Market Pricing of Longevity-linked Securities

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Declaration

I declare that the intellectual content contained in this thesis is an original work of my research. This thesis has not been submitted in support of any other degree or qualification. All sources of information and assistance received to prepare this report have been indicated and acknowledged.

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Summary

One way of mitigating longevity risk is constructing a hedge using longevity- or mortality-linked securities. A fundamental question is how to price these securities in an incomplete life market where liabilities are not liquidly traded. Although there are various pricing methods developed in the literature, there has been no consensus on which one is the best and the choice is often based on user's preference. This article investigates the potential impact of uncertainty arising from the choice of mortality models and pricing rules on the calculation of longevity-linked security prices. Twelve premium principles based on risk-neutral and real-world measures are examined under the Lee-Carter model and a generalisation of the CBD model. The quotations of UK pension annuities are set as the calibration constraints to incorporate the market view of longevity risk. We compare the results between different pricing methods and model assumptions on the prices of S-forwards and longevity swaps with different maturities.

Overall, we find that the pricing rule uncertainty is less material than the mortality model uncertainty. Particularly, the relationships between the results from different premium principles tend to rely on the underlying mortality model assumption. Our results suggest that the Lee-Carter model tends to give higher implied risk premiums than the CBD model does for both S-forwards and longevity swaps. Besides, the risk premiums calculated by the risk-neutral pricing methods are often lower than those by methods with real-world probabilities, while the results are more comparable within each of the two families.

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Table of Contents

1. Introduction

Due to social and economic development, human longevity has been improving over the past several decades. Such a trend has brought financial burdens to pension plans and insurance companies who provide lifetime income for pensioners and annuitants. These institutions are exposed to a risk of people living unexpectedly longer, which is called the longevity risk. According to Blake (2018), the total longevity risk consists of idiosyncratic longevity risk and systematic longevity risk. Thereinto, specific longevity risk coming from the status of different individuals can be mitigated when the population is large enough. However, systematic longevity risk affects the entire population and cannot be fully diversified away.

Solutions to manage longevity risk broadly fall into three categories (Cairns et al., 2008). Firstly, institutions providing pension products or annuities can transfer (at least partially) the unacceptable longevity risk to insurers or reinsurers. For example, insurers may pay a premium and arrange a reinsurance contract; pension funds may purchase annuity products from insurance companies to hedge their longevity risk. A hedge can also take effect when a company sells both life insurances and annuities, which belongs to the second category – natural hedging. The payoffs of the two types of products have opposite movements because their values are tied to the mortality level (death rates) and longevity level (survival rates) respectively. Nevertheless, there are limitations on the first two approaches in practice. The tightened regulatory capital requirements on mortality and longevity risk have limited the demand from reinsurers on accepting the risk. Moreover, many insurance companies do not have access to the resources required in providing both life insurances and annuities. The third category is to mitigate longevity risk by means of the life market where longevity-linked liabilities can be traded. Specifically, one may securitise the risk (Cowley & Cummins, 2005) or use mortality- and longevity-linked products (Blake et al., 2006). Through securitisation, insurers can package their business lines into securities whose payments are linked to the performance of the underlying business. The major problem of life insurances and annuities securitisation is the complexity of their inherent risks, which impedes investors' ability to understand the securitised portfolios. Besides insurance securitisation, mortality- and longevity-linked instruments serve as a vehicle through which the unwanted longevity risk can be transferred. Previously proposed solutions include longevity bonds (Blake, 2001), longevity swaps (Dowd et al., 2006), q-forwards (Coughlan et al., 2007), S-forwards (Life and Longevity Markets Association, 2010), K-forwards (Tan et al., 2014), mortality options (Cairns et al., 2008), and survivor options (Dowd, 2003).

Practitioners have been managing their longevity risk using some of the above methods. For instance, Blake et al. (2018) surveyed that forty-eight longevity swaps were implemented in the UK from 2007 to 2016, with the first transaction completed in April 2007 by Swiss Re and Friends Life Group (acquired by Aviva in 2015). However, the majority of existing transactions through the life market has been bespoke agreements which are illiquid, to some degree. To improve the transparency and attraction of the life market, some focus has been put on designing standardised products which have payments linked to pre-specified longevity or mortality indices. There were two key contributions to longevity-linked instruments, including the 25-year EIB/BNP longevity bond and the 8-year Swiss Re Kortis longevity trend bond. Being the first longevity bond, the EIB/BNP bond was issued in November 2004. This amortising bond had coupon payments proportional to an index linked to the survival rates of English and Welsh males. It was withdrawn one year after the issuance due to the failure of attracting enough investors' interests. In 2010, a longevity trend bond was designed to improve the effectiveness of Swiss Re's natural hedging strategy. The bond paid quarterly coupons and provided a principal repayment with its value depending on the divergence between the longevity trends in the US and the UK. If the longevity trends of the two countries deviated too much from each other, the bond reduced its principal payment. The reduced amount would then be used to cover some of the losses from Swiss Re's natural hedging portfolio.

The two examples above represent the first attempts in developing a liquid life market. However, the EIB/BNP bond was not traded in the market, suggesting that its payment structure and price did not satisfy the market needs. Moreover, the Swiss Re bond would cover only extreme events (i.e., longevity trends in the two countries diverge too much) but not the usual longevity risk facing most populations. Blake et al. (2018) also pointed out that the Swiss Re bond had a low trading volume, and that no similar longevity trend bonds were issued after 2008. Evidently, the life market is still in its infant stage and is lack of market liquidity. Achieving liquidity requires developing standardised (i.e., index-based) longevity- or mortality-linked securities. In the absence of benchmark longevity risk premiums, assigning an appropriate price to these index-linked securities stays challenging. Given the fact that mortality and longevity rates are not tradable in the current market, it is difficult to decide a proper valuation framework.

The price of longevity hedging instruments can be derived from two components – future longevity patterns and a specified pricing formula, so the choice of mortality models and pricing principles is of utmost importance. Yet, different valuation models may not come up with a consistent price of longevity risk, making it a difficult problem to determine a suitable pricing framework. None of the proposed mortality models in the literature consistently outperforms the others in all data sets and time periods in terms of forecasting accuracy. The selection of mortality models can become quite subjective. Also, it is not clear which premium principle is the most appropriate in life market pricing. Researchers have proposed various pricing principles built on theories in different disciplines. For example, risk-neutral pricing came from the no-arbitrage finance theory; the zero-utility principle was developed based on economic utility functions. Loosely speaking, existing valuation methods may be categorised into two types based on their probability measures[1]. The first type is risk-neutral pricing. Principles under this category set a price equal to the expected present value using risk-neutral probabilities. The present value is calculated by discounting future cash flows at a risk-free rate which is the expected return in a risk-neutral world. Nevertheless, when the market is incomplete, there are an infinite number of risk-neutral measures, and the choice becomes rather arbitrary and has to depend on the situation of the problem. The second type is based on a real-world probability measure. Premiums are determined using real-world probabilities derived from historical data and a discount rate determined by the insurers' own targets. The major shortcoming of

[1] One may also classify different pricing rules by the kind of method in developing each of them (Young (2006). There are three categories – Ad-hoc method, Characterisation method, and Economic method from which premium principles are derived based on intuitive grounds, required properties, and economic theories respectively. Yet, under such classification, groups may overlap with one another. For example, the Wang transform discussed in Section 2 belongs to all three methods.

applying this approach is the difficulty in choosing the appropriate discount rate. In practice, companies may incorporate profit loadings or risk premiums into their required rate of return which varies from one company to another. Without the availability of such private information, the use of a particular value of discount rate may be hard to justify in research.

So far, no consensus has been reached on the "right" mortality models and premium principles in valuing mortality- and longevity-linked securities. This issue has aroused our interests in understanding the uncertainty in selecting between different mortality models and pricing rules and discovering their impact on pricing longevity risk.

We examine the mortality model uncertainty by generating future mortality scenarios from two mortality models – Lee-Carter model (Lee & Carter, 1992) and a generalisation of the Cairns-Blake-Dowd (CBD) model (Cairns et al., 2008). These two mortality models have been applied in life market pricing. For instance, Kogure et al. (2014) employed the Lee-Carter model to price reverse mortgages in Japan; Li (2010) compared prices of longevity bonds derived under the Lee-Carter model, the original CBD model (Cairns et al., 2006), and the generalised CBD model. Moreover, the semi-parametric bootstrap (Brouhns et al., 2005) has often been adopted in the simulation process to incorporate parameter risk. Yang et al. (2015) studied the potentially significant impact of model uncertainty on risk-neutral pricing by applying a modified semi-parametric bootstrap process. We also follow their suggestions and integrate model uncertainty into our mortality simulation using the modified bootstrap.

Next, we investigate the uncertainty arising from different pricing rules. There have been some previous comparisons in the literature. Nonetheless, the analysis was often conducted among only four or fewer premium principles. For example, Barrieu and Veraart (2016) analysed the impact of different sources of uncertainty on valuing q-forwards. The model uncertainty and pricing rule uncertainty were studied by applying three premium principles to mortality scenarios simulated from the Lee-Carter model and the original CBD model. This thesis compares a wider range of premium principles both theoretically and empirically. We first begin our study with one of the most famous risk-neutral pricing principles – the Wang transform (Wang, 2000). It is a risk distortion measure which applies a specific distortion function to the cumulative distribution function of the underlying risk. We then consider six other risk distortion measures and review the potential relationships amongst these methods. We also cover the canonical valuation and the Esscher transform which convert the real-world probability density function into a risk-neutral density function. Furthermore, two candidates with real-world probability measures are added to our list, namely the standard deviation principle and variance principle. Lastly, we propose and experiment with one variation of the standard deviation principle and refer to it as the median absolute deviation principle. It holds the same structure as the standard deviation principle, but the median absolute deviation from the median is applied instead of the standard deviation from the mean. Note that we are not aiming at advocating the "best" premium principle, but rather we try to understand the pricing rule uncertainty by comparing a wide range of valuation methods.

In summary, we examine twelve principles including the Wang transform, proportional hazard transform, dual-power transform, Gini principle, Denneberg's absolute deviation principle, exponential transform, logarithmic transform, canonical valuation, Esscher transform, standard deviation principle, variance principle, and median absolute deviation principle. Most pricing principles require a specification of one or more parameter values which can be obtained by making arbitrary choices or setting market price constraints (i.e., calibration). The calibration process requires information about market prices of securities with payments associated with survival or mortality rates. For instance, Kogure and Kurachi (2010) estimated the market annuity price based on the actuarial life table commonly used by Japanese insurers and set the market price as the price constraint; Sherris et al. (2019) calibrated a risk-neutral pricing principle via the BlackRock CoRI Retirement Indices which estimate the retirement costs of 20 cohorts in the US. Yet most of the previous studies chose the EIB/BNP longevity bond price as a constraint (e.g., Chen et al., 2010; Li, 2010; Li & Ng, 2011; Zhou & Li, 2013). As we have discussed earlier, the EIB/BNP bond was not a successful attempt of issuing longevity-linked securities, and the appropriateness of setting this price constraint is doubtful. Therefore, we use market quotations of standard UK pension annuities in our calibration procedures to incorporate the market view of longevity risk. This study focuses on the pricing of S-forwards and longevity swaps[2]. S-forwards involve the exchange of a single cash flow linked to the survival rates of a predetermined population on the maturity date. We believe that studying S-forwards

[2] Some earlier analyses have been performed on q-forwards (Coughlan et al., 2007; Barrieu & Veraart, 2016), longevity bonds (Denuit et al., 2007; Kogure & Kurachi, 2010), and longevity swaps (Dowd et al., 2006; Zhou & Li, 2013; Li & Tan, 2018; Li et al., 2019).

with a simple payment structure helps to understand the impact of pricing rule uncertainty and differentiate it from other sources of uncertainty such as interest rate risk and structural risk. For further demonstration, longevity swaps comprising a set of S-forwards with different maturities are also included in our analysis.

Overall, the main objective of this thesis is to study the influence of mortality model uncertainty and pricing rule uncertainty on pricing S-forwards and longevity swaps. Twelve premium principles are examined both theoretically and empirically based on mortality scenarios simulated from two mortality models. We find that the choice of mortality models plays a greater role than the choice of premium rules, and the relationships between the results from different pricing principles also tend to reply on the underlying mortality model assumption.

The remainder of the thesis is structured as follows. Section 2 reviews the various premium principles which have been applied in the longevity context and also the wider area. In section 3, we introduce the primary modelling components required in our analysis and the implementation steps. Section 4 discusses the differences and similarities between the twelve pricing methods based on nine desirable properties and an empirical study. Section 5 provides a sensitivity analysis on the valuation results. Section 6 gives the concluding remarks.

2. Literature Review

In this section, we provide a review of some existing pricing principles that have been applied in the literature. S-forwards and longevity swaps are similar to classical forward contracts and swaps, except that the underlying assets are linked to survival rates rather than the values of standard securities like stocks. An intuitive way to price these longevity-linked derivatives may be risk-neutral pricing which has been widely applied in financial derivatives valuation. This stream of pricing methods converts the real-world probability measure P into an equivalent risk-neutral (martingale) measure Q. In a risk-neutral world, the security price is equal to the expected present value of future payoffs under the risk-neutral distribution, using a risk-free discount rate. Nevertheless, the uniqueness of the martingale measure Q relies upon the no-arbitrage assumption in a market where numerous securities are traded liquidly. In an incomplete life market, there exist an infinite number of equivalent measures. For instance, Cairns et al. (2006) incorporated separate longevity risk loadings into each of the two time series processes in the CBD model. The identification of the two risk loading parameters was made by setting one market price constraint and one arbitrary constraint (e.g., the two parameters are set as equal). Li (2010) avoided setting arbitrary constraints and employed the canonical valuation method under which the risk-neutral measure is determined by minimising the Kullback-Leibler information criterion (Kullback & Leibler, 1951). This method was initially developed by Stutzer (1996) to evaluate derivatives in financial markets, and it was then extended to applications in the life market (e.g., Kogure & Kurachi, 2010; Leung et al., 2018). Furthermore, Frittelli (2000) has shown that the univariate (one unknown parameter) canonical valuation is equivalent to a pricing method widely applied in the insurance context - the Esscher transform (Gerber & Shiu, 1994). Another example of risk-neutral pricing is the instantaneous Sharpe ratio method studied by Bayraktar et al. (2009). It can be regarded as an alteration from the real-world measure P to a risk-adjusted measure Q, assuming a constant market risk premium (Chen et al., 2010).

