Categories of Mackey Functors

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$$\begin{tikzcd}
M(U) \arrow[rr, "M^*(p_1 s_1)"] \arrow[dr, "M^*(s_1)"'] & & M(P) \arrow[rr, "M_*(t_2 p_2)"] \arrow[dr, "M_*(p_2)"] & & M(W) \\
& M(S) \arrow[ur, "M^*(p_1)"] \arrow[dr, "M_*(s_2)"'] & \mathit{Mackey} & M(T) \arrow[ur, "M_*(t_2)"'] & \\
& & M(V) \arrow[ur, "M^*(t_1)"'] & &
\end{tikzcd}$$

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This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text. This work has not been submitted for a higher degree to any other university or institution.

Elango Panchadcharam

In memory of my Father, T. Panchadcharam
1939 - 1991.

Summary

The thesis studies the theory of Mackey functors as an application of enriched category theory and highlights the notions of lax braiding and lax centre for monoidal categories and more generally for promonoidal categories.

The notion of Mackey functor was first defined by Dress [Dr1] and Green [Gr] in the early 1970's as a tool for studying representations of finite groups. The first contribution of this thesis is the study of Mackey functors on a compact closed category $\mathscr{T}$. We define the Mackey functors on a compact closed category $\mathscr{T}$ and investigate the properties of the category **Mky** of Mackey functors on $\mathscr{T}$. The category **Mky** is a monoidal category and the monoids are Green functors. The category of finite-dimensional Mackey functors $\mathbf{Mky}_{fin}$ is a star-autonomous category. The category $\mathbf{Rep}(G)$ of representations of a finite group G is a full sub-category of $\mathbf{Mky}_{fin}$.

The second contribution of this thesis is the study of lax braiding and lax centre for monoidal categories and more generally for promonoidal categories. The centre of a monoidal category was introduced in [JS1]. The centre of a monoidal category is a braided monoidal category. Lax centres become lax braided monoidal categories. Generally the centre is a full subcategory of the lax centre. However in some cases the two coincide. We study the cases where the lax centre and centre becomes equal. One reason for being interested in the lax centre of a monoidal category is that, if an object of the monoidal category is equipped with the structure of monoid in the lax centre, then tensoring with the object defines a monoidal endofunctor on the monoidal category.

The third contribution of this thesis is the study of functors between categories of permutation representations. Functors which preserve finite coproduct and pullback between the category G-$\mathbf{set}_{fin}$ of finite G-sets to the category H-$\mathbf{set}_{fin}$ of finite H-sets (where G and H are finite groups) give a Mackey functor from G-$\mathbf{set}_{fin}$ to H-$\mathbf{set}_{fin}$ for each Mackey functor on H.

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Chapter 0

Introduction

Groups are used to mathematically understand symmetry in nature and in mathematics itself. Classically, groups were studied either directly or via their representations. In the last 40 years, groups have also been studied using Mackey functors, a concept which arose out of a formalization of representation theory.

Mackey functors were first introduced by J. A. Green [Gr] and A. Dress [Dr1], [Dr2] in the early 1970's as a tool for studying representations of finite groups and their subgroups. The axioms for Mackey functors follow on from earlier ideas of Lam on Frobenius functors [La1] which are described in [CR]. Another structure which appeared early on is Bredon's notion of a coefficient system [Br]. There are (at least) three equivalent definitions of Mackey functor for a finite group G.

The most elementary (in a sense used by some group theorists) definition is due to Green [Gr]. The most complicated axiom in this definition is based on the Mackey Decomposition Theorem (see [Ja, p.300] for example) in representation theory and this is presumably why Mackey's name is attached to the concept. We shall now provide the categorical explanation of this Theorem which is used to characterize when induced characters are irreducible.

The Theorem provides a formula for the restriction, to a subgroup, of a group representation induced from a possibly different subgroup. Restriction is composition with a functor (the inclusion of the subgroup in the group) and inducing is an adjoint process amounting therefore to Kan extension.

Consider a functor $i : H \longrightarrow G$ between small categories H and G, and let $\mathscr{M}$ denote a cocomplete category. We write $[H, \mathscr{M}]$ for the category of functors from H to $\mathscr{M}$ and natural transformations between them. A functor $Res_i : [G, \mathscr{M}] \longrightarrow [H, \mathscr{M}]$ is defined by composition on the right with the functor i. This functor Res_i has a left adjoint $Lan_i : [H, \mathscr{M}] \longrightarrow [G, \mathscr{M}]$ for which there are two (closely related) formulas:

$$Lan_i(W)(c) = \int^{b \in H} G(i(b), c).i(b)$$

and

$$Lan_i(W)(c) = \mathrm{colim}(\ i \downarrow c \longrightarrow H \xrightarrow{W} \mathscr{M}\ )$$

where $S.M$ is the coproduct of S copies of M in $\mathscr{M}$ (for any set S) and $i \downarrow c$ is the comma category; see [Ma] for example.

For any functor $j : K \longrightarrow G$, the comma category $i \downarrow j$ is universal with respect to its being equipped with functors $p : i \downarrow j \longrightarrow H$ and $q : i \downarrow j \longrightarrow K$, and a natural transformation

$$\begin{tikzcd} i \downarrow j \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\lambda}{\Longrightarrow}"] & K \arrow[d, "j"] \\ H \arrow[r, "i"'] & G. \end{tikzcd}$$

The following observation is the basis of the 2-categorical notion of "pointwise left extension" defined in [St2, pp.127-128].

Proposition 0.0.1. *The natural transformation λ induces a canonical natural isomorphism*

$$Res_j \circ Lan_i \cong Lan_q \circ Res_p.$$

The following result is an easy exercise in the defining adjoint property of left Kan extension.

Proposition 0.0.2. *Suppose D is a small category which is the disjoint union of subcategories D_α, $\alpha \in \Lambda$, with inclusion functors $m_\alpha : D_\alpha \longrightarrow D$. For any functor $r : D \longrightarrow K$, there is a canonical natural isomorphism*

$$Lan_r \cong \sum_{\alpha\in\Lambda} Lan_{r\circ m_\alpha} \circ Res_{m_\alpha}.$$

A groupoid is a category in which each morphism is invertible. For each object d of a groupoid D, we obtain a group $D(d) = D(d,d)$ whose elements are morphisms $u : d \longrightarrow d$ in D and whose multiplication is composition. We regard groups as one-object groupoids. Let Λ be a set of representative objects in D for all the isomorphism classes of objects in D. Then there is an equivalence of categories

$$\sum_{d\in\Lambda} D(d) \simeq D;$$

that is, every groupoid is equivalent to a disjoint union of groups.

Now suppose H and K are subgroups of a group G. To apply the above considerations, let $i : H \longrightarrow G$ and $j : K \longrightarrow G$ be the inclusions. The comma category $i \downarrow j$ is actually a groupoid: the objects are elements $g \in G$, the morphisms $(h,k) : g \longrightarrow g'$ are elements of $H \times K$ such that $kg = g'h$, and composition is $(h',k') \circ (h,k) = (h'h, k'k)$. Another name for $i \downarrow j$ might be $K \mathbin{\backslash\!\backslash} G \mathbin{/\!/} H$ since the set of isomorphism classes of objects is isomorphic to the set

$$K\backslash G/H = \{KgH \mid g \in G\}$$

of double cosets $KgH = \{kgh \mid k \in K, h \in H\}$. For each object g of $i \downarrow j$, the projection functor $p : i \downarrow j \longrightarrow H$ induces a group isomorphism

$$(i \downarrow j)(g) \cong H \cap K^g$$

where $K^g = g^{-1}Hg$, and the projection functor $q : i\downarrow j \longrightarrow K$ induces a group isomorphism

$$(i\downarrow j)(g) \cong {}^gH \cap K$$

where ${}^gH = gHg^{-1}$. Thus we can identify $(i\downarrow j)(g)$ with both a subgroup $H\cap K^g$ of H and a subgroup ${}^gH\cap K$ of K. Define p_g, q_g, γ_g by the commutative diagram

$$\begin{tikzcd} {}^gH\cap K \arrow[rr, "\gamma_g"] \arrow[dr, "\cong"'] \arrow[dd, "\leqslant"'] & & H\cap K^g \arrow[dl, "\cong"] \arrow[dd, "\leqslant"] \\ & (i\downarrow j)(g) \arrow[dl, "q_g"] \arrow[dr, "p_g"'] & \\ K & & H. \end{tikzcd}$$

Let $[K\setminus G/H] \subseteq G$ represent all double cosets in the form KgH, where $g \in [K\setminus G/H]$, without repetition. Therefore we have an equivalence of categories

$$\sum_{g\in[K\setminus G/H]} (i\downarrow j)(g) \simeq i\downarrow j.$$

Corollary 0.0.3.

$$Res_j \circ Lan_i \cong \sum_{g\in[K\setminus G/H]} Lan_{q_g} \circ Res_{p_g}.$$

Proof. Take $r = q$ and $\Lambda = [K\setminus G/H]$ in Proposition 0.0.2 and substitute the resultant formula in Proposition 0.0.1. □

To apply this to the theory of linear representations of groups, we put $\mathscr{M} =$ $\mathbf{Mod}_k$ for a commutative ring k. Then Lan_i and Res_i are denoted by Ind_H^G and Res_H^G, and we have the

Mackey Decomposition Theorem. *For subgroups H and K of a group G, there is a canonical natural isomorphism*

$$Res_K^G \circ Ind_H^G \cong \sum_{g\in[K\setminus G/H]} Ind^K_{{}^gH\cap K} \circ Res_{\gamma_g} \circ Res^H_{H\cap K^g}.$$

We now state Green's definition. A *Mackey functor* M for a group G over the commutative ring k consists of

- a function assigning to each subgroup $H \leqslant G$ a k-module $M(H)$,
- for all subgroups $K \leqslant H \leqslant G$, module morphisms

$$t_K^H : M(K) \longrightarrow M(H), \text{ and } r_K^H : M(H) \longrightarrow M(K),$$

- for all subgroups $H \leqslant G$ and $g \in G$, a module isomorphism

$$c_{g,H} : M(H) \longrightarrow M({}^g H),$$

subject to the following axioms:

1. if $L \leqslant K \leqslant H$ then $t_K^H t_L^K = t_L^H$ and $r_L^K r_K^H = r_L^H$,
2. if $H \leqslant G$, $g_1, g_2 \in G$ and $h \in H$ then

$$c_{g_2, {}^{g_1}H} c_{g_1,H} = c_{g_2 g_1,H} \quad \text{and} \quad c_{h,H} = 1_{M(H)},$$

3. if $K \leqslant H \leqslant G$ and $g \in G$ then

$$c_{g,H} t_K^H = t_{{}^gK}^{{}^gH} c_{g,K} \quad \text{and} \quad c_{g,K} r_K^H = r_{{}^gK}^{{}^gH} c_{g,H},$$

4. if $H \leqslant L$ and $K \leqslant L \leqslant G$ then

$$r_K^L t_H^L = \sum_{g \in [K \backslash L / H]} t_{{}^gH \cap K}^K c_{g, H \cap K^g} r_{H \cap K^g}^H.$$

The morphism t_K^H is called *transfer*, *trace*, or *induction*. The morphism r_K^H is called *restriction*. The isomorphism $c_{g,H}$ is called a *conjugation map*. With this terminology, the relation between axiom (4) and the Mackey Decomposition Theorem should be striking, however, we shall explain it further below.

A *morphism* $\theta : M \longrightarrow N$ of Mackey functors is a family of k-module morphisms $\theta_H : M(H) \longrightarrow N(H), H \leqslant G$, satisfying the obvious commutativity conditions with the morphisms t, r and c.

We shall eventually see that a Mackey functor is actually a functor between two categories. In the first instance, we shall see that it is actually a pair of functors agreeing on objects.

For any group G, there is a category $\mathscr{C}_G$ of *connected G-sets*. It is the full subcategory of the category $[G, \mathbf{Set}] = G\text{-}\mathbf{Set}$ of left G-sets consisting of those which are transitive (and non-empty). Every transitive G-set X is isomorphic to the set G/H of cosets of some subgroup $H \leqslant G$ with the obvious action. Using this, we see that there is an equivalence

$$\mathscr{S}(G) \simeq \mathscr{C}_G\,, \quad H \longmapsto G/H,$$

where $\mathscr{S}(G)$ is a category defined by Green [Gr] whose objects are subgroups of G. A *morphism* $g : H \longrightarrow K$ in $\mathscr{S}(G)$ is an element $g \in G$ such that $H^g \leqslant K$; composition $g_2 \circ g_1$ is product $g_1 g_2$ in G in reverse order. Each $g : H \longrightarrow K$ determines a G-set morphism $G/H \longrightarrow G/K$ taking xH to xgK.

Each Mackey functor M on G over k determines two functors

$$M^* : \mathscr{S}(G)^{\mathrm{op}} \longrightarrow \mathbf{Mod}_k \quad \text{and} \quad M_* : \mathscr{S}(G) \longrightarrow \mathbf{Mod}_k$$

with $M^*(H) = M_*(H) = M(H)$. For each $g : H \longrightarrow K$ in $\mathscr{S}(G)$, we define $M^*(g)$ and $M_*(g)$ by the commutative diagrams below.

$$\begin{tikzcd}
M(K) \arrow[r, "r^K_{H^g}"] \arrow[d, "c_{g,K}"'] \arrow[dr, "M^*(g)"] & M(H^g) \arrow[d, "c_{g,H^g}"] \\
M({}^gK) \arrow[r, "r^{{}^gK}_H"'] & M(H)
\end{tikzcd}
\qquad
\begin{tikzcd}
M(H) \arrow[r, "t^{{}^gK}_H"] \arrow[d, "c_{H,g^{-1}}"'] \arrow[dr, "M_*(g)"] & M({}^gK) \arrow[d, "c_{g^{-1},{}^gK}"] \\
M(H^g) \arrow[r, "t^K_{H^g}"'] & M(K)
\end{tikzcd}$$

We shall provide an example of a Mackey functor where the Mackey axiom comes from the Decomposition Theorem.

Each G-set X determines a groupoid $el(X)$ whose objects are the elements $x \in X$ and whose morphisms $g : x \longrightarrow y$ are elements $g \in G$ such that $gx = y$.

In the case of a transitive G-set G/H where $H \leqslant G$, there is an equivalence of groupoids

$$H \simeq el(G/H)\ , \quad (h : a \longrightarrow a) \longmapsto (h : H \longrightarrow H).$$

It follows that we have an equivalence of categories

$$[el(G/H), \mathbf{Mod}_k] \simeq [H, \mathbf{Mod}_k]$$

where the right-hand side is the category of k-linear representations of the group H.

Let $\mathbf{Cat}_{\oplus}$ denote the 2-category whose objects are additive categories with finite direct sums, whose morphisms are additive functors, and whose 2-cells are natural isomorphisms. Let **Gpd** denote the 2-category of small groupoids. We have two 2-functors

$$\mathbf{Rep}^* : \mathbf{Gpd}^{\mathrm{op}} \longrightarrow \mathbf{Cat}_{\oplus} \quad \text{and} \quad \mathbf{Rep}_* : \mathbf{Gpd} \longrightarrow \mathbf{Cat}_{\oplus}$$

defined on objects $D \in \mathbf{Gpd}$ by

$$\mathbf{Rep}^*(D) = \mathbf{Rep}_*(D) = [D, \mathbf{Mod}_k].$$

For $f : D \longrightarrow E$ in **Gpd**, we define

$$\mathbf{Rep}^*(f) = Res_f \quad \text{and} \quad \mathbf{Rep}_*(f) = Lan_f.$$

There is also a 2-functor $K_o : \mathbf{Cat}_{\oplus} \longrightarrow \mathbf{AbGp}$, where $\mathbf{AbGp} = \mathbf{Mod}_{\mathbf{Z}}$ is the category of abelian groups, which assigns to each additive category $\mathscr{A}$ with finite direct sums, the abelian group $K_o\mathscr{A}$ obtained from the free abelian group on the set of isomorphism classes $[A]$ of objects A of $\mathscr{A}$ by imposing the relations

$$[A \oplus B] = [A] + [B];$$

this is called the *Grothendieck group* of $\mathscr{A}$.

An example of a Mackey functor M on G over $\mathbf{Z}$ is obtained by taking M^* to be the composite functor

$$\mathscr{S}(G)^{\mathrm{op}} \xrightarrow{el} \mathbf{Gpd}^{\mathrm{op}} \xrightarrow{\mathbf{Rep}^*} \mathbf{Cat}_{\oplus} \xrightarrow{K_o} \mathbf{AbGp}$$

and taking M_* to be the composite functor

$$\mathscr{S}(G) \xrightarrow{el} \mathbf{Gpd} \xrightarrow{\mathbf{Rep}_*} \mathbf{Cat}_{\oplus} \xrightarrow{K_o} \mathbf{AbGp}$$

so that $M(H) \cong K_o[H, \mathbf{Mod}_k]$. Mackey Decomposition gives the Mackey axiom (4).

Suppose G is finite. We obtain a sub-example of this last example by replacing $\mathbf{Mod}_k$ by the category $\mathbf{mod}_k$ of finitely generated projective k-modules. If k is a field of characteristic zero then $K_o[H, \mathbf{mod}_k]$ is isomorphic to the group of characters of k-linear representations of H.

We now resume our general discussion. A *Green functor* A for G over k is a Mackey functor A for G over k equipped with a k-algebra structure on each k-module $A(H)$ (associative with unit), for $H \leqslant G$, subject to the axioms:

1. the k-module morphisms t_K^H, r_K^H and $c_{g,K}$ for A preserve the algebra multiplication and unit, and

2. if $K \leqslant H \leqslant G,\ a \in A(H)$, and $b \in A(K)$ then

$$a.t_K^H(b) = t_K^H(r_K^H(a).b) \quad \text{and} \quad t_K^H(b).a = t_K^H(b.r_K^H(a)).$$

Axiom (2) is called the *Frobenius condition* since it resembles the following structural version of Frobenius Reciprocity (see [Ja, Theorem 5.17(3), p.292] for example).

Frobenius Reciprocity. *If V is a k-linear representation of a group G and W is a k-linear representation of a subgroup $H \leqslant G$ then*

$$V \otimes Ind_H^G(W) \cong Ind_H^G(Res_H^G(V) \otimes W).$$

A categorical explanation of this reciprocity is as follows.

Proposition 0.0.4. *Suppose $\mathcal{M}$ is a cocomplete monoidal category whose tensor product preserves colimits in each variable. Suppose $i : H \longrightarrow G$ is a functor between small categories. For functors $V : G \longrightarrow \mathcal{M}$ and $W : H \longrightarrow \mathcal{M}$, the left Kan extension of the functor*

$$V \circledast W : G \times H \xrightarrow{V \times W} \mathcal{M} \times \mathcal{M} \xrightarrow{\otimes} \mathcal{M}$$

along $1_G \times i : G \times H \longrightarrow G \times G$ is naturally isomorphic to

$$V \circledast Lan_i(W) : G \times G \xrightarrow{V \times Lan_i(W)} \mathcal{M} \times \mathcal{M} \xrightarrow{\otimes} \mathcal{M}.$$

Proof.

$$\begin{aligned}
Lan_{1_G \times i}(V \circledast W)(c_1, c_2) &\cong \int^{c,a} (G \times G)((c, i(a)), (c_1, c_2)).V(c) \otimes W(a) \\
&\cong \int^{c,a} (G(c, c_1).V(c)) \otimes (G(i(a), c_2).W(a)) \\
&\cong \int^{c} G(c, c_1).V(c) \otimes \int^{a} G(i(a), c_2).W(a) \\
&\cong V(c_1) \otimes Lan_i(W)(c_2) \\
&\cong (V \circledast Lan_i(W))(c_1, c_2).
\end{aligned}$$

□

Proposition 0.0.5. *Let H be a subgroup of a group G, let $i : H \longrightarrow G$ be the inclusion, and let $\Delta : G \longrightarrow G \times G$ be the diagonal. Then the comma groupoid $(1_G \times i) \downarrow \Delta$ is connected and there is an equivalence*

$$H \simeq (1_G \times i) \downarrow \Delta.$$

Proof. Objects of $(1_G \times i) \downarrow \Delta$ are elements $(g_1, g_2) \in G \times G$. A morphism $(g, h, x) : (g_1, g_2) \longrightarrow (g'_1, g'_2)$ is an element of $G \times H \times G$ such that $g'_1 g = x g_1$ and $g'_2 h = x g_2$;

so, for any object (g_1, g_2), we have the morphism $(g_1^{-1}g_2, 1, g_2) : (1,1) \longrightarrow (g_1, g_2)$ proving the comma groupoid connected. The equivalence follows from the group isomorphism

$$H \cong ((1_G \times i) \downarrow \Delta)(1,1), \quad h \longleftrightarrow (h,h,h).$$

□

Proposition 0.0.6. *Suppose $\mathscr{M}$ is as in Proposition 0.0.4 and H is a subgroup of a group G with inclusion $i : H \longrightarrow G$. The categories $[H,\mathscr{M}]$ and $[G,\mathscr{M}]$ are equipped with the pointwise tensor products. For $V \in [G,\mathscr{M}]$ and $W \in [H,\mathscr{M}]$, there is a canonical isomorphism*

$$V \otimes Lan_i(W) \cong Lan_i(Res_i(V) \otimes W).$$

Proof. Contemplate the diagram

$$\begin{tikzcd} H \arrow[r, "i"] \arrow[d, "{(i,1_H)}"'] & G \arrow[d, "\Delta"] \\ G \times H \arrow[r, "1_G \times i"] \arrow[dr, "V \circledast W"'] & G \times G \arrow[d, phantom, "\Longrightarrow"] \arrow[dl, "V \circledast Lan_i(W)"] \\ & \mathscr{M} \end{tikzcd}$$

in the light of Propositions 0.0.1, 0.0.4 and 0.0.5. □

If A is a monoidal additive category with direct sums, $K_o\mathscr{A}$ becomes a ring via

$$[A][B] = [A \otimes B].$$

It follows that the example of the Mackey functor M with $M(H) \cong K_o[H, \mathbf{Mod}_k]$ is actually a Green functor.

Notice that the functor $el : \mathscr{S}(G) \longrightarrow \mathbf{Gpd}$ is the restriction of the coproduct preserving functor $el : [G, \mathbf{Set}] \longrightarrow \mathbf{Gpd}$. This motivates the second definition of Mackey functor (see [Dr1] and [Di]).

We centre attention on the case of a finite group G. We write $\mathbf{set}_{\mathit{fin}}$ for the category of finite sets and G-$\mathbf{set}_{\mathit{fin}} = [G, \mathbf{set}_{\mathit{fin}}]$ for the category of finite G-sets. Every finite G-set is a coproduct (disjoint union) of transitive G-sets. With a little more work we see that G-$\mathbf{set}_{\mathit{fin}}$ is the completion of $\mathscr{C}_G$ under finite coproducts. Therefore, the functors M^* and M_* above extend (uniquely up to isomorphism) to functors

$$M^* : (G\text{-}\mathbf{set}_{\mathit{fin}})^{\mathrm{op}} \longrightarrow \mathbf{Mod}_k \quad \text{and} \quad M_* : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow \mathbf{Mod}_k$$

which respectively preserve finite products and finite coproducts. So here is the second equivalent definition.

A *Mackey functor* M for G over k consists of a pair of functors

$$M^* : (G\text{-}\mathbf{set}_{\mathit{fin}})^{\mathrm{op}} \longrightarrow \mathbf{Mod}_k\ , \quad M_* : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow \mathbf{Mod}_k$$

which agree $M^*(X) = M_*(X) = M(X)$ on objects X of G-$\mathbf{set}_{\mathit{fin}}$ subject to the following axioms:

1. for every pullback diagram

$$\begin{tikzcd} P \arrow[r, "\delta"] \arrow[d, "\gamma"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$

   in G-$\mathbf{set}_{\mathit{fin}}$, the equation

$$M^*(\beta)M_*(\alpha) = M_*(\delta)M^*(\gamma)$$

   holds,

2. for inclusions $i : X \longrightarrow X + Y$ and $j : Y \longrightarrow X + Y$ into the coproduct $X + Y$ of X and Y in G-$\mathbf{set}_{\mathit{fin}}$, the diagram

$$\begin{tikzcd} M(X) \arrow[r, shift right, "M_*(i)"'] & M(X+Y) \arrow[l, shift right, "M^*(i)"'] \arrow[r, shift left, "M^*(j)"] & M(Y) \arrow[l, shift left, "M_*(j)"] \end{tikzcd}$$

is a direct sum situation in $\mathbf{Mod}_k$.

The axiom (1) is now called the *Mackey condition* as it is a more categorically pleasing expression of the previous axiom (4). A *morphism* $\theta : M \longrightarrow N$ of Mackey functors is a family of k-module morphisms

$$\theta_X : M(X) \longrightarrow N(X), \quad X \in G\text{-}\mathbf{set}_{fin},$$

which is natural as both $M^* \longrightarrow N^*$ and $M_* \longrightarrow N_*$. A *Green functor* A for G over k is a Mackey functor for G over k equipped with k-bilinear morphisms

$$A(X) \times A(Y) \longrightarrow A(X \times Y), \quad (a,b) \longmapsto ab$$

which are natural in X and $Y \in G\text{-}\mathbf{set}_{fin}$, and are associative and unital in an obvious way.

A third equivalent definition of Mackey functor appears in [TW] and involves first creating the Mackey algebra $\mu_k(G)$ for the finite group G. The study of Mackey functors becomes the representation theory of this algebra. The Mackey algebra $\mu_k(G)$ of G over k is the associative k-algebra defined by the generators t_K^H, r_K^H and $c_{g,H}$ for subgroups $K \leqslant H$ of G and $g \in G$, satisfying the following relations:

1. if $L \leqslant K \leqslant H$ are subgroups of G, then $t_K^H t_L^K = t_L^H$ and $r_L^K r_K^H = r_L^H$, and if $g, h \in H$ and H is a subgroup of G, then $c_{h,{}^gH} c_{g,H} = c_{hg,H}$,

2. if $g \in G$ and $K \leqslant H$ are subgroups of G, then $c_{g,H} t_K^H = t_{{}^gK}^{{}^gH} c_{g,K}$ and $c_{g,K} r_K^H = r_{{}^gK}^{{}^gH} c_{g,H}$,

3. if $h \in H$ and H is a subgroup of G, then $t_H^H = r_H^H = c_{g,H}$,

4. if $K \leqslant H \geqslant L$ are subgroups of G, then

$$r_K^H t_L^H = \sum_{g \in [K \backslash H / L]} t_{K \cap {}^gL}^K c_{g,K^g \cap L} r_{K^g \cap L}^L$$

where $[K \backslash H/L]$ is a set of representatives of the double cosets modulo K and L in H,

5. all products of generators, different from those appearing in the previous four relations are zero,

6. the sum of the elements t_H^H over all subgroups H of G is equal to the identity element of $\mu_k(G)$.

A *Mackey functor* M for G over k is a $\mu_k(G)$-module and a morphism of Mackey functors is a morphism of $\mu_k(G)$-modules.

The study of Mackey functors on compact Lie groups is described by Lewis [Le]. Many of the fundamental results on Mackey functors for a finite group are extended to Mackey functors for a compact Lie group. Mackey functors have been studied on finite groups for a long time. The study of Mackey functors for an infinite group has appeared recently: references are in [Lü2] and [MN]. There is also a new concept called *globally-defined Mackey functors*. They appeared more recently and were studied in [We]. The main difference is that the globally-defined Mackey functors are defined on all finite groups, where the original Mackey functors are defined on subgroups of a particular group. A second main difference is that the original Mackey functors only possess the inclusion and conjugation operations but the globally-defined Mackey functors possess operations for all group homomorphisms.

Some examples of Mackey functors for finite groups are representations rings, Burnside rings ([Se1],[Di]), group cohomology ([Fe]), equivariant cohomology, equivariant topological K-theory ([Se2]), algebraic K-theory of group rings ([Lü1]), any stable equivariant (co-)homology theories ([LMM]), and higher algebraic K-theory ([Ku]).

One application of Mackey functors to number theory has been to provide relations between λ- and μ-invariants in Iwasawa theory and between Mordell-Weil groups, Shafarevich-Tate groups, Selmer groups and zeta functions of elliptic curves (see [BB]).

This thesis consists of four papers. The first develops the main goal and theory of the thesis: put simply, it develops and extends the theory of Mackey functors as an application of enriched category theory. The other papers arose from specific issues that came up in the preparation of the first paper, particularly, they concern techniques for constructing new Mackey and Green functors from given ones. We saw that, in order for the Dress construction to produce a Green functor from a given one, we needed a monoid in the lax centre of some monoidal category. This led us to a general study of lax braidings and lax centre for monoidal categories and more generally for promonoidal categories. The second and third papers are the outcome; they have application beyond the particular needs of the first paper. The final paper is a categorical treatment of a theorem of Bouc [Bo2] concerning which functors compose with Mackey functors to yield Mackey functors; again this result may be useful in other applications.

The first paper entitled *Mackey functors on compact closed categories*, coauthored with Professor Ross Street, was submitted to the *Journal of Homotopy and Related Structures (JHRS)* to a special volume in memory of Saunders Mac Lane. The second paper entitled *Lax braidings and the lax centre*, coauthored with Dr. Brain Day and Professor Ross Street, will appear in *Contemporary Mathematics*. The third paper entitled *On centres and lax centres for promonoidal categories*, coauthored with Dr. Brain Day and Professor Ross Street, was submitted to "Charles Ehresmann 100 ans", the 100th birthday anniversary conference of Charles Ehresmann which was held at the Universite de Picardie Jules Verne in Amiens between October 7 to 9, 2005. The abstract will appear in

Cahiers de Topologie et Géométrie Différentielle Catégoriques, Volume **XLVI-3**. The fourth paper entitled *Pullback and finite coproduct preserving functors between categories of permutation representations* consists of the paper [PS2] as modified in the light of [PS3]. The papers [PS2] and [PS3] are appearing in the journal of *Theory and Applications of Categories*, Volume **16**, Number **28**, pp. 771–784, (2006) and Volume **18**, Number **5**, pp. 151–156, (2007) respectively.

Chapter 1 consists of the first paper entitled "Mackey functors on compact closed categories". This paper develops the theory of Mackey functors as an application of enriched category theory. Mackey functors on a compact (= rigid= autonomous) closed category $\mathscr{T}$ are defined and the properties of the category **Mky** of Mackey functors on $\mathscr{T}$ are investigated. The category **Mky** is a symmetric monoidal closed abelian category.

We now explain the main constructions and theorems of the sections of this chapter. In Section 1.1 we give an introduction to this paper. In Section 1.2 we define the compact closed category $\mathbf{Spn}(\mathscr{E})$ of spans in a finitely complete category $\mathscr{E}$. The objects of $\mathbf{Spn}(\mathscr{E})$ are the objects of $\mathscr{E}$ and morphisms $U \longrightarrow V$ are the isomorphisms classes of *spans* from U to V in the bicategory of spans in $\mathscr{E}$. The category $\mathbf{Spn}(\mathscr{E})$ is a monoidal category using the cartesian product in $\mathscr{E}$ as the tensor product in $\mathbf{Spn}(\mathscr{E})$. Section 1.3 describes the direct sums in $\mathbf{Spn}(\mathscr{E})$. Here we take $\mathscr{E}$ to be a lextensive category. References for this notion are [Sc1], [CLW], and [CL]. The coproduct $U+V$ in $\mathscr{E}$ is the direct sum of U and V in $\mathbf{Spn}(\mathscr{E})$. The addition of two spans is also defined in $\mathbf{Spn}(\mathscr{E})$. This makes the category $\mathbf{Spn}(\mathscr{E})$ into a monoidal commutative-monoid-enriched category. In Section 1.4 we define the Mackey functors on a lextensive category $\mathscr{E}$ using the approach described by Dress [Dr1] in the G-set case. A Mackey functor M from $\mathscr{E}$ to the category $\mathbf{Mod}_k$ of k-modules consists of two functors $M_* : \mathscr{E} \longrightarrow \mathbf{Mod}_k$, and $M^* : \mathscr{E}^{\mathrm{op}} \longrightarrow \mathbf{Mod}_k$ which coincide on objects and satisfy a couple of conditions. A morphism $\theta : M \longrightarrow N$ of Mackey functors M and N is a family of morphisms $\theta_U : M(U) \longrightarrow N(U)$ for each $U \in \mathscr{E}$ which give two natural transformations $\theta_* : M_* \longrightarrow N_*$ and $\theta^* : M^* \longrightarrow N^*$. We denote the category of Mackey functors from $\mathscr{E}$ to $\mathbf{Mod}_k$ by $\mathbf{Mky}(\mathscr{E}, \mathbf{Mod}_k)$ or simply $\mathbf{Mky}$ when $\mathscr{E}$ and k are understood.

Proposition [1.4.1]. *(Lindner [Li1]) The category* $\mathbf{Mky}(\mathscr{E}, \mathbf{Mod}_k)$ *of Mackey functors, from a lextensive category* $\mathscr{E}$ *to the category* $\mathbf{Mod}_k$ *of* k*-modules, is equivalent to* $[\mathbf{Spn}(\mathscr{E}), \mathbf{Mod}_k]_+$*, the category of coproduct-preserving functors.*

Tensor product of Mackey functors is defined in Section 1.5. Here we work on a general compact closed category $\mathscr{T}$ with finite products in place of $\mathbf{Spn}(\mathscr{E})$. This implies that $\mathscr{T}$ has direct sums (see [Ho]) and $\mathscr{T}$ is enriched in the monoidal category $\mathscr{V}$ of commutative monoids. A Mackey functor on $\mathscr{T}$ is an additive functor $M : \mathscr{T} \longrightarrow \mathbf{Mod}_k$. The tensor product of Mackey functors M and N is defined by:

$$(M * N)(Z) \cong \int^Y M(Z \otimes Y^*) \otimes_k N(Y)$$

using Day's convolution structure ([Da1]). The *Burnside functor* J is defined on objects as the free k-module on $\mathscr{T}(I, U)$ where I is the unit of $\mathscr{T}$ and U is an object of $\mathscr{T}$. It is a Mackey functor and becomes the unit for the tensor product of Mackey functors. The category $\mathbf{Mky}$ becomes a symmetric monoidal closed category. The closed structure is described in Section 1.6. For Mackey functors M and N, the Hom Mackey functor is given by:

$$\mathrm{Hom}(M, N)(V) = \mathbf{Mky}(M(V^* \otimes -), N).$$

There is also another expression for this Hom Mackey functor, which is given by:

$$\mathrm{Hom}(M, N)(V) = \mathbf{Mky}(M, N(V \otimes -)).$$

Green functors are introduced in Section 1.7. A *Green functor* A on $\mathscr{T}$ is a

Mackey functor with a monoidal structure

$$\mu : A(U) \otimes_k A(V) \longrightarrow A(U \otimes V)$$

and a morphism

$$\eta : k \longrightarrow A(1).$$

Green functors precisely become the monoids in the monoidal category **Mky**. In Section 1.8 we describe the Dress construction of Green functors. The Dress construction ([Bo5], [Bo6]) is a process to obtain a new Mackey functor M_Y from a known Mackey functor M, where $M_Y(U) = M(U \otimes Y)$ for fixed $Y \in \mathscr{T}$. We define the *Dress construction*

$$D : \mathscr{T} \otimes \mathbf{Mky} \longrightarrow \mathbf{Mky}$$

by $D(Y, M) = M_Y$. In Proposition 1.8.1 we show that the Dress construction D is a strong monoidal $\mathscr{V}$-functor. We study the centres and the lax centres of the monoidal category $\mathscr{E}/G_C$ (where $\mathscr{E}/G_C$ is the category of crossed G-sets) to obtain the Dress construction for Green functors. The detailed study of centres and lax centres for monoidal categories are in Chapters 2 and 3. We use the following Theorem to induce the Dress construction on Green functors.

Theorem [1.8.4]. *([Bo2], [Bo3]) If Y is a monoid in $\mathscr{E}/G_c$ and A is a Green functor for $\mathscr{E}$ over k then A_Y is a Green functor for $\mathscr{E}$ over k, where $A_Y(X) = A(X \times Y)$.*

Finite dimensional Mackey functors are introduced in Section 1.9. Here we assume the compact closed category is $\mathscr{T} = \mathbf{Spn}(\mathscr{E})$, where $\mathscr{E} = G\text{-}\mathbf{set}_{fin}$ is the category of finite G-sets for a finite group G. Also we assume k is a field and replace $\mathbf{Mod}_k$ by **Vect**, the category of vector spaces. A Mackey functor $M : \mathscr{T} \longrightarrow \mathbf{Vect}$ is called *finite dimensional* when each $M(X)$ is a finite-dimensional vector space. We denote the category of finite dimensional Mackey functors by

$\mathbf{Mky}_{fin}$ which is a full subcategory of **Mky**. We show that the tensor product of finite-dimensional Mackey functors is finite dimensional (Proposition 1.9.1).

Theorem [1.9.2]. *The monoidal category* $\mathbf{Mky}_{fin}$ *of finite-dimensional Mackey functors on* $\mathscr{T}$ *is* $*$*-autonomous.*

In Section 1.10 we study the cohomological Mackey functors and the relation between the ordinary k-linear representations of a finite group G and Mackey functors on G. Let $\mathbf{Rep}_k(G)$ denote the finite-dimensional k-linear representations of G. The relation between $\mathbf{Rep}_k(G)$ and $\mathbf{Mky}(G)$ is shown in the following Proposition:

Proposition[1.10.1]. *The functor* $\widetilde{k_*} : \mathbf{Rep}_k(G) \longrightarrow \mathbf{Mky}(G)$ *is fully faithful.*

In Theorem 1.10.4 we also show that the adjoint functor $\mathbf{Mky}(G)_{fin} \longrightarrow \mathbf{Rep}_k(G)$ is strong monoidal. In Section 1.11 we give examples for the compact closed category $\mathscr{T}$ from a Hopf algebra H (or quantum group). The category $\mathbf{Comod}(\mathscr{R})$ becomes an example of $\mathscr{T}$. The objects of the category $\mathbf{Comod}(\mathscr{R})$ (see [DMS]) are comonoids C in $\mathscr{R}$ (where $\mathscr{R}$ is the category of left H-modules) and morphisms are isomorphisms classes of comodules $S : C \dashrightarrow D$ from C to D. The category $\mathbf{Comod}(\mathscr{R})$ is compact closed and a commutative-monoid enriched category. We also show that $\mathscr{R}^{\mathrm{op}}(\simeq \mathscr{R})$ is another example for $\mathscr{T}$.

