Thomas called this error parsing error.

Students also find it difficult to represent a numerical table of values in the form of an algebraic equation. Many students tend to notice a change in values of one variable only and do not notice the functional relationship between the variables. For example, in a study of approximately 3000 Year 7-10 students in 34 schools, MacGregor and Stacey (1999) found that students made persistent errors in formulating an equation from a table of values. They inferred that difficulties in formulating an equation were due to the inability of students to see the relationships between variables.

2.4.2 Errors in simplification of algebraic expressions

Traditionally, students are introduced to algebra through exercises in which algebraic expressions are evaluated by substituting values for the variables involved. They then move on to simplification of algebraic expressions, which then leads to equation solving. However, it has been argued that this type of traditional approach promotes procedural understanding (Stacey, 1999) as minimal emphasis is placed on understanding the relationships between the meaning of variables and the structure of algebraic expressions (Falle, 2007). When these students come across a familiar problem, they look for visual cues which trigger a sequence of procedural steps they must take to solve that problem. However, in the case of an unfamiliar problem they cannot rely on their previous knowledge and consequently make simplification (manipulation) errors (Falle, 2007).

To simplify accurately, a strong understanding of number properties is essential (Stacey, 1999). When students manipulate algebraic expressions without a thorough understanding of the operations and the structure behind the manipulations, they make characteristic errors in manipulations such as $\frac{x+8}{x+2} = \frac{8}{2}$ (Chalkin & Lesgold, 1984; Collis, 1975; Kieran, 1989; Warren, 2000). Sometimes, student errors in simplification are a reflection of similar errors in purely numerical contexts (Linchevski & Livneh, 1999). For example, students who write $\frac{2+5}{2}$ = 5 in arithmetic, simplify $\frac{a+b}{a}$ = b in algebra (Booth, 1995).

Relying on memorised algebraic rules to manipulate expressions can become a major hurdle in simplifying algebraic expressions. For example, Wong (1997) observed that students find mixed expressions like $(2a^m)^n$ difficult to evaluate compared to expressions such as $(hk)^n$, because students can rely on algebraic rules to solve expressions involving variables alone but not mixed expressions.

Rules can also be misinterpreted. A large number of students who were taught rules such as BODMAS to calculate numerical expressions were not able to decide on the order of operations in an algebraic expression (Watson, 2010). For example, errors like $\frac{2(x+5)}{5}$ = $2x$ were caused either by working inside the brackets first (as required by BODMAS) to (incorrectly) get $5x$ and then multiplying by 2, or by ignoring the brackets and not giving due regard to the order of operations. Stacey and MacGregor (1999) discovered that students do not realise the importance of brackets in algebraic expressions and may not use them at all.

2.4.3 Problems with the interpretation of equality and equivalence

Evidence suggests that students interpret the equal sign in two different ways: firstly as a direction to calculate, and secondly to indicate the equality of two expressions at a particular value (Kieran, 1992). Many students do not interpret equality as an equivalence relation in the sense that one expression can be substituted for another (Kieran, 1992). Knowing that two apparently different expressions can be equivalent and that the process of manipulation is merely a transformation between two algebraic expressions makes a considerable difference in algebraic competence (Watson, 2010).

Kieran and Saldhana (2005) discovered that students interpret equivalence in two different ways: either numerically, in which two expressions are equal at the same value, or algebraically, where one expression is a transformation of another. The algebraic interpretation of equality where students can identify all equivalent forms of an algebraic expression is called structural sense (Kieran, 1988). Even when students learn the techniques of transformation for solving equations they do not realise that the process of transformation generates equivalent expressions (Steinberg, Sleeman, & Ktorza, 1990) or that all these equivalent equations have the same solution (Kieran, 1984). Similar observations were made by Ball, Stacey, and Pierce (2003) when they administered the *Algebraic Expectation Quiz* to Year 11-12 students to assess their ability to recognise equivalent algebraic expressions. They concluded that recognising equivalence is a significant challenge for students. Similar observations were made by Chalkin and Lesgold (1984) when they presented students with different numerical expressions involving addition and subtraction of numbers such as 124 + 215 – 654 and 215 – 654 + 124 and asked them to decide on their equivalence. Many students calculated the answers before deciding on the equivalence of the expressions. If students do not recognise 2 + 3 and 3 + 2 as equivalent arithmetic expressions, they are unlikely to realise that $a + b$ and $b + a$ are algebraically equivalent (Booth, 1984).

Difficulties in recognising equivalent expressions is a significant obstacle for students even when technological tools such as Computer Algebra Systems (CAS) are available (Ball et al., 2003). When an equation is entered in CAS, it displays the solution on the screen. Due to this facility, students are no longer required to solve the equation by lengthy algebraic manipulations and calculations. However, in order to use CAS effectively and correctly, students still need to understand different forms of equivalent equations as well as different methods of solving an equation (Ball & Stacey, 2001).

2.4.4 Moving from arithmetic to algebraic strategies for solving linear equations

Students use different arithmetic and algebraic methods to solve linear equations. These methods can be arranged according to their order of sophistication as follows: guess and check, counting strategies/known basic facts, working backwards, and the balancing method or transformation (transforming an equation into another equivalent equation until the required solution is found) (Linsell, 2009). Examples of these strategies are provided below.

- Guess and check/substitution method;
- Counting strategies/Using known facts (Solving $2x+1 = 5$ by thinking that since $2 \times 2 = 4$ and $4 + 1 = 5$, therefore x must be equal to 2)

- Working backwards/inverse operations method. For example, see Figure 2.1.

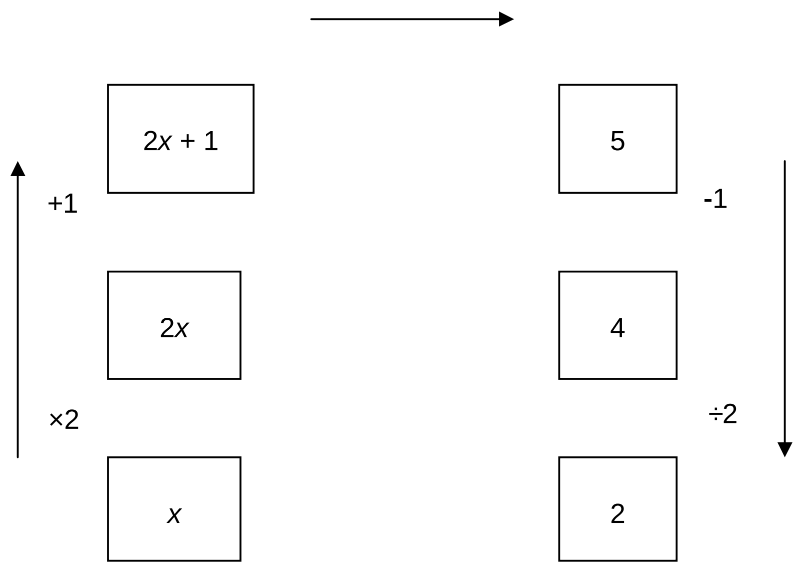


Figure 2.1 Working backwards

- Balancing method or transformations.

$$2x + 1 = 5$$

$$\Leftrightarrow \quad 2x + 1 - 1 = 5 - 1$$

$$\Leftrightarrow \quad 2x = 4$$

$$\Leftrightarrow \quad x = \frac{4}{2} = 2$$

As mentioned above, students prefer to use arithmetic methods (guess and check, counting strategies, and working backwards) over algebraic methods (balancing/transformations) for solving linear equations. For example, Herscovics and Linchevski (1991) noticed that Year 7 students who had not solved equations before and knew only the skill of using numerical substitutions to evaluate unknowns in given formulas, mainly used working backwards/inverse operations for solving addition equations. For multiplication or division problems, students used mixed methods such as working backwards and guess and check and for mixed operations such as addition and multiplication, students turned to systematic substitution or guess and check and working backwards. In particular, 68% to 77% of the students used working backwards for questions involving mixed operations and large numbers. It has also been observed that students find the transformation strategy very difficult and turn towards guess and check for solving equations which have a variable on both sides of the equals sign, such as $2x + 4 = 3x + 7$ (Linsell, 2009; Sfard & Linchevski, 1994; Vlassis, 2002).

Manipulation difficulties sometimes make the substitution method more difficult for students. For example, Filloy, Rojano, and Solares (2003, 2004) found that when 13-14 year-old students solved equations of the type $u = b - v$ and $u^2 + v^2 = 15$, they were not facilitated by the substitution method even when students had the facility of using CAS to solve these linear equations. They also found that students had difficulty in accepting answers of the type $\frac{b \pm \sqrt{30 - b^2}}{2}$.

Many reasons have been suggested for the preference of students to use arithmetic methods. For example, students may prefer arithmetic methods because they like to test actual numbers (Johnson, 1989). Students turn towards arithmetic methods like guess and check as an alternate solution strategy when they forget memorised rules (Herscovics & Linchevski, 1991). Students also use arithmetic solution strategies as they find the balancing method difficult to use (Vlassis, 2002). Moreover, students do not think of using an algebraic method until they come across a linear equation such as $2x + 4 = 3x + 7$ which cannot be easily solved using arithmetic methods (Sfard & Linchevski, 1994). While arithmetic strategies such as guess and check may be a successful solution strategy for simple equations, an over-reliance on this method can become an obstacle to the development of algebraic reasoning (Watson, 2010).

The selection of solution strategy may also be related to the mathematical ability of the students. For example, Linsell (2009) compared students' understanding of solution strategies with item difficulty and mathematical ability. He found that low-ability students preferred to use guess and check for solving linear equations whereas the most sophisticated transformation strategy was only used by very few high-ability students. For solving one-step simple linear equations of the type $x + 6 = 8$, low-ability students used counting strategies or known basic facts whereas high-ability students used inverse operations.

When students use an arithmetic strategy to solve a linear equation, the suggestion is that they are considering the linear equation as a numerical instantiation. For example, when students solve an equation of the type $x + 4 = 6$ by using guess and check, they are in fact considering $x + 4 = 6$ as an image of the arithmetic equation $2 + 4 = 6$ in algebraic form. The answer so obtained is numeric and the procedure is computational. However, when students operate algebraically on an equation they successively transform an equation into equivalent equations. The operations carried out are not computational and the answer obtained is also in the form of an algebraic expression. The same applies when students are required to change the subject of a formula (Kieran, 1991). Kieran (1992) describes the numerical strategy as procedural and the algebraic strategy as structural. She also suggests that the move from procedural to structural operations is not easy for students and takes time.

2.5 ISSUES OF ENGAGEMENT

The purpose of algebraic activity has always been to solve complex problems which are difficult to solve using arithmetic methods. However, the problems presented in algebra are often so simple that students can rely on arithmetic methods to solve them. Therefore, the usefulness and power of algebraic methods is not realised (Stacey & MacGregor, 1999). Also, results of TIMSS in 1999 indicated that a typical Year 8 mathematics lesson in Australia suffers from a "shallow teaching syndrome", where teachers ask students to follow procedures without understanding the meaning behind them (Stacey, 2003). If students do not appreciate the purpose of an algebraic activity, or if teachers ask students to follow procedures and manipulate symbols without understanding the meaning behind them, students become disinterested in studying algebra (Ball et al., 2003; Stacey & Chick, 2004).

Students may also find algebra boring and alienated from real life because school textbooks typically contain a large number of repetitive algorithm-specific symbol manipulation exercises and few contextual problems linked to familiar real life situations. However, real contexts make it easier for students to deal with complexity as it enables them to approach new tasks by using ideas and situations familiar to them (Nemirovsky, 1996).

2.6 SUMMARY

Due to the traditional separation of arithmetic and algebra, students are not provided with opportunities to make links between numbers and variables. The result may be incomplete schemas about algebraic objects such as variables and equations. Their incomplete schemas may result in misconceptions regarding variables since some students ignore variables and some assign arbitrary numerical values to variables, while others use variables as labels or objects.

When students do not understand the meaning of variables, they often rely on memorised rules and operate on numbers without a thorough understanding of operations and the structure of algebraic expressions. This tendency is a major cause of manipulation errors in algebra. Incorrect adaptations from arithmetic to algebra, not understanding the difference between repeated addition and multiplication, and incorrect reasoning about an unfamiliar symbolic system, are also responsible for manipulation errors. Not only do students need a strong conception of a variable, they also require a strong understanding of number properties to succeed in algebra.

Moving from arithmetic to algebraic problem-solving methods is difficult for students. In algebraic problems, students start from unknown quantities (variables), identify relationships between variables to formulate an equation and then solve that equation to find a solution to the problem. When students interpret equality as a direction to calculate, they often use arithmetic methods and consider an equation as a numerical instantiation. Thinking of equality as an equivalence between two linear expressions and using transformations to convert one equation to another is not easy for students.

Another reason for relying on arithmetic is the inclusion in textbooks of many simple problems which can be solved by using arithmetic methods alone. Therefore, students do not appreciate the significance and utility of algebraic methods. Textbooks contain very few contextual problems which can facilitate students in making algebra meaningful and interesting for students. Thus students are unable to maintain interest in algebra.

The teaching approach used for teaching algebra has a very important role in developing students' understanding. In the next chapter, the advantages and disadvantages of the different approaches to teach algebra in secondary schools are described in order to draw out the essential elements of a successful algebra program. These essential elements form the basis of the Multifaceted Variable Approach (MVA), which is also described.

CHAPTER 3

TEACHING APPROACHES

3.1 TEACHING APPROACHES

There is a need to reform the learning and teaching of algebra as students find algebra a difficult subject to study. In particular, students find the concept of a variable problematic and have difficulty in understanding relationships between variables. When students do not understand what variables represent in a particular context they make errors in translating word problems to algebraic expressions. Student difficulties are reflected in their manipulation errors and incorrect solutions of linear equations.

In order to reduce student difficulties in algebra and facilitate learning, different teaching approaches have been suggested by researchers and used by teachers. These can be broadly categorised as the problem-solving and the modelling approach, the generalisation approach and the functional approach (Bednarz, Kieran, & Lee, 1996). Some researchers have also proposed mixed teaching approaches, such as the realistic mathematics education approach (Gravemeijer, 1994), the Elaborated Davydov Approach (Dougherty & Zilliox, 2003) and the 3UV approach (Trigueros & Ursini, 2001). These mixed approaches may contain some elements of the generalisation, functional and problem-solving modelling approaches. Every teaching approach comes with advantages and disadvantages and no approach can claim to solve all student difficulties (Watson, 2010).

In this chapter a brief summary of the various teaching approaches is presented with a focus on the advantages and disadvantages of each approach. The essential elements of a successful algebra program are extracted and integrated to show the design of the MVA which is the focus of this thesis. In the MVA, students study different aspects of variables in parallel with each other with the intention to develop a better understanding of the variable concept.

3.1.1 Problem-solving and modelling approaches

Algebra was historically used as an efficient problem-solving tool for dealing with problems which were difficult to solve using arithmetical methods (Bednarz & Janvier, 1996). As early as 1650 BCE arithmetic word problems were solved by defining unknown quantities even though no formal symbol system existed at that time. During the Babylonian period (1696-

1654 BCE) mathematicians started using similar approaches to solve geometric problems as well.

In the first known algebra text book, written by Al-Khwarizmi in the ninth century, six standard algorithms for solving quadratic equations were presented. Most of the problems presented in his book were not real-life problems, and the only purpose of presenting the problems was to teach the methods for solving quadratic equations. Symbolic algebra and further equation solving techniques were gradually developed by Bombelli, Harriot, Vete and Déscartes after the Renaissance and during the eighteenth and nineteenth centuries (Katz, 2001).

A traditional perspective of problem solving is to start from concepts and procedures, engaging students in the development of skills. Teachers then present students with stories and problems which provide them with an opportunity to use their previously learned skills and algorithms. In the case of an unfamiliar problem or a problem for which no algorithm is available, students have to turn to alternate strategies which are called heuristics (Ormrod, 2008). Students may have to draw a picture to represent the problem, work backwards, look for similar problems, or identify the given and required information to understand the problem (Pôlya, 1957).

A more recent definition of problem solving was given by Lesh and Zawojewski (2007):

> the process of interpreting a system mathematically, which usually involves several iterative cycles of expressing, testing and revising mathematical interpretations and of sorting out, integrating, modifying, revising, or refining clusters of mathematical concepts from various topics within and beyond mathematics (Lesh & Zawojewski, 2007, p. 782)

Similarly, in a call for algebra reform, Kaput (1999) proposed that students should be engaged in interpreting and reasoning about a problem, phenomenon or situation to identify the commonalities, relationships and patterns, and then express the problem using a suitable mathematical model.

There are many advantages of including such an approach to problem solving in school mathematics. Problem solving is not only a source of motivation and recreation for students, but it also develops the skills of reasoning and critical analysis (Stanic & Kilpatrick, 1989). When solving problems, students can express a problem in the form of a representative equation and then use a suitable method to solve that equation. In this way, problem solving provides a platform for developing the mathematical skills of representation and simplification. Moreover, it fosters the language skills of reading and writing (Bell, 1996).

Problem solving has a long history of teaching tradition behind it and many researchers have suggested teaching approaches based on problem solving. For example, Katz (2001) presented a curricular sequence for teaching algebra using problem solving as the main activity. He proposed that any algebra course must start from solving equations since the main purpose of teaching algebra is to learn how to solve an equation. If students learn to solve equations they will appreciate the importance and purpose of algebraic manipulations. All the concepts that need to be a part of an algebra course can be introduced gradually by careful choice of problems. Katz proposed that geometrical ideas should be used wherever possible and that symbolism should be introduced at a later stage when students can appreciate the importance of using symbols over words while solving problems.

Dougherty (2001) likewise suggested that algebraic content should be embedded in problem-solving contexts to promote higher-order thinking. She used the problem-solving processes developed by Krutetskii (1976) for constructing tasks for a teaching approach called the Process Approach (Dougherty, 2001). This approach was used in a project called the Measure Up (MU) project in a primary school in Hawaii. In this project, children compared different quantities such as length, volume, mass or area and developed an understanding of reflexive, transitive and symmetric properties (Dougherty, 2007). Students also compared and discussed quantities of liquids in vessels and soon became able to represent these quantities by symbols. Initial findings from the MU project suggest that children can solve complex problems if they are framed in a context which is familiar and relevant to them.

Sutherland (2004) developed a similar program called the Elaborated Davydov Approach (EDA). EDA starts from the general and moves towards the particular. In EDA, measurement contexts are used to represent mathematical concepts. The focus is on comparing the measurable components of quantities such as length, mass, area and volume. For example, children in Year One compare one, two and three dimensional attributes of objects before working with numbers. Then they use drawings and oral statements such as *more than*, *less than*, *longer than*, or *shorter than* to compare the attributes. Students then begin to use letters to represent the attributes and create statements such as $L > C$ to compare the masses of two bags of rice. Students learn to become comfortable with comparing three quantities without using numbers and they are able to understand and express the relationships without direct comparison. For example, a 6-year-old student may explain that since area G is greater than area L, and area L is the same as area P, therefore area G must be greater than area P.

Another teaching approach based on problem solving is called the Realistic Mathematics Education Approach (RME), which has its origin in the Netherlands. In RME a context is considered realistic not only if it refers to a real context but also if it is meaningful and real to the students. Students use real contexts and models to develop mathematical concepts. They are encouraged to reinvent mathematics and think critically, ask questions, reason, explain and justify their answers using the instructional materials under the guidance of their teachers. Various instructional modes such as group work, individual learning, pairs, with and without technology, are used. Mathematics is learned as one complete strand. The research of Wijers and Reeuwijk (cited in Sutherland, 2004) indicated that when the problem

> Two tee-shirts and two sodas cost $44, and one tee-shirt and three sodas cost $30. Find the cost of one tee-shirt and one soda

was presented using pictorial representations of tee-shirts and sodas to help students learn about simultaneous linear equations, students invented several different ways to solve it, including formal and informal solution strategies. By the end of the unit, all students understood formal notation and were able to solve linear equations. The main difference between the RME and the EDA is that the RME proceeds from the particular to the general while the EDA goes from the general to the particular.

Currently, the focus has shifted from problem solving to modelling and from the problem-solving approach to the modelling approach (Kieran, 2007). In a typical modelling activity students design a model by interpreting a real life situation mathematically. For example one modelling activity designed by Lesh and Harel (2003) is called ‘The big foot’ activity where students develop a procedure to predict the height of a person from the size of their footprints. Another example of a modelling activity is to ask students to develop a procedure for finding the amount of paint needed for a new car when students were given information about the inner and outer dimensions of the car (Mousoulides, Christou, & Sriraman, 2008). Students then test the validity of their procedure by using data from different cars to find a general procedure (for example an algebraic equation) which can be used to find the amount of paint required for any car. The procedure is then refined on the basis of the validity result. During such an activity students learn to sort and select data, create and design mathematical procedures and identify and express relationships between variables. Modelling a problem set within a real and authentic context facilitates students’ appreciation of variables and relationships between variables (Chinnappan, 2010). Even young students of primary school age can be engaged in modelling activities. During a teaching experiment, Lehrer and Schauble (2000) found that students of Years 1-4 were able to create their own language, notation system and analogies to express their thoughts when they were asked to develop their own classification system.

Using realistic situations to model mathematical ideas is challenging for many students (Schliemann, Carraher, & Brizuela, 2001, 2006) as the mathematical abilities required for modelling are quite different from the mathematical abilities usually taught in schools (Lesh, 2003). When students are given real contexts for modelling activities or some contextualised word problems, they prefer to use arithmetic problem-solving methods or sometimes use the given context to find the answer to the problem (Stacey & MacGregor, 1999). One reason for this preference is that expressing word problems algebraically is challenging for many students (Koedinger & Nathan, 2004). Another reason is that realistic problems do not always encourage students to use algebra as it appears that reasoning and symbolising develop as independent capabilities (van Ameron, 2003).

3.1.2 Generalisation approaches

"At the heart of teaching mathematics is the awakening of pupil sensitivity to the nature of mathematical generalisation" (Mason, 1996, p. 65), where generalisation refers to the ability to see "general in particular" and "particular in general". For example, students who notice that 2 + 4 = 4 + 2 and that 1344 + 1442 = 1442 + 1344 may generalise that $a + b = b + a$ is true for any real numbers a and b (Carraher & Schliemann, 2007). The teaching approach which recommends the use of generalisation activities or algebraic reasoning in lessons is classified as the generalisation approach.

Here, *generalisation* means extending the range of students' reasoning and communication beyond the case or cases being considered to start looking at the patterns, relationships and structures among and across the situations. Generalisation can begin from sources inside mathematics such as by reasoning and communicating about a mathematical system, its structure and properties, or it can begin from sources outside mathematics, for example by reasoning quantitatively about the relationships in a modelling situation (Kaput, 1999). The processes in which students generalise from particular instances and then represent those generalisations in appropriate ways are considered *algebraic reasoning* (Blanton & Kaput, 2005).

To understand mathematical generalisations, students can begin their work with a family of number sentences that indicate an underlying mathematical relationship (Fujii & Stephens, 2001). For example, the number sentence 78 – 49 + 49 = 78 belongs to a type of number sentence which remains true if we use any other number instead of 49. This sentence is also true if the first number, 78, is replaced by any other number. Fujii and Stephens (2001) noticed that these numbers are being used just like variables and so they called them *quasi variables.* Their teaching approach, which they called the quasi-variable approach, was based on drawing children's attention to the underlying structure of the number sentence rather than focusing on computation.

Another way to approach generalisation is to express general relationships by translating from words to algebraic symbols (Brown & Coles, 1999). Brown and Coles found that phrases such as 'twice a number plus three', 'is three less than' and 'add three and double the number' encourage the use of algebraic symbols by students. Students in the middle years were able to represent relationships between variables in given situations in problems such as

> How many people can sit around a line of tables given that there can be two people on either side and one at each of the ends?

after using generalisation activities (Arcavi, 1994).

Students can also express algebraic ideas and relationships using tables, graphs, number sentences and words without going into formal algebraic notation (Carraher & Schliemann, 2007). For example, students can use the distributive property to compute 18 × 15 by computing 18 × 10 and 18 × 5 and then adding the results. If children can use reasoning processes to solve problems which mathematicians usually solve by using algebraic notation, Carraher et al. (2007) consider these reasoning processes to be algebraic. They proposed a curriculum for primary school based on promoting algebraic reasoning.

Similar ideas were expressed by Blanton and Kaput (2005) when they recommended that generalisation should be fostered in the primary school years (without going into the formal notations of algebra) so that students could better understand the complexity of mathematics in general and algebra in particular (Blanton & Kaput, 2005). They studied three teachers and their use of algebraic reasoning in their Year 3 classes. They observed that when the teacher of an experimental class integrated algebraic reasoning based on number generalisations and made a sustained effort to give students the opportunity to reason algebraically, the students not only outperformed the control group on selected items of the Massachusetts Comprehensive Assessment System (MCAS), they performed as well as average fourth graders.

When students reason with algebraic relationships, they can notice patterns and relationships between quantities and make generalisations in situations which are meaningful for them (Kaput, 1999). However, this does not mean that students will always make meaningful generalisations from contextually rich problems. For example, while the generalisation of patterns such as match stick patterns can aid in developing the concepts of sequence and function, the link between symbolic representation and the number pattern is not easily established (Healey & Hoyles, 1993). Moreover, while completing generalisation exercises students may concentrate on those aspects of the generalisation technique which can be automatically done without recognising the underlying structure (Mason, 1996).

One reason for student difficulties in generalising patterns is that the generalisation process requires additional skills which are not traditionally associated with algebra (MacGregor & Stacey, 1993; Warren, 2000, 2003). For example, Quinlan (2001) indicated that students find it very difficult to generalise from particular geometric patterns and very few students succeed in expressing pattern generalisations in algebraic form. For example, in the 1998 New South Wales state examinations only 15% of 78 000 students were able to correctly describe a geometric pattern algebraically. Similarly, Blanton and Kaput (2005) found that teachers need time to learn how to use generalisations in the classroom, and sustained professional development is necessary for elementary teachers so that they not only understand the complexity of algebraic reasoning but also learn viable ways to integrate it into their instruction.

3.1.3 Functional approaches

Teaching approaches which use multiple representations define themselves as functional approaches. Mathematical relationships, principals and ideas can be expressed in visual (diagrams and graphs), verbal (words) and symbolic representations (letters) (Mousoulides et al., 2008). Carraher and Schliemann (2007) argue that instead of introducing algebra with symbolic algebraic expressions and manipulations of expressions, which can result in superficial learning, algebra can be introduced using real-valued functions, graphs or tables of values. Algebraic representation of functions can be introduced later on when students are familiar with graphical and numerical representations. By using functions as the fundamental objects of algebra instead of equations, the algebraic character of many mathematical topics becomes more evident (Schwartz & Yerushalmy, 1992). Moreover, a multiple-value image of variables is promoted when graphs are used to display calculations and algebraic expressions are considered as representations of functions (Chazan & Yerushalmy, 2003).

Research shows that multi-representational environments can not only help students in building links between symbolic, numerical and graphical representations in algebra (Balacheff & Kaput, 1996), but they can also have a positive effect on students' understanding of functions (Kieran, 2007). Due to the increased use and availability of computers and hand-held computer algebra system devices, it is now easy to display the algebraic and graphical representations of equations or functions at the same time: Any action performed on the algebraic equation has a corresponding effect on its graphical representation. If students are able to link the change in an algebraic expression with the corresponding change in its graphical representation, they will be better able to understand the meaning of algebraic manipulations and, in turn, algebraic structure (Kaput, 1989). Representational fluency has now become an important skill due to the increased use of computers and related devises for computation, representation and communication.

Technological environments aid in building links between different representations, which helps in developing algebraic thinking (Balacheff & Kaput, 1996; Friedlander & Tabach, 2001). An example of such an approach is the Compu-Math curriculum which was developed by Hershkowitz et al. (2008). In this curriculum, Excel spreadsheets are used as a tool for transformation and generalisation from arithmetic to algebra. Hershkowitz and colleagues found that spreadsheets provided students with an opportunity to use multiple generalisation strategies for solving the same problem. When students were unable to solve a linear or quadratic equation using algebraic methods, they went back to numerical strategies and used spreadsheets as a supportive tool. Spreadsheets were effective for the study of variation, investigating of the properties of linear functions, and for solving linear equations. The use of spreadsheets also helped students to focus on general relationships between variables, which then provided conceptual meaning to algebraic objects such as variables, expressions and equations. However, students faced difficulties when they used spreadsheets for solving inequalities, quadratic equations and rational functions (Dettori, Garuti, & Lemut, 2001; Kieran, 2001). Hershkowitz et al. (2002) also noticed that it was difficult for students to generate a generalised rule from a pattern as students tended to focus on the relation between successive terms rather than the relation between a term and its position in the sequence.

Carraher, Schliemann and Brizuela (2000) used functions to develop algebraic thinking in primary school students. They designed an early algebra program (EA) in which students compared different functions using graphs and tables, and then discussed different representations. Open-ended problems were presented to students who were encouraged to discuss amongst themselves and with their teacher the possible solutions to these problems. After three longitudinal studies, Schliemann et al. (2001, 2006) concluded that reasoning about variable quantities presented a natural setting for introducing functions and variables. At the beginning of the three-year EA program, students gave no answer when they were asked about a variable. Then, gradually they began to consider a variable as an unknown quantity and eventually they moved towards the concept of a variable having multiple values and finally from multiple values to a generalised number (Carraher & Schliemann, 2007). However, when students worked with tables of values, they were only able to recognise the pattern in the values of one variable and they were not able to interpret the functional relationship between the two variables represented in the table (Schliemann et al., 2001, 2006). A solution was to use letters to represent the first variable in the function table, as this was found to enable students to focus on the general rule relating the two variables (Schliemann et al., 2001, 2006).

Nathan, Stephens, Masarik, Alibali and Kroedinger (2002) investigated middle secondary school students' ability to solve problems using different representations—such as tabular, graphical, verbal and symbolic—without using technology. They found that students were more successful in using a given representation rather than a range of representations to solve problems, since they experienced difficulties in translating between different representations. Thus, they suggested that students spend more time on tabular and graphical representations before describing a representation in words or in a symbolic form. Similar results were obtained by Yerushalmy (2000) when she studied the long-term impact of a problem-based functional approach on the teaching of algebra. She found that students initially used numbers for modelling problem representations and moved on to graphs and tables and finally to symbolic representations. However, she also observed that students preferred to use the problem situation to answer the problem instead of using the symbolic representation. She concluded that helping students to appreciate the algebraic symbols requires more than just linking between representations.

There are certain criticisms of the functional approach. Some researchers believe that algebra is not only about functions and that using graphical representations of algebraic forms instead of the algebraic forms could promote the view that all algebraic expressions are functions (Pimm, 1995). Also students sometimes cannot differentiate the equivalence of two equations from the equivalence of two functions (Chazan & Yerushalmy, 2003). According to Wheeler (1996), there is a need to investigate the functional approach further to see what ideas about the variable are being conveyed by this approach and how students interpret the relationships between the variables and the links between the different representations (Wheeler, 1996).

3.2 ESSENTIAL ELEMENTS OF A SUCCESSFUL ALGEBRA PROGRAM

Any successful algebra program should contain aspects of each of the above mentioned approaches in order to develop the concept of a variable and the skills of algebraic reasoning while also ensuring that algebra is interesting and meaningful for students (Bell, 1996). After learning about algebra, students should be able to use the tools of generalisation, equations and functions to express relationships and manipulate algebraic expressions, and they should have the flexibility to work with different representations (Bell, 1996). Moreover, algebra should not be taught as an isolated subject. Instead it should be well integrated with other subject areas within and outside mathematics (Kaput, 1995). These principles guided the design and implementation of the MVA.

3.2.1. Developing the concept of a variable

In a research project supported by the Nuffield foundation, Nunes, Bryant and Watson (2010) recommended that students should be taught to understand the meaning of what is being expressed by the variables. Learning and teaching in algebra should focus on two fundamental ideas. The first is the concept of a variable (as an unknown quantity, as a generalised number, and as a varying quantity) and the second is the use of tables, symbols, graphs, formulas, equations, arrays, identities and relations for discovery and invention (Wheeler, 1996). Instead of emphasising the manipulation of algebraic expressions and the solving of equations, teachers should begin algebra instruction by focusing on the multi-faceted use of letters in algebra (Warren, 2003).

Developing a multifaceted concept of variables (as unknown quantities, generalised numbers and as functional relationships) should thus be an essential element of any algebra program. It is not only necessary for students to consider a variable as a generalised number which can attain decimal and fractional values, but it is also important for students to realise that different letters may not have different values, that a letter can have different values in the same problem, and the same letter may have different values in different problems (Warren, 2003; Watson, 2010). The understanding that x is a generalised number and can attain multiple values, amounts to crossing the bridge from considering a letter as an unknown quantity (as in $x + 5 = 7$) to considering a letter as a variable quantity (as in the equation $2x + 3y = 8$). If the differences between different aspects of variables are not made explicit during instruction, they may cause difficulties later for students (Watson, 2010).

In the “three uses of variable” 3UV teaching model (Trigueros & Ursini, 2001), algebra is introduced through the concept of variables in two phases. In the first phase, three uses of variables (namely, unknown number, generalised number and functional relationship) are studied separately. In the second phase, these different aspects are integrated through the use of problem-solving activities. The two phases are repeated in a spiral of increasing levels of complexity. For example, in the initial stage teachers can use problems such as the following:

1. Find x if $2x + 5 = 0$.

2. What does x represent in the expression $2x + 5$?

3. For what values of x and y is $2x + 5 = y$ true?

In a later stage, teachers may present problems such as:

1. Find x if $(3x + 4)(5x + 7) = 0$.

2. Find an expression equivalent to $(3x + 4)(5x + 7)$.

3. Given $f(x) = (3x + 4)(5x + 7)$, find the interval in which $f(x)$ increases.

The purpose of the 3UV approach is to integrate and differentiate between the different aspects of variables so that a multifaceted conception of variables can evolve. However, the results of the 3UV teaching model have not been reported so it is not possible to be sure whether or not students taught via the 3UV model can differentiate and understand the different facets of variables.

3.2.2. Promoting algebraic reasoning

Although developing the concept of multifaceted variables is very important in any algebra program, fostering algebraic reasoning or algebraic thinking is equally important. The shift from arithmetic thinking to algebraic thinking requires the development of three abilities: First, the ability to represent a problem in words and then in algebraic expressions or equations; second, the ability to shift from using arithmetic problem-solving methods such as guess and check, to algebraic problem-solving methods such as the balancing method; and third, the ability to understand the meaning of the variables used and their relationship to the arithmetic models they represent (Warren, 2003). Learning to reason algebraically could not only develop children's understanding of arithmetic in early grades, it could also help children in the transition from arithmetic to algebra in the later grades.

Warren (2003) argued that many students leave primary school without understanding the commutative and associative properties of numbers. Moreover, students have difficulty in understanding the structure of numbers because they have a limited experience of exploring relationships, including conjecturing and induction. An early start on examining the group properties of numbers and recognising operations as general processes is very important to facilitate algebra learning (Warren, 2003). A deep understanding of arithmetic requires generalisation, which is algebraic in nature. Focusing on generalising activities in order to understand the algebraic structure and the concept of a generalised number should be an important part of any algebra program.

It is also important to note that suitable teaching practices are essential to develop students' algebraic reasoning ability (Blanton & Kaput, 2005). A suitable teaching practice is one in which the teacher starts from particular examples and gradually leads students towards the discovery of a general result. Students then operate on the generalised objects thus obtained and subjugate these objects to further reasoning.

3.2.3. Breadth and integration

The National Council of Teachers of Mathematics (2000) recommended that instead of teaching algebra as an isolated course, algebra learning should be deeply interwoven with other mathematics strands throughout the K-12 curriculum. Further, Blanton and Kaput (2005) proposed that instead of an intensive course targeting algebra in secondary school, students would benefit from being involved in algebraic thinking throughout their school years. Kaput (1995) proposed a three-dimensional algebra reform process focusing on breadth, integration and pedagogy. Breadth refers to all aspects of algebra in the curriculum: generalised arithmetic and quantitative reasoning, functions and relations, structures, modelling, and manipulation of formalisms. Integration of algebra with other mathematical and non-mathematical subjects is the second element of the algebra reform. According to Kaput, making links with other subjects such as statistics, computer science, business studies, and economics would not only make the study of algebra more meaningful but it would also enrich the study of the other subjects. This integration would reveal the similarity of ideas and the common structures across domains. The realisation that algebra can be learned while learning other subject matter would not only help students realise the power of algebra but would also help them in understanding it.

3.2.4. Making algebra meaningful and interesting

What students notice and perceive depends to a large extent on the classroom culture. Thus it is necessary for teachers to build a classroom environment which encourages students to learn with understanding and interest (Piaget & Moreau, 2001). Facilitating productive classroom discourse fosters students' ability to engage with mathematics meaningfully and learn with understanding (Walshaw & Anthony, 2007).

Concrete materials such as cups to denote variables and counters to model the numerical value of the variable can be used to introduce algebra to students, since use of concrete materials helps in maintaining student interest (Quinlan, 2001). Concrete materials can also facilitate students in generalising activities (Pegg & Redden, 1990b) for example, the use of patterns made from counters or match sticks and the use of cups or envelopes to represent the variable *x* and hidden blobs in cups or envelopes to represent the numerical value of the variable *x*

Stacey and MacGregor (1999) noticed that most problems used in school algebra textbooks were very easy and could be solved without using any algebraic method. The exercises tended to place more stress on learning to solve an equation rather than on formulating an equation from a word problem. Therefore, Stacey and MacGregor recommend the use of complex word problems to demonstrate the usefulness and power of algebraic methods and the use of real contexts to provide clear referents for the variables involved, in order to make algebra learning meaningful for students.

Discussing, comparing and talking about the rules governing number patterns can also develop student interest in algebra. Pegg and Redden (1990a) recommended activities based on number patterns to introduce algebra in junior secondary school using multiple representations such as numerical, verbal and algebraic expressions. Number sequences set in pictorial contexts provide students with the facility to interpret and appreciate the structure of the pattern and also give meaning to the variables involved (Bell & Malone, 1992). Patterning activities not only develop interest in algebra, but they can also help students develop mathematical thinking and make them aware of mathematical generalisations (Mulligan & Mitchelmore, 2009).

