Categories of Mackey Functors

Elango Panchadcharam

$$
\begin{tikzcd}
M(U) \arrow[rr, "M^*(p_1 s_1)"] \arrow[dr, "M^*(s_1)"'] & & M(P) \arrow[rr, "M_*(t_2 p_2)"] \arrow[dr, "M_*(p_2)"] & & M(W) \\
& M(S) \arrow[ur, "M^*(p_1)"] \arrow[dr, "M_*(s_2)"'] \arrow[rr, phantom, "\mathit{Mackey}"] & & M(T) \arrow[ur, "M_*(t_2)"'] & \\
& & M(V) \arrow[ur, "M^*(t_1)"'] & &
\end{tikzcd}
$$

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This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text. This work has not been submitted for a higher degree to any other university or institution.

Elango Panchadcharam

In memory of my Father, T. Panchadcharam
1939 - 1991.

Summary

The thesis studies the theory of Mackey functors as an application of enriched category theory and highlights the notions of lax braiding and lax centre for monoidal categories and more generally for promonoidal categories.

The notion of Mackey functor was first defined by Dress [Dr1] and Green [Gr] in the early 1970's as a tool for studying representations of finite groups. The first contribution of this thesis is the study of Mackey functors on a compact closed category $\mathscr{T}$. We define the Mackey functors on a compact closed category $\mathscr{T}$ and investigate the properties of the category **Mky** of Mackey functors on $\mathscr{T}$. The category **Mky** is a monoidal category and the monoids are Green functors. The category of finite-dimensional Mackey functors $\mathbf{Mky}_{fin}$ is a star-autonomous category. The category $\mathbf{Rep}(G)$ of representations of a finite group G is a full sub-category of $\mathbf{Mky}_{fin}$.

The second contribution of this thesis is the study of lax braiding and lax centre for monoidal categories and more generally for promonoidal categories. The centre of a monoidal category was introduced in [JS1]. The centre of a monoidal category is a braided monoidal category. Lax centres become lax braided monoidal categories. Generally the centre is a full subcategory of the lax centre. However in some cases the two coincide. We study the cases where the lax centre and centre becomes equal. One reason for being interested in the lax centre of a monoidal category is that, if an object of the monoidal category is equipped with the structure of monoid in the lax centre, then tensoring with the object defines a monoidal endofunctor on the monoidal category.

The third contribution of this thesis is the study of functors between categories of permutation representations. Functors which preserve finite coproduct and pullback between the category G-$\mathbf{set}_{fin}$ of finite G-sets to the category H-$\mathbf{set}_{fin}$ of finite H-sets (where G and H are finite groups) give a Mackey functor from G-$\mathbf{set}_{fin}$ to H-$\mathbf{set}_{fin}$ for each Mackey functor on H.

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Chapter 0

Introduction

Groups are used to mathematically understand symmetry in nature and in mathematics itself. Classically, groups were studied either directly or via their representations. In the last 40 years, groups have also been studied using Mackey functors, a concept which arose out of a formalization of representation theory.

Mackey functors were first introduced by J. A. Green [Gr] and A. Dress [Dr1], [Dr2] in the early 1970's as a tool for studying representations of finite groups and their subgroups. The axioms for Mackey functors follow on from earlier ideas of Lam on Frobenius functors [La1] which are described in [CR]. Another structure which appeared early on is Bredon's notion of a coefficient system [Br]. There are (at least) three equivalent definitions of Mackey functor for a finite group G.

The most elementary (in a sense used by some group theorists) definition is due to Green [Gr]. The most complicated axiom in this definition is based on the Mackey Decomposition Theorem (see [Ja, p.300] for example) in representation theory and this is presumably why Mackey's name is attached to the concept. We shall now provide the categorical explanation of this Theorem which is used to characterize when induced characters are irreducible.

The Theorem provides a formula for the restriction, to a subgroup, of a group representation induced from a possibly different subgroup. Restriction is composition with a functor (the inclusion of the subgroup in the group) and inducing is an adjoint process amounting therefore to Kan extension.

Consider a functor $i : H \longrightarrow G$ between small categories H and G, and let $\mathscr{M}$ denote a cocomplete category. We write $[H, \mathscr{M}]$ for the category of functors from H to $\mathscr{M}$ and natural transformations between them. A functor $Res_i : [G, \mathscr{M}] \longrightarrow [H, \mathscr{M}]$ is defined by composition on the right with the functor i. This functor Res_i has a left adjoint $Lan_i : [H, \mathscr{M}] \longrightarrow [G, \mathscr{M}]$ for which there are two (closely related) formulas:

$$Lan_i(W)(c) = \int^{b \in H} G(i(b), c).i(b)$$

and

$$Lan_i(W)(c) = \mathrm{colim}(\ i \downarrow c \longrightarrow H \xrightarrow{W} \mathscr{M}\ )$$

where $S.M$ is the coproduct of S copies of M in $\mathscr{M}$ (for any set S) and $i \downarrow c$ is the comma category; see [Ma] for example.

For any functor $j : K \longrightarrow G$, the comma category $i \downarrow j$ is universal with respect to its being equipped with functors $p : i \downarrow j \longrightarrow H$ and $q : i \downarrow j \longrightarrow K$, and a natural transformation

$$\begin{tikzcd} i \downarrow j \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\lambda}{\Longrightarrow}"] & K \arrow[d, "j"] \\ H \arrow[r, "i"'] & G. \end{tikzcd}$$

The following observation is the basis of the 2-categorical notion of "pointwise left extension" defined in [St2, pp.127-128].

Proposition 0.0.1. *The natural transformation λ induces a canonical natural isomorphism*

$$Res_j \circ Lan_i \cong Lan_q \circ Res_p.$$

The following result is an easy exercise in the defining adjoint property of left Kan extension.

Proposition 0.0.2. *Suppose D is a small category which is the disjoint union of subcategories D_α, $\alpha \in \Lambda$, with inclusion functors $m_\alpha : D_\alpha \longrightarrow D$. For any functor $r : D \longrightarrow K$, there is a canonical natural isomorphism*

$$Lan_r \cong \sum_{\alpha\in\Lambda} Lan_{r\circ m_\alpha} \circ Res_{m_\alpha}.$$

A groupoid is a category in which each morphism is invertible. For each object d of a groupoid D, we obtain a group $D(d) = D(d,d)$ whose elements are morphisms $u : d \longrightarrow d$ in D and whose multiplication is composition. We regard groups as one-object groupoids. Let Λ be a set of representative objects in D for all the isomorphism classes of objects in D. Then there is an equivalence of categories

$$\sum_{d\in\Lambda} D(d) \simeq D;$$

that is, every groupoid is equivalent to a disjoint union of groups.

Now suppose H and K are subgroups of a group G. To apply the above considerations, let $i : H \longrightarrow G$ and $j : K \longrightarrow G$ be the inclusions. The comma category $i \downarrow j$ is actually a groupoid: the objects are elements $g \in G$, the morphisms $(h,k) : g \longrightarrow g'$ are elements of $H \times K$ such that $kg = g'h$, and composition is $(h',k') \circ (h,k) = (h'h, k'k)$. Another name for $i \downarrow j$ might be $K \mathbin{\backslash\!\backslash} G \mathbin{/\!/} H$ since the set of isomorphism classes of objects is isomorphic to the set

$$K\backslash G/H = \{KgH \mid g \in G\}$$

of double cosets $KgH = \{kgh \mid k \in K, h \in H\}$. For each object g of $i \downarrow j$, the projection functor $p : i \downarrow j \longrightarrow H$ induces a group isomorphism

$$(i \downarrow j)(g) \cong H \cap K^g$$

where $K^g = g^{-1}Hg$, and the projection functor $q : i \downarrow j \longrightarrow K$ induces a group isomorphism

$$(i \downarrow j)(g) \cong {}^gH \cap K$$

where ${}^gH = gHg^{-1}$. Thus we can identify $(i \downarrow j)(g)$ with both a subgroup $H \cap K^g$ of H and a subgroup ${}^gH \cap K$ of K. Define p_g, q_g, γ_g by the commutative diagram

$$\begin{tikzcd}
{}^gH \cap K \arrow[rr, "\gamma_g"] \arrow[dr, "\cong"] \arrow[dd, "\leqslant"'] & & H \cap K^g \arrow[dl, "\cong"'] \arrow[dd, "\leqslant"] \\
& (i \downarrow j)(g) \arrow[dl, "q_g"] \arrow[dr, "p_g"'] & \\
K & & H.
\end{tikzcd}$$

Let $[K \setminus G/H] \subseteq G$ represent all double cosets in the form KgH, where $g \in [K \setminus G/H]$, without repetition. Therefore we have an equivalence of categories

$$\sum_{g \in [K \setminus G/H]} (i \downarrow j)(g) \simeq i \downarrow j.$$

Corollary 0.0.3.

$$Res_j \circ Lan_i \cong \sum_{g \in [K \setminus G/H]} Lan_{q_g} \circ Res_{p_g}.$$

Proof. Take $r = q$ and $\Lambda = [K \setminus G/H]$ in Proposition 0.0.2 and substitute the resultant formula in Proposition 0.0.1. □

