Pre-Service Teachers' Noticing of Structural Thinking in Mathematics

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Summary

The importance of teachers' understanding and using mathematical structure is recognised but not well researched. Mathematical structure, when understood, connects mathematical concepts and builds powerful understandings of deep mathematical thinking. The research reported in this thesis is based on the premise that educating teachers to consider structural thinking in their teaching and learning processes will improve their mathematical content and pedagogical knowledge. The positive outcome of teachers using mathematical structure is that it encourages students to think structurally.

In this study, I investigated two primary and three secondary mathematics pre-service teachers (PSTs) as they engaged in learning about mathematical structure through a professional learning program during their final undergraduate year.

This study comprised two phases. The first phase involved the primary PSTs, the second involved the secondary mathematics PSTs. All PSTs completed an introductory and exit questionnaire either side of a professional learning program in which they participated in three cycles, each of which comprised a professional learning workshop, teaching a planned mathematics lesson, and a post-lesson interview or post-lesson reflection.

Quantitative and qualitative data were collected from the two questionnaires, audio transcripts of the professional learning workshops and the post-lesson interviews, lesson plans, and videos of the PSTs' mathematics lessons. These data were analysed for the PSTs' understanding and use of a new framework, the **C**onnecting, **R**ecognising Patterns, **I**dentifying Similarities and Differences, **G**eneralising and Reasoning (CRIG) framework, to teach mathematical structure—a framework that refers to "noticing structural thinking".

From the results of this study, I was able to identify that the PSTs came to appreciate the importance of mathematical structure through their familiarity with the new framework. Engaging in the CRIG framework proved to be effective in deepening the PSTs' mathematical content and pedagogical knowledge. However, their use of the CRIG framework when in-the-moment of teaching was not always reflected in their communications or pedagogical approach.

Candidate Declarations

This thesis titled *Pre-Service Teachers' Noticing of Structural Thinking in Mathematics* has not been submitted for a higher or any other degree to any other university or institution.

I certify that the thesis is an original piece of research; all data, references, and other sources of information, including coauthored journal publications and professional editorial support, have been acknowledged.

I declare that the research presented in this thesis complies with requirements of academic ethics. This research was approved by the Human Research Ethics Committee of Macquarie University (Reference number: 5201600943; see Appendix A) and the NSW Department of Education (SERAP reference number: 2016596; see Appendix B).

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Publications Associated with This Research

This thesis is associated with the following two publications:

Gronow, M., Mulligan, J., & Cavanagh, M. (2017). Teachers' understanding and use of mathematical structure. In A. Downton, S. Livy, & J. Hall, (Eds.), *40 years on: We are still learning!: Proceedings of the 40th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 285–292). Adelaide, SA: The Mathematics Education Research Group of Australasia Inc.

Gronow, M., Cavanagh, M., & Mulligan, J. (2019). Primary pre-service teachers noticing of structural thinking in mathematics. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics Education Research: Impacting Practice: Proceedings of the 42nd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 324–332). Perth, WA: The Mathematics Education Research Group of Australasia Inc.

These publications are reproduced as Appendices C and D, respectively.

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Author Background

My professional journey in mathematics education can be chronicled through my experiences over 30 years of teaching mathematics and the executive positions I have held in secondary schools. These experiences, along with three master's degrees, have developed my beliefs in education, especially in mathematics teaching and learning. The early teaching years were formative in establishing my concern that mathematics teaching and learning should focus on deep understanding of mathematical relationships rather than on memorising number facts and formulas. During these early years, I was fortunate to be mentored by an experienced teacher who promoted good mathematics teaching as going beyond the traditional teacher-centred model. At the same time, problem solving in mathematics was appearing in curriculum documents, and mathematics teaching was influenced by the concept of mathematical thinking. The work of Mason, Burton, and Stacey, (1982) was paramount in developing my personal pedagogy and approach to teaching mathematics. By observing my own metacognitive process of thinking mathematically, I was able to comprehend what students were struggling with when solving mathematical problems. Stacey and Groves, (1985) followed Mason et al. (1982) by providing an insightful guide to problem solving and thinking strategies that were relevant. Their strategies for problem solving became my "tool box" for developing an awareness of how to develop my own and my students' mathematical thinking.

During my Master of Education degree, completed in 1992, I developed an interest in educational psychology. In particular, I was interested in students' mathematical self-efficacy, especially students' self-talk as a motivator or inhibitor to mathematical learning. The affective, behavioural, and cognitive components of learning mathematics became a teaching focus in my classroom. My lessons were aimed at engaging students by building their mathematical self-confidence and promoting an intrinsic motivation to learn mathematics through providing all students with opportunities to develop an understanding of mathematical relationships in collaborative and open-learning environments.

John Mason's extensive body of work on the teaching and learning of mathematics continued to be a great inspiration for me in my own understanding of mathematical thinking. Mason's (2004) work on structures of attention was concerned with metacognitive awareness about the kind of thinking that occurs when doing mathematics.

This approach to mathematical thinking developed out of *Researching Your Own Practice* (Mason, 2002), in which Mason introduced the concept of noticing and what he called attention-in-the-moment, which refers to noticing what one attends to when solving, or teaching, mathematical problems and what attention is given to it in one's own thinking. Mason was interested in a person's metacognitive awareness of thinking mathematically, in particular, how an awareness of mathematical relationships connected concepts and procedures.

Stephens (2008) connected Skemp's (1976) relational thinking to structural thinking, which he described as having different ways of thinking about a mathematical property that can develop into an accurate generalisation. Stephens later combined with Mason and Watson as co-authors to promote the case for mathematical structure, referred to simply as structure, being regarded as an "essential part of teaching and learning" (Mason, Stephens, & Watson, 2009, p.10.).

Mason's combined work on mathematical structure and noticing is internationally recognised and shapes the theoretical framework of this thesis: For effective teaching of mathematics, teachers must notice their own structural thinking. Sfard (1994) claimed that mathematicians often cannot explain their own thinking, but Mason et al. (2009) argued that when teachers have an awareness of structure, they are in a position to promote structural thinking in their learners. This metacognitive awareness includes communicating structure to students so that they can develop their own awareness of structural thinking.

Mason et al. (2009) clearly stated that mathematical structure cannot be taught like a mathematical operation. However, teachers can be taught about the components of structure that assist in developing structural thinking skills. Four components of structure can be identified and combined together to form the teaching framework called the CRIG framework (Gronow, Mulligan, & Cavanagh, 2017). CRIG is an acronym used to identify the framework. It represents the following four components:

Connections to other mathematical understandings
Recognising patterns
Identifying similarities and differences
Generalising and reasoning.

Teachers of mathematics should be aware of attending to structure when teaching mathematics. Structure is the bridge between conceptual and procedural knowledge, and awareness of structure promotes mathematical content and pedagogy to communicate mathematical knowledge (Mason et al., 2009). Awareness of structure supports teachers' metacognitive understanding of their own thinking, thus allowing personal insight into communicating so that mathematical concepts and procedures "make sense" to their students. Structural thinking involves mathematical thinking skills required to relate concepts and procedures to solve mathematical problems.

In this study, pre-service teachers learned about structure through the CRIG framework. Then, by attending to the framework when teaching, the pre-service teachers' metacognitive awareness developed to provide greater insight into their structural thinking, and, through noticing structural thinking when attending to the framework, they developed an awareness of how to communicate structure.

Chapter

1

Introduction

1.1 Context of this study

The motivation behind this study is to improve all teachers of mathematics understanding of how to think mathematically (Mason, Burton, & Stacey, 1982). To this end, a teacher of mathematics must notice mathematical relationships between the concepts to understand the structure of mathematics (Mason, Stephens, & Watson, 2009). This implies an ability to notice mathematical thinking through mathematical structure. This study aims to address this issue by supporting pre-service teachers as they learn to notice structural thinking. This introductory chapter identifies the concern of Australian students poor performance on international tests, the decline of students engagement in mathematics and enrolment in advanced mathematics courses in senior school, fewer graduate mathematics teachers and the procedural approach taken to teaching mathematics. The concept of mathematical structure and structural thinking are introduced and a case is made for why teachers' noticing of structural thinking is essential to address some of these concerns.

Australian students' declining performance on the international mathematics assessments *Trends in Mathematics and Science Study* (TIMSS; Mullis, Martin, Goh, & Cotter, 2015) and OECD *Programme for International Student Assessment* (PISA; Thomson, De Bortoli, & Underwood, 2016) raises concerns about the effectiveness of mathematics teaching and students' engagement in and learning of mathematics (Dinham, 2013). Murphy (2019) has raised concern about the declining performance on these international mathematics assessments and refers to how Australian secondary students' (Years 7 and 9) performance on the National Assessment Program—Literacy and Numeracy (NAPLAN) test has stagnated over the last 10 years (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2018).

Mathematics is part of the STEM (science, technology, engineering and mathematics) suite of subjects and is an essential skill that young people need for the future. Internationally, mathematics education is being investigated by many researchers due to the declining numbers of students studying STEM subjects. This decline is a concern for all nations, because STEM skills are needed for a country's future economic prosperity (Office of the Chief Scientist, 2014). Participation and success in mathematics for all is considered fundamental to a country's technological and economic development (Center for Curriculum Redesign, 2013). Governments have promoted increasing student participation in mathematics (Office of the Chief Scientist, 2014). However, this has not correlated with an increase in either participation or achievement in mathematics. The challenge to increase the rates of participation in advanced mathematics courses at senior school levels is an international concern. Marginson, Tytler, Freeman, and Roberts (2013) identified reduced participation rates in mathematics in many English-speaking countries.

Murphy (2019) identified the declining rates of students in Australia attempting the higher levels of mathematics in secondary school. Barrington and Evans (2016) also found that the number of students attempting advanced mathematics courses has declined in the last 10 years, while the overall population of students completing secondary school has increased. This decline is a trend that continues beyond school, with fewer students studying mathematics at university and even fewer entering and completing mathematics teacher education programs (Smith, Ladewig, & Prinsley, 2018; Wilson & Mack, 2014).

The problem of participation in advanced mathematics courses in the senior years of secondary school is connected to students' negative mathematics experience in junior secondary school. Murphy (2019) pointed to student self-efficacy and career aspriations as influences on what levels of mathematics students choose to study in senior secondary school, and Attard (2013) found that student engagement in mathematics lessons begins to decline early in secondary school. Prince (2013) pointed to the declining number of specialised mathematics teachers as part of the problem. Prince described capable mathematics graduates as having the essential knowledge and skills in the subject to increase the regard for mathematics that would, in turn, encourage students to take the advanced courses. A teacher's mathematical and pedagogical content knowledge (PCK) are at the core of Prince's concern. This study addresses Prince's concern. In it, I study how a teacher recognises mathematical thinking, knows the mathematical content, and communicates that content using appropriate pedagogical practices that engage and motivate students mathematically.

The context of this study could have taken place in any New South Wales (NSW) mathematics classroom in the last 50 years. Over this time, successful teaching and learning of mathematics has been associated with final year results in the NSW Higher School Certificate (HSC). Since the introduction of the HSC in 1967, pressure for students to perform well in the HSC mathematics examination encouraged mathematics teachers, at all levels, to teach to tests or examinations. However, according to Skemp (1976) and Boaler (2015), examination performance is not an indicator of mathematical understanding.

For students to succeed in mathematics, they must achieve high results in pen and paper tests and examinations. To achieve this, teachers often teach strategies that allow students to remember rules, facts, and procedures. Skemp (1976) pointed to this conundrum: Students do not want to have a deep conceptual understanding of mathematics if they only need to remember the facts to do well in examinations. The pedagogical approach taken by the teacher to prepare students for examinations is characterised as a procedural understanding of mathematics, and it usually involves a traditional, teacher-centred approach to learning mathematics. Typical of a procedural approach, is a high emphasis on the use of textbooks and worksheets, and memorising rules, facts, and strategies to be recalled during the test or examination (Vincent & Stacey, 2008). Indeed, Lokan, McRae, and Hollingsworth (2003) found that Australia has a high proportion of mathematics teachers who tend to teach using procedural understanding.

1.2 What is mathematical structure?

Mathematical structure has a rich history in mathematics education, but not one that is clearly understood by many teachers of mathematics (Richland, Stigler, & Holyoak, 2012). Mason, et al. (2009) have claimed that appreciation of mathematical structure is essential for all teachers of mathematics. Mathematical structure can be described as connecting mathematical relationships, recognising patterns, identifying similarities and differences, and generalising results. When teachers are aware of structural thinking, they can transform their students' mathematical thinking and disposition to engage.

Mason et al. (2009) defined mathematical structure clearly as "the identification of general properties which are instantiated in particular situations as relationships between elements or subsets of elements of a set" (p. 10). Mason and his colleagues believed that appreciating structure is powerful in developing students' understanding of mathematics and that attention to structure should be an essential part of mathematical teaching and learning. They asserted that you cannot teach mathematical structure; instead, it is an understanding of how procedures and concepts are connected to support student learning.

Historically, Taylor and Wade (1965) proposed a theoretical definition of structure as the formation and arrangement of a system of mathematical properties. The seminal work of Skemp (1976) introduced relational understanding, which is associated with structure. Others have also referenced mathematical structure. Jones and Bush (1996) used a "building blocks" metaphor to describe mathematical structure, stating that mathematical structure is like the foundation of a building, on which the content is built. They identified structural thinking in mathematics as a vehicle for helping students understand and answer the "why" questions in mathematics. Schmidt, Houang, and Cogan (2002) took a different approach to mathematical structure. They identified that a deeper knowledge of mathematical structure enables one to make connections between mathematical concepts. More recent studies have focussed on mathematical structure in research concerning teachers or students. For example, Vale, McAndrew and Krishnan (2011) investigated the developing structural understandings of out-of-field mathematics teachers and Mulligan and Mitchelmore (2009) identified structural thinking in young children's patterning strategies.

1.3 What is structural thinking?

Mathematical structure is a precursor to structural thinking, which can be associated with cognitive structures, producing schemas that are essential in mathematical thinking and successful learning. Stephens (2008) described structural thinking as an awareness of the way different occurrences of a property develop into correct generalisations. Schoenfeld's (1992) metacognitive perspective on mathematical thinking included structural thinking because it involved attending to one's thinking when doing mathematics. Mason et al. (2009) described how structure supports teachers to recognise deep thinking and understanding of mathematics. They argued that mastering procedures is essential when making sense of mathematics, but it is of little use when the procedures increase and memory is overloaded, suggesting that awareness of structure shifts the learning from rote memory to deeper thinking. Structural thinking encompasses mathematical structure by knowing what procedures to use when solving problems while understanding the mathematical concepts (Mason et al., 2009). Mason et al. went on to state that students involved in structural thinking receive an intrinsic reward and that teachers' awareness of structural relationships transforms students' thinking and disposition to engage. They claimed that structure is essential to mathematics teaching and learning because it relates procedures and concepts and promotes structural thinking.

1.3.1 Structural thinking in mathematics curricula

The notion of structure is traced through the development of mathematics curricula. International curricula include structure as an integral component of mathematics teaching. The National Council for Teachers of Mathematics (NCTM) includes mathematical structure as part of the Common Core State Standards for Mathematics (CCSSM) which includes an outcome “Look and make use of structure” (Common Core State Standards for Mathematics [CCSSM], 2010) among the eight standards of practice. In the United Kingdom, the national mathematics curriculum addresses structure in its secondary curriculum, stating that students “use algebra to generalise the structure of arithmetic including to formulate mathematical relationships” and to “make and test conjectures about the generalisations that underlie patterns and relationships” (Department of Education, 2013). In Japan, a core mathematical activity is to “discover and the extend on properties of numbers and geometrical figures based on previously learned mathematics” (Isoda, 2010, p. 78).

Structure is integral to the Australian Curriculum–Mathematics through the four proficiency strands of understanding, fluency, problem-solving, and reasoning (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2019). These proficiency strands reflect the multidimensional aspects of structure, and they support teaching the content and the development of the thinking and doing of mathematics. The proficiency strands are tied to the development of structural thinking skills. In the Australian curriculum, the lack of use of the term *structure* does not mean that the concept of structure is not essential to mathematics teaching and learning. In the NSW mathematics syllabus for the Australian curriculum (NSW Board of Studies, 2012), the proficiency strands of the Australian curriculum are represented as working mathematically. Structure is identified in the working mathematically processes through the communicating, problem solving, reasoning, understanding, and fluency components.

1.3.2 Structural thinking and procedural and conceptual understanding

Skemp (1976) produced his seminal article about instrumental versus relational understanding in connection with the learning of mathematics. He emphasised the need to change mathematics teaching from an instrumental to a relational focus. His ideas about instrumental and relational understanding in mathematics learning remain central to new theories relating to procedural and conceptual understanding. Sullivan (2011) aligned Skemp’s theory of relational understanding with conceptual understanding as an appreciation of ideas and relationships. Kilpatrick, Swafford, and Findell (2001) described procedures as being the ability to use flexible, accurate, efficient, and appropriate methods to solve mathematical problems, and along with these procedures they included the ability to recall mathematical facts readily. This description of learning mathematics describes what most people remember of their mathematical experiences: rote learning facts and procedures to be reproduced in timed tests.

Australian mathematics teachers are identified as teaching predominantly toward a procedural understanding. In the Third International Mathematics and Science Study (TIMMS) 1999 video study, Australia was shown to have a higher proportion of non-qualified secondary mathematics teachers, and teaching methods that were mostly procedural (Lokan, McRae, & Hollingsworth, 2003). There was no identified correlation between nonqualified mathematics teachers and teaching procedurally, although all mathematics teachers need to be aware of the negative effect that a purely procedural approach has on the learning of mathematics. The TIMSS video study identified teachers in countries with the highest scores on TIMMS as teaching for conceptual understanding.

Mason et al. (2009) argued that mathematical thinking is promoted when mathematical structure is connected to mastering procedures and understanding concepts. They stated, further, that learners would understand the relevance of the mathematics taught, rather than relying on memorising, when the teacher's focus is on mathematical structure. Effective mathematical thinking involves being able to use, explain, and connect mathematical properties. Mathematical structure bridges the gap between procedural and conceptual understanding of mathematics in teaching and learning.

Mason et al. (2009) strongly suggested that attention to mathematical structure, as an overarching theory of procedural and conceptual understanding of mathematics, be addressed in every mathematics classroom. They argue that students' mathematical understanding is enhanced when mathematical structure is the focus of learning. To achieve this, teachers need to acknowledge mathematical structure in the content taught and pedagogy employed, and they need to avoid relying on procedural understanding in teaching mathematics.

Research by Prescott and Cavanagh (2006) has shown that new graduate secondary teachers focused on procedural understanding in their teaching. They demonstrated that these teachers, once they began teaching, relied on their own experiences as students about how mathematics should be taught. Similarly, Bobis (2000) found that new graduate primary teachers reverted to a teacher-centred approach described as teaching for procedural understanding.

1.3.3 Structural thinking and pedagogical content knowledge

Teachers' understanding of mathematical structure is a significant component of pedagogical content knowledge (PCK), described by Shulman (1987) as a requirement for good teaching of mathematics (Loewenberg Ball, Thames, & Phelps, 2008).

Teaching requires an awareness of mathematical structure by the teacher for effective communication with learners (Clarke, Clarke, & Sullivan, 2012). Being aware of mathematical structure enables the teacher to explain the content better so students can understand that content. The teacher can apply mathematical structure through making connections with other learning, recognising any existing patterns, identifying similarities and differences, and generalising results in different situations. The ability to demonstrate these relationships is essential in the mathematics teacher's pedagogy. Attention has been given to developing teacher pedagogical content knowledge (PCK) as a means of improving student learning (Bobis, Anderson, Martin, & Way, 2011). Vale, McAndrew, and Krishnan (2011) found that non-qualified teachers' understanding of mathematical content and concepts is improved through an awareness of mathematical structure.

Bobis (2000) reported that effective mathematics teachers understand the inter-connectedness of ideas, can select and use efficient and effective strategies, and challenge students to think and encourage them to explain, listen, and solve problems. A fundamental understanding of mathematical structure can enable the mathematics teacher to use these strategies in the classroom.

1.4 Rationale

A teacher of mathematics must understand mathematical structure and know how to think structurally. Mathematical structure connects different mathematical concepts and promotes one's thinking to be flexible and creative when attempting to solve mathematical problems. Mathematical structure engages the user through less reliance on memorising rules and facts, to an exploration of ideas and experiences that are related to the world around us. An understanding of mathematical structure can support making predictions and establishing relationships between concepts, leading to generalisation.

A study by Vale, McAndrew, and Krishnan (2011) identified that teachers' pedagogical and content knowledge improved when they were able to teach structural awareness. The study provides support for the formulation of new studies that examine both inservice and pre-service teachers' understanding of mathematical structure. Teachers' structural awareness will, in turn, assist their students to develop structural thinking. Mason et al. (2009) believed that when students demonstrate structural thinking when doing mathematics, their engagement and achievement improved.

The concept of teacher noticing is a lens used in the present study for identifying pre-service teachers' understanding and use of structural thinking. The use of noticing as a lens in this study grew out of Mason's concern of how one acts-in-the-moment when doing mathematics (Mason, 2002). Research using teacher professional noticing is a relatively new area of research in mathematics education. Hunter, Hunter, Jorgensen, and Choy (2016) described teacher noticing as not having received the same level of interest in Australia as it has internationally.

Noticing has also appeared as a construct for learning about mathematics teaching. For example, Beswick and Muir (2013) used videos for pre-service teachers to notice effective mathematics teaching. They stressed that structural understanding of mathematics required for PCK is different from that for numerate people or mathematicians. Pre-service teachers in their study were asked questions to respond to what they noticed about students' structural understanding of the mathematical content taught; for example, "What does Aaron's answer of 1.5 tell you about his understanding of decimals/decimal currency?" (p. 32). Results from this study showed that the pre-service teachers developed an interest in their students' structural understanding. Further development of the notion of noticing is shown by Jacobs, Lamb, and Philipp (2010) who conceptualised professional noticing of students' mathematical thinking as the three interrelated skills of attending, interpreting, and deciding.

Ivars, Fernández-Verdú, Llinares, and Choy (2018) found that pre-service teachers involved in a professional discourse improved their professional noticing. The exchanges between the pre-service teachers helped them notice the salient features of their students' mathematical thinking. Ivars et al. revealed that pre-service teachers could learn to interpret students' mathematical thinking. This present study builds on Ivars et al.'s revelation of pre-service teachers learning to interpret students' mathematical thinking. This new investigation considers pre-service teachers learning to notice structural thinking to improve their PCK through the noticing of structural thinking.

1.5 Background to the study

A previous small-scale exploratory study investigated three junior secondary mathematics teachers' understanding of structure and how the same teachers used it when teaching mathematics (Gronow, Mulligan, & Cavanagh, 2017, Appendix C). The results indicated some discrepancies in what teachers said they knew about structure and what they did in their teaching that reflected their understanding of structure. For example, the results from the survey indicated that teachers understood the concept of structure, but, when interviewed, they showed differing interpretations of structure that did not necessarily support relational approaches in their teaching.

The results from the Gronow et al. (2017) study showed that teachers of mathematics had an understanding of mathematical structure. However, when teaching they were not able to use this knowledge as a pedagogical tool. The measure of the teachers' use of mathematical structure was identified in the teachers' references to four components of mathematical structure. These components, given the acronym CRIG, are Connections (C), Recognising patterns (R), Identifying similarities and differences (I) and Generalising and reasoning (G), collectively known as the CRIG framework of mathematical structure. The components of the CRIG framework appear in the research literature, see Chapter Two, Section 3.6, the proficiency strands of the Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2019) and the working mathematically process of the NSW K-10 mathematics syllabus (NSW Board of Studies, 2012). The CRIG framework provides PSTs with a robust workable approach to identifying structural thinking that supports their understanding and use of mathematical structure. In the present study, the PSTs are involved in a professional learning program (PLP). The PLP provides the mechanism for the PSTs to learn about mathematical structural through the CRIG framework and how to notice structural thinking when using the CRIG framework when teaching.

This doctoral research builds on the preliminary study (Gronow et al., 2017) which, in turn, was based on the construct of mathematical structure proposed by Mason.

1.6 Research questions

A key research question is raised: How effective is a professional learning program develop pre-service teachers' (PSTs') structural thinking? Three contributing questions follow:

1. What are the PSTs' understandings of structural thinking (pre and post implementation of a professional learning program?)

2. How do PSTs use structural thinking in their mathematics teaching?

3. How effective is the CRIG framework in helping PSTs to notice structural thinking in their teaching?

This thesis reports on a design-based study of five, fourth-year undergraduate PSTs, two primary and three secondary mathematics students. The study was conducted over two years in two phases. Phase 1 involved the primary PSTs, and Phase 2 involved the secondary PSTs. In the study, I monitored the PSTs during the PLP to identify their ability to notice structural thinking in mathematics during their final year professional teaching experience. The PSTs' learning about mathematical structure is the core of this study. The PSTs' ability to notice their understanding of mathematical structure and to use when teaching is what sets this study apart from others. Many studies have identified PSTs' noticing of students' mathematical thinking, but none has researched PSTs' thinking, in this case, their structural thinking. A report of the Phase 1 findings were presented at the Mathematics Education Research Group of Australasia (MERGA) 2019 annual conference and appears in the conference proceedings (Gronow, Cavanagh, & Mulligan, 2019, Appendix D).

1.7 Thesis structure

This thesis has eight chapters. The second and third chapters comprise a literature review and theoretical perspectives, respectively. The literature review provides details about, and a critique of, the research surrounding mathematical structure, structural thinking, and the construct of noticing. The theoretical perspectives chapter reports on the historical background to mathematical structure, the development of the CRIG framework, its connection to noticing structural thinking and models of teacher noticing. Chapter 4 presents the design and methodological considerations pertinent to this research. Chapters 5 and 6 comprise the results of the two phases of this study. Chapter 7 encompasses a discussion specific to the results in Chapters 5 and 6. In Chapter 8, I summarise the findings and limitations of the study, and I discuss the implications of the results for implementing the CRIG framework into the teaching and learning of mathematics. Testimonials of the participating PSTs' use of the CRIG framework in their first year of teaching are given and, finally, some concluding remarks complete this thesis.

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Chapter

2

Literature Review

2.1 Introduction

In this chapter, I review the literature pertinent to this study. This literature falls under five main headings:

1. Background to mathematical structure
2. Research about mathematical structure and structural thinking
3. Mathematical structure and teacher PCK
4. Studies about pre-service teacher and teacher noticing of structural thinking
5. Studies concerning PSTs' attitudes and beliefs about teaching mathematics.

2.2 Mathematical structure from a relational view

The rationale for the present study is based essentially on the work of Mason, Stephens, and Watson (2009) because they developed a theoretical framework on structural thinking in mathematics in its many forms. Mason et al. considered how the learner articulates the distinction between mathematical relationships in a specific situation or as general properties. In describing structural thinking, they explained that structural thinking exists on a continuum and that it is difficult to tell whether a learner is structurally aware. Ultimately, it is recognising patterns and identifying similarities and differences to predict whether a generalisation has occurred that signifies a learner being structurally aware.

Mason (2004) referred to structures of attention when considering his thinking processes as a demonstration of structural thinking. He identified cognitive processes when learning mathematics as "what learners are attending to" (p. 17). He promoted structures of attention about what learners need to be aware of, what they notice, and how they attend to it when thinking about mathematics. The same focus of attention can be applied to teachers when they consider their structural thinking processes.

In the following sections, I recognise three significant historical influences on the development of structure: Skemp's (1976) seminal work on relational and instrumental thinking, Mason, Burton, and Stacey's (1982) significant progress in highlighting the importance of thinking mathematically, and Hiebert and Lefevre's (1986) later notions of conceptual and procedural knowledge of mathematics teaching and learning.

2.3 Relational and instrumental thinking

Skemp's (1976) early interest in cognitive psychology began with associating levels of intelligence to mathematical learning as a transitioning through Piaget's stage of sensorimotor learning. Through these levels, the learner advances toward reflective intelligence that eventually leads to the ability to generalise conceptual structures. Generalising conceptual structures is an influencing factor in progressing mathematical understanding because it represents a transitioning process from instrumental to relational or structural thinking.

Skemp made a distinction between relational and instrumental learning of mathematics. He explained that instrumental learning was like having a fixed plan that consisted of a starting point and finishing point, with explicit instructions and directions of how to complete the plan. Relational learning involved building up a conceptual structure or a schema that offered an unlimited number of starting points toward any finishing point, with multiple paths to get there.

Skemp also introduced the concepts of relational and instrumental understandings of mathematical learning as identification of how mathematics is being taught and learnt. He drew attention to how memorising of procedures through the teaching of facts did not develop a deep conceptual knowledge of mathematics. His work was ground breaking in the 1970s and received much attention. Skemp argued that there was a distinct difference between relational and instrumental thinking, and he affirmed that a student who learns instrumentally suffers when a teacher's approach is relational, and vice versa. Skemp gave reasons why teachers preferred instrumental understanding over relational understanding. He believed that learning through an instrumental process could be easier to understand, the rewards are immediate, and the process can be faster. He also pointed out that, for the teacher, there were good reasons to avoid relational understanding. Relational thinking can be challenging to understand, and it takes a long time to explain, mainly when students prefer an easily explained procedure. Additionally, a procedure is all that is needed to pass an examination, and it is easier for beginning teachers to teach instrumental understanding when all the other teachers are doing so.

Skemp's work on students' understanding of mathematics created an ongoing interest in the teaching and of learning mathematics. He aligned traditional teacher-centred learning of mathematics to instrumental thinking and considered it detrimental to students' learning. In contrast, relational thinking is similar to structural thinking (Mason et al., 2009; Stephens, 2008).

2.4 Thinking mathematically

Thinking mathematically as a construct became popular with the increased interest in educational psychology through theorists such as Piaget, Pólya, and Dewey in the early twentieth century, and more recently Bruner, Bloom, and Gangé who dealt with a cognitive-based perspective of intellectual development.

Mason, Burton, and Stacey's (1982) book, *Thinking Mathematically*, was revolutionary for mathematics educators when it was first published and was used extensively for educating prospective mathematics teachers (Tall, 2009). Mason et al.'s guide for teachers of mathematics identified how to develop students' mathematical problem-solving skills. The focus in this volume was on mathematical thinking strategies and developing advanced problem-solving proficiencies.

Mason, Burton and Stacey (1982) considered how problem-solving approaches to mathematics required a more in-depth consideration of how one thinks about mathematics. Mason et al.'s (1982) well-respected approach to problem solving considers an awareness of one's thinking and attention to one's affective considerations. Mason et al. (1982) considered the pioneering work of Bruner (1956) in creative thinking, Pólya (1957) in respect to problem solving, and Gattegno (1963) in how young children explore and understand the world. However, their approach to thinking mathematically was groundbreaking. Mason et al. (1982) introduced the idea of "being stuck" when doing mathematics; this process recognised being stuck as a natural part of the mathematical thinking process. Mason et al.'s (1982) approach to thinking mathematically began with one's ability to identify, use and remember mathematical concepts and procedures. These skills supported other structures that underlie mathematical thinking, such as the higher-level mathematical thinking skills of specialising, generalising, conjecturing and convincing.

Thinking Mathematically provided opportunities for mathematics educators to look beyond the content and the procedural approach to solving a mathematical problem. Learning mathematics by rote was supposedly abandoned and replaced by learning through deep thinking.

Tall (2009) wrote a reflection of John Mason's work on thinking mathematically in which he singled out the content of *Thinking Mathematically* as processes derived from attention to recognition, repetition, and language. He recognised similarities and differences as thinkable concepts, repetition as learning to repeat sequences of actions, which he equated to Skemp's procedural knowledge, and language as recognition and repetition. Tall explained that these thinking processes become more sophisticated as language improves. He also explained how students' attitude and confidence grows through these thinking-mathematically experiences.

2.5 Conceptual and procedural understanding

Hiebert and Lefevre (1986) identified a connection between conceptual and procedural understanding of mathematical knowledge, describing how the learning of mathematics is both conceptual and procedural. Conceptual knowledge is rich in relationships, implying a link to existing cognitive structures, making it a higher-order thinking skill, which, in terms of mathematical thinking, refers to the development of generalisation and abstraction. Procedural knowledge is simply a sequence of actions that can be learned with or without meaning. Mason, Stephens, and Watson (2009) asserted that procedural knowledge could be limited to rote learning, which causes a burden on working memory. Boaler (2015) also refers to working memory as limited and showed that when one is stressed or under pressure, working memory becomes blocked so facts cannot be recalled.

Hiebert and Lefevre describe connections between thinking mathematically and conceptual and procedural knowledge. They describe procedural knowledge as knowing how to use a symbol to complete a calculation. A structural understanding is knowing the mathematical relationships between the concepts and making connections between these relationships. For example, the rule of multiplication is an example of repeated addition.

Multiplication is traditionally taught and learnt procedurally by memorising "times tables", yet students' early understanding of multiplicative structures may begin with the idea of a number being added to itself a number of times, such as 3 + 3 + 3 + 3 = 4 × 3.

When learners connect the procedure to a concept, they are involved in structural thinking. Learning a procedure by successfully memorising does not necessarily mean understanding the concept underpinning the procedure. Understanding the concept requires the learner to explain the connection and transfer this knowledge between contexts. Hiebert and Lefevre asserted that without connecting the procedure to the concept it is difficult to employ the same procedure to other problems where the same concept and procedure are similarly linked. An example of this is the distributive law applied to a numerical and algebraic example. The problem 16 × 4, is taught as an algorithm, shown below.

$$\begin{array}{r} 16 \\ \underline{4\times} \\ 24 \\ \underline{40} \\ 64 \end{array}$$

Example 1: Multiplication algorithm 16 × 4

The same problem could be written as 4 × 16 = 4 × (10 + 6) = 40 + 24, which is an application of the distributive law. Example 1 used to solve a problem is an algorithm with no link to the distributive law. The visual representation of the problem and process has no meaning beyond this example and others like it. However, as a procedure, it can be memorised and learnt efficiently. Students, when introduced to expanding algebraic expressions such as 2 (x + 3) = 2 × x + 2 × 3, do not always make the connection to their prior learning, as in the example shown. The concept of expanding brackets is memorised as a new and different procedure. Knowing the concept from connecting to prior learning as a number sentence would require students to seek a pattern to be used in the new context. Through similar examples, students can generalise the rule as a procedure using algebra. The concept of the distributive law applies to both examples. However, when only the procedure is taught and no connection between the concept is made, structural thinking cannot occur.

Mason, Stephens, and Watson (2009) described structure as the bridge that connects conceptual and procedural knowledge. Understanding a concept is attained by connecting the mathematical procedures when solving a problem to achieve deeper thinking about mathematics. An example would be the area of a rectangle. Area is found by multiplying the breadth by the width and can be generalised to $a \times b$, where a is the breadth and b is the width. Although knowing the formula will give the correct answer, it is a procedural process that does not require an understanding of the concept of area. A structural understanding would connect the concept of area to the formula, with the formula coming from the visualisation of an array. Structural thinking is developed when the students can solve different examples by recognising the pattern and developing a generalised rule that involves multiplying the corresponding adjacent sides for the different rectangles, and then the known procedure is recognised as the concept of area.

The synergy between conceptual and relational understanding centres on the deeper cognitive schemas of mathematical thinking that instrumental understanding and procedural knowledge approaches do not develop. Instrumental and procedural approaches tend to be associated with the traditional teacher-centred mathematics lesson, which is widely used in Australian mathematics classrooms (Sullivan, Clarke, & Clarke, 2009). Further, learning to think relationally or structurally is beneficial for mathematical development (Lee, Ng, & Bull, 2018; Mulligan, & Mitchelmore, 2009). The next section describes the CRIG framework designed to assist teachers of mathematics to develop their awareness of mathematical structure.

2.6 Research about mathematical structure and structural thinking

In this section, I report on several significant studies about mathematical structure and students' structural thinking that span a wide range of age levels, from pre-schoolers through to post-secondary school (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2016; Blanton & Kaput, 2004; Ellemor-Collins & Wright, 2009; Empson, Levi, & Carpenter, 2011; Hoch & Dreyfus, 2006; Lee, Ng, & Bull, 2018; Linchevski & Livneh, 1999; Mulligan & Mitchelmore, 2009; Nataraj & Thomas, 2009; Papic, Mulligan, & Mitchelmore, 2011; Richland, Stigler, & Holyoak, 2012; Stephens, 2006). These studies also represent research literature about mathematical structure and structural thinking across different mathematical content domains and the effect of structure on a teacher's PCK. The studies reviewed involve structure and mathematical content areas of number, algebra, and patterns. The studies that pertain to teachers' understanding and use of mathematical structure and its relevance in PCK are reviewed.

2.6.1 Studies about number and mathematical structure

Looking for and recognising the underlying structure of number and arithmetical processes is an essential aspect of mathematical thinking (Bishop et al., 2016). In a two-year study of K-12 students' conceptions of integers and integer arithmetic, Bishop et al. focused on how integer understanding related to structural reasoning. Their study involved interviewing students to ascertain integer conceptions and ways of reasoning when using mathematical structure to solve integer tasks. They made reference to features of mathematical structure such as recognising connections between structures; seeing the fundamental properties of commutativity, associativity, and distributivity as generalised patterns; looking for similarities and differences between known problems; and making generalisations about whole numbers. They found that students used a particular strategy, closely aligned to relational thinking, called "logical necessity" to solve problems. Logical necessity involved the student making connections with what they knew to be true when using number. For example, given a problem such as $-5 + 1 = -4$, a student might solve the problem by any known strategy, such as using a number line or a counting-on procedure. The student may connect this known information to solve another problem, such as $1 + -5$ by making a generalised assumption using the commutative law, although the student may not know the commutative law. In doing so, the student is beginning to generalise the meaning of adding a negative number. By using a series of similar questions, where patterns are recognised, and the numbers and signs were changed to give similar examples with different numbers, the authors determined that some students were able to use mathematical structures to solve many similar problems.

Bishop et al. (2016) found that making comparisons to identify differences between mathematical relationships was a key feature of logical necessity, because making connections is a crucial component of structural thinking, it follows that noticing these connections leads to recognising patterns. By identifying the similarities and differences between numbers in the pattern, a rule or generalisation can develop. In their study, Bishop et al. identified structural reasoning as occurring when comparing alternatives, such as similarities and differences, then making a generalised assumption from the consequences.

Empson, Levi, and Carpenter (2011) reviewed research from the last 14 years on elementary school students' use of relational, or structural thinking as a basis for learning fractions. Relational thinking involved using the fundamental properties of operations and equality as the structure of a problem and using this structure to process a solution. The authors considered relational thinking to be an essential base for children to understand fractions because it involved using fundamental properties of whole number and fractions.

Empson et al. (2011) stated that the teaching of arithmetic and fractions has generally relied upon learning a set of procedures but fails to introduce students to any structural understandings through relational or structural thinking. Understanding the mathematical structure of the number system develops powerful reasoning abilities that are the basis for understanding algebra. Moreover, if students begin to learn algebra with the ability to think structurally about the operations they are using, they are prepared to learn and carry out more important processes that include the ability to generalise their results as algebraic expressions. Empson et al. described relational thinking as powerful in its application of fundamental properties of mathematics, such as associativity.

Ellemor-Collins and Wright (2009) explored students' understanding of the structure of numbers from 1 to 20, through the development of simple addition and subtraction procedures. The project, Numeracy Intervention Research Project (NIRP) involved 25 teachers and 300 third- and fourth-grade students. The authors found that learning about number structures improved the students' arithmetical knowledge. Young students solving simple one and two-digit addition and subtraction problems moved from a counting-on process, which often involved using their fingers to complete a facile addition and subtraction process. This facile process involved understanding the structure of the numbers; for example, in the addition 7 + 5, instead of counting on from 7 to 12 by adding ones, the student can break 5 as 3 + 2, resulting in 7 + 3 + 2. This process involved non-counting, partial number deconstruction, greater number knowledge, application of the associative law, and a higher level of relational number sense. The authors argued that this demonstrates structural thinking because students are making connections by breaking the problem into combinations of other numbers and then making use of differences between the numbers. Students then look for and connect the number relationships rather than merely seeing the problem as a calculation. They can look for patterns of additions to 10, generalise the relationships between the numbers, and break the question down to a more fundamental problem to be solved.

Stephens (2006) studied how year 5, 6, and 7 students solved a missing number problem. He noticed how students used structural thinking when solving arithmetical number problems. Students who demonstrated a higher-level structural understanding solved the missing number problems by looking for the structural relationships between numbers without doing calculations. Stephens asserted that relying upon calculations was not productive mathematical thinking and that powerful thinking does not rely on computation. Stephens later identified this powerful form of thinking as relational thinking and said that it was another form of structural thinking (Stephens, 2008). He was interested in the difference between structural thinking and computational thinking. Stephens' investigation in this difference centred around developing a structural understanding of the relationships between numbers and operations in number sentences. Using the example of why $34 + 29 = 33 + 30$ is true, Stephens found that students who added the numbers on each side got the correct answer by computation. Students who saw 34 was reduced by one to get 33 then 29 should be increased by one to 30, keeping a balance between the two sides of the equation, were demonstrating structural thinking. These students saw the relationships between the numbers by identifying similarities and differences and then used a pattern to generalise the result.

Additionally, students using structural thinking were demonstrating an understanding of equivalence and the closure property of addition. Using a mathematical problem that involves finding a missing number problem, Stephens developed questions to identify whether students were using relational thinking to find the missing number (in the box); for example, $73 + 49 = 72 + \square$. Computational thinkers added or subtracted the numbers and made the difference by another computational process. A structural thinker looked at both sides of the equal sign, then considered the relationship between numbers and operations. In this case, they saw 72 as being one less than 73, so 49 needed to increase by one to maintain equivalence between the two sides.

Nataraj and Thomas (2009) researched the development of understanding number structure from a history of mathematics perspective. Their study of 27, year 9 (13-year-old) students in a secondary school in New Zealand involved showing the students the structure of large numbers in the number system. The results of the study demonstrated that when their attention focused on the structure of the number system, students' competence in naming and using large numbers, positional notation, and the ability to generalise improved.

The results of Nataraj and Thomas' study demonstrate the versatility of structure beyond a purely arithmetic process. Their interest in the structure of the number system included developing students' understanding of the structures of positive and negative numbers. The authors contended that, as students became aware of the structure of the number system, they could assimilate the concepts involved. Through an understanding of the form and structure of large numbers, students learned to make sense of the number system and then began to generalise multiplicative structures. Their learning about number systems that included the notion of place value added to the students' understanding of whole numbers, fractions, and decimals across the four operations. Understanding the number system structure supported the fundamentals of algebra and future mathematical learning because it represented the foundations of abstraction and generalisation.

The studies discussed in this section present structural thinking through various approaches. Bishop et al. (2016) called it logical necessity, and Empson et al. (2011) and Stephens (2008) identified structural thinking as relational thinking. Mason et al. (2009) connected relational thinking to structural thinking but added another dimension by associating structural thinking with the ability to generalise. In each of the studies, structural or relational thinking was regarded as the ability to think flexibly; that is, to understand and use number properties and relationships between the concepts rather than relying on mental arithmetic. In the cases presented, there was similarity in the students' structural thinking to solve the number problems. Empson et al. (2011) and Ellemor-Collins and Wright (2009) demonstrated structural thinking was recognised when students could break a number into smaller components before rearranging the numbers, using the associative law of addition to add the numbers. Bishop et al. demonstrated that students in their study were using relational thinking similarly through the associative law to rearrange numbers before adding them. All these studies are reflective of what led Empson et al. to note that "to understand arithmetic is to think relationally about arithmetic" (2011, p. 412).

2.6.2 Studies about algebra and mathematical structure

Research about mathematical structure is conducted in many different mathematical contexts. A substantial amount of literature surrounds algebra and mathematical structure. It deals with structural thinking about how a learner transitions from arithmetic to algebra, which involves the ability to view algebra as generalised arithmetic. Algebra, as generalised arithmetic, requires an understanding of general, as opposed to particular, statements about numbers and operations. Generalisation is an essential component of mathematical structure, and a skill Mason (2008) declared to be an innate and natural gift of young students. Students who use generalised arithmetic can abstract, generalise, then formalise the structures, principles and properties that are guided by computation with numbers.

Several studies identify the important structural relationships that exist between arithmetic and algebra that learners need to know. Lee, Ng, and Bull (2018) spent four years studying second- to ninth-grade students' mathematical and cognitive capabilities that contributed to their ability to solve algebra word problems. Their findings identified performance on mathematics tasks that involved relational thinking as predicting a student's performance in algebra. The authors asserted the importance of developing students' relational thinking skills as a pathway toward improving an understanding of algebra. They acknowledged that the association between relational thinking and algebra tasks had the same task demands that linked relational thinking to algebra skills. This association included how relational tasks provide a platform for engaging in generalising of mathematical relations.

In their study, Lee et al. (2018) defined relational skills with the same intention as Vale (2013), Mason et al. (2009), and Stephens (2008). In this sense, relational skills can be comparable to structural thinking. Lee et al. investigated relational thinking skills through an understanding of equivalence. They argued that this is prerequisite understanding for progressing from arithmetic to algebra, with the equal sign denoting the relationship between the two sides of the equation. They acknowledged that some students fail to do this and, when moving from one side to the other, they calculate without acknowledging what the problem requires them to do. In the example, 5 + 4 = + 7, a student who does not understand the equal sign may add the five and four, and state that the missing number is nine. Students with a relational understanding are in a better position to see that the two sides of the equal sign must be same, so the answer is whatever adds to seven to give nine.

Lee et al. (2018) used patterning as a process for extending students' algebraic thinking. They claimed that pattern recognition and the ability to change mathematical contexts are elements of algebraic thinking. In the example, "What is the missing number in the sequence 1, 4, 7, 10, , 16, 19?", relational thinking occurs as the sequence requires consideration of the pattern and a reason for choosing the correct missing number. The authors found that there were benefits to learning relational thinking early because relational thinking skills are predictive of success in algebra. Tasks that focus on computational skills were insufficient for success in algebra. Competence on tasks that developed relational thinking had benefits for understanding algebra, compared with just doing algebra questions.

Hoch and Dreyfus (2006) reported the results of a questionnaire given to 165 advanced level mathematics high school students to measure their structural sense. The researchers described structural sense as less reliance on instrumental, and more reliance on relational thinking. They argued that students who have a high level of algebraic skill cannot necessarily think structurally across all mathematical contexts. Students who were able to solve a simple quadratic equation of the form $x^2 + 5x + 6 = 0$ had difficulty in solving the problem when it was of a different form such as $(x^2 + 3x)^2 + 5(x^2 + 3x) + 6 = 0$. From their results, they found that high-achieving students tended to rely on an instrumental approach to learning mathematics and did not think structurally. However, students who did use structural thinking made fewer errors than did those who relied on a rote approach or instrumental understanding. Hoch and Dreyfus argued that developing students' structural thinking would improve overall performance because it would reduce the number of calculations students did when solving problems.