All the above elements are used to varying extent in the design of the Multifaceted Variable Approach (MVA). In the next section the MVA is described, as are the ways in which the current New South Wales 7-10 Mathematics syllabus (Board of Studies NSW, 2002) was rearranged to implement the MVA in the secondary school which participated in the present study.

3.3. THE MULTIFACETED VARIABLE APPROACH

Kieran (2007) classified school algebra activities as generational, transformational and global/meta-level. Generational activities are those in which algebra is used as a language to express meaning for example, the formation of expressions and equations from problem situations and pattern generalisation exercises. Transformational activities are skill-based activities such as the simplification of algebraic expressions, factorising, and the solution of equations and inequalities. Global/meta-level activities are activities which provide meaning and purpose to algebraic activity such as problem solving and modelling and studying change in functional relationships.

In the MVA, meta-level and generational activities (Kieran, 1996) embedded in contextual word problems are used to make variables meaningful for students. Meta-level and generational activities were specifically chosen because they are used to introduce the functional aspect of variables in parallel with generalised numbers. The aspect of variable as an unknown quantity evolves naturally as a consequence of studying these two aspects (generalised number and function). Note that the aspect of a variable as an unknown quantity is a special case of a variable (which can attain more than one value). It is indicated by previous studies (see for example, Perso, 1991) that students who have misconceptions regarding variables make simplification errors. This is the reason that the MVA adopts the use of meta-level and generational activities before getting students involved in transformational activities.

In the MVA, teachers use real contexts as a starting point of discussion about variables, expressions and equations. Real contexts make it easier for students to deal with complexity as they enable them to approach new tasks using ideas and situations familiar to them (Nemirovsky, 1996). In the MVA, word problems framed in real contexts are presented to students so that they can make links between the unknown in a linear equation with a real context. The use of contextual word problems not only makes algebra learning meaningful and interesting (Dougherty, 2007; Stacey, 1999) but also develops the skills of interpreting and representing (for example, describing word problems numerically and in an algebraic form) (Carraher & Schliemann, 2007), reasoning (to select a suitable strategy) (Stanic & Kilpatrick, 1989) and simplifying (to find a solution to the word problem) (Bell, 1996).

MVA activities are also designed to promote higher order thinking. Students are encouraged to use inverse operations and working backwards to solve simple linear equations so that they not only know the “how” but also the “why” of the solution process. The aspect of variables as an unknown quantity is promoted when students solve linear equations based on word problems. When students solve a linear equation, they are in search of a numeric quantity which satisfies the given equation. For example, the possible solutions of the linear equations included in the algebra textbooks (usually used) for Year 7-Year 8 belong to the set of integers. Therefore, the idea that an unknown “*x*” always represents an unknown quantity (for example an integer) is promoted. This necessitates the inclusion of problems whose solution is a generalised number or an algebraic expression. Therefore, some elements of the generalisation approach are also included in the MVA such as the generalisation of number properties, extending a numeric or geometric pattern, finding the general algebraic expression representing that pattern and solving a word problem whose solution is an algebraic expression instead of a number.

According to Mason (1996) generalisation refers to the ability of seeing “general in particular” and “particular in general”. The MVA advocates the development of both these abilities side by side because the concept of variables as generalised numbers is promoted when students move from the general to particular and vice versa. The move from general to particular is made when students identify the relationships between variables in contextual word problems or reason quantitatively about the relationships between the variables (Carraher & Schliemann, 2007). The relationships thus identified are expressed in the form of an expression or equation representing the problem so that it can be solved. This is also in alignment with the RME approach which recommends a move from particular to general.

Students move from particular to general when they extend and generalise patterns (in number and geometry) derived from contextual word problems to find a generalised number representing the pattern. In addition, students study the generalisation of number properties using the generalisation techniques suggested by Blanton and Kaput (2005). For example by generalising number sentences such as $2 + 4 = 4 + 2$, $100 + 400 = 400 + 100$, therefore $m + n = n + m$. The activities used by Carraher and Schliemann (2007) with different representations such as tabulated numbers and graphs of functions to find a relationship between the variables are also included in the MVA. Note that the teaching resources used in the MVA use attributes of objects such as length, area and volume to discuss variables. This is in agreement with the philosophy of the EDA in which students com-are the attributes to learn algebra (Dougherty, 2001).

Reasoning about variable quantities also provide a natural setting to introduce functions and variables (Carraher et al., 2000). In the MVA, Excel spreadsheets are used as tools for transformation and generalisation from arithmetic to algebra. Translating between multiple representations such as table of values, graphs and algebraic expressions promotes the concept of a variable as a functional relationship. When students study variables expressed as functions and translate between numerical, graphical and algebraic representations, the idea that a variable can represent many different numbers at a time (for example tabulated values) is promoted.

Suitable teaching practices are also essential to develop students’ algebraic reasoning ability (Blanton & Kaput, 2005). In the MVA, teachers are expected to incorporate the five *Working Mathematically* processes from the syllabus (Board of Studies NSW, 2002) in their lessons through activities for students to question, apply strategies, communicate, reason and reflect. Note that these five processes are aligned with the idea of reinventing mathematics as recommended by the RME program. RME recommends that every student learns mathematics by same stages and processes through which experienced mathematicians pass when they solve a problem. For example, when students are presented with a real word problem and they think of a suitable strategy to solve that problem, they discuss possible solutions by working with their peers, reason and communicate, refine and reflect their model until a suitable model is obtained which is finally used to find the solution. In this way every student reinvents mathematics and works like a mathematician.

The MVA is also different from the EDA and RME. In the MVA, students use particular examples to find a general algebraic expression representing the problem and also use quantitative reasoning to compare variables given in a problem. In this way the MVA moves from particular to general and also from general to particular. Moreover, the problem presented, in the teaching resources used in the MVA, are more structured than the problems presented in the RME approach. The word problems in the MVA teaching resources are structured in the sense that each problem is further subdivided into smaller problems which gradually lead students towards a final solution to the main problem. There is no evidence that the EDA was used in a secondary school to teach algebra. However, it was not used to facilitate students to move from arithmetic reasoning to algebraic reasoning in primary school.

In the MVA, students learn about the three aspects of variables (unknown quantity, generalised number and function) first before moving on to simplification of algebraic expressions. In this regards the MVA is in agreement with the 3UV approach. However, instead of learning these three meanings one by one, as suggested by the 3UV model, the MVA incorporates the different aspects of variables in parallel with each other. It is anticipated that the experiences of learning all meanings together will promote a richer understanding of variables. Moreover, while the 3UV model is based on a syllabus sequence, the MVA uses a combination of teaching resources, syllabus sequencing, and teaching techniques to promote a multifaceted concept of variables.

The current syllabus sequence for Years 7 and 8 in New South Wales is rearranged in the MVA so that students can study variables and expressions before learning about the simplification of algebraic expressions and the solution of equations. The current New South Wales syllabus and the proposed MVA syllabus sequence are described here.

3.3.1 The MVA in the present study

The Patterns and Algebra outcomes for Years 7 and 8 specified in the New South Wales 7-10 Mathematics Syllabus are as follows.

3.3.1.1 Patterns

1. Recognise, describe, create and continue increasing and decreasing patterns
2. Find missing elements in a pattern, build number relationships
3. Complete simple number sentences by calculating missing values
4. Create, record, analyse and generalise number patterns using words and algebraic symbols, represent number pattern relationships on a grid

3.3.1.2 Algebraic Techniques

1. Addition, subtraction, multiplication and division of algebraic expressions.
2. Use letters to represent numbers, translate between words and algebraic symbols.
3. Expand and factorise simple algebraic expressions
4. Solve linear equations and simple inequalities
5. Apply the index laws to simplify algebraic expressions
6. Simplify, expand and factorise algebraic expressions involving fractions, negative or fractional indices, solve linear and quadratic equations and linear inequalities, solve simultaneous linear equations using graphical and analytical methods.

3.3.1.3 Linear Relationships and Coordinate Geometry

1. Graph and interpret linear relationships created from number patterns and equations in a number plane
2. Graph linear and simple non-linear relationships from equations
3. Apply the gradient intercept form to interpret and graph straight lines

Teachers in New South Wales typically cover Patterns outcomes 1-4 and Algebraic Techniques outcomes 1-2 in Year 7. In Year 8, students study Algebraic Techniques outcomes 3-6 and Linear Relationships and Coordinate Geometry outcomes 1-3.

The MVA was designed to ensure that students would complete the all of the recommended syllabus outcomes in Patterns and Algebra by the end of Year 8; however, they do so in a different order. In the MVA, students study Patterns outcomes 1- 4, Algebraic Techniques outcome 2, and Linear Relationships and Coordinate Geometry outcome 1 in Year 7. In Year 8, they study Algebraic Techniques outcomes 1-4 and Linear Relationships and Coordinate Geometry outcomes 1-2 along with generalisation of number properties. As students study patterns and functions together in Year 7, the multifaceted character of variables is highlighted. Then they move on to the solution of linear equations and the generalising of number properties such as commutative, associative and distributive properties in Year 8. The MVA does not change the content of the current school algebra syllabus but it facilitates students in understanding of the meaning of variables before moving on to the solution of linear equations.

The purpose of the present study is to investigate the effect of the MVA on student understanding of the variable concept and on how this acquired understanding affects the general algebraic competency of students (such as in representing and solving word problems, simplifying algebraic expressions, and solving linear equations). The extent to which the teachers of the experimental group followed the MVA and integrated key elements such as working mathematically, linking algebra within mathematics and outside mathematics with real life to make algebra interesting and meaningful was also investigated. The research questions for the study were presented in Chapter 1 (page 4).

3.4 SUMMARY

Every teaching approach has its own merits as each approach fosters different aspects of algebraic awareness (Bell, 1996). For example, the concept of a variable as an unknown quantity is associated with the problem-solving approach. Students can learn mathematical reasoning, critical analysis, representation skills, different ways of solving equations, and different techniques for solving word problems (including heuristics). Problem solving is an enjoyable activity and this is another reason for including problem solving in the algebra curriculum. Solving real life problems provide a purpose to the mathematical activity. The concept of a variable as a generalised number is associated with the generalisation approach (Radford, 1996). The ability to generalise can be fostered in different ways: using numbers as quasi-variables; drawing students' attention to the associative, commutative and distributive properties; using large numbers; using multiple representations such as graphs, tables of values and algebraic expressions to express relationships; translating from words to algebraic symbols; and by finding rules from patterns. Many researchers recommend inclusion of generalisation activities in lessons and there is evidence that solving problems which require generalisation of relationships facilitates the learning of algebra. The idea of variables as having multiple values and an equation representing a functional relationship is projected when the functional approach is used (Chazan & Yerushalmy, 2003). Functional relationships can be represented by tables of values, graphs or algebraic equations. Due to the increased use of computers and graphing calculators it is possible to see the algebraic, graphical and tabular representations side by side. Thus any change in the algebraic equation generates a corresponding change in the graphical and tabular representations. Recognition of this change can facilitate students in understanding algebraic manipulations (Kaput, 1989).

However, any teaching approach on its own cannot claim to solve all student problems in algebra. For example, word problems and real life modelling problems may not develop the skills of symbolisation as students have been found to turn towards the arithmetic problem solving methods for finding the solution instead of using the algebraic representations of the problem (Stacey & MacGregor, 1999). Some generalisation activities such as generalising from geometric patterns are difficult for students (MacGregor & Stacey, 1993; Quinlan, 2001). Moreover, the Functional Approach may not encourage students to use algebra as students often use numerical and graphical representations of the word problems for finding solutions instead of using algebraic representations of the problem (Yerushalmy, 2000). Also, students may prefer to use one representation for finding a solution instead of using multiple representations since translating between different representations requires additional skills which pose additional difficulties for students (Nathan, et al., 2002).

The Multifaceted Variable Approach uses a combination of the problem-solving generalisation and functional approaches so that a multifaceted concept of variables is promoted. In MVA, students solve word problems based on real contexts so that they can learn mathematical reasoning, critical analysis and representation skills. In addition, students use generalisation of patterns and number properties and numerical, graphical and algebraic representation of functions to experience variables as generalised numbers. Students learn about the multifaceted aspects of variables before learning how to manipulate algebraic expressions and solve linear equations. MVA draws on the essential elements of a successful algebra program as informed by the research.

This study investigates the concept of a variable developed by the MVA and the effect of the acquired concept on students' solution of linear equations and manipulation of algebraic expressions. The next chapter describes the methodology of the study.

CHAPTER 4

METHODOLOGY

4.1 INTRODUCTION

This chapter outlines the methodology used to investigate the effectiveness of the Multifaceted Variable Approach (MVA) to teach algebra in Years 7 and 8. A longitudinal teaching experiment was completed in two phases: Phase I with students and teachers of Year 7 and Phase II with the same cohort of students and teachers in Year 8. In the following sections the methodologies of Phase I and Phase II are described separately. The re-ordering of syllabus outcomes and the tools for data collection are also described.

The purpose of the study was to investigate the effectiveness of using the MVA in developing the students' concept of a variable, and their general algebraic competence, especially in enhancing students' ability to solve linear equations. For this purpose, two groups of students from the same school were taught a beginner algebra course using different teaching approaches. Teachers of one group (the experimental group) used the MVA and teachers of the other group (the comparison group) used a traditional teaching approach. While both groups were taught the content required for Years 7 and 8 of the syllabus of the New South Wales 7-10 Mathematics Syllabus (Board of Studies NSW, 2002), the content was ordered differently. The experimental group used the teaching resource "Activities that Teach Patterns and Algebra" (McMaster & Mitchelmore, 2007a, 2007b) and studied the three aspects of variables (unknown quantity, generalised number and function) in Year 7. They moved on to the manipulation of algebraic expressions and the solution of linear equations in Year 8. The comparison group studied variables using patterns and algebraic manipulations (addition, subtraction, multiplication and division) in Year 7. They then moved on to the solution of linear equations in Year 8. By the end of Year 8, both groups had completed the same content.

4.2 SAMPLE

Seven metropolitan high schools in Sydney were invited to take part in the algebra research study but only one agreed to participate. The participating school, which we will call Cara Girls' High School, is a Kindergarten to Year 12 independent girls' school situated in a high socio-economic area. Every classroom in Cara has an interactive whiteboard which teachers can use to access their teaching plans, lesson resources and the internet. There are also four designated computer laboratories in the school which teachers can reserve for lessons according to their requirements. The school library is well equipped with computers, all with internet access. There is a separate professional library in the mathematics staffroom containing professional and teaching resources, including computer software. New books and teaching resources are continuously added to the stock already available in the library.

Cara operates a regular internal professional development program for teachers and also encourages teachers to participate in external professional development activities. In particular, the head teacher of the Mathematics Department regularly updates the staff on current research developments in the field of mathematics education. Some mathematics teachers also act as mentors to pre-service teachers who are completing their school-based professional experience.

The mathematics classes in Year 7 (ages 13-14 years) are streamed into four graded classes according to students' mathematical ability level as determined by a test of general mathematical understanding which is administered immediately before admission into Year 7. The four classes are referred to here as Set 1 (the high ability class, 28 students), Set 2 (the high-medium ability class, 28 students), Set 3 (the medium ability class, 29 students) and Set 4 (the low ability class, 21 students). No students changed classes during the year, but eleven students were moved between classes after completion of Year 7 according to their results in the yearly examination. These students were removed from the sample for the purposes of data analysis in Phase II.

The experimental group was chosen to be of a lower mathematical ability than the comparison group so that if the experimental group's performance was better after the MVA intervention, it could not be attributed to higher mathematical ability. The two teachers of the experimental group and the two teachers of the comparison group volunteered to participate in this study. Table 4.1 lists the pseudonyms used, and the age range and teaching experience of each participating teacher at the start of the project.

Table 4.1: *The participating teachers*

	Class	Teacher Pseudonym	Age range	Teaching Experience
Experimental Group	Set 2	Rosa	35-39	Between 6 and 7 years
	Set 4	Mona	25-29	Between 1 and 2 years
Comparison Group	Set 1	Amy	25-29	Between 1 and 3 years
	Set 3	Ben	25-29	Less than 1 year

4.3 PHASE I

The school principal and the teachers were contacted by email and were forwarded a Principals' Information and Consent Form (Appendix A) and a Teachers' Information and Consent Form (Appendix B) which described the nature and purpose of the study. Professional Development Workshop I was conducted at the beginning of Phase I of the study, after receipt of the teachers' signed Information and Consent forms in May, 2008. Parents' Information and Consent Forms (Appendix C) were given to the teachers at the start of Phase I so that they could distribute them to their classes. Table 4.2 shows the student and teacher activities and the data collected during Phase I.

4.3.1 Professional Development Workshop I

The purpose of Professional Development Workshop I was to introduce the MVA to the teachers and to plan the experimental algebra teaching program for Year 7 at Cara. Teachers of the experimental group (Set 2 and Set 4), the first author of the resource *Working Mathematically: Activities that Teach Patterns and Algebra* and the researcher participated in this workshop. The teaching resource (both as a book and as a CD) was provided to the teachers as an example of a resource which covered all of the content of the New South Wales 7-10 Mathematics Syllabus (Board of Studies, New South Wales, 2002) and was aligned with the principles of the MVA.

The workshop was held in a meeting room at the university from 9:00 am to 3:00 pm. It was split into two sessions. The first session was conducted by the researcher. In this session the teachers were introduced to MVA and how it had been informed by recent research on the learning and teaching of algebra. The second session was conducted by the first author of the resource book who outlined the structure of the resource book to the teachers, showed the teachers some sample activities, and gave some practical advice concerning their use in the classroom. Later, the teachers worked with the researcher to plan the teaching sequence for Phase I.

Table 4.2 *Student and teacher activities and data collected in Phase I*

Month (2008)	Teacher and Student Activity		Data Collected
	Experimental Group	Comparison Group	
May	Professional Development Workshop I for teachers		Teacher's practice and beliefs questionnaires Field Notes
June - August (Term 3) (13-14 lessons)	Patterns (Outcomes 1-4)* Algebraic Techniques (Outcome 2)* Expressions Linear Relationships and Coordinate Geometry (Outcome 1)*	Patterns (Outcomes 1-4)* Algebraic Techniques (Outcome 1)*	Lesson Observations (Field notes, brief teacher interviews after lessons, lesson plan feedback sheets, student work samples) Video-recordings (Transcriptions of selected segments)
August			First Algebra Test (Student work samples) Student Interview I Audio-recordings (Transcriptions and student work samples)
August –October (Term 4) (10-11 Lessons)	Linear Relationships and Coordinate Geometry (Outcome1)* Rates with continuous and discrete variables	Algebraic Techniques (Outcome 1-2)* Linear Relationships and Coordinate Geometry (Outcome 1)*	Teachers' lesson feedback sheets
December			Second Algebra Test (Student work samples) Yearly Examination (Student work samples)

*The outcomes are listed in Chapter 3.

In preparing for the first session, the teachers' readiness to participate in the study, awareness of student thinking, algebra subject matter knowledge and active involvement were considered (Borko, 2004; Cohen & Hill, 2000; Putnam & Borko, 1997). The teachers of the experimental group wanted to participate in this study as they were aware of the difficulties that their students faced in algebra and were willing to trial a different teaching approach to improve student understanding.

An important aim of the first workshop session was to raise teachers' awareness of student thinking. Awareness of student thinking aids in developing instruction designed to improve student understanding (Fennema et al., 1996), therefore the teachers and the researcher shared their knowledge about students' algebra errors. The discussion was based on their personal experiences of teaching algebra and informed by the literature (Booth, 1984, 1995; Knuth et al., 2005; Küchemann, 1981; Perso, 1991; Warren, 2003). The discussion focused particularly on student misconceptions about the concept of a variable (Perso, 1991) and on possible explanations for these misconceptions, such as those suggested by Booth (1995) and MacGregor and Stacey (1997).

Teachers' knowledge and beliefs about learning, teaching and the nature of mathematics play an important role in the construction of any new knowledge (Bishop, Berryman, Richardson, & Tiakiwai, 2003; Butterfield & Chinnappan, 2011; Carpenter, Franke, & Levi, 1998; Phillips, McNaughton, & MacDonald, 2002; Putnam & Borko, 1997). Therefore, key elements of the MVA (such as the importance of linking different representations of functions, using real contexts to make algebra learning meaningful, and simultaneously studying different aspects of variables) were explained with the help of relevant examples. The rationale and theoretical basis of the MVA was also presented.

The second session of the workshop was designed to provide teachers with an opportunity to discuss and plan the Year 7 algebra lessons they would give as part of the MVA research project. To successfully implement any teaching approach in a school, it is necessary to involve teachers in the process of planning and decision making (Loucks-Horsley et al., 1987). Teachers were free to use as little or as much from the resource *Working Mathematically: Activities that Teach Patterns and Algebra* as they wished however, they were requested to teach the topics according to the syllabus sequence of the MVA and integrate key elements of MVA (such as working mathematically, making links from algebra within mathematics and outside mathematics with real life to make algebra interesting and meaningful) in their teaching. The teachers could select activities from Chapters 1 to 7 of *Working Mathematically: Activities that Teach Patterns and Algebra* in Year 7 and Chapters 8 to 14 of this resource in Year 8 along with any other material of choice in their lessons.

Some activities from Chapters 1 to 7 were demonstrated by the author since demonstrating the use of the teaching resource is known to be a powerful training activity (Sparks, 1983). The teachers worked together to plan their algebra program and received feedback from the author and the researcher during the second session of the workshop.

It was expected that the participating teachers' subject matter knowledge and beliefs about learning and teaching would influence their interpretation and adoption of the MVA in their lessons. Therefore, data about the participating teachers' beliefs and knowledge were collected by means of a Mathematics Teaching Questionnaire (Appendix D) which was completed by the teachers at the end of the workshop. Algebra Questionnaire I (Appendix E) and Algebra Questionnaire II (Appendix F) were also completed by teachers during the first session of the workshop. The algebra questionnaires surveyed the teachers about the algebra errors of their students which they had observed, their techniques for dealing with these errors, and their general ideas about learning and teaching algebra.

At the end of the second session, the researcher and the participating teachers arranged the schedule of lesson observations for Phase I. It was also decided that the teachers and the researcher would meet briefly after each lesson observation to ensure constant support and feedback. Finally, the teachers completed a Workshop Evaluation Form (Appendix G).

4.3.2 The algebra teaching program

According to the school plan, algebra was taught to Year 7 in terms 3 and 4. The topics and outcomes are listed in Table 4.2. The experimental group used Chapters 1 to 7 of *Working Mathematically: Activities that Teach Patterns and Algebra* (McMaster & Mitchelmore, 2007a) and the comparison group used their usual textbook to study algebra. Chapters 1 to 7 of *Working Mathematically: Activities that Teach Patterns and Algebra* are titled as follows:

- Describing patterns
- Describing linear relationships
- Variables in linear relationships
- The number plane
- Patterns in linear graphs
- Patterns in non-linear graphs
- Working Backwards

The book contains pictorial, numerical and graphical representations of patterns in number and space. Problems are framed in real contexts (see Appendix H). These problems are gradually built up through a step by step process to a generalisation of the relationship between the variables. Students are then required to express that relationship in an algebraic form. Different representations of the same relationship between variables are included in the same problem. For example, the relationship between the stair number and the number of strides "*n"* taken by a child going down the stairs is expressed as an algebraic expression 197 - 3n and in words: "The stair number Remy will be on is 197 minus 3 times the number of strides she has taken.". Students interpret and translate the given relationship between variables from one representation to another. Some problems also contain different equivalent algebraic forms of the same relationship.

The teachers of the experimental group followed the sequence used in the resource, selecting activities of their choice. Before the First Algebra Test and the first round of student interviews, the experimental group completed the topics of Patterns, Linear Relationships and The Number Plane. The teachers of the comparison group used their usual textbook and completed the topics of Patterns (recording and extending number patterns, finding missing elements in a pattern, generalising number patterns using words and symbols, and determining an algebraic relationship between variables from a table of values), Linear Relationships and Coordinate Geometry (representing number patterns on a grid) and Algebraic Techniques (addition, subtraction, multiplication and division of algebraic expressions).

After the First Algebra Test (Appendix I), the experimental group moved on to Linear Relationships and Coordinate Geometry (continuous and discrete variables, independent and dependent variables, interpreting and graphing linear relationships). The low ability experimental class did not study non-linear relationships and the high medium ability experimental class solved only one or two problems from the topic of non-linear relationships. However, the technique of working backwards was learned by both the experimental classes.

The comparison group learned Algebraic Techniques (addition, subtraction, multiplication and division of algebraic expressions) and Linear Relationships and Coordinate Geometry (graphing and interpreting linear relationships created from number patterns and equations).

At the end of the year, the Yearly Examination (Appendix J) and the Second Algebra Test (Appendix K) was administered to all participating students successively.

4.3.3 Lesson observations

One algebra lesson from each of the four participating teachers was observed each week (during the algebra teaching period in June-August 2008). Out of these observed lessons, 8 lessons of the experimental group (4 lessons of Rosa and 4 lessons of Mona) and 6 lessons of the comparison group (4 lessons of Ben and 2 lesson of Amy) were video-recorded. All the observed lessons were 50 minutes long.

No lessons were observed after the First Algebra Test (between August and October) as the teachers of the experimental group integrated the remaining algebra lessons with other mathematics lessons between August to October (see Table 4.2) as per their convenience. The researcher was not informed about the timing of these lessons.

A video camera was placed at the back of the room and the researcher also made field notes concerning the student-teacher discourse in each observed lesson. The purpose of the lesson observations in Phase I was to gain insights into students' developing ideas about variables and to investigate the extent to which teachers of the experimental group implemented the MVA. Later, student-teacher discourse about variables was transcribed from the video and students' emerging conceptions of variables were analysed.

The teachers of the experimental group also provided informal verbal feedback about the teaching resources and their algebra lessons after some of the observed lessons. Field notes of these informal conversations were taken by the researcher. In addition the teachers of the experimental group were provided with Lesson Plan Feedback Sheets (Appendix L). There were two main purposes of these Lesson Plan Feedback Sheets. The first purpose was to obtain timely feedback about the algebra lessons and teachers' observations about student learning. The second purpose was to have teachers reflect on the aspect of "variable" they had taught in a lesson. Each sheet listed all aspects of variables and teachers were supposed to circle the aspect taught in that particular lesson.

4.3.4. First and Second Algebra Tests and interviews

At the conclusion of the algebra teaching period in Term 3, the First Algebra Test was administered to all students. The purpose of this written test was to assess the students' ability to extend patterns, to find general rules for patterns, and to write algebraic expressions from word problems. The test consisted of six questions: four questions related to pattern generalisations and pattern extensions, and two questions required translation from words to algebraic expressions.

Two of the questions were adapted from previous research. Question 1 was adapted from Goos, Dole and Makar (2007, p. 243) and Question 4 was adapted from Padula, Lam, and Schmidtke (2001, p. 32). The remaining four questions were designed by the researcher. Student responses were analysed to identify their errors. Students' misconceptions about the concept of a variable and their ability to translate word problems to algebraic expressions were identified through an error analysis.

After the First Algebra Test was administered and marked, the researcher conducted individual interviews with six students from each of the four classes. For this purpose, teachers were requested to select six students of varying mathematical ability from their respective classes: two students of high mathematical ability, two of medium mathematical ability and two of low mathematical ability, relative to the students in their class. Each interview lasted approximately twenty minutes and was audio-recorded and then transcribed. The purpose of Student Interview I (Appendix M) was to further probe students' ideas about variables and to provide them with the opportunity to explain their thinking by justifying their responses. This interview consisted of five items adapted from Perso (1991) that were designed to identify student misconceptions about variables. Interview responses were categorised as indicating a high (correct response with reasoning), medium (correct response on further prompting without reasoning) or low (incorrect response or a single numerical value instead of a multiple value answer) understanding of the concept of a variable.

According to the academic plan for the school, all students completed a Yearly Examination at the end of the year. This Yearly Examination contained some questions assessing the algebraic skills of extending a pattern, expressing a sentence algebraically, explaining the relationship given in a table of values, writing an algebraic equation when a word problem is given, and solving a word problem. The researcher collected these written tests and analysed students' errors to compare the understandings of students in the experimental group with those in the comparison group. Because the Yearly Examination did not assess the students' ability to represent a table of numerical values in graphical form or find a relationship between variables from a graph, a second algebra test was administered to all students after the Yearly Examination to assess these skills. One question on the simplification of algebraic expressions was also included in the Second Algebra Test. The purpose of this short test was to investigate the effect of the MVA on the students' procedural skills and to obtain some initial data about how students in the experimental group were able to perform algebraic operations before studying this topic. Their results were then compared with those of the comparison group who had studied algebraic symbol manipulations in Year 7.

4.4 PHASE II

Phase II was completed with the same cohort of students and teachers the following year when the students were in Year 8. Based on the Yearly Examination results, eleven students were moved from one mathematics class to another at the end of Year 7. Students who moved between classes were removed from the sample for Phase II. Table 4.3 shows the sequence of teaching and the data collected in Phase II.

Table 4.3 *Student-teacher activities and data collection in Phase II*

Month (2009)	Teacher Student Activity		Data Collected
	Experimental Group	Comparison Group	
February	Professional Development Workshop 2		Verbal feedback by the teachers (audio-recording)
February – March (14 lessons)	Linear Relationships and Coordinate Geometry (Outcomes 1-2)* Algebraic Techniques (Outcomes 1-3)*	Algebraic Techniques (Outcomes 3-5)*	Lesson Observations (field notes, brief teacher interviews after lessons, lesson plan feedback sheets, student work samples) Video-recordings (transcriptions of selected segments)
April (9 lessons)	Algebraic Techniques (Outcome 3-4)*	Algebraic Techniques (Outcomes 3-5)* Linear Relationships and Coordinate Geometry (Outcomes 1-3)*	Lesson Observations (field notes, brief teacher interviews after lessons, lesson plan feedback sheets, student work samples) Video-recordings (transcriptions of selected segments) Third Algebra Test
June			Half Yearly Examination, Fourth Algebra Test, Student Interviews II

*See Chapter 3[2]

[2] Algebraic Techniques, Outcome 6 was completed by Set 1 only, therefore this outcome was excluded from Table 4. Moreover, this outcome was not assessed in any algebra assessment as the same assessment was designed for all classes.

4.4.1 Professional Development Workshop II

Another professional development workshop was arranged at the beginning of the academic year in February, 2009. The researcher and the first author of the resource book and the teachers of the experimental group participated. The head teacher of the mathematics department at Cara also attended. One purpose of this meeting was to share some preliminary results of the data analysis of the First Algebra Test and the first round of student interviews, particularly regarding student misconceptions about variables. The other purpose of the workshop was to plan the Year 8 algebra lessons in light of the experience of Phase I.

After the researcher reported the results, the first author of the resource book discussed activities from the book *Working Mathematically: Activities that teach Patterns and Algebra* (McMaster & Mitchelmore, 2007a, 2007b). In particular, the author identified the activities which she believed would be more suitable for the low ability class Set 4 and also the activities which, if excluded, would not affect the completion of the State syllabus by the experimental group. A Lesson Observation schedule for Phase II was also discussed with the teachers.

4.4.2 Algebra teaching

The period for teaching algebra began at the end of February. In all, twenty-three 50-minute lessons were delivered to both the experimental and the comparison groups. The topics are listed in Table 4.3. In Phase II, the experimental group again used *Working Mathematically: Activities that Teach Patterns and Algebra*. Chapters 8 to 14 of this book are:

- Algebra in spreadsheets
- Addition and subtraction
- Multiplication and division
- Algebraic factors
- Algebraic fractions
- Equations and inequations
- Algebraic proof

Students used spreadsheets to plot graphs of functions and learned to read the graphs to answer word problems. To plot graphs of linear functions students completed the horizontal or vertical numerical tables by using the general formula. Students used the generalisation of number properties to learn the operations of addition, subtraction; multiplication and division (see Appendix H) and inverse operations and balancing method to solve linear equations.

Before the Third Algebra Test (Appendix N) the experimental group completed spreadsheets in algebra and addition, subtraction and multiplication and division of algebraic expressions.

The comparison group completed Algebraic Techniques (factorisation and expansion of algebraic expressions and solution of simple linear equations).

After the Third Algebra Test, the experimental group completed the topic of Algebraic Techniques (solution strategies for solving a linear equation). The low ability experimental class Set 4 did not complete the topics of algebraic fractions and inequations. The teacher of Set 4 selected the activities which were comparatively easier to solve. The comparison group studied Algebraic Techniques (solving a linear equation, use of index laws, and simplification of algebraic fractions and solution of inequations). Set 1 completed these topics in comparatively more detail and solved more complex problems as compared to Set 3 and also solved linear equations involving algebraic fractions.

By the end of the algebra teaching period in Year 8, both the experimental and the comparison groups had completed the same syllabus topics. The main differences between the two groups were the use of different teaching resources, the different allocations of time to particular topics, and the order in which the topics were taught. Note that the high ability class (Set 1) also did some advanced work on the same topics as mentioned earlier.

The teachers of the experimental group used spreadsheets to teach about algebraic expressions and functions, and they used real contexts to study generalisations of number properties. The teachers of the comparison group reviewed algebraic operations briefly in the traditional way, and then moved on to factorisation and expansion of linear expressions, followed by the solution of simple linear equations. Set 1 and Set 3 were mainly taught the balancing solution strategy to solve linear equations however, the solution strategies of working backwards and guess and check were also briefly covered in their lessons.

4.4.3 Lesson observations

In Phase II, the purpose of lesson observations was to investigate the influence of the previously acquired variable concept on students' solution of linear equations. Mostly, one lesson of each teacher was observed every week during algebra teaching. Out of these observed lessons, six lessons of the experimental group (2 lessons of Rosa and 4 lessons of Mona) and four lessons of the comparison group (2 lessons of Amy and 2 lessons of Ben) which were video-recorded. Not all lessons were recorded as teachers were very conscious about video recordings and this appeared to influence their teaching styles. They were more comfortable when the researcher took notes while standing at the back of the room.

To investigate the effect of the teachers' method on students' variable concept, the researcher recorded the strategies used by the teachers for solving linear equations and their directions and explanations.

4.4.4 Third Algebra Test and interview

The Third Algebra Test (Appendix N) was administered to all students at the end of the first algebra teaching period in Phase II. The purpose of this test was to analyse the students' skill at manipulating algebraic symbols and making transformations between algebraic, graphical and numerical representations. The test consisted of two sections. Section A consisted of three questions assessing the algebraic skills of factorising and simplifying algebraic expressions involving brackets and expressing a word problem in algebraic form. The algebraic skills of adding, multiplying and dividing algebraic expressions, representing word problems in algebraic form, using algebraic expressions to complete a table of numerical values, plotting a graph using tabulated values, and interpreting a graph to answer a word problem were assessed in Section B.

On advice of the participating teachers, the experimental group attempted Sections A and B together in April, while the comparison group attempted Section A only. Section B was not attempted by the comparison group at that time because the students had not reviewed the topic of interpreting a graph of a linear equation and neither had they solved word problems based on a graph. Instead, the comparison group attempted Section B after the Half Yearly Examinations in June. Unfortunately, while making this decision, the teachers did not consider the fact that the experimental group was relatively disadvantaged by attempting Section A because they had only just been introduced to the techniques of factorisation and simplification of algebraic expressions while the comparison group had spent a considerable amount of time learning these skills.

After the second algebra teaching period, the teachers designed a Half Yearly Examination (Appendix O) for all students. The examination covered the topics of algebra, trigonometry, and area and volume of plane figures and solids. The algebraic skills assessed in this examination were simplification of algebraic expressions, solution of linear equations and solution of word problems. The researcher designed one word problem and three linear equations included in this examination and the teachers of the comparison group designed one word problem and three linear equations.

The linear equations included in the Half Yearly Examination were similar to those solved by all classes during their lessons. In the case of familiar problems, it was observed that students automatically used the learned procedures. Therefore, after the Half Yearly

Examination, a Fourth Algebra Test (Appendix P) which included one unfamiliar problem was administered to all students. This problem required students to identify equivalent equations. The solution strategies used by students were tabulated and students who used algebraic methods of transformation to identify and prove the equivalence of the two equations were identified.

After the Fourth Algebra Test, twenty students (9 students of the experimental group and 11 students of the comparison group) were interviewed individually for twenty minutes. The interviews were audio-recorded. During the interviews, students were given written questions and required to solve these questions on the given sheets and to think aloud as they did so. The researcher also asked questions of the students to help reveal and clarify their thinking. The purpose was to identify the reasons for their manipulation errors and to investigate their thinking behind these errors. These 20 students volunteered to participate in the interview when the Head of Mathematics at the school asked the students in all participating classes to contact her if they wanted to be interviewed. The Student Interview Schedule II is presented in Appendix Q.

4.5 JUSTIFICATION OF THE METHODOLOGY

To study the effect of the MVA on students' conceptions of variables and their general algebraic competency, a two year teaching experiment was conducted in Cara. A teaching experiment is an exploratory tool derived from Piaget's clinical interview which is aimed at exploring children's mathematics (Steffe & Thompson, 2000). While clinical interviews are used to investigate children's current, a teaching experiment is aimed at studying student learning over extended periods of time During a teaching experiment, the actions and language of students, their errors and misconceptions are investigated (Steffe & Thompson, 2000). A teaching experiment was used in this study to collect rich data from classroom episodes over an extended period of time. The MVA includes a teaching approach, a teaching methodology, working mathematically, student engagement and student learning.

In the classical design of a teaching experiment, students are randomly assigned to two groups (an experimental and a control group), a pre-test is administered before the treatment and a post-test is administered after applying the treatment to the experimental group (Newman, 2008, p. 255). Instead of using a traditional teaching experiment design, the current study used a quasi-experimental design (Newman, 2008, p. 256). The quasi-experimental design used in the current study does not contain a pre-test or a random assignment of students. A pre-test was not conducted for two reasons. First, in case of a classical experimental design, similar item types are used in the pre-test and post-test design and sometimes student performance in the post-test is affected due to item familiarity. Second, the research design was planned to confirm to the teaching program of the school; therefore, each of the assessments was administered at the end of a pre-planned study period.