To apply this to the theory of linear representations of groups, we put $\mathscr{M} = \mathbf{Mod}_k$ for a commutative ring k. Then Lan_i and Res_i are denoted by Ind_H^G and Res_H^G, and we have the

Mackey Decomposition Theorem. *For subgroups H and K of a group G, there is a canonical natural isomorphism*

$$Res_K^G \circ Ind_H^G \cong \sum_{g \in [K \setminus G/H]} Ind^K_{{}^gH \cap K} \circ Res_{\gamma_g} \circ Res^H_{H \cap K^g}.$$

We now state Green's definition. A *Mackey functor M* for a group G over the commutative ring k consists of

- a function assigning to each subgroup $H \leqslant G$ a k-module $M(H)$,
- for all subgroups $K \leqslant H \leqslant G$, module morphisms

$$t_K^H : M(K) \longrightarrow M(H), \text{ and } r_K^H : M(H) \longrightarrow M(K),$$

- for all subgroups $H \leqslant G$ and $g \in G$, a module isomorphism

$$c_{g,H} : M(H) \longrightarrow M({}^g H),$$

subject to the following axioms:

1. if $L \leqslant K \leqslant H$ then $t_K^H t_L^K = t_L^H$ and $r_L^K r_K^H = r_L^H$,
2. if $H \leqslant G$, $\quad g_1, g_2 \in G$ and $h \in H$ then

$$c_{g_2, {}^{g_1}H} c_{g_1,H} = c_{g_2 g_1, H} \quad \text{and} \quad c_{h,H} = 1_{M(H)},$$

3. if $K \leqslant H \leqslant G$ and $g \in G$ then

$$c_{g,H} t_K^H = t_{{}^gK}^{{}^gH} c_{g,K} \quad \text{and} \quad c_{g,K} r_K^H = r_{{}^gK}^{{}^gH} c_{g,H},$$

4. if $H \leqslant L$ and $K \leqslant L \leqslant G$ then

$$r_K^L t_H^L = \sum_{g \in [K \backslash L / H]} t_{{}^gH \cap K}^K c_{g, H \cap K^g} r_{H \cap K^g}^H.$$

The morphism t_K^H is called *transfer*, *trace*, or *induction*. The morphism r_K^H is called *restriction*. The isomorphism $c_{g,H}$ is called a *conjugation map*. With this terminology, the relation between axiom (4) and the Mackey Decomposition Theorem should be striking, however, we shall explain it further below.

A *morphism* $\theta : M \longrightarrow N$ of Mackey functors is a family of k-module morphisms $\theta_H : M(H) \longrightarrow N(H), H \leqslant G$, satisfying the obvious commutativity conditions with the morphisms t, r and c.

We shall eventually see that a Mackey functor is actually a functor between two categories. In the first instance, we shall see that it is actually a pair of functors agreeing on objects.

For any group G, there is a category $\mathscr{C}_G$ of *connected G-sets*. It is the full subcategory of the category $[G, \mathbf{Set}] = G\text{-}\mathbf{Set}$ of left G-sets consisting of those which are transitive (and non-empty). Every transitive G-set X is isomorphic to the set G/H of cosets of some subgroup $H \leqslant G$ with the obvious action. Using this, we see that there is an equivalence

$$\mathscr{S}(G) \simeq \mathscr{C}_G\,, \quad H \longmapsto G/H,$$

where $\mathscr{S}(G)$ is a category defined by Green [Gr] whose objects are subgroups of G. A *morphism* $g : H \longrightarrow K$ in $\mathscr{S}(G)$ is an element $g \in G$ such that $H^g \leqslant K$; composition $g_2 \circ g_1$ is product $g_1 g_2$ in G in reverse order. Each $g : H \longrightarrow K$ determines a G-set morphism $G/H \longrightarrow G/K$ taking xH to xgK.

Each Mackey functor M on G over k determines two functors

$$M^* : \mathscr{S}(G)^{\mathrm{op}} \longrightarrow \mathbf{Mod}_k \quad \text{and} \quad M_* : \mathscr{S}(G) \longrightarrow \mathbf{Mod}_k$$

with $M^*(H) = M_*(H) = M(H)$. For each $g : H \longrightarrow K$ in $\mathscr{S}(G)$, we define $M^*(g)$ and $M_*(g)$ by the commutative diagrams below.

$$\begin{tikzcd}
M(K) \arrow[r, "r^K_{H^g}"] \arrow[d, "c_{g,K}"'] \arrow[dr, "M^*(g)"] & M(H^g) \arrow[d, "c_{g,H^g}"] \\
M({}^gK) \arrow[r, "r^{{}^gK}_H"'] & M(H)
\end{tikzcd}
\qquad
\begin{tikzcd}
M(H) \arrow[r, "t^{{}^gK}_H"] \arrow[d, "c_{H,g^{-1}}"'] \arrow[dr, "M_*(g)"] & M({}^gK) \arrow[d, "c_{g^{-1},{}^gK}"] \\
M(H^g) \arrow[r, "t^K_{H^g}"'] & M(K)
\end{tikzcd}$$

We shall provide an example of a Mackey functor where the Mackey axiom comes from the Decomposition Theorem.

Each G-set X determines a groupoid $el(X)$ whose objects are the elements $x \in X$ and whose morphisms $g : x \longrightarrow y$ are elements $g \in G$ such that $gx = y$.

In the case of a transitive G-set G/H where $H \leqslant G$, there is an equivalence of groupoids

$$H \simeq el(G/H)\ , \quad (h : a \longrightarrow a) \longmapsto (h : H \longrightarrow H).$$

It follows that we have an equivalence of categories

$$[el(G/H), \mathbf{Mod}_k] \simeq [H, \mathbf{Mod}_k]$$

where the right-hand side is the category of k-linear representations of the group H.

Let $\mathbf{Cat}_{\oplus}$ denote the 2-category whose objects are additive categories with finite direct sums, whose morphisms are additive functors, and whose 2-cells are natural isomorphisms. Let **Gpd** denote the 2-category of small groupoids. We have two 2-functors

$$\mathbf{Rep}^* : \mathbf{Gpd}^{\mathrm{op}} \longrightarrow \mathbf{Cat}_{\oplus} \quad \text{and} \quad \mathbf{Rep}_* : \mathbf{Gpd} \longrightarrow \mathbf{Cat}_{\oplus}$$

defined on objects $D \in \mathbf{Gpd}$ by

$$\mathbf{Rep}^*(D) = \mathbf{Rep}_*(D) = [D, \mathbf{Mod}_k].$$

For $f : D \longrightarrow E$ in **Gpd**, we define

$$\mathbf{Rep}^*(f) = Res_f \quad \text{and} \quad \mathbf{Rep}_*(f) = Lan_f.$$

There is also a 2-functor $K_o : \mathbf{Cat}_{\oplus} \longrightarrow \mathbf{AbGp}$, where $\mathbf{AbGp} = \mathbf{Mod}_{\mathbf{Z}}$ is the category of abelian groups, which assigns to each additive category $\mathscr{A}$ with finite direct sums, the abelian group $K_o\mathscr{A}$ obtained from the free abelian group on the set of isomorphism classes $[A]$ of objects A of $\mathscr{A}$ by imposing the relations

$$[A \oplus B] = [A] + [B];$$

this is called the *Grothendieck group* of $\mathscr{A}$.

An example of a Mackey functor M on G over $\mathbf{Z}$ is obtained by taking M^* to be the composite functor

$$\mathscr{S}(G)^{\mathrm{op}} \xrightarrow{el} \mathbf{Gpd}^{\mathrm{op}} \xrightarrow{\mathbf{Rep}^*} \mathbf{Cat}_{\oplus} \xrightarrow{K_o} \mathbf{AbGp}$$

and taking M_* to be the composite functor

$$\mathscr{S}(G) \xrightarrow{el} \mathbf{Gpd} \xrightarrow{\mathbf{Rep}_*} \mathbf{Cat}_{\oplus} \xrightarrow{K_o} \mathbf{AbGp}$$

so that $M(H) \cong K_o[H, \mathbf{Mod}_k]$. Mackey Decomposition gives the Mackey axiom (4).

Suppose G is finite. We obtain a sub-example of this last example by replacing $\mathbf{Mod}_k$ by the category $\mathbf{mod}_k$ of finitely generated projective k-modules. If k is a field of characteristic zero then $K_o[H, \mathbf{mod}_k]$ is isomorphic to the group of characters of k-linear representations of H.

We now resume our general discussion. A *Green functor* A for G over k is a Mackey functor A for G over k equipped with a k-algebra structure on each k-module $A(H)$ (associative with unit), for $H \leqslant G$, subject to the axioms:

1. the k-module morphisms t_K^H, r_K^H and $c_{g,K}$ for A preserve the algebra multiplication and unit, and

2. if $K \leqslant H \leqslant G,\ a \in A(H)$, and $b \in A(K)$ then

$$a.t_K^H(b) = t_K^H(r_K^H(a).b) \quad \text{and} \quad t_K^H(b).a = t_K^H(b.r_K^H(a)).$$

Axiom (2) is called the *Frobenius condition* since it resembles the following structural version of Frobenius Reciprocity (see [Ja, Theorem 5.17(3), p.292] for example).

