MACQUARIE UNIVERSITY

Analysis of Pricing Financial Derivatives Under Regime-Switching Economy

by

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Declaration of Authorship

I, Farzad Alavi Fard, declare that this thesis titled, 'Analysis of Pricing Financial Derivatives Under Regime-Switching Economy' and the work presented in it are my own. I confirm that:

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Abstract

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In this thesis we argue that regime-switching models can significantly improve the pricing models for financial derivatives. We use three examples to analyse the valuation of derivative contracts under the Markovian regime-switching framework namely, 1) a European call option 2) a Ruin Contingent Life Annuity and 3) a Participating Product. Such a regime-switching framework unveils a potent class of models. Through the modulation of the model parameters by a Markov chain, they can simultaneously explain the asymmetric leptokurtic features of the returns' distribution, as well as the volatility smile and the volatility clustering effect. The intuition behind regime-switching models is to capture the appealing idea that the macro-economy is subjected to regular, yet unpredictable in time, states, which in turn affect the prices of financial securities.

The market considered in this thesis is incomplete in general due to additional sources of uncertainty, particularly the regime-switching risk. Under these market conditions, a perfectly replicating trading strategy does not exist and there is more than one equivalent martingale measure. As a result, a perfect hedge for derivative contracts is impossible and the holder of the financial derivative needs to impose some testable restrictions to price the residual risk. In this study, we argue that a condition that minimizes the relative entropy between the risk-neutral and the historical probability measures is very suitable. Such condition, determines a price for the derivative contract that maximizes an exponential utility function for the holder. For doing so, we either use the Minimum Entropy Martingale Measure or Esscher Transform to choose the equivalent martingale measure.

Due to the complexity of the pricing models, stemmed from either the modeling assumptions or the path-dependency of the payoff of the derivatives products, there is no known analytical solution to our problems. We employ different numerical methods in each chapter, depending on the respective modeling framework, to approximate the solutions. We also examine numerically the performance of simple hedging strategies by investigating the terminal distribution of hedging errors and the associated risk measures such as Value at Risk and Expected Shortfall. The impacts of the frequency of re-balancing the hedging portfolio and the transition probabilities of the modulating Markov chain on the quality of hedging are also discussed.

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I dedicate this thesis to my wife and my parents, whose support made this journey possible. Without their unwavering love, patience, encouragement and support I would not have accomplished this task.

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Abbreviations

AAA	**A**merican **A**ssociation **A**ctuaries
ALDA	**A**dvanced **L**ife **D**elayed **A**nnuities
AMISE	**A**symptotic **M**ean **I**ntegrated **S**quared **E**rror
ARCH	**A**uto **R**egressive **C**onditional **H**eteroscedasticity
BSM	**B**lack **S**choles **M**erton
CRR	**C**ox **R**oss **R**ubinstein
EMM	**E**sscher **M**artingale **M**easure
ES	**E**xpected **S**hortfall
FD	**F**inite **D**ifference
GARCH	**G**eneralized**A**uto **R**egressive **C**onditional **H**eteroscedasticity
FE	**F**inite **E**lement
GG	**G**eneralized **G**amma
MCMM	**M**ean **C**orrecting **M**artingale **M**easure
MEMM	**M**inimal **E**ntropy **M**artingale **M**easure
MIG	**M**arkov **I**nverse **G**aussian
MMM	**M**inimal **M**artingale **M**easure
MVG	**M**arkov **V**ariance **G**amma
MWG	**M**arkov **W**eighted **G**amma
ODE	**O**rdinary **D**ifferential **E**quation
PDE	**P**artial **D**ifferential **E**quation
PIDE	**P**artial **I**ntegro-**D**ifferential **E**quation
RCLA	**R**uin **C**ontingent **L**ife **A**nnuity
SDE	**S**tochastic **D**ifferential **E**quation
SV	**S**tochastic **V**olatility
TSE	**T**oronto **S**tock **E**xchange

UMM	**U**tility **B**ased **M**artingale **M**easure
VaR	**V**alue **a**t **R**ist
VG	**V**ariance **G**amma
VOMM	**V**ariance **O**ptimal **M**artingale **M**easure

Chapter 1

Introduction

This study concerns the valuation of financial derivatives under the regime-switching framework. We present three examples in order to argue that Markovian regime-switching models can significantly improve the pricing models for financial derivatives, through capturing the impact of the structural changes in the economy. The examples are 1) a European call option, 2) a Ruin Contingent Life Annuity and 3) a Participating Product. We also present robust arguments about different facets of regime-switching models and compare the effectiveness of our models with the similar, but non-regime-switching models in the literature.

Financial derivatives are contracts between two parties that specify conditions under which payments are to be made between themselves. The value of the contracts are based on the value of an underlying asset. The history of financial derivatives can be traced back to ancient history. Thales of Miletus, referred to by Aristotle as the first philosopher in Greek tradition, was a pre-Socratic Greek philosopher and geometrician from Miletus in Asia Minor, and one of the Seven Sages of Greece. At the time of Thales, olive-farming and production of olive oil was the largest component of the Greek economy. For their production, the farmers rented olive presses from local owners. Due to his skills, Thales was approached by the press owners to predict the olive harvest in the next year. He signed a contract that granted him the privilege to use these presses in the upcoming autumn. Thales negotiated low prices successfully because the olive harvest took place in the future and the press owners wanted to hedge against a possible poor yield. With the contemporary definitions of financial derivatives, the contract signed by Thales was a European-style option on the use of olive presses in the future.

Throughout history, contracts similar to options are believed to have been used, inspired by Thales's ingenuity. Real-estate call options were widely used in the Roman Empire to assemble large parcels of land from separate owners. In London, 'puts' and 'refusals' first became well-known contracts in the 1690s during the reign of William and Mary. 'Over the counter' and 'non-tradable' put and call options on shares and commodities grew in popularity in North America during the nineteenth century. (This short history is extracted from Smith [2004])

Market participants mainly engaged with these contracts with the incentive of risk deposition. However, the fair valuation of derivatives contracts remained an open question until 1973, when Fischer Black, Myron Scholes and Robert Merton proposed the risk-neutral valuation methodology, a groundbreaking Nobel prize-wining model (see Black and Scholes [1973] and Merton [1973]). Since the introduction of the Black-Scholes-Merton model (BSM, henceforth), not only did the market for financial derivatives grew immensely in size and volume, but also the ability to determine the fair valuation of conventional option contracts facilitated innovation in modern insurance products such as equity-linked insurance products. Equity-linked or unit-linked products are modern innovations in insurance contracts that link the amount of benefits to a financial asset. This asset could be a certain stock, a stock index, or a foreign currency. For convenience, we refer to the linked investment as the *reference portfolio*.

Compared with traditional life insurance contracts, the main advantage of equity-linked policies from the policyholder's point of view resides in the fact that they have higher flexibility in the determination of their risk exposure to different asset classes. This also implies that they can achieve a better yield, adjusted to their risk preferences. Moreover,there is more transparency about the returns on the policyholder's account and the fee structure. Compared with a direct investment in a mutual fund, the policyholder can benefit from a more favorable tax treatment, in particular tax relief is often available on premiums.

We show that equity-linked products are considered as complex financial derivatives, because their pay-off structure can be decomposed to a combination of vanilla and exotic options. Therefore, the risk-neutral valuation technique can be extended to price equity-linked products. This study demonstrates that when the regime-switching risk is considered, a better pricing model and risk management strategy may be achieved for the issuer of the equity-linked products.

The remainder of this chapter is structured as follows. Section1.1 provides an overview of early option pricing models in complete markets. Section1.2 discusses the notion of market incompleteness, the different sources of incompleteness and their impact on the option pricing models. More specifically, Section1.2 describes the regime-switching framework as a source of market incompleteness. Section1.3 outlines the different remedies as outlined in the previous studies to overcome the modeling difficulties when pricing options in incomplete markets. Section 1.4 provides a brief overview of the conventional numerical methods, available in the literature, to assist analysis of pricing financial derivatives in incomplete markets. Finally, Section 1.6 provides the thesis outline.

1.1 The Early Option pricing models

A crucial element in option pricing is the assumption made about how the price of underlying assets evolve over time. BSM, as the first option valuation model, proposed a geometric Brownian motion with a constant drift and a constant relative volatility for the price process. Furthermore, Black-Scholes assumed a constant risk-free interest rate and a frictionless complete market.

The market completeness assumption ensures that the pay-off of any financial derivative can be constructed by the existing primitive financial instruments (i.e. the risky assets and the risk-free asset). In other words, a financial market is complete if contracts exist to insure against all possible eventualities. Complete markets are desirable because they enable economic agents to allocate scarce resources, invest capital, and share financial risks in a Pareto-efficient way. Additionally, complete markets in the Arrow-Debreu space provide state-of-the-art analysis of capital markets and capital structures. For example, arbitrage-free pricing is feasible only in a complete market and investor expectations are easy to infer from complete market prices. (see Magill and Quinzii [1996] and Magill and Quinzii [2003] for the detailed discussion)

From these assumptions, BSM showed that it is possible to create a risk-free (or hedged) position, consisting of a long position in the stock and a short position in the option, the value of which will not depend on the price of the underlying asset. Based on their assumptions about the dynamics of the price process of the underlying assets, Black-Scholes derived a Stochastic Differential Equation (SDE, henceforth) for the value of the hedged portfolio. Since the portfolio is instantaneously riskless, and as a consequence of the no-arbitrage assumption, it can only appreciate at the risk-free rate. Therefore, the drift term of the SDE must be equal to the risk-free rate of return. This results in the Partial Differential Equation (PDE, henceforth), the solution of which is the price of the option. Hence, the approach is often referred to as the PDE approach. The relationship between the geometric Brownian motion and the BSM PDE is a special case of the SDEs and PDEs. This relationship is more formally established under the Feynman-Kac theorem. Oksendal [2003] provides a detailed discussion and proof of Feynman-Kac theorem in the diffusion case, which were later extended to the jump-diffusion case in Oksendal and Sulem [2005].

In 1979, Harrison [1979] provided an alternative approach to price financial derivatives, using the Martingale theory. A 'martingale' is a stochastic process for which, at a particular time, the expectation of the next value in the sequence is equal to the present observed value, given knowledge of all prior observed values. The concept of martingale is entwined with 'fair game' in probability theory, and corresponds well to the 'efficient market hypothesis' in finance and economics.

To price assets under the modern portfolio theory framework, the calculated expected values need to be adjusted for an investor's risk preferences. Therefore, the discount rates would vary between investors based on their individual risk preference. As an alternative approach, Harrison [1979] argued that in a complete market with no arbitrage opportunities, one can adjust the probabilities of future outcomes such that they incorporate all investors' risk premia and then take the expectation under this new probability distribution, the equivalent martingale measure. As a result, the value of all financial securities is the expected payoff, discounted at the risk-free rate. Since the martingale approach directly corresponds to the risk-neutral valuation, the equivalent martingale measure is also known as the risk-neutral probability measure. More formally, the risk-neutral probability measure is the real probability measure with the expected rate of return on the underlying security replaced by the risk-free rate. The real probability distribution of stock returns can be estimated from the time series of past returns. However, the risk-neutral probability distribution of stock returns can be estimated from the cross-section of option prices.

Consequently, in order to price any financial derivative, it is important to determine the existence and the uniqueness of the equivalent martingale measure. These were discussed in Harrison [1979, 1981] and more comprehensively in Delbaen [1994] through the 'first' and the 'second' fundamental theorems of asset pricing. The former states that the market is arbitrage-free if, and only if, there exists at least one risk-neutral probability measure that is equivalent to the original probability measure. The latter states that the financial market is complete if, and only if, the existing risk-neutral measure is unique.

Work by Ross [1976], Cox and Ross [1976a], Constantinides [1978], Harrison [1979, 1981] provided more insightful analysis into the option pricing formulation, employing more elegant mathematical techniques. In particular, they developed the concept of the *pricing kernel* or *stochastic discount factor*, which is the ratio of the risk-neutral probability density and the real probability density, discounted at the risk-free rate. They argued that the absence of arbitrage implies the existence of a strictly positive pricing kernel.

The concept of arbitrage-free prices and the BSM framework became a standard pricing argument in financial economics. Black [1976] extended the BSM model to price options on future contracts. Heath et al. [1992] developed a model for options on bonds and the market LIBOR models for swaptions, caps and floors. These are options on discretely compounded simple interest rates and are amongst the most traded interest rate options (see also Brace et al. [1997], Jamshidian [1997], amongst many). However, the uniqueness of the risk-neutral probability measure was challenged soon after BSM. Scholars, supported by empirical evidence, argued that modeling incomplete markets is crucial to explain the counterfactual predictions of the complete market models. For instance, as opposed to complete market, where the agents are insured against idiosyncratic risks, in incomplete markets the consumption of individuals are not highly correlated with each other. In addition, the relative position in terms of wealth distribution of agents is volatile. Magill and Quinzii [2003]

In this thesis, we concentrate on predominantly one source of market incompleteness, namely, the Markovian regime-switching framework. We price three different financial derivatives, under the assumption that the structure of the economy can change in accordance to a Markov chain. The general characteristics of this framework, as well as the remedies for the consequent modeling complications are explained in the next section. The specific mathematical modeling for each product is explained in each respective chapter.

1.2 Incomplete Markets

The classic definition of incomplete markets is found in Arrow [1964]: incomplete markets refers to markets in which the number of Arrow-Debreu securities[1] is less than the number of states of nature. In the context of derivatives pricing, the market incompleteness translates to the existence of more than one martingale measure. Under these market conditions, a replicating dynamic trading policy does not exist and the perfect risk transfer[2] is not possible. Despite the ever-increasing sophistication of financial and insurance markets, markets remain significantly incomplete, with important consequences for their participants: workers and homeowners remain exposed to risks involving labor income, property value and taxes. Investors and portfolio managers have limited choices, and traders of derivative securities must bear residual risks. From a theoretical perspective, incomplete markets complicate the study of the financial market equilibrium, portfolio optimization, and derivative securities.

As apposed to complete markets, the theory of derivative securities in incomplete markets is not well understood. This profoundly impacts the practice of trading, speculating, and hedging with derivative securities. Jouini [2001] provides a distinguished survey of derivative security pricing in incomplete markets, covering no-arbitrage bounds, utility maximization, and equilibrium valuation. Cont and Tankov [2004b] provides the topic a text book treatment by expanding the discussion to quadratic and entropy criteria, as well as the model calibration issues.

The instances of market incompleteness are often attributable to the insufficient span of traded assets, market frictions and the model assumptions. In the next 3 sections, the first two are briefly discussed and referenced. Subsequently, the impact of model

[1] An Arrow-Debreu security or a state-price security may be defined as a security which pays one unit of account when a particular state of the nature is realized (see for example, Chapter 1, Duffie [2010]).

[2] Perfect risk transfer in derivatives pricing refers to the situation where the option writer(or holder) can create a self-financing portfolio consisting of the claim, underlying asset and risk free borrowing/lending that is perfectly hedged at all time.

assumptions on the market incompleteness is elaborated and referenced, as it is the focus of this thesis.

1.2.1 Insufficient span of traded assets

A fundamental assumption in risk-neutral valuation is the continuous availability of the underlying asset to trade. This assumption ensures that the payoff of the derivative can dynamically be replicated by a combination of risk-free asset and the underlying asset. When the underlying asset is not available for trading, temporarily or permanently, the holder (or issuer) of the derivative contract would not be able to create hedged positions. The following scenarios instantiate market incompleteness due to the limitations on underlying assets' tradability.

Firstly, the institutional rigidness[3] is a prominent source of incomplete markets (Boehmer and Cocquemas [2011]). Some examples of institutional rigidness includes periodic market closures and discreteness in trading opportunities and prices. Secondly, markets are incomplete when the payoffs of the derivative securities are not entirely determined by market prices, such as weather derivatives, catastrophe bonds, and derivatives written on economic variables such as the gross domestic product (see Alexandridis et al. [2012]). Thirdly, over-the-counter contracts are also issued on no-traded assets (Hung and Liu [2005]). An agent can only trade them through an over-the-counter market-makers, usually at investment banks, and by requesting a quote for 'bid' and 'ask' prices at which the market-makers are willing to buy or sell, respectively.

When there is an insufficient span of traded assets, the market-maker must bear the risk associated with the trades due to the market incompleteness. In order to measure the risk of their portfolio and manage it through hedging, they need to model the future value of the derivatives, and dynamically re-balance their portfolio, using the prices extrapolated from the market prices of some correlated assets. Related problems involving pricing derivative securities on non-traded assets has been examined in the literature, most notably in Duffie et al. [1997], Detemple and Sundaresan [1994], Zariphopoulou [2001], Musiela and Zariphopoulou [2004].

[3] In this context refers to the quality of a financial institution or the financial market, which prevents continuous access to the market or information.

Finally, corporate asset capital investment opportunities in real assets such as land, building, plant and equipment, constitute another instance of market incompleteness. Often there are options embedded in these investment opportunities, known as real options, which are very difficult to value using capital investment appraisal techniques. This is due to the fact that in real options the underlying state variables are non-tradable, so perfect hedging of these real options may be difficult, if not impossible. Some examples of these options include the option to defer or abandon a project, or the option to expand a project. McDonald and Siegel [1985], Titman [1985], Brennan and Schwartz [1985] provided the early analysis for pricing real options (i.e. options on real assets). Furthermore, Schwartz and Trigeorgis [2004] collects classical readings and recent contributions in real options and investment under uncertainty.

1.2.2 Market friction

Market friction is the measure of difficulty with which an asset is traded. Many scholars have studied market micro-structures in order to quantify market friction. A comprehensive survey of the literature is provided in O'Hara [1995]. There are two prominent approaches in measuring friction. Firstly, Demsetz [1968] argues that there are additional costs for being involved in any market, including explicit costs such as the exchange fee, and implicit costs such as *immediacy premium*. The immediacy premium is the additional cost incurred by active buyers to induce the passive sellers to transact. Secondly, Lippman and McCall [1986] measured friction by how long it takes optimally to trade a given amount of an asset. On a more practical level, there are other sources of market imperfection, due to the constraints imposed on the market participants. For example, an employee who is granted stock options is not able to hedge them by selling stock in the company. Different interest rates for borrowing and lending may be modeled by constraints as well.

With frictions in financial markets, the concept of the no-arbitrage option pricing is ill-defined. With market imperfections, the transaction prices of options generally differ from the prices that would prevail in a complete and frictionless market. Under these market conditions, a replicating dynamic trading policy does not exist. Constantindes and Perrakis [2006] provides a comprehensive literature review about different remedies for pricing options in imperfect markets.

1.2.3 Incomplete markets due to model assumptions

In contrast to the BSM model, which is based on the log-normality assumption for the returns distribution, three puzzles emerge from many empirical investigations (see Bouchaud and Potters [2003], amongst many).

- The asymmetric leptokurtic features: the return distribution is skewed to the left, and has a higher peak and two heavier tails than those of the normal distribution.
- The volatility smile: it is widely recognized that the implied volatility is a convex curve of the strike price, which is in contrast to the BSM assumption of volatility being constant.
- The volatility clustering effect: it suggests that returns distributions are best characterized by autocorrelated volatilities, as opposed to asset returns themselves with almost no autocorrelation.

Many researches have addressed the above issues. In order to incorporate the asymmetric leptokurtic features in asset pricing, a variety of models have been proposed. These include, amongst others:

1. Jump-diffusion models. Merton [1976] proposed the first extension of the BSM model, where he used the compound Poisson process to model the random jumps. For this model, an analytical solution for the derivative value is limited to very specific cases such as the European options.
2. Chaos theory, fractal Brownian motion, and stable processes. In these models, the Brownian motion is typically replaced by a fractal Brownian motion which has dependent increments, rather than independent increments. For instance, see Mandelbrot [1963,, 1967], Fama [1963, 1965]. However, Rogers and Chris [1997] pointed out that these models may lead to arbitrage opportunities.
3. Generalized hyperbolic models. These include '$\log t$' model and '$\log -hyperbolic$' model, whereby the normal distribution assumption is replaced by some other distributions; see, for instance, Barndorff-Nielsen and Shephard [2001], Praetz [1972].

4. Variance Gamma models and other time changed Brownian motions. See, for example, Clark [1973], Madan and Eugene [1990], Madan et al. [1998]. An immediate problem with these models is that it may be difficult to obtain analytical solutions for option pricing; more specifically, they might give some analytical formulae for vanilla options, but certainly not for interest rate derivatives and path-dependent options.

In parallel, different models have also proposed to incorporate the volatility smile, and the volatility clustering effect in option pricing, most notably :

1. Stochastic volatility and (generalized) autoregressive conditional heteroskedastic type models; see, Hull and White [1987], Engle [1995], Fouque et al. [2000,], Bollerslev [1986].
2. Constant elasticity variance model as proposed by Cox and Ross [1976b].
3. Affine stochastic-volatility model, in particular that presented by Heston [1993].
4. Regime-Switching models as discussed in Tong [1983], Hamilton [1989], Guo [2001], Buffington and Elliot [2002], Elliott et al. [2005]

In this thesis we investigate certain aspects of three different examples where the incompleteness results from different sources. Nevertheless, the primary source of market incompleteness is regime-switching risk. In what follows, we provide an elaborate literature review for regime-switching models.

1.2.3.1 Regime-Switching models

In this thesis we argue that regime-switching models can significantly improve the pricing models for financial derivatives. We use three examples to analyse the valuation of derivative contracts under Markovian regime-switching framework; namely, 1) a European call option, 2) a Ruin Contingent Life Annuity and 3) a Participating Product. Regime-switching framework is a potent class of models. Through the modulation of the model parameters by a Markov chain, they can simultaneously explain the asymmetric leptokurtic features of the returns' distribution, as well as the volatility smile and the volatility clustering effect. The intuition behind regime-switching models is to capture the appealing idea that the macro-economy is subjected to regular, yet unpredictable in time, states, which in turn affect the prices of financial securities. For example, inflation and recession may induce changes in the stock returns, periods of high market turbulence and liquidity crunches may increase the default risk of financial institutions, and governmental monetary policies may distort equilibrium prices for different asset classes.

The history of regime-switching models can be traced back to Quandt [1958], Goldfield and Quandt [1973] when regime-switching regression models were employed to describe nonlinearity in economic data. The idea of probability switching also appeared in the early development of nonlinear time series analysis. Here Tong [1983] proposed one of the oldest classes of nonlinear time series models, namely the threshold time series models. Subsequently, Hamilton [1989] popularized regime-switching time series models in the economic and econometric literature. Since then, considerable attention has been directed at the investigation of the regime-switching framework to model economic and financial data. Due to the empirical success of regime-switching models, the models have been applied to different areas in banking and finance, including asset allocation, option valuation, risk management, term structure modeling, and others. Recently, scholars have turned their attention to option valuation under regime-switching models, including Naik [1993], Guo [2001], Buffington and Elliot [2002], Elliott et al. [2005], amongst others.

Regime-switching models have become popular in actuarial science in recent years. For instance, Hardy [2001], Siu [2005], Siu et al. [2008] modeled different equity-linked insurance products under regime-switching economy.

In this thesis, the effect of Markovian regime-switching models on pricing options and equity-linked life insurance contracts is studied. In addition, to achieve a more accurate pricing model, the regime-switching risk is coupled with other sources of market incompleteness, in different chapters.

1.3 Remedies for Incomplete Markets

When the market is incomplete, there is not, in general, a unique no-arbitrage price for a given contingent claim. In other words, a perfect replicating strategy is not attainable, thus not all risks can be hedged. The most popular approach to price a divertive under this conditions, is to invoke some other criteria to single out a martingale measure from a continuum of existing martingale measures. As a result of the process, a fair price for contingent claims is determined by selecting a feasible price for risk.

This approach is particularly advantageous, when compared to the early approaches proposed in Jeanblanc-Picqu'e and Pontier. [1990], Shirakawa. [1991]. They addressed the issue of pricing derivatives by adding as many new assets to the sources of uncertainty as possible, in pursuant of completing the market. However, by adding new assets (or new state variables) the dimensions of the problem increases and the possible correlation between assets leads to degenerate PIDEs, with no known solution.

Therefore, we chose the first approach in our pricing methodology. Different criteria have been proposed in the literature to choose the equivalent martingale measure, most prominently:

- Minimal Martingale Measure (MMM) Follmer and Schweizer [1991]
- Variance Optimal Martingale Measure (VOMM) Schweize [1995]
- Mean Correcting Martingale Measure (MCMM)
- Esscher Martingale Measure (EMM) Gerber and Shiu [1994]
- Minimal Entropy Martingale Measure (MEMM) Frittelli [2000]
- Utility Based Martingale Measure (UMM)Davis [1997]

From the list above EMM, MEMM, and MCMM belong to a more general category, known as the *Esscher transformed martingale measures* group. The underlying argument for these methods is based on minimization of relative entropy of a probability measure with respect to another probability measure (also called Kullback-Leibler information number or I-divergence), and its minimization over a convex set of measures.

In our pricing methodologies in the subsequent chapters, EMM and MEMM are our methods of choice. This is because minimization of the relative entropy has been shown to be linked with the maximization of expected utility in the case of an exponential utility function[4]. As a case in point, Buhlmann et al. [1996] showed that in a financial market model maximizing expected exponential utility over a set of admissible trading strategies is dual to finding the entropy minimizing martingale measure for the price process.

The Esscher transformation is a very useful method technique to obtain a reasonable equivalent martingale measure and it is related to the corresponding risk process. Esscher introduced the idea of risk function and transformed risk function for the calculation of collective risk in Esscher [1932]. The idea soon became very popular in actuarial science and risk management. Later, the pioneer work by Gerber and Shiu [1994] extended the implication of the Esscher transform to option pricing, by providing a pertinent solution to the pricing model in incomplete markets. There are two kinds of Esscher transformed martingale measures: 1) The compound return Esscher transformed martingale measure, introduced by Gerber and Shiu [1994]. 2) The simple return Esscher transformed martingale measure, identified with the MEMM (see Elliott et al. [2005]). The latter is particularly useful for investigating and characterizing the existence of the minimal entropy martingale measure in concrete models in an analytically tractable way.

1.4 Numerical methods in options pricing

In the global financial markets, vanilla options are traded along with options that can often have far more complicated payoffs (i.e. exotic options). While sometimes we can find close form solutions for certain exotic derivetives (e.g. European Barriers and Lookback options), for other products a close form solution does not exist. Additionally, more complex model assumptions often results to higher dimensionality of the pricing process, therefore harder to discover analytical solutions. In these cases, the only way a market participant will be able to obtain a price is by using an appropriate numerical method.

Path-dependency is a common reason, for which one may consider a numerical procedure. Path-dependent options are options whose payoffs depend on historical values of the underlying asset over a given time period as well as its current value. Well-known

[4]An exponential utility function may be considered one of the major classes of utility functions used in finance and actuarial science, which utility implies constant absolute risk aversion, with coefficient of absolute risk aversion equal to a constant. In the standard model of one risky asset and one risk-free asset this feature implies that the optimal holding of the risky asset is independent of the level of initial wealth; thus on the margin any additional wealth would be allocated totally to additional holdings of the risk-free asset.(see Cvitanić and Zapatero [2004] for further discussions)

examples are look-backs, which give their owners the right to buy (sell) the asset at an exercise price equal to the minimum (maximum) price of the asset over the life of the option.

Regime-switching models can also produce path-dependency, since the price evolution for the underlying asset could result in different values, depending on the duration that the Markov chain had occupied each state. In this study we examine different numerical schemes to 1) treat the path-dependency issue, imposed by the the Markovian regime-switching model and 2) solve partial differential equations with no known analytical solutions.

Three different numerical procedures are predominantly used in derivatives pricing literature; namely, the lattice model, Monte Carlo simulation, and numerical solutions for the differential equation.

1.4.1 Lattice model

A lattice model can be used to find the fair value of different financial derivatives. The model divides time between now and the option's expiration into a number of discrete periods. For any specific time (or node), the model has a finite number of outcomes in the next period, such that every possible change in the state of the world within the period is captured in a branch. This is an iterative process that maps all the possible paths of the evolution of asset price until the expiration of the option. Accordingly, the risk-neutral probabilities are calculated using the risk-neutral argument, so that through the backward induction of the option prices, the option values could be calculated for each node. The backward induction is the process that starts with the terminal value of the option (intrinsic value of the option at the expiry) for the terminal points and moves backwards in time by calculating the values for the previous points using discounted expectation under risk-neutral probabilities.

The simplest lattice model for options is the binomial options pricing model, which was first introduced by Cox, Ross and Rubenstein in their honored 1979 paper Cox et al. [1979]. For some types of options such as the American options, using an iterative model is the only choice since there is no known closed-form solution that predicts price over time. Ever since Cox et al. [1979], various variations of trees have been proposed in the literature to account for different features of derivatives pricing models. In the second chapter of this thesis, the regime-switching version of the binomial model is proposed for the valuation of vanilla calls. We present the path-dependent structure created due to the state dependency of the variables. Our latices model calculates the fair value of the option through pricing the regime-switching risk.

1.4.2 Monte Carlo simulation

Monte Carlo methods are a broad class of computational algorithms that rely on repeated random sampling from an assumed distribution or model. In financial engineering, Monte Carlo simulation is often designed to generate sample paths of asset prices(or returns) based on the model assumptions for the asset dynamics. In derivatives pricing models, the payoff of the contract is calculated for each generated path, and the fair value of the derivative is approximated when some convergence criteria are met.