Section 1.12 reviews the modules of enriched category theory. Section 1.13 studies the modules over Green functors. A *module* M over a Green functor A or A-*module* means A acts on M via the convolution $*$. We denote the category of left A-modules for a Green functor A by $\mathbf{Mod}(A)$. The objects are A-modules and morphisms are A-module morphisms $\theta : M \longrightarrow N$. The category $\mathbf{Mod}(A)$ is the category of Eilenberg-Moore algebras for the monad $T = A * -$ on $[\mathscr{C}, \mathbf{Mod}_k]$, where $\mathscr{C}$ is a small $\mathscr{V}$-category. In Section 1.14 we study the Morita theory of Green functors. We define the monoidal bicategory $\mathbf{Mod}(\mathscr{W})$ for $\mathscr{W} = \mathbf{Mky}$. The objects are monoids A in $\mathscr{W}$ and morphisms are modules

$M: A \to\!\!\!\!\!\!\!\!\;\shortmid\;\; B$ (that is, algebras for the monad $A * - * B$ on **Mky**) with a two sided action $A * M * B \longrightarrow M$. Composition of morphisms is defined by a coequalizer. Green functors A and B are defined to be *Morita equivalent* when they are equivalent in $\mathbf{Mod}(\mathscr{W})$. In Proposition 1.14.1 we show that if A and B are equivalent in $\mathbf{Mod}(\mathscr{W})$ then $\mathbf{Mod}(A) \simeq \mathbf{Mod}(B)$ as categories. The Cauchy completion $\mathscr{Q}A$ of A is the $\mathscr{W}$-category which consists of the modules $M: J \to\!\!\!\!\!\!\!\!\;\shortmid\;\; A$ with right adjoints $N: A \to\!\!\!\!\!\!\!\!\;\shortmid\;\; J$, where J is the unit of $\mathscr{W}$. In the following Theorem we obtain an explicit description of the objects of the Cauchy completion of a monoid A in the monoidal category $\mathscr{W} = \mathbf{Mky}$.

Theorem[1.14.3]. *The Cauchy completion $\mathscr{Q}A$ of the monoid A in* **Mky** *consists of all the retracts of modules of the form*

$$\bigoplus_{i=1}^{k} A(Y_i \times -)$$

for some $Y_i \in \mathbf{Spn}(\mathscr{E})$, $i = 1, \ldots, k$.

Chapter 2 consists of the paper entitled "Lax braidings and the lax centre". This highlights the notions of lax braiding and lax centre for monoidal categories and more generally for promonoidal categories. Braidings for monoidal categories were introduced in [JS2] and its forerunners. The centre $\mathscr{Z}\mathscr{X}$ of a monoidal category $\mathscr{X}$ was introduced in [JS1] in the process of proving that the free tortile monoidal category has another universal property. The centre of a monoidal category is a braided monoidal category. The centre is generally a full subcategory of the lax centre, but sometimes the two coincide. We examine the cases where these two become equal.

We explain the main constructions and theorems of the sections of this chapter. An introduction is given at the beginning. In Section 2.1 we study the lax braidings for promonoidal categories. Let $\mathscr{V}$ be a complete cocomplete symmetric closed monoidal category and $\mathscr{C}$ be a $\mathscr{V}$-enriched category in the sense of [Ke]. The category $\mathscr{C}$ is called promonoidal when there are two $\mathscr{V}$-

functors $P:\mathscr{C}^{\mathrm{op}}\otimes\mathscr{C}^{\mathrm{op}}\otimes\mathscr{C}\longrightarrow\mathscr{V}$ and $J:\mathscr{C}\longrightarrow\mathscr{V}$ (called a *promagmal* structure on $\mathscr{C}$) satisfying the associative, and left and right unit constraints. Symmetries for promonoidal categories were defined by Day [Da1] and braidings by Joyal and Street [JS2]. A lax braiding for a promonoidal category $\mathscr{C}$ is a $\mathscr{V}$-natural family of morphisms $P(A,B;C)\longrightarrow P(B,A;C)$ which satisfies some commutative diagrams. A braiding is a lax braiding with each $P(A,B;C)\longrightarrow P(B,A;C)$ invertible. We reprove a result of Yetter [Ye] in Proposition 2.1.3 that if $\mathscr{C}$ is a right autonomous (meaning that each object has a right dual) monoidal category then any lax braiding on $\mathscr{C}$ is necessarily a braiding.

In Section 2.2 we define the lax centre and centre of a promonoidal category $\mathscr{C}$. The objects of the lax centre $\mathcal{Z}_l\mathscr{C}$ of a promonoidal category $\mathscr{C}$ are pairs (A,α) where A is an object of $\mathscr{C}$ and α is a $\mathscr{V}$-natural family of morphisms $\alpha_{X,Y}:P(A,X;Y)\longrightarrow P(X,A;Y)$ satisfying a couple of commutative diagrams. The Hom object $\mathcal{Z}_l\mathscr{C}((A,\alpha),(B,\beta))$ is defined to be the equalizer in $\mathscr{V}$ of the two composed paths around the following square.

$$\begin{tikzcd}
\mathscr{C}(A,B) \arrow[r, "P"] \arrow[d, "P"'] & \int_{X,Y}[P(B,X;Y),P(A,X;Y)] \arrow[d, "{[1,\alpha]}"] \\
\int_{X,Y}[P(X,B;Y),P(X,A;Y)] \arrow[r, "{[\beta,1]}"'] & \int_{X,Y}[P(B,X;Y),P(X,A;Y)]
\end{tikzcd}$$

The lax centre $\mathcal{Z}_l\mathscr{C}$ of the promonoidal category $\mathscr{C}$ is often promonoidal. The $\mathscr{V}$-functor $\mathcal{Z}_l\mathscr{C}\longrightarrow\mathscr{C}$ which take (A,α) to A is a strong promonoidal functor. If $\mathscr{C}$ is monoidal then the category $\mathcal{Z}_l\mathscr{C}$ is also a monoidal category and $\mathcal{Z}_l\mathscr{C}\longrightarrow\mathscr{C}$ is strong monoidal. The centre $\mathcal{Z}\mathscr{C}$ of $\mathscr{C}$ is the full sub-$\mathscr{V}$-category of $\mathcal{Z}_l\mathscr{C}$ consisting the objects (A,α) where each $\alpha_{X,Y}:P(A,X;Y)\longrightarrow P(X,A;Y)$ is invertible. Clearly $\mathcal{Z}\mathscr{C}$ is a braided monoidal category.

The lax centre of a monoidal category is studied in Section 2.3. The lax centre $\mathcal{Z}_l\mathscr{C}$ of a monoidal $\mathscr{V}$-category $\mathscr{C}$ has objects pairs (A,u) where A is an object of $\mathscr{C}$ and u is a $\mathscr{V}$-natural family of morphisms $u_B : A \otimes B \longrightarrow B \otimes A$ which satisfy the following two commutative diagrams.

$$\begin{tikzcd} A \otimes B \otimes C \arrow[rr, "u_{B\otimes C}"] \arrow[dr, "u_B \otimes 1_C"'] & & B \otimes C \otimes A \\ & B \otimes A \otimes C \arrow[ur, "1_B \otimes u_C"'] & \end{tikzcd} \qquad \begin{tikzcd} A \otimes I \arrow[rr, "u_I"] \arrow[dr, "\cong"'] & & I \otimes A \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

When $\mathscr{V}$ = **Set** and $\mathscr{C}$ is monoidal, the lax centre of $\mathscr{C}$ was used by P. Schauenburg [Sc1] under the name of "weak centre". One reason for being interested in the lax centre is the following result.

Theorem[2.3.7]. *Suppose an object F of a monoidal $\mathscr{V}$-category $\mathscr{F}$ is equipped with the structure of monoid in the lax centre $\mathcal{Z}_l\mathscr{F}$ of $\mathscr{F}$. Then $- \otimes F : \mathscr{F} \longrightarrow \mathscr{F}$ is equipped with the structure of monoidal $\mathscr{V}$-functor.*

In the following two Corollaries we show two cases in which the lax centre becomes equal to the centre. Corollary 2.3.5 shows that, for any Hopf algebra H, the lax centre of the monoidal category **Comod**H of left H-comodules is equal to its centre. Corollary 2.3.6 shows that, for any finite dimensional Hopf algebra H, the lax centre of the monoidal category **Mod**H of left H-modules is equal to its centre. In Section 2.4 we study the lax centre and centre of cartesian monoidal categories where $\mathscr{V}$ = **Set**. The objects of the lax centre $\mathcal{Z}_l\mathscr{C}$ are pairs (A,ϕ) where A is in $\mathscr{C}$ and ϕ is a family of functions $\phi_X : \mathscr{C}(A,X) \longrightarrow \mathscr{C}(X,X)$ such that the following diagram commutes for all $f : X \longrightarrow Y$ in $\mathscr{C}$.

$$\begin{tikzcd} \mathscr{C}(A,X) \arrow[r, "\phi_X"] \arrow[dd, "\mathscr{C}(1_A,f)"'] & \mathscr{C}(X,X) \arrow[dr, "\mathscr{C}(1_X,f)"] & \\ & & \mathscr{C}(X,Y) \\ \mathscr{C}(A,Y) \arrow[r, "\phi_Y"'] & \mathscr{C}(Y,Y) \arrow[ur, "\mathscr{C}(f,1_Y)"'] & \end{tikzcd}$$

A morphism $g : (A,\phi) \longrightarrow (A',\phi')$ in $\mathcal{Z}_l\mathscr{C}$ is a morphism $g : A \longrightarrow A'$ in $\mathscr{C}$ such that $\phi_X(vg) = \phi'_X(v)$ for all $v : A' \longrightarrow X$. The *core* $C_{\mathscr{X}}$ of the category $\mathscr{X}$ with finite products in the sense of Freyd [Fr2] is precisely a terminal object in $\mathcal{Z}_l\mathcal{X}$. If the core exists, the lax centre can be written as

$$\mathcal{Z}_l\mathcal{X} \cong \mathcal{X}/C_{\mathcal{X}}.$$

In Theorem 2.4.2 we show that for any small category $\mathscr{C}$ equipped with the promonoidal structure whose convolution gives the cartesian monoidal structure on $[\mathscr{C},\mathbf{Set}]$, there is an equivalence and an isomorphism of categories:

$$[\mathcal{Z}_l\mathscr{C},\mathbf{Set}] \xrightarrow{\simeq} [\mathscr{C},\mathbf{Set}]/C_{[\mathscr{C},\mathbf{Set}]} \xrightarrow{\cong} \mathcal{Z}_l[\mathscr{C},\mathbf{Set}].$$

In Theorem 2.4.5 we show that, if $\mathscr{C}$ is a groupoid with a promonoidal structure, then the lax centre of $\mathscr{C}$ is equal to the centre of $\mathscr{C}$. We also show that if the convolution of the promonoidal structure of $\mathscr{C}$ gives a cartesian monoidal structure on $[\mathscr{C},\mathbf{Set}]$ then the lax centre of $[\mathscr{C},\mathbf{Set}]$ is equal to its centre. In the following Theorem we show another case where the lax centre coincides with the centre of the cartesian monoidal category $[\mathscr{C},\mathbf{Set}]$.

Theorem [2.4.4]. *If $\mathscr{C}$ is a category in which every endomorphism is invertible then the lax centre $\mathcal{Z}_l[\mathscr{C},\mathbf{Set}]$ of the cartesian monoidal category $[\mathscr{C},\mathbf{Set}]$ is equal to the centre $\mathcal{Z}[\mathscr{C},\mathbf{Set}]$.*

In Section 2.5 we develop the theory of central cohypomonads for a monoidal $\mathscr{V}$-category $\mathcal{X}$. The lax centre $\mathcal{Z}_l\mathcal{X}$ is the $\mathscr{V}$-category of coalgebras for a cohypomonad. A *cohypomonad* on $\mathcal{X}$ is a monoidal functor $G : \Delta^{\mathrm{op}} \longrightarrow [\mathcal{X},\mathcal{X}]$ where Δ is the category with objects the finite ordinals $\langle n\rangle = \{1,2,\ldots,n\}$. The morphisms of Δ are order-preserving functions. A *coalgebra* for G is an object A of $\mathcal{X}$ together with a coaction morphism satisfying some commutative diagrams.

Proposition [2.5.1]. *Let $\mathcal{X}$ be a complete closed monoidal $\mathscr{V}$-category with a small dense sub-$\mathscr{V}$-category. The structure just defined on $\mathbf{G} : \Delta^{op} \longrightarrow [\mathcal{X},\mathcal{X}]$*

makes it a normal cohypomonad for which $\mathscr{X}^{\mathbf{G}}$ is equivalent to the lax centre of $\mathscr{X}$.

Chapter 3 consists of the paper entitled "On centres and lax centres for promonoidal categories". This reviews the notions of lax braiding and lax centre for monoidal and promonoidal categories and generalizes them to the $\mathscr{V}$-enriched context. To a large extent, this is a conference paper summarizing some results of the last Chapter and of [DS4]. We examine when the centre of $[\mathscr{C},\mathscr{V}]$ with the convolution monoidal structure (in the sense of [Da1]) is again a functor category $[\mathscr{D},\mathscr{V}]$.

We explain the main constructions and theorems of the sections of this chapter. Section 3.1 is the introduction of this paper. Section 3.2 reviews some definitions. A *$\mathscr{V}$-multicategory* is a $\mathscr{V}$-category $\mathscr{C}$ equipped with a sequence of $\mathscr{V}$-functors

$$P_n : \underbrace{\mathscr{C}^{\mathrm{op}} \otimes \ldots \otimes \mathscr{C}^{\mathrm{op}}}_{n} \otimes \mathscr{C} \longrightarrow \mathscr{V},$$

where we write J for $P_0 : \mathscr{C} \longrightarrow \mathscr{V}$, P_1 for $\mathscr{C}(-,\sim) : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \longrightarrow \mathscr{V}$ which is a hom $\mathscr{V}$-functor, and we write P for P_2. Also there are *substitution operations* which are $\mathscr{V}$-natural families of morphisms satisfying the associative and unit conditions. For $\mathscr{V} = \mathbf{Set}$, this is a multicategory in the sense of [La4]. A *promonoidal $\mathscr{V}$-category* [Da1] is a $\mathscr{V}$-multicategory $\mathscr{C}$ for which the substitution operations are invertible. A *monoidal $\mathscr{V}$-category* is a promonoidal $\mathscr{V}$-category $\mathscr{C}$ for which P and J are representable. That is, there are $\mathscr{V}$-natural isomorphisms

$$P(A,B;C) \cong \mathscr{C}(A \boxtimes B, C), \qquad JC \cong \mathscr{C}(U,C).$$

We define lax braiding and braiding for a promonoidal $\mathscr{V}$-category $\mathscr{C}$.

In Section 3.3 we define the lax centre and centre of a monoidal $\mathscr{V}$-category $\mathscr{C}$. The *lax centre* $\mathscr{Z}_l\mathscr{C}$ of a monoidal $\mathscr{V}$-category $\mathscr{C}$ has objects (A,u) where A

is an object of $\mathscr{C}$ and u is a $\mathscr{V}$-natural family of morphisms

$$u_B : A \boxtimes B \longrightarrow B \boxtimes A$$

such that the following two diagrams commute:

$$\begin{tikzcd}
 & (A \boxtimes B) \boxtimes C \arrow[r, "u_B \boxtimes 1"] \arrow[dl, "\cong"'] & (B \boxtimes A) \boxtimes C \arrow[dr, "\cong"] & \\
A \boxtimes (B \boxtimes C) \arrow[d, "u_B \boxtimes C"'] & & & B \boxtimes (A \boxtimes C) \arrow[d, "1 \boxtimes u_C"] \\
(B \boxtimes C) \boxtimes A \arrow[rrr, "\cong"'] & & & B \boxtimes (C \boxtimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \boxtimes U \arrow[rr, "u_U"] \arrow[dr, "\cong"'] & & U \boxtimes A \arrow[dl, "\cong"] \\
 & A\,. &
\end{tikzcd}$$

The monoidal structure on $\mathcal{Z}_l \mathscr{C}$ is defined on objects by

$$(A, u) \boxtimes (B, v) = (A \boxtimes B, w)$$

where $w_C : (A \boxtimes B) \boxtimes C \longrightarrow C \boxtimes (A \boxtimes B)$ is the composite

$$A \boxtimes (B \boxtimes C) \xrightarrow{1 \boxtimes v_C} A \boxtimes (C \boxtimes B) \xrightarrow{\cong} (A \boxtimes C) \boxtimes B \xrightarrow{u_C \boxtimes 1} (C \boxtimes A) \boxtimes B$$

conjugated by canonical isomorphisms. The lax centre $\mathcal{Z}_l \mathscr{C}$ is a lax-braided monoidal $\mathscr{V}$-category. The lax braiding on $\mathcal{Z}_l \mathscr{C}$ is defined to be the family of morphisms

$$c_{(A,u),(B,v)} : (A \boxtimes B, w) \longrightarrow (B \boxtimes A, \tilde{w})$$

lifting $u_B : A \boxtimes B \longrightarrow B \boxtimes A$ to $\mathcal{Z}_l \mathscr{C}$. The *centre* $\mathcal{Z}\mathscr{C}$ of $\mathscr{C}$ is the full monoidal sub-$\mathscr{V}$-category of $\mathcal{Z}_l \mathscr{C}$ consisting of the objects (A, u) with each u_B invertible. Clearly $\mathcal{Z}\mathscr{C}$ is a braided monoidal $\mathscr{V}$-category. We generalize the constructions of the lax centre and the centre to promonoidal $\mathscr{V}$-categories $\mathscr{C}$.

In Section 3.4 we study the lax centre of cartesian monoidal categories $\mathscr{C}$. We identify the objects of $\mathcal{Z}_l\mathscr{C}$ with pairs (A,θ) where A is an object of $\mathscr{C}$ and $\theta_X : A \times X \longrightarrow X$ is a family of morphisms.

Theorem [3.4.1]. *Let $\mathscr{C}$ denote a small category with promonoidal structure such that the convolution structure on $[\mathscr{C},\mathbf{Set}]$ is cartesian product.*

1. *The adjunction $\hat{\Psi} \dashv \tilde{\Psi}$ defines an equivalence of lax-braided monoidal categories*
$$\mathcal{Z}_l[\mathscr{C},\mathbf{Set}] \simeq [\mathcal{Z}_l\mathscr{C},\mathbf{Set}]$$
which restricts to a braided monoidal equivalence
$$\mathcal{Z}[\mathscr{C},\mathbf{Set}] \simeq [\mathcal{Z}\mathscr{C},\mathbf{Set}].$$

2. *If every endomorphism in the category $\mathscr{C}$ is invertible then $\mathcal{Z}_l\mathscr{C} = \mathcal{Z}\mathscr{C}$.*

3. *If $\mathscr{C}$ is a groupoid then*
$$\mathcal{Z}\mathscr{C} = \mathcal{Z}_l\mathscr{C} = [\Sigma\mathbb{Z},\mathscr{C}]$$
(where $\Sigma\mathbb{Z}$ is the additive group of the integers as a one-object groupoid).

In Section 3.5 we study the autonomous case. Here we consider $\mathscr{C}$ to be a closed monoidal $\mathscr{V}$-category with tensor product $\boxtimes$ and unit U.

Theorem [3.5.2]. $(\mathscr{V} = \mathbf{Vect}_k)$ *Suppose $\mathscr{C}$ is a promonoidal k-linear category with finite-dimensional homs. Let $\mathscr{F} = [\mathscr{C},\mathscr{V}]$ have the convolution monoidal structure. Then*
$$\mathcal{Z}\mathscr{F} = \mathcal{Z}_l\mathscr{F} \cong \mathscr{F}^M \simeq [\mathscr{C}_M,\mathscr{V}]$$
where $\mathscr{C}_M$ is the Kleisli category for the promonad M on $\mathscr{C}$.

In Section 3.6 we study monoids in the lax centre of a monoidal $\mathscr{V}$-category $\mathscr{C}$. A monoid (A,u) in the lax centre $\mathcal{Z}_l\mathscr{C}$ determines a canonical enrichment of the $\mathscr{V}$-functor
$$-\boxtimes A : \mathscr{C} \longrightarrow \mathscr{C}$$

to a monoidal functor:

$$X \boxtimes A \boxtimes Y \boxtimes A \xrightarrow{1 \boxtimes u_Y \boxtimes 1} X \boxtimes Y \boxtimes A \boxtimes A \xrightarrow{1 \boxtimes 1 \boxtimes \mu} X \boxtimes Y \boxtimes A$$

$$U \xrightarrow{\eta} A \cong U \boxtimes A.$$

Chapter 4 consists of the paper [PS2] as modified in the light of [PS3]. This studies the finite coproduct and pullback preserving functors between categories of permutation representations of finite groups and gives a categorical explanation of the work of Serge Bouc [Bo1]. A *permutation representation* of a finite group G or a *finite left G-set* is a finite set X together with a function (called action) $G \times X \longrightarrow X$, $(g,x) \longmapsto gx$ such that $1x = x$ and $g_1(g_2x) = (g_1g_2)x$ for $g_1, g_2 \in G$ and $x \in X$. We write G-$\mathbf{set}_{fin}$ for the category of finite left G-sets (that is, of permutation representations of G) with left G-morphisms where a left G-morphism $f : X \longrightarrow Y$ is a function satisfying $f(gx) = gf(x)$. Let M be a Mackey functor on a finite group H. Then $M : \mathbf{Spn}(H\text{-}\mathbf{set}_{fin}) \longrightarrow \mathbf{Mod}_k$ is a coproduct preserving functor. If $F : G\text{-}\mathbf{set}_{fin} \longrightarrow H\text{-}\mathbf{set}_{fin}$ is a pullback and finite coproduct preserving functor (where G is finite) then we get a functor

$$M \circ \mathbf{Spn}(F) : \mathbf{Spn}(G\text{-}\mathbf{set}_{fin}) \longrightarrow \mathbf{Mod}_k$$

which is a Mackey functor on G.

Bouc [Bo2] studied the pullback and finite coproduct preserving functors $F : G\text{-}\mathbf{set}_{fin} \longrightarrow H\text{-}\mathbf{set}_{fin}$ interms of $(G^{\mathrm{op}} \times H)$-sets A (where G^{op}) is G with opposite multiplication). The category $(G^{\mathrm{op}} \times H)$-set of such A is equivalent to the category of finite colimit preserving functors $L : G\text{-}\mathbf{set}_{fin} \longrightarrow H\text{-}\mathbf{set}_{fin}$. In this chapter we explained these two constructions.

Let A be a $(G^{\mathrm{op}} \times H)$-set. For all $(K^{\mathrm{op}} \times G)$-sets B, where K, G, H are all finite groups, Bouc ([Bo1]) defines the $(K^{\mathrm{op}} \times H)$-set

$$A \circ_G B = (A \wedge_G B)/G.$$

Here $A \wedge_G B$ is a $(K^{\mathrm{op}} \times G \times H)$-set given by

$$A \wedge_G B = \{(a,b) \in A \times B \mid g \in G, ag = a \Rightarrow \text{there exists } k \in K \text{ with } gb = bk\}.$$

This paper provides a categorical explanation for the following Theorem of Bouc.

Theorem [4.1.1]. *([Bo5]) Suppose K, G and H are finite groups.*

(i) *If A is a finite $(G^{\mathrm{op}} \times H)$-set then the functor*

$$A \circ_G - : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$$

preserves finite coproducts and pullbacks.

(ii) *Every functor $F : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$ which preserves finite coproducts and pullbacks is isomorphic to one of the form $A \circ_G -$.*

(iii) *The functor F in (ii) preserves terminal objects if and only if A is transitive (connected) as a right $G\text{-}\mathbf{set}_{\mathit{fin}}$.*

(iv) *If A is as in (i) and B is a finite $(K^{\mathrm{op}} \times G)$-set then the composite functor*

$$K\text{-}\mathbf{set}_{\mathit{fin}} \xrightarrow{B \circ_K -} G\text{-}\mathbf{set}_{\mathit{fin}} \xrightarrow{A \circ_G -} H\text{-}\mathbf{set}_{\mathit{fin}}$$

is isomorphic to $(A \circ_G B) \circ_K -$.

We explain the main constructions and theorems of the sections of this chapter. Section 4.1 is the introduction of this paper. In Section 4.2 we provide a direct proof of the well-known representability theorem for the case where “small” means “finite”.

Theorem [4.2.1]. *(Special representability theorem) Suppose $\mathscr{A}$ is a category with the following properties:*

(i) *each homset $\mathscr{A}(A, B)$ is finite;*

(ii) *finite limits exist;*

(iii) *there is a cogenerator Q;*

(iv) *$\mathscr{A}$ is finitely well powered.*

Then every finite limit preserving functor $T : \mathscr{A} \longrightarrow \mathbf{set}_{fin}$ is representable.

In Section 4.3 we study a category $\mathscr{A}$ with finite coproducts. An object C of $\mathscr{A}$ is called *connected* when the functor $\mathscr{A}(C,-) : \mathscr{A} \longrightarrow \mathbf{Set}$ preserves finite coproducts. We write $\mathbf{Conn}(\mathscr{A})$ for the category of connected objects of $\mathscr{A}$ and $\mathbf{Cop}(\mathscr{A},\mathscr{X})$ for the category of finite coproduct preserving functors from $\mathscr{A}$ to $\mathscr{X}$. For any small category $\mathscr{C}$, we write $\mathbf{Fam}(\mathscr{C}^{\mathrm{op}})$ for the free finite coproduct completion of $\mathscr{C}^{\mathrm{op}}$. The objects are families $(C_i)_{i\in I}$ where C_i are objects of $\mathscr{C}$ and I is finite. A morphism $(\xi, f) : (C_i)_{i\in I} \longrightarrow (D_j)_{j\in J}$ consists of a function $\xi : I \longrightarrow J$ and a family $f = (f_i)_{i\in I}$ of morphisms $f_i : D_{\xi(i)} \longrightarrow C_i$ in $\mathscr{C}$. In Proposition 4.3.3 we show that the following is an equivalence of categories

$$\mathbf{Fam}(\mathbf{Conn}(\mathscr{A})^{\mathrm{op}}) \simeq \mathbf{CopPb}(\mathscr{A}, \mathbf{set}_{fin})$$

where the category $\mathscr{A}$ has finite coproducts and the properties of Theorem 4.2.1 and $\mathbf{CopPb}(\mathscr{A},\mathscr{B})$ is the category of finite coproduct and pullback preserving functors from $\mathscr{A}$ to $\mathscr{B}$. In Section 4.4 we study the application to permutation representations. Let $N : \mathscr{C}_G \longrightarrow G\text{-}\mathbf{set}_{fin}$ denote the inclusion functor and define the functor

$$\tilde{N} : G\text{-}\mathbf{set}_{fin} \longrightarrow [\mathscr{C}_G^{\mathrm{op}}, \mathbf{set}_{fin}]$$

by $\tilde{N}X = G\text{-}\mathbf{set}_{fin}(N-, X)$. In Proposition 4.4.4 we show that the functor $\tilde{N}$ induces an equivalence of categories

$$G\text{-}\mathbf{set}_{fin} \simeq \mathbf{Fam}(\mathscr{C}_G).$$

In Section 4.5 we study a factorization for G-morphisms. We use these morphisms of a factorization system on G-set to describe the finite coproduct completion $\mathbf{Fam}(\mathscr{C}_G^{\mathrm{op}})$ of the dual of the category of connected G-sets. For any G-set

X, the set $X/G = \{C \subseteq X : C \text{ is an orbit of} X\}$ is a connected sub-G-set of X. A G-morphism $f : X \longrightarrow Y$ is said to be *slash inverted* when the direct image function $f/G : X/G \longrightarrow Y/G$ of f is a bijection. A G-morphism $f : X \longrightarrow Y$ is said to be *orbit injective* when $orb(x_1) = orb(x_2)$ and $f(x_1) = f(x_2)$ imply $x_1 = x_2$. In Proposition 5.2 we prove that the slash inverted and orbit injective G-morphisms form a factorization system (in the sense of [FK]) on the category of G-sets.

In Section 4.6 we introduce a new category $\mathscr{B}_G$ of G-sets. The objects of $\mathscr{B}_G$ are all the finite G-sets and morphisms are the isomorphisms classes of the span $(u, S, v) : X \longrightarrow Y$ in which $u : S \longrightarrow X$ is slash inverted and $v : S \longrightarrow Y$ is orbit injective. In Proposition 4.6.1 we prove that the subcategory $\mathscr{B}_G$ of $\mathbf{Spn}(G\text{-}\mathbf{set}_{fin})$ is closed under finite coproducts. We obtain a finite coproduct preserving functor $\Sigma : \mathbf{Fam}(\mathscr{C}_G^{\mathrm{op}}) \longrightarrow \mathscr{B}_G$.

Theorem [4.6.3]. The functor $\Sigma : \mathbf{Fam}(\mathscr{C}_G^{\mathrm{op}}) \longrightarrow \mathscr{B}_G$ is an equivalence of categories.

In Corollary 4.6.4 we obtain the following equivalence of categories:

$$\mathscr{B}_G \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{fin}, \mathbf{set}_{fin}).$$

Then we obtain the following corollary:

Corollary [4.6.6]. There is an equivalence

$$\mathscr{B}_{G^{\mathrm{op}}} \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{fin}, \mathbf{set}_{fin}), \quad A \longmapsto A \circ_G -.$$

In Section 4.7 we construct a bicategory **Bouc** of finite groups. We define the category $\mathbf{Bouc}(G, H)$ as the pullback of the inclusion of $\mathscr{B}_{G^{\mathrm{op}}}$ in $\mathbf{Spn}(G^{\mathrm{op}}\text{-}\mathbf{set}_{fin})$ along the forgetful functor $\mathbf{Spn}(G^{\mathrm{op}} \times H\text{-}\mathbf{set}_{fin}) \longrightarrow \mathbf{Spn}(G^{\mathrm{op}}\text{-}\mathbf{set}_{fin})$. That is, $\mathbf{Bouc}(G, H)$ is the subcategory of $\mathbf{Spn}(G^{\mathrm{op}} \times H\text{-}\mathbf{set}_{fin})$ consisting of all the objects yet, as morphisms, only the isomorphism classes of spans (u, S, v) in $G^{\mathrm{op}} \times H\text{-}\mathbf{set}_{fin}$ for which u is slash inverted and v is orbit injective as G-morphisms.

Theorem [4.7.1]. There is an equivalence of categories

$$\mathbf{Bouc}(G,H) \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{fin}, H\text{-}\mathbf{set}_{fin}), \quad A \longmapsto A \circ_G -.$$

In Section 4.8 we study the applications to Mackey functors. If the functor $F : G\text{-}\mathbf{set}_{fin} \longrightarrow H\text{-}\mathbf{set}_{fin}$ preserves pullbacks then this induces a functor $\mathbf{Spn}(F) : \mathbf{Spn}(G\text{-}\mathbf{set}_{fin}) \longrightarrow \mathbf{Spn}(H\text{-}\mathbf{set}_{fin})$ (since composition of spans only involves pullbacks). If F also preserves finite coproducts then $\mathbf{Spn}(F)$ preserves direct sums. Then we can obtain an exact functor

$$\bar{F} : \mathbf{Mky}_{fin}(H) \longrightarrow \mathbf{Mky}_{fin}(G)$$

defined by $\bar{F}(N) = N \circ \mathbf{Spn}(F)$ for all $N \in \mathbf{Mky}_{fin}(H)$, where $\mathbf{Mky}_{fin}$ is the category of finite-dimensional Mackey functors. The functor $\bar{F}$ has a left adjoint

$$\mathbf{Mky}_{fin}(F) : \mathbf{Mky}_{fin}(G) \longrightarrow \mathbf{Mky}_{fin}(H).$$

Let $\mathbf{AbCat}_k$ denote the 2-category of abelian k-linear categories, k-linear functors with right exact right adjoints, and natural transformations. In Corollary 4.8.1 we obtain a homomorphism of bicategories

$$\mathbf{Mky}_{fin} : \ \mathbf{Bouc} \longrightarrow \mathbf{AbCat}_k$$

given by $(A : G \longrightarrow H) \longmapsto (\mathbf{Mky}_{fin}(A \circ_G -) : \mathbf{Mky}_{fin}(G) \longrightarrow \mathbf{Mky}_{fin}(H))$.

This concludes the thesis.

Chapter 1

Paper 1: Mackey functors on compact closed categories

(Coauthored with Professor Ross Street)

This paper was submitted to the *Journal of Homotopy and Related Structures (JHRS)* to a special volume in memory of Saunders Mac Lane.

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MACKEY FUNCTORS ON COMPACT CLOSED CATEGORIES

ELANGO PANCHADCHARAM AND ROSS STREET

Dedicated to the memory of Saunders Mac Lane

ABSTRACT. We develop and extend the theory of Mackey functors as an application of enriched category theory. We define Mackey functors on a lextensive category $\mathscr{E}$ and investigate the properties of the category of Mackey functors on $\mathscr{E}$. We show that it is a monoidal category and the monoids are Green functors. Mackey functors are seen as providing a setting in which mere numerical equations occurring in the theory of groups can be given a structural foundation. We obtain an explicit description of the objects of the Cauchy completion of a monoidal functor and apply this to examine Morita equivalence of Green functors.

1. INTRODUCTION

Groups are used to mathematically understand symmetry in nature and in mathematics itself. Classically, groups were studied either directly or via their representations. In the last 40 years, arising from the latter, groups have been studied using Mackey functors.

Let k be a field. Let $\mathbf{Rep}(G) = \mathbf{Rep}_k(G)$ be the category of k-linear representations of the finite group G. We will study the structure of a monoidal category $\mathbf{Mky}(G)$ where the objects are called Mackey functors. This provides an extension of ordinary representation theory. For example, $\mathbf{Rep}(G)$ can be regarded as a full reflective sub-category of $\mathbf{Mky}(G)$; the reflection is strong monoidal (= tensor preserving). Representations of G are equally representations of the group algebra kG; Mackey functors can be regarded as representations of the "Mackey algebra" constructed from G. While $\mathbf{Rep}(G)$ is compact closed (= autonomous monoidal), we are only able to show that $\mathbf{Mky}(G)$ is star-autonomous in the sense of [Ba].

Mackey functors and Green functors (which are monoids in $\mathbf{Mky}(G)$) have been studied fairly extensively. They provide a setting in which mere numerical equations occurring in group theory can be given a structural foundation. One application has been to provide relations between λ- and μ-invariants in Iwasawa theory and between Mordell-Weil groups, Shafarevich-Tate groups, Selmer groups and zeta functions of elliptic curves (see [BB]).

Our purpose is to give the theory of Mackey functors a categorical simplification and generalization. We speak of Mackey functors on a compact (= rigid = autonomous) closed category $\mathscr{T}$. However, when $\mathscr{T}$ is the category $\mathbf{Spn}(\mathscr{E})$ of spans in a lextensive category $\mathscr{E}$, we speak of Mackey functors on $\mathscr{E}$. Further, when $\mathscr{E}$ is the category (topos) of finite G-sets, we speak of Mackey functors on G.

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Sections 2-4 set the stage for Lindner's result [Li1] that Mackey functors, a concept going back at least as far as [Gr], [Dr] and [Di] in group representation theory, can be regarded as functors out of the category of spans in a suitable category $\mathscr{E}$. The important property of the category of spans is that it is compact closed. So, in Section 5, we look at the category **Mky** of additive functors from a general compact closed category $\mathscr{T}$ (with direct sums) to the category of k-modules. The convolution monoidal structure on **Mky** is described; this general construction (due to Day [Da1]) agrees with the usual tensor product of Mackey functors appearing, for example, in [Bo1]. In fact, again for general reasons, **Mky** is a closed category; the internal hom is described in Section 6. Various convolution structures have been studied by Lewis [Le] in the context of Mackey functors for compact Lie groups mainly to provide counter examples to familiar behaviour.

Green functors are introduced in Section 7 as the monoids in **Mky**. An easy construction, due to Dress [Dr], which creates new Mackey functors from a given one, is described in Section 8. We use the (lax) centre construction for monoidal categories to explain results of [Bo2] and [Bo3] about when the Dress construction yields a Green functor.

In Section 9 we apply the work of [Da4] to show that finite-dimensional Mackey functors form a $*$-autonomous [Ba] full sub-monoidal category $\mathbf{Mky}_{fin}$ of **Mky**.

Section 11 is rather speculative about what the correct notion of Mackey functor should be for quantum groups.

Our approach to Morita theory for Green functors involves even more serious use of enriched category theory: especially the theory of (two-sided) modules. So Section 12 reviews this theory of modules and Section 13 adapts it to our context. Two Green functors are Morita equivalent when their **Mky**-enriched categories of modules are equivalent, and this happens, by the general theory, when the **Mky**-enriched categories of Cauchy modules are equivalent. Section 14 provides a characterization of Cauchy modules.

2. THE COMPACT CLOSED CATEGORY $\mathbf{Spn}(\mathscr{E})$

Let $\mathscr{E}$ be a finitely complete category. Then the category $\mathbf{Spn}(\mathscr{E})$ can be defined as follows. The objects are the objects of the category $\mathscr{E}$ and morphisms $U \longrightarrow V$ are the isomorphism classes of *spans* from U to V in the bicategory of spans in $\mathscr{E}$ in the sense of [Bé]. (Some properties of this bicategory can be found in [CKS].) A *span* from U to V, in the sense of [Bé], is a diagram of two morphisms with a common domain S, as in

$$(s_1, S, s_2): \quad \begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\ U & & V . \end{tikzcd}$$

An isomorphism of two spans $(s_1, S, s_2): U \longrightarrow V$ and $(s_1', S', s_2'): U \longrightarrow V$ is an invertible arrow $h: S \longrightarrow S'$ such that $s_1 = s_1' \circ h$ and $s_2 = s_2' \circ h$.