Frobenius Reciprocity. *If V is a k-linear representation of a group G and W is a k-linear representation of a subgroup $H \leqslant G$ then*

$$V \otimes Ind_H^G(W) \cong Ind_H^G(Res_H^G(V) \otimes W).$$

A categorical explanation of this reciprocity is as follows.

Proposition 0.0.4. *Suppose $\mathcal{M}$ is a cocomplete monoidal category whose tensor product preserves colimits in each variable. Suppose $i : H \longrightarrow G$ is a functor between small categories. For functors $V : G \longrightarrow \mathcal{M}$ and $W : H \longrightarrow \mathcal{M}$, the left Kan extension of the functor*

$$V \circledast W : G \times H \xrightarrow{V \times W} \mathcal{M} \times \mathcal{M} \xrightarrow{\otimes} \mathcal{M}$$

along $1_G \times i : G \times H \longrightarrow G \times G$ is naturally isomorphic to

$$V \circledast Lan_i(W) : G \times G \xrightarrow{V \times Lan_i(W)} \mathcal{M} \times \mathcal{M} \xrightarrow{\otimes} \mathcal{M}.$$

Proof.

$$\begin{aligned} Lan_{1_G \times i}(V \circledast W)(c_1, c_2) &\cong \int^{c,a} (G \times G)((c, i(a)), (c_1, c_2)).V(c) \otimes W(a) \\ &\cong \int^{c,a} (G(c, c_1).V(c)) \otimes (G(i(a), c_2).W(a)) \\ &\cong \int^{c} G(c, c_1).V(c) \otimes \int^{a} G(i(a), c_2).W(a) \\ &\cong V(c_1) \otimes Lan_i(W)(c_2) \\ &\cong (V \circledast Lan_i(W))(c_1, c_2). \end{aligned}$$

□

Proposition 0.0.5. *Let H be a subgroup of a group G, let $i : H \longrightarrow G$ be the inclusion, and let $\Delta : G \longrightarrow G \times G$ be the diagonal. Then the comma groupoid $(1_G \times i) \downarrow \Delta$ is connected and there is an equivalence*

$$H \simeq (1_G \times i) \downarrow \Delta.$$

Proof. Objects of $(1_G \times i) \downarrow \Delta$ are elements $(g_1, g_2) \in G \times G$. A morphism $(g, h, x) : (g_1, g_2) \longrightarrow (g_1', g_2')$ is an element of $G \times H \times G$ such that $g_1' g = x g_1$ and $g_2' h = x g_2$;

so, for any object (g_1,g_2), we have the morphism $(g_1^{-1}g_2,1,g_2):(1,1)\longrightarrow(g_1,g_2)$ proving the comma groupoid connected. The equivalence follows from the group isomorphism

$$H\cong((1_G\times i)\downarrow\Delta)(1,1),\quad h\longleftrightarrow(h,h,h).$$

□

Proposition 0.0.6. *Suppose $\mathscr{M}$ is as in Proposition 0.0.4 and H is a subgroup of a group G with inclusion $i:H\longrightarrow G$. The categories $[H,\mathscr{M}]$ and $[G,\mathscr{M}]$ are equipped with the pointwise tensor products. For $V\in[G,\mathscr{M}]$ and $W\in[H,\mathscr{M}]$, there is a canonical isomorphism*

$$V\otimes Lan_i(W)\cong Lan_i(Res_i(V)\otimes W).$$

Proof. Contemplate the diagram

$$\begin{tikzcd}
H \arrow[rr, "i"] \arrow[d, "{(i,1_H)}"'] & & G \arrow[d, "\Delta"] \\
G\times H \arrow[rr, "1_G\times i"] \arrow[dr, "V\circledast W"'] & \Longrightarrow & G\times G \arrow[dl, "V\circledast Lan_i(W)"] \\
& \mathscr{M} &
\end{tikzcd}$$

in the light of Propositions 0.0.1, 0.0.4 and 0.0.5. □

If A is a monoidal additive category with direct sums, $K_o\mathscr{A}$ becomes a ring via

$$[A][B]=[A\otimes B].$$

It follows that the example of the Mackey functor M with $M(H)\cong K_o[H,\mathbf{Mod}_k]$ is actually a Green functor.

Notice that the functor $el:\mathscr{S}(G)\longrightarrow\mathbf{Gpd}$ is the restriction of the coproduct preserving functor $el:[G,\mathbf{Set}]\longrightarrow\mathbf{Gpd}$. This motivates the second definition of Mackey functor (see [Dr1] and [Di]).

We centre attention on the case of a finite group G. We write $\mathbf{set}_{fin}$ for the category of finite sets and G-$\mathbf{set}_{fin} = [G, \mathbf{set}_{fin}]$ for the category of finite G-sets. Every finite G-set is a coproduct (disjoint union) of transitive G-sets. With a little more work we see that G-$\mathbf{set}_{fin}$ is the completion of $\mathscr{C}_G$ under finite coproducts. Therefore, the functors M^* and M_* above extend (uniquely up to isomorphism) to functors

$$M^* : (G\text{-}\mathbf{set}_{fin})^{\mathrm{op}} \longrightarrow \mathbf{Mod}_k \quad \text{and} \quad M_* : G\text{-}\mathbf{set}_{fin} \longrightarrow \mathbf{Mod}_k$$

which respectively preserve finite products and finite coproducts. So here is the second equivalent definition.

A *Mackey functor* M for G over k consists of a pair of functors

$$M^* : (G\text{-}\mathbf{set}_{fin})^{\mathrm{op}} \longrightarrow \mathbf{Mod}_k\ , \quad M_* : G\text{-}\mathbf{set}_{fin} \longrightarrow \mathbf{Mod}_k$$

which agree $M^*(X) = M_*(X) = M(X)$ on objects X of G-$\mathbf{set}_{fin}$ subject to the following axioms:

1. for every pullback diagram

$$\begin{tikzcd} P \arrow[r, "\delta"] \arrow[d, "\gamma"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$

   in G-$\mathbf{set}_{fin}$, the equation

$$M^*(\beta)M_*(\alpha) = M_*(\delta)M^*(\gamma)$$

   holds,

2. for inclusions $i : X \longrightarrow X+Y$ and $j : Y \longrightarrow X+Y$ into the coproduct $X+Y$ of X and Y in G-$\mathbf{set}_{fin}$, the diagram

$$\begin{tikzcd} M(X) \arrow[r, shift right, "M_*(i)"'] & M(X+Y) \arrow[l, shift right, "M^*(i)"'] \arrow[r, shift left, "M^*(j)"] & M(Y) \arrow[l, shift left, "M_*(j)"] \end{tikzcd}$$

is a direct sum situation in $\mathbf{Mod}_k$.

The axiom (1) is now called the *Mackey condition* as it is a more categorically pleasing expression of the previous axiom (4). A *morphism* $\theta : M \longrightarrow N$ of Mackey functors is a family of k-module morphisms

$$\theta_X : M(X) \longrightarrow N(X), \quad X \in G\text{-}\mathbf{set}_{\mathit{fin}},$$

which is natural as both $M^* \longrightarrow N^*$ and $M_* \longrightarrow N_*$. A *Green functor* A for G over k is a Mackey functor for G over k equipped with k-bilinear morphisms

$$A(X) \times A(Y) \longrightarrow A(X \times Y), \quad (a, b) \longmapsto ab$$

which are natural in X and $Y \in G\text{-}\mathbf{set}_{\mathit{fin}}$, and are associative and unital in an obvious way.

A third equivalent definition of Mackey functor appears in [TW] and involves first creating the Mackey algebra $\mu_k(G)$ for the finite group G. The study of Mackey functors becomes the representation theory of this algebra. The Mackey algebra $\mu_k(G)$ of G over k is the associative k-algebra defined by the generators t_K^H, r_K^H and $c_{g,H}$ for subgroups $K \leqslant H$ of G and $g \in G$, satisfying the following relations:

1. if $L \leqslant K \leqslant H$ are subgroups of G, then $t_K^H t_L^K = t_L^H$ and $r_L^K r_K^H = r_L^H$, and if $g, h \in H$ and H is a subgroup of G, then $c_{h,{}^gH} c_{g,H} = c_{hg,H}$,

2. if $g \in G$ and $K \leqslant H$ are subgroups of G, then $c_{g,H} t_K^H = t_{{}^gK}^{{}^gH} c_{g,K}$ and $c_{g,K} r_K^H = r_{{}^gK}^{{}^gH} c_{g,H}$,

3. if $h \in H$ and H is a subgroup of G, then $t_H^H = r_H^H = c_{g,H}$,

4. if $K \leqslant H \geqslant L$ are subgroups of G, then

$$r_K^H t_L^H = \sum_{g \in [K \backslash H / L]} t_{K \cap {}^gL}^K c_{g, K^g \cap L} r_{K^g \cap L}^L$$

where $[K \setminus H/L]$ is a set of representatives of the double cosets modulo K and L in H,

5. all products of generators, different from those appearing in the previous four relations are zero,

6. the sum of the elements t_H^H over all subgroups H of G is equal to the identity element of $\mu_k(G)$.