However, the adequacy of adopting risk-neutral pricing methods in the life market is questioned by some authors. In a study of q-forwards, Barrieu and Veraart (2016) claimed that the risk-neutral price is a measure of replicating costs, while in an incomplete life market portfolio replication is not always possible. They pointed out the potential inappropriateness of relying on risk-neutral premiums and only examined three pricing rules based on real-world measures, including the fair premium principle, standard deviation principle, and zero-utility principle. The fair premium principle gives a price merely covering the expected payoff, which makes it not very appealing to insurers as it does not include a risk margin to the price. By contrast, the standard deviation principle assigns a premium in which the embedded loading is proportional to the standard deviation of the payoff. The parameter involved in this method is related to the Sharpe ratio of the underlying portfolio. The last method considered in their paper is the zero-utility principle developed from the expected utility theory (Von Neumann & Morgenstern, 1944). In insurance pricing, the zero-utility principle calculates the price that makes the insurer indifferent between providing and not providing the cover, given a specified utility function (Dickson, 2005). For example, Coughlan et al. (2007) adopted the zero-utility principle with an exponential utility function in q-forward valuation. Besides this principle, another economic method which has been examined in pricing mortality-linked securities is the Tatonnement approach (Zhou et al., 2015). This method assumes that the agents of buyer and seller maximise their expected utility by changing the supply and demand function until an agreed price is reached. The main advantage of the Tatonnement method is that inputs of security prices are not necessary, which copes with the issue of the lack of market price data of longevity- or mortality-linked securities. Yet this method still requires a specification of the agents' utility functions. Similar to the zero-utility principle, the choice of utility functions would be arbitrary, and the assumption of a fair market may not be well suited for the life market.

So far, none of the pricing methods above has been found to be the "best" to assess longevity- or mortality-linked securities. In recent years, some focus has been drawn to the comparison among existing methods. Besides the aforementioned comparison between the three real-world principles, Bauer et al. (2010) considered the Wang transform and the instantaneous Sharpe ratio approach, where the calibrated process utilised UK annuity quotations. Moreover, Leung et al. (2018) employed risk-neutral pricing with the risk-adjusted CBD model, Wang transform, canonical valuation, and Tatonnement approach (Zhou et al., 2015) to price the failed EIB/BNP longevity bond under a Bayesian framework. These few studies only included a limited number of pricing rules that have earlier been applied in the longevity context. However, there are many more other methods in the wider literature, and there is no reason to prevent us from considering other sensible premium principles. In this study, we apply a broad collection of valuation methods to price S-forwards and longevity swaps.

We first expand the comparison list by adding candidates with risk-neutral measures. The popular Wang transform belongs to a risk-distortion family whose members apply an increasing and concave distortion function to the (de-)cumulative distribution function of the underlying risk. One of the principles in this family is the proportional hazard transform, which maps the original de-cumulative distribution function to a proportionally distorted one (Wang, 1995). Wang (1996) also analysed some other candidates in the context of insurance pricing, comprising the dual-power transform, Gini principle, Denneberg's absolute deviation principle, exponential transform, and logarithmic transform, which will be covered in this study. Furthermore, two real-world pricing methods with their risk loadings being proportional to dispersion measures, the standard deviation principle and variance principle, are included in our list. When proposing the Denneberg's absolute deviation principle (Denneberg, 1990), one of the advocated reasons is the statistical robustness of the absolute deviation from the median. Inspired by this reasoning, we adjust the standard deviation principle by replacing the mean and standard deviation with the median and median absolute deviation from the median. This proposed alternative determines a risk premium entirely based on a median-related measure, and it is referred to as the median absolute deviation principle. Mathematical formulae of the pricing methods studied in this thesis are given and interpreted in the next section.

3. Methodology

In this section, we introduce the major modelling components involved in our analysis, including two mortality models for simulating future mortality scenarios and a list of twelve premium principles. We also explain the payoff structures of S-forwards and longevity swaps and our estimation process in detail.

3.1 Mortality models

3.1.1 The Lee-Carter model with cohort effect

We first consider the Lee-Carter model (Lee & Carter, 1992) with an additional cohort effect which expresses the natural logarithm of central death rate $m_{x,t}$ as

$ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \gamma_{t-x}$,

where α_x demonstrates the average level of mortality at age x, κ_t is the time-index of mortality improvement, and β_x represents the age sensitivity of mortality to changes in κ_t[3]. The cohort parameter is included and estimated when significant patterns are detected in the residuals plotted against cohort year. To predict future mortality values, we model the mortality index κ_t as a random walk with drift[4]

$\kappa_t = \kappa_{t-1} + \theta + u_t$,

[3] The identification issue exists in the estimation of the model parameters β_x and κ_t. Specifically, there are infinitely many combinations of β_x and κ_t resulting in the same value of $ln(m_{x,t})$. We follow Lee and Carter (1992) and set $\sum_x \beta_x = 1$, $\sum_t \kappa_t = 0$. The constraint on the cohort factor is set as $\sum_{t-x} \gamma_{t-x} = 0$.

[4] The autoregressive integrated moving average (ARIMA(p,1,q)) process is recommended by (Lee & Carter, 1992) to model the period effect κ_t. For comparison purpose, we adopt an ARIMA(0,1,0), i.e., a random walk with drift to eliminate the uncertainty from choices of the order parameters p and q. The random walk with drift has often been employed in previous studies (e.g., Tuljapurkar et al., 2000).

in which θ is the drift term, u_t is a sequence of independent and identically distributed random variables following the standard Gaussian distribution. The cohort parameter γ_{t-x} can be modelled by a weakly stationary AR(1) model[5],

$\gamma_{t-x} = a_0 + a_1\gamma_{t-x-1} + e_t,$

where the Gaussian error term e_t is assumed to be independent of u_t.

3.1.2 The Cairns-Blake-Dowd model with cohort effect and quadratic terms

The CBD model (Cairns et al., 2006) is famous for its advantages in modelling mortality at higher ages. We consider an extension of the original CBD model, which can capture the possible curvature of the mortality curve (e.g. Cairns et al., 2009). It expresses the logit of one-year mortality rate of a life aged *x* in year *t* $q_{x,t}$ as

$$ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \kappa_{t,1} + \kappa_{t,2}(x-\bar{x}) + \kappa_{t,3}((x-\bar{x})^2 - \sigma_x^2) + \gamma_{t-x},$$

where $\kappa_{t,1}$, $\kappa_{t,2}$, and $\kappa_{t,3}$ represent the general level, gradient, and curvature of the mortality curve in year *t*, and $\bar{x}$ and σ_x^2 are the mean of x and $(x-\bar{x})^2$ across the sample age range. Again, a cohort parameter γ_{t-x} is employed to rectify the patterns (if any) in the residual plot. After fitting the model, we can project the time-varying components to generate future mortality scenarios. The three indices $\kappa_{t,1}$, $\kappa_{t,2}$, and $\kappa_{t,3}$ are modelled by a multivariate random walk with drift, that is

$\mathbf{K_t} = \mathbf{K_{t-1}} + \mathbf{\Theta} + \boldsymbol{\varepsilon}_\mathbf{t},$

in which $\mathbf{K_t} = (\kappa_{t,1}, \kappa_{t,2}, \kappa_{t,3})'$, $\mathbf{\Theta}$ is a 3×1 vector which contains three drift terms, and the 3×1 error vector $\boldsymbol{\varepsilon}_\mathbf{t}$ is assumed to follow the standard multivariate Gaussian distribution. Likewise, the cohort parameter is fitted by an AR(1) model. We refer to this model simply as the CBD model in the following of the thesis.

3.2 Premium principles

We will now elaborate the premium principles considered in this research, including nine risk-neutral (arbitrage-free) measures and three valuation methods with real-world probability measures. As mentioned in Section 2, we denote the empirical (real-world)

[5] One may also fit a non-stationary ARIMA process to the cohort effect. However, mortality projection of new cohorts is required in our analysis, the non-stationarity could lead to irrational forecasting results. Considering this issue, we follow Plat (2009) and fit the cohort parameter by a mean-reverting AR(1) model.

and risk-neutral probability distributions by P and Q respectively, where Q is an equivalent martingale measure of P. Risk-neutral principles adjust the real-world probabilities to allow for higher risk, then the premium is equal to the expectation from the risk-neutral distribution. Under the physical measure P, let $\mathbb{Z}$ be the set of non-negative insurance risks on a probability space (Ω, B, P), in which Ω represents the sample space containing all possible outcomes, the event space B comprises all possible events where each event is a collection of outcomes. The probability measure P specifies the likelihood of occurrence of an event. Consider a risk $X \in \mathbb{Z}$, with probability density function (pdf) $f(x)$, cumulative distribution function (cdf) $F(x) = \Pr(X \leq x)$, and de-cumulative function $S(x) = 1 - F(x)$. Let E[X] be the expected value of X, then $\mathrm{E}[X] = \int_0^\infty x\, f(x)\, dx = \int_0^\infty (1 - F(x))\, dx = \int_0^\infty S(x)\, dx$. We use $f^*(x), F^*(x), S^*(x)$ to represent the corresponding risk-neutral functions for measure Q. The risk premium charged for X is denoted by Π_X, which is a function of X. This particular function specifies a premium principle.

Table 1 Summary of twelve premium principles.

Premium Principles	Pricing Formulae
Wang Transform	$\Pi_X = \mathrm{E}^*[X]$, where $F^*(x) = \Phi(\Phi^{-1}(F(x)) - \lambda)$
Proportional Hazard Transform	$\Pi_X = \mathrm{E}^*[X]$, where $F^*(x) = 1 - (1 - F(x))^{1/\lambda}$
Dual-power Transform	$\Pi_X = \mathrm{E}^*[X]$, where $F^*(x) = F(x)^\lambda$
Gini Principle	$\Pi_X = \mathrm{E}^*[X]$, where $F^*(x) = 1 - \left((1+\lambda)(1-F(x)) - \lambda(1-F(x))^2\right)$
Denneberg's Absolute Deviation Principle	$\Pi_X = \mathrm{E}^*[X]$, where $F^*(x) = \begin{cases} 1 - \lambda - (1-\lambda)(1-F(x)), & 0 \leq F(x) \leq 0.5 \\ 1 - (1+\lambda)(1-F(x)), & 0.5 < F(x) \leq 1 \end{cases}$
Exponential Transform	$\Pi_X = \mathrm{E}^*[X]$, where $F^*(x) = 1 - \dfrac{1 - e^{-\lambda(1-F(x))}}{1 - e^{-\lambda}}$
Logarithmic Transform	$\Pi_X = \mathrm{E}^*[X]$, where $F^*(x) = 1 - \dfrac{\ln(1 + \lambda(1 - F(x)))}{\ln(1+\lambda)}$
Canonical Valuation	$\Pi_X = \mathrm{E}^*[X]$, where $f^*(x) = \frac{e^{\lambda x} f(x)}{\int_0^\infty e^{\lambda x} f(x) dx}$
Esscher Transform	$\Pi_X = \mathrm{E}^*[X] = \dfrac{\mathrm{E}[Xe^{\lambda X}]}{\mathrm{E}[e^{\lambda X}]}$
Standard Deviation Principle	$\Pi_X = \mathrm{E}[X] + \lambda \times \mathrm{SD}[X]$

Variance Principle	$\Pi_X = \mathrm{E}[X] + \lambda \times \mathrm{VAR}[X]$
Median Absolute Deviation Principle	$\Pi_X = F^{-1}(0.5) + \lambda \times \mathrm{MAD}[X]$, where $\mathrm{MAD}[X] = \mathrm{median}(\lvert X - F^{-1}(0.5)\rvert)$

The first seven principles in Table 1 can be categorised as distortion risk measures. Although all risk-neutral valuation methods utilise adjusted probabilities to allow for a risk premium, members in this family act on the (de-)cumulative distribution rather than the pdf. The distortion function $g(x)$ applied to the original de-cumulative function is an increasing and concave function (Wang, 1996). Under a distortion method, the resulting risk premium Π_X is equal to the expected value based on the distorted cdf $F^*(x)$. That is, $\Pi_X = E^*[X] = \int_0^\infty (1 - F^*(x))\, dx = \int_0^\infty g(1 - F(x))\, dx = \int_0^\infty g\big(S(x)\big)dx$. Due to the use of the de-cumulative distribution, distortion premium principles can naturally be applied to assign premiums to different layers of insurance (Laeven & Goovaerts, 2008). In the following, we list and discuss the seven distortion risk measures.