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] \arrow[dd, "\cong"', "h"] & \\ U & & V \\ & S' \arrow[ul, "s_1'"] \arrow[ur, "s_2'"'] & \end{tikzcd}$$

The composite of two spans $(s_1, S, s_2) : U \longrightarrow V$ and $(t_1, T, t_2) : V \longrightarrow W$ is defined to be $(s_1 \circ \mathrm{proj}_1, T \circ S, t_2 \circ \mathrm{proj}_2) : U \longrightarrow W$ using the pull-back diagram as in

$$\begin{tikzcd} & & S \times_V T = T \circ S \arrow[dl, "\mathrm{proj}_1"'] \arrow[dr, "\mathrm{proj}_2"] \arrow[dd, phantom, "\mathrm{pb}"] & & \\ & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & & T \arrow[dl, "t_1"'] \arrow[dr, "t_2"] & \\ U & & V & & W . \end{tikzcd}$$

This is well defined since the pull-back is unique up to isomorphism. The identity span $(1, U, 1) : U \longrightarrow U$ is defined by

$$\begin{tikzcd} & U \arrow[dl, "1"'] \arrow[dr, "1"] & \\ U & & U \end{tikzcd}$$

since the composite of it with a span $(s_1, S, s_2) : U \longrightarrow V$ is given by the following diagram and is equal to the span $(s_1, S, s_2) : U \longrightarrow V$

$$\begin{tikzcd} & & S \arrow[dl, "s_1"'] \arrow[dr, "1"] \arrow[dd, phantom, "\mathrm{pb}"] & & \\ & U \arrow[dl, "1"'] \arrow[dr, "1"] & & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\ U & & U & & V . \end{tikzcd}$$

This defines the category $\mathbf{Spn}(\mathscr{E})$. We can write $\mathbf{Spn}(\mathscr{E})(U, V) \cong [\mathscr{E}/(U \times V)]$ where square brackets denote the isomorphism classes of morphisms.

$\mathbf{Spn}(\mathscr{E})$ becomes a monoidal category under the tensor product

$$\mathbf{Spn}(\mathscr{E}) \times \mathbf{Spn}(\mathscr{E}) \xrightarrow{\ \times\ } \mathbf{Spn}(\mathscr{E})$$

defined by

$$(U, V) \longmapsto U \times V$$

$$[U \xrightarrow{S} U', V \xrightarrow{T} V'] \longmapsto [U \times V \xrightarrow{S \times T} U' \times V'].$$

This uses the cartesian product in $\mathscr{E}$ yet is not the cartesian product in $\mathbf{Spn}(\mathscr{E})$. It is also compact closed; in fact, we have the following isomorphisms: $\mathbf{Spn}(\mathscr{E})(U, V) \cong \mathbf{Spn}(\mathscr{E})(V, U)$ and $\mathbf{Spn}(\mathscr{E})(U \times V, W) \cong \mathbf{Spn}(\mathscr{E})(U, V \times W)$. The second isomorphism can be shown by the following diagram

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & & & & S \arrow[dl] \arrow[d] \arrow[dr] & & & & S \arrow[dl] \arrow[dr] & \\ U \times V & & W & \longleftrightarrow & U & V & W & \longleftrightarrow & U & & V \times W . \end{tikzcd}$$

3. Direct sums in $\mathbf{Spn}(\mathscr{E})$

Now we assume $\mathscr{E}$ is lextensive. References for this notion are [Sc], [CLW], and [CL]. A category $\mathscr{E}$ is called lextensive when it has finite limits and finite coproducts such that the functor

$$\mathscr{E}/A \times \mathscr{E}/B \longrightarrow \mathscr{E}/A+B\ ; \quad \begin{array}{c} X \\ \big\downarrow f \\ A \end{array} \ , \ \begin{array}{c} Y \\ \big\downarrow g \\ B \end{array} \longmapsto \begin{array}{c} X+Y \\ \big\downarrow f+g \\ A+B \end{array}$$

is an equivalance of categories for all objects A and B. In a lextensive category, coproducts are disjoint and universal and 0 is strictly initial. Also we have that the canonical morphism

$$(A \times B) + (A \times C) \longrightarrow A \times (B + C)$$

is invertible. It follows that $A \times 0 \cong 0$.

In $\mathbf{Spn}(\mathscr{E})$ the object $U+V$ is the direct sum of U and V. This can be shown as follows (where we use lextensivity):

$$\begin{aligned}
\mathbf{Spn}(\mathscr{E})(U+V,W) &\cong [\mathscr{E}/((U+V)\times W)] \\
&\cong [\mathscr{E}/((U\times W)+(V\times W))] \\
&\simeq [\mathscr{E}/(U\times W)] \times [\mathscr{E}/(V\times W)] \\
&\cong \mathbf{Spn}(\mathscr{E})(U,W) \times \mathbf{Spn}(\mathscr{E})(V,W);
\end{aligned}$$

and so $\mathbf{Spn}(\mathscr{E})(W,U+V) \cong \mathbf{Spn}(\mathscr{E})(W,U) \times \mathbf{Spn}(\mathscr{E})(W,V)$. Also in the category $\mathbf{Spn}(\mathscr{E})$, 0 is the zero object (both initial and terminal):

$$\mathbf{Spn}(\mathscr{E})(0,X) \cong [\mathscr{E}/(0\times X)] \cong [\mathscr{E}/0] \cong 1$$

and so $\mathbf{Spn}(\mathscr{E})(X,0) \cong 1$. It follows that $\mathbf{Spn}(\mathscr{E})$ is a category with homs enriched in commutative monoids.

The addition of two spans $(s_1,S,s_2) : U \longrightarrow V$ and $(t_1,T,t_2) : U \longrightarrow V$ is given by $(\nabla \circ (s_1+t_1), S+T, \nabla \circ (s_2+t_2)) : U \longrightarrow V$ as in

$$\begin{tikzcd}[column sep=small] & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\ U & & V \end{tikzcd} + \begin{tikzcd}[column sep=small] & T \arrow[dl, "t_1"'] \arrow[dr, "t_2"] & \\ U & & V \end{tikzcd} = \begin{tikzcd}[column sep=small] & & S+T \arrow[dl, "s_1+t_1"'] \arrow[dr, "s_2+t_2"] \arrow[ddll, bend right, "{[s_1,t_1]}"'] \arrow[ddrr, bend left, "{[s_2,t_2]}"] & & \\ & U+U \arrow[dl, "\nabla"'] & & V+V \arrow[dr, "\nabla"] & \\ U & & & & V\,. \end{tikzcd}$$

Summarizing, $\mathbf{Spn}(\mathscr{E})$ is a monoidal commutative-monoid-enriched category.

There are functors $(-)_* : \mathscr{E} \longrightarrow \mathbf{Spn}(\mathscr{E})$ and $(-)^* : \mathscr{E}^{\mathrm{op}} \longrightarrow \mathbf{Spn}(\mathscr{E})$ which are the identity on objects and take $f : U \longrightarrow V$ to $f_* = (1_U, U, f)$ and $f^* = (f, U, 1_U)$, respectively.

For any two arrows $U \xrightarrow{f} V \xrightarrow{g} W$ in $\mathscr{E}$, we have $(g\circ f)_* \cong g_* \circ f_*$ as we see from the following diagram

$$\begin{tikzcd}[column sep=small] & & U \arrow[dl, "1"'] \arrow[dr, "f"] & & \\ & U \arrow[dl, "1"'] \arrow[dr, "f"] \arrow[rr, phantom, "\mathrm{pb}"] & & V \arrow[dl, "1"'] \arrow[dr, "g"] & \\ U & & V & & W\,. \end{tikzcd}$$

Similarly $(g \circ f)^* \cong f^* \circ g^*$.

4. MACKEY FUNCTORS ON $\mathscr{E}$

A *Mackey functor* M from $\mathscr{E}$ to the category $\mathbf{Mod}_k$ of k-modules consists of two functors

$$M_* : \mathscr{E} \longrightarrow \mathbf{Mod}_k, \quad M^* : \mathscr{E}^{\mathrm{op}} \longrightarrow \mathbf{Mod}_k$$

such that:

(1) $M_*(U) = M^*(U) \quad (= M(U))$ for all U in $\mathscr{E}$

(2) for all pullbacks

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & V \arrow[d, "s"] \\ U \arrow[r, "r"'] & W \end{tikzcd}$$

in $\mathscr{E}$, the square (which we call a *Mackey square*)

$$\begin{tikzcd} M(P) \arrow[r, "M_*(q)"] & M(V) \\ M(U) \arrow[u, "M^*(p)"] \arrow[r, "M_*(r)"'] & M(W) \arrow[u, "M^*(s)"'] \end{tikzcd}$$

commutes, and

(3) for all coproduct diagrams

$$U \xrightarrow{i} U + V \xleftarrow{j} V$$

in $\mathscr{E}$, the diagram

$$\begin{tikzcd} M(U) \arrow[r, shift right, "M_*(i)"'] & M(U+V) \arrow[l, shift right, "M^*(i)"'] \arrow[r, shift left, "M^*(j)"] & M(V) \arrow[l, shift left, "M_*(j)"] \end{tikzcd}$$

is a direct sum situation in $\mathbf{Mod}_k$. (This implies $M(U+V) \cong M(U) \oplus M(V)$.)

A morphism $\theta : M \longrightarrow N$ of Mackey functors is a family of morphisms $\theta_U : M(U) \longrightarrow N(U)$ for U in $\mathscr{E}$ which defines natural transformations $\theta_* : M_* \longrightarrow N_*$ and $\theta^* : M^* \longrightarrow N^*$.

Proposition 4.1. *(Lindner* [Li1]*) The category* $\mathbf{Mky}(\mathscr{E}, \mathbf{Mod}_k)$ *of Mackey functors, from a lextensive category* $\mathscr{E}$ *to the category* $\mathbf{Mod}_k$ *of k-modules, is equivalent to* $[\mathbf{Spn}(\mathscr{E}), \mathbf{Mod}_k]_+$*, the category of coproduct-preserving functors.*

Proof. Let M be a Mackey functor from $\mathscr{E}$ to $\mathbf{Mod}_k$. Then we have a pair (M_*, M^*) such that $M_* : \mathscr{E} \longrightarrow \mathbf{Mod}_k$, $M^* : \mathscr{E}^{\mathrm{op}} \longrightarrow \mathbf{Mod}_k$ and $M(U) = M_*(U) = M^*(U)$. Now define a functor $M : \mathbf{Spn}(\mathscr{E}) \longrightarrow \mathbf{Mod}_k$ by $M(U) = M_*(U) = M^*(U)$ and

$$M\left(\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\ U & & V \end{tikzcd}\right) = \left(M(U) \xrightarrow{M^*(s_1)} M(S) \xrightarrow{M_*(s_2)} M(V) \right).$$

We need to see that M is well-defined. If $h : S \longrightarrow S'$ is an isomorphism, then the following diagram

$$\begin{tikzcd}
S' \arrow[r, "1"] \arrow[d, "h^{-1}"'] & S' \arrow[d, "1"] \\
S \arrow[r, "h"'] & S'
\end{tikzcd}$$

is a pull back diagram. Therefore $M^*(h^{-1}) = M_*(h)$ and $M_*(h^{-1}) = M^*(h)$. This gives, $M_*(h)^{-1} = M^*(h)$. So if $h : (s_1, S, s_2) \longrightarrow (s'_1, S', s'_2)$ is an isomorphism of spans, we have the following commutative diagram.

$$\begin{tikzcd}
 & M(S) \arrow[dr, "M_*(s_2)"] \arrow[dd, shift left, "M_*(h)"] & \\
M(U) \arrow[ur, "M^*(s_1)"] \arrow[dr, "M^*(s'_1)"'] & & M(V) \\
 & M(S') \arrow[uu, shift left, "M^*(h)"] \arrow[ur, "M_*(s'_2)"'] &
\end{tikzcd}$$

Therefore we get

$$M_*(s_2)M^*(s_1) = M_*(s'_2)M^*(s'_1).$$

From this definition M becomes a functor, since

$$M\left(\begin{tikzcd}
 & & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
 & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \text{pb} & T \arrow[dl, "t_1"'] \arrow[dr, "t_2"] & \\
U & & V & & W
\end{tikzcd}\right) = \begin{tikzcd}
M(U) \arrow[rr, "M^*(p_1 s_1)"] \arrow[dr, "M^*(s_1)"'] & & M(P) \arrow[rr, "M_*(t_2 p_2)"] \arrow[dr, "M_*(p_2)"] & & M(W) \\
 & M(S) \arrow[ur, "M^*(p_1)"] \arrow[dr, "M_*(s_2)"'] & \text{Mackey} & M(T) \arrow[ur, "M_*(t_2)"'] & \\
 & & M(V) \arrow[ur, "M^*(t_1)"'] & &
\end{tikzcd}$$

$$= (\ M(U) \xrightarrow{M(s_1,S,s_2)} M(V) \xrightarrow{M(t_1,T,t_2)} M(W)\ ),$$

where $P = S \times_V T$ and p_1 and p_2 are the projections 1 and 2 respectively, so that

$$M((t_1, T, t_2) \circ (s_1, S, s_2)) = M(t_1, T, t_2) \circ M(s_1, S, s_2).$$

The value of M at the identity span $(1, U, 1) : U \longrightarrow U$ is given by

$$M\left(\begin{tikzcd}
 & U \arrow[dl, "1"'] \arrow[dr, "1"] & \\
U & & U
\end{tikzcd}\right) = (\ M(U) \xrightarrow{1} M(U) \xrightarrow{1} M(U)\ )$$

$$= (1 :\ M(U) \longrightarrow M(U)\ ).$$

Condition (3) on the Mackey functor clearly is equivalent to the requirement that $M : \mathbf{Spn}(\mathscr{E}) \longrightarrow \mathbf{Mod}_k$ should preserve coproducts.

Conversely, let $M : \mathbf{Spn}(\mathscr{E}) \longrightarrow \mathbf{Mod}_k$ be a functor. Then we can define two functors M_* and M^*, referring to the diagram

$$\begin{tikzcd} \mathscr{E} \arrow[r, "(-)_*"] & \mathbf{Spn}(\mathscr{E}) \arrow[r, "M"] & \mathbf{Mod}_k , \\ \mathscr{E}^{\mathrm{op}} \arrow[ur, "(-)^*"'] & & \end{tikzcd}$$

by putting $M_* = M \circ (-)_*$ and $M^* = M \circ (-)^*$. The Mackey square is obtained by using the functoriality of M on the composite span

$$s^* \circ r_* = (p, P, q) = q_* \circ p^*.$$

The remaining details are routine. □

5. Tensor product of Mackey functors

We now work with a general compact closed category $\mathscr{T}$ with finite products. It follows (see [Ho]) that $\mathscr{T}$ has direct sums and therefore that $\mathscr{T}$ is enriched in the monoidal category $\mathscr{V}$ of commutative monoids. We write $\otimes$ for the tensor product in $\mathscr{T}$, write I for the unit, and write $(-)^*$ for the dual. The main example we have in mind is $\mathbf{Spn}(\mathscr{E})$ as in the last section where $\otimes = \times$, $I = 1$, and $V^* = V$. A Mackey functor on $\mathscr{T}$ is an additive functor $M : \mathscr{T} \longrightarrow \mathbf{Mod}_k$.

Let us review the monoidal structure on the category $\mathscr{V}$ of commutative monoids; the binary operation of the monoids will be written additively. It is monoidal closed. For $A, B \in \mathscr{V}$, the commutative monoid

$$[A, B] = \{f : A \longrightarrow B \mid f \text{ is a commutative monoid morphism}\},$$

with pointwise addition, provides the internal hom and there is a tensor product $A \otimes B$ satisfying

$$\mathscr{V}(A \otimes B, C) \cong \mathscr{V}(A, [B, C]).$$

The construction of the tensor product is as follows. The free commutative monoid FS on a set S is

$$FS = \{u : S \longrightarrow \mathbb{N} \mid u(s) = 0 \text{ for all but a finite number of } s \in S\} \subseteq \mathbb{N}^S.$$

For any $A, B \in \mathscr{V}$,

$$A \otimes B = \left(\begin{array}{r} F(A \times B) / (a + a', b) \sim (a, b) + (a', b) \\ (a, b + b') \sim (a, b) + (a, b') \end{array} \right).$$

We regard $\mathscr{T}$ and $\mathbf{Mod}_k$ as $\mathscr{V}$-categories. Every $\mathscr{V}$-functor $\mathscr{T} \longrightarrow \mathbf{Mod}_k$ preserves finite direct sums. So $[\mathscr{T}, \mathbf{Mod}_k]_+$ is the $\mathscr{V}$-category of $\mathscr{V}$-functors.

For each $V \in \mathscr{V}$ and X an object of a $\mathscr{V}$-category $\mathscr{X}$, we write $V \otimes X$ for the object (when it exists) satisfying

$$\mathscr{X}(V \otimes X, Y) \cong [V, \mathscr{X}(X, Y)]$$

$\mathscr{V}$-naturally in Y. Also the coend we use is in the $\mathscr{V}$-enriched sense: for the functor $T : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \longrightarrow \mathscr{X}$, we have a coequalizer

$$\sum_{V,W} \mathscr{C}(V, W) \otimes T(W, V) \rightrightarrows \sum_V T(V, V) \longrightarrow \int^V T(V, V)$$

when the coproducts and tensors exist.

The tensor product of Mackey functors can be defined by convolution (in the sense of [Da1]) in $[\mathscr{T}, \mathbf{Mod}_k]_+$ since $\mathscr{T}$ is a monoidal category. For Mackey functors M and N, the tensor product $M * N$ can be written as follows:

$$\begin{aligned}(M * N)(Z) &= \int^{X,Y} \mathscr{T}(X \otimes Y, Z) \otimes M(X) \otimes_k N(Y) \\ &\cong \int^{X,Y} \mathscr{T}(Y, X^* \otimes Z) \otimes M(X) \otimes_k N(Y) \\ &\cong \int^{X} M(X) \otimes_k N(X^* \otimes Z) \\ &\cong \int^{Y} M(Z \otimes Y^*) \otimes_k N(Y).\end{aligned}$$

the last two isomorphisms are given by the Yoneda lemma.

The *Burnside functor* J is defined to be the Mackey functor on $\mathscr{T}$ taking an object U of $\mathscr{T}$ to the free k-module on $\mathscr{T}(I, U)$. The Burnside functor is the unit for the tensor product of the category **Mky**.

This convolution satisfies the necessary commutative and associative conditions for a symmetric monoidal category (see [Da1]). $[\mathscr{T}, \mathbf{Mod}_k]_+$ is also an abelian category (see [Fr]).

When $\mathscr{T}$ and k are understood, we simply write **Mky** for this category $[\mathscr{T}, \mathbf{Mod}_k]_+$.

6. The Hom functor

We now make explicit the closed structure on **Mky**. The Hom Mackey functor is defined by taking its value at the Mackey functors M and N to be

$$\mathrm{Hom}(M, N)(V) = \mathbf{Mky}(M(V^* \otimes -), N),$$

functorially in V. To see that this hom has the usual universal property with respect to tensor, notice that we have the natural bijections below (represented by horizontal lines).

$$\begin{array}{c} (L * M)(U) \longrightarrow N(U) \text{ natural in } U \\ \hline L(V) \otimes_k M(V^* \otimes U) \longrightarrow N(U) \text{ natural in } U \text{ and dinatural in } V \\ \hline L(V) \longrightarrow \mathrm{Hom}_k(M(V^* \otimes U), N(U)) \text{ dinatural in } U \text{ and natural in } V \\ \hline L(V) \longrightarrow \displaystyle\int_U \mathrm{Hom}_k(M(V^* \otimes U), N(U)) \text{ natural in } V \\ \hline L(V) \longrightarrow \mathbf{Mky}(M(V^* \otimes -), N) \text{ natural in } V \end{array}$$

We can obtain another expression for the hom using the isomorphism

$$\mathscr{T}(V \otimes U, W) \cong \mathscr{T}(U, V^* \otimes W)$$

which shows that we have adjoint functors

$$\begin{tikzcd} \mathscr{T} \arrow[r, bend left, "V\otimes -"] \arrow[r, phantom, "\perp"] & \mathscr{T} \arrow[l, bend left, "V^*\otimes -"] \end{tikzcd} .$$

Since M and N are Mackey functors on $\mathscr{T}$, we obtain a diagram

$$\begin{tikzcd}
\mathscr{T} \arrow[rr, bend left, "V\otimes -"] \arrow[rr, phantom, "\perp"] \arrow[dr, "N"'] & & \mathscr{T} \arrow[ll, bend left, "V^*\otimes -"] \arrow[dl, "M"] \\
& \mathbf{Mod}_k &
\end{tikzcd}$$

and an equivalence of natural transformations

$$\frac{M \Longrightarrow N(V \otimes -)}{M(V^* \otimes -) \Longrightarrow N.}$$

Therefore, the Hom Mackey functor is also given by

$$\mathrm{Hom}(M, N)(V) = \mathbf{Mky}(M, N(V \otimes -)).$$

7. Green functors

A *Green functor* A on $\mathscr{T}$ is a Mackey functor (that is, a coproduct preserving functor $A : \mathscr{T} \longrightarrow \mathbf{Mod}_k$) equipped with a monoidal structure made up of a natural transformation

$$\mu : A(U) \otimes_k A(V) \longrightarrow A(U \otimes V),$$

for which we use the notation $\mu(a \otimes b) = a.b$ for $a \in A(U)$, $b \in A(V)$, and a morphism

$$\eta : k \longrightarrow A(1),$$

whose value at $1 \in k$ we denote by 1. Green functors are the monoids in **Mky**. If $A, B : \mathscr{T} \longrightarrow \mathbf{Mod}_k$ are Green functors then we have an isomorphism

$$\mathbf{Mky}(A * B, C) \cong \mathrm{Nat}_{U,V}(A(U) \otimes_k B(V), C(U \otimes V)).$$

Referring to the square

$$\begin{tikzcd}
\mathscr{T} \otimes \mathscr{T} \arrow[r, "A\otimes B"] \arrow[d, "\otimes"'] & \mathbf{Mod}_k \otimes \mathbf{Mod}_k \arrow[d, "\otimes_k"] \\
\mathscr{T} \arrow[r, "C"'] & \mathbf{Mod}_k ,
\end{tikzcd}$$

we write this more precisely as

$$\mathbf{Mky}(A * B, C) \cong [\mathscr{T} \otimes \mathscr{T}, \mathbf{Mod}_k](\otimes_k \circ (A \otimes B), C \circ \otimes).$$

The Burnside functor J and $\mathrm{Hom}(A, A)$ (for any Mackey functor A) are monoids in **Mky** and so are Green functors.

We denote by $\mathbf{Grn}(\mathscr{T}, \mathbf{Mod}_k)$ the category of Green functors on $\mathscr{T}$. When $\mathscr{T}$ and k are understood, we simply write this as $\mathbf{Grn}(= \mathbf{Mon}(\mathbf{Mky}))$ consisting of the monoids in **Mky**.

8. DRESS CONSTRUCTION

The Dress construction ([Bo2], [Bo3]) provides a family of endofunctors $D(Y,-)$ of the category **Mky**, indexed by the objects Y of $\mathcal{T}$. The Mackey functor defined as the composite

$$\mathcal{T} \xrightarrow{-\otimes Y} \mathcal{T} \xrightarrow{M} \mathbf{Mod}_k$$

is denoted by M_Y for $M \in \mathbf{Mky}$; so $M_Y(U) = M(U \otimes Y)$. We then define the *Dress construction*

$$D : \mathcal{T} \otimes \mathbf{Mky} \longrightarrow \mathbf{Mky}$$

by $D(Y,M) = M_Y$. The $\mathcal{V}$-category $\mathcal{T} \otimes \mathbf{Mky}$ is monoidal via the pointwise structure:

$$(X,M) \otimes (Y,N) = (X \otimes Y, M * N).$$

Proposition 8.1. *The Dress construction*

$$D : \mathcal{T} \otimes \mathbf{Mky} \longrightarrow \mathbf{Mky}$$

is a strong monoidal $\mathcal{V}$-functor.

Proof. We need to show that $D((X,M)\otimes(Y,N)) \cong D(X,M) * D(Y,N)$; that is, $M_X * M_Y \cong (M*N)_{X\otimes Y}$. For this we have the calculation

$$\begin{aligned}
(M_X * N_Y)(Z) &\cong \int^U M_X(U) \otimes_k N_Y(U^* \otimes Z) \\
&= \int^U M(U \otimes X) \otimes_k N(U^* \otimes Z \otimes Y) \\
&\cong \int^{U,V} \mathcal{T}(V, U \otimes X) \otimes M(V) \otimes_k N(U^* \otimes Z \otimes Y) \\
&\cong \int^{U,V} \mathcal{T}(V \otimes X^*, U) \otimes M(V) \otimes_k N(U^* \otimes Z \otimes Y) \\
&\cong \int^{V} M(V) \otimes_k N(V^* \otimes X \otimes Z \otimes Y) \\
&\cong (M*N)(Z \otimes X \otimes Y) \\
&\cong (M*N)_{X\otimes Y}(Z).
\end{aligned}$$

Clearly we have $D(I,J) \cong J$. The coherence conditions are readily checked. □

We shall analyse this situation more fully in Remark 8.5 below.

We are interested, after [Bo2], in when the Dress construction induces a family of endofunctors on the category **Grn** of Green functors. That is to say, when is there a natural structure of Green functor on $A_Y = D(Y,A)$ if A is a Green functor? Since A_Y is the composite

$$\mathcal{T} \xrightarrow{-\otimes Y} \mathcal{T} \xrightarrow{A} \mathbf{Mod}_k$$

with A monoidal, what we require is that $- \otimes Y$ should be monoidal (since monoidal functors compose). For this we use Theorem 3.7 of [DPS]:

if Y is a monoid in the lax centre $\mathcal{Z}_l(\mathcal{T})$ of $\mathcal{T}$ then $- \otimes Y : \mathcal{T} \longrightarrow \mathcal{T}$ is canonically monoidal.

Let $\mathcal{C}$ be a monoidal category. The lax centre $\mathcal{Z}_l(\mathcal{C})$ of $\mathcal{C}$ is defined to have objects the pairs (A,u) where A is an object of $\mathcal{C}$ and u is a natural family of morphisms u_B :

$A\otimes B \longrightarrow B\otimes A$ such that the following two diagrams commute

$$\begin{tikzcd} A\otimes B\otimes C \arrow[rr, "u_{B\otimes C}"] \arrow[dr, "u_B\otimes 1_C"'] & & B\otimes C\otimes A & A\otimes I \arrow[rr, "u_I"] \arrow[dr, "\cong"'] & & I\otimes A \arrow[dl, "\cong"] \\ & B\otimes A\otimes C \arrow[ur, "1_B\otimes u_C"'] & & & A\,. & \end{tikzcd}$$

Morphisms of $\mathcal{Z}_l(\mathscr{C})$ are morphisms in $\mathscr{C}$ compatible with the u. The tensor product is defined by

$$(A,u)\otimes(B,v) = (A\otimes B, w)$$

where $w_C = (u_C\otimes 1_B)\circ(1_A\otimes v_C)$. The centre $\mathcal{Z}(\mathscr{C})$ of $\mathscr{C}$ consists of the objects (A,u) of $\mathcal{Z}_l(\mathscr{C})$ with each u_B invertible.

It is pointed out in [DPS] that, when $\mathscr{C}$ is cartesian monoidal, an object of $\mathcal{Z}_l(\mathscr{C})$ is merely an object A of $\mathscr{C}$ together with a natural family $A\times X \longrightarrow X$. Then we have the natural bijections below (represented by horizontal lines) for $\mathscr{C}$ cartesian closed:

$$\frac{\dfrac{A\times X \longrightarrow X \text{ natural in } X}{A\longrightarrow [X,X] \text{ dinatural in } X}}{A\longrightarrow \int_X [X,X] \text{ in } \mathscr{C}.}$$

Therefore we obtain an equivalence $\mathcal{Z}_l(\mathscr{C}) \simeq \mathscr{C}/\int_X [X,X]$.

The internal hom in $\mathscr{E}$, the category of finite G-sets for the finite group G, is $[X,Y]$ which is the set of functions $r : X \longrightarrow Y$ with $(g.r)(x) = gr(g^{-1}x)$. The G-set $\int_X [X,X]$ is defined by

$$\int_X [X,X] = \Big\{ r = (r_X : X \longrightarrow X) \;\Big|\; f\circ r_X = r_Y\circ f \text{ for all } G\text{-maps } f : X\longrightarrow Y \Big\}$$

with $(g.r)_X(x) = g r_X(g^{-1}x)$.

Lemma 8.2. *The G-set $\int_X [X,X]$ is isomorphic to G_c, which is the set G made a G-set by conjugation action.*

Proof. Take $r\in\int_X [X,X]$. Then we have the commutative square

$$\begin{tikzcd} G \arrow[r, "r_G"] \arrow[d, "\hat{x}"'] & G \arrow[d, "\hat{x}"] \\ X \arrow[r, "r_X"'] & X \end{tikzcd}$$

where $\hat{x}(g) = gx$ for $x\in X$. So we see that r_X is determined by $r_G(1)$ and

$$\begin{aligned} (g.r)_G(1) &= g r_G(g^{-1}1) \\ &= g r_G(g^{-1}) \\ &= g r_G(1) g^{-1}. \end{aligned}$$

□

As a consequence of this Lemma, we have $\mathcal{Z}_l(\mathscr{E}) \simeq \mathscr{E}/G_c$ where $\mathscr{E}/G_c$ is the category of crossed G-sets of Freyd-Yetter ([FY1], [FY2]) who showed that $\mathscr{E}/G_c$ is a braided monoidal category. Objects are pairs $(X, |\;|)$ where X is a G-set and $|\;| : X \longrightarrow G_c$ is a G-set morphism ("equivariant function") meaning $|gx| = g|x|g^{-1}$ for $g\in G$, $x\in X$. The morphisms $f : (X, |\ |) \longrightarrow (Y, |\ |)$ are functions f such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "|\ |"'] & & Y \arrow[dl, "|\ |"] \\ & G_c & \end{tikzcd}$$

That is, $f(gx) = gf(x)$.

Tensor product is defined by

$$(X, |\ |) \otimes (Y, |\ |) = (X \times Y, \|\ \|),$$

where $\|(x, y)\| = |x||y|$.

Proposition 8.3. [DPS, Theorem 4.5] *The centre $\mathscr{Z}(\mathscr{E})$ of the category $\mathscr{E}$ is equivalent to the category $\mathscr{E}/G_c$ of crossed G-sets.*

Proof. We have a fully faithful functor $\mathscr{Z}(\mathscr{E}) \longrightarrow \mathscr{Z}_l(\mathscr{E})$ and so $\mathscr{Z}(\mathscr{E}) \longrightarrow \mathscr{E}/G_c$. On the other hand, let $|\ | : A \longrightarrow G_c$ be an object of $\mathscr{E}/G_c$; so $|ga|g = g|a|$. Then the corresponding object of $\mathscr{Z}_l(\mathscr{E})$ is (A, u) where

$$u_X : A \times X \longrightarrow X \times A$$

with

$$u_X(a, x) = (|a|x, a).$$

However this u is invertible since we see that

$$u_X{}^{-1}(x, a) = (a, |a|^{-1}x).$$

This proves the proposition. □

Theorem 8.4. [Bo3, Bo2] *If Y is a monoid in $\mathscr{E}/G_c$ and A is a Green functor for $\mathscr{E}$ over k then A_Y is a Green functor for $\mathscr{E}$ over k, where $A_Y(X) = A(X \times Y)$.*

Proof. We have $\mathscr{Z}(\mathscr{E}) \simeq \mathscr{E}/G_c$, so Y is a monoid in $\mathscr{Z}(\mathscr{E})$. This implies $- \times Y : \mathscr{E} \longrightarrow \mathscr{E}$ is a monoidal functor (see Theorem 3.7 of [DPS]). It also preserves pullbacks. So $- \times Y : \mathbf{Spn}(\mathscr{E}) \longrightarrow \mathbf{Spn}(\mathscr{E})$ is a monoidal functor . If A is a Green functor for $\mathscr{E}$ over k then $A : \mathbf{Spn}(\mathscr{E}) \longrightarrow \mathbf{Mod}_k$ is monoidal. Then we get $A_Y = A \circ (- \times Y) : \mathbf{Spn}(\mathscr{E}) \longrightarrow \mathbf{Mod}_k$ is monoidal and A_Y is indeed a Green functor for $\mathscr{E}$ over k. □

Remark 8.5. The reader may have noted that Proposition 8.1 implies that D takes monoids to monoids. A monoid in $\mathscr{T} \otimes \mathbf{Mky}$ is a pair (Y, A) where Y is a monoid in $\mathscr{T}$ and A is a Green functor; so in this case, we have that A_Y is a Green functor. A monoid Y in $\mathscr{E}$ is certainly a monoid in $\mathscr{T}$. Since $\mathscr{E}$ is cartesian monoidal (and so symmetric), each monoid in $\mathscr{E}$ gives one in the centre. However, not every monoid in the centre arises in this way. The full result behind Proposition 8.1 and the centre situation is: the Dress construction

$$D : \mathscr{Z}(\mathscr{T}) \otimes \mathbf{Mky} \longrightarrow \mathbf{Mky}$$

is a strong monoidal $\mathscr{V}$-functor; it is merely monoidal when the centre is replaced by the lax centre.

It follows that A_Y is a Green functor whenever A is a Green functor and Y is a monoid in the lax centre of $\mathscr{T}$.

9. Finite dimensional Mackey functors

We make the following further assumptions on the symmetric compact closed category $\mathscr{T}$ with finite direct sums:

- there is a finite set $\mathscr{C}$ of objects of $\mathscr{T}$ such that every object X of $\mathscr{T}$ can be written as a direct sum
$$X \cong \bigoplus_{i=1}^{n} C_i$$
with C_i in $\mathscr{C}$; and
- each hom $\mathscr{T}(X, Y)$ is a finitely generated commutative monoid.

Notice that these assumptions hold when $\mathscr{T} = \mathbf{Spn}(\mathscr{E})$ where $\mathscr{E}$ is the category of finite G-sets for a finite group G. In this case we can take $\mathscr{C}$ to consist of a representative set of connected (transitive) G-sets. Moreover, the spans $S : X \longrightarrow Y$ with $S \in \mathscr{C}$ generate the monoid $\mathscr{T}(X, Y)$.

We also fix k to be a field and write **Vect** in place of $\mathbf{Mod}_k$.

A Mackey functor $M : \mathscr{T} \longrightarrow \mathbf{Vect}$ is called *finite dimensional* when each $M(X)$ is a finite-dimensional vector space. Write $\mathbf{Mky}_{fin}$ for the full subcategory of **Mky** consisting of these.

We regard $\mathscr{C}$ as a full subcategory of $\mathscr{T}$. The inclusion functor $\mathscr{C} \longrightarrow \mathscr{T}$ is dense and the density colimit presentation is preserved by all additive $M : \mathscr{T} \longrightarrow \mathbf{Vect}$. This is shown as follows:

$$\begin{aligned}
\int^{C} \mathscr{T}(C, X) \otimes M(C) &\cong \int^{C} \mathscr{T}(C, \bigoplus_{i=1}^{n} C_i) \otimes M(C) \\
&\cong \bigoplus_{i=1}^{n} \int^{C} \mathscr{T}(C, C_i) \otimes M(C) \\
&\cong \bigoplus_{i=1}^{n} \int^{C} \mathscr{C}(C, C_i) \otimes M(C) \\
&\cong \bigoplus_{i=1}^{n} M(C_i) \\
&\cong M(\bigoplus_{i=1}^{n} C_i) \\
&\cong M(X).
\end{aligned}$$

That is,
$$M \cong \int^{C} \mathscr{T}(C, -) \otimes M(C).$$

Proposition 9.1. *The tensor product of finite-dimensional Mackey functors is finite dimensional.*

Proof. Using the last isomorphism, we have

$$\begin{aligned}
(M * N)(Z) &= \int^{X,Y} \mathscr{T}(X \otimes Y, Z) \otimes M(X) \otimes_k N(Y) \\
&\cong \int^{X,Y,C,D} \mathscr{T}(X \otimes Y, Z) \otimes \mathscr{T}(C, X) \otimes \mathscr{T}(D, Y) \otimes M(C) \otimes_k N(D) \\
&\cong \int^{C,D} \mathscr{T}(C \otimes D, Z) \otimes M(C) \otimes_k N(D).
\end{aligned}$$

If M and N are finite dimensional then so is the vector space $\mathscr{T}(C \otimes D, Z) \otimes M(C) \otimes_k N(D)$ (since $\mathscr{T}(C \otimes D, Z)$ is finitely generated). Also the coend is a quotient of a finite direct sum. So $M * N$ is finite dimensional. $\square$

It follows that $\mathbf{Mky}_{fin}$ is a monoidal subcategory of **Mky** (since the Burnside functor J is certainly finite dimensional under our assumptions on $\mathscr{T}$).

The promonoidal structure on $\mathbf{Mky}_{fin}$ represented by this monoidal structure can be expressed in many ways:

$$\begin{aligned} P(M,N;L) &= \mathbf{Mky}_{fin}(M * N, L) \\ &\cong \mathrm{Nat}_{X,Y,Z}(\mathscr{T}(X \otimes Y, Z) \otimes M(X) \otimes_k N(Y), L(Z)) \\ &\cong \mathrm{Nat}_{X,Y}(M(X) \otimes_k N(Y), L(X \otimes Y)) \\ &\cong \mathrm{Nat}_{X,Z}(M(X) \otimes_k N(X^* \otimes Z), L(Z)) \\ &\cong \mathrm{Nat}_{Y,Z}(M(Z \otimes Y^*) \otimes_k N(Y), L(Z)). \end{aligned}$$

Following the terminology of [DS1], we say that a promonoidal category $\mathscr{M}$ is $*$-*autonomous* when it is equipped with an equivalence $S : \mathscr{M}^{\mathrm{op}} \longrightarrow \mathscr{M}$ and a natural isomorphism

$$P(M,N;S(L)) \cong P(N,L;S^{-1}(M)).$$

A monoidal category is $*$-autonomous when the associated promonoidal category is.