A *Mackey functor* M for G over k is a $\mu_k(G)$-module and a morphism of Mackey functors is a morphism of $\mu_k(G)$-modules.

The study of Mackey functors on compact Lie groups is described by Lewis [Le]. Many of the fundamental results on Mackey functors for a finite group are extended to Mackey functors for a compact Lie group. Mackey functors have been studied on finite groups for a long time. The study of Mackey functors for an infinite group has appeared recently: references are in [Lü2] and [MN]. There is also a new concept called *globally-defined Mackey functors*. They appeared more recently and were studied in [We]. The main difference is that the globally-defined Mackey functors are defined on all finite groups, where the original Mackey functors are defined on subgroups of a particular group. A second main difference is that the original Mackey functors only possess the inclusion and conjugation operations but the globally-defined Mackey functors possess operations for all group homomorphisms.

Some examples of Mackey functors for finite groups are representations rings, Burnside rings ([Se1],[Di]), group cohomology ([Fe]), equivariant cohomology, equivariant topological K-theory ([Se2]), algebraic K-theory of group rings ([Lü1]), any stable equivariant (co-)homology theories ([LMM]), and higher algebraic K-theory ([Ku]).

One application of Mackey functors to number theory has been to provide relations between λ- and μ-invariants in Iwasawa theory and between Mordell-Weil groups, Shafarevich-Tate groups, Selmer groups and zeta functions of elliptic curves (see [BB]).

This thesis consists of four papers. The first develops the main goal and theory of the thesis: put simply, it develops and extends the theory of Mackey functors as an application of enriched category theory. The other papers arose from specific issues that came up in the preparation of the first paper, particularly, they concern techniques for constructing new Mackey and Green functors from given ones. We saw that, in order for the Dress construction to produce a Green functor from a given one, we needed a monoid in the lax centre of some monoidal category. This led us to a general study of lax braidings and lax centre for monoidal categories and more generally for promonoidal categories. The second and third papers are the outcome; they have application beyond the particular needs of the first paper. The final paper is a categorical treatment of a theorem of Bouc [Bo2] concerning which functors compose with Mackey functors to yield Mackey functors; again this result may be useful in other applications.

The first paper entitled *Mackey functors on compact closed categories*, coauthored with Professor Ross Street, was submitted to the *Journal of Homotopy and Related Structures (JHRS)* to a special volume in memory of Saunders Mac Lane. The second paper entitled *Lax braidings and the lax centre*, coauthored with Dr. Brain Day and Professor Ross Street, will appear in *Contemporary Mathematics*. The third paper entitled *On centres and lax centres for promonoidal categories*, coauthored with Dr. Brain Day and Professor Ross Street, was submitted to “Charles Ehresmann 100 ans”, the 100th birthday anniversary conference of Charles Ehresmann which was held at the Universite de Picardie Jules Verne in Amiens between October 7 to 9, 2005. The abstract will appear in

Cahiers de Topologie et Géométrie Différentielle Catégoriques, Volume **XLVI-3**. The fourth paper entitled *Pullback and finite coproduct preserving functors between categories of permutation representations* consists of the paper [PS2] as modified in the light of [PS3]. The papers [PS2] and [PS3] are appearing in the journal of *Theory and Applications of Categories*, Volume **16**, Number **28**, pp. 771–784, (2006) and Volume **18**, Number **5**, pp. 151–156, (2007) respectively.

Chapter 1 consists of the first paper entitled "Mackey functors on compact closed categories". This paper develops the theory of Mackey functors as an application of enriched category theory. Mackey functors on a compact (= rigid= autonomous) closed category $\mathscr{T}$ are defined and the properties of the category **Mky** of Mackey functors on $\mathscr{T}$ are investigated. The category **Mky** is a symmetric monoidal closed abelian category.

We now explain the main constructions and theorems of the sections of this chapter. In Section 1.1 we give an introduction to this paper. In Section 1.2 we define the compact closed category $\mathbf{Spn}(\mathscr{E})$ of spans in a finitely complete category $\mathscr{E}$. The objects of $\mathbf{Spn}(\mathscr{E})$ are the objects of $\mathscr{E}$ and morphisms $U \longrightarrow V$ are the isomorphisms classes of *spans* from U to V in the bicategory of spans in $\mathscr{E}$. The category $\mathbf{Spn}(\mathscr{E})$ is a monoidal category using the cartesian product in $\mathscr{E}$ as the tensor product in $\mathbf{Spn}(\mathscr{E})$. Section 1.3 describes the direct sums in $\mathbf{Spn}(\mathscr{E})$. Here we take $\mathscr{E}$ to be a lextensive category. References for this notion are [Sc1], [CLW], and [CL]. The coproduct $U+V$ in $\mathscr{E}$ is the direct sum of U and V in $\mathbf{Spn}(\mathscr{E})$. The addition of two spans is also defined in $\mathbf{Spn}(\mathscr{E})$. This makes the category $\mathbf{Spn}(\mathscr{E})$ into a monoidal commutative-monoid-enriched category. In Section 1.4 we define the Mackey functors on a lextensive category $\mathscr{E}$ using the approach described by Dress [Dr1] in the G-set case. A Mackey functor M from $\mathscr{E}$ to the category $\mathbf{Mod}_k$ of k-modules consists of two functors $M_* : \mathscr{E} \longrightarrow \mathbf{Mod}_k$, and $M^* : \mathscr{E}^{\mathrm{op}} \longrightarrow \mathbf{Mod}_k$ which coincide on objects and satisfy a couple of conditions. A morphism $\theta : M \longrightarrow N$ of Mackey functors M and N is a family of morphisms $\theta_U : M(U) \longrightarrow N(U)$ for each $U \in \mathscr{E}$ which give two natural transformations $\theta_* : M_* \longrightarrow N_*$ and $\theta^* : M^* \longrightarrow N^*$. We denote the category of Mackey functors from $\mathscr{E}$ to $\mathbf{Mod}_k$ by $\mathbf{Mky}(\mathscr{E}, \mathbf{Mod}_k)$ or simply $\mathbf{Mky}$ when $\mathscr{E}$ and k are understood.

Proposition [1.4.1]. *(Lindner [Li1]) The category* $\mathbf{Mky}(\mathscr{E}, \mathbf{Mod}_k)$ *of Mackey functors, from a lextensive category* $\mathscr{E}$ *to the category* $\mathbf{Mod}_k$ *of* k*-modules, is equivalent to* $[\mathbf{Spn}(\mathscr{E}), \mathbf{Mod}_k]_+$*, the category of coproduct-preserving functors.*

Tensor product of Mackey functors is defined in Section 1.5. Here we work on a general compact closed category $\mathscr{T}$ with finite products in place of $\mathbf{Spn}(\mathscr{E})$. This implies that $\mathscr{T}$ has direct sums (see [Ho]) and $\mathscr{T}$ is enriched in the monoidal category $\mathscr{V}$ of commutative monoids. A Mackey functor on $\mathscr{T}$ is an additive functor $M : \mathscr{T} \longrightarrow \mathbf{Mod}_k$. The tensor product of Mackey functors M and N is defined by:

$$(M * N)(Z) \cong \int^Y M(Z \otimes Y^*) \otimes_k N(Y)$$

using Day's convolution structure ([Da1]). The *Burnside functor* J is defined on objects as the free k-module on $\mathscr{T}(I, U)$ where I is the unit of $\mathscr{T}$ and U is an object of $\mathscr{T}$. It is a Mackey functor and becomes the unit for the tensor product of Mackey functors. The category $\mathbf{Mky}$ becomes a symmetric monoidal closed category. The closed structure is described in Section 1.6. For Mackey functors M and N, the Hom Mackey functor is given by:

$$\mathrm{Hom}(M, N)(V) = \mathbf{Mky}(M(V^* \otimes -), N).$$

There is also another expression for this Hom Mackey functor, which is given by:

$$\mathrm{Hom}(M, N)(V) = \mathbf{Mky}(M, N(V \otimes -)).$$

Green functors are introduced in Section 1.7. A *Green functor* A on $\mathscr{T}$ is a

Mackey functor with a monoidal structure

$$\mu : A(U) \otimes_k A(V) \longrightarrow A(U \otimes V)$$

and a morphism

$$\eta : k \longrightarrow A(1).$$

Green functors precisely become the monoids in the monoidal category **Mky**. In Section 1.8 we describe the Dress construction of Green functors. The Dress construction ([Bo5], [Bo6]) is a process to obtain a new Mackey functor M_Y from a known Mackey functor M, where $M_Y(U) = M(U \otimes Y)$ for fixed $Y \in \mathscr{T}$. We define the *Dress construction*

$$D : \mathscr{T} \otimes \mathbf{Mky} \longrightarrow \mathbf{Mky}$$

by $D(Y, M) = M_Y$. In Proposition 1.8.1 we show that the Dress construction D is a strong monoidal $\mathscr{V}$-functor. We study the centres and the lax centres of the monoidal category $\mathscr{E}/G_C$ (where $\mathscr{E}/G_C$ is the category of crossed G-sets) to obtain the Dress construction for Green functors. The detailed study of centres and lax centres for monoidal categories are in Chapters 2 and 3. We use the following Theorem to induce the Dress construction on Green functors.