Students were not randomly selected and assigned to two groups as all the classes at Cara were already streamed according to the mathematical ability level of the students. Thus it would have been difficult to move students from one class to another without causing undue disturbance to the students and the school. The most appropriate choice in this scenario was to form groups of classes rather than students. Therefore, the high ability class (Set 1) and the medium ability class (Set 3) were chosen as the control group and the high-medium ability class (Set 2) and low ability class (Set 4) was chosen as the experimental group. This choice also ruled out the possibility that any improvements in the algebraic competence of the experimental group may be due to their higher mathematical ability rather than as a result of the proposed intervention.

There are certain practical limitations associated with the selected quasi-experimental design. These include diffusion of treatment or contamination, compensatory behaviour, experimenter expectancy, reactivity and ethical considerations (Newman, 2008). Diffusion of treatment or contamination occurs when research participants in the control and experimental groups communicate with one another and learn about the treatment of the other group. In the current study, all teachers and students were from the same school and there was a possibility that they might communicate with each other about their learning and teaching experiences. Therefore, the teachers of the experimental classes were requested not to share any details about the MVA approach or student learning with the teachers of the comparison classes. However, it was practically impossible to isolate the students. Students of the experimental group were informed that different teaching resources were being used by their teachers to enhance their algebra lessons so that they might improve their algebraic skills.

There was also a risk of compensatory behaviour as it was possible that the participating teachers and students might think that one group was being provided with a more valuable experience compared to other group. The pressure to reduce differences could produce feelings of competitiveness, rivalry or resentful demoralisation between the two groups. Due to ethical considerations it was necessary that students who participated in the study and their parents were informed about the nature and purpose of the study. They were advised that the algebra programs of the experimental and the comparison group were designed to meet all the requirements of the Board of Studies, NSW. The students of the experimental group would not be disadvantaged by the MVA and it was also possible that they might learn algebra with a more complete understanding. The comparison group were following the traditional approach used by the school which they would have used in any case if the study was not taking place. Finally, if the new approach was found to be successful, their teachers could integrate the main ideas of the MVA, such as working mathematically, learning all three aspects of variables concurrently, and the revised teaching resources, in subsequent years.

Another limitation of the quasi-experimental design is the possibility that the researcher may indirectly communicate her expectancy with the participants. For example, she might discuss the expectation that the experimental group would become more competent than the comparison group. This may cause the teachers of the experimental group to work harder than they might otherwise have done to meet the researchers' expectancy. Alternatively, they might react negatively to prove the hypothesis wrong. There is also the possibility that participants may behave differently under observation than in normal circumstances. This reactivity may also affect the learning outcomes. Consequently, the researcher kept her opinions about the study out of her conversation with the teachers at Cara and focused on the observations and feedback from the teachers. Teachers of the experimental group were considered as partners in the implementation and assessment of the proposed teaching approach.

The purpose of the study was to investigate the effectiveness of the MVA compared to the traditional teaching approach in developing a sound conception of variables and general algebraic competence. Students demonstrate a sound understanding when they are able to interpret the meaning of a variable with reference to the given context. For example, in the equation $2x + 3 = 5$, x represents the number 1, while in $3x + 4 > 5$, x may represent any real number greater than $\frac{1}{3}$, and in the functional relationship $y = 2x + 4$, x and y represent the set of ordered pairs that satisfy the equation. By algebraic competency we mean the algebraic skills appropriate for students of Year 7-Year 8, according to the recommendations of the Board of Studies, NSW (Board of Studies NSW, 2002). More specifically, students of Year 7 and 8 are expected to represent word problems in algebraic form, find general expressions to describe patterns, solve word problems, represent table of values in graphical and algebraic form, to add, multiply, factorise and expand algebraic expressions, and to solve linear equations.

In order to compare the students' conceptions of variables and their general algebraic skills, data were collected using six assessment tests and two rounds of student interviews administered at different times throughout the two year study. In addition, lesson observations were made to compare the teachers' and students' explanations and reasoning. In particular, the focus of the lesson observations was on student-teacher discourse regarding variables.

The assessment tests and the student interviews contained problems which required students to translate between different representations (words, numeric, algebraic and graphical), solve word problems and equations. The problems were designed in light of previous studies of student misconceptions regarding variables (for example, (Perso, 1991; Stacey & MacGregor, 1993). Since it was important for the teachers of the experimental group to be active partners in the implementation of the MVA, they designed some of the assessment test items and their suggestions were also included in some of the problems designed by the researcher in assessment tests and student interviews. Each assessment test was designed to cover certain algebraic skills. For example, the purpose of the First Algebra Test (Year 7) was to assess the students' ability to move between different representations (words to algebraic expressions, tables of values to algebraic expressions), to recognise and express relationship between variables (in the form of algebraic expressions and words), and to find algebraic expressions by generalising and extending number patterns. The focus of data analysis was on student errors in representation as informed by the literature, such as conjoining errors, assigning numerical values to variables, or considering variables as objects. Student Interview I (Year 7) was intended to further probe students' ideas about variables. Student responses were categorised as low (incorrect response or a misconception about variables), medium (variable as an unknown quantity) or high (variable as a variable quantity which can attain more has one value depending on the context from which it is derived.

The Yearly Examination (Year 7) also contained problems which required students to identify and translate from one representation to another and to solve word problems. The focus of data analysis was again on student errors in representation and manipulation and to investigate the possible reasons for these errors. In the Second Algebra Test (Year 7) problems regarding students' ability to represent a table of numerical values in graphical form or to find a relationship between variables from a graph were included. In addition, students simplified and evaluated algebraic expressions. Students' working was analysed to investigate the effect of the MVA on the students' procedural skills by collecting some initial data about the simplification skills of students in the experimental group. The purpose was to study their simplification skills even before they formally learned to manipulate algebraic expressions. Student errors and the solution strategies (procedural or analytical) for solving problems, between the two groups, were compared.

The purpose of the Third Algebra Test (Year 8) was to analyse students' skill in manipulating algebraic symbols (simplifying, factorising and expanding) and making transformations between algebraic, graphical and numerical representations (words to algebraic expressions, algebraic expressions to graphical representations, interpreting graphical representations to find numerical solutions to word problems). Student errors in simplification (for example, conjoining errors, incorrect order of operations, and use of incorrect exponentiation), representation (variables as objects or labels) and expressing the given relationships between variables were identified and compared. The role of the given context in assisting students solve a word problem was examined and their simplification strategies were also analysed to study the reasoning used by students. The purpose was to identify the meanings which students associated with the variables in the absence or presence of a context.

The problems included in the Half Yearly Examination (Year 8) required students to solve word problems, simplify algebraic expressions and solve linear equations. The ability of students to represent a word problem in algebraic form, their solution strategies to solve word problems and linear equations (numerical, algebraic, arithmetic or algebraic reasoning), and their interpretation of variables (algebraic or numeric answers) were compared. Student errors in interpretation, representation and simplification were also identified and compared. The Fourth Algebra Test (Year 8) further probed the students' solution strategies (whether algebraic or numeric) to identify and solve equivalent equations. The percentage of students who used algebraic methods to identify and prove the equivalence of the two equations was compared. Finally, Student Interview II was used to identify and investigate Year 8 students' thinking and reasoning, simplification errors and interpretation of variables.

To compare and analyse student learning in each class, a qualitative analysis was used in conjunction with a comparison of mean test scores. A comparison of mean scores in algebra assessments was insufficient to answer since high assessment marks do not necessarily correlate to a sound understanding of algebra concepts (Fujii, 2003). For example, some students might provide correct answers for incorrect reasoning. Thus an analysis of student error patterns in the assessment tests was made to identify student misconceptions. Student conceptions and misconceptions together constitute what students understand about variables, Student misconceptions regarding variables are evidenced in their representation and simplification errors in algebra and their selection of solution strategies (numeric or algebraic). Comparison of the conceptions and misconceptions of variables between the experimental and the comparison group was used to answer whether the MVA leads towards a deeper conception of variables, which is the first research question (p.4). No further statistical analysis was performed as the classes were of varying mathematical ability and the purpose of analysis was to assess student understanding. Therefore it was considered sufficient to compare the mean scores, error patterns, solution strategies and reasoning of students in the experimental and the comparison group.

The teaching styles and beliefs of individual teachers were also considered so that teacher effect on student learning could be identified. The comparison of algebraic skills acquired by students of the comparison and the experimental groups, their representation and simplification errors and their conceptions and misconceptions regarding variables indicated differences in student learning. The reasons for differences between the two groups were analysed to investigate the factors that influenced student learning. The role of teaching and teaching resources along with the answers to the first two research questions were used to identify the aspects of the MVA which tend to promote or hinder student understanding of variables and their algebraic competence and answer the third research question (p.4). The details of the data analysis procedures are presented in chapters 5, 6 and 7.

4.6 SUMMARY

Data collection in Phase I and Phase II was by means of written tests, student interviews and class observations which were analysed to investigate the concept of a variable acquired by the experimental and the comparison groups. The abilities of the comparison and the experimental groups to interpret algebraic expressions and equations, to transform different representations of problems in words and in algebraic, numerical and graphical forms, to simplify algebraic expressions, and to solve linear equations were also compared. Both quantitative methods (mean scores were calculated however no statistical analysis was done as the sample was not random) and qualitative methods were used to analyse the data obtained from the written algebra tests. Qualitative analysis of student interview responses provided insight into student thinking about variables. Student-teacher discussions and teaching strategies helped in identifying reasons behind student thinking and also facilitated the identification of the role of teachers in developing the concept of a variable. Differences between the two groups in their concept of a variable, their ability to move between different representations, their algebraic manipulations and their solution strategies for solving linear equations, indicated differences between the MVA used by the teachers of the experimental group and the traditional teaching approach used by the teachers of the comparison group.

The following three chapters present results following an analysis of the data. Chapter 5 presents data concerning the teachers and their teaching of algebra. Chapter 6 and Chapter 7 present data on the student learning which took place in Phase I and Phase II respectively.

CHAPTER 5

RESULTS: THE TEACHERS

5.1 INTRODUCTION

The purpose of this chapter is to provide background information about the participating teachers, their teaching styles and their algebra lessons.

At the start of the research project, data were collected from the teachers concerning their beliefs about mathematics teaching and learning, about algebra teaching and learning in particular, and about common student errors and preferred teaching approaches in overcoming these errors.

During lesson observations particular attention was given to the concept of a variable portrayed by the teachers. Moreover, teachers of the experimental group provided written and verbal reflections on learning and teaching issues (see p. 47).

5.2 PROFESSIONAL DEVELOPMENT WORKSHOP 1

At the start of the project, a workshop was arranged for the professional development of the two teachers of the experimental group. During the workshop, data were collected from the teachers about their definitions of algebra, the typical sequence of their algebra lessons, what they saw as typical student errors, and the cause of student difficulties in algebra. Teachers also talked about what could be done to reduce student difficulties in algebra. All of the workshop sessions were audio-recorded and conversation that informed the researcher about the beliefs and subject matter knowledge of the teachers was transcribed for analysis.

5.2.1Teachers' beliefs about mathematics teaching and learning

The teachers of the experimental group completed a Mathematics Teaching Questionnaire (Appendix D) during the workshop whereas the teachers of the comparison group filled out the same questionnaire in their own time and returned it to the researcher during the first week of algebra teaching. The Mathematics Questionnaire asked for information about the teachers' beliefs about teaching and learning.

Rosa and Mona were teachers of the experimental group and taught Set 2 and Set 4 respectively while Amy and Ben were teachers of the comparison group and taught Set 1 and Set 3 respectively. A summary of each teacher's questionnaire responses is given here.

5.2.1.1 Rosa

Rosa had been teaching for 6 years, all of which were at Cara School. She believed that mathematics could be used to understand and solve problems that occur in the real world. She liked using concrete materials for teaching mathematics and believed that it was difficult to learn mathematics if you did not know where it would be used. She felt that the problems usually presented in textbooks were artificial and did not represent real life problems.

Rosa thought that effective teachers did not need a good textbook to teach mathematics. She was happy to let students solve problems without explaining the method first as she believed that teachers should never tell students anything they could work out on their own. She was also willing to give students complex problems however she also believed that students were not capable of discovering mathematics without a teacher's guidance.

5.2.1.2. Mona

Mona had one year's teaching experience. She believed that the historical development of mathematics was determined by human needs and that we need mathematics to make sense of our experiences. Mona agreed that it was difficult to learn mathematics without knowing where it would be used. She did not consider mathematics to be a set of rules and procedures and believed that there was room for personal preference in mathematics. However, she also believed that mathematical definitions were fixed and could not change.

Mona thought that the main focus of secondary mathematics should not be on learning to use formulas. Students did not need to solve many practice questions since one or two solved examples were sufficient to learn a rule. She liked using concrete materials for teaching mathematics.

She thought that most students were not capable of discovering mathematics on their own and needed the help of a teacher to learn mathematics. However, she was willing to give students new problems to solve without explaining the solution procedures. She thought that teachers should not tell students anything they could work out on their own.

5.2.1.3 Amy

Amy was also a new teacher and this was her second year of teaching. She attributed the historical development of mathematics to human needs. She considered mathematics to be an integral part of our everyday lives and thought that it could be used to understand our experiences. She was in favour of using concrete materials to teach mathematics. However, she thought that mathematics could be learned easily without linking it to real life scenarios.

Amy considered it the responsibility of teachers to complete the syllabus and to give mathematical knowledge to students. She thought that a teacher must never tell students anything they could work out for themselves. Amy was also willing to give students complex problems and had no objection to giving them problems without explaining the solution technique. She could not imagine learning mathematics without a textbook. However, she also thought that teachers did not need a good textbook to be effective.

She believed that it was not necessary to have a mathematical mind or a good memory to do well at mathematics. Still, she thought that students were not capable of discovering mathematics on their own and they needed the guidance of a teacher. She strongly disagreed that the focus of secondary mathematics should be on using formulas and believed that we should always search for a rule to solve a problem. However, she thought that it was important to teach students techniques to check the correctness of their answers.

5.2.1.4 Ben

Ben was also a new teacher and this was his first year of teaching mathematics. He believed that mathematics is a body of rules, formulae and procedures and that students need numerous practice exercises to learn mathematics. However, he did not agree that the focus of secondary mathematics should be on learning to use formulas. He also believed that mathematics definitions were not fixed and there was room for personal preferences.

He agreed that the historical development of mathematics was determined by human needs and the real purpose of mathematics was to solve real world problems. He considered mathematics to be a part of our everyday life and a tool for understanding our experiences. However, he thought that mathematics could be learned without knowing where it would be used.

Ben believed that people did not need a good memory or a mathematical mind to be successful in mathematics. Ben believed that most students could not learn mathematics without a teacher and it was the responsibility of teachers to cover the syllabus. He thought that teachers needed a good textbook to be effective. He did not agree that it was a teachers' job to give mathematical knowledge to students or that teachers must always be able to answer students' questions.

He was willing to give students problems without first explaining the solution technique and he had no objection to setting problems for students that required lengthy solutions. Ben believed that students should learn to check the correctness of their answers and that it was acceptable to have more than one correct answer to a problem.

In summary, all participating teachers were aware of the role of mathematics in our everyday lives and liked using concrete materials in their lessons. Amy and Ben thought that mathematics can be learned without making links with real life examples whereas Rosa and Mona were in favour of establishing more links with real life during lessons. It is important to remember that Rosa and Mona filled out this questionnaire during the professional development workshop where the importance of linking algebra with real life was stressed with examples from the resource book.

All participating teachers believed that a teacher played a central role in teaching mathematics. They all were in favour of giving students time to solve problems on their own first and offering help as required. All teachers were in favour of letting students solve numerous practice exercises during lessons to master mathematical techniques.

5.2.2 Teacher beliefs about algebra and teaching algebra

Only the teachers of the experimental group shared their ideas about learning and teaching algebra by completing the Algebra Questionnaire (Appendix F) and through informal conversations during the workshop. The ideas of Rosa and Mona about algebra and algebra teaching are described below.

5.2.2.1 Rosa

According to Rosa,

> Algebra is representing a relationship between a pattern. You know, like, you cannot define something just with pure numbers by using actual numbers themselves you can use generalisation. And so you use the concept of a pronumeral or a letter. You could use anything, you could use a picture if you wanted to, to represent algebra to then be able to see a pattern or relationship between things in life that you do.

Rosa wanted her students to know the importance of algebra in solving problems. She liked the idea of introducing algebra lessons by telling a story about the use and importance of algebra. She said that algebra

> …is kind of like our number system. … if you had to think of a number system that didn't have 0 1 2 3 4 what numbers would? How would you have done it?

Rosa emphasised that algebra should not be taught as a set of rules and procedures with no connection to other mathematical concepts and real life. She stressed the importance of making connections between algebra and other topics such as statistics, graphs, rates and directed numbers using patterns. She did not want students to "regurgitate information" though she did want them to "do the mechanics" to solve a problem. She wanted her students to learn with understanding.

Rosa thought that the transition from arithmetic to algebra was difficult for students for several reasons. She agreed that students were expected to follow rules without any real understanding, which was a major reason behind their manipulation errors. Students often considered algebra as a set of meaningless rules and turned towards numerical solutions wherever possible. Moreover, many students were not able to link algebraic expressions with their graphs even though they were often successful in finding rules for patterns. Many students were not able to understand the link between letters and numbers.

Rosa was happy to start algebra with the concept of a variable. She said that it would be difficult for students to solve problems based on algebra if they were not taught about variables first.

5.2.2.2. Mona

Mona considered algebra to be a tool for mathematical modelling. According to Mona algebra was

> just a great mathematical tool that allows you to look at the world around you, build patterns. And because there is that unknown at the end of it you are trying to work towards, well what's happening next? So you are building a pattern you are using all these variables to get this, a model which mimics what we see in real life and just making sense out of it. It's a great tool I think and I often tell my students that if it weren't for algebra a lot of things we take for granted will (not) be proved like weather reports on T.V and sporting events. You know something like that, that's algebra. Models come from somewhere you know.

Mona proposed that we should start algebra teaching with modelling and giving students a contextual situation. However, she agreed that in general it was difficult for students to "contextualise algebra" therefore it was difficult for them to use it in different situations.

Mona believed that it was necessary to teach students about variables. In particular that "*x* should stand for the *number* of apples." Mona proposed it would be beneficial to begin algebra by giving students a scenario and asking them to solve a problem based on that scenario. This would help students in assigning some meaning to the variables involved.

Mona agreed that many students considered algebra to be a study of meaningless rules. Students learned the process of manipulation without assigning any meaning to the symbols involved. Moreover, students were unable to recognise the properties of numbers and this could be another reason for their manipulation errors. She also indicated that it was difficult for students to find rules for patterns or to link an algebraic expression to its graph.

In summary, both Mona and Rosa agreed that using real contexts to begin algebra teaching in junior secondary school would help in associating meaning with the variables involved. Rosa pointed out that making links between algebra and other strands of mathematics was also important. Rosa believed that understanding the meaning of variables and the relationship between variables is important to learning algebra with understanding. It is important to note that Rosa and Mona completed the Algebra Questionnaires I and II and the Mathematics Teaching Questionnaire during the workshop, in which both the researcher and the author of the resource book discussed the importance of making links between algebra and real life and the importance of using real contexts to associate meaning with the variables. This may have affected their responses in both these questionnaires.

5.2.3. Common student errors and the usual sequence of algebra lessons

All participating teachers indicated the usual sequence of their algebra lessons, the student errors they usually watched out for, and some teaching resources they used in their lessons by filling out the Algebra Teaching Questionnaire (Appendix E).

The responses of the teachers of the comparison group indicated that they were aware of some student errors. For example, Amy had observed that students made errors when they operated on algebraic expressions in incorrect order (for example, $2x + 3x \times 5 = 5x \times 5 = 25x$) and also indicated that students made errors when they simplified like terms. Ben indicated that students simplified numbers and variables separately and made errors in the process (for example, $2x + 5 + 3x = 2 + 3 + 5 = 10x$).

Amy liked worded puzzles and Ben liked to use concrete materials in his lessons. Both teachers of the comparison group used individual, paired and group work as per the requirement of the task.

Both Rosa and Mona gave examples of many simplification errors which they had observed (for example, $2a - a = 1$ or 2, $a \times a = a + a = 2a$, $4a + c = 4ac$, $\frac{y^2}{y^4} = y^2$). Both teachers taught rules and procedures before starting formal algebra. Both teachers indicated that they liked to use concrete materials such as cups and counters in their lessons.

Again, it is important to note that Rosa and Mona completed the Algebra Teaching Questionnaire during the Professional Development Workshop where the importance and types of student errors, the use of concrete materials to provide meaningful learning, and the importance of identifying patterns and relationships between quantities were discussed. This experience might have affected their responses. For example, both teachers identified student errors, and indicated the use of cups and counters and patterns to teach algebra. It is difficult to estimate the extent to which these responses reflect actual lessons of the teachers of the experimental group. However, this also suggests that the idea of contextualising algebra presented during the professional development workshop appealed to Rosa and Mona and that they were prepared to trial it in their algebra lessons.

5.2.4 Workshop evaluation

The teachers of the experimental group gave their evaluation of the Professional Development Workshop 1 through the Workshop Evaluation Form (Appendix G).

Mona pointed out that it was very useful to go through the resource book with one of the authors during the workshop. Mona agreed that it was very important to contextualise algebra so that students could be provided with meaningful learning. She expressed her wish to teach mathematics in an integrated way instead of teaching algebra as a separate strand as is the usual practice in the NSW syllabus.

Rosa said that she was looking forward to implementing the ideas discussed during the workshop in her lessons. She added that it was very helpful to see the bigger picture and discuss the details of the research project. However, she was not sure about how she could link the activities in the resource book with drill and practice questions from the textbook.

5.3 LESSON OBSERVATIONS

There were two main objectives of the lesson observations. One purpose was to see how teachers taught the concept of a variable during lessons and the kinds of student activities which they incorporated into their lessons. For the experimental teachers, the second purpose of the lesson observations was to investigate the extent to which they implemented the MVA when they discussed the concept of a variable. The MVA requires the use of real contexts to make learning relevant to the learners, to provide a platform for discussing mathematical ideas and to motivate students to learn algebra. The activities were designed to encourage students to work mathematically that is, students were expected to ask questions, apply strategies, communicate by using mathematical language, reason to explain their choice of strategies, and reflect on their experiences to form generalisations.

Teachers of the experimental classes were also requested to document the particular aspects of variables discussed in each of their algebra lessons on a Lesson Plan Feedback sheet provided by the researcher (see Appendix L).

Some prominent features of each participating teacher's practice and the concept of a variable projected during lessons are described below.

5.3.1 Rosa

Rosa was the teacher of Set 2 and the most experienced teacher in the project. She usually asked her students to explain their ideas instead of taking the lead and defining the concepts herself, and she regularly encouraged her students to ask questions and share their ideas. Most of the time, her students worked in groups or on their own. Rosa encouraged her students to think about the problems however she also explained and answered individual queries by going to each student one by one and explaining common problems to the whole class.

Rosa often pointed out the importance of the topic under discussion by using relevant examples. She also addressed common student errors by drawing students' attention to these errors, for example

> "If that's a half how would you write that in a division form? Would I go 5 divided by 10 or I go 10 divided by....? Which way do I do it?"

The idea that a variable can represent any unknown number was very clearly conveyed during her lessons. She asked her students to think about the meaning of the variables involved. Moreover, she clearly established the link between the context and the variables which were derived from that context by asking suitable questions. For example, she discussed the context of a girl, Remy, who stood on stair number 197 and climbed down the stairs by taking 3 steps in each stride. Rose used this problem to derive the algebraic expression for the stair number r Remy was at after taking n strides, and to explain the meaning of the variables, coefficients and constants in the algebraic expression $r = 197 - 3n$.

Rosa: It says *r* equals, so what do you think the *r* ‘s standing for?
Student_R2: That would be the variable.
Rosa: That would be a variable, but does it ... what does that mean? Does the *r* mean Remy?
Student_R2: No it just means the answer to the output.
Rosa: It means the output doesn’t it? Isn’t that what it’s doing?
Many students: Yeah.
Rosa: So in this case here, in $r = 197 - 3n$, okay? Let’s put ... let’s relate it to the stairs, okay? And we’re talking about where Remy is. What stair number did Remy start on?
Student_R2: One hundred and ninety seven.
Rosa: One hundred and ninety seven, didn’t she okay. What do we also know that’s called in mathematical points? It’s also known as a ..?
Many voices: Constant.
Rosa: Constant, okay? But we also know that it’s the stair number you started on. So can you see there where it says 197 ... *n* is the number of strides. Okay, so what about this bit here, the -3?
Three students: Coefficient.
Rosa: Coefficient? But what does it mean when you’re talking about Remy and the stairs?

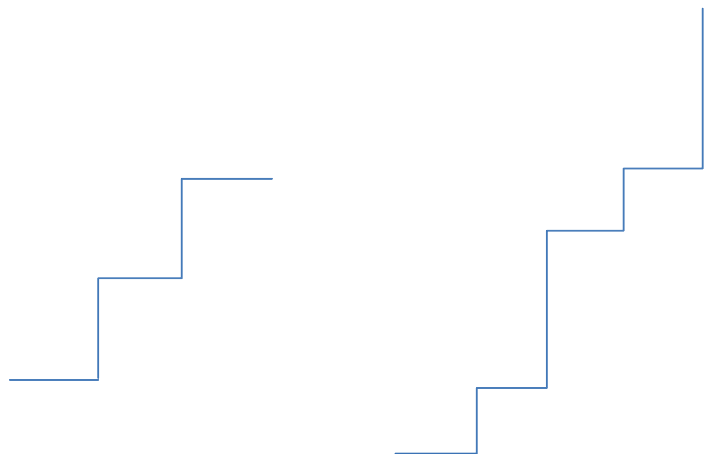
Three or four students together in different words expressed the same idea: “It’s talking about how many strides she’s taking downwards.”

During lessons, students found the general terms for number sequences though the term ‘generalised number’ was not formally mentioned. Similarly, students used tables of values to plot graphs of linear functions and also find a rule relating the two variables in the table. Rosa briefly mentioned that a linear equation in two variables is similar to a linear function. Most of her attention was focused on explaining the relationship between the variables involved using numerical and graphical representations.

Rosa used problems in the resource book to provide links between real life and mathematics, and to make learning relevant to students. She made connections between different representations to clarify concepts. For example, she plotted a graph and made reference to the context of Remy on the stairs to teach the concept of linearity.

Teacher: What if I went across one, up two? Across one up two, across one up two (See Fig 5.1).
Student_R1: It’d just be a more vertical line.
Teacher: It’d be more vertical? It’ll get ...
[Three students raised their hands and one used the word *steeper*.]
Teacher: If it was going at different rates, if it went up ... if I went across one and up one, across one and up three, across one and up one (See Fig *b*).
Student_R1: It wouldn’t be a straight line.

The strands of working mathematically such as questioning, reasoning about mathematical concepts using mathematical language and justifying were quite evident in her lessons.



(a) (b)

Figure 5.1 Graphs drawn on the white board. (a) represents climbing one stair at a time, (b) climbing one stair then two stairs, then repeating this pattern.

During Phase II, students were given some lessons on the topic “algebra in spreadsheets” in the computer laboratory. Rosa was on leave for some of these lessons so students worked on their own for most of the time. Moreover, as students became more familiar with the resource book, Rosa let them work on their own for most of the time and instances where all students participated in class discussion reduced significantly.

After the First Algebra Test, Rosa told the researcher that in the resource book investigation was done using “a lot of words” and there were less “practice exercises and therefore no consolidation”. She preferred more practice exercises and more structure in her teaching. But she liked contextual activities and explaining the bigger picture to the students. Therefore, she would like to include more modelling activities in her lessons.

When the researcher asked her about using the words pronumeral and variable interchangeably in her lessons, Rosa explained the difference between a pronumeral and a variable as follows. “Pronumeral is a symbol which is used instead of a number and the word variable is used in the sense of a variable quantity.” Also “Variable is more like a number that can change or something that represents something that can change. It doesn’t have to represent a number.” At the end of that discussion Rosa came to the conclusion that pronumeral and variable were two different words used for the same concept. However, she still considered it more suitable to use the word variable for quantities which could change and pronumeral in the sense of a place holder for a specific unknown number.

5.3.2 Mona

Mona, the teacher of Set 4, demonstrated a "show and tell" way of teaching. She explained the concepts in detail and then asked suitable questions to further elaborate these concepts. She liked her students to remain attentive during lessons and listen to her carefully. Generally, she presented a problem to students and then asked her students to help her in solving that problem on the whiteboard by asking them suitable questions. During this process of solving problems, she encouraged students to explain their thinking as well.

There were some lessons in which Mona asked her students to work on their worksheets individually while she acted as a facilitator. In particular, when students worked on their computers using spreadsheets, they worked on their own and Mona helped them whenever they asked for her help. However, in most lessons she was the one taking the lead.

Mona liked her lessons to be well structured. There were instances in her lessons when a student asked her a question and she told her that she would come back to it later or after the lesson. However, the moment passed and that question was forgotten. As her class consisted of students of low mathematical ability, during the first term of Year 8 another teacher acted as a teaching aid. That teacher walked around the class and answered student questions and offered help wherever required.

Mona indicated possible student errors during her lessons and advised her students to avoid such errors. For example she told students that $n + n + n = 3n$ and not n^3. She also mentioned the commutative property of numbers in her lessons without giving them the formal definition.

Mona told her students that *x* can stand for anything because it is a variable. The concept of a variable promoted during lessons was that of a quantity which varied. For example, Mona explained the difference between a constant and a variable quantity as follows.

Mona:	So what's the rate of litres?
Student_M1:	Constant.
Mona:	Constant. Exchange rate between Australian and US currency.
Student_M2:	Variable.
Mona:	Variable, yes.
Student_M3:	It changes over ...
Student_M4:	Very able to change
Mona:	Very able to be anything.

Mona encouraged her students to think about the given problem by asking questions so that students could explain their thinking. For example, in one lesson she plotted a graph using a table of values and drew the attention of students towards the idea of linearity in the following manner:

Mona:	Now you've all plotted your points, right? What do you notice about the points you plotted? Student_M1?
Student_M3:	Going up by two.
Mona:	They're going up by twos, but graphically what do you notice about them? Student_M5? They're going up in a ...
Four voices:	A line.
Mona:	A straight line, that's right. If you join them they will form a straight line. Let's write that down. Let's write that down. So yours should look somewhat like this, and they all lie in a straight line. Isn't it fantastic. Could you predict this would happen from the table of values?
Two students:	Yes.
Mona:	How could you tell that?
Student_M4:	Going up at a constant rate.

The activities given in the resource book encouraged both teachers and students to work mathematically. Therefore, to some extent, the working mathematically processes of questioning, reasoning about mathematical concepts, communicating using mathematical language, and justifying were present in Mona's lessons.

In Phase II students spent more time working on their own and Mona's personal style of "show and tell" became more prominent. Although the activities in the resource book encouraged student-teacher discussion, many opportunities to further explore some concepts were lost as Mona preferred students to ask questions after she finished her explanation.

After the First Algebra Test, Mona told the researcher that she liked the resource book and she believed that her students were learning with the help of contextual examples. She said that in the previous year, when she had taught traditionally using the textbook, her students found algebra very difficult and the concept of a variable was problematic. This year her students were able to make links. She said that

> when we went on to functions and did dependent and independent variables they were able to link it back to patterns and make sense of it instead of just looking at the tables of x's and y's … so they followed really well the link between the two.

She also liked the way in which the resource book used "particular scenarios and linked it with different concepts" though she wanted more practice questions to be included in the book.

5.3.3. Amy

Amy, the teacher of Set 1, taught by a "show and tell" technique and gave students many practice questions. She also asked students to explain their thinking and gave them some examples from everyday life to create interest in the subject.

Amy considered mathematics to be an academic subject. The main focus of her lessons was on mastering the procedural techniques. She explained the procedures; students then followed those procedures to solve exercises given in their textbook. Very few real-life or practical problems were included in her lessons. She tried to make links from mathematics to other subjects to motivate students. For example, in one lesson she projected an excerpt from the book “Through the Looking Glass” by Lewis Carroll and said:

Amy:	Okay, so who’s read *Through the Looking Glass*? Who’s read *Alice in Wonderland*?
Student_A1:	Oh yes.
Student_A2:	I’m actually reading it.
Amy:	In his spare time, Lewis Carroll liked to write children’s fantasy books. What do you think he did for a living?
Student_A3:	Maths?
Amy:	Yes he was a mathematician. So in his books he makes lots of reference just to mathematics. A lot of mathematical nonsense. Because if you look at this excerpt closely, there’s only a little bit of algebra. So the first one is addition. Nine from eight, so that is an algebra. Who here’s smarter than Alice?
Student_A4:	I’m smart.
Amy:	I’m smart, not as smart as Alice in Wonderland? So nine from eight is minus one, and we’re going to be doing some algebra with directed numbers today. Where else are we using algebra in this excerpt? Yeah?
Student_A2:	The divide a loaf by a knife.
Amy:	Divide a loaf by a knife, so what did they get when they divided a loaf by a knife? Sammy?
Student_A5:	Bread and butter.
Amy:	Bread and butter. So when we do division, we’re going to see that you can get something different. You can actually find terms that aren’t alike. What about take a bone from a dog? Can we really do that?

However, in this case it was obvious that the only purpose of referring the book was to motivate students. As she pointed out later to her students when referring to the above mentioned discussion,

> I think that’s just gibberish. But anyway what I wanted to do was just show you an excerpt from this, because there’s mathematics in fairy tales, and it’s a little bit relevant to what we’re doing today. Okay, so revision from yesterday. Open your books and do these eight questions please.

Amy’s lessons were assessment-oriented. This was reflected in the questions that students asked (such as “If you wrote x + - 3y in a test, would you lose marks for not simplifying?”) and the responses given by Amy during her lessons. Students were very much focused on getting maximum marks.

The aspect of a variable as an unknown quantity was very prominent during discussions. The idea that a variable can represent different numbers was also there. In some instances it appeared that the fruit-salad algebra approach was being used. For example, she said "Two *y'*s. So there's no way that I can express that as six something because my *x'*s and *y'*s are different. Like when I got my fruit, I can't have orange [and] bananas. So four *x'*s and two *y'*s." and "I have got two *y'*s and then I have got another *x* somewhere" and "one, two, three, four, how many *x*'s?".

Amy wanted her students to focus on learning procedures accurately, so she paid special attention to the rules which could be used to solve a problem. For example, she said "So, again the rules are exactly the same as with these fractions listed except that you're going to have pronumerals involved." and "What's the rule when we move the terms around?" and "You might lose one mark for the whole paper, but you really shouldn't be writing plus minus. Cause we like to write as little as possible." Amy tended to explain the conventions and rules being used without going into the details of why these conventions were in place or how the rules were constructed. For example:

Amy:	Okay who wrote –1*mn*?
Student A1:	Or it could be – *mn*.
Amy:	So would we write –1*mn* or do we write –*mn*, so we know it's minus one, we're not going to bother writing one.
Student A2:	Do you get marked down for that?
Amy:	You could get marked down depending on how nasty we get. So – *mn*.
Student A3:	Should we mark it wrong if we wrote that?
Amy:	Just write next to it equals *mn*... – *mn*.
Student A4	How do you know there is a one there?
Amy:	Because there is only one *mn*.

Amy's class consisted of the high-ability students. They followed rules and solved numerous exercises during lessons and as homework. She also demonstrated different ways to check their answers. There was consistency in Amy's style of teaching and there was no apparent difference in Phase II lessons as compared to Phase I.

5.3.4. Ben

Ben, the teacher of Set 3, believed that mathematics is about rules and their applications and students should do many practice exercises during lessons and as homework. However, he encouraged students to think about the reasons behind the rules. In one lesson while teaching simplification of algebraic expressions, he asked his students:

Ben:	What does minus outside the brackets actually mean?
Student B1:	Minus one.

Ben: Does it mean minus one? Can you explain why?
Student B1: Because the minus one is just like … like *a*1, like *a*. So you … you put the one, and put one minus.
Ben: So, say that again, say
Student B1: You don't need the one 'cause you've got the minus stands for one, minus one.
Ben: So a minus there is just the same as writing minus one?
Student B1: Yeah.

Generally, Ben first demonstrated the method and then asked his students to solve problems working individually or in pairs. After giving them enough time he solved all problematic questions on the board himself or asked a student to come up and solve questions on the whiteboard with the help of other students.

Ben liked to play different mathematical games to motivate students. He tried to create a sense of belonging by celebrating students' birthdays in class. All students sang songs and wished happy birthday to their classmates. He also commented on their extracurricular activities and sometimes used that context to start a new topic. He used examples from real-life scenarios to demonstrate the importance of algebra such as

> When you're making a fence. You're building your backyard fence. Okay. We might need a rule like this. If we're going to need 200 planks of wood, we're not going to count them out individually are we? It's much quicker to use a rule as we just saw. Okay. One and a half minutes, second part, go. In pairs.

He encouraged students to link real life with mathematics. However, if real contexts are not used properly, they can cause misconceptions regarding the concept of a variable such as the misconception of variable as an object or as a label. For example, in one lesson Ben raised two pens in his hands to demonstrate addition of two variables:

> When I talk about adding my pens and I've got one pen and another pen, it's not good enough just to go, oh, two. I have to say two pens. So when I'm doing my algebra and I have one *a* plus another *a*, what's my answer going to be?

Similarly, while teaching addition of two expressions Ben said

> Just a quick example for some of us that are struggling with the issue when there's no number in front, okay. If, for example, like for *c* + *c*, remember how before, okay, if there's no number in front of it there is actually a silence or a one. There's a secret number one. It's like saying if I have a pen plus a pen, how many pens have I got?