As an application of the work of Day [Da4] we obtain that $\mathbf{Mky}_{fin}$ is $*$-autonomous. We shall give the details.

For $M \in \mathbf{Mky}_{fin}$, define $S(M)(X) = M(X^*)^*$ so that $S : \mathbf{Mky}_{fin}^{\mathrm{op}} \longrightarrow \mathbf{Mky}_{fin}$ is its own inverse equivalence.

Theorem 9.2. *The monoidal category* $\mathbf{Mky}_{fin}$ *of finite-dimensional Mackey functors on* $\mathscr{T}$ *is* $*$*-autonomous.*

Proof. With S defined as above, we have the calculation:

$$\begin{aligned} P(M,N;S(L)) &\cong \mathrm{Nat}_{X,Y}(M(X) \otimes_k N(Y), L(X^* \otimes Y^*)^*) \\ &\cong \mathrm{Nat}_{X,Y}(N(Y) \otimes_k L(X^* \otimes Y^*), M(X)^*) \\ &\cong \mathrm{Nat}_{Z,Y}(N(Y) \otimes_k L(Z \otimes Y^*), M(Z^*)^*) \\ &\cong \mathrm{Nat}_{Z,Y}(N(Y) \otimes_k L(Z \otimes Y^*), S(M)(Z)) \\ &\cong P(N,L;S(M)). \end{aligned}$$

$\square$

10. Cohomological Mackey functors

Let k be a field and G be a finite group. We are interested in the relationship between ordinary k-linear representations of G and Mackey functors on G.

Write $\mathscr{E}$ for the category of finite G-sets as usual. Write $\mathscr{R}$ for the category $\mathbf{Rep}_k(G)$ of finite -dimensional k-linear representations of G.

Each G-set X determines a k-linear representation kX of G by extending the action of G linearly on X. This gives a functor

$$k : \mathscr{E} \longrightarrow \mathscr{R}.$$

We extend this to a functor

$$k_* : \mathscr{T}^{\mathrm{op}} \longrightarrow \mathscr{R},$$

where $\mathscr{T} = \mathbf{Spn}(\mathscr{E})$, as follows. On objects $X \in \mathscr{T}$, define

$$k_* X = kX.$$

For a span $(u, S, v) : X \longrightarrow Y$ in $\mathscr{E}$, the linear function $k_*(S) : kY \longrightarrow kX$ is defined by

$$k_*(S)(y) = \sum_{v(s)=y} u(s)\ ;$$

this preserves the G-actions since

$$k_*(S)(gy) = \sum_{v(s)=gy} u(s) = \sum_{v(g^{-1}s)=y} gu(g^{-1}s) \ = \ gk_*(S)(y).$$

Clearly k_* preserves coproducts.

By the usual argument (going back to Kan, and the geometric realization and singular functor adjunction), we obtain a functor

$$\widetilde{k_*} : \mathscr{R} \longrightarrow \mathbf{Mky}(G)_{fin}$$

defined by

$$\widetilde{k_*}(R) = \mathscr{R}(k_* -, R)$$

which we shall write as $R^- : \mathscr{T} \longrightarrow \mathbf{Vect}_k$. So

$$R^X = \mathscr{R}(k_* X, R) \cong G\text{-}\mathbf{Set}(X, R)$$

with the effect on the span $(u, S, v) : X \longrightarrow Y$ transporting to the linear function

$$G\text{-}\mathbf{Set}(X, R) \longrightarrow G\text{-}\mathbf{Set}(Y, R)$$

which takes $\tau : X \longrightarrow R$ to $\tau_S : Y \longrightarrow R$ where

$$\tau_S(y) = \sum_{v(s)=y} \tau(u(s)).$$

The functor $\widetilde{k_*}$ has a left adjoint

$$colim(-, k_*) : \mathbf{Mky}(G)_{fin} \longrightarrow \mathscr{R}$$

defined by

$$colim(M, k_*) = \int^C M(C) \otimes_k k_* C$$

where C runs over a full subcategory $\mathscr{C}$ of $\mathscr{T}$ consisting of a representative set of connected G-sets.

Proposition 10.1. *The functor* $\widetilde{k_*} : \mathbf{Rep}_k(G) \longrightarrow \mathbf{Mky}(G)$ *is fully faithful.*

Proof. For $R_1, R_2 \in \mathscr{R}$, a morphism $\theta : R_1^- \longrightarrow R_2^-$ in $\mathbf{Mky}(G)$ is a family of linear functions θ_X such that the following square commutes for all spans $(u, S, v) : X \longrightarrow Y$ in $\mathscr{E}$.

$$\begin{tikzcd}
G\text{-}\mathbf{Set}(X, R_1) \arrow[r, "\theta_X"] \arrow[d, "(-)_S"'] & G\text{-}\mathbf{Set}(X, R_2) \arrow[d, "(-)_S"] \\
G\text{-}\mathbf{Set}(Y, R_1) \arrow[r, "\theta_Y"'] & G\text{-}\mathbf{Set}(Y, R_2)
\end{tikzcd}$$

Since G (with multiplication action) forms a full dense subcategory of G-$\mathbf{Set}$, it follows that we obtain a unique morphism $f : R_1 \longrightarrow R_2$ in G-$\mathbf{Set}$ such that

$$f(r) = \theta_G(\hat{r})(1)$$

(where $\hat{r} : G \longrightarrow R$ is defined by $\hat{r}(g) = gr$ for $r \in R$); this is a special case of Yoneda's Lemma. Clearly f is linear since θ_G is. By taking $Y = G, S = G$ and $v = 1_G : G \longrightarrow G$, commutativity of the above square yields

$$\theta_X(\tau)(x) = f(\tau(x));$$

that is, $\theta_X = \widetilde{k_*}(f)_X$. □

An important property of Mackey functors in the image of $\widetilde{k_*}$ is that they are *cohomological* in the sense of [We], [Bo4] and [TW]. First we recall some classical terminology associated with a Mackey functor M on a group G.

For subgroups $K \leq H$ of G, we have the canonical G-set morphism $\sigma_K^H : G/K \longrightarrow G/H$ defined on the connected G-sets of left cosets by $\sigma_K^H(gK) = gH$. The linear functions

$$r_K^H = M_*(\sigma_K^H) : M(G/H) \longrightarrow M(G/K) \quad \text{and}$$
$$t_K^H = M^*(\sigma_K^H) : M(G/K) \longrightarrow M(G/H)$$

are called *restriction* and *transfer* (or *trace* or *induction*).

A Mackey functor M on G is called *cohomological* when each composite $t_K^H r_K^H : M(G/H) \longrightarrow M(G/H)$ is equal to multiplication by the index $[H : K]$ of K in H. We supply a proof of the following known example.

Proposition 10.2. *For each k-linear representation R of G, the Mackey functor $\widetilde{k_*}(R) = R^-$ is cohomological.*

Proof. With $M = R^-$ and $\sigma = \sigma_K^H$, notice that the function

$$t_K^H r_K^H = M^*(\sigma)M_*(\sigma) = M(\sigma, G/K, 1)M(1, G/K, \sigma) = M(\sigma, G/K, \sigma)$$

takes $\tau \in \mathscr{E}(G/H, R)$ to $\tau_{G/K} \in \mathscr{E}(G/H, R)$ where

$$\tau_{G/K}(H) = \sum_{\sigma(s)=H} \tau(\sigma(s)) = \sum_{\sigma(s)=H} \tau(H) = \Big(\sum_{\sigma(s)=H} 1\Big)\tau(H)$$

and s runs over the distinct gK with $\sigma(s) = gH = H$; the number of distinct gK with $g \in H$ is of course $[H : K]$. So $\tau_{G/K}(xH) = [H : K]\tau(xH)$. □

Lemma 10.3. *The functor $k_* : \mathscr{T}^{op} \longrightarrow \mathscr{R}$ is strong monoidal.*

Proof. Clearly the canonical isomorphisms

$$k(X_1 \times X_2) \cong kX_1 \otimes kX_2, \quad k1 \cong k$$

show that $k : \mathscr{E} \longrightarrow \mathscr{R}$ is strong monoidal. All that remains to be seen is that these isomorphisms are natural with respect to spans $(u_1, S_1, v_1) : X_1 \longrightarrow Y_1, (u_2, S_2, v_2) : X_2 \longrightarrow Y_2$. This comes down to the bilinearity of tensor product:

$$\sum_{\substack{v_1(s_1)=y_1 \\ v_2(s_2)=y_2}} u_1(s_1) \otimes u_2(s_2) = \sum_{v_1(s_1)=y_1} u_1(y_1) \otimes \sum_{v_2(s_2)=y_2} u_2(y_2).$$

□

We can now see that the adjunction

$$colim(-, k_*) \dashv \widetilde{k_*}$$

fits the situation of Day's Reflection Theorem [Da2] and [Da3, pages 24 and 25]. For this, recall that a fully faithful functor $\Phi : \mathscr{A} \longrightarrow \mathscr{X}$ into a closed category $\mathscr{X}$ is said to be *closed under exponentiation* when, for all A in $\mathscr{A}$ and X in $\mathscr{X}$, the internal hom $[X, \Phi A]$ is isomorphic to an object of the form ΦB for some B in $\mathscr{A}$.

Theorem 10.4. *The functor $colim(-,k_*) : \mathbf{Mky}(G)_{fin} \longrightarrow \mathscr{R}$ is strong monoidal. Consequently, $\widetilde{k_*} : \mathscr{R} \longrightarrow \mathbf{Mky}(G)_{fin}$ is monoidal and closed under exponentiation.*

Proof. The first sentence follows quite formally from Lemma 10.3 and the theory of Day convolution; the main calculation is:

$$\begin{aligned} colim(M*N,k_*)(Z) &= \int^{C} (M*N)(C) \otimes_k k_* C \\ &= \int^{C,X,Y} \mathscr{T}(X\times Y, C) \otimes M(X) \otimes_k N(Y) \otimes_k k_* C \\ &\cong \int^{X,Y} M(X) \otimes_k N(Y) \otimes_k k_*(X\times Y) \\ &\cong \int^{X,Y} M(X) \otimes_k N(Y) \otimes_k k_* X \otimes k_* Y \\ &\cong colim(M,k_*) \otimes colim(N,k_*). \end{aligned}$$

The second sentence then follows from [Da2, Reflection Theorem]. □

In fancier words, the adjunction

$$colim(-,k_*) \dashv \widetilde{k_*}$$

lives in the 2-category of monoidal categories, monoidal functors and monoidal natural transformations (all enriched over $\mathscr{V}$).

11. Mackey functors for Hopf algebras

In this section we provide another example of a compact closed category $\mathscr{T}$ constructed from a Hopf algebara H (or *quantum group*). We speculate that Mackey functors on this $\mathscr{T}$ will prove as useful for Hopf algebras as usual Mackey functors have for groups.

Let H be a braided (semisimple) Hopf algebra (over k). Let $\mathscr{R}$ denote the category of left H-modules which are finite dimensional as vector spaces (over k). This is a compact closed braided monoidal category.

We write $\mathbf{Comod}(\mathscr{R})$ for the category obtained from the bicategory of that name in [DMS] by taking isomorphisms classes of morphisms. Explicitly, the objects are comonoids C in $\mathscr{R}$. The morphisms are isomorphism classes of comodules $S : C \to D$ from C to D; such an S is equipped with a coaction $\delta : S \longrightarrow C \otimes S \otimes D$ satisfying the coassociativity and counity conditions; we can break the two-sided coaction δ into a left coaction $\delta_l : S \longrightarrow C \otimes S$ and a right coaction $\delta_r : S \longrightarrow S \otimes D$ connected by the bicomodule condition. Composition of comodules $S : C \to D$ and $T : D \to E$ is defined by the (coreflexive) equalizer

$$S \otimes_D T \longrightarrow S \otimes T \underset{\delta_r \otimes 1}{\overset{1\otimes\delta_l}{\rightrightarrows}} S \otimes D \otimes T\ .$$

The identity comodule of C is $C : C \to C$. The category $\mathbf{Comod}(\mathscr{R})$ is compact closed: the tensor product is just that for vector spaces equipped with the extra structure. Direct sums in $\mathbf{Comod}(\mathscr{R})$ are given by direct sum as vector spaces. Consequently, $\mathbf{Comod}(\mathscr{R})$ is enriched in the monoidal category $\mathscr{V}$ of commutative monoids: to add comodules

$S_1 : C \rightarrow\!\!\!\!\!+ D$ and $S_2 : C \rightarrow\!\!\!\!\!+ D$, we take the direct sum $S_1 \oplus S_2$ with coaction defined as the composite

$$S_1 \oplus S_2 \xrightarrow{\delta_1 \oplus \delta_2} C \otimes S_1 \otimes D \oplus C \otimes S_2 \otimes D \cong C \otimes (S_1 \oplus S_2) \otimes D.$$

We can now apply our earlier theory to the example $\mathscr{T} = \mathbf{Comod}(\mathscr{R})$. In particular, we call a $\mathscr{V}$-enriched functor $M : \mathbf{Comod}(\mathscr{R}) \longrightarrow \mathbf{Vect}_k$ a *Mackey functor on H*.

In the case where H is the group algebra kG (made Hopf by means of the diagonal $kG \longrightarrow k(G \times G) \cong kG \otimes_k kG$), a Mackey functor on H is not the same as a Mackey functor on G. However, there is a strong relationship that we shall now explain.

As usual, let $\mathscr{E}$ denote the cartesian monoidal category of finite G-sets. The functor $k : \mathscr{E} \longrightarrow \mathscr{R}$ is strong monoidal and preserves coreflexive equalizers. There is a monoidal equivalence

$$\mathbf{Comod}(\mathscr{E}) \simeq \mathbf{Spn}(\mathscr{E}),$$

so $k : \mathscr{E} \longrightarrow \mathscr{R}$ induces a strong monoidal $\mathscr{V}$-functor

$$\hat{k} : \mathbf{Spn}(\mathscr{E}) \longrightarrow \mathbf{Comod}(\mathscr{R}).$$

With $\mathbf{Mky}(G) = [\mathbf{Spn}(\mathscr{E}), \mathbf{Vect}]_+$ as usual and with $\mathbf{Mky}(kG) = [\mathbf{Comod}(\mathscr{R}), \mathbf{Vect}]_+$, we obtain a functor

$$[\hat{k}, 1] : \mathbf{Mky}(kG) \longrightarrow \mathbf{Mky}(G)$$

defined by pre-composition with $\hat{k}$. Proposition 1 of [DS2] applies to yield:

Theorem 11.1. *The functor $[\hat{k}, 1]$ has a strong monoidal left adjoint*

$$\exists_{\hat{k}} : \mathbf{Mky}(G) \longrightarrow \mathbf{Mky}(kG).$$

The adjunction is monoidal.

The formula for $\exists_{\hat{k}}$ is

$$\exists_{\hat{k}}(M)(R) = \int^{X \in \mathbf{Spn}(\mathscr{E})} \mathbf{Comod}(\mathscr{R})(\hat{k}X, R) \otimes M(X).$$

On the other hand, we already have the compact closed category $\mathscr{R}$ of finite-dimensional representations of G and the strong monoidal functor

$$k_* : \mathbf{Spn}(\mathscr{E})^{\mathrm{op}} \longrightarrow \mathscr{R}.$$

Perhaps $\mathscr{R}^{\mathrm{op}} (\simeq \mathscr{R})$ should be our candidate for $\mathscr{T}$ rather than the more complicated $\mathbf{Comod}(\mathscr{R})$. The result of [DS2] applies also to k_* to yield a monoidal adjunction

$$\begin{tikzcd} {[\mathscr{R}^{\mathrm{op}}, \mathbf{Vect}]} \arrow[r, shift right, "{[k_*, 1]}"'] \arrow[r, phantom, "\perp"] & \mathbf{Mky}(G). \arrow[l, shift right, "\exists_{k^*}"'] \end{tikzcd}$$

Perhaps then, additive functors $\mathscr{R}^{\mathrm{op}} \longrightarrow \mathbf{Vect}$ would provide a suitable generalization of Mackey functors in the case of a Hopf algebra H. These matters require investigation at a later time.

12. Review of some enriched category theory

The basic references are [Ke], [La] and [St].

Let $\mathbf{COCT}_{\mathscr{V}}$ denote the 2-category whose objects are cocomplete $\mathscr{V}$-categories and whose morphisms are (weighted-) colimit-preserving $\mathscr{V}$-functors; the 2-cells are $\mathscr{V}$-natural transformations.

Every small $\mathscr{V}$-category $\mathscr{C}$ determines an object $[\mathscr{C},\mathscr{V}]$ of $\mathbf{COCT}_{\mathscr{V}}$. Let

$$Y : \mathscr{C}^{\mathrm{op}} \longrightarrow [\mathscr{C},\mathscr{V}]$$

denote the Yoneda embedding: $YU = \mathscr{C}(U,-)$.

For any object $\mathscr{X}$ of $\mathbf{COCT}_{\mathscr{V}}$, we have an equivalence of categories

$$\mathbf{COCT}_{\mathscr{V}}([\mathscr{C},\mathscr{V}],\mathscr{X}) \simeq [\mathscr{C}^{\mathrm{op}},\mathscr{X}]$$

defined by restriction along Y. This is expressing the fact that $[\mathscr{C},\mathscr{V}]$ is the free cocompletion of $\mathscr{C}^{\mathrm{op}}$. It follows that, for small $\mathscr{V}$-categories $\mathscr{C}$ and $\mathscr{D}$, we have

$$\begin{aligned}\mathbf{COCT}_{\mathscr{V}}([\mathscr{C},\mathscr{V}],[\mathscr{D},\mathscr{V}]) &\simeq [\mathscr{C}^{\mathrm{op}},[\mathscr{D},\mathscr{V}]]\\ &\simeq [\mathscr{C}^{\mathrm{op}}\otimes\mathscr{D},\mathscr{V}].\end{aligned}$$

The way this works is as follows. Suppose $F : \mathscr{C}^{\mathrm{op}}\otimes\mathscr{D}\longrightarrow\mathscr{V}$ is a ($\mathscr{V}$-) functor. We obtain a colimit-preserving functor

$$\widehat{F} : [\mathscr{C},\mathscr{V}] \longrightarrow [\mathscr{D},\mathscr{V}]$$

by the formula

$$\widehat{F}(M)V = \int^{U\in\mathscr{C}} F(U,V)\otimes MU$$

where $M\in[\mathscr{C},\mathscr{V}]$ and $V\in\mathscr{D}$. Conversely, given $G : [\mathscr{C},\mathscr{V}]\longrightarrow[\mathscr{D},\mathscr{V}]$, define

$$\check{G} : \mathscr{C}^{\mathrm{op}}\otimes\mathscr{D}\longrightarrow\mathscr{V}$$

by

$$\check{G}(U,V) = G(\mathscr{C}(U,-))V.$$

The main calculations proving the equivalence are as follows:

$$\begin{aligned}\check{\widehat{F}}(U,V) &= \widehat{F}(\mathscr{C}(U,-))V\\ &\cong \int^{U'} F(U',V)\otimes\mathscr{C}(U,U')\\ &\cong F(U,V) \qquad \text{by Yoneda;}\end{aligned}$$

and,

$$\begin{aligned}\widehat{\check{G}}(M)V &= \int^{U} \check{G}(U,V)\otimes MU\\ &\cong (\int^{U} G(\mathscr{C}(U,-))\otimes MU)V\\ &\cong G(\int^{U}\mathscr{C}(U,-)\otimes MU)V \quad \text{since } G \text{ preserves weighted colimits}\\ &\cong G(M)V \qquad \text{by Yoneda again.}\end{aligned}$$

Next we look how composition of Gs is transported to the Fs. Take

$$F_1 : \mathscr{C}^{\mathrm{op}}\otimes\mathscr{D}\longrightarrow\mathscr{V}, \quad F_2 : \mathscr{D}^{\mathrm{op}}\otimes\mathscr{E}\longrightarrow\mathscr{V}$$

so that $\widehat{F_1}$ and $\widehat{F_2}$ are composable:

$$\begin{tikzcd}
 & {[\mathscr{D},\mathscr{V}]} \arrow[dr, "\widehat{F_2}"] & \\
{[\mathscr{C},\mathscr{V}]} \arrow[ur, "\widehat{F_1}"] \arrow[rr, bend right, "\widehat{F_2}\circ\widehat{F_1}"'] & & {[\mathscr{E},\mathscr{V}].}
\end{tikzcd}$$

Notice that

$$\begin{aligned}
(\widehat{F_2}\circ\widehat{F_1})(M) &= \widehat{F_2}(\widehat{F_1}(M)) \\
&= \int^{V\in\mathscr{D}} F_2(V,-)\otimes\widehat{F_1}(M)V \\
&\cong \int^{U,V} F_2(V,-)\otimes F_1(U,V)\otimes MU \\
&\cong \int^{U}\Big(\int^{V} F_2(V,-)\otimes F_1(U,V)\Big)\otimes MU.
\end{aligned}$$

So we define $F_2\circ F_1 : \mathscr{C}^{\mathrm{op}}\otimes\mathscr{E}\longrightarrow\mathscr{V}$ by

$$(F_2\circ F_1)(U,W) = \int^{V} F_2(V,W)\otimes F_1(U,V); \tag{1}$$

the last calculation then yields

$$\widehat{F_2}\circ\widehat{F_1}\cong\widehat{F_2\circ F_1}.$$

The identity functor $1_{[\mathscr{C},\mathscr{V}]} : [\mathscr{C},\mathscr{V}]\longrightarrow[\mathscr{C},\mathscr{V}]$ corresponds to the hom functor of $\mathscr{C}$; that is,

$$\check{1}_{[\mathscr{C},\mathscr{V}]}(U,V) = \mathscr{C}(U,V).$$

This gives us the bicategory $\mathscr{V}$-**Mod**. The objects are (small) $\mathscr{V}$-categories $\mathscr{C}$. A morphism $F : \mathscr{C}\longrightarrow\!\!\!\!\!\!\!+\;\;\mathscr{D}$ is a $\mathscr{V}$-functor $F : \mathscr{C}^{\mathrm{op}}\otimes\mathscr{D}\longrightarrow\mathscr{V}$; we call this a *module from* $\mathscr{C}$ *to* $\mathscr{D}$ (others call it a *left* $\mathscr{D}$-, *right* $\mathscr{C}$-*bimodule*). Composition of modules is defined by (1) above.

We can sum up now by saying that

$$\widehat{(\;)} : \mathscr{V}\text{-}\mathbf{Mod}\longrightarrow\mathbf{COCT}_{\mathscr{V}}$$

is a pseudofunctor (= homomorphism of bicategories) taking $\mathscr{C}$ to $[\mathscr{C},\mathscr{V}]$, taking $F : \mathscr{C}\longrightarrow\!\!\!\!\!\!\!+\;\;\mathscr{D}$ to $\widehat{F}$, and defined on 2-cells in the obious way; moreover, this pseudofunctor is a local equivalence (that is, it is an equivalence on hom-categories):

$$\widehat{(\;)} : \mathscr{V}\text{-}\mathbf{Mod}(\mathscr{C},\mathscr{D})\simeq\mathbf{COCT}_{\mathscr{V}}([\mathscr{C},\mathscr{V}],[\mathscr{D},\mathscr{V}]).$$

A monad T on an object $\mathscr{C}$ of $\mathscr{V}$-**Mod** is called a *promonad on* $\mathscr{C}$. It is the same as giving a colimit-preserving monad $\widehat{T}$ on the $\mathscr{V}$-category $[\mathscr{C},\mathscr{V}]$. One way that promonads arise is from monoids A for some convolution monoidal structure on $[\mathscr{C},\mathscr{V}]$; then

$$\widehat{T}(M) = A * M.$$

That is, $\mathscr{C}$ is a promonoidal $\mathscr{V}$-category [Da1]:

$$P : \mathscr{C}^{\mathrm{op}}\otimes\mathscr{C}^{\mathrm{op}}\otimes\mathscr{C}\longrightarrow\mathscr{V}$$

$$J : \mathscr{C}\longrightarrow\mathscr{V}$$

so that

$$\widehat{T}(M) = A * M = \int^{U,V} P(U,V;-)\otimes AU\otimes MV.$$

This means that the module $T : \mathscr{C} \longrightarrow\!\!\!\!\!\!\!+\,\mathscr{C}$ is defined by

$$\begin{aligned} T(U,V) &= \widehat{T}(\mathscr{C}(U,-))V \\ &= \int^{U',V'} P(U',V';V) \otimes AU' \otimes \mathscr{C}(U,V') \\ &\cong \int^{U'} P(U',U;V) \otimes AU'. \end{aligned}$$

A promonad T on $\mathscr{C}$ has a unit $\eta : \check{1} \longrightarrow T$ with components

$$\eta_{U,V} : \mathscr{C}(U,V) \longrightarrow T(U,V)$$

and so is determined by

$$\eta_{U,V}(1_U) : I \longrightarrow T(U,U),$$

and has a multiplication $\mu : T \circ T \longrightarrow T$ with components

$$\mu_{U,W} : \int^{V} T(V,W) \otimes T(U,V) \longrightarrow T(U,W)$$

and so is determined by a natural family

$$\mu'_{U,V,W} : T(V,W) \otimes T(U,V) \longrightarrow T(U,W).$$

The *Kleisli category* $\mathscr{C}_T$ for the promonad T on $\mathscr{C}$ has the same objects as $\mathscr{C}$ and has homs defined by

$$\mathscr{C}_T(U,V) = T(U,V);$$

the identites are the $\eta_{U,V}(1_U)$ and the composition is the $\mu'_{U,V,W}$.

Proposition 12.1. $[\mathscr{C}_T, \mathscr{V}] \simeq [\mathscr{C}, \mathscr{V}]^{\widehat{T}}$. *That is, the functor category* $[\mathscr{C}_T, \mathscr{V}]$ *is equivalent to the category of Eilenberg-Moore algebras for the monad* $\widehat{T}$ *on* $[\mathscr{C}, \mathscr{V}]$.

Proof. (sketch) To give a $\widehat{T}$-algebra structure on $M \in [\mathscr{C}, \mathscr{V}]$ is to give a morphism $\alpha : \widehat{T}(M) \longrightarrow M$ satisfying the two axioms for an action. This is to give a natural family of morphisms

$$T(U,V) \otimes MU \longrightarrow MV;$$

but that is to give

$$T(U,V) \longrightarrow [MU, MV];$$

but that is to give

$$\mathscr{C}_T(U,V) \longrightarrow \mathscr{V}(MU, MV). \tag{2}$$

Thus we can define a $\mathscr{V}$-functor

$$\overline{M} : \mathscr{C}_T \longrightarrow \mathscr{V}$$

which agrees with M on objects and is defined by (2) on homs; the action axioms are just what is needed for $\overline{M}$ to be a functor. This process can be reversed. $\square$

13. Modules over a Green functor

In this section, we present work inspired by Chapters 2, 3 and 4 of [Bo1], casting it in a more categorical framework.

Let $\mathscr{E}$ denote a lextensive category and **CMon** denote the category of commutative monoids; this latter is what we called $\mathscr{V}$ in earlier sections. The functor $U : \mathbf{Mod}_k \longrightarrow \mathbf{CMon}$ (which forgets the action of k on the k-module and retains only the additive monoid structure) has a left adjoint $K : \mathbf{CMon} \longrightarrow \mathbf{Mod}_k$ which is strong monoidal for the obvious tensor products on **CMon** and $\mathbf{Mod}_k$. So each category $\mathscr{A}$ enriched in **CMon** determines a category $K_*\mathscr{A}$ enriched in $\mathbf{Mod}_k$: the objects of $K_*\mathscr{A}$ are those of $\mathscr{A}$ and the homs are defined by

$$(K_*\mathscr{A})(A,B) = K\mathscr{A}(A,B)$$

since $\mathscr{A}(A,B)$ is a commutative monoid. The point is that a $\mathbf{Mod}_k$-functor $K_*\mathscr{A} \longrightarrow \mathscr{B}$ is the same as a **CMon**-functor $\mathscr{A} \longrightarrow U_*\mathscr{B}$.

We know that $\mathbf{Spn}(\mathscr{E})$ is a **CMon**-category; so we obtain a monoidal $\mathbf{Mod}_k$-category

$$\mathscr{C} = K_*\mathbf{Spn}(\mathscr{E}).$$

The $\mathbf{Mod}_k$-category of *Mackey functors* on $\mathscr{E}$ is $\mathbf{Mky}_k(\mathscr{E}) = [\mathscr{C}, \mathbf{Mod}_k]$; it becomes monoidal using convolution with the monoidal structure on $\mathscr{C}$ (see Section 5). The $\mathbf{Mod}_k$-category of *Green functors* on $\mathscr{E}$ is $\mathbf{Grn}_k(\mathscr{E}) = \mathbf{Mon}[\mathscr{C}, \mathbf{Mod}_k]$ consisting of the monoids in $[\mathscr{C}, \mathbf{Mod}_k]$ for the convolution.

Let A be a Green functor. A *module* M over the Green functor A, or A-*module* means A acts on M via the convolution $*$. The monoidal action $\alpha^M : A * M \longrightarrow M$ is defined by a family of morphisms

$$\bar{\alpha}^M_{U,V} : A(U) \otimes_k M(V) \longrightarrow M(U \times V),$$

where we put $\bar{\alpha}^M_{U,V}(a \otimes m) = a.m$ for $a \in A(U)$, $m \in M(V)$, satisfing the following commutative diagrams for morphisms $f : U \longrightarrow U'$ and $g : V \longrightarrow V'$ in $\mathscr{E}$.

$$\begin{tikzcd}
A(U) \otimes_k M(V) \arrow[r, "\bar{\alpha}^M_{U,V}"] \arrow[d, "A_*(f)\otimes_k M_*(g)"'] & M(U \times V) \arrow[d, "M_*(f \times g)"] \\
A(U') \otimes_k M(V') \arrow[r, "\bar{\alpha}^M_{U',V'}"'] & M(U' \times V')
\end{tikzcd}
\qquad
\begin{tikzcd}
M(U) \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & A(1) \otimes_k M(U) \arrow[d, "\bar{\alpha}^M"] \\
 & M(1 \times U)
\end{tikzcd}$$

$$\begin{tikzcd}
A(U) \otimes_k A(V) \otimes_k M(W) \arrow[r, "1\otimes\bar{\alpha}^M"] \arrow[d, "\mu\otimes 1"'] & A(U) \otimes_k M(V \times W) \arrow[d, "\bar{\alpha}^M"] \\
A(U \times V) \otimes_k M(W) \arrow[r, "\bar{\alpha}^M"'] & M(U \times V \times W)\,.
\end{tikzcd}$$

If M is an A-module, then M is in particular a Mackey functor.

Lemma 13.1. *Let A be a Green functor and M be an A-module. Then M_U is an A-module for each U of $\mathscr{E}$, where $M_U(X) = M(X \times U)$.*

Proof. Simply define $\bar{\alpha}^{M_U}_{V,W} = \bar{\alpha}^M_{V,W\times U}$. $\square$

Let $\mathbf{Mod}(A)$ denote the category of left A-modules for a Green functor A. The objects are A-modules and morphisms are A-module morphisms $\theta : M \longrightarrow N$ (that is, morphisms of Mackey functors) satisfying the following commutative diagram.

$$\begin{tikzcd}
A(U)\otimes_k M(V) \arrow[r, "\bar{\alpha}^M_{U,V}"] \arrow[d, "1\otimes_k\theta(U)"'] & M(U\times V) \arrow[d, "\theta(U\times V)"] \\
A(U)\otimes_k N(V) \arrow[r, "\bar{\alpha}^N_{U,V}"'] & N(U\times V)
\end{tikzcd}$$

The category $\mathbf{Mod}(A)$ is enriched in $\mathbf{Mky}$. The homs are given by the equalizer

$$\begin{tikzcd}
\mathbf{Mod}(A)(M,N) \arrow[r] & \mathrm{Hom}(M,N) \arrow[rr, "\mathrm{Hom}(\alpha^M,1)"] \arrow[dr, "(A*-)"'] & & \mathrm{Hom}(A*M,N) \\
& & \mathrm{Hom}(A*M,A*N)\,. \arrow[ur, "\mathrm{Hom}(1,\alpha^N)"'] &
\end{tikzcd}$$

Then we see that $\mathbf{Mod}(A)(M,N)$ is the sub-Mackey functor of $\mathrm{Hom}(M,N)$ defined by

$$\begin{aligned}\mathbf{Mod}(A)(M,N)(U) = \{\theta \in \mathbf{Mky}(M(-\times U), N-) \quad &| \ \theta_{V\times W}(a.m) = a.\theta_W(m) \\ &\text{for all } V, W, \text{ and } a \in A(V), m \in M(W\times U)\}.\end{aligned}$$

In particular, if $A = J$ (Burnside functor) then $\mathbf{Mod}(A)$ is the category of Mackey functors and $\mathbf{Mod}(A)(M,N) = \mathrm{Hom}(M,N)$.

The Green functor A is itself an A-module. Then by the Lemma 13.1, we see that A_U is an A-module for each U in $\mathscr{E}$. Define a category $\mathscr{C}_A$ consisting of the objects of the form A_U for each U in $\mathscr{C}$. This is a full subcategory of $\mathbf{Mod}(A)$ and we have the following equivalences

$$\mathscr{C}_A(U,V) \simeq \mathbf{Mod}(A)(A_U, A_V) \simeq A(U\times V).$$

In other words, the category $\mathbf{Mod}(A)$ of left A-modules is the category of Eilenberg-Moore algebras for the monad $T = A * -$ on $[\mathscr{C}, \mathbf{Mod}_k]$; it preserves colimits since it has a right adjoint (as usual with convolution tensor products). By the above, the $\mathbf{Mod}_k$-category $\mathscr{C}_A$ (technically it is the Kleisli category $\mathscr{C}_{\check{T}}$ for the promonad $\check{T}$ on $\mathscr{C}$; see Proposition 12.1) satisfies an equivalence

$$[\mathscr{C}_A, \mathbf{Mod}_k] \simeq \mathbf{Mod}(A).$$

Let $\mathscr{C}$ be a $\mathbf{Mod}_k$-category with finite direct sums and Ω be a finite set of objects of $\mathscr{C}$ such that every object of $\mathscr{C}$ is a direct sum of objects from Ω.

Let W be the algebra of $\Omega\times\Omega$ - matrices whose (X,Y) - entry is a morphism $X \longrightarrow Y$ in $\mathscr{C}$. Then

$$W = \left\{(f_{XY})_{X,Y\in\Omega} \mid f_{XY} \in \mathscr{C}(X,Y)\right\}$$

is a vector space over k, and the product is defined by

$$(g_{XY})_{X,Y\in\Omega}(f_{XY})_{X,Y\in\Omega} = (\sum_{Y\in\Omega} g_{YZ}\circ f_{XY})_{X,Z\in\Omega}.$$

Proposition 13.2. $[\mathscr{C}, \mathbf{Mod}_k] \simeq \mathbf{Mod}_k^W$ *(= the category of left W-modules).*

Proof. Put

$$P = \bigoplus_{X \in \Omega} \mathscr{C}(X, -).$$

This is a small projective generator so Exercise F (page 106) of [Fr] applies and W is identified as End(P). □

In particular; this applies to the category $\mathscr{C}_A$ to obtain the *Green algebra* W_A of a Green functor A: the point being that A and W_A have the same modules.

14. Morita equivalence of Green functors

In this section, we look at the Morita theory of Green functors making use of adjoint two-sided modules rather than *Morita contexts* as in [Bo1].

As for any symmetric cocomplete closed monoidal category $\mathscr{W}$, we have the monoidal bicategory $\mathbf{Mod}(\mathscr{W})$ defined as follows, where we take $\mathscr{W} = \mathbf{Mky}$. Objects are monoids A in $\mathscr{W}$ (that is, $A : \mathscr{E} \longrightarrow \mathbf{Mod}_k$ are Green functors) and morphisms are modules $M : A \looparrowright B$ (that is, algebras for the monad $A * - * B$ on $\mathbf{Mky}$) with a two-sided action

$$\frac{\alpha^M : A * M * B \longrightarrow M}{\bar{\alpha}^M_{U,V,W} : A(U) \otimes_k M(V) \otimes_k B(W) \longrightarrow M(U \times V \times W).}$$

Composition of morphisms $M : A \looparrowright B$ and $N : B \looparrowright C$ is $M *_B N$ and it is defined via the coequalizer

$$\begin{tikzcd} M * B * N \arrow[r, shift left, "\alpha^M * 1_N"] \arrow[r, shift right, "1_M * \alpha^N"'] & M * N \arrow[r] & M *_B N = N \circ M \end{tikzcd}$$

that is,

$$(M *_B N)(U) = \sum_{X,Y} \mathbf{Spn}(\mathscr{E})(X \times Y, U) \otimes M(X) \otimes_k N(Y) / \sim_B \ .$$

The identity morphism is given by $A : A \looparrowright A$.

The 2-cells are natural transformations $\theta : M \longrightarrow M'$ which respect the actions

$$\begin{tikzcd} A(U) \otimes_k M(V) \otimes_k B(W) \arrow[r, "\bar{\alpha}^M_{U,V,W}"] \arrow[d, "1 \otimes_k \theta_V \otimes_k 1"'] & M(U \times V \times W) \arrow[d, "\theta_{U \times V \times W}"] \\ A(U) \otimes_k M'(V) \otimes_k B(W) \arrow[r, "\bar{\alpha}^{M'}_{U,V,W}"'] & M'(U \times V \times W) \ . \end{tikzcd}$$

The tensor product on $\mathbf{Mod}(\mathscr{W})$ is the convolution $*$. The tensor product of the modules $M : A \looparrowright B$ and $N : C \looparrowright D$ is $M * N : A * C \looparrowright B * D$.

Define Green functors A and B to be *Morita equivalent* when they are equivalent in $\mathbf{Mod}(\mathscr{W})$.