Theorem [1.8.4]. *([Bo2], [Bo3]) If Y is a monoid in $\mathscr{E}/G_c$ and A is a Green functor for $\mathscr{E}$ over k then A_Y is a Green functor for $\mathscr{E}$ over k, where $A_Y(X) = A(X \times Y)$.*

Finite dimensional Mackey functors are introduced in Section 1.9. Here we assume the compact closed category is $\mathscr{T} = \mathbf{Spn}(\mathscr{E})$, where $\mathscr{E} = G\text{-}\mathbf{set}_{fin}$ is the category of finite G-sets for a finite group G. Also we assume k is a field and replace $\mathbf{Mod}_k$ by **Vect**, the category of vector spaces. A Mackey functor $M : \mathscr{T} \longrightarrow \mathbf{Vect}$ is called *finite dimensional* when each $M(X)$ is a finite-dimensional vector space. We denote the category of finite dimensional Mackey functors by

$\mathbf{Mky}_{fin}$ which is a full subcategory of **Mky**. We show that the tensor product of finite-dimensional Mackey functors is finite dimensional (Proposition 1.9.1).

Theorem [1.9.2]. *The monoidal category* $\mathbf{Mky}_{fin}$ *of finite-dimensional Mackey functors on* $\mathscr{T}$ *is* $*$*-autonomous.*

In Section 1.10 we study the cohomological Mackey functors and the relation between the ordinary k-linear representations of a finite group G and Mackey functors on G. Let $\mathbf{Rep}_k(G)$ denote the finite-dimensional k-linear representations of G. The relation between $\mathbf{Rep}_k(G)$ and $\mathbf{Mky}(G)$ is shown in the following Proposition:

Proposition[1.10.1]. *The functor* $\widetilde{k_*} : \mathbf{Rep}_k(G) \longrightarrow \mathbf{Mky}(G)$ *is fully faithful.*

In Theorem 1.10.4 we also show that the adjoint functor $\mathbf{Mky}(G)_{fin} \longrightarrow \mathbf{Rep}_k(G)$ is strong monoidal. In Section 1.11 we give examples for the compact closed category $\mathscr{T}$ from a Hopf algebra H (or quantum group). The category $\mathbf{Comod}(\mathscr{R})$ becomes an example of $\mathscr{T}$. The objects of the category $\mathbf{Comod}(\mathscr{R})$ (see [DMS]) are comonoids C in $\mathscr{R}$ (where $\mathscr{R}$ is the category of left H-modules) and morphisms are isomorphisms classes of comodules $S : C \twoheadrightarrow D$ from C to D. The category $\mathbf{Comod}(\mathscr{R})$ is compact closed and a commutative-monoid enriched category. We also show that $\mathscr{R}^{\mathrm{op}} (\simeq \mathscr{R})$ is another example for $\mathscr{T}$.

Section 1.12 reviews the modules of enriched category theory. Section 1.13 studies the modules over Green functors. A *module* M over a Green functor A or A-*module* means A acts on M via the convolution $*$. We denote the category of left A-modules for a Green functor A by $\mathbf{Mod}(A)$. The objects are A-modules and morphisms are A-module morphisms $\theta : M \longrightarrow N$. The category $\mathbf{Mod}(A)$ is the category of Eilenberg-Moore algebras for the monad $T = A * -$ on $[\mathscr{C}, \mathbf{Mod}_k]$, where $\mathscr{C}$ is a small $\mathscr{V}$-category. In Section 1.14 we study the Morita theory of Green functors. We define the monoidal bicategory $\mathbf{Mod}(\mathscr{W})$ for $\mathscr{W} = \mathbf{Mky}$. The objects are monoids A in $\mathscr{W}$ and morphisms are modules

$M: A \twoheadrightarrow B$ (that is, algebras for the monad $A * - * B$ on **Mky**) with a two sided action $A * M * B \longrightarrow M$. Composition of morphisms is defined by a coequalizer. Green functors A and B are defined to be *Morita equivalent* when they are equivalent in $\mathbf{Mod}(\mathscr{W})$. In Proposition 1.14.1 we show that if A and B are equivalent in $\mathbf{Mod}(\mathscr{W})$ then $\mathbf{Mod}(A) \simeq \mathbf{Mod}(B)$ as categories. The Cauchy completion $\mathscr{Q}A$ of A is the $\mathscr{W}$-category which consists of the modules $M: J \twoheadrightarrow A$ with right adjoints $N: A \twoheadrightarrow J$, where J is the unit of $\mathscr{W}$. In the following Theorem we obtain an explicit description of the objects of the Cauchy completion of a monoid A in the monoidal category $\mathscr{W} = \mathbf{Mky}$.

Theorem[1.14.3]. *The Cauchy completion $\mathscr{Q}A$ of the monoid A in* **Mky** *consists of all the retracts of modules of the form*

$$\bigoplus_{i=1}^{k} A(Y_i \times -)$$

for some $Y_i \in \mathbf{Spn}(\mathscr{E})$, $i = 1, \ldots, k$.

Chapter 2 consists of the paper entitled “Lax braidings and the lax centre". This highlights the notions of lax braiding and lax centre for monoidal categories and more generally for promonoidal categories. Braidings for monoidal categories were introduced in [JS2] and its forerunners. The centre $\mathscr{Z}\mathscr{X}$ of a monoidal category $\mathscr{X}$ was introduced in [JS1] in the process of proving that the free tortile monoidal category has another universal property. The centre of a monoidal category is a braided monoidal category. The centre is generally a full subcategory of the lax centre, but sometimes the two coincide. We examine the cases where these two become equal.

We explain the main constructions and theorems of the sections of this chapter. An introduction is given at the beginning. In Section 2.1 we study the lax braidings for promonoidal categories. Let $\mathscr{V}$ be a complete cocomplete symmetric closed monoidal category and $\mathscr{C}$ be a $\mathscr{V}$-enriched category in the sense of [Ke]. The category $\mathscr{C}$ is called promonoidal when there are two $\mathscr{V}$-

functors $P:\mathscr{C}^{\mathrm{op}}\otimes\mathscr{C}^{\mathrm{op}}\otimes\mathscr{C}\longrightarrow\mathscr{V}$ and $J:\mathscr{C}\longrightarrow\mathscr{V}$ (called a *promagmal* structure on $\mathscr{C}$) satisfying the associative, and left and right unit constraints. Symmetries for promonoidal categories were defined by Day [Da1] and braidings by Joyal and Street [JS2]. A lax braiding for a promonoidal category $\mathscr{C}$ is a $\mathscr{V}$-natural family of morphisms $P(A,B;C)\longrightarrow P(B,A;C)$ which satisfies some commutative diagrams. A braiding is a lax braiding with each $P(A,B;C)\longrightarrow P(B,A;C)$ invertible. We reprove a result of Yetter [Ye] in Proposition 2.1.3 that if $\mathscr{C}$ is a right autonomous (meaning that each object has a right dual) monoidal category then any lax braiding on $\mathscr{C}$ is necessarily a braiding.

In Section 2.2 we define the lax centre and centre of a promonoidal category $\mathscr{C}$. The objects of the lax centre $\mathcal{Z}_l\mathscr{C}$ of a promonoidal category $\mathscr{C}$ are pairs (A,α) where A is an object of $\mathscr{C}$ and α is a $\mathscr{V}$-natural family of morphisms $\alpha_{X,Y}:P(A,X;Y)\longrightarrow P(X,A;Y)$ satisfying a couple of commutative diagrams. The Hom object $\mathcal{Z}_l\mathscr{C}((A,\alpha),(B,\beta))$ is defined to be the equalizer in $\mathscr{V}$ of the two composed paths around the following square.

$$\begin{tikzcd}
\mathscr{C}(A,B) \arrow[r, "P"] \arrow[d, "P"'] & \int_{X,Y}[P(B,X;Y),P(A,X;Y)] \arrow[d, "{[1,\alpha]}"] \\
\int_{X,Y}[P(X,B;Y),P(X,A;Y)] \arrow[r, "{[\beta,1]}"'] & \int_{X,Y}[P(B,X;Y),P(X,A;Y)]
\end{tikzcd}$$

The lax centre $\mathcal{Z}_l\mathscr{C}$ of the promonoidal category $\mathscr{C}$ is often promonoidal. The $\mathscr{V}$-functor $\mathcal{Z}_l\mathscr{C}\longrightarrow\mathscr{C}$ which take (A,α) to A is a strong promonoidal functor. If $\mathscr{C}$ is monoidal then the category $\mathcal{Z}_l\mathscr{C}$ is also a monoidal category and $\mathcal{Z}_l\mathscr{C}\longrightarrow\mathscr{C}$ is strong monoidal. The centre $\mathcal{Z}\mathscr{C}$ of $\mathscr{C}$ is the full sub-$\mathscr{V}$-category of $\mathcal{Z}_l\mathscr{C}$ consisting the objects (A,α) where each $\alpha_{X,Y}:P(A,X;Y)\longrightarrow P(X,A;Y)$ is invertible. Clearly $\mathcal{Z}\mathscr{C}$ is a braided monoidal category.

