Optimal Consumption, Investment and Insurance Strategy Applications

by

Jinhui Zhang

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Abstract

Drawing on the existing literature, a utility-maximising agent is studied in the application of a life-cycle optimal strategy of consumption, investment and insurance to different, and unexplored, scenarios. Key factors, including time-inconsistent preferences, an optimal stopping time and a dynamic risk environment, can affect agents' behaviour and thereby influence their financial strategies. In this thesis three research papers are developed to apply optimal strategies in various circumstance.

In the first research paper, an optimal portfolio management model with hyperbolic discounting and luxury-type bequest motives is used to explain the annuity puzzle—the low demand for voluntary life annuities. Using hyperbolic discounting, agents' time-inconsistent preferences can be described and measured in the model. Two extreme types of agents' time-inconsistent behaviours, "naïve" behaviour and "sophisticated" behaviour, are then examined and studied. To build a more realistic model, the luxury-type bequest motives are further incorporated into the model. The model in paper 1 is calibrated to Swiss data to obtain numerical results.

In the second research paper, the financial planning problem of a retiree seeking to enter a retirement village at a future time is studied. As the retiree is assumed to be utility-driven and would fully annuitise her wealth at the time of entry, her optimal strategy is a solution to problems of both optimal control

and optimal stopping. Within the context of dynamic health states, the optimal strategy should include an optimal plan of consumption, investment, bequest and insurance prior to the entry date, and an optimal stopping time to conduct the full annuitisation for entering the retirement village. For a case that has an initial deposit requirement for entering the retirement village, the optimal solution incorporates an American option replication. The model in paper 2 uses Australian data to present our numerical results.

In the final research paper, an optimal strategy is applied in a dynamic risk environment. Jumps and regime switching are incorporated in the risky asset diffusion to describe the dynamic risk environment. By extending the model in Richard (1975), a system of paired Hamilton-Jacobi-Bellman (HJB) equations is obtained and solved. Using numerical methods and calibrating to American data, the numerical results of agents' behaviours for different risk environments are obtained.

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Statement of Originality

This thesis complies with the standards of the Thesis by Publications format which follows the Macquarie University Thesis Submission Guidelines. I hereby certify that this thesis is solely my own research work and that it has not, nor has any part of it, been submitted for a higher degree to any other university or institution. The sources of information or material from where the work of others has been utilised are acknowledged in the thesis.

Jinhui Zhang

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Chapter 1

Introduction

1.1 Research Background

Agents strive to make optimal financial decisions concerning their consumption, investment and insurance. However, compared to finance and economic theories, agents' decisions and behaviours are sometimes found puzzling. The need thus arises to extend existing models to better describe and explain agents' behaviours.

Researchers have increasingly understood the importance of optimal financial strategy and have used various methods to model agent behaviours in order to study optimal strategy. Among these models, one of the most famous is Merton's model (Merton, 1969, 1971) in which the optimal consumption and investment strategy is derived for a utility-maximising agent with assumed constant relative risk averse (CRRA) utility.

Following Merton's model, a great number of optimal control models have been studied in the literature. The model of Richard (1975) is one of the many extensions of Merton's model. Richard has extended Merton's model to incorporate bequest motives and insurance demands in an optimal control model for utility-maximising agents. Merton's model uses the assumption that the agent's lifetime is constant; this has been generalised in Richard's model by treating the agent's lifetime as being uncertain. Compared to Merton's model, Richard's model is more realistic as it reflects an agent's random lifetime. In Richard's model, the lifetime of agents is assumed to be random in a fixed bounded interval (i.e., lifetime is bounded by a maximum age). This extension to lifeyime uncertainty can then complicate an agent's optimal strategy when approaching the end of the life cycle. Although the annuity can be used to hedge the longevity risk in a complete market, the market can still change to become incomplete should hedging activity be obstructed due to the dynamic health status of agents.

In a more realistic optimal control model, agents are assumed to have once-and-for-all irreversible full annuitisation due to institutional anti-selection concerns (Milevsky and Young, 2007; Kingston and Thorp, 2005). Hence, to develop a more advanced optimal control model for utility-maximising agents, we should include optimal strategies for not only consumption, investment and bequest but also for full annuitisation. Furthermore, agents in Richard's model are assumed to receive a deterministic income stream (the value of which reflects their human capital). In perfect and complete market setting in Richard's model, agents are able to capitalise future income stream—such instruments can exist is such settings. Pliska and Ye (2007) have extended Richard's model by allowing lifetime to be unbounded and having a fixed termination time (e.g., retirement) in the model.

However, neither Merton's model or Richard's model can effectively and fully explain various puzzling behaviours of agents in financial markets. The reason is that many complex elements of human behaviour and market movement which will directly affect agents' decision making are not described in either Merton's model or Richard's model.

In contrast to economic theory (Yaari, 1965), which asserts that agents with no bequest motive should fully annuitise and agents with a bequest motive should partially annuitise (or purchase life insurance if their optimal bequest exceeds their wealth), agents in the real world are observed to very infrequently purchase voluntary annuities. This prevalent trend is known as the "annuity puzzle" which has been discussed in empirical studies such as those of Mitchell, Poterba, Warshawsky and Brown (1999) and Bütler and Teppa (2007).

This "annuity puzzle" problem has been widely studied in literature and various explanations have been proposed. One particular explanation for this puzzle is time-inconsistent behaviour[1]. Various studies incorporate agents have time-

[1] The following is a simple example to demonstrate the existence of changing preferences of an individual. Assume there are two options of receiving a payment. Option 1: people need to wait for 1 year to receive $1. Option 2: people need to wait for 1 year plus one day to receive $2. Almost all people will choose option 2, since they will not be bothered to wait for one more day for a doubled payment. From their point of view, the difference between 1 year and 1 year plus one day is relative small compared to the difference in payments. Now, let us modify our example a little bit. Assume there are two options. Option 1: people can receive $1 after waiting for 1 day. Option 2: people can receive $2 after waiting for 2 days. In that case, some people might choose option 1 instead of option 2. The reason is that they are craving for early

inconsistent behaviours (Strotz, 1955; Pollak, 1968) which can be modelled by using hyperbolic discounting factor for utilities (Marín-Solano and Navas, 2010; Marín-Solano, Navas and Roch, 2013). Hyperbolic discounting modulates agent's preference as a function of the time—agent's preference rates experience a dramatic decrease over a short time period. Such a non-constant discounting rate has been introduced in Phelps and Pollak (1968) and Laibson (1997) which linked hyperbolic discounting with agents' time-inconsistent behaviours. Malhotra, Loewenstein and O'Donoghue (2002) state that hyperbolic discounting can more appropriately describe the pattern revealed empirical data. However, the use of the constant exponential discounting factor for utilities in both Merton's model and Richard's model implies the time consistency of agents' preferences, which contradicts such observed time-inconsistent preferences.

Moreover, bequest motives can be regarded as another reason for the "annuity puzzle" (Friedman and Warshawsky, 1990; Bernheim, 1991; Ameriks et al., 2011). Bernheim (1991), Friedman and Warshawsky (1990), Vidal-Meliá and Lejárraga-García (2006), Purcal and Piggott (2008) and Lockwood (2012) have stated that strong bequest desire can banish agent's annuity demand and thus serve as one of the drivers of low annuitisation. Recent empirical data further suggests that bequests have the features of luxury goods (De Nardi, 2004; Ameriks et al., 2011;

payment and probably can not wait for one more day. They believe that the difference between 1 day and 2 days is relatively big compared to the difference in payments. From this example we can see that individual preference may change over time. The preferences over a long period are quite different to the preferences over a short period. We term this changing of preferences as time-inconsistent behaviour of the individual.

Lockwood, 2012).

Other explanations for this "annuity puzzle" include psychological or behavioural bias (Brown and Diamond, 2005), adverse selection (Finkelstein and Poterba, 2004) and the widespread presence of annuitised public social security (Friedman and Warshawsky, 1990).

Thus, one could explain the observed level of voluntary annuitisation by using the bequest motive or time-inconsistent behaviour. However, still, a puzzle remains: voluntary annuitisation *is* prevalent in some countries, such as Switzerland. The "annuity puzzle" might thus be the result of an interaction of several factors, mostly resulting in low voluntary annuity demand, but in certain constellations producing healthy voluntary annuity demand.

In both Merton's model and Richard's model, their studies proceeeded under the assumption that the diffusion of risky assets follows geometric Brownian motion. However, if significant change occurs in the state of the economy, geometric Brownian motion then imperfectly models the dynamics of the risky asset price. To better describe such dynamics, regime switching and jumps can be incorporated into the diffusion process [2].

[2] See, for example, Cont and Tankov (2004), Buffington and Elliott (2002) and Elliott, Aggoun and Moore (1994).

1.2 Contribution of Thesis

This thesis is comprised of three separate papers, each of which form the body of chapters 2–4. Each paper provides a sound understanding of optimal strategy thus contributing to the overall thesis theme. The contribution of this thesis is to develop optimal strategies of consumption, investment and insurance for various situations by extending the model in Richard (1975). Summaries of each research paper are given below.

Three different extensions Richard's models have been presented in this thesis. All three models consider the utility from the consumption and bequest motive. The first model incorporates hyperbolic discounting and luxury bequests. The second model includes medical costs and dynamic health states—it does not include hyperbolic discounting or luxury bequests. The third model considers regime switching and jumps in the financial market—it does not include hyperbolic discounting, luxury bequests, medical costs or dynamic health states.

1.2.1 Paper 1: Optimal Life Insurance and Annuity Demand under Hyperbolic Discounting when Bequests are Luxury Goods

A growing body of literature has studied the low demand for voluntary annuities which is also known as the "annuity puzzle". On the other hand, there are exceptions for this "annuity puzzle" which cause more complications in the research of this issue. In contrast to the prevalent low annuity demand trend, agents from some countries, such as Switzerland, are observed to have a relative high purchase level of voluntary annuities. Hence, in this paper, our motivation is to develop an optimal strategy which can provide explanations for both the "annuity puzzle" and its exception.

From empirical studies, the bequest motive can also be considered as one explanation for the "annuity puzzle" (Lockwood, 2012). In this paper, we use Richard's model (Richard, 1975), which is one extension of Merton's famous model (Merton, 1969, 1971), to study agents' behaviour by incorporating bequest motives. To construct a more realistic model, bequest motives should be recognised as if bequests are luxury goods (De Nardi, 2004; Ameriks et al., 2011; Lockwood, 2012). Inspired by Ding, Kingston and Purcal (2014), Richard's model is extended in this paper by including luxury-type bequests.

Some of the literature, such as the studies of Friedman and Warshawsky (1990), Bernheim (1991) and Finkelstein and Poterba (2004), has also studied and given explanations for the "annuity puzzle". One of the explanations is time-inconsistent behaviour. It is accepted that agents' preferences would alter when they age. Within the context of the optimal strategy for a utility-maximising agent, Marín-Solano and Navas (2010) and Marín-Solano, Navas and Roch (2013) extended Merton's model to incorporate time-inconsistent behaviours by using the hyperbolic discounting factor. One of the features of hyperbolic discounting is that the rate will decline over time and thus it can be used to model the time inconsistency in agents' behaviours. Following Marín-Solano and Navas (2010) and Marín-Solano, Navas and Roch (2013), we modify Richard's model to adopt time-inconsistent behaviours by employing hyperbolic discounting.

With the interaction of time-inconsistent behaviours and luxury-type bequests in our extended optimal life-cycle model, we are able to provide reasonable explanations for the "annuity puzzle" and its exceptions.

The preliminary findings of this paper have been presented at the 18th International Congress on Insurance: Mathematics & Economics, East China Normal University, in July 2014, and at the Quantitative Methods in Finance Conference 2014, University of Technology Sydney, in December 2014.

Paper 1 is a valuable contribution to the optimal life-cycle model literature, as few papers have studed time-inconsistent behaviours and luxury-type bequests using Richard's model.

1.2.2 Paper 2: Optimal Time to Enter a Retirement Village

Population ageing is a widespread global trend and the senior population will outnumber the younger generation in many nations in near future. Compared to previous generations, Australians now live longer. According to the Australian Institute of Health and Welfare (2013), females and males born in 2013 have life expectancies of 84.2 and 79.7 years, respectively. In the light of the potential impacts of population ageing, such as economy stagnation (Bloom, Canning and

Fink, 2010), high demand for social security (Gruber and Wise, 2000) and increased aged care spending (Knickman and Snell, 2002), Australia has now to prepare itself for the consequences of this growing ageing trend. This rise in the number of aged people leading to increasing demand for aged care spending was our motivation for conducting a study on the optimal retirement strategy for retirees.

In Australia, the Australian government spent $13 billion in 2014 to subsidise aged care services for retirees (Australian Institute of Health and Welfare, 2014*a*). This figure will continue to increase with the rising aged population. However, before retirees start to receive aged care services, they can choose to move into retirement villages to improve their well-being and quality of life. Retirement village can help to relieve the government's burden, as the related costs of living in retirement are privately financed by retirees themselves.

It is well-documented in the literature that living in retirement villages with features designed for seniors has a positive influence on retirees' well-being (Buys, 2000; Lord et al., 2003), which can further alleviate the financial burden of aged care services for the Australian government (Towart, 2005). Therefore, in this paper, we assume that retirees will choose to enter retirement villages. Based on this assumption, we developed a life-cycle model for retirees' asset allocation, consumption, bequests and insurance purchases prior to their entry into retirement villages.

Based on the optimal strategies studied in Merton (1969) and Merton (1971), Ding, Kingston and Purcal (2014) incorporated luxury-type bequests in Merton's model by replicating a put option to generate a wealth threshold. Merton's model was also extended in Milevsky and Young (2007) and Kingston and Thorp (2005) to obtain the optimal stopping time for investors to conduct a once-and-for-all annuatisation. In this paper, following Milevsky and Young (2007) and Kingston and Thorp (2005), we modify the extended Merton model in Richard (1975) to study retirees' optimal stopping time and optimal strategy prior to entering retirement villages. As up-front deposits are usually required by retirement villages, we also incorporate the replication of an American put option in our model to create a wealth threshold for this requirement and find increasing investment in the risky asset over time. Interestingly, empirical studies of retirement target saving suggest such targets raise, and not lower, investment in the risky asset (Shum and Faig, 2006).

Inspired by various studies in the literature, such as those by Rosen and Wu (2004), Bernheim, Shleifer and Summers (1985) and Edwards (2008), we consider health status to be a critical factor in retirees' financial decisions. We add the term "dynamic medical cost" to our model to capture the impact of health status.

This paper has been presented at the 2014 Ph.D. workshop, Faculty of Business and Economics, Macquarie University, and at the 19th International Congress on Insurance: Mathematics & Economics, Liverpool University, in July 2014. Inspired by the comments we received at the conference, we modified our model to consider the necessity of consumption.

Paper 2 presents a useful model for analysing retirees' decisions about optimal asset allocation and the optimal time to enter retirement villages. Its incorporation of health status and target retirement saving in an optimal stopping context is novel.

1.2.3 Paper 3: Optimal Life Insurance and Annuity Demand with Jump Diffusion and Regime Switching

Optimal life-cycle models, for example, Merton's model (Merton, 1969, 1971) and Richard's model (Richard, 1975) are usually built under the assumption that the diffusion of risky asset price follows a geometric Brownian motion, which implies that the economic state for those models is stable and constant. However, sudden changes in economic states are observed in empirical studies which contradicts the assumed geometric Brownian motion for the diffusion of risky asset price.

Jumps and regime switching are usually utilised in various studies in the literature, such as those by Hamilton (1989), Bollen (1998), Cont and Tankov (2004) and Elliott et al. (2007), to describe the sudden economic change. In the context of optimal strategy, jumps and regime switching can be incorporated into life cycle modelling. Hanson (2007) has presented an optimal portfolio and consumption policy with log-Uniform jump amplitude; Zhang and Guo (2004) and Sotomayor and Cadenillas (2009) studied optimal strategies in the financial market with regime switching.

With the growth in human life expectancy, retirees have more flexibility for consumption, investment and bequests (Gupta and Murray, 2003), which has motivated us to conduct a study to examine the optimal financial strategies of retirees in a volatile financial market.

In this paper, an optimal strategy is studied in a volatile financial environment. In order to model retiree behaviours in a volatile financial market, we include jumps and regime switching in the diffusion of risky asset price to extend Richard's model (Richard, 1975). Numerical results are then obtained and analysed.

1.3 Structure of Thesis

Following the recommendation of the Higher Degree Research Department, Faculty of Business and Economics, Macquarie University, I adopted the publications format for this thesis. The main body of this thesis consists of three research papers which are:

- Chapter 2 Optimal Life Insurance and Annuity Demand under Hyperbolic Discounting when Bequests are Luxury Goods,
- Chapter 3 Optimal Time to Enter a Retirement Village, and
- Chapter 4 Optimal Life Insurance and Annuity Demand with Jump Diffusion and Regime Switching.

The key findings, limitations and recommendations for future research form the conclusion, comprising the final chapter.

1.4 Table of Notations

Symbol	Definition	Equation
$a(t)$	Assumed parameter in value function	2.2.29
$\bar{a}_t$	Annuity function at time t	3.2.8
$b(t)$	Capitalised value of future income	2.2.10
B_t	Standard Brownian motion	2.2.4
$\mathscr{B}(t)$	Optimal exercise price of an option	3.2.34
$C(t)$	Consumption amount at time t	2.2.1
$D(t)$	Medical cost at time t	3.2.9
$f(t)$	Probability density function of an agent's lifetime at time t	2.2.2
h	Necessity consumption	3.2.3
H_t	Health indicator at time t	3.2.7
J	Uniform distributed jump amplitude	**??**
K	Strike price of an option	3.2.35
$L(t)$	Legacy amount left for bequest at time t	2.2.1
$m(t)$	Annuitisation factor at time t	2.2.9
$P(t)$	Insurance premium at time t	2.2.6
$\tilde{P}_{ij}$	Transition probability from state i to state j	3.2.6
$\mathscr{P}$	American put option price	3.2.32
q_{ij}	Transition intensity from state i to state j	3.2.2
$\mathbf{Q}$	Transition matrix	3.2.2
r	Risk-free rate	2.2.6
R	Required level of wealth for retirees to enter retirement village.	3.2.32
$S(t)$	Survival rate at time t	2.2.2
t	Current time	2.2.1
T	Uncertain time of death	2.2.1
$U(\cdot)$	Utility function	2.2.1
$V(\cdot)$	Value function	2.2.6
$W(t)$	Wealth level at time t	2.2.6
X_t	Risky asset price at time t	2.2.4
Y	Income level	2.2.6
α	Risky asset return rate	2.2.4
γ	Constant that reflects risk-aversion level	2.2.8
$\delta(t)$	Hyperbolic discount rate	2.2.11
ζ	Degree of impatience	2.2.11
$\theta(t)$	Discounting factor	2.2.12
λ	Jump rate of the discontinuous one-dimensional Poisson process	4.2.2
$\mu(t)$	Force of mortality of an agent at time t	2.2.2
ν	Constant that reflects the annuity level to agent's spouse or children	4.2.5
ξ	Constant decline rate of impatience	2.2.12
$\pi(t)$	Proportion of wealth to invest in risky assets at time t	2.2.6
ρ	Long-run rate of time preference	2.2.11
σ	Volatility of risky asset return rate	2.2.4
τ	Maximum possible survival age	2.2.6
υ	Constant within the range of $[0,\ 1]$	2.2.8
$\psi(t)$	Discontinuous one-dimensional Poisson process	**??**

Chapter 2

Paper 1

Optimal Life Insurance and Annuity Demand under Hyperbolic Discounting when Bequests are Luxury Goods

Jinhui Zhang[1 2] , Sachi Purcal[1], Jiaqin Wei[3]

Abstract

In this paper, an optimal portfolio management model with hyperbolic discounting is developed and analysed. Using the hyperbolic discounting factor, the model recognises the time-inconsistency of the strategies that an agent adopts in the case where there is no commitment. Under the framework of time-inconsistent preferences, agents can be categorised into two groups: "naïve", that is, agents

[1]Department of Applied Finance and Actuarial Studies, Faculty of Business and Finance, Macquarie University, Sydney, Australia.

[2]Corresponding author: jinhui.zhang@students.mq.edu.au

[3]School of Statistics, Faculty of Economics and Management, East China Normal University, Shanghai, China

who are not aware of the time inconsistency of their preferences; and "sophisticated", that is, agents who are aware of the time inconsistency of their preferences. For both the naïve and sophisticated cases, we modify Richard's model (Richard, 1975) of optimal life insurance, annuity purchase and investment over the life cycle by using hyperbolic discounting and allowing bequests to be luxury goods. The solution is obtained via numerical methods. We calibrate the model to Swiss data in presenting our results. We note this model contributes to explaining the annuity puzzle—observed low levels of purchases of voluntary life annuities.

Keywords: Annuity puzzle, Time-inconsistent preference, Hyperbolic discounting, Optimal investment, Merton's model, Stochastic optimal control.

2.1 Introduction

Economic theory tells us that the ultimate annuitisation of wealth is often the optimal choice for ageing consumers (Yaari, 1965). However, widespread lack of demand for voluntary annuities can be found around the world, a fact often known as the "annuity puzzle". Exceptions exist complicating the search for explanations: for example, Switzerland still maintains a high demand for voluntary annuities.

The literature suggests several explanations for this puzzle. Friedman and Warshawsky (1990), as well as Bernheim (1991), listed the bequest motive as one potential explanation for the annuity puzzle; access to social security is also mentioned as another potential explanation. Following these studies, Iskhakov, Thorp and Bateman (2015) developed a life-cyle model for optimal annuity purchases and further found that having access to social security, such as a means-tested age pension, can diminish the demand for both immediate and deferred annuities. Ameriks et al. (2011) found that aversion to public care has significant impacts on annuitisation by households. Finkelstein and Poterba (2004) stated that the lack of actuarially fair annuities is a reasonable explanation for low participation in the voluntary annuity market. Explanations, such as, access to social security, annuity loads and bequest motives, for the low annuity demand were analysed in Horneff, Maurer and Stamos (2008) by using a life-cycle model with fixed-payment annuities. Avanzi (2010) listed several potential reasons to explain the high level of voluntary annuitisation in Switzerland.

Another possible explanation is that people realise their preferences will change over time and they do not want to lock themselves into a long-term product like an annuity. It is widely accepted that these individual preferences play a crucial part in solving the optimal life insurance and annuity demand puzzle. Various authors have suggested that the strategies adopted by individuals are time inconsistent. The causality and existence of the time inconsistency of investors have been mentioned and discussed in many areas, for example, psychology (Ainslie, 1992), game theory (Simaan and Cruz Jr., 1973) and behavioural economics (Strotz, 1955).

Where we have the time-inconsistent behaviour of investors, the measure of utility is not equally distributed over time. Under such circumstances, hyperbolic discounting can be used to deal with these preferences. The hyperbolic discount rate is a decreasing function of time. More specifically, the hyperbolic discount rate drops sharply over a short time period and decreases steadily for a long time period. This feature can capture the short-term dramatic change in the individual preferences of time-inconsistent agents.

Using a hyperbolic discount rate in a model which includes life insurance and annuity choice, such as Richard (1975) model, one can study the optimal consumption and life insurance product choices for individuals with time-inconsistent preferences. This approach also offers insight into the annuity puzzle for such agents. Richard's model (Richard, 1975) has its roots in that of Merton (1969, 1971) who proposed a model of stochastic optimal control for maximising the expected utility of intertemporal consumption and bequest wealth. In Merton's model, the return on the risk-free asset is assumed to be constant while the returns on the risky asset are assumed to follow geometric Brownian motion. For investors allocating resources dynamically between consumption and investment who aim to maximise the expected utility, closed-form results for constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA)-type utility functions were obtained by using the model in Merton (1969, 1971).

Following Merton's work, Richard (1975) generalised Merton's model by including life insurance decisions for the investor. The key insight contributed by Richard (1975) is that the life insurance and annuity demands[4] of an investor are

[4]Importantly, note the subtle point that one can view annuity demand as negative life insurance.

related to the investor's legacy and wealth. With a fair and constant discount rate for deterministic wage income (Yaari, 1965), Richard (1975) derived closed-form results for maximising the expected utility of an investor with CRRA by assuming the lifetime of the investor to be stochastic with a known distribution. The insurance demand in Richard's model was further studied by Pliska and Ye (2007) via numerical experiments. Purcal and Piggott (2008) offered a treatment of Richard's model for time-consistent agents with a necessary bequest motive.

In earlier work, bequests were assumed to be luxury goods (Atkinson, 1971). This assumption is backed by empirical data which suggest that the bequest motives of investors are impacted by their wealth levels (Lockwood, 2012). Auten and Joulfaian (1996) and Hurd and Smith (2002) argued that the bequest motive is related to the wealth and the income of investors. Menchik (1980) showed the existence of luxury-type bequests via the estimated elasticity of the bequest. De Nardi (2004), Ameriks et al. (2011) and Lockwood (2012) pointed out that the assumption of luxury-type bequests is matched by empirical data. Ding, Kingston and Purcal (2014) generalised Merton's model by assuming that bequests are luxury goods.

Both Merton's and Richard's models are built based on the time-consistent preferences of investors—the discount factor in the utility function is exponential with a constant rate. Strotz (1955) argued that the preferences of an investor are time inconsistent as the behaviour of the investor is dynamic and their pre-commitment optimal strategy of the investor might be violated. Pollak (1968) categorised agents into two groups: “naïve”, that is, agents who are not aware of their time-inconsistent behaviour and who always try to pre-commit their future behaviours (and fail) and “sophisticated”, that is, agents who recognise their time-inconsistent preferences and adopt a consistent plan. Within the framework of time-inconsistent preferences, Marín-Solano and Navas (2010) and Marín-Solano, Navas and Roch (2013) extended Merton’s model and provided optimal consumption and portfolio rules for “naïve” and “sophisticated” agents via a standard Hamilton-Jacobi-Bellman (HJB) equation and a modified HJB equation, respectively.

In this paper, we further study the explanation for the “annuity puzzle” and its exceptions. We believe this puzzling problem is the result of the interaction of several factors and have completed an analysis of the interaction of two factors: time-inconsistent behaviours and luxury-type bequests. Drawing on the work of Marín-Solano and Navas (2010) and Marín-Solano, Navas and Roch (2013), we extend Richard’s model for the entire lifespan to include a hyperbolic discounting factor to capture the time-inconsistent preference of agents. We further generalise our model by allowing bequests to be luxury goods. Two extreme cases of agent behaviour, that is, naïve and sophisticated are studied. Hump-shaped consumption as evidenced by empirical data (Gourinchas and Parker, 2002; Fernández-Villaverde and Krueger, 2007) is found in the naïve case. Using our model, theoretical explanations are provided for the reasons as well as the exceptions for the “annuity puzzle” which can be used as an inspiration for future empirical research.