Proposition 14.1. *If A and B are equivalent in $\mathbf{Mod}(\mathscr{W})$ then $\mathbf{Mod}(A) \simeq \mathbf{Mod}(B)$ as categories.*

Proof. $\mathbf{Mod}(\mathscr{W})(-, J) : \mathbf{Mod}(\mathscr{W})^{\mathrm{op}} \longrightarrow \mathbf{CAT}$ is a pseudofunctor and so takes equivalences to equivalences. □

Now we will look at the Cauchy completion of a monoid A in a monoidal category $\mathscr{W}$ with the unit J. The $\mathscr{W}$-category $\mathscr{P}A$ has underlying category $\mathbf{Mod}(\mathscr{W})(J,A) = \mathbf{Mod}(A^{\mathrm{op}})$ where A^{op} is the monoid A with commuted multiplication. The objects are modules $M : J \longrightarrow\!\!\!\!\!\!\!+\ A$; that is, right A-modules. The homs of $\mathscr{P}A$ are defined by $(\mathscr{P}A)(M,N) = \mathbf{Mod}(A^{\mathrm{op}})(M,N)$ (see the equalizer of Section 13).

The Cauchy completion $\mathscr{Q}A$ of A is the full sub-$\mathscr{W}$-category of $\mathscr{P}A$ consisting of the modules $M : J \longrightarrow\!\!\!\!\!\!\!+\ A$ with right adjoints $N : A \longrightarrow\!\!\!\!\!\!\!+\ J$. We will examine what the objects of $\mathscr{Q}A$ are in more explicit terms.

For motivation and preparation we will look at the monoidal category $\mathscr{W} = [\mathscr{C},\mathscr{S}]$ where $(\mathscr{C},\otimes,I)$ is a monoidal category and $\mathscr{S}$ is the cartesian monoidal category of sets. Then $[\mathscr{C},\mathscr{S}]$ becomes a monoidal category by convolution. The tensor product $*$ and the unit J are defined by

$$(M * N)(U) = \int^{X,Y} \mathscr{C}(X \otimes Y, U) \times M(X) \times N(Y)$$
$$J(U) = \mathscr{C}(I,U).$$

Write $\mathbf{Mod}[\mathscr{C},\mathscr{S}]$ for the bicategory whose objects are monoids A in $[\mathscr{C},\mathscr{S}]$ and whose morphisms are modules $M : A \longrightarrow\!\!\!\!\!\!\!+\ B$. These modules have two-sided action

$$\frac{\alpha^M : A * M * B \longrightarrow M}{\bar{\alpha}^M_{X,Y,Z} : A(X) \times M(Y) \times B(Z) \longrightarrow M(X \otimes Y \otimes Z)}\,.$$

Composition of morphisms $M : A \longrightarrow\!\!\!\!\!\!\!+\ B$ and $N : B \longrightarrow\!\!\!\!\!\!\!+\ C$ is given by the coequalizer

$$M * B * N \underset{1_M * \alpha^N}{\overset{\alpha^M * 1_N}{\rightrightarrows}} M * N \longrightarrow M *_B N$$

that is,

$$(M *_B N)(U) = \sum_{X,Z} \mathscr{C}(X \otimes Z, U) \times M(X) \times N(Z) / \sim_B$$

where

$$(u, m \circ b, n) \sim_B (u, m, b \circ n)$$
$$(t \circ (r \otimes s), m, n) \sim_B (t, (Mr)m, (Ns)n)$$

for $u : X \otimes Y \otimes Z \longrightarrow U$, $m \in M(X)$, $b \in B(Y)$, $n \in N(Z)$, $t : X' \otimes Z' \longrightarrow U$, $r : X \longrightarrow X'$, $s : Z \longrightarrow Z'$.

For each $K \in \mathscr{C}$, we obtain a module $A(K \otimes -) : J \longrightarrow\!\!\!\!\!\!\!+\ A$. The action

$$A(K \otimes U) \otimes A(V) \longrightarrow A(K \otimes U \otimes V)$$

is defined by the monoid structure on A.

Proposition 14.2. *Every object of the Cauchy completion $\mathscr{Q}A$ of the monoid A in $[\mathscr{C},\mathscr{S}]$ is a retract of a module of the form $A(K \otimes -)$ for some $K \in \mathscr{C}$.*

Proof. Take a module $M : J \longrightarrow\!\!\!\!\!\!\!+\ A$ in $\mathbf{Mod}[\mathscr{C},\mathscr{S}]$. Suppose that M has a right adjoint $N : A \longrightarrow\!\!\!\!\!\!\!+\ J$. Then we have the following actions: $A(V) \times A(W) \longrightarrow A(V \otimes W)$, $M(V) \times A(W) \longrightarrow M(V \otimes W)$, $A(V) \times N(W) \longrightarrow N(V \otimes W)$ since A is a monoid, M is a right A-module, and N is a left A-module respectively.

We have a unit $\eta : J \longrightarrow M *_A N$ and a counit $\epsilon : N * M \longrightarrow A$ for the adjunction. The component $\eta_U : \mathscr{C}(I,U) \longrightarrow (M *_A N)(U)$ of the unit η is determined by

$$\eta' = \eta_U(1_I) \in \sum_{X,Z} \mathscr{C}(X \otimes Z, I) \times M(X) \times N(Z) / \sim_A;$$

so there exist $u : H \otimes K \longrightarrow I$, $\quad p \in M(H)$, $\quad q \in N(K)$ such that $\eta' = [u, p, q]_A$. Then

$$\eta_u(f : I \longrightarrow U) = [fu : H \otimes K \longrightarrow U, p, q]_A.$$

We also have $\bar{\epsilon}_{Y,Z} : NY \times MZ \longrightarrow A(Y \otimes Z)$ coming from ϵ. The commutative diagram

$$\begin{tikzcd} M(U) \arrow[r, "\eta_U * 1"] \arrow[dr, "1"'] & \sum\limits_{X,Y,Z} \mathscr{C}(X \otimes Y \otimes Z, U) \times M(X) \times N(Y) \times M(Z)/\sim \arrow[d, "1 * \epsilon_U"] \\ & M(U) \end{tikzcd}$$

yields the equations

$$\begin{aligned} m &= (1 * \epsilon_U)(\eta_U * 1)(m) \\ &= (1 * \epsilon_U)[u \otimes 1_U, p, q, m]_A \qquad (3) \\ &= M(u \otimes 1_U)(p\, \bar{\epsilon}_{K,U}(q, m)) \end{aligned}$$

for all $m \in M(U)$.

Define

$$\begin{tikzcd} M(U) \arrow[r, bend right, "i_U"'] & A(K \otimes U) \arrow[l, bend right, "r_U"'] \end{tikzcd}$$

by $i_U(m) = \bar{\epsilon}_{K,U}(q, m)$, $r_U(a) = M(u \otimes 1_U)(p.a)$. These are easily seen to be natural in U. Equation (3) says that $r \circ i = 1_M$. So M is a retract of $A(K \otimes -)$. □

Now we will look at what are the objects of $\mathscr{Q}A$ when $\mathscr{W} = \mathbf{Mky}$ which is a symmetric monoidal closed, complete and cocomplete category.

Theorem 14.3. *The Cauchy completion $\mathscr{Q}A$ of the monoid A in* **Mky** *consists of all the retracts of modules of the form*

$$\bigoplus_{i=1}^{k} A(Y_i \times -)$$

for some $Y_i \in \mathbf{Spn}(\mathscr{E})$, $i = 1, \ldots, k$.

Proof. Take a module $M : J \longrightarrow\!\!\!\!\!\!+ A$ in $\mathbf{Mod}(\mathscr{W})$ and suppose that M has a right adjoint $N : A \longrightarrow\!\!\!\!\!\!+ J$. For the adjunction, we have a unit $\eta : J \longrightarrow M *_A N$ and a counit $\epsilon : N * M \longrightarrow A$. We write $\eta_U : \mathbf{Spn}(\mathscr{E})(1, U) \longrightarrow (M *_A N)(U)$ is the component of the unit η and it is determined by

$$\eta' = \eta_1(1_1) \in \sum_{i=1}^{k} \mathbf{Spn}(\mathscr{E})(X \times Y, 1) \otimes M(X) \otimes N(Y)/\sim_A.$$

Put

$$\eta' = \eta_1(1_1) = \sum_{i=1}^{k} [(S_i : X_i \times Y_i \longrightarrow 1) \otimes m_i \otimes n_i]_A$$

where $m_i \in M(X_i)$ and $n_i \in N(Y_i)$. Then

$$\eta_U(T : 1 \longrightarrow U) = \sum_{i=1}^{k} [(S_i \times T) \otimes m_i \otimes n_i]_A.$$

We also have $\bar{\epsilon}_{Y,Z} : NY \otimes MZ \longrightarrow A(Y \times Z)$ coming from ϵ. The commutative diagram

$$\begin{tikzcd}
M(U) \arrow[r, "\eta_U * 1"] \arrow[dr, "1"'] & \displaystyle\sum_{i=1}^{k} \mathbf{Spn}(\mathscr{E})(X_i \times Y_i \times U, U) \otimes M(X_i) \otimes N(Y_i) \otimes M(U) / \sim_A \arrow[d, "1 * \epsilon_U"] \\
 & M(U)
\end{tikzcd}$$

yields

$$m = \sum_{i=1}^{k} [M(P_i \times U) \otimes m_i \otimes \epsilon(n_i \otimes m)]$$

where $m \in M(U)$ and $P_i : X_i \times Y_i \longrightarrow U$.

Define a natural retraction

$$\begin{tikzcd}
M(U) \arrow[r, bend right, "i_U"'] & \displaystyle\bigoplus_{i=1}^{k} A(Y_i \times U) \arrow[l, bend right, "r_U"']
\end{tikzcd}$$

by

$$r_U(a_i) = M(P_{i_k} \times U)(m_i . a_i), \quad i_U(m) = \sum_{i=1}^{k} \bar{\epsilon}_{Y_i,U}(n_i \otimes m).$$

So M is a retract of $\bigoplus_{i=1}^{k} A(Y_i \times -)$.

It remains to check that each module $A(Y \times -)$ has a right adjoint since retracts and direct sums of modules with right adjoints have right adjoints.

In $\mathscr{C} = \mathbf{Spn}(\mathscr{E})$ each object Y has a dual (in fact it is its own dual). This implies that the module $\mathscr{C}(Y,-) : J \nrightarrow J$ has a right dual (in fact it is $\mathscr{C}(Y,-)$ itself) since the Yoneda embedding $\mathscr{C}^{\mathrm{op}} \longrightarrow [\mathscr{C}, \mathbf{Mod}_k]$ is a strong monoidal functor. Moreover, the unit $\eta : J \longrightarrow A$ induces a module $\eta_* = A : J \nrightarrow A$ with a right adjoint $\eta^* : A \nrightarrow J$. Therefore, the composite

$$J \xrightarrow{\mathscr{C}(Y,-)} J \xrightarrow{\eta_*} A \ ,$$

which is $A(Y \times -)$, has a right adjoint. □

Theorem 14.4. *Green functors A and B are Morita equivalent if and only if $\mathscr{Q}A \simeq \mathscr{Q}B$ as $\mathscr{W}$-categories.*

Proof. See [Li2] and [St]. □

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Chapter 2

Paper 2: Lax braidings and the lax centre

(Coauthored with Dr. Brian Day and Professor Ross Street)

This paper will appear in *Contemporary Mathematics*.

Table of Contents

Contemporary Mathematics

Lax Braidings and the Lax Centre

Brian Day, Elango Panchadcharam, and Ross Street

Abstract. The purpose of this work is to highlight the notions of lax braiding and lax centre for monoidal categories and more generally for promonoidal categories. Lax centres are lax braided. Generally the centre is a full subcategory of the lax centre, however we show that it is sometimes the case that the two coincide. We identify lax centres of monoidal functor categories in various cases.

Introduction

Braidings for monoidal categories were introduced in [**JS1**] and its forerunners. The centre $\mathcal{Z}\mathscr{X}$ of a monoidal category $\mathscr{X}$ was introduced in [**JS0**] in the process of proving that the free tortile monoidal category has another universal property. The centre of a monoidal category is a braided monoidal category. What we now call lax braidings were considered tangentially by Yetter [**Yet**]. What we now call the lax centre $\mathcal{Z}_l\mathscr{X}$ of $\mathscr{X}$ was considered under the name "weak centre" by P. Schauenburg [**Sch**]. The purpose of this work is to highlight the notions of lax braiding and lax centre for monoidal categories $\mathscr{X}$ and more generally for promonoidal categories $\mathscr{C}$. Lax centres turn out to be lax braided monoidal categories. Generally the centre is a full subcategory of the lax centre, however it is sometimes the case that the two coincide. We have two such theorems under different hypotheses, one in the case sufficient dual objects exist in the additive context, and the other in the cartesian context. For a promonoidal category $\mathscr{C}$, we relate the lax centre of the [**Day**] convolution on $\mathscr{C}$ to the convolution on the lax centre of $\mathscr{C}$. Indeed, sometimes these are equivalent. One reason for being interested in the lax centre of $\mathscr{X}$ is that, if an object X of $\mathscr{X}$ is equipped with the structure of monoid in $\mathcal{Z}_l\mathscr{X}$, then tensoring with X defines a monoidal endofunctor $-\otimes X$ of $\mathscr{X}$; this has applications in cases where the lax centre can be explicitly identified.

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1. Lax braidings for promonoidal categories

Let $\mathscr{V}$ denote a complete cocomplete symmetric closed monoidal category and let $\mathscr{C}$ be a $\mathscr{V}$-enriched category in the sense of [**Kel**]. A *promagmal structure* on $\mathscr{C}$ consists of two $\mathscr{V}$-functors $P : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \longrightarrow \mathscr{V}$ and $J : \mathscr{C} \longrightarrow \mathscr{V}$ (called the *protensor product* and *prounit*). Recall from [**Day**] that a *promonoidal structure* on $\mathscr{C}$ is a promagmal structure equipped further with $\mathscr{V}$-natural isomorphisms

$$\int^{U} P(U, C; D) \otimes P(A, B; U) \xrightarrow{\text{assoc}} \int^{V} P(A, V; D) \otimes P(B, C; V)$$

$$\int^{U} P(U, A; B) \otimes J(U) \xrightarrow{\text{lun}} \mathscr{C}(A, B) \quad \text{and} \quad \int^{V} P(A, V; B) \otimes J(V) \xrightarrow{\text{run}} \mathscr{C}(A, B)$$

(called the *associativity, left unit* and *right unit constraints*) satisfying two coherence conditions.

The importance of promonoidal structures on $\mathscr{C}$ lies in their equivalence to (left- and -right-) closed monoidal structures on the $\mathscr{V}$-functor category $[\mathscr{C}, \mathscr{V}]$. Given a promonoidal structure on $\mathscr{C}$, we obtain a closed monoidal structure on $[\mathscr{C}, \mathscr{V}]$ where the tensor product $*$ is defined by the *convolution formula*

$$(M * N)C = \int^{X,Y} P(X, Y; C) \otimes MX \otimes NY$$

and the unit is J. Conversely, given a closed monoidal structure on $[\mathscr{C}, \mathscr{V}]$, we obtain a promonoidal structure on $\mathscr{C}$ by defining

$$P(A, B; C) = (\mathscr{C}(A, -) * \mathscr{C}(B, -))C$$

and taking the unit as the prounit.

By way of example, every monoidal structure on $\mathscr{C}$ determines a promonoidal one by defining $P(A, B; C) = \mathscr{C}(B \otimes A, C)$ and $JC = \mathscr{C}(I, C)$. Another example, for any comonoidal $\mathscr{C}$, is defined by $P(A, B; C) = \mathscr{C}(B, C) \otimes \mathscr{C}(A, C)$ and $JC = I$; the comonoidal structure includes $\mathscr{V}$-functors $\mathscr{C} \longrightarrow \mathscr{C} \otimes \mathscr{C}$ and $\mathscr{C} \longrightarrow I$ which are used to make P and J into $\mathscr{V}$-functors in the C variable. These two examples agree in case $\mathscr{V} = \mathbf{Set}$(so that every $\mathscr{C}$ is comonoidal) and the monoidal structure on $\mathscr{C}$ is coproduct.

Symmetries for promonoidal structures were defined by [**Day**] and braidings by [**JS1**]. We generalize this slightly. A *lax braiding* for a promonoidal structure on $\mathscr{C}$ is a $\mathscr{V}$-natural family of morphisms $c_{A,B;C} : P(A, B; C) \longrightarrow P(B, A; C)$ such that the following four diagrams commute.

$$\begin{tikzcd}
\int^{U} P(U, C; D) \otimes P(A, B; U) \arrow[r, "\int^{U} c\otimes 1"] \arrow[d, "\text{assoc}"'] & \int^{U} P(C, U; D) \otimes P(A, B; U) \\
\int^{V} P(A, V; D) \otimes P(B, C; V) \arrow[d, "\int^{V} 1\otimes c"'] & \int^{W} P(W, B; D) \otimes P(C, A; W) \arrow[u, "\text{assoc}"'] \\
\int^{V} P(A, V; D) \otimes P(C, B; V) \arrow[r, "\text{assoc}^{-1}"] & \int^{W} P(W, B; D) \otimes P(A, C; W) \arrow[u, "\int^{W} 1\otimes c"']
\end{tikzcd}$$

$$\begin{tikzcd}
\int^V P(A,V;D)\otimes P(B,C;V) \arrow[r, "\int^V c\otimes 1"] \arrow[d, "\mathrm{assoc}^{-1}"'] & \int^V P(V,A;D)\otimes P(B,C;V) \\
\int^U P(U,C;D)\otimes P(A,B;U) \arrow[d, "\int^U 1\otimes c"'] & \int^W P(B,W;D)\otimes P(C,A;W) \arrow[u, "\mathrm{assoc}^{-1}"'] \\
\int^U P(U,C;D)\otimes P(B,A;U) \arrow[r, "\mathrm{assoc}"] & \int^W P(B,W;D)\otimes P(A,C;W) \arrow[u, "\int^W 1\otimes c"']
\end{tikzcd}$$

$$\begin{tikzcd}
\int^U P(U,A;B)\otimes JU \arrow[rr, "\int^U c\otimes 1"] \arrow[dr, "\mathrm{lun}"'] & & \int^U P(A,U;B)\otimes JU \arrow[dl, "\mathrm{run}"] \\
& \mathscr{C}(A,B) &
\end{tikzcd}$$

$$\begin{tikzcd}
\int^U P(A,U;B)\otimes JU \arrow[rr, "\int^U c\otimes 1"] \arrow[dr, "\mathrm{run}"'] & & \int^U P(U,A;B)\otimes JU \arrow[dl, "\mathrm{lun}"] \\
& \mathscr{C}(A,B) &
\end{tikzcd}$$

A *braiding* is a lax braiding for which each $c_{A,B;C} : P(A,B;C) \longrightarrow P(B,A;C)$ is invertible. In particular, by regarding a monoidal category as a promonoidal one in the manner described above, we obtain the notion of lax braiding and braiding for a monoidal category; by Yoneda's Lemma in this case, we can regard the lax braiding as a morphism $c_{A,B} : A\otimes B \longrightarrow B\otimes A$ satisfying four conditions; then $c_{A,B;C} : \mathscr{C}(B\otimes A, C) \longrightarrow \mathscr{C}(A\otimes B, C)$ is $\mathscr{C}(c_{A,B}, C)$.

We can easily adjust the results of [**Day**] on symmetries to obtain the following for lax braidings.

PROPOSITION 1.1. *Let $\mathscr{C}$ be a promonoidal $\mathscr{V}$-category and regard $[\mathscr{C},\mathscr{V}]^{op}$, under the convolution monoidal structure, as promonoidal. Then the Yoneda embedding $Y : \mathscr{C} \longrightarrow [\mathscr{C},\mathscr{V}]^{op}$ preserves promonoidal structures. Moreover, there is a bijection between lax braidings on $\mathscr{C}$ and those on $[\mathscr{C},\mathscr{V}]^{op}$ defined by the requirement that Y should preserve lax braidings; the bijection restricts to braidings and to symmetries.*

EXAMPLE 1.2. Let $\mathscr{V}$ be the monoidal category of vector spaces over the complex number field $\mathbf{k}$. Let $\mathscr{A}$ be an abelian category. We write $\mathscr{A}_g$ for the subcategory of $\mathscr{A}$ with the same objects but with only the invertible morphisms. We write $\mathbf{k}_*\mathscr{A}_g$ for the free $\mathscr{V}$-category on $\mathscr{A}_g$; it has the same objects as $\mathscr{A}_g$ and its hom vector spaces have the homs of $\mathscr{A}_g$ as bases. A promonoidal structure on $\mathbf{k}_*\mathscr{A}_g$ is obtained by defining $P(A, B; C)$ to have basis

$$\{(f, g) \mid 0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0 \text{ is a short exact sequence in } \mathscr{A}\}$$

and defining

$$JC = \begin{cases} \mathbf{k} & \text{for } C = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The associativity constraints come from contemplation of the following 3×3 diagram of short exact sequences.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \arrow[d] \\ D \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & V \arrow[r] & C \end{tikzcd}$$

A lax braiding is obtained by defining $c_{A,B;C} : P(A, B; C) \longrightarrow P(B, A; C)$ to take the basis element (f, g) to the sum of all those pairs (h, k) such that

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "k"] \arrow[r, shift left, "g"] & B \arrow[l, shift left, "h"] \end{tikzcd}$$

is a direct sum situation; the abelian category $\mathscr{A}$ must be restricted so that this sum is finite. This lax braiding is generally not invertible; however, in the case where $\mathscr{A}$ is the category of finite vector spaces over a fixed finite field, it was shown in [**JS3**] that it is a braiding.

In the presence of duals, various unexpected things can be proved invertible; see [**JS2**, Section 10, Proposition 8], [**Yet**, Proposition 7.1], and [**JS1**, Propositions 7.1 and 7.4].

Proposition 1.3. *If $\mathscr{C}$ is a right autonomous (meaning that each object has a right dual) monoidal category then any lax braiding on $\mathscr{C}$ is necessarily a braiding.*

Proof. If B has right dual C then the mate of $c_{A,C}$ is an inverse for $c_{A,B}$. While the proof of this is in [**JS2**, Section 10, Proposition 8], we shall repeat it below squeezing out a little more in the form of our Proposition 3.1 below. □

We use the terminology of [**Kel**] so that a monoidal functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is equipped with a natural family of morphisms $FA \otimes FB \longrightarrow F(A \otimes B)$ and a morphism $I \longrightarrow FI$; these morphisms satisfy coherence conditions but are not necessarily invertible; in the case where they are all invertible we say the monoidal functor is *strong*.

Proposition 1.4. *Any lax braiding of a monoidal $\mathscr{V}$-category $\mathscr{C}$ equips the tensor product $\mathscr{V}$-functor $\otimes : \mathscr{C} \otimes \mathscr{C} \longrightarrow \mathscr{C}$ with a monoidal structure. Since monoidal functors preserve monoids, it follows that the tensor product of two monoids in $\mathscr{C}$ is again a monoid.*

2. The lax centre of a promonoidal category

For each promonoidal $\mathscr{V}$-category $\mathscr{C}$, we shall construct a promagmal $\mathscr{V}$-category $\mathscr{Z}_l\mathscr{C}$ which we call the *(left) lax centre* of $\mathscr{C}$. It is quite often canonically promonoidal in which case it is lax braided.

The objects of $\mathscr{Z}_l\mathscr{C}$ are pairs (A, α) where A is an object of $\mathscr{C}$ and α is a $\mathscr{V}$-natural family of morphisms $\alpha_{X;Y} : P(A, X; Y) \longrightarrow P(X, A; Y)$ such that the following two diagrams commute.

$$\begin{tikzcd}
\int^V P(A,V;Z) \otimes P(X,Y;V) \arrow[r, "\int^V \alpha\otimes 1"] \arrow[d, "\mathrm{assoc}^{-1}"'] & \int^V P(V,A;Z) \otimes P(X,Y;V) \\
\int^U P(U,Y;Z) \otimes P(A,X;U) \arrow[d, "\int^U 1\otimes\alpha"'] & \int^W P(X,W;Z) \otimes P(Y,A;W) \arrow[u, "\mathrm{assoc}^{-1}"'] \\
\int^U P(U,Y;Z) \otimes P(X,A;U) \arrow[r, "\mathrm{assoc}"] & \int^W P(X,W;Z) \otimes P(A,Y;W) \arrow[u, "\int^W 1\otimes\alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
\int^U P(A,U;X) \otimes JU \arrow[rr, "\int^U \alpha\otimes 1"] \arrow[dr, "\mathrm{run}"'] & & \int^U P(U,A;X) \otimes JU \arrow[dl, "\mathrm{lun}"] \\
& \mathscr{C}(A,X) &
\end{tikzcd}$$

The hom object $\mathscr{Z}_l\mathscr{C}((A,\alpha),(B,\beta))$ is defined to be the equalizer in $\mathscr{V}$ of the two composed paths around the following square.

$$\begin{tikzcd}
\mathscr{C}(A,B) \arrow[r, "P"] \arrow[d, "P"'] & \int_{X,Y} [P(B,X;Y), P(A,X;Y)] \arrow[d, "{[1,\alpha]}"] \\
\int_{X,Y} [P(X,B;Y), P(X,A;Y)] \arrow[r, "{[\beta,1]}"'] & \int_{X,Y} [P(B,X;Y), P(X,A;Y)]
\end{tikzcd}$$

Composition in $\mathscr{Z}_l\mathscr{C}$ is defined so that we have the obvious faithful $\mathscr{V}$-functor $\mathscr{Z}_l\mathscr{C} \longrightarrow \mathscr{C}$ taking (A, α) to A.

The promagmal structure on $\mathscr{Z}_l\mathscr{C}$ is defined by taking $P((A,\alpha),(B,\beta);(C,\gamma))$ to be the equalizer of the two composed paths around the following square in which the top and left sides are transforms under the tensor-hom adjunction of the associativity constraint and its inverse.

$$\begin{tikzcd}
P(A,B;C) \arrow[r] \arrow[d] & \Big[P(C,Y;Z), \int^X P(A,X;Z) \otimes P(B,Y;X)\Big] \arrow[d, "{[1, \int^X \alpha \otimes \beta]}"] \\
\Big[P(Y,C;Z), \int^X P(X,A;Z) \otimes P(Y,B;X)\Big] \arrow[r, "{[\gamma,1]}"'] & \Big[P(C,Y;Z), \int^X P(X,A;Z) \otimes P(Y,B;X)\Big]
\end{tikzcd}$$

We take $J(A,\alpha)$ to be the equalizer of the two legs around the following triangle in which the top side and left side come from the unit constraints.

$$\begin{tikzcd}
JA \arrow[rr] \arrow[dr] & & {[P(A,X;Y), \mathscr{C}(X,Y)]} \\
& {[P(X,A;Y), \mathscr{C}(X,Y)]} \arrow[ur, "{[\alpha_{X;Y},1]}"'] &
\end{tikzcd}$$

It is frequently the case that $\mathcal{Z}_l\mathscr{C}$ is promonoidal in such a way that the forgetful $\mathscr{V}$-functor $\mathcal{Z}_l\mathscr{C} \longrightarrow \mathscr{C}$ is strong promonoidal. For example, if $\mathscr{C}$ is monoidal then so too is $\mathcal{Z}_l\mathscr{C}$ and $\mathcal{Z}_l\mathscr{C} \longrightarrow \mathscr{C}$ is strong monoidal.

The lax braiding on $\mathcal{Z}_l\mathscr{C}$ is defined by taking the unique $c = c_{(A,\alpha),(B,\beta);(C,\gamma)}$ such that the following square commutes.

$$\begin{tikzcd}
P((A,\alpha),(B,\beta);(C,\gamma)) \arrow[r, "\text{equalizer}"] \arrow[d, "c"'] & P(A,B;C) \arrow[d, "\alpha"] \\
P((B,\beta),(A,\alpha);(C,\gamma)) \arrow[r, "\text{equalizer}"'] & P(B,A;C)
\end{tikzcd}$$

The *centre* of $\mathscr{C}$ is the full sub-$\mathscr{V}$-category $\mathcal{Z}\mathscr{C}$ of $\mathcal{Z}_l\mathscr{C}$ consisting of the objects (A,α) for which each $\alpha_{X;Y} : P(A,X;Y) \longrightarrow P(X,A;Y)$ is invertible.

There is a fully faithful $\mathscr{V}$-functor $\Psi : (\mathcal{Z}_l\mathscr{C})^{\mathrm{op}} \longrightarrow \mathcal{Z}_l[\mathscr{C},\mathscr{V}]$ defined by

$$\Psi(A,\alpha) = \Big(\mathscr{C}(A,-), \mathscr{C}(A,-) * F \xrightarrow{\theta_F} F * \mathscr{C}(A,-)\Big)$$

where

$$\theta_F = \left(\int^U P(A,U;-) \otimes FU \xrightarrow{\int^U \alpha_{U;-} \otimes 1_{FU}} \int^U P(U,A;-) \otimes FU\right).$$

In fact, the promagmal structure on $\mathcal{Z}_l\mathscr{C}$ is obtained by restriction along Ψ of the promonoidal (actually monoidal) structure on $\mathcal{Z}_l[\mathscr{C},\mathscr{V}]$. The following diagram of $\mathscr{V}$-functors and $\mathscr{V}$-categories is a pullback.

$$\begin{tikzcd}
(\mathcal{Z}_l\mathscr{C})^{\mathrm{op}} \arrow[r, "\Psi"] \arrow[d] & \mathcal{Z}_l[\mathscr{C},\mathscr{V}] \arrow[d] \\
\mathscr{C}^{\mathrm{op}} \arrow[r, "\text{Yoneda}"'] & {[\mathscr{C},\mathscr{V}]}
\end{tikzcd}$$

The $\mathscr{V}$-functor Ψ induces an adjunction

$$\mathcal{Z}_l[\mathscr{C}, \mathscr{V}] \underset{\tilde{\Psi}}{\overset{\hat{\Psi}}{\rightleftarrows}} [\mathcal{Z}_l\mathscr{C}, \mathscr{V}]$$

defined by

$$\widehat{\Psi}(G) = \int^{(A,\alpha)} G(A,\alpha)\otimes\Psi(A,\alpha) \quad \text{and} \quad \widetilde{\Psi}(F,\theta)(A,\alpha) = \mathcal{Z}_l[\mathscr{C},\mathscr{V}](\Psi(A,\alpha),(F,\theta));$$

this last object can be obtained as the equalizer of two morphisms out of $F(A)$. In later sections we shall see that this adjunction can be a lax-braided monoidal equivalence.

3. The lax centre of a monoidal category

Let $\mathscr{C}$ denote a monoidal $\mathscr{V}$-category. The *lax centre* $\mathcal{Z}_l\mathscr{C}$ of $\mathscr{C}$ is the lax centre of $\mathscr{C}$ as a promonoidal category with promonoidal structure defined by

$$JC = \mathscr{C}(I, C) \qquad \text{and} \qquad P(A, B; C) = \mathscr{C}(B \otimes A, C).$$

Using the Yoneda lemma, we identify objects of $\mathcal{Z}_l\mathscr{C}$ with pairs (A, u) where A is an object of $\mathscr{C}$ and u is a $\mathscr{V}$-natural family of morphisms $u_B : A \otimes B \longrightarrow B \otimes A$ such that the following two diagrams commute.

$$\begin{tikzcd} A \otimes B \otimes C \arrow[rr, "u_{B\otimes C}"] \arrow[dr, "u_B\otimes 1_C"'] & & B \otimes C \otimes A \\ & B \otimes A \otimes C \arrow[ur, "1_B\otimes u_C"'] & \end{tikzcd} \qquad \begin{tikzcd} A \otimes I \arrow[rr, "u_I"] \arrow[dr, "\cong"'] & & I \otimes A \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

In the case where $\mathscr{V} = \mathbf{Set}$ and $\mathscr{C}$ is monoidal, the lax centre of $\mathscr{C}$, under the name "(left) weak centre", was used in Section 4 of [**Sch**] where it is shown to be related to Yetter-Drinfeld modules.

We shall see that the lax centre can be equal to the centre. As a preliminary to this, we note the following result which implies Proposition 1.3 since every object of a lax braided monoidal category is equipped with a canonical structure of object in the lax centre.

PROPOSITION 3.1. *If (A, u) is an object of the lax centre of a monoidal $\mathscr{V}$-category $\mathscr{C}$ and X is an object of $\mathscr{C}$ with a right dual X^* then the mate of u_{X^*} : $A \otimes X^* \longrightarrow X^* \otimes A$ is an inverse for $u_X : A \otimes X \longrightarrow X \otimes A$.*

PROOF. The mate of u_{X^*} is the composite

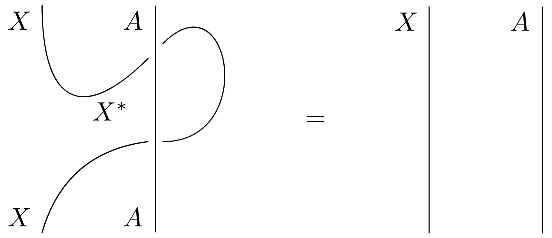
$$X \otimes A \xrightarrow{1_X\otimes 1_A\otimes\eta} X \otimes A \otimes X^* \otimes X \xrightarrow{1_X\otimes u_{X^*}\otimes 1_X} X \otimes X^* \otimes A \otimes X \xrightarrow{\varepsilon\otimes 1_A\otimes 1_X} A \otimes X$$

where η and ε are the unit and the counit for the duality $X \dashv X^*$. The proof that this is a right inverse uses the naturality of u with respect to the morphism

$\eta : I \longrightarrow X^* \otimes X$ and the axioms for u_I and $u_{X^* \otimes X}$:

$$\begin{tikzcd}
X \otimes A \arrow[rr, "1_X \otimes 1_A \otimes \eta"] \arrow[drr, "1_X \otimes \eta \otimes 1_A"] \arrow[dd, "1_{X \otimes A}"'] & & X \otimes A \otimes X^* \otimes X \arrow[d, "1_X \otimes u_{X^* \otimes X}"] \arrow[dr, "1_X \otimes u_{X^*} \otimes 1_X"] & \\
& & X \otimes X^* \otimes X \otimes A \arrow[dll, "\epsilon \otimes 1_X \otimes 1_A"] & X \otimes X^* \otimes A \otimes X \arrow[l, "1_X \otimes 1_{X^*} \otimes u_X"'] \arrow[dl, "\epsilon \otimes 1_A \otimes 1_X"] \\
X \otimes A & & A \otimes X \arrow[ll, "u_X"] &
\end{tikzcd}$$

Alternatively, we can prove it using string diagrams:



Similarly, the proof that the mate of u_{X^*} is a left inverse uses the naturality of u with respect to the morphism $\varepsilon : X^* \otimes X \longrightarrow I$ and the axioms for u_I and $u_{X \otimes X^*}$. □

Proposition 3.2. *Suppose $\mathscr{F}$ is a monoidal $\mathscr{V}$-category such that, for each object F, the functor $F \otimes - : \mathscr{F} \longrightarrow \mathscr{F}$ preserves (weighted) colimits. If $K : \mathscr{C} \longrightarrow \mathscr{F}$ is a dense $\mathscr{V}$-functor then, for each object F of $\mathscr{F}$ and endo-$\mathscr{V}$-functor T of $\mathscr{F}$, restriction along K provides a bijection between $\mathscr{V}$-natural transformations*

$$u : F \otimes - \Rightarrow T : \mathscr{F} \longrightarrow \mathscr{F}$$

and $\mathscr{V}$-natural transformations

$$t : F \otimes K- \Rightarrow TK- : \mathscr{C} \longrightarrow \mathscr{F}.$$

The components of u are induced on colimits by the components of the corresponding t; so that, if t is invertible, so is u.

Proof. The density of K means that each M in $\mathscr{F}$ is the $\mathscr{F}(K-, M)$-weighted colimit $\mathrm{colim}(\mathscr{F}(K-, M), K)$ of K. Since $F \otimes - : \mathscr{F} \longrightarrow \mathscr{F}$ preserves colimits, we have

$$F \otimes M \cong \mathrm{colim}(\mathscr{F}(K-, M), F \otimes K-).$$

It follows that $\mathscr{V}$-natural families of morphisms $u_M : F \otimes M \longrightarrow TM$ are in bijection with $\mathscr{V}$-natural families of morphisms $\mathscr{F}(K-, M) \longrightarrow \mathscr{F}(F \otimes K-, TM)$ which, by Yoneda, are in bijection with $\mathscr{V}$-natural families of morphisms $t_A : F \otimes KA \longrightarrow TKA$. □

Proposition 3.3. *Suppose $\mathscr{F}$ is a monoidal $\mathscr{V}$-category such that, for each object F, the functors $- \otimes F$ and $F \otimes - : \mathscr{F} \longrightarrow \mathscr{F}$ preserve (weighted) colimits.*

If $K : \mathscr{C} \longrightarrow \mathscr{F}$ is a dense $\mathscr{V}$-functor and $u : F \otimes - \Rightarrow - \otimes F : \mathscr{F} \longrightarrow \mathscr{F}$ is a $\mathscr{V}$-natural transformation then, in order for the triangle

$$\begin{tikzcd}
F \otimes M \otimes N \arrow[rr, "u_{M \otimes N}"] \arrow[dr, "u_m \otimes 1_N"'] & & M \otimes N \otimes F \\
& M \otimes F \otimes N \arrow[ur, "1_M \otimes u_N"'] &
\end{tikzcd}$$

to commute for all M and N in $\mathscr{F}$, it suffices that it commute for all M and N equal to values of K.