Warren (2005) gave a written test to 672 students aged 11–14 years to investigate their understanding of number laws that assist in the transition from arithmetic to algebra. She argued that mathematical structures are essential for students to make the transition from arithmetic to algebra and that the reason many students experience problems with algebra is because of inadequate arithmetic knowledge. In her study, students did a written test of six tasks to ascertain an understanding of the arithmetic properties such as the ability to break addition and division into essential components and their associative and commutative properties.

Warren (2005) found that students did not have an understanding of addition or division as a generalised process, and they did not understand the associative or commutative laws. She claimed that students leave primary school with a limited notion of mathematical structure or the general processes of arithmetic operations. Therefore, from their arithmetic experiences, students fail to develop an abstract experience that will develop the relationships and principles needed for algebra.

Linchevski and Livneh (1999) interviewed 53 fifth- and sixth-grade students in Canada and Israel in a study about how students inherited algebra sense-making from structural properties with no exposure to either integers or algebra. The researchers identified structural sense as the students' ability to think flexibly and creatively. Linchevski and Livneh saw the lack of structural thinking abilities as reasons why students have trouble learning algebra. Students were given simple numerical tasks to complete and then asked about the best way to find the answer, such as when the subtraction problem $12 - 5$ is solved as $12 - 2 - 3$. Alternatively, when given the problem $50 - 10 - 10 - 10$, students may take 10 from 50, three times. On the other hand, students who identified the structure may have used a model of multiplication as repeated addition and noticed the three tens as 3×10. By recognising the structure of the problem, students using structural thinking would reorganise the problem as $50 - (3 \times 10)$.

Linchevski and Livneh (1999) found that most students used known calculations and did not look for structural relationships between the numbers and operations when doing simple order-of-operations problems. Their results confirmed that students' underlying problems with algebra stem from a lack of understanding of the mathematical structures of the number system. The researchers concluded that it is necessary to look for pedagogical ways to establish a connection between arithmetic and algebra, possibly through solving and modelling concrete situations.

The common theme throughout these studies is that, before competency in algebra is achieved, a structural understanding of number is necessary. Linchevski and Livneh (1999) saw algebra as having the structural properties attributed to operations on real numbers. Lee et al. (2018) identified a structural understanding of equivalence and associativity as important before students have sufficient understanding to apply algebra to solve problems. They were able to show that relational or structural thinking skills were predictors of performance on algebraic word problems. Hoch and Dreyfus (2006) and Linchevski and Livneh (1999) reported students tending to remember procedures and calculations to solve problems and not looking for structural relationships. Warren (2003) found that students have a limited structural understanding when they leave primary school.

The studies reviewed in this section indicate that, although relational thinking is a prerequisite for students to process algebraic skills, most students do not have sufficient structural thinking skills. All authors promoted the teaching of structural or relational thinking skills when learning arithmetic procedures because it develops a deeper understanding of algebraic structures.

2.6.3 Studies about pattern and mathematical structure

The inclusion of patterning in learning mathematics has been the focus of many related studies (Blanton & Kaput 2004; Cooper & Warren, 2011; Mulligan & Mitchelmore, 2009; Papic et al., 2011; Warren & Cooper, 2008). Schoenfeld (1992) described mathematics as the "science of patterns" because it relates to the sciences in using empirical evidence to establish rules based on patterns. The inclusion of patterning in the number and algebra strand of the NSW K–10 mathematics syllabus (NSW Board of Studies, 2012) and other international curricula (Common Core State Standards, 2010) signifies the importance of patterning in establishing number sense, which leads to generalising and abstract thinking.

Papic et al. (2011) studied the development of patterning strategies of 53 children from two preschools through an intervention program involving a wide range of patterning tasks, that included repeating patterns, spatial structure, and growing patterns. Using an interview-based assessment, the Early Mathematical Patterning Assessment (EMPA), pre- and post-intervention students were monitored during the year prior to formal school.

Papic et al. (2011) identified young children as being capable of generalising and abstracting mathematical concepts. They found that an intuitive awareness of structural relationships in patterning is fundamental to learning mathematics and early algebraic thinking. They reported on the importance of recognising pattern structure, even at a simple level, as part of developing multiplicative reasoning. Papic et al. were able to demonstrate from their results that an early intervention program for pre-school children's pattern development supports understanding of a unit of repeat and elementary spatial concepts. The researchers also found that young children were able to generalise and symbolise pattern structure.

In their Early Algebra Thinking Project (EATP), Cooper and Warren (2011) followed 220 students over five years, from year 2 to year 6 investigating their ability to generalise how a pattern grows. The researchers described how the development of EATP allowed students to generalise arithmetical structure and to think algebraically through the comprehension of expressions and equations. Cooper and Warren described generalising arithmetic structure as essential for developing algebraic understanding. They viewed algebra in terms of mathematical structures, principles, and behaviours, not as the manipulation of letters. Like other researchers, they proposed that generalisation is a significant determinant of algebraic thinking, particularly in pattern rules with growing patterns. In the EATP project, the generalisation of patterns in tables was a significant outcome in terms of generalising principles and abstract representations. The results demonstrated that young students "can generalise relationships between different materials within repeating patterns across many repeats" (p. 197).

Cooper and Warren found from their EATP study that early- and middle-years students can learn to understand mathematical structures with appropriate instruction and teaching that dealt with structure. In particular, they focused on critical components of mathematical structure such as making connections between generalisations and abstract representations. Cooper and Warren studied generalisations across a variety of contexts and a range of abstract mathematical representations with the intention to identify the relationship between these representations and algebraic thinking. They identified mathematical representations in language, diagrams, figures, symbols, and graphs, and generalisations in patterns, tables, and abstractions. They discovered that generalisation is a significant determinant of the growth in algebraic thinking. An example of this was demonstrated by the students generalising the compensation principle for addition. That is, when adding two numbers, if one number is increased by a certain amount then the second number is decreased by the same amount to keep the sum of the two numbers the same, shown as $a + b = (a + k) + (b - k)$.

In another study, Warren and Cooper (2008) investigated instruction that can assist students to generalise how a pattern grows. The researchers concluded that abstract patterning is the basis of structural knowledge and that students will understand mathematical structures when structure is a focus of patterning tasks. They also found that students often have trouble making the transition from patterns to functions because of their inability to use appropriate language to describe the relationships. They tended to rely on an additive strategy as a generalisation. This inability represents a failure of linking position to pattern or to convert the pattern to a table and generalise the result. These difficulties can persist in higher levels of mathematics.

Mulligan and Mitchelmore (2009) identified structural awareness as a crucial component in the development of mathematical concepts in young children. They coined the term *awareness of mathematical pattern and structure* (AMPS) which was associated with a conceptual understanding of pattern and structure that could be generalised across other mathematics concepts. They described two interdependent components of structure, one as the knowledge of structure, and the other as the ability to see and analyse patterns. They considered these to be general features of how students perceive and react to their environment.

A suite of Australian studies developed, implemented, and evaluated an early mathematics intervention, the Pattern and Structure Mathematics Awareness Program (PASMAP) with four to eight-year-olds focused on the integrated development of core mathematical concepts and processes. A quasi-experimental evaluation study of 319 students revealed highly significant differences in mathematics performance in favour of the PASMAP group at the end of kindergarten and a year later, measured by the Pattern and Structure Assessment (PASA) and a Rasch scale. The PASMAP components and pedagogy comprised repetitions, structured counting and grouping, shape and alignment, partitioning, additive and multiplicative structures, unitising, measurement, and transformations. Students' growth in structural development was described through an analysis of their mathematical representations and explanations.

The significance of the development of AMPS was that the researchers could demonstrate that early learners' structural responses to mathematical tasks can be classified and, importantly, that the level of structural development can indicate overall mathematical achievement. The findings from this study demonstrate that AMPS provides insights into early mathematical understanding. AMPS provides a focus on a deep understanding of the students' structural understanding rather than recognising their procedural skills. This research has contributed to our understanding of early algebraic thinking through an understanding of pattern structure.

Blanton and Kaput (2004) studied how elementary grade students develop and express algebraic reasoning through functional thinking. The researchers found that the beginning of these young students' early algebra understanding comes from recognising mathematical structure. They found that children could use functional or structural thinking to generalise and could solve arithmetic problems without calculating. Blanton and Kaput argued that a classroom culture that promotes structural thinking is developed by a teacher who understands "sociomathematical norms of conjecturing, arguing and generalising" (p. 20). The teacher must not only understand mathematical structure but must make it a standard daily practice related to arithmetic procedures and calculations.

Research about three aspects of students' structural understanding of arithmetic, algebra, and patterning has been discussed in this section. Papic et al. (2011) investigated patterning in developing early structural thinking skills, as a precursor for arithmetic and algebra. They found that young children who receive instruction on patterning developed structural thinking skills. Cooper and Warren (2011) found that students can learn to understand mathematical structures with appropriate guidance. They concluded that abstract patterning is the basis of structural knowledge and that students will understand mathematical structures when the teacher focuses on structure when teaching patterning. Mulligan and Mitchelmore (2009) developed an assessment protocol to measure students' structural understandings through recognising patterns. Their findings demonstrate that a program based on developing patterns and relationships across mathematical concepts develops students' structural understandings rather than memorising procedures. The studies on pattern and mathematical structure demonstrate how pattern generalisation can emphasise students' structural thinking

The teachers' understanding of mathematical structure and ability to embed structural thinking skills, identified in these research studies, are essential to developing appropriate pedagogy. Teachers' use of structure in the classroom must reflect a commitment to developing structural thinking skills and they need to know why structural thinking is essential in learning mathematics. The following section reviews the mathematics teacher's role in embedding structure into their PCK.

2.7 Mathematical structure and teacher mathematical content and pedagogical knowledge

Research that links teachers' PCK to mathematical structure is limited. Mason et al. (2009) presented an essential case for teachers to appreciate mathematical structure. They asserted that, for students to think structurally, teachers should not only be aware of structure but have the strategies to make structural relationships the focus of their lessons. Mason et al. stated that "teachers who are themselves explicitly aware of structural relationships, who are aware of perceiving situations as instances of properties (rather than as surprising and unique events), are in a position to promote similar awareness in their learners" (p. 29).

The studies reviewed here include teachers' understanding and use of mathematical structure (Gronow, Mulligan, & Cavanagh, 2017), professional learning for primary teachers on noticing relational or structural thinking (Vale, 2013), how teachers overuse instrumental or procedural practice in their teaching (Richland, Stigler, & Holyoak, 2012), out-of-field teachers' of mathematics professional development (Vale, McAndrew, & Krishnan, 2011) and primary teachers use of functional thinking to build algebraic reasoning into their instruction (Blanton & Kaput, 2011).

2.7.1 Teachers' understanding and use of mathematical structure

In an exploratory study, Gronow, Mulligan, and Cavanagh (2017) researched teachers' understanding and use of structure in junior secondary mathematics classrooms. Three teachers were interviewed and observed during their teaching.

This study investigated the concept of structure as a form of pedagogical practice that will support students' mathematical understanding and engage students in learning mathematics. The research questions of this study focused on two main aspects, namely what teachers said they knew about structure and how they used language in the classroom to promote structural thinking.

Gronow et al. observed how the teachers used mathematical structure and encouraged structural thinking through four components of mathematical structure known as the CRIG framework (connections to prior learning and other mathematical relationships, recognising patterns, identifying similarities and differences, and generalising and reasoning [see the theoretical framework, Chapter 3]). Transcripts from interviews and lesson observations were examined for teachers' understanding of mathematical structure and teachers' utterances of the CRIG framework.

Teachers' understanding of structure, identified in an initial survey and later articulated in the interviews, was inconsistent. The survey results alone indicated that teachers were aware of the nature and value of mathematical structure, but they did not provide adequate explanations or examples of structure when interviewed. The observation records of teachers' pedagogical practices that included their use of language to promote structure did not match the survey data. Analysis of the observations revealed a limited reference to mathematical structure in their language, which was predominantly characterised by the use of procedural terms. The researchers identified that teachers regard conceptual understanding as important but have difficulty distinguishing it from procedural understanding and tend to rely on a procedural approach when teaching mathematics. This reliance was particularly relevant for the teachers when preparing their students for upcoming examinations.

The conclusions drawn from the interview and observation data were that the teachers did not have a deep understanding of mathematical structure. The benefits of structural thinking in students' learning were acknowledged in the teachers' initial views but were absent when talking about the nature and value of structure in teaching and learning. From this study, it became clear that mathematics teachers do refer to mathematical structure in their pedagogy when teaching mathematics; however, the teachers' awareness of doing so was not apparent. This study formed the background research for this PhD thesis which focuses on developing pre-service teachers' ability to think structurally.

2.7.2 Teacher professional development and mathematical structure

Vale (2013) designed a professional learning program for primary teachers to investigate students' thinking when solving arithmetic questions that involve finding a missing number. She asked the teachers in a workshop to have students solve the missing number problems, for example, "What is the missing number for ☐ in 17 + 24 = ☐+ 21?" Vale identified three successful strategies. The first of these, a balance strategy, involved the use of addition to make the left-hand side of the equation equal the right-hand side. The second strategy, transformation, involved students finding the sum of the left-hand side, then subtracting the value of the number on the right-hand side to find the missing number. The third strategy, relational or structural thinking, involved finding the relationship between the numbers on the opposite side of the equal sign. Teachers were required to categorise students' mathematical thinking as computational or relational. Students who used relational thinking recognised the relationships between numbers and used equivalence instead of calculation to solve problems.

Richland, Stigler, and Holyoak (2012) studied the mathematical knowledge of students at a post-secondary community college, believing that students who used relational thinking when doing mathematics would have an enhanced ability to transfer mathematical knowledge and engage in the learning of mathematics. Richland et al.'s report suggested that there is a connection between relational thinking and enhanced ability for flexible thinking and reasoning that was associated with students' high performance in mathematics.

Richland et al. (2012) argued that developing students' awareness of mathematical structures builds the students' understanding to make sense of the mathematics by developing their structural thinking and enhancing their mathematical understanding. This structural understanding is more easily remembered and allows for flexible transfer of knowledge across contexts to solve problems, notice mathematical connections and different representations of the same concept, and to reason through the problem without remembering the procedure. Richland et al. concluded that further research into teacher professional development would assist teachers to develop pedagogical strategies that support students to develop relational thinking and the ability to transfer their thinking across different mathematical contexts.

Richland et al. identified that a teacher's traditional beliefs about learning mathematics limit their teaching to a procedural manner. Although many teachers espouse the importance of mathematical learning through conceptual understanding, for many teachers, the meaning of a conceptual understanding is not clear. They see success at mathematics as being both conceptual and procedural but find it difficult to distinguish between the two. The result is that the teacher will often lean toward the comfort of a procedure that has been proven to be successful for students achieving success in examinations.

In an intensive 3-year project integrating the development of algebraic reasoning into elementary school mathematics, Kaput and Blanton (2000) involved approximately twenty grade 2–5 teachers from eight different primary schools. Their quest was to move algebra beyond the traditional view of symbolic language and move toward seeing algebra as the generalising of patterns to a structural form of algebraic reasoning. Their approach was to work with teachers to "algebrafy" their instructional materials through building algebraic reasoning opportunities, especially generalising and formalising, then creating a pedagogy that supports students generalising and formalising.

Blanton and Kaput (2011) later conducted a 5-year study involving a professional development project in an urban school district within a graduate course for elementary teachers. They argued that mathematics education needed to go beyond calculations and procedural learning of rules that are characteristic of instrumental learning. The researchers highlighted the importance of examining the structure of mathematics and developing deeper understanding of the concepts and relationships involved when doing mathematics. Blanton and Kaput (2011) found that teachers can provide opportunities for students to make mathematical generalisations through arithmetic tasks. The students, when provided with these opportunities, begin to develop mathematical reasoning, as they start to recognise mathematical structure in their thinking.

Blanton and Kaput (2011) recommended that teachers build structural thinking skills through patterns, conjecturing, generalising, and reasoning. They also recommended embedding structural thinking skills into mathematics pedagogy as a normal part of the mathematical activity. Structural thinking allows students to build on their natural thinking, providing them with a more profound mathematical experience.

In a study of Australian teachers, Vale et al. (2011) used the expression *connects with the horizon* to describe how the teachers' knowledge of mathematical structure connects concepts to future mathematical learning. In doing so, Vale et al. identified components of their professional learning program with what Shulman (1987) defined as pedagogical content knowledge (PCK), which refers to "the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organised, represented, and adapted to the diverse interest and ability of learning and presented for instruction" (Shulman, 1987, p. 8). In their study, Vale et al. devised a professional learning program for mathematics teachers who either had not been trained in mathematics pedagogy or content or were out-of-field teachers. Their program focused on developing and furthering teacher mathematical pedagogical and content knowledge through an awareness of structure.

Vale et al. (2011) identified the use of structure as being essential pedagogical content knowledge for teaching mathematics. Their study consisted of 10 out-of-field teachers of mathematics; these teachers were of varying ages and experience in teaching mathematics. The teachers attended 21 face-to-face seminars each lasting three hours and also worked in a mentor relationship with an experienced senior secondary mathematics teacher. They team-taught senior mathematics lessons, observed and reflected on students' working, reviewed the schools' teaching and assessment resources and participated in the moderation of senior mathematics assessment tasks. Vale et al. identified teachers' awareness of structure through analysis of semi-structured interviews with the teachers taken 10 weeks after completion of the program. The teachers gave personal accounts of how they had applied the structural components of making connections and applying a "mathematical horizon" to the content in their teaching.

The researchers found that, through an understanding of structure, the teachers developed a deeper understanding of the mathematical content and a heightened awareness of pedagogical content knowledge when teaching mathematics. Appreciation of structure added to the teachers' insight into how mathematical content connects future learning and how patterns support generalisations such as the formula for the derivative of a polynomial. The teachers' understanding of students' thinking also developed. This understanding included their knowing about potential student misconceptions and encouraging students' disposition to persevere when solving problems.

Vale et al.'s professional development program focused on both mathematical content and pedagogy. At the end of the program, teachers reflected on the mathematical connections and their appreciation of pedagogical content knowledge. Their reflections indicated they could both deepen and broaden their knowledge of teaching junior secondary mathematics and develop a capacity to support students' learning of mathematics. The researchers indicated the need for further research into teachers' awareness of structure and how teachers could be encouraged to embed structure in their teaching practice.

An essential outcome of the study by Vale et al. (2011) was that mathematics teachers developed an awareness of structure as a part of their pedagogical content knowledge for classroom practice. The teachers reported positive experiences of working in a learning community, and they developed a community response and received support when seeking assistance about content, explanations about mathematical methods, and tasks for teaching. The researchers showed that, when teachers collaborated in a learning community, they improved their understanding of mathematical structure and depth of structural thinking.

The literature reviewed in this section accounts for the various studies that have aimed to improve teachers' understanding of structure. Teacher professional development in mathematical structure and structural thinking impacts on teachers' mathematical content knowledge. Vale's (2013) program for teachers to identify relational thinking in students' solutions to missing number problems reflects Stephens' (2008) study, discussed earlier in this chapter. Vale was able to demonstrate the importance of teachers being able to identify students' ability to solve problems through structural understanding, instead of using calculations. Richland et al. (2012) supported the idea that structural or flexible thinking was important for student understanding of mathematics and identified that teachers of mathematics agree that this approach would improve students' mathematical understandings. Their findings were similar to those of Gronow et al. (2017) where teachers tended toward a procedural approach instead of a conceptual approach because it was easier and students preferred it in their examination preparation.

Blanton and Kaput (2011) found that teachers can deliver pedagogy that focuses on structural relationships. They proposed framing classroom activities around students' natural play and encouraging teachers to use tasks so students can contextualise and generalise. Vale et al.'s (2011) professional development program exposed out-of-field teachers of mathematics to mathematical structure. The teachers in this program improved their mathematical understandings and were successful in deepening their pedagogical approach.

2.8 Studies about pre-service teacher and teacher noticing structural thinking

Despite the relatively short time that the term noticing has been highlighted in mathematics education research, there is growing interest in this construct (Scheiner, 2016). Scheiner announced in a comprehensive review of teacher noticing that descriptions of noticing are varied, describing it as having "many faces" and that a single definition should not limit it. Scheiner explained that noticing is an amalgamation of skills or processes that differ in the terms and the assumptions of how they relate to one another, thus reflecting a wide range of contexts for noticing.

Mason (2002) viewed the practice of noticing as helping one to appreciate the complexities of teaching mathematics. He described awareness as a form of noticing and argued that awareness was required to make informed choices when responding to situations as they arise by acting in-the-moment. Mason (2011) talked about the discipline of noticing as acting-in-the-moment. He referred to a collection of techniques for pre-paring and post-paring to notice what happened, to select what is needed to be noticed and then to act freshly, rather than out of habit. The talent of noticing allows the teacher to "notice" the different ways in which students attend to the mathematical content.

Sherin, Jacobs, and Philipp (2011) regarded teacher noticing as a powerful construct of mathematics teaching that has its foundation in the question: "Where do teachers look, what do they see, and what sense do they make of what they see?" (p. 3). Sherin et al. (2011) emphasised the importance of noticing as central to student-centred mathematics learning, and they argued that teachers cannot act on students' ideas if they do not notice those ideas. Sherin et al. (2011) pointed out that using the concept of teacher noticing is not always consistent and that most researchers describe teacher noticing based on two facets: attending to, and making sense of, events in an instructional setting:

> … attending to particular events in an instructional setting. To manage the complexity of the classroom, teachers must pay attention to some things and not to others. In other words, they must choose where to focus their attention and for how long and where their attention is not needed … (p. 5)

Mason (2002) was instrumental in drawing attention to the idea teacher professional noticing. Teachers' noticing students' mathematical thinking accounts for much of the research in this area Bragg and Vale (2014) observed teachers' noticing students' mathematical reasoning; Choy (2016) introduced productive noticing when teachers' plan, teach, and review their mathematics lessons; Goldsmith and Seago (2011) used classroom artefacts to notice mathematical thinking; Jacobs and Empson (2016) researched one teacher's response to children's mathematical thinking; Levin, Hammer and Coffey (2009) found teachers' noticing as being more productive when teachers are working in a learning community than when working individually; LaRochelle et al. (2018) researched inservice teachers' initial professional noticing of students mathematical thinking; Star, Lynch and Perova (2011) used video to help teachers attend to noticing students' mathematical thinking; and, van Es (2011) developed a framework for noticing students' mathematical thinking.

Additionally, many studies have focused on PSTs' noticing students' mathematical thinking (Ivars, Fernández-Verdú, Llinares, & Choy, 2018; Callejo & Zapatera, 2017; Fernández, Llinares & Valls, 2012; Jacobs, Lamb, Philipp, & Schappelle, 2011; Simpson & Haltiwanger, 2016; Stockero, Rupnow, & Pascoe, 2017).

Scheiner (2016) identified the dynamic interactions of teacher noticing that offer further opportunities to explore the ideas related to teacher noticing. These include PSTs' professional learning (Anthony, Hunter, & Hunter, 2015; Ivars, Fernández, Llinares & Choy, 2018; Lee, 2019), teacher lesson preparation and task design (Choy, 2014; Choy, 2016; Lee & Choy, 2017), and teacher competency (Kaiser, Busse, Hoth, König, & Blömeke 2015). Use of videos has become a standard component in research involving teacher noticing because it enables reflection opportunities as well as collaboration in a professional learning community (Beswick & Muir, 2013; Fernández, Llinares & Valls, 2012; Kaiser, Busse, Hoth, König, & Blömeke, 2015; Miller, 2011; Rosaen, Lundeberg, Cooper, Fritzen, & Terpstra, 2008; Russ, Sherin, & Colestock, 2011; Star, Lynch, & Perova, 2011; Star & Strickland, 2008; van Es, 2011; van Es & Sherin, 2008; Walkoe, 2015). In this chapter, the literature examined present facets of teacher noticing relevant to this study.

2.8.1 Pre-service teachers' noticing mathematical thinking

Schoenfeld (1988) viewed mathematical thinking broadly. He defined learning to think mathematically as developing a mathematical viewpoint by valuing and applying the mathematical processes of mathematisation and abstraction. Schoenfeld's definition of mathematical thinking aligns with that of structural thinking. The literature discussed here refers to teachers' noticing mathematical thinking when interpreting students' task solutions, such as the study by Ivars, Fernández-Verdú, Llinares, and Choy (2018) who focused on fraction and pattern generalisation tasks (Callejo & Zapatera, 2017; Simpson; & Haltiwanger, 2016).

In their article about noticing students' mathematical thinking, Ivars, Fernández-Verdú, Llinares, and Choy (2018) introduced a hypothetical learning trajectory as a guide to improving PSTs' professional discourse. They asserted that teacher noticing would improve if the PSTs had a framework to follow or at least be given focus points about what to notice. They introduced a structured framework for PSTs to focus their attention on to notice students' thinking.

Ivars et al. (2018) identified verbal or written professional discourse as improving teachers' noticing expertise. They refer to the work of Mason (2011) who claimed that "noticing is a movement or shift of attention" (p. 45) and a mathematical thinking model proposed by Jacobs, Lamb, and Philipp (2010). This model, discussed further in Chapter 3, Section 3.8.1, is used to identify teachers' noticing skills through attending (A) to the details of students work, interpreting (I) students' mathematical understandings and deciding (D) how to respond to students based on their understandings. By integrating Mason's (2004) structures of attention and the Jacobs et al. perspective of noticing, teachers can focus their attention on noticing students' mathematical thinking.

Ivars et al.'s research involved 29 primary PSTs who attended a course on the teaching and learning of mathematics in primary school. Part of the course was on noticing students' fractional thinking. The researchers designed a hypothetical learning trajectory of the part-whole meaning of fraction as a guide for the PSTs to analyse students' thinking. The course comprised six sessions; the first two introduced the part-whole concept of a fraction; the PST participants completed some fraction activities and analysed students' work. In the last four sessions, the PSTs were introduced to a hypothetical learning trajectory for part–whole meaning of the fraction concept, and used the hypothetical learning trajectory to consider students' mathematical thinking. The hypothetical learning trajectory helped PSTs' professional discourse and linked the skill of noticing to the PSTs' mathematical content knowledge. The research showed that the PSTs' difficulty in interpreting students' mathematical thinking was related to weak mathematical content knowledge.

Stockero, Rupnow, and Pascoe (2017) described teachers' noticing students' mathematical thinking as a fundamental and critical responsibility of mathematics teachers. However, it is not something that PSTs or novice teachers easily do. In their research, 17 secondary mathematics PSTs with little to no teaching experience were involved in a professional learning program. The PSTs observed and videoed experienced teachers' interactions with students in the classrooms. The PSTs cited instances they engaged in noticing through analysis of the video recordings. A goal of the experience was to improve the PSTs' ability to notice students' mathematical thinking and to understand how noticing students' mathematical thinking supports students' learning. The results showed that the intervention was successful in developing PSTs' noticing skills. It was also evident that the PSTs were able to understand students' mathematical thinking at a high level.

Stockero et al. (2017) pointed out that focusing on one mathematical idea at a time may have constrained studies in noticing students' mathematical thinking. They noted that, although focusing on one event had its advantages, it is limited in its transferability to the classroom. Limiting noticing to small chunks of learning allows for an in-the-moment assessment on a specific aspect of students' mathematics thinking. However, it does not allow for a broader understanding of how students' think mathematically.

Callejo and Zapatera (2017) researched PSTs' competence in noticing students' mathematical thinking in pattern generalisation. Research participants were thirty-eight PSTs in the first year of their teacher education program studying pattern generalisation, number operations, operations with number, and divisibility. The PSTs were required to use their professional noticing skills to analyse three problems completed by three primary students. The PSTs had completed the same problems before and observed the students' responses.

The data were analysed in two phases. First, the PSTs described the students' answers and how they interpreted them. In the second phase, the researchers characterised the different stages of teaching competence. PSTs identified the different elements the students used to solve the problems but did not always use these elements to interpret students' understanding of pattern generalisation. The researchers identified five levels of competencies. These levels referred to the range of PSTs' competence to notice the degree of the students' mathematical understanding. The researchers reported that the PSTs' knowledge of mathematics did not guarantee competency in noticing skills. PSTs knew how to solve the problems but could not always identify the mathematical elements in the students' responses or interpret students' mathematical understanding. The researchers also reported that noticing students' mathematical understanding requires more than identifying what is correct or incorrect.

Simpson and Haltiwanger (2016) were interested in secondary mathematics PSTs' development, at differing stages of their studies, in noticing students' mathematical thinking. Thirty secondary mathematics PSTs, ranging from first year to final year in a teaching degree, volunteered to participate in the study. The researchers examined how participants made sense of professional noticing skills when observing their students' mathematical thinking. Data were collected from a two-phase study; phase one consisted of an open-ended questionnaire followed by a semi-structured interview in phase two. A subsample of phase one participants volunteered to participate in phase two of this research. In this mixed-methods study, Simpson and Haltiwanger implemented Jacobs et al.'s (2010) teacher professional noticing model to investigate ways that PSTs responded to students' mathematical thinking. Data from both phases were used to determine whether the PSTs made sense of the students' mathematical thinking, and in phase two, the researchers additionally analysed the participants' perceived weakness in analysing the students' mathematical thinking. During phase one, the PSTs reviewed three senior secondary students' mathematics work samples. The researchers analysed PSTs' responses by using an adapted rubric from the Jacobs et al.'s framework for professional modelling of noticing students' mathematical thinking. During phase two, the interviews were used to explore ways the PSTs attended to, interpreted, and responded to students' mathematical thinking and to determine the PSTs' self-perceived strengths and weaknesses concerning the process. Results indicated that final-year PSTs responded to students' thinking in significantly different ways than did PSTs who were early in their degree. Combining the data highlighted inconsistencies between how the PSTs made sense of students' mathematical thinking, as well as their self-perceived strengths and weaknesses.

In their research, Simpson and Haltiwanger (2016) found that final year mathematics education undergraduates could notice students' thinking better than could first-year PSTs and that PSTs can develop noticing skills. PSTs' strengths were reflected in their ability to understand students' mathematical thinking and their misconceptions. The PSTs felt they were open-minded about different approaches students use and that they had the mathematical content knowledge to know how students solve problems. The PSTs identified their weaknesses as a lack of confidence transitioning from a PST to a novice teacher due to a lack of PCK.

The studies of both Stockero et al. (2017) and Simpson and Haltiwanger (2016) into PSTs' learning to notice students' mathematical thinking led the researchers to conclude that PSTs do not have a natural gift or talent for noticing students' mathematical thinking. However, with supervision and guidance, the PSTs were able to do so. Simpson et al. discovered that final year PSTs had better noticing skills than did first year PSTs. Callejo and Zapatera (2017) identified what PSTs notice in students' mathematical thinking and demonstrated that simply knowing or understanding mathematics was not enough to recognise students' mathematical understandings. They found that noticing students' mathematical thinking required PSTs to understand what is meaningful in a student's thinking.

2.8.2 Teachers' noticing mathematical thinking

LaRochelle et al. (2019) investigated inservice teachers' initial professional noticing expertise. They believed that, by learning about teachers' noticing of children's mathematical thinking, they could better prepare professional development programs. LaRochelle et al. identified the lack of research about secondary teachers' professional noticing of students' mathematical thinking. Their focus was on secondary teachers' professional noticing of students' algebraic thinking in pattern generalisation. Generalisation was chosen because of its foundation in algebraic reasoning.

This study involved 16 mathematics teachers with between two and 30 years of teaching experience who participated in a professional learning project. Participants watched an 8-minute video of a mathematics lesson pertinent to the study. Before watching the video, the teachers solved a task similar to that undertaken in the video. After watching the video, the participants completed a written assessment in response to three writing prompts.

The video showed students aged 13 to 14 years, engaging in a pattern-building task. Students were given a diagram of a pattern and in small groups discussed the patterns they noticed, which they shared with the class. Some students identified the repeating pattern. The teacher in the video used this information to pose the main task for the class; that is, to create a formula for the pattern.

LaRochelle et al. (2019) used the Jacobs et al. (2010) AID model of teachers' professional noticing of students' mathematical thinking. The authors discovered that three-quarters of the teacher participants showed evidence that they could attend to the detail of the students' strategies in their mathematical thinking, one half could interpret students' mathematical understandings, and one-quarter could decide how to respond to students' mathematical understanding. Overall, teachers in this study noticed some of the students' mathematical understandings in their solutions but tended to make general claims by overemphasising these understandings. The authors found that the teachers tended to address other mathematical aspects of the students' learning and did not focus on their ability to make generalisations. After considering their results, the researchers suggested that further professional learning to support teachers' noticing abilities might include a framework of students' algebraic thinking to support teachers in developing their noticing expertise, working collaboratively on a mathematical task, and exploring different solutions.

Jacobs and Empson's (2016) case study research involved one primary school teacher, with 29 years' teaching experience, teaching fourth and fifth graders. In this study, the authors aimed to characterise teaching that is responsive to children's mathematical thinking. They defined responsive teaching as when teachers are continually making and adjusting their instructional decisions according to students' thinking. The data collected in this case study were from video-recorded one-on-one interviews with five children, a video of two mathematics lessons and field notes. The "moves" teaching framework was used to recognise four categories of teaching opportunities that extended the students' mathematical thinking. These are, ensuring the child understands the problem, exploring the child's strategy, encouraging other strategies and connecting symbolic notation.

In another study using video clips, van Es and Sherin (2008) examined how teachers' thinking changed as a result of their participation in a video club to learn to notice and interpret students' mathematical thinking. Seven elementary school teachers with experience ranging from one to 10 years participated in this study. Before each of the 10 video-club meetings, the researchers would videotape two of the teachers' classrooms, then select a five to seven-minute section of the videoed lesson for discussion at the next video-club meeting. During the video club, a researcher acted as the facilitator and prompted the teachers to notice aspects of the students' interpretations and understanding of mathematics and to use evidence to support their claims about students' thinking. The data from this research suggest that teachers developed their noticing skills, especially in conversations with peers in the video club. Noticing was different for individuals—some teachers noticed something that others did not. Van Es and Sherin found that, as individuals gained experience, they made more sense of situations as they arose.

Noticing was also identified by Choy (2014) as having the potential to improve mathematics teaching and learning. In a study of six mathematics teachers in a professional learning group, Choy investigated what mathematics teachers notice about students' mathematical reasoning through lesson planning. The concept of "productive noticing", was observed as teachers' noticing improved when the teachers considered the reasons for what they noticed. Choy found that teacher noticing is best utilised when responding to observations involving student interactions. Choy (2013) and Levin, Hammer, and Coffey (2009) identified noticing as being more productive when teachers are working in a learning community than when working individually. Using noticing helped teachers to understand student reasoning and assisted teachers to explore their teaching and in-the-moment responses to student thinking.

Choy (2016) introduced the FOCUS framework, which highlighted two aspects of productive noticing when teachers' plan, teach, and review lessons. One of these aspects referred to what to notice (focus on noticing), and the other referred to how to notice (pedagogical reasoning). Six teachers participated in this study as a mathematics lesson study group in which the teachers collaboratively planned a mathematics lesson. Through teachers discussing the design of a lesson on fractions, Choy illustrated how teachers' noticing could be analysed and represented through these snapshots. The snapshots proved useful in the analysis of teaching competencies of listening, responding, and reflecting on students' thinking. Additionally, the snapshots proved valuable in identifying teachers' noticing when interacting with students.

Bragg and Vale (2014) observed teachers' noticing students' mathematical reasoning in the *Mathematical Reasoning Professional Learning Research Program* [MRPLRP] conducted in Australia and Canada. This research program provided teachers with opportunities to notice students' reasoning and to develop their understanding of mathematical reasoning. Seventeen Australian and seven Canadian teachers participated in this research program, which consisted of an initial interview, observation of a demonstration lesson, audio-taped post-lesson group discussion, a trial of the same lesson, and a repeat of the program in the next term. The teachers received a detailed lesson plan several days before observing the demonstration lesson of 60 minutes duration designed for a Grade 3/4 class. Three to four participant teachers observed the demonstration lessons, completed an observation schedule and took notes while observing the students. After the demonstration, the participating teachers met for a group interview. The meetings consisted of a 30–40 minute audio-recorded group interview in which the teachers shared their observations of students' reasoning, lesson objectives, and teachers' actions that elicited reasoning.

From the observation schedule and group discussion data, the authors found that the teachers noticed students' language and conceptual understandings. Teachers' noticing of reasoning occurred when teachers were talking to each other about students' inability to express their reasoning. The teachers in the MRPLRP had the opportunity to meet as a collaborative learning community (Lewis, Perry, & Hurd, 2004). The results of this research showed that teachers could perceive reasoning in mathematics when the students communicated their thinking. Further research using this approach may enable opportunities for further professional learning about noticing students' reasoning.

A summary of the research in this section shows that teachers benefit from developing professional noticing skills. Jacobs and Empson (2016) showed that the teachers respond to children's thinking differently and that the teacher–student relationship is important in children's mathematical thinking. Van Es and Sherin (2008) showed how teachers learn to notice and interpret students' mathematical thinking through conversations with peers. Choy (2014) found that teachers' use of noticing is best when responding to observations involving student interactions. Choy (2016) highlighted teachers' productive noticing when planning, teaching, and reviewing lessons concerning what to notice and how to notice. Productive noticing was useful in analysing teacher competencies of listening, responding, and reflecting on students' thinking. The studies of Bragg and Vale (2014) and Lewis, Perry, and Hurd (2004) identified teachers' noticing explicit structural thinking as a type of reasoning in students' mathematical thinking. Lewis, Perry, and Hurd (2004) showed that teachers were able to perceive reasoning in students' communications of their thinking.

2.8.3 Noticing in teacher education programs

Anthony, Hunter, and Hunter (2015) trialled and evaluated new instructional strategies and tools used to support pre-service mathematics teachers at a New Zealand university. The aim was to observe if the PSTs' pedagogical practices that are used to engage learners in mathematical activities develop school students' thinking and reasoning. The research used several scenarios in which PSTs and teacher educators worked together to rehearse student and teacher interactions in a mathematics classroom. The authors in this study used rehearsals, as the pedagogical design, to view the complex notion of an ambitious teacher. The data consisted of 16 video recordings of in-class rehearsal and 16 audio records of post-rehearsal interviews. The rehearsal activity began with the PSTs' observation and analyses of the teacher educator's teaching. Next, the PSTs taught a group of peers, acting as the students, with the teacher educator acting as the coach. The modelling/observation process provided opportunities for demonstration and breakdown of the observed practice. The analysis concerned the identification and coaching of professional noticing. Instances of professional noticing identified were making students' thinking visible, eliciting an explanation, building on the reason, unpacking incomplete or erroneous responses, and connecting mathematical ideas.

Anthony et al. (2015) assigned roles to individuals in the rehearsal: the mathematics educator as the coach, and the PSTs were the teacher, school students, and observers. Through the rehearsal process, the mathematics educator, as the coach, helped the school students make sense of mathematical explanations. The acting teacher, observing this, used this knowledge to incorporate it in new instructions. The acting school student's mathematical knowledge and reasoning became expanded. This study demonstrated how the PST, as the rehearsing teacher, saw the student's thinking through gestures and active listening, enabling the PST to interpret the student's thinking.

This study involved only PSTs, but there were some other limitations to the study; such as, it was not conducted in a real classroom situation so there may have been other factors that a teacher would be noticing about student engagement. However, the study provided valuable insight into how PSTs notice aspects of mathematical thinking where the PSTs observed and talked about mathematical thinking, then explained and justified their reasoning. The researchers focused on the benefits for teacher educators to support learning opportunities for PSTs to develop meta-noticing skills.

2.8.4 Noticing and videos

Studies about teacher noticing in mathematics classrooms have centred around inservice and pre-service teachers watching video clips of classroom teaching, then being asked to comment on features of instruction that they noticed. Beswick and Muir's (2013) research simulated how PSTs would react in-the-moment to noticing students' mathematical thinking. Their research required PSTs to view a video excerpt of the teaching of a mathematical concept to identify the focus of the lesson and aspects of the teaching that were effective. Beswick and Muir concluded from the results of this research that PSTs struggle to see effective teaching beyond what is obvious.

Beswick and Muir did identify potential benefits of the use of videos to develop PSTs PCK. The authors explained that the mathematical knowledge required for teaching is different from mathematical knowledge used by mathematicians. Their description of mathematical knowledge for teaching has structure at its core. The definition of mathematical thinking is reflective of structure in its reference to connections, relationships and sequencing, "knowledge of ways in which mathematical concepts are represented, how they interconnect, the underpinning understanding upon which they depend, and their place in the overall development of mathematical competence." (p. 28).

Through observing a teacher in practice on video, the PSTs revealed the nature and scope of their PCK (Shulman, 1987). They attended to specific aspects of the teacher's role and the awareness of pedagogy, content, and how to teach the content required for teaching mathematics. From the reflection on the video experience, the PSTs were able to grasp the range of demands that require teachers' attention and decision making.

Beswick and Muir (2013) found that identifying teaching that facilitates students' mathematical understanding is not sufficient to understand what the teacher does. They found that the PSTs were interested in developing student understanding but struggled to identify it or how the teacher's actions contributed to it. PSTs required more time and reflection of the teacher's actions and utterances in order to notice students' mathematical understanding.

Van Es (2011) developed a framework for noticing students' mathematical thinking. In a video club, teachers analysed short videos of the teaching of content that the researcher felt demonstrated students' thinking. The facilitator used prompts to encourage teachers to notice students' thinking. From the results, she identified what teachers notice and how they notice using four levels: baseline, mixed, focused and extended. Through a professional learning community (video club), van Es investigated teachers' learning to notice students' mathematical thinking and how the learning community influenced teacher learning. Van Es stated that the professional learning community supports teachers' learning to notice, and teachers' thinking changed as a result of learning community discussions.

The research in this section deals with teacher noticing through the use of videos. In all cases the use of videos was a positive approach for teacher professional noticing. Beswick and Muir (2013) found that PSTs were able to notice structure in video-recorded examples of mathematics lessons. The video enabled the PSTs to observe and reflect on what they noticed. In the van Es (2011) video club, teachers were able to come together in a learning community to view videos of each other teaching. Through the video viewing, van Es was able to develop her framework of noticing students' mathematical thinking. However, the benefits for the teachers were observed through the support and professional learning advice gained from working in a professional learning community.

2.9 Studies concerning PSTs' attitudes and beliefs about teaching mathematics

This section pertains to the role of the PSTs in this study. The PSTs are inexperienced teachers. Their university courses have prepared them for the theoretical aspects of teaching mathematics to some extent, but their practical experience is limited. Given the PSTs are inexperienced teachers of mathematics and that many other factors influence their teaching, these factors may impact on the PSTs' ability to notice structural thinking. This section presents research that identifies some of these factors.

PSTs have set ideas about the teaching and learning of mathematics. These ideas may relate to an established understanding of mathematical content and how that content is taught. PSTs' beliefs about mathematics and mathematics teaching can impact significantly on their future practice. In their study, Cavanagh and Prescott (2007) involved 16 Graduate Diploma of Education students from two universities. The participants were interviewed individually for about 20 minutes before the start of the diploma course. The semi-structured interviews investigated the participants' memories as school students and their beliefs about teaching mathematics. Results indicated that the participants, in general, had no problems recounting their memories from school. In general, these memories recounted a traditional teacher-centred classroom in which the teacher was the authority and holder of all mathematical knowledge. Participants reported achieving excellent results in school, being in the higher-level mathematics classes, and that lessons were quiet and orderly. The participants indicated they were happy with the straightforward nature of the mathematics lessons and that they responded well to this style of teaching and achieved excellent results.

In a follow-up study, Prescott and Cavanagh (2007) tracked the same group of secondary mathematics PSTs, as this group were asked at the beginning of their professional experience how they would teach mathematics as a first-year teacher. Results from this study showed that the PSTs recognised many influences on their teaching. The PSTs regarded their supervising teacher's practices as similar to their own experiences from school. The pattern of the lessons also fitted to what they remembered in their mathematics lessons. The PSTs noted that the teacher was the focus of the lesson who delivered the work in a procedural approach of chalk and talk, and textbook exercises. The PSTs noted that the professional experience supervisors were more influential in shaping their teaching styles, particularly as the supervising teacher would be determining the PSTs' professional experience report. In respect to classroom management, most of the PSTs commented that textbook-based lessons were not useful in student learning but made lesson preparation easier. The PSTs maintained the authoritative role in the classroom and kept students on task with copying from the board to maintain classroom management.

When looking ahead to their first year of teaching, the PSTs believed that they would eventually follow the reform practices they had been introduced to at university, although they expected that other mathematics teachers would follow the traditional teaching approach. Prescott and Cavanagh (2007) pointed out the demands placed upon the PSTs during their professional experience, mainly because they were following the supervising teacher's approach.

Grootenboer (2006a) evaluated PSTs' teaching skills and content knowledge in a classroom setting. These experiences were valuable in shaping PSTs' views of teaching because they represented real experiences. Grootenboer explored the affective responses of 29 primary PSTs over their initial teacher education (ITE) program of 3 years. During the ITE program, the PSTs completed a 6- and 11-week school-based professional experience placement where they worked with an experienced teacher. Grootenboer adopted a mixed-method approach, with data collected on three occasions: before the commencement of the ITE program, during the first-year course of teaching, and during the school-based professional experience. PSTs were interviewed in small groups of between two and five people and were asked to describe their mathematical experiences during the corresponding period. A questionnaire was used to collect quantitative data during the three phases of the study.

The results from the quantitative analysis indicated that the PSTs' affective responses to mathematics after their tertiary course were more positive than their experiences as a student. However, their responses after the professional experience were not as high as they were after the tertiary course, but higher than their school experience. Results from Grootenboer's research reported a mixture of positive and negative affective experiences from PSTs' professional experience. Over 40% of the participants regarded their professional experience as positive, but nearly 50% considered their experience as not being so. Grootenboer found that the professional experience was crucial in the PSTs' development of attitudes toward mathematics. The experiences of some PSTs were that teaching mathematics reflected their beliefs as a student, which reinforced beliefs and attitudes held before commencing the ITE program. PSTs, who had a negative experience, worked in streamed classes and felt the lessons were routine. PSTs who had a positive attitude, in general, regarded the professional experience as though they were qualified teachers. This study highlighted the importance of the professional experience in developing PSTs' affective views of mathematics, and how a poor professional experience can be detrimental to the PSTs' positive tertiary experience.

2.10 Summary

Studies discussed in this chapter have highlighted the importance of developing students' structural thinking (Cooper & Warren, 2008; Lee et al., 2018; Mulligan & Mitchelmore, 2009; Papic et al., 2011). Procedures and calculations are preferred by many students and teacher. However, these skills do not lead to a structural understanding of mathematics (Blanton & Kaput, 2004; Ellemor-Collins & Wright; 2009; Vale, 2013). Researchers have reported teachers' ability to notice students' mathematical thinking (Ivars et al., 2018; Jacobs et al., 2010; La Rochelle et al., 2018). For students to develop structural thinking skills, teachers must be aware of structure (Mason et al., 2009).