3.2.1 Wang transform

One of the most popular methods applied in pricing longevity and mortality instruments is the Wang transform (Wang, 2000, 2002). It embeds a Gaussian-related distortion function which gives a distorted cdf $F^*(x) = \Phi(\Phi^{-1}(F(x)) - \lambda)$ $(\lambda \geq 0)$, in which $\Phi(x)$ represents the cdf of the standard Gaussian distribution, and $\Phi^{-1}(x)$ is its inverse function. There is no initial specification of the distribution of the underlying risk *X*, but when *X* is normally or lognormally distributed, the Wang transform recovers the Capital Asset Pricing Model and the Black-Scholes Model (Wang, 2003). Due to its elegant structure and theoretical support, this method has been widely employed by academics in life market pricing. For example, Lin and Cox (2005) and Cox et al. (2006) applied it to price mortality bonds by distorting distributions of mortality rates. Assuming that asset prices follow a Geometric Brownian Motion, Wang (2002) demonstrated that the Wang transform forms a universal pricing framework. Nonetheless, Pelsser (2008) proved that the Wang transform does not always produce consistent results with those from no-arbitrage pricing unless certain restrictive constraints are satisfied. Following the analysis, the author concluded that this principle is not a universal pricing method in general. Moreover, one may extend the one-factor Wang transform to a two-factor model with $F^*(x) = Q(\Phi^{-1}(F(x)) - \lambda)$ $(\lambda \geq 0)$, where Q refers to a cdf of *t*-distribution with *v* degrees of freedom. This extension has been found suitable to model the yield spread premium of catastrophe bonds and corporate bonds (e.g., (Wang, 2004). For comparison purpose, this study only focuses on the one-factor Wang transform.

3.2.2 Proportional hazard transform

The proportional hazard transform has one of the simplest distortion functions $g(x) = x^{1/\lambda}$ $(\lambda \geq 1)$. Wang (1995) applied this function to the hazard rate of an insurance risk, which results in a distorted distribution with $S^*(x) = S(x)^{1/\lambda}$. The distorted cdf is given by $F^*(x) = 1 - S^*(x) = 1 - S(x)^{1/\lambda} = 1 - (1 - F(x))^{1/\lambda}$. Researchers have applied the proportional hazard transform in pricing insurance risks. Maria de and João Andrade (2005) analysed the application of this principle under the assumptions of exponential, Pareto, and uniform distributions with resampling techniques. Necir and Meraghni (2009) proposed an estimator of the proportional hazard premium based on extreme quantiles to price heavy-tailed insurance claims.

3.2.3 Dual-power transform

The dual-power function (Wang, 1996) is similar to the proportional hazard function, but it transforms *F*(*x*) rather than *S*(*x*). Specifically, $S^*(x) = 1 - (1 - S(x))^{\lambda}$ and $F^*(x) = F(x)^{\lambda}$. The implied distortion function is $g(x) = 1 - (1 - x)^{\lambda}$ $(\lambda \geq 1)$. This principle was applied by Lynn Wirch and Hardy (1999) to calculate the guarantee liability of segregated funds.

3.2.4 Gini principle

Denneberg (1990) introduced the Gini principle involving the use of the Gini coefficient – a measure of wealth inequality. Its distortion function is $g(x) = (1 + \lambda)x - \lambda x^2$ $(0 \leq \lambda \leq 1)$, which implies distorted $S^*(x) = (1 + \lambda)S(x) - \lambda(S(x))^2$ and $F^*(x) = 1 - ((1 + \lambda)(1 - F(x)) - \lambda(1 - F(x))^2)$. As one candidate of distortion measures, the Gini principle was applied in estimating premiums for excess-of-loss reinsurance with high retention levels (Vandewalle & Beirlant, 2006).

3.2.5 Denneberg's absolute deviation principle

The median and mean absolute deviation from the median are two alternatives of the mean and standard deviation respectively. The former two statistics about the median are less affected by outliers than the latter two; also, the median-related measures can reflect asymmetry of the underlying distributions (Denneberg, 1988). Accordingly, Denneberg (1990) argued that the average absolute deviation from the median is more appropriate than the standard deviation from the mean in determining insurance risk loadings. The author derived the absolute deviation principle using a piecewise distortion function

$$g(x) = \begin{cases} (1+\lambda)x, & 0 \le x < 0.5 \\ \lambda + (1-\lambda)x, & 0.5 \le x \le 1 \end{cases} \quad (0 \le \lambda \le 1).$$

Then $g(x)$ is applied to $S(x)$ and this produces a distorted piecewise cdf

$$F^*(x) = \begin{cases} 1-\lambda-(1-\lambda)\big(1-F(x)\big), & 0 \le F(x) \le 0.5 \\ 1-(1+\lambda)\big(1-F(x)\big), & 0.5 < F(x) \le 1 \end{cases}.$$

Subject to the increasing and concave constraints on distortion functions, one can develop many other distortion members. We include two more distortion candidates mentioned in Wang (1996) as below.

3.2.6 Exponential transform

The distortion function is formed by two exponential functions. It maps [0,1] to [0,1] with weighted probabilities. The distortion function and the risk-adjusted cdf are $g(x) = \frac{1-e^{-\lambda x}}{1-e^{-\lambda}}$ $(\lambda > 0)$ and $F^*(x) = 1 - \frac{1-e^{-\lambda(1-F(x))}}{1-e^{-\lambda}}$ respectively.

3.2.7 Logarithmic transform

This principle adopts a similar transformation to the exponential one, with the distortion function and the cdf being $g(x) = \frac{\ln(1+\lambda x)}{\ln(1+\lambda)}$ $(\lambda > 0)$ and $F^*(x) = 1 - \frac{\ln\left(1+\lambda\left(1-F(x)\right)\right)}{\ln(1+\lambda)}$.

Instead of distorting the cdf, some risk-neutral principles act on the pdf. We first consider one representative among the variations within this category – the canonical valuation.

3.2.8 Canonical valuation

Stutzer (1996) proposed the canonical valuation stemming from the Shannon entropy used in Physics. Li (2010) applied this method to price longevity risk using simulations from parametric bootstrap (Brouhns et al., 2005); Li and Ng (2011) employed it in valuing mortality risk with nonparametric bootstrap (Efron, 1979). The canonical valuation identifies the equivalent martingale measure Q by maximising the Shannon entropy, which is the same as minimising the Kullback-Leibler information criterion (Kullback & Leibler, 1951). For this reason, the canonical valuation is often called the maximum entropy principle. The implementation of canonical valuation is demonstrated as follows.

The risk-neutral probability function $f^*(x)$ is solved by minimising the Kullback-Leibler information criterion $\int_0^\infty f^*(x) \ln \frac{f^*(x)}{f(x)} dx$, subject to the constraint $\int_0^\infty f^*(x)\, dx = 1$. Then the premium is equal to the expected value under the risk-neutral measure, i.e., $\Pi_X = E^*[X] = \int_0^\infty x f^*(x)\, dx$. The solution is given by $f^*(x) = \frac{e^{\lambda x} f(x)}{\int_0^\infty e^{\lambda x} f(x) dx}$, where λ can be obtained by setting a market price constraint. Specifically, λ is estimated as the parameter value that equates the simulated mean price (risk-neutral) for Π_X and the market price.

When there is more than one security price available in the market, this approach can be readily extended to incorporate the additional information. Given m security prices, the risk-neutral probability is $f^*(x) = \frac{e^{\sum_{i=1}^m \lambda_i x_i} f(x)}{\int_0^\infty e^{\sum_{i=1}^m \lambda_i x_i} f(x) dx}$, where x_i is the present value of payoffs from security i, and λ_i $(i = 1, 2, \dots, m)$ can be obtained by setting multiple market price constraints. Again, we only consider the univariate canonical valuation with one parameter to make the results comparable among different principles. Moreover, it can be shown that the univariate canonical valuation gives the same solution as the Esscher principle – a popular valuation method in actuarial science (Gerber & Shiu, 1994).

3.2.9 Esscher transform

The Esscher transform was proposed by Esscher (1932). It was applied in economic pricing by Bühlmann (1980) and in option pricing by Gerber and Shiu (1994) respectively. In the context of actuarial science, the Esscher transform can serve as an insurance pricing method (refer to Dickson (2005) for more details). One can obtain the premium under the Esscher principle by $\Pi_X = \frac{\mathrm{E}[Xe^{\lambda X}]}{\mathrm{E}[e^{\lambda X}]} = \frac{\int_0^\infty xe^{\lambda x} f(x)dx}{\mathrm{E}[e^{\lambda X}]}$ $(\lambda > 0)$. With an appropriate parameter value λ, the loss process after transform X^* becomes a unique martingale under no-arbitrage assumptions. The premium function can also be written as the expected value of the risk-adjusted variable X^* with $f^*(x) = \frac{e^{\lambda x} f(x)}{\mathrm{E}[e^{\lambda X}]} = \frac{e^{\lambda x} f(x)}{\int_0^\infty e^{\lambda x} f(x)dx}$, which is an exponential tilting to the original pdf. It can be seen that the risk-adjusted pdf is identical to that under the univariate canonical valuation.

Next, we discuss some valuation methods formed by real-world probability measures.

3.2.10 Standard deviation and variance principle

The price under the standard deviation principle is given by $\Pi_X = \mathrm{E}[X] + \lambda * \mathrm{SD}[X]$. It is equal to the pure premium plus a risk loading proportional to the standard deviation of the underlying risk. The loading parameter λ is related to the well-known Sharpe ratio (Sharpe, 1966). As noted by Buhlmann (1970), this method has been commonly applied in casualty and property insurance. A similar branch is called the variance principle and its risk loading is proportional to the variance of the risk, that is, $\Pi_X = \mathrm{E}[X] + \lambda * \mathrm{VAR}[X]$.

3.2.11 Median absolute deviation principle

As demonstrated by Denneberg (1988), statistics about the median tend to be more robust than mean-variance measures (e.g., mean, standard deviation, and variance) for asymmetric distributions and samples with outliers. For example, Leys et al. (2013) advocated the use of the median absolute deviation from the median (MAD) because the standard deviation is not resilient to outliers. More importantly, they stated that the MAD is less affected by the sample size. Such properties may be desirable in life market pricing. Not all available mortality data are adequate for estimating mortality models, as a prolonged period may involve structural changes. To avoid the inclusion of shifts in the longevity trend, a shorter sample period starting from more recent years is often utilised. Besides, there could be mortality jumps resulted from catastrophe events and medical breakthroughs included in sample data. Although such incidents may not lead

to permanent changes in mortality trends, they could produce outliers in the short run. Therefore, the statistical robustness of the MAD makes it better suited for mortality data with relatively small sample size and potential outliers. The aforementioned Denneberg's absolute deviation principle utilises the mean absolute deviation about the median[6], while one may construct a premium principle which is entirely based on the median. Accordingly, we modify the standard deviation principle by replacing the mean and standard deviation with the median and MAD as
$\Pi_X = F^{-1}(0.5) + \lambda \times \mathrm{MAD}[X]\ (\lambda > 0)$, where $\mathrm{MAD}[X] = \mathrm{median}(|X - F^{-1}(0.5)|)$.
Given the long-term nature of the underlying risk, coping with possible permanent shifts in longevity trends (i.e., structural changes) is vital in pricing longevity-linked products. The standard deviation and variance principles may fail to price possible structural changes fairly, since these measures around the mean may not truly reflect the real uncertainty of future mortality. By contrast, any asymmetry in the mortality distribution can be more properly accommodated by the median absolute deviation principle. In Section 4, we shock our mortality models by arbitrarily including permanent shifts in the drift terms of the (multivariate) random walk processes to further investigate the performance of these variations.
We will discuss the properties of each principle and some empirical results in Section 4.

3.3 Longevity-linked instruments

3.3.1 S-forwards (Life and Longevity Markets Association, 2010)
Similar to classical forward contracts, two counterparties of an S-forward agree to exchange two payments on a predetermined future date T (maturity date). The cash flows are linked to indices of survival rates rather than security prices. The buyer of an S-forward pays a fixed rate $K(T)$ (forward rate) to the seller and receives a floating rate $S(T)$ (realised survival rate) in return. Thereby, one counterparty needs to pay an amount that is linked to the difference between the forward rate and realised rate on the maturity date. The forward rate is specified at the outset of the contract and reflects the

[6] Note that we denote the median absolute deviation from the median by MAD, but not the mean absolute deviation from the median or mean.

expectation of future longevity level. A fair (long) S-forward has zero value at the inception of the contract, i.e.,

$V_0[\text{notional principal} \times (S(T) - K(T))] = 0,$

where $V_0[X]$ represents the time-0 value of the risk X under a specific valuation formula. For convenience, we assume that the notional principal agreed in the contract is one unit. Each premium principle involves a different function for $V_0[X]$, which would produce a different forward rate of $K(T)$.

3.3.2 Longevity swaps (Dowd et al., 2006)

In a longevity swap, two counterparties exchange a stream of future cash flows depending on the differences between the floating rates and fixed rates in regular periods. In other words, it consists of a series of S-forwards with different maturities. A risk premium is incorporated into the fix-leg payments such that the swap value on the issue date is zero for both parties.

3.4 Estimation process

To investigate the impact of using different premium principles under different mortality models on the calculation of S-forward and longevity swap prices, we need to forecast the empirical distribution of mortality rates using past mortality data and estimate the parameters in the pricing formulae with market information. Our estimation process can be split into three general steps. Figure 1 shows a flow diagram of the process.