The method was first introduced to finance in Boyle [1977] and is generally quite easy to implement as it can be used without too much difficulty to value a large range of European style exotic options. The Monte-Carlo simulation is reliable, and providing enough sample paths are taken, there can be a high statistical confidence in the accuracy of the prices. It is, therefore, often used as the benchmark valuation technique for many complex, European-style exotic options.

However, this accuracy can often come at a large computational cost. To generate sufficiently accurate prices, a large number of paths must be generated. If some kind of path-dependent exotic is to be priced, then the value of the asset on each point of the paths need to be calculated. For instance, in a typical option valuation problem, one must consider storing and evaluating 10^9 asset values. This creates an immense amount of computational time and processing expense, which is not attractive from a practical point of view. Another draw-back using the Monte Carlo simulation is the fact that it is very difficult to be implemented for American style contracts, due to the early exercise feature.

After the seminal paper by Boyle [1977], the paper by Boyle et al. [1997] may perhaps be a more recent representative literature on the use of Monte Carlo method for option pricing. This paper also discussed the use of Quasi Monte Carlo method for option pricing which was introduced by Joy et al. [1996]. The Quasi Monte Carlo method is based on low discrepancy sequences, which are deterministic, instead of pseudo random numbers. The rate of convergence of Quasi Monte Carlo method is the reciprocal of the number of simulations while the rate of convergence of Monte Carlo method is the reciprocal of square root of the number of simulations. Furthermore, in recent years, there have been a number of advancements in more efficient Monte Carlo based techniques for derivatives valuation. For instance, Fouque and Han [2004] used the variance reduction technique for Monte Carlo methods to evaluate option prices under multi-factor stochastic volatility models. McLeish [2011] and Schlogl [2013] provide a comprehensive review of literature and discuss some modern applications of Monte Carlo simulation in derivatives pricing.

1.4.3 Numerical solutions for the differential equation

The third class of numerical methods for options pricing targets solving the pricing partial differential equations (PDE). These numerical schemes, which were originated from applied mathematics and physics, have rapidly grown in popularity, due to their reasonable accuracy of the results, the simplicity in the implementation and the speed of execution (particulary, compared to Monte Carlo based algorithms). The methods rely on discretization of the pricing PDEs and calculation of each nodal value using the boundary conditions, through backward induction. The discretization can be in both times and space dimensions (as in Finite Difference Method, FDM) or just the space dimension (as in Finite Element Method, FEM).

The FDM was first applied in Brennan and Schwartz [1978] to solve derivatives-pricing problems with jump. In this method the derivative terms (both time and space)are replaced by their respective numerical approximation values (or the Finite Difference approximations). Therefore, the by-product of the procedure is the numerical approximation of the so-called Greeks, which are the sensitivity of the option value to time and different state variables. This is a, particularly, attractive feature for practical purposes, due to the importance of Greeks in risk management. Due to the simplicity and the popularity of the method, many textbooks on option pricing, such as Wilmott [1998], provide thorough introductions to the approximate solution of differential equations arising in finance with FDM.

FEM is a more powerful procedure, compared to FDM, especially, when dealing with irregular shapes of partial integro-differential equations or high curvature in the value function. Additionally, the method computes a solution for the entire domain, instead of isolated nodes as in the case of FDM. The basic idea of FEM is to approximate the solution of a given differential equation with a set of algebraically simple functions. The spatial domain of the differential equation is divided into sub-domains called elements. For each element, the parameters of this function are usually different. The functions are equal with respect to the function type but different with respect to the values of the parameters. Each of these functions has only local support, that is, outside a small number of elements it takes on the value zero. The elements are non-overlapping and cover the domain on which the differential equation is defined. Topper [2005] provides a comprehensive discussion on different approaches to Finite Elements Method and their application in economics and finance.

In this thesis we employ both FDM and FEM for evaluating the value functions in Chapter3 and Chapter4, respectively.

1.5 Valuation of Equity-linked insurance products

This study gives a 'financial engineering' treatment to valuation of equity-linked insurance products. One distinguishing characteristic of equity-linked products, compared to the traditional insurance contracts, is the *principle of equivalence*, which states that the company's income and expense should balance in the long term. In the case of traditional insurance products, actuarial techniques are used in tandem with financial valuation theories to price such products. However, under the principle of equivalence, it is assumed that the financial risk factors, mortality and risk neutrality are independent. That is the insurance company is not expected to receive any compensation for accepting the mortality risk. The assumption is also justified by the *pooling argument* which states that the insurer can eliminate the mortality risk by adequately increasing the number of identical and independent contracts in his portfolio.

The above argument allow us to treat equity-linked contacts as a combination of some vanilla or exotic financial derivatives. In Chapter3 and Chapter4 we price two different equity-linked contracts. We show that similar to Chapter2 standard no-arbitrage arguments do not provide unique prices, hence the market is incomplete.

Early work on pricing equity-linked contracts using a combination of no-arbitrage arguments and actuarial principles was carried out by Brennan and Schwartz [1976], Boyle and Schwartz [1977]. They recognized that the payoff from an individual equity-linked contract at expiration is identical to the payoff from an European call option plus a certain amount (the guarantee amount) or to the payoff from an European put option plus the value of the reference portfolio. Further considerations with deterministic interest rate have been discussed in Aase and Persson [1994] and Persson [1993], where they applied a continuous-time model of mortality. These two papers established a connection between the classical Thiele equation of the actuarial sciences and the BSM equation.

The advancements in the option pricing literature, simultaneously benefited the models for pricing equity-linked products. For example, Delbaen [1986] used Monte Carlo simulation for pricing the contracts, where a close form solution could not be found. Bacinello and Ortu [1993c,a,b], Nielsen and Sandmann [1995] applied the martingale-pricing theory credited to Harrison [1979]. The main achievement by this methodology is that the single premium of an insurance contract may be calculated as an expectation under a risk-adjusted probability measure. Further, Bacinello and Ortu [1993b] showed that the assumption of deterministic interest rates cannot conform interest rate risk in the real market. The paper allowed for interest rate risk by assuming an Ornstein-Uhlenbeck process implying a closed form solution of the single premium endowment policy. Many other researchers tried to achieve a better price, by taking more complex model assumptions into consideration. For instance, Kassberger et al. [2008], Le Courtois and Quittard-Pinon [2008] studied the problem under the jump-diffusion framework. Wilkie [1995], Hardy et al. [2006] considered (Generalized) Autoregressive Conditional Heteroscedastic models. Additionally, there have been a number of researches on pricing the equity-linked products under the regime-switching framework, including, Hardy [2001], Siu [2005], Siu et al. [2008].

1.6 Research Question and Thesis Outline

In this thesis, we price three different financial derivatives under the Markovian regime-switching framework. We primarily focus on 1) the analysis of market incompleteness, due to the regime-switching risk; 2) the usefulness of some numerical methods, when pricing financial derivatives under the Markovian regime-switching framework.

We investigate the problem of market incompleteness, appeared under different model assumptions. The only source of market incompleteness in Chapter2 is the regime-switching risk. In Chapter3 and Chapter4, nevertheless, our models also incorporate random jumps, which is an additional source of market incompleteness. As discussed in Section1.3, in our pricing methodologies for all chapters, we only consider methods that minimizes the entropy between the equivalent martingale measure and the historical measure. Buhlmann et al. [1996] showed that the method is dual to maximizing expected exponential utility over a set of admissible trading strategies. Therefore, we either use the Minimum Entropy Martingale Measure or Esscher Transform for this purpose. As previously discussed, these two methods correspond to the same martingale measured, therefore, it maintains the cohesion of the pricing methodology across the thesis.

In this study, we also investigate the usefulness of some numerical methods, when pricing financial derivatives under the Markovian regime-switching framework. In Chapter2, we develop a lattice approach for the valuation of a vanilla call option. In Chapter3, we utilize the Finite Difference method for pricing ruin contingent life annuities. In Chapter4, we consider the Finite Element method to approximate the solution for a partial integro-differential equation and ultimately value a participating life-insurance product.

1.6.1 Thesis Outline

In the subsequent chapters, we analysed the problem of pricing financial derivatives from three different angels. Nevertheless, every chapter is closely bonded to the others, from the methodological point of view.

In Chapter2 we consider the pricing of vanilla European call options. We implement the regime-switching version of the lattice approach in the valuation of the option. More importantly, we develop a method to price the regime-switching risk, through calculating the risk-neutral transition probabilities of the Markov chain (in addition to the risk-neutral probabilities of the evolution of the asset), using MEMM. Finally, due to the market incompleteness, there is no strategy available, consisting of the primitive assets, that perfectly replicates the payoff of the option. Consequently, the imperfect nodal hedging results in a nodal hedging error, which has important implications from the risk management point of view. We evaluate the nodal hedging error, through the implementation of a simple Delta hedging strategy. Then we approximate the terminal distribution of hedging error, with which we calculate two important risk metrics, namely Value at Risk and Expected Shortfall.

In Chapter3, we consider the pricing of a Ruin Contingent Life Annuity, RCLA, which is a modern innovation in the equity-linked life insurance products. We use the regime-switching version of the Variance Gamma process to model the evolution of the underlying reference portfolio. The source of incompleteness is not only the regime-switching risk, but also the random jump process. We employ the Esscher Transform to choose the equivalent martingale measure, under which we derive the risk-neutral dynamics for the asset. Then we determine the PIDE for pricing the contract and we solve it numerically, using the FD method. Finally, we implement a simple Delta hedging strategy, using the FD approximation for the Delta and Monte Carlo Simulation for the underlying portfolio. Then we approximate the terminal distribution of hedging error, with which we calculate two important risk metrics, namely Value at Risk and Expected Shortfall.

In Chapter4, we propose a model for the valuation of the participating life insurance product under a generalized jump-diffusion model with a Markov-switching compensator. We employ the Esscher transform to determine an equivalent martingale measure in the incomplete market. The results are further manipulated through the utilization of the change of numeraire technique to reduce the dimensions of the pricing formulation. This paper is the first that extends the technique for a generalized jump-diffusion process with a Markov-switching kernel-biased completely random measure which nests a number of important and popular models in finance. Similar to Chapter3, due to the path dependency for the payoff of the contract, no close form solution exists for the problem. Therefore, we implement the collocation method, a subclass of the FE method, to find the numerical approximation for the value function. Finally, the results are compared to the conventional Merton model, through a numerical example.

Our study shows that 1) the regime-switching risk can be priced using MEMM method, 2) Esscher transform is a computationally convenient and an economically efficient method of pricing financial derivatives in incomplete markets, 3) under certain conditions, similar to Chapter4, the utilization of the change of numeraire technique can simplify the value function, through the reduction of dimensionality; and 4) an appropriate numerical method may approximate the value function very efficiently.

Chapter5 concludes the thesis, and provides thorough discussion for the in future possibilities in the research area. Despite the recent attention to the regime-switching literature, not all facets of it has been thoroughly understood. Further research may shed light on different theoretical concepts and practical implications of this class of models.

Chapter 2

A Regime-Switching Binomial Model for Pricing and Risk Management of European Options

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Abstract

We discuss the pricing and risk management problems of standard European-style options in a Markovian regime-switching binomial model. Due to the presence of an additional source of uncertainty described by a Markov chain, the market is incomplete, so the no-arbitrage condition is not sufficient to fix a unique pricing kernel, hence, a unique option price. Using the minimal entropy martingale measure, we determine a pricing kernel. We examine numerically the performance of a simple hedging strategy by investigating the terminal distribution of hedging errors and the associated risk measures such as Value at Risk and Expected Shortfall. The impacts of the frequency of re-balancing the hedging portfolio and the transition probabilities of the modulating Markov chain on the quality of hedging are also discussed.

Keywords Binomial tree - Regime-switching - Minimum entropy martingale measure

2.1 Introduction

The history of regime-switching models can be dated back to the early works of Quandt [1958], Goldfield and Quandt [1973] where regime-switching regression models were used to describe nonlinearity in economic data. The idea of probability switching also appeared in early development of nonlinear time series analysis, where Tong [1983] proposed one of the oldest classes of nonlinear time series models, namely the threshold time series models. Hamilton [1989] popularized regime-switching time series models in the economic and econometric literature. Since then, considerable attention has been paid to investigate the use of regime-switching models to model economic and financial data. Regime-switching models have become popular in actuarial science in recent years. Hardy [2001] provided sound empirical evidence to support the use of a discrete-time regime-switching lognormal model for fitting long-term investment returns using the S&P 500 and the TSE 300 indices.

Due to their empirical success, regime-switching models have been applied to different important areas in banking and finance, including asset allocation, option valuation, risk management, term structure modeling, and others. Recently the spotlight seems to turn to option valuation under regime-switching models. Early works in this area include Naik [1993], Guo [2001], Buffington and Elliot [2002], Elliott et al. [2005], amongst others. These works deal with the option valuation problem under continuous-time regime-switching models. The option valuation problem in a discrete-time regime-switching model has received relatively little attention in the literature, (see, for example, Liew and Siu [2010]). Nevertheless, the investigation of the option pricing problem in a discrete-time regime-switching framework is certainly important from the practical perspective, since in practice data is monitored discretely over time and a discrete-time model is relatively easy to estimate than its continuous-time counterpart.

The main challenge of option valuation in regime-switching models is that the market in a regime-switching model is, in general, incomplete. This is attributable to the additional source of uncertainty described by the modulating Markov chain. Consequently, there is more than one equivalent martingale measure, and hence, more than one no-arbitrage price for an option. Different approaches have been proposed for pricing and hedging derivative securities in incomplete financial markets. Follmer and Sondermann

[1986], Follmer and Schweizer [1991] and Schweize [1995] selected an equivalent martingale measure by minimizing the quadratic utility of the terminal hedging errors. Davis [1997] adopted an economic approach based on the marginal rate of substitution to pick a pricing measure via a utility maximization problem. Gerber and Shiu [1994] pioneered the use of the Esscher transform, a time-honored tool in actuarial science, for option valuation in an incomplete market. Elliott et al. [2005] proposed the use of the Esscher transform to select a martingale pricing measure in a continuous-time, Markovian regime-switching market. The Esscher transform provides a convenient way to select a pricing measure in the regime-switching market. One of the key features of the Esscher transform in Elliott et al. [2005] is that the regime-switching risk was not priced. This is evidenced by the fact that the probability laws of the modulating Markov chain does not change after the measure change by the Esscher transform. Siu and Yang [2009] extended the Esscher transform in Elliott et al. [2005] to take into account explicitly the regime-switching risk in the specification of a pricing kernel. Elliott and Siu. [2011b,a] considered a general pricing kernel which is defined by the product of two density processes, one for a measure change for a diffusion process and another one for a measure change for a Markov chain. It appears that the existing literature focuses on investigating the pricing of regime-switching risk in a continuous-time regime-switching market. However, a relatively little attention has been paid to discussing the pricing of regime-switching risk in a discrete-time regime-switching model.

The purpose of this paper is to investigate the pricing and hedging of vanilla European options in a discrete-time Markovian regime-switching binomial model. Particular focus is placed on pricing regime-switching risk. Here we specify the parametric form of a pricing kernel by a product of two density processes, one for a measure change for the discrete-time binomial model and another one for a discrete-time Markov chain. This pricing kernel allows the pricing of both market risk due to price movements and the regime-switching risk. In this situation, the martingale condition in the fundamental theorem of asset pricing is not sufficient to fix a pricing kernel. Here we adopt the minimal entropy martingale measure (MEMM) approach to fix a pricing kernel and obtain a closed-form expression for the 'risk-neutral' transition probabilities of the modulating Markov chain. We present a computationally efficient method to price options in such a modeling framework. For the hedging of options, we investigate consequences of not hedging the regime-switching risk. In particular, we consider a simple delta hedging strategy, which does not take into account regime-switching risk, and document the potential risk inherent from adopting such a simple hedging strategy. We examine the risk measures such as Value at Risk (VaR) and the Expected Shortfall (ES), for the terminal hedging errors associated with the delta hedging strategy.

The rest of this paper is organized as follows. The next section presents the discrete-time, Markovian regime-switching binomial model. In Section2.3, we discuss the specification of the parametric form or a pricing kernel using the product of two density processes. The use of the MEMM approach to pick a pricing kernel is discussed in Section2.4. We also describe the dynamic delta hedging strategy to be used in our numerical experiment. We present the numerical results of option valuation and the risk measures for the hedging errors associated with the delta hedging strategy in Section2.6. We conclude our paper by discussing a heuristic hedging strategy which takes into account the regime-switching risk.

2.2 The Markovian Regime-Switching Binomial Tree

We consider a discrete-time, Markovian regime-switching, binomial model consisting of a bond B and a share S. To describe uncertainty, we consider a complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$, where $\mathbb{P}$ is the historical probability measure. Let $\mathcal{T} := \{0, 1, 2, \cdots, T\}$ be the time parameter set, where transactions take place in the time points in $\mathcal{T}$.

To describe the evolution of economic conditions over time, we consider a discrete-time, finite-state, Markov chain $\{\mathbf{X}_t\}_{t\in\mathcal{T}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in the canonical state space $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N\} \subset \mathbb{R}^N$, where the j^{th}-component of $\mathbf{e}_i$ is the Kronecker product δ_{ij}, for each $i, j = 1, 2, \cdots, N$. To describe the probability laws of the Markov chain under $\mathbb{P}$, we define the transition probability matrix $\mathbf{A} := [a_{ji}]_{i,j=1,2,\cdots,N}$, where

$$a_{ji} := \mathbb{P}(\mathbf{X}_{t+1} = \mathbf{e}_j | \mathbf{X}_t = \mathbf{e}_i) \ .$$

That is, a_{ji} is the transition probability of the chain $\{\mathbf{X}_t\}_{t\in\mathcal{T}}$ from state $\mathbf{e}_i$ to state $\mathbf{e}_j$.

Let r be the constant risk-free interest rate of the bond B, where $r > 0$. Then the bond price process $\{B_t\}_{t\in\mathcal{T}}$ evolves over time as:

$$B_{t+1} = B_t e^r \ , \quad B_0 = 1 \ .$$

We now specify the share price process $\{S_t\}_{t\in\mathcal{T}}$. For each $t \in \mathcal{T}\backslash\{0\}$, let $\omega_t := S_t/S_{t-1}$ be the return from the share S from time $t-1$ to time t. We suppose that for each $t \in \mathcal{T}\backslash\{0\}$,

$$\omega_t \in \{u(\mathbf{X}_t), d(\mathbf{X}_t)\} \ ,$$

where $u(\mathbf{X}_t)$ and $d(\mathbf{X}_t)$ are modulated by the chain as follows:

$$\begin{aligned} u(\mathbf{X}_t) &:= \langle \mathbf{u}, \mathbf{X}_t \rangle \ , \\ d(\mathbf{X}_t) &:= \langle \mathbf{d}, \mathbf{X}_t \rangle \ . \end{aligned}$$

Here $\mathbf{u} := (u_1, u_2, \cdots, u_N)' \in \mathbb{R}^N$ and $\mathbf{d} := (d_1, d_2, \cdots, d_N)' \in \mathbb{R}^N$; $\mathbf{y}'$ is the transpose of a matrix, or a vector, $\mathbf{y}$; $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^N$; u_i (resp. d_i) is an upward jump (resp. a downward jump) of the share price at time t when the chain $\mathbf{X}_t = \mathbf{e}_i$. In other words, the upward jump and the downward jump of the share price depend on the state of the chain $\{\mathbf{X}_t\}_{t\in\mathcal{T}}$.

Let $\mathbb{F}^\omega := \{\mathcal{F}_t^\omega\}_{t\in\mathcal{T}}$ be the $\mathbb{P}$-completed, natural filtration generated by the return process $\{\omega_t\}_{t\in\mathcal{T}\backslash\{0\}}$, where for each $t \in \mathcal{T}\backslash\{0\}$, $\mathcal{F}_t^\omega := \sigma\{\omega_1, \omega_2, \cdots, \omega_t\} \vee \mathcal{N}$, $\mathcal{F}_0^\omega = \sigma\{\emptyset, \Omega\}$ and $\mathcal{N}$ is a $\mathbb{P}$-null set. We suppose that for each $t \in \mathcal{T}$ and each $j = 1, 2, \cdots, N$,

$$p_{t,j} := \mathbb{P}(\omega_t = u(\mathbf{X}_t)|\mathbf{X}_t = \mathbf{e}_j, \mathcal{F}_t^\omega) = \mathbb{P}(\omega_t = u_j) \ ,$$

so that $\{\omega_t\}_{t\in\mathcal{T}}$ is a sequence of independent random variables conditional on the sample path of the Markov chain $\{\mathbf{X}_t\}_{t\in\mathcal{T}}$.

Then for each $t \in \mathcal{T}$, the probability of a downward jump is:

$$1 - p_{t,j} = \mathbb{P}(\omega_t = d(\mathbf{X}_t)|\mathbf{X}_t = \mathbf{e}_j, \mathcal{F}_t^\omega) = \mathbb{P}(\omega_t = d_j) \ .$$

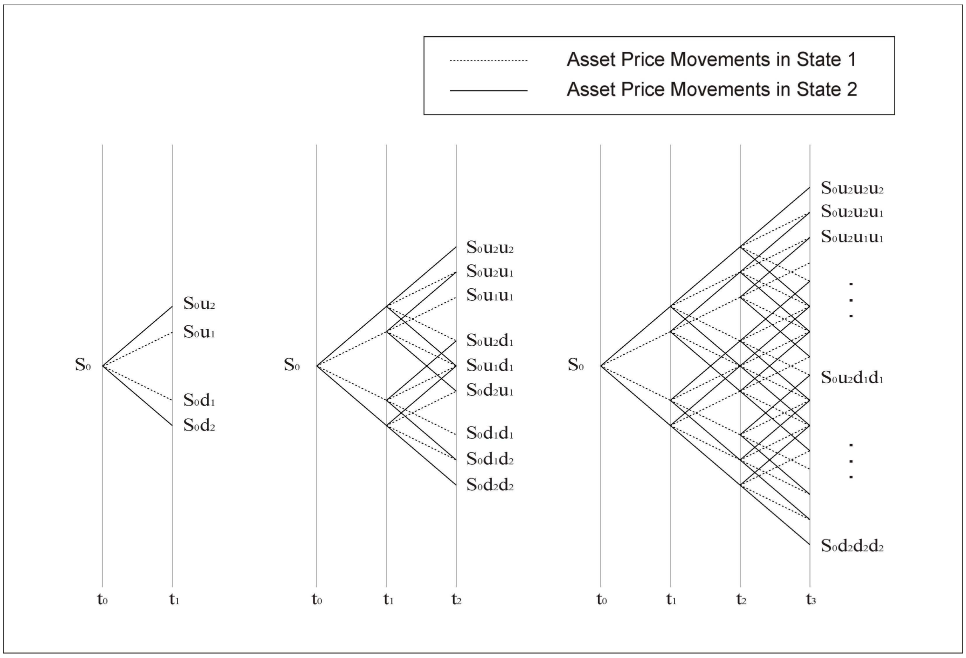


FIGURE 2.1: Evolutions of the price of the underlying asset under Markovian Regime-Switching Binomial Tree, with two states in the economy

Let $\mathbb{F}^{\mathbf{X}} := \{\mathcal{F}_t^{\mathbf{X}}\}_{t\in\mathcal{T}}$ be the $\mathbb{P}$-completed, natural filtration generated by the chain $\{\mathbf{X}_t\}_{t\in\mathcal{T}}$, where for each $t \in \mathcal{T}\backslash\{0\}$, $\mathcal{F}_t^{\mathbf{X}} := \sigma\{\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_T\} \vee \mathcal{N}$ and $\mathcal{F}_0^{\mathbf{X}} := \sigma\{\emptyset, \Omega\}$.

For each $t \in \mathcal{T}$, we define:

$$\mathcal{G}_t := \mathcal{F}_t^{\omega} \vee \mathcal{F}_t^{\mathbf{X}} .$$

This is the minimal σ-field containing both $\mathcal{F}_t^{\omega}$ and $\mathcal{F}_t^{\mathbf{X}}$. Write $\mathbb{G} := \{\mathcal{G}_t\}_{t\in\mathcal{T}}$.

The Markovian regime-switching binomial model consists of two sources of risk, one inherent from the binomial process and another one inherent from the Markov chain. Given that there are two primitive securities in the model, the market model is, in general, incomplete. Unlike the situation of the standard CRR binomial model, there is more than one equivalent martingale measure, or pricing kernel, in the Markovian regime-switching binomial model. Consequently, a key question is how we can select an equivalent martingale measure in such a market model.

The structure of the Markovian regime-switching binomial model is depicted in Figure (2.1), where a three-step, two-regime, binomial tree is displayed. The tree is recombining at both the levels of the binomial process and the Markov chain.

2.3 A Pricing Kernel and MEMM

In this section, we shall discuss a pricing kernel which is defined by the product of two density processes, one for a measure change for the binomial process and another one for a measure change for the Markov chain. This pricing kernel can price both the price risk and the regime-switching risk. Since the parametric form of the pricing kernel involves two 'prices of risk', the martingale condition is not sufficient to fix a pricing kernel. Additional criteria are required. Here we employ the minimal equivalent martingale measure (MEMM) to determine a pricing kernel. We obtain a closed-form expression for the 'risk-neutral' transition probability matrix for the Markov chain.

Firstly, we define the density process for the measure change for the binomial process. Consider the $\mathbb{G}$-adapted process $\{\lambda_t^S\}_{t\in\mathcal{T}\setminus\{0\}}$ defined by:

$$\lambda_t^S := \sum_{j=1}^{N} \frac{(q_{t,j})^{\mathbb{I}_{\{w_t=u(\mathbf{e}_j)\}}}(1-q_{t,j})^{1-\mathbb{I}_{\{w_t=u(\mathbf{e}_j)\}}}}{(p_{t,j})^{\mathbb{I}_{\{w_t=u(\mathbf{e}_j)\}}}(1-p_{t,j})^{1-\mathbb{I}_{\{w_t=u(\mathbf{e}_j)\}}}} \times \langle \mathbf{X}_t, \mathbf{e}_j\rangle \ .$$

Here $q_{t,j} \in (0,1)$ and $\mathbb{I}_{\{w_t=u(\mathbf{e}_j)\}}$ is the indicator function of the event $\{w_t = u(\mathbf{e}_j)\}$.

Furthermore, we define the $\mathbb{G}$-adapted process $\{\Lambda_t^S | t \in \mathcal{T}\}$ by putting:

$$\Lambda_t^S := \prod_{u=1}^{t} \lambda_u^S \ , \quad \Lambda_0^S = 1 \ .$$

It is then easy to check that $\{\Lambda_t^S\}_{t\in\mathcal{T}}$ is a $(\mathbb{G},\mathbb{P})$-martingale. This is the density process for a measure change for the binomial process.

We now define the density process for a measure change for the Markov chain. Let $\mathbf{C} := [c_{ji}]_{i,j=1,2,\cdots,N}$, where

1. for each $i, j = 1, 2, \cdots, N$, with $i \neq j$, $\sum_{j=1}^{N} c_{ji} = 1$;
2. $c_{ji} \geq 0$, so $c_{ii} \leq 0$.

Consider the $\mathbb{F}^{\mathbf{X}}$-adapted process $\{\lambda_t^{\mathbf{X}}\}_{t\in\mathcal{T}\setminus\{0\}}$ defined by:

$$\lambda_t^{\mathbf{X}} := \prod_{i,j=1,i\neq j}^{N} \left(\frac{c_{ji}}{a_{ji}}\right)^{\langle \mathbf{X}_{t-1,\mathbf{e}_j}\rangle\langle \mathbf{X}_{t,\mathbf{e}_i}\rangle} \ .$$

Then we define the $\mathbb{F}^{\mathbf{X}}$-adapted process $\{\Lambda_t^{\mathbf{X}}\}_{t\in\mathcal{T}}$ by setting:

$$\Lambda_t^{\mathbf{X}} := \prod_{u=1}^{t} \lambda_u^{\mathbf{X}} \ , \quad \Lambda_0^{\mathbf{X}} = 1 \ .$$

Again it is easy to see that $\{\Lambda_t^{\mathbf{X}}\}_{t\in\mathcal{T}}$ is an $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingale. This is the density process for a measure change for the Markov chain.

Define the $\mathbb{G}$-adapted process $\Lambda := \{\Lambda_t\}_{t\in\mathcal{T}}$ by putting:

$$\Lambda_t := \Lambda_t^{\mathbf{X}} \cdot \Lambda_t^{S} \ ,$$

so Λ is a $(\mathbb{G}, \mathbb{P})$-martingale. This then implies that $\mathrm{E}[\Lambda_T] = 1$.

A probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{G}_T$ is then defined by setting:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{G}_T} := \Lambda_T \ .$$

Then we have the following lemma.