In section 2.2, the HJB equations for naïve and sophisticated agents are derived. Inspired by Ekeland, Mbodji and Pirvu (2012), a numerical scheme is developed to generate numerical results in section 2.3. Finally, conclusion is presented in section 2.4.

2.2 Model and Method

Richard (1975) proposed a generalised version of Merton (1969) model. In Richard's model, an agent is assumed to maximise intertemporal consumption and bequest utility

$$\max\left\{E\left[\int_t^T U_1(C(s),s)ds + U_2(L(T),T)\right]\right\}, \tag{2.2.1}$$

where T is the uncertain time of death, and U_1, C, L and U_2 are utility, consumption, legacy at death and utility from the bequest of the agent, respectively. The legacy amount, L, quantitatively describes agent's desired amount of bequest. Both C and L must be positive.

The key insight here is that the demand for insurance products can be described by the interaction between legacy cost $L(t)$ and total wealth $W(t)$. When legacy cost exceeds wealth, that is, $L(t) > W(t)$, individuals should purchase life insurance for bequest purposes. When wealth is greater than legacy cost, that is, $W(t) > L(t)$, individuals have more wealth than they need for bequest purposes and they would then purchase an annuity to maximise their utility (Yaari, 1965). In Richard (1975), the lifetime of an agent is assumed to follow a known distribution with this denoted by $\mu(t)$ and $S(t)$, the force of mortality and survival probability, respectively. Both $\mu(t)$ and $S(t)$ have the condition of being non-negative. The density function of mortality $f(t)$ is defined as

$$f(t) = \mu(t) \cdot S(t). \tag{2.2.2}$$

Therefore, $\mu(t) = f(t)/S(t)$ can be regarded as the conditional instantaneous probability of death.

Hence, following Purcal and Piggott (2008), the premium rate $P(t)$ of actuarially fair insurance is

$$P(t) = \mu(t) \left[L(t) - W(t)\right]. \tag{2.2.3}$$

When $L(t) > W(t)$ (i.e., we have $P(t) > 0$), that is, the wealth amount is below the desired bequest amount, then retirees must purchase life insurance to cover this shortage. On the other hand, when $W(t) > L(t)$ (i.e., we have $P(t) < 0$), that is, wealth exceeds the optimal legacy/bequest amount then this means the agent has surplus funds. These can be annuitised via the variable annuity and, we assume, will be done so—as the expected return on the annuity, $r + \mu(t)$, exceeds the safe rate of return, r.

We assume that two assets, that is, one risky and one risk-free, are available in which the agent can invest. The dynamics of the price of the risky asset $X_t,\ t \geq 0$

are given by the following geometric Brownian motion,

$$dX_t = \alpha X_t dt + \sigma X_t dB_t, \quad X_0 = X, \tag{2.2.4}$$

where α is the risky asset return rate, σ is the volatility of risky asset return rate and B_t is a one-dimensional Brownian motion defined on a complete probability space ($\mathbf{\Omega}$, $\mathbf{F}$, $\mathbf{P}$). Here, $\mathbf{\Omega}$ is the sample space, $\mathbf{F}$ is the σ-algebra and $\mathbf{P}$ is a real-world probability measure. Both α and σ must be non-negative.

At each moment, the agent chooses $C(t)$, the consumption amount, $P(t)$, the insurance premium amount, and $\pi(t)$, the proportion of wealth to invest in risky assets. The dynamics of the wealth $W(t)$ process can then be described as

$$\begin{aligned} dW(t) &= -C(t)dt - P(t)dt + Y(t)dt + rW(t)dt \\ &\quad +(\alpha - r)\pi(t)W(t)dt + \sigma\pi(t)W(t)dB_t, \end{aligned} \tag{2.2.5}$$

where $Y(t)$ is the deterministic labour income rate and r is the risk-free rate. $W(t)$ is constrained to be non-negative.

By introducing a hyperbolic discounting factor, the model from Richard (1975) can be re-expressed as

$$V(W(t), t) = \max_{C(t),\pi(t),L(t)} E\int_t^\tau \frac{S(s)}{S(t)}\theta(s-t)\left[U_1(C,s) + \mu(s)U_2(L,s)\right] ds, \tag{2.2.6}$$

where τ is the maximum possible survival age. Here, $V(W(t), t)$ is the value function and $\theta(t)$ is the hyperbolic discount factor. Equation (2.2.6) is then subject to the dynamic wealth constraint

$$dW(t) = [(\alpha - r)\pi(t)W(t) + r \cdot W(t) + Y(t) - C(t) - P(t)]dt + \pi(t)\sigma W(t)dB_t. \tag{2.2.7}$$

Power utility functions are adopted in this paper for both the utility of consumption and bequest, that is,

$$\begin{aligned} U_1(C(t), t) &= \frac{C(t)^\gamma}{\gamma}, \\ U_2(L(t), t) &= m(t)^{1-\gamma} \frac{[L(t) + m(t)\upsilon \cdot (W(t) + b(t))]^\gamma}{\gamma}, \end{aligned} \tag{2.2.8}$$

where γ is a constant reflecting the risk-aversion level of an agent,

$$m(t) \equiv \frac{2}{3} \int_t^\tau e^{-r(u-t)} du \tag{2.2.9}$$

is employed as an annuitisation factor for the bequest motive, $\upsilon \in [0, 1]$ is a constant and

$$b(t) = \int_t^\tau Y \frac{S(s)}{S(t)} e^{-r(s-t)} ds \tag{2.2.10}$$

represents the capitalised value of future income.

The assumed value of $m(t)$ is from Purcal and Piggott (2008). This definition assumes that the bequest motive of the agent is to leave an annuitised payment to the surviving spouse (assumed to be the same age as the agent) from the date of death to the maximum age of the mortality table. The annuitised payment is used to cover two-thirds of the amount of optimal consumption.

From equation (2.2.8), a threshold item $m(t)v \cdot (W(t)+b(t))$ with the condition of being non-negative is shown in the utility function of bequest. This threshold is designed to be the annuitised desired level of consumption which reflects the feature of the luxury bequests. Here, we assume that the desired level of consumption is a percentage of the sum of wealth and the capitalised value of future income. A non-negative optimal bequest will be motivated only if optimal consumption is greater than the desired consumption level. In other words, we treat bequests as luxury goods and agents with more wealth intend to leave more bequests.

We adopt the model proposed in Barro (1999) for the hyperbolic discount rate $\delta(t)$ and hyperbolic discounting factor $\theta(t)$,

$$\begin{aligned} \delta(t) &= \rho + \zeta \exp(-\xi t), && (2.2.11) \\ \theta(t) &= \exp(-\int_0^t \delta(u)du) = \exp\left\{ -\rho t - \frac{\zeta}{\xi}\left[1 - \exp(-\xi t)\right] \right\}, && (2.2.12) \end{aligned}$$

where ρ is the long-run rate of time preference, ζ is the degree of impatience and ξ is the constant decline rate of impatience.

From equation (2.2.6), we have

$$V(W(t),t) = \max_{C(t),\pi(t),L(t)} E\left[\int_t^{\tau} \frac{S(s)}{S(t)}\theta(s-t)\mathscr{F}(C,L,s)ds\right], \quad (2.2.13)$$

where $\mathscr{F}(C, L, s) = U_1(C, s) + \mu(s)U_2(L, s)$.

2.2.1 Dynamic programming: the two cases

In the naïve case, the agent solves equation (2.2.6) to obtain a strategy of consumption, insurance and investment at any time t. With unrealised time-inconsistent preferences, the previous decided strategy of this naïve agent is always reconsidered and modified after a very short period of time. Following Marín-Solano and Navas (2010) and Marín-Solano, Navas and Roch (2013), the derivation of the HJB equation for the naïve case is given below.

At time t, the agent has the value function

$$\begin{aligned} V(W(t), t) &= \max_{C(t),\pi(t),L(t)} E\left[\int_t^{t+\epsilon} \frac{S(s)}{S(t)}\theta(s-t)\mathscr{F}(C, L, s)ds\right] \\ &\quad + \max_{C(t),\pi(t),L(t)} E\left[\int_{t+\epsilon}^{\tau} \frac{S(s)}{S(t)}\theta(s-t)\mathscr{F}(C, L, s)ds\right]. \end{aligned} \tag{2.2.14}$$

For small ϵ, that is, $\epsilon^2 \approx 0$, we can have, recalling equations (2.2.11) and (2.2.12),

$$S(t+\epsilon) = S(t)(1-\mu(t)\epsilon)$$

and

$$\begin{aligned}\theta(s-t) &= \exp\left\{-\rho(s-t) - \frac{\zeta}{\xi}\left[1-\exp\left(-\xi(s-t)\right)\right]\right\} \\ &= \exp\left\{-\rho\epsilon + \frac{\zeta}{\xi}\exp(\xi t)\left[\exp(-\xi s) - \exp(-\xi(s-\epsilon))\right]\right\} \\ &\quad \times \exp\left\{-\rho(s-t-\epsilon) - \frac{\zeta}{\xi}[1-\exp\left(-\xi(s-t-\epsilon)\right)]\right\} \\ &\approx \exp\left[-\rho\epsilon - \zeta\exp\left(\xi t\right)\epsilon\right]\theta(s-t-\epsilon) \\ &\approx (1-\delta(t)\epsilon)\theta(s-t-\epsilon). \qquad (2.2.15)\end{aligned}$$

Hence,

$$\begin{aligned}\frac{S(s)}{S(t)}\theta(s-t) &= \frac{S(s)}{S(t+\epsilon)}(1-\mu(t)\epsilon)(1-\delta(t)\epsilon)\theta(s-t-\epsilon) \\ &= (1-\mu(t)\epsilon-\delta(t)\epsilon)\frac{S(s)}{S(t+\epsilon)}\theta(s-t-\epsilon). \qquad (2.2.16)\end{aligned}$$

Equation (2.2.14) can now be written as

$$\begin{aligned}V(W(t),t) &= \max_{C(t),\pi(t),L(t)} E\left[\int_t^{t+\epsilon}\frac{S(s)}{S(t)}\theta(s-t)\mathscr{F}(C,L,s)ds\right] \\ &\quad +(1-\delta(t)\epsilon-\mu(t)\epsilon)V(W(t+\epsilon),t+\epsilon). \qquad (2.2.17)\end{aligned}$$

Using Itô's lemma and equation (2.2.29), $V(W(t+\epsilon), t+\epsilon)$ can be shown to be

$$\begin{aligned} V(W(t+\epsilon), t+\epsilon) &= V(W(t), t) + V_t(W(t), t)\epsilon \\ &\quad + V_W(W(t), t)[(\alpha - r)\pi^*(t)W(t) + r \cdot W(t) + Y(t) \\ &\quad - C^*(t) - P^*(t)]\epsilon + \frac{1}{2}V_{WW}(W(t), t)\pi^{*2}W(t)^2\sigma^2\epsilon, \quad (2.2.18) \end{aligned}$$

where $C^*(t)$, $\pi^*(t)$ and $P^*(t)$ are the optimal consumption, optimal portfolio investment proportion and optimal insurance premium, respectively.

By substituting equation (2.2.18) into equation (2.2.17), the standard HJB equation for the naïve agent at time s is now

$$\begin{aligned} &\delta(s)V(W(s), s)\epsilon + \mu(s)V(W(s), s)\epsilon - V_s(W(s), s)\epsilon \\ &\quad = \max_{C(s),\pi(s),L(s)} E\left[\int_s^{s+\epsilon} \frac{S(u)}{S(s)}\theta(u-s)\mathscr{F}(C, L, u)du\right] \\ &\quad\quad + V_W(W(s), s)[(\alpha - r)\pi^*(s)W(s) + r \cdot W(s) + Y \\ &\quad\quad - C^*(s) - P^*(s)]\epsilon + \frac{1}{2}V_{WW}(W(s), s)\pi^{*2}W(s)^2\sigma^2\epsilon. \quad (2.2.19) \end{aligned}$$

By dividing equation (2.2.19) by ϵ and taking the limit $\epsilon \to 0$, we now have

$$\begin{aligned} &\delta(s)V(W(s), s) + \mu(s)V(W(s), s) - V_s(W(s), s) \\ &\quad = \mathscr{F}(C^*, L^*, s) + V_W(W(s), s)[(\alpha - r)\pi^*(s)W(s) + r \cdot W(s) \\ &\quad\quad + Y(s) - C^*(s) - P^*(s)] + \frac{1}{2}V_{WW}(W(s), s)\pi^{*2}W(s)^2\sigma^2, \quad (2.2.20) \end{aligned}$$

where L^* is the optimal legacy cost.

If there is no commitment, the naïve agent at time $t = 0$ will take the action based on the solution of equation (2.2.20) for a short time period, that is, ϵ. Then, at time $t = \epsilon$, the naïve agent will change this decision to the solution of

$$\begin{aligned}
&\delta(s-\epsilon)V(W(s),s) + \mu(s)V(W(s),s) - V_s(W(s),s) \\
&\quad = \mathscr{F}(C^*, L^*, s) + V_W(W(s),s)[(\alpha - r)\pi(s)W(s) + r\cdot W(s) \\
&\qquad + Y(s) - C^*(s) - P^*(s)] + \frac{1}{2}V_{WW}(W(s),s)\pi^{*2}W(s)^2\sigma^2.
\end{aligned}$$

Therefore, generally, the solution of the naïve case is this solution from the family of HJB equations,

$$\begin{aligned}
&\delta(s-t)V(W(s),s) + \mu(s)V(W(s),s) - V_s(W(s),s) \\
&\quad = \mathscr{F}(C^*, L^*, s) + V_W(W(s),s)[(\alpha - r)\pi^*(s)W(s) + r\cdot W(s) \\
&\qquad + Y(s) - C^*(s) - P^*(s)] + \frac{1}{2}V_{WW}(W(s),s)\pi^{*2}W(s)^2\sigma^2. \qquad (2.2.21)
\end{aligned}$$

Sophisticated agents notice that their behaviour can be time-inconsistent. Their optimal strategy is time-consistent which is based on an equilibrium of a series of future behaviours. For the sophisticated case, however, the strategy adopted by the agent at one time point is related to the strategy at the next time point (Marín-Solano and Navas, 2010). Hence, we first study the sophisticated case in discrete time and then move on to consider continuous time. We can discretise equation (2.2.14) by dividing $(0, \tau)$ into N time periods of a very short length ϵ, where $\epsilon^2 = 0$. Defining $dt = \epsilon$ and $t = j\epsilon$, we have

$$\begin{aligned} V(W(j), j) &= \max E\left[\sum_{i=0}^{N-j} \frac{S(i+j)}{S(j)} \theta(i) \mathscr{F}(C, L, i+j)\right] \epsilon \\ &= \max E\left[\mathscr{F}(C, L, j)\right] \epsilon \\ &\quad + \max E\left[\sum_{i=1}^{N-j-1} \frac{S(i+j)}{S(j)} \theta(i) \mathscr{F}(C, L, i+j)\right] \epsilon \\ &\quad + \max E\left[\frac{S(N)}{S(j)} \theta(N-j) \mathscr{F}(C, L, N)\right] \epsilon. \end{aligned} \tag{2.2.22}$$

Also, we have

$$\begin{aligned} V(W((j+1), j+1) = \max E &\left[\sum_{k=0}^{N-j-2} \frac{S(k+1+j)}{S(j+1)} \theta(k) \mathscr{F}(C, L, k+1+j)\right] \epsilon \\ &+ \max E\left[\frac{S(N)}{S(j+1)} \theta(N-j-1) \mathscr{F}(C, L, N)\right] \epsilon. \end{aligned} \tag{2.2.23}$$

By substituting equation (2.2.23) into equation (2.2.22), it can be shown that

$$\begin{aligned} &S(j)\theta(N-j-1)V(W(j), j) - S(j+1)\theta(N-j)V(W(j+1), j+1) \\ &= S(j)\theta(N-j-1) \max E\left[\mathscr{F}(C, L, j)\right] \epsilon \\ &\quad + \theta(N-j-1) \max E\left[\sum_{i=1}^{N-j-1} S(i+j)\theta(i) \mathscr{F}(C, L, i+j)\right] \epsilon \\ &\quad - \theta(N-j) \max E\left[\sum_{k=0}^{N-j-2} S(k+1+j)\theta(k) \mathscr{F}(C, L, k+1+j)\right] \epsilon. \end{aligned} \tag{2.2.24}$$

Using Itô's lemma, $V(W(j+1), j+1)$ then becomes

$$\begin{aligned} V(W(j+1), j+1) = V(W(j), j) + V_t(W(j), j)\epsilon \\ + V_W(W(j), j)[(\alpha - r)\pi^*(j)W(j) + r \cdot W(j) + Y \\ - C^*(j) - P^*(j)]\epsilon + \frac{1}{2}V_{WW}(W(j), j)\pi^2(j)W^2(j)\sigma^2\epsilon, \end{aligned} \tag{2.2.25}$$

With equation (2.2.25), we can simplify equation (2.2.24) to

$$\begin{aligned} &(S(j)\theta(N-j-1) - S(j+1)\theta(N-j))\, V(W(j), j) \\ &= \mathscr{K}\epsilon + S(j)\theta(N-j-1) \max E\left[\mathscr{F}(C, L, j)\right]\epsilon \\ &\quad + S(j+1)\theta(N-j)V_W(W(j), j)[(\alpha - r)\pi^*(j)W(j)\epsilon \\ &\quad + r \cdot W(j) + Y - C^*(j) - P^*(j)] + V_t(W(j), j)\epsilon \\ &\quad + \frac{1}{2}S(j+1)\theta(N-j)V_{WW}(W(j), j)\pi^{*2}(j)W^2(j)\sigma^2\epsilon, \end{aligned} \tag{2.2.26}$$

where

$$\mathscr{K} = \max E \sum_{i=1}^{N-j-1} S(i+j)\left[\theta(N-j-1)\theta(i) - \theta(N-j)\theta(i-1)\mathscr{F}(C, L, i+j)\right].$$

Here, we define $\theta(N-j) = \theta(N-j-1)[1-\delta(N-j)\epsilon]$ and $\theta(k) = \theta(k-1)[1+\delta(k)\epsilon]$. By dividing equation (2.2.26) by ϵ and taking the limit $\epsilon \to 0$, we then have the modified HJB equation for the sophisticated case,

$$
\begin{aligned}
&\delta(\tau-t)V(W(t),t)+\mu(t)V(W(t),t)-V_t(W(t),t)\\
&\qquad=U_1(C^*,t)+\mu(t)U_2(L^*,t)+\mathscr{K}(t)\\
&\qquad\quad+V_W(W(t),t)[(\alpha-r)\pi^*(t)W(t)+r\cdot W(t)+Y(t)-C^*(t)-P^*(t)]\\
&\qquad\quad+\frac{1}{2}V_{WW}(W(t),t)\pi^{*2}(t)W^2(t)\sigma^2, \qquad (2.2.27)
\end{aligned}
$$

where

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{S(j)\theta((N-j-1))-S((j+1))\theta(N-j)}{\epsilon}\\
&\quad=\lim_{\epsilon\to 0}\frac{\theta(N-j-1)(S(j)-S(j+1))+\theta(N-j-1)\delta(N-j)S(j+1)\epsilon}{\epsilon}\\
&\quad=\lim_{\epsilon\to 0}\left[\theta(N-j-1)\frac{(S(j)-S(j+1))}{\epsilon}+\theta(N-j-1)\delta(N-j)S(j+1)\right]\\
&\quad=\theta(\tau-t)S(t)\mu(t)+\theta(\tau-t)\delta(\tau-t)S(t),
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{\theta(N-j-1)\theta(i)-\theta(N-j)\theta(i-1)}{\epsilon}\\
&\quad=\lim_{\epsilon\to 0}\frac{\theta(N-j-1)\theta(i)-\theta(N-j-1)\theta(i)(1-\delta(N-j)\epsilon)(1+\delta(i)\epsilon)}{\epsilon}\\
&\quad=\lim_{\epsilon\to 0}\theta(N-j-1)\theta(i)(\delta(N-j)-\delta(i))\\
&\quad=\theta(\tau-t)\theta(s-t)(\delta(\tau-t)-\delta(s-t))
\end{aligned}
$$

and

$$\mathscr{K}(t) = E\int_t^{\tau} \frac{S(s)}{S(t)}\theta(s-t)\left[\delta(\tau - t) - \delta(s-t)\right]\left[U_1(C^*, s) + \mu(s)U_2(L^*, s)\right] ds. \tag{2.2.28}$$

Different to the naïve agent, who modifies decisions at each time point without the consideration of future inconsistent behaviours, sophisticated agent makes the decision at the start of the decision period (i.e.,$\tau - t$) and consider the future utility[5].

2.2.2 Optimal behaviours for time-inconsistent agents

Here we consider the ansatz for the value function,

$$V(W, t) = a(t)\frac{(W(t) + b(t))^{\gamma}}{\gamma}, \tag{2.2.29}$$

where $a(t)$ is defined as an annuitisation factor or marginal propensity of consumption.

From equations (2.2.21) and (2.2.27) and the first-order condition, the optimal consumption, optimal portfolio investment proportion and optimal legacy amount

[5] $\mathscr{K}(t)$ can be defined as the optimal expected discounted future utility.

can be obtained as follows,

$$\begin{aligned}
C^*(t) &= a(t)^{\frac{1}{\gamma-1}}(W(t)+b(t)), \\
\pi^*(t) &= \frac{\alpha-r}{(1-\gamma)\sigma^2}\frac{(W(t)+b(t))}{W(t)} \\
\text{and} \quad L^*(t) &= m(t)[C^*(t)-\upsilon(W(t)+b(t))] \\
&= m(t)a(t)^{\frac{1}{\gamma-1}}(W(t)+b(t))-m(t)\upsilon\cdot(W(t)+b(t)). \qquad (2.2.30)
\end{aligned}$$

As a negative legacy, that is, a negative bequest motive, is not allowed in most jurisdictions, we adopt the constraint: $\upsilon \leq a(t)^{\frac{1}{\gamma-1}}$.

Hence, with equation (2.2.8), the utility functions of optimal consumption and legacy can be shown to be,

$$\begin{aligned}
U_1(C^*(t),t) &= \frac{a(t)^{\frac{\gamma}{\gamma-1}}(W(t)+b(t))^\gamma}{\gamma}, \\
\text{and} \quad U_2(L^*(t),t) &= \frac{m(t)a(t)^{\frac{\gamma}{\gamma-1}}(W(t)+b(t))^\gamma}{\gamma}. \qquad (2.2.31)
\end{aligned}$$

Based on equation (2.2.3), the optimal premium amount $P^*(t)$ is then,

$$\begin{aligned}
P^*(t) &= (L^*(t)-W(t))\mu(t) \\
&= \mu(t)m(t)(a(t)^{\frac{1}{\gamma-1}}-\upsilon)(W(t)+b(t))-\mu(t)W(t). \qquad (2.2.32)
\end{aligned}$$

We define $V_t(W,t)=\partial V(W,t)/\partial t$, $V_W(W,t)=\partial V(W,t)/\partial W$ and $V_{WW}(W,t)=$

$\partial^2 V(W,t)/\partial W^2$, respectively. Then, from equation (2.2.29), we have

$$\begin{aligned}
V_W(W,t) &= a(t)(W(t)+b(t))^{\gamma-1}, \\
V_{WW}(W(t),t) &= (\gamma-1)a(t)(W(t)+b(t))^{\gamma-2} \\
\text{and} \quad V_t(W,t) &= a'(t)\frac{(W(t)+b(t))^{\gamma}}{\gamma} + a(t)(W(t)+b(t))^{\gamma-1}b'(t) \\
&= a'(t)\frac{(W(t)+b(t))^{\gamma}}{\gamma} \\
&\quad +a(t)(W(t)+b(t))^{\gamma-1}[-Y+(r+\mu(t))b(t)]. \quad (2.2.33)
\end{aligned}$$

For the naïve case, let us put equations (2.2.30), (2.2.31), (2.2.32) and (2.2.33) into equation (2.2.21). Following same simplifications, we have

$$\begin{aligned}
&a'(s)+a(s)\left[\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)}\gamma + r\cdot\gamma - \mu(s)(1-\gamma-\gamma m(s)\upsilon) - \delta(s-t)\right] \\
&= -\left(1+\mu(s)m(s)\right)(1-\gamma)a(s)^{\frac{\gamma}{\gamma-1}}, \qquad (2.2.34)
\end{aligned}$$

where $a(0)=1$ and $a'(0)=0$.

Equation (2.2.34) is a Bernoulli differential equation, and we define

$$a(s) = d(s)^{1-\gamma}, \qquad (2.2.35)$$

where $a'(s) = (1-\gamma)d(s)^{-\gamma}d'(s)$.