The lax centre of a monoidal category is studied in Section 2.3. The lax centre $\mathcal{Z}_l\mathscr{C}$ of a monoidal $\mathscr{V}$-category $\mathscr{C}$ has objects pairs (A,u) where A is an object of $\mathscr{C}$ and u is a $\mathscr{V}$-natural family of morphisms $u_B : A \otimes B \longrightarrow B \otimes A$ which satisfy the following two commutative diagrams.

$$\begin{tikzcd} A \otimes B \otimes C \arrow[rr, "u_{B\otimes C}"] \arrow[dr, "u_B \otimes 1_C"'] & & B \otimes C \otimes A \\ & B \otimes A \otimes C \arrow[ur, "1_B \otimes u_C"'] & \end{tikzcd} \qquad \begin{tikzcd} A \otimes I \arrow[rr, "u_I"] \arrow[dr, "\cong"'] & & I \otimes A \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

When $\mathscr{V} = \mathbf{Set}$ and $\mathscr{C}$ is monoidal, the lax centre of $\mathscr{C}$ was used by P. Schauenburg [Sc1] under the name of "weak centre". One reason for being interested in the lax centre is the following result.

Theorem[2.3.7]. *Suppose an object F of a monoidal $\mathscr{V}$-category $\mathscr{F}$ is equipped with the structure of monoid in the lax centre $\mathscr{Z}_l\mathscr{F}$ of $\mathscr{F}$. Then $-\otimes F : \mathscr{F} \longrightarrow \mathscr{F}$ is equipped with the structure of monoidal $\mathscr{V}$-functor.*

In the following two Corollaries we show two cases in which the lax centre becomes equal to the centre. Corollary 2.3.5 shows that, for any Hopf algebra H, the lax centre of the monoidal category **Comod**H of left H-comodules is equal to its centre. Corollary 2.3.6 shows that, for any finite dimensional Hopf algebra H, the lax centre of the monoidal category **Mod**H of left H-modules is equal to its centre. In Section 2.4 we study the lax centre and centre of cartesian monoidal categories where $\mathscr{V} = \mathbf{Set}$. The objects of the lax centre $\mathscr{Z}_l\mathscr{C}$ are pairs (A,ϕ) where A is in $\mathscr{C}$ and ϕ is a family of functions $\phi_X : \mathscr{C}(A,X) \longrightarrow \mathscr{C}(X,X)$ such that the following diagram commutes for all $f : X \longrightarrow Y$ in $\mathscr{C}$.

$$\begin{tikzcd} \mathscr{C}(A,X) \arrow[r, "\phi_X"] \arrow[dd, "\mathscr{C}(1_A,f)"'] & \mathscr{C}(X,X) \arrow[dr, "\mathscr{C}(1_X,f)"] & \\ & & \mathscr{C}(X,Y) \\ \mathscr{C}(A,Y) \arrow[r, "\phi_Y"'] & \mathscr{C}(Y,Y) \arrow[ur, "\mathscr{C}(f,1_Y)"'] & \end{tikzcd}$$

A morphism $g : (A,\phi) \longrightarrow (A',\phi')$ in $\mathscr{Z}_l\mathscr{C}$ is a morphism $g : A \longrightarrow A'$ in $\mathscr{C}$ such that $\phi_X(vg) = \phi'_X(v)$ for all $v : A' \longrightarrow X$. The *core* $C_{\mathscr{X}}$ of the category $\mathscr{X}$ with finite products in the sense of Freyd [Fr2] is precisely a terminal object in $\mathcal{Z}_l\mathcal{X}$. If the core exists, the lax centre can be written as

$$\mathcal{Z}_l\mathcal{X} \cong \mathcal{X}/C_{\mathcal{X}}.$$

In Theorem 2.4.2 we show that for any small category $\mathcal{C}$ equipped with the promonoidal structure whose convolution gives the cartesian monoidal structure on $[\mathcal{C},\mathbf{Set}]$, there is an equivalence and an isomorphism of categories:

$$[\mathcal{Z}_l\mathcal{C},\mathbf{Set}] \xrightarrow{\simeq} [\mathcal{C},\mathbf{Set}]/C_{[\mathcal{C},\mathbf{Set}]} \xrightarrow{\cong} \mathcal{Z}_l[\mathcal{C},\mathbf{Set}].$$

In Theorem 2.4.5 we show that, if $\mathcal{C}$ is a groupoid with a promonoidal structure, then the lax centre of $\mathcal{C}$ is equal to the centre of $\mathcal{C}$. We also show that if the convolution of the promonoidal structure of $\mathcal{C}$ gives a cartesian monoidal structure on $[\mathcal{C},\mathbf{Set}]$ then the lax centre of $[\mathcal{C},\mathbf{Set}]$ is equal to its centre. In the following Theorem we show another case where the lax centre coincides with the centre of the cartesian monoidal category $[\mathcal{C},\mathbf{Set}]$.

Theorem [2.4.4]. *If $\mathcal{C}$ is a category in which every endomorphism is invertible then the lax centre $\mathcal{Z}_l[\mathcal{C},\mathbf{Set}]$ of the cartesian monoidal category $[\mathcal{C},\mathbf{Set}]$ is equal to the centre $\mathcal{Z}[\mathcal{C},\mathbf{Set}]$.*

In Section 2.5 we develop the theory of central cohypomonads for a monoidal $\mathcal{V}$-category $\mathcal{X}$. The lax centre $\mathcal{Z}_l\mathcal{X}$ is the $\mathcal{V}$-category of coalgebras for a cohypomonad. A *cohypomonad* on $\mathcal{X}$ is a monoidal functor $G : \Delta^{\mathrm{op}} \longrightarrow [\mathcal{X},\mathcal{X}]$ where Δ is the category with objects the finite ordinals $\langle n\rangle = \{1,2,\ldots,n\}$. The morphisms of Δ are order-preserving functions. A *coalgebra* for G is an object A of $\mathcal{X}$ together with a coaction morphism satisfying some commutative diagrams.

Proposition [2.5.1]. *Let $\mathcal{X}$ be a complete closed monoidal $\mathcal{V}$-category with a small dense sub-$\mathcal{V}$-category. The structure just defined on $\mathbf{G} : \Delta^{op} \longrightarrow [\mathcal{X},\mathcal{X}]$*

makes it a normal cohypomonad for which $\mathscr{X}^{\mathbf{G}}$ is equivalent to the lax centre of $\mathscr{X}$.

Chapter 3 consists of the paper entitled "On centres and lax centres for promonoidal categories". This reviews the notions of lax braiding and lax centre for monoidal and promonoidal categories and generalizes them to the $\mathscr{V}$-enriched context. To a large extent, this is a conference paper summarizing some results of the last Chapter and of [DS4]. We examine when the centre of $[\mathscr{C},\mathscr{V}]$ with the convolution monoidal structure (in the sense of [Da1]) is again a functor category $[\mathscr{D},\mathscr{V}]$.

We explain the main constructions and theorems of the sections of this chapter. Section 3.1 is the introduction of this paper. Section 3.2 reviews some definitions. A *$\mathscr{V}$-multicategory* is a $\mathscr{V}$-category $\mathscr{C}$ equipped with a sequence of $\mathscr{V}$-functors

$$P_n : \underbrace{\mathscr{C}^{\mathrm{op}} \otimes \ldots \otimes \mathscr{C}^{\mathrm{op}}}_{n} \otimes \mathscr{C} \longrightarrow \mathscr{V},$$

where we write J for $P_0 : \mathscr{C} \longrightarrow \mathscr{V}$, P_1 for $\mathscr{C}(-,\sim) : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \longrightarrow \mathscr{V}$ which is a hom $\mathscr{V}$-functor, and we write P for P_2. Also there are *substitution operations* which are $\mathscr{V}$-natural families of morphisms satisfying the associative and unit conditions. For $\mathscr{V} = \mathbf{Set}$, this is a multicategory in the sense of [La4]. A *promonoidal $\mathscr{V}$-category* [Da1] is a $\mathscr{V}$-multicategory $\mathscr{C}$ for which the substitution operations are invertible. A *monoidal $\mathscr{V}$-category* is a promonoidal $\mathscr{V}$-category $\mathscr{C}$ for which P and J are representable. That is, there are $\mathscr{V}$-natural isomorphisms

$$P(A,B;C) \cong \mathscr{C}(A \boxtimes B, C), \qquad JC \cong \mathscr{C}(U,C).$$

We define lax braiding and braiding for a promonoidal $\mathscr{V}$-category $\mathscr{C}$.