An extensive literature search did not reveal studies that considered teachers' noticing of structural thinking. This new study was designed to fill the gap in the research to focus explicitly on pre-service teachers' noticing structural thinking. This study develops a framework for noticing structural thinking that reflects similar components (Ivars et al., 2018; LaRochelle et al., 2018). The CRIG framework's components of mathematical structure are: connections (C), recognising patterns (R), identifying similarities and differences (I) and generalising and reasoning (G). Mason's (2003) structures of attention reinforced the development of the CRIG framework and Mason's (2002) ideas on noticing support the PSTs' focus on structural thinking through the CRIG framework.

Chapter

3

Theoretical Perspectives

3.1 Introduction

Chapter 2 presented a review of the work of seminal theories related to relational thinking (Skemp) and structural thinking (Mason and colleagues). In this chapter, I discuss the development of a framework for mathematical structure (CRIG) and approaches to teacher professional noticing.

3.2 Components of the CRIG framework

The CRIG framework of mathematical structure was used in a small research study in teachers' understanding and use of mathematical structure, see Appendix C (Gronow et al., 2017). In that study, Gronow et al. used the theoretical framework developed by John Mason and colleagues. Mason (2004) claimed that in order "to appreciate learners' experience of mathematics it is vital to become aware first of how my own attention and that of learners is differently and variously structured at different times when focusing on mathematical ideas, problems and tasks" (p. 1). His personal awareness in thinking mathematically made him attentive towards the form and structure of the learner's mathematical thinking. Gronow et al.'s. study builds upon the five modes of what people attend to or notice, developed by Mason (2004) as 'structures of attention' when learning mathematics. Mason's (2004) structures of attention considered how one thinks mathematically. The extra demand for teachers is to know how to communicate their mathematical thinking so that others can make sense of it. That is, the teacher has to know how others think about the mathematics and what is happening when they are doing mathematics. Mason's (2004) five modes of structures of attention are given here:

- Holding wholes (gazing);
- Recognising relationships (part–part, part-whole);
- Discerning details (features and attributes);
- Perceiving properties (leading to generalisation); and,
- Deducing from definitions (axioms and definitions stated independently of particular objects).

Mason identified these modes as what teachers attend to, and then designed problems that embedded features of mathematical structure. He utilised diagrams or expressions to draw the learner's attention towards structural thinking skills to solve the problems. He did this by asking probing questions, such as: *What do you see when you look at this picture? What do you look at? What questions come to mind?* (Mason, 2004, p. 9.) Mason invites the learner to look for relationships between numbers, symbols and shapes. The learner is directed to reflect on the structure of the problem by considering the key components of connecting relationships, recognising patterns, identifying similarities and differences, and generalising and reasoning about a result. Mason viewed these key components from a *thinking mathematically* perspective and not from mathematics as procedural learning. He argued that these key components of structure need to be identified and promoted by the teacher as mathematical thinking skills that can support students' structural thinking. In noticing his own mathematical thinking, Mason raises the key question—*What do mathematics teachers think when communicating with learners?* (Mason, 2018).

Mason (2004) promoted structures of attention as what to be aware of and how to attend to thinking about mathematics. The same focus of attention can be applied to teachers when they consider structural thinking processes. These structures of attention are preferred over other frameworks of mathematical thinking as they promote thinking about mathematical thinking. However, for teachers in-the-field, they may not be easy to use. For teachers to consider metacognitive aspects of mathematical thinking, they need a framework that is easily accessible in the busy-ness of the mathematics classroom. Mason's modes are used here as they form the basis of the CRIG teaching framework that teachers can use to attend to their own or the students' structural understandings.

The CRIG framework is identified in Mason's (2004) forms. The first form of holding wholes (gazing) is connection because the reference here is between viewing or gazing by making connections to what is known, what can be seen or interpreted. The second form of recognising relationships is associated with patterning and equivalence. Mathematical relationships are often recognised within patterns and the concept of equivalence identifies similarity and difference. The third form, discerning details, also relates patterns and equivalences, and deeper thinking looks for relationships mostly identified in patterns or equivalences. The two forms: perceiving properties and reasoning, are identified as generalising and reasoning.

This framework was developed during Gronow et al.'s study. Observing the teachers required a guiding framework to notice structural thinking but Mason's structures of attention do not represent a workable model for teachers. Mason's underlying principles of structures of attention are used as the basis of the development of four components of mathematical structure that were developed as a framework to identify teachers' understanding and use of mathematical structure. These components represent practical and identifiable aspects of mathematics teaching and learning that a teacher needs to understand and use when teaching mathematics. The use of the CRIG framework in this study is considered essential in that it focuses the attention of the PSTs towards structural thinking. The PSTs can begin to understand their thinking and teaching through noticing the components of the CRIG framework.

3.2.1 Connections

Connections between contexts or concepts to knowledge allow for an informal process of mathematical understanding. Making connections with prior learning of mathematics supports students' reasoning. Albert (2012) recognised the importance of making connections between past, present, and future learning experiences and the knowledge acquired from those experiences. Albert encouraged teachers to ask questions of the students—questions that were beyond the students' level of development—while asking them to apply their prior knowledge. Albert found that when the teachers facilitated a connection with the students' prior knowledge in this way, the students could apply it to new problems. Connecting a context or concept to previous knowledge allowed for an informal process of understanding a situation. Making connections with prior learning of mathematics represents a component of structure that supports students' reasoning.

Structural thinking allows for flexible thinking, which encourages the development of connections between the different representations—a development that results in understanding. The different representations have connecting elements. These connections give fluidity between different situations so that knowledge learnt can be applied to other contexts. An example of this is a mathematical concept given in a visual form that moves toward a physical form through symbols and language. Albert gave the example of the numerals 1 to 10. They are represented as quantity in a visual or concrete form but can also connect symbols, written words, and sounds.

Jones and Bush (1996) noted that there is a limited understanding of how connections help students, but once made, there is a deepening of mathematical understanding. Vale, McAndrew, and Krishnan (2011) recognised that connections are fundamental to structural thinking, that connections between various representations of mathematics are central to learning, and that teachers of mathematics should be familiar with methods that can connect the different representations. The various forms of connections between the different mathematical representations are visual, symbolic, verbal, contextual, and physical. These forms provide different representations of how to teach content and the many ways in which learners may understand the required knowledge.

Beswick and Muir (2013) stated the importance of connections between mathematical topics. Teachers emphasised the effectiveness of using examples that placed emphasis on existing knowledge. Connections require the recalling of a piece of mathematical knowledge and then reapplying it or adapting it to a new piece of knowledge. The NSW Mathematics K–10 syllabus recognises that "students develop understanding and fluency in mathematics through inquiry and exploring and connecting mathematical concepts" (NSW Board of Studies, 2012). Connections in structural understanding of concepts occur by recalling and reapplying a fact, procedure, or method used in a new context, not only connecting content knowledge but also the procedures and concepts behind the content. In the case of a procedure, the student needs to connect each step involved when solving a mathematical problem coherently. The concepts taught might appear with one piece of content knowledge, but students need to see how it can connect to new content. When the teacher can associate an example with prior experience, that teacher is making connections, reinforcing the students' understanding. The NSW Mathematics K–10 syllabus has as one of its outcomes for working mathematically that a student "communicates and connects mathematical ideas using appropriate terminology, diagrams and symbols"—working mathematical outcome MA4-1WM, (NSW Board of Studies, 2012).

3.2.2 Recognising patterns

Recognising patterns identifies the importance of patterning, awareness of patterns, and reproducing patterns as essential for mathematical development. The association with structure generates mathematical knowledge and understanding. Recognising patterns occurs as an innate observation of the natural world. Children can recognise and observe patterns before reaching school. Once introduced to mathematics at school, children use patterns through formalised learning processes that follow the content strands of the syllabus. Patterning is identified extensively throughout primary and junior secondary education (early Stage 1 to Stage 4) to support student mathematics learning. In the NSW mathematics syllabus (NSW Board of Studies, 2012), patterns are associated with the *Number and Algebra* strand with its stated aim to “develop efficient strategies for numerical calculation, recognise patterns, describe relationships and apply algebraic techniques and generalisation”. Early Stage 1–Stage 3 has the content strand of *Patterns and Algebra* that has recognising patterns as an outcome for students’ mathematical learning, and in Stage 4, the *Number and Algebra* content strand has “create and displays number patterns” (MA4-11NA) as an outcome. The overarching statement reads: “Students develop efficient strategies for numerical calculation, recognise patterns, and describe relationships” (NSW Board of Studies, 2012, p. 18).

The importance of patterning, awareness of patterns, and reproducing patterns is considered essential for mathematical development, which can be associated with structure because it generates mathematical knowledge and understanding. Papic et al. (2011) found that awareness of patterning and structural relationships is essential in mathematical learning. They found that pre-schoolers’ identification of pattern structure was crucial in concept formation in future learning.

Mulligan and Mitchelmore (2009) identified structural thinking in young students as a requirement for developing mathematical competence. They proposed a construct, namely awareness of mathematical pattern and structure (AMPS) that was common across mathematical concepts. Patterning was found to be essential in developing mathematical understanding of other concepts. Generalising here relates to one aspect of patterning.

Stephens (2008) noted that structural thinking is much more than simply seeing a pattern. Merely recounting a pattern without the ability to replicate it is not demonstrating awareness of the property. The ability to generate a pattern and applying it to other examples illustrates a feature of structural thinking.

3.2.3 Identifying similarities and differences

Identifying similarities and differences is primarily built on sorting and classifying objects into like or unlike categories. Equivalence, as a different notion of sameness, can also develop through experience and extends into more subtle differences in mathematical representations. Early mathematics includes making decisions about similarities and differences—whether things are equal or unequal, bigger or smaller—and how to recognise these differences. Identifying similarities and differences is primarily built on sorting and classifying objects into like or unlike categories. Equivalence, as a different notion of sameness, can also develop through experiencing and extends into more subtle differences in mathematical representations. Warren and Cooper (2009) identified primary school children as often misrepresenting the equal sign by not identifying the symbol as representing sameness, but as an operator, meaning to do something, just as an addition sign means to sum. They noted that this confusion carries through to secondary and tertiary studies, which affects overall mathematics learning.

Identifying similarities and differences as a component of structure is essential in developing students' deeper structural awareness. It helps reveal that essential features of mathematical ideas persist despite their various forms, which empowers one to consider similarities and differences when regarding other concepts. Jones and Bush (1996) recommended visual forms of diagrams, charts, tables, mind maps, and flowcharts as sources that can be used to help develop structure. Students can identify similarities and differences as concepts developed through hierarchies that allow for the addition of more complicated concepts.

3.2.4 Generalising and reasoning

Of all the components of the CRIG framework, the combination of generalising and reasoning is the most universal. Generalising and Reasoning as Mason (2008) described, generalising is an innately human activity that should be built upon to develop a more in-depth and exciting experience of mathematics. This component encompasses the other three components as a higher-order thinking skill. Through identifying similarities and differences, one can recognise patterns connected to a general rule.

Concept formation is a process involving generalising because of the interaction between a concrete and abstract association with structure. Albert et al. (2012) indicated that conceptual mastery is an ability to generalise learning from one situation and extrapolate it to a different situation, or the "transition from one structure of generalisation to another" (p. 21). Unlike connections, generalising involves developing a *what happens next* scenario.

Mason et al. (2009) wrote that appreciation of structure involves the experience of generality. The NSW mathematics syllabus K–10 (NSW Board of Studies, 2102) *Number and Algebra* content stipulates that "students develop efficient strategies for numerical calculation, recognise patterns, describe relationships and apply algebraic techniques and generalisation". Warren and Cooper (2008) found that generalising to a real-world problem was an essential aspect of identifying structural thinking. Watson and Mason (2005) focused on learners generating their mathematical examples from given situations. They identified these as examples of anything the learner might be able to take from a given situation to generalise into a new idea. They stress that, although each mathematical problem has its particularity, the solution process evolves into the generalisation.

Stephens (2008) applied structural thinking to designing arithmetic questions. He explained that structural thinking involves being able to go from several instances of the same thing and then being able to generalise the property. He asserted that children could articulate a generalised structural principle underlying a whole problem.

The identification of the CRIG framework is vital in this thesis because it forms the basis for PSTs' awareness of structural thinking. PSTs can acknowledge the CRIG framework in communication with colleagues or with students in the classroom. The framework enables them to be attentive to structure in their pedagogical practice when teaching mathematics.

In the next section, I will outline the significance of noticing as a construct that will be used as the lens to identify the PSTs' understanding and use of structural thinking through CRIG framework.

3.3 Noticing structural thinking

The framework of noticing supports the identification of structural thinking in mathematics. Noticing in this respect can be viewed from two perspectives: that of the researcher, and that of the PST. Scheiner (2016) identified how the complex process of noticing takes in various forms and its application is not restricted to a single process. Chapter 2 presented a review of previous studies that have focused on the other forms of teacher noticing.

The PSTs' noticing of structural thinking is a form of metacognitive awareness of their mathematical thinking. Mason (2002) and Schoenfeld (1992) identified that teachers need to reflect on their mathematical thinking to be able to explain mathematics adequately. The PSTs' language that refers to mathematical structure in their pedagogy identifies their use of structural thinking. Mason (2002) asserted that "every act depends on noticing" (p.7), and he used the term "awareness" to characterise the ability to notice. Mason referred to noticing as an awareness of what one is attending to. In this study, noticing structural thinking implies an awareness of understanding and using mathematical structure. Mason (2002) alleged that the practice of noticing helps appreciate the complexities of teaching mathematics, and that awareness, as a form of noticing, is required to make informed choices when responding to these complexities. Mason termed this as "in-the-moment" and recognised that when acting in-the-moment people may not always be aware of what they are attending to.

Scheiner (2016) pointed out that not all we attend to is consciously perceived, and Lamme (2003) argued that there are various levels of noticing what we do. Noticing can include implicitly attending to an action or behaviour without awareness of it. Equally, we can explicitly attend to an action or behaviour with complete awareness. Scheiner pointed out that teacher noticing is not just about what one sees through the teacher's eyes, but also includes what teachers are noticing with their own "mind's eye". Choy and Dindyal (2017) believed that developing teachers' eyes to see and developing their minds to make sense of mathematical relationships, are crucial for developing learning experiences. Teacher noticing is essential for teachers' awareness of what they attend to and the sense they make of what they notice.

By adopting Mason's (2002) approach to noticing, the development and use of mathematical structure has emerged as a form of directing PSTs' attention to deep mathematical thinking. Mason's (2003) structures of attention considered how one thinks mathematically. The extra demand for teachers is to know how to communicate their mathematical thinking so that others make sense of it. That is, the teacher has to know how others think about the mathematics and what is happening when they are doing mathematics: "When teachers are themselves thinking mathematically, whether alone or collectively, there is an ethos and a sensitivity to learners that fades when teachers stop doing mathematics themselves" (Mason, 2018, p. 333).

For the PSTs in the forthcoming study, this awareness comes from the ability to notice structural relationships in their own mathematical thinking. Once they recognise and understand this, they can include these mathematical understandings in their teaching. It follows that teachers should aim to make understanding clear and to align their structural thinking with students' thinking and understandings (Krupa, Huey, Lesseig, Casey, & Monson, 2017).

3.4 Models of teacher noticing

In this section, I outline two models of noticing that have been developed by groups of researchers focused on teachers' professional noticing. These two models are similar, and have consistent approaches to noticing, but differ in their research focus. These two models involve the approaches taken by German and American mathematics educators. In America, the approach to noticing by the team of Jacobs et al. (2010) dealt with the concept as professional noticing of children's mathematical thinking, the AID model. The German researchers, Kaiser et al. (2015), introduced noticing of situation-specific skills as teachers' professional competencies, the PID model.

In the design of this study (see Chapter 4), both models are considered in terms of noticing structural thinking observed through the PSTs' understanding and use of the CRIG framework. This noticing can be observed in the PSTs' thinking and behaviour or in observing students' thinking and behaviour when doing mathematics.

3.4.1 Attending, Interpreting, and Deciding (AID)

Van Es & Sherin (2002) introduced noticing as a framework for assessing teachers' competencies. They described this aspect of expertise as "attention-dependent knowledge". Three critical components of teachers' ability to notice were proposed: noticing what is essential when teaching, noticing connections between specific classroom interactions involving teaching and learning, and teachers' noticing what they know about their teaching.

Jacobs et al. (2010) conceptualised professional noticing of children's mathematical thinking through an in-the-moment decision-making process. The practice of in-the-moment decision was a result of teachers making responses to children's mathematical responses. Teachers attended to children's strategies, interpreted the students' understandings, and decided how to respond to these understandings. Teachers' attention to children's strategies were generalised to a set of three interrelated skills identified as the AID model of attending (A) to children's strategies, interpreting (I) children's understandings, and deciding (D) how to respond based on the children's understandings. The AID model is a framework for professional noticing of children's mathematical thinking.

Jacobs et al. used data from the Studying Teachers' Evolving Perspectives (STEP) project to study teachers' professional noticing in a professional learning program (PLP) that focused on children's mathematical thinking. There were 131 practising elementary teachers who differed in their teaching experiences and a group of PSTs who attended five full days of workshops throughout the year. The workshops occurred before the study with goals to help teachers learn about how children think and develop mathematical understandings and how teachers' use this knowledge to inform instruction. In the workshops, teachers analysed videos and written samples of students' work and explored mathematical concepts and teachers' mathematical understanding of these concepts and how these understandings inform instruction. A goal of professional development was to help the teacher learn how children think about and develop their mathematical understandings. Additionally, teachers had opportunities to respond to and support students' understandings.

The authors developed a written assessment to capture participants' professional noticing expertise. Participants watched a video of an interview between a teacher and kindergarten child and then wrote their reaction. From the responses of watching the video, two skills were identified: deciding how to respond to the student's understandings and attending to the student's strategies. Teachers collaborated in developing their framework based on the children's strategies. During the professional development workshops, the teachers were solving mathematical problems, researching and analysing video and written work samples from their classrooms as well as those provided by the facilitator. Together the teachers worked to make sense of the students' thinking. For each artefact, the teachers were asked to write a response to the three prompts of attending, interpreting, and deciding-how-to-respond. The attending prompt asked the teachers to describe what the child did in responding to the question. For the interpreting prompt, the teachers were asked to explain what they had learned from the children's understanding, and for the final deciding prompt, the teachers were asked to pose a problem for the child to do next.

Analysis of the three professional noticing skills of attending, interpreting, and deciding revealed that teaching experiences supported individual development in attending to children's strategies. Teachers were able to interpret children's understanding but were not necessarily able to decide how to respond to children's thinking. However, professional development did provide support for developing skills in all three areas.

From the data collected, the researchers were able to recognise that decision making, when deciding how to respond, was focused on extending the students' understanding rather than helping them to solve the problem. For the connection between participants' expertise in deciding how to respond and attending to the children strategies, the results indicated that attending to the children's strategies was the basis for deciding how to respond to children's understandings. The researchers argued that teachers need support in learning how to attend to children's strategies and that deciding how to respond to student learning remains central. The findings helped clarify the skill of teacher noticing and the varying levels of expertise; however, given time, teachers do learn to develop and improve their noticing skills.

3.4.2 Perception, Interpretation and Decision making (PID)

Researchers in the area of teacher competency have used teacher noticing as a theoretical framework. Blömeke, Gustafsson, and Shavelson (2015) detailed an amalgamated view of competency as part of a continuum of dispositions and performance. Their continuum model consisted of disposition, situation-specific skills, and performance, with the situation-specific processes of perception, interpretation, and decision-making being the intercession between disposition and performance. Kaiser, Busse, Hoth, König, and Blömeke (2015) investigated teacher competency concerning teacher PCK in their follow-up international study of Teacher Education and Development Study in Mathematics (TEDS-M) called TEDS-FU. They incorporated new and innovative ways to assess teachers' competency. These ways included a cognitive-affective component and situation facts. Video-based situation evaluation was used to capture teachers' noticing classroom events. A situation-specific component of the Blömeke et al. model was used to capture teacher noticing as a trait of perception, interpretation, and decision making (PID). This PID-model comprised perception of particular events in the classroom, interpreting these perceived events, and decision-making as anticipating a response to students' activities or as proposing alternative instructional strategies.

Kaiser et al. (2015) were interested in teacher competencies and making a comparison between expert and novice teachers. They found that the construct of noticing is vital for distinguishing novices from experts. They saw noticing in line with Sherin, Jacobs, and Philipp (2011). However, Kaiser et al. identified noticing through the PID-model, which was closely connected to the approach by Blömeke et al., which Kaiser et al. had used in their teacher competency research.

The PID model identifies how teachers can selectively perceive (P) events in the class and draw on existing knowledge to interpret these events. Interpretation (I) of what they perceive is a skill that experts have over novices. Finally, based on perception and interpretation, teachers have to make a decision (D) about what to do based on what they perceived and how they interpreted it, either as anticipating a response to students' activities or as proposing alternative instructional strategies.

This PID-model comprises a broad understanding of noticing and does not limit noticing to particular incidents. In this model, noticing comprises all essential aspects involved in achieving quality mathematics teaching.

3.5 Summary

In this chapter, I discussed the theoretical perspectives of this study. Skemp's instrumental and relational understanding introduced a new approach to teaching mathematics that challenged traditional methods. Mason, Burton, and Stacey provided educators with mathematical thinking in new ways. The four components of the CRIG framework of mathematical structure are given for teachers to use to promote structural thinking. The theory of noticing discussed in terms of Mason's in-the-moment approach and two noticing models as options for PSTs' noticing of structural thinking.

The two noticing models identified here provide a lens for PSTs to notice structural thinking. Although established for different reasons, noticing students' mathematical thinking and identifying teacher awareness of structural thinking is the approach taken in this study because it combines both processes. The focus on the PSTs' noticing structural thinking represents, first, the development of the PSTs' professional competence through learning about the CRIG framework. The second focus is that the PSTs' noticing of structural thinking begins with the PSTs' awareness about how people process mathematical concepts and procedures.

The use of these models in this study represents the flexibility of the construct of noticing. Scheiner (2016) indicated that noticing takes many forms and in this study the form is on PSTs' noticing of structural thinking. To that end, the PSTs' noticing involves a metacognitive approach to their mathematical thinking. Ivars et al. (2018) pointed to the need for PSTs to have a specific framework to support their professional noticing so in the present study the PSTs learn about mathematical structure through the CRIG framework. Studies have identified noticing students' mathematical thinking in various forms, but noticing structural thinking through the CRIG framework is unique to this study. Given that the components of the CRIG framework are specific, identifiable and relate to the mathematical relationships, the PSTs' ability to notice these components is a strength of this study. The PID and the AID noticing models were used as guides in this study to support PSTs' noticing of structural thinking through the CRIG framework. However, ultimately, the PSTs are noticing the CRIG framework with or without these noticing models. Their ability to identify the components of the CRIG framework is evident not only in their communication of mathematical content to the students during the mathematics lesson, but also in their pedagogical content knowledge, especially when asking students questions and listening to their responses.

Chapter

4

Design and Methodology

4.1 Introduction

In this chapter, a description is given of the design and methodology of this study. This comprises descriptions of the study's context, the participants, ethical considerations and the role of the researcher. The remainder of the chapter describes the data sources, research process, the instruments and methods for collecting and analysing the data.

4.2 Research design

The design of this study follows the design-based research approach of Cobb, Confrey, DiSessa, Lehrer, and Schauble (2003). Cobb et al. (2003) described design experiments as pragmatic and theoretical; that is, the experiment is engineered to develop and support forms of learning. In regard to the present study, the experiment undertaken relates to observing and developing pre-service teachers' learning about mathematical structure. This design comprises of a professional learning program (PLP) aimed at PSTs learning about mathematical structure to enhance their PCK. The CRIG framework, described in Chapter 3, is introduced to provide the PSTs with strategies to develop a deep understanding of mathematical structure and to notice structural thinking when teaching and learning about mathematics.

Swan (2014) described design-based research as transformative. In this study, noticing structural thinking is presented as both a progressive and transformational approach to mathematics teaching and learning. For growth and change to occur, the PSTs learn to notice structural thinking through their awareness and application of the CRIG framework during their planning, teaching, and review of their teaching practice.

4.3 Participants

Phase 1 participants, known as Ms S and Ms N volunteered for this study after an open invitation to all ORS (Opening Real Science) project participants (Opening Real Science, 2015). Ms S and Ms N were final-year students in the Bachelor of Arts/Bachelor of Education degree at Macquarie University Sydney in 2017. Recruitment for participants in Phase 2 of this study began with an open invitation to all final-year secondary mathematics students enrolled in the Bachelor of Arts/Bachelor of Education (Secondary) program at Macquarie University in 2018. Three of these PSTs, referred to as Ms K, Ms M, and Mr T, agreed to participate in the study that would take place during the final professional experience component of their degree. Table 4.1 contains background information on the PSTs taken from the introductory questionnaire.

Table 4.1
Phases 1 and 2 PST Participants' Background Information

Phase	PST	Age (years)	Level of mathematics studied in the final year of secondary school in NSW[a]	Professional experience teaching mathematics	Other experience in mathematics	School age taught
1	Ms S	32	General Mathematics	4 weeks	ORS project[b]	7–11 yo
	Ms N	23	Mathematics (2 Unit)	4 weeks	ORS project[b]	5–11 yo
2	Ms K	21	Extension 1	4 weeks	Tertiary mathematics	12–16 yo
	Ms M	23	Extension 2	4 weeks	Tertiary mathematics	10–16 yo
	Mr T	45	Extension 2	4 weeks	Bachelor of Engineering	12–16 yo

[a] General Mathematics is a non-calculus course, Mathematics (2 Unit) is a calculus-based course and Extension 1 and Extension 2 are advanced calculus-based courses.
[b] ORS (Opening Real Science) project participants (Opening Real Science, 2015)

All PSTs were allocated to a participating New South Wales Department of Education school for their professional experience by Macquarie University's Professional Experience Office. Table 4.2 presents details of the schools where the PSTs were placed for their professional experience.

Table 4.2
PSTs' Professional Experience Placements

PST	No of Professional experience weeks	Sydney Metropolitan Location	School Gender	Stage	Class allocated	Age in years	Students in class
Ms S	4	North	Co-educational	1	Year 1	6–7	22
Ms N	4	North	Co-educational	1	Year 1	6–7	23
Ms K	6	North	Co-educational	5	Year 9	14–15	21
Ms M	6	Outer west	Co-educational	4	Year 8	13–14	18
Mr N	6	Inner west	All Girls	4	Year 7	12–13	20

4.4 Ethical considerations

In accordance with ethical guidelines, this study was approved by the Macquarie University Faculty of Human Sciences Ethics Committee (Reference number 5201600943). Appendix A contains the ethics letter issued by the committee. Because the research was to take place in NSW Department of Education schools, additional ethics approval was sought and obtained through the *State Education Research Applications Process* (SERAP; Reference number 2016595). Appendix B contains the SERAP approval letter.

Before the commencement of each phase, all participants were informed about the study, its duration, and their involvement. Signed consent was obtained from the PSTs, principals of the professional experience schools, professional experience supervising teachers, the students' parents/carers in Phase 1, and the students and their parents/carers in Phase 2 (Appendix E).

4.5 Role of the researcher

In this research, my role was as a participant researcher. Cohen, Manion, and Morrison (2013) indicate that the researcher's involvement in the design, implementation, and interpretation of a study is a feature of the qualitative interpretive research paradigm. Goodchild, Fuglestad, and Jaworski (2013) adopted a study design that involved a professional learning community that involved establishing a relationship between the researcher and the research participants. In this study, I was the researcher, but the study design, based on the PLP allowed for a community of learning to emerge. In this study, I acted as a mentor to the PSTs because, as an experienced mathematics teacher, I was able to offer advice about the mathematical content to be taught. Moreover, I acted as the facilitator during the cycle at the same time as I taught the PSTs the concepts of mathematical structure and structural thinking and provided instruction about how to include structural thinking in their pedagogical practice. I continually reviewed and redesigned the investigation to support the participants' understanding and use of structural thinking. In this research, I was active in working with the PSTs to improve their understanding of structural thinking and their awareness of using it when teaching.

Acting as the sole researcher meant that researcher bias was a consideration with respect to my observations of practice, data coding and analysis. The process employed to counter any researcher bias involved three experienced mathematics education professionals acting as interrater coders for the data analysis. Section 4.9 details how the data were coded and the process of interrater reliability to ensure the validity of the coding process was effective for this study.

4.6 Study design

The study involved two phases; each phase began with an introductory questionnaire, followed by a PLP of three cycles. Each cycle consisted of a professional learning workshop (PLW), PSTs' planning and teaching a mathematics lesson, a post-lesson interview for the primary PSTs and a post-lesson reflection for the secondary PSTs. At the end of the third cycle of the PLP, the PSTs completed an exit questionnaire. Figure 4.1 shows flowcharts representing the study design for each phase. While the design of each phase was similar, some modifications were made in Phase 2 after a review of the Phase 1 experience and unforeseen circumstances. Phase 1 took place between 14 July and 11 August 2017, and Phase 2 took place between 24 April and 27 August 2018.

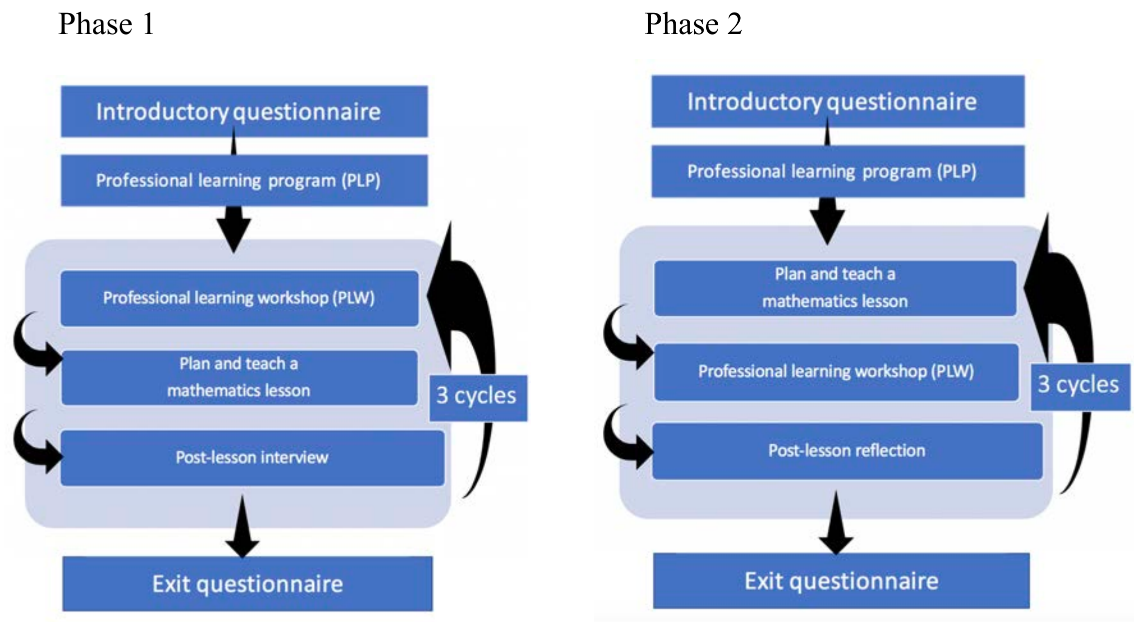

Figure 4.1. Study design in relation to data collection procedures.

4.7 Data sources

The data collected from each of the sources are, introductory and exit questionnaire results, audiotaped PLWs, lesson plans and videoed mathematics lessons, audiotaped post-lesson interviews or reflections from the three cycles of the PLP.

4.7.1 Introductory questionnaire

For both phases, PSTs completed an introductory questionnaire focusing on their educational background and their beliefs about, and attitudes toward, mathematics education and mathematical structure. Table 4.3 contains the introductory questionnaire questions in conjunction with an indication of how those questions were aligned to the research questions. Item 9 responses were measured as a score on a zero to 100 scale, all other items involved written responses.

Table 4.3

Introductory Questionnaire Questions Aligned to Research Questions

Nº	Question	Research question
1.	What is your name?	Participant background information
2.	What is your age?	
3.	What is your gender?	
4.	What degree are you completing at Macquarie University?	
5.	What stage of your degree are you at?	
6.	What other mathematics courses/degrees have you completed?	
7.	What experience have you had teaching mathematics?	
8.	At what primary school stage level according to the NSW syllabus have you taught mathematics?	
9.	What is easy/difficult about teaching mathematics? • Lesson preparation • Creating an engaging lesson • Teaching strategies that engage students • Engaging all students in the activities • Students' understanding of mathematics	
10.	How do you describe your teaching strategy when teaching mathematics? (You can choose more than one): Procedural, Teacher centred, Inquiry based, Collaborative learning, Conceptual	
11.	What is mathematical thinking?	What is the PSTs' understanding of structural thinking?
12.	Give an example of a students' mathematical thinking?	
13.	What is mathematical structure?	
14.	Give an example of mathematical structure.	
15.	What is structural thinking?	
16.	Give an example of structural thinking.	

The introductory and exit questionnaires were conducted online using Qualtrics online (Qualtrics, 2018). The introductory questionnaire link was delivered to the PSTs before the first workshop in both phases. Participants were provided with the following definition of mathematical structure and structural thinking (see Gronow, Mulligan, & Cavanagh, 2017):

> Some authors describe mathematical structure as the building blocks of mathematical learning. Mathematical structure can be found in connecting mathematical concepts, recognising and reproducing patterns, identifying similarities and differences, and generalising and reasoning results. Students who perform structural thinking use these skills without always considering them when solving problems. Many students need to be taught these skills when introduced to concepts as a reminder of how to think mathematically.

4.7.2 Professional learning program (PLP)

This section describes each data-collecting element in the three cycles of the PLP for each phase of the study. Participants in the PLP were the two primary PSTs in Phase 1 and three secondary mathematics PSTs in Phase 2.

4.7.2.1 Professional learning workshops

The PLWs were recorded on a personal recording device (iPhone 6 mobile phone) and, subsequently, these audio recordings were professionally transcribed.[1] Phase 1 PLW occurred at the beginning of each cycle. In Phase 2, the PLW for Cycle 1 occurred at the beginning of the cycle, and for Cycles 2 and 3 the PLW occurred at the end of the cycle. The agenda of each PLW is detailed in Table 4.4, on the next page, with an appendix reference for relevant documents.

[1] Transcribing was organised through www.transcribeme.com.

Table 4.4

Agenda for Each Professional Learning Workshop

Phase	Agenda
1	1. Professional learning workshop 1
	1.1. Presentation of the CRIG framework (Appendix F)
	1.2. Videoed lesson - Teaching channel video
	1.3. Arithmetic number sentences worksheet (Appendix G)
	1.4. Prepared lesson plan 1 (Appendix H)
	1.5. Lesson plan template distributed (Appendix I)
	2. Professional learning workshop 2
	2.1. Review of interview questions
	2.2. Mathematics lesson reflection (Ms N Lesson 1)
	2.3. Prepared lesson plan 2 - (Appendix G)
	3. Professional learning workshop 3
	3.1. Mathematics lesson reflection (Ms S Lesson 1, Ms N Lesson 2)
2	1. Professional learning workshop 1
	1.1. Presentation of the CRIG framework (Appendix F)
	1.2. Videoed lesson - Teaching channel video
	1.3. Written observation of teaching channel video
	1.4. Prepared lesson plan 1 (Appendix J)
	2. Professional learning workshop 2
	2.1. Noticing structural thinking video
	2.2. Arithmetic number sentences worksheet (Appendix G)
	2.3. Mathematics reflection (Ms K Lesson 2)
	2.4. Prepared lesson plan 2 (Appendix J)
	3. Professional learning workshop 3
	3.1. Mathematics reflection (Ms K Lesson 2, Ms M Lesson 2, Mr T Lesson 2)
	3.2. Pedagogical Content Knowledge (PCK) worksheet (Appendix K)

Phase 1 Cycle 1 professional learning workshop

In the Phase 1 Cycle 1 PLW, I introduced the CRIG framework and how it related to working mathematically processes of the NSW K–10 mathematics syllabuses (see NSW Board of Studies, 2012). A Teaching Channel (Teaching Channel, 2017) video was shown. The teaching channel is an American-based organisation intended to provide professional learning for teachers. This video titled *Reasoning About Addition Through Related Problems* (https://www.teachingchannel.org/videos/reasoning-about-addition-nsf) was of a Grade 1 class (6- and 7-year-old students) being taught to add numbers. The video was chosen because teachers' pedagogical approach to teaching addition unknowingly uses the CRIG framework. The teacher in the video embedded the components of the CRIG framework in the lesson through instructions, questions, and prompts. In a discussion that followed, the PSTs and I considered how the teacher used the CRIG framework and what the students' responses were.

The PSTs were given an arithmetic number sentence activity adapted from Stephens (2008), designed to invoke the PSTs' structural thinking capabilities. The problems involved the PSTs using structural thinking to look for patterns in the number sentences. In the final part of this PLW, we reviewed a prepared lesson plan of a Stage 1 topic from the number and algebra strand of the NSW K–10 Mathematics syllabus (see NSW Board of Studies, 2012). The lesson topic involved adding numbers in groups of 10. A lesson plan template was given to the PSTs to use for preparing their mathematics lessons.

Phase 1 Cycle 2 professional learning workshop

In the second PLW, the PSTs were asked for feedback given from the Cycle 1 post-lesson interview questions. A video segment of Ms N's first mathematics lesson was shown, and a conversation followed about Ms N's use of the CRIG framework. A second example of a mathematics lesson plan with the CRIG framework was presented. This lesson incorporated the Stage 1 strand of number and algebra of the NSW K–10 mathematics syllabus (see NSW Board of Studies, 2012) using a problem-solving activity of "squares in a grid." The activity was modified from a Stacey and Groves' (1985) example and was intended to examine structural thinking through recognising patterns and generalising the result.

Phase 1 Cycle 3 professional learning workshop

The final PLW involved the PSTs watching, then discussing, the use of the CRIG framework in two video segments from each of the PSTs' mathematics lessons.

Phase 2 Cycle 1 professional learning workshop

In the Phase 2 Cycle 1 PLW, the CRIG framework presentation was shown, then a viewing of the same Teaching Channel video shown to the primary PSTs, titled *Reasoning About Addition Through Related Problems* (Teaching Channel, 2017). After the video, the PSTs completed an online questionnaire (Qualtrics, 2018) about when the teacher made reference to the components of the CRIG framework and how the teacher encouraged students to look for the components during his communication.

In the next part of the workshop, the PSTs reviewed and discussed a prepared lesson plan for a Stage 4, Year 7 (12–13-year old) class. The lesson was about the subtraction of integers, taken from the Stage 4 number and algebra strand of the NSW K–10 mathematics syllabus (see NSW Board of Studies, 2012). PSTs were asked to describe how their experiences of learning to subtract directed numbers differed from the lesson plan.

Phase 2 Cycle 2 professional learning workshop

In the second PLW, the PSTs were asked to read the article *Equivalence and Relational Thinking: Opportunities for Professional Learning* (Vale, 2013). The article identified students' solutions to missing number problems using relational thinking or as a balance or transformational strategy. The PSTs viewed a video recording of a child attempting several different arithmetic problems. The PSTs were asked to identify the child's mathematical thinking as balance, transformational, or relational/structural when solving the arithmetic problems and then comment on the thinking the child undertook when solving the problems.

The next part of the PLW involved the PSTs doing the arithmetic number sentence (ANS) activities that the Phase 1 PSTs had completed in their PLW 1. The third component of the workshop involved viewing Ms K's video segment from her mathematics lesson during Cycle 1. The PSTs were reminded about the AID teacher model (attending, interpreting, deciding) of noticing and were asked what components of CRIG framework were used. The final component of the PLW comprised a review of a prepared mathematics lesson plan. This was a Stage 4 geometry strand of the NSW K–10 mathematics syllabus (see NSW Board of Studies, 2012) Year 7 (12–13-year-old) lesson on the properties of geometrical figures, tessellating squares, and rectangles.

Phase 2 Cycle 3 professional learning workshop

In the final PLW in Phase 2, the PSTs reviewed the video segments of their mathematics lessons from Cycle 2. The PSTs answered questions that identified where they used the CRIG framework. The questions used the Jacobs et al. (2010) AID teacher noticing model. The subsequent discussion dealt with the PSTs' use of the CRIG framework in each video segment. The final component of this PLW was the completion of a worksheet involving the PSTs' application of the CRIG framework to an algebraic procedure. The procedure known as "expanding binomial products" is part of the number and algebra stand for Stage 5 Years 9 and 10 (14–15-year-old) students in the NSW K–10 mathematics syllabus (see NSW Board of Studies, 2012). The PSTs used structural thinking to deepen their PCK by connecting binomial products to other mathematical content relationships.

4.7.2.2 Mathematics lessons and planning

PSTs designed lesson plans for each of their mathematics lessons. The lesson plans were presented on a template that included sections to acknowledge the CRIG components. All PSTs' lessons, apart from one secondary PST's final lesson, were submitted as data.

The PSTs taught three mathematics lessons in which the professional experience supervising teacher determined the topic in accordance with the school's mathematics program. Table 4.5 contains the details of the videoed lessons and the mathematics lesson topics taught.

Table 4.5

Schedule of Videoed Mathematics Lessons by Lesson Topic

			Videoed mathematics lessons date and topic					
Phase	Year	PST	Date	Topic	Date	Topic	Date	Topic
1	2017	Ms N	31 July	Multiplication using arrays	3 August	Estimating length	7 August	Division by sharing
		Ms S	26 July	Developing patterns	2 August	Adding 2-digit numbers	8 August	Classifying 3-dimensional objects
2	2018	Ms K	9 May	Simultaneous equations with non-linear equations	21 May	Interior and exterior angles of polygons	12 June	Quadratic equations
		Ms M	11 May	Circumference of a circle	18 May	Area of composite shapes	22 June	Volume of a cylinder
		Mr T	4 May	Ordering fractions	14 May	Adding and subtracting fractions	23 July	Stem and leaf plot graphs

Video recording of lessons

During Phase 1, a digital SLR camera (Canon 60d) was used with a professional camera operator focused on the PST. Some inefficiencies using this camera were noted during Phase 1, so for Phase 2, a GoPro Hero 5 video camera was used. A lapel microphone connected to a personal recording device (iPhone 6 mobile phone) was attached to the PST. The video and audio recordings were synchronised. Table 4.6 shows each PSTs' lesson topics and the videoed lesson duration. The length of the videos ranged from 27 to 80 minutes.

Table 4.6

Lesson Topic, Year Level, Streaming Level and Videoed Lesson Length

Phase	PST	Cycle	Lesson topic	Year level	Stream level	Videoed lesson length (min:sec)
1	Ms S	1	Multiplication using arrays			40:13
		2	Estimating measurement	1	Mixed ability	27:13
		3	Division by sharing			50:18
	Ms N	1	Developing patterns			35:04
		2	Adding 2-digit numbers	1	Mixed ability	61:51
		3	Classifying 3-dimensional (3D) objects			43:09
2	Ms K	1	Simultaneous equations with non-linear equations			61:49
		2	Interior and exterior angles of polygons	9	Acce-lerated	61:49
		3	Quadratic equations			80:17
	Ms M	1	Circumference of a circle			56:04
		2	Area of composite shapes	8	Top	71:16
		3	Volume of a cylinder			73:41
	Mr T	1	Ordering fractions			73:11
		2	Adding and subtracting fractions	7	Middle	56:21
		3	Stem-and-leaf plot graphs			61:13

4.7.2.3 Post-lesson interviews and reflections

Phase 1 post-lesson interviews

For Phase 1, the PSTs participated in a post-lesson interview. This interview was held immediately after the mathematics lesson. PSTs were asked a series of questions during the interview, and their responses were audio recorded on a personal recording device (iPhone 6 mobile phone). The interviews were between 5 and 10 minutes in duration.

In Phase 1, the post-lesson interview questions were open ended, without prompts. However, when a question was not well understood, it was rephrased. The post-lesson interview reflective questions are presented in Table 4.7 and aligned to the supporting research questions.

Table 4.7

Phase 1 Interview Questions Aligned to Research Supporting Questions

	Interview question for the PST post videoed lesson	Supporting question
1.	What is noticing structural thinking?	What are the PSTs' understandings of structural thinking?
2.	Give an example of where you noticed a structural thinking.	
3.	How do the CRIG framework assist in noticing structural thinking?	
4.	What teaching action did you use when mentioning a CRIG component?	How do PSTs use structural thinking in their mathematics teaching?
5.	Which teaching action was effective for you to notice structural thinking?	
6.	Give an example of how you involved the CRIG framework in your lesson?	
7.	How effective was using the CRIG framework in developing structural thinking?	How effective is the CRIG framework in helping PSTs to notice structural thinking in their teaching?
8.	Did the CRIG framework help you understand the thinking?	
9.	When involving the CRIG framework, how did you notice structural thinking in the students' responses? Can you give an example?	

After Cycle 1, Question 3 was eliminated, and three new questions (Questions 10–12) were added to the interview schedules for inclusion in Cycles 2 and 3 post-lesson interviews. Table 4.8 contains those three additional questions.

Table 4.8

Additional Post-Lesson Interview Questions Aligned to Research Supporting Questions

	Post-lesson interview - Additional questions	Supporting question
1.	What use of the CRIG framework do you see to be useful in your teaching for the future?	How do PSTs use structural thinking in their mathematics teaching?
2.	Does knowledge of the CRIG framework support your pedagogical practices when teaching mathematics?	
3.	How has the CRIG component helped you understand students' understanding of mathematics?	How effective is the CRIG framework in helping PSTs to notice structural thinking in their teaching?

Phase 2 post-lesson reflections

For Phase 2, a post-lesson reflection on the PSTs' mathematics lesson was introduced, and the PSTs viewed a short segment of their videoed mathematics lesson. After viewing the video, the PSTs answered a series of questions focused on noticing structural thinking. I chose video segments of the PSTs using the CRIG framework. Table 4.9 shows the lesson topic and length of each videoed segment.

Table 4.9

Phase 2 Post-Lesson Reflection and Video Segment Length

Phase	PST	Cycle	Lesson topic	Post-lesson reflection length, min:sec	Video segment viewed length, min:sec
2	Ms K	1	Simultaneous equations with non-linear equations	12:47	2:11
		2	Interior and exterior angles of polygons	10:32	3:43
		3	Quadratic equations	15:25	4:00
	Ms M	1	Circumference of a circle	4:26	2:31
		2	Area of composite shapes	8:02	3:35
		3	Volume of a cylinder	15:27	3:52
	Mr T	1	Ordering fractions	6:31	4:00
		2	Adding and subtracting fractions	10:38	5:01
		3	Stem and leaf plot graphs	16:41	3:01

In Chapter 3, two models of teacher professional noticing were described. These were the Kaiser et al. (2015) perception, interpretation, decision making (PID) model of teacher competency and the Jacobs et al. (2010) attending, interpreting, and deciding (AID) model of teacher professional noticing of students' mathematical thinking. The PID model was used in Cycle 1, and the AID model was used in Cycles 2 and 3 post-lesson reflection activity on noticing.