Proof. Using the density of K and the colimit preservation properties of the tensor, we have an isomorphism

$$F \otimes M \otimes N \cong \int^{A,B} \mathscr{F}(KA, M) \otimes \mathscr{F}(KB, N) \otimes F \otimes KA \otimes KB$$

which is $\mathscr{V}$-natural in M and N. There are two similar isomorphisms for the other two vertices of the triangle in the proposition. By $\mathscr{V}$-naturality, the triangle itself transports across the isomorphisms to the triangle

$$\begin{tikzcd}
\int^{A,B}\mathscr{F}(KA, M) \otimes \mathscr{F}(KB, N) \otimes F \otimes KA \otimes KB \arrow[rr, "\int^{A,B} 1 \otimes 1 \otimes u_{KA \otimes KB}"] \arrow[dr, "\int^{A,B} 1 \otimes 1 \otimes u_{KA} \otimes 1"'] & & \int^{A,B}\mathscr{F}(KA, M) \otimes \mathscr{F}(KB, N) \otimes KA \otimes KB \otimes F \\
& \int^{A,B}\mathscr{F}(KA, M) \otimes \mathscr{F}(KB, N) \otimes KA \otimes F \otimes KB \arrow[ur, "\int^{A,B} 1 \otimes 1 \otimes 1 \otimes u_{KB}"'] &
\end{tikzcd}$$

which commutes since it is induced on colimits by triangles that commute by hypothesis. So the triangle of the proposition commutes. □

Theorem 3.4. *Suppose $\mathscr{F}$ is a monoidal $\mathscr{V}$-category such that, for each object F, the functor $F \otimes - : \mathscr{F} \longrightarrow \mathscr{F}$ preserves (weighted) colimits. If the full sub-$\mathscr{V}$-category of $\mathscr{F}$ consisting of the objects with right duals is dense in $\mathscr{F}$ then the lax centre of $\mathscr{F}$ is equal to the centre: $\mathcal{Z}_l\mathscr{F} = \mathcal{Z}\mathscr{F}$.*

Proof. Let $\mathscr{C}$ be the full sub-$\mathscr{V}$-category of $\mathscr{F}$ consisting of the objects with right duals, and let K denote the inclusion. Suppose (F, u) is an object of the lax centre of $\mathscr{F}$. Let t correspond to u under the bijection of Proposition 3.2. By Proposition 3.1, t is invertible. By Proposition 3.2, u is invertible so that (F, u) is in the centre of $\mathscr{F}$. □

Corollary 3.5. *For any Hopf algebra H, the lax centre of the monoidal category* **Comod**H *of left H-comodules is equal to its centre.*

Proof. For any coalgebra H, every comodule is the directed union of its finite dimensional subcomodules (see Section 7, Proposition 1 of [**JS2**]). It follows that the comodules which are finite dimensional (as vector spaces) are dense in the category **Comod**H . The bialgebra structure on H provides the monoidal structure on **Comod**H which is preserved by the underlying functor into vector spaces. Since

H is a Hopf algebra, the objects of $\mathbf{Comod}H$ with right duals are those whose underlying vector spaces are finite dimensional (see Section 9, Proposition 4 of [**JS2**]). So Theorem 3.4 applies. □

Corollary 3.6. *For any finite dimensional Hopf algebra H, the lax centre of the monoidal category $\mathbf{Mod}H$ of left H-modules is equal to its centre.*

Proof. Since Yoneda embeddings are dense, the object H of $\mathbf{Mod}H$ (where the action is the algebra multiplication) is dense in $\mathbf{Mod}H$. Since H is finite dimensional, it has a right dual in $\mathbf{Mod}H$. So the objects of $\mathbf{Mod}H$ with right duals are dense and Theorem 3.4 applies. □

Theorem 3.7. *Suppose an object F of a monoidal $\mathscr{V}$-category $\mathscr{F}$ is equipped with the structure of monoid in the lax centre $\mathscr{Z}_l\mathscr{F}$ of $\mathscr{F}$. Then $-\otimes F : \mathscr{F} \longrightarrow \mathscr{F}$ is equipped with the structure of monoidal $\mathscr{V}$-functor.*

Proof. Let (F, u) be a monoid in $\mathscr{Z}_l\mathscr{F}$. So we have a monoid structure on F with multiplication $\mu : F \otimes F \longrightarrow F$ and unit $\eta : I \longrightarrow F$ such that the following two diagrams commute.

$$\begin{tikzcd}
F \otimes F \otimes X \arrow[r, "1\otimes u_X"] \arrow[d, "\mu\otimes 1"'] & F \otimes X \otimes F \arrow[r, "u_X\otimes 1"] & X \otimes F \otimes F \arrow[d, "1\otimes\mu"] \\
F \otimes X \arrow[rr, "u_X"'] & & X \otimes F
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes X \arrow[r, "u_I", "\cong"'] \arrow[d, "\eta\otimes 1"'] & X \otimes I \arrow[d, "1\otimes\eta"] \\
F \otimes X \arrow[r, "u_X"'] & X \otimes F
\end{tikzcd}$$

The monoidal structure on the functor $-\otimes F : \mathscr{F} \longrightarrow \mathscr{F}$ is defined as follows: $\phi_0 : I \longrightarrow F$ is equal to η and $\phi_{2;X,Y} : X \otimes F \otimes Y \otimes F \longrightarrow X \otimes Y \otimes F$ is the composite

$$X \otimes F \otimes Y \otimes F \xrightarrow{1\otimes u_Y\otimes 1} X \otimes Y \otimes F \otimes F \xrightarrow{1\otimes 1\otimes\mu\otimes 1\otimes 1} X \otimes Y \otimes F.$$

The following diagrams commute:

$$\begin{tikzcd}
X{\otimes}F{\otimes}Y{\otimes}F{\otimes}Z{\otimes}F \arrow[r, "1\otimes u_Y\otimes 1\otimes 1\otimes 1"] \arrow[dd, "1\otimes 1\otimes 1\otimes u_Z\otimes 1"'] & X{\otimes}Y{\otimes}F{\otimes}F{\otimes}Z{\otimes}F \arrow[r, "1\otimes 1\otimes\mu\otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes 1\otimes u_Z\otimes 1"'] & X{\otimes}Y{\otimes}F{\otimes}Z{\otimes}F \arrow[dd, "1\otimes 1\otimes u_Z\otimes 1"'] \\
& X{\otimes}Y{\otimes}F{\otimes}Z{\otimes}F{\otimes}F \arrow[dd, "1\otimes 1\otimes u_Z\otimes 1\otimes 1"'] & \\
X{\otimes}F{\otimes}Y{\otimes}Z{\otimes}F{\otimes}F \arrow[ur, "1\otimes u_Y\otimes 1\otimes 1\otimes 1"] \arrow[dr, "1\otimes u_{Y\otimes Z}\otimes 1\otimes 1"'] \arrow[dd, "1\otimes 1\otimes 1\otimes 1\otimes\mu"'] & & X{\otimes}Y{\otimes}Z{\otimes}F{\otimes}F \arrow[dd, "1\otimes 1\otimes 1\otimes\mu"'] \\
& X{\otimes}Y{\otimes}Z{\otimes}F{\otimes}F{\otimes}F \arrow[ur, "1\otimes 1\otimes 1\otimes\mu\otimes 1"] \arrow[d, "1\otimes 1\otimes 1\otimes\mu"'] & \\
X{\otimes}F{\otimes}Y{\otimes}Z{\otimes}F \arrow[r, "1\otimes u_{Y\otimes Z}\otimes 1"'] & X{\otimes}Y{\otimes}Z{\otimes}F{\otimes}H \arrow[r, "1\otimes 1\otimes 1\otimes\mu"'] & X{\otimes}Y{\otimes}Z{\otimes}F
\end{tikzcd}$$

$$
\begin{tikzcd}
X \otimes F \otimes F \arrow[r, "1\otimes\mu"] & X \otimes F \\
& X \otimes F \arrow[ul, "1\otimes 1\otimes\eta"] \arrow[u, "1\otimes 1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F \otimes Y \otimes F \arrow[r, "u_Y\otimes 1"] & Y \otimes F \otimes F \arrow[r, "1\otimes\mu"] & Y \otimes F \\
& Y \otimes F \arrow[ul, "\eta\otimes 1\otimes 1"] \arrow[u, "1\otimes\eta\otimes 1"'] \arrow[ur, "1\otimes 1"']
\end{tikzcd}
$$

which completes the proof. □

4. The cartesian example

For this section we take $\mathscr{V} = \mathbf{Set}$ and study the lax centre of any category $\mathscr{C}$ equipped with the promonoidal structure defined by $P(A, B; C) = \mathscr{C}(B, C) \times \mathscr{C}(A, C)$ and $JC = 1$. Then the corresponding convolution monoidal structure on the functor category $[\mathscr{C}, \mathbf{Set}]$ is none other than (pointwise cartesian) product.

Consider an object (A, α) of $\mathcal{Z}_l\mathscr{C}$. In order that the natural family of morphisms

$$\alpha_{X;Y} : \mathscr{C}(X, Y) \times \mathscr{C}(A, Y) \longrightarrow \mathscr{C}(A, Y) \times \mathscr{C}(X, Y)$$

should satisfy the second condition for an object of $\mathcal{Z}_l\mathscr{C}$, it must be determined by its second projection; that is,

$$\alpha_{X;Y}(f, g) = (g, \overline{\alpha}_{X;Y}(f, g))$$

for a unique natural family of morphisms

$$\overline{\alpha}_{X;Y} : \mathscr{C}(X, Y) \times \mathscr{C}(A, Y) \longrightarrow \mathscr{C}(X, Y)$$

The first condition on α then follows automatically from naturality. Now we can apply the Yoneda Lemma to see that such families $\overline{\alpha}$ are in bijection with dinatural transformations ϕ (in the sense of [**DuSt**]) from the representable functor $\mathscr{C}(A, -)$, thought of as constant in a contravariant variable, to the hom functor $\mathscr{C}(-, \sim) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathbf{Set}$ of $\mathscr{C}$. In other words, we have a family ϕ of functions $\phi_X : \mathscr{C}(A, X) \longrightarrow \mathscr{C}(X, X)$ such that, for all $f : X \longrightarrow Y$ in $\mathscr{C}$, the following diagram commutes.

$$
\begin{tikzcd}
\mathscr{C}(A, X) \arrow[r, "\phi_X"] \arrow[dd, "\mathscr{C}(1_A, f)"'] & \mathscr{C}(X, X) \arrow[dr, "\mathscr{C}(1_X, f)"] & \\
& & \mathscr{C}(X, Y) \\
\mathscr{C}(A, Y) \arrow[r, "\phi_Y"'] & \mathscr{C}(Y, Y) \arrow[ur, "\mathscr{C}(f, 1_Y)"'] &
\end{tikzcd}
$$

In other words, $f\phi_X(u) = \phi_Y(fu)f$ for all morphisms $f : X \longrightarrow Y$ and $u : A \longrightarrow X$. The bijection is obtained by $\alpha_{X;Y}(f, u) = (u, \phi_Y(u)f)$. We therefore identify objects of $\mathcal{Z}_l\mathscr{C}$ with pairs (A, ϕ). A morphism $g : (A, \phi) \longrightarrow (A', \phi')$ in $\mathcal{Z}_l\mathscr{C}$ is a morphism $g : A \longrightarrow A'$ in $\mathscr{C}$ such that $\phi_X(vg) = \phi'_X(v)$ for all $v : A' \longrightarrow X$.

For a moment let us look at the special case where $\mathscr{C}$ has finite coproducts. Then, in the above notation, $\overline{\alpha}_{X,Y} : \mathscr{C}(X, Y) \times \mathscr{C}(A, Y) \longrightarrow \mathscr{C}(X, Y)$ is determined by its composite with the natural bijection $\mathscr{C}(X+A, Y) \cong \mathscr{C}(X, Y) \times \mathscr{C}(A, Y)$

so that the Yoneda Lemma can be applied. Thus we have a bijection between the $\overline{\alpha}$ and the natural transformations $\theta : (-) \longrightarrow (-) + A$ defined by the equations

$$\theta_X = \overline{\alpha}_{X;X+A}(\mathrm{copr}_1, \mathrm{copr}_2) = \phi_{X+A}(\mathrm{copr}_2)\mathrm{copr}_1 : X \longrightarrow X + A.$$

We therefore identify objects of $\mathcal{Z}_l\mathscr{C}$ with pairs (A, θ); morphisms $g : (A, \theta) \longrightarrow (A', \theta')$ are morphisms $g : A \longrightarrow A'$ in $\mathscr{C}$ such that $\theta'_X = (1_X + g)\theta_X$. For a category $\mathscr{X}$ with finite products, we can take $\mathscr{C} = \mathscr{X}^{\mathrm{op}}$ in the above to see that the lax centre $\mathcal{Z}_l\mathscr{X} = (\mathcal{Z}_l\mathscr{X}^{\mathrm{op}})^{\mathrm{op}}$ of the cartesian monoidal category $\mathscr{X}$ has objects pairs (A, θ) where $\theta : (-) \times A \longrightarrow (-)$ is a natural transformation. The tensor product in $\mathcal{Z}_l\mathscr{X}$ is given by

$$(A, \theta) \otimes (A', \theta') = \Big(A \times A', (-) \times A \times A' \xrightarrow{\theta \times 1_{A'}} (-) \times A' \xrightarrow{\theta'} (-)\Big).$$

The lax braiding $c_{(A,\theta),(A',\theta')} : (A, \theta) \otimes (A', \theta') \longrightarrow (A', \theta') \otimes (A, \theta)$ is the morphism

$$(\theta_{A'}, \mathrm{pr}_1) : (A \times A', \theta'(\theta \times 1_{A'})) \longrightarrow (A' \times A, \theta(\theta' \times 1_A)).$$

The *core* $C_{\mathscr{X}}$ of the category $\mathscr{X}$ in the sense of [**Fre**] is precisely a terminal object in $\mathcal{Z}_l\mathscr{X}$; it may not exist in general. Although we shall often write $C_{\mathscr{X}}$ for the underlying object of $\mathscr{X}$, as an object of $\mathcal{Z}_l\mathscr{X}$ it is equipped with a natural transformation $(-) \times C_{\mathscr{X}} \longrightarrow (-)$; however, it is also a monoid in $\mathscr{X}$ whose multiplication is the morphism $C_{\mathscr{X}} \times C_{\mathscr{X}} \longrightarrow C_{\mathscr{X}}$ into the terminal object in $\mathcal{Z}_l\mathscr{X}$. If the core exists, we have the identification of the lax centre with a slice category:

$$\mathcal{Z}_l\mathscr{X} \cong \mathscr{X}/C_{\mathscr{X}}.$$

The monoid structure on $C_{\mathscr{X}}$ defines an obvious monoidal structure on the slice category and the isomorphism is in fact monoidal. If $\mathscr{X}$ is cartesian closed (with internal hom written as $[X, Y]$), we have the formula

$$C_{\mathscr{X}} \cong \int_X [X, X];$$

but in general this end may not exist either.

Proposition 4.1. *If $\mathscr{X}$ is a complete cartesian closed category and $K : \mathscr{D} \longrightarrow \mathscr{X}$ is a dense functor from a small category $\mathscr{D}$ then $\mathscr{X}$ has a core $C_{\mathscr{X}} \cong \int_D [KD, KD]$.*

Proof. The denseness of K amounts to the natural isomorphism

$$\mathscr{X}(X, Y) \cong \int_D \mathbf{Set}(\mathscr{X}(KD, X), \mathscr{X}(KD, Y)).$$

Since $\mathscr{D}$ is small and $\mathscr{X}$ is complete, $\int_D [KD, KD]$ exists. We have the calculation:

$$\mathscr{X}\Big(Z, \int_D [KD, KD]\Big) \cong \int_D \mathscr{X}(Z, [KD, KD]) \cong \int_D \mathscr{X}(KD, [Z, KD])$$

$$\cong \int_{X,D} \mathbf{Set}(\mathscr{X}(KD, X), \mathscr{X}(KD, [Z, X])) \cong \int_X \mathscr{X}(X, [Z, X]) \cong \int_X \mathscr{X}(Z, [X, X]),$$

from which it follows that $\int_X [X, X]$ exists and is isomorphic to $\int_D [KD, KD]$. $\square$

We return now to our arbitrary small category $\mathscr{C}$, equipped with the promonoidal structure defined by $P(A, B; C) = \mathscr{C}(B, C) \times \mathscr{C}(A, C)$ and $JC = 1$, so that the corresponding convolution monoidal structure on the functor category $[\mathscr{C}, \mathbf{Set}]$ is the product. Recall that the internal hom for $[\mathscr{C}, \mathbf{Set}]$ is given by the formula

$$[F,G](A) \cong \int_V \mathbf{Set}(\mathscr{C}(A,V) \times FV, GV).$$

Applying Proposition 4.1 with K equal to the Yoneda embedding $\mathscr{C}^{\mathrm{op}} \longrightarrow [\mathscr{C}, \mathbf{Set}]$, we obtain

$$C_{[\mathscr{C},\mathbf{Set}]}(A) \cong \int_{W,V} \mathbf{Set}(\mathscr{C}(A,V) \times \mathscr{C}(W,V), \mathscr{C}(W,V)) \cong \int_V \mathbf{Set}(\mathscr{C}(A,V), \mathscr{C}(V,V))$$

where the second isomorphism uses the Yoneda Lemma. In other words, interpreting the last end and using our previous notation, we have a connection between the core of $[\mathscr{C}, \mathbf{Set}]$ and the lax centre of $\mathscr{C}$:

$$C_{[\mathscr{C},\mathbf{Set}]}(A) \cong \{\phi \mid (A,\phi) \text{ is an object of } \mathcal{Z}_l\mathscr{C}\}$$

The canonical function $C_{[\mathscr{C},\mathbf{Set}]}(A) \times F(A) \longrightarrow F(A)$ takes (ϕ, a) to $F(\phi_A(1_A))(a)$. The monoid structure $*$ on the functor $C_{[\mathscr{C},\mathbf{Set}]}$ is given by $(\phi * \phi')_U(h) = \phi_U(h)\phi'_U(h)$.

Recall from folklore that the *category* el$F$ *of elements* of a functor $F : \mathscr{C} \to \mathbf{Set}$ has objects pairs (A,a) where A is an object of $\mathscr{C}$ and a is an element of $F(A)$; a morphism $g : (A,a) \longrightarrow (B,b)$ is a morphism $g : A \longrightarrow B$ in $\mathscr{C}$ such that $F(g)(a) = b$. There is an equivalence of categories

$$[\mathscr{C}, \mathbf{Set}]/F \xrightarrow{\sim} [\mathrm{el}F, \mathbf{Set}]$$

taking each object $\rho : T \longrightarrow F$ over F to the functor whose value at (A,a) is the fibre of the component function $\rho_A : T(A) \longrightarrow F(A)$ over $a \in F(A)$. If F is a monoid in $[\mathscr{C}, \mathbf{Set}]$ (that is a functor from $\mathscr{C}$ to the category **Mon** of monoids) then the obvious monoidal structure on $[\mathscr{C}, \mathbf{Set}]/F$ transports to a monoidal structure on $[\mathrm{el}F, \mathbf{Set}]$ which is obtained by convolution from the promonoidal structure on el$F$ defined by

$$P((A,a),(B,b);(C,c)) = \left\{ A \xrightarrow{u} C \xleftarrow{v} B \mid F(u)(a) * F(v)(b) = c \right\}$$

where $*$ is multiplication in the monoid $F(C)$.

As a particular case, we see that the category of elements of $C_{[\mathscr{C},\mathbf{Set}]}$ is $\mathcal{Z}_l\mathscr{C}$ and the monoid structure on $C_{[\mathscr{C},\mathbf{Set}]}$ corresponds to the promagmal structure on $\mathcal{Z}_l\mathscr{C}$.

Putting all this together, we have proved the following result.

THEOREM 4.2. *For any small category $\mathscr{C}$ equipped with the promonoidal structure whose convolution gives the cartesian monoidal structure on $[\mathscr{C}, \mathbf{Set}]$, there is an equivalence and an isomorphism of categories:*

$$[\mathcal{Z}_l\mathscr{C}, \mathbf{Set}] \xrightarrow{\simeq} [\mathscr{C}, \mathbf{Set}]/C_{[\mathscr{C},\mathbf{Set}]} \xrightarrow{\cong} \mathcal{Z}_l[\mathscr{C}, \mathbf{Set}].$$

The promagmal category $\mathcal{Z}_l\mathscr{C}$ is lax-braided promonoidal resulting in a lax-braided convolution monoidal structure on $[\mathcal{Z}_l\mathscr{C}, \mathbf{Set}]$ for which the above composite equivalence is lax-braided monoidal.

The objects of $[\mathscr{C}, \mathbf{Set}]/C_{[\mathscr{C},\mathbf{Set}]}$ can also be interpreted in terms of dinatural transformations. A natural transformation $F \longrightarrow C_{[\mathscr{C},\mathbf{Set}]}$ has components

$$FA \longrightarrow \int_U \mathbf{Set}(\mathscr{C}(A,U), \mathscr{C}(U,U))$$

which are in natural bijection with families of morphisms

$$\mathscr{C}(A,U) \longrightarrow \mathbf{Set}(FA, \mathscr{C}(U,U))$$

natural in A and dinatural in U. By Yoneda, these families are in natural bijection with families of morphisms

$$\rho_U : FU \longrightarrow \mathscr{C}(U,U)$$

dinatural in U. Write $\mathrm{Hom}_{\mathscr{C}}$ for the set-valued hom functor of the category $\mathscr{C}$.

Proposition 4.3. *For any small category $\mathscr{C}$, the lax centre $\mathcal{Z}_l[\mathscr{C}, \mathbf{Set}]$ of the cartesian monoidal category $[\mathscr{C}, \mathbf{Set}]$ is equivalent to the category of dinatural transformations $\rho : F \longrightarrow \mathrm{Hom}_{\mathscr{C}}$ over $\mathrm{Hom}_{\mathscr{C}}$. Given such a dinatural ρ, the corresponding object of $\mathcal{Z}_l[\mathscr{C}, \mathbf{Set}]$ is (F, u) where*

$$u_M : F \times M \longrightarrow M \times F$$

is defined by $(u_M U)(x, m) = (M(\rho_U(x))(m), x)$ for all x in FU and m in MU.

Theorem 4.4. *If $\mathscr{C}$ is a category in which every endomorphism is invertible then the lax centre $\mathcal{Z}_l[\mathscr{C}, \mathbf{Set}]$ of the cartesian monoidal category $[\mathscr{C}, \mathbf{Set}]$ is equal to the centre $\mathcal{Z}[\mathscr{C}, \mathbf{Set}]$.*

Proof. Notice in Proposition 4.3 that each $\rho_U(x)$ is an endomorphism, so under the present hypotheses, an inverse for u_M is defined by

$$(u_M^{-1} U)(m, x) = \Big(x, M(\rho_U(x)^{-1})(m)\Big).$$

$\square$

Before closing this section, let us consider the case where $\mathscr{C}$ is a groupoid. Then the equation $f\phi_X(u) = \phi_Y(fu)f$ can be rewritten $f\phi_X(u)f^{-1} = \phi_Y(fu)$ so that

$$\phi_X(f) = f\phi_A(1_A)f^{-1}.$$

In other words, objects of $\mathcal{Z}_l\mathscr{C}$ can be identified with automorphisms $s : A \longrightarrow A$; the corresponding ϕ is defined by the conjugation formula $\phi_X(f) = fsf^{-1}$. So $\mathcal{Z}_l\mathscr{C} = \mathscr{C}^{\mathbf{Z}}$ is the category of automorphisms in $\mathscr{C}$. As described in Example 9 of [**DaSt**], the promonoidal structure is defined by

$$P((A,s),(B,t);(C,r)) = \Big\{ A \xrightarrow{u} C \xleftarrow{v} B \mid {}^u s {}^v t = r \Big\}.$$

The lax braiding $P((A,s),(B,t);(C,r)) \longrightarrow P((B,t),(A,s);(C,r))$ takes (u,v) to $({}^u sv, u)$. The family of morphisms $\alpha_{X;Y} : \mathscr{C}(X,Y) \times \mathscr{C}(A,Y) \longrightarrow \mathscr{C}(A,Y) \times \mathscr{C}(X,Y)$ corresponding to the ϕ corresponding to s is then defined by $\alpha_{X;Y}(f,u) = (u, usu^{-1}f)$ which is obviously invertible (the inverse takes (u,g) to $(us^{-1}u^{-1}g, u)$). This implies that the lax centre of $\mathscr{C}$ is equal to the centre of $\mathscr{C}$ and that the lax braiding is a braiding. It also follows that $C[\mathscr{C}, \mathbf{Set}] = \mathrm{Aut}_{\mathscr{C}}$ where $\mathrm{Aut}_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathbf{Set}$ is the functor taking the object A to $\mathscr{C}(A,A)$ and the morphism f to conjugation by f.

Theorem 4.5. *If $\mathscr{C}$ as in Theorem 4.2 is a groupoid then*

$$\mathcal{Z}_l\mathscr{C} = \mathcal{Z}\mathscr{C} = \mathscr{C}^{\mathbf{Z}}, \quad \mathcal{Z}_l[\mathscr{C}, \mathbf{Set}] = \mathcal{Z}[\mathscr{C}, \mathbf{Set}], \quad C_{[\mathscr{C}, \mathbf{Set}]} = \mathrm{Aut}_{\mathscr{C}}$$

and there is a braided monoidal equivalence

$$\mathcal{Z}[\mathscr{C}, \mathbf{Set}] \xrightarrow{\sim} [\mathscr{C}^{\mathbf{Z}}, \mathbf{Set}].$$

5. The central cohypomonad

The lax centre of a monoidal $\mathscr{V}$-category $\mathscr{X}$ can be, in very special cases, monadic over $\mathscr{X}$ or comonadic over $\mathscr{X}$. However, with the mere assumption of left closedness, we find that the lax centre $\mathscr{Z}_l\mathscr{X}$ is the $\mathscr{V}$-category of coalgebras for a "cohypomonad", a concept we shall now define.

Let Δ denote the category whose objects are finite ordinals $\langle n\rangle = \{1, 2, \dots, n\}$ and whose morphisms are order-preserving functions. It becomes strict monoidal under the tensor product defined by ordinal sum: $\langle m\rangle + \langle n\rangle = \langle m+n\rangle$. Recall that a comonad on the $\mathscr{V}$-category $\mathscr{X}$ can be identified with a strict monoidal functor $\mathbf{G} : \Delta^{\mathrm{op}} \longrightarrow [\mathscr{X}, \mathscr{X}]$ where the endo-$\mathscr{V}$-functor category $[\mathscr{X}, \mathscr{X}]$ is monoidal under composition. A monad on $\mathscr{X}$ is a strict monoidal functor $\Delta \longrightarrow [\mathscr{X}, \mathscr{X}]$; a mere monoidal functor is something less, so we call it a *hypomonad.*

A *cohypomonad* on $\mathscr{X}$ is a monoidal functor $\mathbf{G} : \Delta^{\mathrm{op}} \longrightarrow [\mathscr{X}, \mathscr{X}]$. More explicitly, it is an augmented simplicial endo-$\mathscr{V}$-functor

$$\begin{tikzcd}
G_0 & G_1 \arrow[l, "\epsilon"'] \arrow[r, "\delta" description] & G_2 \arrow[l, shift right=2.5, "\epsilon_0"'] \arrow[l, shift left=2.5, "\epsilon_1"] \arrow[r, shift left=1.5, "\delta_0"] \arrow[r, shift right=1.5, "\delta_1"'] & \cdots \arrow[l, shift right=4, "\epsilon_0"'] \arrow[l, "\epsilon_1" description] \arrow[l, shift left=4, "\epsilon_2"]
\end{tikzcd}$$

on $\mathscr{X}$ together with $\mathscr{V}$-natural transformations $\gamma_{2;m,n} : G_m \circ G_n \longrightarrow G_{m+n}$ and $\gamma_0 : 1_{\mathscr{X}} \longrightarrow G_0$ satisfying naturality of $\gamma_{2;m,n}$ in $\langle m\rangle$ and $\langle n\rangle$, plus associativity and unit conditions. A cohypomonad is called *normal* when $\gamma_0 : 1_{\mathscr{X}} \longrightarrow G_0$ is invertible.

A *coalgebra* for $\mathbf{G}$ is an object A of $\mathscr{X}$ together with a morphism $\alpha : A \to G_1A$ (called the *coaction*) such that the following two diagrams commute.

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[dr, "\gamma_{0;A}"'] & G_1A \arrow[d, "\epsilon_A"] \\
& G_0A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & G_1A \arrow[dr, "\delta_A"] & \\
& & G_2A \\
G_1A \arrow[r, "G_1\alpha"'] & G_1G_1A \arrow[ur, "\gamma_{2;1,1;A}"'] &
\end{tikzcd}$$

Such a coalgebra gives rise to an extended simplicial diagram on the value of $\mathbf{G}$ at A; we omit the details. A *coalgebra morphism* is a morphism in $\mathscr{X}$ which commutes with the coactions. We obtain a $\mathscr{V}$-category $\mathscr{X}^{\mathbf{G}}$ of $\mathbf{G}$-coalgebras by taking the obvious equalizer in $\mathscr{V}$ to define the $\mathscr{V}$-valued homs

We now turn to our principal example of a cohypomonad. Suppose $\mathscr{X}$ is a left-closed monoidal $\mathscr{V}$-category. For each natural number n, define the endo-$\mathscr{V}$-functor G_n of $\mathscr{X}$ by the end formula

$$G_nA = \int_{X_1,\dots,X_n} [X_1 \otimes \cdots \otimes X_n, X_1 \otimes \cdots \otimes X_n \otimes A],$$

where the square brackets denote the left internal hom. The end exists when, for example, we assume $\mathscr{X}$ is complete, right closed, and has a small dense full sub-$\mathscr{V}$-category. (Alternatively, we could avoid the internal homs and these size problems by looking at modules (= distributors) from $\mathscr{X}$ to $\mathscr{X}$ rather than functors.)

The functor $\mathbf{G} : \Delta^{\mathrm{op}} \longrightarrow [\mathscr{X}, \mathscr{X}]$ is defined as follows. The value at the object $\langle n\rangle$ is of course G_n. Let $\xi : \langle m\rangle \longrightarrow \langle n\rangle$ be an order-preserving function and suppose the fibre of ξ over $k \in \langle n \rangle$ has cardinality m_k. The $\mathscr{V}$-natural transformation $G_\xi : G_n \longrightarrow G_m$ has its component at A defined by commutativity of the triangle

$$\begin{tikzcd}
G_n A \arrow[rr, "G_\xi A"] \arrow[drr, "\mathrm{proj}_{Y_1 \otimes \cdots \otimes Y_{m_1}, \ldots, Y_{m-m_n+1} \otimes \cdots \otimes Y_{m_1}}"'] & & G_m A \arrow[d, "\mathrm{proj}_{Y_1, \ldots, Y_m}"] \\
& & {[Y_1 \otimes \cdots \otimes Y_m, Y_1 \otimes \cdots \otimes Y_m \otimes A]}
\end{tikzcd}$$

for all choices of objects $Y_1, \ldots, Y_m$.

We now describe the monoidal structure on the functor $\mathbf{G}$. In fact, it is normal; there is an obvious canonical $\mathscr{V}$-natural isomorphism $\gamma_0 : 1_{\mathscr{X}} \longrightarrow G_0$. The component of the $\mathscr{V}$-natural transformation $\gamma_{2;m,n} : G_m \circ G_n \longrightarrow G_{m+n}$ at A is defined by commutativity of the diagram

$$\begin{tikzcd}
\int_{\mathbf{Y}} \Big[\bigotimes_m \mathbf{Y}, \bigotimes_m \mathbf{Y} \otimes \int_{\mathbf{X}} \Big[\bigotimes_n \mathbf{X}, \bigotimes_n \mathbf{X} \otimes A\Big]\Big] \arrow[r, "\gamma_{2;m,n}A"] \arrow[d, "\int_{\mathbf{Y}}[1, 1\otimes \mathrm{proj}_{\mathbf{X}}]"'] & \int_{\mathbf{Y},\mathbf{X}} \Big[\bigotimes_m \mathbf{Y} \otimes \bigotimes_n \mathbf{X}, \bigotimes_m \mathbf{Y} \otimes \bigotimes_n \mathbf{X} \otimes A\Big] \arrow[d, "\mathrm{proj}_{\mathbf{Y},\mathbf{X}}"] \\
\int_{\mathbf{Y}} \Big[\bigotimes_m \mathbf{Y}, \bigotimes_m \mathbf{Y} \otimes \Big[\bigotimes_n \mathbf{X}, \bigotimes_n \mathbf{X} \otimes A\Big]\Big] \arrow[d, "\mathrm{proj}_{\mathbf{Y}}"'] & \Big[\bigotimes_m \mathbf{Y} \otimes \bigotimes_n \mathbf{X}, \bigotimes_m \mathbf{Y} \otimes \bigotimes_n \mathbf{X} \otimes A\Big] \arrow[d, "\cong"] \\
\Big[\bigotimes_m \mathbf{Y}, \bigotimes_m \mathbf{Y} \otimes \Big[\bigotimes_n \mathbf{X}, \bigotimes_n \mathbf{X} \otimes A\Big]\Big] \arrow[r, "{[1,\mathrm{canon}]}"'] & \Big[\bigotimes_m \mathbf{Y}, \Big[\bigotimes_n \mathbf{X}, \bigotimes_m \mathbf{Y} \otimes \bigotimes_n \mathbf{X} \otimes A\Big]\Big]
\end{tikzcd}$$

for all lists $\mathbf{Y} = (Y_1, \ldots, Y_m)$ and $\mathbf{X} = (X_1, \ldots, X_n)$ of objects, where the map canon: $Y \otimes [X, Z] \longrightarrow [X, Y \otimes Z]$ corresponds, under the tensor-hom adjunction to $1 \otimes \mathrm{eval} : Y \otimes [X, Z] \otimes X \longrightarrow Y \otimes Z$.

PROPOSITION 5.1. *Let $\mathscr{X}$ be a complete closed monoidal $\mathscr{V}$-category with a small dense sub-$\mathscr{V}$-category. The structure just defined on $\mathbf{G} : \Delta^{op} \longrightarrow [\mathscr{X}, \mathscr{X}]$ makes it a normal cohypomonad for which $\mathscr{X}^{\mathbf{G}}$ is equivalent to the lax centre of $\mathscr{X}$.*

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Chapter 3

Paper 3: On centres and lax centres for promonoidal categories

(Coauthored with Dr. Brian Day and Professor Ross Street)

This paper was submitted to “Charles Ehresmann 100 ans”, the 100th birthday anniversary conference of Charles Ehresmann which was held at the Universite de Picardie Jules Verne in Amiens between October 7 to 9, 2005. This paper is now at *http://perso.orange.fr/vbm-ehr/ChEh/articles/articlesFrT.htm*

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On centres and lax centres for promonoidal categories

Brian Day, Elango Panchadcharam, and Ross Street

Abstract

Our purpose is to highlight the notions of lax braiding and lax centre for a monoidal category and more generally for a promonoidal category. Lax centres are lax braided. Generally the centre is a full subcategory of the lax centre, however it is sometimes the case that the two coincide. There is always an adjunction involving the (lax) centre of a presheaf category and the presheaf category on the (lax) centre. In important cases the adjunction is an equivalence of (lax) braided monoidal categories. One reason for being interested in the lax centre of a monoidal category is that, if an object of the monoidal category is equipped with the structure of monoid in the lax centre, then tensoring with the object defines a monoidal endofunctor on the monoidal category. This has applications in cases where the lax centre can be explicitly identified (as in the presheaf cases mentioned above).

On centres and lax centres for promonoidal categories *

Brian Day, Elango Panchadcharam, and Ross Street

In celebration of the hundreth anniversary of Charles Ehresmann's birth

1 Introduction

Braidings for monoidal categories were introduced in [7] and its forerunners. The centre $\mathcal{Z}\mathscr{X}$ of a monoidal category $\mathscr{X}$ was introduced in [6] in the process of proving that the free tortile monoidal category has another universal property. The centre of a monoidal category is a braided monoidal category. What we now call lax braidings were considered tangentially by Yetter [13]. What we now call the lax centre $\mathcal{Z}_l\mathscr{X}$ of $\mathscr{X}$ was considered under the name "weak centre" by P. Schauenburg [12]. The purpose of this work is to highlight the notions of lax braiding and lax centre for monoidal categories $\mathscr{X}$ and more generally for promonoidal categories $\mathscr{C}$. Indeed we further generalize to the $\mathscr{V}$-enriched context. Lax centres turn out to be lax braided monoidal categories. Generally the centre is a full subcategory of the lax centre, however it is sometimes the case that the two coincide. We have two such theorems under different hypotheses, one in the case sufficient dual objects exist in the additive context, and the other in the cartesian context. We examine when the centre of $[\mathscr{C}, \mathscr{V}]$ with a convolution monoidal structure (in the sense of [1]) is again a functor category $[\mathscr{D}, \mathscr{V}]$.

One reason for being interested in the lax centre of $\mathscr{X}$ is that, if an object X of $\mathscr{X}$ is equipped with the structure of monoid in $\mathcal{Z}_l\mathscr{X}$, then tensoring with X defines a monoidal endofunctor $-\otimes X$ of $\mathscr{X}$; this has applications in cases where the lax centre can be explicitly identified.

2 Review of definitions

The context in which we work is enriched category theory in the sense of [10]. The base monoidal category $\mathscr{V}$ is symmetric, closed, complete and cocomplete. The tensor product of $\mathscr{V}$ is denoted by $\otimes : \mathscr{V} \times \mathscr{V} \longrightarrow \mathscr{V}$, the unit by I, and the associativity and unital isomorphisms will be regarded as canonical (and so unnamed).

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A $\mathscr{V}$*-multicategory* is a $\mathscr{V}$-category $\mathscr{C}$ equipped with a sequence of $\mathscr{V}$-functors

$$P_n : \underbrace{\mathscr{C}^{\mathrm{op}} \otimes \ldots \otimes \mathscr{C}^{\mathrm{op}}}_{n} \otimes \mathscr{C} \longrightarrow \mathscr{V},$$

where we write J for $P_0 : \mathscr{C} \longrightarrow \mathscr{V}$, where P_1 is the hom $\mathscr{V}$-functor $\mathscr{C}(-, \sim) : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \longrightarrow \mathscr{V}$, and where we write P for P_2. Furthermore, there are *substitution operations*, which include $\mathscr{V}$-natural families

$$\int^X P(X, C; D) \otimes P(A, B; X) \xrightarrow{\mu_1} P_3(A, B, C; D) \xleftarrow{\mu_2} \int^Y P(A, Y; D) \otimes P(B, C; Y)$$

$$\int^X P(X, A; B) \otimes JX \xrightarrow{\eta_1} \mathscr{C}(A, B) \xleftarrow{\eta_2} \int^Y P(A, Y; B) \otimes JY,$$

satisfying associativity and unital conditions. For $\mathscr{V} = \mathbf{Set}$, this is a multicategory in the sense of [11].