In Section 3.3 we define the lax centre and centre of a monoidal $\mathscr{V}$-category $\mathscr{C}$. The *lax centre* $\mathscr{Z}_l\mathscr{C}$ of a monoidal $\mathscr{V}$-category $\mathscr{C}$ has objects (A,u) where A

is an object of $\mathscr{C}$ and u is a $\mathscr{V}$-natural family of morphisms

$$u_B : A \boxtimes B \longrightarrow B \boxtimes A$$

such that the following two diagrams commute:

$$\begin{tikzcd}
& (A \boxtimes B) \boxtimes C \arrow[r, "u_B \boxtimes 1"] \arrow[dl, "\cong"'] & (B \boxtimes A) \boxtimes C \arrow[dr, "\cong"] & \\
A \boxtimes (B \boxtimes C) \arrow[d, "u_B \boxtimes C"'] & & & B \boxtimes (A \boxtimes C) \arrow[d, "1 \boxtimes u_C"] \\
(B \boxtimes C) \boxtimes A \arrow[rrr, "\cong"'] & & & B \boxtimes (C \boxtimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \boxtimes U \arrow[rr, "u_U"] \arrow[dr, "\cong"'] & & U \boxtimes A \arrow[dl, "\cong"] \\
& A \,. &
\end{tikzcd}$$

The monoidal structure on $\mathcal{Z}_l\mathscr{C}$ is defined on objects by

$$(A, u) \boxtimes (B, v) = (A \boxtimes B, w)$$

where $w_C : (A \boxtimes B) \boxtimes C \longrightarrow C \boxtimes (A \boxtimes B)$ is the composite

$$A \boxtimes (B \boxtimes C) \xrightarrow{1 \boxtimes v_C} A \boxtimes (C \boxtimes B) \xrightarrow{\cong} (A \boxtimes C) \boxtimes B \xrightarrow{u_C \boxtimes 1} (C \boxtimes A) \boxtimes B$$

conjugated by canonical isomorphisms. The lax centre $\mathcal{Z}_l\mathscr{C}$ is a lax-braided monoidal $\mathscr{V}$-category. The lax braiding on $\mathcal{Z}_l\mathscr{C}$ is defined to be the family of morphisms

$$c_{(A,u),(B,v)} : (A \boxtimes B, w) \longrightarrow (B \boxtimes A, \tilde{w})$$

lifting $u_B : A \boxtimes B \longrightarrow B \boxtimes A$ to $\mathcal{Z}_l\mathscr{C}$. The *centre* $\mathcal{Z}\mathscr{C}$ of $\mathscr{C}$ is the full monoidal sub-$\mathscr{V}$-category of $\mathcal{Z}_l\mathscr{C}$ consisting of the objects (A, u) with each u_B invertible. Clearly $\mathcal{Z}\mathscr{C}$ is a braided monoidal $\mathscr{V}$-category. We generalize the constructions of the lax centre and the centre to promonoidal $\mathscr{V}$-categories $\mathscr{C}$.

In Section 3.4 we study the lax centre of cartesian monoidal categories $\mathscr{C}$. We identify the objects of $\mathcal{Z}_l\mathscr{C}$ with pairs (A,θ) where A is an object of $\mathscr{C}$ and $\theta_X : A \times X \longrightarrow X$ is a family of morphisms.

Theorem [3.4.1]. *Let $\mathscr{C}$ denote a small category with promonoidal structure such that the convolution structure on $[\mathscr{C}, \mathbf{Set}]$ is cartesian product.*

1. *The adjunction $\hat{\Psi} \dashv \tilde{\Psi}$ defines an equivalence of lax-braided monoidal categories*

$$\mathcal{Z}_l[\mathscr{C}, \mathbf{Set}] \simeq [\mathcal{Z}_l\mathscr{C}, \mathbf{Set}]$$

which restricts to a braided monoidal equivalence

$$\mathcal{Z}[\mathscr{C}, \mathbf{Set}] \simeq [\mathcal{Z}\mathscr{C}, \mathbf{Set}].$$

2. *If every endomorphism in the category $\mathscr{C}$ is invertible then $\mathcal{Z}_l\mathscr{C} = \mathcal{Z}\mathscr{C}$.*

3. *If $\mathscr{C}$ is a groupoid then*

$$\mathcal{Z}\mathscr{C} = \mathcal{Z}_l\mathscr{C} = [\Sigma\mathbb{Z}, \mathscr{C}]$$

(where $\Sigma\mathbb{Z}$ is the additive group of the integers as a one-object groupoid).

In Section 3.5 we study the autonomous case. Here we consider $\mathscr{C}$ to be a closed monoidal $\mathscr{V}$-category with tensor product $\boxtimes$ and unit U.

Theorem [3.5.2]. $(\mathscr{V} = \mathbf{Vect}_k)$ *Suppose $\mathscr{C}$ is a promonoidal k-linear category with finite-dimensional homs. Let $\mathscr{F} = [\mathscr{C}, \mathscr{V}]$ have the convolution monoidal structure. Then*

$$\mathcal{Z}\mathscr{F} = \mathcal{Z}_l\mathscr{F} \cong \mathscr{F}^M \simeq [\mathscr{C}_M, \mathscr{V}]$$

where $\mathscr{C}_M$ is the Kleisli category for the promonad M on $\mathscr{C}$.

In Section 3.6 we study monoids in the lax centre of a monoidal $\mathscr{V}$-category $\mathscr{C}$. A monoid (A, u) in the lax centre $\mathcal{Z}_l\mathscr{C}$ determines a canonical enrichment of the $\mathscr{V}$-functor

$$- \boxtimes A : \mathscr{C} \longrightarrow \mathscr{C}$$

to a monoidal functor:

$$X \boxtimes A \boxtimes Y \boxtimes A \xrightarrow{1 \boxtimes u_Y \boxtimes 1} X \boxtimes Y \boxtimes A \boxtimes A \xrightarrow{1 \boxtimes 1 \boxtimes \mu} X \boxtimes Y \boxtimes A$$

$$U \xrightarrow{\eta} A \cong U \boxtimes A.$$

Chapter 4 consists of the paper [PS2] as modified in the light of [PS3]. This studies the finite coproduct and pullback preserving functors between categories of permutation representations of finite groups and gives a categorical explanation of the work of Serge Bouc [Bo1]. A *permutation representation* of a finite group G or a *finite left G-set* is a finite set X together with a function (called action) $G \times X \longrightarrow X$, $(g, x) \longmapsto gx$ such that $1x = x$ and $g_1(g_2x) = (g_1g_2)x$ for $g_1, g_2 \in G$ and $x \in X$. We write G-$\mathbf{set}_{fin}$ for the category of finite left G-sets (that is, of permutation representations of G) with left G-morphisms where a left G-morphism $f : X \longrightarrow Y$ is a function satisfying $f(gx) = gf(x)$. Let M be a Mackey functor on a finite group H. Then $M : \mathbf{Spn}(H\text{-}\mathbf{set}_{fin}) \longrightarrow \mathbf{Mod}_k$ is a coproduct preserving functor. If $F : G\text{-}\mathbf{set}_{fin} \longrightarrow H\text{-}\mathbf{set}_{fin}$ is a pullback and finite coproduct preserving functor (where G is finite) then we get a functor

$$M \circ \mathbf{Spn}(F) : \mathbf{Spn}(G\text{-}\mathbf{set}_{fin}) \longrightarrow \mathbf{Mod}_k$$

which is a Mackey functor on G.

Bouc [Bo2] studied the pullback and finite coproduct preserving functors $F : G\text{-}\mathbf{set}_{fin} \longrightarrow H\text{-}\mathbf{set}_{fin}$ interms of $(G^{\mathrm{op}} \times H)$-sets A (where G^{op}) is G with opposite multiplication). The category $(G^{\mathrm{op}} \times H)$-set of such A is equivalent to the category of finite colimit preserving functors $L : G\text{-}\mathbf{set}_{fin} \longrightarrow H\text{-}\mathbf{set}_{fin}$. In this chapter we explained these two constructions.

Let A be a $(G^{\mathrm{op}} \times H)$-set. For all $(K^{\mathrm{op}} \times G)$-sets B, where K, G, H are all finite groups, Bouc ([Bo1]) defines the $(K^{\mathrm{op}} \times H)$-set

$$A \circ_G B = (A \wedge_G B)/G.$$

Here $A \wedge_G B$ is a $(K^{\mathrm{op}} \times G \times H)$-set given by

$$A \wedge_G B = \{(a,b) \in A \times B \mid g \in G, ag = a \Rightarrow \text{there exists } k \in K \text{ with } gb = bk\}.$$

This paper provides a categorical explanation for the following Theorem of Bouc.

Theorem [4.1.1]. *([Bo5]) Suppose K, G and H are finite groups.*

(i) *If A is a finite $(G^{\mathrm{op}} \times H)$-set then the functor*

$$A \circ_G - : G\text{-}\mathbf{set}_{fin} \longrightarrow H\text{-}\mathbf{set}_{fin}$$

preserves finite coproducts and pullbacks.