With equation (2.2.35), equation (2.2.34) can be written as

$$(1-\gamma)d(s)^{-\gamma}d'(s)+\Bigg[\left(\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)}+r\right)\gamma - \mu(s)(1-\gamma-m(s)\upsilon)-\delta(s-t)\Bigg]d(s)^{1-\gamma} = -\left[1+\mu(s)m(s)\right](1-\gamma)d(s)^{-\gamma}. \tag{2.2.36}$$

Dividing equation (2.2.36) by $(1-\gamma)d(s)^{-\gamma}$ and multiplying ς, we have

$$d'(s)\varsigma+\left[\left(\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)}+r\right)\frac{\gamma}{1-\gamma}-\mu(s)(1-\gamma-m(s)\upsilon)-\frac{\delta(s-t)}{1-\gamma}\right]d(s)\varsigma = -\left[1+\mu(s)m(s)\right]\varsigma, \tag{2.2.37}$$

where $\varsigma=\exp\left[\int_0^s\left(\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)}+r\right)\frac{\gamma}{1-\gamma}-\mu(u)+\frac{\gamma\mu(u)m(u)\upsilon}{1-\gamma}-\frac{\delta(u-t)}{1-\gamma}du\right]$. Integrating both sides of equation (2.2.37) from t to τ yields that

$$d(s)\exp\left[\int_0^s\left(\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)}+r\right)\frac{\gamma}{1-\gamma}-\mu(u)+\frac{\gamma\mu(u)m(u)\upsilon}{1-\gamma}-\frac{\delta(u-t)}{1-\gamma}du\right]\Bigg|_t^\tau = -\int_t^\tau\left[1+\mu(s)m(s)\right]\exp\left[\int_0^s\left(\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)}+r\right)\frac{\gamma}{1-\gamma}-\mu(u) + \frac{\gamma\mu(u)m(u)\upsilon}{1-\gamma}-\frac{\delta(u-t)}{1-\gamma}du\right]ds. \tag{2.2.38}$$

Equation (2.2.38) can be simplified as

$$
\begin{aligned}
&- d(t)S(t)\exp\left\{\left(\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)}+r\right)\frac{\gamma}{1-\gamma}t\exp\left[\frac{-\rho t-\frac{\zeta}{\xi}(1-\exp(-\xi t))}{1-\gamma}\right]\right.\\
&\qquad\qquad\left.\times\exp\left(\int_0^t\frac{\gamma\mu(u)m(u)\upsilon}{1-\gamma}du\right)\right\}\\
&\qquad= -\int_t^\tau[1+\mu(s)m(s)]\,S(s)\exp\left[\left(\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)}+r\right)\frac{\gamma}{1-\gamma}s\right]\\
&\qquad\qquad\times\exp\left\{\frac{-\rho s-\frac{\zeta}{\xi}[1-\exp(-\xi s)]}{1-\gamma}\right\}\exp\left(\int_0^s\frac{\gamma\mu(u)m(u)\upsilon}{1-\gamma}du\right)ds.
\end{aligned}
\tag{2.2.39}
$$

Expressing equation (2.2.39) in terms of $d(t)$ gives

$$
\begin{aligned}
d(t) = &\int_t^\tau(1+\mu(s)m(s))\frac{S(s)}{S(t)}\theta(s-t)^{\frac{1}{1-\gamma}}\\
&\times\exp\left[\left(\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)}+r\right)\frac{\gamma}{1-\gamma}(s-t)\right]\exp\left(\int_t^s\frac{\gamma\mu(u)m(u)\upsilon}{1-\gamma}du\right)ds.
\end{aligned}
$$

Based on equation (2.2.35), the value of $a(t)$ can now be found,

$$
\begin{aligned}
a(t) = &\left\{\int_t^\tau(1+\mu(s)m(s))\frac{S(s)}{S(t)}\theta(s-t)^{\frac{1}{1-\gamma}}\right.\\
&\left.\times\exp\left[\left(\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)}+r\right)\frac{\gamma}{1-\gamma}(s-t)\right]\exp\left(\int_t^s\frac{\gamma\mu(u)m(u)\upsilon}{1-\gamma}du\right)ds\right\}^{1-\gamma}.
\end{aligned}
$$

With the closed-form result for $a(t)$ and given the initial wealth level, we can obtain the optimal strategy for the naïve agent via equations (2.2.29), (2.2.30) and (2.2.32).

For the sophisticated case, substituting the optimal consumption and bequest equation into the modified HJB equation, equation (2.2.27) and multiplying both sides by γ, we have the following equation

$$\begin{aligned}0 = & a'(t)(W(t)+b(t))^{\gamma} \\ & + a(t)^{\frac{\gamma}{\gamma-1}}(W(t)+b(t))^{\gamma}[(1-\gamma)+\mu(t)m(t)(1-\gamma)] \\ & + a(t)(W(t)+b(t))^{\gamma}\left[\frac{1}{2}\frac{(\alpha-r)^2}{\sigma^2(1-\gamma)}\gamma-(\mu(t)+\delta(\tau-t))\right. \\ & \left. +(r+\mu(t))\gamma+\gamma\mu(t)m(t)\upsilon\right]+\gamma\mathscr{K}(t), \end{aligned} \tag{2.2.40}$$

together with the wealth constraint

$$\begin{aligned} dW(t) &= [(\alpha-r)\pi(t)W(t)+r\cdot W(t)+Y-C(t)-P(t)]dt+\pi(t)\sigma W(t)dB_t \\ &= \left[\frac{(\alpha-r)^2}{(1-\gamma)\sigma^2}+\mu(t)m(t)\upsilon-(1+\mu(t)m(t))a(t)^{\frac{1}{\gamma-1}}\right](W(t)+b(t))dt \\ &\quad +[(r+\mu(t))W(t)+Y]\,dt+\frac{\alpha-r}{(1-\gamma)\sigma}(W(t)+b(t))dB_t. \end{aligned} \tag{2.2.41}$$

By letting $\tilde{W}(t)=W(t)+b(t)$, we then have

$$\begin{aligned} d\tilde{W}(t) &= \left[(r+\mu(t))+\frac{(\alpha-r)^2}{(1-\gamma)\sigma^2}+\mu(t)m(t)\upsilon-(1+\mu(t)m(t))a(t)^{\frac{1}{\gamma-1}}\right]\tilde{W}(t)dt \\ &\quad +[-(r+\mu)b(t)+b'(t)+Y]\,dt+\frac{\alpha-r}{(1-\gamma)\sigma}\tilde{W}(t)dB_t. \end{aligned} \tag{2.2.42}$$

From equation (2.2.10), it follows that $-(r+\mu(t))b(t)+b'(t)+Y=0$. Therefore, we obtain

$$\begin{aligned} d\tilde{W}(t) &= \left[(r+\mu(t)) + \frac{(\alpha-r)^2}{(1-\gamma)\sigma^2} + \mu(t)m(t)\upsilon - (1+\mu(t)m(t))a(t)^{\frac{1}{\gamma-1}}\right]\tilde{W}(t)dt \\ &\quad + \frac{\alpha-r}{(1-\gamma)\sigma}\tilde{W}(t)dB_t. \end{aligned} \tag{2.2.43}$$

The above equation is a geometric Brownian motion differential equation. Hence, we have

$$\begin{aligned} \frac{\tilde{W}(s)}{\tilde{W}(t)} = \exp\Bigg\{ \int_t^s (r-\mu(t)) + \frac{(\alpha-r)^2}{(1-\gamma)\sigma^2} - \frac{(\alpha-r)^2}{2(1-\gamma)^2\sigma^2} + \mu(u)m(u)\upsilon \\ - [1+\mu(u)m(u)]a(u)^{\frac{1}{\gamma-1}}du + \frac{\alpha-r}{(1-\gamma)\sigma}(B(s)-B(t))\Bigg\}. \end{aligned} \tag{2.2.44}$$

Putting equations (2.2.31) and (2.2.44) into equation (2.2.28), it follows that

$$\begin{aligned} \mathscr{K}(t) = (W(t)+b(t))^\gamma \int_t^\tau \frac{S(s)}{S(t)}\theta(s-t)(\delta(\tau-t) \\ - \delta(s-t))\frac{1}{\gamma}(1+\mu(s)m(s))a(s)^{\frac{\gamma}{\gamma-1}}g(s)ds, \end{aligned} \tag{2.2.45}$$

where $g(s) = \exp\left\{ \int_t^s (r+\mu(t))\gamma + \frac{(\alpha-r)^2}{(1-\gamma)\sigma^2}\gamma + \mu(u)m(u)\upsilon\cdot\gamma - \gamma[1+\mu(u)m(u)]a(u)^{\frac{1}{\gamma-1}}du \right\}$.

Based on equations (2.2.40) and (2.2.45), we obtain the integro-differential equation

$$
\begin{aligned}
0 = {} & a'(t) + a(t)^{\frac{\gamma}{\gamma-1}}[(1-\gamma) + \mu(t)m(t)(1-\gamma)] \\
& + \int_t^\tau \frac{S(s)}{S(t)}\theta(s-t)(\delta(\tau-t) - \delta(s-t))(1+\mu(s)m(s))a(s)^{\frac{\gamma}{\gamma-1}}g(s)ds \\
& + a(t)\left\{\frac{1}{2}\frac{(\alpha-r)^2}{\sigma^2(1-\gamma)}\gamma - [\mu(t) + \delta(\tau-t)] + (r+\mu(t))\gamma + \gamma\mu(t)m(t)\upsilon\right\}.
\end{aligned}
\tag{2.2.46}
$$

Drawing on the work of Ekeland, Mbodji and Pirvu (2012), we develop a numerical scheme to approximate the integro-differential equation, equation (2.2.46). The existence and uniqueness and convergence of the numerical scheme have been discussed in Ekeland, Mbodji and Pirvu (2012), where they point out proofs of the existence and uniqueness are an ongoing research project. Nevertheless, these authors have used these techniques in their published work, as do we.

We write

$$
g(s) = \exp\left(\int_t^s l + \mu(u)(1+m(u)\upsilon)\gamma du\right)\left(\frac{A(s)}{A(t)}\right), \tag{2.2.47}
$$

where

$$
l = r\cdot\gamma + \frac{(\alpha-r)^2}{(1-\gamma)\sigma^2}\gamma
$$

and

$$A(s) = \exp\left\{ \int_s^{\tau} \gamma[1 + \mu(u)m(u)]a(u)^{\frac{1}{\gamma-1}} du \right\}. \tag{2.2.48}$$

We define $t_n = \tau - n\Delta t$, where $\Delta t = \tau/N$ and the total number of time units, N, are used to discretise the time interval $[0, \tau]$; Equation (2.2.46) can be written as follows

$$\begin{aligned} a'(t_n) &= a(t_n)^{\frac{\gamma}{\gamma-1}}[\gamma - 1 - \mu(t_n)m(t_n)(1-\gamma)] \\ &\quad - \int_{t_n}^{\tau} \frac{S(s_n)}{S(t_n)} \theta(s_n - t_n)(\delta(\tau - t_n) - \delta(s_n - t_n))(1 + \mu(s_n)m(s_n))a(s_n)^{\frac{\gamma}{\gamma-1}} g(s_n) ds_n \\ &\quad + a(t_n)\left[-\frac{1}{2}\frac{(\alpha - r)^2}{\sigma^2(1-\gamma)}\gamma + \mu(t_n) + \delta(\tau - t_n) - (r + \mu(t_n))\gamma + \gamma\mu(t_n)m(t_n)\upsilon \right]. \end{aligned}$$

We can write

$$\begin{aligned} a'(t_n) &= a(t_n)^{\frac{\gamma}{\gamma-1}}[\gamma - 1 - \mu(t_n)m(t_n)(1-\gamma)] \\ &\quad + a(t_n)\left[-\frac{1}{2}\frac{(\alpha - r)^2}{\sigma^2(1-\gamma)}\gamma + \mu(t_n) + \delta(\tau - t_n) - (\tau + \mu(t_n))\gamma + \gamma\mu(t_n)m(t_n)\upsilon \right] \\ &\quad - \int_{t_n}^{\tau} I(s_n, t_n)(1 + \mu(s_n)m(s_n))a(s_n)^{\frac{\gamma}{\gamma-1}} \left(\frac{A(s_n)}{A(t_n)} \right) ds_n, \end{aligned} \tag{2.2.49}$$

where

$$I(s_n, t_n) = \frac{S(s_n)}{S(t_n)} \theta(s_n - t_n)(\delta(\tau - t_n) - \delta(s_n - t_n)) \exp(l(s_n - t_n) + \int_{t_n}^{s_n} \mu(u)(1 + m(u)\upsilon)\gamma du).$$

For the sequence a_{t_n} and A_{t_n} we have

$$a(t_{n+1}) = a(t_n) - a'(t_n)\Delta t,$$

and

$$\begin{aligned} A(t_{n+1}) &= A(t_n) - A'(t_n)\Delta t \\ &= A(t_n) + \gamma(1 + \mu(t_n)m(t_n))a(t_n)^{\frac{1}{\gamma-1}} A(t_n)\Delta t, \end{aligned} \tag{2.2.50}$$

where $A'(t_n) = -\gamma(1 + \mu(t_n)m(t_n))a(t_n)^{\frac{1}{\gamma-1}} A(t_n)$. Equation (2.2.49) can then be discretised as follows

$$\begin{aligned} a(t_{n+1}) = a(t_n) &- a(t_n)^{\frac{\gamma}{\gamma-1}}[\gamma - 1 - \mu(t_n)m(t_n)(1-\gamma)]\Delta t \\ &- a(t_n)\left[-\frac{1}{2}\frac{(\alpha - r)^2}{\sigma^2(1-\gamma)}\gamma + (\mu(t_n) + \delta(\tau - t_n)) - (r + \mu(t_n))\gamma + \gamma\mu(t_n)m(t_n)v\right]\Delta t \\ &- (\Delta t)^2 \sum_{j=0}^{n-1} I(t_j, t_n)(1 + \mu(t_j)m(t_j))a(t_j)^{\frac{\gamma}{\gamma-1}}\left(\frac{A(t_j)}{A(t_n)}\right). \end{aligned} \tag{2.2.51}$$

Based on equations (2.2.29), (2.2.30) and (2.2.32), an optimal strategy for the sophisticated case can be determined using the result of $a(t)$ from (2.2.51).

Table 2.1: Parameters used in the numerical results

$t = 30$	$\tau = 100$
$\alpha = 0.0757$	$r = 0.0244$
$\rho = 0.001925$	$\sigma = 0.18$
$Y = 81,012$ Swiss francs	$\gamma = -0.5$
$\xi = 0.5$	$\zeta = 0.25$
$\upsilon = 0$ or 0.02	Retirement age=65

2.3 Numerical Results

2.3.1 Parameter values

In this paper, we calibrate our model to Swiss data and present numerical results from the starting age $t = 30$ to the limiting age $\tau = 100$. We use the most recent male population mortality in Swiss Statistics from Swiss Federal Statistical Office (2005) to calibrate the survival rate and force of mortality.

According to Barro (1999), the parameters used in equation (2.2.12) can be calibrated to match the exponential model. Specifically, the present values of a perpetuity contract via the exponential model or hyperbolic model should be no different. With the exponential rate in Purcal and Piggott (2008), that is, 0.005 and the suggested ζ and ξ value from Barro (1999), we follow Tang (2009) to calibrate parameter values, finding $\rho = 0.001925$, $\zeta = 0.25$ and $\xi = 0.5$.

We set the risky asset return rate at $\alpha = 0.0757$ (p.a.) and the risk-free rate at $r = 0.0244$ (p.a.), based on the geometric mean of the SIX Swiss Exchange Index (1990–2014) and the yield of Swiss federal bonds over the last 25 years (1990–2014). From the Six Index, we also calculated the average volatility over the last 25 years, finding that, $\sigma = 0.18$ (p.a.). In addition, we use the Swiss average monthly wage (2009–2011) from the Swiss Federal Statistical Office to obtain the annual income, $Y = 81,012$ Swiss francs[6] which is also assumed to be the initial wealth amount. In our model, agents are set to receive a constant annual income, that is, $Y = 81,012$ Swiss francs, before the retirement age and have no income stream, that is, $Y = 0$, after retirement. Here, we assume υ to be 0 or 0.02 to demonstrate the effect of luxury-type bequests. We set the compulsory retirement age at age 65.

The closed-form of Richard's model for the control variables is stated in equation (2.2.30), where $W(t)$ is the state variable and $a(t)$ is still unknown. In section 2.2.2, we give analytic solutions for $a(t)$ for both the naïve and sophisticated cases stated in section 2.2. Hence, the value of the control variables can be determined for a given $W(t)$.

From equation (2.2.43), the dynamics of combined wealth and capitalised income follow the geometric Brownian motion; the wealth path $W(t)$ is not deterministic. Thus, we adopt a simulation approach to determine the expected value of the state and control variables. Specifically, following Purcal and Piggott (2008), we average 10,000 simulations to generate the expected time path of all variables. It can be shown that relative mean error[7] (RME) is small given that 10,000 simulations are used.

[6]This figure is the median male gross income in 2014 from the Swiss Federal Statistical Office. Due to the complexity of the Swiss tax system, we use the gross income instead of net income.

[7]See table 2.2 for these values. Note the relative mean error is defined as the standard error divided by the mean.

Table 2.2: RME for both naïve and sophisticated agents

Age	Naïve case with $\zeta = 0.25$ and $\upsilon = 0$	Naïve case with $\zeta = 0.25$ and $\upsilon = 0.02$	Sophisticated case with $\zeta = 0.25$ and $\upsilon = 0$	Sophisticated case with $\zeta = 0.25$ and $\upsilon = 0.02$
40	0.006782	0.006533	0.006506	0.006399
50	0.01071	0.009778	0.010186	0.010025
60	0.015043	0.013078	0.013746	0.013345
70	0.021945	0.017227	0.017844	0.018040
80	0.026585	0.020455	0.020894	0.020936
90	0.027674	0.024114	0.026247	0.028249

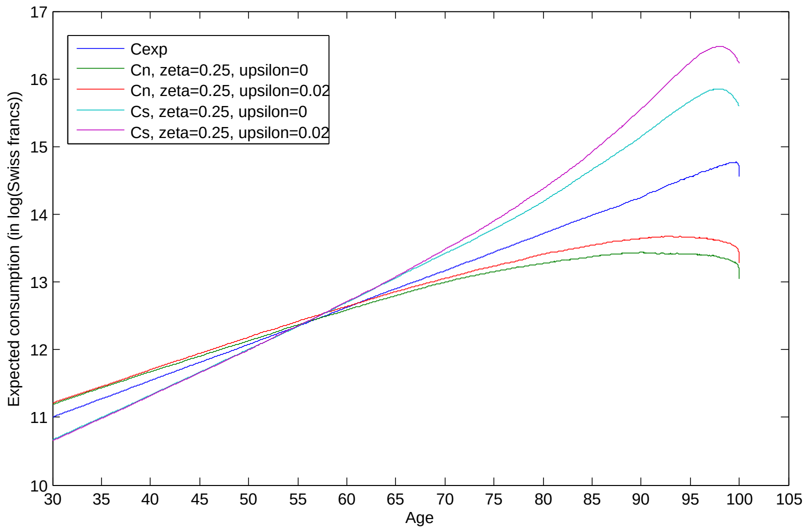

Figure 2.1: Expected consumption comparison between exponential discounting case (Cexp in legend) and hyperbolic discounting cases (Cn and Cs in legend)

2.3.2 Comparison with exponential discounts

To explore the time-inconsistent preferences inherent in hyperbolic discounting, we conduct a comparison with the time-consistent preferences case, that is, the exponential discounting case, and the hyperbolic discounting cases, including the naïve, naïve with luxury-type bequest, sophisticated and sophisticated with luxury-type bequest cases as shown in figures 2.1 and 2.2. In figure 2.1 (which is our only diagram on a log scale), utility-maximising naïve agents consume more than both sophisticated agents and agents with exponential discounting in the early stages of life. The result, therefore, is that we expect those naïve agents to have less wealth as they age which leads to their low consumption afterwards. In contrast, the sophisticated agents who realise their time-inconsistent preferences would consciously reduce their consumption in their early years. Their sophisticated behaviour even allows them to consume more in the later stages of their life than agents with time-consistent preferences (the exponential discounting case).

In figure 2.2, naïve agents continue to indicate their demand for life insurance while sophisticated agents and agents with time-consistent preferences utilise their wealth ultimately for annuitisation. As will be recalled, in our model, a positive premium amount means demand for life insurance while a negative premium amount means annuitisation. Again, with their awareness of time-inconsistent preferences, sophisticated agents end up with wealth that is surplus to bequests with this used to purchase annuities.

Figures 2.1 and 2.2 indicate the luxury bequest effect would induce both higher consumption and demand for annuities. This arises from the reduced need to leave a legacy.

2.3.3 Legacy amount comparison

As will be recalled, Richard's model provides the insight that agents buy life insurance when the legacy amount, $L(t)$, exceeds wealth, $W(t)$; the sum assured

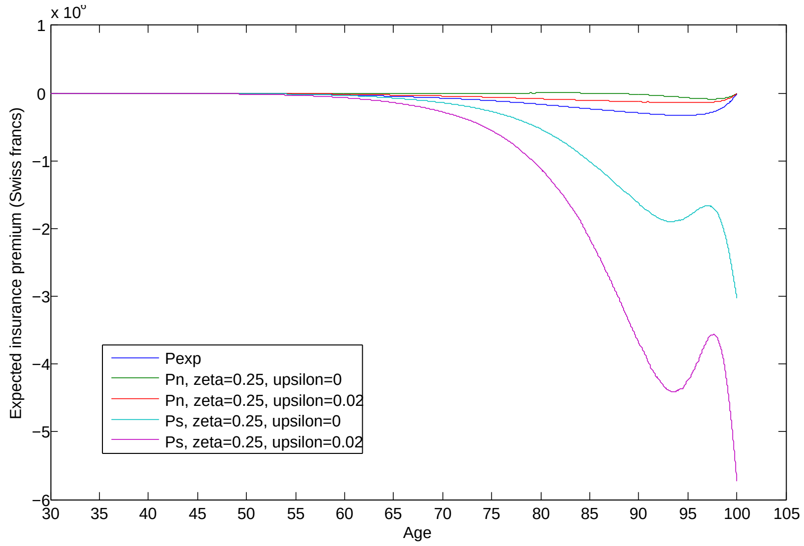

Figure 2.2: Expected insurance premium comparison between exponential discounting case (Pexp in legend) and hyperbolic discounting cases (Pn and Ps in legend)

amount is measured by $L(t) - W(t)$.

Drawing on Purcal and Piggott (2008), we assume that desired legacy (bequest motive) is to leave a payment commencing at the death of the legator and payable to the limiting age of the legatee. As can be seen from figure 2.3, there is a noticeable difference in legacy behaviours between the two forms of time-inconsistent agents studied. When bequests are treated as luxury goods, naïve agents, compared to sophisticated agents, have greater legacy amounts, that is, stronger bequest motives, in the early stage of life together with lower legacy amounts, that is, weaker bequest motives, in the later stage of life. This can be explained by the difference in consumption patterns between naïve and sophisticated agents, particularly as the legacy amount is described in this paper as two-thirds of optimal consumption from current age to the limiting age. When

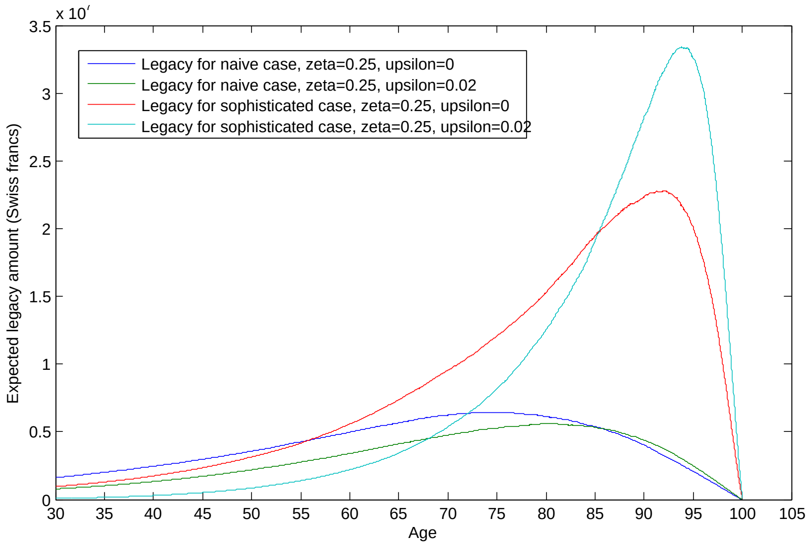

Figure 2.3: Expected legacy amount for naïve case and sophisticated case

bequests are treated as luxury goods, we continue to observe this pattern: in comparison with sophisticated agents, naïve agents have higher legacy amounts in the earlier stage of life while having lower legacy amounts in the later stage of life.

When there are luxury-type bequests, agents are less motivated to provide bequests and prefer to annuitise their wealth—due to the effects of the consumption threshold in the bequest utility function. As a result, the legacy amount is lower for both naïve and sophisticated agents over most of their lifespan compared to the no-luxury-goods case. The exceptions for agents at senior ages arise from superior returns from annuitisation of wealth towards the end of life.

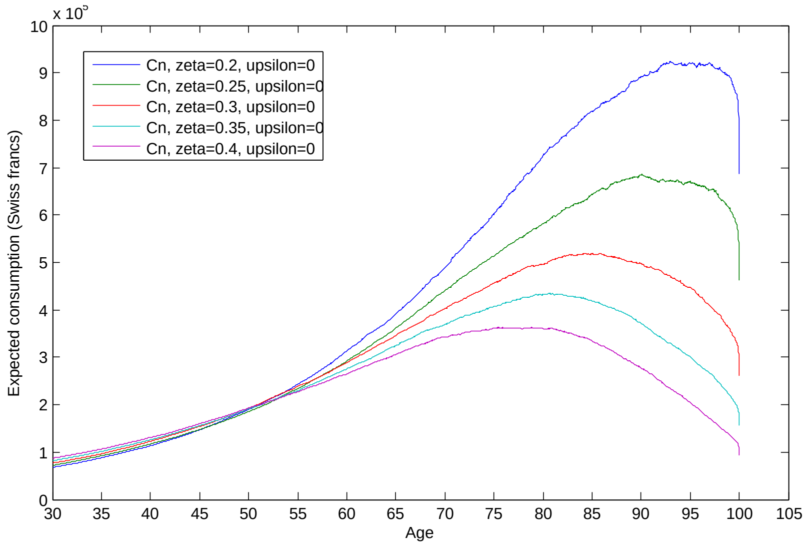

Figure 2.4: Expected consumption for naïve case without luxury-type bequest (Cn in legend)

2.3.4 Expected consumption paths

Equation (2.2.12) links time-inconsistent preferences with the value of the degree of impatience, ζ. Higher ζ values means that agents are more impatient in their behaviours. Hence, with different values of ζ, we can explore the impact of different time-inconsistent preferences on the patterns of consumption and life insurance demand.