A *promonoidal* $\mathscr{V}$*-category* [1] is a $\mathscr{V}$-multicategory $\mathscr{C}$ for which $\mu_1, \mu_2, \eta_1, \eta_2$ are invertible. In this case, P_n is determined up to isomorphism by P_0, P_1, P_2.

A *monoidal* $\mathscr{V}$*-category* is a promonoidal $\mathscr{V}$-category $\mathscr{C}$ for which P and J are representable. That is, there are $\mathscr{V}$-natural isomorphisms

$$P(A, B; C) \cong \mathscr{C}(A \boxtimes B, C), \qquad JC \cong \mathscr{C}(U, C)$$

for some $A \boxtimes B$ (depending on the choice of A and B) and some U. Monoidal structures on $\mathscr{C}$ are in bijection with monoidal structures on $\mathscr{C}^{\mathrm{op}}$.

For any small promonoidal $\mathscr{V}$-category $\mathscr{C}$, there is a *convolution* monoidal structure on the $\mathscr{V}$-functor $\mathscr{V}$-category $\mathscr{F} = [\mathscr{C}, \mathscr{V}]$ defined (following [1]) by

$$(F * G)C = \int^{A,B} P(A, B; C) \otimes FA \otimes GB.$$

The unit J and $\mathscr{F}$ is closed (by which we always mean "on both sides"):

$$\mathscr{F}(F, [G, H]_l) \cong \mathscr{F}(F * G, H) \cong \mathscr{F}(G, [F, H]_r)$$

where

$$[G, H]_l A = \int_{B,C} \mathscr{V}(P(A, B; C) \otimes GB, HC) \quad \text{and}$$

$$[F, H]_r B = \int_{A,C} \mathscr{V}(P(A, B; C) \otimes FA, HC).$$

Conversely, every closed monoidal structure $*$, J on $\mathscr{F} = [\mathscr{C}, \mathscr{V}]$ for a small $\mathscr{V}$-category $\mathscr{C}$ defines a promonoidal structure on $\mathscr{C}$ where

$$P_n(A_1, \ldots, A_n; B) = P_n(\mathscr{C}(A_1, -), \ldots, \mathscr{C}(A_n, -); \mathscr{C}(B, -)).$$

That is, we restrict the promonoidal structure along the Yoneda embedding $Y : \mathscr{C} \longrightarrow \mathscr{F}^{\mathrm{op}}$.

A *lax braiding* for a promonoidal $\mathscr{V}$-category $\mathscr{C}$ is a $\mathscr{V}$-natural family of morphisms

$$c_{A,B;C} : P(A,B;C) \longrightarrow P(B,A;C)$$

such that the following diagrams commute.

$$\begin{tikzcd}
\int^U P(U,C;D) \otimes P(A,B;U) \arrow[r, "\int^U c\otimes 1"] \arrow[d, "\cong"'] & \int^U P(C,U;D) \otimes P(A,B;U) \\
\int^V P(A,V;D) \otimes P(B,C;V) \arrow[d, "\int^V 1\otimes c"'] & \int^W P(W,B;D) \otimes P(C,A;W) \arrow[u, "\cong"'] \\
\int^V P(A,V;D) \otimes P(C,B;V) \arrow[r, "\cong"] & \int^W P(W,B;D) \otimes P(A,C;W) \arrow[u, "\int^W 1\otimes c"']
\end{tikzcd}$$

$$\begin{tikzcd}
\int^V P(A,V;D) \otimes P(B,C;V) \arrow[r, "\int^V c\otimes 1"] \arrow[d, "\cong"'] & \int^V P(V,A;D) \otimes P(B,C;V) \\
\int^U P(U,C;D) \otimes P(A,B;U) \arrow[d, "\int^U 1\otimes c"'] & \int^W P(B,W;D) \otimes P(C,A;W) \arrow[u, "\cong"'] \\
\int^U P(U,C;D) \otimes P(B,A;U) \arrow[r, "\cong"] & \int^W P(B,W;D) \otimes P(A,C;W) \arrow[u, "\int^W 1\otimes c"']
\end{tikzcd}$$

$$\begin{tikzcd}
\int^U P(U,A;B) \otimes JU \arrow[rr, "\int^U c\otimes 1"] \arrow[dr, "\cong"'] & & \int^U P(A,U;B) \otimes JU \arrow[dl, "\cong"] \\
& \mathscr{C}(A,B) &
\end{tikzcd}$$

$$\begin{tikzcd}
\int^U P(A,U;B) \otimes JU \arrow[rr, "\int^U c\otimes 1"] \arrow[dr, "\cong"'] & & \int^U P(U,A;B) \otimes JU \arrow[dl, "\cong"] \\
& \mathscr{C}(A,B) &
\end{tikzcd}$$

When $\mathscr{C}$ is monoidal, the lax braiding is induced by a $\mathscr{V}$-natural family of morphisms

$$c_{A,B} : A \boxtimes B \longrightarrow B \boxtimes A$$

which we also call the lax braiding in this case. For general promonoidal $\mathscr{C}$, lax braidings on the convolution monoidal $\mathscr{V}$-category $\mathscr{F} = [\mathscr{C}, \mathscr{V}]$ are in bijection with lax braidings on $\mathscr{C}$: the Yoneda embedding $Y : \mathscr{C} \longrightarrow \mathscr{F}^{\mathrm{op}}$ is a lax-braided promonoidal functor.

A *braiding* is a lax braiding for which each $c_{A,B;C}$ (and hence each $c_{A,B}$ in the monoidal case) is invertible. The third and fourth conditions on a lax braiding are automatic in this case.

In the presence of duals in a monoidal $\mathscr{C}$ (more precisely, $\mathscr{C}$ should be right autonomous in the sense of [7]), *every lax braiding is automatically a braiding* (see [8, Section 10, Proposition 8], [13, Proposition 7.1], [7, Propositions 7.1 and 7.4]).

3 Lax centres

The *lax centre* $\mathcal{Z}_l\mathscr{C}$ of a monoidal $\mathscr{V}$-category $\mathscr{C}$ is the lax-braided monoidal $\mathscr{V}$-category defined as follows. The objects are pairs (A, u) where A is an object of $\mathscr{C}$ and u is a $\mathscr{V}$-natural family of morphisms

$$u_B : A \boxtimes B \longrightarrow B \boxtimes A$$

such that the following two diagrams commute:

$$\begin{tikzcd}
 & (A \boxtimes B) \boxtimes C \arrow[r, "u_B \boxtimes 1"] \arrow[dl, "\cong"'] & (B \boxtimes A) \boxtimes C \arrow[dr, "\cong"] & \\
A \boxtimes (B \boxtimes C) \arrow[d, "u_B \boxtimes C"'] & & & B \boxtimes (A \boxtimes C) \arrow[d, "1 \boxtimes u_C"] \\
(B \boxtimes C) \boxtimes A) \arrow[rrr, "\cong"'] & & & B \boxtimes (C \boxtimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \boxtimes U \arrow[rr, "u_U"] \arrow[dr, "\cong"'] & & U \boxtimes A \arrow[dl, "\cong"] \\
 & A &
\end{tikzcd}$$

(where the marked isomorphisms are induced by the substitution operations μ and η and their inverses). The hom object $\mathcal{Z}_l\mathscr{C}((A, u), (A', u'))$ is defined to be the equalizer in $\mathscr{V}$ of the two composed paths around the following square.

$$\begin{tikzcd}
\mathscr{C}(A, A') \arrow[rr, "- \boxtimes B"] \arrow[d, "B \boxtimes -"'] & & \mathscr{C}(A \boxtimes B, A' \boxtimes B) \arrow[d, "\mathscr{C}(1, u'_B)"] \\
\mathscr{C}(B \boxtimes A, B \boxtimes A') \arrow[rr, "\mathscr{C}(u_B, 1)"'] & & \mathscr{C}(A \boxtimes B, B \boxtimes A')
\end{tikzcd}$$

Composition in $\mathcal{Z}_l\mathscr{C}$ is defined so that we have the obvious faithful $\mathscr{V}$-functor $\mathcal{Z}_l\mathscr{C} \longrightarrow \mathscr{C}$ taking (A, u) to A.

The monoidal structure on $\mathcal{Z}_l\mathscr{C}$ is defined on objects by

$$(A, u) \boxtimes (B, v) = (A \boxtimes B, w)$$

where $w_C : (A \boxtimes B) \boxtimes C \longrightarrow C \boxtimes (A \boxtimes B)$ is the composite

$$A \boxtimes (B \boxtimes C) \xrightarrow{1 \boxtimes v_C} A \boxtimes (C \boxtimes B) \xrightarrow{\cong} (A \boxtimes C) \boxtimes B \xrightarrow{u_C \boxtimes 1} (C \boxtimes A) \boxtimes B$$

conjugated by canonical isomorphisms. The unit object is U equipped with the family of canonical isomorphisms $U \boxtimes C \cong C \boxtimes U$. The faithful $\mathscr{V}$-functor $\mathcal{Z}_l\mathscr{C} \longrightarrow \mathscr{C}$ is strong monoidal.

The lax braiding on $\mathcal{Z}_l\mathscr{C}$ is defined to be the family of morphisms

$$c_{(A,u),(B,v)} : (A \boxtimes B, w) \longrightarrow (B \boxtimes A, \tilde{w})$$

lifting $u_B : A \boxtimes B \longrightarrow B \boxtimes A$ to $\mathcal{Z}_l\mathscr{C}$.

The *centre* $\mathcal{Z}\mathscr{C}$ of $\mathscr{C}$ is the full monoidal sub-$\mathscr{V}$-category of $\mathcal{Z}_l\mathscr{C}$ consisting of the objects (A, u) with each u_B invertible. Clearly $\mathcal{Z}\mathscr{C}$ is a braided monoidal $\mathscr{V}$-category.

There are interesting cases where the centre $\mathcal{Z}\mathscr{C}$ is actually equal to the lax centre. Much as a lax braiding in the presence of duals is a braiding, we have that, if $\mathscr{C}$ is right autonomous, then $\mathcal{Z}\mathscr{C} = \mathcal{Z}_l\mathscr{C}$ (see [2, Proposition 3.1]).

It is worth noting that for a closed monoidal $\mathscr{C}$ with a dense full sub-$\mathscr{V}$-category, the objects (A, u) of $\mathcal{Z}_l\mathscr{C}$ are determined by the restriction of u_B to those B in the dense sub-$\mathscr{V}$-category (see [2, Proposition 3.1 and 3.3]).

We can generalize the lax centre construction to promonoidal $\mathscr{V}$-categories $\mathscr{C}$. It is defined as a $\mathscr{V}$-multicategory to be the pullback $\mathcal{Z}_l\mathscr{C}$

$$\begin{tikzcd}
(\mathcal{Z}_l\mathscr{C}) \arrow[r, "\Psi"] \arrow[d] & (\mathcal{Z}_l\mathscr{F})^{\mathrm{op}} \arrow[d] \\
\mathscr{C} \arrow[r, "\mathrm{Y}"'] & \mathscr{F}^{\mathrm{op}}
\end{tikzcd}$$

of $\mathscr{V}$-categories and $\mathscr{V}$-functors. The multicategory structure is defined by restriction along the fully faithful Ψ (where $\mathscr{F} = [\mathscr{C}, \mathscr{V}]$ with convolution).

Similarly $\mathcal{Z}\mathscr{C}$ is defined by replacing $\mathcal{Z}_l\mathscr{F}$ by $\mathcal{Z}\mathscr{F}$ in the pullback.

It is frequently the case that $\mathcal{Z}_l\mathscr{C}$ is promonoidal, not merely a multicategory; moreover, the forgetful $\mathscr{V}$-functor $\mathcal{Z}_l\mathscr{C} \longrightarrow \mathscr{C}$ is strong promonoidal. If $\mathscr{C}$ is monoidal, this $\mathcal{Z}_l\mathscr{C}$ agrees with the definition in Section 2.

The objects of $\mathcal{Z}_l\mathscr{C}$ are pairs (A, α) where A is an object of $\mathscr{C}$ and α is a $\mathscr{V}$-natural family of morphisms $\alpha_{X;Y} : P(A, X; Y) \longrightarrow P(X, A; Y)$ such that the pair $(\mathscr{C}(A, -), u)$ is an object of $(\mathcal{Z}_l\mathscr{F})^{\mathrm{op}}$, where u is determined by

$$u_{\mathscr{C}(X,-)} = \alpha_{X;-} : \mathscr{C}(A, -) * \mathscr{C}(X, -) \longrightarrow \mathscr{C}(X, -) * \mathscr{C}(A, -).$$

If $\mathcal{Z}_l\mathscr{C}$ is promonoidal, it has a lax braiding

$$c_{(A,\alpha),(B,\beta);(C,\gamma)} : P((A, \alpha), (B, \beta); (C, \gamma)) \longrightarrow P((B, \beta), (A, \alpha); (C, \gamma))$$

obtained by restriction of $\alpha_{B;C} : P(A, B; C) \longrightarrow P(B, A; C)$ to the equalizers.

The $\mathscr{V}$-functor Ψ induces an adjunction $\hat{\Psi} \dashv \tilde{\Psi}$:

$$\mathcal{Z}_l[\mathscr{C}, \mathscr{V}] \underset{\tilde{\Psi}}{\overset{\hat{\Psi}}{\rightleftarrows}} [\mathcal{Z}_l\mathscr{C}, \mathscr{V}]$$

where

$$\hat{\Psi}(G) = \int^{(A,\alpha)} G(A, \alpha) \otimes \Psi(A, \alpha) \quad \text{and} \quad \tilde{\Psi}(F, \theta)(A, \alpha) = \mathcal{Z}_l[\mathscr{C}, \mathscr{V}](\Psi(A, \alpha), (F, \theta)).$$

The last object can be obtained as the equalizer of two morphisms out of FA. If $\mathcal{Z}_l\mathscr{C}$ is promonoidal, this is a lax-braided monoidal adjunction. We shall see that the adjunction can be an equivalence of lax-braided monoidal $\mathscr{V}$-categories.

Similar remarks apply to $\mathcal{Z}\mathscr{C}$.

4 The cartesian case

For this section, suppose $\mathscr{V} = \mathbf{Set}$ with cartesian monoidal structure. Our concern is with the lax centre of cartesian monoidal categories $\mathscr{C}$: that is, $\mathscr{C}$ is a category with finite products regarded as a monoidal category whose tensor is product.

It is an easy exercise to see that an object (A, u) of $\mathcal{Z}_l\mathscr{C}$ is such that $u_X : A \times X \longrightarrow X \times A$ is determined by its first projection $A \times X \longrightarrow X$. In fact, every natural family of morphisms $\theta_X : A \times X \longrightarrow X$ determines an object (A, u) of $\mathcal{Z}_l\mathscr{C}$ via

$$u_X = (\theta_X, \mathrm{pr}_1).$$

So we identify objects of $\mathcal{Z}_l\mathscr{C}$ with pairs (A, θ).

We therefore see that the *core* $C_{\mathscr{C}}$ of the category $\mathscr{C}$ in the sense of [5] is precisely a terminal object of $\mathcal{Z}_l\mathscr{C}$. If this core exists, we have the identification of the lax centre with a slice category

$$\mathcal{Z}_l\mathscr{C} \cong \mathscr{C}/C_{\mathscr{C}}.$$

The monoidal structure on $\mathscr{C}/C_{\mathscr{C}}$ arises from a monoidal structure on $C_{\mathscr{C}}$ in $\mathscr{C}$: the multiplication $C_{\mathscr{C}} \times C_{\mathscr{C}} \longrightarrow C_{\mathscr{C}}$ in $\mathscr{C}$ is the unique morphism into the terminal object in $\mathcal{Z}_l\mathscr{C}$.

If $\mathscr{C}$ is cartesian closed (with internal hom written as [X,Y]), we have the formula

$$C_{\mathscr{C}} \cong \int_X [X, X]$$

provided the end exists; it does when $\mathscr{C}$ has a small dense subcategory and is complete.

Now suppose $\mathscr{C}$ is any small category and we shall apply the considerations of this section to the cartesian monoidal category $\mathscr{F} = [\mathscr{C}, \mathbf{Set}]$.

The promonoidal structure on $\mathscr{C}$ that leads to the cartesian structure on $\mathscr{F}$ via convolution is defined by

$$P(A, B; C) = \mathscr{C}(A, C) \times \mathscr{C}(B, C).$$

(This is monoidal if and only if $\mathscr{C}$ has finite coproducts.) We can obtain the following explict descriptions of $\mathcal{Z}_l\mathscr{C}$ and $\mathcal{Z}\mathscr{C}$ in this case. The objects of $\mathcal{Z}_l\mathscr{C}$ are pairs (A, ϕ) where A is an object $\mathscr{C}$ and ϕ is a family of morphisms

$$\phi_X : \mathscr{C}(A, X) \longrightarrow \mathscr{C}(X, X)$$

dinatural in X in the sense of [4]; that is,

$$f \circ \phi_X(u) = \phi_Y(f \circ u) \circ f$$

for $f : X \longrightarrow Y$. A morphism $g : (A, \phi) \longrightarrow (A', \phi')$ in $\mathcal{Z}_l\mathscr{C}$ is a morphism $g : A \longrightarrow A'$ in $\mathscr{C}$ such that $\phi_X(v \circ g) = \phi'_X(v)$. The promonoidal structure on $\mathcal{Z}_l\mathscr{C}$ is defined by

$$P((A, \phi), (B, \psi); (C, \chi)) =$$
$$\left\{ A \xrightarrow{u} C \xleftarrow{v} B \mid \chi_X(f) = \phi_X(f \circ u) \circ \psi_X(f \circ v) \quad \text{for all} \quad C \xrightarrow{f} X \right\}.$$

The lax braiding on $\mathcal{Z}_l\mathscr{C}$ is defined by

$$P((A, \phi), (B, \psi); (C, \chi)) \xrightarrow{c_{(A,\phi),(B,\psi);(C,\chi)}} P((B, \psi), (A, \phi); (C, \chi))$$
$$(A \xrightarrow{u} C \xleftarrow{v} B) \longmapsto (\ B \xrightarrow{\alpha_C(u) \circ v} C \xleftarrow{u} A\ ).$$

An object (A, ϕ) of $\mathcal{Z}_l\mathscr{C}$ is in $\mathcal{Z}\mathscr{C}$ if and only if the function

$$\mathscr{C}(A, C) \times \mathscr{C}(B, C) \longrightarrow \mathscr{C}(B, C) \times \mathscr{C}(A, C) \ ; \ (u, v) \longmapsto (\alpha_C(u) \circ v, u)$$

is bijection for all B, C.

Theorem 4.1. *Let $\mathscr{C}$ denote a small category with promonoidal structure such that the convolution structure on $[\mathscr{C}, \mathbf{Set}]$ is cartesian product.*

(a). The adjunction $\hat{\Psi} \dashv \tilde{\Psi}$ defines an equivalence of lax-braided monoidal categories

$$\mathcal{Z}_l[\mathscr{C}, \mathbf{Set}] \simeq [\mathcal{Z}_l\mathscr{C}, \mathbf{Set}]$$

which restricts to a braided monoidal equivalence

$$\mathcal{Z}[\mathscr{C}, \mathbf{Set}] \simeq [\mathcal{Z}\mathscr{C}, \mathbf{Set}].$$

(b). If every endomorphism in the category $\mathscr{C}$ is invertible then $\mathcal{Z}_l\mathscr{C} = \mathcal{Z}\mathscr{C}$.

(c). If $\mathscr{C}$ is a groupoid then

$$\mathcal{Z}\mathscr{C} = \mathcal{Z}_l\mathscr{C} = [\Sigma\mathbb{Z}, \mathscr{C}]$$

(where $\Sigma\mathbb{Z}$ is the additive group of the integers as a one-object groupoid).

5 The autonomous case

Suppose $\mathscr{C}$ is a closed monoidal $\mathscr{V}$-category with tensor product $\boxtimes$ and unit U. We write $[Y, Z]_l$ for the left internal hom. Put $X^l = [X, U]_l$. We have a canonical isomorphism $U^l \cong D$ and a canonical morphism $Y^l \boxtimes X^l \longrightarrow (X \boxtimes Y)^l$.

Define a $\mathscr{V}$-functor $M : \mathscr{C} \longrightarrow \mathscr{C}$ by

$$M(A) = \int^X X^l \boxtimes A \boxtimes X$$

when the coend exists (which it does when $\mathscr{C}$ is cocomplete and has a small dense sub-$\mathscr{V}$-category). Using the canonical isomorphism and morphism just mentioned, we obtain a monad structure on M. Notice that M preserves colimits.

Proposition 5.1. *If $\mathscr{C}$ has a small dense sub-$\mathscr{V}$-category of objects with left duals then $\mathcal{Z}_l\mathscr{C}$ is isomorphic to the $\mathscr{V}$-category $\mathscr{C}^M$ of Eilenberg-Moore algebras for the monad M.*

We can apply this in the case where $\mathscr{C}$ is replaced by $\mathscr{F}$.

Theorem 5.2. $(\mathscr{V} = \mathbf{Vect}_k)$ *Suppose $\mathscr{C}$ is a promonoidal k-linear category with finite-dimensional homs. Let $\mathscr{F} = [\mathscr{C}, \mathscr{V}]$ have the convolution monoidal structure. Then*

$$\mathcal{Z}\mathscr{F} = \mathcal{Z}_l\mathscr{F} \cong \mathscr{F}^M \simeq [\mathscr{C}_M, \mathscr{V}]$$

where $\mathscr{C}_M$ is the Kleisli category for the promonad M on $\mathscr{C}$.

6 Monoids in the lax centre

Let $\mathscr{C}$ be a monoidal $\mathscr{V}$-category. Each monoid (A, u) in $\mathcal{Z}_l\mathscr{C}$ determines a canonical enrichment of the $\mathscr{V}$-functor

$$- \boxtimes A : \mathscr{C} \longrightarrow \mathscr{C}$$

to a monoidal functor:

$$X \boxtimes A \boxtimes Y \boxtimes A \xrightarrow{1 \boxtimes u_Y \boxtimes 1} X \boxtimes Y \boxtimes A \boxtimes A \xrightarrow{1 \boxtimes 1 \boxtimes \mu} X \boxtimes Y \boxtimes A$$

$$U \xrightarrow{\eta} A \cong U \boxtimes A.$$

This becomes useful when $\mathscr{Z}_l \mathscr{C}$ can be explicitly identified as in the last two sections.

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Chapter 4

Paper 4: Pullback and finite coproduct preserving functors between categories of permutation representations

(Coauthored with Professor Ross Street)

This chapter is a modified version of the paper [PS2] in accord with the correction and addition explained in [PS3].

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PULLBACK AND FINITE COPRODUCT PRESERVING FUNCTORS BETWEEN CATEGORIES OF PERMUTATION REPRESENTATIONS

ELANGO PANCHADCHARAM AND ROSS STREET

Abstract. Motivated by applications to Mackey functors, Serge Bouc [Bo] characterized pullback and finite coproduct preserving functors between categories of permutation representations of finite groups. Initially surprising to a category theorist, this result does have a categorical explanation which we provide.

1. Introduction

For a finite group G, we write $G\text{-}\mathbf{set}_{\mathit{fin}}$ for the category of finite (left) G-sets (that is, of permutation representations of G) and equivariant functions. We write $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ for the category whose morphisms are isomorphism classes of spans between finite G-sets. Coproducts in $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ are those of $G\text{-}\mathbf{set}_{\mathit{fin}}$ and composition in $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ involves pullbacks in $G\text{-}\mathbf{set}_{\mathit{fin}}$.

According to Harald Lindner [Li], a Mackey functor M on a finite group H is a finite coproduct preserving functor $M : \mathbf{Spn}(H\text{-}\mathbf{set}_{\mathit{fin}}) \longrightarrow \mathbf{Mod}_k$[1]. A functor $F : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$ which preserves pullbacks and finite coproducts will induce a functor

$$\mathbf{Spn}(F) : \mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}}) \longrightarrow \mathbf{Spn}(H\text{-}\mathbf{set}_{\mathit{fin}})$$

preserving finite coproducts. By composition with $\mathbf{Spn}(F)$, each Mackey functor M on H will produce a Mackey functor $M \circ \mathbf{Spn}(F)$ on G.

This observation led Bouc [Bo] to a systematic study of pullback and finite coproduct preserving functors $F : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$. He characterized them in terms of $G^{\mathrm{op}} \times H$-sets A (where G^{op} is G with opposite multiplication). This perplexed us initially, as the category $(G^{\mathrm{op}} \times H)\text{-}\mathbf{set}$ of such A is equivalent to the category of finite colimit preserving functors $L : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$; these L generally do not preserve pullbacks, while the F generally do not preserve coequalizers. Of course, Bouc's construction of L from a left H-, right G-set A is quite different from the standard module theory construction of F from A. We shall explain the two constructions.

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[1]Actually, Lindner asked for finite product preserving; but, in our case, the categories have direct sums.

We put $(g,h)a = hag$ for $g \in G, h \in H$ and a in the $(G^{\mathrm{op}} \times H)$-set A, so that A becomes a left H-set and a right G-set. For each left G-set X, define the left H-set $A \otimes_G X$ to be the quotient of the set $A \times X = \{(a,x) \mid a \in A, x \in X\}$ by the equivalence relation generated by

$$(ag, x) \sim (a, gx), \quad a \in A, x \in X, g \in G.$$

Write $[a,x]$ for the equivalence class of (a,x) and define $h[a,x] = [ha,x]$. For A finite, this defines our functor $L = A \otimes_G - : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$ on objects; it is defined on morphisms $f : X \longrightarrow X'$ by $L(f)[a,x] = [a, f(x)]$. Certainly L preserves all colimits that exist in $G\text{-}\mathbf{set}_{\mathit{fin}}$ since it has a right adjoint $R : H\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow G\text{-}\mathbf{set}_{\mathit{fin}}$ defined on the left H-set Y by $R(Y) = H\text{-}\mathbf{set}_{\mathit{fin}}(A, Y)$ with action $(g, \theta) \longmapsto g\theta$ where $(g\theta)(a) = \theta(ag)$. All this is classical "module" theory.

Now we turn to Bouc's construction. Again let A be a $(G^{\mathrm{op}} \times H)$-set. Rather than a mere G-set X, we define a functor on all $(K^{\mathrm{op}} \times G)$-sets B where K, G, H are all finite groups. Put

$$A \wedge_G B = \{(a,b) \in A \times B \mid g \in G, ag = a \Longrightarrow \text{there exists } k \in K \text{ with } gb = bk\}.$$

This becomes a $(K^{\mathrm{op}} \times G \times H)$-set via the action

$$(k,g,h)(a,b) = (hag^{-1}, gbk).$$

Then Bouc defines the $(K^{\mathrm{op}} \times H)$-set

$$A \circ_G B = (A \wedge_G B)/G,$$

to be the set of orbits $orb(a,b) = [a,b]$ of elements (a,b) of $A \wedge_G B$ under the action of G. In particular, when $K = \mathbf{1}$ and $B = X \in G\text{-}\mathbf{set}_{\mathit{fin}}$, we obtain $F(X) = A \circ_G X \in H\text{-}\mathbf{set}_{\mathit{fin}}$. This defines the functor

$$F : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}.$$

1.1. Theorem. *[Bo] Suppose K, G and H are finite groups.*

(i) If A is a finite $(G^{\mathrm{op}} \times H)$-set then the functor

$$A \circ_G - : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$$

preserves finite coproducts and pullbacks.

(ii) Every functor $F : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$ which preserves finite coproducts and pullbacks is isomorphic to one of the form $A \circ_G -$.

(iii) The functor F in (ii) preserves terminal objects if and only if A is transitive (connected) as a right G-$\mathbf{set}_{\mathit{fin}}$.

(iv) If A is as in (i) and B is a finite $(K^{\mathrm{op}} \times G)$-set then the composite functor

$$K\text{-}\mathbf{set}_{\mathit{fin}} \xrightarrow{B \circ_K -} G\text{-}\mathbf{set}_{\mathit{fin}} \xrightarrow{A \circ_G -} H\text{-}\mathbf{set}_{\mathit{fin}}$$

is isomorphic to $(A \circ_G B) \circ_K -$.

Our intention in the present paper is to provide a categorical explanation for this Theorem.

In Section 2, before turning to the problem of preserving pullbacks, we examine finite limit preserving functors from categories like G-$\mathbf{set}_{\mathit{fin}}$ to $\mathbf{set}_{\mathit{fin}}$. We adapt the appropriate classical adjoint functor theorem to this "finite" situation. To make use of this for the purpose in hand, in Section 3, we need to adapt the result to include preservation of finite coproducts and reduce the further preservation of pullbacks to the finite limit case.

Section 4 interprets the work in the finite G-set case. In Section 5 we express the conclusions bicategorically. Implications for our original motivating work on Mackey functors are explained in the final Section 6.

2. Special representability theorem

In this section we provide a direct proof of the well-known representability theorem (see Chapter 5 [Ma]) for the case where "small" means "finite".

Recall that an object Q of a category $\mathscr{A}$ is called a *cogenerator* when, for all $f, g : A \longrightarrow B$ in $\mathscr{A}$, if $uf = ug$ for all $u : B \longrightarrow Q$, then $f = g$.

A *subobject* of an object A of $\mathscr{A}$ is an isomorphism class of monomorphisms $m : S \longrightarrow A$; two such monomorphisms $m : S \longrightarrow A$ and $m' : S' \longrightarrow A$ are isomorphic when there is an invertible morphism $h : S \longrightarrow S'$ with $m' \circ h = m$. We call $\mathscr{A}$ *finitely well powered* when each object A has only finitely many subobjects. Write $\mathit{Sub}(A)$ for the set of subobjects $[m : S \longrightarrow A]$ of A.

For each set X and object A of $\mathscr{A}$, we write A^X for the object of $\mathscr{A}$ for which there is a natural isomorphism

$$\mathscr{A}(B, A^X) \cong \mathscr{A}(B, A)^X$$

where Y^X is the set of functions from X to Y. Such an object may not exist; if $\mathscr{A}$ has products indexed by X then A^X is the product of X copies of A.

We write $\mathbf{set}_{\mathit{fin}}$ for the category of finite sets and functions. A functor $T : \mathscr{A} \longrightarrow \mathbf{set}_{\mathit{fin}}$ is *representable* when there is an object $K \in \mathscr{A}$ and a natural isomorphism $T \cong \mathscr{A}(K, -)$.

2.1. Theorem. *(Special representability theorem) Suppose $\mathscr{A}$ is a category with the following properties:*

(i) each homset $\mathscr{A}(A, B)$ is finite;

(ii) finite limits exist;

(iii) there is a cogenerator Q;

(iv) $\mathscr{A}$ *is finitely well powered.*

Then every finite limit preserving functor $T : \mathscr{A} \longrightarrow \mathbf{set}_{fin}$ *is representable.*

PROOF. Using (ii) and (iv), we have the object

$$P = \prod_{[S] \in Sub(Q^{TQ})} S^{TS}.$$

We shall prove that, for each $A \in \mathscr{A}$ and $a \in TA$, there exists $p \in TP$ and $w : P \longrightarrow A$ such that $(Tw)p = a$. The following diagram defines δ, ι and S_0.

$$\begin{tikzcd}
S_0 \arrow[r, "m"] \arrow[d, "t"'] \arrow[dr, phantom, "\mathrm{pb}"] & Q^{TQ} \arrow[d, "\iota"] \arrow[ddr, bend left, "\mathrm{pr}_{(Tf)a}"] & \\
A \arrow[r, "\delta"'] \arrow[drr, bend right, "f"'] & Q^{\mathscr{A}(A,Q)} \arrow[dr, "\mathrm{pr}_f"] & \\
& & Q
\end{tikzcd}$$

Now $\delta u = \delta v$ implies $fu = fv$ for all $f : A \longrightarrow Q$, so $u = v$ by (iii). So δ is a monomorphism. So $[S_0] \in Sub(Q^{TQ})$. Since T preserves pullbacks, there is a unique $s \in TS_0$ such that $(Tt)s = a$ and $(Tm)s$ transports to 1_{TQ} under $T(Q^{TQ}) \cong (TQ)^{TQ}$. Let p transport to $(1_{TS})_{[S]}$ under $TP \cong \prod_{[S]}(TS)^{TS}$. Then we can define w to be the composite

$$P \xrightarrow{\mathrm{pr}_{[S_0],s}} S_0 \xrightarrow{t} A$$

with $(Tw)p = a$.

Now let K be the equalizer of all the endomorphisms e (including 1_P) of P for which $T(e)(p) = p$ (we are using (i)):

$$\begin{tikzcd}
P \arrow[rr, "e"] & & P \\
& K. \arrow[ul, "l"] \arrow[ur, "l"'] &
\end{tikzcd}$$

Since T preserves limits, there is a unique $k \in TK$ with $(Tl)k = p$. Define

$$\theta_A : \mathscr{A}(K, A) \longrightarrow TA$$

by $\theta_A(r) = (Tr)k$; this is natural in A. Moreover, θ_A is surjective since (K, k) clearly has the same property that we proved for (P, p).

It remains to prove θ_A injective. Suppose r and $r' : K \longrightarrow A$ are such that $(Tr)k = (Tr')k$.

Let $n : U \longrightarrow K$ be the equalizer of r and r', and let $u \in TU$ be unique with $(Tn)u = k$. By the property of (P, p), there exists $w : P \longrightarrow U$ with $(Tw)p = u$.

$$\begin{tikzcd} P \arrow[r, "w"] & U \arrow[r, "n"] & K \arrow[r, "l"] & P \\ & K \arrow[ul, "l"] \arrow[urr, "l"'] & & \end{tikzcd}$$

From the definition of K, we have $lnwl = l$. Yet l is a monomorphism (since it is an equalizer), so $nwl = 1$ and $r = rnwl = r'nwl = r'$, as required. ∎

For categories $\mathscr{A}$ and $\mathscr{X}$ admitting finite limits, write $\mathbf{Lex}(\mathscr{A}, \mathscr{X})$ for the full subcategory of the functor category $[\mathscr{A}, \mathscr{X}]$ consisting of the finite limit preserving functors.

2.2. Corollary. *For a category $\mathscr{A}$ satisfying the conditions of Theorem 2.1, the Yoneda embedding defines an equivalence of categories*

$$\mathscr{A}^{\mathrm{op}} \simeq \mathbf{Lex}(\mathscr{A}, \mathbf{set}_{\mathit{fin}}) \ , \quad A \longmapsto \mathscr{A}(A, -).$$

3. Finite coproducts

Suppose the category $\mathscr{A}$ has finite coproducts. An object C of $\mathscr{A}$ is called *connected* when the functor $\mathscr{A}(C, -) : \mathscr{A} \longrightarrow \mathbf{Set}$ preserves finite coproducts. Write $\mathbf{Conn}(\mathscr{A})$ for the full subcategory of $\mathscr{A}$ consisting of the connected objects.

Write $\mathbf{Cop}(\mathscr{A}, \mathscr{X})$ for the full subcategory of $[\mathscr{A}, \mathscr{X}]$ consisting of the finite coproduct preserving functors. Also $\mathbf{CopLex}(\mathscr{A}, \mathscr{X})$ consists of the finite coproduct and finite limit preserving functors. As an immediate consequence of Corollary 2.2 we have

3.1. Corollary. *For a category $\mathscr{A}$ with finite coproducts and the properties of Theorem 2.1, the Yoneda embedding defines an equivalence of categories*

$$\mathbf{Conn}(\mathscr{A})^{\mathrm{op}} \simeq \mathbf{CopLex}(\mathscr{A}, \mathbf{set}_{\mathit{fin}}).$$

Suppose $\mathscr{A}$ is a finitely complete category and $T : \mathscr{A} \longrightarrow \mathbf{set}_{\mathit{fin}}$ is a functor. For each $t \in T1$, define a functor $T_t : \mathscr{A} \longrightarrow \mathbf{set}_{\mathit{fin}}$ using the universal property of the pullback

$$\begin{tikzcd} T_t A \arrow[r, "\iota_t A"] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & TA \arrow[d, "T\,!"] \\ 1 \arrow[r, "t"'] & T1. \end{tikzcd}$$

Clearly $T \cong \sum_{t \in T1} T_t$. Taking $A = 1$ in the above pullback, we see that each T_t preserves terminal objects. The following observations are obvious.

3.2. PROPOSITION. *Suppose $\mathscr{A}$ is a finitely complete category, $T : \mathscr{A} \longrightarrow \mathbf{set}_{fin}$ is a functor, and the functors $T_t : \mathscr{A} \longrightarrow \mathbf{set}_{fin}$, $t \in T1$, are defined as above.*

(i) If T preserves pullbacks then each T_t preserves finite limits.

(ii) Each T_t preserves whatever coproducts that are preserved by T.

For any small category $\mathscr{C}$, we write $\mathbf{Fam}(\mathscr{C}^{\mathrm{op}})$ for the free finite coproduct completion of $\mathscr{C}^{\mathrm{op}}$. The objects are families $(C_i)_{i \in I}$ of objects C_i of $\mathscr{C}$ with indexing set I finite. A morphism $(\xi, f) : (C_i)_{i \in I} \longrightarrow (D_j)_{j \in J}$ consists of a function $\xi : I \longrightarrow J$ and a family $f = (f_i)_{i \in I}$ of morphisms $f_i : D_{\xi(i)} \longrightarrow C_i$ in $\mathscr{C}$.

There is a functor

$$\mathscr{Z}_{\mathscr{C}} : \mathbf{Fam}(\mathscr{C}^{\mathrm{op}}) \longrightarrow [\mathscr{C}, \mathbf{Set}]$$

defined by

$$\mathscr{Z}_{\mathscr{C}}(C_i)_{i \in I} = \sum_{i \in I} \mathscr{C}(C_i, -)$$

which is fully faithful. So $\mathbf{Fam}(\mathscr{C}^{\mathrm{op}})$ is equivalent to the closure under finite coproducts of the representables in $[\mathscr{C}, \mathbf{Set}]$.

We write $\mathbf{Pb}(\mathscr{A}, \mathscr{X})$ for the full subcategory of $[\mathscr{A}, \mathscr{X}]$ consisting of pullback preserving functors. Also $\mathbf{CopPb}(\mathscr{A}, \mathscr{X})$ has objects restricted to those preserving finite coproducts and pullbacks.