(ii) *Every functor $F : G\text{-}\mathbf{set}_{fin} \longrightarrow H\text{-}\mathbf{set}_{fin}$ which preserves finite coproducts and pullbacks is isomorphic to one of the form $A \circ_G -$.*

(iii) *The functor F in (ii) preserves terminal objects if and only if A is transitive (connected) as a right G-$\mathbf{set}_{fin}$.*

(iv) *If A is as in (i) and B is a finite $(K^{\mathrm{op}} \times G)$-set then the composite functor*

$$K\text{-}\mathbf{set}_{fin} \xrightarrow{B \circ_K -} G\text{-}\mathbf{set}_{fin} \xrightarrow{A \circ_G -} H\text{-}\mathbf{set}_{fin}$$

is isomorphic to $(A \circ_G B) \circ_K -$.

We explain the main constructions and theorems of the sections of this chapter. Section 4.1 is the introduction of this paper. In Section 4.2 we provide a direct proof of the well-known representability theorem for the case where "small" means "finite".

Theorem [4.2.1]. *(Special representability theorem) Suppose $\mathscr{A}$ is a category with the following properties:*

(i) *each homset $\mathscr{A}(A, B)$ is finite;*

(ii) *finite limits exist;*

(iii) *there is a cogenerator Q;*

(iv) *$\mathscr{A}$ is finitely well powered.*

Then every finite limit preserving functor $T : \mathscr{A} \longrightarrow \mathbf{set}_{fin}$ is representable.

In Section 4.3 we study a category $\mathscr{A}$ with finite coproducts. An object C of $\mathscr{A}$ is called *connected* when the functor $\mathscr{A}(C,-) : \mathscr{A} \longrightarrow \mathbf{Set}$ preserves finite coproducts. We write $\mathbf{Conn}(\mathscr{A})$ for the category of connected objects of $\mathscr{A}$ and $\mathbf{Cop}(\mathscr{A}, \mathscr{X})$ for the category of finite coproduct preserving functors from $\mathscr{A}$ to $\mathscr{X}$. For any small category $\mathscr{C}$, we write $\mathbf{Fam}(\mathscr{C}^{\mathrm{op}})$ for the free finite coproduct completion of $\mathscr{C}^{\mathrm{op}}$. The objects are families $(C_i)_{i\in I}$ where C_i are objects of $\mathscr{C}$ and I is finite. A morphism $(\xi, f) : (C_i)_{i\in I} \longrightarrow (D_j)_{j\in J}$ consists of a function $\xi : I \longrightarrow J$ and a family $f = (f_i)_{i\in I}$ of morphisms $f_i : D_{\xi(i)} \longrightarrow C_i$ in $\mathscr{C}$. In Proposition 4.3.3 we show that the following is an equivalence of categories

$$\mathbf{Fam}(\mathbf{Conn}(\mathscr{A})^{\mathrm{op}}) \simeq \mathbf{CopPb}(\mathscr{A}, \mathbf{set}_{fin})$$

where the category $\mathscr{A}$ has finite coproducts and the properties of Theorem 4.2.1 and $\mathbf{CopPb}(\mathscr{A}, \mathscr{B})$ is the category of finite coproduct and pullback preserving functors from $\mathscr{A}$ to $\mathscr{B}$. In Section 4.4 we study the application to permutation representations. Let $N : \mathscr{C}_G \longrightarrow G\text{-}\mathbf{set}_{fin}$ denote the inclusion functor and define the functor

$$\tilde{N} : G\text{-}\mathbf{set}_{fin} \longrightarrow [\mathscr{C}_G^{\mathrm{op}}, \mathbf{set}_{fin}]$$

by $\tilde{N}X = G\text{-}\mathbf{set}_{fin}(N-, X)$. In Proposition 4.4.4 we show that the functor $\tilde{N}$ induces an equivalence of categories

$$G\text{-}\mathbf{set}_{fin} \simeq \mathbf{Fam}(\mathscr{C}_G).$$

In Section 4.5 we study a factorization for G-morphisms. We use these morphisms of a factorization system on G-set to describe the finite coproduct completion $\mathbf{Fam}(\mathscr{C}_G^{\mathrm{op}})$ of the dual of the category of connected G-sets. For any G-set

X, the set $X/G = \{C \subseteq X : C \text{ is an orbit of} X\}$ is a connected sub-G-set of X. A G-morphism $f : X \longrightarrow Y$ is said to be *slash inverted* when the direct image function $f/G : X/G \longrightarrow Y/G$ of f is a bijection. A G-morphism $f : X \longrightarrow Y$ is said to be *orbit injective* when $orb(x_1) = orb(x_2)$ and $f(x_1) = f(x_2)$ imply $x_1 = x_2$. In Proposition 5.2 we prove that the slash inverted and orbit injective G-morphisms form a factorization system (in the sense of [FK]) on the category of G-sets.

In Section 4.6 we introduce a new category $\mathscr{B}_G$ of G-sets. The objects of $\mathscr{B}_G$ are all the finite G-sets and morphisms are the isomorphisms classes of the span $(u, S, v) : X \longrightarrow Y$ in which $u : S \longrightarrow X$ is slash inverted and $v : S \longrightarrow Y$ is orbit injective. In Proposition 4.6.1 we prove that the subcategory $\mathscr{B}_G$ of $\mathbf{Spn}(G\text{-}\mathbf{set}_{fin})$ is closed under finite coproducts. We obtain a finite coproduct preserving functor $\Sigma : \mathbf{Fam}(\mathscr{C}_G^{\mathrm{op}}) \longrightarrow \mathscr{B}_G$.

Theorem [4.6.3]. The functor $\Sigma : \mathbf{Fam}(\mathscr{C}_G^{\mathrm{op}}) \longrightarrow \mathscr{B}_G$ is an equivalence of categories.

In Corollary 4.6.4 we obtain the following equivalence of categories:

$$\mathscr{B}_G \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{fin}, \mathbf{set}_{fin}).$$

Then we obtain the following corollary:

Corollary [4.6.6]. There is an equivalence

$$\mathscr{B}_{G^{\mathrm{op}}} \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{fin}, \mathbf{set}_{fin}), \quad A \longmapsto A \circ_G -.$$

In Section 4.7 we construct a bicategory **Bouc** of finite groups. We define the category $\mathbf{Bouc}(G, H)$ as the pullback of the inclusion of $\mathscr{B}_{G^{\mathrm{op}}}$ in $\mathbf{Spn}(G^{\mathrm{op}}\text{-}\mathbf{set}_{fin})$ along the forgetful functor $\mathbf{Spn}(G^{\mathrm{op}} \times H\text{-}\mathbf{set}_{fin}) \longrightarrow \mathbf{Spn}(G^{\mathrm{op}}\text{-}\mathbf{set}_{fin})$. That is, $\mathbf{Bouc}(G, H)$ is the subcategory of $\mathbf{Spn}(G^{\mathrm{op}} \times H\text{-}\mathbf{set}_{fin})$ consisting of all the objects yet, as morphisms, only the isomorphism classes of spans (u, S, v) in $G^{\mathrm{op}} \times H\text{-}\mathbf{set}_{fin}$ for which u is slash inverted and v is orbit injective as G-morphisms.

Theorem [4.7.1]. There is an equivalence of categories

$$\mathbf{Bouc}(G,H) \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{fin}, H\text{-}\mathbf{set}_{fin}), \quad A \longmapsto A \circ_G -.$$

In Section 4.8 we study the applications to Mackey functors. If the functor $F : G\text{-}\mathbf{set}_{fin} \longrightarrow H\text{-}\mathbf{set}_{fin}$ preserves pullbacks then this induces a functor $\mathbf{Spn}(F) : \mathbf{Spn}(G\text{-}\mathbf{set}_{fin}) \longrightarrow \mathbf{Spn}(H\text{-}\mathbf{set}_{fin})$ (since composition of spans only involves pullbacks). If F also preserves finite coproducts then $\mathbf{Spn}(F)$ preserves direct sums. Then we can obtain an exact functor

$$\bar{F} : \mathbf{Mky}_{fin}(H) \longrightarrow \mathbf{Mky}_{fin}(G)$$

defined by $\bar{F}(N) = N \circ \mathbf{Spn}(F)$ for all $N \in \mathbf{Mky}_{fin}(H)$, where $\mathbf{Mky}_{fin}$ is the category of finite-dimensional Mackey functors. The functor $\bar{F}$ has a left adjoint

$$\mathbf{Mky}_{fin}(F) : \mathbf{Mky}_{fin}(G) \longrightarrow \mathbf{Mky}_{fin}(H).$$

Let $\mathbf{AbCat}_k$ denote the 2-category of abelian k-linear categories, k-linear functors with right exact right adjoints, and natural transformations. In Corollary 4.8.1 we obtain a homomorphism of bicategories

$$\mathbf{Mky}_{fin} : \ \mathbf{Bouc} \longrightarrow \mathbf{AbCat}_k$$

given by $(A : G \longrightarrow H) \longmapsto (\mathbf{Mky}_{fin}(A \circ_G -) : \mathbf{Mky}_{fin}(G) \longrightarrow \mathbf{Mky}_{fin}(H))$.

This concludes the thesis.