The main idea about variables promoted in his lessons was that of an unknown quantity. For example, Ben used the symbol "*x*" as well as a smiley face to represent an unknown number in a lesson when he asked students to evaluate

$x + 5 +$ ☺ where $x = 3$ and ☺ $= 1$

Some responses from students during lessons indicated that they thought that variables stood for unknown numbers. For example:

> 'Cause pronumerals represent numbers they can be added or subtracted just like numbers. Addition can be done in any order; for example, five plus four equals four plus five.

In Phase I, Ben was able to maintain the interest of students in lessons. However, in Phase II it appeared that students learned mathematics without any real interest and motivation. Some students told the researcher that their lessons were "boring" and they "don't like algebra".

In summary, Rosa and Mona had their own unique teaching styles. Rosa had more variety in her lessons and she encouraged student discussion. Mona preferred a silent and involved audience and liked to be in control. She also liked to solve problems on the whiteboard with the help of students. Amy and Ben also demonstrated solution strategies in front of the class and then encouraged students to work on their own on practice exercises. Ben had more variety in his lessons as he played number games, asked students to solve problems on the board with the help of other students and celebrated their birthdays by singing songs.

Ben and Amy used real-life applications of mathematics to create interest when students appeared bored and lost interest in solving practice questions. The resource book, *Working Mathematically: Activities that Teach Patterns and Algebra,* used by Rosa and Mona contained a variety of real-life examples which made algebra learning interesting and meaningful for students. During lessons students seemed involved in their work and there was no need to make an extra effort to engage students.

5.4 PROFESSIONAL DEVELOPMENT WORKSHOP II

A professional development workshop was arranged at the beginning of Phase II. This meeting lasted for two hours. Rosa, Mona, Elle (the mathematics head teacher from Cara), the first author of the resource book, the researcher and her supervisor all participated in this meeting. The purpose of the meeting was to discuss results obtained from the data analysis in Phase I and to provide Rosa and Mona the opportunity to discuss Chapters 8 to 14 of *Working Mathematically: Activities that Teach Patterns and Algebra* with the author. This discussion was intended to facilitate the teachers in planning their lessons for Phase II.

In the first half of this meeting, the researcher summarised algebra errors found in the First Algebra Test. Different errors, along with examples from the papers, were presented to the teachers.

When Rosa saw the errors regarding variables, she said that this had been an eye opener for her. Rosa also shared that at the beginning some parents were concerned about the difference in the algebra syllabus being studied by their daughters as compared to what they had studied as students. She told the parents that “we are putting it in context first and doing the mechanics later. They won’t get any worse but should even get better”.

Both Rosa and Mona said that using real contexts and learning the method of working backwards helped students in learning mathematics in general. Rosa said that, in a unit on Circles, she was concerned that her students would not be able to solve problems requiring the solution of linear equations. However, she was surprised that her students used the given context and worked backwards to solve the problems. Mona told us that most students in her class were comfortable with using the method of working backwards.

Rosa also said that she thought that teaching using lots of activities during lessons, as compared to solving exercises, was hard. She said that “a lot of discussion and not enough practice” worried her. She felt that during discussions “some of them are listening and some did not seem to be”. At least when all students did practice questions she knew that everyone was working. With discussions she was not sure whether everyone understood or not. Mona agreed that teaching according to the MVA was difficult as she had to work hard to get all the definitions of pronumeral, variable and functions straight in her head before going to teach students.

At the end of the meeting, Rosa said that her participation in this research project had totally changed her thinking and, as much as she wanted to go back to her old way of teaching, she felt she could not do so now. The head teacher, who was also present, said that she would like to share the information about student errors in algebra with all teachers in her department. She said that she was surprised to see the marks of the experimental classes and their errors and that a change of criteria in marking (for example, deducting marks for algebraic errors rather than incorrect answers) could make such a big difference in the marks. She also said that she would like to tell her high ability students that if the papers were marked according to different criteria, they might not stay in the same class.

5.5 SUMMARY

From two different questionnaires, data were collected from all participating teachers concerning their beliefs regarding mathematics teaching and learning, what they believed to be common student errors in algebra, and what their usual algebra lessons were like. In addition, the teachers of the experimental group shared their ideas about algebra and completed an additional written questionnaire about approaches to teaching algebra.

Every school and mathematics department has their own culture and teaching style which is reflected in the teaching practices of their teachers. In the present study, all participating teachers had certain common practices which could be attributed to the teaching culture of the school. Firstly, all teachers encouraged group work to facilitate learning, used number games to make lessons interesting and used worksheets for individual student work.

The teachers also have individual teaching characteristics. For example, Rosa encouraged students to justify and reason about mathematical concepts, Mona liked to control and structure in her lessons, Ben used examples from everyday life to explain concepts, and Amy taught students how to study for the assessments.

The teaching experience and ages of Amy, Mona and Ben were very similar. Rosa was the more experienced teacher compared to the other three teachers. Teachers of the experimental group were given professional development and knew about three different aspects of a variable, the advantages of using meaningful contexts and student manipulation errors. All teachers were aware of advantages of working mathematically as the working mathematically process strand is an essential component of the mathematics curriculum in New South Wales.

If the teachers of the comparison group had taught the students of the experimental group, the results might not have been much different. It was evident from the lesson observations that the teachers of the experimental group followed a more conceptual approach and integrated the elements of working mathematically in their lessons. There could be three possible reasons for this difference in teaching. Firstly, the teachers of the experimental group used the teaching resources, which contained problems which required students to work mathematically. Secondly, teachers of the experimental group were provided with additional professional development which reminded them about the benefits of working mathematically and linking algebra with real life and other subjects. Thirdly, the school where the teachers worked also provided regular professional development to their teachers. The school arranged for their teachers to participate in mathematics education conferences and also regularly arranged to have experienced mathematics researchers give presentations at the school. Therefore, all participating teachers were already aware of the benefits of concrete materials, real contexts, and working mathematically in mathematics. This awareness was also reflected in their responses to the Mathematics Teaching Questionnaire and the Algebra Questionnaires. Therefore, if the teachers of the comparison group were provided with additional professional development and given the same teaching resources as the teachers of the experimental group then it is possible that they might have been more successful in integrating the elements of working mathematically in their lessons.

The essential elements of working mathematically such as questioning, reasoning, justifying and communicating were clearly reflected in lessons given by Rosa and Mona in Phase I. Although Rosa's teaching style was more student-centred and Mona's teaching style was more teacher-centred, the teaching resources facilitated both of them to integrate the elements of working mathematically in their lessons. However, with the passage of time, teachers reverted back to their previous teaching styles and instances of concepts and mathematical ideas being discussed and students working mathematically declined. Instead, students were left to work on their own and share ideas with each other without their teacher's influence and guidance.

Test results in Phase I surprised Rosa and Mona as they were not expecting their classes to perform better than usual. They were also unaware of student errors that indicated their misunderstanding of the concept of a variable. Rosa and Mona also told the researcher that doing more contextual activities, discussing concepts with students and doing fewer practice exercises was different and more challenging than their usual teaching practice. However during their lessons, they realised that the use of contexts had helped their students solve problems and that they were able to solve linear equations through working backwards, before having been taught this technique.

The next two chapters present results from the algebra tests and student interviews in Phase I and Phase II successively. The data analysis provides information about student learning during this research project.

CHAPTER 6

RESULTS: THE STUDENTS (PHASE I)

6.1 INTRODUCTION

To study the effect of the MVA on student learning in Phase I, data were collected from three student assessments and one set of student interviews. The main purpose of the data analysis was to find differences, if any, between the comparison and the experimental groups regarding the students' conception of variables, their skill in moving between different representations (such as numeric, graphic and algebraic), and their ability to represent a word problem algebraically and then solve it.

This chapter presents an analysis of student learning in algebra in Phase I. Data presented in this chapter were collected from student tests and interview responses in Phase I only. The main focus of Phase I was to identify student misconceptions about variables and to analyse the differences if any, between the comparison group in translating word problems, graphs and table of values into algebraic form and vice versa. Some observations regarding differences between the comparison and the experimental group in simplification of algebraic expressions are also made in this chapter.

6.2 LEARNING (PHASE I)

The students at the participating school scored above the state average in reading, writing, speaking, grammar and punctuation, and numeracy. Since the mathematics classes at Cara are graded according to ability levels, the comparison group (Set 1; the high mathematical ability class and Set 3; the low medium ability class) had an overall higher mathematical ability than the experimental group (Set 2; the high medium ability class and Set 4; the low ability class).

The algebra teaching for Phase I began in Term 3. By that time, students of Year 7 had studied number and angles and their topic test results were available for analysis. Being a low mathematical ability class, Set 4 had been given a separate and easier test than the other classes. The average marks for Set 1, Set 2, Set 3 and Set 4 (as a percentage) for the topic tests in numbers and angles are listed in Table 6.1.

6.2.1 First Algebra Test

Directly after the algebra teaching in Term 3, the First Algebra Test was administered to all participating students. The results of this test are presented here. The First Algebra Test (see Appendix I) was taken by 49 students from the experimental group (27 students of Set 2 and 22 students of Set 4) and 54 students from the comparison group (27 students of Set 1 and 27 students of Set 3). The percentage mean marks for each class are tabulated in Table 6.1.

Table 6.1
Mean marks (as percentage) in number, angles and algebra

Topic	Comparison Group		Experimental Group	
	Set 1 (n=27)	Set 3 (n=27)	Set 2 (n=27)	Set 4 (n=22)
Number	91	77	88	56*
Angles	98	85	92	70*
Algebra	88	67	83	53

*Marks on a different and easier test assessing the same skills

As is evident from Table 6.1, the mean marks of all classes were lower in algebra than in number and angles. However, the order of algebra marks is the same as the order of the marks for number skills and angles as well as the test used for sorting students into the four sets. Despite the fact that Set 4 was administered the same assessment in algebra and an easier assessment in numbers and angle, the mean mark of Set 4 in algebra when compared to the other classes, was similar to their previous marks in number and angles.

The mean marks for specific algebraic skills were also calculated by adding together the marks of individual test questions relating to each specific skill. The mean number of correct responses (expressed as percentage) in Set 1, Set 2, Set 3 and Set 4 for these algebraic skills are shown in Table 6.2.

The experimental group (Set 2 and Set 4) were each at least as successful as the corresponding comparison class (Set 1 and Set 3) in all skills except solving a linear equation, despite their lower overall mathematical ability (see Table 6.2). In particular, the lowest ability class, Set 4, performed considerably better in writing algebraic expressions and describing a tabular relationship in words than the medium ability class, Set 3.

Table 6.2
Mean marks (as percentage) in algebraic skills for the First Algebra Test

Skill	Questions	Comparison group		Experimental group	
		Set 1 (n =27)	Set 3 (n=27)	Set 2 (n=27)	Set 4 (n=22)
Write an algebraic expression from a sentence	Q2, Q4 (d)	70	23	81	40
Write an algebraic expression from a sentence accompanied by a table of values	Q5 (b), Q6 (a)	93	73	91	77
Describe a relationship in words from values given in a table	Q5 (a)	90	24	91	55
Solve a linear equation	Q5 (c), Q6 (b)	87	86	78	61
Extend a pattern	Q1, Q3, Q4 (b), Q6 (table)	99	84	93	80
Solve a word problem	Q4 (a)	100	98	100	98

The comparison group performed better in the algebraic skill of solving a linear equation to find the unknown variable. However, it is important to note that the comparison group had solved many problems in which they were required to find the unknown quantity in a linear expression or an equation whereas the experimental group had no previous experience of solving linear equations. As indicated in Table 6.2, all classes performed very well in extending a pattern and solving word problems.

To explore in greater depth the students' conception of variables, an error analysis of the students' responses to Question 2 and Question 4 of the First Algebra Test was conducted using the error categories reported by MacGregor and Stacey (1997). These two questions were chosen as student errors in these two questions were indicative of their perceptions regarding variables. The results are shown in Table 6.3.

Table 6.3
Percentage of students who made errors in Question 2 and Question 4 by category

Error	Comparison group		Experimental group	
	Set 1 (n=27)	Set 3 (n=27)	Set 2 (n=27)	Set 4 (n=22)
Variable considered as an object or label	44	88	0	27
Numerical values assigned to variables	11	15	4	9
Expressions conjoined incorrectly	19	22	4	5
Incorrect exponential notation	4	30	0	9

To consider a variable as an object or a label was a common error in the comparison group. This error was not made by any student of Set 2 and was found in only 27% students of the low ability class Set 4. For example, Question 2 was as follows:

> Q2 Sarah's mother gave her 2 times more chocolates than Hannah.
> a) If Hannah has x chocolates. Then Sarah will have ………. chocolates.
> b) When her father came home, he gave each of the girls 5 more chocolates. Describe the number of chocolates each girl has using x. Show your working.
> Sarah has ………………… chocolates. Hannah has ………………….chocolates.

One student of Set 1 (the high ability class) may have imagined two chocolates arranged side by side as she represented *twice as many chocolates* as xx instead of $2x$ and after Sarah's father gave them five more chocolates each, she represented Hannah's share as $x + 5 = 5x$ and Sarah's share as $x^2 + 5 = 5x^2$ (chocolates). Note that the student also conjoined the terms to arrive at her final answers.

Another student of Set 1 answered as follows:

> $x = 1$ chocolate, Hannah's share is $x + 5x = 6x$ and Sarah's share is $2x + 5x = 7x$,
>
> $x = 2$ chocolates Hannah's share is $2x + 5x = 7x$ and Sarah's share is $4x + 5x = 9x$

This student considered the variable x as representing one chocolate. So, when $x = 1$, she represented one chocolate as x, and when $x = 2$, she represented two chocolates as $2x$. She further used x as a label and represented five chocolates as $5x$. Her answers indicate that she felt compelled to write x alongside every number which represented the number of chocolates.

Similarly, students in Set 3 also gave answers which indicated the letter-as-object misconception. For example, one student of Set 3 represented Sarah's chocolates as xx or x^2 and after receiving five more chocolates, Sarah had x^7 and Hannah had x^6 chocolates. The student showed her working as H = $xxxxx = x^5$ and S = $xxxxxxx = x^7$ which suggests that she had arranged x chocolates side by side to calculate her answer. This misconception may have originated from using the cups and counters model during lessons in which their teacher gave examples of cups arranged side by side to represent unknown variables and counters to represent their values.

Notice that the letter-as-object misconception was not present in Set 2, although it did appear in the low ability experimental class, Set 4. However, the occurrence of this misconception in Set 4 is considerably lower than in the medium ability comparison class, Set 3.

A number of students unnecessarily assigned numerical values to variables, giving in Question 2 the answers 2, 7 and 9 instead of $2x$, $2x + 5$ and $x + 5$ respectively. As the data in Table 6.3 indicates, this error was uncommon in the experimental group.

Students often conjoined algebraic expressions incorrectly, for example, reasoning that $2x + 5 = 7x$, $x + 5 = 6x$ or $x^2 + 5x = 6x^2$. Others used exponential notation incorrectly. For example, $x + x = x^2$ and $x + 5x = 7x = x^7$. These errors often occurred when variables were considered as objects (see above) or when students were not satisfied with the answer being an expression involving the sum of two terms instead of a single term. Table 6.3 shows that these errors occurred frequently in the comparison group but rarely in the experimental group.

Question 4 also led to some interesting errors. In this question, it was possible for students to use letters as abbreviations (for example to use *b* for bus and *p* for passenger instead of using x and y as the <u>number</u> of buses and <u>number</u> of passengers as directed in part d) or translate directly from words to algebra and make a reversal error (see p. 12). Question 4 was as follows:

> Q4 There are 20 passengers for every bus.
> a) How many passengers are there when there are 2 buses? Show your working.
> b) How many passengers are there when there are 5 buses? Show your working.
> c) Express the relationship in words between the number of passengers and the number of buses.
> d) Write the relationship in part 'c' algebraically using x for the number of buses and y for the number of passengers.

In Question 4c, students were required to express the relationship between the number of buses and the number of passengers in their own words. The word statement indicating the relationship was also stated in the problem. But the terms “number of buses”, and “number of passengers”, were deliberately avoided. Some students in every class expressed the relationship in their own words and other students just rephrased or repeated the given statement. For example, in Set 1, 21 students used “number of passengers” or “number of buses” in their statement. Four students did not use the word “number” in their response at all and just rephrased the statement. Most of the students who did not use the word “number” when they expressed the relationship between buses and passengers made a letter as object error (see p. 8) in Question 1. Two students of Set 1 rephrased the given statement and made the reversal error (see p. 12) in representation. In Set 3, eleven students used the terms “number of buses” or “number of passengers” in their statement and sixteen students did not use the word *number* at all. Of these sixteen students, eleven students considered x as a chocolate in Question 2.

One student of Set 1 answered as $20p \times 2b = 40p$ and $20p \times 5b = 100p$, using b for buses and p for passengers. Here, the student used letters as labels to identify buses and passengers. Another student of Set 1 explained her answer as, “Per bus there is 20 passengers. You are able to multiply the number of passengers (20) to the amount of buses $b = 20p$, $2b = 40p$ etc” and represented the statement algebraically as “ $x = 20y$” which is the reversal error identified by Clement (1982). This error was present in the comparison group, but was not found in either of the experimental classes.

All students in the experimental group, except one student in Set 4 and two students in Set 2, used the terms “number of buses” and “number of passengers” in their word statements expressing the relationship. However, these three students of the experimental group who did not use the word number in Question 4 made no error in Question 2.

Another question which revealed student misconceptions regarding variables was Question 6 part c. The purpose of including Question 6 in the First Algebra Test was to identify the students who had the misconception that different variables (such as x and y) always represent different numerical values (see p.9).

Question 6 and student responses to Question 6 part c are described here.

Q6: Complete the table using the rule.

Rule: Add 3 to the input number and then multiply by 5.

Input (*x*)	2	3	5	12	43
Output (*y*)					

a) State the relationship between *x* and *y* algebraically. Explain how you got your answer?

b) If *y* = 15, what is *x* =? Show your working.

c) Can *x* be equal to *y*? Please explain.

To answer Question 6c, “Can *x* be equal to *y*? Please explain.” students could use the word statement, the linear equation [their answer to part a, $y = 5(x + 3)$] or the table of values.

Some students used the relationship expressed in the word statement or the linear equation to answer part c of Question 6, and other students in each class based their reasoning on the numerical values of the variables in the table. Student responses indicated that no one considered the possibility that *x* or *y* could represent a fraction instead of a whole number. Moreover, no student of either the comparison group or the experimental group was able to find the point where *y* was equal to *x*.

All students of Set 1 used the linear equation to answer part c. Twenty four students stated the reason explicitly. For example, one student of Set 1 wrote, “no, because once you add 3 and times *x* by 5, *x* will be smaller than *y*”. Two students presented their answer as a generalised statement. For example, “no, because you can't start with a number then times it by another and then end with the starting number”. One student gave an incorrect response, “*x* cannot be equal to *y*, because addition and multiplication cannot cancel each other out and neither can 3 and 5”.

In Set 3, sixteen students based their reasoning on the relationship expressed in the linear equation and stated the reason explicitly. For example, “no, *x* cannot be equal to *y* because 3 must be added to the number and the number must be multiplied by 5”. One student stated the relationship in a general way as “no, it can't, because you’re adding up to bigger numbers not subtracting” Three students gave incorrect reasons. For example, “no, because they cannot add up to the same number because they are different” and “no, if *x* was equal to *y* the rule won’t work”, and “no, because you can divide *x* by *y* which equals 5 and 5 does not equal *y*”. Four students used the table of values to explain their response. For example, "I do not think *x* can be equal to *y* ….unless….you can't really have them equal because in this table *y* is always a lot larger than *x*".

In Set 2, 20 students based their response on the relationship expressed in the word problem. One student expressed the relationship in the form of a generalised statement and six students used the table of values. Three students of Set 2 thought that *x* could be equal to *y*; however, they did not use any of the stated relationships and relied on their previous knowledge of input-output numbers. For example, one student responded as, "yes, because if you minus a number from your constant e.g., 2 from 4 and your input number was 2 then your input number would be equal to your output number", another student of Set 2 responded as "yes at the beginning it can be as the input can start at 0 and so can the output so they are equal". A third student answered as "*x* can never be equal to *y* as *y* is always a larger number and *x* does not advance by enough numbers. If *y* started off positive/above *x* and it was being subtracted each time then it is possible at one point they would be equal". The responses of these students indicated that they were learning to reason about the variables and relationships and they considered *x* as a variable which could attain different values.

In Set 4, 11 students used the word statement or the linear equation to answer the question. Three students used the table of values and three gave incorrect answers. Students who used the table of values simply compared the values of *x* and *y* listed in the table to decide whether or not *x* and *y* attain equal values. For example, one student responded as "no, because the numbers will always be different. The output will always be different from the input". Three students of Set 4 thought that *x* and *y* were different numbers and one of these three students responded as "no, because they are two different variables standing for different numbers". These three students were not able to identify the relationship between the variables *x* and *y* and thought that different variables represent different numbers. This misconception about variables was also identified by MacGregor and Stacey (1997).

In summary, although there was no significant difference in student achievement in the First Algebra Test between the experimental and the comparison groups, there appeared to be a considerable difference in the concept of a variable that students in the two groups had acquired. As student responses to Question 2 and Question 4 indicate, the misconceptions of variables as objects, considering variables as specific numbers and conjoining expressions in translating from words to algebra were exhibited by many students of the comparison group. Very few students of the experimental group demonstrated such misconceptions. In addition, student responses to Question 4 also suggested that students who do not use the words "number of" to refer to the value represented by the variable, may be more inclined to consider variables as representing objects instead of the number of objects.

The responses of the comparison group to part c of Question 6 indicated that they considered the linear equation as a rule (a procedure to be performed) to calculate the value of the dependent variable. In comparison, many students of the experimental group (in particular the students of Set 2) explained the relationship between the variables *x* and *y* to answer part c and some considered the general characteristics of input-output numbers to answer part c. A few students of Set 4 did not interpret the relationship between the variables and looked at the tabulated values. Moreover, the responses of majority of students in the experimental group indicated that they considered the unknown variable to represent more than one value. In conclusion, the students in the experimental group exhibited far fewer of the traditional misconceptions than the comparison group, the difference being particularly noticeable in the lowest ability group.

6.2.2 Student Interview 1

After the First Algebra Test, six students from each class were individually given a 20 minute audio-recorded interview. During this interview students were asked to give verbal responses, however they could also explain their answers on a piece of paper which was placed in front of them at the time of interview. The purpose of the student interview was to probe the students' concepts of a variable in more detail and to provide students with the opportunity to explain and justify their answers.

During the interview, the researcher asked students to solve five problems (See Appendix M) and also asked some students, if time permitted, to explain their responses to Question 2 and Question 4 of the First Algebra Test. Student responses for each question given in Student Interview 1 were categorised as low, medium and high reasoning and were allocated 0, 1 and 2 marks respectively. To explain this categorisation, examples of student responses to Question 3 for each category are given in Table 6.4. Question 3 was as follows:

Q3 *a* and *b* are numbers and $a = 28 + b$. Which of the following must be true?

a) *a* is larger than *b*
b) *b* is larger than *a*
c) you cannot tell which number is larger
d) *a* is equal to 28

Table 6.4
Categories of student responses

Reasoning ability level/category	Examples of student responses for Question 3.
Incorrect response Low (score = 0)	"*b* is larger than *a* because you are adding 28 to *b* to get *a* so *b* would be larger than *a*" "*a* = 28" "you can't tell"
Correct response on prompting but no explanation given Medium (score = 1)	"I don't know what *b* is so you can't really tell I think." When the researcher asked student to think again about the relationship of *a* and *b* even if she did not know the exact value of *b*, student replied "well then I would say that *a* is greater than *b*". She gave no further reason for this statement.
Correct response with a suitable explanation High (score = 2)	"*a* is higher than *b* because you have to plus two things together including *b* to get to *a* so *b* can't really be higher than *a* at least because it is +28… you will always have to plus 28 no matter what number *b* is but then if you know *b* then you have to minus 28 from *a* to figure out what *b* is."

The mean scores in each interview question were calculated for the four classes. The mean scores are represented in Table 6.5.

Table 6.5
Mean interview scores

Group	Class	Question Number					Total
		Q1	Q2	Q3	Q4	Q5	
Comparison	Set 1	1.3	1.5	1.0	1.3	1.8	6.9
	Set 3	1.2	0.7	1.2	0.8	1.3	5.2
Experimental	Set 2	1.8	1.7	1.5	1.8	1.8	8.6
	Set 4	1.5	0.5	1.5	1.5	1.8	6.8

Table 6.5 indicates that the responses of the experimental group indicated a higher reasoning ability compared to the comparison group. In particular, the responses of students in Set 4 were at least as high as the high ability class, Set 1, in all questions except Question 2. It seems that the students of the experimental group were better able to communicate and justify their responses in the interview because the teachers of the experimental group

involved their students in justifying and reasoning about mathematical concepts in their algebra lessons.

A more detailed analysis of the interview responses was made to determine how students in each class understood the concept of a variable. Students in Set 1 generally considered variables as unknown quantities. In response Question 5 (What can you say about c if $c + d = 10$ and $c < d$?) most students were content to give one answer for c and not to consider any more values. However, when the researcher suggested the possibility of c having more values, the students agreed that this was possible but they could not provide an adequate explanation for this.

Misconceptions about variables (such as considering a variable x to stand for one object or ignoring variables) were also evident in student responses. When the researcher asked one student of Set 1 to solve Question 4 of the First Algebra Test again during the interview, she represented 5 buses as $5x$ and when she was asked for a reason she replied "because x is equal to 1 bus and you are timesing it by how many buses are there". In reply to Question 4 ("Can you tell me which one is larger, 3n or n + 6? Please explain your answer.") another student of Set 1 completely ignored the variable n and gave her answer by looking at the relationship between 3 and 6 as "since 6 is greater than 3 therefore $6 + n$ would be greater than $3n$".

Students in Set 3 could not understand the relationship between letters and numbers and expressed different ideas about variables. For example, a letter standing alone is equal to one. In Set 3, students most often considered the given variable as representing an unknown quantity or a letter as indicated by their comments such as "n stands for 1, $6 + n$ is 7 but $3n$ on its own is just $3n$", and "a letter doesn't stand for a number unless a specific rule is given" and "it doesn't say what b is equal to, b is equal to b" and "you can't really times letters by letters to get a number. Say if you times like m times m it would be m^2 or n times m is nm, so you can't really do two letters to equal a number or two numbers to equal a letter". One student was unsure about the difference between the product and sum of two algebraic expressions. For example, in response to Question 3, she said "well $n + 6$ is $6n$ and that is double $3n$" and if n is 10 then "$3n$ would be equal to 13 if you plus it together and $n + 6$ is equal to 16". Only two students gave high reasoning responses to Question 1and only three gave high reasoning ability responses to Question 5. All but one student gave a low reasoning response to Question 3.

Set 2 students were more aware of the fact that variables can have multiple values, as five out of six students gave responses that were categorised as high reasoning in Questions 1 and 5. Four students correctly interpreted the equation $a = 28 + b$ in Question 3 and three of these students also verified their answers using multiple values. Their responses indicated a deeper understanding of algebraic equations. For example, "you have to plus two things together including b to get to a so b can't really be higher than a because it is always going to be higher than a at least because it is + 28… you will always have to plus 28 no matter what number b is but then if you know b then you have to minus 28 from a to figure out what b is". This student not only understood the meaning of this algebraic equation but she was also able to transform this equation into another equivalent algebraic equation. It is also worth noting that these students had not yet covered linear equations in their algebra course. However, not everyone had an accurate idea about algebraic equations in Set 2, as one student responded that an equation is a "math thing like" while an "algebraic expression uses letters to symbolise other things".

Students in Set 4 readily chose multiple values for variables when they were answering Question 5 (mean score 1.8) and Question 1 (mean score 1.5); however, most gave a low level response to Question 3. For example, one student said "I thought $a = 28 + b$ so $a = 28$ and $28 + b$ = something". Another student replied "No actually I think cause it says $a = 28$ but then it says $+ b$ so I actually I think b is larger than a". It appeared that students looked at the algebraic expression $28 + b$ and did not consider the equality of a and $28 + b$ in choosing their answer. This was not surprising as the experimental group had not yet studied algebraic equations at that time.

While answering Question 5, five out of six students of Set 4 chose multiple values for c and d, implying that they did not consider a variable merely as an unknown quantity. Only one student ignored the variable n while choosing the larger expression from $3n$ and $n + 6$ and said that "like it says $n + 6$, couldn't it be 6?" Some students in this class were confused about the operations of addition and multiplication as two students chose $6k = m$ instead of $6 + k = m$ in Question 2.

Question 2. I have m dollars and you have k dollars. I have \$6 more than you. Which equation must be true?

a) $6k = m$ b) $6m = k$ c) $k + 6 = m$
d) $m + 6 = k$ e) $6 – m = k$.

In summary, students of Set 1 considered variables as unknown quantities; however they also used letters as labels or sometimes just ignored them. Many students of Set 3 could not understand the relationship between letters and numbers and some students demonstrated misconceptions regarding variables, such as regarding a variable standing alone as equal to one. However, students of Set 3 were aware that they could solve an equation to find the value for a letter. In contrast, students of the experimental group knew that a variable could have many values and they gave higher reasoning responses than the comparison group.

6.2.3 Yearly Examination

At the end of the year, all students were given a Yearly Examination prepared by their teachers (attached as Appendix J) which contained some questions on algebra. These questions assessed the algebraic skills of extending a pattern, expressing a sentence algebraically, explaining the relationship given in a table of values, writing an algebraic equation when a word problem and a table of values are given, and solving a word problem. Table 6.6 represents the percentage mean mark of students in individual algebraic skills and the questions used to assess that skill.

Table 6.6
Mean marks (as percentage) in algebraic skills for algebra problems in the Yearly Examination

Algebraic Skill	Questions	Comparison		Experimental	
		Set 1 (n=27)	Set 3 (n=29)	Set 2 (n=29)	Set 4 (n=19)
Extend a pattern	Q27, Q47(a, d)	98	84	92	63
Write an algebraic expression from a phrase	Q42(a, b, c)	99	81	94	77
Express relationship in words when table of values is given	Q47(b)	96	76	100	68
Write an algebraic expression when word problem and table of values is given	Q47(c)	100	83	100	63
Solve a word problem	Q41	96	97	100	79
Total scores		98	97	84	70

Set 1 and Set 2 performed equally well in the algebraic skills assessed in the yearly examination despite differences in their learning programs and mathematical ability levels. Set 1 had solved many practice questions while Set 2 had focused more on working mathematically using contextual word problems and had not spent time on practice exercises. It is also interesting to note that Set 2 was as successful as the high ability class Set 1 in solving a word problem, which could be due to the more frequent use of real life examples by their teacher.

Students of Set 1 made no errors which could reveal any misconceptions about variables. However, such errors were present in the answers of Set 3. For example, one student expressed "A increased by 2" as $A + A + A$ which indicated the use of a letter as an object. Two students of Set 3 answered Question 47, part d (How many pieces would there be if 50 cuts were made) as $50c \times 2 = 100p$ and one student further expressed the table of values (see Appendix J, Question 47, part c) algebraically as c = cuts p = pieces, $1c \times 2p = 2p$, $2c \times 2p = 4$. This error indicated the use of letters as labels. There were some manipulation errors as well. For example, four students of Set 3 conjoined terms and expressed "10 more than the product of 3 and B" as $13B$ and one student conjoined the terms 2 and A and gave the answer $2A$.

In Set 2, with the exception of one student who conjoined terms (representing "A increased by 2" as $2A$ instead of $A + 2$), all errors were due to misinterpretation of the given text. For example, in response to "10 more than the product of 3 and b", incorrect algebraic representations were $3b \times 10$, and $3 + b + 10$. No student in this class represented variables as objects or labels.

There were only two students in Set 4 who made errors in algebraic representation. One student conjoined terms and gave the answer to "10 more than the product of 3 and B" as $13B$. Another student expressed the rule given by the table of values in Question 47 as $c = 2p$ instead of $p = 2c$ (the reversal error).

Errors in the experimental group appeared to arise from misinterpretation of the word sentences and unawareness of the algebraic techniques of addition, multiplication and division rather than from any misconception about the meaning of variables. On the other hand there was some evidence of misconceptions about variables (such as variables as objects or labels) in the responses of Set 3 in the comparison group.

6.2.4 Second Algebra Test

The Yearly Examination did not assess the skills of representing a table of numerical values in graphical form, interpreting a graph to write an algebraic equation representing the relationship between dependent and independent variables, and algebraic manipulation (addition, subtraction, multiplication and division of algebraic expressions). Therefore, soon after the yearly examination, the researcher gave students a Second Algebra Test (Appendix K) assessing these skills.

It is important to note here that the comparison group had studied algebraic manipulation in Year 7, whereas the experimental group had not done so. The MVA program for the experimental group required them to solve word problems by questioning and reasoning, and was based on word problems in which students worked mathematically. The comparison group learned manipulation of expressions involving addition, subtraction, multiplication and division and solved some word problems as well. Both groups had learned how to plot a graph from a given table of numerical values; however, none of the classes had attempted problems of the type "Simplify $2p\ (3y + 7)$". The mean marks (as percentage) in algebraic skills were calculated for each class. The mean marks (as percentage) and the questions used to calculate student ability in that particular skill is shown in Table 6.7.

These mean marks are comparatively lower than the Yearly Examination, which was understandable as some of these skills had not been taught, as explained earlier.

Table 6.7
Mean marks (as percentage) in algebraic skills for the Second Algebra Test

Algebraic Skill	Questions	Comparison		Experimental	
		Set 1 (n=25)	Set 3 (n=25)	Set 2 (n=25)	Set 4 (n=18)
Plot a graph from a table of values	Q1(a)	65	42	64	41
Complete a table of values from a graph	Q4(a)	86	92	94	97
Use a table of values and its graphical representation to express a relationship between two variables in the form of an algebraic equation	Q1(b), Q4(b)	60	29	34	11
Simplify algebraic expressions	Q2	68	53	45	26
Evaluate an algebraic expression	Q3	100	35	44	17
Overall		71	57	53	37

Both the comparison and the experimental groups were equally successful in plotting a graph from a table of values. The experimental group performed slightly better in completing a table of values from a given graph with both experimental classes out-performing both the comparison classes. The comparison group performed better than the experimental group in the algebraic skills of simplification, but it is important to note that a sizeable proportion of students of Set 2 and Set 4 were able to add and multiply algebraic expressions without having learned the formal procedures for doing so.

Table 6.8 indicates the percentage of students who conjoined terms for addition due to brackets such as $2p\ (3y + 7) = 2p \times 10y$); for multiplication such as $2p\ (3y + 7) = 5py + 7$; for addition such as $14p + 6py = 20py$; or made an operational error such as

$2p\ (3y + 7) = 6py + 7$. These results indicate that many students were not aware that multiplication is distributive over addition. The conjoining error for brackets was very noticeable in Set 2 and also in Set 1 and Set 3. This error appeared less in students of Set 4, but 44% of students of the set conjoined for multiplication.

Table 6.8
Percentage of conjoining errors in Question 2 (part b and part c)

Errors	Comparison Group		Experimental Group	
	Set 1 (n = 25)	Set 3 (n = 25)	Set 2 (n = 25)	Set 4 (n = 18)
Conjoining error for addition due to bracket	32	48	60	22
Conjoining error for multiplication	12	8	4	44
Conjoining error for addition only	32	40	4	6
Operational error	44	16	4	6

Table 6.8 also indicates that about one-third of students of Sets 1 and 2 conjoined for addition when there were no brackets involved. Note that students of the comparison group had been taught the procedure of addition, multiplication, and division of algebraic expressions; however, the lessons on simplification had been delivered two to three months earlier than the Second Algebra Test. During this short period of two months, some students were not able to remember the practised techniques. Many students of the comparison group were not aware that multiplication is distributive over addition so they conjoined terms which resulted in operational errors such as $2 \times p\ (3 \times y + 7) = 6 \times p \times y + 7 = 6py + 7$. In the high ability class 40% of students made this error, and this result suggests that these students were not able to use the distributive property of multiplication over addition accurately.

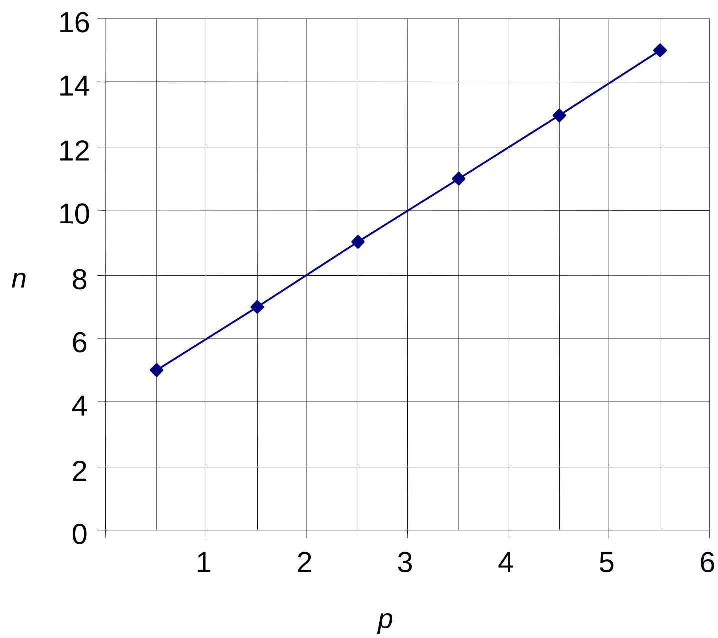
Very few students of the experimental group conjoined for addition but most of the students in the experimental group conjoined the terms within brackets. The solution procedure employed by one student of Set 2 indicated that she related the sign written at the right side of the terms to decide which operation would be used, for example, to simplify

> $5x + 2y + 3x + 4 + 6y =$
> $5 \times x + 2 \times y + 3 \times x + 4 + 6 \times y$ [She identified the operators]
> actions towards x: $\times 5 \times 3$, [She described her actions and separated like terms]
> actions towards y: $\times 2 \times (6 + 4)$.
> $= 15x + 20y$ [Answered the problem]

Question 1 and Question 4 required students to translate between different representations such as words, numerical, graphic and algebraic representations. Solution strategies used by students and their worked solutions indicated the differences between the students of the experimental and the comparison groups. The statements of Question 1 and Question 4 and student responses are described here.