Lemma 2.1. *Under $\mathbb{Q}$, the transition probability matrix of the chain $\mathbf{X}$ is $\mathbf{C}$.*

Proof: Let $\mathrm{E}^{\mathbf{C}}$ be the expectation under $\mathbb{Q}^{\mathbf{C}}$. Write I_E for the indicator function of an event E. Then by a version of the Bayes' rule, the Markovian property of the chain $\mathbf{X}$ and the martingale property of $\Lambda^{\mathbf{X}}$,

$$\begin{aligned}
& \mathbb{Q}^{\mathbf{C}}[\mathbf{X}_{t+1} = \mathbf{e}_j | \mathbf{X}_t = \mathbf{e}_i] \\
= \ & \mathrm{E}^{\mathbf{C}}[I_{\{\mathbf{X}_{t+1}=\mathbf{e}_j\}} | \mathbf{X}_t = \mathbf{e}_i] \\
= \ & \frac{\mathrm{E}[\Lambda_{t+1}^{\mathbf{X}} I_{\{\mathbf{X}_{t+1}=\mathbf{e}_j\}} | \mathbf{X}_t = \mathbf{e}_i]}{\mathrm{E}[\Lambda_{t+1}^{\mathbf{X}} | \mathbf{X}_t = \mathbf{e}_i]} \\
= \ & \mathrm{E}[\lambda_{t+1}^{\mathbf{X}} I_{\{\mathbf{X}_{t+1}=\mathbf{e}_j\}} | \mathbf{X}_t = \mathbf{e}_i] \\
= \ & \mathrm{E}\Bigg[\prod_{i,j=1,,i\neq j}^{N} \left(\frac{c_{ji}}{a_{ji}}\right)^{\langle \mathbf{X}_t, \mathbf{e}_i\rangle \langle \mathbf{X}_{t+1}, \mathbf{e}_j\rangle} I_{\{\mathbf{X}_{t+1}=\mathbf{e}_j\}} | \mathbf{X}_t = \mathbf{e}_i \Bigg] \\
= \ & \left(\frac{c_{ji}}{a_{ji}}\right) \mathrm{E}[I_{\{\mathbf{X}_{t+1}=\mathbf{e}_j\}} | \mathbf{X}_t = \mathbf{e}_i] \\
= \ & c_{ji} \ .
\end{aligned}$$

Hence, the result follows. □

By the fundamental theorem of asset pricing, (see Harrison and Kreps [1979], Harrison and Pliska [1981, 1983]), the absence of arbitrage opportunities in a market is essentially equivalent to the existence of an equivalent martingale measure under which discounted asset prices are martingales. The latter condition is called a martingale condition. In our current context, the martingale condition states that there is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{G}_T$ under which the following condition is satisfied:

$$\mathrm{E}^{\mathbb{Q}}[e^{-r}S_{t+1}|\mathcal{G}_t] = S_t \ , \quad \mathbb{P}\text{-a.s} \ , \quad t \in \mathcal{T} \ .$$

Conditional on $\mathbf{X}_{t+1} = \mathbf{e}_j$, the martingale condition becomes:

$$\mathrm{E}^{\mathbb{Q}}[e^{-r}S_{t+1}|\mathcal{G}_t, \mathbf{X}_{t+1} = \mathbf{e}_j] = S_t \ , \quad \mathbb{P}\text{-a.s} \ , \quad t \in \mathcal{T} \ .$$

We call this the conditional martingale condition given $\mathbf{X}_{t+1} = \mathbf{e}_j$. Then we have the following lemma.

Lemma 2.2. *The conditional martingale condition holds if and only if for each* $t \in \mathcal{T}\backslash\{0\}$ *and* $j = 1, 2, \cdots, N$,

$$q_{t,j} = \frac{e^r - d_j}{u_j - d_j} \ .$$

Proof: The conditional martingale condition is given by:

$$\mathrm{E}^{\mathbb{Q}}[e^{-r}S_{t+1}|\mathcal{G}_t, \mathbf{X}_{t+1} = \mathbf{e}_j] = S_t \ .$$

This, if and only if,

$$\mathrm{E}^{\mathbb{Q}}[\omega_{t+1}|\mathcal{G}_t, \mathbf{X}_{t+1} = \mathbf{e}_j] = e^r \ .$$

By a version of the Bayes' rule,

$$
\begin{aligned}
& \mathrm{E}^{\mathbb{Q}}[\omega_{t+1}|\mathcal{G}_t, \mathbf{X}_{t+1} = \mathbf{e}_j] \\
= & \frac{\mathrm{E}[\Lambda_{t+1}\omega_{t+1}|\mathcal{G}_t, \mathbf{X}_{t+1} = \mathbf{e}_j]}{\mathrm{E}[\Lambda_{t+1}|\mathcal{G}_t, \mathbf{X}_{t+1} = \mathbf{e}_j]} \\
= & \frac{\mathrm{E}[\Lambda^S_{t+1}\omega_{t+1}|\mathcal{G}_t, \mathbf{X}_{t+1} = \mathbf{e}_j]}{\mathrm{E}[\Lambda^S_{t+1}|\mathcal{G}_t, \mathbf{X}_{t+1} = \mathbf{e}_j]} \\
= & \mathrm{E}[\lambda^S_{t+1}\omega_{t+1}|\mathcal{G}_t, \mathbf{X}_{t+1} = \mathbf{e}_j] \\
= & q_{t+1,j}u_j + (1 - q_{t+1,j})d_j \ .
\end{aligned}
$$

Consequently, the conditional martingale condition holds true if and only if

$$
q_{t+1,j}u_j + (1 - q_{t+1,j})d_j = e^r \ .
$$

Hence, the result follows. □

Note that the martingale condition is not sufficient to determine a risk-neutral transition probability matrix $\mathbf{C}$ of the chain $\mathbf{X}$. Additional criteria are required to determine the matrix $\mathbf{C}$.

2.4 Option Valuation by MEMM and Dynamic Hedging

In this section, we adopt the minimal entropy martingale measure (MEMM) approach to determine a risk-neutral transition probability matrix $\mathbf{C}$. Indeed, the MEMM approach is a popular approach for option valuation in an incomplete market. The basic idea is to select an equivalent martingale measure so that it minimizes the relative entropy between an equivalent martingale measure and the physical probability measure, (i.e. the distance between the two probability measures.) From a statistical perspective, the MEMM approach selects an equivalent martingale measure which is closest to the physical one so that the probabilistic features of the physical probability measure are preserved when changing the measures from the physical measure to a risk-neutral one. From an economic perspective, the choice of an equivalent martingale measure by the MEMM approach can be justified by the maximization of an expected exponential utility. For more details, interested readers may refer to Frittelli [2000].

Consider the probability measure $\mathbb{Q}^{\mathbf{C}}$ absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{F}_T^{\mathbf{X}}$ defined by putting:

$$\left.\frac{d\mathbb{Q}^{\mathbf{C}}}{d\mathbb{P}}\right|_{\mathcal{F}_T^{\mathbf{X}}} := \Lambda_T^{\mathbf{X}} \ .$$

For each $t \in \mathcal{T}$, let $\mathbb{Q}_t^{\mathbf{C}}$ and $\mathbb{P}_t$ be the restriction of $\mathbb{Q}^{\mathbf{C}}$ and $\mathbb{P}$ on $\mathcal{F}_t^{\mathbf{X}}$, respectively. That is, $\mathbb{Q}_t^{\mathbf{C}} := \mathbb{Q}^{\mathbf{C}}|\mathcal{F}_t^{\mathbf{X}}$ and $\mathbb{P}_t := \mathbb{P}|\mathcal{F}_t$. Here we suppose that $\mathbb{Q}^{\mathbf{C}}$ is locally absolutely continuous with respect to $\mathbb{P}$. That is, for each $t \in \mathcal{T}$, $\mathbb{Q}_t^{\mathbf{C}}$ is absolutely continuous with respect to $\mathbb{P}_t$ on $\mathcal{F}_t^{\mathbf{X}}$, so

$$\left.\frac{d\mathbb{Q}_{t+1}^{\mathbf{C}}}{d\mathbb{Q}_{t+1}}\right|_{\mathcal{F}_{t+1}^{\mathbf{X}}} := \Lambda_{t+1}^{\mathbf{X}} \ .$$

Then we define a one-step-ahead conditional relative entropy between $\mathbb{Q}_{t+1}^{\mathbf{C}}$ and $\mathbb{P}_{t+1}$ given $\mathcal{F}_t^{\mathbf{X}}$ as follows:

$$\begin{aligned} I(\mathbb{Q}_{t+1}^{\mathbf{C}}, \mathbb{P}_{t+1}|\mathcal{F}_t^{\mathbf{X}}) &:= \mathrm{E}\left[\left(\frac{d\mathbb{Q}_{t+1}^{\mathbf{C}}}{d\mathbb{P}_{t+1}}\right)\ln\left(\frac{d\mathbb{Q}_{t+1}^{\mathbf{C}}}{d\mathbb{P}_{t+1}}\right)|\mathcal{F}_t^{\mathbf{X}}\right] \\ &= \mathrm{E}[\Lambda_{t+1}^{\mathbf{X}} \ln \Lambda_{t+1}^{\mathbf{X}} | \mathcal{F}_t^{\mathbf{X}}] \ . \end{aligned}$$

Our object is to determine the risk-neutral transition matrix $\mathbf{C}$ so as to minimize $I(\mathbb{Q}_{t+1}^{\mathbf{C}}, \mathbb{P}_{t+1}|\mathcal{F}_t^{\mathbf{X}})$. That is to solve the following optimization problem:

$$I(\mathbb{Q}_{t+1}^{\mathbf{C}^\dagger}, \mathbb{P}_{t+1}|\mathcal{F}_t^{\mathbf{X}}) = \min_{\mathbf{C}} I(\mathbb{Q}_{t+1}^{\mathbf{C}}, \mathbb{P}_{t+1}|\mathcal{F}_t^{\mathbf{X}}) \ ,$$

subject to the constraints:

$$\sum_{j=1}^{N} c_{ji} = 1 \ , \quad c_{ji} \geq 0 \ , \quad \forall i, j = 1, 2, \cdots, N \ .$$

In the sequel we assume that there are only two states in the Markov chain $\mathbf{X}$, (i.e. the slow regime denoted by 1 and the fast regime denoted by 2) [1] We further assume that the probability of a transition from state $\mathbf{e}_i$ to state $\mathbf{e}_j$ is equal to that of remaining in state $\mathbf{e}_j$. Then we have the following result.

[1]The slow regime is characterized by a low volatility state and is referred to as *stable markets* and *trending markets*. On the other hand, in the fast regime, we observe a high level of volatility. The term *jumpy regime* is often used by practitioners to refer to the fast regime.

Theorem 2.3. *Under the assumptions described above, an optimal risk-neutral transition probability matrix* $\mathbf{C}^{\dagger} := [c^{\dagger}_{ji}]_{i,j=1,2}$ *is given by:*

$$c^{\dagger}_{1i} = \frac{\exp\left(1 - 2a_{1i} - \ln\left(\frac{1-a_{1i}}{a_{1i}}\right)\right)}{1 + \exp\left(1 - 2a_{1i} - \ln\left(\frac{1-a_{1i}}{a_{1i}}\right)\right)} \tag{2.1}$$

$$c^{\dagger}_{2i} = \frac{1}{1 + \exp\left(1 - 2a_{1i} - \ln\left(\frac{1-a_{1i}}{a_{1i}}\right)\right)}, \tag{2.2}$$

for each $i = 1, 2$.

Proof: By a version of the Bayes' rule,

$$\begin{aligned} & \mathrm{E}[\Lambda^{\mathbf{X}}_{t+1} \ln \Lambda^{\mathbf{X}}_{t+1} | \mathcal{F}^{\mathbf{X}}_t] \\ = \; & \mathrm{E}^{\mathbf{C}}[\ln \Lambda^{\mathbf{X}}_{t+1} | \mathcal{F}^{\mathbf{X}}_t] \Lambda^{\mathbf{X}}_t \\ = \; & \Lambda^{\mathbf{X}}_t \ln \Lambda^{\mathbf{X}}_t + \mathrm{E}^{\mathbf{C}}[\ln \lambda^{\mathbf{X}}_{t+1} | \mathcal{F}^{\mathbf{X}}_t] \Lambda^{\mathbf{X}}_t \end{aligned}$$

Consequently, the original optimization problem

$$\min_{\mathbf{C}_{t+1}} \mathrm{E}[\Lambda^{\mathbf{X}}_{t+1} \ln \Lambda^{\mathbf{X}}_{t+1} | \mathcal{F}^{\mathbf{X}}_t]$$

is equivalent to the following simplified optimization problem:

$$\min_{\mathbf{C}_{t+1}} \mathrm{E}^{\mathbf{C}}[\ln \lambda^{\mathbf{X}}_{t+1} | \mathcal{F}^{\mathbf{X}}_t] \; .$$

Note that

$$\mathrm{E}^{\mathbf{C}}[\ln \lambda^{\mathbf{X}}_{t+1} | \mathcal{F}^{\mathbf{X}}_t] \;=\; \sum_{i,j=1, i \neq j}^{2} \ln\left(\frac{c_{ji}(t+1)}{a_{ji}}\right) c_{ij}(t+1) \left\langle \mathbf{X}_t, \mathbf{e}_i \right\rangle \; .$$

The result then follows by differentiating $\mathrm{E}^{\mathbf{C}}[\ln \lambda^{\mathbf{X}}_{t+1} | \mathcal{F}^{\mathbf{X}}_t]$ with respect to $c_{ij}(t+1)$, for $i, j = 1, 2$ with $i \neq j$, and setting the derivatives equal to zero. $\square$

Then from Lemma (2.2) and equations (2.1) and (2.2), as well as using standard combinatorial arguments, it is not difficult to show that given the initial state of the chain $\mathbf{X}_0 = \mathbf{e}_i$, $i = 1, 2$, a price of a standard European call option with maturity at time T

and strike price K is given by:

$$\begin{aligned} V_0^i &= e^{-rT} \sum_{k_1+k_2+k_3+k_4=T} \binom{T}{k_1, k_2, k_3, k_4} \left(c_{1i}q_1\right)^{k_1} \left(c_{1i}(1-q_1)\right)^{k_2} \left(c_{2i}q_2\right)^{k_3} \left(c_{2i}(1-q_2)\right)^{k_4} \\ &\times \max\{0, S_0 u_1^{k_1} d_1^{k_2} u_2^{k_3} d_2^{k_4} - K\} \; . \end{aligned} \tag{2.3}$$

Here the number of terms in the summation corresponds to the non-negative roots of $k_1 + k_2 + k_3 + k_4 = T$.

In what follows, we shall describe the dynamic hedging Delta strategy to be used in our numerical experiment. One of the fundamental problems in mathematical finance is how the issuer of an option can hedge the resulting exposure by trading in the underlying. However in an incomplete market there are no perfect hedging strategies for all options, so a second question always follows. This is how should one partially hedge an option? There is always a trade-off between the simplicity and the realism of a hedging strategy. Here we subscribe to the view of simplicity and to assess the performance of a simple Delta hedging strategy in the regime-switching binomial model.

The initial value of an option found from (2.3) has a practical significance apart from being a fair selling price. At time zero the option writer establishes a hedge portfolio, at the cost of V_0, consisting of the primitive assets. This portfolio is continuously rebalanced throughout the life of the option such that the amount of stock held at any instant is given by the delta, calculated based on the current state of the economy. The delta hedging portfolio is self financing, which means that no infusion or withdrawal of money is admissible. We assume that the option holder adapts a strategy to trade Δ amount of the stock, calculated based on the current state of the economy, in order to dynamically hedge his portfolio. Mathematically:

$$\Delta_t\big|_{\mathbf{X_t}=\mathbf{e_i}} = \frac{V^u_{t+1,\mathbf{e_i}} - V^d_{t+1,\mathbf{e_i}}}{S^u_{t+1,\mathbf{e_i}} - S^d_{t+1,\mathbf{e_i}}} \tag{2.4}$$

This may clearly show the source of risk for an option issuer. If the market was complete, the hedging portfolio value at the expiry would be exactly enough to cover the payoff if the option is exercised. However, in the regime-switching binomial model, there is an anticipation of cumulative discrepancy at the maturity time between the payoff of the option and the hedging portfolio value. The source of the hedging error is the inability of the hedger to predict the next step's regime. In other words, when the hedger is adapting the delta strategy, he would require to trade an amount of the underlying asset equal to the difference between the prospective high and low values of the asset in the next step, and their corresponding option prices. Nevertheless, in this paradigm, he is restricted to make a decision based on the state of the economy, in which he operates at the time. Now, it is clear that if the economy stays in the same state in the next period, there would be no local hedging error. However, since the economy switches over time throughout the life of the option, there will exist local hedging errors that will accumulate to the maturity time. Our aim is then to examine the performance of the dynamic Delta hedging strategy in the regime-switching binomial tree by evaluating some important risk metrics, such as Value at Risk (VaR) and Expected Shortfall (ES), of the terminal hedging errors of the imperfectly Delta hedging portfolio in various scenarios, where the terminal hedging errors are evaluated as the differences between the terminal payoffs of the option and the terminal values of the dynamic Delta hedging portfolio. The uses of VaR and ES for evaluating option risks have been discussed in the paper by Boyle et al. [2002]. However, they considered unhedged option positions rather than partially hedged option positions as we consider here.

2.5 Numerical Results

In this section we present the numerical results for the option prices and the risk metrics, say VaR and ES, of the hedging errors arising from the dynamic Delta hedging strategy presented in the previous two sections.

2.5.1 Option Prices

We present numerical results of European call option prices arising from the standard CRR binomial model and the regime-switching binomial model. Assume that an underlying risky asset is currently traded at \$10, and money is worth 4.5% per annum. An option writer is considering a one year to maturity option with exercise price of \$11. The economy can either be in the slow state with 30% volatility or in the fast state with 60% volatility. Based on Moody's average one-year rating transition rates for the period 1970en2009, we calculate the historical transition probability from any state to state slow to be 89.28% and from any state to state fast to be 10.72%. If the writer uses the CRR model he is limited to choose only one state, hence one volatility, to price the option. However, the reigme-switching binomial model allows the writer to incorporate the possibility of regime switches (and the corresponding volatility of each regime) to calculate the option price. Figure (2.2) depicts the price of the call option calculated for different term maturities and moneyness. We can see that the option prices arising from the regime-switching binomial model fall between prices calculated by the standard CRR binomial model with high and low volatility. This result makes intuitive sense.

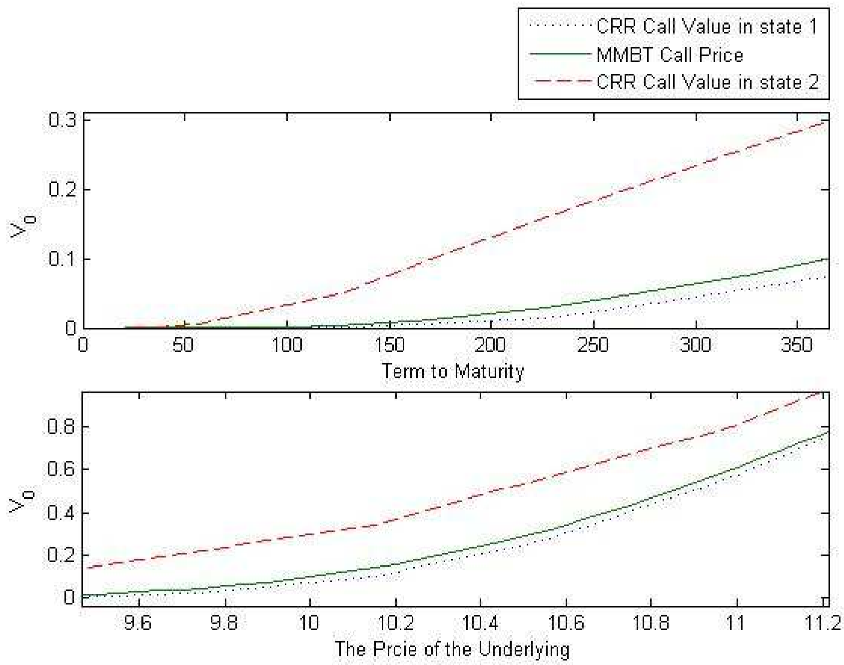


FIGURE 2.2: Call prices for different values of the maturity time and the moneyness. Here, Moody's transition probabilities are used for calculating the option prices in the regime-switching binomial model

Furthermore, to illustrate the sensitivity of the option's prices to changes in the values of the transition probabilities, we re-plot the prices of the option described above; however, this time we assume the unrealistic transition probabilities of 60% changing to state slow, and 40% changing to state fast. From Figure (2.3), we observe that the option prices arising from the regime-switching binomial model are more distanced from the CRR option prices in the slow economy. We, therefore, conclude that if the option writer uses the CRR model, he is bound to either underprice or overprice the option in comparison to the regime-switching binomial model. Furthermore, the degree of the pricing error is directly related to the transition probabilities.

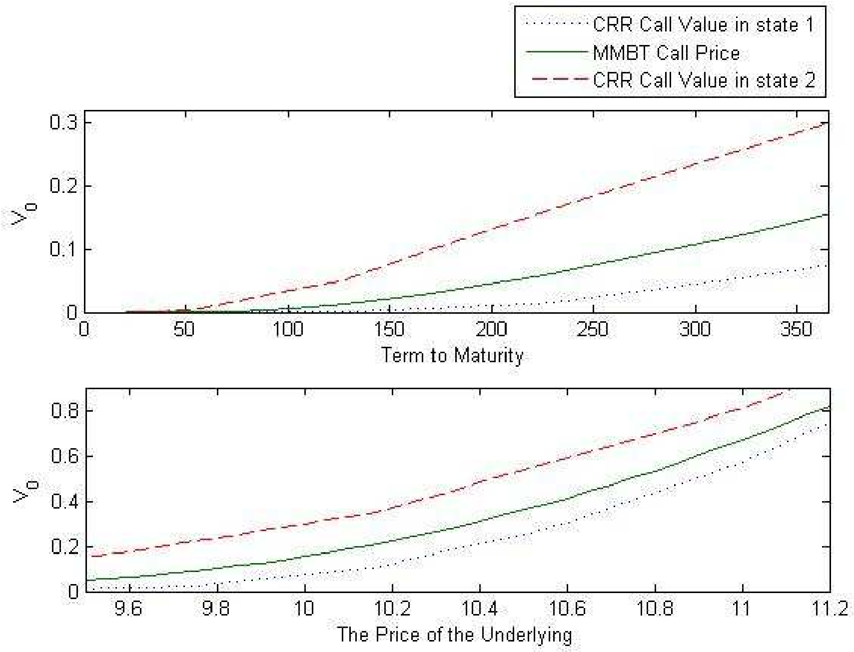


FIGURE 2.3: Call prices for different maturity time and moneyness. Here, Hypothetical transition probabilities of 40% and 60%, (for changing to regime fast and slow, respectively) are used for calculating the option prices in MRSBT framework

2.5.2 Risk Metrics for Dynamic Delta Hedging

We now present and discuss the numerical results of the VaR and ES for dynamic Delta hedging portfolios. In this case, we value an option on an underlying risky asset that is currently traded at \$8 with strike price \$10. The option has 90 days to maturity and the risk-free interest rate is 6% p.a. The volatility of the underlying asset price would be 30%, when the economy is in the state slow, and 60% when the economy is in state fast. We convey a sensitivity analysis for different transition probabilities and different hedging frequencies. Our aim is to investigate the behavior of the hedging errors distributions for these two variables.

To approximate the distribution of the hedging errors, we use a kernel density estimation with Epanechnikov as our choice of kernel function. The performance of a kernel function is measured by Asymptotic Mean Integrated Squared Error (AMISE), and Epanechnikov kernel minimizes AMISE, hence is optimal. The validity of the density approximation is tightly entwined with the optimal choice bandwidth. If we take a very small bandwidth the approximation errors would be small; nevertheless, this is restricted to the number of data points in the local neighborhood. Besides, the variance of the estimated local parameters is often large for small number of data points. On the other hand, large bandwidth would create large modeling bias depending on the underlying function. Hence, the determination of the best bandwidth involves a trade-off between bias and variance. In this paper we use the approach introduced in Shimazaki and Shinomoto [2010] in order to determine the optimal bandwidth.

FIGURE 2.4: Distributions of hedging errors for different transition probabilities, when the option is hedged in 8 and 9 (respectively from right to left) equal time intervals, starting from the beginning

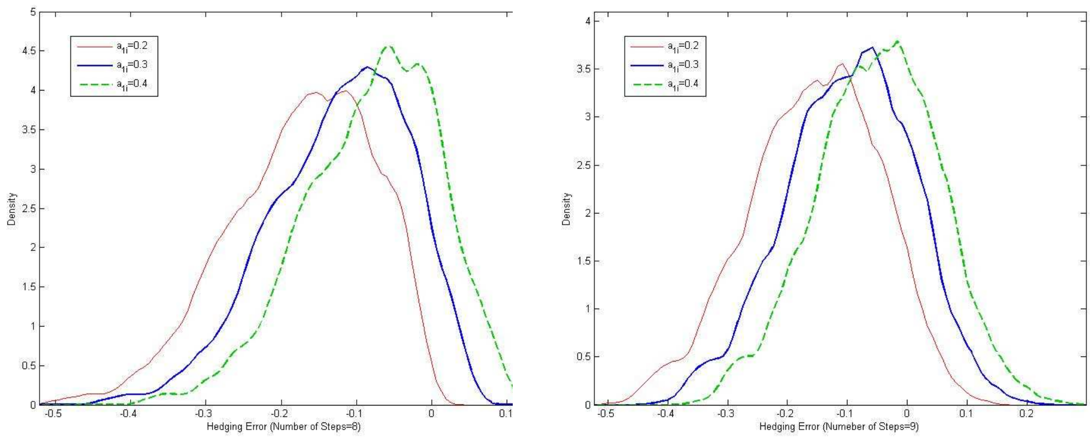
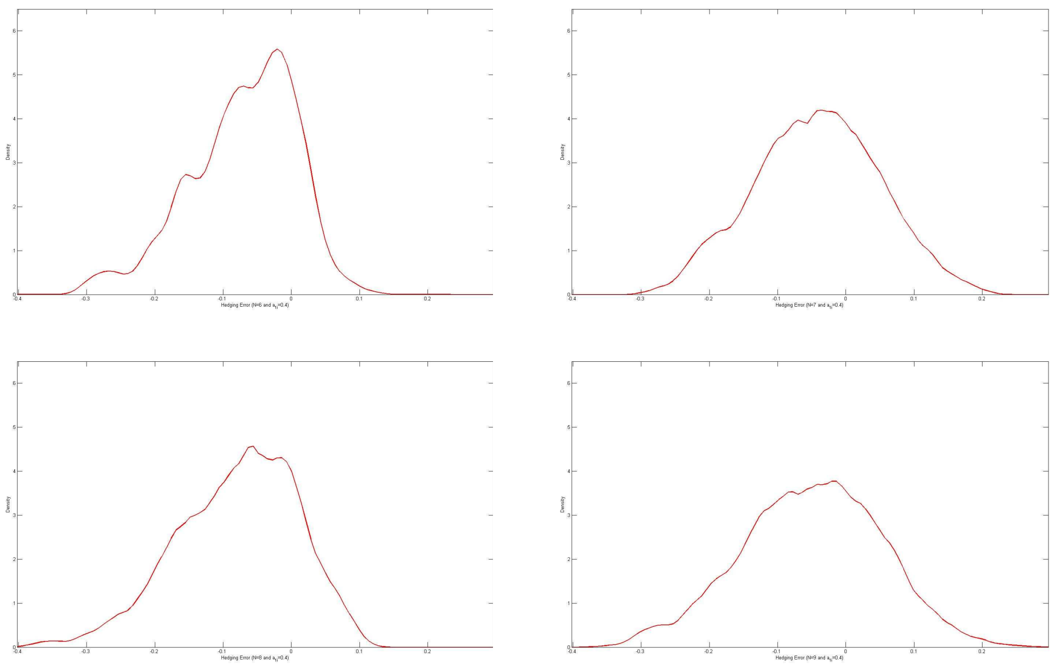


Figure 2.4 illustrates the sensitivity of the density of hedging errors to the transition probabilities. Each of the figures show that as the probability of a switch to regime one ($a_{1i} = 1 - a_{2i}$) increases the mean of the terminal hedging errors approaching to zero from negative values, and the variance of the distribution decreases. Assume that an agent issues a standard European option, and at the same time, he constructs a self-financing portfolio to hedge his risk exposure. As the price of the underlying evolves over time, the agent requires to re-balance his positions, based on the delta calculated for the current state. This is indeed the source of hedging errors; because if the state switches, the state delta would not be the sufficient amount to hedge all sources of risk. Consequently, similar to any speculative position, the un-hedged amount might either result in unanticipated loss or profit. On the other hand, when the values of a_{1i} and a_{2i} are equal or very close to each other, the frequency of having positive and negative hedging errors would also get closer, hence some of the losses would be compensated along the tree.

Figure 2.5 shows the sensitivity of the hedging errors density to the frequency of hedging. Our result reveals that the distribution of hedging errors is negatively skewed. However, we observe that as the frequency increases, the distribution tends to be more symmetric.

This provides evidence for reasonably good performance of the regime-switching to value contingent claims, especially when the hedging portfolio is frequently balanced. Of course due to the presence of transactions costs, the frequent rebalancing of hedging portfolios becomes very costly in practice. To further examine the hedging performance, the VaR and ES of the hedging errors are calculated for different hedging frequencies and transition probabilities. Table 2.1 and Table 2.2 report the results for the 95% and 99% thresholds, respectively. It can be observed that there is a consistent declining trend of risk metrics as the transition probability approaches to 50%, which further justifies the discussion above.