During the secondary PSTs' (Phase 2) Cycle 1, which used the PID model, the PSTs chose where to stop the video and were asked to provide an answer to this question: "Describe what you notice at that point in the video segment." They were asked not to discuss classroom management or student behaviour, but rather to focus on what they were teaching and how they were teaching it. The PSTs were not prompted to look for the CRIG framework because the intention was to see if they had noticed the CRIG components without being prompted.

The second question on interpretation was asked at the end of the video. The PSTs commented on what they were noticing during this part of the lesson. Again, they were not asked to identify the components of the CRIG framework specifically. However, they were asked to consider the framework in their discussion. The final question on decision making required the PSTs to consider whether there were any missed opportunities. Table 4.10 presents the PID questions given to the PSTs before watching the video segment.

Table 4.10

Cycle 1 Noticing CRIG Framework Questions Using the PID Model

PID	Question
1. Perception	Watch the video and then stop at any point where you want to make a comment about what you're doing, and, if you can, keep it within the reference of the project.
2. Interpretation	So, can you just give an overview of that section that we watched of the lesson in relation to the CRIG framework?
3. Decision	Do you see opportunities that were missed there?

For the Phase 2 post-lesson reflection, I decided that the PSTs required more direction about noticing the CRIG framework in the video segment. The AID model was considered appropriate because it allowed me to direct the PSTs toward what to notice about structural thinking in their teaching, particularly about the CRIG framework.

In Cycle 2, the PSTs watched the video segment, and during the video segment, they were asked to stop the video at any point where they were using the CRIG framework. The attending, interpreting, and deciding questions are displayed in Table 4.11.

Table 4.11

Cycle 2 PSTs' Noticing CRIG Framework Questions Using the AID Model

AID	Question
Attending	Please explain what component of the CRIG framework you are using and when you are using it.
Interpreting	Please explain how you are using the component of the CRIG framework at this point in the class.
Deciding	Pretend you were to teach this again. Name another component of the CRIG framework you could incorporate, and how you would incorporate it.

During Cycle 3, the approach to the PSTs viewing the video segment was altered. An additional component to the PSTs' noticing structural thinking was added to the post-lesson reflections. To begin, the PSTs watched the whole video segment before answering the three AID questions. They then viewed the video segment for a second time. During this second viewing, they were asked to stop the video at any point and comment on the component of the CRIG framework they were using. Table 4.12 contains the instructions and questions for the PSTs during Cycle 3.

Table 4.12

Cycle 3 PSTs' Noticing CRIG Framework Instructions and Questions Using the AID Model

Section	Instruction and questions
PART A	Watch the video all the way through once. Then consider these questions.
Question One: Attending	Please explain what CRIG component of mathematical structure you are using and when you were using it.
Question Two: Interpreting	Please explain how you are using this CRIG component at this point in the class.
Question Three: Deciding	Pretend you were to teach this again. What other CRIG components could you incorporate, and how?
PART B	Now, watch the video a second time. During this time, stop the video at any point and mention what CRIG component you notice. Consider CRIG components that you notice while reflecting on this video but did not think about when you were teaching this class.

Both the noticing models used in the post-lesson reflections had the intention of identifying the PSTs' noticing of structural thinking through the CRIG framework. In this study, Mason's (2002) premise that teachers are not always aware of what they attend to when teaching, is investigated. The PSTs' initial awareness of the CRIG framework was limited due to their inexperience. Through noticing the components of the CRIG framework in the post-lesson reflections, the PSTs' experience and familiarity with using the CRIG framework when teaching was expected to improve.

4.7.3 Exit questionnaire

An exit questionnaire was conducted after Cycle 3 in both phases. The exit questionnaire included the same questions as the introductory questionnaire to assess any changes in the PSTs' responses, but there were additional questions regarding their understanding of the CRIG framework and noticing structural thinking. Table 4.13 presents the additional questions added to the exit questionnaire to identify any changes in the PSTs understanding of structural thinking. The questions are aligned to the research supporting questions.

Table 4.13

Exit Questionnaire Questions Aligned to the Research Questions

Number	Question	Supporting question
1.	Do you think your teaching strategy when teaching mathematics has changed during your participation in this project? If so how?	How do PSTs use structural thinking in their mathematics teaching?
2.	How has your understanding of structural thinking grown during this study?	What are the PSTs' understandings of structural thinking?
3.	How has your involvement in this study improved your noticing of structural thinking?	
4.	Explain what you understand to be the CRIG framework of mathematical structure. Did you find this framework difficult to comprehend?	How effective is the CRIG framework in helping PSTs to notice structural thinking in their teaching?
5.	Explain how learning about the CRIG framework of mathematical structure have added to your understanding of structural thinking.	
6.	Explain how learning about the CRIG framework of mathematical structure have added to your ability to notice structural thinking.	
7.	What teaching actions did you find most effective to notice structural thinking? What was the role of the CRIG framework to achieve this?	How do PSTs use structural thinking in their mathematics teaching?

4.8 Data analysis

All data collected in Phase 1, apart from the videoed mathematics lessons, were uploaded to NVivo (QSR International, 2017). NVivo is a qualitative data analysis (QDA) computer software package. This software is suitable for analysing data collected in qualitative research because it allows for the analysis of rich text-based and multimedia data. NVivo allows for analysis of a variety of objects. In the case of this research, that included transcriptions of audio recordings from PLWs, pre- and post-lesson interviews, and lesson plans. In Phase 1, the mathematics lesson videos were not uploaded to NVivo because NVivo could not load the video format. As a result, the videos were coded manually using an Excel spreadsheet.

4.9 Coding process

The NVivo coding provided quantitative measures of the PSTs’ time spent in each TDC category and the frequencies refer to a component of the CRIG framework in the mathematics lessons. Frequencies of components of the CRIG framework were also recorded for post-lesson interviews and reflections. NVivo coding also provided the qualitative references to components of the CRIG framework during the PLW in the mathematics lesson, post-lesson interviews and reflections. The videos of the mathematics lessons were coded for the type of teacher-directed communication (TDC) the PSTs were involved in and the time spent using each form of TDC during their lessons. In Phase 1 the videoed mathematics lessons were coded for the amount of time the PST spent using the components of the CRIG framework. Some video segments coded for a component of the CRIG framework were used as exemplars for the qualitative analysis of the mathematics lessons.

For Phase 2, NVivo analysis involved setting up a hierarchy of coding nodes for coding the data from the data source at four levels. Table 4.14 shows this hierarchy.

Selected components of the transcripts were allocated to the appropriate node. These components were tallied for quantitative results that represented references to the components of the CRIG framework in the mathematics lessons and the post-lesson reflections. The coding hierarchy allowed for closer scrutiny of the data by generating themes from the various data types. The selected components provided rich qualitative data that provided exemplars of the PSTs' understanding and use of the CRIG framework (see Chapters 5 and 6).

Table 4.14
Phase 2 NVivo Hierarchy of Coding Nodes

Coding hierarchy nodes			
Level 1	Level 2	Level 3	Level 4
Component of the CRIG framework	Data collecting instrument	Cycle	PST
Connections	PLW	1	Ms K
Recognising patterns	Lesson plan	2	Ms M
Identifying similarities and differences	Mathematics lesson	3	Mr T
Generalising and reasoning	Post-lesson reflection		

Additionally, transcripts of the data collection instruments (PLWs, lesson plans, mathematics lessons, and post-lesson interviews or reflections) were analysed manually for the components of the CRIG framework. Constant comparative analysis (Bakker,

2019) was used to interpret the transcripts and videos for the CRIG framework, and a peer consultation process was employed to clarify and support my findings. Consistency in coding the data was achieved through collegial observations and discussions (see Section 4.10). Three independent coders, two for Phase 1 and one for Phase 2, were employed to provide interrater reliability categorisation of the data for CRIG components.

4.9.1 Professional learning workshops

Transcripts of the audio recordings from PLWs in both phases were uploaded to NVivo. Coding in NVivo involved selecting sections of the transcript that reflected a component of the CRIG framework and allocating it to the corresponding node. NVivo tallies the number of sections for each node and groups these sections together for closer qualitative analysis. Selected sections of the transcripts from each component of the CRIG framework were chosen as exemplars presented in the results.

4.9.2 Mathematics lessons

In Phase 1, I viewed each videoed mathematics lesson and entered the teacher-directed communication (TDC) into a spreadsheet as a time component of the lesson. Three categories of TDC were identified: (1) the PST teaching the whole class with no student interaction, (2) the PST teaching the whole class with student interaction, and (3) the students working independently with the PST assisting. There was another category that allocated time to when the PST was not teaching mathematics. The videos were coded for a component of the CRIG framework to which the PST was attending. The times for TDC and the component of the CRIG framework were recorded, giving the PSTs' total time in TDC and use of the CRIG framework.

In Phase 2, the mathematics lesson videos and lesson transcripts were uploaded to NVivo. The videos were coded using PST time in each TDC category. The same three categories of TDC were identified as for Phase 1. However, an additional non-TDC category was included with the category: no mathematics being taught. This category was used when the students were working without any assistance from the PST. The TDC times were recorded, giving the PSTs' total amount of time using TDC.

The lesson transcripts were coded using NVivo for components of the CRIG framework. These data were not coded by time, as in Phase 1, but for references to a component of the CRIG framework. The frequencies of the PSTs' use of a component of the CRIG framework were coded for subsequent analysis, and some sections of extended dialogue were identified as exemplars of the PSTs' communication using the CRIG framework during their TDC.

4.9.3 Post-lesson interviews and reflections

In Phase 1, the audiotaped interviews were transcribed and uploaded into NVivo for coding and analysis. A qualitative coding process was employed to identify statements made by a PST that represented any of one of the four components of the CRIG framework. This coding process also formed the basis of the quantitative analysis of the PSTs' understanding of the CRIG framework. Additionally, some of each PST's responses to the interview questions became exemplars for the qualitative component of the study.

In Phase 2, the CRIG framework of mathematical structure and the PID/AID noticing framework were used separately but were also combined as the lens to notice structural thinking in the PSTs' reflections on the videos viewed. The CRIG framework of mathematical structure identified whether the PSTs were aware of attending to structural thinking when teaching mathematics. The PID (Kaiser et al., 2015) framework focused on general process of noticing structural thinking in the CRIG framework and the AID (Jacobs et al., 2010) framework focused directly on the CRIG components. Post-lesson reflections were transcribed and uploaded into NVivo for coding and analysis of components of the CRIG framework.

4.10 Reliability and validity

Morse, Barrett, Mayan, Olson, and Spiers, (2002) argue that qualitative researchers should be responsible for reliability and validity in their research by constantly verifying and correcting their strategies for making judgments. During the analysis of this study, I achieved this by continually checking and rechecking the consistency of the categorisation. As an experienced mathematics teacher and also having had the experience of using the CRIG framework in a previous research project (see Gronow, Mulligan, & Cavanagh, 2017), I was familiar with the components in a mathematics teacher's lexicon of content and pedagogy, and I felt confident about coding the data for each CRIG component.

The process of coding for the CRIG framework involved checking the context of the teacher's use of the framework against the lesson topic, lesson activity, and the TDC to ensure the use of the component was related to the lesson topic. I reviewed manually printed copies of the PLWs, the mathematics lessons, and transcripts of the post-lesson interviews and reflections to identify the CRIG components. Any anomalies were resolved (under supervision), so that the categorisation was consistent.

When coding the data, I was consistent in coding the type of utterance or comment that was assigned to each CRIG component. In some cases, the coded response came from the PST's single statements consisting of a few words; in others, it was a lengthy description or instruction. There were also examples of the PST involved in discussion with an individual student, a group of students, or the whole class. In some instances, more than one component was identified. In this situation, the statement was coded as characteristic of the most significant component that appeared in the communication.

When coding the data for each component of the CRIG framework, I looked for a similar theme that was reflective of that component. I developed a simple process that allowed me to consistently code the data (see Table 4.15).

Table 4.15
Questions Asked to Allocate Data to a CRIG Component

Component of the CRIG framework	Questions asked for coding
Connections	Is there a connection of learning to past experiences?
	Is there a connection to real experiences?
	Is there a connection to known facts?
	Is there a connection to prior learning?
Recognising patterns	Is a pattern identified?
	Is a pattern represented?
	Is a pattern being created?
Identifying similarities and differences	Is same and difference mentioned?
	Is there same and difference in what is seen?
	Is same and difference inferred?
Generalising and reasoning	Is 'what happens next' considered?
	Is there a rule?
	Is this a reasonable statement?
	Is there a reflective response?

Table 4.16 contains exemplars from Phases 1 and 2 of specific statements assigned to components of the CRIG framework.

Table 4.16

Exemplars from PSTs Mathematics Lessons for Coding to a Component of the CRIG Framework

CRIG component	PST	Phase & Lesson	Description of what the PST did	Example of what the PST said
Connections	Ms S	Phase 1 Lesson 1	PST connects to prior learning, recall of knowledge or known facts.	What maths we are learning today? What does multiplication mean? What is the cross?
	Ms M	Phase 2 Lesson 1	Connect the learning to a real-world example.	And then you would also call this one a sector. So, it's kind of like your pizza; that would be your sector.
Recognising patterns	Ms N	Phase 1 Lesson 1	Makes a pattern Acknowledges a pattern Agrees on a pattern	Oh, my goodness look at these patterns, tell me about this one. What type of pattern does it make? Is it an increasing pattern? How does it increase, what changes happen every time? What changes would make it a decreasing pattern?
	Ms K	Phase 2 Lesson 2	Directs instruction for the students to look for a pattern in the mathematical relationships.	Now, can you find the pattern of what's going on between the relationship of the sides, the number of triangles?
Identifying similarities and differences	Ms N	Phase 1 Lesson 3	PST encourages students to look for features of 3D objects. Identifies same and different and states reason for difference.	Where would you put this object? Does it have curved and flat surfaces?
	Mr T	Phase 2 Lesson 3	Asks students to make a comparison between two objects to identify the differences.	We're going to compare these four graphs, and I want you to tell me what things you see that are the same and what are the things that are different that you see between these four graphs?

Table 4.16 (*cont.*)

CRIG component	PST	Phase & Lesson	Description of what the PST did	Example of what the PST said
Generalising and reasoning	Ms N	Phase 1 Lesson 1	What happens next? Is there are rule? Makes a prediction.	Ms N: Why do you think there is going to be 5 in the next pattern? You're not sure? Student: Because it's going up 1,3, 5 Ms N: I like the way you think there that's good so if I draw the next pattern here. I am going to start with the one I had originally like this with three triangles and then I am going to draw some more triangles. Let's see what I come up with.
	Ms K	Phase 2 Lesson 1	Asks the students if the approach is a reasonable way to solve the problem.	But the exercise was just to help you realise that you can't use it all the time. Sometimes substitution is easier.
	Ms M	Phase 2 Lesson 1	PST identifies the student's ability to generalise the relationship between the components of the circle.	He said that what he found out today was that knowing the relationship between the circumference, pi and the diameter, you can find out what the circumference is.

Two independent coders were employed to provide interrater reliability on the data coding for the Phase 1 categorisation of the lesson transcripts and the pre-and post-lesson interview data for CRIG components. Each coder was given a briefing on the components of the CRIG framework. Examples were provided along with instructions about coding for the CRIG framework (Appendix L).

Each coder was given ten examples from Phase 1 PSTs' mathematics lessons. Of these, the first coder agreed with my coding of the CRIG components in nine out of ten of the examples. After reviewing the example coded differently, the coder reassessed, which resulted in consistent categorisation for all ten examples.

For coding of Phase 2 data, a similar procedure as Phase 1 was followed, except a different coder was employed. Ten examples from the PSTs' mathematics lesson were coded. The coder agreed with my coding for eight of the ten examples. After a review of the two examples coded differently, we were able to agree on one of the examples, but not both, leaving a 0.9 agreement rate. Over the two phases with two different coders, a 0.95 agreement of coding for the CRIG components was achieved.

4.11 Summary

In this chapter, I have outlined the design and context of the study, described the participants', and referred to ethical considerations and the methodological approaches used to conduct the study. Additionally, I described the data collecting procedures and processes for analysing the data. I concluded the chapter with an outline of the reliability and validity measures employed to ensure of the integrity of the data analysis.

The following two chapters focus on the results from the data collected during the two phases of this study. Chapter 5 presents the data collected the primary PSTs and the secondary PSTs data is presented in Chapter 6.

Chapter

5

Results: Phase 1

5.1 Introduction

This chapter is concerned with results obtained from two primary PSTs—Ms S and Ms N. The results are presented in four main sections: an introductory questionnaire, a quantitative integration of the PSTs' lesson and interview data and use of CRIG across the three cycles, a qualitative analysis of the PSTs' use of CRIG in lesson and interview data, and an exit questionnaire.

5.2 Introductory questionnaire

In this section, I describe the PSTs' views about mathematics and fundamental understanding of mathematical structure gained from the introductory questionnaire (see Chapter 4 Table 4.3). Table 5.1 presents the PSTs' scores indicating their self-perceived views about various processes when teaching mathematics using scores between 0 (easiest) and 100 (most difficult).

Table 5.1

Introductory Questionnaire Scores for PSTs

	Score	
Mathematics-teaching category	Ms S	Ms N
1. Lesson preparation	70	40
2. Creating an engaging lesson	70	32
3. Teaching strategies that engage students	70	37
4. Engaging all students in the activities	92	48
5. Students' understanding of mathematics	61	50

Entries in Table 5.1 reveal that Ms S scored all categories closer to 100, i.e., toward being "most difficult." She indicated that identifying students' understanding of mathematics was the easiest aspect of teaching mathematics, followed by preparing and creating a lesson that contained strategies to engage students. However, she was more uncomfortable with engaging the students in doing mathematics when teaching. In the last two categories of the questionnaire, Ms S described her lessons as teacher-centred and her teaching as procedural, although she indicated that she fostered collaborative learning and attempted to develop students' conceptual understanding.

Table 5.1 reveals that Ms N reported she had moderate difficulty in teaching by scoring categories as mid-range or toward being "easiest." Ms N scored preparing a mathematics lesson and engaging students as not difficult for her, nor was engaging students or recognising their understanding. Ms N reported that she developed students' conceptual understanding through inquiry-based and collaborative pedagogical practice.

The second part of the questionnaire, Questions 11 to 16 are aligned to the research contributing question of what are the PSTs' understanding of structural thinking. Ms S described mathematical thinking as identifying patterns and applying knowledge, and she believed mathematical structure comprised recognising patterns and connecting concepts. Ms S described structural thinking as applying mathematical relationships across concepts and how solving problems involves understanding concepts and using facts to support understanding. She explained this as, "Understanding then organising of new mathematical concepts to be transferred to a new mathematical concept. This can include breaking down the problem and looking at what they know and build onto it." Ms N described mathematical structure through some components of the CRIG framework. According to her, "Mathematical structure describes a means of learning mathematical concepts by discovering and applying mathematical patterns, generalising results and drawing connections."

5.3 Quantitative integration of findings across all cycles

This section contains quantitative analyses from the mathematics lessons and the post-lesson interviews. These analyses are based on:

- the amount of time that each PST engaged in Teacher-Directed Communication TDC during the mathematics lessons,
- the amount of time that each PST spent using the components of the CRIG framework during the mathematics lessons, and
- the frequency of PSTs' references to the components of the CRIG framework in the post-lesson interviews.

5.3.1 In-class time spent on teacher-directed communication (TDC)

Analysis of the mathematics lessons for TDC is presented in Table 5.2. Analysis of time spent in each TDC category was based on the video data of the PSTs' teaching. The type of TDC indicates the PSTs' mode of communication when teaching. Three categories were used to code TDC in the observed mathematics lessons (see Chapter 4, Section 4.9.2):

Category 1: TDC to the whole class, with no student discussion

Category 2: TDC to the whole class, with student discussion

Category 3: TDC is not to the whole class but between the PST and an individual or a small group of students

An additional category of non-TDC represents the time during which the PST was not communicating about the mathematics in the lesson, such as giving introductory directions to the students at the start of the lesson.

Table 5.2 displays the proportion of time each PST spent teaching according to each TDC category. Proportions of time are shown as whole percentages and whole minutes, except for lesson length. For example, Ms S's lesson 1 has TDC for Category 1 as 59(24) which represents 59% or 24 minutes of the lesson time. Categories 1 and 2 involved the PST communicating to the whole class, which is indicative of a teacher-centred pedagogical practice. Category 3 was considered student-centred.

Table 5.2

Teacher-Directed Communication by Lesson[a]

PST	Lesson	Topic	Lesson length min: sec	TDC Category 1	TDC Category 2	TDC Category 3	TDC time	Non-TDC
Ms S	1	Multiplication	40:13	59(24)	15(6)	16(6)	90(36)	10(4)
	2	Measurement	27:13	24(7)	21(6)	34(9)	97(22)	3(6)
	3	Division	50:18	44(22)	21(10)	21(10)	85(43)	15(7)
Ms N	1	Patterns	35:04	27(10)	17(6)	43(15)	88(31)	12(4)
	2	Addition	61:51	39(24)	15(10)	28(17)	82(51)	18(11)
	3	3D objects	43:09	27(11)	16(7)	21(9)	64(27)	36(15)

[a] Rounding of the percentage to a whole number and time, in parentheses, to the nearest minute. The percentages and times do not always total 100% due to rounding.

The main finding from Table 5.2 is that both PSTs tended toward teacher-centred pedagogical practice. The exception was Ms N's lesson 1, which had an even distribution between teacher-centred (Categories 1 and 2 combined) and student-centred (Category 3).

Table 5.2 also indicates that overall lesson time may have influenced the TDC. Ms S's lessons ranged between 27 and 50 minutes and Ms N's lessons between 35 and 61 minutes. The average percentage of class time on TDC for Ms S was 91% and 78% for Ms N. Ms S had the highest time in Category 1 in lesson 1 and lesson 3 and the least time in lesson 2, and she had the most time for Category 3 in lesson 2. Ms S's time in Category 2 was lower than Category 1 or 3 for all three lessons, although, as her lessons became longer, her TDC shifted more to Category 1. Ms N spent more time on Category 3 in lesson 1, but in lesson 2, most of her time was in Category 1. In lesson 3, she had the same amount of time in Categories 1 and 3, but less time in Category 2.

Lesson topics might have influenced TDC. For example, lessons involving arithmetic number skills such as multiplication (Ms S, lesson 1), division (Ms S, lesson 2), and addition (Ms N, lesson 2) were taught in a teacher-centred manner, as indicated in Table 5.2. The PSTs' teaching time for these lessons was highest in the teacher-centred categories of Category 1 and Category 2. Lesson topics that involved learning attributes of patterns (Ms N, lesson 1), 3D objects (Ms N, lesson 3) and length (Ms S, lesson 2) were student-centred, which reflected more time in Category 3.

5.3.2 Teacher-directed communication time referring to the CRIG framework

An analysis of the PSTs' use of the CRIG framework as a proportion of TDC time that the PSTs spent on each component in the mathematics lesson is shown in Table 5.3. These times are shown as percentages of the total TDC time and in minutes. Values are rounded to whole amounts with minutes shown in parentheses.

Table 5.3 shows that both PSTs used the CRIG components to some degree, but there was some variability in the frequency of their use. Over the three lessons, Ms S's highest proportion of references to a single component of the CRIG framework during TDC time over the three lessons were: lesson 1 where she referred to *Identifying Similarities and*

Differences, *Generalising and Reasoning* in lesson 2 and *Connections* in lesson 3. There was a different pattern of responses found for Ms N where the highest proportion of TDC time was attributed to *Recognising Patterns* in lessons 1 and 2 and *Identifying Similarities and Differences* in lesson 3. Neither PST showed a clear preference for any single CRIG component, but it was clear that the lesson topic influenced the PSTs' use of the CRIG framework.

Table 5.3

Teacher-Directed Communication Time and Components of the CRIG Framework[a]

				Component of CRIG framework as a percentage of TDC time				
PST	Lesson	Topic	TDC time	Connections	Recognising patterns	Identifying similarities and differences	Generalising and reasoning	Non-CRIG
Ms S	1	Multiplication	36	25(9)	8(3)	43(16)	14(5)	11(4)
	2	Measurement	22	20(4)	0(0)	24(5)	48(10)	9(2)
	3	Division	43	38(16)	22(10)	22(9)	4(2)	15(6)
Ms N	1	Patterns	31	7(2)	59(18)	16(5)	7(2)	11(3)
	2	Addition	51*	26(14)	38(20)	7(4)	2(1)	27(14)
	3	3D objects	26*	14(4)	0(0)	42(11)	11(3)	33(9)

[a] Rounding applied to the CRIG and non-CRIG time in these lessons.

5.3.3 References to the CRIG framework in post-lesson interviews

Table 5.4 displays the PSTs' references to the CRIG framework retrieved from the analysis of interview transcripts (see Chapter 4, Section 4.7.2.3).

The main finding from Table 5.4 is that although the PSTs referred to the components of the CRIG framework, there was no distinct pattern across the lessons for either PST. The many references to the CRIG components indicate the PTSs' overall familiarity with the CRIG framework. Ms S's references to the CRIG components increased significantly in Cycle 3, possibly as a result of her developing a deeper understanding of the CRIG framework. Ms S referred only minimally to *Recognising Patterns* and *Identifying*

Similarities and Differences in her post-lesson interviews following lessons 1 and 2. However, in the interview following lesson 3, her references to these components increased significantly. Ms S made fewer references to *Connections* after the lesson 1 interview, but she made a similar number of references to *Generalising and Reasoning* during all three interviews.

Table 5.4

PSTs' Reference to the CRIG Framework in the Post-Lesson Interviews

			Component of CRIG framework				
PST	Cycle	Lesson Topic	Connection	Recognising Patterns	Identifying Similarities and Differences	Generalising and Reasoning	Total
Ms S	1	Multiplication	7	1	2	5	15
	2	Measurement	2	1	1	4	8
	3	Division	3	9	10	4	26
Ms N	1	Patterns	2	11	4	2	19
	2	Addition	6	5	2	4	17
	3	3D objects	3	1	5	7	16
		Total	23	28	24	26	

Ms N's overall references to the CRIG components were evident across all three cycles. Her references to *Recognising Patterns* in lesson 1 were the highest, possibly due to the lesson topic as these were lower in lessons 2 and 3. References to *Connections* was marginally higher than *Recognising Patterns* in lesson 2. However, in lesson 3, Ms N referred to *Identifying Similarities and Differences*, and *Generalising and Reasoning*, where she linked these components to the lesson topic of classifying 3D objects.

5.4 Professional learning program (PLP)

This section comprises qualitative analyses of the three cycles within the PLP: three components of each cycle were analysed to describe the PSTs' noticing of structural thinking through the CRIG framework. These three components are:

- professional learning workshops (PLWs),

- mathematics lesson observations and lesson plans, and
- post-lesson interviews.

Pertinent examples are drawn from each of these components.

5.4.1 Cycle 1

The first cycle of the PLP occurred following the PSTs' completion of the introductory questionnaire. The PLP component for this cycle consisted of the first Professional Learning Workshop (PLW1). The PLW was followed by each PST teaching a mathematics lesson using the CRIG framework, and a post-lesson interview was conducted between the PST and the researcher immediately after teaching the mathematics lesson.

5.4.1.1 Professional learning workshop

The agenda for the Phase 1 PLW1, displayed in Chapter 4, Section 4.7.2.1, Table 4.4, consisted of a presentation by the researcher about the CRIG framework, a videoed lesson from the Teaching Channel titled *Related Problems: Reasoning About Addition* (Teaching Channel, 2017), an arithmetic number sentence (ANS) worksheet and a sample mathematics lesson plan using the CRIG framework.

This section focuses on the PSTs' completion of the arithmetic number sentence (ANS) task. During PLW1, Ms S and Ms N completed the task to investigate their ability to use relational thinking. The first ANS task involved the PSTs looking for patterns of numbers that added to 10, designed to encourage structural/relational thinking rather than numerical calculations. A series of number sentences were given, comprising two numbers that summed to 10 as follows:

$$10 + 0 = \square$$

$$9 + 1 = \square$$

$$8 + 2 = \square$$

$$7 + 3 = \square$$

$$6 + 4 = \square$$

PSTs were asked to complete a further six lines with the pattern established above. The PSTs' responses displayed structural thinking in the form of *Recognising Patterns* and *Generalising and Reasoning*. For example, Ms S recognised the pattern, "They can see that there's a pattern. One side is adding, one side is subtracting, and they switch it around and they can see that numbers are moved around." Ms N explained how to approach the pattern by saying, "When you've switched it around, it's the first time that the students had to stop and say, 'Okay, there are two numbers missing, how do I have to start with this pattern? How's this pattern going to work? I have to decide'."

In another example, the PSTs were asked to find the missing number in the number sentence 13 + 66 + □ + 27 = 40 + 80 + 15. In solving the problem, Ms S stated, "13 plus 27. That makes it 40. [Then] 66 plus 14 is 80. And then it's 15." In doing so, Ms S demonstrated relational thinking by comparing both sides of the equation and looking for patterns of numbers adding to the multiples of 10 on the right-hand side of the equation. Ms N's response to the question was, "So the tens column and the ones column are instantly match[ed] up. Sounds like you did match the three and the seven." Ms N's response indicated that she noticed the structural thinking process Ms S used to solve the problem.

5.4.1.2 Mathematics lesson observations

All of the mathematics lesson observations involved two aspects: reviewing the PST's written lesson plan and then observing the lesson. Ms S's first lesson plan identified *Connections* to prior learning as a "recap on what they know about multiplication" and *Recognising Patterns* as well as *Identifying Similarities and Differences* in representing multiplication as repeated addition, grouping, and arrays.

In lesson 1, Ms S started her lesson on multiplication by communicating with the whole class to connect one student's knowledge about multiplication through the multiplication symbol ×. Using a procedural approach, she prompted the students: "What is the cross? What do we call that sign, the cross?" However, she did attempt to develop the CRIG components throughout the lesson. For example, she made *Connections* with prior learning and encouraged the students to *Recognise Patterns* as well as *Identify Similarities and Differences* when communicating with a student who confused multiplication with addition. The student said, "Multiplication is almost like a plus, but a little different. Seven times seven is probably 14." Another student responded by connecting multiplication to addition and identified multiplication as a pattern of repeated addition by describing 7 × 7 as 7 + 7+ 7 + 7 + 7 + 7 + 7. Ms S used *Connections* with prior learning and *Recognising Patterns* and *Identifying Similarities and Differences* when she connected the expression 7 × 7 and repeated addition of 7 + 7+ 7 + 7 + 7 + 7 + 7 to a diagram showing seven groups of seven and extended that into drawing arrays (see Figure 5.1).

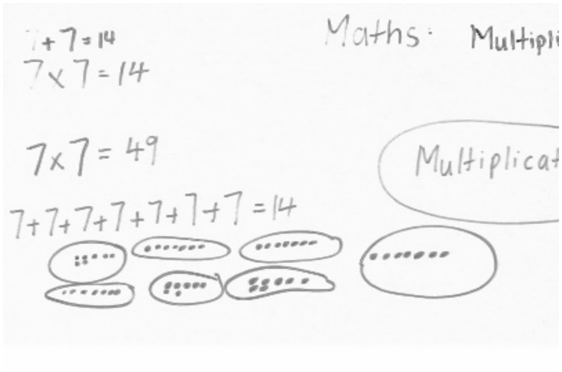

Figure 5.1. Example from Ms S's lesson 1 – multiplication using arrays.

In Ms N's lesson on patterns, her lesson plan identified *Connections* to prior learning by her asking the students what they already knew about patterns. She also identified *Recognising Patterns* in her observation of the students' work, *Identifying Similarities and Differences* in different types of increasing patterns, for example, by demonstrating a pattern with laminated squares or as a staircase going up by one step at a time, and *Generalising* and *Reasoning* when applying the pattern rule.

Ms N used the components of the CRIG framework in a class discussion on patterns which she described as repetitions when she introduced increasing and decreasing staircase number patterns. The remainder of the lesson involved students building, identifying, and describing growing patterns using coloured blocks.

5.4.1.3 Post-lesson interviews

From an analysis of the interview transcriptions, Ms S initially explained that noticing structural thinking was achieved through her questioning. When asked to consider where the CRIG framework assisted her in noticing students' structural thinking, she responded by saying, "It would be effective. I'll be honest, when I'm teaching it, I don't think about these CRIG components." She believed the CRIG framework was like a set of procedures: "It goes in a process. It's like they connect it, then recognise and identify, and then generalise it."

From the analysis of Ms N's interview transcript, it was clear that she encouraged students to see the differences between each other's patterns when she commented that "students could recognise the patterns themselves … they could start to identify similarities and differences between patterns." Ms N also showed insight when students did not demonstrate structural thinking as they were constructing a staircase pattern of increasing by threes. "Students struggle to translate that knowledge of increasing by one, to increase by three. So, that is an example where their structural thinking might not have been as good." Ms N identified a potential relationship between *Recognising Patterns* and *Generalising and Reasonin*g when the height of the staircase consistently increased by three units (blocks): "Generalising and reasoning would be more to do with how the pattern shape changed every time."

5.4.2 Cycle 2

The second cycle of the Phase 1 PLP consisted of the second PLW (PLW2), PSTs' planning and teaching another mathematics lesson and a post-lesson interview between the PST and the researcher.

5.4.2.1 Professional learning workshop

The agenda for the Phase 1 PLW2 is displayed in Chapter 4, Section 4.7.2.1, Table 4.4. PLW2 consisted of a review of the post-lesson interview questions, a video reflection activity from a segment from Ms N's lesson 1 and a sample mathematics lesson plan using the CRIG framework.

The video segment from Ms N's patterns lesson involved Ms N changing the orientation of the groups of blocks to notice structural thinking through *Identifying Similarities and Differences*. Ms N encouraged students to recognise growing patterns and communicate the unit of repeat in the pattern. For example, "You can still use colours to make an increasing pattern. How might you change that?" Ms N encouraged the students to create an increasing pattern with coloured blocks. By putting the blocks into a horizontal row, Ms N suggested, "Let us start with black, and with an increasing pattern, we add one block every time." The students were not sure what to do, so Ms N changed the orientation of the blocks from a horizontal row to separate vertical stacks, saying, "What if we do it this way?" (Figure 5.2). Ms N encouraged the students to produce a fifth column, and they immediately started to build the new column without any assistance from her. Ms N's communications with the students focused on *Identifying Similarities and Differences* to demonstrate different arrangements of the blocks which helped her to notice the students' pattern reproduction as an indicator of their structural thinking.

Figure 5.2. Students' increasing patterns using colours on a linear (horizontal) progression.

Ms N's response after viewing the video segment identified how she noticed the students' structural thinking change: "I tried doing it in a line, and they weren't getting it. As soon as I put it in vertical columns, they saw it was increasing." Ms S explained how the students were able to replicate the increasing pattern and that the students may have been generalising. "So, they self-corrected themselves. So, they helped each other out. So, I think that could be an example of generalising."

Ms S believed the CRIG framework was useful in noticing structural thinking, even when not she was not aware of doing so. She commented that the components, "were effective because, without us knowing, they were implemented in the teaching." Ms N explained how she was developing an understanding of the CRIG framework. She focused on the individual components but also thought of all the components as a collective:

> Recognising that this is an increasing pattern whether it be with numbers of cubes or colours was something that I needed to conquer or achieve first before I could progress. So, I was focusing on parts of the CRIG components, but on a whole I wasn't thinking about all of them at the same time.

Ms N followed this comment with a statement that she considered the CRIG components to be a natural part of a teacher's pedagogy: "They could become a natural part (of your teaching) where you do not think about using them, but they just are implicit in your teaching."

5.4.2.2 Mathematics lesson observations

In Ms S's lesson plan on estimating length, she introduced the CRIG framework through all four components. She referred to *Connections* to prior learning of addition and estimation. She also referred to *Recognising Patterns* in how a length can be divided up and re-formed to produce the same length. *Identifying Similarities and Differences* was identified as the difference between the width and length of a shape; and *Generalising and Reasoning* as making reasonable decisions of knowing the units to use when measuring smaller or longer lengths.

From an analysis of time spent referring to the CRIG framework, Ms S made more references to the component *Generalising and Reasoning* than to other components. Closer analysis of these references indicates that Ms S aligned these responses closely with encouraging students to make reasonable decisions in their estimations of length and use of informal units of measure. Ms S embedded reasoning in her questioning and prompted the students about the object to be measured and the appropriate unit of measure, for example: "What is the length of the table? Choose informal units to measure the table" and "What happens if I measure like that? Is that an accurate reading?" She encouraged structural thinking by asking the students to estimate lengths using informal units of measurement and used reasoning to make decisions about measuring the length. She said: "Did you guess how many of your handspans you need? So, you're going to change your estimate, have a guess first. [and] So, you have estimated it, and now you measure it." Ms S used the language of "have a guess" to encourage students' reasoning when making estimations that were then checked by measuring, and she prompted the students with statements such as, "You have guessed it and then measured it" so students could compare the reasonableness of their estimate to the measure. Ms S was encouraging the students to use informal units of length to measure. However, she was also prompting them to consider why these measures were inaccurate. She questioned the students to consider whether their responses were reasonable, and she probed them to generalise why a formal unit of length was needed.

In Ms N's lesson plan on adding 2-digit numbers, she explored concepts of addition of two single-digit numbers using the jump strategy on an open number line. She also made specific references to the CRIG framework. For example, she identified *Connections* to prior learning of addition of two-digit numbers. She referred to *Recognising Patterns* as the relationship between numbers when decomposing two-digit numbers as multiples of five and 10 when adding them on a number line. In the lesson plan, Ms N's references to *Identifying Similarities and Differences* were linked to the different sized jumps of five and 10 on the number line. Her reference to *Generalising and Reasoning* was made when students could strategically apply the jump strategy without needing to use a number line to add two-digit numbers.

During the lesson, Ms N referred to *Recognising Patterns* and *Connections* as related to prior learning. Use of these two components far outweighed the use of the other two components, indicating Ms N's reliance on connecting students' knowledge of one-digit addition, multiples of five, and the patterns that are formed by decomposing two-digit numbers and using the jump strategy. Thus Ms N made *Connections* to extend students' knowledge of one-digit numbers to two-digit numbers, and encouraged *Recognising Patterns* of multiples of five and 10 to decompose two-digit numbers, and "jump" using these patterns on the number line. Ms N gave the students an extension activity that required students to add two-digit numbers that were not multiples of five. As a result, *Identifying Similarities and Differences* extended students' structural thinking. Without direct instruction from Ms N, the students applied *Generalising and Reasoning* from the first activity and decomposed these new numbers into tens and units rather than fives, using the same general strategy of adding to the tens.

5.4.2.3 Post-lesson interviews

In the interview following lesson 2, Ms S considered the CRIG framework to be a regular part of teaching: "You naturally teach it in the lesson." She also experienced students' structural thinking as "noticing their process of … working things out from step one to step two." Ms S used the framework directly in her lesson planning: "When I am planning the lesson and I follow the CRIG framework, I think students can understand it a little bit better." In her lesson plan, Ms S referenced the CRIG framework and these references were observed in the lesson. There were clear examples of where she used the CRIG components. However, when asked during the interview if she used the framework, she hesitated: "I couldn't really. Like I said, I didn't use any of the CRIG, well, most of the

CRIG components." Ms S regarded her lesson as procedural and that she did not consider the CRIG framework: "When I was doing the lesson, I think it was all about explaining what to do rather than identifying the CRIG framework."

During the lesson, Ms N's approach was to teach the students to use the jump strategy as a demonstration of decomposing numbers. In her interview, Ms N described how students use structural thinking as a mental process that needs to be applied as a procedure. In the case of decomposing numbers, it was by completing the jump strategy. Ms N described structural thinking as subconscious processing of mathematics: "Structural thinking is something that students are often subconsciously using, and it is something a pre-service teacher can either pick-up or not." Ms N noticed this happening when a student completed mental calculations for the number line tasks. She believed the students did not understand the concept and would not be able to apply the process to more challenging problems. She reflected on the processes used by students: "When it got to the harder questions at the top of the tower activity, he didn't apply the same addition strategies to complete that answer" and "Students were jumping straight to mental strategies without using the number line. So, they were not able to show that they understood the empty number line concept." However, Ms N may not have realised that the students' strategies had advanced beyond the number line concept, and they were using their structural thinking skills to solve the problem.

5.4.3 Cycle 3

The third and final cycle of the PLP began with the third PLW (PLW3), PSTs planning and teaching a final mathematics lesson and a final post-lesson interview between the PST and the researcher.

5.4.3.1 Professional learning workshop

The agenda for the Phase 1 PLW3 is displayed in Chapter 4, Section 4.7.2.1, Table 4.4. In PLW3, the PSTs viewed two video segments of Ms S's cycle 1 lesson 1 and Ms N's cycle 2, lesson 2 and they participated in a reflection activity.

In the first video, a student from Ms S's class accurately drew a five by three array as five rows of three dots, shown in Figure 5.3, although Ms S had asked the student to draw an array three rows of five.

Figure 5.3. Student draws a five-by-three array.

After viewing the video, Ms S indicated that she would have changed her teaching strategy and made *Connections* to the commutative law for multiplication by *Identifying Similarities and Differences* in an array of 3×5 and 5×3; but, during the lesson, her focus was the representation of the diagram. Ms N did notice the students' structural thinking in drawing the array by stating, "She knew that an array was in columns or in rows … maybe she may not have known columns. However, she knew that that's how an array looked, and she did demonstrate that she understood."

The second video reflection was from a segment of Ms N's second lesson on addition. In the video, Ms N asked a student who had solved 10 + 15 using mental calculation how she had done so. The student explained that 15 became $10 + 5$ and that 10 + 15 is $10 + 10 + 5$, which the student re-ordered as $10 + 5 + 10$. The student used the associative law of addition to solve the problem. By breaking down 15 and adding the five to the first 10, the problem became 15 + 10. However, Ms N was not satisfied with the student's approach despite them showing structural thinking in applying the associative law. She wanted the student to transfer this thinking to using the number line. However, as the student had already solved the problem, Ms N questioned the student about how to solve it using the jump strategy: "If you were to use the number line, how would we do it?" Ms N was concerned that the student needed to demonstrate this skill to transfer his thinking to more difficult number combinations.

Ms N was attempting to develop students' ability to notice the structural features of relating numbers. The example of 15 involves adding 10 and 5, which uses a base-10 approach, and the structural thinking is incorporated into mental computation strategies such as 23 + 5 could be 20 + 5 + 3. However, from the PLW transcript, a conversation between the PSTs revealed that they preferred students to use the jump strategy rather than mental arithmetic because it enabled students to use a representational strategy that the PSTs were able to observe. Ms N stated that she believed that relying on mental arithmetic was not beneficial for the student when faced with harder problems to solve. She considered competence at the jump strategy, through the decomposition process, was necessary to progress to more difficult questions, even though the students were displaying structural thinking. So, although Ms N could have been encouraging mental computation, she felt students needed the jump strategy as an alternative: "Use this when you get stuck. When you get to the double-digit numbers, it gets harder. Try solving it by using the jumping strategy." Ms S identified *Connections* to new situations: "They didn't make that connection themselves because those ones who understood would have applied what they've learned to the next task."

5.4.3.2 Mathematics lesson observations

In Ms S's lesson plan on division, she introduced the CRIG framework through the four components. She identified *Connections* to a knowledge of arrays to represent groups in division. *Recognising Patterns* was shown as a strategy where increasing the number of groups reduces the number of units in each group. Students used *Identifying Similarities and Differences* between multiplying units by groups to get a total and dividing a total into groups to find the number of units. In this lesson plan, Ms S claimed that knowing the number of groups and number of units in each group always gives the product as an example of *Generalising and Reasoning* multiplicatively. Ms S described generalising as being in every division situation. She thought it was always a matter of finding the number of groups or the number in each group by using the total, that is, by using grouping as a general strategy that leads to inverse relationships. This process is considered as generalising when recognising the situation is partitive or quotitive.

Ms S began lesson 3 by explaining how division is the sharing of equal amounts into groups. She wrote 12 ÷ 4 = ? on the board and promoted a discussion about the numbers and symbols. Ms S referred to the *Connections* between division and multiplication, *Recognising Patterns* as sharing between groups, and *Identifying Similarities and Differences* between division and multiplication. Students solved word problems by drawing groups, then using counters to construct the groups, and finally writing a number sentence.

In Ms N's lesson plan about 3D objects, she introduced the CRIG framework through *Connections*. Students completed a 3D objects pre-test including *Recognising Patterns* in 3D objects that have curved and flat surfaces, and *Identifying Similarities and Differences* in comparing and classifying 3D objects. The idea of *Generalising and Reasoning* the properties of 3D objects such as a cylinder having curved and flat surfaces and a rectangular prism having all flat surfaces was also examined.

In lesson 3, Ms N's students recognised, described, and categorised 3D objects. She referred to *Identifying Similarities and Differences* between the different 3D objects. She encouraged the students to create a Venn diagram using two hoops as a representation of what is common between the objects so that students could generalise about the groups and categorise 3D objects. Ms N asked the students how to place the hoops in a way that accommodated all the 3D objects. Her communication with the students encouraged structural thinking through the placement of the hoops: "Can everyone see what is going on here? You used the word overlap. Is there another way?" The hoops were placed with an overlap to accommodate 3D objects with flat and curved surfaces. A simple form of *Generalising and Reasoning* was identified by the students as knowing that not all 3D objects have only flat surfaces, or, conversely, as knowing that 3D objects can have curved surfaces or a combination of both flat and curved surfaces. She encouraged students' reasoning in their emerging understandings about the categories and the appropriateness of the hoops to classify the objects: "Can you show me what to do with these hoops to show what is the same and what's different?"

5.4.3.3 Post-lesson interviews

An analysis of the interview transcripts indicated how Ms S noticed *Identifying Similarities and Differences* in the different forms of grouping of counters the students created to show sharing in division: "They arranged the counters differently. Some of them arranged them in columns; some of them in circles." Ms S described the components of the CRIG teaching framework to support students' thinking: "I used the same terms as the CRIG framework, as it triggered the students into thinking about the relationship between multiplication and division." Ms S talked about the CRIG framework when teaching in the following way: "This was not planned. I just thought I might test it [CRIG framework] out and see what they see." Her attention to the CRIG framework became evident when Ms S was initially describing her understanding of mathematics. Without giving specific mathematical examples, she described knowing maths as "all about identifying patterns and transferring it to new unfamiliar concepts." So, she transferred this understanding through the CRIG framework to her lessons. She explained: "That's what I try to push 'What patterns do you see? Do you see any similarities?' In that way then hopefully, they can generalise it." Ms S was then able to directly apply components of the CRIG framework by "trying to test if they could identify the similarities and differences in multiplication and division." Ms S felt that the framework had helped her: "The CRIG framework fits into my teaching."

Analysis of Ms N's interview transcript revealed a focus on the CRIG framework to help her in lesson planning "to design a lesson to ensure that they were generalising" and to claim that "identifying similarities and differences was throughout the lesson." Ms N said her structural thinking benefited from the CRIG framework when applying new concepts to different contexts: "It helped me to understand where the students are coming from and their process of understanding a new concept and then being able to put that into a new context." Ms N noticed structural thinking when students compared the surfaces of the objects: "Structural thinking came into play when they applied the concept of a flat versus a curved surface."