Intuitively, we believe that agents with higher impatience are more "short-sighted or myopic", thus, there should be some over-consumption in the earlier life stage. As shown in figure 2.4, when bequests are not luxury goods, consumption is greater among naïve agents with a higher ζ value in the early stage of life. For agents who are less "short sighted or myopic", that is, who have a lower ζ value, consumption increases substantially and, in the later stages of life, overwhelms

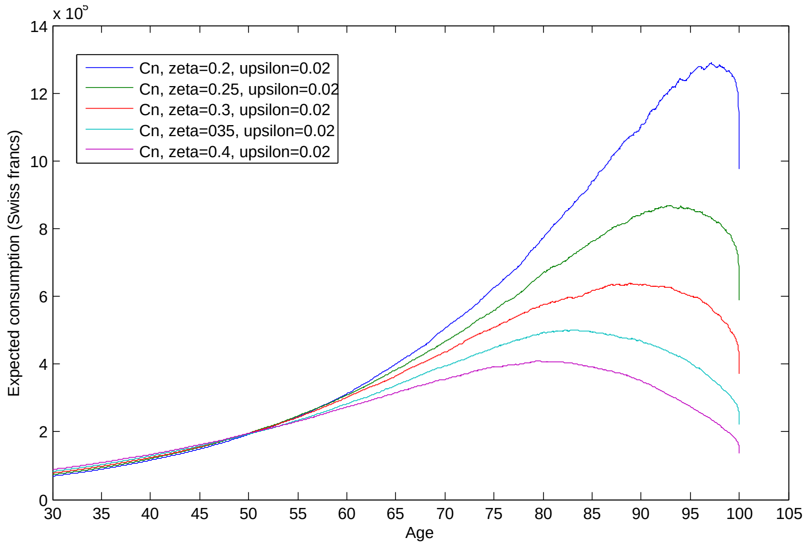

Figure 2.5: Expected consumption for naïve case with luxury-type bequest (Cn in legend)

that of agents with higher ζ values. Hump-shaped consumption is exhibited for some ζ values which matches the empirical data[8].

The consumption of naïve agent with luxury-type bequests is demonstrated in figure 2.5. When bequests are treated as luxury goods, we can still observe similar patterns of consumption. These are now at higher levels, with the peaks almost 60% above their level in figure 2.4.

In figure 2.6 and figure 2.7, sophisticated agents are shown to have patterns different from those of natïve agents. Sophisticated agents with a higher impatience level, that is, a higher ζ value, would consume less in the earlier stages of life and consume more in the later stages whether or not bequests are luxury goods. This phenomenon can be explained by the fact that sophisticated agents have the ability to notice their hyperbolic preference and then deliberately choose to hold

[8]See, for example, Gourinchas and Parker (2002) and Fernández-Villaverde and Krueger (2007).

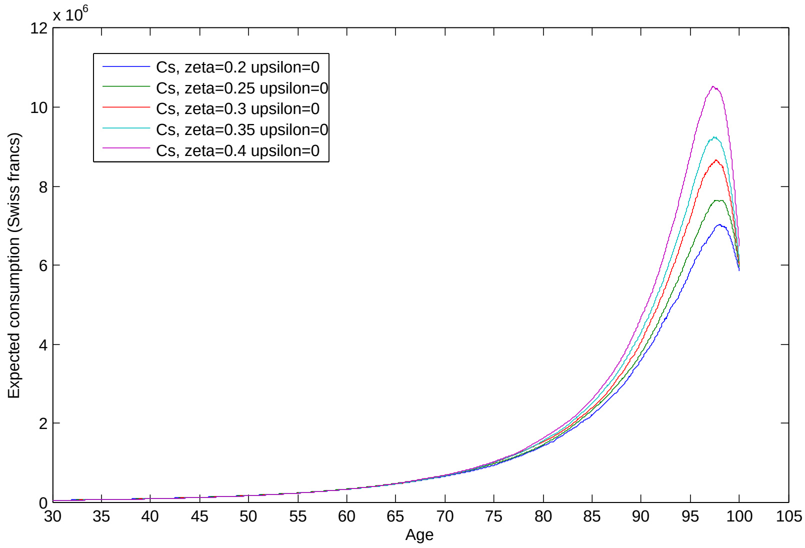

Figure 2.6: Expected consumption for sophisticated case without luxury-type bequest (Cs in legend)

their consumption when they are impatient. However, the impatience impact becomes greater in the later life stage for cases both with or without luxury-type bequests. As mentioned before, when bequests are luxury goods, agents allocate less wealth to bequests and more to consumption. Compared to naïve agents, using time-consistent strategies, sophisticated agents have higher consumption levels[9]. For the sophisticated case, hump-shaped consumption can be observed where the turning point for a decrease is in the range from age 95 to age 100.

[9]The high consumption levels in the later life state that are shown in figure 2.6 and figure 2.7 indicate that agents who survive to that age would like to consume a large proportion of their wealth for utility maximisation.

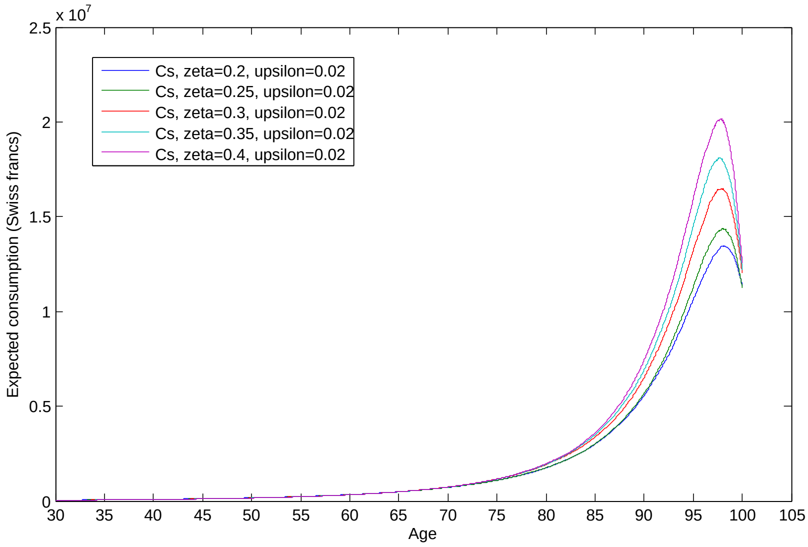

Figure 2.7: Expected consumption for sophisticated case with luxury-type bequest (Cs in legend)

2.3.5 Expected insurance premium paths

The expected insurance premium paths for naïve and sophisticated agents with different ζ[10] and υ values are shown in figures 2.8–2.11. In our model, positive insurance premiums follow from the requirement that the sum assured should cover the legacy cost while negative premiums mean that wealth in excess of the legacy cost is used to purchase annuities to maximise utility. For naïve agents with or without luxury-type bequests, their higher impatience levels, that is, higher ζ values, mean higher life insurance premiums; that is, they require more life insurance payouts. For different ζ values, most of the time, naïve agents without luxury-type bequests do not have the excess wealth needed for annuitisation and require

[10]As stated in Barro (1999), the ζ value should be around 0.5. We choose the ζ value to be either 0.2, 0.25, 0.3, 0.35 or 0.4 for our calculations.

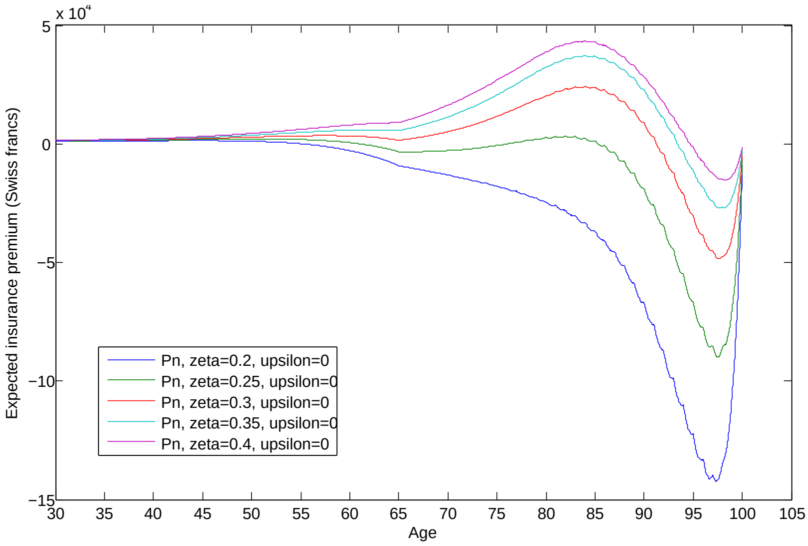

Figure 2.8: Expected insurance premium for naïve case without luxury-type bequest (Pn in legend)

positive insurance premiums, as shown in figure 2.8. However, this requirement is lower when bequests are luxury goods, as shown in figure 2.9. Naïve agents with luxury-type bequests have less bequest motive and more excess wealth and start to purchase annuities for utility maximisation. The transition from working to compulsory retirement for the naïve case both with and without luxury-type bequests are seen in the kinks around age 65 in figures 2.8 and 2.9.

As indicated on figures 2.10 and 2.11, sophisticated agents with or without luxury-type bequests spend part of their wealth to purchase life insurance in their earlier life stage and then annuitise strongly. Larger annuity amounts are associated with sophisticated agents with luxury-type bequests compared to those sophisticated agents without luxury-type bequests. The impacts of luxury-type bequests which increase wealth annuitisation are also identified in figures 2.10

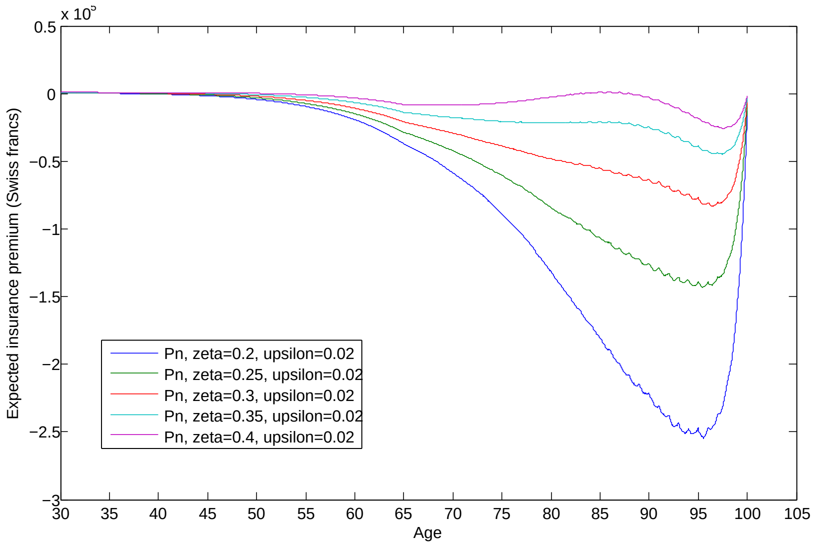

Figure 2.9: Expected insurance premium for naïve case with luxury-type bequest (Pn in legend)

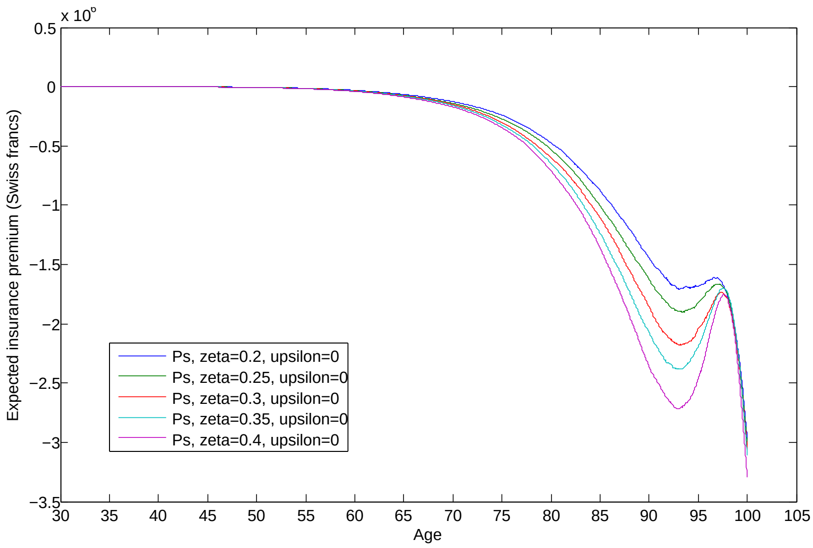

Figure 2.10: Expected insurance premium for sophisticated case without luxury-type bequest (Ps in legend)

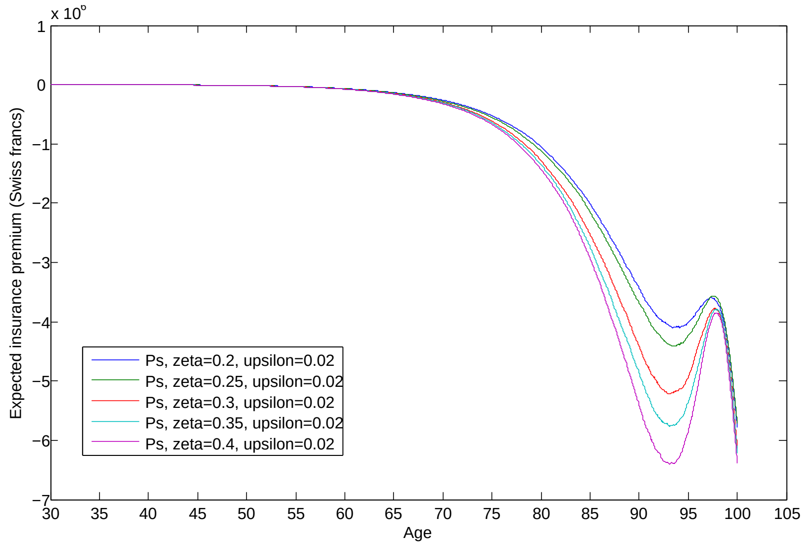

Figure 2.11: Expected insurance premium for sophisticated case with luxury-type bequest (Ps in legend)

and 2.11. The transition from working to retirement for the sophisticated case is also shown in figures 2.10 and 2.11. The non-monotonic patterns of annuitisation, which include the reduction in annuitisation from roughly age 93 to 97, have been observed at the end life in both figures 2.10 and 2.11. We believe that the reduction in annuitisation is due the wealth constraint and increasing consumption during that age range. When the consumption starts to decline after reaching its peak, retirees then have the enough wealth to increase the annuitisation[11] level again which results a peak in premium amount around age 97.

[11]Note that we assume retirees can purchase a variable annuity to annuitise their wealth at any age, that is, the annuity market is complete.

2.4 Conclusion

In this paper, we have sought to provide a novel explanation for the "annuity puzzle". In our view, the "annuity puzzle" and its exceptions results from the interaction of several factors. The combined effects of two factors, time-inconsistent preferences and luxury-type bequest motives, are examined in our numerical demonstration.

We extend Richard's model to explain the thinness of the voluntary annuity market and the high demand for voluntary annuities in Switzerland. One of the explanations for the "annuity puzzle" is the existence of agents' time-inconsistent behaviours of the agents. Here, we use the hyperbolic discounting factor to capture time-inconsistent behaviours.

Note that there are other influences, such as psychological bias, trust of the annuity provider and public social security[12], on agent's choice of annuitisation which is not captured in out model.

Marín-Solano, Navas and Roch (2013) have also studied the optimal strategies for agents with time-inconsistent behaviours using the Richard model. In their work, a simplified discounting function[13] is used in order to obtain analytic solutions for sophisticated agents. Compared to their work, we use a more theocratically appropriate discount function and then have developed a numerical solution

[12]Please refer to Avanzi (2010), Brown and Diamond (2005) and Friedman and Warshawsky (1990).

[13]In fact, they use a a rate of simple discount, not compound discount. Our approach uses a more appropriate compound discount approach.

based on hyperbolic discounting. To obtain a more generalised result and further explain the exceptions to the "annuity puzzle", we further extend Richard model by allowing bequests to be luxury goods. This generalisation of our model also brings us closer to reality. In the real world, agents are found to treat bequests as luxury goods, as demonstrated by the work of Lockwood (2012).

In relation to agent behaviour, we examine two extreme cases: naïve and sophisticated. The dynamic programming equations of both cases have been derived in the paper. We then use numerical schemes and simulation of the wealth path, that is, the state variable, to obtain the numerical results.

Our results show different behaviours between naïve agents and sophisticated agents in terms of consumption and insurance demand. The behaviour of naïve agents with or without luxury-type bequests modelled in this paper demonstrates over-consumption in earlier life stages and under-consumption in later life stages compared to sophisticated agents. Demand for life insurance to fulfil the bequest motive is shown for naïve agents without luxury-type bequests. Hump-shaped consumption has also been exhibited in the naïve case with or without luxury-type bequests. In contrast, the behaviour of sophisticated agents results in reasonable consumption, enough wealth for the bequest motive and for annuitisation of excess wealth to maximise utility. The sophisticated agents in the model are aware of their inconsistent behaviours and impatience level and would accordingly adjust their strategy.

From our results, the model of naïve agents can be used to replicate the behaviours of the real-world agents. Based on the empirical data of some countries, for example, Japan[14] or Australia[15], agents are observed to have hump-shaped consumption and they are not intending to purchase voluntary annuities. These phenomena are shown in the behaviours of naïve agents without luxury-type bequests[16]. Based on the empirical data of some countries, for example, Switzerland, agents are observed to have hump-shaped consumption and yet they annuitise wealth, which is reflected in the behaviours of naïve agents with luxury-type bequests. We can conclude that, like the naïve agents, agents in the real world could have time-inconsistent behaviours. Future research can use our results to further study the "annuity puzzle".

[14]See Abe, Inakura and Yamada (2007).

[15]See Beech, Dollman, Finlay and LaCava (2014).

[16]Note that there are many other factors can impact agents' behaviours. Our model just provides another explanation for the discrepancy in agents' annuitisation behaviours among different countries.

Chapter 3

Paper 2

Optimal Time to Enter a Retirement Village

Jinhui Zhang[1 2] , Sachi Purcal[1], Jiaqin, Wei[3]

Abstract We consider the financial planning problem of a retiree wishing to enter a retirement village at a future uncertain date. The date of entry is determined by the retiree's utility and bequest maximisation problem within the context of uncertain future health states. In addition, the retiree must choose optimal consumption, investment, bequest and purchase of insurance products prior to her full annuitisation on entry to the retirement village. A hyperbolic absolute risk-aversion (HARA) utility function is used to allow necessary consumption for basic living and medical costs. The retirement village will typically require an

[1]Department of Applied Finance and Actuarial Studies, Faculty of Business and Finance, Macquarie University, Sydney, Australia.
[2]Corresponding author: `jinhui.zhang@students.mq.edu.au`
[3]School of Statistics, Faculty of Economics and Management, East China Normal University, Shanghai, China

initial deposit upon entry. This threshold wealth requirement leads to exercising the replication of an American put option at the uncertain stopping time and results in an increasing proportion invested in the risky asset over time. From our numerical results, active insurance and annuity markets are shown to be a critical aspect in retirement planning.

Keywords: Retirement village, Optimal control, Optimal stopping, HARA, American put option, Long-term care needs, costs and products for the elderly, Disability/health state transitions, Life-cycle modelling related to the retirement phase.

3.1 Introduction

With the reduced mortality rate, life expectancy is continuing to increase globally (World Health Organization, 2015). In next 40-50 years, the percentage of people aged over 60 years will nearly double all over the world. People are predicted to have longer lives and extended retirement living.

Australia has one of the longest life expectancies in the world, that is, 79.7 years for males and 84.2 years for females (Australian Institute of Health and Welfare, 2013). With the growing ageing population, Australia is now facing a more profound ageing problem. The potential impact includes economy stagnation, high demand for pensions and increased aged care spending, which has caught Australian Government's attention (Australian Government, 2004).

As reported by the Australian Institute of Health and Welfare (2014*a*), 28.31% of the population aged 65 or over receive aged care services. This requires recurrent annual expenditure of more than A$13 billion for the Australian federal, state and territory governments. Almost 70% of the total spending on aged care is allocated to residential aged care services, that is, aged care homes (Australian Institute of Health and Welfare, 2014*b*). The increasing demand for aged care has become a burden for the Australian government. Hence, improving wellness during retirement living has become a more profound topic.

For the growing senior population, retirement villages which are linked with "active ageing" and "community support" present an alternative high-quality retirement living option. A retirement village or retirement community can be defined as an organised residential place with a certain level of service for a voluntary age-specified retired or partially retired person (Glass and Skinner, 2013). The retirement village should provide its residents with shared activities and facilities in a community that offers secured living (Bernard et al., 2007).

In the United States, a retirement village is usually called as retirement community. According to the size, scale, location, and facilities and activities provided, the retirement community can be classified into different categories, such as senior apartments, continuing-care retirement communities, leisure-oriented retirement communities, congregate housing, etc. (Glass and Skinner, 2013). In the United Kingdom (UK), the retirement village is now growing as a new growing long-term residential option for retirees (Bernard et al., 2007).

It is well documented in the literature that residing in a retirement village can improve well-being. Factors that contribute to well-being include community facilities, accessibility features and 24-hour emergency assistance (Retirement Living Council, 2015*b*), social contact (McDonald, 1996; Buys, 2000), living independence (Kingston et al., 2001) and organised group activity and exercise (Lord et al., 2003).

The Retirement Living Council (2015*a*) states that currently over 177,000 seniors aged 65 and over (i.e., only 5% of the total number) reside in an Australian retirement village. However, as stated by the Australian Bureau of Statistics (2013*a*), males have recently been stated to close the life expectance gap. This prevailing tendency implies that retirees are expected to live a longer time as a part of a couple. As an alternative retirement living option for a spouse, retirement villages would attract more demand (Glass and Skinner, 2013).

Optimal strategies of consumption and investment have been studied in the life cycle model literature, while modelling the optimal strategies for retirees can help us to achieve a clear vision of the financial problems associated with ageing. Merton (1969, 1971) developed a well-known optimal asset allocation and consumption model for an investor with a fixed lifetime. In the model, utility is measured by a constant relative risk-aversion (CRRA) function and is maximised by the investor to determine her optimal strategy. Ding, Kingston and Purcal (2014) used put option replication to create a wealth threshold in Merton's model to allow for a luxury bequest. Noting the conclusion from Yaari (1965) that investors benefit from a life annuity, Merton's model was extended in Richard (1975) in which investors were assumed to have a stochastic lifetime and access to the purchase of insurance products, that is, life insurance and life annuities.

Within the framework of Merton's model, Milevsky and Young (2007) studied an optimal stopping problem for investors seeking a once-and-for-all annuitisation. Kingston and Thorp (2005) extended the work of Milevsky and Young (2007) to the more general case of hyperbolic absolute risk-aversion (HARA) utility.

Health status is another aspect which impacts on financial decision. Rosen and Wu (2004) showed that self-rated health status is a profound indicator for portfolio choice. Bernheim, Shleifer and Summers (1985) studied the circumstances under which health status can initiate bequest motives. Edwards (2008) explored the link between health status and portfolio selection. Specifically, in Edwards (2008), the decline of financial risk observed after investors' retirement is partially explained by investors' health risk which usually increases along with age. Furthermore, the existence of medical costs associated with their health risk can vary retirees' financial strategy. Retirees who pay out-of-pocket medical costs consequently have less wealth (Yogo, 2016) and tend to save more (De Nardi, French and Jones, 2010).

With the growing numbers of retirees choosing to use retirement village as their living option (Hu et al., 2017), we are interested in examining the optimal retirement strategies for retirees seeking long-term retirement village living. Our research takes into account retirees' health care needs over the life cycle, since medical costs related to the health state can impact retirees' annuitisation levels[4]. Since there are various types of retirement living options with different services provided, the retirement village fee schemes are complicated (Kyng and Stolz, 2016; Hu et al., 2017; Jones et al., 2010). For certain retirement living options, we use our model to link the entry deposit to prudent investment behaviour and financial instruments that can be harnessed to improve financial outcomes.

This arising ageing problem provided us with the motivation to develop a life-cycle model involving retirement living choices while considering asset allocation, consumption, bequests and insurance purchase, thus contributing to our understanding of the optimal financial behaviour of retirees. In our model, retirees are found to have an increasing proportion of wealth invested in risky assets in line with their increasing age when there is a wealth requirement threshold to enter a retirement village. This increasing proportion trend during retirement is also noted in Kingston and Fisher (2014), Ding, Kingston and Purcal (2014) and Pfau and Kitces (2013). By allowing for dynamic health states, our model can also be more attuned to ageing problems.

In this paper, we study retirees' optimal strategies for different cases in section 3.2. Numerical demonstrations are presented in section 3.3, followed by the conclusion in section 3.4.

[4]Reichling and Smetters (2015) studied the impacts of health shocks that are related to mortality risk and related lump sum medical costs on agents' annuitisation decisions.

3.2 Model and Method

We assume that risky assets available in the market follow the geometric Brownian motion:

$$dX_t = \alpha X_t dt + \sigma X_t dB_t, \tag{3.2.1}$$

where α and σ are the expected return rate and volatility of the risky assets X_t and B_t is the standard Brownian motion.

In this paper, we use a HARA utility function for consumption, that is,

$$U_1(C) = \frac{(C-h)^\gamma}{\gamma},$$

where C is consumption, h is consumption of necessities for basic living (not including medical costs) and γ is a constant that reflects the individual's level of risk aversion.

With inspiration from Haberman and Pitacco (1998), we assume the retiree's health status is stochastic and is modelled by a continuous Markov chain process with the transition matrix shown as follows

$$\mathbf{Q} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}, \tag{3.2.2}$$

where q_{11} is the intensity of staying in a healthy state, q_{12} is the intensity of becoming sick from a healthy state, q_{21} is the intensity of recovery from a sick state to a healthy state and q_{22} is the intensity of staying in a sick state. Here state 1 represents a healthy condition and state 2 represents a sick condition. For the homogeneity case, we have the transition probability of staying healthy, being sick from a healthy state, recovery from being sick to a healthy state and staying in a sick state, that is, $\tilde{P}_{11}$, $\tilde{P}_{12}$, $\tilde{P}_{21}$ and $\tilde{P}_{22}$

$$\tilde{P}_{11}(t,T) = \frac{1}{q_{12}+q_{21}}[q_{21} + q_{12}e^{-(q_{12}+q_{21})(T-t)}] \tag{3.2.3}$$

$$\tilde{P}_{12}(t,T) = 1 - \tilde{P}_{11}(t,T) \tag{3.2.4}$$

$$\tilde{P}_{21}(t,T) = 1 - \tilde{P}_{22}(t,T) \tag{3.2.5}$$

$$\tilde{P}_{22}(t,T) = \frac{1}{q_{12}+q_{21}}[q_{12} + q_{21}e^{-(q_{12}+q_{21})(T-t)}], \tag{3.2.6}$$

where $\tilde{P}_{ij}(t,T)$ is the transition probability from state i to state j with time interval (t,T).

In our model, a known distribution is assumed to describe the lifetime of retirees. The density function of mortality $f_x(t)$ is defined as follows,

$$f_x(t) = \mu(t) \cdot S(t),$$

where $\mu(t)$ is the force of mortality and $S(t)$ is the survival probability.