FIGURE 2.5: Distribution of hedging errors for different hedging frequencies: N equals to 6, 7, 8, and 9 respectively from left to right and top to bottom. The transition probability to state 1 for all of these graphs equal to 40%.



2.6 Conclusion

We discussed the option pricing and hedging in a regime-switching binomial tree, where there are two sources of uncertainties, namely, the risk due to binomial movements of the underlying risky asset price and the risk due to transitions in economic states. We adopted the MEMM approach to price the two sources of risk and examined the pricing and hedging performance of this approach using numerical examples. A simple dynamic

TABLE 2.1: VaR and ES calculated for the distribution of hedging errors of the regime-switching binomial model at the 95% threshold. The risk metrics are calculated for different hedging frequencies (N) and transition probabilities ($a_{1i} = 1 - a_{2i}$).

	Transition Probabilities		
Hedging Frequency	$a_{1i} = 0.2$	$a_{1i} = 0.3$	$a_{1i} = 0.4$
N=4	VaR=0.2573 ES=0.2720	VaR=0.2247 ES= 0.2379	VaR=0.1962 ES= 0.2080
N=5	VaR=0.2412 ES= 0.2629	VaR=0.2049 ES= 0.2215	VaR=0.2049 ES= 0.2215
N=6	VaR=0.2904 ES= 0.3413	VaR= 0.2422 ES= 0.2944	VaR=0.2065 ES= 0.2544
N=7	VaR=0.2964 ES= 0.3266	VaR=0.2457 ES= 0.2719	VaR=0.2013 ES= 0.2263
N=8	VaR= 0.3429 ES= 0.3903	VaR=0.2841 ES= 0.3312	VaR=0.2376 ES= 0.2819
N=9	VaR=0.3424 ES= 0.3894	VaR=0.2787 ES= 0.3225	VaR=0.2258 ES=0.2668

TABLE 2.2: VaR and ES calculated for the distributions of hedging errors in the regime-switching binomial model at the 99% threshold. The risk metrics are calculated for different hedging frequencies (N) and transition probabilities ($a_{1i} = 1 - a_{2i}$).

	Transition Probabilities		
Hedging Frequency	$a_{1i} = 0.2$	$a_{1i} = 0.3$	$a_{1i} = 0.4$
N=4	VaR=0.2812 ES= 0.2812	VaR=0.2462 ES= 0.2462	VaR=0.2154 ES= 0.2154
N=5	VaR=0.2858 ES=0.2920	VaR=0.2404 ES= 0.2457	VaR=0.2404 ES= 0.2457
N=6	VaR=0.3776 ES= 0.3876	VaR=0.3266 ES= 0.3377	VaR=0.2834 ES=0.2948
N=7	VaR=0.3437 ES= 0.3660	VaR=0.2881 ES= 0.3098	VaR=0.2398 ES= 0.2634
N=8	VaR=0.4228 ES= 0.4596	VaR= 0.3618 ES= 0.3980	VaR=0.3082 ES= 0.3457
N=9	VaR= 0.4207 ES= 0.4424	VaR=0.3520 ES= 0.3721	VaR=0.2941 ES= 0.3132

Delta hedging was considered and its performance was examined by evaluating the VaR and ES of the terminal hedging errors arising from the dynamic Delta hedging strategy. Numerical results were provided which reveal that the impact of pricing regime-switching risk is significant and that both the hedging frequencies and transition probabilities of regime switches have significant impacts of the performance of the delta hedging strategy.

For future research, one may further investigate the hedging strategies in the regime-switching binomial model. In our current paper, we illustrated the risk of the option issuer based on the terminal hedging error, if they conveniently ignore the regime-switching risk. Such a hedger observes the regime at each node and choose the appropriate delta accordingly. However, the issuer is able to reduce the terminal hedging error (in the case of vanilla European options, potentially to zero), by implementing a trading strategy that takes the delta of the all states into consideration, concurrently.

Assume that an agent is issuing a standard European option in an economy where all of the assumptions presented in Section2.3 hold true. Given the information at time t, the model suggests that the economy may be either in regime 1 or regime 2, (i.e. the process is governed by the Markov chain). Therefore, the agent may have two choices of delta, namely, $\Delta_{t,1} = (V^u_{t+1,1} - V^d_{t+1,1}) \times (S^u_{t+1,1} - S^d_{t+1,1})^{-1}$ and $\Delta_{t,2} = (V^u_{t+1,2} - V^d_{t+1,2}) \times (S^u_{t+1,2} - S^d_{t+1,2})^{-1}$.

Thus, at time t, the agent has to choose an appropriate delta. In Section2.4 we presented the result of the case where the hedger chooses the delta corresponding to the current regime. However, the terminal hedging error for a standard European option can potentially be reduced to zero, if the hedger selects a delta hedge calculated based on the weighted average of $\Delta_{t,1}$ and $\Delta_{t,2}$. In this strategy the weights are calculated according to the risk-neutral transition probabilities. Consequently, the value at time t of an option can be written as:

$$V_t = c_{1i}\{\Delta_{t,1}S_t - B_{t,1}\} + c_{2i}\{\Delta_{t,2}S_t - B_{t,2}\}. \tag{2.5}$$

where $B_{t,i}$ denotes the amount of riskless borrowing and lending at time t when the economy is in the state $\mathbf{e_i}$. Hence, the hedging strategy involves trading $c_{1i}\Delta_{t,1} + c_{2i}\Delta_{t,2}$ of the underlying asset and $-(c_{1i}B_{t,1} + c_{2i}B_{t,2})$ riskless borrowing and lending. One may expand the equation (2.5) as follows, which suggests the possibility of complete risk transfer or a perfect hedge under our incomplete market framework.

$$V_t = c_{1i}q_{t,1}V^u_{t+1,1} + c_{2i}q_{t,2}V^u_{t+1,2} + c_{1i}(1 - q_{t,1})V^d_{t+1,1} + c_{2i}(1 - q_{t,2})V^d_{t+1,2}$$

We re-emphasize that this only show that there exist a trading strategy which can perfectly hedge all the risks in a regime-switching economy. However, the remaining question would be: out of the infinite number of martingale measures, which one would provide the perfect hedge? In other words, whilst we can show the possibility of perfect hedge in the regime-switching economy, we are unable to find the specific $[c_{ji}]$ that perfectly hedges the contingent claim.

Chapter 3

Ruin Contingent Life Annuities under Regime-Switching Variance Gamma Process

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Abstract

We propose a model for valuing ruin contingent life annuities under the regime-switching variance gamma process. The Esscher transform is employed to determine the equivalent martingale measure. The PIDE approach is adopted for the pricing formulation. Due to the path dependency of the payoff of the insurance product and the non-existence of a closed-form solution for the PIDE, the finite difference method is utilized to numerically calculate the value of the product. To highlight some practical features of the product, we present a numerical example. Finally, we examine numerically the performance of a simple hedging strategy by investigating the terminal distribution of hedging errors and the associated risk measures such as the value at risk and the expected shortfall.

Keywords Ruin contingent life annuity - Regime-switching variance gamma - Esscher transform - Pricing and risk management

3.1 Introduction

The Ruin Contingent Life Annuity (RCLA, henceforth) contract is a modern innovation in bundle insurance contracts that allows the near to retirement population to hedge their retirement risks. This is delivered by the means of insuring the joint occurrence of two independent events; namely, an above average survival rate and bear market during the retirement risk zone.

The concept of RCLA, as well as the preliminary pricing formulations of the contract, was first introduced in Huang et al. [2009]; with the purpose of improving more conventional products such as advanced life-delayed annuities (ALDA). They demonstrated that the life-long income provided by RCLA should come at much less cost to the annuitant.

The RCLA contract, as a contingent annuity contract, is bought by the insured by a lump sum payment. The insurance contract entitles the insured to receive a series of periodic payments from the insurer. One key feature of the policy is the sharing of profits from an investment portfolio between the policyholder and the insurer. Nevertheless, the policyholder can manage his exposure to the risk of an underlying reference portfolio through their choice of the asset mix.

An optional feature that a RCLA product might offer is periodic withdrawals from the fund before retirement, provided that the mark-to-market value of the reference portfolio is larger than the size of the withdrawal. In this paper we assume no withdrawals prior to the retirement for two reasons. First, there is no economic advantage for the insured to withdraw from the fund, since the accumulated money is to remain as the bequest for the descendants. Second, the withdrawals create discontinuity in the mark-to-market value of the reference portfolio, hence, unnecessary modeling complications. Nevertheless, the findings of this paper could be extended to accommodate for the withdrawals.

Huang et al. [2009] developed a risk-neutral pricing model for the RCLA contracts using the PDE approach assuming a complete market. They then describe some efficient numerical techniques and provide estimates of a typical RCLA under a variety of realistic parameters. Under the same framework, Huang et al. [2012] further investigated the hedging strategy of the contract and a thorough comparison is made across different options embedded variable annuity products. Rong and Fard [2013] considered the valuation of the contracts using stochastic volatility for the dynamics of the underlying portfolio. In this paper, we consider the valuation of RCLA, under the regime-switching Variance Gamma process(MVG, henceforth).

Since the mid-1990s, the pure-jump Levy processes have been growing in popularity amongst scholars as an alternative to the Black-Scholes economy (e.g. Eberlein and Prause [1998], Madan et al. [1998], Elliott and Siu [2013]). In particular, Geman et al. [2001] noted that the pure-jump Levy processes are appropriate, when in the process of an asset evolution the time varies with the martingale component. The Variance Gamma process (VG, henceforth) is shown by Madan et al. [1998] to be a class of the pure-jump Levy processes, thus it inherits all the attributes. As a result the VG processes can be advantageous to use when pricing options and equity linked insurance products, since it allows for a wider modeling of skewness and kurtosis than a diffusive model does. In particular, in the case of equity linked products, such as RCLA, where the payoff is decomposed to different options with different strikes and maturities, the VG process provides a more consistent pricing framework.

Madan et al. [1998] showed that the VG process can be represented in a number of equivalent ways, including, the time-changed Brownian motion, the difference between two Gamma processes, the Levy measure representation and the predictable compensator representation. In this paper, we use the regime-switching version of the Levy measure representation. We first specify the model parameters such that the impact of regime-switching is captured by the modulation of the parameters of the VG process by a Markov chain. Then we take the approach presented by Madan et al. [1998] to link the MVG process with the regime-switching pure-jump Levy processes. Subsequently, we define the dynamics of the underlying portfolio, similar to Elliott and Siu [2013] and use the Esscher transform to find the equivalent martingale measure. Finally, the pricing PIDE is driven and solved using a numerical method.

The history of regime-switching models can be traced back to Quandt [1958], Goldfield and Quandt [1973] where they employed regime-switching regression models to describe nonlinearity in economic data. The idea of probability switching also appeared in the early development of nonlinear time-series analysis, where Tong [1983] proposed one of the oldest classes of nonlinear time series models, namely the threshold time series models. Hamilton [1989] popularized regime-switching time series models in the economic and econometric literature. Since then, considerable attention has been paid to the use of regime-switching framework in modeling economic and financial data. Due to the empirical success of regime-switching models, the models have been applied to different areas in banking and finance, including asset allocation, option valuation, risk management, term structure modeling, and others. Recently, scholars have turned their attention to option valuation under regime-switching models. This includes Naik [1993], Guo [2001], Buffington and Elliot [2002], Elliott et al. [2005] , and Fard and Siu [2013] amongst others. These studies analyse the option valuation problem under continuous-time regime-switching models. Regime-switching models have become popular in actuarial science in recent years, for example, in Hardy [2001], Siu [2005], Siu et al. [2008] work.

This article is structured as follows: Section 3.2 presents the Markov-modulated Variance Gamma process. The section continues by presenting the pricing formulation through the Esscher transform. Section 3.3 provides the numerical solution for the underlying PIDE through a version of the Finite Differences method. The section continues by a numerical experiment, as well as the analysis of the hedging error, under dynamic delta hedging.

3.2 Modeling Framework

3.2.1 The Pricing Dynamics

We fix a complete probability space $(\Gamma, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the real-world probability measure. Let $\mathcal{T}$ denote the time index set $[0, T]$ of the economy. We describe the state of the economy by a continuous-time Markov chain $\{X_t\}_{t\in\mathcal{T}}$ on $(\Gamma, \mathcal{F}, \mathbb{P})$ with a finite state space $\mathcal{S} := (s_1, s_2, \cdots, s_N)$. Without loss of generality, we can identify the state space of the process $\{X_t\}_{t\in\mathcal{T}}$ to be a finite set of unit vectors $\{e_1, e_2, \cdots, e_N\}$, where $e_i = (0, \cdots, 1, \cdots, 0) \in \mathcal{R}^N$. From Elliott et al. [1995] we present the following semi-martingale decomposition for the process $\{X_t\}_{t\in\mathcal{T}}$:

$$X_t = X_0 + \int_0^t \mathcal{Q} X_s ds + M_t. \tag{3.1}$$

Here $\mathcal{Q}$ is called the 'generator' and is defined as $\mathcal{Q} = [q_{ij}]_{i,j=1,2,\cdots,N}$ and M_t is a $\mathcal{R}^N$-valued martingale with respect to the filtration generated by $\{X_t\}_{t\in\mathcal{T}}$. In the remainder of this paper, any parameter P modulated by the Markov chain X_t is denoted by P_{X_t}, and defined as follows:

$$P_{X_t} = \langle \mathbf{P}, X_t \rangle = \sum_{i=1}^{N} P_i \langle X_t, e_i \rangle, \quad t \in \mathcal{T}, \tag{3.2}$$

where $\mathbf{P} := (P_1, P_2, \cdots, P_N)$ with $P_j > 0$ for each $j = 1, 2, \cdots, N$ and $\langle ., . \rangle$ denotes the inner product in the space $\mathcal{R}^N$.

Let $\{r_{X_t}\}_{t\in\mathcal{T}}$ be the instantaneous market interest rate of a money market account, which depends on the state of the economy. Hence, r_{X_t} is defined as per (3.2) and the dynamic of the value of the risk-free asset, $\{B_t\}_{t\in\mathcal{T}}$ would be:

$$dB_t = r_{X_t} B_t dt. \tag{3.3}$$

Let Z_t to be an observation process that follows a MVG process. Now let $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$ denote a measurable space, where $\mathcal{B}(\mathcal{T})$ is the Borel σ-field generated by the open subset of $\mathcal{T}$. Then the measurable space is given by $(\mathcal{T} \times \mathcal{R}, \mathcal{B}(\mathcal{T}))$.

Then for a family of Borel sets $\mathcal{U} \in \mathcal{R}$, let $N_{X_t}(dz, dt)$ denote a differential form of a Markov-switching jump measure and a compensator measure under the historical measure $\mathbb{P}$ given by

$$\nu_{X_t}(dz, dt) = k_{X_t}(z) dz dt, \tag{3.4}$$

where,

$$k_{X_t} = \frac{C}{z} \Big(\exp(-z M_{X_t}) \mathbb{I}_{\{z>0\}} - \exp(z G_{X_t}) \mathbb{I}_{\{0<z\}} \Big), \tag{3.5}$$

and $C, M, G \in \mathcal{R}^+$ are parameters of the MVG process. The definitions for N_{X_t} , ν_{X_t}, k_{X_t}, and G_{X_t} are as per (3.2).

Proposition 1. Let b_{X_t} (defined as in (3.2)) denote the drift of the process. By setting to

$$b_{X_t} = -C\int_{-1}^{0} e^{zG_{X_t}}dz + C\int_{0}^{1} e^{-zM_{X_t}}dz, \tag{3.6}$$

Z_{X_t} would become a uniquely defined Markov-modulated Levy process, with the Levy triple $(b_{X_t}, 0, \rho_{X_t})$.

Proof. Set $\phi_j, j = 1, ..., N$ to be the the characteristic function for the time t level of the MVG process under the historical measure, given that $X_t = e_j$. Then consistent with Madan et al. [1998] write:

$$\begin{aligned} \phi_j(u) &= \Big(\frac{1}{1 - iu\rho_j\vartheta_j + \sigma_j^2 u^2\rho/2}\Big)^{\frac{t}{\rho}} \\ &= \Big(1 - \frac{iu}{M_j}\Big)^{-Ct}\Big(1 + \frac{iu}{G_j}\Big)^{-Ct} \end{aligned}$$

Then by similar calculations as in the chapter 8 of Sato [2004], we continue:

$$\begin{aligned} \phi_j(u) &= \exp\Big(Ct\int_{\mathcal{R}^+}(e^{iuz}-1)\frac{e^{-zM_j}}{z}dz\Big)\exp\Big(-Ct\int_{\mathcal{R}^-}(e^{iuz}-1)\frac{e^{zG_j}}{z}dz\Big) \\ &= \exp\Big(\int_{\mathcal{R}}(e^{iuz}-1)\rho_j(dz)\Big) \\ &= \exp\Big(tiub_j + t\int_{\mathcal{R}}(e^{iuz} - 1 - iuz\mathbb{I}_{|z|<\geq 1})\rho_j(dz)\Big), \end{aligned}$$

where $\rho_j(dz) = k_j(z)dz$ is the Levy measure, with k defined in (3.5). Then by the Levy-Khintchine formula the results follow.□

Notice that, as in Madan et al. [1998], the volatility in the Levy triple is set to zero, which sets the (M)VG process as a pure jump process with no continuous Brownian motion component. Then the process Z can be expressed in terms of the random measure $N_{X_t}(dz, dt)$ as follows:

$$Z_t = \int_0^t\int_{\mathcal{R}} zN_{X_t}(dz, ds), \qquad t \in \mathcal{T}. \tag{3.7}$$

such that the compensator $\nu_{X_t}(dz, dt)$ specifies the probability law of the jump process Z under the historical measure. For the modeling to be well-posed we must further assume that:

$$\int_{\mathcal{R}} z^2 k_j(dz) < \infty, \qquad j = 1, 2, \cdots, N.$$

Let $\widetilde{N}_{X_t}(dt, dz)$ denote the compensated Poisson random measure defined by

$$\widetilde{N}_{X_t}(dz, dt) = N_{X_t}(dz, dt) - \nu_{X_t}(dz, dt). \tag{3.8}$$

Let μ_{X_t} (defined as in (3.2))be the appreciation rate of the reference portfolio A, where $\mu_{X_t} > r_{X_t}$. Then, as in Elliott and Siu [2013] we define the following return process for the portfolio:

$$Y_t = \mu_{X_t} t + \int_0^t \int_{\mathcal{R}} z \widetilde{N}_{X_t}(dz, ds). \tag{3.9}$$

It is conventional to define $A_t = A_0 \exp(Y_t)$. Then, by Ito's differentiation rule we have:

$$\begin{aligned} A_t &= A_0 + \mu_{X_t} \int_0^t A(s) ds + \int_0^t \int_{\mathcal{R}} A(s-)(e^z - 1) \widetilde{N}_{X_t}(dz, ds) \\ &\quad + \int_0^t \int_{\mathcal{R}} A(s)(e^z - 1 - z) \nu_{X_t}(dz, ds). \end{aligned} \tag{3.10}$$

3.2.2 Pricing through the Esscher Transform

The RCLA contract can be viewed as a combination of a European call option and a down-and-in barrier option. The provisions of these featured options are typically financed by the lump sums at the initiation of the contract; nevertheless, it could be replaced by continuous proportional fees to the insurer, or a combination of the two. Instead of evaluating the fair value of the terminal payoff of the policy, we consider the fair valuation for each of the components of the terminal payoff. We present the procedure for the fair valuation based on an equivalent martingale measure chosen by the regime-switching Esscher transform.

In accordance with the first fundamental asset pricing theorem, for the fair valuation of the policy we need to ensure there is no arbitrage opportunities in the market, through the determination of the equivalent risk-neutral martingale measure. In incomplete markets, as is the case in this paper, there is more than one equivalent martingale measure, and hence, more than one no-arbitrage price. Different approaches have been proposed for pricing and hedging derivative securities in incomplete financial markets. To name a few, Follmer and Sondermann [1986], Schweize [1995], Follmer and Schweizer [1991]selected an equivalent martingale measure by minimizing the quadratic utility of the terminal hedging error. Davis [1997]adopted an economic approach based on the marginal rate of substitution to pick a pricing measure via a utility maximization problem. Gerber and Shiu [1994] pioneered the use of the Esscher transform, a time-honored tool in actuarial science. The Esscher transform provides market practitioners with a convenient and flexible way to value options. More importantly, Elliott et al. [2005] showed that the result achieved from the Esscher transform corresponds directly to the Minimum Entropy Martingale Measure method.

Let $\mathcal{F}^X := \{\mathcal{F}_t^X\}_{t\in\mathcal{T}}$ and $\mathcal{F}^Y := \{\mathcal{F}_t^Y\}_{t\in\mathcal{T}}$ denote the $\mathbb{P}$-augmentation of the natural filtration generated by X and Y, respectively. Then, define $\mathcal{G}_t$ for the σ-algebra $\mathcal{F}^X \vee \mathcal{F}^Y$ for each $t \in \mathcal{T}$. Further, let $L(Y)$ be the space of processes $\theta_{X_t}, \forall t \in \mathcal{T}$, such that it is integrable with respect to the return process.

For each $t \in \mathcal{T}$, write

$$(\theta.Y)_t := \int_0^t \theta_{X_u} dY(u).$$

This is the stochastic integral of θ with respect to Y. Let $\{\Lambda_t\}_{t\in\mathcal{T}}$ denote a $\mathcal{G}$-adapted stochastic process defined as below:

$$\Lambda_t := \frac{e^{-(\theta.Y)_t}}{\mathcal{M}(\theta)_t}, \qquad t \in \mathcal{T}, \tag{3.11}$$

where $\mathcal{M}(\theta)_t := E^{\mathbb{P}}[e^{-(\theta.Y)_t}|\mathcal{F}_t^Y]$, is a Laplace cumulant process. Apply Ito's differentiation rule for jump diffusion

$$\begin{aligned} e^{-(\theta.Y)_t} &= 1-\int_0^t e^{-(\theta.Y)_s}\theta_{X_s}\mu_{X_s}ds-\int_0^t\int_{\mathcal{R}} e^{-(\theta.Y)_{s-}}z\theta_{X_{s-}}\widetilde{N}_{X_s}(dz,ds) \\ &+\int_0^t\int_{\mathcal{R}} e^{-(\theta.Y)_{s-}}\Big(e^{z\theta_{X_s}}-1\Big)\widetilde{N}_{X_s}(dz,ds) \\ &+\int_0^t\int_{\mathcal{R}} e^{-(\theta.Y)_{s-}}\Big(e^{z\theta_{X_s}}-1+z\theta_{X_s}\Big)\nu_{X_t}(dz,ds). \end{aligned} \tag{3.12}$$

Conditioning on $\mathcal{F}^X$ for both sides of (3.12),

$$\begin{aligned} \mathcal{M}_Y(\theta)_t &= \exp\Bigg[-\int_0^t \theta_{X_s}\mu_{X_s}ds \\ &+\int_0^t\int_{\mathcal{R}}\Big(e^{-z\theta_{X_s}}-1+z\theta_{X_s}\Big)\nu_{X_t}(dz,ds)\Bigg]. \end{aligned} \tag{3.13}$$

Therefore,

$$\begin{aligned} \Lambda_t &= \exp\Bigg[-\int_0^t\int_{\mathcal{R}} z\theta_{X_s}\widetilde{N}_{X_s}(dz,ds) \\ &-\int_0^t\int_{\mathcal{R}}(e^{-z\theta_{X_s}}-1+z\theta_{X_s})\nu_{X_s}(dz,ds)\Bigg]. \end{aligned} \tag{3.14}$$

Proposition 2. Λ_t is martingale with respect to the enlarged filtration $\mathcal{G}$.

Proof:

$$\begin{aligned} E[\Lambda_t|\mathcal{G}_t] &= E\Bigg\{\exp\Bigg[-\int_0^t\int_{\mathcal{R}} z\theta_{X_s}\widetilde{N}_{X_s}(dz,ds) \\ &-\int_0^t\int_{\mathcal{R}}(e^{-z\theta_{X_s}}-1+z\theta_{X_s})\nu_{X_s}(dz,ds)\Bigg]\Bigg|\mathcal{G}_t\Bigg\}. \end{aligned}$$

Note that by James [2002, 2005],

$$E\Big[\exp\Big(-\int_0^t\int_{\mathcal{R}} z\theta_{X_s}\widetilde{N}_{X_s}(dz,ds)\Big)\Big|\mathcal{G}_t\Big]=\exp\Big(\int_0^t\int_{\mathcal{R}}(e^{-z\theta_{X_s}}-1+z\theta_{X_s})\nu_{X_s}(dz,ds)\Big)$$

Hence:

$$E[\Lambda_t|\mathcal{G}_t] = 1 \qquad \mathbb{P} - a.s.$$

□

Then, following Elliott et al. [2005], we define the regime-switching Esscher transform $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{G}_t$ as:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}_t} = \Lambda_t, \quad t \in \mathcal{T}. \tag{3.15}$$

Elliott et al. [2005] presented that following the fundamental theorem of asset pricing, the parameter θ must satisfy the marginal condition for the discounted market value of the underlying asset. Hence:

$$\widetilde{A}_s = E^{\mathbb{Q}}[\tilde{A}_t|\mathcal{G}_s], \quad \forall t, s \in \mathcal{T}, \ \textit{with } t \geq s, \tag{3.16}$$

where $E^{\mathbb{Q}}$ denotes the expectation under $\mathbb{Q}$, and $\tilde{A}_t := \exp(-\int_0^t r_{X_s} ds)A_t$. The filtration $\mathcal{G}_t$ ensures that the Markov Chain process is accessible to the market's agents in advance. By the tower law, if one can find a probability measure $\mathbb{Q}$ satisfying the martingale condition on $\mathcal{G}_t$, $\mathbb{Q}$ also satisfies the martingale condition without knowing $\mathcal{G}_t$.

Therefore, using the Bayes' rule and (4.5) as:

$$\begin{aligned}
\widetilde{A}_s &= \exp\left\{-\int_0^t (r_{X_s} ds) E^{\mathbb{P}}[\Lambda_t A_t|\mathcal{G}_0]\right\} \\
&= \exp\left\{-\int_0^t (r_{X_s} ds) E^{\mathbb{P}}\left[\frac{\exp(-(\theta.Y)_t) \times \exp(\int_0^t dY_s)}{\mathcal{M}(\theta)_t}\right]\right\} \\
&= \exp\left\{\int_0^t (\mu_{X_s} - r_{X_s}) ds - \int_0^t \int_{\mathcal{R}} \left[z - e^{(1-\theta_{X_s})z} + e^{-z\theta_{X_s}}\right] \nu_{X_t}(dz, ds)\right\}
\end{aligned} \tag{3.17}$$

Then by setting $s = 0$, the martingale condition implies that:

$$E^{\mathbb{Q}}[\tilde{A}_t|\mathcal{G}_0] = 1$$

Therefore, θ could be determined from:

$$\mu_{X_t} - r_{X_t} - \int_{\mathcal{R}} \left[z - e^{z(1-\theta_{X_t})} - e^{-z\theta_{X_t}}\right]\nu_{X_t}(dz, dt) = 0, \qquad \forall t \in \mathcal{T}. \tag{3.18}$$

It is noteworthy that despite taking a different path for the proofs, our results are in full agreement with Elliott and Siu [2013]. Consequently, we must attain same price dynamics for the reference portfolio under a risk-neutral measure specified by the Esscher transform. We shall present the result and refer the readers to Elliott and Siu [2013] for the proof.

$$\begin{aligned} A_t &= r_{X_t} \int_0^t A(s)ds \\ &\quad - \int_0^t \int_{\mathcal{R}} A(s-)\Big\{z - e^{z(1-\theta_{X_{s-}})} + e^{-z\theta_{X_{s-}}}\Big\} N^{\mathbb{Q}}_{X_t}(dz, dt), \end{aligned} \tag{3.19}$$

where $\widetilde{N}^{\mathbb{Q}}_{X_t}$ denote a compensated Markov-modulated random measure with compensator $\nu^{\mathbb{Q}}_{X_t}(dz, dt) := e^{-\theta_s\gamma(z)}\nu_{X_t}(dz, dt)$. Additionally, for convenience in the notation let:

$$f(t, z, \theta, X) := e^{z(1-\theta_{X_{s-}})} - z - e^{-z\theta_{X_{s-}}}.$$

3.2.3 Fair Valuation

Let $\tau_0 := \inf\{t : A_t = 0\}$ denote the first passage time of the value process A_t hitting zero, after which the value of A_t remains zero forever (i.e. $t \geq \tau_0$). The RCLA policy stipulates that within this time until the policyholder deceases, T_d, he will receive the life annuity from the insurance company. The payoff at ruin time τ_0 is defined as in Huang et al. [2009], for a typical \$1 annuity value as:

$$F_{X_t}(\tau_0) = \int_{\tau_0}^{T_d \vee \tau_0} \exp(-r_{X_s}s)ds. \tag{3.20}$$

Define the survival probability as:

$$\Pr(T_d > t) = \exp(-\int_0^t \lambda_s ds) \tag{3.21}$$

Consistent with Huang et al. [2009] we show that, for the regime-switching version of Gompertz-Makeham continuous law of mortality that λ_t obeys, the payoff equation (3.20) can be defined as:

$$F_{X_t}(\tau_0) = \frac{\beta\Gamma\Big(-(\lambda_t + r_{X_t})\beta, \exp\left(\tau_0 - \beta^{-1}m_{gm}\right)\Big)}{\exp\left\{(m_{mg} - \tau_0)(\lambda + r_{X_t}) - \exp\left(\tau_0 - \beta^{-1}m_{gm}\right)\right\}} \tag{3.22}$$

where λ is the constant non-age-dependent hazard rate, which is time dependent but there is no economic justification for it to be state dependent. Furthermore, $\beta > 0$ denotes the dispersion coefficient, $m_{gm} > 0$ is the modal value and $\Gamma(.)$ is the gamma function. Let $V(A, X, t)$ denote the no-arbitrage value of the RCLA contract and $\widehat{V}(A, X, t)$ denote the discounted value of $V(A, X, t)$, both under $\mathbb{Q}$-measure. Thus

$$\begin{aligned}
\widehat{V}(A, X, t) &= \exp\Big(-\int_t^{T_d}(r_{X_s} + \lambda_s)ds\Big)V(A, X, t) \qquad (3.23)\\
&= E^{\mathbb{Q}}\Big[\exp\Big(-\int_t^{\tau_0}(r_{X_s} + \lambda_s)ds\Big)F(\tau_0)\mathbb{I}_{\tau_0 < T_d}\\
&\quad + \exp\Big(-\int_t^{T_d}(r_{X_s} + \lambda_s)ds\Big)\Big(A_{T_d} - A_0\Big)\mathbb{I}_{\tau_0 \geq T_d}\Big|\mathcal{F}_t\Big].
\end{aligned}$$

The above representation indicates that the value of the RCLA contract can be decomposed into a combination of:

1. European call option maturing at the time of death of the policyholder with strike price equals to the initial investment amount A_0.