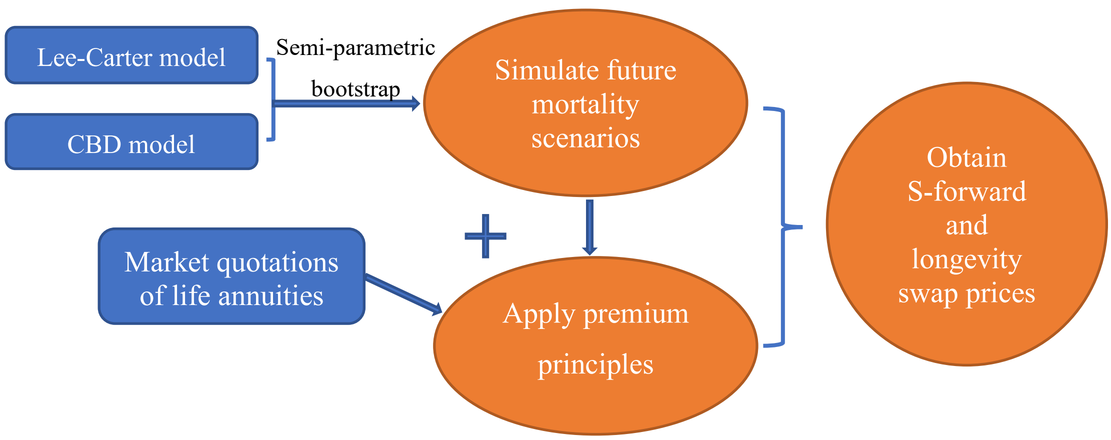


Fig. 1 Estimation process

First, two sets of future mortality scenarios are generated from the CBD model and the Lee-Carter model. We fit the two mortality models to our mortality data using an iterative updating method under the Poisson assumption (Brouhns et al., 2002; Li, 2013). The number of deaths at age x in year t $D_{x,t}$ is assumed to follow a Poisson distribution Poisson($E_{x,t}m_{x,t}$) with the mean equal to the expected number of deaths. This setting allows us to estimate the model parameters using the maximum likelihood approach. Then under each of the two mortality models, we project and simulate future values of time series to generate a large number of mortality scenarios. The semi-parametric bootstrap proposed by Brouhns et al. (2005), which incorporates both process error (uncertainty in the time series) and parameter error (uncertainty in the parameter estimation), is used here. Specifically, we obtain a pseudo sample by simulating the number of deaths from a Poisson distribution with a mean equal to the observed death counts. Then the mortality model is fit to this pseudo sample and the fitted model is used to forecast future mortality distributions. The same process is repeated 10,000 times to generate 10,000 scenarios under each model. After obtaining future mortality scenarios, the second step is to calibrate the parameters of the twelve pricing principles by setting market price constraints. Under each premium principle, we find the parameter value such that the pricing formula gives a price that is equal to the reference market price. Lastly, we apply the calibrated premium principles to calculate the prices of S-forwards and longevity swaps with different maturities.

3.5 Data

We have collected UK mortality data by single age and year from the Human Mortality Database (HMD, 2018). The number of deaths and exposed to risk of the UK population aged 60 to 89 between 1965 and 2016 are chosen for our analysis. The starting age of 60 is selected because we are interested in longevity risk of pensioners; the ending age of 89 is chosen to avoid using the volatile data at higher ages (Thatcher, 1999). We extend the projected and simulated mortality rates to a prescribed maximum age using the Coale-Kisker method (Coale & Kisker, 1990). Following Gampe (2010), we assume an ultimate value of 0.7 at age 110 and obtain the central death rates between ages 90 and 109 accordingly. Data before 1965 are not included, as structural changes in the mortality trend have been detected around that time (Li et al., 2011). Note that we do not split our data by sex because it can no longer be a pricing factor for annuities in Europe after 21st December 2012 under the revised legislation Council Directive 2004/113/EC (2004). As at 20th October 2018, the standard annuity rate for a £100,000 pension fund was £5,563 for a single life aged 65[7]. Although the quoted annual income is paid monthly in advance, for convenience, we treat it as an annual payment. For example, an annuitant aged 65 will receive a payment of £5,563 in the middle of each future year on survival. One problem with incorporating the market annuity quotations into the constraints is that these prices usually include loadings other than the market perception of longevity risk. To filter out the impact of non-longevity components, we exploit the concept of money's worth (MW) which is a reflection of an annuity's "true value". The money's worth is defined as the expected present value of annuity payments as a proportion of the annuity price. Annuity providers require a non-negative loading of $(1 - MW)\times 100\%$, comprising longevity loadings and non-longevity loadings such as expenses. We can subtract the proportion of non-longevity loadings from the market price and obtain a risk premium into which only longevity risk is incorporated. Nevertheless, the size and type of loading factors embedded in annuity prices are usually not publicly available. In this thesis, we assume that money's worth is 94%, and half of the loading $(1 - MW)$ comes from longevity risk[8], implying 3% non-longevity loadings. Under the current economic environment and general perception of future outlook, interest rates are expected to remain low. We then use the UK gilt rate as the risk-free rate and the discount rate. As at 20th October 2018, the 15-year gilt rate was 1.7%.

[7] Annuity quotations were obtained from https://www.sharingpensions.co.uk.

[8] Aquilina et al. (2017) investigated that the money's worth for 65-year-old male annuitants is 94% in the UK for a £50,000 pension from 2006 to 2014. We acknowledge that this information does not match with our data exactly (e.g., the size of the pension pot), while it is the most relevant one we can find as a proxy. Gallagher (2003) reported in his conference paper that the expense assumption of UK annuities is usually 1% to 3% of the total price. An analysis of the European insurance industry by Oliver Wyman (Whitworth & Byron, 2012) also suggested an equal annuity margin for operation expense and longevity risk. We borrow this information and adopt a 3% expense loading in our analysis.

4. Empirical and theoretical analyses of pricing principles under different mortality model assumptions

4.1 Fitting results

Based on the sample range described earlier, we fit the Lee-Carter model and the CBD model with a curvature term, under which the estimated time series components are plotted in Fig. 2.

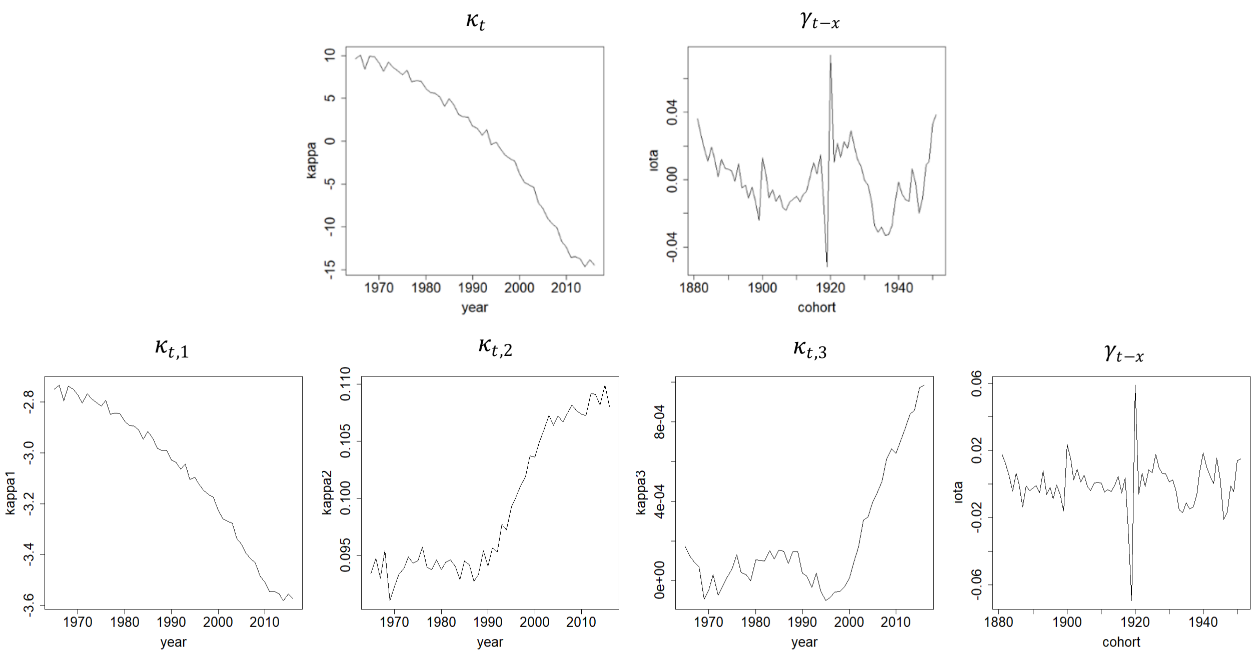


Fig. 2 Parameter estimates of the Lee-Carter model (top) and the generalised CBD model (bottom) with cohort effect

We can observe that both the κ_t and $\kappa_{t,1}$ in the Lee-Carter and CBD models have a downward trend, indicating the overall improvement in mortality rates over time. Also, the other two time-varying components ($\kappa_{t,2}$ and $\kappa_{t,3}$) of the CBD model show a generally increasing trend. It may be reasonable to fit a (three-dimensional) random walk with drift to the time-related parameters in the two models[9]. A cohort effect γ_{t-x} is included in each of the two mortality models given the significant patterns detected in the residuals plots against cohort year (left diagrams of Fig. 3). As displayed on the right side of Fig. 3, the patterns and spikes are mostly removed after incorporating cohort terms. Furthermore, we examine the goodness-of-fit of our models using the Bayesian Information Criterion (BIC), which penalises using additional parameters. After removing the systematic patterns, despite a larger number of parameters, the BIC value of the Lee-Carter (CBD) model reduces from 34,530 (28,805) to 25,609 (23,112), which again advocates the use of cohort factors in modelling the UK population.

Next, we discuss some desirable premium principle properties which may or may not be satisfied by the twelve pricing methods under study.

[9] Cairns et al. (2008) pointed out the potential biological inappropriateness resulted from fitting a random walk with drift to parameters in the CBD model, and suggested to remove the drift term of the slope and curvature coefficients. One may also try other time series choices such as the (vector) autoregressive integrated moving average model. Yet, we are not targeting at discovering the uncertainty arising from time series models. We follow Cairns et al. (2011) and employ a multivariate random walk with drift.

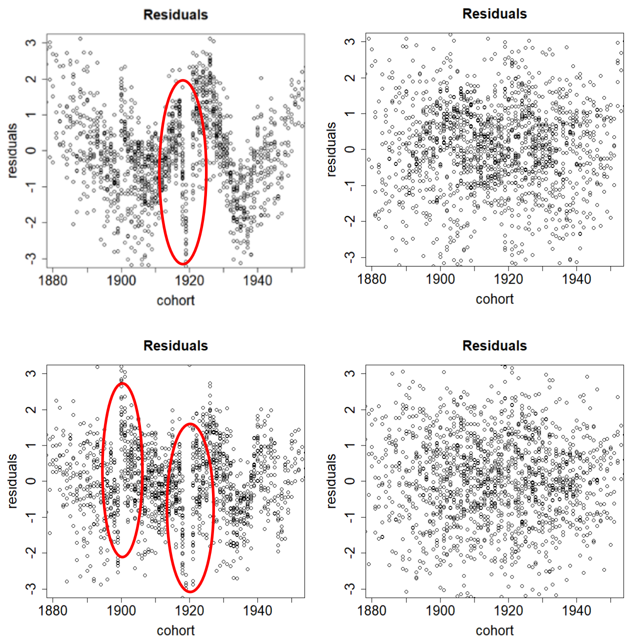


Fig. 3 Standardised deviation residuals against cohort year without (left column) and with (right column) cohort effect from the Lee-Carter model (top panel) and the generalised CBD model (bottom panel)

4.2 Desirable properties satisfied by each premium principle

In practice, a pricing method may be chosen for its particular properties. For example, scale invariance ensures that when claims are expressed in another currency, which is effectively a scale transformation, the premium in that currency is the same scale transformation of the previous premium. Therefore, insurers may prefer a scale-invariant method in dealing with international business. To provide a more comprehensive comparison between different pricing principles, we list some popular properties below and demonstrate which of those properties are satisfied by each principle (see Young (2004) for an extended list of properties). We use the same notation Π_X, $\mathrm{E}[X]$, and $S(x)$ to denote the risk premium, expected value, and de-cumulative distribution function of a non-negative risk $X \in \mathbb{Z}$ with the probability space (Ω, B, P).

1. **Non-negative loading:** $\Pi_X \geq E[X]$ for all $X \in \mathbb{Z}$.

It is reasonable to expect that the premium charged for a risk *X* should at least cover the expected payment of *X*. Otherwise, insurers would suffer a loss on average.

2. **No unjustified risk loading:** $\Pi_X = c$ if $X(\omega) = c$ for all $\omega \in \Omega$, where c is a non-negative constant.
   When there is no uncertainty about the payout of *X*, an appropriate premium is just the constant payout of c.

3. **Additivity for independent risks:** $\Pi_{X_1+X_2} = \Pi_{X_1} + \Pi_{X_2}$ for all $X_1, X_2 \in \mathbb{Z}$, where X_1 and X_2 are independent.
   This property requires that the premium for the sum of independent risks is equal to the sum of individual premiums. From the point of view of premium size, there is no incentive to combine or separate independent risks, given that additivity is satisfied.

4. **Sub-additivity:** $\Pi_{X_1+X_2} \leq \Pi_{X_1} + \Pi_{X_2}$ for all $X_1, X_2 \in \mathbb{Z}$.
   Taking insurance as an example, sub-additivity states that it would be cheaper to insure multiple (possibly dependent) risks together than individually.

5. **Scale invariance:** $\Pi_Z = c\Pi_X$ for all $X \in \mathbb{Z}$, where $Z = cX$ and c is a non-negative constant.
   This scale invariance property can be justified in some circumstances. For instance, customers would expect to pay double the price (i.e., $2\Pi_X$) when their policy size is doubled, otherwise they could buy two separate policies and pay a total of $2\Pi_X$. However, when the size of risk *X* is huge, insurers may apply a greater loading for the doubled risk 2*X* considering their capital adequacy (Wang, 2004).

6. **Translation invariance:** $\Pi_Y = \Pi_X + c$ for all $X \in \mathbb{Z}$, where $Y = X + c$, and c is a non-negative constant.
   This property states that when a risk *X* changes by a constant amount *c*, its premium should be adjusted by the same amount.

7. **No rip-off:** $\Pi_X \leq x_{\max}$, where $x_{\max}$ is the finite maximum value of *X* (if it exists).

If the premium exceeds the maximum possible payout of the underlying risk, rational customers would not have the motivation to buy such a product.

Suppose all insurers (or investors) are risk-averse, then higher risks should be compensated by higher premiums. The following properties refer to two fundamental orderings of risks.

8. Monotonicity:

If $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$, then $\Pi_X \leq \Pi_Y$.