Ms N referred to how the CRIG framework supports structural thinking by stating that the "CRIG framework helped understand the students' thinking." Ms N's attention to decision making demonstrated her ability to notice student learning: "It has drilled home that the most amount of thinking and learning is happening in the moment." Ms N's summation clarified her awareness of structural thinking: "So, it [CRIG framework] just helped me to understand where the students are coming from."

5.5 Exit questionnaire

In this section. I summarise the results from the exit questionnaire. Table 5.5 contains the PSTs' scores relating to what they regarded as easy or difficult about teaching mathematics using a score between 0 (easiest) and 100 (most difficult).

Table 5.5

Introductory and Exit Questionnaire Scores for PSTs

	Introductory Score		Exit Score	
Mathematics-teaching category	Ms S	Ms N	Ms S	Ms N
1. Lesson preparation	70	40	61	21
2. Creating an engaging lesson	70	32	50	31
3. Teaching strategies that engage students	70	37	27	32
4. Engaging all students in the activities	92	48	82	35
5. Students' understanding of mathematics	61	50	60	44
6. Using the CRIG components in teaching mathematics			60	36
7. Using the CRIG components to notice students' structural thinking			48	36

In the exit questionnaire responses, categories one to five showed both PSTs' scores changed from the introductory questionnaire and the results indicated that they felt mathematics teaching had become less difficult. Ms N's scores on the introductory questionnaire showed that she thought mathematics teaching was not difficult, and on the exit questionnaire her scores decreased slightly, showing she still regarded mathematics to be reasonably easy to teach. Both PSTs responded with scores to the statements about using the CRIG framework in the mid-range. Ms S thought that using the components of the CRIG framework when teaching was slightly more difficult than Ms N did. Both PSTs' scores indicated that using the framework as useful for noticing structural thinking as neither difficult nor easy. Ms N's overall scores for the using the CRIG framework were more towards being easier, possibly an indication of Ms N's overall satisfaction with her ability to incorporate the CRIG framework into her teaching and use it to notice structural thinking.

The second part of the questionnaire, Questions 11 to 16 are aligned to the research contributing question of what are the PSTs' understanding of structural thinking. These responses are to be used to identify the shift in the PSTs thinking regarding mathematical structure and structural thinking. Both PSTs related the components of the CRIG framework to mathematical thinking. When asked to define structural thinking, Ms S said it involved moving from concrete to abstract thinking, and Ms N considered structural thinking to be the way numbers and patterns are formed, organised and related. She gave an example of structural thinking as knowing the properties of geometrical figure.

Additional questions to ascertain PSTs' understanding of the CRIG framework were included in the exit questionnaire. Analysis of Ms S's exit questionnaire transcript revealed that she believed her lesson planning improved when she included the CRIG framework: "The CRIG framework helps me to better plan the lessons." It also helped her to deliver a more sequenced lesson: "The CRIG framework has helped me to break down my lessons into logical steps."

Ms S included the CRIG framework in her thoughts about mathematical structure: "Connecting mathematical concepts through recognising patterns, identifying similarities and differences, and generalising the results to an unfamiliar concept." However, Ms N considered an understanding of mathematical structure to be more than mathematical skills used when doing mathematics. She regarded mathematical structure to be a deep understanding of mathematical relationships and properties: "Mathematical structure refers to more than simply recognising mathematical regularities or properties, but rather to have a deeper understanding of how those properties are connected and used in new and varied contexts." Ms S's view of mathematical structure was related closely to the CRIG framework, whereas Ms N regarded it as a general thinking skill that related mathematical properties.

Ms S and Ms N both regarded the CRIG framework as developing students' understanding. Ms S explained: "When students were able to see these differences and similarities, they were able to generalise certain rules and patterns to help them see the relationship of mathematical concepts." As an example of this, in Ms S's lesson 3, students' use of different types of patterns to divide counters supported their ability to generalise a number sentence and show a different representation of a division expression. Ms N explained: "Mathematical thinking refers to the way that students identify similarities and differences and make connections in mathematics to develop and generalise abstract mathematical ideas." This statement represented Ms N's commitment to pursuing the CRIG framework to develop her ability to notice structural thinking. As evident in lesson 1, she observed how students were not able to reproduce an increasing pattern alone. However, by adding a different colour to create the increasing pattern, and by repositioning the groups of blocks from horizontal rows to vertical towers, she noticed how the students developed an understanding and could generalise the pattern.

Ms N regarded the CRIG framework as helping her to understand students' thinking: "The CRIG framework has helped me to understand better the thought processes that students go through when learning new mathematical concepts." She went on to state: "I am now better able to notice how students switch between learning and engaging with mathematical concepts that are presented in an abstract form." Ms N was aware of the importance of students' use of structural thinking by being able to move between contexts and adapt their learning to different scenarios. "I have learned that it is very important to ensure that structural thinking is encouraged with explicit instruction of new structures, formulas, or methods." Ms N indicated her personal growth in understanding of structural thinking: "My understanding of structural thinking has grown as I have a better understanding of the weighting students place on mathematical structures when learning new mathematical concepts." This statement indicates that Ms N believed that she could understand students' structural thinking through her awareness of mathematical structures.

5.6 Summary

The data analyses have provided indicators of the PSTs' use of the CRIG framework to notice structural thinking. The results described here provide a summary of the findings from the introductory questionnaire, the quantitative analysis of TDC, and the qualitative analyses of three cycles of PLWs, mathematics lessons, post-lesson interviews, and the exit questionnaire.

This chapter began with a review of the introductory questionnaire that identified the PSTs' awareness of structural thinking at the start of Phase 1. The results of this questionnaire showed that the PSTs had some knowledge of mathematical structure through the components of the CRIG framework. This review was followed by a quantitative analysis of the proportion of each mathematics lesson time that each PST spent in TDC. The qualitative analysis of each PST's teaching time was beneficial in identifying the TDC time in each lesson. The results indicated that most of the PSTs' TDC time was spent in teacher-centred instruction. When these results were correlated with the lesson transcripts, evidence of the PSTs use of the CRIG framework during each TDC category was identified. As expected, the PSTs use of the CRIG framework was predominantly during teacher-centred instruction without student communication. This was expected as the PSTs were involved in this category more than the others. However, the results showed that the PSTs used the CRIG framework in their instructions during the mathematics lessons demonstrating their deepening PCK. A further examination of transcriptions of the mathematics lessons and post-lesson interviews identified PSTs' frequency of references to the CRIG framework. A qualitative analysis of the data from the PLW transcripts, mathematics lesson observations, and post-lesson interview transcripts of the three PLP cycles presented exemplars of the PSTs' use of CRIG. This analysis identified how the PSTs' noticed structural thinking through their references to the components of the CRIG framework. The PSTs' reflections of their mathematics lessons proved beneficial in supporting the PSTs mathematical content knowledge. During the PLWs there was robust discussions between the PSTs about structural relationships of the content being taught. This helped support the PSTs mathematical content knowledge. The specific examples from the lessons, given in this chapter, demonstrate how the PSTs had developed a heightened awareness of noticing structural thinking through attending to the CRIG framework, improving their PCK. The exit questionnaires provided a view of how the PSTs' awareness of structural thinking had changed from the start to the end of Phase 1. These questionnaires showed that the PSTs had developed an understanding of the importance of mathematical structure. The responses from the questionnaires results showed that the CRIG framework had provided the PSTs with a mechanism to develop structural thinking in their mathematics lessons, thus indicating their improving PCK.

The following chapter presents the data collected from the secondary mathematics PSTs during Phase 2 of this study. This includes a qualitative and quantitative view of the PSTs' noticing of structural thinking through the CRIG framework.

CHAPTER

6

Results: Phase 2

6.1 Introduction

This chapter is concerned with results obtained from three secondary mathematics PSTs—Ms K, Ms M, and Mr T. The results are presented in four main sections: an introductory questionnaire, a quantitative integration of the PSTs' lesson and interview data and use of CRIG across the three cycles of a professional learning program (PLP), a qualitative analysis of the PSTs' use of CRIG in lesson and interview data, and an exit questionnaire.

6.2 Introductory questionnaire

In this section, I describe the PSTs' views about mathematics and fundamental understanding of mathematical structure gained from the introductory questionnaire (see Chapter 4, Table 4.3). Table 6.1 presents the PSTs' scores, indicating their self-perceived views about teaching mathematics using scores between 0 (easiest) and 100 (most difficult).

Table 6.1

Introductory Questionnaire scores for PSTs

	Score		
Mathematics-teaching category	Ms K	Ms M	Mr T
1. Lesson preparation	61	50	70
2. Creating an engaging lesson	100	72	80
3. Teaching strategies that engage students	94	74	80
4. Engaging all students in the activities	100	52	90
5. Students' understanding of mathematics	30	40	80

Entries in Table 6.1 reveal that the PSTs generally regarded mathematics teaching as more difficult than easy. The only scores that were below 50 were for students' understanding of mathematics for Ms K, and Ms M. All PSTs regarded lesson preparation as slightly difficult in comparison to other aspects of mathematics teaching. Ms K scored 'creating an engaging lesson', 'teaching strategies that engage students', and 'engaging all students in the activities' as the most difficult, as did Mr T, but his scores were not as high. Ms M's scores indicated that teaching mathematics was not as difficult for her as the other PSTs.

Each PST described a different approach to their teaching. Ms K and Mr T defined themselves as using a teacher-centred approach, with Ms K as 'procedural' and Mr T as 'collaborative', while Ms M stated she used 'conceptual learning'. Each PST gave a different definition of mathematical thinking. Ms K wrote that it was combining mathematical concepts to solve problems, while Ms M said it involved applying logic and reasoning to solve problems. Mr T referred to the Australian Curriculum: Mathematics proficiency strands (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2019): "When students can use a variety of techniques to solve problems."

PSTs' responses to Questions 11 – 16, which were aligned to Research Question 1 about PSTs' understanding of mathematical structure indicated that Mr T had an existing knowledge of mathematical structure through the CRIG framework. All three PSTs gave a basic definition of structural thinking. Ms K thought it was "using a variety of methods to find a solution to a problem", Ms M stated that it "justified how she solved a problem with a particular solution", and Mr T recognised it as "applying skills learnt prior to a new situation."

6.3 Quantitative integration of findings across all cycles

This section contains the quantitative analyses from the mathematics lessons and the post-lesson interviews. These analyses are based on:

- the amount of time that each PST engaged in Teacher-Directed Communication (TDC) during the mathematics lessons,
- the frequency of the PST's references to the components of the CRIG framework during the mathematics lessons, and
- the frequency of PSTs' references to the components of the CRIG framework in the post-lesson reflections.

6.3.1 In-class time spent on teacher-directed communication (TDC)

Analysis of the mathematics lessons for TDC is presented in Table 6.2. Analysis of time spent in each TDC category was based on the video data of the PSTs' teaching. The type of TDC indicates the PSTs' mode of communication when teaching. There were three categories assigned for coding TDC in the observed mathematics lessons (See Chapter 4, Section 4.9.2):

Category 1: TDC to the whole class, with no student discussion

Category 2: TDC to the whole class, with student discussion

Category 3: TDC is not to the whole class but between the PST and an individual or small group of students

Two categories of non-TDC included in the table present the times in the lesson where the PSTs were not communicating about mathematics:

Category 1: Students are involved in individual or group mathematics work. The PST is not communicating about mathematics with the students during this time.

Category 2: Students are not involved directly in any mathematics work.

Table 6.2 displays the proportion of teaching time PSTs spent in TDC and non-TDC modes of communication, with proportions shown as whole percentages and whole minutes. For example, Ms K's Lesson 1 shows TDC for Category 2 as 50(31), which represents 50%, or 31 minutes of the 62-minute lesson time. The column headed total TDC for Ms K in lesson 1 shows 96(56), which represents that Ms K was involved in TDC 96% of the time or 56 minutes of the 62-minute lesson. The percentages and time totals may slightly differ to the totals of 100% and lesson time due to the rounding.

Table 6.2

Teacher-Directed Communication (TDC) by Lesson [a]

PST	Lesson	Topic	Lesson length min:sec	TDC Category 1	TDC Category 2	TDC Category 3	TDC time	Non-TDC Category 1	Non-TDC Category 2	Non-TDC time
Ms K	1	Simultaneous equations with non-linear equations	61:49	0(0)	50(31)	46(28)	96(59)	2(2)	2(1)	4(3)
	2	Interior and exterior angles of polygons	80:17	5(4)	19(15)	36(29)	60(48)	0(0)	40(32)	40(32)
	3	Quadratic equations	56:04	0(0)	75(42)	16(9)	91(51)	0(0)	9(5)	9(5)
Ms M	1	Circumference of a circle	71:16	13(9)	42(30)	30(21)	85(60)	6(4)	8(6)	16(10)
	2	Area of composite shapes	73:41	11(8)	8(6)	69(51)	88(65)	1(1)	11(8)	12(9)
	3	Volume of a cylinder	73:11	3(2)	14(11)	65(47)	82(60)	3(2)	14(10)	17(12)
Mr T	1	Ordering fractions	56:21	4(2)	43(24)	35(20)	82(46)	6(4)	13(7)	19(10)
	2	Adding and subtracting fractions	61:13	25(15)	31(19)	36(22)	92(56)	5(3)	3(2)	8(5)
	3	Stem-and-leaf plot graphs	56:49	4(2)	69(39)	18(10)	90(51)	5(3)	5(3)	10(6)

[a] Rounding of percentage to a whole number and time, in parentheses, to the nearest minute. The percentages and times do not always total 100% due to rounding.

An examination of TDC in Table 6.2 reveals the diversity in the PSTs' type of communication. Table 6.2 also shows variations in the duration of each of the PSTs' lessons. Overall, the PSTs spent a minimal amount of time communicating with the whole class without any student communication. In Category 1, which refers to teacher-centred communication, Ms K demonstrated only 5% of TDC in Lesson 2; Ms M used TDC only 13% and 11% of class time in Lesson 1 and 2 respectively and only 3% in Lesson 3. Mr T demonstrated a higher proportion of TDC with 25% of class time in Lesson 2, but only 4% of TDC time in his other two lessons.

TDC Category 2 represented a teacher-centred approach but with the PST communicating with the whole class with student discussion. Overall, the secondary PSTs spent more time in teacher-centred lessons communicating to the whole class when engaged in student discussion. For Category 3, the students were working individually or in groups, and the PST was assisting them. Overall, the time PSTs spent in Category 3 fluctuated depending on the lesson topic.

6.3.2 Teacher-directed communication time referring to the CRIG framework

An analysis of the PSTs' use of the CRIG framework as a proportion of TDC time that the PSTs spent on each component in the mathematics lesson is shown in Table 5.3. These times are shown as percentages of the total TDC time and in minutes. All values are rounded to whole amounts with minutes shown in parentheses.

Table 6.3

Teacher-Directed Communication Time and the CRIG Framework

				Frequency of CRIG category in each TDC time			
PST	Lesson	Topic	TDC time min:sec	Category 1	Category 2	Category 3	Total
Ms K	1	Simultaneous equations	58:47	0	32	5	37
	2	Angles of polygons	48:08	0	21	17	38
	3	Quadratic equations	50:45	0	43	6	49
Ms M	1	Circumference	60:32	0	35	13	48
	2	Area	64:46	0	3	41	44
	3	Cylinder	60:21	4	3	29	36
Mr T	1	Fractions 1	46:18	3	9	5	17
	2	Fractions 2	55:46	6	10	7	23
	3	Graphs	50:23	1	23	10	34

Analysis of Table 6.2 identifies the PSTs' use of the CRIG framework. As expected, these data showed that most of the PSTs' references to the CRIG framework occurred in TDC Categories 2 and 3 when the PSTs were communicating with the students. Ms K's highest number of references to CRIG was in TDC Category 2, where she was communicating to the whole class with an average of 31 references. Ms M's highest number of references to CRIG was in TDC Category 3 when students were working independently, with an average of 28 references. Mr T made 23 CRIG references in TDC Category 2 in lesson 3. These frequencies indicate the diversity in the PSTs' use of the CRIG framework, but it appeared that the lesson topic and pedagogical approach taken by the PSTs influenced their use of the CRIG framework.

Table 6.4 contains the frequencies of references to the four components of the CRIG framework as they occurred during TDC in each of the PSTs' lessons.

Table 6.4

PSTs' references to the Components of the CRIG Framework in Teacher-Directed Communication Time

				Components of CRIG framework				
PST	Lesson	Topic	TDC time (min)	Connections	Recognising Patterns	Identifying Similarities and Differences	Generalising and Reasoning	Total
Ms K	1	Simultaneous equations	59	6	1	13	17	37
	2	Angles of polygons	48	4	4	6	24	38
	3	Quadratic equations	51	13	2	18	16	49
Ms M	1	Circumference	61	3	2	23	20	48
	2	Area	65	7	0	20	17	44
	3	Cylinder	60	13	0	3	20	36
Ms T	1	Fractions 1	46	3	0	12	2	17
	2	Fractions 2	56	2	5	11	5	23
	3	Graphs	51	6	0	17	11	34
		Total		57	14	123	132	

The contents of Table 6.4 provide insight into how the PSTs structured their lessons. PSTs explicitly referred to *Generalising and Reasoning* more than the other components of the CRIG framework, followed by *Identifying Similarities and Differences*. This reference implies that the PSTs were aware of making *Connections* to prior learning and the role of *Recognising Patterns*, but they made fewer direct references to these components.

6.3.3 References to the CRIG framework in post-lesson reflections

Table 6.5 displays the PSTs' references to the CRIG framework retrieved from the analysis of interview transcriptions (see Chapter 4, Section 4.7.2.3).

Table 6.5

PSTs' Reference to CRIG Framework in the Post-lesson Reflections

			CRIG components				
PST	Lesson	Topic	Connections	Recognising Patterns	Identifying Similarities and Differences	Generalising and Reasoning	Total
	1	Simultaneous equations	5	2	12	8	27
Ms K	2	Angles of polygons	8	12	5	4	29
	3	Quadratic equations	6	4	7	7	24
	1	Circumference of a circle	1	1	1	2	5
Ms M	2	Area of shapes	0	0	0	0	0
	3	Volume of a cylinder	6	7	4	10	27
	1	Fractions 1	2	3	3	3	11
Mr T	2	Fractions 2	1	2	5	2	10
	3	Stem-and-leaf plot graphs	3	4	2	3	12
		Total	32	35	39	39	

Table 6.5 shows that the PSTs referred to the components of the CRIG framework during the post-lesson reflections. However, each of the PSTs' references varied considerably depending on the lesson topic and the order of the lesson. Ms K consistently referred to the CRIG framework after each lesson in her reflections. Ms M made few references after her first lesson, none after the second, but made a significant number of references after the third lesson. Mr T made relatively fewer references than Ms K and Ms M in her third lesson, but a consistent number of references throughout the post-lesson reflections.

There was no distinct pattern in the frequencies of the PSTs' references to the CRIG framework. All PSTs demonstrated variation in their use of CRIG without any one component as a focus. PSTs' references to the CRIG framework in the post-lesson reflections are summarised briefly as:

1. Ms K demonstrated a consistently high awareness of the CRIG framework;
2. Ms M's references were negligible initially, but increased at the end; and
3. Mr T's overall references were consistent across all reflections.

This section has provided an overview of the analysis of the PSTs' references to the CRIG framework for the lesson observations and post-lesson reflections.

6.4 Professional learning program (PLP)

This section comprises a qualitative analysis of the three cycles within the PLP. Three components of each cycle were analysed to describe the PSTs' noticing of structural thinking through the CRIG framework. These three components are:

- professional learning workshops (PLWs),
- mathematics lesson observations and lesson plans, and
- post-lesson interviews.

Pertinent examples are drawn from each of these components.

6.4.1 Cycle 1

The first cycle of the PLP occurred following the PSTs' completion of the introductory questionnaire. The PLP component for this cycle consisted of the first PLW (PLW1). This PLP was followed by each PST teaching a mathematics lesson using the CRIG framework and a post-lesson reflection where the PST viewed a videoed segment of their mathematics lesson.

6.4.1.1 Professional learning workshop

The agenda for the Phase 2 PLW1 is shown in Chapter 4, Section 4.7.2.1, Table 4.4. This PLW began with a presentation on the CRIG framework by the researcher, followed by a viewing of the Teaching Channel video, titled *Related Problems: Reasoning About Addition* (Teaching Chanel, 2017) and the question "Where do you notice the teacher using CRIG components?" All PSTs noticed the teacher's use of patterns. Ms K said the teacher used different techniques to identify patterns and made suggestions for the students to look for patterns. Ms M recognised that patterns engaged the students and gave opportunities to develop reasoning skills. Mr T noticed how students saw similarities in the patterns and generalised them.

The PSTs identified where the CRIG framework appeared in the lesson plan designed for this PLW. Ms K noticed how the framework facilitated students' thinking, Ms M felt that the CRIG framework was evident in this plan, and Mr T stated that it allowed the students to interact and make generalisations. Further responses from Ms K were that *Recognising Patterns* helped students' understandings and that when teachers use patterns, they do not rely solely on formulas. Ms M thought the lesson plan encouraged deeper reasoning and understanding and that the use of *Recognising Patterns* and *Identifying Similarities and Differences* engaged students. Mr T mentioned that the CRIG framework was vital for teachers so that they would know what to focus on during the lesson.

6.4.1.2 Mathematics lesson observations

In this section, I review the PSTs' Cycle 1 lesson plans and mathematics lessons.

In Ms K's lesson plan on simultaneous equations, she identified *Connections* to prior learning through students' knowledge of solving simultaneous equations and that the square root of a negative number is not a real number. Additionally, she identified *Connections, Recognising Patterns* and *Identifying Similarities and Differences* of non-linear equations in connection with their graphs. She highlighted *Generalising and Reasoning* in the process of finding points of intersection through algebra.

In her lesson, Ms K made *Connections* to the relationship between the intersection points of two graphs and the solutions found when solving the graphs' equations simultaneously. *Recognising Patterns* appeared when Ms K introduced the power of x in non-linear equations to determine the shape of a curve. Ms K used *Identifying Similarities and Differences* on several occasions. For example, discussing linear and non-linear equations, she asked the students, "What do you notice about the shape of the lines? One of them is curved. Which one would represent that curved line?" In relating the shape of non-linear graphs to the power of x in the equation, she said, "They are all curves, they're all similar in their curves. What is similar in the equations?" Also, when describing the intersection of a straight line and a curved line (parabola) she discussed aspects of the straight-line graph, such as gradient and the shape of a non-linear graph: "You will have different solutions depending on the gradient and on the shape of the line, if it is linear or non-linear." Ms K encouraged *Generalising and Reasoning* to identify how the highest power of x changed the shape of the curve, and this enabled the students to make predictions about the number of intersection points between the two graphs.

In Ms M's lesson plan on circumference of a circle, a reference to the CRIG components appeared as *Connections* to prior learning about circles, *Recognising Patterns* and *Identifying Similarities and Differences* in the ratio of the circumference and the diameter of the circle, and *Generalising and Reasoning* in developing a rule to find the circumference of the circle.

At the beginning of the lesson, Ms M made *Connections* with prior learning about a sector of a circle compared to a slice of a pizza: "You would also call this one a sector. So, it's kind of like your pizza." During the lesson, Ms M used *Generalising and Reasoning* to establish the relationship between the radius and the diameter: "Can anyone see a relationship between the radius and the diameter?" She also referred to *Identifying Similarities and Differences* in order to differentiate between the radius and the diameter: "So it's the distance from the centre to the other side of the circle. So, it's only this part that's the same."

Ms M modelled finding the ratio of the circumference to the diameter. She measured the circumference of a circle by rolling it along a ruler and then measured the diameter of that circle. She drew attention to *Recognising Patterns* in the form of ratios from different circles: "Can you see a pattern in the numbers when I give you another circle?" Ms M used *Identifying Similarities and Differences* in reviewing the results: "Have you guys found something similar in your ratios?" Ms M introduced pi (π) through *Generalising and Reasoning* by establishing a rule for finding the circumference of a circle: "So we just found the relationship between pi, the diameter and the circumference."

Mr T's lesson plan on fractions identified *Connections* to prior learning of equivalent fractions and *Recognising Patterns* as comparing numerators of fractions with "same" denominators. *Identifying Similarities and Differences* was identified in ordering fractions by denominators and *Generalising and Reasoning,* as a rule, to make the denominators the same when ordering fractions.

Mr T used *Identifying Similarities and Differences* to discriminate between the size of fractions: “When you look at this, which one’s bigger? Or, which one’s smaller?” He used *Generalising and Reasoning* to define the rule about using numerators and denominators.

Mr T considered *Identifying Similarities and Differences* to compare equivalent fractions with different denominators: “Because the size of the denominators are [sic] different. So how do you think it would make it easier for us to compare these?” He applied *Identifying Similarities and Differences* by making the denominators the same to compare the fractions: “So if we made these denominators the same, would it make it easier?” He followed this up by asking how to make the denominators the same,” How do you think we can make these denominators the same?” Mr T used *Generalising and Reasoning* to identify the size of fractions through a diagram: “Two shaded parts and this one remains the same, three over four. Now, can we tell which one is bigger?”

Mr T used *Identifying Similarities and Differences* to compare fractions and used a chocolate bar *to make Connections* with a real example: “Hands up if you want one-third of my chocolate bar. Hands up if you want four-sixths of my chocolate bar.” Mr T encouraged *Generalising and Reasoning* when choosing the bigger portion of the chocolate bar: “How do you think we can make the two denominators the same? We come up with the generalisation that the size of the parts needs to be the same.”

6.4.1.3 Post-lesson reflections

In this section, I analyse the transcriptions of the Cycle 1 post-lesson reflections. A description of the reflection process is described in Chapter 4, Section 4.9.3.

While watching her videoed lesson segment, Ms K noticed *Connections* in the equation to the shape of the graph: “I think to show how the y^2 and the x^2 is giving us part of the circle, that relationship.” She identified *Recognising Patterns* as follows: “So, there I was helping them find the pattern and then I challenged what pattern they were seeing.” She later drew attention to *Identifying Similarities and Differences* between equations: “But that last one didn’t have a square, but it is still non-linear. So, they could see that all of them had a square except the last one, which had -1.”

After watching the video, Ms K gave an overview of her use of the CRIG framework, and she described her aim to make *Connections* between equations and graphs. “The aim was for them to link the form of the equations to the graphs." She used *Identifying Similarities and Differences* to distinguish between the graphs: "So, that was the whole; noticing the similarities of them but then also the difference with the hyperbola." She also made *Connections* to prior learning: "There was a little segment there where it was connecting to the past knowledge of index laws."

When asked if there were missed opportunities to use the CRIG framework, Ms K said she could have made further *Connections* to prior learning: "Last year they did linear equations where they would use tables of values to graph. I could have incorporated more of what they learned."

Ms M's video segment was about finding the ratio of the circumference and diameter to discover an approximation of pi. She recognised *Connections* through the questions she asked the students: "I'm asking and prompting questions, which is good, so I can get the answer out of them rather than just feed it to them myself." Ms M also identified *Generalising and Reasoning* through students' discussion when dividing the circumference by the diameter then using the results to make generalisations: "I'm looking at what they just did. I'm asking them to contribute what they found and see what they conclude from what they've done."

In his video segment, Mr T used circle diagrams to compare two fractions with different denominators. He identified *Connections* that could have been made when expressing his concern about the way he asked questions: "I should have maybe worded the question more openly because this was something that we did in the last lesson." Mr T used *Identifying Similarities and Differences* in acknowledging the representation of equivalent fractions: "I had the approach of trying to show the diagram of shaded fractions first rather than show it symbolically."

6.4.2 Cycle 2

The second cycle of Phase 2 PLP consisted of the PSTs' planning and teaching another mathematics lesson using the CRIG framework. This PLW was followed by the second post-lesson reflection activity of the PST watching and commenting on a video segment of their mathematics lesson. A second PLW (PLW2) was held after all PSTs completed the post-lesson reflection.

6.4.2.1 Mathematics lesson observations

In this section, I review PSTs' Cycle 2 lesson plans and mathematics lessons. In her lesson plan, Ms K aimed to deduce a general formula for the interior and exterior angle sum of polygons. She made *Connections* to prior learning of geometrical properties such as angles on parallel lines and used *Recognising Patterns* to develop a pattern to prove the angle sum of polygons. *Identifying Similarities and Differences* was noted in defining convex, non-convex, regular, and non-regular polygons. *Generalising and Reasoning* were used in developing a formula for the interior and exterior angle sum of polygons.

Ms K introduced *Generalising and Reasoning* to develop a formula for the angle sum of any polygon, stating: "I want you guys to have a go at forming a proof. So, what we want to do is calculate the interior sum of any polygon." She made *Connections* to prior learning when developing the formula, by saying: "So how did we prove the interior angle sum of the quadrilateral?" Ms K used *Recognising Patterns* to develop the angle sum of a polygon formula: "Can you find the pattern of what is going on between the relationship of the sides, the number of triangles?" Then she asked: "Now, we take away two for each of the sides. Have you been able to see a pattern between them?" Ms K used *Identifying Similarities and Differences* when modelling an alternative process for the students to identify the angle sum of a polygon formula. She encouraged students to consider *Generalising and Reasoning* when deriving the formula, stating: "You need to say why you are doing things. So here, what is your reasoning for putting this in?"

In Ms M's lesson plan on area of shapes, she identified *Connections* as a revision of perimeter and the relationship between perimeter and area. She referred to *Generalising and Reasoning* through the area of a circle formula. In the lesson, Ms M used *Identifying Similarities and Differences* to distinguish between regular and irregular shapes. She said: "A composite shape is a shape that looks like this. It doesn't look like a regular shape." When explaining the types of composite shapes, Ms M again acknowledged *Identifying Similarities and Differences* by stating: "So we have a composite shape. You can see that there are two different shapes in this. You have your rectangle and you have a triangle."

Ms M made *Connections* to prior learning when learning about the perimeter of a circle. She said: "Remember the perimeter is around only the outside of the shape." In the revision of pi (π) to circles, she stated, "Do you remember when we did the pi investigation? How did we find that every single circle has some sort of relationship with pi?" Ms M encouraged *Generalising and Reasoning* about the circumference of the circle when mentioning, "It's a semi-circle, so how do we know if that was a full circle? How would we find the perimeter of that?" *Identifying Similarities and Differences* were also used to explain the formula of the area of circles. Ms M said, "Area equals πr^2 which is the same as saying $\pi \times r \times r$", and this is in the two formulas for the circumference of a circle. So, you have the circumference equals π times the diameter, so this is the diameter so it's πd but another way of writing this. Can you see the radius is half of the diameter? So, you can either write $2\pi r$ because the radius is half of the diameter." Ms M supported *Generalising and Reasoning* when presenting π as the ratio of the circumference and diameter of a circle. "How come we have π for every single circle? Because the circumference divided by the diameter was always equal to π."

In Mr T's lesson plan on adding and subtracting fractions, he made *Connections* to prior learning in defining a numerator and denominator, *Recognising Patterns* when adding fractions, *Identifying Similarities and Differences* in knowing that adding fractions is different to adding whole numbers, and *Generalising and Reasoning,* as a rule to add numerators when denominators are the same.

In the lesson, Mr T used *Recognising Patterns* and *Generalising and Reasoning* when adding fractions. He said, "I would like you to predict what the next pattern will be." Mr T referred to *Identifying Similarities and Differences* to distinguish differences between fractions. He said, "And what else did you notice? What do you notice about the numerators?" Mr T encouraged *Generalising and Reasoning* to consider if adding whole numbers can be applied to adding fractions. He used an example of $\frac{1}{2}+\frac{1}{2}$, and asked the class if there was an error when the denominators were added: "So if a $1+1=2$, then, if I use the same thing, for a $\frac{1}{2}+\frac{1}{2}$, is $1+1=2$, and $2+2=4$, so it's over 4, right? Should it?" Using this example, he challenged the students about applying the incorrect use of whole number thinking to fractions.

6.4.2.2 Post-lesson reflections

In her video segment, Ms K developed the interior angle sum of a polygon formula, where she made a *Connection* between the angle sum of a polygon to the angle sum of a triangle and quadrilateral. Ms K commented on how she was making *Connections* to prior learning: "So, that was me trying to connect it back to their prior learning because we actually did a proof of a quadrilateral." Ms K also identified *Connections* to prior learning and *Similarities and Differences*: "He's found that each of them has a triangle in them, so that's not only the connecting prior learning but also the similarity between them." She saw *Recognising Patterns* in the number of sides of the polygon to the number of triangles formed: "So that's just me using the visual representations of getting them to show the pattern that minusing two from the number of sides and then, timesing it by 180."

Ms K used *Recognising Patterns* when she noticed students' structural thinking in finding the formula by developing a different pattern. "Because after seeing all the kids' responses, they found the pattern first before they found the point." She promoted *Identifying Similarities and Differences* when developing formulas, noticing how the students' different patterns helped her thinking. "I only had the triangles meeting at a point in my notes. So, I actually adjusted as I went. Because I saw the pattern they were working out and then linked it myself."

Ms K stated she would use the CRIG framework more: "If I were to do this again, I'd teach the patterning way, and I would incorporate the CRIG more." She identified *Recognising Patterns* as helping students to develop the formula: "I think they understood it better with the pattern—the relationship between the amount of sides and the number of triangles."

Ms M's reflection on the video focused on the lesson of finding the area of a bullseye on a dart board. She made *Connections* to prior learning of the area of a circle: "At the beginning, I did ask them to recall what the area of a circle was first." Ms M responded to *Recognising Patterns* in the dartboard as "asking them how to figure out the area of a dartboard without including the bullseye. That could have been kind of recognising patterns." She noticed *Identifying Similarities and Differences* between the exact form and the decimal form of the answer: "Kind of how to write something in exact form and not in exact form using your calculator." She made reference to *Generalising and Reasoning* in students' results: "I'm asking them to see what they can conclude from what they've done."

In Mr T's videoed segment on adding fractions, he acknowledged *Recognising Patterns*: "I tried to set up some patterns and then asking the kids to try to recognise the patterns." He also encouraged *Generalising and Reasoning*: "I've tried to incorporate generalisation in terms of asking them, 'What do you think would be the next pattern?'." Mr T referenced *Identifying Similarities and Differences* when asking students about the patterns: "I'm also asking them if they see any similarities in the patterns." He stated his intention for students to "identify specific similar features" by asking them, "What do you notice about the denominators and what do you notice about the numerators?" Mr T continued to use *Generalising and Reasoning* to question students' understanding of whole number addition concerning fractions: "I'm trying to consolidate the similarities that they've identified with fractions and generalise in a way that it will only work for fractions and not whole numbers." Mr T stated that to repeat the lesson, he would use *Connections* and *Identifying Similarities and Differences*: "I'd try to use connections to whole numbers and ask them to identify similarities and differences."

6.4.2.3 Professional learning workshop

The agenda for the Phase 2 PLW2 is shown in Chapter 4, Section 4.7.2.1, Table 4.4. In this PLW, the PSTs continued to learn about noticing structural thinking through the CRIG framework. The PLW began with the PSTs watching a video of a child solving missing number problems. PSTs were asked to identify the approach taken by the child to solve the problem. Vale (2013) identified that some students would use calculations to solve such problems, while relational, or structural thinkers would look for relationships between the numbers. Following the video, the PSTs completed an arithmetic number sentence (ANS) worksheet and then viewed the reflections video segment of Ms K's lesson 1. The PSTs then reviewed a sample lesson plan with the CRIG components.

The PSTs read an article by Vale (2013) about arithmetic number sentences before the PLW. On viewing the missing number problem video, the PSTs referenced the CRIG framework indicating where they noticed the student's structural thinking. Ms K noticed that the child relied on calculation and did not *Identify Similarities and Differences* between the numbers: "She's not seeing the difference between the two sides. She's just gone straight to the calculation." Ms K saw the student's structural thinking through *Recognising Patterns*: "She's showing a bit more relational thinking when she stated that you just go up and then you come back down the same amount."

Mr T noticed the student's lack of structural thinking when the numbers became larger: "I noticed when it gets to three-digit numbers, she's afraid of using relational thinking." He saw the student's reaction after a prompt that involved the CRIG framework: "The fact that she got it straight away after the CRIG prompt means that she does have relational understanding."

The PSTs considered how students could solve the ANS worksheet questions using a relational/structural thinking approach. Ms K noticed *Recognising Patterns* when partitioning numbers rather than making calculations: "The numbers and they go up or down by a certain amount. You can figure it out without doing massive calculations." Ms M considered *Identifying Similarities and Differences* as helpful: "It's thinking of the number on the left-hand side and the right-hand side as a combination of numbers that will add up on either side. So, you need to figure out the difference between one of the numbers." Mr T also saw *Identifying Similarities and Differences* to manipulate the numbers: "You make a number on the right-hand side that is similar to a number on left-hand side."

PSTs reviewed Ms K's video segment on finding the formula for the interior sum of a polygon. They discussed what CRIG framework component Ms K was attending to, what their interpretation of its use was, and what decision was made from it. Ms K mentioned the relationship between the number of sides and triangles drawn inside each polygon: "I was going through the relationship between the amount of sides and the number of triangles there, and how can we make a formula from that." She stated that the students had used an alternative pattern to derive the formula: "I wasn't anticipating them to show me the pattern result." Ms K's interpretation of the students' pattern changed her structural thinking: "They had seen the pattern when I hadn't. I had to change my explanation to make sure that I satisfied those who were doing the pattern." Ms M realised that her students had *Recognised Patterns*: "Try to see if they could figure it out, the gist of the general pattern from a table of values." Mr T acknowledged *Connections* with students' recent learning: "I saw the connection: 'We did the sum of the triangles in the previous lesson.' So, she's linking it back to that." He also gave an example of *Identifying Similarities and Differences*: "She's questioning them and asking what similarities can you use."

6.4.3 Cycle 3

The third and final cycle of the PLP began with the PSTs planning and teaching their final mathematics lesson using the CRIG framework. After teaching their lesson, the PSTs completed the post-lesson reflection activity of the PST watching and commenting on a video segment of their mathematics lesson. The third PLW (PLW3) in this cycle occurred after the post-lesson reflections.

6.4.3.1 Mathematics lessons observations

In this section, I review the PSTs' Cycle 3 lesson plans and mathematics lessons. In Ms K's lesson plan on quadratic equations, she identified the aim of the lesson as being to identify forms of a quadratic equation and to determine solutions to the equation. She made *Connections* to solving the quadratic equations from recognising where the graph of the parabola cuts the x-axis and then using algebra to find the solution to the equation when $y = 0$. She informed the class: "That's what our quadratics are like— parabolas. It crosses the x-axis here and here. So those two values are the solutions to the quadratic." *Recognising Patterns* was demonstrated in pointing out the similar structure of all quadratic equations and *Identifying Similarities and Differences* in quadratics was related to having a squared term but not necessarily having other terms that were the same. *Generalising and Reasoning* were developed in creating a general form of quadratic equations and recognising the shape of the graph.

Ms K's lesson used *Connections* to the real-world application of parabolas to strengthen students' understanding of quadratic equations. She related the quadratic equation to the graphs of a parabola: "Quadratics and parabolas go hand-in-hand: The visual representation of a quadratic is a parabola." Ms K reinforced the x^2 term in the quadratic as a *Connection* to the parabola: "We are doing quadratics. We are doing squares. We are doing parabolas." Ms K used *Identifying Similarities and Differences* of different forms of the quadratic equation to algebraically manipulate them to the generalised quadratic expression $ax^2 + bx + c$: "What we want to do is we want to create a generalised form of what a quadratic looks like, all right?" She also used *Identifying Similarities and Differences* to show that if the x^2 component of each equation is the highest power of x, then the curve is a parabola, but if this term has a different power of x, then it is a different non-linear graph: "This is not of degree two; it is a power of negative two. So, this one is not a quadratic." Ms K made *Connections* to prior knowledge of index laws and used *Identifying Similarities and Differences* between equations to determine whether it was a quadratic: "If you think back to index laws, we have got $\frac{1}{x^2}$ which is actually x^{-2}. Which isn't of degree two anymore. It is not the power of two. It is power of negative two. So, this one is, no, it is not a quadratic."

Ms M did not submit a lesson plan for this lesson. In the lesson on volume of a cylinder, she introduced revision questions starting with *Generalising and Reasoning* of the formula for the volume of the prism: "Calculate the volume of a prism. You have to tell me the formula." Ms M referred to the general formula of a prism, but it was not entirely clear whether she was encouraging students to generalise a formula or to remember the formula. She said: "What's the general formula? What am I doing normally to find the volume of any prism?" Ms M referred to the general formula continuously during the lesson: "Well, what was the formula we said? We said it was area of the base times the height." She made *Connections* between the prism and its name, stating, "Normally the name of that prism will give you a clue of what that base is."

The main lesson activity involved *Connections* of a real world problem to the volume of a cylinder: "This is a picture of the sinkhole. These are some pictures of what it looks like. What shape does it look like?" She continued to use *Connections* to prior knowledge: "What do we already know from the problem?" When solving the problem, Ms M encouraged *Generalising and Reasoning* to think about how to fill the sinkhole. She asked the class to consider this: "What do we need to know to solve this problem? What are we trying to find in the end?" Ms M's lesson outcome focused on students generalising an approach to solving a real world problem by connecting their prior learning of the volume of a cylinder formula to the problem and then making decisions about the reasonableness of their answer.

In Mr T's third lesson plan on stem-and-leaf plot graphs, he identified *Connections* to prior learning of column graphs to represent data and *Recognising Patterns* as how numbers are ordered on a stem-and-leaf plot. Mr T identified *Similarities and Differences* between column graphs and stem-and-leaf plots, and he used *Generalising and Reasoning* to show how a stem-and-leaf plot is useful for analysing data. Mr T collected students' birthdates to create a stem-and-leaf plot, the stem being the tens digit of the date and the units digit being the leaf. He encouraged the students to make generalisations about the graph from observing the results displayed.

In the lesson, Mr T helped the students see *Connections* to prior learning where they completed two questionnaires and identified *Similarities and Differences* between the questionnaire results: "Now what were the same things compared to our class. What's similar?" He used this form of questioning in several instances during the lesson when he asked, "If we took a similar survey in our class, do you think it'll be similar or different?" and when considering the different types of graphs to represent the data: "So there are different graphs that can answer the same question." Mr T encouraged *Generalising and Reasoning* when discussing graphs: "I want you to take a look at your graph and talk to the other person and tell them what the graph tells you?" When interpreting graphs, he asked the students to make generalised statements about the class: "What does what information say about the class? What does it say about the students?"

6.4.3.2 Post-lesson reflections

In the reflections in this cycle, the PSTs reviewed a video segment from their third mathematics lesson and answered questions based on Jacobs et al.'s (2010) AID model of noticing students' mathematical thinking which includes attention, interpretation, and decision making (see Chapter 3, Section 3.8.1). In these questions, the PSTs were given the opportunity to notice structural thinking through the CRIG framework:

1. Attention: Explain what component of the CRIG framework you are using and when you were using it.
2. Interpretation: Please explain how you are using this component of the CRIG framework at this point in the class.
3. Decision: Pretend you were to teach this again. Is there another component of the CRIG framework you could incorporate, and how?

Ms K's video segment was on quadratic equations. Responding to the attention question, Ms K acknowledged *Connections*: "I definitely saw connecting to prior learning because I was connecting it back to when we did the non-linear simultaneous." For the interpreting question, Ms K again cited *Connections* as helpful in understanding equations: "Mainly just to consolidate their understanding and linking things together, really. Just to deepen their understanding of what's going on within the equations." In the decision question, Ms K demonstrated an association between the graphs and the equations in finding the x-intercepts. Ms K acknowledged *Recognising Patterns*, suggesting that she would do this by noticing a pattern in graphs and their equations: "Rather than just doing three random graphs on the board, I'd probably link them to the forms of the equations and see if they can recognise any patterns from a factorised quadratic."

In the second stage of the post-lesson reflection, the PSTs were asked to watch the video again and stop the video at any time to indicate where the CRIG framework could be incorporated into that point of the lesson.

When Ms K watched the video again, she identified *Connections* to prior learning, that the equation $y = 0$, is the x-axis. “Well, that's creating a link there to prior learning as well. They’ve learnt that $y = 0$, and so if it crosses the x-axis it has to be $y = 0$.” She indicated *Recognising Patterns* between the different forms of the equations and their graphs: “They could find a pattern between a worded problem and put it into an equation and a graphical form, that’s a pattern.” She also indicated *Identifying Similarities and Differences* as “to see the different forms, the similarities and differences of how it could be represented.” Ms K considered *Generalising and Reasoning* as solving a quadratic equation: “That could be considered generalising the solutions of when crossing the x-axis.”

In Ms M’s reflection, she identified *Connections* to prior learning when answering the attention question: “Okay, so I think I was probably helping students, first of all, connect their learning.” She referred to *Generalising and Reasoning* as: “Generalising and reasoning by using information that we collected from the articles. They try to put that together to answer the problem and see how they could find the relationship.”

For the interpreting question, Ms M again referred to *Connections* to prior learning: “I think that they would have been connecting their learning by interpreting the data and trying to understand how they could use previous things they've learnt.” She also noticed how the students were *Recognising Patterns*: “They were definitely recognising (patterns) because they had to identify the relationships between the different pieces of information.” She again referred to *Generalising and Reasoning* in how students were able to put the information together to solve the problem: “They were definitely generalising and reasoning when they put all that information together. When they put all of that together, they generalised it and they solved the problem so that’s how they did it.”

In the deciding question, Ms M responded with a reference to *Generalising and Reasoning*: “I just should have said, Give me some suggestions of how we could use this information to solve the problem.” Ms M identified *Connections* “to help them focus more on connecting their learning with what they've learned” and *Recognising Patterns* as beneficial when working mathematically “by helping them recognise patterns so that they're working mathematically.”

Ms M watched the video for the second time and identified *Recognising Patterns* in the following way: “You can see as they’re recognising patterns when they’re discussing what they’re trying to find.” She also made *Connections* to the real world: “So they’re starting to understand the relationship between volume and different shapes and what that might look like in the real world.” *Identifying Similarities and Differences* was referred to when she described the difference between volume and capacity: “They're trying to understand what volume and capacity is. Are we finding the volume of the cement? What does it mean? Or is it the volume of the cylinder?” She identified *Generalising and Reasoning* as deeper conceptual understanding: “Thinking more about what volume means and conceptualising. So that would have been really helpful for Generalising and Reasoning. What is volume? What's capacity?” She expressed her understanding of *Generalising and Reasoning* in the following way: “Recognising the meaning and interpreting the information and trying to write that problem in their own words so that they’re prepared to think mathematically when they solve the problem and work mathematically.”