2. a 'down-and-in' barrier option with payoff structure defined in equation (3.22), depending on the ruin time of the investment value process.

The path dependency of the state variable $F(\tau_0)$ does not allow us to get an analytical formulation for the value of the policy. On the other hand, the Monte Carlo based simulation methods are too slow for this type of question. As an alternative, the option value can be determined by numerically solving the PIDE.

Write V_i for $V(A, e_i, t)$, where $i = 1, 2, ..., N$ and $\mathbf{V} := \{V_1, V_2, ..., V_N\}$. Then, as in Buffington and Elliot [2002] $\mathbf{V}$ satisfies the following N coupled PIDEs:

$$\mathcal{L}_{A,e_i}(V_i) + \langle \mathbf{V}, \mathcal{Q}e_i \rangle = \big(r_{X_t} + \lambda\big)V_i, \qquad i = 1, 2, ..., N. \tag{3.24}$$

Where $\mathcal{L}_{A,e_i}(V_i)$ is the partial-integro differential operator. By applying the Ito's lemma:

$$\begin{aligned} dV(A, X, t) \;=\; & \Bigg[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial A} r_{X_t} A_{t-} \\ & + \int_{\mathcal{R}^+} \Big\{ V\Big(A_{t-} + f(t, z, \theta, X)\Big) - V(A_{t-}) - \frac{\partial V}{\partial A} f(t, z, \theta, X) \Big\} \nu^{\mathbb{Q}}_{X_t} dz \Bigg] dt \\ & + \int_{\mathcal{R}} \Big\{ V\Big(A_{t-} + f(t, z, \theta, X)\Big) - V(A_{t-}) \Big\} \widetilde{N}^{\mathbb{Q}}_{X_t}(dt, dz) \end{aligned}$$

hence:

$$\begin{aligned} \mathcal{L}_{A,e_i}(V_i) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial A} r_{X_t} A_{t-} + \int_{\mathcal{R}^+} \Big\{ \quad & V\Big(A_{t-} + f(t, z, \theta, X)\Big) - V(A_{t-}) \\ & - \frac{\partial V}{\partial A} f(t, z, \theta, X) \Big\} \nu^{\mathbb{Q}}_{X_t} dz \end{aligned} \tag{3.25}$$

3.3 Numerical Analysis for Valuation and Hedging

3.3.1 Explicit finite difference scheme

While the continuous version of the pricing model has infinite domain, the discretized computational domain must be limited by finite boundaries. Let $[0, A^{\max}] \times [0, T]$ denote the finite computational domain, where the width of the spatial interval is chosen to be sufficiently large. We may obtain a solution to equation (3.24) by specifying the following boundary conditions:

$$V(0,X,T) = F_{\tau_0}, \tag{3.26}$$

$$V(A,X,T) = (A_{T_d} - A_0)^+, \tag{3.27}$$

$$\frac{dV(A^{\max},X,t)}{dA} = 1. \tag{3.28}$$

The derivatives of the value function $V(A,X,t)$ in equations (4.11) can be replaced by the finite differences and the integral terms are approximated by using the trapezoidal rule at first. Then, the problem then can be solved by using an explicit scheme. The computational domain is discretized into a finite difference mesh, where ΔA and Δt are the step-width and time step, respectively. Let $U_j^m(X)$ denote the numerical approximation to $U(j\Delta A, m\Delta t)$, where $m = 0,1,2,...,M$ and $M\Delta t = T$, as well as $j = 0,1,2,...,J$ and $J\Delta A = A^{\max}$.

Instead of prescribing the boundary conditions along the numerical boundaries, corresponding to $j = 0$ and $j = A^{\max}$, we enforce the satisfaction of the discretized version of the governing equation along the boundaries. This is done by using one-sided difference operators to approximate the differential operators in the differential equation so that fictitious mesh points outside the computational domain are avoided. We approximate the differential terms by the following difference:

$$\frac{\partial U}{\partial A}(J\Delta A, X, M\Delta t) = \frac{U_m^j(X) - U_m^{j-1}(X)}{\Delta A}, \tag{3.29}$$

$$\frac{\partial U}{\partial t}(J\Delta A, X, M\Delta t) = \frac{U_m^j(X) - U_{m-1}^j(X)}{\Delta t} \tag{3.30}$$

In order to approximate the integral term, we adopt the trapezoidal rule used by Cont and Tankov [2004a], with the same spatial grids as in (3.29). The domain of the integral is truncated to a bounded interval, with $[B_l, B_r]$ denotes the jump in the investment value. Then we choose integers K_r and K_l such that:

$$[B_l, B_r] \subset [(K_l - 0.5)dA, (K_r + 0.5)dA]. \tag{3.31}$$

Then the integral term in equation (4.11) is approximated by:

$$\sum_{h=K_l}^{K_r} U_{i+h} \int_{(h-0.5)dA}^{(h+0.5)dA} \widetilde{N}^{\mathbb{Q}}(dt, dz). \tag{3.32}$$

By the explicit finite difference scheme, we start at $T = T_d$ with terminal values, and move backwards on the time dimension so that we can calculate the value function for the fair value of RCLA contact.

3.3.2 Numerical Example

Let's assume an economy with two states, where $'X_t = 1'$ represents 'Good' economy, and$'X_t = 2'$ represents 'Bad' economy. Let $\mathcal{P}(t)$ be the transition probability matrix for time t. Write:

$$\mathcal{P}(t) = \begin{pmatrix} 1-p_1 & p_1 \\ p_2 & 1-p_2 \end{pmatrix}, \tag{3.33}$$

where p_i is the probability in which the economy switches from i to state $3-i$, for $i = 1, 2$. Since $\mathcal{P}(t) = e^{\mathcal{Q}t}$, we have by simple calculation that:

$$\mathcal{P}(t) = \frac{1}{\pi} \begin{pmatrix} q_{22} & q_{22} \\ q_{11} & q_{11} \end{pmatrix} + \frac{e^{-\pi t}}{\pi} \mathcal{Q} \tag{3.34}$$

where $\pi = q_{11} + q_{22}$.

In the following numerical example, we assume that the probability for which the economy switches from state one to state two is $p_1 = 40\%$, and from state two to state one is $p_2 = 30\%$.

Further, suppose $T_0 = 50$ and $T_d = 100$, and divide one year into $n = 252$ time-interval, each representing one business day. Also, consider some specimen values for model parameters. These values are in a reasonable range of magnitude from a practical point of view, and are consistent with the magnitudes of model parameters in empirical literature.

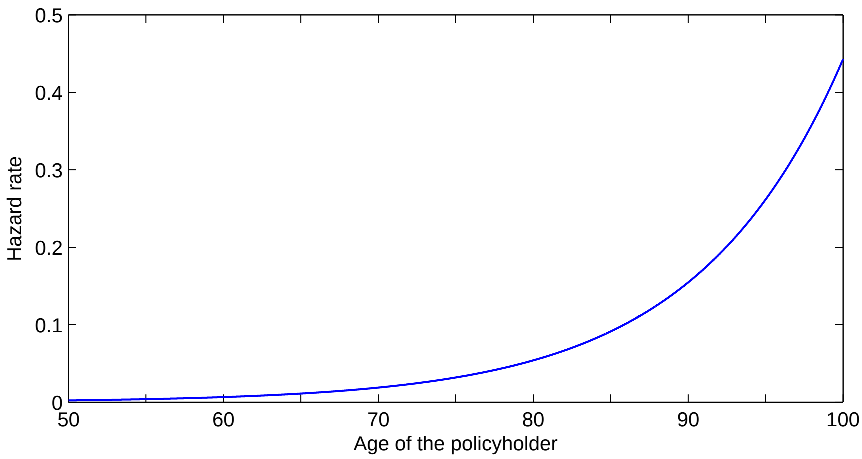

FIGURE 3.1: Age-dependent hazard rate λ from age 50 to 100.

$$r_1 = 0.05; \quad r_2 = 0.01; \quad ; C = 1;$$
$$M_1 = 200; \quad M_2 = 250; \quad G_1 = 500; \quad G_2 = 600;$$

In addition, in (3.22) we defined the age-dependent hazard rate λ such that it follows the Gomperts-Makeham continuous law. For this example, let us assume $m_{mg} = 86.34$ and $\beta = 9.5$. Figure3.1 presents the hazard rate for agents between $T_0 = 50$ and $T_d = 100$.

In order to discretize the space domain, we assume that $A^{\max} = \$400$, with $dA = 1$. In addition, we choose the value of dt to be sufficiently small, to avoid instability. Figure3.2, displays a comparison between the value of a RCLA contract under two different scenarios; namely with Markov regime-switching and without Markov regime-switching. We assume that the parameter values of the no-regime-switching version of our model match with those in the corresponding regime-switching model, when the economy is in state one. For both cases, we calculate the value of a policy that pays \$1 per annum life annuity.

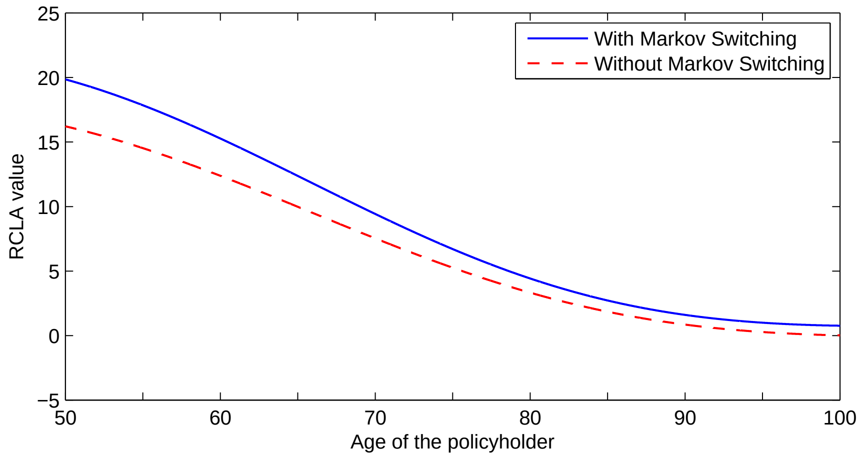

FIGURE 3.2: The value of $1p.a. life annuity for policyholder purchases at age from 50 to 100.

3.3.3 Dynamic Delta hedging

In addition to the fair valuation of the RCLA contracts, it is interesting to investigate how the risks inherent in these products can be hedged once the policy has been sold from a risk management perspective. Despite our main focus being the fair valuation issue of the policy, we highlight the practical importance of the hedging and risk management issues.

There are different ways to hedge the risks inherent in the options embedded in the policy. Hedging via the Greeks and the risk-minimizing hedging represent two popular approaches to hedging these risks. However, due to the fact that hedging using the Greeks is only an approximating hedging strategy, and also due to the market incompleteness, a perfect hedge strategy is not attainable. There is a wealth of literature about different methods of calculation of the Greeks. Some prominent examples are the efficient Monte Carlo simulation method proposed by Broadie and Glasserman [1996], the Mallivian calculus approach introduced by Fournie et al. [2001], and a combination of the two methods in a more recent paper by Davis and Johansson [2006], in particular for jump-diffusion models.

In this paper we choose to use a conventional method to numerically calculate the values for Delta, at discrete equidistance time intervals. The numerical hedging analysis is widely used in the practical arena, due to its efficiency and simplicity in implication. We compute the Delta at each point using the Finite Difference values from (3.29) to evaluate the sensitivity of the RCLA contracts to the changes in the value of the underlying portfolio. Then, we employ the Monte Carlo algorithm to simulate a dynamic Delta hedging strategy for our numerical example. We expect nodal hedging errors due to the market incompleteness. Finally, we approximate the distribution of the terminal hedging error, based on which we calculate non-parametrically the Value at Risk (VaR) and the Expected Shortfall (ES) of the hedging error.

The nodal hedging error is the profit or loss arisen from the difference between the nodal value of the insurance product, and the nodal value of the replicating portfolio of the primitive assets. In a complete market such as Black-Scholes economy, the difference is zero. In contrast, in an incomplete market we always have a none-zero amount for the hedging error that accumulates to the terminal time. We let $\Pi(T, N)$ denote the accumulated value, which is known as the terminal hedging error, and can be defined as:

$$\begin{aligned}\Pi(T, N) = \sum_{n=1}^{N} \Big[\quad & (A_n - A_{n-1}))\Delta(t_{n-1}, A_{n-1}) \\ & -\big(U(t_n, A_n) - U(t_{n-1}, A_{n-1})\big)\Big] \end{aligned} \tag{3.35}$$

where the path of $\{A_n\}, n = 1, ..., N$ is simulated by a simulation experiment, $\Delta(t, A_n)$ is calculated by (3.29) and the path of $U(t_n, A_n)$ is calculated by the Finite Difference method as explain in section3.3.1.

For the purpose of simulating the path of $\{A_n\}$, we employ the Monte Carlo simulation. The time intervals for the simulation is chosen such that it corresponds to the FD mesh. We perform the simulation for the numerical example in section3.3.2 and we consider daily, weekly, monthly and quarterly hedging frequencies. The purpose of this exercise is to investigate how much of the hedging error is due to the market incompleteness, and how much is due to the frequency of the hedging strategy.

To approximate the distribution of the hedging errors, we use a kernel density estimation with Epanechnikov as our choice of kernel function. The performance of a kernel function is measured by the Asymptotic Mean Integrated Squared Error (AMISE), and the Epanechnikov kernel minimizes the AMISE. The optimal choice of the bandwidth is critically important for the accuracy of the density approximation. The determination

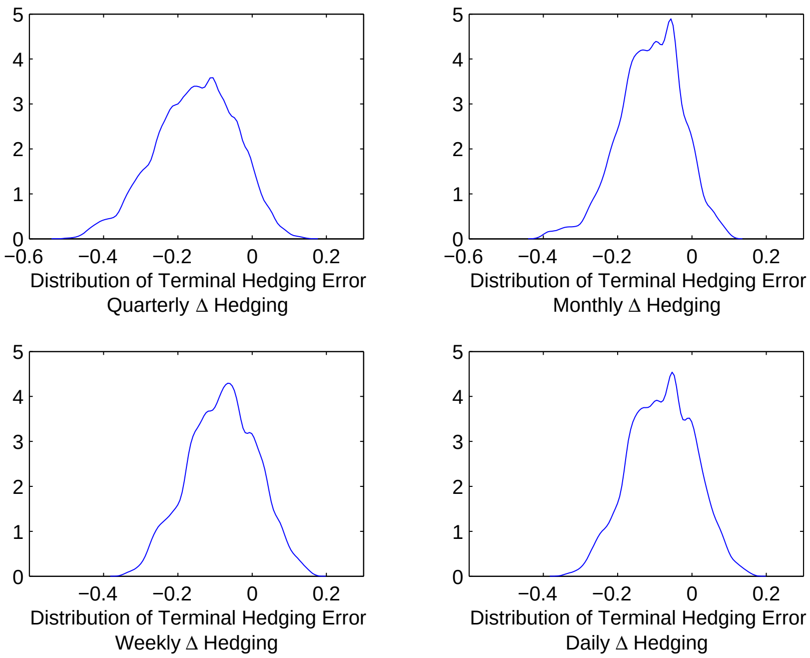

FIGURE 3.3: Distribution of hedging error for different hedging frequencies.

of the best bandwidth involves a trade-off between bias and variance. A very small bandwidth would result in small errors in the approximation; however, this is limited to the number of data points in the local neighborhood. Besides, the variance of the estimated local parameters is often large for a small number of data points. On the other hand, large bandwidth would create a large modeling bias depending on the underlying function. In this paper we use the approach introduced in Shimazaki and Shinomoto [2010] in order to determine the optimal bandwidth.

To summarize the results, in figure3.3 we plot the distribution of the hedging error, for different hedging frequencies. In addition, table3.1 reports the key statistics for the distribution plots in figure3.3. The results demonstrate that by increasing the frequency of hedging, we can reduce the hedging error. Particulary, figures in table3.1 show improvements in the VaR and ES measures, after increasing the hedging frequency from quarters to weeks. Nevertheless, in line with our expectations, we see that after a certain threshold the hedging error cannot be reduced. The residual hedging error is due to the market incompleteness. In this paper, due to the long life of the insurance contracts, we believe the threshold could be chosen between one day to one week.

Table 3.1: The 95% Value at Risk and Expected Shortfall, as well as the mean and the standard deviation for the distribution of hedging error, using different Δ-hedging frequencies

	Statistics			
Hedging Frequency	VaR (95%)	ES (95%)	Mean	Standard Deviation
Quarterly	0.3412	0.3883	-0.1533	0.1087
Monthly	0.2584	0.3067	-0.1119	0.0849
Weekly	0.2449	0.2711	-0.0805	0.0925
Daily	0.2314	0.2611	-0.0806	0.0875

3.4 Conclusion

In this paper we consider pricing and managing the hedging risk of Ruin Contingent Life Annuities, under the regime-switching Variance Gamma process. The RCLA contract is a modern equity-linked insurance contract that allows the near-to-retirement population to hedge two types of retirement risks; namely, an above average survival rate and bear market during the retirement risk zone.

The Variance Gamma process is a class of the pure-jump Levy processes, with growing popularity as an alternative to the Black-Scholes economy. We use the Levy measure representation of the VG process and modulate its parameters by a Markov chain to capture the impacts of the regime-switching risk.

We employ the Esscher transform to find an equivalent martingale measure in our incomplete market and derive the risk-neutral dynamics of the underlying portfolio. Subsequently, we derive the pricing PIDE, and solve for the value function numerically. To highlight the practical implications of our model, we conduct a numerical example, through which we calculated the value of the RCLA product. Then we conduct a Monte Carlo simulation experiment for a simple dynamic Delta hedging strategy. The performance of the strategy is analyzed by examining the VaR and ES of the terminal hedging errors, arisen from the dynamic Delta hedging.

Chapter 4

Pricing Participating Products under a Generalized and Regime-Switching Jump-Diffusion Model

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Abstract

We propose a model for the valuation of participating life insurance products under a generalized jump- diffusion model with a Markov-switching compensator. The Esscher transform is employed to determine an equivalent martingale measure in the incomplete market. The results are further manipulated through the utilization of the change of numeraire technique to reduce the dimensions of the pricing formulation. This paper is the first that extends the technique for a generalized jump-diffusion process with a Markov-switching kernel-biased completely random measure, which nests a number of important and popular models in finance. A numerical analysis is conducted to illustrate the practical implications.

Keywords: Participating products - Generalized jump-diffusion - Esscher transform - Reduction of dimensionality - Collocation method

4.1 Introduction

Participating life insurance products are a popular class of equity linked insurance products around the world. In these policies the insured not only receives the guaranteed annual minimum benefit, but also receives proceeds from an investment portfolio. At the initiation of the contract, the policyholder pays the first annual premium to the insurance company, who will manage a well diversified reference portfolio. The insurer employs a surplus distribution mechanism to credit interest at or above a specified amount of guaranteed rate annually to the insured. The net difference between the market value of the asset portfolio and the book value of the policyholder's account is called the bonus reserve or buffer. The bonus reserve is used to provide stable and smooth returns to the policyholder in the future and protect against insolvency. If the terminal surplus of the fund is positive, the policyholder can also receive a terminal bonus. The insurer has an option to default at the maturity of the policy, in which case the insured will receive the outstanding assets. Grosen and Jorgensen [2000] provided a comprehensive discussion on different contractual features of participating policies. It is important to develop a mathematical model for the fair valuation of these policies, due to the international trend of using the market-based and fair valuation accountancy standards for the implementation of risk management practices.

Accurate pricing of life insurance participating policies, through the fair valuation of the embedded options, can be traced back to Wilkie [1987]. Grosen and Jorgensen [2000] analyzed the minimum rate guarantee and surplus distribution mechanism, and modeled the surrender risk as an American early exercise feature in their contingent claims model. Bacinello [2001, 2003a,b] employed binomial models to construct pricing models of participating policies with different types of embedded features. Prieul et al. [2001] and Siu [2005] priced the participating policy using a partial differential equation (PDE) approach. In both papers, the authors utilized the method of similarity transformations of variables to reduce the dimensions of the PDE. In addition, Siu [2005] and Siu et al. [2008] considered the pricing of the participating policies when different parameters of the dynamics of the underlying reference portfolio are Markov-modulated, in order to model the regime-switching risk for the economy. While the former, employed a Markov-modulated diffusive process, the latter considered the valuation of the products under a generalized jump-diffusion model with a Markov-switching compensator.

In the present paper, similar to Siu et al. [2008], we propose a model for the valuation of participating life insurance products under a generalized jump-diffusion model with a Markov-switching compensator. We make the assumption that the parameters of the market values of the reference portfolio, namely, the risk-free interest rates, the expected growth rate and the volatility of the risky asset, depend on the state of the economy, which is modeled by a continuous-time Markov-chain process. In addition, we specify the jump component by a class of Markov-modulated kernel-biased completely random measures. Our model is a modified version of the kernel-biased representation of James [2002, 2005]. Here the kernel function allows different forms of distortion of jump sizes into the model. Incorporation of the Markov chain process to this framework provides further flexibility to describe the impact of structural changes in macroeconomic conditions and business cycles on the valuation model. Hence, we utilize the Esscher transform to determine the equivalent martingale measure and price the participating product under the generalized jump-diffusion model.

Siu et al. [2008] solve the risk-neutral stochastic differential equation (SDE) using a robust Monte Carlo based simulation, whereas, in this paper we derive the pricing formulation using the partial integro-differential equation (PIDE) approach. For this approach we face the problem of the high dimensionality of the PIDE, because of the embedded credit scheme in the policy. However, we overcome the problem by reduceing the dimensions of the PIDE by using a version of the change of numeraire technique, through which we avoid the evolution of the joint distribution of two stochastic variables.

The technique was introduced in Hansen and Jorgensen [2000], however, to the knowledge of the authors, this is the first paper that extends the technique for a generalized jump-diffusion process with a Markov-switching kernel-biased completely random measure, which nests a number of important and popular models in finance. Finally, we employ a numerical scheme, namely the collocation method, to approximate the solution of the valuation differential equation.

The concept of regime-switching can be traced back to Quandt [1958] and Goldfield and Quandt [1973] where they employed regime-switching regression models to describe nonlinearity in economic data. The idea of probability switching appeared in the early development of nonlinear time series analysis, where Tong [1983] proposed one of the oldest classes of nonlinear time series models, namely the threshold models. Regime-switching models aim to capture the appealing idea that the macro-economy is subject to regular, yet unpredictable in time, regimes which in turn affect the prices of financial securities. For example, structural changes of macro-economic conditions, such as inflation and recession, may induce changes in the stock returns or in the term structure of interest rates. Similarly, periods of high market turbulence and liquidity crunches may increase the default risk of financial institutions.

Hamilton [1989] popularized regime-switching time series models in the economic and econometric literature and since then, considerable attention has been paid to investigate the use of regime-switching to model economic and financial data. Due to the empirical success of regime-switching models, they have been applied to different areas in banking and finance; including asset allocation, option valuation, risk management, term structure modeling. Recently, scholars have turned their attention to option valuation under regime-switching model, including, Naik [1993], Guo [2001], Buffington and Elliot [2002], Elliott et al. [2005] , and Fard and Siu [2013]. Additionally, regime-switching models have become popular in actuarial science in recent years. For example, Hardy [2001], Siu [2005], and Siu et al. [2008] used the regime-switching models to capture the impact of the structural changes in the economy on the value of different equity linked insurance products.

This article is structured as follows. Section 4.2 presents the Markov-modulated pure-jump asset price model, the pricing formulation through the Esscher transform, as well as the change of numeraire technique for the dimensionality reduction. Section 4.3, provides some important parametric cases of the model, namely, Markov-modulated generalized gamma processes, as well as the scale-distorted and power-distorted versions. Finally, Section 4.4.2 provides the numerical solution for the underlying PIDE through a class of the finite element method, known as the collocation method.

4.2 Modeling Framework

Let us suppose that the economy can switch between a finite number of states each of which are characterized by their respective parameters. In this economy, we consider a financial market, where an agent can either invest in a risk-free money market account or choose from a range of risky assets. All the parameters of the risk-free asset, as well as the risky assets, vary as the economy switches regimes, a process governed by a Markov-chain. We assume that the market is frictionless, the mortality risk is absent, and there are no taxes. In this market, an insurer is considering issuing a participating life insurance policy, linked to a well diversified portfolio of the risky assets, known as the reference portfolio, as well as a money market account. We assume that the market value of the reference portfolio is governed by a jump-diffusion model with the jump component being specified as a kernel-biased completely random measure with a Markov-switching compensator.

The purpose of this paper is to develop a fair valuation model for the insurance product, where we are challenged by:

- the market incompleteness, due to the jump competent and the regime-switching framework;
- the high dimensionality of the pricing PIDE, due to the attached credit scheme; and
- the complex pricing PIDE, with no known analytical solutions.

The first two challenges are addressed in this section, and the latter is discussed in section 4.4.2.

4.2.1 The Pricing Dynamics

We fix a complete probability space $(\Gamma, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the real-world probability measure. Let $\mathcal{T}$ denote the time index set $[0, T]$ of the economy. We describe the states of the economy by a continuous-time Markov chain $\{X_t\}_{t\in\mathcal{T}}$ on $(\Gamma, \mathcal{F}, \mathbb{P})$ with a finite state space $\mathcal{S} := (s_1, s_2, \cdots, s_N)$. Without loss of generality, we can identify the state space of the process $\{X_t\}_{t\in\mathcal{T}}$ to be a finite set of unit vectors $\{e_1, e_2, \cdots, e_N\}$, where $e_i = (0, \cdots, 1, \cdots, 0) \in \mathcal{R}^N$.

Let $\mathcal{Q}(t) = [q_{ij}(t)]_{i,j=1,2,\ldots,N}, t \in \mathcal{T}$, denote a family of generators, or rate matrices, of the chain $\{X_t\}_{t\in\mathcal{T}}$ under $\mathbb{P}$. Here, $q_{ij}(t)$ represents the instantaneous intensity of the transition of the chain $\{X_t\}_{t\in\mathcal{T}}$ from state j to state i at time t . Note that for each $t \in \mathcal{T}$, $q_{ij}(t) > 0$, for $i \neq j$ and $\sum_i^N q_{ij}(t) = 0$, so $q_{ii}(t) \leq 0$. We assume that $q_{ij}(t) > 0$, for each $i, j = 1, 2, ..., N$ and $i \neq j$ and each $t \in \mathcal{T}$. For any such matrix $\mathcal{Q}(t)$, write $q(t) := (q_{11}(t), ..., q_{ii}(t), ..., q_{NN}(t))'$. With the canonical representation of the state space of the chain, Elliott et al. [1995] provide the following semi-martingale decomposition for $\{X_t\}_{t\in\mathcal{T}}$:

$$X_t = X_0 + \int_0^t \mathcal{Q}X_s ds + M_t. \tag{4.1}$$

Further, M_t is a $\mathcal{R}^N$-valued martingale with respect to the filtration generated by $\{X_t\}_{t\in\mathcal{T}}$. Let $\{r(t, X_t)\}_{t\in\mathcal{T}}$ be the instantaneous market interest rate of a money market account, which depends on the state of the economy; that is,

$$r(t, X_t) \quad = \quad \langle \mathbf{r}, X_t \rangle = \sum_{i=1}^N r_i \langle X_t, e_i \rangle, \qquad t \in \mathcal{T},$$

where $\mathbf{r} := (r_1, r_2, \cdots, r_N)$, with $r_i > 0$ for each $i = 1, 2, \cdots, N$. Additionally, $\langle ., . \rangle$ denotes the inner product in the space $\mathcal{R}^N$. Thus, the dynamics of the value of the risk-free asset, $\{B_t\}_{t\in\mathcal{T}}$ would be

$$\frac{dB_t}{B_t} \quad = \quad r(t, X_t)dt,$$

with $B_0 = 1$.