Further, retirees are assumed to have short-sighted or myopic vision about their future health state. That is, although their health state can continually change, reflected in the modelling above, our myopic agents make their plans assuming their current health state will continue indefinitely into the future. We make this assumption to reduce the complexity of our already complex model. Allowing retirees to plan their future while being aware of future health changes is recognised as a mathematically difficult problem (Guo, 2001), and we leave this task for future research. Unlike Milevsky and Young (2007), we do not explore asymmetric information between insured/annuitant and insurer, and so we assume insurers share the retirees' myopia.

With an assumed two-state health stochastic process $\{H(s),\ t \leq s \leq \tau\}$, the presence of myopic retirees making financial plans at some time t implies

$$\mu(s) = \begin{cases} \mu_1(s), & H(t) = h_t = 1,\ t \leq s \leq \tau \\ \mu_2(s), & H(t) = h_t = 2,\ t \leq s \leq \tau \end{cases} \tag{3.2.7}$$

with its corresponding $S(s)$, and so myopia implies $H(s) = h_t,\ t \leq s \leq \tau$, for some deterministic maximum age τ, and where h_t is the realisation of the individual's current health state at time t, $\mu_i(s)$ and its corresponding $S_i(s)$ are the future force of mortality and survival probability at time s for health state i. In this paper, state 1 represents the healthy state and state 2 represents the sick state. Hence, $\mu_1(s) \leq \mu_2(s)$ and $S_1(s) \geq S_2(2)$.

3.2.1 Case 1: no bequest and incomplete insurance market

In Australia, several types of housing are offered by retirement villages (Kyng and Stolz, 2016; Jones et al., 2010). One common type is the serviced apartment offered by a lease contract, such as, leasehold discussed in Hu, Xia, Skitmore, Buys and Zuo (2017). These apartment-type residential options for seniors can also be found in other countries, as the UK and the United States. For this case, the retirees rent the apartment on a pay-as-you-go basis to move into a retirement village. According to Iskhakov, Thorp and Bateman (2015), an owner-occupied house asset can be treated as a bequest. As retirees do not own the residential property in the retirement village, we assume that these retirees have no bequest motive. Consequently, as these retirees have no bequest motive they have no need to access the insurance market prior to their full annuitisation on their entry to the retirement village.

Meanwhile, retirees are assumed to maximise their utilities by consumption and investment before the optimal time $\tilde{\tau}$, that is, the chosen optimal time to enter retirement villages. At time $\tilde{\tau}$, retirees without a bequest motive would use all their remaining wealth to purchase a life annuity at the time they enter retirement villages.

Therefore, the value function of this optimal problem is as follows:

$$
\begin{aligned}
V &= \max_{\pi,C,\tilde{\tau}} E\Big\{ \int_t^{\tilde{\tau}} \frac{S(s)}{S(t)} e^{-\rho(s-t)} U_1(C(s))ds + \int_{\tilde{\tau}}^{\tau} \frac{S(s)}{S(t)} e^{-\rho(s-t)} U_1(\frac{W(\tilde{\tau})}{\bar{a}_{\tilde{\tau}}})ds \Big| H_t \Big\} \\
&= \max_{\pi,C,\tilde{\tau}} E\Big\{ \int_t^{\tilde{\tau}} \frac{S(s)}{S(t)} e^{-\rho(s-t)} U_1(C(s))ds + e^{-\rho(\tilde{\tau}-t)} \frac{S(\tilde{\tau})}{S(t)} \bar{a}_{\tilde{\tau}} U_1(\frac{W(\tilde{\tau})}{\bar{a}_{\tilde{\tau}}}) \Big| H(s) = h_t,\ t \le s \le \tau \Big\},
\end{aligned}
\tag{3.2.8}
$$

with the wealth dynamics as

$$
dW(t) = (rW(t) - D(t)W(t) - C(t) + (\alpha - r)\pi(t)W(t))dt + \sigma\pi(t)W(t)dB_t,
$$

where t is the starting age, $\pi(t)$ is the proportion of total wealth invested in risky assets, $\bar{a}_t = \int_t^{\tau} \frac{S(s)}{S(t)} e^{-(r-D(t))(s-t)} ds$ is the annuity function which is net of medical expense and $D(t)$ is the medical cost represented by a percentage of wealth. As with the force of mortality and survival rate,

$$
D(s) = D_{h_t}(s),\ t \le s \le \tau. \tag{3.2.9}
$$

Following Purcal and Piggott (2008) and the discussion in their paper, we set the time preference rate equal to the risk free rate, $\rho = r$.

From Milevsky and Young (2007) and Kingston and Thorp (2005), the optimal stopping time $\tilde{\tau}$ has been proven to be deterministic for CRRA utility and HARA utility. Based on Milevsky and Young (2007) and Øksendal (2003), the variational inequality is shown as follows,

$$(\rho + \mu(t))V \geq V_t + (r - D(t))W(t)V_W + \max_c [U_1(C(t)) - C(t)V_W] + \max_\pi [(\alpha - r)\pi W(t)V_W + \frac{1}{2}\sigma^2\pi^2 W(t)^2 V_{WW}],\ t \in [0, \tilde{\tau}] \quad (3.2.10)$$

and

$$V \geq \bar{a}_t U_1(\frac{W(t)}{\bar{a}_t}),\ t \in (\tilde{\tau}, \tau). \quad (3.2.11)$$

We consider the following ansatz for the value function V,

$$V = \frac{1}{\gamma}(W(t) - \hat{W}(t))^\gamma a(t)^{1-\gamma}, \quad (3.2.12)$$

where

$$\hat{W}(t) = \frac{h}{r - D(t)}(1 - e^{-(r-D(t))(\tau - t)})$$

is the 'floor' or 'protected' wealth, and $r - D(t)$ reflects the continuous compounding rate of interest to give the retirees an income stream covering health costs up to the maximum possible age τ. Such protection is needed as they are assumed to have no access to insurance markets prior to entry to the retirement village. Here, $a(t)$ is defined as an annuitisation factor and further note that $a(t)$ is health state dependent.

We also write $\tilde{W}(t) = W(t) - \hat{W}(t)$ as the difference between wealth and protected wealth which is known as 'surplus' wealth.

The derivatives of the value function are then

$$\begin{aligned} V_t &= \frac{1-\gamma}{\gamma}\tilde{W}(t)^{\gamma}a(t)^{-\gamma}a'(t) + \tilde{W}(t)^{\gamma-1}a(t)^{1-\gamma}h, \\ V_W &= \tilde{W}(t)^{\gamma-1}a(t)^{1-\gamma}, \\ \text{and} \qquad V_{WW} &= (\gamma-1)\tilde{W}(t)^{\gamma-2}a(t)^{1-\gamma}. \end{aligned} \tag{3.2.13}$$

Following Milevsky and Young (2007), and Kingston and Thorp (2005), we can use the first order derivative condition of equation (3.2.10) to show that the optimal consumption $C^*(t)$ and optimal proportion invested in risky assets are

$$\begin{aligned} C^*(t) &= \tilde{W}(t)a(t)^{-1} + h, \\ \pi^*(t) &= \frac{\alpha - r}{\sigma^2(1-\gamma)}\frac{\tilde{W}(t)}{W(t)}. \end{aligned} \tag{3.2.14}$$

and $C^*(t)$ depends on the current health state.

We substitute equations (3.2.12), (3.2.13) and (3.2.14) into (3.2.10) and (3.2.11):

for $t \leq \tilde{\tau}$, we have

$$-1 \geq a'(t) + \frac{1}{1-\gamma}\Big[\gamma r - \gamma D(t) - \rho - \mu(t) + \frac{1}{2}\frac{(\alpha-r)^2\gamma}{\sigma^2(1-\gamma)}\Big]a(t),\ t \in [0, \tilde{\tau}], \tag{3.2.15}$$

while for $t > \tilde{\tau}$, we have

$$a(t) \geq \bar{a}_t,\ t \in (\tilde{\tau}, \tau). \tag{3.2.16}$$

We adopt the hypothesis from Milevsky and Young (2007) which assumes the time before full annuitisation is of the form $(0, \tilde{\tau})$[5]. With this hypothesis, $\tilde{\tau}$ is set to be deterministic and we write $\phi(t)$ as the solution of equations (3.2.15) and (3.2.16) and let $\eta_1(t) = \frac{1}{1-\gamma}\Big[\gamma r - \gamma D(t) - \rho - \mu(t) + \frac{1}{2}\frac{(\alpha-r)^2\gamma}{\sigma^2(1-\gamma)}\Big]$. Hence, for $t \leq \tilde{\tau}$, we have

$$-1 = \phi'(t) + \eta_1(t)\phi(t). \tag{3.2.17}$$

Multiplying equation (3.2.17) by $e^{\int_0^t \eta_1(u)du}$, the equation can be shown as

$$-e^{\int_0^t \eta_1(u)du} = \phi'(t)e^{\int_0^t \eta_1(u)du} + \eta_1(t)\phi(t)e^{\int_0^t \eta_1(u)du}. \tag{3.2.18}$$

[5]Similar to Milevsky and Young (2007), the optimal stopping time in our model for this case is treated as the full annuitisation time—but with the addition that retirees have a dynamic health status and related medical costs. That is, the model of these authors and the model at hand have similar structures and so similar approaches are appropriate.

Integrating equation (3.2.18) from t to $\tilde{\tau}$, we can have

$$\int_t^{\tilde{\tau}} -e^{\int_0^s \eta_1(u)du} ds = \left[\phi(s) e^{\int_0^s \eta_1(u)du}\right]_t^{\tilde{\tau}}$$

and

$$\phi(t) = \bar{a}_{\tilde{\tau}} e^{\int_t^{\tilde{\tau}} \eta_1(u)du} + \int_t^{\tilde{\tau}} e^{\int_t^s \eta_1(u)du} ds. \tag{3.2.19}$$

For $t > \tilde{\tau}$, the solution ϕ is

$$\phi(t) = \bar{a}_{\tilde{\tau}}.$$

To find the optimal stopping time, we can differentiate the value function with respect to $\tilde{\tau}$:

$$\frac{\partial V}{\partial \tilde{\tau}} = \frac{1-\gamma}{\gamma} W^{\gamma}(t) \phi^{-\gamma}(t, \tilde{\tau}) \frac{\partial \phi(t, \tilde{\tau})}{\partial \tilde{\tau}},$$

and noting $\partial \bar{a}_{\tilde{\tau}} / \partial \tilde{\tau} = (\mu(\tilde{\tau}) + \rho) \bar{a}_{\tilde{\tau}} - 1$ and $\partial \phi(t, \tilde{\tau}) / \partial \tilde{\tau} = [(\mu(\tilde{\tau}) + \rho) \bar{a}_{\tilde{\tau}}] e^{\int_t^{\tilde{\tau}} \eta_1(u)du} + \bar{a}_{\tilde{\tau}} e^{\int_t^{\tilde{\tau}} \eta_1(u)du} \eta_1(\tilde{\tau})$, with $W(t)$, $\phi(t)$ and $\bar{a}_{\tilde{\tau}}$ always positive, then

$$\frac{\partial V}{\partial \tilde{\tau}} \propto \mu(\tilde{\tau}) + \rho + \eta_1(\tilde{\tau}) \tag{3.2.20}$$

and the optimal stopping time is given when this expression is zero.

3.2.2 Case 2: with bequest and complete insurance market

The most common housing type offered by retirement villages in Australia is the resident-funded unit, such as, loan/licence title as discussed in Hu, Xia, Skitmore, Buys and Zuo (2017). Retirees need to purchase a licence to reside in the retirement village and can sell the licence when they exit. This type of agreement is similar to a purchase in the real-estate market. Retirees who have a licence to live in a resident-funded unit can be regarded as house owners. Similarly, in the UK, retirees can purchase retirement housing on a leasehold basis[6] or as a property owner. In the United States, it is also common for retirees to purchase properties in leisure-oriented retirement communities for retirement living. Following the assumption by Iskhakov, Thorp and Bateman (2015)—that the owner-occupied house can be treated as a bequest—we can assume that those retirees have bequest motives and access to the insurance market prior to full annuitisation.

In this case, retirees are assumed to have bequest motives from time t to $\tilde{\tau}$. We continue to use the power utility function for the bequest motive U_2

$$U_2(L(t)) = m(t)^{1-\gamma}\frac{L(t)^\gamma}{\gamma},$$

where $L(t)$ is the legacy amount and $m(t) = \frac{2}{3}\int_t^\tau e^{-r(u-t)}du$. In our calculation, we use τ to represent the deterministic maximum age.

We also assume that insurance products, that is, life insurance and annuities,

[6]Retirees need to pay a large amount in upfront fees to live in such community and have the right to re-sell the occupation right of the property.

are available in the market. Before the optimal time to enter a retirement village $\tilde{\tau}$, retirees use consumption, bequests and the purchase of insurance products to maximise their utility. From Richard (1975), the insurance premium is related to $L(t)$ and wealth $W(t)$

$$P(t) = \mu(t)[L(t) - W(t)].$$

At time $\tilde{\tau}$, retirees split their wealth into two parts: $\upsilon W_{\tilde{\tau}}$ and $(1-\upsilon)W_{\tilde{\tau}}$. The first part, $\upsilon W_{\tilde{\tau}}$ is used to purchase lifetime annuity products with this being similar to the behaviour of retirees without a bequest motive. The second part, $(1-\upsilon)W_{\tilde{\tau}}$ is planned to be delivered to their heirs at time $\tilde{\tau}$ as a pre-inheritance. Hence, the value function is

$$\begin{aligned}
V = & \max_{\pi,C,L,\tilde{\tau}} E\Big\{ \int_t^{\tilde{\tau}} \frac{S(s)}{S(t)} e^{-\rho(s-t)} \left[U_1(C(s)) + \mu(s)U_2(L(s))\right] ds \\
& + \int_{\tilde{\tau}}^{\tau} \frac{S(s)}{S(t)} e^{-\rho(s-t)} U_1(\frac{\upsilon W(\tilde{\tau})}{\bar{a}_{\tilde{\tau}}}) ds + \frac{S(\tilde{\tau})}{S(t)} e^{-\rho(\tilde{\tau}-t)} U_2((1-\upsilon)W(\tilde{\tau})) \Big| H(s) = h_t,\ t \le s \le \tau \Big\} \\
= & \max_{\pi,C,L,\tilde{\tau}} E\Big\{ \int_t^{\tilde{\tau}} \frac{S(s)}{S(t)} e^{-\rho(s-t)} \left[U_1(C(s)) + \mu(s)U_2(L(s))\right] ds + e^{-\rho(\tilde{\tau}-t)} \frac{S(\tilde{\tau})}{S(t)} \bar{a}_{\tilde{\tau}} U_1(\frac{\upsilon W(\tilde{\tau})}{\bar{a}_{\tilde{\tau}}}) \\
& + \frac{S(\tilde{\tau})}{S(t)} e^{-\rho(\tilde{\tau}-t)} U_2((1-\upsilon)W(\tilde{\tau})) \Big| H(s) = h_t,\ t \le s \le \tau \Big\}
\end{aligned}$$

with the wealth dynamics

$$dW(t) = (rW(t) - D(t)W(t) - C(t) + (\alpha - r)\pi(t)W(t) - P(t))dt + \sigma\pi(t)W(t)dB_t.$$

The variational inequality is then shown as

$$
\begin{aligned}
(\rho+\mu(t))V \geq V_t + rWV_W - P(t)V_W + \max_{C,L}[U_1(C(t)) + \mu(t)U_2(L(t)) - C(t)V_W] \\
+ \max_{\pi}[(\alpha-r)\pi W(t)V_W + \frac{1}{2}\sigma^2\pi^2 W(t)^2 V_{WW}],\ t \in [0,\tilde{\tau}] \qquad (3.2.21)
\end{aligned}
$$

and

$$
V \geq \frac{(\upsilon\frac{W(t)}{\bar{a}_{\tilde{\tau}}})^{\gamma}}{\gamma}\bar{a}_{\tilde{\tau}} + \frac{((1-\upsilon)W(t))^{\gamma}m(t)}{\gamma},\ t \in (\tilde{\tau},\tau). \qquad (3.2.22)
$$

Similar to the case in section 3.2.1, we have

$$
V = \frac{1}{\gamma}(W(t) - \hat{W}(t))^{\gamma} a(t)^{1-\gamma},
$$

where

$$
\hat{W}(t) = h\int_t^{\tau} \frac{S(s)}{S(t)} e^{-(r-D(t))(s-t)} ds.
$$

For the time $t \leq \tilde{\tau}$, the value function reduces to Richard's model (Richard, 1975) in which the optimal consumption $C^*(t)$, optimal legacy amount $L^*(t)$, optimal proportion invested in risky assets $\pi^*(t)$ and optimal insurance premium

$P^*(t)$ are shown as follows

$$
\begin{aligned}
C^*(t) &= \tilde{W}(t)a(t)^{-1} + h, \\
L^*(t) &= m(t)\tilde{W}(t)a(t)^{-1}, \\
\pi^*(t) &= \frac{\alpha - r}{\sigma^2(1-\gamma)}\frac{\tilde{W}(t)}{W(t)}, \\
\text{and} \qquad P^*(t) &= (L^*(t) - W(t))\mu(t) \\
&= \mu(t)m(t)\tilde{W}(t)a(t)^{-1} - \mu(t)W(t).
\end{aligned}
\tag{3.2.23}
$$

The utility function with optimal consumption and optimal legacy is then shown as

$$
\begin{aligned}
U_1(C^*) &= \frac{\tilde{W}(t)^{\gamma}a(t)^{-\gamma}}{\gamma}, \\
U_2(L^*) &= \frac{m(t)\tilde{W}(t)^{\gamma}a(t)^{-\gamma}}{\gamma}.
\end{aligned}
\tag{3.2.24}
$$

By substituting equations (3.2.12), (3.2.13), (3.2.23) and (3.2.24) into equations (3.2.21) and (3.2.22), for $t \leq \tilde{\tau}$, we have

$$
-(1+\mu(t)m(t)) \geq a'(t) + \Big[\frac{\gamma}{1-\gamma}(r - D(t)) - \frac{1}{1-\gamma}\rho - \mu(t) + \frac{1}{2}\frac{(\alpha - r)^2\gamma}{(1-\gamma)^2\sigma^2}\Big]a(t)
\tag{3.2.25}
$$

and for $t > \tilde{\tau}$, we have

$$a(t) \geq \left[\upsilon^{\gamma}\bar{a}_t^{1-\gamma} + (1-\upsilon)^{\gamma}m(t)\right]^{\frac{1}{1-\gamma}}. \tag{3.2.26}$$

We write ϕ as the solution of a and $\eta_2 = \frac{\gamma}{1-\gamma}(r - D(t)) - \frac{1}{1-\gamma}\rho - \mu(t) + \frac{1}{2}\frac{(\alpha-r)^2\gamma}{(1-\gamma)^2\sigma^2}$. Hence, for $t \leq \tilde{\tau}$,

$$-(1 + \mu(t)m(t)) = \phi'(t) + \eta_2(t)\phi(t). \tag{3.2.27}$$

Multiplying equation (3.2.27) by $e^{\int_0^t \eta_2(u)du}$, it can be shown as

$$-(1 + \mu(t)m(t))e^{\int_0^t \eta_2(u)du} = \phi'(t)e^{\int_0^t \eta_2(u)du} + \eta_2(t)\phi(t)e^{\int_0^t \eta_2(u)du}. \tag{3.2.28}$$

Integrating equation (3.2.28) from time t to $\tilde{\tau}$, the equation can be shown as

$$-\int_t^{\tilde{\tau}} (1 + \mu(s)m(s))e^{\int_0^s \eta_2(u)du}ds = \left[\phi(s)e^{\int_0^s \eta_2(u)du}\right]_t^{\tilde{\tau}}.$$

and

$$\phi(t) = \left[\upsilon^{\gamma}\bar{a}_{\tilde{\tau}}^{1-\gamma} + (1-\upsilon)^{\gamma}m(\tilde{\tau})\right]^{\frac{1}{1-\gamma}} e^{\int_t^{\tilde{\tau}} \eta_2(u)du} + \int_t^{\tilde{\tau}} [1 + \mu(s)m(s)]e^{\int_t^s \eta_2(u)du}ds. \tag{3.2.29}$$

For $t > \tilde{\tau}$, the solution ϕ is

$$\phi(t) = \left[\upsilon^{\gamma}\bar{a}_t^{1-\gamma} + (1-\upsilon)^{\gamma}m(t)\right]^{\frac{1}{1-\gamma}}. \tag{3.2.30}$$

To find the optimal stopping time, we can differentiate the value function with respect to $\tilde{\tau}$:

$$\frac{\partial V}{\partial \tilde{\tau}} = \frac{1-\gamma}{\gamma}\tilde{W}^{\gamma}(t)\phi^{-\gamma}(t,\tilde{\tau})\frac{\partial \phi(t,\tilde{\tau})}{\partial \tilde{\tau}},$$

where

$$\begin{aligned}\frac{\partial \phi(t,\tilde{\tau})}{\partial \tilde{\tau}} &= \eta_2(\tilde{\tau})[\upsilon^{\gamma}\bar{a}_{\tilde{\tau}}^{1-\gamma} + (1-\upsilon)^{\gamma}m(\tilde{\tau})]^{\frac{1}{1-\gamma}}e^{\int_t^{\tilde{\tau}}\eta_2(u)du} + [1+\mu(\tilde{\tau})m(\tilde{\tau})]e^{\int_t^{\tilde{\tau}}\eta_2(u)du}\\ &\quad + \frac{1}{1-\gamma}[\upsilon^{\gamma}\bar{a}_{\tilde{\tau}}^{1-\gamma} + (1-\upsilon)^{\gamma}m(\tilde{\tau})]^{\frac{\gamma}{1-\gamma}}\Big\{\upsilon^{\gamma}(1-\gamma)\bar{a}_{\tilde{\tau}}^{-\gamma}[(\mu(\tilde{\tau})+\rho)\bar{a}_{\tilde{\tau}}-1]\\ &\quad + (1-\upsilon)^{\gamma}[rm(\tilde{\tau}) - \frac{2}{3}]\Big\}e^{\int_t^{\tilde{\tau}}\eta_2(u)du}.\end{aligned}$$

It then follows, noting our approach for case 1 above, that

$$\frac{\partial V}{\partial \tilde{\tau}} \propto \frac{\partial \phi(t,\tilde{\tau})}{\partial \tilde{\tau}} \tag{3.2.31}$$

and so we can determine our optimal stopping time.

3.2.3 Case 3: with bequest, complete insurance market and wealth floor

In addition to resident-funded unit and serviced apartment, some non-profit Australian retirement villages offer a type of housing (e.g., independent living unit) with an entry contribution (Jones et al., 2010). To reside in such place, retirees are required to make a contribution deposit. This deposit might contribute to the maintenance or improvement of a retirement village. In the United States, an entry contribution with monthly fees is a payment option for continuing-care retirement community living. We can treat this contribution requirement as a threshold for the wealth level for retirees to enter a retirement village, that is,

$$W(t) \geq R,$$

where R is the certain level of wealth required for retirees to enter a retirement village. This R can be explained as a combination of the management fee, upfront loading fee of the retirement village or the transaction cost of asset relocation.

Inspired by Ding, Kingston and Purcal (2014), retirees in our model are assumed to dynamically allocate assets to achieve their saving targets. Specifically, we assume that retirees would still follow the optimal strategy of consumption, bequests and entering a retirement village as in case 2 but that they would change the proportion of wealth invested in risky assets to meet the new threshold. Asset allocation paths over the life cycle has been discussed in Kingston and Fisher

(2014).

In letting $W(t)$ can fulfil this requirement, we are inspired by Ding, Kingston and Purcal (2014) and assume that retirees would separate their wealth into two parts: surplus wealth $\tilde{W}(t)$ and protected wealth $\hat{W}(t)$:

$$W(t) = \hat{W}(t) + \tilde{W}(t),$$

where $\hat{W}(t) = h\bar{a}_t$. The protected wealth is used for necessity consumption h, which can be basic living costs and medical costs.

In terms of their surplus wealth, retirees can use it for consumption and bequest purposes. To ensure that $\tilde{W}(t)$ is greater than the certain required level R, retirees can replicate a put option by separating their surplus wealth into two parts:

$$\tilde{W}(t) = \tilde{W}_\kappa(t) + \mathscr{P}(\tilde{W}_\kappa(t), R, t). \tag{3.2.32}$$

The first part $\tilde{W}_\kappa(t)$ is the remaining wealth used for consumption, investment and insurance and the second part is used to replicate an American put option: $\mathscr{P}(W_\kappa(t), R, t))$, with the underlying asset $W_\kappa(t)$ and strike price R.