In his reflection, when Mr T viewed a lesson video segment on stem-and-leaf plots, he responded to the attention question in that he had not made any references to the CRIG framework during this part of the lesson. For the interpreting question, he did not give any further detail regarding the CRIG framework in his instructions. For the deciding question, he suggested that he would have used *Connections* to prior learning: “I should have made it more explicit connecting to their prior experience”, and he also acknowledged *Recognising Patterns*: “So I should have put one number on, and then got the students to see a pattern.” He distinguished *Identifying Similarities and Differences* in the following way: “With similarities and differences, I should have started the conversation of how this was different to other graphs. With the similarities, what sort of information or data type are we putting on this graph that was the same as the other graphs?”

When reviewing the video for the second time, Mr T made many comments about where the CRIG framework could be applied. He first identified *Connections* with prior learning about place value of a number on the stem-and-leaf plot: "I should have made the connection [that] the number 70 is comprised of the tens column, 7, and the ones digit column, zero. And I should have made it more explicit as a connection to a CRIG component." He realised that *Recognising Patterns* could have been used to demonstrate number positions on the stem-and-leaf plot: "I think when I've done three numbers, I should have stopped at that point and used the recognised patterns, and asked: 'What do you notice what I'm doing with these three numbers?'" Mr T noticed how he missed an opportunity to develop *Recognising Patterns* after a student identified a pattern: "That student just spotted a pattern there and I missed the opportunity to get the class to actually listen to her and actually draw out that particular." Mr T noticed *Identifying Similarities and Differences* in the different way numbers were placed on the stem-and-leaf plot: "With those three numbers, I should have asked the students about the placement of these three numbers: 'How are they different? Why is this number here and why is this number here?' and used that student's question about what if we had the same number as 'identifying' and use it as an opportunity to identify 'similarities'."

6.4.3.3 Professional learning workshop

The agenda for the phase 2 PLW3 is shown in Chapter 4, Section 4.7.2.1, Table 4.4. In this PLW, the PSTs undertook two activities as a final opportunity to recognise how they noticed structural thinking. The first activity involved viewing video segments and responding to three questions related to the Jacobs et al. (2010) AID model. The second activity required the PSTs to apply the CRIG framework to a mathematical question. The activity titled *Mathematical and Pedagogical Content Knowledge Worksheet* (Appendix K) required the PSTs to consider the components of the CRIG framework to teach the given question. The question given to the PSTs on this worksheet was on expanding a binomial product. This is a topic the PSTs would be familiar with from the Number and Algebra strand in years 9 or 10 of the NSW K-10 mathematics syllabus (NSW Board of Studies, 2012).

Activity One: Attending, interpreting, and deciding (AID) questions to video segments.
In the first activity, the PSTs watched three video segments and answered three attending, interpreting, and deciding (AID) questions. The following gives a brief outline of the PSTs comments from each video.

Video 1 Ms M: Volume of a sinkhole. The PSTs saw the *Connections* to prior learning of volume and the real-world problem. *Recognising Patterns* was considered when using formulas and *Identifying Similarities and Differences* recognised in the 3D solids that could be used to solve the problem. *Generalising and Reasoning* were identified in the students' understanding of what "calculating the volume" meant and recognising it as a rule. Ms M discussed how the CRIG framework was beneficial for her students in developing an understanding of volume.

Video 2 Mr T: Comparing graphs. The PSTs were able to make *Connections* between the graphs, they were *Recognising Patterns* in the data displayed, and they were *Identifying Similarities and Differences* between the types of graphs and how the data were displayed in different forms. *Generalising and Reasoning* were recognised when analysing the data in the graphs.

Video 3 Ms K: Quadratics. The PSTs made *Connections* to prior work on parabolas and linked the quadratic to the concept of area. *Recognising Patterns* was found in identifying the pattern of the squared term to create the parabola and *Generalising and Reasoning* was noted in how squaring the x-term changes the negative number into a positive value which develops a symmetrical graph to create the parabola.

Activity Two: CRIG in mathematical and pedagogical content knowledge.

The worksheet examples encouraged the PSTs to think more deeply about the alternative approaches to solving a problem using the CRIG framework. The PSTs were asked to describe how they were taught and how they would teach the question: *Expand* $(x+2)(x+3)$, which requires the use of the distributive law. Further questions explored how they would incorporate the CRIG framework into their lessons and what differences they believed using the CRIG framework would have in their teaching and the students' learning.

PSTs were familiar with the question and recognised the FOIL method to expand binomial expressions. FOIL is an acronym used by teachers for students to remember that to expand the binomial they must multiply the first (F), outer (O), inner (I) and last (L) terms in the brackets. They described that this was how they were taught to expand binomial expressions and how they would teach it. All PSTs commented that they would give more explanation about why the FOIL method be used, such as using a rectangle divided into sections and use the concept of area to multiply the terms together. The PSTs were able to identify how this question made *Connections* to other mathematical domains.

The PSTs showed insightful responses using components of the CRIG framework. Examples of the PSTs' comments include: making *Connections* to the distributive law of expanding the expression using the FOIL method and by connecting the distributive law to other mathematical relationships such as two-digit multiplication, probability and parabolic graphs. The process of *Recognising Patterns* related to the FOIL method, which Ms M called the "eye-brow" method. *Identifying Similarities and Differences* was shown in how changing numbers, pronumerals, signs and coefficients in the binomial expression can produce a different expansion; and *Generalising and Reasoning* as an ability to summarise the process and concept of expansion and to see it applied in other mathematical contexts.

When describing how the CRIG framework changed their teaching, Ms K stated that it made her think more deeply about the concepts and helped her to explain concepts to the students, so the concepts made sense. Ms M commented that it made her consider prior and future learning that connects to the topic she is teaching. The PSTs all tended to agree that the CRIG framework supported students' conceptual understanding, prevented them from following set procedures, and helped students to break down a problem by looking for prior learning, patterns, similarities and differences, and then make generalisations.

6.5 Exit questionnaire

In this section, I summarise the results from the exit questionnaire. Table 6.6 contains the PSTs' scores relating to what they regarded as easy or difficult about teaching mathematics using a score between 0 (easiest) and 100 (most difficult).

Table 6.6

Introductory and Exit Questionnaire Scores for PSTs

	Introductory Score			Exit Score		
Mathematics-teaching category	Ms K	Ms M	Mr T	Ms K	Ms M	Mr T
1. Lesson preparation	61	50	70	70	65	71
2. Creating an engaging lesson	100	72	80	80	65	62
3. Teaching strategies that engage students	94	74	80	50	65	61
4. Engaging all students in the activities	100	52	90	72	64	80
5. Students' understanding of mathematics	30	40	80	69	50	72
6. Using the CRIG components in teaching mathematics				51	39	63
7. Using the CRIG components to notice students' structural thinking				60	60	73

The PSTs' exit questionnaire responses differed from their introductory questionnaire scores. Ms K and Ms M scored 'preparing lessons' as slightly more difficult in the exit questionnaire, Mr T's score remained the same. All three scored this category in the mid-range (65 to 70) of difficulty. The PSTs scored 'creating an engaging lesson' and 'strategies to engage students' as less difficult aspects of teaching mathematics than they had indicated on the introductory questionnaire. Ms K's score for 'teaching strategies to engage students' decreased from a very high level of difficulty toward being easier (94 to 50), and her scores dropped from being the highest level of difficult (100) towards being easier for the categories 'creating an engaging lesson' (80) and 'engaging all students in activities'(72). For 'engaging all students in the activities', Mr T's scored this the most difficult of all the mathematics-teaching categories, his introductory questionnaire score was 90 which decreased in the exit questionnaire to 80. Ms M, who had initially scored this category as neither easy nor hard increased her score in the exit questionnaire toward being slightly more difficult.

Scores for the additional questions regarding the CRIG framework were varied. The PSTs did not score ‘using the CRIG components in teaching mathematics’ as difficult. Ms K scored it as 51 on the scale, Ms M scored it as the easiest of all her categories on the exit questionnaire, and Mr T scored it slightly towards difficult. All PSTs scored ‘using the CRIG framework to notice students’ structural thinking’ as more toward slightly difficult, although Mr T scored it as more difficult than did Ms K and Ms M.

The second part of the questionnaire, Questions 11 to 16 are aligned to the research contributing question of what are the PSTs’ understanding of structural thinking. These responses are to be used to identify the shift in the PSTs thinking regarding mathematical structure and structural thinking. Ms K saw structural thinking as being able to recognise interconnections between mathematics and using this understanding to manipulate problems compared to following a procedure to get an answer. Ms M described structural thinking as the reasoning that coincides with existing mathematical concepts that solidify their existence in facts brought about by processing, gathering and understanding the world around us. Mr T saw strucxtural thinking in using the CRIG framework.

Additional questions to ascertain PSTs’ understanding of the CRIG framework were included in the exit questionnaire. Ms K felt the CRIG framework was beneficial in helping her understand students’ thinking. Ms M stated that the CRIG framework gave her a deeper awareness of how mathematical structure is included in mathematical conceptual understanding. Mr T said the CRIG framework provided him with a scaffold for structural thinking and that they were the basics of mathematical structure.

6.6 Summary

The data analyses have provided indicators of the PSTs’ use of the CRIG framework to notice structural thinking. The results described here provide a summary of the findings from the introductory questionnaire, the quantitative analysis of TDC, and the qualitative analyses of three cycles of PLWs, mathematics lessons, post-lesson reflections and activities, and the exit questionnaire.

This chapter began with a review of the introductory questionnaire that identified the PSTs’ awareness of structural thinking at the start of the Phase 2. The results of this questionnaire showed that the PSTs had a simplistic understanding of mathematical structure at the beginning of this study. Following this review was a quantitative analysis of a general overview of the mathematics lessons, and the PSTs' reflections. This analysis described the proportion of mathematics lesson time that each PST spent in TDC. The qualitative analysis of each PST's teaching time was beneficial in identifying the TDC time in each lesson. The results indicated that most of the PSTs' TDC was spent in teacher-centred instruction with student communication. When these results were correlated with the lesson transcripts evidence of the PSTs use of the CRIG framework during each TDC category was identified. As expected, the PSTs mostly used the CRIG framework during teacher-centred instruction with student communication. This was expected as the PSTs were involved in this category more than the others. However, the results showed that the PSTs used the CRIG framework in their pedagogical strategies to explain the mathematical content during the mathematics lessons, demonstrating their deepening mathematical and pedagogical content knowledge. A further examination of transcripts of the mathematics lessons and post-lesson reflections identified PSTs' frequency of references to the CRIG framework. A qualitative analysis of the data from the PLW transcripts, mathematics lesson observations, and post-lesson reflection transcripts of the three PLP cycles presented exemplars of the PSTs' use of CRIG. This analysis identified how the PSTs' noticed structural thinking through their use of and references to the components of the CRIG framework. The PSTs' reflections of their mathematics lessons from the videos proved beneficial in supporting the PSTs noticing of structural thinking. The PSTs were able to notice when they used a component of the CRIG framework and how the effective it was in supporting their pedagocial content knowledge. Examples from the lessons demonstrated how the PSTs had developed a heightened awareness of noticing structural thinking through attending to the CRIG framework. The exit questionnaires provided a view of how the PSTs' awareness of structural thinking through the CRIG framework had changed from the start to the end of Phase 2. The PSTs' responses to these questions demonstrate their understanding of mathematical structure had improved over the course of the PLP. In particular, their ability to recognise the importance of mathematical structure as a key element in mathematical content and pedagogical knowledge..

The results of this chapter indicated that the secondary PSTs were able to use the CRIG framework to notice structural thinking. The development of their mathematical and pedagogical content knowledge improved over the course of the PLP, identifying that knowledge and use of mathematical structure through the CRIG framework is a viable and robust approach suitable to primary teachers

The next chapter is a discussion of the results from Phase 1 and 2 of this study. This chapter provides an overview of the two primary and three secondary PSTs' learning and understanding structural thinking through the CRIG framework. In particular, this chapter gives an account of how the PSTs learn to notice structural thinking in what they say and what they do and how undersrtanding structural thinking impacts on their pedagogy when teaching. The discussion in Chapter 7 illustrates the effectiveness of the CRIG framework as a mechanism to understand and use mathematical structure, but also to enhance the PSTs' ability to notice structural thinking, thus improving their PCK.

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Chapter

7

Discussion

7.1 Introduction

This chapter contains a discussion of the findings given the three contributing research questions:

1. What are the PSTs' understandings of structural thinking, pre- and post-implementation of the professional learning program (PLP)?

2. How do PSTs use structural thinking in their mathematics teaching?

3. How effective is the CRIG framework in helping PSTs to notice structural thinking in their teaching?

7.2 PSTs' understandings of structural thinking

In this section, I discuss the PSTs' initial ideas about mathematical structure and the impact on their teaching. Further, I discuss how these initial views of structural thinking changed as a result of the PLP and lesson cycles.

The PLP, consisting of the cycle of PLWs, mathematics lessons, primary PSTs' post-lesson interviews, and secondary PSTs' post-lesson reflections, contributed to the PSTs' understanding of structural thinking. The PSTs' initial perceptions obtained from the introductory questionnaire when compared with the exit questionnaire, as reported in Chapter 5, Section 5.5, and Chapter 6, Section 6.5, demonstrated the PSTs' growth in their understanding of structural thinking.

During the PLP, the PSTs' exposure to structural thinking was through the CRIG framework. The conversations during the learning community experience of PLWs exposed the PSTs' understanding of structural thinking. Planning and teaching mathematics lessons enabled the PSTs to develop further an understanding of structural thinking through the CRIG framework. Reflections on their teaching when viewing the videos gave further evidence of the PSTs' understanding of structural thinking when noticing the CRIG framework in their teaching.

The introductory questionnaire served to gain initial insights into the PSTs' understanding of structural thinking. Neither group of PSTs had been exposed to structural thinking or the CRIG framework directly in their teaching practice before the study commenced, but they were familiar to some extent with the concept of mathematical structure through their university undergraduate program and the definition given in the introductory questionnaire.

The secondary PSTs could not articulate an understanding of structural thinking as clearly as could the primary PSTs. The primary PSTs' views of structural thinking aligned more with understanding mathematical thinking as being relational and included references to some components of the CRIG framework. The secondary PSTs' definitions were not indicative of a sophisticated understanding of structural thinking. Their views were that structure was used as a method to find the solution to a problem.

The PSTs' teaching experience was limited to their university professional experience program. Researchers have identified how PSTs' limited experiences influence what they attend to when teaching. Cavanagh and Prescott (2007) found that PSTs plan lessons that are teacher-centred and focus on their teaching actions rather than their students' learning. Similarly, Star and Strickland (2008) found that PSTs were not good at noticing mathematical content. Mason (2002) also asserted that PSTs lack experience in recognising and using classroom interactions effectively to promote mathematical understanding. The PSTs' lack of professional experience before this study most likely influenced their fundamental understanding of the CRIG framework and their ability to notice structural thinking in themselves and their students. More teaching experience would provide opportunities to recognise the framework components in their lessons and to communicate it when teaching. However, it cannot be assumed that the PSTs could sustain the use of the CRIG framework without further assistance. This assumption is reflective of the research of Prescott and Cavanagh (2006) and Grootenboer (2006a). Prescott and Cavanagh's findings indicated that secondary mathematics PSTs had fixed views about mathematics teaching that were inclined toward procedural methods. Similarly, Grootenboer found in his study that primary PSTs reverted to their views and beliefs about mathematics from their schooling when teaching, which was often procedural.

When initially considering their teaching practice, three PSTs, Ms S, Ms K, and Mr T, described themselves as adopting a teacher-centred and procedural pedagogical approach. Only Ms N and Ms M considered their pedagogy to be focused on conceptual understanding. Mason et al. (2009) maintained that the notion of structure bridges the gap between procedural and conceptual learning. However, at the initial stage of the study, the PSTs did not know how mathematical structure was relevant in their teaching. Their inexperience with applying the notion of structure to their teaching limited their ability to consider the link between procedural and conceptual understanding that Mason et al. had identified.

Improvement in the PSTs' understanding of structural thinking was evident in references to the CRIG framework drawn directly from the statements they made during the PLWs, mathematics lessons, the interviews and reflections. Ms S's frequency of references to the CRIG framework increased significantly in her final mathematics lessons, as shown in Table 5.3 (Chapter 5). In her final interview, Ms S showed more confidence in talking about how she applied the CRIG framework. She did so after stating she did not think she had used it in her first two lessons. Ms M frequency of CRIG references in her lessons was consistent throughout; however, there were limited references to the framework in her reflections during Cycles 1 and 2. In her final reflection, Ms M spoke in detail about the CRIG framework and her frequency of references to the CRIG framework increased significantly in this reflection activity, as shown in Table 6.5 (Chapter 6).

In the final PLW, Ms K, who had applied the CRIG framework consistently in her lessons and demonstrated her ability to notice structural thinking in her video reflections, showed deep insight about structural thinking. In her descriptions of teaching mathematics, she wrote: "You guide the students to the concepts, and structure your practice to facilitate deeper thought as to what and how things made sense." Ms N had also shown a consistent appreciation of structure throughout the study, and in her final interview she made several insightful statements reflective of Mason's (2002) response to acting in-the-moment. She also referred to the benefits of structural thinking for students. In line with Mason's acting in-the-moment, Ms N's insightfulness was noted in her comment regarding the PSTs' awareness of structure when teaching mathematics. She stated: "It has drilled home that the most amount of thinking and learning is happening in the moment."

In the exit questionnaire, both groups of PSTs were able to articulate positive aspects of structural thinking. Ms S's view of structural thinking was related to how the CRIG framework promoted relational thinking, whereas Ms N regarded it as a thinking skill that related mathematical properties. Ms K described how the CRIG framework had been beneficial for her understanding of students' structural thinking. Ms M saw structural thinking at the core of how students reason and Mr T noted that watching and reflecting on his teaching helped him to understand structural thinking.

When comparing the changing views about teaching mathematics between the introductory and exit questionnaire, the PSTs regarded teaching mathematics as being less difficult when using the CRIG framework. It seemed that the PSTs believed that the study had improved their understanding of mathematical structure, and the CRIG framework helped them to teach mathematics less procedurally.

Awareness of the CRIG framework encouraged the PSTs to reflect on their understanding of the lesson content and the best pedagogical approach for delivering the content. The PSTs' thinking shifted as their pedagogies changed to accommodate the CRIG framework and a deeper understanding of the mathematical content. However, the procedural approach demonstrated by the high proportion of TDC teacher-centred time remained, despite the PSTs' increasing structural awareness.

7.3 PSTs' use of structural thinking in their teaching

In this section, I discuss the PSTs' use of structural thinking and explain how attending to the CRIG framework when teaching caused a shift in their PCK, resulting in a greater understanding of structure. Also explained are the factors and constraints that contributed to PSTs' use, or lack of use, of the CRIG framework in their lesson planning and teaching. This section concludes with an overview of the efficacy of the CRIG framework and the PSTs' views of the CRIG framework as strategy to develop effective

7.3.1 PSTs' pedagogical shift

The CRIG framework encouraged the PSTs to give more consideration to their PCK PCKin their lessons. Star and Strickland (2008) suggested that primary PSTs lack mathematical content knowledge, and Prescott and Cavanagh (2007) found that secondary mathematics PSTs tended toward a traditional teaching pedagogy. However, the results suggest that the primary PSTs in this study improved their conceptual understanding of mathematical content because they focused on the CRIG principles. While, the secondary PSTs attempted alternative less teacher-centred pedagogical practices when teaching.

The primary PSTs' pedagogy was predominantly a traditional teacher-centred approach, for Ms S more so than for Ms N. Traditional teacher-centred pedagogy is usually associated with a procedural manner of teaching that is typically identified in secondary mathematics teaching (Lokan et al., 2003). Use of the CRIG framework in each of the primary PSTs' lessons demonstrated that they were attempting to teach for conceptual understanding but within a teacher-centred learning approach. This attempt reflects the PSTs' use of the CRIG framework as a reason for their pedagogical shift from a procedural to a conceptual approach within what was essentially a teacher-centred classroom.

The primary PSTs' knowledge of mathematical content was not measured directly, but from the exit questionnaire, the primary PSTs' scores reinforced the positive experience the PSTs had in improving their confidence in mathematics teaching. This positive result could reflect their improved content knowledge assisted by using the CRIG framework. Grootenboer (2006b) found the primary PSTs tended to be anxious and to dislike mathematics based on their own experiences as a learner. In this study, the CRIG framework enabled them to develop more confidence in developing pedagogies to better reflect the concepts.

For example, Ms S struggled with implementing the CRIG framework in her first two lessons. Her focus was on explaining the content, and she felt that she was not using the CRIG framework at all. The primary PST's focus on content over pedagogy was noted in the second post-lesson interview with Ms S. When asked if she used the framework, she hesitated and said, "I couldn't really, like I said, I didn't use any of the CRIG, well, most of the CRIG components". Ms S considered the CRIG framework as assisting her pedagogy and did not realise she was using it while teaching the content, albeit in a predominantly procedural manner. Ms S had been applying the CRIG framework despite thinking she was not: "When I was doing the lesson, I think it was all about explaining what to do rather than identifying the CRIG framework."

By the end of the research, Ms S had developed an understanding of structural thinking through the CRIG framework. She explained that structural thinking involved knowing the relationships between mathematical concepts and that the CRIG framework helped her to do this. In her third lesson, Ms S demonstrated a better understanding of the CRIG framework and how using the components of the CRIG framework had helped her teaching. In the exit questionnaire, Ms S identified the components of the CRIG framework as supporting her understanding of the structural thinking. She articulated this concisely when describing the CRIG components as being an integrated approach when noticing students' structural thinking. In the interview after her first mathematics lesson, Ms S believed that the CRIG framework was a sequence of procedures that needed to be followed. However, her statement in the exit questionnaire "that when students were able to see these differences and similarities, they were able to generalise certain rules and patterns to help them see the relationship of mathematical concepts" was a significant transitional shift in Ms S's initial understanding of the CRIG framework.

Two of the secondary mathematics PSTs, Ms K and Ms M, focused on *Generalising and Reasoning* as a single component of the CRIG framework, suggesting their pedagogical practices were less traditionally procedural. This focus on *Generalising and Reasoning* is considered the result of the PSTs encouraging the students in the lessons to interact in the discussions and posing questions that encouraged *Generalising and Reasoning*. The results suggest that the student-centred learning environment promoted by the secondary PSTs focused more on pedagogy than on delivery of the mathematical content. This finding supports the PSTs' high proportion of time in TDC with student communication. These results showed that the CRIG framework supported the secondary PSTs' pedagogical practices to deepen structural awareness.

7.3.2 Lesson planning and teaching

Given the insights of their peers and my influence as a mentor, the discussion in the PLWs involving the lesson plans enabled the PSTs to include the CRIG framework explicitly in their teaching. When planning their lessons, the PSTs used a lesson-planning template with the CRIG framework identified. Even so, there were some constraints regarding the implementation of the framework in practice.

One constraint was the PSTs' difficulty in differentiating the learning experiences for the varying ability levels within the class. Given the PSTs' lack of teaching experience, they found it challenging to differentiate their lessons for all ability levels in the class and their lesson planning often focused on one ability level. Ms K's accelerated class was taught challenging content above what the average cohort for this year group would be taught. Ms M gave her top-streamed class opportunities to be involved in collaborative work, but Mr T's middle-ability class was given mostly teacher-centred instruction. The two primary classes were of mixed ability, but there was a tendency for the primary PSTs to focus their attention on the same groups of students. Ms S indicated she only asked questions of students she knew could give the correct answer, which calls into question how the PSTs could represent the CRIG framework in their lessons while accommodating different mathematical abilities.

Another issue regarding the PSTs' delivery of the planned lesson was time allocation. The PSTs attempted to develop structural thinking. However, this was difficult given the need to complete the prepared lesson content within the allocated time. The PSTs focused on completing the lesson as planned and pushing forward to complete the content. In their study, Cavanagh and Prescott (2010) expressed the PSTs' concern that they needed to follow the supervising teacher's instructions and complete the lesson on time. This concern was evident in Ms N's second lesson, where the lesson continued for an extended amount of time to complete all the planned activities.

In all lesson plans, there was evidence of the PSTs' attempts to align the content with CRIG components, but overall there was a lack of detail about where the CRIG processes would be reflected explicitly in the lessons. In the PLWs, Ms S and Ms N explained that the CRIG framework had helped them in preparing the mathematics lessons and in the lesson plans they made statements that included the CRIG components. However, there was little evidence in their lesson plans of how they intended to use the framework in their lessons or when evaluating student learning. Practical constraints such as time limited the PSTs' application of CRIG in the lesson implementation, but it was more evident that the PSTs lacked mathematical content knowledge and pedagogical experience as teachers to fully use the CRIG framework (Star & Strickland, 2008). The lack of experience limited their ability to identify where the CRIG components could be applied, and where they did refer to CRIG, these references were quite superficial. Choy (2013) also found that teachers tended to focus on the superficial aspects of a task when planning a lesson. He used the notion of 'productive mathematical noticing' to identify whether primary mathematics teachers' concerns in lesson planning were worthwhile. Choy discovered that teachers changed their focus on the task to be more productive when working in a learning community. This focus was achieved in this study to some extent. However, although the PLWs enabled discussion of lessons and observation of peer teaching, there may have been insufficient time devoted to peer and researcher review of their developing lesson plans. Further collaboration time within their learning community in reviewing lesson plans could have supported the PSTs further in using the CRIG framework effectively.

All PSTs may have been limited in their use of the CRIG framework because they did not have sufficient familiarity with the mathematical content and relationships to embed the CRIG components in their lessons. Choy (2013) found that primary teachers do have difficulty in making sense of the mathematics they are to teach, which explains some of the issues that the primary PSTs had in incorporating the CRIG framework into their lessons. The secondary PSTs had better content knowledge but lacked the pedagogical content knowledge to make full use of the framework in their teaching.

The PSTs' ability to implement the CRIG framework into their mathematics lessons could have been improved through close attention lesson preparation of planning, teaching, reflecting, and revising the lesson before teaching the same or similar lessons. Choy (2014) found lesson preparation necessary for teachers to develop noticing skills, yet there remains a lack of research about how to effectively develop noticing skills (Lee & Choy, 2017). However, Lee (2019) did find that PSTs involved in a lesson study cycle improved their noticing expertise in reviewing and planning lessons. Hence, the additional focus and time devoted to noticing structural thinking in the PLWs may have enhanced CRIG understanding and use in the mathematics lessons.

7.4 Efficacy of the CRIG framework to develop PSTs' pedagogical content knowledge

The PSTs' understanding of mathematical structure and ability to notice structural thinking demonstrated that the CRIG framework of mathematical structure provided these PSTs, as novice teachers, with a mechanism to develop their PCK. Some of the PSTs in this study had some prior knowledge of the CRIG framework and were able to articulate the framework in their description of mathematical structure in the introductory questionnaire. Ms S and Ms N identified making connections, patterns and generalising in their definition of mathematical structure, and Mr T identified all components of the CRIG framework in his definition. At times, in this study, the PSTs were not always articulating the components directly in their mathematics lessons. However, they were using the components when delivering the mathematical content through their pedagogical practice. The primary PSTs' use of the CRIG framework was dominated by a single CRIG component that was associated with the lesson topic; for example, Ms N's first lesson on patterns and her third lesson classifying 3D objects which focused on identifying similar and different qualities of each object. In Ms S's second lesson on estimating length, she used generalising and reasoning far more than the other components by encouraging the students to make assumptions and then to check the accuracy of their assumptions.

The secondary PSTs used the CRIG framework in their pedagogical practices. Ms M, for example, would continually use connections to remind students that the content knowledge for each lesson was built on from the previous lesson. Ms K's pedagogical practice in her second lesson used patterns to find the formula for the sum of interior angles of a polygon and Mr T's pedagogical approach in his final lesson on stem-and-leaf plot graphs used similarities and differences to compare different forms of a graph.

By the end of the study, the PSTs had demonstrated an understanding of the CRIG framework in further development of their PCK, particularly in their use of the components when instructing or communicating with students. The use of the CRIG framework was helpful in developing Ms K's PCK. In her second mathematics lesson, Ms K used the CRIG component of recognising patterns, instead of a learning a rule, for students to develop a structural thinking approach to learning a formula for the angle sum of a polygon. As the students had discovered a different pattern, not considered by Ms K, she was required to reflect on the mathematical and pedagogical content of this problem and act in-the-moment to determine the correctness of this new approach. The CRIG framework proved useful for Ms S in building a deeper structural understanding of number properties, demonstrated when a student drew a five by three array of dots instead of a three by five array. Ms S was forced to consider the commutative law for multiplication and the students' structural thinking in knowing that the two different expressions represented the same amount.

The simplicity of PSTs' learning to understand and use the components of the CRIG framework supported their confidence to notice structural thinking. Ivars et al. (2018) identified the need for a specific framework for PSTs to help them learn to notice effectively. The CRIG framework provided this point of focus. Ms S stated in her exit questionnaire "I did not find this framework difficult to comprehend. When I think about the lessons that I have taught, there are many aspects of the lesson that use either one or most of the components". The CRIG framework also proved to be robust in its versatility and application to different mathematical content as well as being applied to different pedagogical practices. For example, when the primary PSTs in Phase 1 of the study used the framework to teach content for year 1 students, their pedagogical practices were more often teacher-centred with some student-centred activities. Ms S and Ms N both identified questions that included the CRIG components as effective pedagogical practices to noticing structural thinking.

The ability of the all the PSTs to understand the CRIG framework and to use it as support for understanding the content taught or as a part of the pedagogical approach taken demonstrated its simplicity as a practical and useful tool for teachers of mathematics. The secondary PSTs content knowledge was established from their extensive mathematical background in their university studies. The CRIG framework, however, proved useful for them in communicating this knowledge. The PSTs were communicating the mathematical relationships of content through their pedagogical strategies. Ms K actively encouraged students to be involved in the lesson by asking questions that encouraged them to explain their thinking. She felt that she could probe students' structural thinking through the connections, patterns, and similarities and differences. Ms M created activities where the students were involved in group work and interacted with each other; she felt that the using the CRIG framework provided an opportunity to easily apply positive pedagogical practice. Mr T stated that his focus on using the CRIG framework in his mathematics lessons allowed for a constructivist pedagogical approach. He believed that pedagogy involving the CRIG framework allowed students to make conjectures about the mathematical content, as opposed to a traditional teacher-centred, transmissionist pedagogical approach.

The PSTs identified the effectiveness of the CRIG framework in the exit questionnaires. Ms K stated that the components of the CRIG framework were beneficial to developing student understanding of mathematics and helped her when sequencing her lessons. In particular, the framework facilitated students' deep understanding of the relationships between mathematical concepts. Ms K said that the four components were effective in helping her to notice structural thinking by allowing her to determine if the students understood the mathematical concepts or were following a procedure.

Ms M believed the CRIG framework should be familiar to all teachers of mathematics. She stated that the CRIG framework helped her notice students' structural thinking through students' ability to conceptually understand maths. Ms N believed that the CRIG components had helped her to understand students' structural thinking when learning mathematical concepts. She said that CRIG framework highlighted that every student processes and understands new concepts differently.

7.5 Efficacy of the professional learning workshops in helping PSTs to notice structural thinking

In this section, I discuss the effectiveness of the PLWs to develop the PSTs' ability to notice structural thinking through the CRIG framework. Following this is a discussion of how video reflections improved the PSTs' ability to notice structural thinking through the CRIG framework.

7.5.1 PLWs and noticing structural thinking

The PLWs were integral in introducing the PSTs to the notion of structural thinking that was well beyond their pre-conceived ideas portrayed in the introductory questionnaire. The pedagogical model employed in the PLWs allowed for authentic sharing of ideas between me as a mentor and the PSTs as a small learning community, in a similar way as reported by Star and Strickland (2008). The PSTs had opportunities to discuss different forms of pedagogy in a supportive environment. The sharing through discussion of videoed segments of each PST's mathematics lessons provided powerful opportunities to compare and contrast each other's understanding and application of the CRIG framework.

Goos and Bennison (2004) found that PSTs working in a learning community were able to define their own academic goals and helped each other to link theory and practice. Similarly, Cavanagh and Garvey (2013) found that PSTs working in a learning community developed a bond where PSTs learned from each other, were reflective about pedagogies, and could connect theory to practice. The PSTs benefited from listening to the views of their peers. For example, the primary PSTs, Ms S and Ms N, shared their thoughts on the commutative law during the third PLW. Both PSTs were able to recognise the structural representation of drawing arrays to demonstrate the commutative law while improving their structural understanding.

As beginning teachers, the PSTs were applying new pedagogical skills to content that they had not taught before. The introduction to structural thinking through the CRIG framework could be regarded as an extra burden for the PSTs to consider in-the-moment in their teaching. However, the evidence in this study indicates that the PSTs were comfortable with identifying and including structural thinking in their lessons. The comments from the secondary mathematics PSTs in their final PLW indicated they could apply the framework to an unseen mathematical question. These comments demonstrated how their understanding of structural thinking afforded the PSTs a deeper understanding of PCK.

During the final PLW, the secondary PSTs developed new pedagogical approaches by applying the CRIG framework to a common algebraic skill of using the distributive law to expand a binomial product from the K–10 mathematics syllabus (NSW Board of Studies, 2012). The PSTs related the CRIG framework to the binomial expansion question by making connections to the distributive law, quadratic equations, and parabolic graphs. They acknowledged the expressions when signs are different, and they generalised the result with the general quadratic form.

The secondary PSTs were able to articulate how they would teach a binomial expansion using the CRIG framework: how CRIG would be used, how it changed their pedagogy, and how they could notice students' structural thinking. The secondary PSTs could also explain how the framework developed students' and teachers' conceptual understanding and influenced their own pedagogical practice. Ms K referred to her use of the CRIG framework in helping students to make connections with other learning and looking for patterns in order to seek generalisation. Ms M explained how CRIG improved conceptual understanding because it prepares a teacher to teach holistically by causing the teacher to consider mathematical relationships. Mr T's central insight was that it forces teachers to look for the structure of a problem rather than using a procedural learning process.

The data presented in Table 5.3 (Chapter 5) and Table 6.4 (Chapter 6), of the PSTs' references to the CRIG framework in the TDC, exposed some differences between the primary and secondary mathematics PSTs' use of the CRIG framework. The lesson topic appears to have influenced the primary PSTs' use of the CRIG framework as there was a tendency to focus on a single component of the CRIG framework, which varied for each lesson. It is worth acknowledging that the lesson topic was not the choice of the PSTs, as it was determined by the teaching program of the professional experience supervising teacher. There was no evidence that the supervising teacher had expectations that the

PSTs had to follow; however, as Cavanagh and Prescott (2010) claimed, the PSTs may have been concerned that they were following their supervisor's guidance.

7.5.2 Reflections on video segments of PSTs' teaching

The primary PSTs viewed, then discussed their peer's video segments of lessons during the second and third PLW. Secondary mathematics PSTs initially viewed their video segments with me as the mentor during reflection and then viewed and discussed their peers' video segments in the second and third PLW.

Video, when used as a reflective tool for PSTs to view themselves and their peers' teaching, has produced positive learning outcomes for PSTs. A number of studies have supported the use of video reflections in undergraduate teacher education programs (Beswick & Muir, 2013; Santagata, Zannoni, & Stigler, 2007; Star & Strickland, 2008) Other researchers have found that video reflections were helpful for teachers to learn to notice their students' mathematical thinking (Philipp et al., 2007; van Es & Sherin, 2008). In the present study, PSTs focused on analysing their practice through video, which assisted in developing their knowledge and identification of CRIG.

Primary PSTs referred to the CRIG framework superficially in their lesson plans and the interviews. However, they were able to articulate a greater understanding of CRIG when viewing the corresponding video segments in the PLWs. During the third PLW, the primary PSTs demonstrated an increased awareness of the CRIG framework. After watching the video segment, Ms S identified having missed an opportunity to use the CRIG framework when she failed to notice a student's structural thinking. She drew upon her new-found knowledge of the commutative law of multiplication to explain the generalisation in the lesson. Ms S had not previously noticed the commutative law in her teaching. Discussion between the two primary PSTs demonstrated that both had gained a deeper structural understanding of the number laws and relationships, which provided insight into structural thinking afforded by the lessons.

In her mathematics lessons, Ms N appeared to be implicitly using the CRIG framework in her teaching without making direct references to the individual components. Ms N indicated that her failure to identify the CRIG framework while teaching resulted from her inexperience as a classroom teacher, reinforcing what Star and Strickland (2008) found that PSTs are not good at noticing classroom events, mainly when dealing with mathematical content. Ms N's observation was that further experience would develop her understanding and familiarity with the CRIG framework so that it would become "implicit in your teaching".

The secondary mathematics PSTs also benefitted from the reflection on video segments. Being able to think flexibly and consider alternative approaches to solving problems, rather than relying on a standard procedure, is an essential consideration of mathematical structure (Richland et al., 2012). When viewing her videoed segments, Ms K noticed that her thinking was more flexible and indicated how it improved her understanding of structural thinking. In her second lesson on the angle sum of a polygon, Ms K considered an alternative solution to a problem presented by a student, digressing from her original plan. After the video reflection, she recognised how she diverged from her prepared lesson plan and adjusted her communications with the students to include the alternative approach. Ms K asserted that the student had seen a pattern that she had not and that she had changed her thinking to accommodate the alternative pattern, fundamental to CRIG. Ms K confirmed that she had to be flexible in her thinking to acknowledge the student's alternative approach. In this sense, she was adapting her teaching to act-in-the-moment (Mason, 2002).

In Ms K's videoed segment of her third lesson on quadratic equations, she recognised, on reflection, that she had considered an alternative procedure for teaching quadratic graphs. Ms K was able to notice the structural relationship between a parabolic graph and a quadratic equation during the video reflection. Ms K's pedagogical approach involved identifying the relationships between quadratic equations and associated parabolic graphs from the patterns representing different forms of the equation's general and factorised form. In essence, she was making meaningful connections that lead to a generalisation between a parabola's x-intercepts and finding the solution of a quadratic equation when $y = 0$.

Ms M demonstrated further evidence from the video reflections of her structural understanding. After watching the video segment of her final lesson, she noticed that the students were developing a greater conceptual understanding of the content through their ability to generalise and reason. Ms M stated in her reflection that the students were demonstrating structural thinking when they could recognise, then interpret, mathematical information in their words.

7.6 Summary

The PSTs' involvement in this study enabled them to notice and develop structural thinking through the CRIG framework. The PSTs' responses given in the interviews, reflections, and questionnaires indicated that they understood the concept of mathematical structure and could notice features of structural thinking by providing exemplars in their teaching and that of their peers. However, in many instances, the PSTs were not aware they were using the CRIG framework. They unconsciously used the CRIG framework as a part of their pedagogical practice, and in doing so, they were developing personal structural understandings while identifying and encouraging students to think structurally.

The CRIG framework contributed to PSTs' deeper awareness of mathematical structure. Their definitions of mathematical structure and structural thinking at the exit questionnaire included their views about making sense of mathematics, developing conceptual rather than procedural understanding, seeking mathematical relationships behind the concepts, and manipulating problems to connect mathematical properties.

The framework also helped the PSTs to understand their students' thought processes and to notice how students' learning and levels of engagement changed, especially when dealing with abstract mathematical concepts. It was also helpful in lesson planning because it provided a useful scaffold for PSTs to sequence mathematical concepts that would facilitate noticing structural thinking. The PSTs' use of the CRIG framework supported their students' reasoning, problem solving, generalising of rules from patterns and seeing relationships between mathematical concepts.

This study presents the CRIG framework of mathematical structure as a viable and effective approach to the teaching of mathematics. From the results of this study, a teacher's PCK can benefit from adapting the CRIG framework to their thinking and learning when teaching mathematics. A mathematics teacher's metacognitive approach to thinking mathematically is enhanced through an understanding of structural thinking. The CRIG framework can enable this understanding, demonstrating not only benefits for the teacher, but also for students. The PSTs in this study stated they noticed their students' structural thinking when they applied CRIG framework in their pedagogical practices.

The following chapter concludes this thesis. A summary of the main findings, the limitations of this research and implications for further research are given. The thesis finishes with an outline of implementing the CRIG framework and testimonials from the two participating PSTs as practicing teachers on their current use of the CRIG framework.

Chapter

8

Conclusion

8.1 Introduction

This chapter begins with a summary of the main findings of the study. Next, the limitations of the study are presented, along with the study's significance and implications for further research. The thesis concludes with testimonials from some of the participating PSTs, as beginning teachers, followed by concluding remarks from me as the researcher.

The present study, as described in Chapter 4, Section 4.6, was completed in two phases over two years. Phase 1 involved two primary PSTs and Phase 2 involved three secondary mathematics PSTs, all of whom were in the final year of their teacher education degree. PSTs participated in one of two PLPs, one for primary PSTs and one for secondary PSTs, during their final professional experience. Each PLP consisted of three cycles of a PLW, planning and teaching a mathematics lesson that was videoed and a post-lesson interview for the primary PSTs and a post-lesson reflection for the secondary PSTs. At the end of the third cycle the PSTs completed an exit questionnaire.

8.2 Summary of the main findings

Overall, the findings from this study indicate that engaging with the CRIG framework proved to be effective in deepening the PSTs' mathematical content and pedagogical content knowledge. However, although components of the CRIG framework were evident in the PSTs' mathematics lessons, their awareness of using the framework when in-the-moment (Mason, 2002) of teaching was not always reflected in their communications or pedagogical approach.

8.2.1 Individual PST's understanding of structural thinking

There were varying levels of understanding of structural thinking among the five PSTs. One primary PST, Ms S, and one secondary PST, Ms M, did not demonstrate an understanding of the framework until the end of the study. Ms S's final lessons recorded the highest frequency of the CRIG framework of all three lessons, as did her post-lesson interview and Ms M's final post-lesson reflection recorded her highest number of references to the framework. One primary PST, Ms N, and one secondary PST, Ms K, used language in the post-lesson interviews or reflections that demonstrated an understanding of structural thinking from the outset of the study, and they used the CRIG framework in each lesson. The remaining secondary PST, Mr T, showed some understanding of structural thinking. He consistently used the CRIG framework in his lessons and demonstrated awareness when he used the CRIG framework in the post-lesson reflections of his video segments. However, in the post-lesson reflection of the video segment from his final lesson, Mr T commented that he had not used the CRIG framework and had missed opportunities to use the CRIG framework during this video segment of his lesson.

8.2.2 Differences between the primary and secondary PSTs

The primary PSTs were very focused on mathematical content and delivered it through a predominantly procedural approach. However, the primary PSTs' structural thinking improved as a result of using the CRIG framework, and this also improved their conceptual understanding of the mathematics content taught. In their lessons, the primary PSTs used the CRIG framework to develop students' structural awareness through a pedagogy that focused on developing a conceptual understanding of the mathematical content. The secondary PSTs did not generally use a procedural approach even though there was significant teacher-centred time. The secondary PSTs appeared to break down the traditional mathematics lesson. For example, while spending time at the front of the class, the secondary PSTs, Ms K and Mr T, encouraged student participation, and Ms M had more of her lesson time in student-centred communication than any of the five PST participants. The primary PSTs' focus on the delivery of content tended to dictate their pedagogical approach. In particular, lessons involving number were more teacher-centred. In contrast, lessons that had a practical application such as measurement and patterns were more student-centred.

The pedagogical approach was similar among the PSTs when using the CRIG framework. This approach featured students involved in meaningful discussions with the PST. The PSTs' use of the CRIG framework in the lessons occurred mostly during this teacher-centred time, demonstrating that teachers can develop their own and their students' structural awareness through a traditional mathematics teaching approach.

8.2.3 Noticing students' structural thinking

This study provided PSTs with opportunities to notice their students' structural thinking and there were cases where the PSTs observed the students and identified their structural thinking through using the CRIG framework. When the PSTs were noticing the students' structural thinking, their communications with the students tended to involve conceptual explanations of the mathematical content rather than procedural practices. Noticing the students' structural thinking promoted the PSTs' understanding of the mathematical content and pedagogy that effectively communicated the mathematical content.

8.2.4 Promoting structural thinking

Both primary PSTs promoted structural thinking through lessons that focused on one component of the CRIG framework to describe the concepts and procedures of the lesson topic. For two secondary PSTs, Ms K and Ms M, pedagogical practices were aimed toward their higher-ability classes. In these classes, Ms K and Ms M promoted structural thinking through the *Generalising and Reasoning* component of the CRIG framework. These two PSTs used this component of the CRIG framework more often than the other components to develop their students' higher-order thinking skills. These PSTs tended toward a student-centred pedagogy, where open discussion and collaboration were encouraged.

8.2.5 PSTs' noticing of structural thinking through the CRIG framework

The CRIG framework proved to be very useful in helping the PSTs to notice structural thinking in their teaching. The PSTs did not always notice the CRIG framework when in-the-moment of teaching but they did reflect on what they used and what they could have used. In the five cases in this study, there were situations in which the PSTs were using the CRIG framework; however, they were not aware of doing so. Despite this, as the study progressed, the PSTs became more aware of using the CRIG framework and mentioned the components of the CRIG framework by name during the lesson.

8.3 Limitations

8.3.1 Participants

The scope of the study was limited to a small number of participants (i.e., five cases), and although the PLWs operated as a small learning community, it may have been more effective to include more PST participants. The data collected from a larger group that implemented CRIG across a broader range of year levels and topics may have given more insight into PSTs' noticing structural thinking through the CRIG framework. A cross-comparison of pre-service teachers in their first, second, or third year of initial teacher education, or from different teacher education programs, might have provided rich opportunities to measure their developing structural thinking over time or investigate how different teacher education programs develop PSTs' structural thinking.

Recruiting practising teachers into the study may have given a different perspective on the results. The diversity of experienced teachers of mathematics as well as primary, secondary, and out-of-field teachers would all allow for an investigation of how all teachers of mathematics could use the CRIG framework.

8.3.2 Additions to the study

The PLP provided an opportunity for the PSTs to learn about structural thinking through the CRIG framework. In another study, the PSTs' development of structural thinking capabilities could have been evaluated longitudinally over a more extensive PLP. Such a study would involve more cycles of practice and review with the time between each cycle allowing the PSTs to consolidate their experiences. However, in this study, more time between each cycle was not possible given the constraints of PSTs' university schedule of professional experience.

In addition, a further study could involve extending the research into the PSTs' first year of teaching. This extension of the study would give the PSTs opportunities to use the CRIG framework in a broader range of lessons with different cohorts of students. Further to this, by offering a professional learning program involving the CRIG framework to the PSTs as novice teachers, there is potential for the PSTs to continue to develop both their mathematical content and pedagogical content knowledge.

8.3.3 Lesson planning

Given the success of noticing in lesson planning (Choy, 2014), it would have been worthwhile to give the PSTs more assistance in developing and reviewing their lesson plans. Such assistance could be achieved in the PLWs by allowing the PSTs to collaborate on writing a lesson plan or to participate in a peer review of each other's lesson plans. Alternatively, each PST could communicate with me, as a mentor, to review their lesson plans.