James [2002, 2005] proposed a kernel-biased representation of completely random measures, which provided a great deal of flexibility in modeling different types of finite and infinite jump activities by choosing different kernel functions. The approach is an amplification of Bayesian techniques developed by Lo and Weng [1989] for the gamma-Dirichlet processes. Perman et al. [1992] considered applications to the models, which all fall within an inhomogeneous spatial extension of the size-biased framework. In this sequel, we adopt the Markov-modulated version of the kernel-biased representation of completely random measures proposed by James [2002, 2005].

Let $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$ denote a measurable space, where $\mathcal{B}(\mathcal{T})$ is the Borel σ-field generated by the open subsets of $\mathcal{T}$. Write $\mathcal{B}_0$ for the family of Borel sets $U \in \mathcal{R}^+$, whose closure $\bar{U}$ does not contain the point 0. Let $\mathcal{X}$ denote $\mathcal{T} \times \mathcal{R}^+$. The measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is then given by $(\mathcal{T} \times \mathcal{R}^+, \mathcal{B}(\mathcal{T}) \otimes \mathcal{B}_0)$.

For each $U \in \mathcal{B}_0$, let $N_{X_t}(., U)$ denote a Markov-switching Poisson random measure on the space $\mathcal{X}$. Write $N_{X_t}(dt, dz)$ for the differential form of measure $N_{X_t}(t, U)$. Let $\rho_{X_t}(dz|t)$ denote a Markov-switching Levy measure on the space $\mathcal{X}$ depending on t and the state X_t; η is a σ-finite (nonatomic) measure on $\mathcal{T}$. As in James [2005], the existence of the kernel-biased completely random measure is ensured by supposing an arbitrary positive function on $\mathcal{R}^+, h(z), \rho_i$ and η are selected in such a way that for each bounded set $\mathcal{B}$ in $\mathcal{T}$

$$\sum_{i=1}^{N} \int_{\mathcal{B}} \int_{\mathcal{R}^+} \min(h(z), 1) \rho_i(dz|t) \eta(dt) < \infty.$$

As in Siu et al. [2008], assume the Markov-switching intensity measure

$$\nu_{X_t}(dt, dz) := \rho_{X_t}(dz|t)\eta(dt) = \sum_{i=1}^{N} \big(\rho_i(dz|t)\langle X_t, e_i\rangle\big)\eta(dt).$$

Define a Markov-modulated kernel-biased completely random measure

$$\mu_{X_t}(dt) := \int_{\mathcal{R}^+} h(z) N_{X_t}(dt, dz),$$

which is a kernel-biased Markov-modulated Poisson random measure $N_{X_t}(dt, dz)$ over the state space of the jump size $\mathcal{R}^+$ with the mixing kernel function $h(z)$. We can replace the Poisson random measure with a random measure and choose some quite exotic functions for $h(z)$ to generate different types of finite and infinite jump activities. Let $\{W_t\}_{t \in \mathcal{T}}$ denote a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the $\mathbb{P}$-augmentation of its natural filtration $\mathcal{F}^W := \{\mathcal{F}_t^W\}_{t \in \mathcal{T}}$. Let $\widetilde{N}_{X_t}(dt, dz)$ denote the compensated Poisson random measure defined by

$$\widetilde{N}_{X_t}(dt, dz) = N_{X_t}(dt, dz) - \rho_{X_t}(dz|t)\eta(dt).$$

Let μ_t and σ_t denote the drift and volatility of the market value of the reference asset, respectively, and define

$$\begin{aligned}\mu_t &:= \langle \boldsymbol{\mu}, X_t \rangle = \sum_{i=1}^{N} \mu_i \langle X_t, e_i \rangle, \\ \sigma_t &:= \langle \boldsymbol{\sigma}, X_t \rangle = \sum_{i=1}^{N} \sigma_i \langle X_t, e_i \rangle,\end{aligned}$$

where $\boldsymbol{\mu} := (\mu_1, \mu_2, ..., \mu_N)$, $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, ..., \sigma_N)$, $\mu_i \in \mathcal{R}$ and $\sigma_i > 0$ for each $i = 1, 2, ..., N$. To focus on modeling the impact of transitions of economic states on the price dynamics of the reference portfolio and the fair value of the policy, we assume here that μ_t and σ_t depend on the current economic state X_t only. Consider a generalized jump-diffusion process $A := \{A(t)|t \in \mathcal{T}\}$, such that

$$dA_t = A_{t-}\left[\mu_t dt + \sigma_t dW_t + \int_{\mathcal{R}^+} h(z)\widetilde{N}_{X_t}(dt, dz)\right], \tag{4.2}$$

where $A_0 = 0$. We assume under $\mathbb{P}$ the price process $\{S_t\}_{t\in\mathcal{T}}$ is defined as $S_t := \exp(A_t)$, so that

$$\begin{aligned}dS_t &= \left(\mu_t + \frac{1}{2}\sigma_t^2\right)dt + \sigma_t dW_t + \int_{\mathcal{R}^+}\left(e^{h(z)} - 1\right)\widetilde{N}_{X_t}(dt, dz) \\ &\quad - \int_{\mathcal{R}^+}\left\{h(z) - e^{h(z)} + 1\right\}\rho_{X_t}(dz|t)\eta(dt)\end{aligned} \tag{4.3}$$

with $S_0 = 1$.

4.2.2 The Credit Scheme

We now describe the dynamics of the liability side of the balance sheet. For each time $t \in \mathcal{T}$, let R_t and D_t denote the book value of the policy reserve and the bonus buffer, respectively. The purpose of the bonus buffer is to iron out the fluctuations in the stream of the cash flows to the policyholders, inherited from the fluctuations of the reference portfolio. R_t is considered as the policyholder's account balance. Let S_t denote the market value of the asset backing the policy, so that

$$S_t = R_t + D_t, \quad t \in \mathcal{T},$$

where $R(0) := \alpha_p S(0), \alpha_p \in (0,1]$. $R(0)$ is the single initial premium paid by the policyholder for acquiring the contract and α_p is the cost allocation parameter. The funds are distributed between two components of liability over time according to the bonus policy described by the continuously compounded interest rate credited to the policy reserve c_R,

$$dR_t = c_R(S,R)R_t dt.$$

In practice, $c_R(S,R)$ is specified by the management level based on the rule of bonus distribution. Here we adopt the interest rate crediting scheme used in Chu and Kwok [2006]

$$c_R(S,R) = \max\left(r_g, \ln\frac{S_t}{R_t} - \beta\right),$$

where β is the reversionary bonus, which is a long-term constant target ratio specified by the management, and is the r_g for the minimum interest rate credited to the policyholder's account.

4.2.3 Fair Valuation

The fair value of the participating product can be decomposed into the fair value of a guaranteed benefit, a surrender option, and a default option. Let $g(S,R,X,T)$ denote the terminal payoff of the participating policy's maturity date T, when the state of the economy at time T is X. Then

$$V(S_T, R_T, X_T) = R_T + \gamma P_{1T} - P_{2T}, \tag{4.4}$$

where γ is the terminal bonus distribution rate, $P_{1T} := \max(\alpha_p S_T - R_T, 0)$ is the terminal bonus option, $P_{2T} := \max(R_T - S_T, 0)$ represents the terminal default option, and R_T is the guaranteed benefit. The bonus option can be viewed as a vanilla European call option that grants the policyholder the right to pay the policy value as a strike price to receive α_p-portion of the asset portfolio.

4.2.4 Pricing by the Esscher Transform

For the fair valuation of the policy we need to ensure that there is no arbitrage opportunities in the market through the determination of the equivalent risk- neutral martingale measure (Pilska [1997]). In incomplete markets, as is the case in this paper, there are more than one equivalent martingale measure, and hence, more than one no-arbitrage price. Different approaches have been proposed for pricing and hedging derivative securities in incomplete financial markets. For instance, Follmer and Sondermann [1986], Schweize [1995], and Follmer and Schweizer [1991] selected an equivalent martingale measure by minimizing the quadratic utility of the terminal hedging errors. Davis [1997] adopted an economic approach based on the marginal rate of substitution to pick a pricing measure via a utility maximization problem. Avellaneda [1998], Frittelli [2000], and Fard and Siu [2013] employed the minimum entropy martingale measure method to choose the equivalent martingale measure. Gerber and Shiu [1994] pioneered the use of the Esscher transform, a popular tool in actuarial science. The Esscher transform provides market practitioners with a convenient and flexible way to value options. Elliott et al. [2005] demonstrated that the results driven from Esscher transform, when pricing a contingent claim, is equivalent to that driven from the minimum entropy martingale measure.

In this paper, we employ the regime-switching Esscher transform to determine an equivalent martingale measure for the valuation of the policy. Let $\mathcal{F}^X := \{\mathcal{F}_t^X\}_{t\in\mathcal{T}}$, $\mathcal{F}^A := \{\mathcal{F}_t^A\}_{t\in\mathcal{T}}$ and $\mathcal{F}^S := \{\mathcal{F}_t^S\}_{t\in\mathcal{T}}$ denote the $\mathbb{P}$-augmentation of the natural filtration generated by A and S, respectively. Since, $\mathcal{F}^A$ and $\mathcal{F}^S$ are equivalent, we can use either one of them as an observed information structure. Define $\mathcal{G}_t$ for the σ-algebra $\mathcal{F}^X \vee \mathcal{F}^A$ for each $t \in \mathcal{T}$. Write $B(\mathcal{T})$ for the Borel σ-field of $\mathcal{T}$ and let $BM(\mathcal{T})$ denote the collection of $B(\mathcal{T})$-measurable and non-negative functions with compact support on $\mathcal{T}$. For each processes $\theta \in BM(\mathcal{T})$, write

$$(\theta.A)_t \quad := \quad \int_0^t \theta(u)dA(u), \qquad t \in \mathcal{T},$$

such that

1. for each $t \in \mathcal{T}$, $\theta_t := \langle \boldsymbol{\theta}, X_t \rangle$, where $\boldsymbol{\theta} := (\theta_1, \theta_2, ..., \theta_N) \in \mathcal{R}^N$,

2. θ is integrable with respect to the return process.

Let $\{\Lambda_t\}_{t \in \mathcal{T}}$ denote a $\mathcal{G}$-adapted stochastic process

$$\Lambda_t := \frac{e^{(\theta.A)_t}}{\mathcal{M}(\theta)_t}, \qquad t \in \mathcal{T},$$

where $\mathcal{M}(\theta)_t := E[e^{(\theta.A)_t} | \mathcal{F}_t^X]$ is a Laplace cumulant process and takes the following form

$$\begin{aligned} \mathcal{M}(\theta)_t &= \exp\Bigg[\int_0^t \theta_s \Big(\mu_s - \frac{1}{2}\sigma_s^2 \Big) ds + \frac{1}{2} \int_0^t \theta_s^2 \sigma_s^2 ds \\ &\quad + \int_0^t \int_{\mathcal{R}^+} \Big(e^{\theta_s h(z)} - 1 - \theta_s h(z) \Big) \rho_{X_s}(dz|s) \eta(ds) \Bigg]. \end{aligned}$$

(see Siu et al. [2008] for similar calculations). Therefore

$$\begin{aligned} \Lambda_t &= \exp\Bigg[\int_0^t \theta_s \sigma_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 \sigma_s^2 ds + \int_0^t \int_{\mathcal{R}^+} \theta_s h(z) \widetilde{N}_{X_s}(dz, ds) \\ &\quad - \int_0^t \int_{\mathcal{R}^+} \Big(e^{\theta_s h(z)} - 1 + \theta_s h(z) \Big) \rho_{X_s}(dz|s) \eta(ds) \Bigg]. \end{aligned} \tag{4.5}$$

Equation (4.5) is an essential part of our pricing formulation, since we aim to use Λ_t as the Radon-Nikodym Process to change the measure from the historical measure to the risk-neutral measure. One key characteristic of risk-neutral measure is that under this measure, every discounted price process is a martingale. So it is also essential to demonstrate that (4.5) is $\mathcal{G}_t$-martingale.

Lemma 1. Λ_t is $\mathbb{P}$ martingale w.r.t $\mathcal{G}_t$.

Proof. James [2002, 2005] showed that

$$\begin{aligned} E\Big[\ \exp\Big(\int_0^t \int_{\mathcal{R}^+} \theta_s h(z) \widetilde{N}_{X_s}(dz, ds) \Big) \Big| \mathcal{G}_t \Big] \\ = \exp\Big(\int_0^t \int_{\mathcal{R}^+} \Big(e^{\theta_s h(z)} - 1 + \theta_s h(z) \Big) \rho_{X_s}(dz|s) \eta(ds) \Big). \end{aligned}$$

Then, by taking the conditional expectations of (4.5), the results follow.$\square$

For each $\theta \in L(A)$ define a new probability measure $\mathbb{P}^{\theta} \sim \mathbb{P}$ on $\mathcal{G}(T)$ by the Radon-Nikodym derivative

$$\left.\frac{d\mathbb{P}^{\theta}}{d\mathbb{P}}\right|_{\mathcal{G}(T)} := \Lambda_T. \tag{4.6}$$

This new measure $d\mathbb{P}^{\theta}$ is defined by the Esscher transform Λ_T associated with $\theta \in L(A)$.

According to the fundamental theorem of asset pricing, the absence of arbitrage means there exists an equivalent martingale measure under which discounted asset prices are local-martingales, which is widely known as the *local-martingale condition*. Now we stipulate a necessary and sufficient condition for the local martingale condition.

Proposition 3. For each $t \in \mathcal{T}$, let the discounted price of the risky asset at time t be

$$\widetilde{S}(t) := e^{-rt}S(t).$$

Then the discounted price process $\widetilde{S} := \{\widetilde{S}(t)|t \in \mathcal{T}\}$ is an $\mathbb{P}^{\theta}$-local-martingale if and only if $\theta_t := \langle \theta, X_t \rangle,\ \ t \in \mathcal{T}$, is such that $\theta := (\theta_1, \theta_2, ..., \theta_N) \in \mathcal{R}^N$ satisfies the following equation

$$\theta_t \sigma_t^2 \quad + \int_{\mathcal{R}+} \left\{ e^{\theta_t h(z)}(e^{h(z)} - 1) - h(z) \right\} \rho_{X_t}(dz|t)\eta'(t) = r_t - \mu_t. \tag{4.7}$$

Proof. Since $\widetilde{S}$ is $\mathcal{F}^A$-adapted, $\widetilde{S}$ is an $(\mathcal{F}^Y, \mathbb{P}^{\theta})$-local-martingale if and only if it is a $(\mathcal{G}, \mathbb{P}^{\theta})$-local-martingale. By Lemma 7.2.2 in Elliott and Kopp [2005], $\widetilde{S}$ is an $(\mathcal{G}^Y, \mathbb{P}^{\theta})$-local-martingale if and only if $\Lambda\widetilde{S} := \{\Lambda(t)\widetilde{S}(t)|t \in \mathcal{T}\}$ is a $(\mathcal{G}, \mathbb{P}^{\theta})$-local-martingale.

First, by Bayes' rule

$$
\begin{aligned}
E^{\theta}\Big[\exp\Big(-\int_0^t r_s ds\Big)S_t\Big|\mathcal{G}_0\Big] &= \exp\Big(-\int_0^t r_s ds\Big)E\Big[\Lambda_t \exp\Big(\int_0^t dA_u\Big)|\mathcal{G}_0\Big] \\
&= \exp\Big(-\int_0^t r_s ds\Big)\frac{E\Big[\exp\Big(\int_0^t(\theta_u+1)dA_u\Big)|\mathcal{G}_0\Big]}{\mathcal{M}(\theta)_t} \\
&= \exp\Big(-\int_0^t r_s ds\Big)\frac{\mathcal{M}(\theta+1)_t}{\mathcal{M}(\theta)_t} \\
&= \exp\Big(\int_0^t(\mu_s - r_s - \frac{1}{2}\sigma_s^2)ds + \frac{1}{2}\int_0^t(2\theta_s+1)\sigma_s^2 ds \\
&\quad + \int_0^t\int_{\mathcal{R}^+}\big\{e^{\theta_s h(z)}(e^{h(z)}-1)-h(z)\big\}\rho_{X_s}(dz|s)\eta(ds)\Big).
\end{aligned}
$$

Then by setting time $s = 0$, and applying the martingale condition we achieve

$$
\int_0^t(\mu_s - r_s - \frac{1}{2}\sigma_s^2)ds + \frac{1}{2}\int_0^t(1+2\theta_s)\sigma_s^2 = -\int_0^t\int_{\mathcal{R}^+}\big\{e^{\theta_s h(z)}(e^{h(z)}-1)-h(z)\big\}\rho_{X_s}(dz|s)\eta'(s)ds.
$$

Hence, for each $t \in \mathcal{T}$, (4.7) must hold.□

The results from the Lemma 1, Equation (4.6), and Proposition 3, allow us to use (4.5) to drive the risk-neutral dynamics of the return process.

Proposition 4. Suppose $\widetilde{W}_t = W_t - \int_0^t \sigma_s\theta_s ds$ is a $\mathbb{P}^{\theta}$-Browning motion, $\rho^{\theta}_{X_t}(dz|t) := e^{\theta_s h(z)}\rho_{X_t}(dz|t)$ is the $\mathbb{P}^{\theta}$ compensator of $N^{\theta}_{X_t}(dz, dt)$ then

$$
\begin{aligned}
dA_t &= \big(\mu_t + 2\theta_t\sigma_t^2 - \frac{1}{2}\sigma_t^2\big)dt + \sigma_t d\widetilde{W}_t + \int_{\mathcal{R}^+} h(z)\big(1-e^{-\theta_t h(z)}\big)\rho^{\theta}_{X_t}(dz|t)\eta(dt) \\
&\quad + \int_{\mathcal{R}^+} h(z)\widetilde{N}^{\theta}_{X_t}(dz, dt).
\end{aligned} \tag{4.8}
$$

Proof. Assume that $\mathbb{P} \sim \mathbb{P}^\theta$ with density process Λ_t. Suppose $\mathcal{Z}_u \in BM(\mathcal{T})$. Then by Bayes' rule

$$\begin{aligned}
\mathcal{M}^\theta_A(\mathcal{Z})_t &:= E^\theta[e^{(\mathcal{Z}.A)_t}|\mathcal{G}_0] = E[\Lambda_t.e^{(\mathcal{Z}.A)_t}|\mathcal{G}_0] \\
&= \exp\Bigg(\int_0^t \mathcal{Z}_s(\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_0^t \frac{1}{2}(\mathcal{Z}_s+\theta_s)^2\sigma_s^2 ds \\
&\quad + \int_0^t\int_{\mathcal{R}^+}\Big\{e^{(\mathcal{Z}_s+\theta_s)h(z)} - 1 - (\mathcal{Z}_s+\theta_s)h(z)\Big\}\rho_{X_s}(dz|s)\eta(ds) - \int_0^t \frac{1}{2}(\theta.\sigma_s)^2 ds \\
&\quad - \int_0^t\int_{\mathcal{R}^+}\Big\{e^{\theta_s h(z)} - 1 - \theta_s h(z)\Big\}\rho_{X_s}(dz|s)\eta(ds)\Bigg) \\
&= \exp\Bigg(\int_0^t \mathcal{Z}_s(\mu_s + 2\theta_s\sigma_s^2 - \frac{1}{2}\sigma_s^2)ds \\
&\quad + \int_0^t\int_{\mathcal{R}^+}\Big(e^{\theta_s h(z)} - 1\Big)\rho_{X_s}(dz|s)\eta(ds) + \frac{1}{2}\int_0^t \mathcal{Z}_s^2\sigma_s^2 ds \\
&\quad + \int_0^t\int_{\mathcal{R}^+}\Big\{e^{\mathcal{Z}_s h(z)} - 1 - \mathcal{Z}_s h(z)\Big\}e^{\theta_s h(z)}\rho_{X_s}(dz|s)\eta(ds)\Bigg).
\end{aligned}$$

Then under $\mathbb{P}^\theta$, (4.8) holds. □

Similarly, we can derive the risk-neutral price process of the reference portfolio.

Proposition 5. The price process of the reference portfolio S under $\mathbb{P}^\theta$ is

$$dS_t = (r_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t d\widetilde{W}_t + \int_{\mathcal{R}^+}\Big(e^{\theta_t h(z)}(e^{h(z)} - 1) - h(z)\Big)\widetilde{N}^\theta_{X_t}(dt, dz).$$

For the proof, recall $S_t := \exp(A_t)$. Then the proof can easily follow by applying Ito's lemma and the martingale condition (4.7) to (4.8).

4.2.5 Reduction of Dimensionality

The fair value of the participating product V_t has three state variables. As in Hansen and Jorgensen [2000], we choose an alternative numeraire which will result in the reduction of one state variable. We will show that by the change of numeraire, we will avoid the evolution of a joint distribution of two stochastic variables. To begin, define a new state variable $Z := \ln(\frac{S}{R})$, so that

$$C_Z(Z) = C_R(S, R, X, t),$$

and

$$V_Z(Z_t, X_t) = \frac{V(S_t, R_t, X_t)}{R_t}.$$

As a result, the intrinsic value of the policy would be

$$V_Z(Z, X, T) = 1 + \gamma P_{1T}^{Z} - P_{2T}^{Z_T}, \tag{4.9}$$

where $P_{1T}^{Z} := \max(\alpha_p e^{Z_T} - 1, 0)$ and $P_{2T}^{Z} := \max(1 - e^{Z_T}, 0)$. By Ito's lemma, the dynamics of Z under $\mathbb{P}$ is given by

$$dZ_t = \big(\mu_t - C_Z(Z_t)\big)dt + \sigma_t dW_t + \int_{\mathcal{R}^+} h(z)\widetilde{N}_{X_t}(dt, dz).$$

Under $\mathbb{P}^\theta$ and with respect to $\mathcal{G}_t$ we define

$$\begin{aligned}
\mathcal{E}(t) \quad &:= \quad \exp\left(- \int_0^t (r_s - \frac{1}{2}\sigma_s^2) ds \right) \frac{S_t}{S_0} \qquad (4.10) \\
&= \quad \exp\Bigg(- \int_0^t \frac{1}{2}\sigma_s^2 ds + \int_0^t \sigma_s d\widetilde{W} \\
&\quad + \int_{\mathcal{R}^+} \int_0^t \{\ln(1 + f(z, \theta_s) - f(z, \theta_s)\} \rho_{X_s}^{\theta}(dz|s)\eta(ds) \\
&\quad + \int_{\mathcal{R}^+} \int_0^t \ln\big(f(z, \theta_s) + 1\big) \widetilde{N}_{X_s}^{\theta}(ds, dz) \Bigg),
\end{aligned}$$

where $f(z, \theta_t) := e^{\theta_t h(z)}(e^{h(z)} - 1) - h(z)$ for convenience in presentation. We notice that $\mathcal{E}(t)$ is martingale w.r.t $\mathcal{G}_t$ (Oksendal and Sulem [2005], chapter 1).

Define a new equivalent measure $\mathbb{Q}$ as follows

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}^\theta}\right|_{\mathcal{G}(T)} := \mathcal{E}(T),$$

By a version of the Girsanov theorem

$$W_t^{\mathbb{Q}} := \widetilde{W}_t - \int_0^t \sigma_s ds,$$

and

$$N^{\mathbb{Q}}_{X_t}(dz,dt) := N^{\theta}_{X_t}(dz,dt) - f(z,\theta)\rho^{\theta}_{X_t}(dz|t)\eta(dt).$$

Then,

$$dS_t = \quad (r_t + \tfrac{1}{2}\sigma_t^2)dt + \sigma_t dW_t^{\mathbb{Q}} + \textstyle\int_{\mathcal{R}^+}\Big(e^{\theta_t h(z)}(e^{h(z)}-1) - h(z)\Big)\widetilde{N}^{\mathbb{Q}}_{X_t}(dz,dt).$$

By Ito's lemma, the dynamic of Z under $\mathbb{Q}$ is

$$\begin{aligned} dZ_t &= \Big(r_t - C_Z(Z_t)\Big)dt + \sigma_t dW_t^{\mathbb{Q}} \\ &\quad + \int_{\mathcal{R}^+}\Big(\ln(1+f(z,\theta_t)) - f(z,\theta_t)\Big)\rho^{\mathbb{Q}}_{X_t}(dz|t)\eta(dt) \\ &\quad + \int_{\mathcal{R}^+}\ln(1+f(z,\theta_t))\widetilde{N}^{\mathbb{Q}}_{X_t}(dt,dz), \end{aligned}$$

Where, $\rho^{\mathbb{Q}}_{X_t}(dz|t)\eta(dt)$ is defined under $\mathbb{Q}$ for $\widetilde{N}^{\mathbb{Q}}(dt,dz)$.

Hence, given the information of the chain process X, the new state variable Z is a Markov process on its natural filtration. Then, the new valuation problem, with two variables, is much easier than the original problem with three state variables.

Proposition 6. The valuation of V_t using the process Z under $\mathbb{Q}$, is equivalent to that from process S under $\mathbb{P}^\theta$.

Proof. Let $E^{\mathbb{Q}}$ and E^θ be the expectation operator under $\mathbb{Q}$ and $\mathbb{P}^\theta$, respectively. Then, by Bayes' rule

$$\begin{aligned} V_t &= E^{\theta}\Big[\exp\Big(-\int_t^T r_s ds\Big)V(S,R,X,T)\Big|\mathcal{G}_t\Big] \\ &= \frac{E^{\mathbb{Q}}\Big[E^{\mathbb{Q}}\Big(\frac{d\mathbb{P}^\theta}{d\mathbb{Q}}\Big|\mathcal{G}_t\Big)\exp\Big(-\int_t^T r_s ds\Big)V(S,R,X,t)\Big|\mathcal{G}_t\Big]}{E^{\mathbb{Q}}\Big[\Big(\frac{d\mathbb{P}^\theta}{d\mathbb{Q}}\Big)\Big|\mathcal{G}_t\Big]} \\ &= E^{\mathbb{Q}}\Big[\frac{\mathcal{E}(t)}{\mathcal{E}(T)}\exp\Big(-\int_t^T r_s ds\Big)V(S,R,X,t)\Big|\mathcal{G}_t\Big] \\ &= S_t E^{\mathbb{Q}}\Big[\Big(\frac{R_T}{S_T}\Big)\frac{V(S,R,X,t)}{R_T}\Big|(S_t,R_t,X_t) = (S,R,X)\Big] \\ &= S_t E^{\mathbb{Q}}\Big[e^{-Z_T}V_Z(Z,X,t)\Big|(Z_t,X_t) = (Z,X)\Big]. \end{aligned}$$

As in Hansen and Jorgensen [2000] let $\bar{V}_Z(Z,X,t)$ denote the value of the participating product denominated by the asset price S. We call $\bar{V}_Z(Z,X,t)$, S-denominated value of the contract; that is

$$\bar{V}_Z(Z,X,t) = E^{\mathbb{Q}}\Big[e^{-Z_T}V_Z(Z,X,T)\Big|(Z_t,X_t)=(Z,X)\Big].$$

In this paper, we provide the analysis for the S-nominated value of the participating product $\bar{V}_Z(Z,X,t)$ instead of $V(Z,X,t)$. □

Corollary 1. (Z_t,X_t) is a two-dimensional Markov process with respect to the enlarged filtration $\mathcal{G}_t$.

Corollary 2. The S-denominated value of the participating product $\bar{V}_Z(Z,X,t)$ is $\mathbb{Q}$ martingale.