At the optimal time of entering a retirement village, retirees will then exercise the option to let wealth $W(t)$ have the minimum value R:

$$\tilde{W}(t) = \tilde{W}_\kappa(t) + \max(0, R - \tilde{W}_\kappa(t)) = \max(\tilde{W}_\kappa(t), R).$$

We now define the value function as

$$
\begin{aligned}
V = & \max_{\pi,\tilde{C}_\kappa,\tilde{L}_\kappa,\tilde{\tau}} \left\{ \int_t^{\tilde{\tau}} \frac{S(s)}{S(t)} e^{-\rho(s-t)} \left[U_1(\tilde{C}_\kappa(s)) + \mu(s) U_2(\tilde{L}_\kappa(s)) \right] ds \right. \\
& \left. + \int_{\tilde{\tau}}^{\infty} \frac{S(s)}{S(t)} e^{-\rho(s-t)} U_1(\frac{\upsilon \tilde{W}_\kappa(\tilde{\tau})}{\bar{a}_{\tilde{\tau}}}) ds + \frac{S(\tilde{\tau})}{S(t)} e^{-\rho(\tilde{\tau}-t)} U_2((1-\upsilon)\tilde{W}_\kappa(\tilde{\tau})) \middle| H(s) = h_t,\ t \leq s \leq \tau \right\} \\
= & \max_{\pi,\tilde{C}_\kappa,\tilde{L}_\kappa,\tilde{\tau}} \left\{ \int_t^{\tilde{\tau}} \frac{S(s)}{S(t)} e^{-\rho(s-t)} \left[U_1(\tilde{C}_\kappa(s)) + \mu(s) U_2(\tilde{W}_\kappa(s)) \right] ds + e^{-\rho(\tilde{\tau}-t)} \frac{S(\tilde{\tau})}{S(t)} \bar{a}_{\tilde{\tau}} U_1(\frac{\upsilon \tilde{W}_\kappa(\tilde{\tau})}{\bar{a}_{\tilde{\tau}}}) \right. \\
& \left. + \frac{S(\tilde{\tau})}{S(t)} e^{-\rho(\tilde{\tau}-t)} U_2((1-\upsilon)\tilde{W}_\kappa(\tilde{\tau})) \middle| H(s) = h_t,\ t \leq s \leq \tau \right\}
\end{aligned}
$$

with the wealth dynamics

$$
d\tilde{W}_\kappa(t) = (r\tilde{W}_\kappa(t) - D(t)\tilde{W}_\kappa(t) - \tilde{C}_\kappa(t) + (\alpha - r)\pi(t)\tilde{W}_\kappa(t) - \tilde{P}_\kappa(t))dt + \sigma\pi(t)\tilde{W}_\kappa(t)dB_t,
$$

where $\tilde{C}_\kappa(t)$ and $\tilde{P}_\kappa(t)$ are the consumption and insurance premium at time t by using the surplus wealth $\tilde{W}_\kappa(t)$. The form of the value function is assumed be

$$
V = \frac{1}{\gamma} \tilde{W}_\kappa(t)^\gamma a(t)^{1-\gamma},
$$

in which the solution of a is in equations (3.2.29) and (3.2.30). Then for $\tilde{W}_\kappa(t)$ the optimal consumption $C^*_\kappa(t)$, optimal legacy amount $L^*_\kappa(t)$, optimal proportion invested in risky assets $\pi^*_\kappa(t)$ and optimal insurance premium $P^*_\kappa(t)$ are shown as follows

$$C_\kappa^*(t) = \tilde{W}_\kappa(t)a(t)^{-1} + h,$$

$$L_\kappa^*(t) = m(t)\tilde{W}_\kappa(t)a(t)^{-1},$$

$$\pi_\kappa^*(t) = \frac{\alpha - r}{\sigma^2(1-\gamma)} \frac{\tilde{W}(t)}{\hat{W}(t) + \tilde{W}_\kappa(t)},$$

$$P_\kappa^*(t) = (L_\kappa^*(t) - \hat{W}(t) - \tilde{W}_\kappa(t))\mu(t)$$

and where all controls, apart from $\pi_\kappa^*(t)$, depend on the health state at time t.

To replicate an American put option, we use the delta hedging defined in Huang, Subrahmanyam and Yu (1996),

$$Delta = \frac{\partial \mathscr{P}}{\partial X} = -N(-d_1(X, K, T-t)) - \int_t^T \frac{r}{\sigma\sqrt{2\pi u}} e^{-\frac{\tilde{d}_1}{2}} du, \tag{3.2.33}$$

where $\tilde{d}_1 = \left(\ln \frac{X}{\mathscr{B}} + (r + \frac{\sigma^2}{2})u\right)/\sigma\sqrt{u}$ and $\mathscr{B}(t)$ is defined as the optimal exercise price for underlying asset X.

Based on the definition of $\mathscr{B}(t)$, the dynamics of the American put option price are the same as those for the price of the European put option, when the $S(t)$ is greater than $\mathscr{B}(t)$. Hence, from Black and Scholes (1973), we have

$$\frac{\partial \mathscr{P}}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 \mathscr{P}}{\partial X_t^2} + rS_{\tilde{\tau}} - r\mathscr{P} = 0, \; X \in (\mathscr{B}(t), \infty). \tag{3.2.34}$$

The American put options should be exercised at the strike price K when the $S(t)$

is less than $\mathscr{B}(t)$

$$\mathscr{P}(X(t), K, t) = K - X(t), X \in (0, \mathscr{B}(t)). \tag{3.2.35}$$

The boundary condition of the American put option is

$$\lim_{X(t)\to\infty} \mathscr{P}(X(t), K, t) = 0. \tag{3.2.36}$$

The American put option price $\mathscr{P}(X_t, K, t)$ also has the following conditions at the fixed exercise boundary $\mathscr{B}(t)$,

$$\mathscr{P}(\mathscr{B}(t), K, t) = K - \mathscr{B}(t),\ \frac{\partial \mathscr{P}(\mathscr{B}(t), K, t)}{\partial X} = -1. \tag{3.2.37}$$

At the time of expiration, all unexercised American put options will be exercised or expired. As $\mathscr{B}(\tilde{\tau})$ is the optimal exercise price, the terminal condition is provided by

$$\mathscr{P}(\mathscr{B}(\tilde{\tau}), \tilde{\tau}, K) = 0,\ X \in (\mathscr{B}(\tilde{\tau}), \infty) \text{ with } \tilde{\tau} = 0 \text{ and } \mathscr{B}(0) = K. \tag{3.2.38}$$

To obtain the optimal exercise price $\mathscr{B}(t)$, we use the front fixing finite difference method from Wu and Kwok (1997). We transform the option price $\mathscr{P}(S_t, K, t)$, asset price S_t and the fixed boundary $\mathscr{B}(t)$ respectively, as follows

$$\tilde{\mathscr{P}} = \frac{\mathscr{P}}{K}, \quad \tilde{\mathscr{B}}(\tilde{\tau}) = \frac{\mathscr{B}(t)}{K}, \quad \tilde{X}(t) = \frac{X(t)}{K}, \quad \tilde{K} = \frac{K}{K} = 1,$$

where $\mathscr{P}$ represents the American put option price $\mathscr{P}(X(t), t, K)$ and $\tilde{\mathscr{P}}$ represents the transformed American put option price at time $\tilde{\tau}$ about the underlying asset $\tilde{X}(t)$ and the strike price $\tilde{K}$.

Here the dynamics of $\tilde{\mathscr{P}}$ are described by equations (3.2.34) and (3.2.35) with $K = 1$. Equations (3.2.36), (3.2.37) and (3.2.38) still hold for $\tilde{\mathscr{P}}$, $\tilde{X}$ and $\tilde{\mathscr{B}}$ with $K = 1$.

In Wu and Kwok (1997), a new variable $\tilde{y}$ at time $\tilde{\tau}$ which was introduced to transform the the unknown boundary to a known fixed one is defined as

$$\tilde{y}(t) = \ln \frac{\tilde{X}(t)}{\tilde{\mathscr{B}}(t)}. \tag{3.2.39}$$

The process of $\tilde{y}$ is shown as follows:

$$\begin{aligned}
\tilde{y}(t) &= \ln\tilde{X}(t) - \ln\tilde{\mathscr{B}}(t), \\
d\tilde{y}(t) &= d\ln\tilde{X}(t) - d\ln\tilde{\mathscr{B}}(t) \\
&= \left(r - \frac{\sigma^2}{2} + \frac{\tilde{\mathscr{B}}'(t)}{\tilde{\mathscr{B}}(t)} \right) dt + \sigma dB_t.
\end{aligned}$$

Following Wu and Kwok (1997), the partial differential equation (PDE) of the new variable $\tilde{y}$ is obtained by forming a direct substitution to equation (3.2.34):

$$\frac{\partial \tilde{\mathscr{P}}}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 \tilde{\mathscr{P}}}{\partial y^2(t)} + (r - \frac{\sigma^2}{2})\frac{\partial \tilde{\mathscr{P}}}{\partial y(t)} - r\tilde{\mathscr{P}} + \frac{\tilde{\mathscr{B}}'(t)}{\tilde{\mathscr{B}}(t)}\frac{\partial \tilde{\mathscr{P}}}{\partial y(t)} = 0. \tag{3.2.40}$$

Equation (3.2.40) is the PDE of a transformed American put option price $\tilde{\mathscr{P}}$ with fixed boundary $\tilde{\mathscr{B}}(t)$. Using the finite difference scheme defined in Wu and Kwok (1997), we can explicitly solve equation (3.2.40) and obtain the numerical result for $\mathscr{B}(t)$.

Substituting the $\mathscr{B}(t)$ value into equation (3.2.33), we can obtain the delta value of an American put option. With this delta value, an American put option can be replicated by risky assets in the market.

3.3 Numerical Results

In this paper, we calibrate our parameters to Australian data to obtain numerical results for a starting age of $t = 65$ to a maximum age of $\tau = 109$. Survival probabilities and the force of mortality are from the Australian Government Actuary (2014). In particular, we use the tabulated values from Australian Government Actuary (2014) for $S_1(s)$ and $\mu_1(s)$. To determine survival rates and force of mortality for the sick state, we adopt the frailty model from Su and Sherris (2012). For $S_2(s)$ and $\mu_2(s)$, we simply set $S_2(s) = S_1^u(s)$ and $\mu_2(s) = u \times \mu_1(s)$, where u is defined as a frailty factor and is assumed to be a constant here.

The risky return rate, $\alpha = 8.112\%$, and volatility of risky assets, $\sigma = 0.15685$, are based on the 5-year average rate (from 2009 to 2014) of the ASX 200 (http://www.asx.com.au/). We use the average of five years' of cash rates (from 2009 to 2014) from the Reserve Bank of Australia (http://www.rba.gov.au/statistics/cash-rate/) as our risk free rate, that is, $r = 3.4\%$ p.a.[7]. As was done by Milevsky and Young (2007) and Kingston and Thorp (2005), we set the rate of time preference to be equal to the risk-free rate, $\rho = r$. The average annual income, $Y =$ AUD\$47 736, is from Australian Bureau of Statistics (2013*b*). Retirees in our model are assumed to have total wealth of $10Y$ from previous savings and have no future income. Following Purcal and Piggott (2008), the risk-aversion parameter γ is set to be -0.5. In this paper, retirees with bequest motives are assumed to use 80% of their wealth, $\upsilon = 0.8$[8], to annuitise and use the rest as a pre-inheritance disbursement at the time of entering the retirement village. In this paper, we set the frailty factor u to be 1.2[9]. Medical costs are assumed to be 1% of total wealth for agents in the healthy state and be 2% of total wealth for agents in the sick state, that is, $D_1 = 0.01$ and $D_2 = 0.02$, respectively. With expenditure as estimated by Australian Bureau of Statistics (2011), the necessary consumption amount h is set to be AUD\$12 000 per annum. As mentioned above,

[7]Note that the cash rate is the overnight money market interest rate determined by the Reserve Bank of Australia.

[8]Following consideration of the expected wealth levels at stopping times for all cases, we set $\upsilon = 0.8$ to ensure that retirees have the enough money for future living. Higher values of υ imply higher annuitised income stream; conversely, low values of υ are associated with meager future income streams. Having regard to retirement village implied rents, as discussed in Kyng and Stolz (2016), a value of $\upsilon = 0.8$ appears appropriate for current Australian retirement village conditions.

[9]Note that a higher frailty factor in the sick state is linked with lower level of wealth, consumption and annuitisation. We have chosen a value of 1.2 as illustrative–this level produces behaviours which are clearly distinguished from the healthy state. We have not determined this value by calibrating to data at this time and leave this task for future research.

we set the maximum survival age to 109.

Table 3.1: Parameters used in the numerical simulation

t=65	τ=109
q_{12}=0.04	q_{21}=0.4
α=0.08112	r=0.034
ρ=0.034	σ=0.15685
Y=AUD$47,736	γ=−0.5
υ=0.8	u=1.2
D_1=0.01	D_2=0.02
h=AUD$12,000 p.a.	τ=109

In our numerical demonstration, three cases are studied. For the serviced apartment case (case 1), there is no bequest motive and agents have no access to the insurance market prior to entering the retirement village; retirees can purchase neither life insurance nor a variable annuity. Retirees are assumed to be fully annuitisated (purchase of a fixed annuity) at the time of entering the retirement village. For the resident-funded unit case (case 2) and the early contribution unit case (case 3), retirees have bequest motives and can purchase life insurance or a variable annuity in the insurance market prior to entering the retirement village. In addition, retirees are assumed to leave part of their wealth as a pre-inheritance disbursement and use the rest for full annuitisation when entering the retirement village. Furthermore, in the entry contribution case (case 3), a minimum wealth requirement is a prerequisite for retirement village entry. These retirees are then assumed to replicate an American put option to clear this financial hurdle.

We present the expected consumption path for case 1 in Figure 3.1. From the plot, we see the expected consumption path is hump-shaped—similar to con-

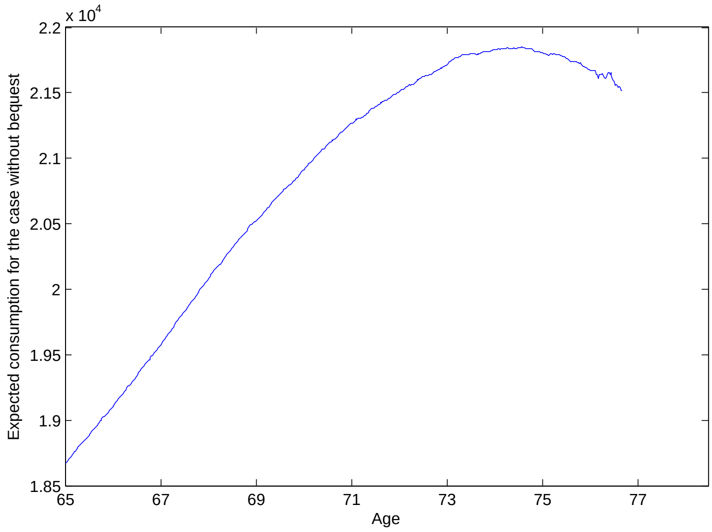

Figure 3.1: Expected consumption path for case 1 retirees, starting in the healthy state at age 65 with total wealth of $10Y$, and truncated at the optimal case 1 stopping times. This captures the expected consumption outcomes of agents with no bequest motive. Note that these agents have no access to insurance markets, and are assumed to purchase a term certain annuity to protect their basic consumption needs—which is much more expensive than a life annuity, particularly at older ages.

sumption observed in empirical studies (Gourinchas and Parker, 2002; Fernández-Villaverde and Krueger, 2007). This phenomenon can be attributed to both market incompleteness (lack of access to insurance markets) and low wealth levels in the later life stages.

In figure 3.2, the expected consumption for cases 2 and 3 rises in line with increasing age. Due to uncertainty arising from the unknown future health state, the market is not entirely complete and thus expected consumption is slightly convex. Compared to figure 3.1, figure 3.2 reflects the ability of retirees in cases 2 and 3 who have bequest motives to spend more on consumption as they have access to an active insurance market to carry out annuitisation or to purchase insurance. Figure 3.2 also shows that retirees in case 3 have less consumption than those in case 2, due to the cost of replication of the American put option to ensure they can clear the wealth hurdle required for entry.

Our calculations indicate health changes impact optimal consumption decisions. As one would expect, agents in the poorer health state consume more than those in the healthy state[10].

We display the expected wealth path for cases 1, 2 and 3 in figure 3.3. For most of time, retirees in case 1 are in possession of more expected wealth than those in case 2 and case 3. As there is no active insurance market in case 1, self-insurance due to precautionary motives is found to be another driver for holding wealth (Ameriks et al., 2011). Hence, figure 3.3, suggests retirees tend to draw on their wealth more cautiously when there is no active insurance market. Moreover, the wealth floor requirement in case 3 demands more outgoes and results in less wealth.

The expected insurance premiums for life insurance or receipt of variable annuity income for cases 2 and 3 are displayed in figure 3.4. A positive or negative premium value is linked to the demand for life insurance or a variable annuity, respectively. In figure 3.4, retirees in cases 2 and 3 are shown to purchase a variable annuity in order to maximise utility. Compared to those in case 2, retirees in case 3 have a lower annuitisation amount, reflecting the resources they have to

[10]Agents in the poorer health state have less incentive for partial annuitisation but higher consumption desires prior to the entering to a retirement village, since their expected lifespan is shorter.

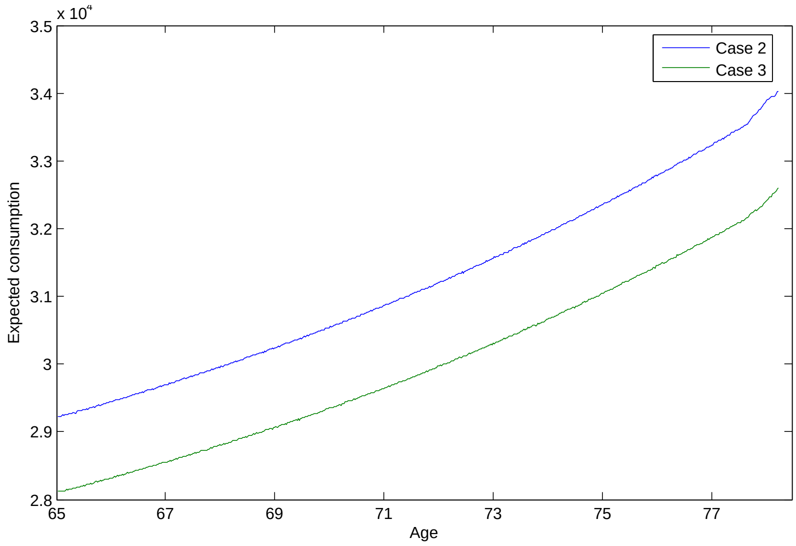

Figure 3.2: Expected consumption, truncated at optimal stopping times, for case 2 and case 3 agents commencing at age 65 in the healthy state with total wealth of $10Y$. The figure captures the expected consumption paths for agents with bequest motives. These agents, in contrast to case 1, have access to perfect insurance markets.

put toward replicating the American put option to secure their retirement village entry.

We calculate the proportions of total wealth in risky assets for cases 2 and 3, and display the expected paths of the proportion of surplus wealth, $\tilde{W}$, invested in the risky assets in figure 3.5. Retirees in case 2 appear to invest a constant proportion of surplus wealth in risky assets, very much in line with the Merton ratio Merton (1969, 1971). Indeed, the high values seen are characteristic of the Merton ratio for the parameters chosen and also reflect the lack of short-selling/borrowing restrictions in the modelling. The situation is very different for retirees in case 3 who are target savers and who are assumed to replicate an American put option to meet their target. These retirees, who want to hedge

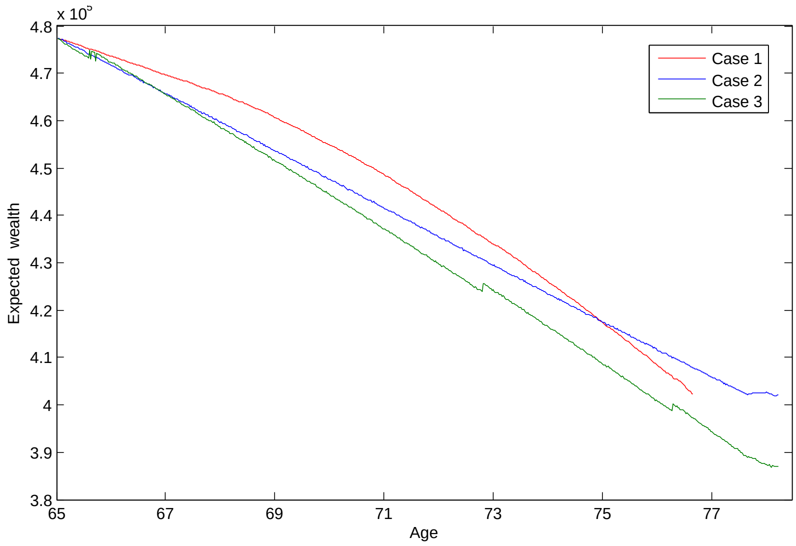

Figure 3.3: Expected wealth, truncated at optimal stopping times, for case 1, 2 and 3 agents commencing at age 65 in the healthy state with total wealth of $10Y$. Recall case 1 agents have no access to insurance markets, while case 3 agents replicate an American put option to ensure their savings target is met.

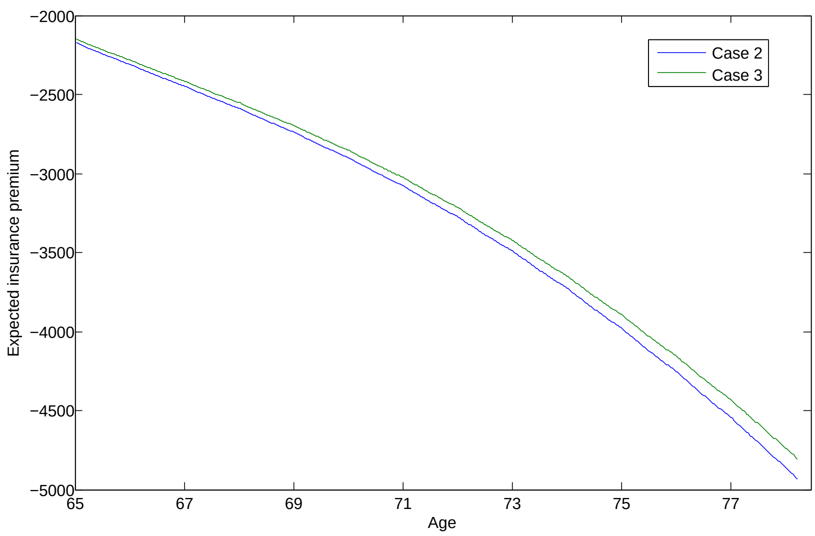

Figure 3.4: Expected insurance premiums paid by case 2 and case 3 agents, truncated at optimal stopping times, for those commencing at age 65 in the healthy state with a total wealth of $10Y$. Negative insurance premiums mean the agents are receiving funds from the insurers, that is, they are in receipt of an annuity.

risk, are encouraged to have an increasing risk exposure while they are ageing [11]. Interestingly, in an empirical study Shum and Faig (2006) find that retirees in case 3, who have a retirement saving target, have higher levels of investment in risky assets. In figure 3.5, the proportion of surplus wealth for case 3 rises along with age, producing a convex shape. The trend in figure 3.5 is similar to that reported in the study by Ding, Kingston and Purcal (2014), in which retirees are assumed to replicate a European put option for their wealth requirement.

Health changes are seen not to impact investment decisions for our health myopic agents as we chose a level of risk aversion, γ, that was constant between health states. It should be clear from our myopic health modelling above that if this value differed between states then this would lead to investment behaviour that differed between states. That is, if investors were more risk averse in the sick state, then they would also invest less in the risky assets (compare Merton ratios).

We also test the impacts of some variables on optimal stopping times. As shown in table 3.2, we try different risk-aversion parameter values for case 1, that is, no bequest motive and an incomplete insurance market, and case 2, that is, with bequest motives and a complete insurance market, respectively. With an increasing risk-aversion level for both cases, retirees are shown to be more afraid of potential risks in the markets and prefer an earlier stopping time. The stopping times for case 2 are more sensitive to change in the risk-aversion parameter value. This phenomenon can be explained by the extra risk aversion generated by the

[11] Retirees are also found to use increasing risk exposure to hedge against risk in other studies, such as Hulley et al. (2013) and Thorp, Kingston and Bateman (2007).

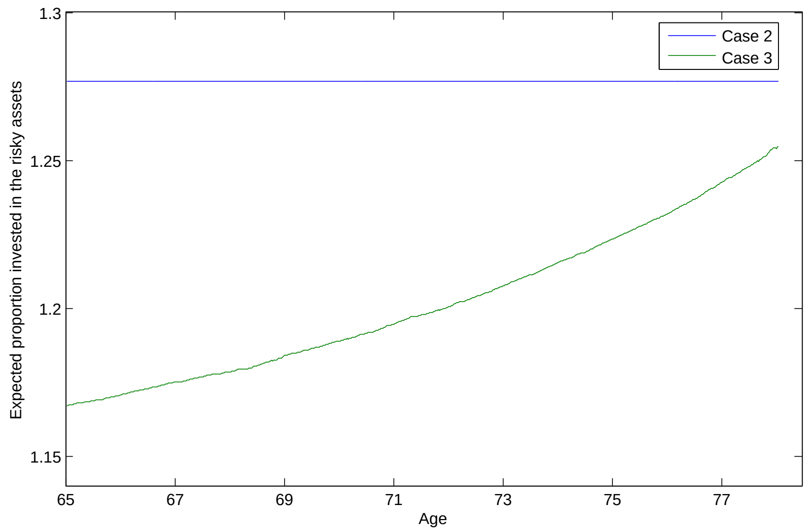

Figure 3.5: Expected proportion of surplus wealth, $\tilde{W}$, invested in the risky assets, or $\pi^* W/\tilde{W}$, by agents starting at age 65 in the healthy state with a total wealth of $10Y$. The expected paths are truncated at the optimal stopping times for case 2 and case 3, respectively. The differing behaviour of the case 3 target savers is clear.

bequest motive utility function.

Table 3.3 shows the results of our tests on the impact of excess returns, $\alpha - r$, on the stopping time for case 1, that is, no bequest motive and an incomplete insurance market, and case 2, that is, with bequest motives and a complete insurance market, respectively. As we expected, higher excess returns are more attractive to retirees and defer the stopping time for both cases. This trend can be also found in Kingston and Thorp (2005).

The impact of volatility, σ, on the stopping time for case 1 and case 2 is demonstrated in table 3.4. For both two cases, retirees are seen to enter the retirement village earlier when the market is more volatile.

As shown in table 3.5, we also study the impact of the frailty factor on the

Table 3.2: Expected stopping times by level of risk aversion for case 1 and case 2 agents aged 65, in the healthy state, and with total wealth of $10Y$—and other parameters as given in table 3.1. Increasing levels of risk aversion lead to falling stopping times. Also, the bequest motives of case 2 agents produce results much more sensitive to the level of risk aversion. Indeed, at $\gamma = -0.5$ case 2 agents abandon their conservative behaviour and embrace the risky investment environment.

	Expected stopping time (years)	
Gamma	Case 1	Case 2
-0.5	11.65	13.22
-0.6	11.04	10.53
-0.7	10.56	8.28
-0.8	10.04	6.23
-0.9	9.54	4.50
-1	9.13	2.99

Table 3.3: Expected stopping times by level of equity premium for case 1 and case 2 agents aged 65, in the healthy state, and with total wealth of $10Y$—and other parameters as given in table 3.1. Increasing the equity premium results in longer stopping times, as agents exploit the more profitable investment environment. The situation illustrated is for agents with risk aversion of $\gamma = -0.5$, where case 2 agents are less conservative than case 1 agents. With more risk averse agents, this boldness of agents with bequest motives over those without is reversed.