9. First stochastic dominance (FSD) ordering:

If $S_X(z) \leq S_Y(z)$ for all $z \geq 0$, then $\Pi_X \leq \Pi_Y$.

Each premium principle may or may not meet all the nine properties discussed above. Table 2 lists the twelve pricing methods under the nine desirable properties. We use Y (N) to represent that the corresponding property is satisfied (not satisfied) by each method.

Table 2 Principle × property matrix, Y (N) indicates satisfaction (non-satisfaction) of the property[10].

Premium principles	Non-negative loading	No unjustified loading	Additivity	Sub-additivity	Scale invariance	Translation invariance	No rip-off	Monotone	FSD
Wang	Y	Y	N	Y	Y	Y	Y	Y	Y
PH	Y	Y	N	Y	Y	Y	Y	Y	Y
DP	Y	Y	N	Y	Y	Y	Y	Y	Y
Gini	Y	Y	N	Y	Y	Y	Y	Y	Y
Denne	Y	Y	N	Y	Y	Y	Y	Y	Y
Exp	Y	Y	N	Y	Y	Y	Y	Y	Y
Log	Y	Y	N	Y	Y	Y	Y	Y	Y
ME	Y	Y	Y	N	N	Y	Y	N	N

[10] For presentation purpose, we only show the abbreviation of each principle in the following tables and figures. Specifically, the Wang transform, proportional hazard transform, dual-power transform, Gini principle, Denneberg's absolute deviation principle, exponential transform, logarithmic transform, canonical valuation (maximum entropy), Esscher transform, standard deviation principle, variance principle, and median absolute deviation principle are denoted by Wang, PH, DP, Gini, Denne, Exp, Log, ME, Ess, sd, var, and mad, respectively.

Ess	Y	Y	Y	N[11]	N	Y	Y	N	N
sd	Y	Y	N	Y	Y	Y	N	N	N
var	Y	Y	Y	N	N	Y	N	N	N
mad	N	Y	N	N	Y	Y	N	N	N

As shown in Table 2, all pricing principles considered above are **translation invariant**, and they do not have **unjustified loadings**. Translation invariance can readily be justified. The risk premium calculated by the first nine principles is the expected value from a risk-neutral distribution. Adding a constant to the underlying risk would shift the risk premium by the same amount. Due to the zero dispersion of a constant, the last three principles also possess the translation invariance property. For the same reason, when there is no uncertainty in the risk (i.e., payout is equal to a constant), the resulting risk premium from all twelve methods is simply equal to the constant payout. In addition, the first eleven valuation methods ensure that the price of a risk covers at least the expected payment (**non-negative loading**), given the non-negative parameter value λ. However, the median absolute deviation principle does not satisfy this property in all cases. Since its lower bound is the median rather than the mean, it is possible to have a premium below the mean for positively skewed distributions. One may set a lower bound for λ to guarantee this property. When the median $F^{-1}(0.5)$ is less than the mean $E[X]$, enforcing the constraint $\lambda \geq \frac{E[X]-F^{-1}(0.5)}{MAD[X]}$ can yield a premium with non-negative loadings. Besides the lower bound, all principles except the last three with real-world probability measures produce a premium subject to an upper bound – the maximum payout from the underlying risk (**no rip-off**). Again, charging a premium exceeding the upper limit can be avoided by restricting the domain of λ. Furthermore, only the canonical valuation, Esscher transform, and variance principle are **additive** for independent risks. Without the independence assumption, the seven candidates with distortion risk measures and the standard deviation principle are **sub-additive**[12]. The same set of candidates and the newly proposed principle satisfy the **scale-invariant** property. For example, the variance principle violates this property because its risk loading is proportional to the variance

[11] See Wang (2003) for a counter example.

[12] (Wang, 1995) proved the sub-additivity for the proportional hazard transform, which can be generalised to any distortion measures with increasing and concave distortion functions (Wang, 1996).

which is not in the same scale as the payout. By comparison, the standard deviation principle and median absolute deviation principle may be more suitable than the variance principle when scale-invariance is favoured. Besides, only those principles with distortion measures preserve the two orderings of risks – **monotonicity** and **first stochastic dominance**. It is not surprising to observe that these seven candidates satisfy the same set of properties. The premium principles with distortion risk measures are fundamentally similar, as they all add a risk loading by applying an increasing and concave distortion function to the underlying cdf. The only property which is not preserved by them is the additivity. Although the Esscher transform and canonical valuation are also based on risk-neutral distributions, they distort the pdf rather than the cdf. Such differences in techniques make them fail to satisfy the two orderings of risks.

4.3 Pricing S-forwards and longevity swaps

This section presents the implied risk premiums of S-forwards and longevity swaps calculated by the twelve principles under the two mortality models. The premium principles are calibrated by setting the market annuity price with a starting age of 65 as a constraint. Each calibrated pricing principle is then applied to the simulated mortality scenarios to produce a set of forward rates with different maturities. Instead of comparing forward rates, we display the implied risk premiums over the central estimates of survival rates. For longevity swaps with a term T, the risk premium δ per annum can be solved by the following equation: $\sum_{t=1}^{T} \frac{K(t)}{(1+r)^t} = \sum_{t=1}^{T} \frac{S^c(t)}{(1+r)^t} e^{\delta t}$, where r is the discount rate over the term of the contract, $K(t)$ is the forward rate at time t, and $S^c(t)$ is the central estimate (projected value) of the survival rate at time t. Since S-forwards only involve one cash flow on the maturity date, the implied risk premium can be found from the equation[13] $K(T) = S^c(T)e^{\delta T}$. Fig. 4 plots the implied risk premiums of S-forwards and longevity swaps with different maturities. The numerical results are illustrated in Tables 3 and 4. One would expect a positive relationship between longevity risk premiums and the contract term due to the increased uncertainty in longevity levels in the more distant future. In general, all of the valuation methods give higher premiums to longer contracts.

[13] The original form is $\frac{K(T)}{(1+r)^T} = \frac{S^c(T)}{(1+r)^T} e^{\delta T}$, where the discounting factors can be cancelled.

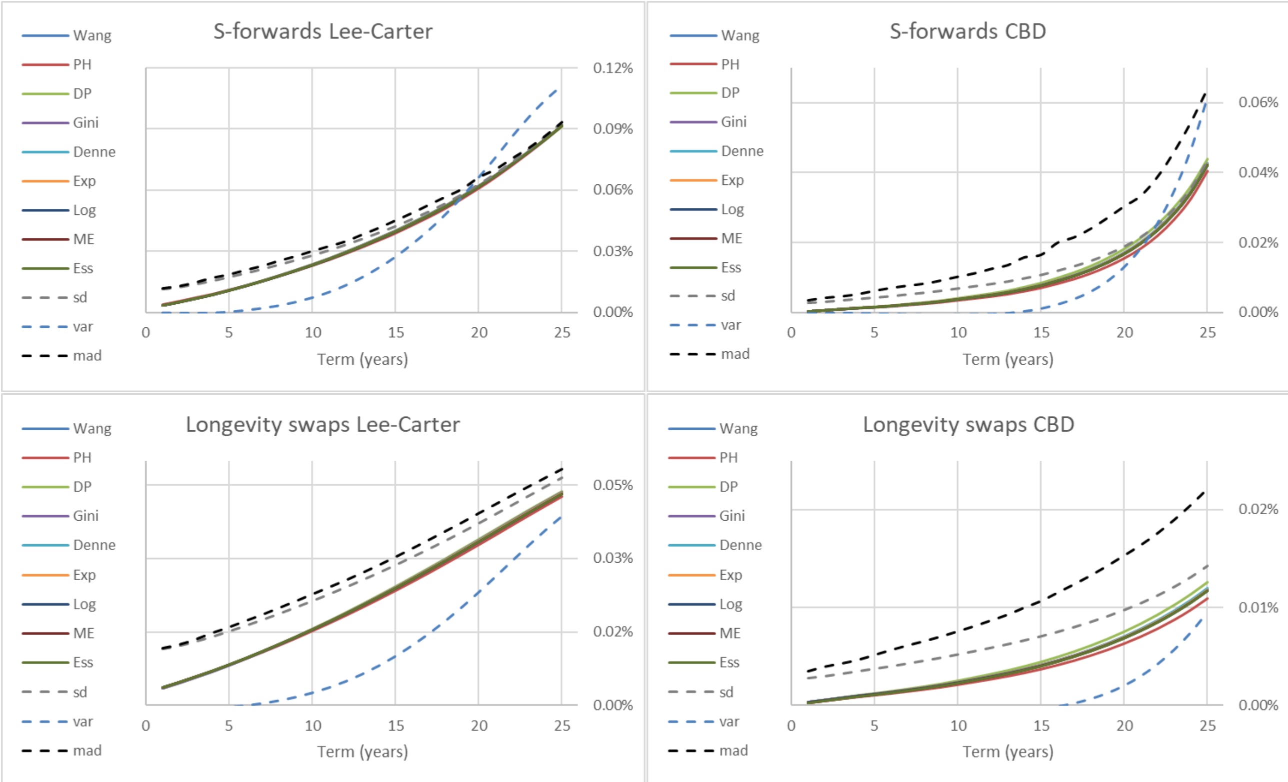


Fig. 4 Annual risk premiums of S-forwards (top panel) and longevity swaps (bottom panel) calculated by different principles under the Lee-Carter model (left column) and CBD model (right column)[14]

Table 3 Annual risk premiums of S-forwards calculated by different principles under the Lee-Carter and CBD model

Model	Lee-Carter			CBD		
Term	15	20	25	15	20	25
Wang	0.04%	0.06%	0.09%	0.01%	0.02%	0.04%
PH	0.04%	0.06%	0.09%	0.01%	0.02%	0.04%
DP	0.04%	0.06%	0.09%	0.01%	0.02%	0.04%
Gini	0.04%	0.06%	0.09%	0.01%	0.02%	0.04%
Denne	0.04%	0.06%	0.09%	0.01%	0.02%	0.04%
Exp	0.04%	0.06%	0.09%	0.01%	0.02%	0.04%
Log	0.04%	0.06%	0.09%	0.01%	0.02%	0.04%
ME	0.04%	0.06%	0.09%	0.01%	0.02%	0.04%
Ess	0.04%	0.06%	0.09%	0.01%	0.02%	0.04%

[14] Note that the variance principle (dashed blue curve) assigns a premium close to zero (even slightly negative) to short-term contracts, especially for the CBD model.

sd	0.04%	0.06%	0.09%	0.01%	0.02%	0.04%
var	0.03%	0.07%	0.11%	0.00%	0.01%	0.06%
mad	0.05%	0.07%	0.09%	0.02%	0.03%	0.06%

Table 4 Annual risk premiums of longevity swaps calculated by different principles under the Lee-Carter and CBD model

Model	Lee-Carter			CBD		
Term	15	20	25	15	20	25
Wang	0.02%	0.03%	0.04%	0.00%	0.01%	0.01%
PH	0.02%	0.03%	0.04%	0.00%	0.01%	0.01%
DP	0.02%	0.03%	0.04%	0.00%	0.01%	0.01%
Gini	0.02%	0.03%	0.04%	0.00%	0.01%	0.01%
Denne	0.02%	0.03%	0.04%	0.00%	0.01%	0.01%
Exp	0.02%	0.03%	0.04%	0.00%	0.01%	0.01%
Log	0.02%	0.03%	0.04%	0.00%	0.01%	0.01%
ME	0.02%	0.03%	0.04%	0.00%	0.01%	0.01%
Ess	0.02%	0.03%	0.04%	0.00%	0.01%	0.01%
sd	0.03%	0.04%	0.05%	0.01%	0.01%	0.01%
var	0.01%	0.02%	0.04%	0.00%	0.00%	0.01%
mad	0.03%	0.04%	0.05%	0.01%	0.02%	0.02%

Several interesting observations can be made from Fig. 4 and Tables 3 and 4. Firstly, S-forwards have higher implied risk premiums than longevity swaps. The former exchanges a single cash flow on the maturity date, while the latter swaps a set of cash flows linked to the difference between the forward and realized survival rate on each exchange date. Given the same underlying cash flows, the implied risk premiums of longevity swaps are weighted by those of S-forwards whose term is shorter than or equal to that of the corresponding longevity swap. For instance, the premium of a 10-year longevity swap is affected by that of 10 S-forward contracts (with identical characteristics) with terms ranging from 1 to 10 years. Since longevity risk premiums tend to increase by time, one would expect S-forwards to produce higher implied premiums than longevity swaps with the same maturity. Our results suggest that for 25-year contracts, the difference in the implied risk premiums between longevity swaps and S-forwards is around 0.04% to 0.05% under the Lee-Carter model and 0.03% to 0.04% under the CBD model.

Secondly, the distinctions in annual risk premiums are minimal between the nine risk-neutral candidates. The premium curves are almost identical under the Lee-Carter model, and under the CBD model the differences are slightly more obvious. Despite the small differences (less than 0.01%), the dual-power principle tends to produce the highest risk premiums and the proportional hazard principle the lowest for both S-forwards and longevity swaps. Besides, the canonical valuation and Esscher transform result in the same parameter value and risk premiums.

The risk premiums of the two securities obtained from the standard deviation principle and median absolute deviation principle tend to be higher than those from risk-neutral measures. Such divergence is more apparent for longer-term longevity swaps which involve the accumulation effect of a series of cash flows exchanged before maturity. Also, the median absolute deviation principle results in similar risk premiums to those from the standard deviation principle under the Lee-Carter model, while the two methods deviate more under the CBD model.