Further, write $\bar{V}_i$ for $\bar{V}(Z,e_i,t)$, where $i=1,2,...,N$ and $\bar{\mathbf{V}} := \{\bar{V}_1,\bar{V}_2,...,\bar{V}_N\}$. Then, as in Buffington and Elliot [2002], $\bar{\mathbf{V}}$ satisfies the following N PIDEs:

$$\mathcal{L}_{Z,e_i}(\bar{V}_i) + \langle\bar{\mathbf{V}},\mathcal{Q}e_i\rangle = 0, \qquad i=1,2,...,N, \tag{4.11}$$

where

$$\mathcal{L}_{Z,e_i}(\bar{V}_i) = \mathcal{L}^T_{Z,e_i}(\bar{V}_i) + \mathcal{L}^C_{Z,e_i}(\bar{V}_i) + \mathcal{L}^J_{Z,e_i}(\bar{V}_i). \tag{4.12}$$

Here, the first term is the time differential operator, the middle term is the continuous part of the space differential operator $\mathcal{L}_{Z,e_i}(.)$, and the latter is the integral term. By applying Ito's rule to $\bar{V}_Z(Z,X,t)$, the the partial differential operators would be distinguished as

$$\mathcal{L}^T_{Z,e_i}(\bar{V}_i) = \frac{\partial\bar{V}_Z}{\partial t} - \int_{\mathcal{R}^+}\frac{\partial\bar{V}_Z}{\partial t}\ln\big(1+f(z,\theta_t)\big)\rho^{\mathbb{Q}}_{e_i}(dz|t)\eta'(t)), \tag{4.13}$$

$$\begin{aligned}\mathcal{L}^C_{Z,e_i}(\bar{V}_i) \;&=\; \frac{1}{2}\sigma^2\frac{\partial^2\bar{V}_Z}{\partial Z^2} - \frac{\partial\bar{V}_Z}{\partial Z}\Big(r_g\mathbb{I}_{\{Z\le r_g+\beta\}} + (Z_t-\beta)\mathbb{I}_{\{Z>r_g+\beta\}}\Big)\\ &\quad+\frac{\partial\bar{V}_Z}{\partial Z}\Big(r_t + \int_{\mathcal{R}^+}\Big(\ln(1+f(z,\theta_t)) - f(z,\theta_t)\Big)\rho^{\mathbb{Q}}_{e_i}(dz|t)\eta'(t)\Big),\end{aligned}$$

$$\mathcal{L}^{J}_{Z,e_i}(\bar{V}_i) = \int_{\mathcal{R}^+} \left\{ \bar{V}_Z\Big(Z_{t^-} + \ln(1+f(z))\Big) - \bar{V}_Z(Z_{t^-}) \right\} \rho^{\mathbb{Q}}_{e_i}(dz|t)\eta'(t).$$

Further, as a result of the Corollary2, we have

$$\begin{aligned} d\bar{V}_Z(Z,X,t) &= \int_{\mathcal{R}^+} \left\{ \bar{V}_Z\Big(Z_{t^-} + \ln(1+f(z,\theta_t))\Big) - \bar{V}_Z(Z_{t^-}) \right\} \widetilde{N}^{\mathbb{Q}}_{X_t}(dt,dz) \\ &+ \frac{\partial \bar{V}_Z}{\partial Z}(Z,X,t)\sigma dW^{\mathbb{Q}}_t + \langle \bar{\mathbf{V}}, \mathcal{Q}X_t \rangle dt. \end{aligned}$$

Notice that in (4.14) we use $C_Z(Z) = \max(r_g, Z-\beta)$. Also, recall that the auxiliary condition for (4.11) is:

$$\begin{aligned} \bar{V}_Z(Z,X,T) &= e^{-Z_T} V_Z(Z,X,T) \\ &= e^{-Z_T} + \gamma \max(\alpha - e^{-Z_T}, 0) + \max(e^{-Z_T} - 1, 0). \end{aligned} \qquad (4.14)$$

There is no known analytical solution for the PIDE (4.11). For a similar problem, Siu et al. [2008] implemented a robust, Monte Carlo based simulation method. In this paper, we utilize a numerical scheme to solve the PIDE, which may be a more efficient and faster method to reach the solution.

4.3 Parameter Specifications

In this section, we consider some parametric cases of the general jump process by specifying some particular forms of the kernel function and the Markovian regime-switching intensity measure. Since we aim to maintain our results comparable to Siu et al. [2008], we use their parameter specifications. These parametric cases include the Markov-modulated generalized gamma (MGG) process, as well as the the scale-distorted and power-distorted versions of the MGG Process.

The generalized gamma (GG) process is a special case of the kernel-biased completely random measure and can be obtained by setting the kernel function $h(z) = z$. However, in this paper we are seeking a specific class of the GG processes that assist us in describing the impact of the states of an economy on the jump component. Hence, we use a MGG process, whose compensator switches over time, according to the states of the economy. Following, we present how to derive the intensity process for different classes of the MGG process. The result will be used in the next section to approximate the market price of risk from (4.7), as well as the dynamics of S_t under $\mathbb{P}^\theta$ and $\mathbb{Q}$.

Let $\alpha \geq 0$ and $b(t)$ denote the shape parameter and the scale parameter of the MGG process, respectively. Further, define

$$b(t) \quad := \quad \langle \mathbf{b}, X_t \rangle, \tag{4.15}$$

where $\mathbf{b} := (b_1, b_2, ..., b_N) \in \mathcal{R}^N$ and $b_i \geq 0$, for each $i = 1, 2, ..., N$. Then, the intensity process of the MGG process is

$$\begin{aligned} \rho_{X_t}(dz|t)\eta(dt) &= \frac{1}{\Gamma(1-\alpha)z^{(1+\alpha)}} e^{-z\langle \mathbf{b}, X_t \rangle} dz\eta(dt) \\ &= \frac{1}{\Gamma(1-\alpha)z^{(1+\alpha)}} \sum_{i=1}^{N} e^{-b_i z} \langle \mathbf{b}, X_t \rangle dz\eta(dt). \end{aligned} \tag{4.16}$$

When $\alpha = 0$, the MGG process reduces to a Markov modulated weighted gamma(MWG) process. When $\alpha = 0.5$ the MGG process becomes the Markov modulated inverse Gaussian (MIG) process.

Another class of the MGG process is the distorted MGG, which includes scale-distorted and power-distorted versions of the MGG processes. They can describe the overstate and understate of jump amplitudes due to overreaction and underreaction of market participants to extraordinary events, respectively. For the scale-distorted version of the MGG process, the kernel function takes the form of $h(z) = cz$, where c is a positive constant. When $c > 1$ the jump sizes are overstated and when $0 < c < 1$, jump sizes are understated. For the power-distorted MGG, the kernel function takes the form of $h(z) = z^q$, where q is a positive constant. When $q > 1$ the small jump sizes are overstated and large jump sizes are understated; when $0 < q < 1$, the large jump sizes are overstated and small jump sizes are understated.

In general, the scale-distorted and power-distorted versions of the MGG process can assist modeling different behaviors of market participants when they react to extraordinary events, hence, they are important from a behavioral finance perspective. In addition, similar to the MGG processes, when $\alpha = 0$, the distorted MGG process reduces to the distorted MWG process. When $\alpha = 0.5$ the distorted MGG processes become distorted MIG processes.

As an extension to this framework, one might also like to consider modulating the distortion constants c and q, by a Markov chain, to reflect the dependency of the behavior of market participants to the state of the economy. This, in particular, could be beneficial for modeling the behavior of markets which become overzealous, when the state of the economy switches.

4.4 Numerical Analysis

In this section, we employ an elegant class of finite element (FE) methods, namely *the collocation method*, to numerically calculate the value of the participating policy. FE methods have been growing in popularity in the finance arena as a strong alternative to finite difference (FD) methods. Some examples could be found in the scholarly work of Forsyth et al. [1999], Holmesa et al. [2012], and Matache et al. [2005].

When the underlying problem is a PIDE or contains irregular shapes of differential equations, placing FD type grid points is difficult and provides poor approximation of the solution. FE methods not only handle these complex models easier, but also provide a solution for the entire domain, instead of isolated nodes as in the case of FD. In addition, FE techniques can incorporate boundary conditions involving derivatives easier than FD.

Although the Galerkin approach appears to be the most popular FE method in finance, in this paper we employ the collocation method, for a number of reasons. First, the implementation of the method is easier and more natural than the Galerkin method. For example, the integration of the second order terms of the PIDEs is not required, as is the case in the derivation of the variational formulation for the Galerkin method. More importantly, one of the reasons for the popularity of the Galerkin approach is the equivalence of this method and the approach by Ritz when applied to self-adjoint operators, which is not the case in (4.11).

4.4.1 The formulation of the Collocation Method

In order to render the problem well-posed, we also need two boundary conditions for the cases when the value of the reference portfolio hits zero, or when it grows to infinity. This can be delivered by financial insight. If the value of the reference portfolio hits zero, the price can never rise above zero again. Thus, from (4.4), zero should be an absorbing boundary for $V(S, X, t)$. On the other hand, if the value of S_t approaches infinity, the value of the default option in (4.4) will approach zero and the value of the bonus option will approach $\gamma\alpha S_t$. Consequently, the boundary conditions for the S-denominated value of the policy could be written as

$$\bar{V}_Z(0, X, t) = 0, \tag{4.17}$$

and

$$\bar{V}_Z(Z^{\max}, X, t) = \gamma\alpha + e^{-Z_t}. \tag{4.18}$$

Suppose $\widetilde{V}(Z, X, t)$ is the approximation solution for $\bar{V}(Z, X, t)$. Then, by the separation of variables technique, if $\{\phi\}_i^M$ denotes a set of nodal shape functions, we can expand $\widetilde{V}(Z, X, t)$ as

$$\widetilde{V}(Z, X, t) = \sum_{k=1}^{M+1} \xi_k(t, X)\phi_k(Z, X), \qquad k = 1, 2, ..., M. \tag{4.19}$$

Notice that the coefficients $\xi_i(t, X)$ are time-dependent but not space-dependent functions.

The collocation method is used either with a shape function representing a higher degree polynomial and possessing global support, or with a shape function with higher order continuity and local support. The shape function must have a continuous first order derivative, because the collocation equation supplies information about the second order derivative of the PIDE. Examples for shape functions with global support are Legendre or Chebyshev; employed in this setting the approach is called the *spectral method*.

Fixing the state of the economy to $X = e_i \;\; i = 1, 2, ..., N$, and inserting (4.19) into (4.11) leads to the following system of ordinary differential equations (ODEs)

$$0 = \mathbf{M}\frac{d\boldsymbol{\xi}_l}{dt}(t, e_i) + \mathbf{A^C}\boldsymbol{\xi}_l(t, e_i) + \mathbf{A^J}\boldsymbol{\xi}_l(t, e_i) + \langle \mathbf{D}, \mathcal{Q}e_i\rangle\boldsymbol{\xi}_l(t, e_i) = 0, \tag{4.20}$$

Here, $\boldsymbol{\xi}(t, X)$ denotes a vector holding the nodal values $\xi_l(t, X)$, and the entries of matrices $\mathbf{M}$, $\mathbf{A^C}$, $\mathbf{A^J}$ and $\mathbf{D}$ are given by

$$M_{lk} = \Big\{1 - \int_{\mathcal{R}^+} \ln\big(1 + f(z, \theta_t)\big)\rho^{\mathbb{Q}}_{e_i}(dz|t)\eta'(t)\Big\}\phi_k(Z_l, e_i),$$

$$\begin{aligned} A^C_{lk} \;\; &= \;\; \frac{1}{2}\sigma^2\frac{\partial^2\phi_k(Z_l, e_i)}{\partial Z^2} + r_t\frac{\partial\phi_k(Z_l, e_i)}{\partial Z} - \Big(r_g\mathbb{I}_{\{Z\leq r_g+\beta\}} + (Z_t - \beta)\mathbb{I}_{\{Z>r_g+\beta\}}\Big)\frac{\partial\phi_k(Z_l, e_i)}{\partial Z} \\ &\quad + \Big(\int_{\mathcal{R}^+}\Big(\ln(1 + f(z, \theta_t)) - f(z, \theta_t)\Big)\rho^{\mathbb{Q}}_{e_i}(dz|t)\eta'(t)\Big)\frac{\partial\phi_k(Z_l, e_i)}{\partial Z}, \end{aligned}$$

$$A^J_{lk} \;\; = \;\; \int_{\mathcal{R}^+}\phi_k\Big(Z_l + \ln(1 + f(z, \theta_t)), e_i\Big)\rho^{\mathbb{Q}}_{e_i}(dz|t)\eta'(t) - \int_{\mathcal{R}^+}\phi_k(Z_l, e_i)\rho^{\mathbb{Q}}_{e_i}(dz|t)\eta'(t),$$

and

$$D_{lk} = \phi_k(Z_l, e_i),$$

where $l, k = 1, 2, ..., M + 1$.

Note that for a problem with N states of economy and M spatial nodes, there will be a system of $N \times (M + 1)$ ODEs. In addition, each system of ODEs fixed for state $X = e_i, \;\; i = 1, 2, ..., N$, contains values of ϕ from other states (i.e. $\phi(Z, X = e_j), \;\; j \neq i$ and $j = 1, 2, ..., N$), therefore, it is not possible to solve the system of ODEs from each state, stand alone. In order to find the solutions, we will need to replace the first and the last equation for the system of ODEs of each state by the boundary conditions (4.17) and (4.18).

4.4.2 Numerical Example

In this section, we conduct a numerical example to calculate the fair value of the participating policy, implied by various parametric specifications of our generalized jump-type model described in section 4.3. We report the impact of the regime-switching effect in the price dynamics of the reference portfolio on the fair value of the policy. In addition, we compare our results with the no-regime-switching version of the model, as well as the Merton model. The readers could also compare the results with Siu et al. [2008], with caution since their specifications of the the dynamics of the reference portfolio (Equation (4.3)) are slightly different.

Our programs were written in Matlab, and we completely vectorized the codes to increase the computational speed. Let us assume an economy with two states where an insurance company is considering issuing a participating life insurance contract. Hence, we consider a two state Markov-chain model X with $N = 2$, where $X_t = 1$ represents a 'Good' economy while $X_t = 2$ represents a 'Bad' economy. Let $\mathcal{P}(t)$ be the transition probability matrix for time t. Write

$$\mathcal{P}(t) = \begin{pmatrix} 1 - p_1 & p_1 \\ p_2 & 1 - p_2 \end{pmatrix},$$

where p_i is the probability in which the economy switches from i to state $3 - i$, for $i = 1, 2$. Note that in section-4.2.1, we characterize the Markov chain using the matrix of transition rates, $\mathcal{Q}$. For simplicity in notation, for our two-states model, we write $q_{11} = -q_1$ and $q_{22} = -q_2$, thus $\mathcal{Q} = \begin{pmatrix} q_1 & -q_2 \\ -q_1 & q_2 \end{pmatrix}$.

The reason to calculate $\mathcal{P}$-matrix from the $\mathcal{Q}$-matrix is that for the numerical analysis it is more natural to consider the transition probabilities. The process is justified with the fact that in the continuous-time Markov chain, the probability of a particular transition is roughly proportional to the duration of the infinitesimally small time interval. Thus, we shall first calculate the $\mathcal{Q}$-matrix.

Solving the semi-martingale representation (4.1), assuming X_0 the initial value of $(X_t)_{t\geq 0}$ is known, yields the following

$$\mathcal{P}(t) \quad = \quad \exp(\mathcal{Q}t) = \frac{1}{\pi}\begin{pmatrix} q_2 & q_2 \\ q_1 & q_1 \end{pmatrix} + \frac{e^{-\pi t}}{\pi}\mathcal{Q},$$

where $\pi = q_1 + q_2$.

We need to choose some specimen values for our model to illustrate the practical implications. We choose the same values for the model parameters and the policy parameters as in Siu et al. [2008] and Bacinello [2003a]:

$$\begin{array}{llll} p_1 = p_2 = 0.40; & r_1 = 0.035; & r_2 = 0.015; & \\ \sigma_1 = 0.2; & \sigma_2 = 0.2; & \mu_1 = 0.10; & \mu_2 = 0.05; \\ b_1 = 200.00; & b_2 = 500.00. & & \end{array}$$

The term to maturity of the contract is $T = 20$ years, Δt is assumed to be one trading day ($\Delta t = 1/252$), and, we select the following values for the parameters of the policy:

$$r_g = 0.04; \quad \beta = 0.5; \quad \gamma = 0.7; \quad \alpha_p = 0.6; \quad S_0 = 100.$$

The (M)GG processes

Since we employ the collocation method, the choice of the shape function ϕ, might be critical. So we use two different functions for ϕ, namely Legendre and Chebyshev, in order to examine the sensitivity of the results to the choice of the shape function. For this example we select the MGG process with the shape parameter of $\alpha = 0.1$, and we calculate the approximation value for the S-denominated fair value of the participating policy. The approximate solutions for $\bar{V}_Z(Z, X, t)$ using the two different shape functions differ no more than 0.3679 between Z Values -1.00 to $+1.00$. We also test the impact of the different values of the shape parameter on $\bar{V}_Z(Z, X, t)$, using the two different shape function ϕ. Our results show a very good agreement of the solution over $0 \leq \alpha < 1$, with maximum divergence of 0.3803. In order to concentrate on the analysis of the participating policy with respect to the model parameters, we choose the Legendre function as our choice of the shape function for the rest of the paper. In addition, we convert the values of $\bar{V}_Z(Z, X, t)$ to $V(S, R, X, t)$ for the purpose of presentation, so that our results could be easily compared to other models, such as the Merton jump diffusion model.

We consider the MGG process with the shape parameter α that ranges from 0.0 to 0.9, with increments of 0.1. When $\alpha = 0.0$, the MGG process becomes the MWG process. When $\alpha = 0.5$ the MGG process becomes the MIG process. Other values of α generate different parametric forms of the MGG processes. The parameter values of the no-regime-switching version of the model correspond to a regime-switching model with only one state.

For the Merton jump diffusion model, we consider the drift and the dispersion of the reference portfolio as well as the risk-free rate to be equal to the corresponding parameters in the no-regime-switching version of the model. In addition, we assume the intensity parameter of the model to be 60%, and the jump size of the compound Poisson process follows a normal distribution of $N(-0.05, 0.49)$.

Figure 4.1 presents the impact of α on the fair values of the participating policy, calculated with the above model specifications. The graph shows a meaningful difference between the fair values of the policy, with and without switching regimes. For example, when $\alpha = 0.2$, the fair value calculated without regime-switching is 16.01% lower than the fair value of the contract with regime-switching. This difference is as high as 73.33% for the fair values under the two scenarios with $\alpha = 0.9$. We also document the significant effect of α on the values of the contracts for both cases. For instance, under regime-switching scenario the fair value of the policy reduces from \$85.4763 to \$0.1092 in the domain of $0 \leq \alpha \leq 0.9$.

The scale-distorted (M)IG processes

In this section we make a comparison between scale-distorted IG and MIG processes, as well as the Merton jump-diffusion model. We examine how the variations of the distortion parameter, c, impacts the underlying price behaviors and the fair values of the participating policies. We consider that c takes values from 0.5 to 3.0, with increments of 0.5. The parameter values for the Merton jump-diffusion model are those in

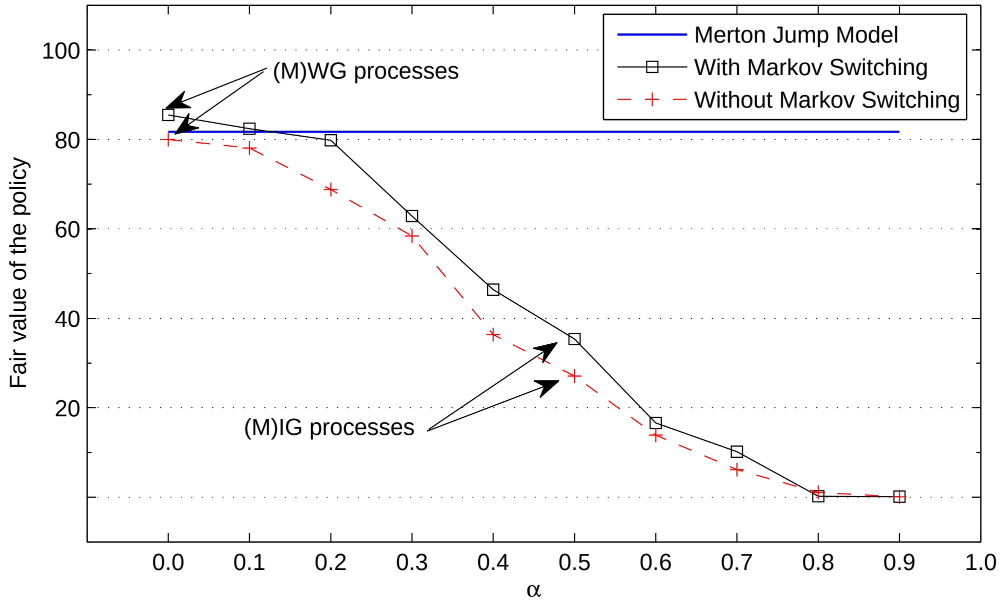

FIGURE 4.1: The fair value of the participating policy, with general (M)GG processes.

the previous part. Figure 4.2 displays the numerical results for the fair values of the participating products for the power-distorted version of the (M)IG processes ((M)GG with $\alpha = 0.5$). The results are also compared to the Merton jump diffusion model. It is observed that the effect of switching regimes on the fair values of the participating policy is still significant. We document that the value of the policy under the scenario with regime-switching is always larger than the scenario without regime-switching. The difference is as low as 10.93% when $c = 1.5$, and is as high as 74.57% when $c = 3.0$.

The power-distorted (M)IG processes

In this section, we compare the power-distorted MIG process with the IG version of the process, as well as the Merton's jump-diffusion model. We examine how the changes in the distortion parameter, q, impact the underlying price behaviors and the fair values of the participating policies. We consider that q takes values from 0.8 to 1.4, with

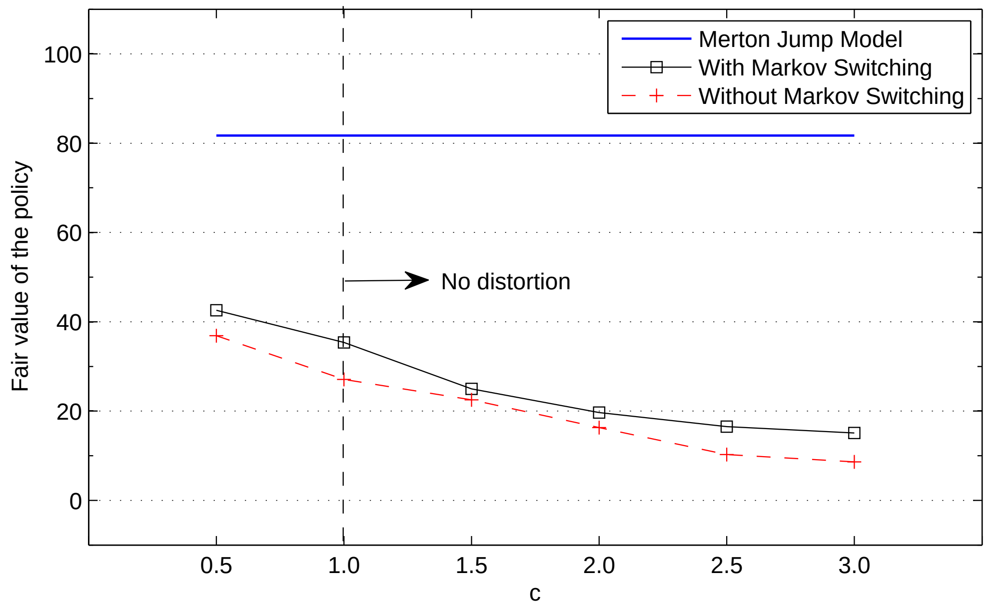

FIGURE 4.2: The fair value of the participating policy, with scale-distorted M(IG) processes.

increments of 0.1. The parameter values for the Merton jump-diffusion model are given by those in the previous parts.

Figure 4.3 displays the numerical results for the fair values of the participating policy for the power-distorted (M)IG processes. The results are also compared to the Merton's jump diffusion model. It is observed that the effect of switching regimes on the fair values of the participating policy is significant.

We document that the value of the policy under the scenario with regime-switching is always larger than the scenario without regime-switching. The difference is as low as 3.19% when $q = 0.8$, and is as high as 30.56% when $q = 1.0$. These results, compared to those presented for the scale-distorted scenario, reveal that the impact of changes in q is less significant that changes in c.

In addition to the fair valuation of the contracts, it is interesting to investigate how the risks inherent in these products can be hedged once the policy has been sold from a risk management perspective. Hedging via the Greeks and the risk-minimizing hedging

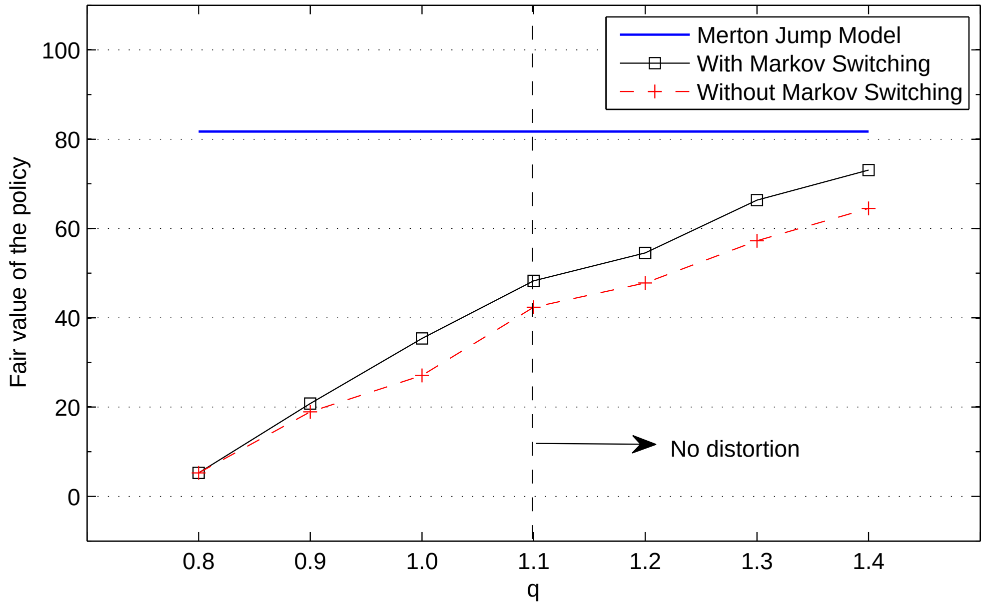

FIGURE 4.3: The fair value of the participating policy, with power-distorted M(IG) processes.

represent two popular approaches to hedging these risks. However, hedging using the Greeks is only an approximating hedging strategy and it cannot provide a perfect hedging result due to the market incompleteness, and time discretization. There is a wealth of literature about different methods of calculation of the Greeks. Some prominent examples are the efficient Monte Carlo simulation method proposed by Broadie and Glasserman [1996], the Malliavin calculus approach introduced by Fournie et al. [2001], and a combination of the two methods in a more recent paper by Davis and Johansson [2006], in particular for jump-diffusion models. In this paper, our main focus has been the fair valuation issue of the policy; however, approximate values of different Greeks can be derived directly from the PIDE grid (see Karatzas and Shreve [1998] for further details).

4.5 Conclusion

In this article, we have proposed a model for the valuation of participating life insurance products under a generalized jump-diffusion model with a Markov-switching compensator. Participating life insurance products are a class of equity linked insurance products, which provide the insured a certain guaranteed annual minimum benefit, as well as the proceeds from an investment portfolio. We demonstrated that the payoff of a participating contract can be decomposed to the guaranteed benefit, a European-style default option, and a European-style bonus option.

Using the insights of the previous research on the pricing of the contracts with similar model assumptions for the reference portfolio (Siu et al. [2008]), we employed the Esscher transform to determine an equivalent martingale measure under the incomplete market setting. However, due to the high dimensionality of the pricing formulation under the new probability measure, there is not any close form solutions using the PIDE approach. Siu et al. [2008] showed that the price for the participating contract could be approximated using a Monte Carlo based simulation algorithm, although, the process can be cumbersome and time consuming.

With the intention to use the PIDE method to increase the accuracy and the computational speed, we proposed a change of measure technique by defining an artificial numeraire. The technique reduced the dimensions of the regime-switching PIDE. The regime-switching PIDE derived from the change of measures depends on two state variables including a new observable state variable and the state of the economy. This paper is the first that extends the technique for a generalized jump-diffusion process with a Markov-switching kernel-biased completely random measure, which nests a number of important and popular models in finance, including the classes of jump-diffusion models and Markovian regime-switching models. We also considered a number of parametric cases of the Markov-modulated kernel-biased completely random measure; namely, Markov-modulated generalized gamma, as well as the scale-distorted and power-distorted Markov-modulated generalized gamma.

We solved the PIDE numerically using the collocation method, to calculate the fair value of the participating policy. In particular, we utilize the spectral method, through experimentation and comparing two different shape functions (i.e. Legendre and Chebyshev).

Our numerical results exhibit a very good agreement of the solution over a reasonable range of values of the space variable. The collocation method is a class of FE methods, which compared to the conventional FD method provide higher level flexibility and accuracy in the numerical approximation process.

Further, we extended the numerical analysis by examining the sensitivity of the fair values to various model parameters. In order to highlight the features of our model, we have also compared our results to the fair value of the policy obtained from the Merton's jump-diffusion model. The numerical analysis reveals that the impact of various specifications of the jump component and the switching regimes on the fair value of the participating policy are large enough to be of practical importance.

Chapter 5

Summary and Conclusion

5.1 Summary

In this thesis, we analysed pricing and risk management of financial derivatives under regime-switching framework, captured by an observed Markov chain. The market considered in this thesis is incomplete, because of the additional source of risk described by the switching regimes. Under these market conditions, a replicating dynamic trading policy does not exist and there is more than one equivalent martingale measure.

In Chapter1, we briefly reviewed the literature on option pricing models, and the challenges in valuation and risk management of contingent claims. In particular, we discussed different forms of market incompleteness, including the Markov regime-switching framework. Additionally, we reviewed the literature on different remedies for pricing derivatives in incomplete markets, and we discussed the advantages of choosing MEMM and Esscher transform over other methodologies. We also reviewed the prominent numerical methods commonly used in financial engineering, and their specific applications in derivatives-pricing methodologies. Finally, we outlined the subsequent chapters, where we analysed the problem of pricing three different derivative contracts; namely, 1) a European call option, 2) a ruin contingent life annuity, and 3) a participating product.

In Chapter2, we discussed the pricing and risk management problems of standard European-style options in a Markovian regime-switching binomial model. Due to the market incompleteness, we found that the no-arbitrage condition is not sufficient to fix a unique pricing kernel. Using the minimal entropy martingale measure, we determined a pricing kernel. Additionally, we examined numerically the performance of a simple hedging strategy by investigating the terminal distribution of hedging errors and the associated risk measures such as Value at Risk and Expected Shortfall. The impact of the frequency of re-balancing the hedging portfolio and the transition probabilities of the modulating Markov chain on the quality of hedging were also discussed.