	Expected stopping time (years)	
$\alpha - r$	Case 1	Case 2
0.02	5.49	6.38
0.03	7.76	9.02
0.04	10.07	11.60
0.05	12.27	13.87
0.06	14.31	16.06

Table 3.4: Expected stopping times by level of market volatility for case 1 and case 2 agents aged 65, in the healthy state, and with total wealth of $10Y$—and other parameters as given in table 3.1. Increasing market volatility results in shorter stopping times, as agents shy away from the riskier environment. The situation illustrated is for agents with risk aversion of $\gamma = -0.5$, where case 2 agents are less conservative than case 1 agents. With more risk averse agents, this boldness of agents with bequest motives over those without is reversed.

	Expected stopping time (years)	
σ	Case 1	Case 2
0.12	14.62	16.45
0.13	13.67	15.46
0.14	13.44	14.51
0.15	12.10	13.70
0.16	11.42	12.99

stopping time. In both case 1 and case 2, when retirees are more frail in the sick state, and consequently have more mortality risk, they intend to stop earlier. These findings are in line with Kyng and Stolz (2016), who uses a very different (actuarial) approach, to discover that retirees entering retirement villages when they are younger and healthier are financially better off.

Table 3.5: Expected stopping times by frailty factor u for case 1 and case 2 agents aged 65, in the healthy state, and with total wealth of $10Y$—and other parameters as given in table 3.1. Increasing frailty results in shorter stopping times, as less healthy agents choose the safer retirement village world sooner. The situation illustrated is for agents with risk aversion of $\gamma = -0.5$, where case 2 agents are less conservative than case 1 agents. With more risk averse agents, this boldness of agents with bequest motives over those without is reversed.

	Expected stopping time (years)	
u	Case 1	Case 2
1.1	13.40	14.15
1.2	11.65	13.22
1.3	10.15	12.38
1.4	8.85	11.58

3.4 Conclusion

This paper provides an innovative contribution in its investigation of several cases of retirees entering retirement villages by using Richard's model with a HARA utility function and a dynamic health state. In our research, in which the time of entering the retirement village is the stopping time, we study the optimal strategy with the optimal stopping time for retirees.

We make several different assumptions for bequest motives and the insurance market to resemble the options faced by retirees when entering retirement villages in the real world. To address those problems, we obtain numerical results of consumption, wealth, insurance premiums and stopping times. In our generalised model, retirees are assumed to have the necessary consumption, dynamic health status and medical costs.

Retirees are found to have divergent consumption and stopping time trends, when the assumptions of bequest motives and the insurance market change. If retirees are assumed to have a bequest motive and access to insurance and annuity products, they are found to annuitise their excess wealth and to have a higher level of consumption. Otherwise, retirees are shown to have less consumption and to hold more wealth for precautionary purposes. Our numerical results indicate the importance of complete insurance markets for self-reliance in retirement—for increasing the consumption level prior to full annuatisation. This finding implies that the existence of a life insurance market for retirees is essential and critical for retirees' financial strategy. Our finding supports the argument of Blake (1999) and others for deepening insurance and annuity markets. A new research direction is then suggested in the insurance market in relation to the ageing problem. Stopping times are also impacted by the risk-aversion parameter, excess returns and the frailty factor.

In this paper, we also study the investment proportion in risky assets. In the case where there is a wealth requirement (wealth floor), retirees are assumed to replicate an American put option. Those retirees with a retirement savings target are shown to have an increasing proportion invested in risky assets over time. Shum and Faig (2006)[12], in empirical work, find agents with a retirement savings motive invest more in stocks. In our numerical results, retirees are shown to be more conservative and have an increasing proportion of wealth invested in risky assets over the life cycle. These differences merit further investigation.

[12]Note that Shum and Faig (2006) found that the increased investment in risky assets, which is positively related to the retirement saving target, is for those holding stocks. The investment level will fall for other risky assets.

Chapter 4

Paper 3

Optimal Life Insurance and Annuity Demand with Jump Diffusion and Regime Switching

Jinhui Zhang[1 2] , Sachi Purcal[1], Jiaqin Wei[3]

Abstract Classic Merton optimal life-cycle portfolio and consumption models are based on diffusion models for risky assets. In this paper, we extend the life-cycle model in Richard (1975) by allowing jumps and regime switching in the diffusion of risky assets within a model including life insurance and annuities. Agents are then exposed to varying degrees of background risk (jumps) as well as a business cycle (regimes). We develop a system of paired Hamilton-Jacobi-Bellman (HJB) equations. Using numerical methods, we obtain results for agents'

[1]Department of Applied Finance and Actuarial Studies, Faculty of Business and Finance, Macquarie University, Sydney, Australia.
[2]Corresponding author: jinhui.zhang@students.mq.edu.au
[3]School of Statistics, Faculty of Economics and Management, East China Normal University, Shanghai, China

behaviour. Agents are more conservative in consumption and annuitisation when the economic environment is more volatile and the bequest motive is stronger. Interestingly, under certain conditions, agents increase their exposure to risky assets in the face of increasing background risk.

Keywords: Stochastic optimal control, Richard's model, Optimal investment, Jumps, Regime switching.

4.1 Introduction

In this paper, we extend Richard's model (Richard, 1975) to study investors' behaviour by allowing jumps and regime switching in the underlying asset dynamics. Empirical evidence that underlying asset dynamics are impacted by changes in the state of the economy drives our motivation for assuming that underlying asset dynamics reflect the sudden changes in the economy which could be attributed to jumps and regime switching.

The presence of regime switching and jumps in the underlying asset dynamics is widely studied in the context of option pricing. A discontinuous model was proposed and examined in Merton (1976) for pricing options with an assumed log-normal distributed jump size. Cont and Tankov (2004) described jumps in the underlying asset dynamics for option pricing via an exponential Lévy process model. Elliott et al. (2007) utilised a Markov-modulated pure jump process to derive the regime-switching partial differential equations for European options, barrier options and American options. In Hamilton (1989), a discrete-state Markov process was introduced for regime-switching parameter values to allow endogenous structural breaks. Following Hamilton (1989), various numerical methods were presented in the literature for pricing options with regime switching. The lattice-based method, quadratic approximation method and front-fixing method were presented and studied in Bollen (1998), Brown (2001) and Wu and Kwok (1997), respectively.

Optimal investment strategy has been studied in the existing literature. The classic Merton model was developed in Merton (1969) with the assumed constant relative risk aversion (CRRA) utility function. Richard (1975) generalised the Merton model by including the bequest motive and insurance demand in the model. To capture market dynamics, jumps and regime switching have been studied in the literature for optimal investment strategy. Based on Richard (1975), Wang and Purcal (2005) introduced a jump-diffusion environment with a fixed jump size. Hanson (2007) utilised CRRA utility and log-uniform jump amplitude to present optimal portfolio and consumption policies. Song, Yin and Zhang (2006) developed a numerical scheme for controlled regime-switching jump diffusions. For financial markets with regime switching, Zhang and Guo (2004) introduced nearly optimal strategies and Sotomayor and Cadenillas (2009) presented explicit solutions for the optimisation problem of consumption and investment.

As they are made on a daily basis, financial decisions are evidently impacted by the changing market. According to Heaton and Lucas (2000), investor behaviours would deviate due to changes in the market risk. Shocks, such as the global financial crisis are documented as having influence on the portfolio choices of investors (Bateman, Islam, Louviere, Satchell and Thorp, 2011). Therefore, investors, especially retirees, face risks not only from consumption, investment and longevity but also from rapid changes in the market or economic state. However, compared to the large amount of research on the investment optimisation problem, the post-retirement optimal financial strategy problem has not received much attention (Gupta and Murray, 2003). This motivates us to build a post retirement-model to study the retiree behaviours of consumption, investment and bequests when there are regime switching and jumps in the financial market.

Drawing on the existing literature, we seek to determine agents' optimal strategies. With an extension to Richard's model, retirees' optimal consumption, investment and insurance decisions can be obtained within a financial environment containing jumps and regime switching. Numerical results are obtained for the optimal consumption, investment and insurance strategies in order to study the bequest motive and market risk effects. The addition of regime switching, capturing the economy's movements through the business cycle, result in a finding that elderly investors should still embrace risk.

This paper is organised as follows. Section 4.2 extends the Richard's model to the regime-switching jump diffusion environment. Section 4.3.2 demonstrates the numerical results and analyses the findings while section 4.4 concludes the paper.

4.2 Model and Method

From Hanson (2007), the dynamics of the risky asset price, $X(t)$, are assumed to be

$$dX(t) = X(t)\left(\alpha(t)dt + \sigma(t)dB(t) + Jd\psi(t)\right),$$

where $\alpha(t)$ and $\sigma(t)$ are the return rate and volatility of the risky asset price, $dB(t)$ is the standard Brownian motion, J is a uniform distributed jump size on $[\mathscr{G}_1(t), \mathscr{G}_2(t)]$ and $\psi(t)$ is a discontinuous one-dimensional Poisson process with a jump rate λ.

The agent is assumed to have a random time of death which is modelled by the survival rate, $S(t)$, and the force of mortality, $\mu(t)$, with the density function of mortality, $f(t) = \mu(t) \cdot S(t)$.

Here we assume the agent has utility from consumption, that is, U_1, as well as utility from leaving bequests, that is, U_2. Then the objective of a utility-maximising agent is

$$\max_{C(t),\pi(t),Z(t)} E_t\left[\int_t^{\tau} \frac{S(T)}{S(t)}\left(\frac{\theta(T)}{\theta(t)}U_1(C,T) + \mu(T)\frac{\theta(T)}{\theta(t)}U_2(L,T)\right)dT\right],$$

which is subject to the dynamics of wealth W,

$$dW(t) = [(\alpha(t) - r(t))\pi(t)W(t) + r \cdot W(t) + Y - C(t) - P(t)]\, dt + \pi(t)\sigma(t)W(t)dq(t)$$
$$+ \pi W(t) \sum_{k=1}^{d\psi(t)} J(T_k^-),$$

where $r(t)$ is the risk-free rate, consumption, $C(t)$, proportion of wealth invested in risky assets, $\pi(t)$, and legacy amount, $L(t)$, are the control variables, $P(t)$ is the insurance premium, $P(t) = \mu(t)(L(t) - W(t))$, Y is the deterministic income which is set to be zero for retirees, and T_k^- is the pre-jump time.

A continuous-time Markov chain process $\mathscr{X} := \{\mathscr{X}_t\}_{t\in\mathscr{T}}$ is defined here with a finite state space $\{e_1, e_2, ..., e_N\}$, where $e_i = (0, ..., 1, ..., 0)' \in \mathscr{R}^N$. The element $\mathscr{X}_t = e_i$ of the Markov chain demonstrates that, at time t, the economy is in the ith state. Elliott, Aggoun and Moore (1994) showed that the Markov chain process $\mathscr{X} = \{\mathscr{X}_t,\ t \in \mathscr{T}\}$ satisfies the following semi-martingale representation theorem:

$$\mathscr{X}_t = \mathscr{X}_0 + \int_0^t \mathbf{Q}\mathscr{X}_u du + M_t$$

where $M = \{M_t,\ t \in \mathscr{T}\}$ is a martingale with respect to the filtration generated by $\mathscr{X}$ and $\mathbf{Q}$ is the intensity matrix for N number of regimes,

$$\mathbf{Q} = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1N} \\ q_{21} & q_{22} & \cdots & q_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ q_{N1} & q_{N2} & \cdots & q_{NN} \end{pmatrix}. \tag{4.2.1}$$

The risk-free rate $\{r_t\}_{t\in\mathscr{T}}$, risky asset return rate $\{\alpha_t\}_{t\in\mathscr{T}}$ and volatility $\{\sigma_t\}_{t\in\mathscr{T}}$ in the underlying asset dynamics are defined as:

$$\begin{aligned} r(t) &:= r(t, \mathscr{X}_t) = \langle r, \mathscr{X}_t\rangle = \sum_{i=1}^{N} r_i\langle \mathscr{X}_t,\ e_i\rangle, \\ \alpha(t) &:= \alpha(t, \mathscr{X}_t) = \langle \alpha, \mathscr{X}_t\rangle = \sum_{i=1}^{N} \alpha_i\langle \mathscr{X}_t,\ e_i\rangle, \\ \sigma(t) &:= \sigma(t, \mathscr{X}_t) = \langle \sigma, \mathscr{X}_t\rangle = \sum_{i=1}^{N} \sigma_i\langle \mathscr{X}_t,\ e_i\rangle, \end{aligned}$$

where $r := (r_1, r_2, \ldots, r_N)$, $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_N)$, $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)$ with $\sigma_i > 0$ for all regimes $i = 1, 2, \ldots, N$ and $\langle\cdot,\cdot\rangle$ denotes the inner product in space $\mathscr{R}^N$. We use V_i to denote the objective function for regime i. Then the dynamic programming equation is

$$
\begin{aligned}
0 &= (U_1(C,t) + \mu(t)U_2(L,t)) - \mu(t)V(W(t),t) - r(t)V(W(t),t) \\
&+ V_W(W(t),t)\left[(\alpha - r)\pi(t)W(t) + r \cdot W(t) + Y - C(t) - P(t)\right] \\
&+ \frac{1}{2}V_{WW}(W(t),t)\pi^2 W(t)^2\sigma^2(t) + V_t(W(t),t) \\
&+ \frac{\lambda(t)}{\mathscr{G}_2(t) - \mathscr{G}_1(t)}\int_{\mathscr{G}_1(t)}^{\mathscr{G}_2(t)} [V(W(t) + \pi(e^u - 1)W) - V(W)]du + \sum_j q_{ji}V_j(W(t),t),
\end{aligned}
\tag{4.2.2}
$$

where q_{ji} is the intensity rate from regime j to i. We use the power utility function for consumption and bequests,

$$
U_1(C(t)) = \frac{C(t)^\gamma}{\gamma}, \tag{4.2.3}
$$

$$
U_2(L(t)) = m(t)^{1-\gamma}\frac{L(t)^\gamma}{\gamma}, \tag{4.2.4}
$$

where γ is the risk aversion parameter and $m(t)^{1-\gamma}$ is the discount function for bequests,

$$
m(t) = e^{-\rho t/(1-\gamma)}\nu \int_t^\tau e^{-r(u-t)}du \tag{4.2.5}
$$

and ν is a constant that reflects the annuity level an agent's spouse or children compared to the current consumption amount.

From Richard (1975) and Purcal and Piggott (2008), the original value function V has the assumed form,

$$V(W,t) = a(t)\frac{W^{\gamma}}{\gamma}. \tag{4.2.6}$$

Following Song, Yin and Zhang (2006), we can rewrite equation (4.2.6) as

$$V_i(W,t) = \sum_j \tilde{P}_{ij} a_j(t)\frac{W^{\gamma}}{\gamma},\ i = 1,2,..., \tag{4.2.7}$$

where $V_i(W,t)$ is the value function for regime i, $\tilde{P}_{ji}$ is the transition probability of state j switching to state i and $a_i(t)$ is the coefficient in the value function at time t for regime i.

Applying the first-order condition and substituting equation (4.2.6) into equation (4.2.2) in each regime, we have the optimal control variables

$$C^*(t) = \sum_i a_i(t)^{\frac{1}{\gamma-1}} W \cdot 1_{\{\mathscr{X}_t=i\}}, \tag{4.2.8}$$

$$\pi^*(t) = \frac{1}{(1-\gamma)\sigma^2(t)}\left[\alpha(t) - r(t) + \frac{\lambda(t)}{\mathscr{G}_2(t)-\mathscr{G}_1(t)}\int_{\mathscr{G}_1(t)}^{\mathscr{G}_2(t)} G(u)^{\gamma-1}(e^u-1)du\right]\cdot 1_{\{\mathscr{X}_t=i\}}, \tag{4.2.9}$$

and

$$L^*(t) = \sum_i m(t)a_i(t)^{\frac{1}{\gamma-1}} W \cdot 1_{\{\mathscr{X}_t=i\}}, \tag{4.2.10}$$

where $G(u) = 1 + \pi^*(t)(e^u - 1)$. Equation (4.2.9) does not have a closed-form result. We need to use a numerical method to obtain the value of π for the given parameter values.

Given that the current regime is i, then substituting equations (4.2.6)–equation (4.2.10) into equation (4.2.2) and dividing it by W^γ/γ, we can have

$$\begin{aligned} 0 =& \mu(t)m(t)a_i(t)^{\frac{\gamma}{\gamma-1}} + a_i(t)^{\frac{\gamma}{\gamma-1}} - (\mu(t)+\rho)a_i(t) + a_i'(t) \\ &- \gamma(1+\mu(t)m(t))a_i(t)^{\frac{\gamma}{\gamma-1}} + (r(t)+\mu(t))a_i(t)\gamma + (\alpha(t)-r(t))\pi^* a_i(t)\gamma \\ &+ \frac{1}{2}\sigma^2(t)(\pi^*)^2(\gamma-1)a_i(t)\gamma + \frac{\lambda(t)a_i(t)}{\mathscr{G}_2(t)-\mathscr{G}_1(t)}\int_{\mathscr{G}_1(t)}^{\mathscr{G}_2(t)}(G^\gamma - 1)du + \sum_j q_{ij}a_j(t). \end{aligned} \tag{4.2.11}$$

By rearranging equation (4.2.11), we can have

$$\begin{aligned} &a_i(t)^{\frac{\gamma}{\gamma-1}}[(\gamma-1)(1+\mu(t)m(t))] \\ &\quad = a_i'(t) + a_i(t)\Big[-\mu(t) - \rho + (r+\mu(t))\gamma + (\alpha(t)-r(t))\gamma\pi^* \\ &\quad + \frac{1}{2}\sigma^2(t)(\pi^*)^2(\gamma-1)\gamma + \frac{\lambda(t)}{\mathscr{G}_2(t)-\mathscr{G}_1(t)}\int_{\mathscr{G}_1(t)}^{\mathscr{G}_2(t)}(G^\gamma-1)du + \sum_j q_{ij}\frac{a_j(t)}{a_i(t)}\Big]. \end{aligned} \tag{4.2.12}$$

As we have the dynamic programming equation as equation (4.2.12) for each regime, we end up with a system of equations. Through the numerical schemes from Song, Yin and Zhang (2006), we can solve equation (4.2.12) and obtain the numerical solution for coefficient $a_i(t)$. The consumption, investment and insurance premiums can then be calculated. We can show the existence and uniqueness of the solution under some certain conditions[4].

4.3 Numerical Results

4.3.1 Parameter values

Table 4.1: Parameters used in the numerical results

t=65	τ=109	ρ=r_1 = 0.04187
Y=USD\$52,250	γ=−2 or −0.8	ν=2/3 or 1
$\mathscr{G}_1$=−0.5	$\mathscr{G}_2$=0.5	λ=0, 0.2, 0.4 and 0.6
r_1=0.04187	α_1=0.0592	σ_1=0.1853
r_2=0.03	α_2=0.04	σ_2=0.3
r_3=0.025	α_3=0.03	σ_3=0.4

In this paper, the American survival probabilities and forces of mortality are obtained from the American 2010 life table for males Human Mortality Database (N.d.). We calibrate our parameters to the American data to obtain numerical results from the starting age $t = 65$ to the maximum age $\omega = 109$.

We calculate the regime 1 risk-free rate, r_1, by using the 1-year yield rate (from 1990 to 2010) from the US Department of the Treasury (2015). The agent's time preference is assumed to be same as the risk-free rate, $\rho = r_1$ for simplicity. Meanwhile, the regime 1 risky asset return rate, α_1, and volatility, σ_1 are calibrated to Standard & Poor's (S&P) 500 data from 1990 to 2010.

We also adopt the average income, $Y = \$52\,250$ from Noss (2013). Agents in our model are assumed to have total wealth of $10Y$ from previous savings and

[4]See, e.g., Sotomayor and Cadenillas (2009).

have no future income. The risk aversion parameter γ is set to be -2 for the jump case and -0.8 for the regime switching case.[5] Here, the ν value illustrates the annuity level left for the surviving spouse or children which is an indicator for a bequest motive. From Purcal and Piggott (2008), we set ν to be two-thirds for a low bequest motive, that is, agents will only leave two-thirds of their current consumption level for the surviving spouse or children. As we want to further study the effects from a higher bequest motive, we also calculate our result for $\nu = 1$ for a high bequest motive.

For the jump case, we assume the values of $\mathscr{G}_1$ and $\mathscr{G}_2$ to be -0.5 and 0.5, respectively. To test the jump effects, different values are tried for the jump frequency, that is, $\lambda = 0, 0.2, 0.4$ and 0.6. When $\lambda = 0$, this means that there is no jump in the risky asset. Higher frequency rate means the financial market becomes more volatile.

To demonstrate the numerical results for the case with jumps and regime switching, we assume there are three regimes: regime 1 “good”, regime 2 “okay” and regime 3 “bad”. We assume that agents initially start in the good regime. For regime 2 and 3, we assume the following risk-free rates, risky asset return rates and volatility rates, regime 2: $r_2 = 0.03$ $\alpha_2 = 0.04$ $\sigma_2 = 0.3$; regime 3: $r_3 = 0.025$ $\alpha_3 = 0.03$ $\sigma_3 = 0.4$. With these preset values, a worse economic environment is described by the lower risk-free rate, lower risky asset return rate and higher

[5]This change in risk aversion was done for computational reasons. Adding regime switching to our numerical solution with jumps added more complexity to an already complex model. In this final environment, we were not able to find numerical solutions with $\gamma = -2$; solutions were forthcoming, however, with $\gamma = -0.8$.

volatility. We assume the preset intensity rates for switching are as follows,

$$\mathbf{Q} = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}. \tag{4.3.13}$$

This implies a regularity of movement between states: from 1 to 2, then 2 to 3, then 3 back to 1, and so on[6]. That is, the agents find themselves in a business cycle, moving progressively from good to bad, then returning immediately to good and repeating. The annual probability of staying in a particular state (and not moving to the next) is $1/e$ or 0.37. The preset intensity rates used to obtain numerical results are for illustrative purposes. More realistic assumption (e.g., state 1 to 2, then 2 to 3, then 3 back to 2, then 2 back to 1 and so on.) bring difficulties in calculation with the solution method failing to return results. We will leave more realistic assumptions for future research.

4.3.2 Case with jumps and no regime switching

Here, we compare the cases with different levels of jump frequency and bequest motive, that is, $\lambda = 0, 0.2, 0.4$ or 0.6 and $\nu = 2/3$ or 1. Specifically, we calculate the expected proportions of wealth, consumption, insurance premium and risky asset investment for each λ and ν value, as shown in figures 4.1–4.4.

As our model is to study the post-retirement period, the wealth level will decline

[6]For this particular case, the duration of this cycle is 3 years.