Another real-world measure, the variance principle, gives a more convex curve, which behaves differently compared to the other eleven methods. For S-forwards, the curve produced from the variance principle stays below the others for shorter maturities but goes above them after the crossover at the 20- and 22-year maturity under the Lee-Carter and CBD model, respectively. However, its curve for longevity swaps is always below those from the other eleven principles. The reason may be that the higher premiums under the variance principle at long maturities are not enough to compensate for the effect of the lower premiums at short maturities. On the one hand, the discounting effect has a greater impact on longer-term cash flows, reducing the weights of those higher premiums. On the other hand, the crossover point (around year 20) may be too late to balance the influence of risk premiums below and above those calculated by the other principles. Lastly, it can be seen that the CBD model gives more convex risk premium curves than the Lee-Carter model does, which agrees with the observation on longevity bond risk premiums by Li (2010).

Note that the above results of the three physical measures are derived by discounting future cash flows at a discount rate equal to the risk-free rate. In practice, the real-world discount rate is usually equal to the risk-free rate plus a loading required by insurers. It is hard to determine a particular value for the real-world discount rate in our analysis without such private information. Therefore, we add a range of arbitrary loadings and examine the impact. Table 5 presents the incremental annual risk premiums of 25-year contracts under the two models for each additional 0.1% (10 basis points) loading, and the results for different maturities under the CBD model are plotted in Fig. 5. When a 10-basis point loading is included in the real-world discount rate, the annual risk premium of 25-year S-forwards calculated by the standard deviation principle and median absolute deviation principle increases by about 15 to 16 (17 to 18) basis points under the Lee-Carter (CBD) model. A greater impact is observed under the variance principle, with an incremental premium of around 19 to 20 (23 to 25) basis points under the Lee-Carter (CBD) model. For longevity swaps, an additional 10-basis point increase in loadings results in a rise of 7 to 9 basis points in the annual risk premiums under the Lee-Carter model, and the figures are about 2 basis points lower under the CBD model. Unlike the case for S-forwards, the risk premiums of longevity swaps derived from the variance principle are less affected by additional loadings. This observation is further underlined by Fig. 5 which displays the risk premiums over different terms under the CBD model.

Table 5 Incremental annual risk premiums of 25-year contracts from additional loadings in the real-world discount rate

Derivatives	Mortality models	Lee-Carter			CBD		
	Premium principles	sd	var	mad	sd	var	mad
S-forwards	baseline risk premiums[15]	0.09%	0.11%	0.09%	0.04%	0.06%	0.06%
	$r + 0.1\%$	0.16%	0.19%	0.16%	0.18%	0.23%	0.18%
	$r + 0.2\%$	0.16%	0.20%	0.16%	0.17%	0.24%	0.18%
	$r + 0.3\%$	0.15%	0.20%	0.16%	0.17%	0.24%	0.17%
	$r + 0.4\%$	0.15%	0.20%	0.15%	0.17%	0.25%	0.17%
	$r + 0.5\%$	0.15%	0.20%	0.15%	0.17%	0.25%	0.17%
Longevity swaps	baseline risk premiums	0.05%	0.04%	0.05%	0.01%	0.01%	0.02%
	$r + 0.1\%$	0.08%	0.07%	0.08%	0.06%	0.05%	0.06%
	$r + 0.2\%$	0.09%	0.08%	0.08%	0.06%	0.05%	0.06%
	$r + 0.3\%$	0.09%	0.08%	0.09%	0.06%	0.06%	0.07%
	$r + 0.4\%$	0.09%	0.08%	0.09%	0.07%	0.06%	0.07%

[15] The baseline risk premiums refer to those calculated using the risk-free discounts rate.

$r + 0.5\%$	0.09%	0.09%	0.09%	0.07%	0.06%	0.07%

Under the variance principle, short-term (shorter than 10 years) S-forwards are assigned with risk premiums close to zero. Although the risk premiums are higher than those calculated by the other two principles for longer-term S-forwards, the aggregate effect over time causes the consistently lower (incremental) risk premiums of longevity swaps. To make a straightforward comparison between mortality models and valuation methods, we plot the results calculated by all the three principles under the two mortality models in Fig. 6, given a 10-basis point increase. The graphs show that the premium curve of the variance principle crosses over the other two for S-forwards, while the curve remains the lowest for longevity swaps. As noted, the risk premiums obtained by the standard deviation principle and median absolute deviation principle are close to each other under the Lee-Carter model, while the latter produces slightly higher values under the CBD model.

Fig. 5 Annual risk premiums of S-forwards (top panel) and longevity swaps (bottom panel) calculated by real-world measures under the CBD model, assuming different loadings in the discount rate

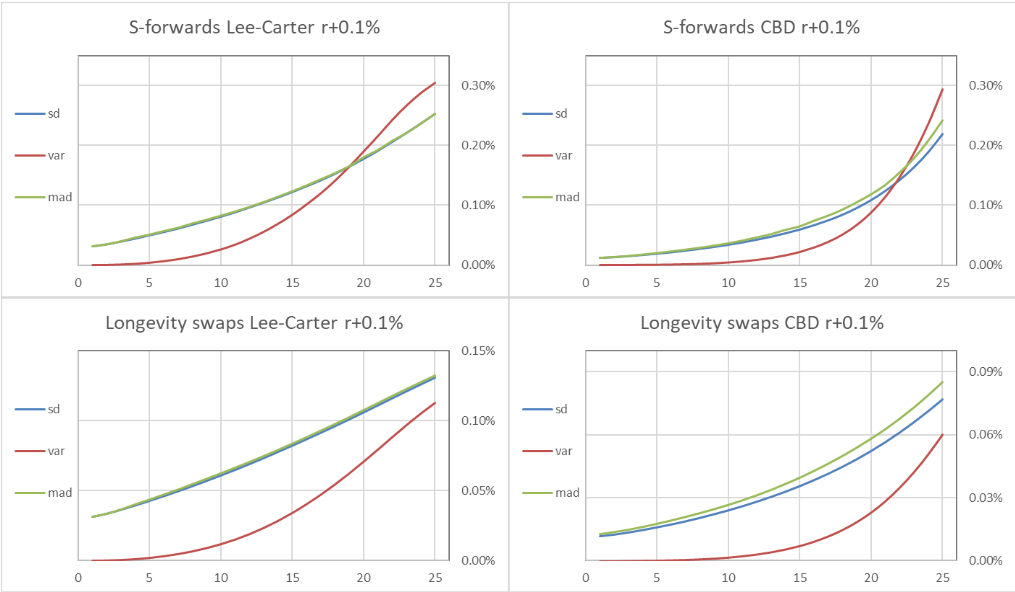


Fig. 6 Annual risk premiums of S-forwards (top panel) and longevity swaps (bottom panel) calculated by real-world measures under the Lee-Carter model (left column) and the CBD model (right column), with 0.1% loading in the discount rate

Overall, the results suggest that the choice of mortality models matters more than that of premium principles. For example, the Lee-Carter model gives 4 to 5 (3 to 4) basis points higher in annual risk premiums for 25-year S-forwards (longevity swaps) compared to those from the CBD model. By contrast, the variations in the results between different premium principles seem to be less material. Between the risk-neutral and real-world families, the former often produces lower risk premiums than the latter, by less than 2 basis points. Within the risk-neutral family, we observe rather similar results between the nine candidates; the deviations are relatively greater under the CBD model but are still negligible (less than 1 basis point). Among the three real-world measures, the variance principle displays more convex premium curves, while the other two perform comparably. As aforementioned, the CBD model leads to clearer distinctions between the standard deviation principle and median absolute deviation principle than the Lee-Carter model does. Based on all our numerical results, it appears that in general the pricing rule uncertainty is less significant than the mortality model uncertainty. Moreover, the specific impact of choosing a particular premium principle depends on the underlying assumption of the mortality model.

In the next section, we conduct a sensitivity test to further investigate the impact of mortality model uncertainty. A robustness test allowing for structural changes in mortality improvement is also provided and discussed.

5. Sensitivity tests

5.1 Allowance for model uncertainty

We have demonstrated that the selection of mortality model plays a vital role in pricing longevity-linked securities. Nonetheless, no consensus has been reached on which mortality model serves best in life market pricing. To better account for model uncertainty in our pricing analysis, we employ the modified semi-parametric bootstrap method proposed by Yang et al. (2015). The idea is to select the "best" mortality model within the bootstrapping process, given predetermined list of model candidates and selection criteria. One may set selection criteria based on particular needs. For instance, the Bayesian Information Criterion (BIC) or Akaike Information Criterion (AIC) may be applied when a balance between goodness-of-fit and parameter parsimony is required. Our objective is to examine the effect of mortality model uncertainty on pricing which involves predicting future mortality distributions, so the forecasting accuracy is one major concern. Therefore, in each iteration step, we select the mortality model which produces the lowest mean absolute percentage error (MAPE) value in a backtest for specified fitting and forecasting periods. For convenience, we split the total data period into halves, while one may use other splits where appropriate. The implementation is briefly described as follows. After performing a backtest using the pseudo sample generated from Poisson distributions, the model with the lowest MAPE value is selected for that particular sample. Then we estimate the parameters of the chosen model using the entire pseudo sample and simulate future values of the time-varying components and mortality rates. The selection-fitting-simulation process is repeated 10,000 times to get different paths of mortality scenarios. This modified bootstrapping method allows us to integrate process error, parameter error and model error into a comprehensive simulation framework.

Among 10,000 pseudo samples, 4,192 of them called for the Lee-Carter model, and the other 5,808 samples pointed to the CBD model. Fig. 7 illustrates the annual risk premiums of S-forwards and longevity swaps obtained under different models. An enlarged version for the modified bootstrap is given in Fig. 8. In general, the figures from the modified semi-parametric bootstrap are somewhere between those from the Lee-Carter model and CBD model on their own. For example, the premiums of a 25-year S-forward contract after incorporating the model risk are between 6 and 9 basis points, compared with the values of 4 to 6 basis points under the CBD model and 9 to 10 basis points under the Lee-Carter model. This observation is in line with our expectation because the results from the modified bootstrap may be regarded as some form of a "weighted" average between those from the two models, with weights depending on the forecast accuracy. Besides, the nine candidates of the risk-neutral family present more variations under the modified bootstrap approach. It can be seen from Fig. 8 that the risk premiums from the Denneberg's absolute deviation principle and dual-power principle tend to rank highest, followed closely by the Gini principle, exponential principle, logarithm principle, Wang transform, and Esscher transform (canonical valuation). The proportional hazard transform still tends to produce the lowest premiums. Again, the impact of mortality model uncertainty is greater than that of pricing rule uncertainty.

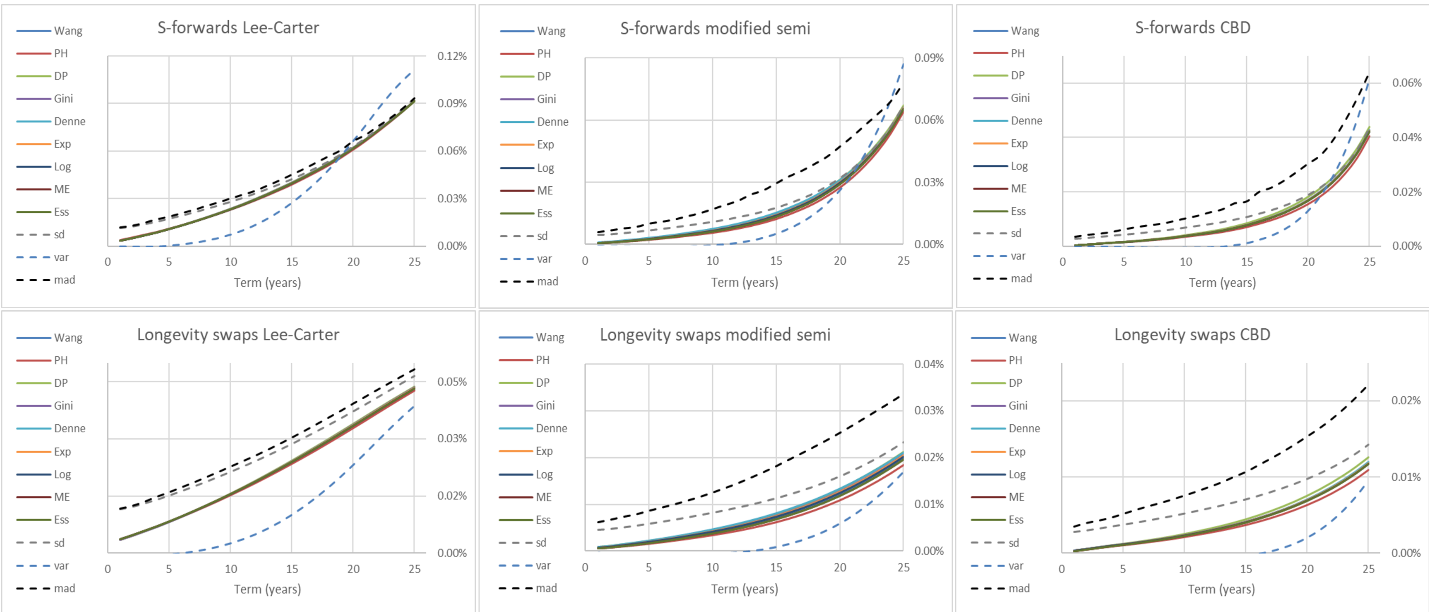
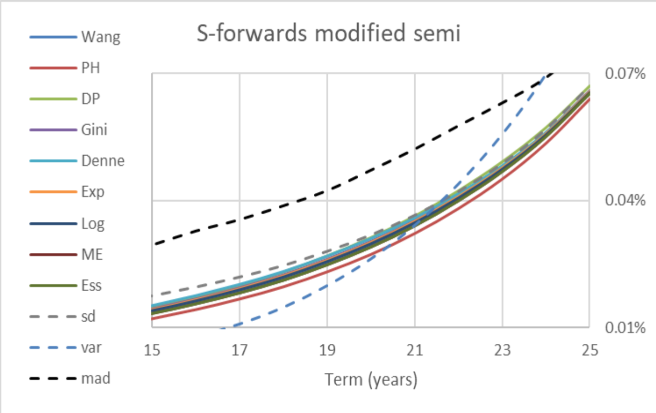