In Chapter3, we proposed a model for valuing ruin contingent life annuities under the regime-switching variance gamma process. The Esscher transform was employed to determine the equivalent martingale measure. The PIDE approach was adopted for the pricing formulation. Due to the path dependency of the payoff of the insurance product and the non-existence of a closed-form solution for the PIDE, the finite difference method was utilized to numerically calculate the value of the product. To highlight some practical features of the product, we presented a numerical example. Finally, we examined numerically the performance of a simple hedging strategy by investigating the terminal distribution of hedging errors and the associated risk measures such as the value at risk and the expected shortfall. The impacts of the frequency of re-balancing the hedging portfolio on the quality of hedging were also discussed.

In Chapter4, we proposed a model for the valuation of participating life insurance products under a generalized jump-diffusion model with a Markov-switching compensator. The Esscher transform was employed to determine an equivalent martingale measure in the incomplete market. The results were further manipulated through the utilization of the change of numeraire technique to reduce the dimensions of the pricing formulation. This paper is the first that extends the technique for a generalized jump-diffusion process with a Markov-switching kernel-biased completely random measure, which nests a number of important and popular models in finance. The collocation method was utilized to numerically analysis the model and to illustrate the practical implications.

5.2 Conclusion and Future Research

5.2.1 Chapter2: A Regime-Switching Binomial Model for Pricing and Risk Management of European Options

In Chapter2, we discussed the option pricing and hedging in a regime-switching binomial tree, where we assume two sources of uncertainties; namely, the risk due to binomial movements of the underlying risky asset price and the risk due to transitions in economic states. We adopted the MEMM approach to price the two sources of risk and examined the pricing and hedging performance of this approach using numerical examples. A simple, dynamic Delta hedging was considered and its performance was examined by evaluating the VaR and ES of the terminal hedging errors arising from the dynamic Delta hedging strategy. Numerical results were provided which reveal that the impact of pricing regime-switching risk is significant and that both the hedging frequencies and transition probabilities of regime switches have significant impacts on the performance of the delta hedging strategy.

Future research, may further address the hedging strategies in the regime-switching binomial model. In the present paper, we illustrated the risk of the option issuer based on the terminal hedging error, if they conveniently ignore the regime-switching risk. Such a hedger observes the regime at each node and chooses the appropriate delta accordingly. However, the issuer is able to reduce the terminal hedging error (in the case of vanilla European options, potentially to zero), by implementing a trading strategy that takes the delta of all states into consideration, concurrently.

It is assumed that an agent is issuing a standard European option in an economy where all of the assumptions presented in Section2.3 hold true. Given the information at time t, the model suggests that the economy may be either in regime 1 or regime 2 (i.e. the process is governed by the Markov chain). Therefore, the agent may have two choices of delta; namely, $\Delta_{t,1} = (V^u_{t+1,1} - V^d_{t+1,1}) \times (S^u_{t+1,1} - S^d_{t+1,1})^{-1}$ and $\Delta_{t,2} = (V^u_{t+1,2} - V^d_{t+1,2}) \times (S^u_{t+1,2} - S^d_{t+1,2})^{-1}$.

Thus, at time t, the agent has to choose an appropriate delta. In Section2.4 we presented the result of the case where the hedger chooses the delta corresponding to the current regime. However, the terminal hedging error for a standard European option can potentially be reduced to zero, if the hedger selects a delta hedge based on the weighted average of $\Delta_{t,1}$ and $\Delta_{t,2}$. In this strategy the weights are calculated according to the risk-neutral transition probabilities. Consequently, in equation (2.5) we show that the value at time t of an option can be written as:

$$V_t = c_{1i}\{\Delta_{t,1}S_t - B_{t,1}\} + c_{2i}\{\Delta_{t,2}S_t - B_{t,2}\}.$$

where $B_{t,i}$ denotes the amount of riskless borrowing and lending at time t when the economy is in the state $\mathbf{e_i}$. Hence, the hedging strategy involves trading $c_{1i}\Delta_{t,1} + c_{2i}\Delta_{t,2}$ of the underlying asset and $-(c_{1i}B_{t,1} + c_{2i}B_{t,2})$ riskless borrowing and lending. One may expand the equation (2.5) as follows, which suggests the possibility of complete risk transfer or a perfect hedge under our incomplete market framework.

$$V_t = c_{1i}q_{t,1}V^u_{t+1,1} + c_{2i}q_{t,2}V^u_{t+1,2} + c_{1i}(1 - q_{t,1})V^d_{t+1,1} + c_{2i}(1 - q_{t,2})V^d_{t+1,2}$$

We re-emphasize that this only shows that there exist a trading strategy which can perfectly hedge all the risks in a regime-switching economy. However, the question remains: out of the infinite number of martingale measures, which one would provide the perfect hedge? In other words, whilst we can show the possibility of the perfect hedge in the regime-switching economy, we are unable to find the specific $[c_{ji}]$ that perfectly hedges the contingent claim.

5.2.2 Chapter3: Ruin Contingent Life Annuities under Regime-Switching Variance Gamma Process

The motivations underlying this study are twofold: 1) The paucity of literature on pricing of RCLA contract and with respect to hedging implications for the issuer of the contract; and 2) the significance of markets whereby RCLA contracts are able to provide flexible hedging solutions to both financial market risk and personal longevity risk.

In Chapter3, we introduce a regime-switching variance gamma model, for pricing the Ruin Contingent Life Annuity (RCLA) contract. This study, also, addresses different shortcomings of Huang et al. [2009], which was based on a complete market model. To price the RCLA contract, the payoff is decomposed into a down-and-in barrier option and a European call option. Therefore, we derive the no-arbitrage value of RCLA using the PIDE method. The analytic solution to the PIDE is not known, hence we employ the explicit finite difference method to numerically approximate the solution. The results indicate that the model assumptions for the process driving the price of the underlying asset may have a tremendous impact on the fair value of the contract.

The VG process is a class of jump Levy processes, with growing popularity as an alternative to the Black-Scholes economy. We use the Levy measure representation of the VG process and modulate its parameters by a Markov chain to capture the impacts of the regime-switching risk. We employ the Esscher transform to find an equivalent martingale measure in our incomplete market and derive the risk-neutral dynamics of the underlying portfolio. Subsequently, we derive the pricing PIDE, and solve for the value function numerically. To highlight the practical implications of our model, we conduct a numerical example through which we calculate the value of the RCLA product. Then we conduct a Monte Carlo simulation experiment for a simple, dynamic delta hedging strategy. The performance of the strategy is examined by examining the VaR and ES of the terminal hedging errors, arisen from the dynamic delta hedging.

Despite our analysis of the risk inheritance in the embedded option using two conventional methods, the literature may benefit from further research on more complex risk evaluation methodologies. There is a wealth of literature about different methods of calculation of the Greeks that could be extended to the RCLA contracts. Some prominent examples are the efficient Monte Carlo simulation method proposed by Broadie and Glasserman [1996], the Mallivian calculus approach introduced by Fournie et al. [2001], and a combination of the two methods, in particular for jump-diffusion models, in a more recent paper by Davis and Johansson [2006].

5.2.3 Chapter4:Pricing Participating Products under a Generalized and Regime-Switching Jump-Diffusion Model

In this article, we have proposed a model for the valuation of participating life insurance products under a generalized jump-diffusion model with a Markov-switching compensator. Participating life insurance products are a class of equity-linked insurance products, which provides the insured a certain guaranteed annual minimum benefit, as well as the proceeds from an investment portfolio. We demonstrate that the payoff of a participating contract can be decomposed to the guaranteed benefit, a European style default option, and a European style bonus option.

Using the insights provided by the previous study on the pricing of the contracts with similar model assumptions (Siu et al. [2008]), we employed the Esscher transform to determine an equivalent martingale measure under the incomplete market setting. However, due to the high dimensionality of the pricing formulation under the new probability measure, there is not any close form solutions using the PIDE approach. Siu et al. [2008] showed that the price for the participating contract could be approximated using a Monte Carlo based simulation , despite the process being cumbersome and time consuming.

With the intention to use the PIDE method to increase the accuracy and the computational speed, we proposed a change of measure technique by defining an artificial numeraire. The technique reduces the dimensions of the regime-switching PIDE. The regime-switching PIDE derived from the change of measures depends on two state variables including a new observable state variable and the state of the economy. This paper is the first that extended the technique for a generalized jump-diffusion process with a Markov-switching kernel-biased completely random measure, which nests a number of important and popular models in finance, including the classes of jump-diffusion models and Markovian regime-switching models. We also considered a number of parametric cases of the Markov-modulated kernel-biased completely random measure; namely, Markov-modulated generalized gamma as well as the scale-distorted and power-distorted Markov-modulated generalized gamma.

We solved the PIDE numerically using the collocation method, to calculate the fair value of the participating policy. The collocation method is a class of FE methods, which compared to the conventional FD method provide higher level flexibility and accuracy in the numerical approximation process. We utilized the spectral method, through experimentation and comparing two different shape functions (i.e. Legendre and Chebyshev). Our numerical results exhibited a very good agreement of the solution over a reasonable range of values of the space variable.

Further, we extend the numerical analysis by examining the sensitivity of the fair values to various model parameters. In order to highlight the features of our model, we have also compared our results to the fair value of the policy obtained from the Merton's jump-diffusion model. The numerical analysis revealed that the impact of various specifications of the jump component and the switching regimes on the fair value of the participating policy are large enough to be of practical importance.

Besides fair valuation of the options embedded in the participating products, it would also be interesting to investigate how the risk inheritance in these options can be hedged once the policy has been sold. In this pricing model since we assumed two sources of incompleteness, further research would shed light on the hedging strategies for issuers of the policies from a risk management standpoint.

Bibliography

K. Aase and S.A. Persson. Pricing of unit-linked insurance policies. *Scandenavian Actuarial Journal*, pages 26–52, 1994.

A. K. Alexandridis, A. Alexandridis, and A. Zapranis. *Weather Derivatives: Modeling and Pricing Weather-Related Risk*. Springer, 2012.

K. Arrow. The role of securities in the optimal allocation of risk bearing. *Review of Economic Studies*, 31:91–96, 1964.

M Avellaneda. Minimum relative entropy calibration of asset pricing models. *International Journal of Theoretical and Applied Finance*, 1:447–472, 1998.

A.R. Bacinello. Fair pricing of life insurance participating policies with a minimum interest rate guaranteed. *ASTIN Bulletin*, 31(2):275–297, 2001.

A.R. Bacinello. Fair valuation of a guaranteed life insurance participating contract embedding a surrender option. *Journal of Risk and Insurance*, 70(3):461–87, 2003a.

A.R. Bacinello. Pricing guaranteed life insurance participating policies with annual premiums, and surrenders option. *North American Actuarial Journal*, 7(3):1–17, 2003b.

A.R. Bacinello and F. Ortu. Pricing equity-linked life insurance with endogeneous minimum guarantees. *Insurance: Mathematics and Economics*, 12:245–257, 1993a.

A.R. Bacinello and F. Ortu. Single and periodic premiums for guaranteed equity-linked life insurance under interest-rate risk: the lognormal+vasicek case. *Financial Modelling Physica-Verlag*, pages 1–25, 1993b.

A.R. Bacinello and F. Ortu. Pricing guaranteed securities-linked life insurance under interest-rate risk actuarial approach for financial risks. *Transactions of the 3rd AFIR International Colloquium*, pages 35–55, 1993c.

O.E. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck based models and some of their uses in financial economics. *Journal of Royal Statistical Society, Series B*, 63:167–241, 2001.

F. Black. The pricing of commodity contract. *Journal of Financial Economics*, 3: 167–179, 1976.

F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–654, 1973.

E. Boehmer and F. Cocquemas. *Market incompleteness and market microstructure.* Springer, 2011.

T. Bollerslev. Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, 31:307–327, 1986.

J.P. Bouchaud and M. Potters. *Theory of Financial Risk and Derivative Pricing : From Statistical Physics to Risk Management.* New York : Cambridge University Press, 2003.

P. Boyle. A Monte Carlo approach. *Journal of Financial Economics*, 4(3):323–338, 1977.

P. Boyle and E. Schwartz. Equilibrium pricing of guarantees under equity-linked contracts. *The Journal of Risk and Insurance*, pages 639–660, 1977.

P. Boyle, M. Broadie, and P. Glasserman. Monte carlo methods for security pricing. *Journal of economic dynamics and control*, 21(8):1267–1321, 1997.

P.P. Boyle, T.K Siu., and H. Yang. Risk and probability measures. *Risk*, 15:53–57, 2002.

A. Brace, D. Gatarek, and M. Musiela. The market model of interest rate dynamics. *Math. Finance*, 7:127–155, 1997.

M. Brennan and E. Schwartz. The pricing of equity-linked life insurance policies with an asset value guarantee. *Journal of Financial Economics*, pages 195–213, 1976.

M. Brennan and E. Schwartz. Finite difference methods and jump processes arising in the pricing of contingent claims. *Journal of Financial and Quantitative Analysis*, 5(4): 461–474, 1978.

M. J. Brennan and E.S. Schwartz. Evaluating natural resource investments. *Journal of Business*, 58:135–157, 1985.

M. Broadie and P. Glasserman. Estimating security price derivatives using simulation. *Management Science*, 42(2):269–285, 1996.

J. Buffington and R.J. Elliot. American options with regime switching. *International Journal of Theoretical and Applied Finance*, 5:497–514, 2002.

H. Buhlmann, F. Delbaen, P. Embrechts, and A.N. Shiryaev. No-arbitrage, change of measure and conditional Esscher transforms,. *CWI Quarterly,*, 9(4):291–317, 1996.

C.C. Chu and Y.K. Kwok. Pricing participating policies with rate guarantees. *International Journal of Theoretical and Applied Finance*, 9:517–532, 2006.

P.K. Clark. A subordinated stochastic process model with finite variance for speculative prices. *Econometrica.*, 41:135–155, 1973.

G.M. Constantindes and S. Perrakis. Stochastic dominance bounds on american option prices in markets with frictions. *Review of Finance*, 10:1–43, 2006.

G. M. Constantinides. Market risk adjustment in portfolio valuation. *Journal of Finance*, 33:603–616, 1978.

R. Cont and P. Tankov. *Financial Modelling with Jump Process.* Chapman & Hall/CRC Press, Boca Raton, FL, 2004a.

R. Cont and P. Tankov. *Financial Modelling with Jump-Processes.* Financial Mathematics SeriesChapman & Hall/CRC, London, 2004b.

J. Cox and S.A. Ross. The valuation of options for alternative stochastic processes. . *Journal of Financial Economics*, 3:145–166, 1976a.

J. Cox, S. Ross, and M. Rubinstein. Option pricing: A simplified approach. *Journal of Financial Economics*, 7:229–263, 1979.

J.T. Cox and S.A. Ross. The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3:145–166, 1976b.

J. Cvitanić and F. Zapatero. *Introduction to the economics and mathematics of financial markets.* MIT press, 2004.

M.H.A. Davis. *Option Pricing in Incomplete Market.* In Mathematics of Derivative Securities, ed. M. A. H. Dempster and S. R. Pliska, Cambridge: Cambridge University Press, 1997.

M.H.A. Davis and M.P. Johansson. Malliavin Monte Carlo Greeks for jump diffusions. *Stocchastic Processes and Their Applications,*, 116:101–129, 2006.

F. Delbaen. Equity linked policies. *Bulletin Association Royal Actuaires*, pages 33–52, 1986.

Schachermayer W. Delbaen, F. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300:463–520, 1994.

D. Demsetz. The cost of transacting. *Quarterly Journal of Economics*, 82:33–53, 1968.

J. Detemple and S. Sundaresan. Nontraded asset valuation with portfolio constraints: A Binomial approach. *Review of Financial Studies*, 12(4):835–872, 1994.

D. Duffie. *Dynamic asset pricing theory*. Princeton University Press, 2010.

D. Duffie, W.H. Fleming, H.M. Soner, and T. Zariphopoulou. Hedging in incomplete markets with hara utility. *Journal of Ecconomic Dynamic Control*, 21:753–782, 1997.

U.K. Eberlein and K. Prause. New insights into smile, mispricing and value at risk. *Journal of Business*, 71:371–406, 1998.

R. Elliott and T.K. Siu. Option pricing and filtering with hidden Markov-modulated pure-jump process. *Applied Mathematical Finance*, 25:1–25, 2013.

R. J. Elliott and P.E. Kopp. *Mathematics of Financial Market*. Springer, 2005.

R. J. Elliott and T.K. Siu. A stochastic differential game for optimal investment of an insurer with regime switching. *Quantitative Finance*, 11(3):365–80, 2011a.

R. J. Elliott, L. Aggoun, and J. B. Moore. *Hidden Markov Models: Estimation and Control, Applications of Mathematics*. Springer, 1995.

R. J. Elliott, L. Chan, and T. K. Siu. Option pricing and Esscher transform under regime switching. *Annals of Finance*, 4:423–32, 2005.

R.J. Elliott and T.K. Siu. Pricing and hedging contingent claims with regime switching risk. *Communications in Mathematical Sciences*, 9(2):477–98, 2011b.

R. Engle. *ARCH, Selected Readings*. Oxford, Oxford University Press., 1995.

F. Esscher. On the probability function in the collective theory of risk. *Scandinavian Actuarial Journal*, 15:175–9, 1932.

E. Fama. Mandelbrot and the stable Peritian hypothesis. *Journal of Business*, 36: 297–306, 1963.

E. Fama. The behavior of stock-market prices. *Journal of Business*, 38:34–105, 1965.

F.A. Fard and T.K. Siu. Pricing and managing risks of european-style options in a Markovian regime-switching binomial model. *Annals of Finance*, 9, 2013. ISSN 421-438.

H. Follmer and M. Schweizer. *Hedging of Contingent Claims under Incomplete Information.* M. H. A. Davis and R. J. Elliot. London: Gordon and Breach., 1991.

H. Follmer and D. Sondermann. Hedging of contingent claims under incomplete information. *In Contributions to Mathematical Economics*, ed. W. Hildenbrand & A. Mas-Colell:205–23, 1986.

P.A. Forsyth, K.R. Vetzal, and R. Zvan. A finite element approach to the pricing of discrete lookbacks with stochastic volatility. *Applied Mathematical Finance*, 9(2): 87–106, 1999.

J.P. Fouque and C. Han. Variance reduction for monte carlo methods to evaluate option prices under multi-factor stochastic volatility models. *Quantitative Finance*, 4(5): 597–606, 2004.

J.P. Fouque, G. Papanicolaou, and K.R. Sircar. *Derivatives in Financial Markets with Stochastic Volatility.* Cambridge University Press., 2000,.

E. Fournie, J.M. Lasry, J. Lebuchoux, and P.L. Lions. Applications of Malliavin calculus to Monte Carlo methods in finance. *Financial and Stochastic*, 5(2):201–236, 2001.

M. Frittelli. The Minimal Entropy Martingale Measure and the valuation problem in incomplete markets. *Mathematical Finance*, 10:39–52, 2000.

H. Geman, D. Madan, and Y. Marc. Time changes for levy processes. *Mathematical Finance*, 11:79–96, 2001.

H. U. Gerber and E. S. W. Shiu. Option pricing by Esscher transforms. *Transactions of the Society of Actuaries*, 46:99–191, 1994.

S. M. Goldfield and R.E. Quandt. A Markov model for switching regressions. *Journal of Econometric*, 1:3–16, 1973.

A. Grosen and P.L. Jorgensen. Fair valuation of life insurance liabilities: the impact of interest rate guarantees, surrender options, and bonus policies. *Insurance: Mathematics and Economics*, 26:37–57, 2000.

X. Guo. An explicit solution to an optimal stopping problem with regime switching. *Journal of Applied Probability*, 38(2):464–81, 2001.

J. D. Hamilton. A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica*, 57:357–84, 1989.

M. Hansen and P.L. Jorgensen. Analytical valuation of american-style Asian options. *Management Science*, 46:1116–1136, 2000.

M. R. Hardy. A regime-switching model of long-term stock returns. *North American Actuarial Journal*, 5:41–5, 2001.

M. R. Hardy, K. Freeland, and M.C. Till. Valuation of equity return model for long term equity linked guarantees. *North American Actuarial Journal*, 10(4):28–47, 2006.

J. M. Harrison and D. M. Kreps. Martingales and arbitrage in multi-period securities markets. *Journal of Economic Theory*, 20(3):381–408, 1979.

J. M. Harrison and S. R. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications*, 11:215–260, 1981.

J. M. Harrison and S. R. Pliska. A stochastic calculus model of continuous trading: Complete markets. *Stochastic Processes and their Applications*, 15:313–316, 1983.

Kreps D.M. Harrison, J.M. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20:381–408, 1979.

Pliska S.R. Harrison, J.M. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl*, 11:215–260, 1981.

D. Heath, R. Jarrow, and A. Morton. Bond pricing and the term structure models: a new methodology for contingent claims valuation. *Econometrica.*, 60:77–105, 1992.

S. Heston. A closed-form solution for options with stochastic volatility with applications to bond ans currency options. *Review of Financial Studies*, 6:327–343, 1993.

A. Holmesa, H. Yangb, and S. Zhangc. A front-fixing finite element method for the valuation of American options with regime switching. *International Journal of Computer Mathematics*, 89(9):1094–1111, 2012.

H. Huang, M.A. Milevsky, and T.S. Salisbury. A different perspective on retirement income sustainability: introducing the ruin-contingent life annuity. *Journal of Wealth Management*, arXiv:1205.2513:89–97, 2009.

H. Huang, M.A. Milevsky, and T.S. Salisbury. Valuation and hedging of the ruin-contingent life annuity (RCLA). *Working paper*, 2012.

J. Hull and A. White. The pricing of options with stochastic volatilities. *Journal of Finance*, 42:281–300, 1987.

M.W. Hung and Y. Liu. Pricing vulnerable options in incomplete markets. *Journal of Futures markets*, 25(2):135–170, 2005.

L.F. James. Poisson process partition calculus with applications to exchangeable models and Bayesian nonparametrics. *arXiv:math/0205093v1 [math.PR]*, 2002.

L.F. James. Bayesian Poisson process partition calculus with an application to Bayesian Levy moving averages. *The Annals of Statistics*, 33:1771–1799, 2005.

F. Jamshidian. LIBOR and swap market models and measures. *Finance and Stochastic*, 1:293–330, 1997.

M. Jeanblanc-Picqu'e and M. Pontier. Optimal portfolio for a small investor in a market model with discontinuous prices. *Applied Mathematics and Optimization*, 22(3):287–310, 1990.

E. Jouini. Arbitrage and control problems in finance: A presentation. *Journal of Mathematical Economics*, 31:167–183, 2001.

C. Joy, P. Boyle, and K.S. Tan. Quasi-monte carlo methods in numerical finance. *Management Science*, 42(6):926–938, 1996.

I. Karatzas and S.E Shreve. *Methods of Mathematical Finance*. Springer, 1998.

S. Kassberger, R. Kiesel, and T. Liebmann. Fair valuation of insurance contracts under Levy process specifications. *Insurance: Mathematics and Economics*, 42(1):419–443, 2008.

O. Le Courtois and F. Quittard-Pinon. Fair valuation of participating life insurance contracts with jump risk. *Geneva Risk and Insurance Review*, 106-136:33(2), 2008.

C.C Liew and K.T. Siu. A hidden Markov regime-switching model for option valuation. *Insurance: Mathematics and Economics*, 47(3):374–384, 2010.

S. Lippman and J.J. McCall. An operational measure of liquidity. *American Economic Review*, 76:43–55, 1986.

A. Y. Lo and C. S. Weng. On a class of Bayesian nonparametric estimates: II. hazard rates estimates. *Annals of Institute of Statatistical Mathematics*, 41:227–245, 1989.

D.B. Madan and S. Eugene. The variance gamma (v.g.) model for share market returns. *Journal of Business*, 63:511–524, 1990.

D.B Madan, P.P. Carr, and E.C. Chang. The variance gamma process and option pricing. *Europian Finance Revision*, 2:79–105, 1998.

M. Magill and M. Quinzii. *Theory of Incomplete Markets*. MIT Press, 2003.

M.P. Magill and M. Quinzii. *Theory of Incomplete Markets*. The MIT Press, 1996.

B. Mandelbrot. The variation of certain speculative prices. *Journal of Business*, 36: 394–419, 1963,.

B. Mandelbrot. The variation of some other speculative prices. *Journal of Business,*, 40:511–524, 1967.

A.-M. Matache, P.-A. Nitsche, and C. Schwab. Wavelet Galerkin pricing of American options on Levy driven assets. *Quantitative Finance*, 5/4:403–424, 2005.

R. McDonald and D.R. Siegel. Investment and the valuation of firms when there is an option to shut down. *International Economic Review*, 26:331–349, 1985.

D.L. McLeish. *Monte Carlo simulation and finance*, volume 276. John Wiley & Sons, 2011.

R. Merton. Option pricing when the underlying stock returns are discontinuous. *Journal of Financial Economics*, 5:125–144, 1976.

R.C. Merton. Theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4:141–183, 1973.

M. Musiela and T. Zariphopoulou. A valuation algorithm for indifference prices in incomplete markets. *Finance and Stochastic*, 8:399–414, 2004.

V. Naik. Option valuation and hedging strategies with jumps in the volatility of asset returns. *The Journal of Finance*, 48(5):1964–1984, 1993.

J.A. Nielsen and K. Sandmann. Equity-linked life insurance - A model with stochastic interest rates. *Insurance: Mathematics and Economics*, 16(3):225–253, 1995.

M. O'Hara. *Market Microstructure Theory*. Blackwell, 1995.

B. Oksendal. *Stochastic Differential Equations: An Introduction With Applications*. Springer, 2003.

B. Oksendal and A. Sulem. *Applied Stochastic Control of Jump-Difussion*. Springer, 2005.

M. Perman, J. Pitman, and M. Yor. Size-biased sampling of Poisson point processes and excursions. *Probabability Theory and Related Fields*, 92:21–39, 1992.

S. Persson. Valuation of a multistate life insurance contract with random benefit. *The Scandinavian Journal of Management*, 9S:73–86, 1993.

S.R. Pilska. *Intorduction to Mathematical Finance*. (Malden, MA: Blackwell), 1997.

P.D. Praetz. The distribution of share price changes. *Journal of Business*, 45:49–55, 1972.

D. Prieul, V. Putyatin, and T. Nassar. On pricing and reserving with-profits life insurance contracts. *Applied Mathematical Finance*, 8(3):145–166, 2001.

R. E. Quandt. The estimation of parameters of linear regression system obeying two separate regime. *Journal of the American Statistical Association*, 55:873–80, 1958.

L. Rogers and G. Chris. Arbitrage from fractional Brownian motion. *Mathematical Finance*, 7:95–105, 1997.

N. Rong and F.A. Fard. Pricing ruin-contingent life annuity under stochastic volatility. *The Journal of Risk Finance*, 14:35–48, 2013.

S.A. Ross. Options and efficiency. *Quarterly Journal of Economics*, 90:75–89, 1976.

K.I. Sato. *Levy Process and Infinitely Divisible Distributions.* Cambridge University Press, 2004.

E. Schlogl. *Quantitative Finance: An Object-Oriented Approach in C++.* CRC Press, 2013.

E.S. Schwartz and L. Trigeorgis. *Real Options and Investment under Uncertainty: Classical Readings and Recent Contributions,.* MIT Press, 2004.

M. Schweize. On minimal martingale measure and and Follmer-Schweizer decomposition. *Stochastic Analysis and Applications*, 13:573–599, 1995.

H. Shimazaki and S. Shinomoto. Kernel bandwidth optimization in spike rate estimation. *Journal of Computational Neuroscience*, 29:171–182, 2010.

H. Shirakawa. Interest rate option pricing with Poisson-Gaussian forward rate curve processes. *Mathematical Finance*, 1(4):77–91, 1991.

T. K. Siu. Fair valuation of participating policies with surrender options and regime switching. *Insurance: Mathematics and Economics*, 37:533–52, 2005.

T.K. Siu and H. Yang. Option pricing when the regime-switching risk is priced. *Special issue of ACTA Mathematicae Applicatae Sinica*, 25(3):369–388, 2009.

T.K. Siu, J.W. Lau, and H. Yang. Pricing participating products under a generalized jump-diffusion model. *Journal of Applied Mathematics and Stochastic Analysis*, doi:10.1155/2008/474623:30, 2008.

B.M. Smith. *A History of the Global Stock Market: From Ancient Rome to Silicon Valley.* Chicago University Press, 2004.

S. Titman. Urban land prices under uncertainty,. *American Economic Review,*, 75: 505–514, 1985.

H. Tong. *Threshold Models in Nonlinear Time Series Analysis.* Springer, 1983.

T. Topper. *Financial Engineering with Finite Elements.* John Wiley & Sons Ltd, Chichester, 2005.

A.D. Wilkie. An option pricing approach to bonus policy. *Journal of the Institute of Actuaries*, 114:21–77, 1987.

D. Wilkie. More on a stochastic asset model for actuarial use. *British Actuarial Journal*, 1(5):777–964, 1995.

P. Wilmott. *Derivatives*. John Wiley & Sons Ltd, Chichester, 1998.

T Zariphopoulou. A solution approach to valuation with unhedgeable risks. *Finance and Stochastic*, 5:61–82, 2001.