> Q1 Complete the table using the rule
> "Rule: Subtract 3 from the input number and then divide by 2".
> a) Graph the data in the table on the number plane.
> b) Write the relationship between *x* and *y* algebraically.
>
> Q4 Complete the table of values using this graph.

x					
y					



> Express this relationship algebraically.

During their algebra lessons, all students of the comparison group were taught to use the expression $\frac{\Delta y}{\Delta x} = \frac{\text{change in value of y}}{\text{change in value of x}}$ to find the gradient of a linear function from a table of values and then use $y = mx + b$ to write the equation. This procedure called MSD (Method of Successive Differences by Amy (the teacher of Set 1) was taught during algebra lessons two to three months before the assessment, so not all students were able to remember and use it to find a linear equation from tabulated values. As Table 6.7 indicates, 60% of students in Set 1 were able to find an algebraic equation from the given table of values, their working indicating that most of them had used the procedure described above to find the linear equation. In Set 3, only 29% of students were able to represent the table of values in the form of an algebraic equation. Many students forgot MSD and only saw *x* and *y* as independent sequences with no relation with each other.

Some students described the relationship instead of writing an algebraic equation. For example, one student of Set 1 explained that "*x* and *y* are the coordinates of the grid" and another student of the same class explained her ideas as "these points on the graph showing the data makes a horizontal line when joined together". Two students in Set 3 noticed the corresponding increase in one variable with the increase in the other variable. For example, one student expressed the relationship between *p* and *n* given in Question 4 (b) as "whenever *p* goes up by one so does the number of *n*" and the other student wrote "adds 1 more every time for the amount to be added to *p* to equal *n*". However, they were not able to write relationship between *p* and *n* in the form of an algebraic equation.

The students of the experimental group were not taught the same procedure for finding an algebraic equation from a table of values as the students of the comparison group. However, their teachers encouraged the students to compare and contrast the relationships between variables from their graphical representations and diagrams (see Figures 5.1a and 5.1b) and express the relationship in the form of algebraic expressions. This was reflected in the responses of students from Set 2 when they expressed the relationship between variables in different ways. For example, one student identified the relationship between *p* and *n* as linear. Another student described the relationship in these words, "The relation between *x* and *y* is that they are on the same diagonal line on the graph as they go up by the same value". Two students pointed out that *x* (value) and *y* (value) intersects each other (on the graph). One student pointed out that *x* and *y* represent the two axes. Two students tried to describe the pattern. For example, "adds on one every time; 1 + 4 = 5, 2 + 5 = 7", and "when *p* is changed to *n* every value has the next consecutive number added on; for example, in the *p*: 1 + 4, 2 + 5, 3 + 6 et cetera". Even though only 34% of students of Set 2 accurately expressed the table of values in the form of an algebraic equation, it was clear that students were thinking about the relationships between the variables. Similarly, only 11% students of Set 4 were able to write the answer in the form of a linear equation in Question 1 (b) and Question 4 (b), but most of the students interpreted the variables as representing the coordinates of the graphed line even though they were not able to express the relationship in algebraic form.

Student responses indicated that the procedure taught to the comparison group is effective in finding the relationship between variables when a table of values is given. Whether they understood the relationship between the variables or were just using a learned procedure to calculate the answer is not known. Also, it is important to note that even though teachers of the comparison group taught their classes MSD to calculate an algebraic equation from a table of values, only Set 1 used the technique successfully. The other three classes gave diversified answers, and were less successful in representing the relationship algebraically.

The comparison group was also able to use learned procedures to simplify algebraic expressions. However, in the case of unfamiliar problems they made similar errors to the experimental group who had not been taught these. This result indicates that procedural knowledge gained by solving many problems facilitated the comparison group in solving problems; however, the retention of such knowledge and its application to unfamiliar problems was difficult for them.

Even though the experimental group had gained a sound knowledge of the variable concept, they were not able to apply it to the simplification of algebraic expressions when no context was given. The students of the experimental group had solved very few non-contextual problems during their algebra lessons. In the absence of a context, it may have been difficult for students to assign any meaning to the variables. The contextual problems facilitated the experimental group in thinking about the meaning of the variables involved; however, to translate those ideas into algebraic equations required the additional skill of selecting the relevant information and then translating that information into an algebraic form. The worked solutions indicated that these skills were developing though not fully developed in all students of the experimental group at that time.

6.3 SUMMARY

Students of the experimental classes demonstrated a deeper understanding of the variable concept compared to the comparison classes. Student-error patterns indicated that students in the comparison classes more often interpreted variables as unknown quantities or as objects. This could be due to the focus of the lessons given to the comparison classes on drill and practice exercises rather than reasoning and discussion. Moreover, examples used by teachers of the comparison classes such as, "let x represent a table and y represent a chair" may have contributed to the students' understanding of variables as objects. Also, the textbooks used by the comparison group contained many questions which required students to evaluate x as an unknown quantity (For example, Find $2x + 5$ when $x = 6$) however such questions in which students were required to calculate more than one value for the variables in one equation (for example, Find all possible values of x and y that satisfy the linear equation $x + y = 20$ were non-existent.

The teachers of the experimental classes were concerned that their students spent too much time in discussion and not enough time on drill and practice exercises therefore, they would not obtain good marks in algebra. The results of First Algebra Test were surprising for the teachers as the students in the experimental group scored as well as expected in the assessments. In addition, the high levels of discussion and reasoning between teachers and students appears to have contributed the development of reasoning ability in students and fewer misconceptions regarding variables.

The comparison group spent time on simplification exercises and had also learned the MSD for finding an algebraic equation when a table of values is given. Therefore, the comparison group performed better than the experimental group when familiar problems were given. However, the experimental group was still able to solve simple addition and multiplication problems without having learnt the formal procedures of algebraic simplifications.

The main errors in students of the comparison group were to conjoin for addition and to conjoin for terms within brackets, whereas the majority of students in Set 2 conjoined for brackets and Set 4 conjoined for multiplication. Operational errors were also more prevalent in the comparison group than the experimental group. The error patterns of the comparison and the experimental group were similar on unfamiliar problems. This was reflected in the Second Algebra Test, when all classes made similar errors while expanding $2p\ (3y + 7)$. Moreover, in the same assessment when all students needed to find the relationship between independent and dependent variables, the high ability class in the comparison group relied on the learned procedure, while the medium ability class got side tracked by the graphical representation and forgot to use the learned procedure of finding an algebraic equation from a table of values. The students of the experimental group were able to plot a graph from a table of values and find the numerical values from the graph. However, not all students were successful in representing the relationship between the variables in algebraic form. The skill of selecting relevant information from the graph and then translating that information into an algebraic form was not demonstrated by all students in the experimental group.

CHAPTER 7

RESULTS: THE STUDENTS (PHASE II)

7.1 INTRODUCTION

This chapter presents the results and analysis of algebra learning in Phase II. In Phase I, the purpose of data analysis was to identify student misconceptions about variables and to find the differences, if any, between the comparison group and the experimental group in translating word problems, graphs and tables of values into algebraic form and vice versa. The main purpose of data analysis in Phase II was to find differences, if any, between the comparison and the experimental groups in their ability to represent a word problem algebraically, and to determine the solution strategies selected by students to solve simple linear equations. Algebraic skills of translating between different representations (numeric, graphical and algebraic) and of addition, multiplication and division of algebraic expressions are also compared, along with student errors.

During Phase II, data were collected by means of three assessments and an audio-recorded one-on-one interview of 19 students (10 students of the comparison group and 9 students of the experimental group).

According to the policy of the school, at the end of Year 7 some students were moved between classes according to their test results during the year. Between Phase I (Year 7) and Phase II (Year 8), one girl from Set 1 moved to Set 2; two girls from Set 2 moved to Set 1 and one moved to Set 3; one girl from Set 3 moved to Set 2 and three moved to Set 4; and three girls from Set 4 moved to Set 3. Since these 11 girls had changed from an experimental class to a comparison class or vice versa they were removed from the sample in Phase II.

By the time the Third Algebra Test (Appendix N) was administered to all the participating classes, the high ability class (Set 1) had completed all the pre-planned algebra content while the other three classes still required two more weeks of algebra teaching to complete their lesson topics. After the Third Algebra Test, Set 1 commenced two weeks of advanced algebra lessons and the other three classes completed their remaining algebra lessons. By the Half Yearly Examination, all students had completed their algebra lessons for the year.

7.2 THIRD ALGEBRA TEST

The Third Algebra Test (see Appendix N) consisted of two sections. In Section A, students were required to solve three questions which assessed the algebraic skills of factorisation and simplification of algebraic expressions involving brackets, and expressing a word problem in algebraic form. The algebraic skills assessed in Section B were addition, multiplication and division of algebraic expressions; representing word problems in algebraic form; using algebraic expressions to complete a table of values; plotting a graph using tabulated values; and interpreting a graph to answer a word problem.

On the advice of the participating teachers, the experimental group attempted Sections A and B together in April (2009), while the comparison group attempted Section A only. Section B was not attempted by the comparison group at that time because the students had neither reviewed the topics of interpreting graphs of linear equations nor had they solved word problems based on graphs. Instead, the comparison group attempted Section B after the Half Yearly Examinations in June. The experimental group was disadvantaged by attempting Section A early because they had only been given two lessons on factorisation and expansion of algebraic expressions while the comparison group had spent considerable time learning these skills. The results for the Third Algebra Test are reported in Table 7.1.

Table 7.1
Mean marks (as percentage) in algebraic skills for the Third Algebra Test

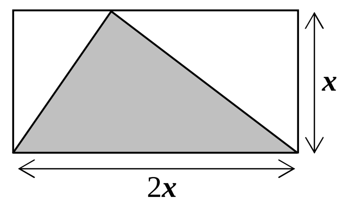
Algebraic Skill	Questions	Comparison		Experimental	
		Set 1 (n = 23)	Set 3 (n = 23)	Set 2 (n = 24)	Set 4 (n = 17)
Simplify an algebraic expression	Q1, Q7	87	62	66	51
Factorise/expand an algebraic expression	Q2, Q3	80	50	41	21
Express a word problem in an algebraic form	Q4	65	49	51	11
	Q5	99	71	83	59
	Q6(i)	91	41	90	47
Plot a graph using an algebraic equation	Q6(ii & iii)	94	86	94	70
Interpret a graph to answer a word problem	Q6(iv)	91	74	92	63

Table 7.1 shows that the comparison group was more successful in factorisation and simplification problems than the experimental group. This is understandable as they had concentrated on learning these two skills immediately prior to the Third Algebra Test.

The percentage of correct responses in the skill of expressing a word problem in an algebraic form depended on the complexity of the statement, the prerequisite knowledge required and the familiarity of the given problem. This was reflected in the students' solutions to the given word problems, Questions 4, 5 and 6. The questions and the solutions are presented here.

Question 4 Here is a diagram of a triangle inside a rectangle.
The length of the rectangle is twice as long as its height.

Write an algebraic expression for:
the perimeter of the rectangle, and the area of the triangle.



Question 5 (a) In a game of cards everyone starts with a score of 100 points.
Each time you win a round of the game, you gain x points.
Each time you lose a round of the game, you lose y points.
Anne won 2 rounds and lost 5 rounds. Write an expression for Anne's score in terms of x and y.

(b) Nelly collected P number of eggs during an Easter egg hunt. Mandy collected half as many eggs as her older sister Nelly. Then Mandy dropped 3 of her eggs. Write an expression for the number of eggs that Mandy has at the end of the hunt.

Questions 4 and 5 are similar to problems given in the textbook used by the comparison group. Table 7.1 indicates that Question 4 was more challenging for all classes as it presented the additional task of interpreting the diagram and then using the formulae for the perimeter of a rectangle and area of a triangle. It was obvious from the worked solutions that students substituted the given values into the formulas for the perimeter and area. For example, common responses were

P = L × 2 + H × 2

P = $2x \times 2 + x \times 2$

P = $4x + 2x$

P = $6x$, or

Area of rectangle = $2x \times x$, and area of triangle = $\frac{2x \times x}{2} = \frac{2x^2}{2}$.

Errors in algebraic representation were due to incorrect simplification, use of an incorrect formula for the area of a triangle and misinterpretation of the diagram. Some students in each class (43% in Set 1, 57% in Set 3, 17% in Set 2 and 65% in Set 4) left the perimeter and area in expanded form (for example $2x \times 2 + x \times 2$ and $2x \times \frac{x}{2}$). It is possible that some of these students may have thought that it is sufficient to find the representation without the need to simplify it.

Table 7.1 indicates that all classes were facilitated by the contextual nature of Question 6. In Question 6, two payment options for a girl who distributed leaflets were stated in the form of a word sentence (Option A: $2 per leaflet; Option B: $6 plus $1 per leaflet). Students were required to represent the money received under each option in the form of two linear expressions. They then had to use these linear expressions to complete two tables of values and plot both lines on the same graph. Finally, students were required to interpret the graphs to choose the better payment option for the girl if she delivered seven leaflets.

The context for Question 6 was familiar to the students. Also, the design of the problem assisted the students as it subdivided into smaller parts which guided students towards the solution of the main problem. Question 6 was also similar to the problems included in the resource book used by the experimental group. The percentage of correct responses in Set 4 was higher in Question 6 than Set 3. Both the comparison and the experimental groups were equally successful in plotting a graph and using the graph to solve the problem.

In Question 6, students were required to represent the payment options as algebraic expressions. However, some students wrote two equations (such as $*x* = 2*L* and $*x* = 6 +1*L*) instead of two expressions (2*L* and 6 + *L*). 41% of students in the comparison group (Set 1: 17%, Set 3: 65%) but only 5% of students in the experimental group (Set 2: 4%, Set 4: 6%) did so. This means that the most students of the experimental group accepted algebraic expressions as an answer to a word problem.

Two students of Set 1 used the dollar symbol as a variable ($ = 2*L* and $ = 6 +1*L*). One student misinterpreted the statement and wrote the equation 1*L*/6 = 2, 1*L* = 6. One student of Set 1 gave a numerical answer, possibly considering variables as specific unknown numbers. The percentage of students who gave numerical answers was highest in Set 3 where 43% wrote numerical values (for example *L* = 2, *L* = 6 + 1). In Set 3, 17% of students misinterpreted the given statement or did not know the meaning of the variables and gave responses such as *L* = 2*L* and *L* = 6 + 1*L*. In comparison, only two students of the experimental group wrote equations. One student of Set 2 gave a numerical answer and one student of Set 4 wrote *L* = $ × 2, *L* = $ × 6 + 1 as her answer.

Some students did not simplify the algebraic expressions in both Question 6 and Question 4. For example, some students represented the payment received by the girl as 2L, and 6+1 L and the area and perimeter in Question 4 as $2x \times 2 + x \times 2$ and $2x \times \frac{x}{2}$. At the time this test was administered every student had learned addition and multiplication of algebraic expressions. Therefore it was expected that students would simplify the terms and write the final answers as $6x$ and x^2. However, in the comparison group, 26% students of Set 1 and 13% students of Set 3 did not simplify their answers. In the experimental group only 12% students of Set 4 and none in Set 2 gave answers of this type.

To investigate the understanding of the comparison and the experimental group in more detail, simplification errors made by both groups were also analysed (see Table 7.2). The errors of conjoining terms in translating from word statements to algebraic expressions and in simplification of algebraic expressions, were identified earlier in the First and the Second Algebra Tests.

Table 7.2
Percentage of students making various errors in the Third Algebra Test

No.	Errors	Comparison		Experimental	
		Set 1 (n=23)	Set 3 (n=23)	Set 2 (n=24)	Set 4 (n=17)
1	Conjoining	61	91	75	94
2	Conjoining in addition or subtraction	61	83	71	94
3	Conjoining in multiplication	17	39	38	35
4	Conjoining terms inside bracket	4	9	38	29
5	Conjoining for division	0	35	17	29
4	$0 \times 4y^2t \neq 0$	61	70	38	59
5	Confusing y^3 and $3y$	0	17	0	29
6	Incorrect order of operations	87	64	17	18
7	Object/Label	26	13	0	12
8	Equation instead of expression	17	65	4	6

As Table 7.2 indicates, the conjoining error was very widespread among students of all classes. In particular, the number of students who conjoined for addition (such as $x + y + 2z = 2xyz$, $x + x + x + x = x^4$, and $ab + ba = a^2b^2$) was very high and increased in the lower-ability classes. The number of students who conjoined for multiplication (such as $x \times x \times x = 3x$ and $2p \times 4p = 6p^2$) was lower than the number of students who conjoined for addition.

The tendency of students to present all their answers as a single term could be an over-generalisation of the appropriate way of writing a product of two variables. For example, in all classes, students were taught to write $x \times y$ as xy as there was no need to write the multiplication sign between two variables. Over-generalisation of this rule to conjoining terms for addition may also have resulted in the addition of two variables.

Students' interpretations of the rules and simplification techniques learned during algebra lessons were reflected in their worked solutions. For example, one student of Set 3 seemed to have interpreted her teacher's advice to "simplify like terms" as follows,

$2a + 4b + 6c = 12(abc)$ (collected variables and added numbers)

$4xy + 1 + x = 5x + 1y$ (added $4x$ to x and multiplied y by 1)

$-6y + 6x = -y(x)$ (cancelled 6 and -6 and conjoined x and y)

$ab + ba = 2a + 2b$ (added a's and b's separately)

Note that this student did not see any link between numbers and variables as she added numbers together and collected variables together quite independently. The fourth example is particularly revealing.

Some students appeared to pick and simplify numbers and variables, or two different variables, separately. For example, one student of Set 1 explained her working as $2x + 9 + 5x = 2 + 9 + 5 + x + x = 16x$. Another student of Set 1 showed her working as $ab + ba = a + a + b + b = 2ab$. She was accidently successful in using this strategy but some students made errors and wrote final answers as $2a + 2b$, a^2b^2 or $a^2 + b^2$.

The idea that numbers and variables can be collected separately was very widespread among students of Set 3 as well. Students conjoined terms in different ways, for example:

$4xy + 4x^2y = 8x^3y^2$

$8x^2 - 2x = 6x$,

$ca + cab = 2c2ab$,

$x + y + 2z = 2xyz$

The reason for picking and choosing like terms could be due to the instructions "circle variables" and "circle constants", "how many x's are there" and "how many y's are there", then "add or subtract the variables together or the constants together" that were given by all participating teachers. However, the frequency of these errors seems to depend on the ability level of the student.

Some students thought that subtraction operates differently on numbers and variables. For example, one student of Set 1 simplified $4x^2y - 4xy^2$ as xy. Another student of Set 1 simplified the same expression as $(xy)^2$. Both students subtracted the numbers only. The former student divided instead of subtraction and the latter collected terms instead of subtracting.

Similar conjoining errors were made by students of Set 2 and Set 4. The working of one student of Set 4 indicated that she simplified the algebraic expressions adjacent to each another from left to right and made an operational error and then conjoined for addition:

$c(a + ab) \div ac$

$= c \times a + a \times b \div ac$ (operational error)

$= (ca^2 \times b) \div ac$ (since $ca + a = ca^2$ therefore $c \times a + a \times b \div ac$)

$= a^2bc \div ac$ (since $(ca^2 \times b) \div ac$ are conjoined for addition)

$= (a^2bc \div a)\ c$

$= (abc) \times c$

$= abc^2$

Some students who conjoined for multiplication did not differentiate between addition and multiplication of variables. For example, one student of Set 1 explained that $(p \times p) \times (p \times p \times p \times p) = 8p$ indicating that she considered $p \times p$ was equal to $2p$ and that $p \times p \text{ x } p \times p$ was equal to $4p$ and then that $2p \text{ x } 4p$ was equal to $8p$. The students who were not able to differentiate between y^3 and $3y$ also conjoined terms for multiplication or addition.

The percentage of students who conjoined for terms inside the brackets (for example $2x - (4x + 1) = 2x - 5x = -2x$) was higher in the experimental group than the comparison group. Note that the students of the experimental group conjoined for brackets in Year 7 as well, as indicated by the Second Algebra Test. This could be due to two reasons. Firstly, they had a limited experience with brackets in Year 7; secondly, they had attended only two lessons on factorization and expansion before the Third Algebra Test in Year 8.

As Table 7.2 indicates, students of Set 2, Set 3 and Set 4 also conjoined terms for division. For example, they wrote $12y \div y = 3y$ or $3y^2$ and $xy \div x = x^2y$ or xy. Six students of Set 3 were not sure how division operates on variables as they wrote two answers, xy and x, for $xy \div x$. Note that no student in the low ability class, Set 4, wrote two answers for any problem. All the errors mentioned here originated from the use of incorrect procedures rather than misconceptions about the meaning of variables.

When students of the comparison group were required to simplify expressions such as $2p \times 4$ or $7 + 7x + 5$, they were usually able to use the simplification procedures accurately. However, when an unfamiliar problem was presented to them, such as $0 \times 4y^2t$, they misapplied learned procedural rules. In this example, they presented two different incorrect solutions. Some students (Set 1: 39%, Set 3: 39%, Set 2: 33%, Set 4: 41%) ignored the 0 and simplified $0 \times 4y^2t$ as $4y^2t$ (Error a) while other students (Set 1: 8%, Set 3: 26%, Set 2: 4%, Set 4: 0%) multiplied 0 by 4 incorrectly and simplified $0 \times 4y^2t$ as y^2t (Error b). Some students (Set 1: 13%, Set 3: 0%, Set 2: 0%, Set 4: 18%) simply omitted this question. One reason for ignoring the zero may be an over-generalisation of the observation that since zero leaves the number unchanged when it is added to another number then it is also possible that zero may leave the number unchanged after multiplication. Solution error (*b*) may be an attempt to simplify numbers and variables separately.

An incorrect order of operations error, such as simplifying $y \div 2 \div 2$ by dividing 2 by 2 first and giving the final answer as y, was very common in the comparison group. For example, all students in Set 1 first divided 2 by 2 and then divided y by 1 (or 0 in some cases). This type of error was far less prevalent in the experimental group than in the comparison group, as indicated in Table 7.2. This error of not using the correct order of operations could be due to the tendency to simplify similar terms first.

Some students in Set 3 and in Set 4 did not appear to know the difference between y^3 and $3y$ although this misconception was absent in Set 1 and Set 2. It is important to note that the simplification questions $y \div 2 \div 2$, $0 \times 4y^2t$ and "explain the difference between y^3 and $3y$" were included in Section B (attempted by the comparison group in June, after completing all their algebra lessons). However, the experimental group attempted this Section in April, when algebraic simplification was a relatively new topic for them.

In summary, students' ability to represent a word problem in algebraic form depended on the familiarity of the problem, the complexity of the statement, the previous knowledge required and the context used in the problem. Both the comparison and the experimental group were equally successful in plotting and interpreting a graph.

The comparison group was more successful than the experimental group in simplification and factorization of algebraic expressions. They relied more on learned procedural rules as their errors increased whenever an unfamiliar problem was presented to them. In comparison, the complexity of the problem affected the experimental group more than the unfamiliarity of the problems as they made more errors in factorisation and expansion of terms than in simple addition, subtraction and multiplication.

The most common error found in all classes was the conjoining error, with the number of students who conjoined terms for addition in each class being far greater than the number of students who conjoined terms for multiplication. There were many reasons for conjoining terms. For example, in an attempt to simplify like terms, students chose numbers or similar variables and added or multiplied them by taking cues from the related operations. Some students thought that operators such as addition, subtraction, multiplication and division act differently on numbers and variables. Students who were not able to differentiate between addition, multiplication and exponentiation also conjoined terms. Some students may have thought that since we do not write the multiplication sign between two algebraic terms, there is no need to write the sign of addition. The main reason for conjoining terms was the lack of understanding about the use of operators such as addition, subtraction, multiplication, division on variables rather than the meaning of the variables involved. The presence of a context made no difference to simplification errors, as students made similar errors in the contextual problems (Question 4, 5 and 6) as they made when no context was given.

7.3 HALF-YEARLY EXAMINATION

According to the academic plan of the school, all classes were required to sit for a Half-Yearly Examination in June. The Half-Yearly Examination covered algebra; trigonometry; the area of plane figures such as rectangles, triangles and circles; and the volume of simple solids. The analysis presented here was performed only on the items which related to algebra. The algebraic skills assessed were word problems, the simplification of algebraic expressions, and the solution of linear equations.

There were two word problems included in this Half-Yearly Examination:

> Question 1 Anne bought 5 burgers and six doughnuts for $31.
> If a burger costs $$x$ then Anne spends…on burgers.
> If a doughnut costs $$y$ then Anne spends…on doughnuts.
> Write an equation that represents total spending using x, y and 31.
> What is the cost of 2 doughnuts if 1 burger costs $5?
>
> Question 2. The length of a rectangle is three times its width (let t be the width).
> Draw and label a diagram showing the above information.
> Write an algebraic expression for the perimeter in terms of t.
> If the perimeter is 24cm calculate the value of t.
> What would be the length of this rectangle?

The percentages of correct responses in all classes are represented in Table 7.3. The high ability comparison class and the high medium ability experimental class were equally successful in representing and solving both word problems. All students of Set 1 were able to represent word problems in the form of an algebraic equation or expression accurately.

Table 7.3

Percentage of correct responses in algebraic skills in the Half-Yearly examination

Algebraic Skill	Comparison		Experimental	
	Set 1 (n = 27)	Set 3 (n = 24)	Set 2 (n = 25)	Set 4 (n = 17)
Write an algebraic expression (Question1, part i and part ii)	100	67	100	53
Write an algebraic equation (Question 1)	100	83	92	41
Solve a word problem (Question 1)	93	88	100	47
Write an algebraic expression or equation (Question 2)	100	79	92	41
Draw a diagram (Question 2)	100	92	100	94
Solve a word problem (Question 2)	98	59	96	47

Students of Set 3 made various errors, such as writing numerical answers in Question 1 instead of algebraic expressions or calculating perimeter inaccurately, and two students used exponential notation incorrectly. A total of eight students in Set 3 first solved Question 1 algebraically and then wrote numerical answers in place of their algebraic expressions. Three students out of these eight gave an incorrect answer. One student did not attempt this problem and one student used incorrect exponential notation by representing the cost of doughnuts and cost of burgers as x^6 and x^5 respectively.

Three students in Set 3 also used incorrect exponential notation in Question 2. Two students labelled the sides of the rectangle as t and t^3 and represented the perimeter as $p = 2t + t^6$ and $p = 3 \times t \times 2$ respectively. Similar errors were made by two further students. One represented the perimeter as $t = 3^2 \times t^2$ and the other labelled the sides of the rectangle as $3 \times t$ and $t \times t \times t$ and represented the perimeter as $3 (t \times 2) + t$. These students could not make a clear distinction between addition, multiplication and exponentiation.

All students of Set 2 were able to express the price of the burgers and the doughnuts in the form of algebraic expressions accurately. However, two students made errors in writing linear equations. For example, one student of Set 2 conjoined terms and represented the linear equation $5x + 6y = 31$ as $31 = 11xy$ and another student used x for both variables and wrote the equation $5x + 6x = 31$. In Question 2, one student of Set 2 used incorrect exponential notation and represented the perimeter as $p = t + t^3$ and two students represented the perimeter as $p = t + 3t$ and t × (t × 3) respectively.

Students of Set 4 found both questions equally challenging although only two students wrote numerical answers in place of algebraic expressions. However, other errors such as misinterpretation of the word problem and incorrect use of exponential notation were also present. Note that the style of the two word problems was different. Question 2 explicitly described the relationship between length and width (*length is three times the width*), whereas in Question 1 students were required to identify the relationship between money (as represented by the variables x and y) and the number of burgers and doughnuts and then represent it appropriately in an algebraic form. Student errors in Set 4 indicated that they were able to represent the additive relationship but found the multiplicative relationship difficult to identify. For example, four students misinterpreted Question 1 and represented the cost of burgers and doughnuts as x and y; three of these four students then wrote the equation $x + y = 31$. One student used letters p and q to represent the cost of burgers and doughnuts instead of using the given variables x and y, but then wrote the equation as $x + y = 31$.

Five students of Set 4 also used incorrect exponential notation. One student represented the cost of burgers and doughnuts in Question 1 as x^5 and y^6, four other students represented length in Question 2 as t^3, and one student wrote the equation as $t = t^2 \times 3$.

The use of incorrect exponential notation seemed to arise from not making a clear distinction between multiplication and exponentiation. In particular, students represented “length is three times the width (let t be the width)” as $l = t^3$. Also, some other errors seemed to arise as students did not have the pre-requisite knowledge regarding perimeters. For example, some students assumed that perimeter can be calculated by adding two sides only, such as $t + 3t$ or by multiplying two sides such as $t \times (t \times 3)$.

7.3.1 Errors in simplification

The distribution of simplification errors in the Half-Yearly Examination are reported in Table 7.4. It is important to remember that Section B of the Third Algebra Test was taken by the comparison group after the Half-Yearly Examination. Therefore, the percentage of students conjoining terms shown in Table 7.2 and Table 7.4 taken together reflects the percentage of students who conjoined terms in the comparison group. The experimental group had taken the Third Algebra Test in April and the Half-Yearly Examination in June. Therefore, Table 7.2

and Table 7.4 to some extent may indicate the difference in their learning between these two tests.

Overall the conjoining errors in the Half-Yearly Examination appear lower than the errors in the Third Algebra Test. This could be due to the following two reasons. Firstly, in the Half-Yearly Examination there were fewer simplification problems than in the Third Algebra Test. Secondly, students cancelled algebraic expressions instead of conjoining when they multiplied or divided algebraic fractions. These errors are listed as cancellation errors in Table 7.4.

In the Half Yearly Examination, the conjoining errors for addition were aligned with the students' mathematical ability. Conjoining errors for multiplication and division were uncommon in Set 1, Set 2 and Set 3; however, about one-third of students of Set 4 conjoined for division (such as $6ac \div 6c \times 3d = 3acd$) as well. The reasons for conjoining errors were discussed earlier in the results of the Third Algebra Test (pp. 102-103). Conjoining errors are persistent and difficult to reduce particularly in students of low mathematical ability (Booth, 1981).

The percentage of students who conjoined for fractions was higher in Set 2 and Set 3 than Set 1. The high ability class Set 1 had studied simplification of algebraic fractions and indices for an additional two weeks, which was probably why they did not make any conjoining errors in addition and multiplication and only one student conjoined fractions ($\frac{p}{3} \times \frac{1}{3} = \frac{p}{3}$). The low ability class, Set 4, did not attempt the extended response problems which involved simplification of fractions.

The majority of students in the low ability class, Set 4. made operational errors for brackets. No student of the experimental group conjoined terms within brackets in the Half Yearly Examination though this error was made by many students in the Third Algebra Test (see Table 7.2).

In all three of Set 1, Set 2 and Set 3, students cancelled numbers and variables without any regard for the operations joining them, for example writing $\frac{2a+10}{3} \times \frac{9\ 3}{a+5} = \frac{36}{5}$. Problems involving fractions were not included in the test of Set 4. As indicated in Table 7.4, some students in each class used incorrect order of operations and simplified $p + 2p \times 3 = 3p \times 3 = 9p$ incorrectly. Their written work indicated that either these students simplified from left to right or simplified like terms first. Some students of Set 3 and Set 4 gave numerical answers instead of algebraic expressions in Question 1 as indicated in Table 7.4. These students may have thought that the blank spaces were provided to write the final cost for burgers and doughnuts or perhaps they were not willing to accept algebraic expressions as an answer to a word problem.

Table 7.4

Percentage of students who made simplification errors in the Half-Yearly Examination

No.	Errors	Comparison		Experimental	
		Set 1 (n=23)	Set 3 (n=23)	Set 2 (n=24)	Set 4 (n=17)
1	Conjoining	4	52	33	76
2	Conjoining in addition	0	39	25	76
3	Conjoining in multiplication	0	0	8	12
4	Conjoining for division	0	4	4	35
5	Conjoining in fractions	4	30	17	----*
6	Operational error for brackets	0	4	16	71
7	Incorrect cancellation	39	57	63	6
8	Order of operations	39	61	42	41
9	Numerical answers instead of algebraic expressions in Question 1	0	26	0	12

* Set 4 students did not attempt fractions questions

7.3.2 Solution of linear equations

In the Half-Yearly Examination, students were also required to solve a total of six linear equations. The percentage of correct responses in the comparison and the experimental groups in each linear equation are reported in Table 7.5.

Student performance on the last three linear equations was lower than the first three linear equations (as listed in Table 7.5). The first three equations involved one operation and could be solved numerically or by intuition; whereas the last three items required students to use additional knowledge and skills, such as how to expand brackets, the meaning of equality, operating on fractions and the use of operations on variables.

The experimental class, Set 2, was as successful as the corresponding higher ability class, Set 1, of the comparison group in solving the linear equations except equation 4 (see Table 7.4). The medium ability class, Set 3, performed better than the low ability class, Set 4, except in equation 5. As identified earlier in the Third Algebra Test, students find it difficult to operate (add, multiply, or divide) on variables as compared to numbers. This was evidenced again when students solved the linear equation, $7x - 2 = 5x + 8$, which involved numbers and variables on both sides. Many students made errors in the simplification of variables or omitted the second step after simplifying the numbers.

Table 7.5

Percentage of correct responses of each class in solving linear equations

No.	Linear Equation	Comparison group		Experimental group	
		Set 1 (n = 26)	Set 3 (n = 25)	Set 2 (n = 24)	Set 4 (n = 18)
1	$n + 6 = 4$	100	100	100	89
2	$7x = 56$	100	88	96	89
3	$\frac{p}{5} = 9$	100	88	96	83
4	$3(m - 1) = 18$	96	56	79	39
5	$\frac{p}{5} + 6 = 9$	77	48	75	72
6	$7x - 2 = 5x + 8$	85	44	92	22

To investigate the differences between the comparison and the experimental group further, their selection and use of solution strategies for solving a linear equation were compared. A brief description of the solution strategies used by students to solve the linear equations is described here and the number of students who used that particular strategy and the correct responses are reported in Table 7.6.

Students who used the balancing method (see p. 16) added, subtracted, multiplied or divided each side of the linear equation with the same term. The balancing method used by the experimental group evolved from the strategies of working backwards/inverse operations. It was not possible to determine from the students' work which method (balancing, working backwards, or inverse operations) they had used so all of these methods (balancing, working backwards, or inverse operations) were classified as balancing in Table 7.6. In appearance, their method appears like the ordinary balancing method however the thinking process employed by the students of the experimental group is different. This difference was clearly reflected in student responses. For example, a student who used this modification of the balancing method to solve the equations might proceed as follows:

$\frac{p}{5} + 6 = 9$ inverse of + is – (therefore) 9 – 6 = 3

$\frac{p}{5}$ = 3 inverse of ÷ is × (therefore) $p = 3 \times 5 = 15$.

In the method of guess and check, students think about the number which can replace the letter to make the equation correct. For example, to solve $n + 6 = 4$, since $-2 + 6 = 4$, therefore $n = -2$.

The balancing method was the solution strategy most frequently used by the teachers of the comparison group and this was the only solution method used by students in Set 1. Some students in Set 3 avoided the balancing method and used guess and check instead. Note that the teacher of Set 3 sometimes used the guess and check strategy as an alternative way to find whether a given numerical value satisfies an equation and also explained the balancing method by demonstrating the procedure of working backwards.

The most frequently used solution strategy by the teachers of the experimental group was inverse operations. Also, both teachers of the experimental group used the balancing method. This may explain the difference in the relative frequencies of the balancing strategy between the comparison and the experimental groups and may have facilitated the experimental group in learning the balancing method more accurately. For example, only two students of Set 2 made an error when they used the balancing method to solve $7x - 2 = 5x + 8$ while four students of Set 1 made errors in solving this equation.

The experimental group did not use the same automatic strategy to solve every linear equation; instead, they dealt with each equation on its own merit. Some solution strategies facilitated the experimental group. For example, to solve $3(m - 1) = 18$, some students divided 18 by three to get $m – 1 = 6$ and then directly wrote the answer $m = 7$ thus avoiding the error $3(m - 1) = 3m – 1$ which was made by several comparison group students. In fact, the incorrect use of distributive property of multiplication over addition was the main reason for student errors in the comparison group.

Table 7.6
*The number of students who used the solution strategy and the number of correct responses.**

Linear Equation	Solution Strategy	Comparison Group		Experimental Group	
		Set 1 (n = 26)	Set 3 (n = 25)	Set 2 (n = 24)	Set 4 (n = 18)
$n + 6 = 4$	Balancing (B)	26(26)	20(20)	24(24)	11(10)
	Guess and check		5(5)		5(5)
	Answer only				1(1)
	Omitted				1
$7x = 56$	Balancing (B)	26(26)	20(18)	24(23)	8(7)
	Guess and check		5(4)		5(5)
	Answer only				4(4)
	Omitted				1
$\frac{p}{5} = 9$	Balancing (B)	26(26)	19(19)	23(22)	13(11
	Guess and check		4(2)		3(3)
	Answer only		1(1)	1(1)	1(1)
	Omitted		1		1
$3(m - 1) = 18$	Balancing	26(25)	20(13)	23(18)	6(6)
	Guess and check		3(1)	1(1)	6(1)
	Answer only				
	Omitted		1		6
$\frac{p}{5} + 6 = 9$	Balancing (B)	26(20)	21(11)	21(15)	7(5)
	Guess and check		2(1)	3(3)	8(7)
	Answer only				1(1)
	Omitted		2		2
$7x - 2 = 5x + 8$	Balancing (B)	26(22)	19(10)	23(21)	5(1)
	Guess and check		3(1)		6(3)
	Answer only			1(1)	
	Omitted		3		7

*The numbers inside the brackets indicate the number correct responses.