Fig. 7 Annual risk premiums of S-forwards (top panel) and longevity swaps (bottom panel) calculated by different principles using the Lee-Carter model (left column), modified bootstrap (middle column), and CBD model (right column)



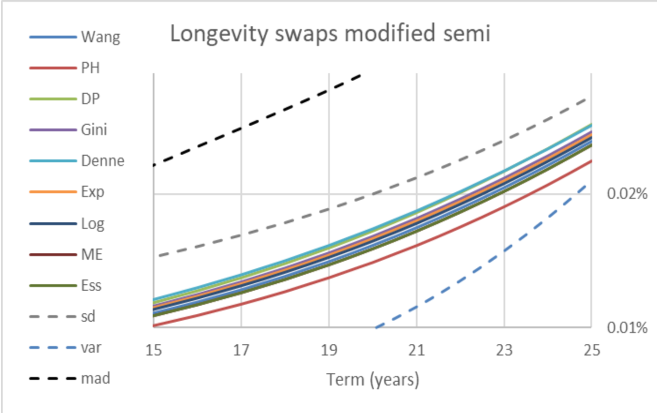


Fig. 8 Annual risk premiums of S-forwards and longevity swaps calculated by different principles using the modified bootstrap, with a term of 15, 16, ... , 25 years

5.2 Incorporation of structural changes

Barrieu and Veraart (2016) stressed the impact of longevity trend risk in pricing longevity-linked securities. They suggested that there could be permanent changes in the trend of longevity improvements, which is often referred to as structural changes. However, the mortality models considered in our analysis are rather data-driven, and they extrapolate future mortality rates by assuming a continuity of past trends. Based on data from only the latest few decades, it is difficult to allow for rare structural changes in mortality forecasting. We follow Li et al. (2019) and adjust the drift terms of random walk processes in the two mortality models to incorporate arbitrary changes in longevity trends. They set three potential regimes for future longevity levels - low, moderate, and high mortality improvements by allowing time-varying drift terms. For the Lee-Carter model, we modify the drift term to $\Theta_t = (0.5\theta, \theta, 1.5\theta)'$, where θ is the original drift estimated from past data. For the CBD model, the drift vector of the multivariate random walk is adjusted to $\Theta_t = (0.5\Theta, \Theta, 1.5\Theta)'$, where $\Theta = (\theta_1, \theta_2, \theta_3)$ contains three estimated drifts of the three time-specific components. The drift term is increased (decreased) by 50% to reflect high (low) mortality improvements. The transition matrix of structural changes is assumed to be $\begin{pmatrix} 0.99 & 0.01 & 0 \\ 0.01 & 0.98 & 0.01 \\ 0 & 0.01 & 0.99 \end{pmatrix}$. It is difficult to determine the frequency of transitions between states without studying the underlying causes of structural changes such as urbanisation, medical breakthroughs or a change in life style. An arbitrary value 0.01 is assigned to the probability of switching to the adjacent state, considering the once-in-a-century occurrence of permanent longevity shocks in historical data.

Fig. 9 shows the risk premiums of longevity swaps under the two models after allowing for structural changes. Compared with the previous pattern without structural changes (Fig. 4), both the magnitude and the shape of the curves are quite similar. With this allowance for structural changes, the simulated paths of future longevity would trend differently, which then affect the calibration of premium principles using the market price. Table 6 gives the estimated parameters of the twelve pricing methods with and without structural changes under the Lee-Carter model. It can be seen that the loading factors of the three real-world principles under the Lee-Carter model decrease after allowing for varying drift terms. The impact of more volatile mortality scenarios turns out to be compensated by the decrease in magnitude of the loading factor, which then results in a similar level of risk premiums. For the risk-neutral methods, the connection between the risk parameter values and the resulting risk premiums is not as straightforward as that for the real-world methods. Instead of altering the explicit risk loading, the influence of more fluctuating longevity trends lies on the distorted distribution. Fig. 10 plots the risk-neutral probabilities from the Wang transform under the Lee-Carter model. We can observe that the allocated weights to the 10,000 simulated scenarios (left graph) have very different patterns under the two assumptions. This is a direct consequence of having to match the same market price. The result is that the two corresponding distorted distributions are comparable with each other (right graph) and they give similar risk premiums. In practice, nevertheless, one may expect higher risk premiums of longevity-linked securities if the market has incorporated such structural changes into their consideration.

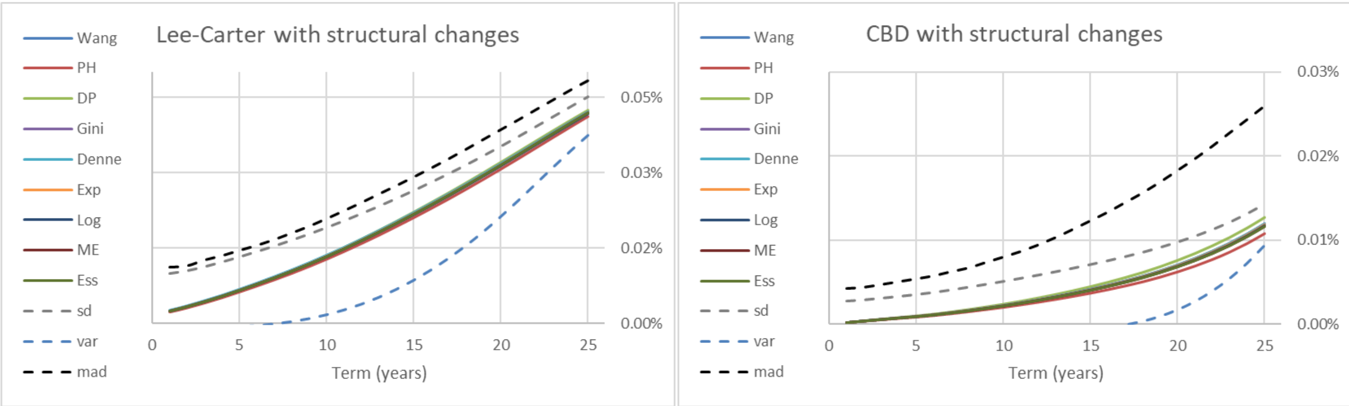


Fig. 9 Annual risk premiums of longevity swaps calculated by different principles after incorporating possible structural changes

Table 6 Parameter estimates under the Lee-Carter model (with and without structural changes)

Model		Wang	PH	DP	Gini	Denne	Exp	Log	ME	Ess	sd	var	mad
Lee-Carter	original	0.37	1.48	1.55	0.66	0.47	1.36	2.80	1.16	1.16	0.37	1.14	0.54
	structural changes	0.34	1.42	1.48	0.59	0.42	1.21	2.31	0.95	0.95	0.33	0.94	0.51

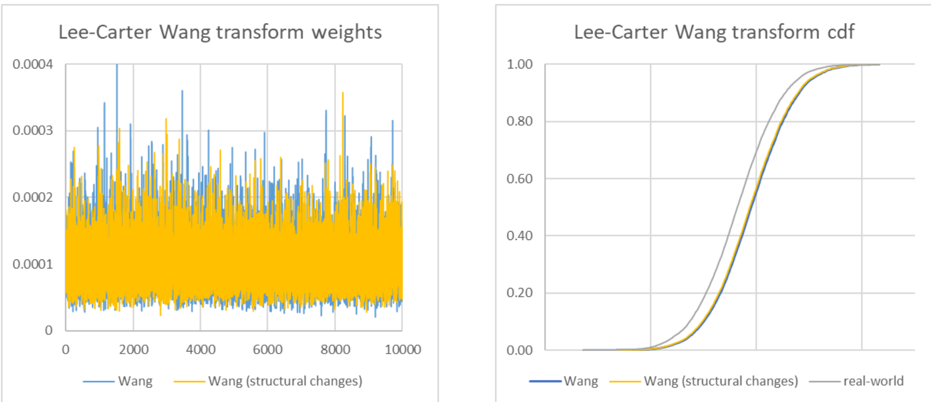


Fig. 10 Risk neutral probabilities from the Wang transform under the Lee-Carter model, with (yellow) and without (blue) structural changes[16]

[16] Note that the empirical cumulative distribution is based on scenarios sorted by simulated annuity prices. The real-world probability of each scenario is equal to $\frac{1}{10000}$ (given 10,000 simulations).

6. Concluding remarks

In this thesis, we have investigated the impact of mortality model uncertainty and pricing rule uncertainty on pricing S-forwards and longevity swaps by comparing risk premiums calculated by twelve premium principles under two mortality models. The mortality models are fitted to UK mortality data and the valuation methods are calibrated using the market quotation of pension annuities. Our empirical results indicate that the uncertainty arising from the choice of mortality models dominates that of premium principles, and the relationships in the results between different pricing methods tend to rely on the underlying mortality model assumption. Considering the Lee-Carter model with a cohort parameter and the generalised CBD model, the risk premiums obtained under the former are generally higher than the latter. Regarding premium principles, those based on real-world measures produce greater premiums than the risk-neutral methods, assuming the same discount rate. When loadings are added to the real-world discount rate, figures from the three real-world pricing methods become even higher. Among this class, the standard deviation principle gives lower risk premiums than the median absolute deviation principle under the CBD model, while their pricing results are quite comparable under the Lee-Carter model. Moreover, the variance principle results in the most convex premium curves among all twelve candidates. Within the risk-neutral family, variations between the nine members are more detectable under the CBD model, although the magnitude is rather minimal.

Given the significant influence of the choice of mortality model on the prices of longevity-linked securities, it is imperative to highlight the effect of incorporating model uncertainty into the pricing process. Our sensitivity test of model uncertainty demonstrates that the risk premiums calculated from the modified semi-parametric bootstrap may be viewed as a “weighted” average of those computed from individual models, where the "weights" can be determined by the forecast accuracy on simulated pseudo samples.

In practice, insurers may have particular preferences on certain mortality models, for instance, the user is more familiar with a specific model. A more comprehensive approach is to blend potential candidates together and obtain more balanced results by means of the modified bootstrap. Regarding pricing principles, both risk-neutral and real-world methods have their difficulties to implement in reality such as theoretical appropriateness and justifiable discount rates. Despite that, based on our numerical study, applying methods with real-world measures could result in higher premiums. Within each of the two categories, the results are more similar, and so insurers may select a principle based on its theoretical properties. For example, the seven candidates with distortion risk measures satisfy all the properties considered except additivity, which makes them qualitatively attractive. Nevertheless, when additivity is favored, one may adopt the variance principle.

There are some future directions for research. When the life market transits to a more mature stage, more longevity products will be available. One can then employ those security prices with more matching features (e.g., matched term to maturity) to set the calibration constraints rather than using annuity quotations as in this study. However, one can imagine that even the securities whose payoffs are linked to the same population may not carry the same market view of longevity risk premium at times. In such situation, practitioners need to select the most suitable market price or they may apply those pricing principles with multiple parameters to incorporate all market information. We have only considered pricing principles in the univariate case for comparison purposes, while some pricing principles can be extended to multivariate versions. It would be interesting to examine the pricing rule uncertainty by integrating more market information. Moreover, given the long-term nature of longevity risk, it may not be realistic to assume a constant discount rate for all cash flows. One may incorporate the interest rate uncertainty and investigate the combined effect on the pricing of longevity-linked securities. Lastly, the primary concern of pension plan sponsors and annuity providers is to manage longevity risk. It would be useful to integrate pricing and hedging into a comprehensive system. One could apply calibrated pricing methods to mortality scenarios of the underlying reference population of longevity products, then embed the cost of hedging (risk premiums) in assessing hedging effectiveness.

References

Aquilina, M., Baker, R., & Majer, T. (2017). The Money's Worth of annuities in the UK between 2006 and 2014. *The Journal of the Economics of Ageing, 9*, 208-218. doi:https://doi.org/10.1016/j.jeoa.2016.10.002

Barrieu, P. M., & Veraart, L. A. M. (2016). Pricing q- forward contracts: an evaluation of estimation window and pricing method under different mortality models. *Scandinavian Actuarial Journal, 2016*(2), 146-166. doi:10.1080/03461238.2014.916228

Bauer, D., Börger, M., & Ruß, J. (2010). On the pricing of longevity-linked securities. *Insurance Mathematics and Economics, 46*(1), 139-149. doi:10.1016/j.insmatheco.2009.06.005

Bayraktar, E., Milevsky, M. A., David, P. S., & Young, V. R. (2009). Valuation of mortality risk via the instantaneous Sharpe ratio: Applications to life annuities. *Journal of Economic Dynamics and Control, 33*(3), 676-691. doi:https://doi.org/10.1016/j.jedc.2008.09.004

Blake, D. (2001). Survivor bonds helping to hedge mortality risk. *The journal of risk and insurance : the journal of the American Risk and Insurance Association, 68*(2), 339-348. doi:10.2307/2678106

Blake, D. (2018). Longevity: a new asset class. *Journal of Asset Management, 19*(5), 278-300. doi:10.1057/s41260-018-0084-9

Blake, D., Cairns, A. J. G., & Dowd, K. (2006). Living with Mortality: Longevity Bonds and Other Mortality-Linked Securities. *British Actuarial Journal, 12*(1), 153-197. doi:10.1017/S1357321700004736

Blake, D., Cairns, A. J. G., Dowd, K., & Kessler, A. R. (2018). Still living with mortality: the longevity risk transfer market after one decade. *British Actuarial Journal, 24*(1), 1-80. doi:10.1017/S1357321718000314

Brouhns, N., Denuit, M., & Van Keilegom, I. (2005). Bootstrapping the Poisson log-bilinear model for mortality forecasting. *Scandinavian Actuarial Journal, 2005*(3), 212-224. doi:10.1080/03461230510009754

Brouhns, N., Denuit, M., & Vermunt, J. K. (2002). A Poisson log-bilinear regression approach to the construction of projected lifetables. *Insurance: Mathematics and Economics, 31*(3), 373-393. Retrieved from https://EconPapers.repec.org/RePEc:eee:insuma:v:31:y:2002:i:3:p:373-393

Buhlmann, H. (1970). *Mathematical models in risk theory*: Springer-Verlag.