8.3.4 Video reflections

The video reflections provided an ideal opportunity for the PSTs to reflect on their teaching, both individually and in the PLW learning communities. However, the analysis of the video segment from each PST's lesson was limited by not having multiple videoed segments from the lesson. The video segments chosen may or may not have demonstrated the PSTs' use or lack of use of the CRIG framework. Some segments showed the PST explicitly using the CRIG framework, while others had the PST implying a component of the framework. There were many scenarios that could have been videoed; unfortunately, not all could be used. Also, given the time constraints of the study, there were no opportunities for showing any other video segments. Benefits of reflecting on more videoed segments of the lessons would be the PSTs' further exposure to alternative scenarios involving teaching the CRIG framework with the additional benefits of sharing ideas with peers about using the CRIG framework.

8.4 Significance of the research

This research extends on the work of John Mason and his extensive body of work on mathematical thinking. *Thinking Mathematically* (Mason, Stacey, & Burton, 1982) gave teachers of mathematics opportunities to engage in a concerted effort to understand the thinking processes involved when solving mathematical problems. These advances represented opportunities for teachers of mathematics to teach students how to apply mathematical thinking when solving problems, and not just state the procedures of how to solve a problem. Mason (2002) introduced the concept of noticing into the lexicon of mathematics education, and, the collaboration of Mason et al. (2009), allowed the notion of teachers' noticing of structural thinking to emerge as a significant contribution to mathematics education. Teachers' noticing of structural thinking has been the focus of this study and, as evident from the results, there is potential to advance the discourse of mathematics education in this area. Teacher noticing and structural thinking are two essential components of mathematics education that, when combined as teachers' noticing of structural thinking, can improve teachers' mathematical content and pedagogical content knowledge.

8.5 Implications for further research

The introduction of the notion of mathematical structure as a consideration in the teaching and learning of mathematics at all age and ability levels has implications for future mathematics education research. This study has opened up the possibility of research about how noticing structural thinking through the CRIG framework may benefit pre-service and in-service teachers as well as students of mathematics. The CRIG framework has been identified in this study as a workable and robust tool that could be easily identified and implemented by all teachers of mathematics. Further research could consider how the CRIG framework could be adapted to the proficiency strands of the Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2019) and the working mathematically processes in the NSW K-10 mathematics syllabus (NSW Board of Studies, 2012). Such studies would look at how to support teachers of mathematics no matter what their teaching background.

Learning mathematical structure through the CRIG framework has been demonstrated in the present study as effective in supporting all teachers' PCK. The focus of the research could be broadened to identify how particular groups of teachers (e.g., primary, secondary, pre-service, novice, experienced, and out-of-field teachers) can best be supported to develop insight into noticing structural thinking and use the CRIG framework when teaching mathematics. Additionally, future research could explore how using the CRIG framework to develop structural thinking can develop students' engagement and success in mathematics.

8.6 Implementing the CRIG framework

This study identified some strategies and procedures that integrated the CRIG framework into teaching and learning mathematics. To successfully assimilate the CRIG framework into a mathematics teacher's 'toolbox', the framework would need to be embedded into the mathematics curriculum. Additionally, professional learning programs for pre-service and more experienced teachers would need to espouse the benefits of using the CRIG framework to develop teachers' and students' structural thinking skills. The following sections discuss four areas of mathematics education that may benefit from implementing the CRIG framework: pre-service teacher education programs, mathematics teacher professional learning programs, students' mathematics learning, and the mathematics curriculum.

8.6.1 Pre-service teacher education programs

Introducing the CRIG framework into undergraduate primary and secondary mathematics programs represents an opportunity to consolidate PSTs' mathematical content and pedagogical content knowledge. PSTs' exposure to the CRIG framework before and during their professional experience could support their view of teaching mathematics through a structural approach.

The CRIG framework could be used in conjunction with video to assist in developing PSTs in teacher education programs. PSTs could be videoed in micro-teaching lessons using the framework, and they could analyse videos of other PSTs and experienced teachers using the CRIG framework.

8.6.2 Mathematics teacher professional learning programs

This study has indicated that pre-service teachers with knowledge of structural thinking can improve their mathematical content and pedagogical content knowledge and can also support students' engagement and success in mathematics. Although the study focused on pre-service teachers, the PLP could be used to develop other professional learning programs that support inservice teachers of mathematics. The CRIG framework may prove to be attractive to teachers because it does not necessarily require any immediate direct change to current pedagogical practices. However, it does contribute to developing teacher's mathematical content and pedagogical content knowledge through deepening understandings of mathematical structures and relationships.

8.6.3 Students' mathematical learning

School students' mathematical thinking may improve through their understanding of structural thinking. This understanding would be achieved by students by utilising the components of the CRIG framework to solving mathematical problems. The CRIG framework could be offered to students at all levels of schooling as an engaging and helpful tool to help them develop structural thinking of mathematical concepts and processes. Teachers could also use the CRIG framework to encourage students' mathematical thinking.

8.6.4 Mathematics curriculum

The working mathematically processes in the NSW Mathematics Syllabus for the Australian Curriculum (NSW Board of Studies, 2012) and the proficiency strands of the Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2019) identify mathematical structures and relationships. The four components of the CRIG framework are found in the working mathematically processes, the outcomes for each stage, and the course content of the NSW mathematics syllabus (NSW Board of Studies, 2012). These components are connected to the mathematical content and teachers' pedagogical approach. For example, the Stage 4 working mathematically outcomes acknowledge the CRIG framework components of *Connections* as "communicates and connects mathematical ideas using appropriate terminology, diagrams and symbols" and *Generalising and Reasoning* as "recognises and explains mathematical relationships using reasoning". The Stage 4 Number and Algebra outcome states: "generalises number properties to operate with algebraic expressions" and "creates and displays number patterns; graphs and analyses linear relationships; and performs transformations on the cartesian plane". The identification of the CRIG framework in the syllabuses reflects the importance of mathematical understanding through structural thinking. What is not apparent in the NSW mathematics syllabus is an overarching statement about how essential mathematical structure is in the teaching and learning of mathematics. Australian curricula do not use the term structure; however, there is acknolwegdment of the importance of developing students' understanding of the structural relationships involved in learning mathematical concepts. The Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2019) identifies the components of the CRIG framework in the understanding and reasoning proficiencies. In the United States, the *Common Core State Standards for Mathematics* (CCSSM) includes "Look and make use of structure" (Common Core State Standards, 2010) as one of the eight standards of practice, thus demonstrating the value placed on structure in the learning of mathematics.

8.7 Testimonials

All five PSTs were contacted during their first year of teaching to offer a testimonial about their use of the CRIG framework. Two of the five responded with a report of how the CRIG framework forms a part of their lesson planning, how it informs their pedagogy practice, and how they use the framework to encourage students' structural awareness.

Ms K wrote in an email, as a first-year mathematics teacher at a rural secondary school, that she was using the CRIG framework in her mathematics lessons.

> I was first introduced to the CRIG framework on my last education studies practicum. Throughout this time, this framework stood as a foundation to guide my lessons and thus create a more meaningful learning sequence for students. The framework was easy to follow with a high impact on student engagement and understanding.
>
> The CRIG framework connects to prior understanding to establish a firm foundation and starting point for future learning. Students are then encouraged to recognise patterns and identify any similarities or differences. This is a key component of the learning process in mathematics as learners need to create their own meaning and make sense of the mathematics themselves for greater result understanding [sic]. Only then can students generalise their thoughts, the final step, and be able to apply their understanding to new contexts.
>
> Throughout my current practice as a first-year teacher, I have been incorporating this framework in most of my lessons with automaticity. CRIG is a tool which has been embedded into my daily teaching and is appearing to positively encourage deeper understanding in my mathematics classrooms. The CRIG framework has been establishing a classroom culture for students to think mathematically on a day to day basis.

Mr T offered two lesson observations where he used the CRIG framework in his Year 10 and Year 12 Standard Mathematics classes. Mr T reported his use of the CRIG framework when teaching "Graphs of practical situations" to his Year 12 class:

> The greatest benefit of CRIG helped me engaged students in the learning of mathematics. I was able to take the students outside the classroom to conduct the experiment, in which the students recorded their own measurements. CRIG helped me create a fun and meaningful lesson which was especially important in establishing the teacher relationship with students for the first time. CRIG helped me get the 'big ideas' in my lesson planning that aligned to the learning outcomes of the syllabus. CRIG gave me further teaching opportunities that arose throughout the lesson, for example, using c from the equation $y = \mathrm{m}x + \mathrm{c}$ to describe the scenario of a walker with a 3 second head start analogy. I had opportunities to discuss rate of change behaviour in the graph in a Standard 1 course which is normally reserved for the advanced calculus maths course. CRIG provided positive test results from the students. I collected the test that was administered to students at the end of the lesson; the results show that 13 of the 17 students correctly answered Q1 and half answered correctly the harder Q2.

For his Year 10 class, Mr T made the following comment about his use of CRIG:

> CRIG has helped me to pose investigative questioning for students. I have been able to provide guided practice for students on the difficult questions that they raise by connecting to their prior learning on what they already know. The biggest benefit of CRIG is that I am able to scaffold for students that avoids transmission teaching (i.e. avoid telling students the answer and letting them have the opportunity to work it out themselves).

8.8 Concluding remarks

In this study, PSTs had opportunities to think structurally using the CRIG framework. The four components each represent mathematical thinking processes and provide strategies for PSTs to think deeply about mathematical content. Teachers need to understand mathematical structure so they can teach mathematical concepts more effectively, and the CRIG framework can help teachers to improve their pedagogical practice and engage students in learning mathematics. The CRIG framework provides a set of strategies to encourage structural thinking, develop mathematical understanding, and help teachers and students to articulate this understanding. In doing so, teachers can better promote and develop their own and their students' engagement and understanding in mathematics learning.

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APPENDIX A

Macquarie University Ethics Approval Letter

Dear Professor Mulligan,

Re: "Preservice mathematics teachers' noticing of structural thinking in a community of inquiry" (5201600943)

Thank you very much for your response. Your response has addressed the issues raised by the Faculty of Human Sciences Human Research Ethics Sub-Committee and approval has been granted, effective 16th February 2017. This email constitutes ethical approval only.

This research meets the requirements of the National Statement on Ethical Conduct in Human Research (2007). The National Statement is available at the following web site:

https://www.nhmrc.gov.au/book/national-statement-ethical-conduct-human-research

The following personnel are authorised to conduct this research:

Professor Joanne Mulligan

Dr Michael Cavanagh

Mr Mark Thomas Gronow

Please note the following standard requirements of approval: 1. The approval of this project is conditional upon your continuing compliance with the National Statement on Ethical Conduct in Human Research (2007). 2. Approval will be for a period of five (5) years subject to the provision of annual reports. Progress Report 1 Due: 16th February 2018

Progress Report 2 Due: 16th February 2019

Progress Report 3 Due: 16th February 2020

Progress Report 4 Due: 16th February 2021

Final Report Due: 16th February 2022

NB. If you complete the work earlier than you had planned you must submit a Final Report as soon as the work is completed. If the project has been discontinued or not commenced for any reason, you are also required to submit a Final Report for the project. Progress reports and Final Reports are available at the following website: http://www.research.mq.edu.au/current_research_staff/human_research_ethics/resources 3. If the project has run for more than five (5) years you cannot renew approval for the project. You will need to complete and submit a Final Report and submit a new application for the project. (The five year limit on renewal of approvals allows the Sub-Committee to fully re-review research in an environment where legislation, guidelines and requirements are continually changing, for example, new child protection and privacy laws). 4. All amendments to the project must be reviewed and approved by the Sub-Committee before implementation. Please complete and submit a Request for Amendment Form available at the following website: http://www.research.mq.edu.au/current_research_staff/human_research_ethics/managing_approved_research_projects

5. Please notify the Sub-Committee immediately in the event of any adverse effects on participants or of any unforeseen events that affect the continued ethical acceptability of the project. 6. At all times you are responsible for the ethical conduct of your research in accordance with the guidelines established by the University. This information is available at the following websites: http://www.mq.edu.au/policy http://www.research.mq.edu.au/current_research_staff/human_research_ethics/managing_approved_research_projects If you will be applying for or have applied for internal or external funding for the above project it is your responsibility to provide the Macquarie University's Research Grants Management Assistant with a copy of this email as soon as possible. Internal and External funding agencies will not be informed that you have approval for your project and funds will not be released until the Research Grants Management Assistant has received a copy of this email. If you need to provide a hard copy letter of approval to an external organisation as evidence that you have approval, please do not hesitate to contact the Ethics Secretariat at the address below. Please retain a copy of this email as this is your official notification of ethics approval.

Yours sincerely,
Dr Naomi Sweller
Chair
Faculty of Human Sciences
Human Research Ethics Sub-Committee

FHS Ethics
Faculty of Human Sciences Ethics
C5C-17 Wallys Walk L3
Macquarie University, NSW 2109, Australia
T: +61 2 9850 4197 | http://www.research.mq.edu.au/
Ethics Forms and Templates
http://www.research.mq.edu.au/current_research_staff/human_research_ethics/resources
The Faculty of Human Sciences acknowledges the traditional custodians of the Macquarie University Land,
the Wattamattagal clan of the Darug nation, whose cultures and customs have nurtured and continue to
nurture this land since the Dreamtime. We pay our respects to Elders past, present and future.

APPENDIX B

NSW Department of Education SERAP Approval Letter

NSW GOVERNMENT | Education Public Schools

Mr Mark Gronow
19/12-16 Chelsea Street
REDFERN NSW 2016

DOC17/522543
SERAP 2016595

Dear Mr Gronow

I refer to your application to conduct a research project in NSW government schools entitled *Mathematics preservice teachers' noticing structural thinking through a community of inquiry*. I am pleased to inform you that your application has been approved.

You may contact principals of the nominated schools to seek their participation. **You should include a copy of this letter with the documents you send to principals.**

This approval will remain valid until 30-May-2018.

The following researchers or research assistants have fulfilled the Working with Children screening requirements to interact with or observe children for the purposes of this research for the period indicated:

Researcher name	WWCC	WWCC expires
Mark Gronow	WWC0862911E	01-Dec-2020

I draw your attention to the following requirements for all researchers in NSW government schools:

- The privacy of participants is to be protected as per the NSW Privacy and Personal Information Protection Act 1998.
- School principals have the right to withdraw the school from the study at any time. The approval of the principal for the specific method of gathering information must also be sought.
- The privacy of the school and the students is to be protected.
- The participation of teachers and students must be voluntary and must be at the school's convenience.
- Any proposal to publish the outcomes of the study should be discussed with the research approvals officer before publication proceeds.
- All conditions attached to the approval must be complied with.

When your study is completed please email your report to: serap@det.nsw.edu.au
You may also be asked to present on the findings of your research.

I wish you every success with your research.

Yours sincerely

Dr Robert Stevens
Manager, Research
30 May 2017

School Policy and Information Management
NSW Department of Education
Level 1, 1 Oxford Street, Darlinghurst NSW 2010 – Locked Bag 53, Darlinghurst NSW 1300
Telephone: 02 9244 5060 – Email: serap@det.nsw.edu.au

Pages 184-191 of this thesis have been removed as they contain published material. Please refer to the following citation for details of the article contained in these pages:

Gronow, M., Mulligan, J., & Cavanagh, M. (2017). Teachers' understanding and use of mathematical structure. In A. Downton, S. Livy, & J. Hall, (Eds.), 40 years on: We are still learning!: Proceedings of the 40th Annual Conference of the Mathematics Education Research Group of Australasia (pp. 285–292). Adelaide, SA: The Mathematics Education Research Group of Australasia Inc.

Available in the 2017 MERGA conference proceedings available at https://www.merga.net.au/Public/Publications/Annual_Conference_Proceedings/2017_MERGA_annual_conference_proceedings.aspx

Pages 191-200 of this thesis have been removed as they contain published material. Please refer to the following citation for details of the article contained in these pages:

Gronow, M., Cavanagh, M., & Mulligan, J. (2019). Primary pre-service teachers noticing of structural thinking in mathematics. In G. Hine, S. Blackley, & A. Cooke (Eds.), Mathematics Education Research: Impacting Practice: Proceedings of the 42nd Annual Conference of the Mathematics Education Research Group of Australasia (pp. 324–332). Perth, WA: The Mathematics Education Research Group of Australasia Inc.

Available in the 2019 MERGA conference proceedings available at https://www.merga.net.au/Public/Publications/Annual_Conference_Proceedings/2019-MERGA-conference-proceedings.aspx

APPENDIX E

Participant Consent Letters

Department of Educational Studies
Faculty of Human Sciences
MACQUARIE UNIVERSITY NSW 2109

Phone: +61 9850 8621
Email: joanne.mulligan@mq.edu.au
Chief Investigator: Professor Joanne Mulligan Co-investigator: Dr Michael Cavanagh
Co-investigator: Mark Gronow (PhD Student)

Pre-service Teacher Information and Consent Form

Name of Project: Pre-service teachers' noticing of structural thinking in mathematics.

You are invited to participate in a research project involving pre-service primary teachers. The purpose of the research is to discover if pre-service teachers can improve their ability to notice students' structural thinking in mathematics by sharing their insights in discussions with colleagues. Structural thinking involves deep thinking and understanding of mathematical processes and concepts, rather than memorising facts and procedures. Developing structural thinking helps students learn mathematics.

This project is being conducted by Mark Gronow (mark.gronow@hdr.mq.edu.au) to meet the requirements of PhD program under the supervision of Professor Joanne Mulligan (phone: 9850 8621, joanne.mulligan@mq.edu.au), and Dr Michael Cavanagh (phone: 9850 8239, michael.cavanagh@mq.edu.au) from Macquarie University's Department of Educational Studies.

If you decide to participate, you will be required to: attend a workshop lasting about 60 minutes, before teaching mathematics to a Stage 3 class during your July/August 2017 Professional Experience. You will also be asked to submit your lesson preparation notes, attend three individual interview sessions lasting about 30 minutes each, participate in the pre-service teachers' community of inquiry online group established for this research, attend two pre-service teacher community of inquiry group meetings lasting about 60 minutes each, and participate in a post-research questionnaire.

Three of your Stage 3 mathematics lessons will be video recorded, and after each videoed lesson you will be asked to view this video and reflect on the lesson for about 20 minutes. The pre-service teachers' community of inquiry group will also meet after the videoed lesson to discuss noticing students' structural thinking, and to prepare for the next videoed lesson. This group will consist of the pre-service teachers participating in the research and the researcher. You will also be required to undertake a pre- and post-research questionnaire. The questionnaires should each take about 15 minutes to complete.

There are no discernable risks, yet there are considerable benefits for your teaching experience. Any information or personal details gathered in the course of the research are confidential, except as required by law. All participating pre-service teachers will be given pseudonyms and no individual will be identified in any publication of the results. Only the chief investigator (Mulligan) and the co-investigators (Cavanagh and Gronow) will have access to the data. All video and audio data collected will be transcribed by a professional transcription service with a strict privacy policy. A summary of the results of the data can be made available to you on request once the project is completed.

This research will form the basis of a future conference presentations and publications. No names or personal information would be given out, however, a part of the videoed lesson recorded may be used during the presentation.

Participation in this research is entirely voluntary. You are not obliged to participate. If you decide to participate, you are free to withdraw at any time without having to give a reason and without consequence. If you complete all participation requirements you will receive a certificate from the Open Real Science project acknowledging your participation and tasks undertaken, as well as a $100 voucher from The Co-op bookshop.

If you have any queries about the research, please do not hesitate to contact me at any time on [redacted] or by email at mark.gronow@hdr.mq.edu.au .

Yours sincerely, Mark Gronow

PRINCIPAL INTRODUCTION LETTER

Department of Educational Studies
Faculty of Human Sciences
MACQUARIE UNIVERSITY NSW 2109
Phone: +61 9850 8621
Email: joanne.mulligan@mq.edu.au

Chief Investigator: Professor Joanne Mulligan
Co-investigator: Dr Michael Cavanagh
Co-investigator: Mark Gronow

Name of Research: Pre-service teachers' noticing of structural thinking in mathematics.

Dear *Name of principal*,
My name is Mark Gronow, and I am a PhD student at Macquarie University undertaking a research study with primary pre-service teachers. My research involves observing primary teacher education students teaching mathematics. The aim of the research is to identify whether pre-service teachers notice students' mathematical thinking, known as structural thinking, and how noticing of students' structural thinking can be improved through the pre-service teachers sharing their experiences in a collegial discussion group, known as a community of inquiry.
The pre-service teacher involved in this research is a primary teacher education student from Macquarie University. The researcher will observe the pre-service teacher while on their Professional Experience at your school during July and August 2017.
This letter, as a letter of introduction, is a request for permission for your school to participate in this research. Participation involves the pre-service teacher teaching mathematics lessons. Three of the lessons taught over the period of the Professional Experience are video recorded. Students' parents or carers are asked to complete a consent form allowing their child to be a participant, not a subject, in the research. The videoed lessons will focus on the pre-service teacher, however, students may appear in the video. Footage of students whose parents/carers do not give consent to be involved will be edited out of the video. There is no requirement for these students to be removed from the classroom. The only requirement for the students will be collection of their work samples.
Additionally, the supervising/mentor teacher of the Professional Experience student will be asked to complete an observational schedule of the pre-service teacher when teaching this same class. The supervising teacher will not be videoed.
Attached is a consent form for you to sign if you agree that the research can take place at your school. Once the consent form is signed and returned to me, I will contact the supervisor of the Professional Experience student.
This research will form the basis of future conference presentations and publications. No names or personal information will be given out, however, a part of the videoed lesson recorded may be used during presentations.
Thank you for your time to consider this request. If you require any further information, please do not hesitate to contact me via email (mark.gronow@students.mq.edu.au) or on my mobile [redacted]
Yours sincerely,

Mark Gronow

Department of Educational Studies
Faculty of Human Sciences
MACQUARIE UNIVERSITY NSW 2109
Phone: +61 9850 8621
Email: joanne.mulligan@mq.edu.au
Chief Investigator: Professor Joanne Mulligan
Co-investigator: Dr Michael Cavanagh
Co-investigator: Mark Gronow

MACQUARIE University SYDNEY·AUSTRALIA

School Principal Information and Consent Form

Name of Project: Pre-service teachers' noticing of structural thinking in mathematics.

Your school is invited to participate in a study involving pre-service primary teachers. The purpose of the study is to discover if pre-service teachers can improve their ability to notice students' structural thinking in mathematics by sharing their insights in a discussion with colleagues. Structural thinking involves deep thinking and understanding of mathematical processes and concepts, rather than memorising of facts and procedures. Developing structural thinking helps students learn mathematics.

This project is being conducted by Mark Gronow (mark.gronow@students.mq.edu.au) to meet the requirements of PhD program under the supervision of Professor Joanne Mulligan (phone: 9850 8621, joanne.mulligan@mq.edu.au), and Dr Michael Cavanagh (phone: 9850 8239, michael.cavanagh@mq.edu.au) from Macquarie University's Department of Educational Studies.

If you decide to participate, a pre-service teacher (PST) from Macquarie University enrolled in the BABEd or BEd (Primary) degree who has been allocated to your school to complete Professional Experience from July to August 2017 will be videoed teaching three mathematics classes. The focus of the research is the PST; other participants in the research are the students in the class, and the PST's supervising/mentor teacher. The PST, parents/carers and supervising/mentor teacher are asked to complete an Information and Consent Form before participating in the research. If any parents/carers do not give permission for their child to participate, then footage of their child will be edited out of the video. Students will also be asked to volunteer work samples from the videoed lessons. All video and audio data collected will be transcribed by a professional transcription service with a strict privacy policy.

Supervising/mentor teachers who agree to participate will be asked to complete an observation schedule while observing the PST teaching the three videoed classes. Additionally, the supervising teacher will be asked to complete a weekly observation schedule of the PST teaching this class for the duration of the Professional Experience.

There are no discernable risks, yet there are considerable benefits for participants. Any information or personal details gathered in the course of the study are confidential, except as required by law. No individual or school will be identified in any publication of the results. Only the chief investigator and the co-investigators will have access to the data. All participants will be given pseudonyms. A summary of the results of the data can be made available to you on request once the project is completed.

This research will form the basis of a future conference presentation and publications. No names or personal information will be given out, however, a part of the videoed lesson may be used during presentations.

Participation in this study is entirely voluntary. You are not obliged to participate and if you decide for your school to participate, you are free to withdraw at any time without having to give a reason and without consequence.

If you have any queries about the research, please do not hesitate to contact me at any time on [redacted].

Yours sincerely,

Mark Gronow

Department of Educational Studies
Faculty of Human Sciences
MACQUARIE UNIVERSITY NSW 2109
Phone: +61 9850 8621
Email: joanne.mulligan@mq.edu.au
Chief Investigator: Professor Joanne Mulligan
Co-investigator: Dr Michael Cavanagh
Co-investigator: Mark Gronow

MACQUARIE University SYDNEY · AUSTRALIA

Parent/carer - Information and Consent Form

Name of Research: Pre-service teachers' noticing of structural thinking in mathematics.

Your child is invited to participate in a research study of pre-service primary students, known as pre-service teachers. The purpose of the study is to discover if pre-service teachers when involved in their Professional Experience notice students' mathematical thinking. The research involves preservice teachers working as a group called a community of inquiry in receiving support and advice from the other pre-service teachers. By participating in the community of inquiry the pre-service teachers learn about noticing students in the classroom and understanding of structural thinking as part of mathematical thinking. Structural thinking involves deep thinking and understanding of mathematical processes and concepts, rather than memorising of facts and procedures.

This project is being conducted by Mark Gronow (mark.gronow@students.mq.edu.au) to meet the requirements of PhD program under the supervision of Professor Joanne Mulligan (phone: 9850 8621, joanne.mulligan@mq.edu.au), and Dr Michael Cavanagh (phone: 9850 8239, michael.cavanagh@mq.edu.au) from Macquarie University's Department of Educational Studies.

If you give permission for your child to participate, he or she will participate in their regular mathematics lessons, which will be taught by a pre-service teacher undertaking Professional Experience. Three lessons that the pre-service teacher will teach during July and August 2017 will be video recorded for research purposes only. Students in the videoed class are participants, not subjects of the research. There will be one video camera and audio recording unit in operation. The video camera will focus on the pre-service teacher, and their interactions with the students. If you do not wish your child to be videoed, then footage where he or she appears will be edited from the video. The video will be analysed by the co-investigator, Mr Gronow and a small vignette, approximately 3-5 minutes will be viewed by a small group of pre-service teachers who are participating in the research.

Additionally, your child may be asked to participate in a focus group interview of five students from the class, for approximately 30 minutes, after each of the videoed lessons. The focus group will meet with the co-investigator of the research, Mr Gronow, and the interview will be audio recorded. Students in the focus group will be asked to volunteer work samples from the video recorded lessons.

There are no discernable risks for any student involved in this research. Any information or personal details gathered in the course of the research are confidential, except as required by law. No individual will be identified in any publication of the results. Only the chief investigator and the co-investigators will have access to the data. Results can be made available to you on request once the project is completed.

This research will form the basis of a future conference presentations and publications. No names or personal information would be given out, however, a part of the videoed lesson recorded may be used during the presentation. All video and audio data collected will be written up by a professional transcription service with a strict privacy policy.

Participation in this study is entirely voluntary: you are not obliged to give permission for your child to participate and if you decide to give permission for their participation, you are free to withdraw your child at any time without having to give a reason and without consequence.

If you have any queries about the research, please do not hesitate to contact me at any time on [redacted].

Yours sincerely,

Mark Gronow

Department of Educational Studies
Faculty of Human Sciences
MACQUARIE UNIVERSITY NSW 2109

Phone: +61 9850 8621

Email: joanne.mulligan@mq.edu.au

Chief Investigator: Professor Joanne Mulligan

Co-investigator: Dr Michael Cavanagh

Co-investigator: Mark Gronow (PhD Student)

Supervising/Mentor Teacher Information and Consent Form

Name of Project: Pre-service teachers' noticing of structural thinking in mathematics.

You are invited to participate in a study involving pre-service primary teachers. The purpose of the study is to discover if pre-service teachers can improve their ability to notice students' structural thinking in mathematics by sharing their insights in a discussion with colleagues. Structural thinking involves deep thinking and understanding of mathematical processes and concepts, rather than memorising facts and procedures. Developing structural thinking helps students learn mathematics.

This project is being conducted by Mark Gronow (mark.gronow@students.mq.edu.au) to meet the requirements of PhD program under the supervision of Professor Joanne Mulligan (phone: 9850 8621, joanne.mulligan@mq.edu.au), and Dr Michael Cavanagh (phone: 9850 8239, michael.cavanagh@mq.edu.au) from Macquarie University's Department of Educational Studies.

If you decide to participate, you will be required to supervise a pre-service teacher enrolled in the BABEd or BEd (Primary) degree at Macquarie University assigned to your school for Professional Experience from July to August 2017. The pre-service teacher will be required to teach mathematics lessons class during their Professional Experience. Three lessons taught to the class during the Professional Experience will be video and audio recorded. Students in the class will be participants, and their parents/carers will be required to submit a consent form giving permission to participate. Parents/carers will also be required to give consent for their child to be videoed.

As the supervising/mentor teacher, you are asked to submit a weekly observational schedule of the pre-service teacher, teaching this class that you observe for the duration of the Professional Experience, including the three videoed classes.

There are no discernable risks, yet there are considerable benefits for the pre-service teacher in developing the ability to notice students' structural thinking in mathematics. Any information or personal details gathered in the course of the study are confidential, except as required by law. All participants will be given pseudonyms. No individual will be identified in any publication of the results. Only the chief investigator and the co-investigators will have access to the data. A summary of the results of the data can be made available to you on request once the project is completed.

This research will form the basis of a future conference presentations and publications. No names or personal information would be given out, however, a part of the videoed lesson recorded may be used during the presentation.

Participation in this study is entirely voluntary. You are not obliged to participate and if you decide to participate, you are free to withdraw at any time without having to give a reason and without consequence.

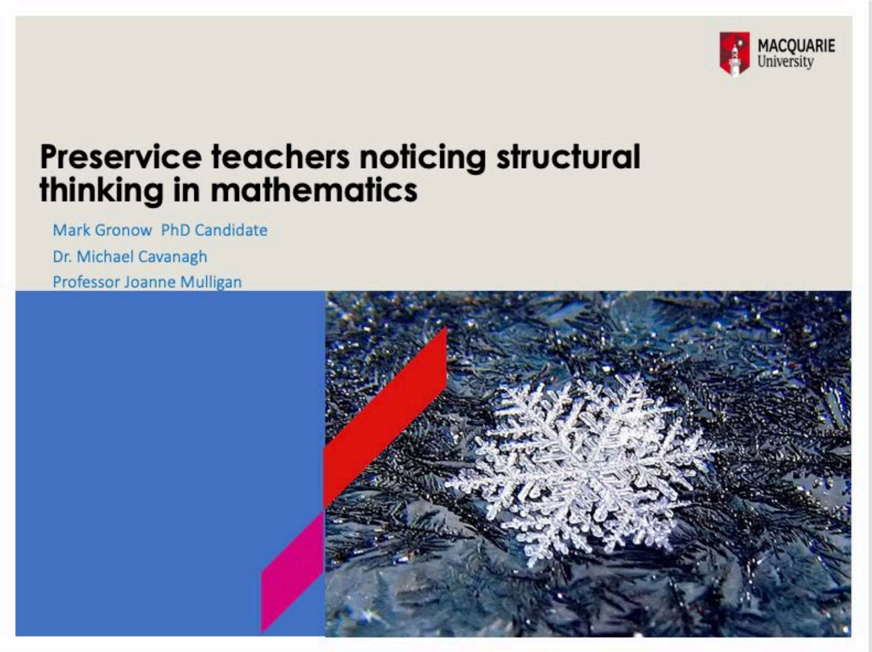
If you have any queries, please do not hesitate to contact me at any time on [redacted].

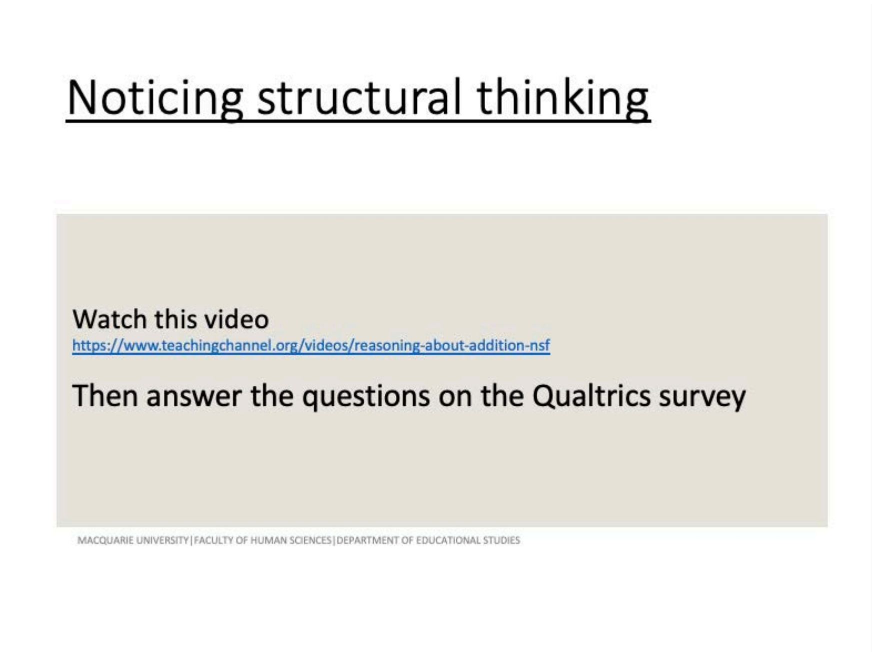
Yours sincerely,

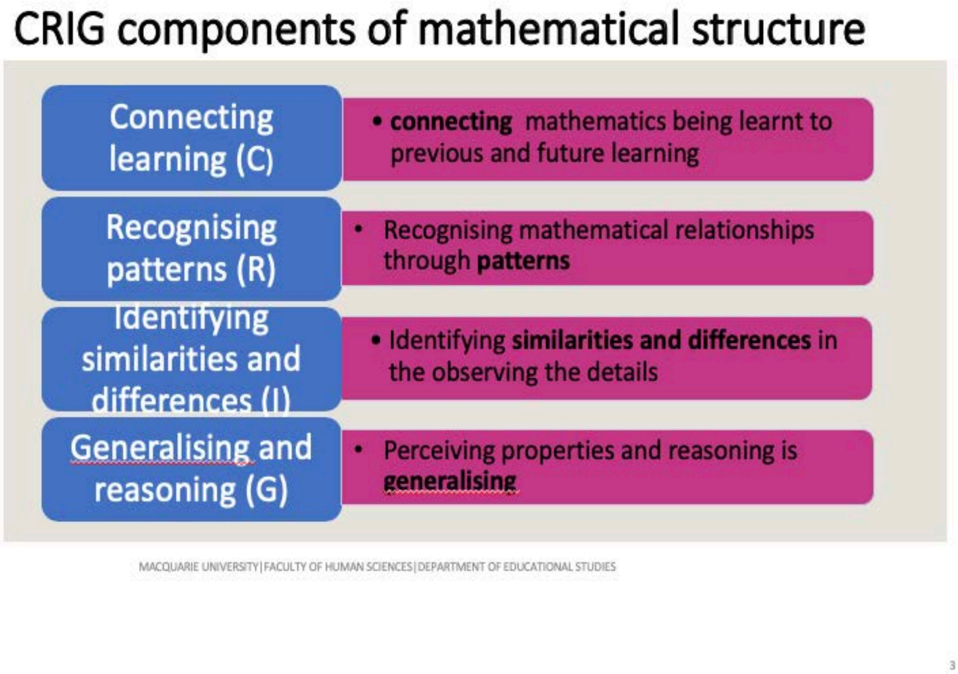
Mark Gronow

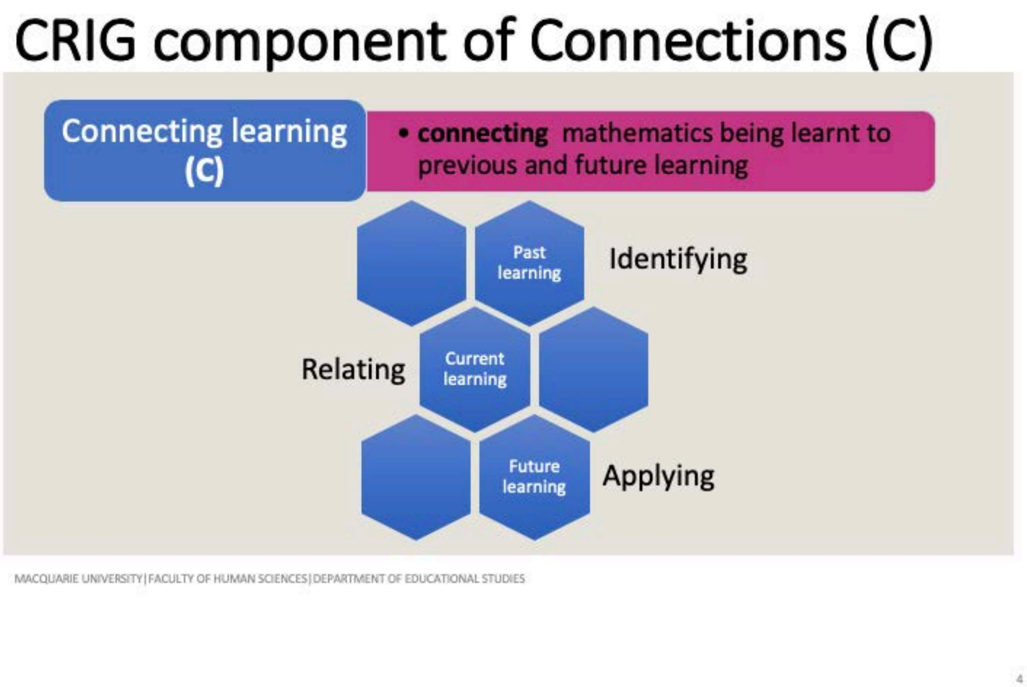
APPENDIX F

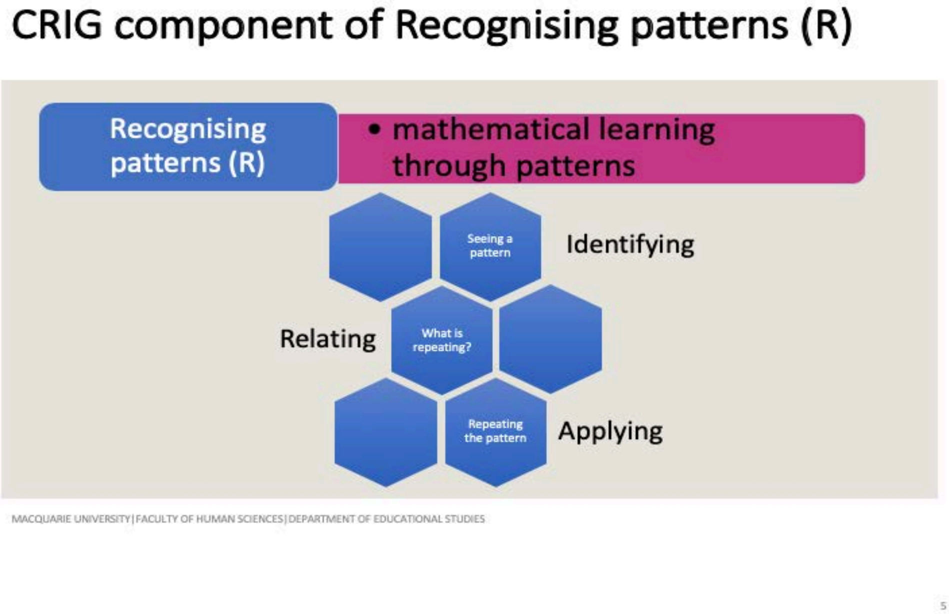
CRIG framework Presentation

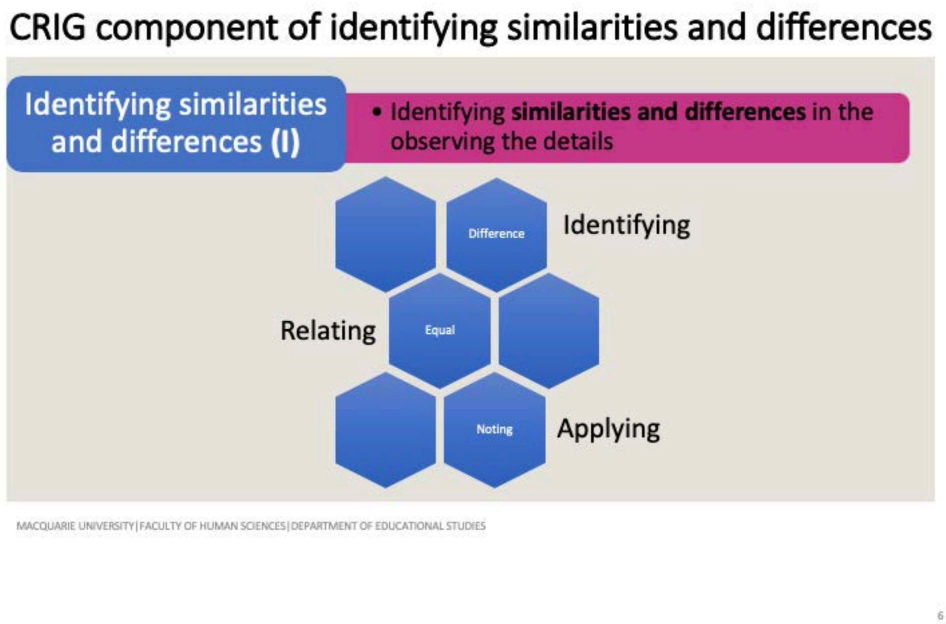


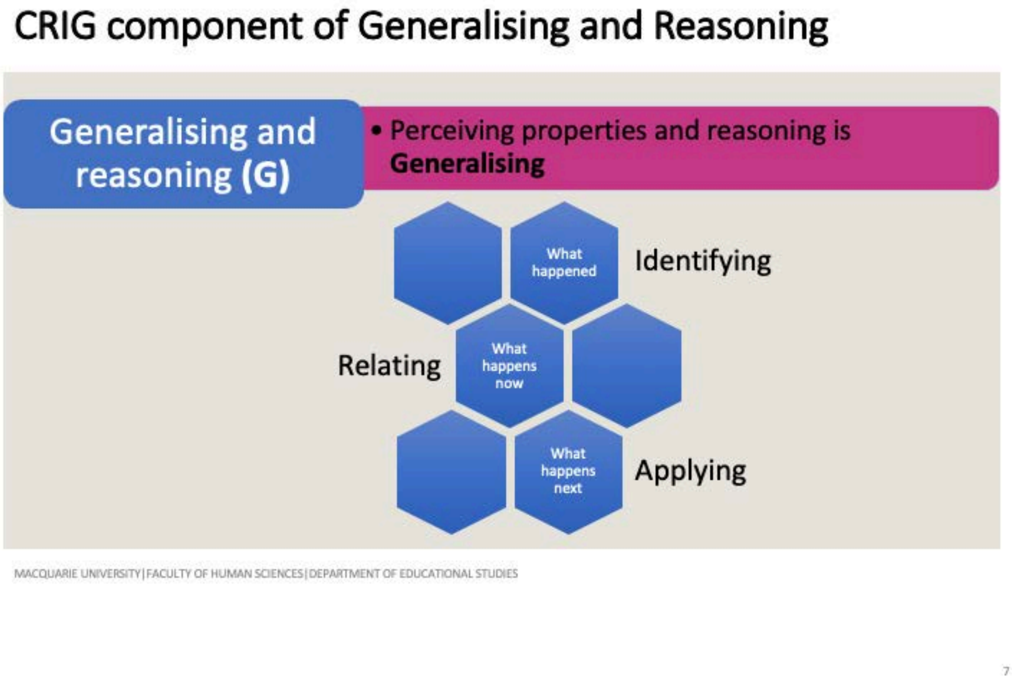


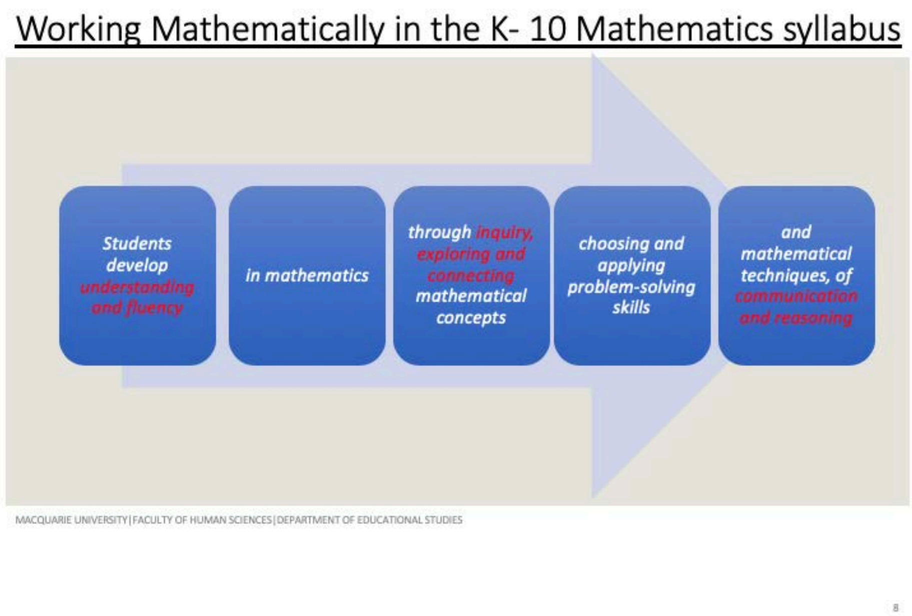

CRIG components of mathematical structure
Connecting learning (C)
• connecting mathematics being learnt to previous and future learning
Recognising patterns (R)
• Recognising mathematical relationships through patterns
Identifying similarities and differences (I)
• Identifying similarities and differences in the observing the details
Generalising and reasoning (G)
• Perceiving properties and reasoning is generalising
MACQUARIE UNIVERSITY|FACULTY OF HUMAN SCIENCES|DEPARTMENT OF EDUCATIONAL STUDIES
3


CRIG component of Connections (C)
Connecting learning (C)
• connecting mathematics being learnt to previous and future learning
Past learning
Identifying
Relating
Current learning
Future learning
Applying
MACQUARIE UNIVERSITY|FACULTY OF HUMAN SCIENCES|DEPARTMENT OF EDUCATIONAL STUDIES
4

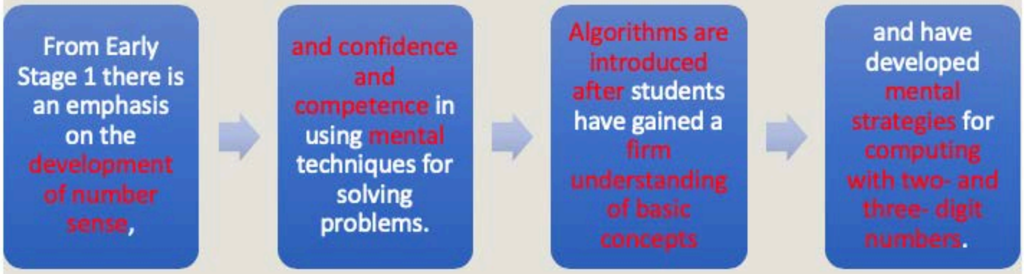

CRIG component of Recognising patterns (R)
Recognising patterns (R)
• mathematical learning through patterns
Seeing a pattern
Identifying
Relating
What is repeating?
Repeating the pattern
Applying
MACQUARIE UNIVERSITY | FACULTY OF HUMAN SCIENCES | DEPARTMENT OF EDUCATIONAL STUDIES
5


CRIG component of identifying similarities and differences
Identifying similarities and differences (I)
• Identifying **similarities and differences** in the observing the details
Difference
Identifying
Relating
Equal
Noting
Applying
MACQUARIE UNIVERSITY | FACULTY OF HUMAN SCIENCES | DEPARTMENT OF EDUCATIONAL STUDIES
6


CRIG component of Generalising and Reasoning
Generalising and reasoning (G)
• Perceiving properties and reasoning is **Generalising**
What happened
Identifying
Relating
What happens now
What happens next
Applying
MACQUARIE UNIVERSITY | FACULTY OF HUMAN SCIENCES | DEPARTMENT OF EDUCATIONAL STUDIES
7


Working Mathematically in the K- 10 Mathematics syllabus
Students develop understanding and fluency
in mathematics
through inquiry, exploring and connecting mathematical concepts
choosing and applying problem-solving skills
and mathematical techniques, of communication and reasoning
MACQUARIE UNIVERSITY | FACULTY OF HUMAN SCIENCES | DEPARTMENT OF EDUCATIONAL STUDIES
8

Number and Algebra Strand overview from the K- 10 Mathematics syllabus states



MACQUARIE UNIVERSITY | FACULTY OF HUMAN SCIENCES | DEPARTMENT OF EDUCATIONAL STUDIES

9

Noticing structural thinking in CRIG

Watch this video
https://www.teachingchannel.org/videos/teaching-middle-school-math-styles

MACQUARIE UNIVERSITY | FACULTY OF HUMAN SCIENCES | DEPARTMENT OF EDUCATIONAL STUDIES

APPENDIX G

Arithmetic Number Sentence (ANS) Worksheet

Arithmetic Number Sentences (ANS) for primary pre-service teachers to notice students' structural thinking

The following outlines the content of the lessons to be taught by the primary PSTs for the research

Strand overview states:

From Early Stage 1 there is an emphasis on the development of number sense, and confidence and competence in using concrete materials and mental, written and calculator techniques for solving appropriate problems. Algorithms are introduced after students have gained a firm understanding of basic concepts, including place value, and have developed mental strategies for computing with two- and three-digit numbers. Approximation is important and the systematic use of estimation is to be encouraged always. Students should always check that their answers 'make sense' in the contexts of the problems that they are solving.