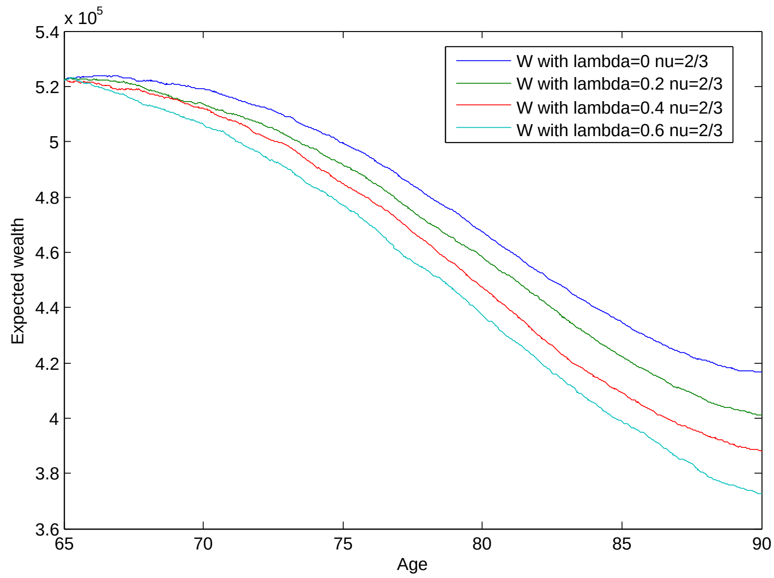

Figure 4.1: Expected wealth with jumps (no regime switching)
$\gamma = -2,\ \nu = 2/3,\ \lambda = 0, 0.2, 0.4$ or 0.6

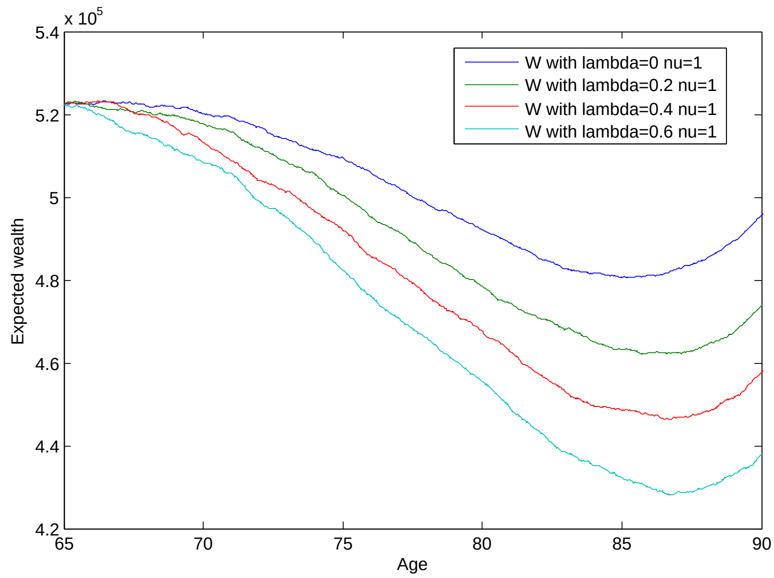

Figure 4.2: Expected wealth with jumps (no regime switching)
$\gamma = -2,\ \nu = 1,\ \lambda = 0, 0.2, 0.4$ or 0.6

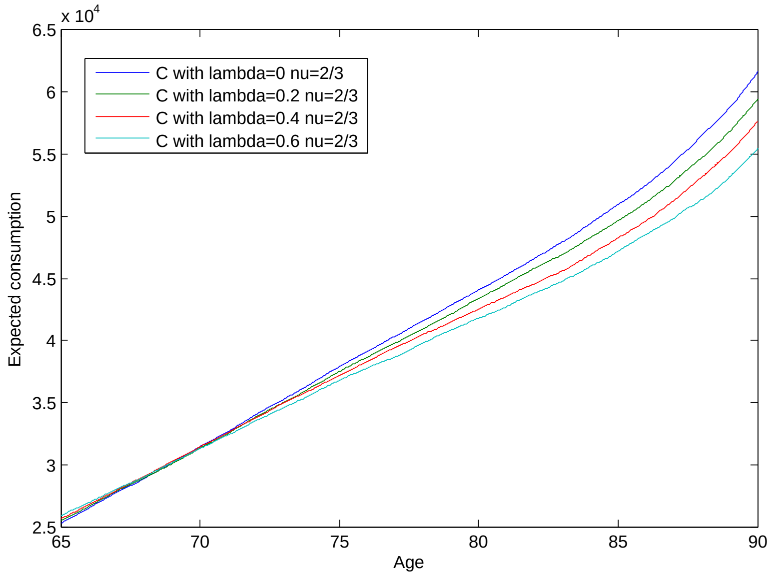

Figure 4.3: Expected consumption with jumps (no regime switching)
$\gamma = -2$, $\nu = 2/3$, $\lambda = 0, 0.2, 0.4$ or 0.6

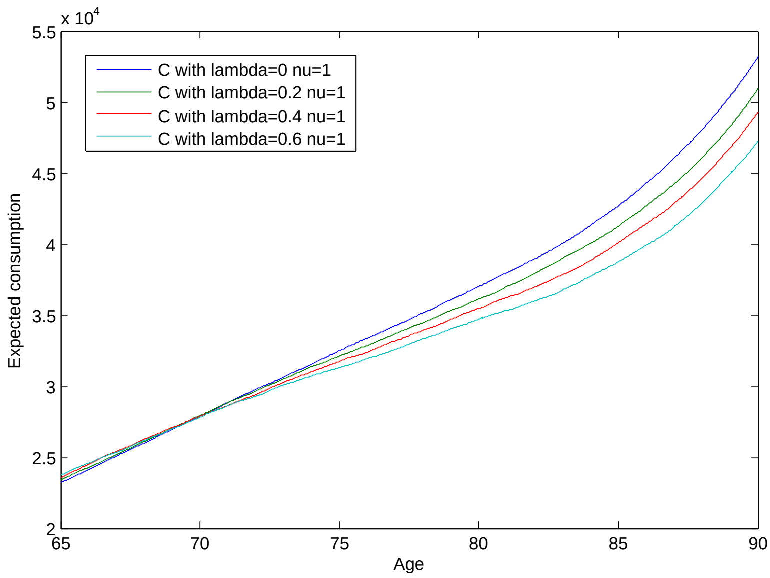

Figure 4.4: Expected consumption with jumps (no regime switching)
$\gamma = -2$, $\nu = 1$, $\lambda = 0, 0.2, 0.4$ or 0.6

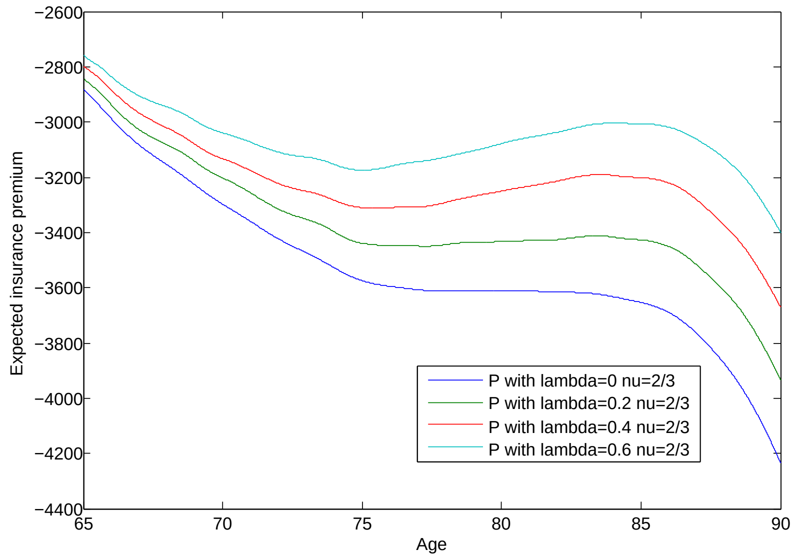

Figure 4.5: Expected insurance premium with jumps (no regime switching) $\gamma = -2,\ \nu = 2/3,\ \lambda = 0, 0.2, 0.4$ or 0.6

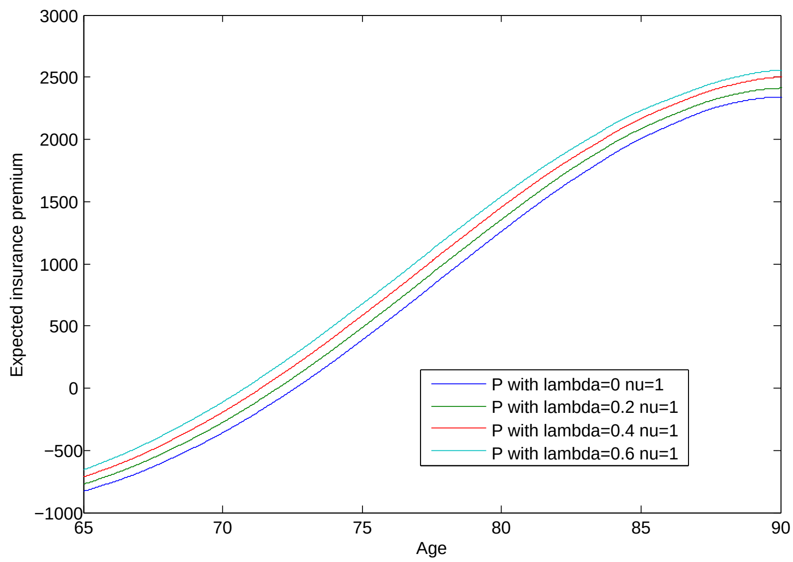

Figure 4.6: Expected insurance premium with jumps (no regime switching) $\gamma = -2,\ \nu = 1,\ \lambda = 0, 0.2, 0.4$ or 0.6

when agents are ageing. From figures 4.1 and 4.2, we observe that a higher jump frequency can result in less wealth for both levels of bequest motive. This phenomenon can be linked with variations in the proportion of wealth invested in risky assets, that is, π due to different jump frequencies. Table 4.2 shows that agents would reduce their exposure to risky assets when jumps are more frequent. Comparing the results in figure 4.1 and 4.2, agents with a higher bequest motive are willing to hold more wealth.

From figure 4.3 and 4.4, the expected increases in consumption level in line with increasing age for different λ values can be found for both levels of bequest motive. With a higher jump frequency, that is, a higher λ value, the expected consumption level is lower in line with increasing age. We also notice that agents have a propensity to consume less when they have a higher bequest motive.

In our model, the insurance premium is the indicator for life insurance or annuity demand. A positive insurance premium, that is, P, indicates agent's demand for life insurance while a negative insurance premium indicates the demand for annuitisation. In figure 4.5, we have all negative insurance premiums which illustrate agents' annuitisation intention. With more frequent potential jumps in the risky assets, agents reduce their annuitisation amount due their lower level of wealth. However, as shown in figure 4.6, with a higher bequest motive, agents' demand for life insurance starts a few years after retirement for all market environments, with the higher level of demand corresponding to the higher λ value.

Table 4.2: Investment proportion in risky assets for different jump frequencies (no regime switching present). Results are independent of the degree of altruism.

λ	π with $\nu = 2/3$ or 1
0	0.1680
0.2	0.1668
0.4	0.1662
0.6	0.1659

4.3.3 Case with jumps and regime switching

We conduct another set of numerical experiments for the case with jumps, but this time augmented by the presence of regime switching. To study the effects of regime switching and a bequest motive, we calculate the numerical results with or without regime switching for different levels of bequest motive, as shown in figures 4.7–4.10, when there are jumps in the financial market. For reasons discussed above, we alter our focus from an agent with a risk aversion parameter of $\gamma = -2$ to one who is less risk averse, with $\gamma = -0.8$.

From figure 4.7 and 4.8, agents with both low and high bequest motives are found to have less consumption motivation when there are additional risks present due to regime switching. Bequest motive effects are also detected here, as agents tend to consume less for higher ν value in an effort to meet their chosen level of responsibility to their legatees.

Similarly, as shown in figures 4.9–4.10, agents choose lower annuitisation amounts when they are faced with additional risks from regime switching. Furthermore, agents reduce their annuitisation intention if they have a higher level of bequest

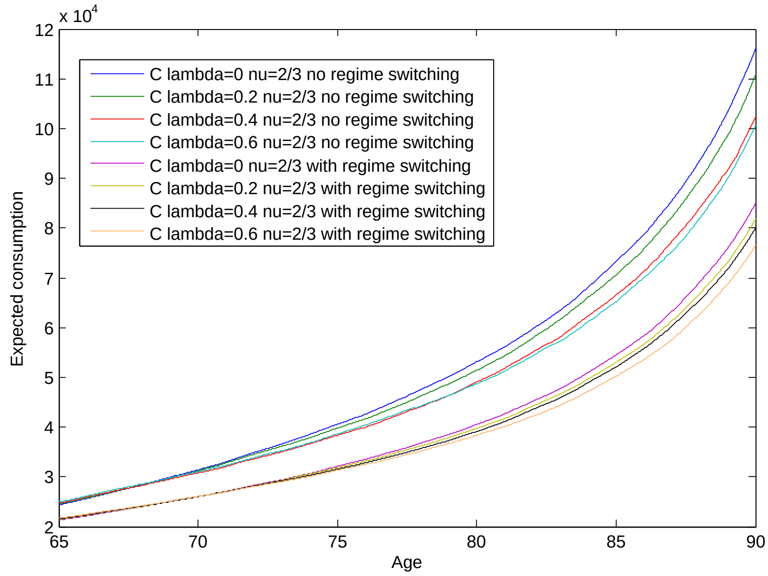

Figure 4.7: Expected consumption with jumps and regime switching
$\gamma = -0.8$, $\nu = 2/3$, $\lambda = 0, 0.2, 0.4$ or 0.6

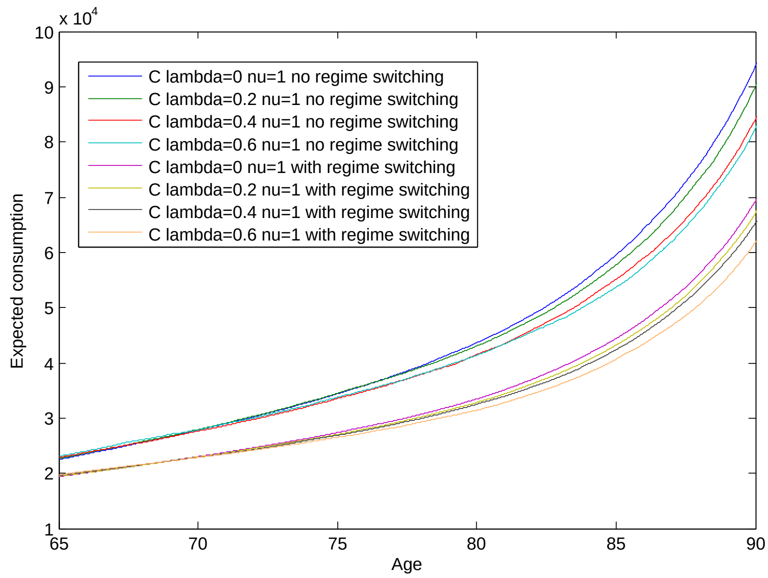

Figure 4.8: Expected consumption with jumps and regime switching
$\gamma = -0.8$, $\nu = 1$, $\lambda = 0, 0.2, 0.4$ or 0.6

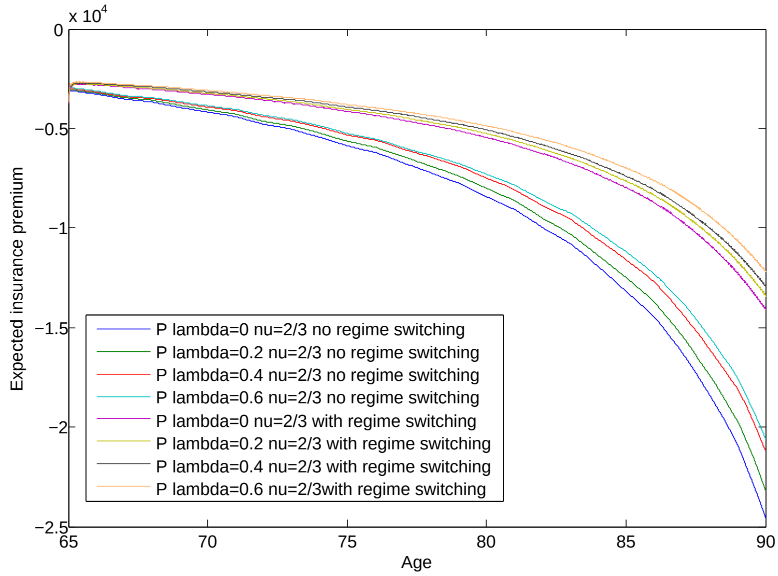

Figure 4.9: Expected insurance premium with jumps and regime switching $\gamma = -0.8$, $\nu = 2/3$, $\lambda = 0, 0.2, 0.4$ or 0.6

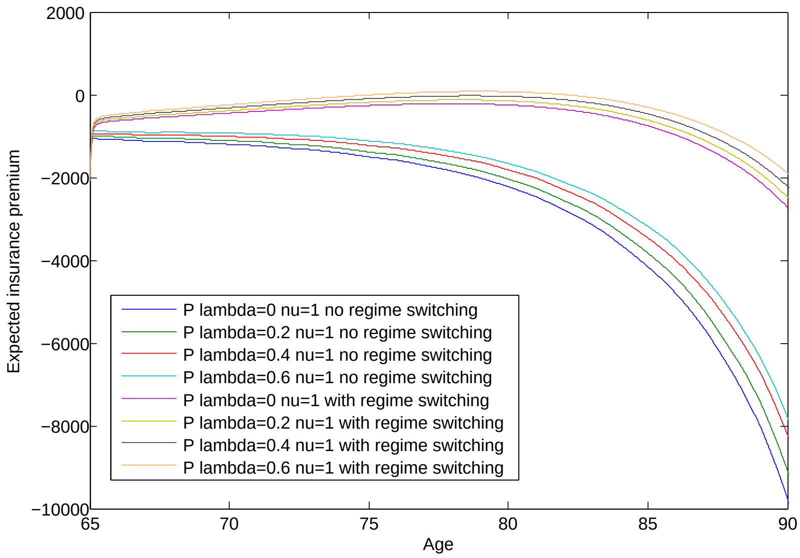

Figure 4.10: Expected insurance premium with jumps and regime switching $\gamma = -0.8$, $\nu = 1$, $\lambda = 0, 0.2, 0.4$ or 0.6

motive.

In table 4.3 we also calculate the proportion of agents' wealth invested in risky assets in the economic environment with jumps and in the economic environment with jumps and regime switching (the latter in expection: over the different states). Comparing the π values for the jump only case with the expected π values for the jumps with regime switching case we see agents tend to reduce their exposure to risky assets when regime switching occurs. Interesting, we see a very different pattern of behaviour for the two different cases. In the jump-only environment agents react with apparent indifference to the increases in background risk. With the introduction of regime switching, however, the increasing probability of the potentially worse economic environment that is captured by more frequent jumps leads agents to increase their investment proportion [7]. That is, in the face of increasing background risk, the agents choose to invest more in the risky asset. We explore this further below.

Table 4.3: Investment proportion in risky assets with jumps and regime switching

λ	π with jumps	$\mathbf{E}[\pi]$ with jumps and regime switching
0	0.2800	0.1256
0.2	0.2786	0.1438
0.4	0.2779	0.1579
0.6	0.2774	0.1691

In figure 4.11 we tease out the regime-specific behaviour of π in the face of increasing background risk. Wang and Purcal (2005) show that negative (non-

[7]Please refer to figure 4.11 for the investment proportion in specific economic states.

symmetric) jumps in risky asset returns leads to uniformly lower investment in risky, with increasing jump frequency exacerbating the result. Their model included no regime-switching. The regime switching in this model, and the jump symmetry (negative and positive), are the point of departure from Wang and Purcal (2005). Their inclusion generates an interesting result in terms of increasing background risk—seen when the jump frequency increases. The jump symmetry[8] and promise of better economic conditions as one moves through the business cycle[9] leads agents, in the two worst economic states, to increase their investment in risky assets as background risk increases, while in the best economic state agents temper their exposure to risky somewhat with the increasing background risk. While it is known that under certain circumstances increases in background risk reduce demand for other independent risks, this is not true in general (Gollier, 2001), and our above result is a case in point. While empirical studies have found increasing background risk leads to less investment in stocks (Dimmock, 2012; Palia, Qi and Wu, 2014), these studies do not control for the stage of the business cycle.

4.4 Conclusion

In this paper, we extend Richard's model (Richard, 1975) to examine agents' investment behaviour during changes in the economic state. Using our model,

[8]Our results are driven by the constant jump frequency and symmetric jump size distribution in each economic state.
[9]Regime switching between different states can be regarded as the business cycle movement.

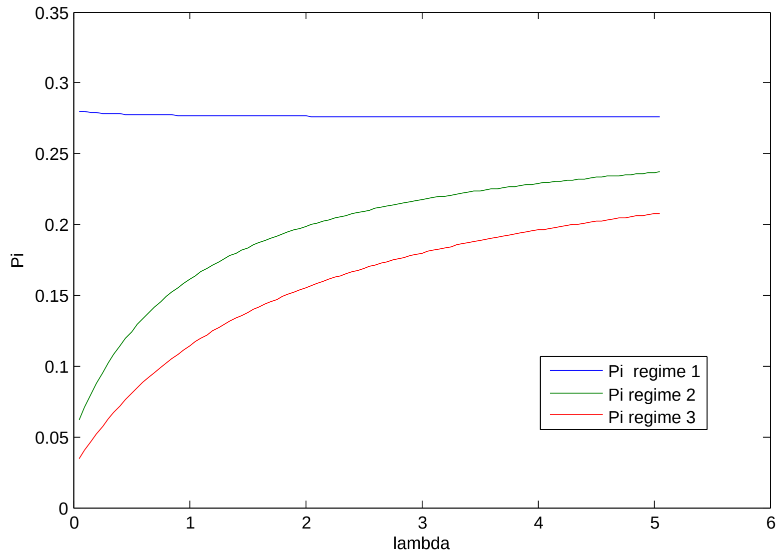

Figure 4.11: π value for three regimes
$\gamma = -0.8,\ \lambda = 0, 0.05, 0.1, ..., 4.95, 5$

we study agents' post-retirement investment behaviour in relation to risky assets when economic dynamics are described by jumps and regime switching.

Based on our numerical results for the case with jumps, agents' behaviour deviates when they experience changes in the economic state. When agents detect potential future jumps, they will reduce their exposure to risky assets which will result in lower wealth, consumption and annuitisation. This type of behaviour will become more substantial with a higher jump frequency. When the agents' bequest motive is higher, they might further reduce their consumption and annuitisation. In fact, agents will seek life insurance, if their wealth cannot cover the legacy amount.

When there are both potential jumps and regime switching in the market, agents have the tendency to further reduce their consumption and annuitisation compared to when the market has jumps only. With a higher bequest motive, this trend is more obvious. However, according to our calculations, agents would plan to enhance their exposure to risky assets when jumps are more frequent.

We can conclude that agents' behaviour will be different when they are in the presence of risks from a changing economic state. With reduced wealth, they will be more conservative which is shown in their reduced consumption and annuitisation. From agents' behaviour, we find that having a bequest motive has negative effects on consumption and annuitisation. However, in our model for different volatility values, agents can either exhibit their preference for risk taking or risk aversion when the market is more volatile. Specifically, agents who are in regime 1, that is in a good economic state, reduce their proportion of wealth invested in risky assets along with increasing jump frequency. Meanwhile agents who are in regime 2 and 3, which are worse economic states, raise their proportion of wealth invested in risky assets along with increasing jump frequency. When agents are confronting a market that has a high degree of variation, it is more common for us to anticipate risk averse behaviour rather than the risk taking behaviour. However, in our model, jumps are assumed to be both negative and positive. Agents in regime 2 and 3 are confronting lower return rates but higher volatility. Those agents then take on more risk due to the possibility of positive jumps and probability of switching to a better regime. On the other hand, agents in regime 1 hold fears for downward movements in returns and choose to be more conservative.

Chapter 5

Conclusion

5.1 Summary and Findings

In this thesis, we use the life-cycle model to investigate the optimal strategies for consumption, investment and insurance demand. Our optimal strategy study is developed with the motive being to provide explanations for investors' behaviours in relation to real-world problems.

Many research papers have studied optimal financial strategy via the life-cycle model. Among these studies, the famous Merton's model (Merton, 1969, 1971) and its extension, Richard's model (Richard, 1975), bring us important insights on agents' behaviours in relation to consumption, investment and insurance.

Although these two models provide us with useful information about agent's behaviours, certain limitations are generated by the assumptions used. In these two models, utility-maximising agents are assumed to have constant preferences and consistent behaviours. In addition, health status and the financial market are set to be stable without change. However, to develop optimal strategy for practical problems, we need to integrate extra elements in the life-cycle model to describe the time-inconsistent behaviours, dynamic health status and unstable financial market which are observed in the empirical data.

5.1.1 Paper 1

The first research paper provides a novel explanation of the "annuity puzzle" by using the life-cycle model. The interactions of time inconsistency and luxury-type bequest motives that are integrated into our extended life-cycle model help us to describe possible reasons behind the "annuity puzzle" and its exceptions.

In this paper, Richard's model (Richard, 1975) is generalised by adopting the hyperbolic discounting factor as well as a luxury-type legacy amount to describe agents' time-inconsistent behaviours and luxury-type bequest motives. These two factors are observed in empirical data and have been studied in the literature.

Based on the type of time-inconsistent behaviour, we categorise agents into two groups, "naïve" and "sophisticated", and calculate numerical results for optimal decisions for each group. The consumption pattern of naïve agents is observed to be hump-shaped which is similar to the empirical data. The demand of naïve agents for insurance or annuities is impacted by luxury-type bequests. The naïve agents with hyperbolic preferences may switch from purchasing life insurance to annuitisation at stages of the life cycle when they possess luxury-type bequest motives. In contrast to naïve agents, sophisticated agents have more rational behaviours and exhibit a larger amount of consumption as well as annuitisation of excess wealth.

In paper 1, we find that our modelled behaviours of naïve agents resemble the behaviours of agents in the real world. With the interaction of hyperbolic discounting and luxury-type bequest motives, naïve agents can manifest the demand for life insurance or annuitisation. This bi-directional behaviour can clearly explain the "annuity puzzle" observed in most developed countries and the exceptions in Switzerland and the Netherlands. Based on our research, the behaviours of real-world agents could be justified by hyperbolic discounting and luxury-type bequests.

5.1.2 Paper 2

In paper 2, we develop a useful model to analyse retirees' decisions about their optimal strategy and the optimal time to enter a retirement village. Using different bequest and insurance market assumptions, we replicate retirement village options faced by a utility-maximising retiree with an uncertain future health status. Retirees determine the entry date as an optimal stopping time for full annuitisation.

We generalise our model by including dynamic health status. For each health state, retirees are assumed to have a specific mortality rate and medical costs that reflect the impact of health status. Following the idea of the frailty model, a healthier state is linked with a lower mortality rate and lower medical costs.

We find that retirees' consumption and wealth are influenced by bequest motives and the existence of an active insurance market. When retirees with bequest motives can purchase life insurance or annuities in the market, they will annuitise their excess wealth and will have a higher consumption level. Hence, these retirees need to hold less accumulated wealth in their optimal strategies. The availability of the insurance market initiates wealth annuitisation and, furthermore, generates better outcomes.

When retirees are faced with a wealth requirement threshold to enter the retirement village (a retirement savings target), they are assumed to replicate an American put option. Consequently, the proportion of wealth invested in risky assets is found to have an increasing trend. Interestingly, this is in contrast to some empirical work, which finds agents with a retirement savings target will tend to invest more in stocks (Shum and Faig, 2006).

5.1.3 Paper 3

In paper 3, we investigate the optimal strategy for an agent in volatile circumstances—with bequest motives and being within a risky financial environment that contains jumps and regime switching. To obtain optimal strategies, we extend the Richard (Richard, 1975) model to determine retirees' optimal consumption, investment and insurance decisions.

Based on our numerical results, when the economic state experiences jumps, agents would accordingly adjust their optimal strategy by reducing the proportion they have invested in risky assets. Falling wealth, consumption and annuitisation are consequently observed. This trend would be more obvious when the jump frequency as well as the bequest motive is higher.

We find that agents lower their levels of consumption and annuitisation when there are not only jumps but also regime switching in the financial environment. Furthermore, our results show that agents with a higher bequest motive are more impacted by such a risky environment. For our chosen parameter set, agents, on average, exhibit a greater tendency to invest in risky assets when they are facing an economic state with a higher frequency of potential jumps (that is, higher levels of background risk).

In our model, agents show different behaviours within the two risky modelled financial environments. Agents can either demonstrate aggressive or conservative investment behaviour in reaction in increasing background risk, depending on the presence of regime switching. Empirical studies suggest increasing background risk results in more conservative behaviour (Dimmock, 2012; Palia, Qi and Wu, 2014).

5.2 Limitations and Recommendations for Future Research

- The retirees in paper 2 are assumed to be “myopic” or short-sighted, that is, they naïvely believe that their future health state and optimal strategy will not change. To obtain more generalised results, the "sophisticated" case, that is, retirees who realise that their health status is uncertain and that they consequently need to adjust their optimal strategies, could be included in the model for future research.

- The risk-aversion parameter γ is set to be constant and is applied to all health states in paper 2. In future research, different risk-aversion parameters could be applied for dynamic health states to resemble real-world retirees' behaviours.

- In paper 2, we establish a financial threshold combined with option replication to meet the wealth floor requirement. The replication can result in investment behaviour that differs from the constant Merton ratio. This observation merits further exploration, particularly as it contrasts to empirical findings (Shum and Faig, 2006).

- The bequest motive utility function used in paper 3 implies that the bequest is a necessity. This utility function could be extended in future to incorporate the luxury-type bequest feature as suggested by Lockwood (2012).

- In paper 3, we determine that in the presence of jumps and a business cycle agents will increase their exposure to risky assets, on average, as background risk increases. This observation merits further investigation, particularly as it contrasts to empirical findings (Dimmock, 2012; Palia, Qi and Wu, 2014).

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