Bühlmann, H. (1980). *An Economic Premium Principle* (Vol. 11).

Cairns, A. J. G., Blake, D., & Dowd, K. (2006). A Two-Factor Model for Stochastic Mortality with Parameter Uncertainty: Theory and Calibration. *Journal of Risk and Insurance, 73*(4), 687-718. doi:10.1111/j.1539-6975.2006.00195.x

Cairns, A. J. G., Blake, D., & Dowd, K. (2008). Modelling and management of mortality risk: a review. *Scandinavian Actuarial Journal, 2008*(2-3), 79-113. doi:10.1080/03461230802173608

Cairns, A. J. G., Blake, D., Dowd, K., Coughlan, G. D., Epstein, D., & Khalaf-Allah, M. (2011). Mortality density forecasts: An analysis of six stochastic mortality models. *Insurance: Mathematics and Economics, 48*(3), 355-367. doi:https://doi.org/10.1016/j.insmatheco.2010.12.005

Cairns, A. J. G., Blake, D., Dowd, K., Coughlan, G. D., Epstein, D., Ong, A., & Balevich, I. (2009). A Quantitative Comparison of Stochastic Mortality Models Using Data From England and Wales and the United States. *North American Actuarial Journal, 13*(1), 1-35. doi:10.1080/10920277.2009.10597538

Chen, B., Zhang, L., & Zhao, L. (2010). On the robustness of longevity risk pricing. *Insurance Mathematics and Economics, 47*(3), 358-373. doi:10.1016/j.insmatheco.2010.08.002

Coale, A. J., & Kisker, E. (1990). *Defects in data on old-age mortality in the United States: New procedures for calculating mortality schedules and life tables at the highest ages* (Vol. 4).

Coughlan, G., Epstein, D., Sinha, A., & Honig, P. (2007). *q-Forwards: Derivatives for Transferring Longevity and Mortality Risks*. Retrieved from www.lifemetrics.com

Council Directive 2004/113/EC. (2004). *Implementing the principle of equal treatment between women and men in the access to and supply of goods and services*. OJ 21.12.04

Cowley, A., & Cummins, J. (2005). Securitization of Life Insurance Assets and Liabilities. *Journal of Risk & Insurance, 72*(2), 193-226. Retrieved from https://EconPapers.repec.org/RePEc:bla:jrinsu:v:72:y:2005:i:2:p:193-226

Cox, S., Lin, Y., & Wang, S. (2006). Multivariate Exponential Tilting and Pricing Implications for Mortality Securitization. *Journal of Risk & Insurance, 73*, 719-736. doi:10.1111/j.1539-6975.2006.00196.x

Denneberg, D. (1988). Keynes' decision rule, on "expectation" in the general theory. *European Journal of Political Economy, 4*(2), 205-223. doi:https://doi.org/10.1016/0176-2680(88)90001-8

Denneberg, D. (1990). Premium Calculation: Why Standard Deviation Should be Replaced by Absolute Deviation. *ASTIN Bulletin, 20*(2), 181-190. doi:10.2143/AST.20.2.2005441

Denuit, M., Devolder, P., & Goderniaux, A.-C. (2007). Securitization of Longevity Risk: Pricing Survivor Bonds With Wang Transform in the Lee-Carter Framework. *Journal of Risk and Insurance, 74*(1), 87-113. doi:doi:10.1111/j.1539-6975.2007.00203.x

Dickson, D. C. M. (2005). *Insurance Risk and Ruin*. Cambridge: Cambridge University Press.

Dowd, K. (2003). Survivor Bonds: A Comment on Blake and Burrows. *Journal of Risk and Insurance, 70*(2), 339-348. doi:10.1111/1539-6975.00063

Dowd, K., Blake, D., Cairns, A. J. G., & Dawson, P. (2006). Survivor Swaps. *Journal of Risk and Insurance, 73*(1), 1-17. doi:10.1111/j.1539-6975.2006.00163.x

Efron, B. (1979). Bootstrap Methods: Another Look at the Jackknife. *The Annals of Statistics, 7*(1), 1-26. Retrieved from http://www.jstor.org/stable/2958830

Esscher, F. (1932). On the probability function in the collective theory of risk. *Scandinavian Actuarial Journal, 1932*(3), 175-195. doi:10.1080/03461238.1932.10405883

Frittelli, M. (2000). The Minimal Entropy Martingale Measure and the Valuation Problem in Incomplete Markets. *Mathematical Finance, 10*(1), 39-52. doi:doi:10.1111/1467-9965.00079

Gallagher, G. (2003). *Pricing Annuities*. Paper presented at the Pensions Convention 2003, Brighton, England. https://www.actuaries.org.uk/system/files/documents/pdf/pricingannuitieshandout.pdf

Gampe, J. (2010). Human mortality beyond age 110. In H. Maier, J. Gampe, B. Jeune, J.-M. Robine, & J. W. Vaupel (Eds.), *Supercentenarians* (pp. 219-230). Berlin, Heidelberg: Springer Berlin Heidelberg.

Gerber, H. U., & Shiu, E. S. W. (1994). Option pricing by Esscher transforms. *Transactions of the Society of Actuaries, 46*, 99-191.

Human Mortality Database. (2018). University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org.

Kogure, A., & Kurachi, Y. (2010). A Bayesian approach to pricing longevity risk based on risk-neutral predictive distributions. *Insurance: Mathematics and Economics, 46*(1), 162-172. doi:https://doi.org/10.1016/j.insmatheco.2009.10.005

Kogure, A., Li, J., & Kamiya, S. (2014). A Bayesian Multivariate Risk-Neutral Method for Pricing Reverse Mortgages. *North American Actuarial Journal, 18*(1), 242-257. doi:10.1080/10920277.2013.872983

Kullback, S., & Leibler, R. A. (1951). On Information and Sufficiency. *Ann. Math. Statist., 22*(1), 79-86. doi:10.1214/aoms/1177729694

Laeven, R. J. A., & Goovaerts, M. J. (2008). Premium Calculation and Insurance Pricing. In.

Lee, R. D., & Carter, L. R. (1992). Modeling and Forecasting U.S. Mortality. *Journal of the American Statistical Association, 87*(419), 659-671. doi:10.1080/01621459.1992.10475265

Leung, M., Fung, M. C., & O'hare, C. (2018). A comparative study of pricing approaches for longevity instruments. *Insurance Mathematics and Economics, 82*, 95-116. doi:10.1016/j.insmatheco.2018.06.010

Leys, C., Ley, C., Klein, O., Bernard, P., & Licata, L. (2013). Detecting outliers: Do not use standard deviation around the mean, use absolute deviation around the median. *Journal of Experimental Social Psychology, 49*(4), 764-766. doi:https://doi.org/10.1016/j.jesp.2013.03.013

Li, J. (2013). A Poisson common factor model for projecting mortality and life expectancy jointly for females and males. *Population Studies, 67*(1), 111-126. doi:10.1080/00324728.2012.689316

Li, J., Li, J. S.-H., Tan, C. I., & Tickle, L. (2019). Assessing basis risk in index-based longevity swap transactions. *Annals of Actuarial Science, 13*(1), 166-197. doi:10.1017/S1748499518000179

Li, J., & Tan, C. I. (2018). Assessing basis risk for longevity transactions – phase 2 presented by Dr Jackie Li and Dr Chong It Tan, IFoA Longevity Basis Risk

Working Group - Abstract of the London Discussion. *British Actuarial Journal, 23*, e25. doi:10.1017/S1357321718000235

Li, J. S.-H. (2010). Pricing longevity risk with the parametric bootstrap: A maximum entropy approach. *Insurance Mathematics and Economics, 47*(2), 176-186. doi:10.1016/j.insmatheco.2010.05.004

Li, J. S.-H., Chan, W.-S., & Cheung, S.-H. (2011). Structural Changes in the Lee-Carter Mortality Indexes. *North American Actuarial Journal, 15*(1), 13-31. doi:10.1080/10920277.2011.10597607

Li, J. S.-H., & Ng, A. C. Y. (2011). Canonical Valuation of Mortality-Linked Securities. *Journal of Risk and Insurance, 78*(4), 853-884. doi:10.1111/j.1539-6975.2010.01394.x

Life and Longevity Markets Association. (2010). *Technical Note: The S-forward*. Retrieved from www.llma.org

Lin, Y., & Cox, S. H. (2005). Securitization of Mortality Risks in Life Annuities. *The Journal of Risk and Insurance, 72*(2), 227-252. Retrieved from http://www.jstor.org/stable/3519949

Lynn Wirch, J., & Hardy, M. R. (1999). A synthesis of risk measures for capital adequacy. *Insurance: Mathematics and Economics, 25*(3), 337-347. doi:https://doi.org/10.1016/S0167-6687(99)00036-0

Maria de, L. C., & João Andrade, e. S. (2005). Applying the Proportional Hazard Premium Calculation Principle. *ASTIN Bulletin, 35*(2), 409-425. doi:10.1017/S0515036100014318

Necir, A., & Meraghni, D. (2009). Empirical estimation of the proportional hazard premium for heavy-tailed claim amounts. *Insurance: Mathematics and Economics, 45*(1), 49-58. doi:https://doi.org/10.1016/j.insmatheco.2009.03.004

Pelsser, A. (2008). On the Applicability of the Wang Transform for Pricing Financial Risks. *ASTIN Bulletin, 38*(1), 171-181. doi:10.1017/S0515036100015129

Plat, R. (2009). On stochastic mortality modeling. *Insurance: Mathematics and Economics, 45*, 393-404. doi:10.2139/ssrn.1362487

Sharpe, W. F. (1966). Mutual Fund Performance. *The Journal of Business, 39*(1), 119-138. Retrieved from http://www.jstor.org/stable/2351741

Sherris, M., Xu, Y., & Ziveyi, J. (2019). Market Price of Longevity Risk for a Multi-Cohort Mortality Model with Application to Longevity Bond Option Pricing. *Journal of Risk & Insurance*. doi:10.2139/ssrn.3121520

Stutzer, M. (1996). A Simple Nonparametric Approach to Derivative Security Valuation. *Journal of Finance, 51*(5), 1633-1652. doi:10.1111/j.1540-6261.1996.tb05220.x

Tan, C. I., Li, J., Li, J. S.-H., & Balasooriya, U. (2014). Parametric mortality indexes: From index construction to hedging strategies. *Insurance Mathematics and Economics, 59*(C), 285-299. doi:10.1016/j.insmatheco.2014.10.005

Thatcher, A. R. (1999). The long-term pattern of adult mortality and the highest attained age. *Journal of the Royal Statistical Society: Series A (Statistics in Society), 162*(1), 5-43. doi:10.1111/1467-985x.00119

Tuljapurkar, S., Li, N., & Boe, C. (2000). A universal pattern of mortality decline in the G7 countries. *Nature, 405*(6788), 789-792. doi:10.1038/35015561

Vandewalle, B., & Beirlant, J. (2006). On univariate extreme value statistics and the estimation of reinsurance premiums. *Insurance: Mathematics and Economics, 38*(3), 441-459. doi:https://doi.org/10.1016/j.insmatheco.2005.11.002

Von Neumann, J., & Morgenstern, O. (1944). *Theory of games and economic behavior*. Princeton, NJ, US: Princeton University Press.

Wang, S. (1995). Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance: Mathematics and Economics, 17*(1), 43-54. doi:https://doi.org/10.1016/0167-6687(95)00010-P

Wang, S. (1996). Premium Calculation by Transforming the Layer Premium Density. *ASTIN Bulletin, 26*(1), 71-92. doi:10.2143/AST.26.1.563234

Wang, S. (2000). A Class of Distortion Operators for Pricing Financial and Insurance Risks. *The Journal of Risk and Insurance, 67*(1), 15-36. doi:10.2307/253675

Wang, S. (2002). A Universal Framework for Pricing Financial and Insurance Risks. *ASTIN Bulletin, 32*(2), 213-234. doi:10.2143/AST.32.2.1027

Wang, S. (2003). Equilibrium Pricing Transforms: New Results Using Buhlmann's 1980 Economic Model. *ASTIN Bulletin, 33*(1), 57-73. doi:10.1017/S0515036100013301

Wang, S. (2004). CAT Bond Pricing Using Probability Transforms. *278*.

Whitworth, J., & Byron, E. (2012). *The €200 Billion Opportunity: Why Insurers Should Lend More*. Retrieved from https://www.oliverwyman.com/content/dam/oliverwyman/global/en/files/archive/2012/Oliver_Wyman_EUR200_BN_Opportunity_Screen.pdf

Yang, B., Li, J., & Balasooriya, U. (2015). Using bootstrapping to incorporate model error for risk-neutral pricing of longevity risk. *Insurance Mathematics and Economics, 62*, 16-27. doi:10.1016/j.insmatheco.2015.02.004

Young, V. R. (2004). *Encyclopedia of actuarial science* (J. L. Teugels & B. Sundt Eds.). Chichester, England: John Wiley & Sons.

Young, V. R. (2006). Premium Principles. In (Vol. 3).

Zhou, R., & Li, J. S.-H. (2013). A cautionary note on pricing longevity index swaps. *Scandinavian Actuarial Journal, 2013*(1), 1-23. doi:10.1080/03461238.2010.507582

Zhou, R., Li, J. S.-H., & Tan, K. S. (2015). Economic Pricing of Mortality-Linked Securities: A Tâtonnement Approach. *Journal of Risk and Insurance, 82*(1), 65-96. doi:10.1111/j.1539-6975.2013.12008.x