Some strategies were not well chosen. For example, some students of Set 4 used the guess and check strategy to solve 3(*m* – 1). This strategy did not facilitate students (one response out of six correct) and students made operational errors in expanding brackets even when they used a number instead of the variable *m*. This was because these students were not familiar with the distributive property of multiplication over addition for numbers.

In summary, the students of Set 1 and Set 2 were able to represent and solve both word problems. However, the students of the medium ability comparison class (Set 3) were more successful in the questions which were familiar to them. There was no difference in the attempts of students in Set 4 on familiar and unfamiliar word problems. Student errors in algebraic representations were due to incorrect use of exponential notation, incorrect definition of perimeter and misinterpretation of the word problem.

The percentage of correct responses of the experimental group (Set 2, 88%, Set 4, 65%) as compared to the comparison group (Set 1, 92%, Set 3, 71%) indicates that the students of the experimental group were about as successful as the comparison group in solving linear equations. The comparison group used only one solution strategy (the balancing method) to solve all linear equations whereas the experimental group appeared to have thought carefully about which method would be more suitable for each equation. Moreover, using alternative solution strategies facilitated the experimental group in solving some linear equations (for example, $\frac{p}{5} + 6 = 9$) as they avoided making the errors which students of the comparison group made while using the balancing method.

7.4 FOURTH ALGEBRA TEST

One week after the Half-Yearly Examination, the Fourth Algebra Test (Appendix P) was administered to all students. There was only one question in the Fourth Algebra Test which required students to select equations which were equivalent to a given equation and justify their answers. The question was as follows:

> Which of the following equations can be transformed to $x - 2 = 0$? For the ones that can be transformed to $x - 2 = 0$, show how you realised this.
>
> a. $2x = 4$ b. $4 = 2x$ c. $\frac{x}{2} = 4$
>
> d. $4x = 2$ e. $x + 1 = 3$ f. $x - 3 = 1$

Three equations out of the six given equations were equivalent to the equation $x - 2 = 0$. Some students (Set 1; 72%, Set 2; 56%, Set 3; 30%) in the high ability class and the medium ability classes recognised all three equivalent equations. No student in Set 4 recognised all three, but one student recognised two and one recognised one equivalent equation. Note that Set 1 and Set 3 had studied equivalent equations just before the Fourth

Algebra Test, Set 2 had also completed the topic of equivalent equations during their algebra lessons, but Set 4 had not studied this topic explicitly.

The following are the main categories of response.

Transformation method: Transform each equation one by one (using balancing method or inverse operations/working backwards). For example, for $x + 1 = 3$, subtract 3 from both sides to get $x - 2 = 0$. Or solve the equation first and then transform the equation to $x - 2 = 0$.

Substitution method: Solve the equation. For example, solve $x - 2 = 0$ as $x = 2$, then substitute $x = 2$ in the given equations one by one; if the equation is true, then mark it as equivalent. Or solve each equation and substitute the answer in the equation $x - 2 = 0$ to verify the equivalence.

Comparison method: Solve the equation $x - 2 = 0$ as $x = 2$ first, then solve the other equations one by one using any method; if the answer is $x = 2$, mark that equation as equivalent.

Answer without justification: Solve the equations by any method then select some equations as equivalent to the given equation $x - 2 = 0$ without giving any justification.

Omitted or incorrectly solved: Equation not solved or incorrectly solved and no equation selected as answer.

The distribution of solution strategies is listed in Table 7.7.

Table 7.7
The mean (as percentage) of students who used the solution strategies

Solution Strategies	Comparison Group		Experimental Group	
	Set 1 (n = 25)	Set 3 (n = 23)	Set 2 (n = 25)	Set 4 (n = 15)
Transformation Method	13	7	17	9
Substitution Method	65	21	29	2
Comparison Method	5	1	6	2
Answer without justification	16	70	34	20
Omitted/Incorrect/Not solved	1	1	14	67

The selection of solution strategies indicates that very few students preferred to use the algebraic method of transformations to prove the equivalence of two linear equations. Students who used the transformation method were operating on an equation algebraically rather than numerically, and on average 13% of students in the experimental group used the transformation method as compared to 10% of students in the comparison group. Some students in every class used the transformation method. In particular, one student of the low mathematical ability class (Set 4) also used the transformation method, which indicates that the low ability students can also learn to use algebraic methods.

The most frequently used method to show equivalence in the comparison group was the substitution method, which reflects the solution strategy taught in class. Note that the comparison classes started algebraic manipulations with finding unknowns in an algebraic expression in Year 7 and then moved on to the balancing method to solve equations. Thus their inclination to solve each equation and then use a substitution method to answer the question was most likely due to the frequency of these two methods in their lessons.

The mean percentage of students selecting an equation as their answer without giving any justification was highest in Set 3 (70%). Out of these 70% students in Set 3, 37% students gave an inaccurate answer. This indicates that either these students could not recognise the equivalent equations or they were just guessing the answer. The percentage of students who solved the equations and did not give any answer was highest in Set 4 (51%). Out of these 51% students in Set 4, 76% solved the equations accurately however since these students had not studied equivalent equations, therefore they could not select any equation as equivalent.

Table 7.8 shows for each class, the percentage of responses using each strategy that correctly identified the equivalence or otherwise of the given equation.

Table 7.8
Percentage of correct responses by solution strategy

Solution Strategies	Comparison group		Experimental group	
	Set 1 (n = 25)	Set 3 (n = 23)	Set 2 (n = 25)	Set 4 (n = 15)
Transformation Method	70	30	72	13
Comparison Method	88	----	100	----
Substitution Method	97	97	93	100
Answer without justification	83	64	84	35
Overall	90	67	79	50

Table 7.8 indicates that Set 2 was as successful as Set 1 in the use of the transformation, comparison and substitution methods. Set 3 was not as successful in using the transformation method however used the substitution method more accurately. In fact, all four classes used the substitution method accurately as compared to any other method. It is important to note here that the comparison classes had used substitution to verify the solution of linear equations in their lessons while the experimental group had not solved many problems in which they were required to substitute a number in an algebraic expression and then find its numerical value. However, both groups were equally successful in using the substitution method. Moreover, the simplification of the given linear equations after substituting the value $x = 2$ was very easy, which could be a reason for the accurate use of this method.

7.5 STUDENT INTERVIEW II

After the Fourth Algebra Test, student interviews were arranged in which 9 students from the experimental group (6 students from Set 2 and 3 students from Set 4) and 11 students from the comparison group (6 students from Set 1 and 5 students from Set 3) participated. These students were not selected by the researcher; they volunteered to participate in this interview at the request of their teachers therefore the results may not be generalisable. The Head of Mathematics Department made an announcement in all classes that students who would like to participate in this interview may contact her. The 20 students who were interviewed, all came forward willingly to participate in this interview.

The purpose of Student Interview 2 (Appendix Q) was to investigate the reasons behind the students' algebraic manipulation errors and to examine how well the students could interpret the meaning of a linear equation. For this purpose, students were asked to link two linear equations with their graphs and to find the point of intersection of two straight-line graphs. A summary of the main ideas expressed by students in the comparison and the experimental groups is given below.

7.5.1 The experimental group

Students in both Set 2 and Set 4 generally considered a variable as a place holder which could attain any value. For example, in responding to the Q2 (part a) [Is n + 1 odd or even? Explain your answer], one student of Set 4 explained that "since n is a variable it could be either even or odd" and another student of Set 4 explained that "2n would be an even number and n could be both: even or odd."

Students of the experimental group were able to interpret the algebraic expressions and were also able to provide a context for the algebraic expressions. For example, one student of Set 2 explained the difference between $6x$ and $12x$ as "$6x$ means six times something and $12x$ means 12 times something". When another student of Set 2 was asked to express $3x + 4 - (12 - x)$ in words, she explained the term $3x$ as follows: "If you had like nine oranges in a bag, you bought three bags with nine oranges in each of them. Like nine oranges in a bag. Three bags, each with nine oranges in it".

In Question 4, students were required to interpret some word sentences and then represent them in algebraic form. The word sentences are stated here:

(a) My cousin is 3 times as old as I am.
(b) Gregory has \$6 more than Penny, and Phillip has \$13 less than Gregory. How much does each person have if they have \$29 altogether?
(c) The result of adding 12 to a certain number is the same as multiplying the number by 4.

All students of the experimental group wrote algebraic expressions and equations in (a) and (c) correctly, except for one student who solved the word problem and gave a numerical answer. Part (b) presented a greater challenge and though students were able to represent the shares of each individual person, no one was able to write the final algebraic equation by finding an expression for the total amount of money. It appeared from student responses that it was difficult for students to see the whole picture while writing algebraic expressions, as one student of Set 4 explained:

> …since Gregory has 6 more than Penny, so $x + 6$ would be Gregory. Penny has x and Gregory has 6 more than Penny. Phillip has 13 less than Gregory and $x + 6$ is what Gregory has, so $x + 6 - 13$ is the answer.

This indicates that students directly translate from words to symbols and due to a high cognitive load, tend to make errors in the case of a complex word problem.

Students of the experimental group also made errors in simplifying algebraic expressions. For example, all students in Set 2 cancelled 2 in the algebraic expression $\frac{x+2}{2}$. A reason for cancelling 2 could be that students of the experimental group had learned to read the algebraic expressions from left to right. Therefore, they read $\frac{x+2}{2}$ as x plus 2 divided by 2 which compelled them to divide 2 by 2. For example, one student of Set 2 explained that since 2 is being divided by 2 so we can cancel these 2's. Most of the students in Set 2 were able to interpret algebraic expressions in the presence of a context. However, when they simplified without a context the algebraic expression lost its meaning and they simplified by using procedures for simplification such as, cancel like terms.

Two students of Set 4 did not simplify this algebraic expression further. One student explained that she knew that she could not cancel 2; however, she did not know the reason. The other student was not sure how to simplify it further. The responses of these two students indicated that they do not understand the operations and their effect on algebraic expressions and were relying on their previous knowledge of operating on similar expressions. A third student of Set 4 conjoined terms and simplified the algebraic fractions as $\frac{x+2}{2} = \frac{2x}{2} = x$ and $\frac{6(x+3)}{3} = \frac{18x}{3} = 6x$. She appeared to be among the students in

Set 4 who could not differentiate between addition and multiplication which is a reason for these students conjoining terms.

All students of the experimental group were able to link the given graphs with the linear equations. Students were aware that $4x$ goes up at a faster rate than $x + 12$. For example, one student of Set 4 stated that "$4x$ goes up more than $x + 12$" and another explained that "it has to be bigger or equal to 12 to start with because in the equation it says + 12". Students were also able to find the point of intersection as (4, 16) with one student of Set 4 and three students of Set 2 representing the point in this way. Three students out of five in Set 4 were able to link the point with the equation and the given word sentence. One student of Set 2 explained that "$x + 12$ is the same as $4x$ then x is 4 and 4 + 12 is 16, 4 × 4 = 16 then 16 is where they are going to meet on". Since the students of the experimental group translated between different representations such as graphic, numeric and algebraic, they were able to link the graphs with the linear equations. Moreover, they had learned to interpret and generalise relationships in their lessons which helped them to notice the differences between the gradients of the two lines $x + 12$ and $4x$ and the relationship between the point of intersection and the algebraic equation $x + 12 = 4x$, though it is possible that some students may have made the connection by looking at the surface characteristics of the equation only.

One student of Set 4 misinterpreted the question and treated $3x + 4 - (12 - x)$ as a linear equation $3x + 4 = (12 - x)$ which she attempted to solve.

7.5.2 The comparison group

Students of the comparison group considered variables as objects/labels or unknown quantities. All students of Set 1 used abbreviations to represent variables in Question 4. For example, they used the abbreviation *P* for Penny's money, *Ph* for Philip's money and represented “my cousin is three times as old as I am”, by $c = 3i$ (i stands for I), or $c = 3m$ (m stands for me).

The use of x as a label was also reflected in the responses of two students of Set 3. For example, they represented consecutive terms starting from x as x, $2x$ and $3x$. One student explained her answer as “I don’t see how x got involved if we know that 1, 2, 3 are consecutive. If we start with x the next consecutive number is x, $2x$, $3x$.” She used x as a label with 1, 2 and 3 to make the answer look like algebra. When students of Set 3 were asked to decide whether $n + 1$ was even or odd, two students conjoined for addition and based their responses on the answers thus obtained (“$n + 1 = 2n$ therefore n is even”, “$n + 1 = 1n$ therefore $n + 1$ was odd”). Only one student of Set 3 out of the five interviewed was aware that n could represent an even or an odd number. This indicates that the former student considered the value of n as equal to one and wrote it in the answer as a label and the latter student used n as a label only. Both these students treated variables as meaningless and based their decision about $n + 1$ as being even or odd on the coefficients only.

The idea of a letter as a specific unknown quantity was reflected in the response of one student in Set 1 when she answered Question 2. For example,

> Question 2 “Write down three consecutive whole numbers starting with x”

She wrote the answer as x, 4 and 5 and explained that since x could be any number she could assume that x was equal to 3. Therefore, she used x as a specific unknown “3” in x, 4, 5.

Another misconception evident in student responses (3 students out of 5 interviewed) of Set 3 was that the variables and numbers in algebraic expressions represented different quantities. For example, in one question students were asked to express $3x + 4 - (12 - x)$ in words. One student in Set 3 suggested,

> “When there are 3 Indians and then 4 bananas minus 12 bananas minus 1 Indian”.

Another student of Set 3 expressed the same algebraic expression as

> 3 apples + 4 - (12 - x apples)

and a third student of Set 3 said,

> “Three girls (x) plus four boys minus twelve boys minus one girl”

This misconception was not present in the experimental group as all students of the experimental group knew that all terms in an expression represent the same quantity.

A fourth student of Set 3 expressed the expression $3x + 4 - (12 - x)$ as:

> "I have \$$x$ but Jane has three times as much as I do. Tim adds \$$4x$ to James \$$3x$ but then Sarah subtracts the $(12x – x)$".

This student wanted to treat each term as money and used the symbol x as a label in every term to show that they all represent money. On another question, she represented a point of intersection as $(4x, 16y)$ instead of (4, 16) and explained that she wrote "x with 4 and y with 16 to show that 4 lies on the x [axis] and 16 lies on the y [axis]".

Students often used memorised simplification rules incorrectly, which resulted in simplification errors such as incorrect cancellation or incorrect order of operations. For example, five out of six students in Set 1 cancelled the twos in the algebraic expression $\frac{x+2}{2}$ and three students of Set 3 cancelled 2's in the fraction $\frac{x+2}{2}$.

One student of Set 1 explained the reason for cancellation as

> "if any of the top numbers is a multiple of the bottom number you can divide… Like terms don't matter when you are dividing, so you divide both".

Her response indicates that she applied this rule automatically without real understanding. As the main direction given in lessons was to simplify like terms first, the student automatically used this direction and ignored the last part of the rule stated above "Like terms don't matter when you are dividing, so you divide both". This also indicates that she did not notice the order of operations when she simplified. For example, that x should be added to 2 before she could divide it with the denominator "2" or alternatively that she needs to use the distributive property of multiplication over addition here. As indicated by the results of the Half Yearly Examination, many students in the comparison group made errors in the use of distributive property of multiplication over division. Not understanding the distributive property could also be a reason for this error.

One student in Set 3 wrote two answers when she simplified $\frac{6(x+3)}{3}$

$\frac{6(x+3)}{3} = \frac{6x+18}{3} = 6x + 6$ (by dividing 18 by 3)

and also $\frac{6(x+3)}{3} = 6x$ (by cancelling the 3's).

She said she was not sure which answer was correct; however, she chose $6x$ as her final answer. Firstly, her response indicates that she used the procedures automatically and secondly she had no idea about equivalent expressions as she equated $\frac{6(x + 3)}{3}$ to both

$6x + 6$ and $6x$.

Some students of Set 3 also conjoined terms, for example, $x + 2 = 2x$, $x + 6 – 13 = -7x$, and $6(x + 3) = 6 \times 3x = 18x$; however, they could not offer any reason for conjoining these terms.

This idea that x and y in an equation represent integers rather than real numbers was reflected in student responses to Question 5. In Question 5 students were required to link the given linear equations with their plotted graphs and find the coordinates of their point of intersection. Student responses in Set 1 indicated that students were aware of only integer values plotted on a line and also knew that values of x and y were incremental. This was reflected in comments such as “so we are going up in 2’s or, wait, 3’s or 1’s; I don’t know” and “we can see by the number they start on. These are multiples of 4. This is going in increments of 1 and this is going in increments of 4” and “you can see the gaps to see what the difference is to see how many 4’s or how many jumps there are”.

Also, some students of Set 1 were not sure whether the point of intersection could be represented by only the x coordinate or only the y coordinate or both; some students represented the point of intersection as just 16, or (4, 16) or 4 at 16. One student was not sure whether the point of intersection of two linear equations represented in the graph was 4 or 16 and another student was not sure whether the x in the equations $y = 4x$ and $y = x + 12$ represented the same value or not. All students of Set 1 knew the procedure for solving linear equations and they had studied simultaneous linear equations during their algebra lessons; however, they could not relate the solution of the two linear equations $y = 4x$ and $y = x + 12$ with the point of intersection of two lines plotted on a graph. This indicates that they were not made aware of the geometrical meaning of the solution of simultaneous linear equations during their lessons.

In Set 3, three students identified the point of intersection of the two lines as 4, 16, or $4x$, $16y$ or “the point of intersection was 16 point 4”. One student could not identify the point of intersection and she only saw the line, as she said that “it was going up in 4’s”. No student interviewed from Set 3 realised that the point of intersection represented the solution of the simultaneous linear equations $y = 4x$ and $y = x + 12$. All five students of Set 3 were able to solve two-step linear equations with the variable on one side using the balancing method but only one was able to solve the linear equation with the variable on both sides. Students were comfortable with operating on numbers, however they found operating on variables difficult.

7.5.3 Comparing the two groups

In summary, since the students were not randomly selected the results may not be generalisable to the groups. However, the responses of students from the experimental group who were interviewed indicate that they considered variables as having multiple values while most of the students from the comparison group who were interviewed generally considered variables as unknown quantities. Also, some students in the comparison group demonstrated certain misconceptions regarding variables. For example, x as a specific unknown quantity, using variables as labels. Some students of Set 3 thought that the numbers and variables constituting a single algebraic expression represent different objects and some students did not recognise that the various terms in an algebraic expression represent different parts of a single quantity.

Students of the experimental group studied three aspects of variables using numerical, graphical and algebraic representations. They solved word problems in which they learned to identify the variables and the relationships between variables embedded in real contexts. This facilitated the students in understanding the meaning of variables and algebraic expressions with reference to a context. In contrast, the comparison group studied the variables as unknown quantities briefly and then move on to manipulations of algebraic expressions. This brief experience with variables was not sufficient to acquire a sound concept of a variable and thus some students of the comparison group demonstrated misconceptions such as numbers and variables in one algebraic expression represent different quantities. Some students of Set 3 were not able to operate on variables and numbers with the same comfort level.

The students of the experimental group were also facilitated by the problems included in the resource book which required them to translate between different representations. These translation problems helped students to interpret the graph and link the linear equations with the graphs. In contrast, the students of the comparison group knew the procedure of solving simultaneous linear equations and had learned to plot graphs from tabulated numerical values and linear equations. However, their interview responses indicated that they were not aware of the link between the solution of simultaneous equations and their point of intersection on the graphs.

Errors such as incorrect cancellations, incorrect use of exponential notation and conjoining errors were present to some extent in all classes. Reasons for these errors appeared to be the use of simplification rules without understanding. Some students of the comparison group had difficulty in understanding the distributive property and some students had difficulty in understanding equivalent expressions which resulted in incorrect cancellations. Similarly, some students of the experimental group also cancelled terms incorrectly as they read and simplified the algebraic fraction from left to right ignoring the structure of the algebraic fraction. Some students of the experimental group could not differentiate between addition and multiplication and conjoined terms as a result.

7.6 SUMMARY

Most of the students in the comparison group considered a variable as an unknown quantity whereas most of the students in the experimental group considered a variable as a quantity which varies and could attain any value. Certain misconceptions regarding variables were also found in some students of the comparison group such as using variables as labels, considering x to stand for one quantity or simply the number 1, and thinking that the numbers and variables constituting one expression come from different sources. Such misconceptions were few, if any, in the experimental group.

Student responses indicated that the ability to represent a word problem in algebraic form depends on the familiarity of the problem, the complexity of the statement, the previous knowledge required and the suitability of the context used in the problem. Students in both the high ability comparison class, Set 1, and medium ability experimental class, Set 2, were generally able to write an algebraic expression or equation from a word problem and then solve that word problem. The students of Set 3 found familiar word problems easiest and the students of Set 4 found simple word problems easiest. It was difficult for students of Set 4 to link multiple ideas together for complex word problems involving several components. The errors in algebraic representations by Set 3 and Set 4 were mainly due to incorrect use of exponential notation, lack of required knowledge, and misinterpretation of the word problem.

The comparison group relied more on learned procedural rules and their errors increased whenever an unfamiliar problem was presented before them. In contrast, the complexity of the problem affected the experimental group more than the unfamiliarity of the problems as they made more errors in factorisation and expansion of terms than in simple addition, subtraction and multiplication.

The most common error found with all students was the conjoining error for addition, subtraction or multiplication. The reasons for conjoining terms were the tendency to pick and choose terms and to take cues from the given operation and the tendency to give the final answer as a single term. Students incorrectly applied the rule that “we do not write a multiplication sign between two algebraic terms” to omit the addition sign between two algebraic terms as well. With special attention given by the teachers and increased familiarity of algebraic simplifications, the conjoining error reduced in the high ability and medium ability classes during Phase I.

The comparison group learned and used the balancing method to solve linear equations during their lessons and they used the same strategy to solve all given linear equations in the assessment. The experimental group had solved problems based on real contexts which gave them an opportunity to study a greater variety of solution strategies. They thought more about the variable involved and decided on a solution procedure accordingly.

The use of alternative solution strategies such as working backwards, facilitated the experimental group to learn the balancing method and apply it correctly. Moreover, the use of alternative solution strategies facilitated the experimental group in solving the same linear equations, as they avoided making errors which some students of the comparison group made while using the balancing method.

Most students used numerical methods such as the substitution method to identify equivalent equations and very few students transformed linear equations algebraically to prove equivalence. However, some students in every class used the transformation method despite the difference in their mathematical abilities.

Both the comparison and the experimental group were able to plot graphs of linear equations and then interpret those graphs to answer word problems. The students of the high ability comparison class were not aware that a line represented all real numbers as they saw only the integer values on the graph of an equation. They were not sure how to represent a point using coordinates. There was no such confusion that in the experimental group. It is not known whether the students of the experimental group could relate the real points with the line, however it was clear from their responses that they were aware of the gradient and could use that knowledge to match a line graph with its equation.

The conjoining error was very widespread among all students. Students conjoined terms for addition, multiplication, inside brackets, or for division. Conjoining errors due to multiplication and division increased with the decrease in the mathematical ability of students. There were different reasons for conjoining algebraic terms. For example, some students thought that operators such as addition, multiplication and division acted differently on numbers than on variables. Therefore students simplified numbers by taking cues from the operations and wrote all variables adjacent to each other in the answer to make the answer look like algebra, or they multiplied instead of adding, or added instead of multiplying. Students who were not able to differentiate between addition, multiplication and exponentiation also conjoined terms. There was one more difference between the comparison and the experimental groups: Most of the students in the comparison group seemed to pick and simplify numbers and similar variables without any concern for the operations joining them, whereas in the experimental group most of the students simplified from left to right by adding or multiplying successive terms.

The results indicate that the students of the experimental group demonstrated greater flexibility in selecting solution strategies for solving linear equations and a deeper understanding regarding variables. This facilitated them in solving linear equations as they avoided some errors which the students of the comparison group made. Simplification errors such as conjoining numbers and variables, incorrect use of distributive property of multiplication over addition, and incorrect order of operations were made by all students. The main reason for simplification errors was difficulty in interpreting the use of operators on variables. The simplification errors of the experimental group were less, not more than the comparison group, with the exception of the conjoining error for brackets.

CHAPTER 8

DISCUSSION

This chapter brings together the results presented in Chapters 5, 6 and 7. It includes discussion on the concept of a variable acquired by students and their algebraic competence. Differences between the comparison and the experimental groups in their conception of variables, how they solved linear equations, represented word problems algebraically and simplified algebraic expressions are identified. The role of the teachers and the effectiveness of the teaching resources used by the participating teachers are also discussed. Finally, the effectiveness of the Multifaceted Variable Approach (MVA) compared to the traditional teaching approach used by the school is summarised.

8.1 STUDENTS' CONCEPTIONS OF VARIABLES

The students' conceptions of variables were reflected in their interview responses, in their worked solutions and representations of word problems, in their selection of solution strategies to solve linear equations, and in the strategies they selected to verify the equivalence of two equations. The results of the study indicate that students in both the experimental and comparison groups most commonly considered letters as unknown numbers or as variable quantities. Certain misconceptions regarding variables were also evident, such as considering variables as objects or labels, considering that the variable x represents the number 1 or just one quantity, or considering that different letters stand for different quantities. Most of these misconceptions were identified earlier by Booth (1984), Küchemann (1981), Perso (1991) and MacGregor and Stacey (1997) with the exception of one misconception: that numbers and variables in the same algebraic expression represent different quantities. The various conceptions of variables found in the students are described here.

8.1.1 Specific unknown quantity

Many students considered the variable x as an unknown whole number. Some students thought that x was a specific unknown number. This was reflected in the reasoning provided and processes used by students to describe the relationship between two variables represented as an inequality. For example, in Student Interview I, 67% students (out of 24) were content to answer questions (such as “if 5 + y is larger than 14, then what could y be equal to?”) by simplifying one number only.

Students who considered variables as specific unknown numbers were not prepared to accept algebraic expressions as answers to word problems. In different assessments, instead of giving an answer in the form of a variable or an algebraic expression, some students gave numerical answers. For example, 10% of the students assigned numerical values to the variable in the First Algebra Test, and 14% and 7% respectively wrote numerical answers instead of algebraic expressions in the Third Algebra Test and in the Half Yearly Examination.

8.1.2 Variable quantity (generalised number)

According to Küchemann (1981) very few students aged 13-15 years reach the stage where they consider a variable as a generalised number that can attain multiple values. In Student Interview I, most students thought that the value of the unknown in an inequality must be just one number. However, some students, in particular the high ability students, began to consider more values as they came across different problems. For example, in Student Interview I, 42% of the students were able to suggest more than one value which satisfies the inequality “if 5 + y is larger than 14, then what could y be equal to?” However, only 33% of students were able to fully explain and justify their answers.

In Student Interview 2, students were asked to talk about their thoughts and the method that they used to solve each step of the problem. In all, 8 out of 11 students considered more than one value for the variables when they answered the question “Is n + 1 even or odd?” and some students also justified their response as “It depends on what number n is. If n is an even number then the numbers added together will be odd” or “Since n is a variable it could be either even or odd”.

8.1.3 Letter as a label (to represent an object).

Certain student misconceptions about the letters used to represent variables were also observed during the study. For example, some students used letters as labels to represent physical objects instead of a number of objects. Some students also used a variable x as a label with a number to indicate that the numerical value represents that quantity. For example, one student translated the algebraic expression $3x + 4 - (12 - x)$ into words as “I have \$$x$ but Jane has three times as much as I do. Tim adds \$4x to Jane’s \$3x but then Sarah subtracts the (12x – x)”. She used x as a label with every number to indicate that each number in the expression represented the same quantity. These two misconceptions were identified earlier by Booth (1984) and MacGregor and Stacey (1997).

The misconception of considering a variable represented by x as an label or object was revealed when some students arranged "chocolates" side by side (such as $xxxxx = x^5$, using x as a label for chocolate) to calculate the total number of chocolates in the First Algebra Test. Another student used the label x for a bus and explained that five buses can be represented as $5x$, "because x is equal to 1 bus and you are timesing it by how many buses are there". This student not only used x as a label, she also thought that x represented one object: a bus.

Some students used letters as labels to represent association. For example, one student used b for a bus and p for passengers when she used information "there are 20 passengers for every bus" to calculate the number of passengers in 2 buses and 5 buses respectively as $20p \times 2b = 40p$ and $20p \times 5b = 100p$. Similarly, another student used x and y as labels to associate numbers with axes. For example, she represented the point of intersection (4, 16) of two straight line graphs as ($4x$, $16y$) and explained that she wrote "x with 4 and y with 16 to show that 4 lies on x [axis] and 16 lies on y [axis]".

Some students also used letters as labels to make their answer look like algebra (MacGregor and Stacey, 1997). For example, one student wrote three consecutive numbers starting with x as x, $2x$, $3x$ and explained that "I don't see how x got involved if we know that 1, 2, 3 are consecutive. If we start with x, the next consecutive number is x, $2x$, $3x$".

In the First Algebra Test, 38% out of 103 students considered letters as an object or as a label. However, the percentage of students who considered letters as objects decreased and only 2% out of 104 students in the Yearly Exam at the end of Year 7 used letters as labels when representing word problems in algebraic form. This result indicates that the error of using letters as objects or labels is likely to reduce in frequency as greater familiarity with variables develops.

8.1.4 Letters and numbers detached

Students think that variables and numbers are detached (Perso, 1991). Numbers are equal to variables under certain conditions and otherwise they are separate entities. For example one student explained during Student Interview I that "a letter doesn't stand for a number unless a specific rule is given" and "it doesn't say what b is equal to, b is equal to b". She further explained that "you can't really times letters by letters to get a number. Say if you times like *m* times *m* it would be m^2 or *n* times *m* is *nm*, so you can't really do two letters to equal a number or two numbers to equal a letter".

Some students further extended the idea that letters and numbers are different, into the idea that letters and numbers can represent different quantities in one algebraic expression. For example, one student translated the expression $4x + 9 + x - 15$ as "four lollies plus nine bananas + one lolly minus fifteen bananas" and another student translated $3x + 4 – (12 – x)$ as "when there are 3 Indians and then 4 bananas minus 12 bananas minus 1 Indian" during Student Interview II. The misconception that different letters represent different quantities has been identified earlier by MacGregor and Stacey (1997) and Perso (1991), however the misconception that numbers and letters represent different quantities in one algebraic expression does not appear to have been reported previously.

8.2 COMPARISON BETWEEN THE EXPERIMENTAL AND THE COMPARISON GROUP

Many students of the experimental group reached the stage of considering *x* as an unknown quantity in Year 7, and some students moved on to accept a variable as a generalised number by the middle of Year 8. This was reflected in Student Interview II and the results of the Third Algebra Test (mid Year 8). The percentage of students in the experimental group (Set 2: 98%, Set 4: 94%) who accepted answers in the form of algebraic expressions was higher than in the comparison group (Set 1: 98%, Set 3: 66%) (the average of Third Algebra Test and Half Yearly Examination).

Many students in the comparison group reached the stage of accepting a variable as an unknown quantity by the middle of Year 8. However, there were students who still used variables as objects or labels. There is no evidence which could indicate that the students of the comparison group were accepting variables as generalised numbers by the middle of Year 8.

The percentage of students who demonstrated misconceptions regarding variables when they represented word problems in algebraic form was higher in the comparison group. For example, in the First Algebra Test (mid Year 7); 65% students of the comparison group demonstrated the misconception of variables as an object or a label as compared to 12% students of the experimental group (see Table 6.3). In Year 8, the misconception of letter as object or label appeared in the work samples of 26% of students of Set 1 and 13% of students of Set 3 while only 12% of students of Set 4 and none of Set 2 made this error in the Third Algebra Test (mid Year 8).

Simplification of algebraic expressions, in particular the evaluation exercises, promoted the conception of a variable as an unknown quantity in the comparison group. However, the comparison group started simplification of algebraic expressions soon after being introduced to variables, which may have promoted the idea of a variable as an object or a label. On the other hand, the experimental group started algebra from the representation of linear relations, which required them to give answers in the form of algebraic expressions. The solution of problems in the form of algebraic expressions may have promoted the idea of a variable as a generalised number.

8.3 ALGEBRAIC COMPETENCE OF STUDENTS

A detailed analysis was made of representation errors (e.g. in translating from words to algebra or from tabulated values to line graphs and linear equations and vice versa), simplification errors, and the choice of solution strategies for linear equations, to identify the algebraic proficiency of students. The percentage of correct responses reported is based on the average score of the experimental and the comparison group in that particular skill during the study

8.3.1 Translating word problems to algebraic form

The percentage of correct responses in translating from a word problem to an algebraic form (Set 1: 91%, Set 2: 85%, Set 3: 62%, Set 4: 37%) and finding a solution to a word problem (Set 1: 97%, Set 2: 99%, Set 3: 86%, Set 4: 68%) were aligned with their mathematical abilities. The higher ability experimental class, Set 2, was about as successful as the higher ability comparison class, Set 1, in translating word problems to equations and solving them, despite the fact that Set 1 was supposed to have been of higher general ability than Set 2. By contrast, the percentage of errors in the lower ability experimental class, Set 4, was markedly lower than that in the lower ability comparison class, Set 3, in line with their general ability.

Students' success in translating a word problem into an algebraic expression was dependent upon certain factors, such as the familiarity of the context or the problem, the complexity of the word statement, the selection of relevant information and the prerequisite knowledge required. The comparison group tended to use practised procedures and memorised formulas; therefore, the percentage of correct responses improved with the familiarity of the given problems. On the other hand, the students in the experimental group attempted to interpret the relationships from the text; therefore, the complexity of the word problem affected the experimental group more than the unfamiliarity of the problem. In particular, although the low ability experimental class, Set 4, could express simple word sentences in algebraic form, they made errors in the more complex word problems which required establishing links between multiple relationships. In the case of complex problems, students of Set 4 often selected the most obvious information from the text instead of spending sufficient time trying to understand the problem.

Stacey and MacGregor (1999) pointed out students' preference to use arithmetical methods for solving contextual word problems. The results of the Year 8 Half Yearly Examination indicate that most students (Set 1: 100%, Set 2: 92%, Set 3: 81%, Set 4: 41%) in every class had learnt to express word problems in algebraic form. However, only some (Set 1: 83%, Set 2: 33%, Set 3: 36%, Set 4: 0%) used algebraic methods to solve the resulting equations. Irrespective of the solution strategy used, most students (Set 1: 96%, Set 2: 98%, Set 3: 74%, Set 4: 47%) solved the word problems accurately. This finding also reinforces the idea that symbolism and reasoning develop as independent capabilities (van Ameron, 2003).

The average percentage of students who gave numerical answers instead of algebraic expressions in the First, Second, Third and Fourth Algebra tests in the experimental group (Set 2: 1%, Set 4: 5%) was lower than in the comparison group (Set 1: 4%, Set 3: 23%). More students in the experimental group accepted answers in the form of algebraic expressions than the comparison group. The students in the experimental group accepted algebraic expressions as they had used spreadsheets to study the numerical, graphical and algebraic representations of functions. This agrees with the finding of Sutherland (1991) that students who used Logo and spreadsheets readily accepted algebraic expressions such as $x + 2$ as objects.

The students of the experimental group made fewer errors than the students of the comparison group in translating word problems into algebraic expressions as they had fewer misconceptions regarding variables. For example, in translating a word problem into algebraic form, some students used variables as objects or labels (average percentages in First, Second, Third and Fourth Algebra tests: Set 1: 19%, Set 2: 0%, Set 3: 29%, Set 4: 10%). Some students were not able to make a clear distinction between repeated addition and multiplication (average percentages: Set 1: 4%, Set 2: 0%, Set 3: 22%, Set 4: 29%) and process $3 \times y$ as y^3. Not making a clear distinction between addition and multiplication is one reason for conjoining terms (Stacey & MacGregor, 1994). The percentage of conjoining errors in students of the comparison group was higher than the experimental group in word problems which required students to translate from words to algebra (Set 1: 19%, Set 2: 4%, Set 3: 22%, Set 4: 5%) in the First Algebra Test (mid-year 7). The reason indicated earlier was not completely responsible for the conjoining errors of Set 1 as the percentage of students (19%) who conjoined terms was higher than the percentage of students (4%) students who could not differentiate between addition and multiplication. It appears that students in Set 1 conjoined terms as they did not consider $2x + 5$ as an acceptable answer, another reason of conjoining indicated by Booth (1986) and Küchemann (1981).

8.3.2 Translating tabulated values to line graphs and vice versa

In the Second and Third Algebra Tests (see Table 6.7 and Table 7.1), the percentage of correct responses for plotting a graph from a table of values and vice versa was similar in the experimental group (Set 2: 84%, Set 4: 69%) and the comparison group (Set 1: 82%, Set 3: 73%).

Previous research has shown that even if students can plot a line graph accurately, many students have a superficial concept of points on a straight line (Knuth, 2000). In this study, many students of the comparison group were not aware that the graph of a linear equation represented real ordered pairs. When asked to identify the plotted graphs of the linear equations $y = 4x$ and $y = x + 12$, students of the comparison group justified and explained their answers as “you can see the gaps to see what the difference is to see how many 4’s or how many jumps are”. The view of lines as being comprised of only points with whole number coordinates is not surprising as most of the problems they studied in class involved positive whole numbers. The effect of such a limited type of examples has previously been identified by Warren (2003).

By contrast, students of the experimental group considered not only the table of values but also the gradient of the line, the linear equation and the intercept. For example, one student of Set 4 reasoned, “$4x$ goes up more than $x + 12$” and another student of Set 4 explained that “It has to be bigger or equal to 12 to start with, because in the equation it says plus 12”. Students of the experimental group may have been encouraged to think about relationships and characteristics of the line graphs as a result of the many contextual problems presented in the algebra resource book they used. The function machine and the table of values representing linear functions, as studied by the experimental group, may also have contributed to the idea that a variable may take several different values (see section 8.1.2).

8.3.3 Translating tabulated numerical values to linear equations

The average percentage of correct responses in representing a table of numerical values in the form of an algebraic equation was similar in the experimental group (Set 2: 96%, Set 4: 70%) and the comparison group (Set 1: 97%, Set 3: 78%) despite the fact that the experimental group had a lower general ability than the comparison group. The MVA had facilitated the identification of the relationships between variables in contextual word problems and the representation of tabulated numerical values in the form of algebraic expressions.