Making mental calculations easier by knowing and applying the associative and commutative laws.

Aim: Students:

- Become familiar with the flexibility, and ease, of manipulating numbers using the associative and commutative laws.
- Confidently use the associative and commutative laws for calculations to ten.
- Confidently solve numbers sentence problems by number manipulations of adding to ten, without relying upon closure to solve the problem.

NOTES for the pre service teacher.

Use the lesson content to demonstrate the associative and commutative number laws.

As the class teacher, you can choose your preferred pedagogical approach to trial these questions.

- The lesson content is divided into Parts. Each Part may be a lesson or part of a lesson. The number work is followed by a series of questions requiring worded answers from the students.
- These questions are aimed at identifying students' mathematical thinking, called structural thinking. Structural thinking identifies how students can make:
    - **C**onnections to other mathematics,
    - **R**ecognise patterns,
    - **I**dentify similarities and differences, and
    - **G**eneralise

Lesson Content:

PART A

MA1-5NA uses a range of strategies and informal recording methods for addition and subtraction involving one- and two-digit numbers

Addition arithmetic number sentence problems, that demonstrate the associative law and commutative laws.

Complete the patterns of pairs by putting a number for the □ and Δ

1) $10 + 0 = \square$
2) $9 + 1 = \square$
3) $8 + 2 = \square$
4) $7 + 3 = \square$
5) $6 + 4 = \square$

Complete the remainder of the pattern

6) $\Delta + \square = 10$
7) $\Delta + \square = 10$
8) $\Delta + \square = 10$
9) $\Delta + \square = 10$
10) $\Delta + \square = 10$

Some questions to consider:

1. Where you able to find the missing numbers?

___

2. What did you do after completing the first three?

___

3. Did it become easier for you? Why or why not?

___

4. Did you add numbers together to get the missing number? If you didn't why not?

___

5. Describe a pattern that you followed?

___

6. What is similar about the questions 1) – 5) and questions 6) – 11)?

___

7. What is different about the questions 1) – 5) and questions 6) – 11)?

___

8. Why is question 6 different to the others?

___

9. Does order matter when adding the same numbers together?

Here is a pattern of triples that add to 10	Can you create your own pattern of triples below
1) $\square + 0 + 5 = 10$, $\square = ?$	
2) $\square + 1 + 5 = 10$, $\square = ?$	
3) $\square + 2 + 5 = 10$, $\square = ?$	
4) $\square + 3 + 5 = 10$, $\square = ?$	
5) $\square + 4 + 5 = 10$, $\square = ?$	
6) $\square + 5 + 5 = 10$, $\square = ?$	
7) $\square + 5 + 4 = 10$, $\square = ?$	
8) $\square + 5 + 3 = 10$, $\square = ?$	
9) $\square + 5 + 2 = 10$, $\square = ?$	
10) $\square + 5 + 1 = 10$, $\square = ?$	
11) $\square + 5 + 0 = 10$, $\square = ?$	

Some questions to consider:

1. Where you able to find the missing numbers?
2. What did you do after completing the first three?
3. Did it become easier for you? Why or why not?
4. Did you add numbers together to get the missing number? If you didn't why not?
5. Describe a pattern that you followed?
6. What is different about this group of questions to the first group?
7. What did you notice was different about 1) – 5) when compared to 6) – 11)?
8. What is similar about the questions 1), 6) and 11)?
9. Does order matter when adding the same numbers together?
10. Can you write a generalising statement about what you have discovered?

Complete the following addition by using pairs that add to 10

$6 + 3 + 4\ + 7 = 6 + \square + 3 + 7, \quad \square\ = ?$

$= 10 + 10$

$= 20$

Some questions to consider:

1. Where you able to find the missing number?
2. What did you do to the number sentence?
3. Did it become easier for you to calculate the answer? Why or why not?
4. Is this a better way to add numbers? Why or why not?
5. Describe a pattern that you followed?

Rearrange the numbers so you can create pairs that add to 10.

$9 + 8 + 2\ + 1 = \square + \square + \square + \square,$

$= 10 + 10$

$= 20$

Some questions to consider:

1. Do you begin to add from left to right or do you look for pairs first?
2. What would you do if there were no pairs that add to ten?

Rearrange the numbers so you can create triples that add to 10.

$9 + 8 + 7 + 5 + 6 + 3 = \square + \square + \square + \square + \square + \square,$

$= \square$

Some questions to consider:

1. What groupings did you look for first?
2. Where you tempted to add the numbers left to right?
3. Was it easy to find a triple that added to ten?
4. Is this a quicker way of adding numbers mentally?
5. Create three of your own number sentences that relies upon pairings to ten to get an answer?

This next section introduces work for Stages 2 and 3 or for extension work on strategy development ie sum to 10, 20, 30, … (multiples of ten) and decomposition of numbers to create pairs that add to ten. Hint: Always encourage the use of pen and paper to reorder the numbers into groups that sum to 10

Group 1: Find the missing value

$\square + 2 + 8 = 6 + 10,$	$\square = ?$	Create your own here
$\square + 2 + 8 = 8 + 10,$	$\square = ?$	____________________
$\square + 2 + 19 + 8 = 20 + 10$	$\square = ?$	____________________
$12 + \square + 7 + 8 = 20 + 10$	$\square = ?$	____________________
$13 + 6 + \square + 4 = 20 + 10$	$\square = ?$	____________________

Group 2: Problems involving sums to multiples of ten

$13 + 66 + \square + 14 + 27 = 40 + 80 + 15,$	Create your own here
$13 + 48 + \square + 17 + 32 = 30 + 80 + 46,$	____________________
$23 + 62 + 15 + 28 + 17 = 90 + 40 + \square$	____________________

Group 3: Find the sum of these number sentences by using pairs of numbers that add to a mulitples of ten.

$27 + 15 + 32 + 23 + 18 + 35 = \square, \square = ?$

$45 + 36 + 32 + 58 + 29 + 31 + 44 = \square, \quad \square = ?$

Create your own below

____

____

Group 4: Decompose numbers to create pairs that add to ten

$8 + 7 = 5 + \square + 7,$	$5 + \square = 8$
$4 + 9 = 4 + \square + 3,$	$9 = \square + 3$
$6 + 8 = 6 + 4 + \square,$	$8 = 4 + \square$

Create your own below

____

____

____

APPENDIX H

Phase 1 Sample Lesson Plans

PLW1 Sample Lesson Plan

MACQUARIE University SYDNEY·AUSTRALIA

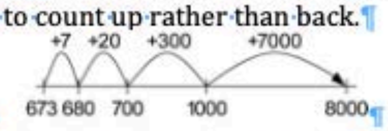
LEARNING EXPERIENCE (LESSON PLAN)

<table>
<tr><td>Student teacher's name</td><td colspan="2">Mr X</td><td>School</td><td colspan="2">Z public school</td></tr>
<tr><td>Year / Stage</td><td>1</td><td>Date</td><td>14/7/17</td><td>Duration of lesson</td><td>30 mins</td></tr>
<tr><td colspan="2">Key Learning Area(s)</td><td colspan="4">Mathematics – Number and Algebra</td></tr>
<tr><td colspan="6">Main aim of lesson</td></tr>
<tr><td colspan="6">To be able to combine numbers that add to ten.</td></tr>
<tr><td colspan="6">CRIG components considered</td></tr>
<tr><td colspan="6">Connection to previous learning</td></tr>
<tr><td colspan="6">From the NSW K-10 mathematics syllabus
Counting to ten, counting on, counting back,
Stage one: create and recognise combinations for numbers to at least 10, eg 'How many more to ten?
compare two groups of objects to determine 'how many more'
use visual representations of numbers to assist with addition and subtraction, eg ten frames
create and recognise combinations for numbers to at least 10, eg 'How many more make 10?'</td></tr>
<tr><td colspan="6">Recognising patterns</td></tr>
<tr><td colspan="6">Grouping patterns 3+7, 4+6, 5+5, one goes up one goes down
Repeating patterns in columns add one, take away one
Ten frame patterns, pairs, other</td></tr>
<tr><td colspan="6">Identifying similarities and differences</td></tr>
<tr><td colspan="6">What is the same and what is the difference between each combination of pairs
How many ways can pairs be written?
Can groups of three be made?
How many ways can the groups be made</td></tr>
<tr><td colspan="6">Generalising</td></tr>
<tr><td colspan="6">Can pairs be broken into groups of three?
Can a pattern be produced?
How many numbers can be used at once to add to ten.
How can this help when adding up numbers?
If they have to be different?, or
if they can be the same?</td></tr>
<tr><td colspan="3">Syllabus outcomes</td><td colspan="3">Summary of content</td></tr>
<tr><td colspan="3">MA1-5NA uses a range of strategies and informal recording methods for addition and subtraction involving one- and two-digit numbers</td><td colspan="3">combining numbers that add to 10,
eg 4 + 7 + 8 + 6 + 3:
first combine 4 and 6, and 7 and 3,
then add 8</td></tr>
<tr><td colspan="6">Resources required (e.g. ICT tools, ipad, IWB, A3 paper, colouring pencils, glue...)</td></tr>
<tr><td colspan="6">Ten Frame grids one per student</td></tr>
</table>

Coloured counters a range of colours
Recording sheets
White board and markers with a ten frame already drawn on the whiteboard.
Teacher magnetic counters or other source to demonstrate groups of ten on the whiteboard.

Links with further learning (where to next? Include outcomes)

- In Stage 3, mental strategies need to be continually reinforced.
- Students may find recording (writing out) informal mental strategies to be more efficient than using formal written algorithms, particularly in the case of subtraction.
- For example, 8000 – 673 is easier to calculate mentally than by using a formal algorithm. Written strategies using informal mental strategies (empty number line): The jump strategy can be used on an empty number line to count up rather than back.



- The answer will therefore be 7000 + 300 + 20 + 7 = 7327. Students could share possible approaches and compare them to determine the most efficient. The difference can be shifted one unit to the left on an empty number line, so that 8000 – 673 becomes 7999 – 672, which is an easier subtraction to calculate.

- Written strategies using a formal algorithm (decomposition method):

$$\begin{array}{r} {}^{7}\cancel{8}{}^{9}\cancel{0}{}^{9}\cancel{0}{}^{1}0\,- \\ 673 \\ \hline 7327 \end{array}$$

- An inverse operation is an operation that reverses the effect of the original operation. Addition and subtraction are inverse operations; multiplication and division are inverse operations.

Notes

Sequence of teaching / learning experiences (Guidelines: Sections can be expanded or adapted for appropriate use)	Teaching strategies (e.g. Whole class, pairwork, groupwork	Assessment (e.g. how do you know that learning is occurring ?	Time
Engagement: Link to prior learning Teacher begins lesson with students on the floor at the front of the room. Teacher asks the students to raise their arm if they can count to ten Teacher asks for the class to count to ten together	Whole class together Social learning between students Role model learning	Teacher is noticing: • students raising or not raising their arm. • Students who are counting in the group.	3
Description of the lesson (will you be ***modelling*** *a strategy? will there be a task to solve? an investigation?)* The teacher will indicate numbers that add to ten on the board by placing two colours of magnetic counters in a ten frame. Teacher will ask a volunteer to write the numbers in a table form. Teacher will ask another volunteer to create a new pair in the ten frame from the suggestions from the other students. Volunteers can rotate with other students. Teacher poses questions to the group such as: • How many pairs are there? • Is a pattern appearing? • What is different about each of the pairs? • What is the same? • Would a similar pattern appear you added to eleven or twelve? • What would be different or the same?	Whole class together Students in pairs	Teacher is noticing: Students who volunteer answers (students who understand offer answers those who don't- don't Accuracy in students answer (all correct or all wrong- tells about what needs to be learnt or not!)	5
Students actively engage in the activity Students move to work in pairs with a ten-frame grid and counters. Students in pairs create their own groups of tens Students can record the pairs Students can put the pairs in order to present pattern Students describe and record the pattern Students move around and report to other groups what their pattern is Students report any differences in the other groups' pattern Students report how patterns were achieved	Students in pairs, Whole class interaction	Teacher is applying assessment as Learning technique. Having students explain their own and others work demonstrates an understanding. Teacher identifies components of CRIG to consider students understanding of the content.	7
Teacher ***guiding*** *scaffolding, observing* Teacher is the facilitator and mentor. Asking questions of the students to	Students are working in pairs	Teacher is listening to students' responses to the questions asked and	5

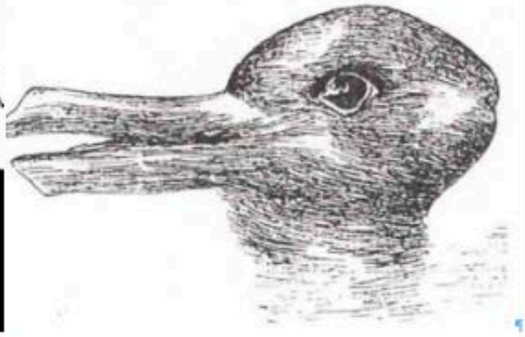
encourage structural thinking such as: Teacher is promoting student thinking by asking scaffolded questions such as: What made you think that? Did you see a pattern? Did you notice anything different? What did you see was like something else? What will happen next? How are the numbers discovered? What were you thinking to put them like that? Did you think of something else you had done that helped you? What was the pattern you noticed?	and groups. Teacher is visible as moving between groups and individuals	their discussions with other students.	
Representation of learning and sharing of ideas Students are interacting as a whole group (teacher instruction and whole class work) and in small groups discussions. This encourages communication and explanation of work. Opportunities for all students to express their opinions openly without public display	In pairs and groups students can share their ideas	Teachers is able to monitor student discussions. Notice who is more involved in the discussion and is the talk demonstrating understanding of the concept.	2
Teacher summary and scaffold for further learning. Teacher calls the class to the front of the room to sit on the floor. When the teacher has the classes attention then the teacher can proceed with a summary of the lesson and how this knowing this can help in completing more challenging problems such as: Pairing numbers that use adding to ten for ease in larger addition. Teacher writes a problem on the board that involves 4 numbers such as: 4+9 + 6 +1 = ? Ask the students how they would find the answer Explore the possible answers Ask for volunteers to write their own sequence of numbers on the white board. Ask students to discuss with the person next to them what way they would find the answer. Ask for another student to solve on the board using the pairing, ask for different ways to find the answer. Discuss the difference and how pairing to ten is a good way to do it in your head. Teacher can explain how it is OK to move the numbers around to pair to groups of ten This can be enhanced if students are aware of other manipulations of number such as 10=5+3+2 triples of numbers add to give 10, or more Possible for advanced students to rearranging numbers to add give ten 23 +54 + 26 + 17 =(23 +17) + (54+ 26) = 40 + 80	Whole group work	Teacher is able to monitor student involvement through attention and involvement. Teacher should have a good idea from the whole lesson of students overall understanding as well as individual understanding.	8

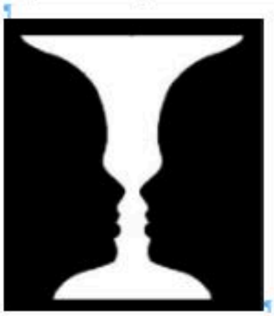
Phase 1 PLW2 Sample Lesson Plan

MACQUARIE University
SYDNEY·AUSTRALIA

LEARNING EXPERIENCE (LESSON PLAN)

Student teacher's name		Mr Gronow	School	XXX ps	
Year / Stage	1/stage1	Date	31/7/17	Duration of lesson	30
Key Learning Area(s)		Patterns and Algebra/Measurement and Geometry/ Problem solving			
Main aim of lesson					
How many squares – students can recognise and count many squares in a square grid pattern					
CRIG components considered					
Connection to previous or other mathematics learning					
MAe-15MG manipulates, sorts and describes representations of two-dimensional shapes, including circles, triangles, squares and rectangles, using everyday language. Students have worked on one dimensional patterns and repeating patterns. https://topdrawer.aamt.edu.au/Patterns/Big-ideas/Repeating-patterns/Repeating-patterns-and-multiplication Students already know about squares Students have seen a grid (ten frame) Questions to ask students could include: Can you see patterns in shapes? Give some examples. Brickwork, tiles When did we look at patterns and shapes?					
Connection with further learning (where to next?)					
MA2-15MG manipulates, identifies and sketches two-dimensional shapes, including special quadrilaterals, and describes their features Students will be required to manipulate shapes to create bigger patterns, their learning will involve angles and length, through matching same shapes together to form similar shape with differing sizes, with equilateral triangles and squares.					
Recognising patterns					
Rectangular arrays https://topdrawer.aamt.edu.au/Patterns/Misunderstandings/Rectangular-grids Show different patterns and ask to identify the shapes that make up the patterns					
Identifying similarities and differences					
Use the optical illusion picture to show differences in the drawings. How things are seen differently by different people. What we see is not always everything that is there. Demonstrate how you can look more carefully at a picture and find out more about it.					
Generalising					
Can students predict how many squares in bigger grid? Can students reproduce the problem for another shape (triangle, rectangle)? Can students come up with a rule?					
Syllabus outcomes			Summary of content		
MA1-1WM describes mathematical situations and methods using everyday and some mathematical language, actions, materials, diagrams and symbols **MA1-2WM** uses objects, diagrams and technology to explore mathematical problems **MA1-3WM** supports conclusions by explaining or demonstrating how answers were obtained **A1-8NA** creates, represents and continues a variety of patterns with numbers and objects **MA1-15MG** manipulates, sorts, represents, describes and explores two-dimensional shapes, including quadrilaterals, pentagons, hexagons and octagons			In this lesson students will be asked to find the number of squares in a 4x4 square grid. They may easily recognise the 16 1x1 small squares and the one 4x4 large square. But they not easily see the 2x2 and 3x3 squares. The aim is to extend their thinking into developing their problem solving skills. Teacher needs to set the scene so students can look deeply at the shape and notice more than the obvious. Then record what they see, try and see a pattern and generalise the pattern.		
Resources required (e.g. ICT tools, ipad, IWB, A3 paper, colouring pencils, glue…)					
Multiple Square shapes different colours Ten frame grid on a worksheet Recording sheet 4X4 square grid Teacher manipulable magnetic squares for the whiteboard or prepared copies of one square, two squares joined to form a rectangle, a four square grid and a 3x3 square grid. Copies of the optical illusion pictures enough for on between two					
Notes					





Sequence of teaching / learning experiences (Guidelines: Sections can be expanded or adapted for appropriate use)	Teaching strategies (e.g. Whole class, pairwork, groupwork	Assessment (e.g. how do you know that learning is occurring?)	Time
Engagement: Link to prior learning Go over connections to previous learning of what is a square, what is a rectangle Ask students to identify earlier patterns and counting patterns either as number or in pattern order Where are patterns in shapes, give examples Can you remember when we have looked at patterns in shapes, this could be from other disciplines outside maths. Ask of their understanding of what is a pattern. The teacher can show pictures of patterns in numbers, nature, tiles, architecture, clothing…	Teacher at the front of the room students on the floor listening.	Teacher is assessing the students prior learning by asking questions that connect their current learning to previous learning When did you learn about patterns? What sort of patterns do you're remember using? Do you remember seeing patterns in things that are around you outside of the classroom (nature, architecture, art, clothing)? What are some examples Do you remember who we described a pattern? Can you tell me what a pattern is? What is next in the pattern?	
*Description of the lesson (will you be **modelling** a strategy? will there be a task to solve? an investigation?)* Teacher presents students with the optical illusion pictures and asks what they see. Start with the old lady/young woman or duck/rabbit. Some students will not be able to see both images. Ask ones who can to show a friend where the image is seen as two different pictures. Talk to the students about the need to look deeply and carefully at things before accepting what they see. Teacher puts a magnetic square on the board or shows a picture of two adjacent squares on a printed paper Aim here is to see if the students can make the connection between one square/two squares and a rectangle. Will they identify the rectangle only or the two squares.		Teacher is encouraging the students to look beyond what they see and try to think more about what is presented to them. Questions that connect their learning to previous and current learning. Questions about the optical illusions could include: What is different about the pictures? What is the same? Can you think of things that look like something else? Ask questions about the two squares and ask what they see in the picture. Do they see more than the rectangle? Are there other shapes? Make the connection to the optical illusion picture that they may need to look more closely	
Students actively engage in the activity Students are asked to return to their tables where there are the square tiles and printed square grids and rectangular square grids such as the ten frame. Students are asked to build a big square using 16 tiles and count the number of different size squares.	Students are at their desks and working in their groups	Teacher moves between the groups and looks for students' progress. Identifying if students are recognising 1x1, 2x2, 3x3 and 4x4 squares. Teacher asks questions and gives instructions like: How big can a square be? Is there a different size square? Can the squares be in different positions?	
*Teacher **guiding** scaffolding, observing* Teacher is monitoring groups and using CRIG questions and instructions to support student thinking		Teacher needs to introduce the concept of recording to enable students to generalise further. This may be difficult for the students at this stage. Teacher may need to set up a scaffold to lead the students thinking.	
Representation of learning and sharing of ideas Students are asked to come to the front of the class and sit on the floor and bring. Teacher plans to have students explain their learning, demonstrate their recording techniques and to generalise to other examples.	Teacher at the front of the room students on the floor listening.	Teachers asks students to explain how many square they found in the 4x4 grid and how they found it. Ask students for different explanation. Invite a student to come to the front of the class and show how they recorded their findings. Ask the class if the can predict (generalise) home many squares in a 5x5 square grid.	
Teacher summary and scaffold for further learning. Students could be given other examples such as a triangular sheet. See if the experience translates that they now know how to look beyond the surface and think about alternatives		Encourage the students to look beyond what they first see and look for more, use the optical illusions to demonstrate how what they see can change and then the grid examples. Explain, things that are the same can be different as well. Then promote a generalising through the pattern identified.	

APPENDIX I

Lesson Plan Template

MACQUARIE University
SYDNEY · AUSTRALIA

LEARNING EXPERIENCE (LESSON PLAN)

<table>
<tr><td colspan="2">Student teacher’s name</td><td></td><td>School</td><td colspan="2"></td></tr>
<tr><td>Year / Stage</td><td></td><td>Date</td><td></td><td>Duration of lesson</td><td>20 mins</td></tr>
<tr><td colspan="2">Key Learning Area(s)</td><td colspan="4">Mathematics</td></tr>
<tr><td colspan="6">Main aim of lesson</td></tr>
<tr><td colspan="6">To be able to combine numbers that add to ten</td></tr>
<tr><td colspan="6">Connection to previous learning</td></tr>
<tr><td colspan="6"></td></tr>
<tr><td colspan="6">Recognising patterns</td></tr>
<tr><td colspan="6"></td></tr>
<tr><td colspan="6">Identifying similarities and differences</td></tr>
<tr><td colspan="6"></td></tr>
<tr><td colspan="6">Generalising</td></tr>
<tr><td colspan="6"></td></tr>
<tr><td colspan="3">Syllabus outcomes</td><td colspan="3">Summary of content</td></tr>
<tr><td colspan="3"></td><td colspan="3"></td></tr>
<tr><td colspan="6">Resources required (e.g. ICT tools, ipad, IWB, A3 paper, colouring pencils, glue…)</td></tr>
<tr><td colspan="6"></td></tr>
<tr><td colspan="6">Links with further learning (where to next? Include outcomes)</td></tr>
<tr><td colspan="6"></td></tr>
</table>

Sequence of teaching / learning experiences (Guidelines: Sections can be expanded or adapted for appropriate use)	Teaching strategies (e.g. Whole class, pairwork, groupwork	Assessment (e.g. how do you know that learning is occurring ?	Time
Engagement: Link to prior learning			
Description of the lesson (will you be ***modelling*** *a strategy? will there be a task to solve? an investigation?*)			
Students actively engage in the activity			
Teacher ***guiding*** *scaffolding, observing*			
Representation of learning and sharing of ideas			
Teacher summary and scaffold for further learning.			

APPENDIX J

Phase 2 Sample Lesson Plans

PLW1 Sample Lesson Plan

<table>
<tr><td colspan="2">Student teacher's name</td><td>Mr X</td><td>School</td><td colspan="2">Z public school</td></tr>
<tr><td>Year / Stage</td><td>4</td><td>Date</td><td>xx/xx/18</td><td>Duration of lesson</td><td>30 mins</td></tr>
<tr><td colspan="2">Key Learning Area(s)</td><td colspan="4">Mathematics – Number and Algebra</td></tr>
<tr><td colspan="6">Main aim of lesson</td></tr>
<tr><td colspan="6">To be able add and subtract integers. Apply the associative, commutative and distributive laws to aid mental and written computation</td></tr>
<tr><td colspan="6">CRIG components considered</td></tr>
<tr><td colspan="6">Connection to previous learning</td></tr>
<tr><td colspan="6">From the NSW K -10 mathematics syllabus
Recognise and describe integers as direction and magnitude
Apply a directed number sentence to real-life eg going up and down stairs
Able to place directed numbers on number line
Compare size of integers (bigger and smaller) using < and ></td></tr>
<tr><td colspan="6">Recognising patterns</td></tr>
<tr><td colspan="6">See patterns in number sentences
Identify numbers going up and going down
Know the patterns in subtraction and addition
See the differences between two numbers on a number line</td></tr>
<tr><td colspan="6">Identifying similarities and differences</td></tr>
<tr><td colspan="6">Know difference between numbers does not change when the numbers are increased or decreased by the same amount
A number and its negative add to zero 2+(-2)=0
What is the same and different about adding a negative and subtracting a positive
3-(+2)=3+(-2)</td></tr>
<tr><td colspan="6">Generalising</td></tr>
<tr><td colspan="6">Develop the rule that addition of adding of a negative is same as subtracting a positive +(-2)= -(+2)
Develop a rule that subtracting a negative is adding a positive –(-2)=+(+2)
Know subtraction is adding a negative
Rules of opposites (positive/negative, plus/minus, up/down, right/left etc)</td></tr>
<tr><td colspan="3">Syllabus outcomes</td><td colspan="3">Summary of content</td></tr>
<tr><td colspan="3">MA4-4NA compares, orders and calculates with integers, applying a range of strategies to aid computation
MA4-2WM compares, orders and calculates with integers, applying a range of strategies to aid computation</td><td colspan="3">Show that adding a negative number and subtracting a positive can be the same
Show that subtracting an negative is the same as adding the positive</td></tr>
</table>

MA4-3WM recognises and explains mathematical relationships using reasoning	
Resources required (e.g. ICT tools, ipad, IWB, A3 paper, colouring pencils, glue…)	
Teacher has white board and markers. Students may have tablets or personal whiteboard, and counters Teacher can demonstrate using a number line on the white board or outside in the garden	
Links with further learning (where to next? Include outcomes)	
• In Stage 4, mental strategies need to be continually reinforced. • Students may find recording (writing out) informal mental strategies to be more efficient than using formal written algorithms, particularly in the case of subtraction. • For example, 72-90 could be written as 70-88 = 70-70-12 • Students need to visualise numbers and manipulate them by knowing number laws and solve without computation	

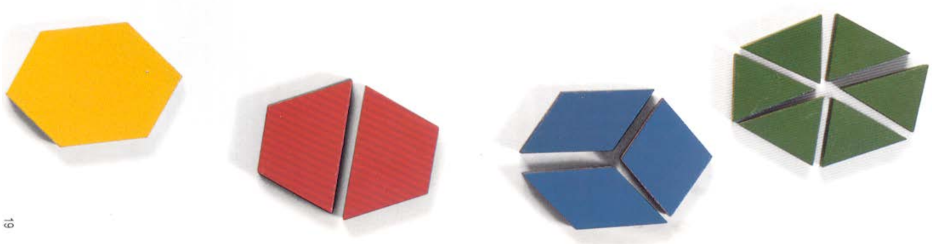
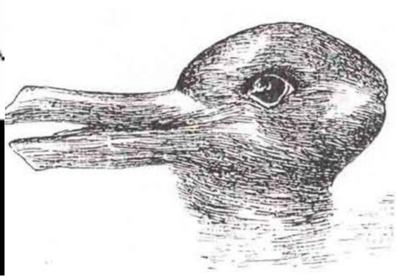
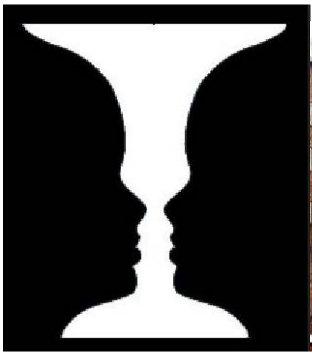
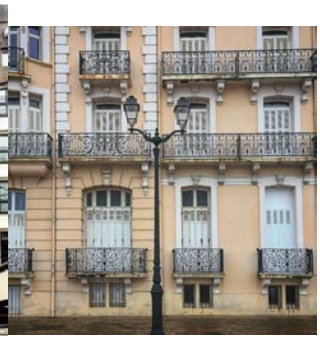
Sequence of teaching / learning experiences (Guidelines: Sections can be expanded or adapted for appropriate use)	Teaching strategies (e.g. Whole class, pairwork, groupwork	Assessment (e.g. how do you know that learning is occurring ?	Time
Engagement: Link to prior learning Teacher begins lesson with students at their desks. Teacher asks the students how can numbers be used when giving directions Teacher asks the students give different examples where numbers are used for directions going up two flights of stairs, turning to the left at the corner and walking two blocks,	Whole class together Social learning between students Role model learning	Teacher is noticing: • students raising or not raising their arm. • Students who are paraticipating in activity • Student responses	3
*Description of the lesson (will you be **modelling** a strategy? will there be a task to solve? an investigation?)* The teacher will draw a number line on the board and point to a position and call it zero. Teacher will ask for volunteers to write the numbers on the number line, on the right and left side of zero Teacher will ask another volunteers for suggestions that could give names to the numbers and their position from zero. Teachers encourages words like left/right, east/west as direction. Teacher asks the class what is a good "mathematical" name for numbers on the left and numbers on the right Teacher poses questions to the group such as: • What are numbers that are equal distance away from zero? • What is the same about these pairs of numbers?	Whole class together Students in pairs	Teacher is noticing: Students who volunteer answers (students who understand offer answers those who don't- don't Accuracy in students answer (all correct or all wrong- tells about what needs to be learnt or not!)	5

• What is different about each of the pairs? • What is the same? Teacher asks the students what is the difference between two numbers. Teacher waits until difference is equated with subtraction and then asks questions like: • How do you find the difference between my height and your height? • What is the difference between your age and your parents age? Teacher asks student to play a game, it involves putting counters on a number line and working out the difference between them. Teacher models the activity and writes a number sentence. Then asks students to play the game themselves using their own sensible numbers.			
Students actively engage in the activity Students move to work in pairs with a number line and counters. Students in pairs create their own subtraction number sentence Students can record the number sentence Students can write a number sentence to present pattern Students describe and record the pattern Students move around and report to other groups what their pattern is Students report any differences in the other groups' pattern Students report how patterns were achieved	Students in pairs, Whole class interaction	Teacher is applying assessment as Learning technique. Having students explain their own and others work demonstrates an understanding. Teacher identifies components of CRIG to consider students understanding of the content.	7
*Teacher **guiding** scaffolding, observing* Teacher is the facilitator and mentor. Asking questions of the students to encourage structural thinking such as: Teacher is promoting student thinking by asking scaffolded questions such as: What made you think that? Did you see a pattern? Did you notice anything different? What did you see was like something else? What will happen next? How are the numbers discovered? What were you thinking to put them like that? Did you think of something else you had done that helped you? What was the pattern you noticed?	Students are working in pairs and groups. Teacher is visible as moving between groups and individuals	Teacher is listening to students' responses to the questions asked and their discussions with other students.	5
Representation of learning and sharing of ideas	In pairs and groups students can share their ideas	Teachers is able to monitor student discussions. Notice who is more involved in the discussion and is the	2

Students are interacting as a whole group (teacher instruction and whole class work) and in small groups discussions. This encourages communication and explanation of work. Opportunities for all students to express their opinions openly without public display		talk demonstrating understanding of the concept.	
Teacher summary and scaffold for further learning. Teacher calls the class to get their attention. When the teacher has the classes attention then the teacher can proceed with a summary of the what they have done and how that subtraction is the difference between the numbers and this can be seen on the number line. Teacher writes a problem on the board that involves such as 4 – (-6) Ask the students how they would find the answer Explore the possible answers Teacher does not draw a number line, but starts with 4 – (+2) Asks students for the answer Then underneath writes 4 - (+1) Then follows 4 – (0) 4 - (-1) 4 - (-2) By now the teacher is asking students to see a pattern in the numbers and the answers. Ask the students is there a rule they can see, what is happening with the –(+) and –(-) Ask for volunteers to write their own number sentence involving the subtraction of a negative number on their white board. Ask students to discuss with the person next to them what way they would find the answer. Ask for another student to solve on the board using the method, ask for different ways to find the answer. Lesson Summary Discuss the difference ways you may understand that subtracting a negative is a adding the number by the two different approaches demonstrated this lesson	Whole group work	Teacher is able to monitor student involvement through attention and involvement. Teacher should have a good idea from the whole lesson of students overall understanding as well as individual understanding.	8

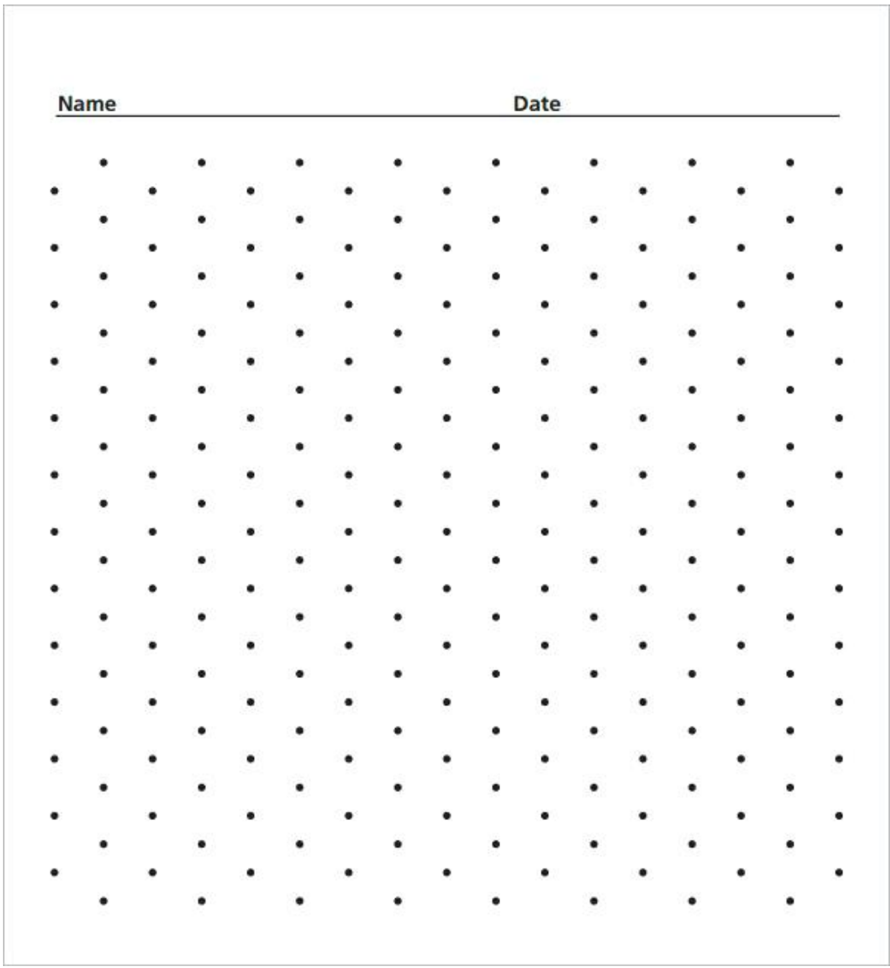
Phase 2 PLW 2 Sample Lesson Plan

<table>
<tr><td colspan="2">Student teacher’s name</td><td>Mr X</td><td>School</td><td colspan="2">XXX</td></tr>
<tr><td>Year / Stage</td><td>Year 7 Stage 4</td><td>Date</td><td>24/1/18</td><td>Duration of lesson</td><td>75 mins</td></tr>
<tr><td colspan="2">Key Learning Area(s)</td><td colspan="4">Properties of Geometrical Figures/ tessellating squares and rectangles</td></tr>
<tr><td colspan="6">Main aim of lesson</td></tr>
<tr><td colspan="6">How many squares – students can recognise and count many squares in a square grid pattern</td></tr>
<tr><td colspan="6">CRIG components considered</td></tr>
<tr><td colspan="6">Connection to previous or other mathematics learning</td></tr>
<tr><td colspan="6">Properties of two dimensional shapes – What are 2d shapes? Where are they in our world? Name some of them?</td></tr>
<tr><td colspan="6">Connection with further learning (where to next?)</td></tr>
<tr><td colspan="6">Visual representation, be able to see shapes and patterns in things, show some optical illusion pictures
How this is important to see 3d shapes and perspectives</td></tr>
<tr><td colspan="6">Recognising patterns</td></tr>
<tr><td colspan="6">Tessellated Patterns in brickworks, tiles, paving, windows in buildings, escher drawings, islamic art</td></tr>
<tr><td colspan="6">Identifying similarities and differences</td></tr>
<tr><td colspan="6">Difference in 2d shapes, sides, angles, convex, regular, irregular. Shapes in a tangram, make the patterns what tessellates and what doesn’t</td></tr>
<tr><td colspan="6">Generalising</td></tr>
<tr><td colspan="6">What shapes can tesselate and why?</td></tr>
<tr><td colspan="4">Syllabus outcomes</td><td colspan="2">Summary of content</td></tr>
<tr><td colspan="4">MA4-1WM
communicates and connects mathematical ideas using appropriate terminology, diagrams and symbols
MA4-2WM
applies appropriate mathematical techniques to solve problems
MA4-3WM
recognises and explains mathematical relationships using reasoning
MA4-17MG
classifies, describes and uses the properties of triangles and quadrilaterals, and determines congruent triangles to find unknown side lengths and angles</td><td colspan="2">This lesson students are going to tesselate shapes by working out what shapes can fit into a 360-degree configuration. Students will see different shapes by looking beyond what they see, and play with shapes using a tangram. Then looking at how shapes tessellate. Can the repeat the pattern and create a tessellating pattern with squares, rectangles, triangles, hexagons, develop a sense of shape</td></tr>
</table>

Resources required (e.g. ICT tools, ipad, IWB, A3 paper, colouring pencils, glue…)
Multiple geometric shapes different colours  Tangram worksheet https://en.wikipedia.org/wiki/Tangram and https://www.tangram-channel.com/ A4 paper for distribution Optical illusion (pictures below) Escher diagrams http://www.mcescher.com (pictures below) Islamic art https://en.wikipedia.org/wiki/Islamic_geometric_patterns (pictures below) Isometric and square dot grid paper
Notes
     

Sequence of teaching / learning experiences (Guidelines: Sections can be expanded or adapted for appropriate use)	Teaching strategies (e.g. Whole class, pairwork, groupwork	Assessment (e.g. how do you know that learning is occurring?)	Time
Engagement: Link to prior learning Go over what is a 2d shape, recognise all the different shapes, Then classify by regular and irregular. Work out all the connections of shapes in their world? Why are shapes important in art, architecture, design, …	Teacher prepares the class for group work	Students are in groups or pairs and drawing up their own classification sheet. Using an A4 paper. Ask "what's the same and different about the shapes? classify by size and angle?"	10
Description of the lesson (will you be ***modelling*** *a strategy? will there be a task to solve? an investigation?)* Teacher presents students with the optical illusion pictures and asks what they see. Start with the old lady/young woman or duck/rabbit. Some students will not be able to see both images. Ask ones who can to show a friend where the image is seen as two different pictures. Talk to the students about the need to look deeply and carefully at things before accepting what they see. Show the pictures of the windows what are the shapes that can see? How many different shapes area there? Can you see shapes inside other shapes? How many different sized squares and rectangles?		Teacher is encouraging the students to look beyond what they see and try to think more about what is presented to them. Questions that connect their learning to previous and current learning. Questions about the optical illusions could include: What is different about the pictures? What is the same? Can you think of things that look like something else? Ask questions about the two squares and rectangle? Are there other shapes? Make the connection to the optical illusion picture that they may need to look more closely	20
Students actively engage in the activity Give students the tangram outline what shapes do they have create a tangram image. Different tangram images are observed. Students use the tangram shapes to fit the pattern.	Students are at their desks and working in their groups	Teacher moves between the groups and looks for students' progress. Encourages students to get to know the shapes look what is the same about them angle and side size. Check what shapes have same sized sides and angles. What shapes fill and don't fit. Change the orientation of the shape	10
Students actively engage in the activity Teacher shows examples of tessellated patterns of tiles and paving. Asks the students what patterns of shapes they can think of from home (connections) what makes up the shapes (same different) Teacher shows some Escher patterns or Islamic art patterns and Tesselating Pavement (Tasmania) Asks the students to take squares and triangles from the shapes provided, give out an isometric grid paper and square grid paper create a grid pattern with the squares and rows of triangles	Students are at their desks and working in their groups	Teacher moves between the groups and looks for students' progress. Encourages students to get to know the shapes look what is the same about them angle and side size. Check what shapes have same sized sides and angles. What shapes fit and don't fit. Change the orientation of the shape Ask what shapes fit together around point Why do they do that? Ask about the size of the angle? Use the isometric grid paper and square grid paper to create tessellations by drawing the shapes.	10

		Ask about the angle sum at a point Why do some shapes don't tessellate What does to tessellate mean? Can you consider two or more shapes that could tessellate? (Hexagon and Square) Can they see other shapes inside the shapes drawn eg square to rectangles, triangles to hexagons, rhombus How many shapes are with in shapes?	
*Teacher **guiding** scaffolding, observing* Teacher is monitoring groups and using CRIG questions and instructions to support student thinking What have we done today that connects your learning to what you know already? How important were the patterns in the lesson, Why is this so? Did you notice what was the same and different in the shapes that tessellate? Can you generalise to create a rule that is true for all shapes to be able to tessellate?	Teacher at the front of the room students at their desks	Teacher needs to introduce the concept of recording to enable students to generalise further. This may be difficult for the students at this stage. Teacher may need to draw a table of shapes that shape and do not shape. Table headings, shape name, number of sides, number angles, size of angles, tessellate or not	



APPENDIX K

Pedagogical Content Knowledge (PCK) Worksheet
Binomial Products

You are to teach expanding binomial products to a year 9 class.

Consider the question given is: Expand $(x+2)(x+3)$.

Consider these questions, make reference to your mathematical content and pedagogical knowledge.

1. What procedure were you taught to answer this question?

2. What procedure would you teach to answer this question?

3. What explanation would you give to your students?

4. Would you just teach the procedure?

5. What other mathematical content does this relate to conceptually or procedurally?

6. Where you could use CRIG to explain and model to students how to answer this question?

7. What differs when you use CRIG in your teaching?

8. What understandings do your students develop from using CRIG?

APPENDIX L

Inter-rater Coding Instructions

Instructions for coding video commentary and transcripts for CRIG components.

From the transcripts given identify the CRIG components of mathematical structure.

What is a CRIG components of mathematical structure.
The key components of mathematical structure are:

- Connecting prior and future learning (C);
- Recognising and producing patterns (R);
- Identifying similarities and difference (I); and,
- Generalising and reasoning (G).

These are referred to by the acronym CRIG components.

An example, a pattern of numbers such as 2, 4, 6, … is immediately recognised as the even numbers (connections), the pattern (recognising patterns) is the difference between each consecutive number (identifying similarities and differences) and knowing that the pattern progresses to 8,10,12, … leads to (generalising).

Teachers may use CRIG in their pedagogical practices with or without awareness. Sometimes terms or phrases are used that clearly identify a CRIG component, other times they may be used without clearly identifying the actual component.

In the commentary/dialogues of the pre-service teachers (PST) the CRIG components are identified in the PSTs' communications in what is said, and in the "turns" that take place between the PST and a student. CRIG components are noticed in the PPST pedagogical practice.

Read the commentary/dialogue which may include a dialogue between the PST and a student.

- Decide what CRIG component is being used.
- This may not be obvious or clearly stated on first reading and.
- Use your professional judgement.

Write one of the letters C, R, I or G in the right-hand column as an identifier to what CRIG is present. If more than one CRIG is present, then write the letter for each component. If no CRIG component is present then leave the section blank