The comparison group mainly used the procedure of calculating $\frac{y_2 - y_2}{x_2 - x_1}$, finding the value of the *y*- intercept from the table, and then using the gradient-intercept form to write an equation for a table of values of a linear function. Some students in Set 3 instead attempted to reason out the relationship between the variables. By contrast, the experimental group had not learned the gradient-intercept form and they all reasoned out the relationship between the variables. On this task, there was no difference in the percentage of correct responses in the high ability class (Set 1) and the high medium ability class (Set 2) although the students used different methods.

(M. MacGregor & Stacey, 1993) found that students who are able to apply a correct method to formulate an equation often find articulating a relationship in words and expressing a relationship in symbolic form to be more problematic. This observation agrees with the results obtained in the First Algebra Test and the Yearly Examination in this study. On average, 78% students of the medium ability comparison class (Set 3) used the table of numerical values and the gradient intercept formula to formulate the linear equation in the First Algebra Test as well as in the Yearly Examination (see table 6.2 and 6.6). However on average, only 50% students of Set 3 were able to express the relationship in words (see table 6.2 and 6.6). A reason for this low percentage could be the unfamiliarity of this task. Students of Set 3 had not solved any problem in their algebra lessons which required them to express a relationship in algebraic form before the First Algebra Test. Therefore, the percentage of correct responses of Set 3 in writing a relationship in words was only 24% in the First Algebra Test, increasing to 76% in the Yearly Examination. The second difficulty faced by the students of Set 3 was in translating from words to algebraic expressions: on average only 52% students of Set 3 were able to express the relationship in algebraic form. The reason for this difficulty may be associated with the difficulties in understanding the meaning of variables. Many students in Set 3 considered variables as specific unknown numbers and some even considered variables as physical objects as indicated earlier. This may have made it difficult for the students of Set 3 to translate a relationship from words to an algebraic expression or equation.

In comparison, 70% students of Set 4 formulated the linear equation from a table of values and 62% students of Set 4 were able to express the relationship given in a table of numerical values in words despite their lower mathematical ability (average of this skill in Table 6.2 and

6.6). Students of Set 4 did not use a learned method to formulate an algebraic equation; instead they relied on recognising the relationship between variables. The students of Set 4 often selected the most visible information. Therefore they were affected by the complexity of the task rather than the unfamiliarity of the task. For example, in the First Algebra Test 27% students of Set 4 identified the recursive relationship only and did not see the functional relationship between the variables. The tendency of students to see the recursive rather than the functional relationship is a persistent difficult of students that has been identified by many researchers (e.g., (Orton & Orton, 1994; Stacey, 1989). However, the ability to see the recursive relationship cannot be interpreted as a student failure, as students behave differently when non-linear relationships are represented in numerical tables (Noss, Healy, & Hoyles, 1997).

8.3.4 Simplification of algebraic expressions

The percentage of correct responses in the simplification of algebraic expressions was aligned with the mathematical ability of the students (Set 1: 91%, Set 2: 74%, Set 3: 68%, Set 4: 43%) based on the average of the Third and Fourth Algebra Tests.

The students of the comparison group were able to solve familiar problems using learned simplification procedures; however, their errors increased with unfamiliar problems. For example, a high percentage of students in the comparison group (Set 1: 61%, Set 3: 70%) were not able to simplify $0 \times 4y^2t$ accurately. Some students multiplied 0 with 4 to give an answer of y^2t and others ignored 0 to give their final answer as $4y^2t$. This error was considerably lower in the experimental group (Set 2: 38%, Set 4: 59%). This finding is consistent with previous reports that when students solve familiar problems they usually rely on visual cues which trigger a sequence of steps they may undertake, whereas for unfamiliar problems they cannot rely on their previous knowledge and hence they make simplification errors (Falle, 2007).

The main simplification errors made by students were conjoining algebraic expressions and simplifying numbers and variables separately without any regard of the operations joining them (for example by cancelling 3's in $\frac{6(x+3)}{3}$). The most widespread error in all classes was the error of conjoining terms for addition (such as $7x + 9 = 16x$), multiplication (such as $2p \times 6p = 8p$), or division ($xy \div x = xy$). Some students also conjoined terms within brackets (such as $2p\ (3y + 7) = 2p \times 10y$).

Since students read from left to right, they also process algebraic expressions in the same way without any concern for the operations joining the terms. For example, many students conjoined because they processed from left to right as $p + 2p \times 3 = 3p \times 3 = 9p$. Tall and Thomas (1991) called this obstacle, which results in conjoined terms in simplification, the *parsing obstacle.*

Some students picked and simplified like terms by simplifying like terms and interpreting the signs of addition as the direction to calculate. For example, by adding 4 to 4 and multiplying x by x^2 and y by y in $4xy + 4x^2y = 8x^3y^2$.This technique of picking and collecting was prominent in Set 2 (50%), Set 3 (65%) and Set 4 (41%) in the Third Algebra Test. In comparison, only 6% of students in Set 1 picked and collected numbers and variables in this way. This reason for conjoining terms was partly due to the interpretation of addition or multiplication in an algebraic expression as the direction to calculate, which Tall and Thomas (1991) called the *process-product obstacle.*

In Set 3 (17%) and Set 4 (18%), several students could not differentiate between y^3 and $3y$. During Student Interview I, one student of Set 3 explained that if n is 10 then "$3n$ would be equal to 13 if you plus it together and $n + 6$ is equal to 16". The simplification errors of students in algebra assessments (Table 6.3, Table 7.2, and in the Half-Yearly Examination) indicate that some students could not differentiate between addition, multiplication and exponentiation, which may be another reason why they conjoined terms for addition or multiplication. This reason of conjoining terms was also indicated by Stacey and Macgregor (1994).

Some students use rules such as BODMAS (calculate what is in the brackets first) without understanding the structure of the algebraic expressions, may conjoin terms inside the brackets. For example, some students simplified the fraction $\frac{6(x+3)}{3} = \frac{6 \times 3x}{3} = 6x$ by combining 3 with x first and then cancelling the 3's or dividing 18 by 3. One student of Set 1 explained, "If any of the top numbers is a multiple of the bottom number you can divide". The report of the Nuffield foundation (2009) also agrees with this observation that students who used rules such as BODMAS to simplify algebraic or numerical expressions are often not able to decide on the order of operations.

The percentage of students who conjoined terms within parentheses was higher in the experimental group than the comparison group in the Second Algebra Test (Set 1: 32%, Set 3: 48%, Set 2, 60%, Set 4: 22%) and in the Third Algebra Test (Set 1: 4%, Set 3: 9%, Set 2: 38%, Set 4: 35%), although the number of students making errors decreased in most classes. The reason for conjoining terms inside parentheses appears to come from the strong perception that terms inside brackets need to be simplified first. The error of conjoining terms within parentheses reduced after the students became familiar with factorisation and expansion of algebraic expressions and no student conjoined terms inside brackets in the Half-Yearly Examination in Year 8.

The percentage of students who conjoined terms in the comparison group (Set 1: 32%, Set 3: 51%) was similar to the experimental group (Set 2: 36%, Set 4: 53%) [averaged from Tables 6.8, 7.2 and 7.5]. The percentage of students who made errors in expanding parentheses in the comparison group (Set 1: 35%, Set 3: 45%) was also similar to the experimental group (Set 2: 24%, Set 4: 65%) [averaged from Tables 6.8, 7.2 and 7.5]. The error of incorrect order of operations was higher in the comparison group (Set 1: 54%, Set 3: 62%) than the experimental group (Set 2: 39%, Set 4: 30%) [averaged from order of operations and cancellation errors, Tables 7.2 and 7.5].

The many practice exercises they had completed, no doubt facilitated the comparison group in simplification of algebraic expressions. In comparison, the students of the experimental group did not solve many simplification exercises. They used the teaching *resource Activities that Teach Patterns and Algebra* which used the generalisations of number properties for simplification of algebraic expressions. Generalisation of number properties facilitated the experimental group in learning the use of correct order of operations. Many students in both the comparison and the experimental groups conjoined terms. Students conjoined terms when they simplified algebraic expressions from left to right without taking into account the structure of algebraic expressions (Tall & Thomas, 1991), when they applied procedures without understanding, and when they could not differentiate between addition and multiplication (Stacey & MacGregor, 1994). Some students also conjoined terms inside brackets to simplify them first (Brown & Coles, 1999).

8.3.5 Solving linear equations

The percentage of correct responses in the Half Yearly Examination in the experimental group (Set 2: 88%, Set 4: 65%) as compared to the comparison group (Set 1, 92%, Set 3, 71%) indicates that the students of the experimental group were as successful as the comparison group in solving linear equations, although they tended to use rather different solution strategies.

The comparison group used the balancing method to solve all linear equations as this was the main solution strategy they were taught. In comparison, the experimental group had solved problems based on a variety of real contexts and were also exposed to a greater range of solution strategies such as guess and check, the balancing method, and inverse operations. As a result, they used different solution strategies to solve different linear equations.

Alternative solution strategies, such as guess and check and working backwards, facilitated the experimental group in avoiding certain simplification errors. For example, the students of the comparison group used the distributive property of multiplication over addition incorrectly when they solved $\frac{p}{5}+6=9$ and $3(m-1)=18$. The students of the experimental group had learned the balancing method by building on the solution strategies of working backwards and inverse operations, and some students of the experimental group used this version of the balancing method to undo the addition of 6 in $\frac{p}{5}+6=9$ and the multiplication in $3(m-1)=18$, which helped them in accurately solving these linear equations.

Most of the students in Set 2 used the balancing method to solve the equation $7x-2=5x+8$ because this problem could not be solved by using numerical strategies. Some students of the experimental group used the numerical solution procedures such as guess and check to solve the linear equations, $7x-2=5x+8$ and $3(m-1)=18$. Their use of guess and check to solve these linear equations indicated that these students were aware of the meaning of equality in a linear equation in a numerical sense.

Some students of Set 2 made errors in expanding brackets even when they replaced *m* by different numbers. This indicates that they were not aware of the distributive property of numbers. This also indicates that learning the generalisation of number properties using numbers in arithmetic may facilitate students in simplifying brackets.

Students of the comparison group were taught only one strategy, the balancing method, to solve all linear equations. However, in the Fourth Algebra Test most of the students of the comparison group as well as the experimental group used numerical methods to prove the equivalence of two linear equations. This agrees with the findings of Linsell (2009) that very few students use the transformation method to solve linear equations.

The experimental group was facilitated by learning different solution strategies and they also naturally used the balancing method when it was required. For example, students of the experimental group used different solution strategies to solve different problems. In comparison, the comparison group always used the learned method and often made errors when they did not think about the most efficient method of solving an equation. Moreover, when faced with an unfamiliar problem, they relied on numerical methods. This finding indicates that the shift of moving from arithmetic to an algebraic method is a gradual process that requires a full understanding of variables and equations.

8.3.6 Reasoning and justifying

A significant difference between the responses given by students of the experimental and comparison groups was in the reasoning they gave to justify their responses. Students of the experimental group provided more high quality reasoning responses during Student Interview I (Set 2: 8.6; Set 4: 6.8) than the comparison group (Set 1: 6.9, Set 3: 5.2). Similarly, during Student Interview II, the students of the experimental group readily justified their responses and provided reasons for their answers willingly as compared to the comparison group who frequently needed to be prompted to explain their answers.

The students of the comparison group generally based their reasoning on the procedure used to calculate an answer (see p. 81 – 83). In contrast, most (65%) students in the experimental group based their reasoning on the relationship between *x* and *y* and most of these (69%) wrote a generalised statement to explain their reasoning.

8.4 THE ROLE OF THE TEACHERS AND TEACHING RESOURCES

Teachers and the teaching resources have a profound effect on students' learning, as confirmed in the present study. The role of the teachers of the experimental and the comparison group is described here.

8.4.1 The experimental teachers

Both teachers of the experimental group (Rosa and Mona) were informed about the different aspects of variables (specific unknown, generalised number and as a functional relationship) and the importance of learning about all three aspects together in order to develop a multifaceted conception of variables. The effect of the Professional Development Workshop I was felt as both teachers involved their students in discussion about algebraic objects. They encouraged their students to think about the procedures and explain their thinking by reasoning and justifying their point of view. In particular, both teachers promoted the conception of a variable as a quantity which can attain many different values. Rosa (Set 2) used the word pronumeral for the letter *x* when used as a place holder for any unknown number and the word variable for a quantity that varied. She used both these two terms along with their associated meanings throughout her lessons. Mona (Set 4) used the term variable in her lessons and explained it as a quantity that was able to change. Therefore, to some extent, students were aware that the three aspects of variable were all related.

Teachers of the experimental group used the resource book *Working Mathematically*: *Activities that Teach Patterns and Algebra* to teach algebra. This book contains word problems derived from real contexts which require students to identify and represent relationships between variables in algebraic form. The aspects of a variable as a specific unknown, as a generalised number and as a functional relationship are developed through different types of problem situations. Teachers and students used the resource to relate variables with the sources from which they were derived and identify the relationships between the variables. Numerical, graphical and algebraic representations facilitated students in seeing expressions and equations from different perspectives. The problems given in the resource include discussion prompts and both Rosa and Mona occasionally used these prompts to encourage summative discussions regarding variables and expressions.

Both teachers encouraged students to link algebraic expressions with the given context. For example, while using the context of a girl climbing stairs, students interpreted the expression $3x + 5$ as, "Amy started at stair number 5 and climbed three stairs (in one step) at a time" and then used this interpretation to calculate the stair number at which Amy would be after taking 12 steps. The reason that some students of the experimental group were able to consider the variable n as a generalised number could be due to the problems solved by the experimental group, many of which required them to generalise algebraically.

Both teachers encouraged students to think about each problem in order to select a suitable solution strategy. They used a combination of the working backwards method and inverse operations to teach the balancing method. This technique probably facilitated the experimental group in learning the accurate use of the balancing method. (Recall that Set 2 made fewer errors than Set 1 in solving $7x - 2 = 5x + 8$).

The teachers of the experimental group also encouraged their students to explain and justify their solutions. For example, in one lesson about patterns, Rosa asked her students "Do you think this is a pattern?" and then asked students to explain the characteristics of that pattern. In another lesson Mona asked students to explain a procedure, asking "What did I do to get from the 7th to 15th position?" and "How are you going to find the next pair of values?"

In Year 8, many such instances when teachers could have involved students in summative discussions were lost because students solved problems on their own more frequently in Phase II than in Phase I and teachers helped them only occasionally. Moreover, both teachers of the experimental group were unable to alter their belief that students needed many practice exercises to learn algebraic simplification, despite having observed the quality of their students' conceptions of variables and algebraic ability that they had attained without practice exercises in Phase I. Therefore they tended to focus on teaching the correct procedures for simplification. The contextual exercises provided in the resource book alone were not sufficient for this purpose. Therefore, after formulating an equation or an expression, students completely ignored the context and focused on the algebraic simplification procedures.

Both teachers used realistic contexts to teach students the meaning of number properties. However, there was no instance when any teacher linked the simplification process with a contextual representation. Teachers also let students explore spreadsheets to study graphical representations of functions on their own, when a guided demonstration of how a graph changes when the algebraic representation changes may have facilitated the experimental group. Unfortunately, there had been no activity in the Professional Development Workshop which could have drawn teachers' attention towards this important teaching technique.

Professional Development Workshop 1 seemed to have a stronger effect on both teachers than Professional Development Workshop 2. There were certain differences between the two workshops. During Professional Development Workshop 1, student errors and possible solutions as informed by research were discussed along with the multifaceted aspects of variables using examples from textbooks. The important elements of algebra teaching, such as linking algebra with real life examples, with other subjects and with other strands of mathematics, along with the purpose and need of working mathematically in algebra, were also discussed. In comparison, the focus of the Professional Development Workshop 2 was a reflection of student learning in the previous year, and student errors regarding variables were discussed. The elements of teaching practice as discussed earlier were not reinforced because it was expected that teachers would continue with the same practice as in Year 7 once they had seen the positive effect of their teaching reflected in student learning.

8.4.2 The comparison teachers

The teachers of the comparison group (Amy and Ben) defined variables as unknown quantities or letters evaluated (Küchemann, 1981, Chapter 2) and further developed this concept by teaching students that they could substitute different numerical values in place of a letter in an equation to verify which number satisfied that equation. The teachers of the comparison group used the class textbook, which contained many problems on generalising patterns and evaluating unknown variables in linear expressions and equations. The concept of a variable promoted by the many evaluation exercises which students solved (such as "Evaluate $2x + 5$ if $x = 2$") promoted the concept of variables as letters to be evaluated. The exercises in which students plotted graphs of linear equations from tabulated (usually positive) numerical values reinforced the idea that an unknown variable in a linear equation can attain only positive integer values. Ben (Set 3) also promoted the concept of a variable as an object by using different objects to demonstrate addition of like terms during his lessons. Amy (Set 1) preferred her students to work on their own for the major portion of their algebra lessons. Although she explained simplification procedures in detail, she did not explain the reasons behind the procedures, whereas Ben explained the procedures and sometimes also explained the reasons why he was simplifying in this manner, however such instances were few and far between.

The teachers of the comparison classes were very conscious of teaching correct procedures during their algebra lessons, and students spent considerable time practising simplification techniques. They primarily used the class textbook, which contained many problems on evaluating unknown variables in linear expressions and solving equations. There were few, if any, problems in the textbook where the value of x was a fraction, and exercises based on graphs of linear equations typically included tabulated numerical values which were positive integers. These restricted values reinforced the idea that an unknown variable in a linear equation can attain only positive integer values.

The conception that letters and variables represent different quantities may have been unwittingly promoted by the examples Ben used in teaching Set 3. For example, he used the examples of physically different objects, such as a table and a chair or two different pens, to illustrated the difference between x and y. Combining this idea that different letters represent different quantities coupled with the direction “simplify like terms only” may have been responsible for students’ thinking that letters and variables represented different quantities in the same algebraic expression.

Ben also tended to circle numbers and like variables to teach students the simplification procedures for addition and multiplication. (This technique was also used by the teachers of Set 2 and Set 4, though not as frequently.) Circling like terms does help students to focus on the like terms. However, it also draws students’ attention away from the operations joining the terms. It is possible that the reason for picking and choosing like terms in students of Set 3, Set 2 and Set 4 were due to the circling technique used in their lessons. Amy did not use this technique with Set 1, but instead rearranged the like terms together with the operations. This could explain why very few students in her class appeared to pick and solve numbers and variables separately.

All teachers were from the same school, therefore it was possible that the teachers may have shared some of their observations regarding student learning with each other. The results of the experimental and the comparison group were discussed during the Professional Development Workshop II, where it was pointed out that the experimental classes had outperformed the comparison group in algebraic skills and conceptions of the variables. Somehow this message may have reached the teachers of the comparison group, who made extra efforts in Phase II to maintain their students' grading. For example, both teachers revised the solution strategies and concepts before the tests (for example, revising equivalent equations before Fourth Algebra Test) and made the decision that the comparison group would be take Section B of the Third Algebra Test in June instead of April as students had not revised this topic during algebra lessons at that time. This may have affected the students' results in Phase II.

8.5 SUMMARY

Students of the comparison group mainly considered a variable as an unknown quantity. Some misconceptions regarding variables, such as thinking that variables are objects or as an abbreviation with a number to identify a quantity, were also found in some students of the comparison group. The ideas that letters and numbers represent different quantities and different letters represented different quantities were also present in some students in Set 3.

The students of the experimental group considered variables as a variable quantity and some students also considered variables as a generalised number. A few students of Set 4 in the experimental group demonstrated the misconception of letter as a label and that different letters represent different quantities. However, the misconception that numbers and letters represent different quantities in one algebraic expression was not found in any student of the experimental group.

Despite their lower general mathematical ability, the experimental group was as successful as the comparison group in solving word problems, representing word problems or tabulated numerical values in algebraic forms, plotting graphs from tabulated numerical values and vice versa, simplifying algebraic expressions and solving linear equations. In addition, the experimental group demonstrated a better reasoning and justification ability.

The comparison group mainly used procedures and rules to solve linear equations and word problems and made more errors on unfamiliar problems. In comparison, the experimental group was more flexible in their choice of a solution strategy and used the given context to identify and reason about the relationships. They therefore found complex problems more difficult. Students of the experimental group treated each equation on its own merit and chose a solution strategy accordingly. They were as successful as the comparison group in solving linear equations. The knowledge and understanding of various strategies such as the working backwards/inverse operations method and the guess and check method facilitated the experimental group in avoiding some simplification errors and also helped them in the accurate use of the balancing method.

Overall, very few students changed their preference from using numerical methods to algebraic methods during this two year study. Most students in the comparison group in particular, despite spending so much time learning the balancing method, continued to use numerical methods such as substitution to verify the equivalence of two linear equations.

The most common error in all students was the conjoining error for addition, multiplication or division. The main reason was the lack of understanding about operations on variables. Students who learned by circling like terms often picked and simplified like algebraic expressions without any regard for the operations joining them. Students made fewer conjoining errors in translating from word to algebraic expressions and more conjoining errors when they simplified algebraic expressions. When students simplified they focused on the procedures of simplification and not on the meaning of the variables. This could be due to the focus of all their teachers on teaching procedures for simplification.

CHAPTER 9

CONCLUSIONS AND IMPLICATIONS

This chapter will discuss the limitations of the study and address the research questions. Some implications of the study for teaching and research are also included in this chapter.

9.1 SUMMARY

The purpose of this investigation was to compare the effectiveness of the MVA with the traditional teaching approach being used by the participating school in terms of promoting a deeper conception of a variable and improved algebraic competence. The sample consisted of four graded Year 7 classes in a girls' school located in a high socioeconomic area in Sydney. The students and the teachers of Set 2 and Set 4 comprised the experimental group while the teachers and students of Set 1 and Set 3 comprised the comparison group. Overall, the experimental group was of a lower mathematical ability than the comparison group. The case study was a longitudinal teaching experiment completed in two phases: Phase I with the students and teachers of Year 7 and Phase II with the same cohort of students and teachers in Year 8.

Two professional development workshops were arranged for the teachers of the experimental group during the study. Professional Development Workshop I was conducted at the start of Phase I in June, 2008, and Professional Development Workshop II was conducted at the beginning of Phase II in February, 2009.

Data were collected from six student assessments and two rounds of individual student interviews with 5-6 students from each of the participating classes. For each teacher, one lesson was observed each week during the algebra teaching period and five to six of these lessons were video-recorded.

9.2 LIMITATIONS OF THE STUDY

This study was a small case study conducted in only one school and the results cannot be generalised to other schools. The participants were all girls and the school is in a high socioeconomic area. So, whether or not the same results would be obtained with boys or in a co-educational school, or for a school situated in a low socioeconomic area with limited facilities, needs more investigation.

At the end of Year 7, a small number of students who obtained good marks on the end of year examinations were promoted to the high ability classes and vice versa. In this way, some students moved between the comparison and the experimental classes. These changes may have affected the algebra learning of both groups, as these students contributed to class discussions during group work.

The study was limited to four teachers. There may have been some mutual discussion about the work done in the two groups, thus threatening the integrity of the MVA.

The students who were interviewed in Year 7 were selected by their teachers and the students who were interviewed in Year 8 volunteered for the interview, so the interview results may not have been an accurate reflection of each group. It was not possible to interview all students due to time constraints.

For a similar reason, it was not possible to observe more than a few lessons for each teacher. The impression gained from the teacher observations might therefore have been unrepresentative.

The tests and the duration of algebra lessons needed to conform to the school program, which caused a rather uneven implementation of the MVA. Students were not able to complete the algebra lessons in the resource book as planned and rushed through some of the topics. It might have been useful to re-test Year 8 students at the end of the year to see if students had retained what they had learnt, but this was not possible.

9.3 THE RESEARCH QUESTIONS

The study set out to answer three research questions (see page 4). The results are summarised for each research question here.

Does the MVA lead to a deeper conception of variables by students than the traditional approach to teacher algebra in Years 7-8?

Most students in the experimental group developed a sound concept of a variable. By the end of Year 8 they considered the variable *x* as an unknown quantity which could represent more than one value. There were also some students who used variables as generalised numbers (e.g., 2*n* represents all even numbers). Very few students demonstrated misconceptions regarding variables such as the use of the letter *x* as a label, or thinking that different letters must represent different numbers.

In contrast, most students of the comparison group considered variables as an unknown quantity and there was evidence that students of the high ability class were willing to consider more than one value for the unknown variables. However, there was no evidence that they considered variables as generalised numbers (in the sense of *n* representing all natural numbers). Students in the comparison group demonstrated more misconceptions than the experimental group, for example, considering the letter *x* as representing a physical object, or the number 1, or using letters as labels to identify quantities. Some students in Set 3 also thought that letters and numbers must represent different quantities. They also thought that different letters always represent different quantities.

These results indicate that studying all three aspects of variables together through the MVA did facilitate the experimental group in attaining a deeper conception of a variable than the comparison group who were taught by a traditional teaching approach.

Does the MVA result in superior algebraic competence (in terms of representation of word problems in algebraic form, simplification of algebraic expressions, and solution of linear equations) at the end of Year 8, when compared to the results of traditional algebra teaching?

Students in the experimental group were facilitated by their deeper understanding of the concept of a variable in interpreting and translating word problems as algebraic expressions and equations. Many students explained the meaning of algebraic expressions with reference to the context in which they were framed during lessons. Students of the experimental group had encountered algebraic expressions in a variety of contextual situations and many problems given in their teaching resource required them to write an algebraic expression as an answer. Therefore, they accepted algebraic expressions as objects in their own right. In contrast many students of the comparison group considered variables as specific unknown numbers, which may be a reason that they wrote numerical values even if the problem required an algebraic expression. Also the unacceptability of an algebraic expression as an answer to a word problem may also have compelled the students of the high ability group to conjoin terms for addition when they translated from words to algebra.

The students of the experimental group were comfortable with translating between different representations (graphic, numeric and algebraic) of linear functions. The understanding of variables and the skill of identifying the relationship between variables in the given context facilitated students in representing tabulated values in the form of algebraic equations and interpreting graphs to answer the given word problems. The students of the comparison group also easily transformed from graphical to numerical representation of a linear function. However, they relied on learned procedures to write an equation from a numerical table and some students who did not use the procedure found it challenging to express the relationship in algebraic form.

Students of the experimental group did not rely on learned procedures, therefore they dealt with each problem on its own merit and their percentage of accurate responses did not decline with the difficulty of the equation. In comparison, the medium ability class was more successful on familiar problems and their percentage of correct responses declined on unfamiliar problems. However, the emphasis by teachers of the experimental group on teaching accurate simplification procedures in Phase II shifted the focus of students towards learning procedures and away from the meaning of variables. Therefore, students of the experimental group made similar conjoining errors as the comparison group. It did not matter whether the simplification tasks were contextualised or presented as abstract manipulations because the students were unable to relate the actions they performed on the algebraic symbols back to the context. Therefore, it appeared that representation and simplification were developing as separate capabilities.

The use of different numerical solution strategies such as guess and check and inverse operations facilitated students in the experimental classes in learning the algebraic solution strategy of transformations. The knowledge of different strategies facilitated students in solving linear equations with greater accuracy. In contrast, students of the comparison group mostly practiced and used the balancing method. Some students made errors in solving equations due to their lack of understanding of the distributive property.

The integration of “working mathematically” into the learning and teaching of algebra encouraged the development of logical reasoning in students of the experimental group. The encouragement of the teachers of the experimental group by questioning and reasoning developed the students’ communication and algebraic thinking skills. Many students of the experimental group were able to express their thinking and explained and justified their answers during lessons and in the assessments. The use of real and familiar contexts in the resource book made algebra learning meaningful and interesting for students whereas the students in the comparison group often complained that their lessons were “boring”. Most students of the comparison group gave brief responses and the students who were interviewed needed extra encouragement to explain their answers as compared to the experimental group.

In conclusion, the MVA was a more successful than the approach traditionally used by the school. The MVA facilitated students in acquiring a deeper concept of a variable and reduced their misconceptions. Within the framework of the algebra standards for Years 7 and 8, all students learned the skills appropriate for this stage and the students who used the MVA demonstrated a deeper learning as compared to the students taught by the traditional teaching approach. However, all students made errors in symbol manipulations and frequently conjoined terms.

What aspects of algebra teaching promote students' conceptual understanding of a variable and develop their algebraic competence?

The learning sequence of algebra facilitated the students and the teachers of the experimental group in developing a better understanding of variables. Teachers of the experimental group spent sufficient time in Year 7 on teaching the three aspects of variables in parallel with each other, using different real contexts, before moving on to simplification of algebraic expressions and solution of linear equations in Year 8. This teaching sequence and the teaching resources used facilitated the teachers in promoting a broader concept of a variable in the experimental group.

In contrast, the learning sequence of the comparison group developed a narrow concept of a variable in the students. The teachers taught them to evaluate algebraic expressions first and then quickly moved on to the simplification of algebraic expressions. The limited experience of students with variables and their early involvement in simplification procedures promoted many misconceptions regarding variables. Many students considered variables as specific unknown numbers and some were not able to link variables with numbers at all. Students operated on variables procedurally by treating them as objects which can be added, multiplied or divided with other similar objects almost at random. The insufficient time spent on understanding variables and their concentration on only one aspect of variables (as unknown quantities) appeared to be responsible for the development of misconceptions in many students of the comparison group.

Linking algebra to other mathematical concepts and real-life contexts makes algebra meaningful and interesting for students. Linking algebra within mathematics and to realistic contexts was not only familiar to the teachers of the experimental group. It was also compatible with their beliefs about learning and teaching mathematics. Therefore, both teachers readily integrated these aspects into their teaching practice. The teachers of the comparison group were aware of the importance of mathematics in real life; however, they also believed that mathematics could be learned without making these links. One teacher of the comparison group used examples from real life to explain mathematical concepts and sometimes explained the reasons behind the construction of rules. This practice facilitated his students as they became comfortable with solving word problems using logical arithmetic reasoning. The other comparison teacher only used examples to make her lesson interesting and stressed the teaching of procedures. This tendency may have contributed to the monotonous procedural approach of students to making algebraic simplifications, and the similarity of their errors to those in the low ability classes on unfamiliar problems.

Integrating aspects of “working mathematically” in the teaching not only made students think about the concepts but also facilitated students of the experimental group in learning the skills of communicating and reasoning. The teaching resources used by the experimental group and the two Professional Development Workshops, facilitated the teachers of the experimental group in integrating the aspects of “working mathematically” such as reasoning and questioning into their lessons. Instances where the teachers of the comparison group worked mathematically in their lessons were rare. Therefore, the skills of communicating, reasoning and mathematical thinking developed in students of the experimental group and were not so obvious in students of the comparison group.

However, the belief of all the participating teachers that students best learn the processes of algebraic simplification by completing many practice exercises continued to dominate their teaching practice, and they focused on teaching their students procedures to simplify algebraic expressions in Phase II. The teachers did not relate the simplification exercises to the contexts in which they were set in the teaching materials. Also, the teachers of the experimental group let students work on their own when they used spreadsheets to study graphs of linear functions. Thus opportunities to indicate the change in the graphical representation with the corresponding change in the linear equations were lost. This may be a reason that students were not able to link the change in the equation with a change in the context. In hindsight, such activities should have been included in the Professional Development Workshops.

Some words that teachers use to refer to variables and processes are interpreted differently than intended by students. For example, one teacher of the comparison group explained the difference between the variables x and y by referring to different objects. This explanation was misinterpreted by some students and they thought that letters must represent physical objects or that different letters represent different quantities.

The misconception of using letters as labels came under discussion during Professional Development Workshop I. Therefore, teachers of the experimental group were careful in using the word "number of" with the variables representing a quantity. However, the teachers of the comparison group were not aware that students may consider the letter used as a variable to representing the object itself. Therefore, they did not stress the words "number of" when they used a variable to represent some quantity. This may for example have contributed to the perception that the variable *a* represents an apple instead of the number of apples as originally intended by the teachers.

Some actions and words may have contributed to simplification errors. For example, one teacher of the experimental group frequently used the phrase "get rid of" to indicate that a number or an algebraic expression needed to be moved from one side of a linear equation to another. Some students interpreted "get rid of" as a direction to subtract and some interpreted it as a direction to divide which made them subtract instead of divide and vice versa.

The process of simplification taught by three participating teachers may also have contributed to some students' tendency to ignore linking operations. For example, to simplify $2x + 4y - 6x + 2y + 8$ they circled or ticked x's and y's separately, then processed similar terms as $2x - 6x = -4x$, $4y + 2y = 6y$, then rearranged the terms back together again as $-4x + 6y + 8$. Some students used this procedure accurately, however many students could not put the expressions back together again and as a result conjoined terms or otherwise obtained incorrect answers. The process of taking similar variable terms out of an expression to process and then putting them back together again, may facilitate students in breaking down a bigger problem into subparts. However, it also sends the message that it is acceptable to take out like variables and numbers from an algebraic expression to process them individually. Hence when students do not understand the relationship between the terms, they are not able to reconstruct the algebraic expressions from the parts. In contrast, the process of rearranging the terms instead of taking them out, keeps individual terms linked together.

9.4 IMPLICATIONS FOR THE ALGEBRA CURRICULUM

This study has shown that studying the three aspects of variables in parallel with each other can promote a deeper conception of a variable. Therefore the proposed syllabus arrangement of the MVA in which patterns and functions are learned in parallel with each other is preferable to the previous syllabus arrangement used by the comparison group.

The time allocated to algebra in the school's program was not sufficient to complete all the topic exercises of generalisation of number properties. Since students had an insufficient knowledge of number properties, they required more time to complete this topic. The new

Australian Curriculum integrates algebra with number beginning from primary school. Therefore, it may be possible that students who had learned the generalisation of number properties before formal algebra may be facilitated in Year 7. This conjecture needs further investigation.

Generalisation of number properties facilitates students' simplification of algebraic expressions, such as their ability to expand brackets and factorise. Therefore, generalisation of number properties should be included in the pre-algebra curriculum for primary schools.

9.5 IMPLICATIONS FOR TEACHING

The results of the study confirm that teachers' awareness of the different facets of variables and likely student misconceptions regarding variables facilitate teachers in planning their lessons. The study also indicates that teachers' actions and words during lessons can be misinterpreted by the students.

Care should be exercised by teachers when they use an abbreviated name to represent a variable as this can promote the concept of a variable as an object. Instead, frequent use of the word "number" to mention an unknown quantity may facilitate students in avoiding the "letter as object" misconception.

Linear equations which have only integer solutions can restrict students' understanding of variables so that they view variables as always representing whole numbers. Therefore, linear equations which have fractional and negative solutions should also be included in algebra lessons to promote an understanding that the unknown variables can represent real numbers.

Circling like terms to draw students' attention to them may distract students from the operations which join them. If students simplify expressions by rearranging like terms, they may be better able to keep their focus on the operations joining the terms and as a result make fewer errors.

Transformation exercises facilitate students in thinking about the structure of equations instead of their numerical equivalences. Inclusion of problems involving transformations between equivalent expressions may facilitate students in recognising equivalence and learning about the structure of expressions.

9.6 IMPLICATIONS FOR RESEARCH

The MVA was a combination of teaching by working mathematically; syllabus arrangement in which all aspects of variables are learned together in parallel with each other; and the use of algebra word problems based on real contexts. The MVA can be improved in two ways. Firstly, by making arrangements for sustained professional development for the teachers so that, although they are used to teaching the three aspects of variables separately, they can utilise the teaching resources to integrate the three aspects of variables. Secondly, by including more complex problems in which the answers appear as fractions or negative integers. Thirdly, sustained professional development for the teachers is also essential so that they can be facilitated in integrating the elements of the MVA in their teaching practice.

It would also be useful to compare the different ways of teaching simplification of algebraic expressions. The strategy of learning simplifications by rearrangement of like variables instead of circling or highlighting like terms appears to have merit. More research is needed to investigate this approach and its link to the conjoining error.

It would also be useful to investigate how the deep-seated belief of teachers that students learn best by many simplification exercises, can be amended.

In this study students used the MVA and the resource book *Activities that Teach Patterns and Algebra*. This book is one of a series of five books which together cover all the topics in the NSW syllabus for Stage 4 (normally Years 7 and 8). It would be useful to study the effect on learning and teaching of mathematics if students use this series of resource books instead of the usual textbook to study the complete mathematics syllabus. Also, it would be interesting to follow the same cohort of experimental and the comparison groups in Years 9 and 10 to study the difference in learning between the two groups, especially if the students in the experimental group continued to use the MVA in Years 9 and 10.

Since the use of real contexts and translating between different representations facilitates students' understanding of the meaning of variables and algebraic expressions, teachers could also use real contexts and different representations to teach simplification procedures, for example, by representing the effect of an operation on the graphical and numerical representations along with the algebraic representation. If students are taught simplification of algebraic expressions in this manner, then it is possible that students may link the meaning of the operations with the effect that operation has on the algebraic expressions. However, this approach needs further investigation.

Concluding Remarks

This study has provided evidence that it is possible to minimise student misconceptions by using the MVA in a beginner algebra course in Year 7 and Year 8. It has also provided evidence that a deeper conception of a variable, use of contextual word problems and integration of the principles of “working mathematically” in lessons facilitates students in learning algebra and develops their communicating and reasoning abilities. This study also provides evidence that the use of patterns and structure, functions and logical reasoning, and the representation of relationships, develops students’ algebraic competence.

This study has also identified a student misconception that some students think that letters and numbers represent different quantities in the same algebraic expression.

Finally, this study has further strengthened the proposition that teachers’ beliefs are not easy to change and have a strong influence on their teaching.

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Appendices

Appendix A

Investigator's Copy / Participant's Copy

Principal's information and consent form

I give my permission to Mrs. Salma Tahir in the Australian Centre for Educational Studies at Macquarie University to undertake the research project titled "Teaching and Learning Algebra in the Junior Secondary Years".

In 2008, Mrs. Tahir has permission to observe and videotape two participating teachers of Year 7 in four algebra lessons each and to undertake and audio-tape teacher interviews of approximately 20 minutes duration after each lesson. She also has permission to interview six Year 7 students from each experimental and control classes, randomly selected by their teachers, individually once for 20 minutes and to audio-tape the interviews. She also has my permission to collect the algebra work samples of the participating students after each observed lesson.

In 2009, Mrs. Tahir has permission to observe and videotape two participating teachers of Year 8 in four algebra lessons each and to undertake and audio-tape teacher interviews of approximately 20 minutes duration after each lesson. She also has permission to interview six Year 8 students, from each experimental and control class, randomly selected by their teachers, individually once for 20 minutes and to audio-tape the interviews. She also has my permission to collect the algebra work samples of the participating students after each observed lesson.

I also give permission to Mrs Tahir to collect the class tests and the final examination papers in 2008 for Year 7 Mathematics and the final examination papers for Year 8 Mathematics in 2009 respectively for data analysis.

I understand that the anonymity of the school, the teachers and the students will be preserved at all times in any future publication of the results of this study. I also acknowledge that the school, the teachers and the students may withdraw consent at any time, without reason and without penalty. I understand that the results of the study and feedback on students and teachers' performance will be provided to me by the researcher.

Principal's Name……………………………………………..

Signed…………………………………………. Date…………………………………...

The ethical aspects of this study have been approved by the Macquarie University Research Review Committee (Human Research). If you have any complaints or reservations about any ethical aspect of your participation in this research, you may contact the Ethics Review Committee through its secretary (Tel 9850 7854; ethics@mq.edu.au). Any complaint you make will be treated in confidence and investigated, and you will be informed of the outcome.

Appendix B

Investigator's Copy / Participant's Copy

Teacher's information and consent form

This is a longitudinal study of two years. It will commence in April 2008 and data collection will be completed in December 2009. If you agree to participate in the research, you will attend a professional development workshop in April 2008 for Phase 1 and another workshop in March 2009 for Phase 2 of the study. Parts of each workshop will be videotaped. You will be asked to submit the general mathematics ability test results of the participating classes at the beginning of the study.

I will observe and videotape two of your Year 7 algebra lessons each week during the Algebra teaching unit in Phase 1 and again for your Year 8 lessons in Phase 2. You will be interviewed individually for 20 minutes at the conclusion of each observed lesson and the interview will be audio-taped.

I would also like to have access to the Year 7 students' work samples, their algebra test papers, and their end-of-year Mathematics examination papers for analysis. The same procedures will be followed for the Year 8 students in Phase 2.

You will also be required to select a random sample of six students from each experimental and control Year 7 classes at the end of the algebra teaching intervention in 2008 and from Year eight in 2009 for an audio-taped interview by the researcher. The data gathered as a result of this research will be confidential and will be used for research and training purposes only.

I …………………………………. have read the information about Teaching and Learning Algebra in the Junior Secondary Years. I agree to participate in this research, knowing that I can withdraw from further participation at any time without consequence. I have been given a copy of this form to keep.

Participant's Name:………………………………..

(Block Letters)

Participant's signature:…………………………….. Date………………………..

Investigator's Name……………………………………………..

(Block Letters)

Investigator's Signature:……………………………Date……………………….

The ethical aspects of this study have been approved by the Macquarie University Research Review Committee (Human Research). If you have any complaints or reservations about any ethical aspect of your participation in this research, you may contact the Ethics Review Committee through its secretary (Tel 9850 7854; ethics@mq.edu.au). Any complaint you make will be treated in confidence and investigated, and you will be informed of the outcome.

Appendix C

Investigator's Copy / Participant's Copy

Parent's Information and Consent Form

I have read and understood the information about the study "Teaching and Learning Algebra in the Junior Secondary Years" and I have discussed it with my child. Any questions we have asked have been answered to our satisfaction. I have been given a copy of this form to keep.

I agree to my child participating in this research, realizing that I can withdraw permission at any time and that withdrawal will not affect my child's relationship with the school in any way.

I agree that the research data collected for the research may be used in scholarly publications provided neither my child's nor my child's school will be identified.

I agree to the researcher interviewing my child if he or she is randomly chosen to take part in an audio-taped interview. I also permit the researcher to analyse my child's work samples, test papers and final examination paper in December 2008 and again in December 2009 for research purposes.

Student's Name (Block Letters):…………………………………………………

Parent's/Guardian's Name (Block Letters):…………………………………………

Parent's/Guardian's Sigmature:……………………………Date…………………

Investigator's Name…………………………………………………………….

Investigator's Signature…………………………………Date.………………

Investigator's Tel : 9850 8352

The ethical aspects of this study have been approved by the Macquarie University Research Review Committee (Human Research). If you have any complaints or reservations about any ethical aspect of your participation in this research, you may contact the Ethics Review Committee through its secretary (Tel 9850 7854; ethics@mq.edu.au). Any complaint you make will be treated in confidence and investigated, and you will be informed of the outcome.

Appendix D

Mathematics Teaching Questionnaire

Name_________________________________

Please circle one of the responses to each question

SA	Strongly agree
A	Agree
N	Neutral
D	Disagree
SD	Strongly Disagree

What do you believe mathematics to be?

Mathematics is a body of rules, formulae and procedures.	SD D N A SA
The main purpose of mathematics is to solve real world problems.	SD D N A SA
Mathematical definitions are fixed and cannot change.	SD D N A SA
In mathematics, there is never any room for personal preferences.	SD D N A SA
The historical development of mathematics was determined by human needs.	SD D N A SA
Almost everything we do involve mathematics.	SD D N A SA
In mathematics, there is only one correct answer.	SD D N A SA
Mathematics is a tool for understanding our experiences.	SD D N A SA
To solve a mathematical problem, you first need to find which rule to apply.	SD D N A SA
Mathematics is a search for patterns in number and space.	SD D N A SA

How do students learn mathematics?

Students learn best by doing lots of practice exercises.	SD D N A SA
I cannot imagine learning mathematics without a textbook.	SD D N A SA
All mathematics can be learned better using concrete materials.	SD D N A SA
To learn a rule in mathematics, all you need is one or two worked examples.	SD D N A SA
Only people with a mathematical mind can really understand mathematics.	SD D N A SA
You need a good memory to do well at mathematics.	SD D N A SA
Students learn mathematics better through cooperation than competition.	SD D N A SA
Most students are capable of discovering mathematics for themselves.	SD D N A SA
Most students cannot learn mathematics without a teacher.	SD D N A SA
It is difficult to learn mathematics if you don’t know what it is useful for.	SD D N A SA

How should mathematics be taught?

The main focus of primary mathematics should be computational skills. SD D N A SA

The main focus of secondary mathematics should be using formulae. SD D N A SA

The teachers' job is to give mathematical knowledge to students. SD D N A SA

Teachers have a responsibility to cover the syllabus. SD D N A SA

Teachers should never give problems before explaining how to solve them. SD D N A SA

Teachers should never set problems that take more than 10 min to solve. SD D N A SA

Students should be taught to decide for themselves if their answers are correct.

SD D N A SA

Teachers must always be able to answer students' questions. SD D N A SA

A teacher needs a good textbook to be effective. SD D N A SA

Teachers should never tell students anything they can't work out for themselves.

SD D N A SA

Appendix E

Algebra Questionnaire (I)

Name:…………………………………………….

Given name:…………………………………….

Qualifications:…………………………………..

Age: 25- 29 ☐ 30- 34 ☐ 35-39 ☐ 40- 44 ☐ 45-50 ☐bove 50 ☐

Email:…………………………………………….

Telephone (Work):……………………..Office:………………………Mob:……………..

Position:………………………………………………………………………..

Teaching middle school mathematics (Number of years):…………………………………

Membership of a professional organization:……………………………………………….

What is the usual sequence of your algebra lesson?

In teaching algebra what student misconceptions do you usually watch out for?

What are typical student activities in your algebra lessons?

List some resources you use in your algebra lesson.

Appendix F

Algebra Questionnaire (II)

What is algebra?

What algebra should we teach?

Should algebra be taught as a separate strand in school?

Can you give some examples of how to make connections between algebra and other strands of mathematics for e.g., geometry, statistics etc? And may be to other subjects?

What broad underlying concepts, skills and attitudes of algebra you want your students to learn?

- Concepts

- Skills

- Attitudes

What is the cause of student difficulties?

How can students create meaning in what they are learning in algebra?

Student understanding of algebra shows that

- Transition from arithmetic to algebra is difficult for students.

1) Strongly agree 2) Agree 3) Neutral 4) Disagree 5) Strongly disagree

- Students interpret letters as specific unknowns and not as generalized numbers.

1) Strongly agree 2) Agree 3) Neutral 4) Disagree 5) Strongly disagree

- Students learn the procedure of manipulation without assigning any meaning to the symbols.

1) Strongly agree 2) Agree 3) Neutral 4) Disagree 5) Strongly disagree

- Students believe that letters and numbers are detached.

1) Strongly agree 2) Agree 3) Neutral 4) Disagree 5) Strongly disagree

- Students are unable to recognize the underlying properties of numbers.

1) Strongly agree 2) Agree 3) Neutral 4) Disagree 5) Strongly disagree

- Students frequently do these characteristic errors in manipulation:
  $\frac{x+8}{x+2}$ = $\frac{8}{2}$, or x – 5 = 7 giving x = 2 or $\frac{a+b}{a}$ = b

1) Strongly agree 2) Agree 3) Neutral 4) Disagree 5) Strongly disagree

- Students regard algebra as a set of meaningless rules which are hard to remember.

1) Strongly agree 2) Agree 3) Neutral 4) Disagree 5) Strongly disagree

- Students are asked to follow procedures without reasons.

1) Strongly agree 2) Agree 3) Neutral 4) Disagree 5) Strongly disagree

- Students are generally successful in finding rules from patterns.

1) Strongly agree 2) Agree 3) Neutral 4) Disagree 5) Strongly disagree

- Students prefer numerical methods over algebraic methods for solving equations.

1) Strongly agree 2) Agree 3) Neutral 4) Disagree 5) Strongly disagree

- Students can easily link an algebraic expression to its graph.

1) Strongly agree 2) Agree 3) Neutral 4) Disagree 5) Strongly disagree

Appendix G

Workshop Evaluation Form

What do you think about this research? Has this workshop served its purpose of clarifying the aims and objectives of this research?

Is the information about the book given to you is helpful in lesson planning? Do you want to add any comment?

What were your expectations from this workshop?

Give your suggestions by which we can make a professional development workshop more useful for you in the future?

Any comment you want to add?

Examples of Activities from the Resource Book:

An activity developing the concept of a variable quantity.

Activity 2 - 4

Different starting points

One day Remy's younger sister, Gina, came to the football stadium to help her carry a rolled-up banner to the top of the stairs. One girl held each end of the roll.

Gina started walking up the stairs when Remy was on stair number 4.

When the banner starts its way up the stairs, Remy is on stair number 4.
Gina is at the bottom of the stairs (i.e. on stair number 0).
Together they walk up the stairs, climbing one stair with each stride.

There are two number lines below.
On the top line, show Remy making 10 strides up from stair number 4.
On the bottom line, show Gina making 10 strides up from the bottom.

Remy

0

Gina

0

Complete the table below to show which stairs Remy and Gina will be on after each stride. (Remember that 1 stride is 1 stair in this example.)

Number of strides	0	1	2	3	4	5	6	7	8	9	10
Stair number of Remy											
Stair number of Gina											

What stair number will Gina be on after she has made 65 strides?......
What stair number will Remy be on after she has made 65 strides?......

After both the girls have made 65 strides, how many stairs higher is Remy than Gina?.....

 30 www.workingmaths.net

Activity 2 - 4

What stair number will Gina be on after she has made n strides?...
What stair number will Remy be on after she has made n strides?......

After both the girls have made n strides, how many stairs higher is Remy than Gina?......

On the number lines below, show what happens when the girls make 10 strides if they both climb <u>2 stairs with each stride</u>.
Remy still starts from stair number 4.

<u>Remy</u>

<u>Gina</u>

Complete the table below to show which stairs Remy and Gina will be on after each stride (i.e. when 1 stride is 2 stairs).

Number of strides	0	1	2	3	4	5	6	7	8	9	10
Stair number of Remy											
Stair number of Gina											

What stair number will Gina be on after she has made 65 strides?.....
What stair number will Remy be on after she has made 65 strides?.....

After both the girls have made 65 strides, how many stairs higher is Remy than Gina?......

What stair number will Gina be on after she has made n strides?.......
What stair number will Remy be on after she has made n strides?........

After both the girls have made n strides, how many stairs higher is Remy than Gina?......

When both girls take the same number of equal strides, why does the difference in the number of stairs between Remy and Gina <u>not</u> change?

....

....

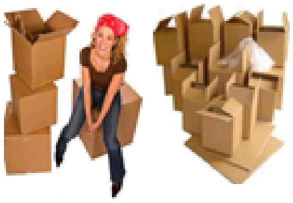
 31 www.workingmaths.net

An activity developing the concept of a generalised number.

Like terms

Resources required:
a calculator.

Packing boxes



Suzie was going overseas to study. She packed her belongings into boxes for storage while she was away.

Suzie wanted to find out what volume of storage space she needed to buy.

Suzie packed her belongings into 5 types of boxes (labelled from A to E). She measured the dimensions of each type of box in centimetres. She also wrote down how many of each type of box she filled. This information is in the table below.

Box Type	Length	Width	Height	Number of this box type
A	22	32	40	5
B	40	32	22	3
C	63	63	48	4
D	48	63	48	7
E	48	63	63	2
		Total number of boxes:		21

To calculate the total volume occupied by all 21 boxes, Suzie wrote a term for each type of box, so there were 5 terms altogether.

Complete her calculation below.

Total volume = 5(22 x 32 x 40) cm^3
+ 3(...............) cm^3
+ 4(...............) cm^3
+ 7(...............) cm^3
+ 2(...............) cm^3

Sarah told Suzie that there was an easier way to do the calculation.

What box type has the same volume as a Type **A** box?.................

How do you know they have the same volume without calculating it?

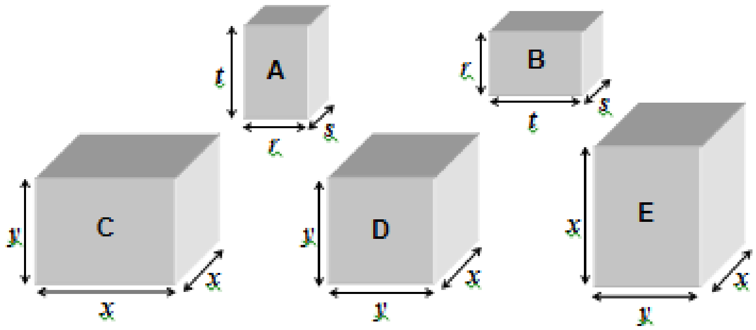
....

What other box types have the same volume as each other?..........

Sarah combined boxes with the same volume in the same term. On the next page, re-write Sarah's calculation (using a line to write each term).

Total volume = ...
+..............................
+.............................cm^3

The boxes below have their dimensions written as variables.
Note: The same length is represented by the same variable.



Box Type	Length	Width	Height	Number of this box type
A	r	s	t	5
B	t	s	r	3
C	x	x	y	4
D	y	x	y	7
E	y	x	x	2
		Total number of boxes:		21

Write an expression for the total volume of the boxes with 5 terms (one term for each type of box).

Total volume = $5rst$ +..........+..........+..........+..........

Like terms are terms with the same set of algebraic factors (in any order). They may or may not have the same numerical factor.

In your algebraic expression, which term is like $5rst$?............

Which other two terms are like terms?................................

Why can like terms be added? (Hint: See Activity 11 - 3 Grain in the silos)..

....

Simplify the algebraic expression above by adding like terms.

Total volume = ...+.........+.........

Why can the terms $4x^2y$ and $7xy^2$ not be added to make a single term?

....

Appendix I

First Algebra Test

Name____________________________

Duration___________

Class____________________________ Marks_____________

Show your working where necessary.

Q1: a) Complete the next line in the number pattern: (1+1+1)

$(4 + 3) \times (4 - 3) = 4^2 - 9$

$(5 + 3) \times (5 - 3) = 5^2 - 9$

$(6 + 3) \times (6 - 3) = 6^2 - 9$

$(7 + 3) \times$ ____________

b) Now write the sentence in this number pattern which begins with:

(59 + __________________________

c) Now write the sentence in this number pattern which begins with:

(123456 __________________________

Q2: Sarah's mother gave her 2 times more chocolates than Hannah. (1+1+1)

a) If Hannah has *x* chocolates. Then Sarah will have ………. chocolates.

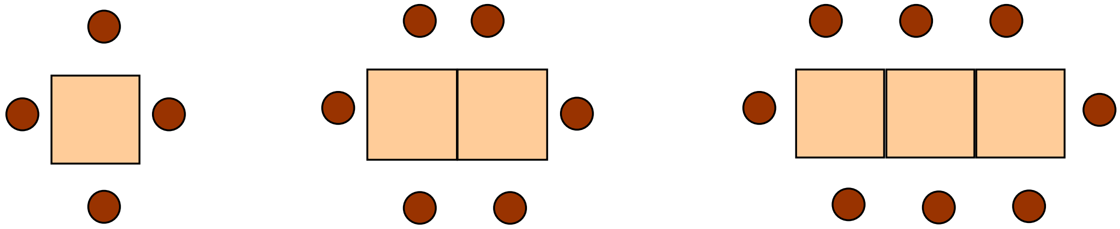
b) When her father came home, he gave each of the girls 5 more chocolates. Describe the

number of chocolates each girl has using *x*. Show your working.

Sarah has ………………… chocolates.

Hannah has ………………….chocolates.

Q3: The desks in a classroom are square in shape and students can sit around the desks on circular chairs as shown in the diagram. (1+1+1+2)



a) Complete the table for the number of desks (N) and the number of students (M)

N	M
1	4
2	
3	

b) How many students can sit if 4 desks are joined together in a row? Show your working.

c) How many students can sit if 5 desks are joined together in a row? Show your working.

d) How many students can sit if 60 desks are joined together? Show your working.

Q4: There are 20 passengers for every bus. (1+1+2+2)

a) How many passengers are there when there are 2 buses? Show your working.

b) How many passengers are there when there are 5 buses? Show your working.

c) Express the relationship in words between the number of passengers and the number of buses.

d) Write the relationship in part 'c' algebraically using x for the number of buses and y for the number of passengers.

(2+2+1)

Q5: Look at the number pattern shown in the table and then answer the questions.

First Number (n)	1	3	5	7	9
Second Number (m)	3	9	15	21	27

a) Describe the relationship between first number 'n' and second number 'm' in words.

b) State the pattern (rule) algebraically using n and m. Explain how you got your rule?

c) Find m if $n = 23$. Show your working.

Q6: Complete the table using the rule. (2+1+2)

Rule: Add 3 to the input number and then multiply by 5.

Input (x)	2	3	5	12	43
Output (y)					

a) State the relationship between x and y algebraically. Explain how you got your answer?

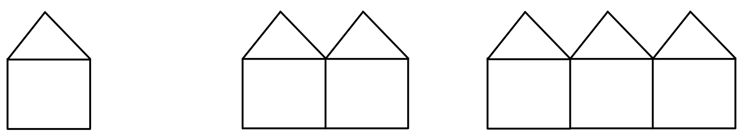
b) If $y = 15$, what is x =? Show your working.

c) Can x be equal to y? Please explain.

Appendix J

Yearly Examination (Algebra Problems)

Q27: Look at the pattern of these house shapes.



1 House 2 Houses 3 houses

6 toothpicks 11 toothpicks 16 tooth picks

How many toothpicks were used to make the shape for 100 houses?

A. 100	B. 501	C. 401	D. 500

Q41: Shania earns \$4.50 per hour working in a pizza shop. In one week she worked for 20 hours. How much did she earn in that week?

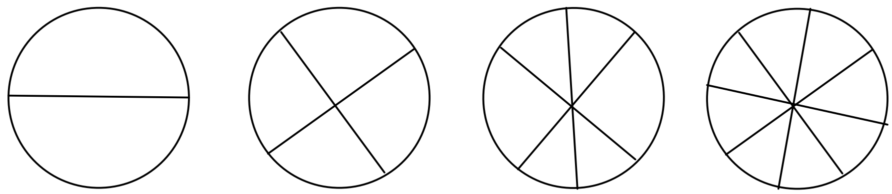
Q42: Write an algebraic expression for:

a) *A* increased by 2

b) *P* lots of *Q*

c) 10 more than the product of 3 and *B.*

Q47: Andrew has the job of cutting up a birthday cake by making straight cuts through the middle of the cake. Below is a diagram of the cake in several stages.



Complete the table below:

c	1	2	3	4	5	7
p	2	4				

a) Write the rule for the pattern in words

b) Write the rule as a formula using the pronumerals *c* and *p*

c) How many pieces would be there if 50 cuts were made?
(Show your working mathematically)

Appendix K

Second Algebra Test

Name______________________________

Date___________

Class______________________________

Marks_____________

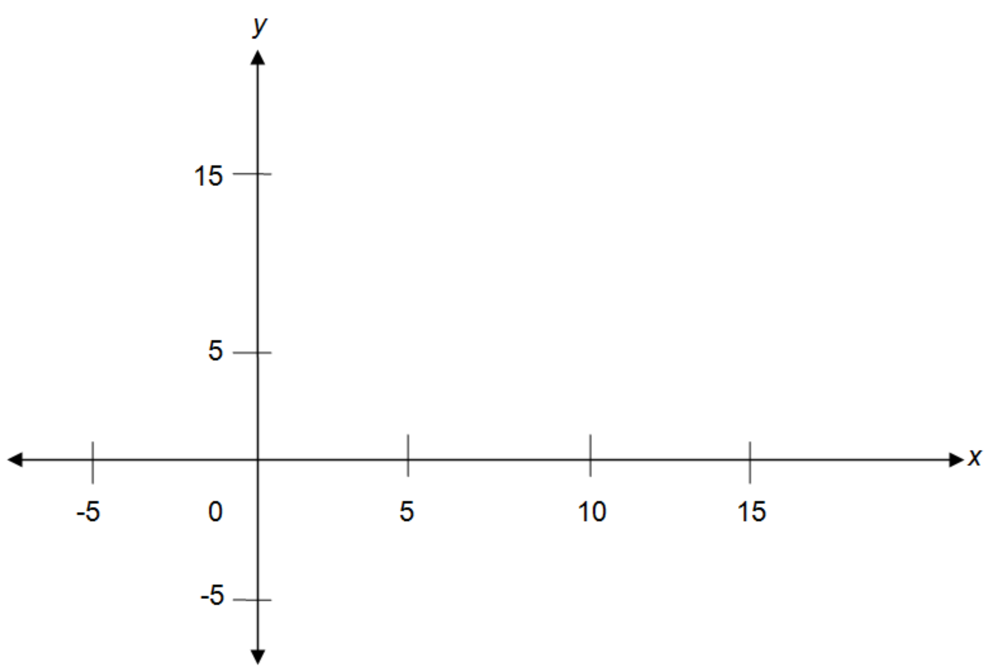
Write all your answers on this question sheet.

Q1: Complete the following table of values: (3+1)

Rule: Subtract 3 from the input number and then divide by 2.

x	1	3	5	7	9
y	-4	0	4	8	12

a) Graph the data in the table on the number plane below.



b) Write the relationship between x and y algebraically.

Q2: Simplify: (5)

a) $2x \times 5y =$

b) $2x(3y + 7) =$

c) $5x + 2y + 3x + 4 + 6y =$

d) $\frac{2abc}{ac} =$

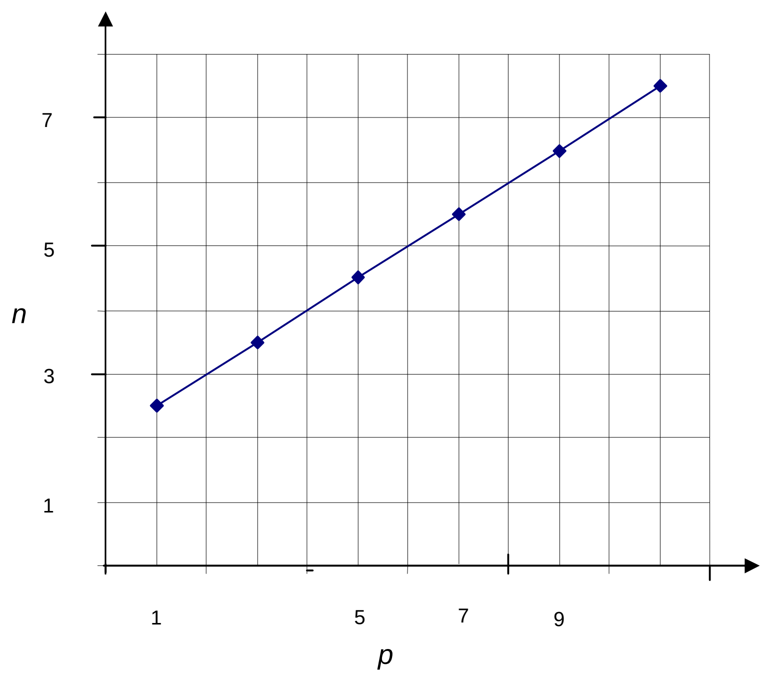
Q3: Evaluate $\frac{a^2}{bc}$ if $a = 5$, $b = -4$ and $c = 1$. (1)

Q4:

a) Complete the table of values using the graph given below? (2 + 1)

p					
n					

b) Write the relationship between *p* and *n.*



Appendix L

Lesson Plan Feedback Sheet

Algebra Lesson No:

Date:

Lesson Duration:

Objectives of lesson:

1.

2.

3.

Aspect of variable to be discussed: Tick one

Unknown quantity   Generalised Number   Function   Variable Quantity

Resources used:

1.

2.

3.

Activity selected from book "Activities that Teach Patterns and Algebra"

Chapter: Page No:

Teachers Comments:

Appendix M

Student Interview Schedule I

Q1: If 5 + *y* is larger than 14, then what could *y* be equal to?

a) 8

b) 10

c) *y* can have any value

d) *y* can have any value greater than 9

Q2: I have *m* dollars and you have *k* dollars. I have $6 more than you! Which equation must be true?

a) $6k = m$

b) $6m = k$

c) $k + 6 = m$

d) $m + 6 = k$

e) $6 – m = k$

{Bell, 1992 #136}(Bell & Malone, 1992)

Q3: *a* and *b* are numbers and $a = 28 + b$. Which of the following must be true?

a) *a* is larger than *b*

b) *b* is larger than *a*

c) you cannot tell which number is larger

d) *a* is equal to 28

Q4: Can you tell me which one is larger $3n$ or $n + 6$? Please explain your answer.

(Knuth et al., 2005)

Q5: What can you say about c if $c + d = 10$ and $c < d$?

Appendix N

Third Algebra Test

Name ________________________________

Date___________

Class___________ Marks__________

Section A

Show all your working.

1) Simplify the following expressions:

   i. $2x + 9 + 5x$

   ii. $6x + 3y + x - 6$

   iii. $2x - (4x + 1)$

   iv. $6x - (4 + 2x - 3)$

   v. $12y \div 3$

   vi. $xy \div x$

   vii. $12y \div 4y$

2) Factorise (group) these expressions fully:

   i. $ab + ac$

   ii. $5x + 15y$

   iii. $6y + 6x$

   iv. $2a + 4b + 6c$

   v. $ab^2 - abc$

   vi. $2ab^2 - 4b^2 + 6b^2$

3) Expand and simplify these expressions:

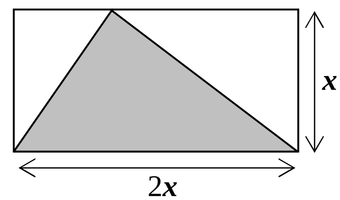
i. $4(1 + x)$

ii. $4xy(1 + x)$

iii. $4xy(1 + x) - y(4xy - x)$

iv. $c(a + b) \div ac$

4) Here is diagram of a triangle inside a rectangle.
The length of the rectangle is twice as long as its height.



Write an algebraic expression for:

i. The perimeter of the rectangle

ii. The area of the rectangle.

Section B

5) a) In a game of cards everyone starts with a score of 100 points.

Each time you win a round of the game, you gain x points.

Each time you lose a round of the game, you lose y points.

Anne won 2 rounds and lost 5 rounds. Write an expression for Anne's score in terms of x and y.

b) Nelly collected P number of eggs during an Easter egg hunt. Mandy collected half as many eggs as her older sister Nelly.

Then Mandy dropped 3 of her eggs.

Write an expression for the number of eggs that Mandy has at the end of the hunt.

6) Hope is offered a job distributing advertising leaflets in her area.

Each week, groups of leaflets need to be put in letter boxes.

Hope has a choice of how she would like to be paid each week.

Her choices are:

A. \$2 per leaflet in the group.
B. \$6 plus another \$1 per leaflet in the group.

Let ***L*** be the number of leaflets in the group.

(i) Write an algebraic expression for:

The money Hope receives if she chooses payment method A: ....

The money Hope receives if she chooses payment method B: ....

Use your algebraic expressions for method A and method B to complete the tables below:

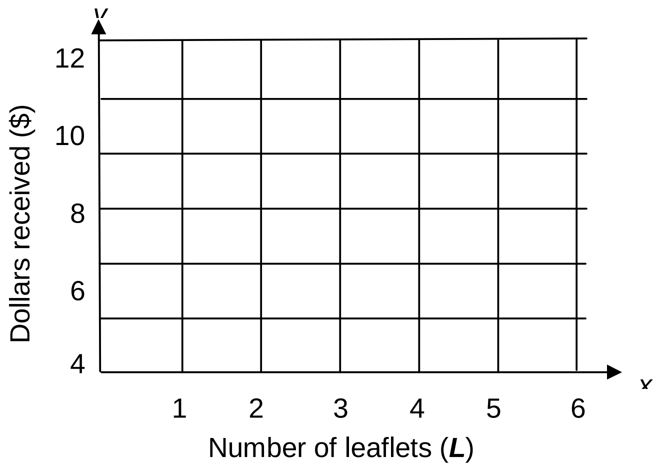
L	1	2	3	4	5	6
\$						

L	1	2	3	4	5	6
\$						

Graph the table of values for **each** of the methods.

(ii) Use a dot (•) for plotting payments if she chose payment method A.

(iii) Use a cross (x) for plotting payments if she chose payment method B.



(iv) Which payment method is better for Hope if there are 7 leaflets to deliver?___________

How can you tell this from the graph?____

____

7) Simplify fully where possible:

i. $x + x + x + x$

ii. $x \times x \times x$

iii. $y \div 2 \div 2$

iv. $x + y + 2z$

v. $4 \times y \times \mathrm{t}^2 \times 0$

vi. $2p \times 4p$

vii. $2p + 4p$

viii. $7 + 7x + 5$

ix. $ab + ba$

8) Explain why $y^3 \neq 3y$

Appendix O

Half Yearly Examination (Algebra Problems)

Q2: Multiple choice. Circle your answers.

ii. Which of these terms are like terms?

a) b^2a & a^2b

b) ab & ba

c) a^2b & $2ab$

d) $2ab$ & ba

viii. a × a fully simplified…

a) a^2

b) $2a^2$

c) $2a$

d) $a(a)^2$

Q11. Simplify fully:

i. $3a + 4a$

ii. $3d \times 4$

iii. $p + 2p \times 3$

iv. $6ac \div 6c \times d$

v. $p \div 3 \div 3$

vi. $3a + 12 – 5a$

Q14. Expand and simplify fully if necessary:

i. $3(p + 2) = 3\mathrm{p} + 6$

ii. $6(3a +2) + a$

Q17. Factorise fully:

i. $3p + 6$

ii. $6by^2 - 2by$

Q19. T*he* length of a rectangle is three times its width (Let *t* be the width).

i. Draw and label a diagram showing the above information.

ii. Writs an algebraic expression for the perimeter in terms of *t.*

iii. If the perimeter is 24cm calculate the value of *t.*

iv. What would be the length of this rectangle?

Q21. Anne bought 5 burgers and six doughnuts for $31.

i. If a burger costs $x then Anne spends …… on burgers.

ii. If a doughnut costs $y then Anne spends ….. on doughnuts

iii. Write an equation that represents total spending using *x*, *y* and 31.

iv. What is the cost of two doughnuts if one burger costs $5?

Q23. Solve:

i. $n + 6 = 4$

ii. $7x = 56$

iii. $\frac{p}{5} = 9$

iv. $3(m - 1) = 18$

v. $\frac{p}{5} + 6 = 9$

vi. $7x - 2 = 5x + 8$

Appendix P

Fourth Algebra Test

Which of the following equations can be transformed to $x - 2 = 0$?

a) $2x = 4$

b) $4 = 2x$

c) $\frac{x}{2} = 4$

d) $4x = 2$

e) $x + 1 = 3$

f) $x - 3 = 1$

Appendix Q

Student Interview Schedule II

Q1. Simplify:

a) $\frac{x+2}{2}$

b) $\frac{6(x+3)}{3}$

Q2.

a) Is $n + 1$ odd or even? Explain your answer.

b) Write down three consecutive numbers starting with *x*

Q3. Solve the following equations:

$4x – 12 = x$

$3x – 1 = 29$

Can you give a real life situation which can be represented by the following algebraic expressions?

3x + 4 - (12 - x)

4x + 9 + x -15

Q4. Write an algebraic expression for each of the following statements. Clearly state what each symbol represents.

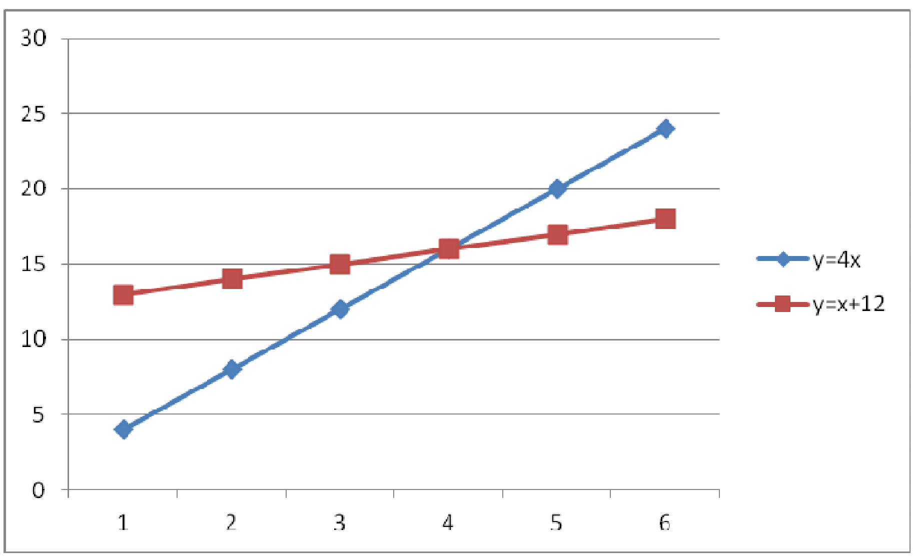
a) My cousin is 3 times as old as I am.

b) Gregory has \$6 more than Penny, and Phillip has \$13 less than Gregory. How much does each person have if they have \$29 altogether?

c) The result of adding 12 to a certain number is the same as multiplying the number by 4. Find the number.

Q5. Given below is a graph of following equations:

$y = 4x, \qquad y = x + 12$



a) Which line represents $y = 4x$?

b) Which line represents $y = 12 + x$?

c) Find the point of intersection of these lines.

Can you choose a suitable question from the ones given above whose solution is represented by the given graph?

Appendix R

MACQUARIE
UNIVERSITY ~ SYDNEY

13 March 2008

Dr Michael Cavanagh
Department of School of Education
Australian Centre for Educational Studies
College of Humanities and Social Sciences

Reference: HE22FEB2008-D05638

Dear Mrs Tahir

FINAL APPROVAL

Title of project: ***Teaching and learning algebra in the junior secondary years***

Thank you for your recent correspondence. Your responses have satisfactorily addressed the outstanding issues raised by the Committee. You may now proceed with your research.

Please note the following standard requirements of approval:

1. Approval will be for a period of twelve months. At the end of this period, if the project has been completed, abandoned, discontinued or not commenced for any reason, you are required to submit a Final Report on the project. If you complete the work earlier than you had planned you must submit a Final Report as soon as the work is completed. The Final Report is available at http://www.ro.mq.edu.au/ethics/human/forms
2. However, at the end of the 12 month period if the project is still current you should instead submit an application for renewal of the approval if the project has run for less than five (5) years. This form is available at http://www.ro.mq.edu.au/ethics/human/forms. If the project has run for more than five (5) years you cannot renew approval for the project. You will need to complete and submit a Final Report (see Point 1 above) and submit a new application for the project. (The five year limit on renewal of approvals allows the Committee to fully re-review research in an environment where legislation, guidelines and requirements are continually changing, for example, new child protection and privacy laws).
3. Please remember the Committee must be notified of any alteration to the project.
4. You must notify the Committee immediately in the event of any adverse effects on participants or of any unforeseen events that might affect continued ethical acceptability of the project.
5. At all times you are responsible for the ethical conduct of your research in accordance with the guidelines established by the University (http://www.ro.mq.edu.au/ethics/human).

If you will be applying for or have applied for internal or external funding for the above project **it is your responsibility** to provide Macquarie University's Research Grants Officer with a copy of this letter as soon as possible. The Research Grants Officer will not inform external funding agencies that you have final approval for your project and funds will not be released until the Research Grants Officer has received a copy of this final approval letter.

Yours sincerely

PR Dr Margaret Stuart
Director of Research Ethics
Chair, Ethics Review Committee [Human Research]

CRO File: 07/1595

ETHICS REVIEW COMMITTEE (HUMAN RESEARCH)
MACQUARIE UNIVERSITY (E11A)
SYDNEY, NSW, 2109 AUSTRALIA
Secretary: Ph: (02) 9850 7850 Fax: (02) 9850 4465 E-mail: ethics.secretariat@vc.mq.edu.au
http://www.ro.mq.edu.au/ethics/human

Portrait (85%)