3.3. PROPOSITION. *Suppose the category $\mathscr{A}$ is as in Corollary 3.1. The functor $\mathscr{Z}_{\mathscr{A}}$ induces an equivalence of categories*

$$\mathbf{Fam}(\mathbf{Conn}(\mathscr{A})^{\mathrm{op}}) \simeq \mathbf{CopPb}(\mathscr{A}, \mathbf{set}_{fin}).$$

PROOF. Clearly $\mathbf{Fam}(\mathbf{Conn}(\mathscr{A})^{\mathrm{op}})$ is a full subcategory of $\mathbf{Fam}(\mathscr{A}^{\mathrm{op}})$ and $\mathscr{Z}_A$ restricts to a fully faithful functor

$$\mathbf{Fam}(\mathbf{Conn}(\mathscr{A})^{\mathrm{op}}) \longrightarrow [\mathscr{A}, \mathbf{set}_{fin}].$$

It remains to identify the essential image of this functor as those $T : \mathscr{A} \longrightarrow \mathbf{set}_{fin}$ which preserve finite coproducts and pullbacks. However, we have seen in Proposition 3.2 that such a T has the form $T \cong \sum_{t \in T1} T_t$ where each T_t preserves finite coproducts and is left exact. By Corollary 3.1, we have

$$T_t \cong \mathscr{A}(C_t, -)$$

where each C_t is connected. ∎

4. Application to permutation representations

A *permutation representation* of a finite group G, also called a *finite left G-set*, is a finite set X together with a function $G \times X \longrightarrow X$, $(g, x) \longmapsto gx$, called the *action* such that

$$1x = x \quad \text{and} \quad g_1(g_2x) = (g_1g_2)x.$$

If X and Y are such G-sets, a *(left) G-morphism* $f : X \longrightarrow Y$ is a function satisfying $f(gx) = gf(x)$. We write G-$\mathbf{set}_{\mathit{fin}}$ for the category of finite left G-sets and left G-morphisms.

The terminal object of G-$\mathbf{set}_{\mathit{fin}}$ is the set $\mathbf{1}$ with only one element with its unique action. The pullback in G-$\mathbf{set}_{\mathit{fin}}$ of two morphisms $f : X \longrightarrow Z$ and $k : Y \longrightarrow Z$ is given by $\{(x, y) \in X \times Y | f(x) = k(y)\}$ with componentwise action $g(x, y) = (gx, gy)$. So G-$\mathbf{set}_{\mathit{fin}}$ has finite limits.

Since the set G-$\mathbf{set}_{\mathit{fin}}(X, Y)$ is a subset of the set $\mathbf{set}_{\mathit{fin}}(X, Y) = Y^X$, it is finite.

The set $\mathscr{P}G$ of subsets of G becomes a G-set by defining the action as

$$gS = \{h \in G \mid hg \in S\}$$

for $S \subseteq G$ and $g \in G$. For each $x \in X \in G$-$\mathbf{set}_{\mathit{fin}}$, we can define a G-morphism $\chi_x : X \longrightarrow \mathscr{P}G$ by

$$\chi_x(z) = \{h \in G \mid hz = x\}.$$

If $x, y \in X \in G$-$\mathbf{set}_{\mathit{fin}}$ then $\chi_x(x) = \chi_x(y)$ implies $1 \in \chi_x(y)$, so $y = x$. It follows that $\mathscr{P}G$ is a cogenerator for G-$\mathbf{set}_{\mathit{fin}}$.

Subobjects of $X \in G$-$\mathbf{set}_{\mathit{fin}}$ are in bijection with sub-G-sets of X. So G-$\mathbf{set}_{\mathit{fin}}$ is finitely well powered.

From Section 2, we therefore have:

4.1. Corollary. *Every limit preserving functor $T : G$-$\mathbf{set}_{\mathit{fin}} \longrightarrow \mathbf{set}_{\mathit{fin}}$ is representable. The Yoneda embedding induces an equivalence of categories*

$$G\text{-}\mathbf{set}_{\mathit{fin}}^{\mathrm{op}} \simeq \mathbf{Lex}(G\text{-}\mathbf{set}_{\mathit{fin}}, \mathbf{set}_{\mathit{fin}}).$$

Recall that a G-set X is called *transitive* when it is non-empty and, for all $x, y \in X$, there exists $g \in G$ with $gx = y$.

For any G-set X and any $x \in X$, we put

$$\mathit{stab}(x) = \{g \in G \mid gx = x\}$$

which is a subgroup of G called the *stabilizer* of x. We also put

$$\mathit{orb}(x) = \{gx \mid g \in G\}$$

which is a transitive sub-G-set X called the *orbit* of x. We write X/G for the set of orbits which can be regarded as a G-$\mathbf{set}_{\mathit{fin}}$ with trivial action so that $\mathit{orb} : X \longrightarrow X/G$ is a surjective G-morphism. If $u \in X/G$, we also write X_u for the orbit u as a sub-G-set of X. So every G-set is the disjoint union of its orbits.

The empty coproduct is the empty set $\mathbf{0}$ with its unique action. The coproduct of two G-sets X and Y is their disjoint union $X + Y$ with action such that the coprojections $X \longrightarrow X + Y$ and $Y \longrightarrow X + Y$ are G-morphisms. So every G-set X is a coproduct

$$X \cong \sum_{u \in X/G} X_u$$

of transitive G-sets (the orbits X_u).

Each subgroup H of G determines a transitive G-set

$$G/H = \{xH \mid x \in G\}$$

where $xH = \{xh \mid h \in H\}$ is the left coset of H containing x, and where the action is

$$g(xH) = (gx)H.$$

Every transitive G-set X is isomorphic to one of the form G/H; we can take $H = \mathit{stab}(x)$ for any $x \in X$.

The G-morphisms $f : G/H \longrightarrow X$ are in bijection with those $x \in X$ such that $H \leq \mathit{stab}(x)$. The G-sets G/H and G/K are isomorphic if and only if the subgroups H and K are conjugate (that is, there exists $x \in G$ with $Hx = xK$).

We provide a proof of the following well-known fact.

4.2. Proposition. *A finite G-set X is transitive if and only if X is a connected object of G-$\mathbf{set}_{\mathit{fin}}$.*

Proof. A G-set X is non-empty if and only if G-$\mathbf{set}_{\mathit{fin}}(X, \mathbf{0})$ is empty; that is, if and only if G-$\mathbf{set}_{\mathit{fin}}(X, -)$ preserves empty coproducts.

A morphism $G/H \longrightarrow Y + Z$ is determined by an element of $Y + Z$ stable under H; such an element must either be an element of Y or an element of Z stable under H. So transitive G-sets are connected.

Assume X is connected. We have already seen that X is non-empty so choose $x \in X$. Then

$$X = \mathit{orb}(x) + U$$

for some sub-G-set U of X. We therefore have the canonical function

$$G\text{-}\mathbf{set}_{\mathit{fin}}(X, \mathit{orb}(x)) + G\text{-}\mathbf{set}_{\mathit{fin}}(X, U) \longrightarrow G\text{-}\mathbf{set}_{\mathit{fin}}(X, X)$$

which is invertible since X is connected. So the identity function $X \longrightarrow X$ is in the image of the canonical function and so factors through $\mathit{orb}(x) \subseteq X$ or $U \subseteq X$. Since $x \notin U$, we must have $\mathit{orb}(x) = X$. So X is connected. ∎

We have thus identified $\mathbf{Conn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ as consisting of the transitive G-sets. This category has a finite skeleton $\mathscr{C}_G$ since there are only finitely many G-sets of the form G/H. Corollary 3.1 yields:

4.3. Corollary. *The Yoneda embedding induces an equivalence of categories*

$$\mathscr{C}_G^{\mathrm{op}} \simeq \mathbf{CopLex}(G\text{-}\mathbf{set}_{\mathit{fin}}, \mathbf{set}_{\mathit{fin}}).$$

Let $N : \mathscr{C}_G \longrightarrow G\text{-}\mathbf{set}_{\mathit{fin}}$ denote the inclusion functor and define the functor

$$\tilde{N} : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow [\mathscr{C}_G^{\mathrm{op}}, \mathbf{set}_{\mathit{fin}}]$$

by $\tilde{N}X = G\text{-}\mathbf{set}_{\mathit{fin}}(N-, X)$.

4.4. Proposition. *The functor $\tilde{N}$ induces an equivalence of categories*

$$G\text{-}\mathbf{set}_{\mathit{fin}} \simeq \mathbf{Fam}(\mathscr{C}_G).$$

Proof. We first prove that N is dense; that is, that $\tilde{N}$ is fully faithful. Let $\theta : \tilde{N}X \longrightarrow \tilde{N}Y$ be a natural transformation. For each $u : C \longrightarrow D$ in $\mathscr{C}_G$ we have a commutative square

$$\begin{tikzcd}
G\text{-}\mathbf{set}_{\mathit{fin}}(D, X) \arrow[r, "\theta_D"] \arrow[d, "-\circ u"'] & G\text{-}\mathbf{set}_{\mathit{fin}}(D, Y) \arrow[d, "-\circ u"] \\
G\text{-}\mathbf{set}_{\mathit{fin}}(C, X) \arrow[r, "\theta_C"'] & G\text{-}\mathbf{set}_{\mathit{fin}}(C, Y).
\end{tikzcd}$$

Since the single-object full subcategory of $G\text{-}\mathbf{set}_{\mathit{fin}}$ consisting of G is dense ($G^{\mathrm{op}} \longrightarrow G\text{-}\mathbf{set}_{\mathit{fin}}$ is a Yoneda embedding), by restricting C and D to be equal to $G \in \mathscr{C}_G$, we obtain a G-morphism $f : X \longrightarrow Y$ defined uniquely by $f(x) = \theta_G(\hat{x})(1)$ where $\hat{x} : G \longrightarrow X$ is given by $\hat{x}(g) = gx$. Then, for all $w : D \longrightarrow X$ and $d \in D$, the above commutative square, with $C = G$, yields

$$\theta_D(w)(d) = (\theta_D(w) \circ \hat{d})(1) = \theta_G(w \circ \hat{d})(1) = \theta_G(\widehat{w \circ d})(1) = (f \circ w)(d).$$

So $\theta_D = G\text{-}\mathbf{set}_{\mathit{fin}}(D, f)$ for a unique G-morphism f.

The proof of the equivalence of categories will be completed by characterizing the essential image of $\tilde{N}$ as finite coproducts of representables in $[\mathscr{C}_G^{\mathrm{op}}, \mathbf{set}_{\mathit{fin}}]$. If $F \in [\mathscr{C}_G^{\mathrm{op}}, \mathbf{set}_{\mathit{fin}}]$ is a finite coproduct of representables then we have a finite family $(C_i)_{i \in I}$ of objects of $\mathscr{C}_G$ and an isomorphism $F \cong \sum_i \mathscr{C}(-, C_i)$. We have the calculation:

$$\begin{aligned}
\sum_i \mathscr{C}(-, C_i) &\cong \sum_i G\text{-}\mathbf{set}_{\mathit{fin}}(N-, C_i) \\
&\cong G\text{-}\mathbf{set}_{\mathit{fin}}(N-, \sum_i C_i) \\
&\cong \tilde{N}(\sum_i C_i).
\end{aligned}$$

So F is in the essential image of $\tilde{N}$. Conversely, every $X \in G\text{-}\mathbf{set}_{\mathit{fin}}$ is a coproduct $X \cong \sum_i C_i$ of connected G-sets. So the same calculation, read from bottom to top, shows that $\tilde{N}(X)$ is a finite coproduct of representables. ■

5. A factorization for G-morphisms

For any G-set X, we have the set $X/G = \{C \subseteq X : C \text{ is an orbit of} X\}$ of connected sub-G-sets of X. We have the function $orb : X \longrightarrow X/G$ taking each element $x \in X$ to its orbit $orb(x) = \{gx : g \in G\}$. Each G-morphism $f : X \longrightarrow Y$ induces a direct image function $f/G : X/G \longrightarrow Y/G$ defined by $(f/G)(C) = f_*(C)$.

A G-morphism $f : X \longrightarrow Y$ is said to be *slash inverted* when $f/G : X/G \longrightarrow Y/G$ is a bijection.

5.1. Proposition. *A G-morphism $f : X \longrightarrow Y$ is slash inverted if and only if it is surjective and $f(x_1) = f(x_2)$ implies $orb(x_1) = orb(x_2)$.*

Proof. Suppose f is slash inverted. For each $y \in Y$ there exists $x \in X$ with $f_*(orb(x)) = orb(y)$. So $f(x) = gy$ for some $g \in G$. It follows that $y = f(g^{-1}x)$, so f is surjective. If $f(x_1) = f(x_2)$ then $(f/G)(orb(x_1)) = (f/G)(orb(x_2))$; so $orb(x_1) = orb(x_2)$. For the converse, take $orb(y) \in Y/G$. Then $y = f(x)$ for some $x \in X$ and so $orb(f(x)) = orb(y)$. Also, if $orb(f(x_1) = orb(f(x_2))$, then $f(x_1) = gf(x_2) = f(gx_2)$ for some $g \in G$. So $orb(x_1) = orb(gx_2) = orb(x_2)$. ■

A G-morphism $f : X \longrightarrow Y$ is said to be *orbit injective* when $orb(x_1) = orb(x_2)$ and $f(x_1) = f(x_2)$ imply $x_1 = x_2$. Orbit injective morphisms were considered by Bouc [Bo].

5.2. Proposition. *The slash inverted and orbit injective G-morphisms form a factorization system (in the sense of [FK]) on the category of G-sets.*

Proof. To factor a G-morphism $f : X \longrightarrow Y$, construct the G-set $S = \sum_{C \in X/G} f_*(C)$ and define G-morphisms $u : X \longrightarrow S$ and $v : S \longrightarrow Y$ by

$$u(x) = f(x) \in f_*(orb(x)) \text{ and } v(y \in f_*(C)) = y.$$

Then $f = v \circ u$ while u is slash inverted and v is orbit injective.

The only other non-obvious thing remaining to prove is the diagonal fill-in property. For this, suppose $k \circ u = v \circ h$ where u is slash inverted and v is orbit injective.

If $u(x_1) = u(x_2)$ then $orb(x_1) = orb(x_2)$, so $orb(h(x_1)) = orb(h(x_2))$. Yet we also have $v(h(x_1)) = k(u(x_1)) = k(u(x_2)) = v(h(x_2))$. Since v is orbit injective, we deduce that $h(x_1) = h(x_2)$.

Since u is surjective, for each $s \in S$ there is an $x \in X$ with $u(x) = s$. By the last paragraph, the value $h(x)$ is independent of the choice of x. So we obtain a function r by defining $r(s) = h(x)$. Clearly r is a G-morphism with $r \circ u = h$ and $v \circ r = k$; and r is unique since u is surjective. ■

This factorization system has a special property.

5.3. Proposition. *The pullback of a slash inverted G-morphism along an orbit injective G-morphism is slash inverted.*

Proof. Suppose the G-morphisms $u : X \longrightarrow S$ and $v : Y \longrightarrow S$ are slash inverted and orbit injective, respectively. Let P be the pullback of u and v with projections $p : P \longrightarrow X$ and $q : P \longrightarrow Y$. We claim that q is slash inverted. It is clearly surjective so suppose that $q(x_1, y_1) = q(x_2, y_2)$ where $u(x_1) = v(y_1)$ and $u(x_2) = v(y_2)$. So $y_1 = y_2$ and $u(x_1) = u(x_2)$. Since u is slash inverted, $orb(x_1) = orb(x_2)$; so there exists $g \in G$ with $x_2 = gx_1$. Since y_1 and gy_1 are in the same orbit, the calculation

$$v(gy_1) = gv(y_1) = gu(x_1) = u(gx_1) = u(x_2) = v(y_2) = v(y_1)$$

implies that $gy_1 = y_1 = y_2$. So $g(x_1, y_1) = (x_2, y_2)$, which implies that (x_1, y_1) and (x_2, y_2) are in the same orbit. ∎

6. A new category of G-sets

For a finite group G, we write $G\text{-}\mathbf{set}_{fin}$ for the category of finite G-sets and G-morphisms. We write $\mathcal{C}_G$ for (a skeleton of) the category of connected G-sets and all G-morphisms between them. There is also the category $\mathbf{Spn}(G\text{-}\mathbf{set}_{fin})$ whose objects are finite G-sets and whose morphisms are isomorphism classes of spans between finite G-sets. All these categories are important for the study of Mackey functors.

However, we now wish to introduce another category $\mathcal{B}_G$ whose objects are all the finite G-sets. In fact, $\mathcal{B}_G$ is the subcategory of $\mathbf{Spn}(G\text{-}\mathbf{set}_{fin})$ whose morphisms are the isomorphism classes of those spans $(u, S, v) : X \longrightarrow Y$ in which $u : S \longrightarrow X$ is slash inverted and $v : S \longrightarrow Y$ is orbit injective. It follows from Proposition 5.2 and Proposition 5.3 that these particular spans are closed under span composition.

6.1. Proposition. *The subcategory $\mathcal{B}_G$ of $\mathbf{Spn}(G\text{-}\mathbf{set}_{fin})$ is closed under finite coproducts.*

Proof. The coproduct in $\mathbf{Spn}(G\text{-}\mathbf{set}_{fin})$ is that of $G\text{-}\mathbf{set}_{fin}$, namely, disjoint union. For G-sets X and Y, the coprojections $X + Y$ in $\mathbf{Spn}(G\text{-}\mathbf{set}_{fin})$ are the spans

$$(1_X, X, \mathrm{copr}_1) : X \longrightarrow X + Y \text{ and } (1_Y, Y, \mathrm{copr}_2) : Y \longrightarrow X + Y$$

which clearly yield morphisms in $\mathcal{B}_G$. Moreover, if $(u, S, v) : X \longrightarrow Z$ and $(h, T, k) : Y \longrightarrow Z$ are spans with u and h slash inverted and with v and k orbit injective then $u + h : S + T \longrightarrow X + Y$ is slash inverted and $[v, k] : S + T \longrightarrow Z$ orbit injective. So the induced span $(u + h, S + T, [v, k]) : X + Y \longrightarrow Z$ yields a morphism in $\mathcal{B}_G$. ∎

6.2. Remark. While the coproduct in $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ is also the product (since $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ is self dual), this is no longer true in $\mathcal{B}_G$.

There is a functor $\mathcal{C}_G^{\mathrm{op}} \longrightarrow \mathcal{B}_G$ taking each connected G-set to itself and each G-morphism $f : C \longrightarrow D$ between connected G-sets to the isomorphism class of the span $(f, C, 1_C) : D \longrightarrow C$; clearly f must be slash inverted since C and D each have one orbit. Using the universal property of $\mathbf{Fam}(\mathcal{C}_G^{\mathrm{op}})$ and Proposition 6.1, we obtain a finite coproduct preserving functor $\Sigma : \mathbf{Fam}(\mathcal{C}_G^{\mathrm{op}}) \longrightarrow \mathcal{B}_G$ extending $\mathcal{C}_G^{\mathrm{op}} \longrightarrow \mathcal{B}_G$.

6.3. Theorem. *The functor* $\Sigma : \mathbf{Fam}(\mathcal{C}_G^{\mathrm{op}}) \longrightarrow \mathcal{B}_G$ *is an equivalence of categories.*

Proof. Each object of $\mathcal{B}_G$ is a coproduct of connected G-sets so Σ is certainly essentially surjective on objects. To prove Σ fully faithful we need to use the description of $\mathbf{Fam}(\mathcal{C}_G^{\mathrm{op}})$ just after Proposition 3.2. The objects are finite families $(C_i)_{i\in I}$ of connected G-sets and a morphism $(\xi, f) : (C_i)_{i\in I} \longrightarrow (D_j)_{j\in J}$ consists of a function $\xi : I \longrightarrow J$ and a family $f = (f_i)_{i\in I}$ of morphisms $f_i : D_{\xi(i)} \longrightarrow C_i$. The functor Σ takes the morphism $(\xi, f) : (C_i)_{i\in I} \longrightarrow (D_j)_{j\in J}$ to the isomorphism class of the span $(u, S, v) : X \longrightarrow Y$ where $X = \sum_{i\in I} C_i$, $Y = \sum_{j\in J} D_j$, $S = \sum_{i\in I} D_{\xi(i)}$,

$$u \circ \mathrm{copr}_i = \mathrm{copr}_i \circ f_i \quad \text{and} \quad v \circ \mathrm{copr}_i = \mathrm{copr}_{\xi(i)}.$$

Notice that u induces a bijection $S/G \longrightarrow X/G$ and that both of these sets are isomorphic to I. It is then clear that u is slash inverted and v is orbit injective. Yet this process can be inverted as follows. Given any span $(u, S, v) : X \longrightarrow Y$ for the same X and Y, with u slash inverted and v orbit injective, the direct image $v_* u^*(C_i)$ of the inverse image $u^*(C_i)$ of C_i must be an orbit $D_{\xi(i)}$ of Y. This defines a function ξ while f_i is the composite of the restriction of u to $u^*(C_i)$ with the inverse of the isomorphism $u^*(C_i) \cong v_* u^*(C_i)$ induced by v. ∎

6.4. Corollary. *There is an equivalence*

$$\mathcal{B}_G \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{\mathit{fin}}, \mathbf{set}_{\mathit{fin}})$$

taking the left G-set C to the functor

$$\sum_{w\in C/G} G\text{-}\mathbf{set}_{\mathit{fin}}(C_w, -)$$

where C_w is the orbit w as a sub-G-set of C.

There is an isomorphism of categories

$$\ell_G : G^{\mathrm{op}}\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow G\text{-}\mathbf{set}_{\mathit{fin}}$$

which preserves the underlying sets. If A is a right G-set then $\ell_G A = A$ as a set with left action ga in $\ell_G A$ equal to ag^{-1} in A. As a special case of the construction in the Introduction, for a right G-set A and a left G-set X, we have

$$A \wedge_G X = \{(a, x) \in \ell_G A \times X \mid \mathit{stab}(a) \leq \mathit{stab}(x)\} \text{ and}$$
$$A \circ_G X = (A \wedge_G X)/G.$$

Each $(a,x) \in A \wedge_G X$ defines a G-morphism

$$\theta_X(a,x) : \ell_G A_u \longrightarrow X$$

where $u = orb(a)$ and $\theta_X(a,x)(ag^{-1}) = gx$ (which is well defined since $ag_1^{-1} = ag_2^{-1} \Longrightarrow g_2^{-1}g_1 \in stab(a) \Longrightarrow g_2^{-1}g_1 \in stab(x) \Longrightarrow g_1x = g_2x$). This defines a function

$$\theta_X : A \wedge_G X \longrightarrow \sum_{u \in G \backslash A} G\text{-}\mathbf{set}_{\mathit{fin}}(\ell_G A_u, X)$$

naturally in $X \in G\text{-}\mathbf{set}_{\mathit{fin}}$ (where $G\backslash A$ is the set of orbits of the right action). Clearly

$$\theta_X(a,x) = \theta_X(b,y) \quad \text{if and only if} \quad orb(a,x) = orb(b,y).$$

This proves:

6.5. Proposition. *For all $A \in G^{\mathrm{op}}\text{-}\mathbf{set}_{\mathit{fin}}$, the natural transformation θ induces a natural isomorphism*

$$\bar{\theta} : A \circ_G - \cong \sum_{u \in G \backslash A} G\text{-}\mathbf{set}_{\mathit{fin}}(\ell_G A_u, -)$$

between functors from $G\text{-}\mathbf{set}_{\mathit{fin}}$ to $\mathbf{set}_{\mathit{fin}}$.

6.6. Corollary. *There is an equivalence*

$$\mathcal{B}_{G^{\mathrm{op}}} \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{\mathit{fin}}, \mathbf{set}_{\mathit{fin}}), \quad A \longmapsto A \circ_G -,$$

and on morphisms, takes a span (u, S, v) from A to B with u slash inverting and v orbit injective, to the natural transformation whose component at X is the function $A \circ_G X \longrightarrow B \circ_G X$ taking $[a,x]$ to $[v(s),x]$ where $u(s) = a$.

7. A bicategory of finite groups

The goal of this section is to consolidate our results in terms of a homomorphism of bicategories which is an equivalence on homcategories. We construct a bicategory whose objects are finite groups and whose morphisms are permutation representations between them. This bicategory is the domain of the homomorphism. The codomain is the 2-category of categories of the form $G\text{-}\mathbf{set}_{\mathit{fin}}$ and pullback-and-finite-coproduct-preserving functors between them.

For any finite monoid H and any category $\mathcal{X}$ with finite coproducts, there is a monad $H\cdot$ on $\mathcal{X}$ whose underlying endofunctor is defined by $H \cdot X = \sum_H X$ (the coproduct of H copies of X); the unit and multiplication are induced in the obvious way by the unit and multiplication of H. The category $\mathcal{X}^{H\cdot}$ of Eilenberg-Moore algebras for the monad is none other than the functor category $[H, \mathcal{X}]$ where H is regarded as a category with one object and with morphisms the elements of H. We are interested in the particular case where H is a finite group and $\mathcal{X} = \mathcal{B}_{G^{\mathrm{op}}}$.

Define $\mathbf{Bouc}(G,H)$ to be the category obtained as the pullback of the inclusion of $\mathcal{B}_{G^{\mathrm{op}}}$ in $\mathbf{Spn}(G^{\mathrm{op}}\text{-}\mathbf{set}_{fin})$ along the forgetful functor $\mathbf{Spn}(G^{\mathrm{op}} \times H\text{-}\mathbf{set}_{fin}) \longrightarrow \mathbf{Spn}(G^{\mathrm{op}}\text{-}\mathbf{set}_{fin})$. That is, $\mathbf{Bouc}(G,H)$ is the subcategory of $\mathbf{Spn}(G^{\mathrm{op}} \times H\text{-}\mathbf{set}_{fin})$ consisting of all the objects yet, as morphisms, only the isomorphism classes of spans (u, S, v) in $G^{\mathrm{op}} \times H\text{-}\mathbf{set}_{fin}$ for which u is slash inverted and v is orbit injective as G-morphisms.

There is an isomorphism of categories $\Gamma : \mathbf{Bouc}(G,H) \longrightarrow [H, \mathcal{B}_{G^{\mathrm{op}}}]$ defined as follows. For each $G^{\mathrm{op}} \times H$-set A, the left action by H provides injective right G-morphisms $h : A \longrightarrow A$ for all $h \in H$; so the isomorphism class of the span $(1_A, A, h) : A \longrightarrow A$ is a morphism in $\mathcal{B}_{G^{\mathrm{op}}}$. So the right G-set underlying A becomes a left H-object ΓA in $\mathcal{B}_{G^{\mathrm{op}}}$. Conversely, each left H-object X in $\mathcal{B}_{G^{\mathrm{op}}}$ has, for each $h \in H$, an invertible morphism $[u_h, M_h, v_h] : X \longrightarrow X$ in $\mathcal{B}_{G^{\mathrm{op}}}$; it follows that u_h and v_h are invertible and so $[u_h, M_h, v_h] = [1_X, X, w_h]$ where the $w_h : X \longrightarrow X$ define a left H-action on X making it a $G^{\mathrm{op}} \times H$-set A with $\Gamma A = X$. That Γ is fully faithful should now be obvious.

7.1. Theorem. *There is an equivalence of categories*

$$\mathbf{Bouc}(G,H) \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{fin}, H\text{-}\mathbf{set}_{fin}), \quad A \longmapsto A \circ_G -.$$

Parts $(i), (ii)$ and (iii) of Theorem 1.1 follows from Theorem 7.1. It is also clear, in the setting of Theorem 1.1(iv), that there exists a $(K^{\mathrm{op}} \times H)$-set C such that

$$A \circ_G (B \circ_K -) \cong C \circ_K -$$

since the composite of pullback and finite coproduct preserving functors also preserves them. It remains to identify C as $A \circ_G B$. We do this directly.

7.2. Proposition. *If A, B and Z are respectively $(G^{\mathrm{op}} \times H)$-, $(K^{\mathrm{op}} \times G)$-, and K-sets then*

$$A \circ_G (B \circ_K Z) \cong (A \circ_G B) \circ_K Z\ , \quad [a,[b,z]] \longmapsto [[a,b],z]$$

is an isomorphism of H-sets.

Proof. To say $(a,[b,z]) \in A \wedge_G (B \circ_K Z)$ is to say that $ag = a$ implies $g[b,z] = [b,z]$; that is, there exists $k \in K$ such that $gb = bk$ and $hz = z$. In particular, this means that $(a,b) \in A \wedge_G B$. We need to see that $([a,b],z) \in (A \circ_G B) \wedge_K Z$. So suppose $[a,b]k = [a,b]$. Then there exists $g \in G$ with $a = ag$ and $bk = gb$. The former implies there is $k_1 \in K$ such that $gb = bk_1$ and $k_1 z = z$. Then $bk = gb = bk_1$, so $bkk_1^{-1} = b$. Since $(b,z) \in B \wedge_K Z$, we have $kk_1^{-1}z = z$; so $kz = z$. One also sees that

$$\begin{aligned}[a,[b,z]] = [a',[b',z']] &\Longleftrightarrow \exists g \in G, k \in K \ : \ a = a'g,\ gb = b'k,\ kz = z' \\ &\Longleftrightarrow [[a,b],z] = [[a',b'],z'].\end{aligned}$$

This proves the bijection. Clearly the H-actions correspond. ∎

Chapter 5

Conclusion

In this thesis we study Mackey functors as an application of enriched category theory and give a categorical simplification and generalization. The first paper *Mackey functors on compact closed categories*, which constitutes Chapter 1, is the main paper of this thesis. The papers *Lax braidings and the lax centre* and *On centres and lax centres for promonoidal categories*, which are Chapters 2 and 3 respectively, are the supporting papers of our main paper. The last paper *Pullback and finite coproduct preserving functors between categories of permutation representations*, which is in Chapter 4, is the extension of the work of our main paper.

In Chapter 1, we define a compact closed category $\mathbf{Spn}(\mathscr{E})$ of spans for a lextensive category $\mathscr{E}$. We show that the category $\mathbf{Spn}(\mathscr{E})$ is a commutative-monoid-enriched category. We use the approach of Dress [Dr1] on Mackey functors and define a Mackey functor M form the lextensive category $\mathscr{E}$ to the category $\mathbf{Mod}_k$ of k modules, where k is a commutative ring. A *Mackey functor* $M : \mathscr{E} \longrightarrow \mathbf{Mod}_k$ consists of a couple of functors M^* and M_* satisfying certain axioms. We show that a Mackey functor $M : \mathscr{E} \longrightarrow \mathbf{Mod}_k$ is equivalent to a coproduct preserving functor $M : \mathbf{Spn}(\mathscr{E}) \longrightarrow \mathbf{Mod}_k$, using Linder's [Li1] result. Then we work on a general compact closed category $\mathscr{T}$ with direct sums and develop the theory of Mackey functors as an enriched categorical study.

The category **Mky** of Mackey functors from the category $\mathscr{T}$ to the category $\mathbf{Mod}_k$ is a symmetric monoidal closed category. The tensor product of the Mackey functors M and N is defined by a Day [Da1] convolution structure and is given by

$$(M * N)(Z) \cong \int^{Y} M(Z \otimes Y^{*}) \otimes_k N(Y).$$

The unit J of this tensor product is the *Burnside functor*. The monoids of the monoidal category **Mky** are defined to be *Green functors*. The Dress construction ([Bo5], [Bo6]) is a process to obtain a new Mackey functor M_Y from a known Mackey functor M, where $M_Y(U) = M(U \otimes Y)$ for fixed $Y \in \mathscr{T}$. We define the *Dress construction*

$$D : \mathscr{T} \otimes \mathbf{Mky} \longrightarrow \mathbf{Mky}$$

by $D(Y, M) = M_Y$ and show that the Dress construction D is a strong monoidal $\mathscr{V}$-functor. To apply the Dress construction to a Green functor A, that is to make A_Y a Green functor, we use the results form our supporting papers which are in Chapters 2 and 3. If A is a Green functor and Y is in the lax centre $\mathscr{Z}_l(\mathscr{E})$ of $\mathscr{E}$, where $\mathscr{E} = G\text{-}\mathbf{set}_{fin}$ is the category of finite G-sets for a finite group G, then we show that A_Y is a Green functor.

A Mackey functor $M : \mathscr{T} \longrightarrow \mathbf{Vect}$ is called *finite dimensional* when each $M(X)$ is a finite-dimensional vector space. Here we assume $\mathscr{T} = \mathbf{Spn}(\mathscr{E})$, where $\mathscr{E} = G\text{-}\mathbf{set}_{fin}$, and k is a field and replace $\mathbf{Mod}_k$ by **Vect** which is the category of vector spaces. We show that the category $\mathbf{Mky}_{fin}$ of finite dimensional Mackey functors is a star-autonomous full sub-monoidal category of **Mky**. Let $\mathbf{Rep}_k(G)$ be the category of finite-dimensional k-linear representations of G. Then the functor $\mathbf{Mky}(G)_{fin} \longrightarrow \mathbf{Rep}_k(G)$ is strong monoidal and has an adjoint functor $\mathbf{Rep}_k(G) \longrightarrow \mathbf{Mky}(G)_{fin}$ which is monoidal and closed under exponentiation.

We give notions of Mackey functors for *quantum groups*. We show that, the category **Comod**($\mathscr{R}$) and $\mathscr{R}^{\text{op}}(\simeq \mathscr{R})$ are two examples of a compact closed category $\mathscr{T}$ from a Hopf algebra H (or quantum group). Here $\mathscr{R}$ is the category of left H-modules and **Comod**($\mathscr{R}$) is obtained from [DMS].

We work out the Morita Theory for Green functors using enriched category theory especially the theory of (two-sided) modules rather than *Morita contexts* as in [Bo3]. Green functors A and B are defined to be *Morita equivalent* when they are equivalent in the **Mky**-enriched category **Mod**($\mathscr{W}$), where **Mod**($\mathscr{W}$) is the category of (two-sided) modules of $\mathscr{W}$ = **Mky**. The *Cauchy completion* $\mathscr{Q}A$ of the monoid A in the category **Mky** of Mackey functors consists of all the retracts of modules of the form

$$\bigoplus_{i=1}^{k} A(Y_i \times -)$$

for some $Y_i \in \mathbf{Spn}(\mathscr{E})$, $i = 1, \ldots, k$. Green functors A and B are Morita equivalent if and only if their **Mky**-enriched categories $\mathscr{Q}A$ and $\mathscr{Q}B$ are equivalent.

We define the notions of lax braiding and lax centre for monoidal categories and more generally for promonoidal categories in this thesis. These are studied in the papers entitled “Lax braidings and the lax centre”, and “On centres and lax centres for promonoidal categories” which are in Chapters 2 and 3 respectively.

A lax braiding for a promonoidal category $\mathscr{C}$ is a $\mathscr{V}$-natural family of morphisms $P(A,B;C) \longrightarrow P(B,A;C)$ which satisfies some commutative diagrams. A braiding is a lax braiding with each $P(A,B;C) \longrightarrow P(B,A;C)$ invertible. If $\mathscr{C}$ is a monoidal category, then we can regard the lax braiding as a morphism $A \otimes B \longrightarrow B \otimes A$. A braiding is a lax braiding in which each $A \otimes B \longrightarrow B \otimes A$ invertible. We reprove a result of Yetter [Ye] that if $\mathscr{C}$ is a right autonomous (meaning that each object has a right dual) monoidal category then any lax braiding on $\mathscr{C}$ is necessarily a braiding.

The objects of the lax centre $\mathscr{Z}_l\mathscr{C}$ of a promonoidal category $\mathscr{C}$ are pairs

(A, α) where A is an object of $\mathscr{C}$ and α is a $\mathscr{V}$-natural family of morphisms $\alpha_{X,Y} : P(A, X; Y) \longrightarrow P(X, A; Y)$ satisfying a couple of commutative diagrams. The Hom object $\mathcal{Z}_l\mathscr{C}((A, \alpha), (B, \beta))$ is defined to be an equalizer in $\mathscr{V}$. The lax centre $\mathcal{Z}_l\mathscr{C}$ of the promonoidal category $\mathscr{C}$ is often promonoidal. The centre $\mathcal{Z}\mathscr{C}$ of a promonoidal category $\mathscr{C}$ consists of objects (A, α) for which each $\alpha_{X,Y} : P(A, X; Y) \longrightarrow P(X, A; Y)$ is invertible. The centre $\mathcal{Z}\mathscr{C}$ of $\mathscr{C}$ which is a sub-$\mathscr{V}$-category of $\mathcal{Z}_l\mathscr{C}$, is a braided monoidal category.

The lax centre $\mathcal{Z}_l\mathscr{C}$ of a monoidal $\mathscr{V}$-category $\mathscr{C}$ has objects pairs (A, u) where A is an object of $\mathscr{C}$ and u is a $\mathscr{V}$-natural family of morphisms $u_B : A \otimes B \longrightarrow B \otimes A$ satisfying a couple of commutative diagrams.The centre $\mathcal{Z}\mathscr{C}$ of a monoidal $\mathscr{V}$-category $\mathscr{C}$ is a lax centre with each $u_B : A \otimes B \longrightarrow B \otimes A$ invertible. The lax centre of a monoidal category is a lax braided monoidal category and the centre of a monoidal category is a braided monoidal category. Generally the centre is a full subcategory of the lax centre but in some cases the two coincide. In these two Chapters we identify cases where these two coincide. One reason for being interested in the lax centre of a monoidal category $\mathscr{C}$ is that, if an object X of $\mathscr{C}$ is equipped with the structure of monoid in $\mathcal{Z}_l\mathscr{C}$, then tensoring with X defines a monoidal endofunctor $- \otimes X$ of $\mathscr{C}$; we use this result for the Dress construction of Green functors in Chapter 1.

Let $\mathscr{F}$ be a monoidal $\mathscr{V}$-category with the functor $F \otimes - : \mathscr{F} \longrightarrow \mathscr{F}$ preserveing (weighted) colimits for each object F of $\mathscr{F}$. If the full sub-$\mathscr{V}$-category of $\mathscr{F}$ consisting of the objects with right duals is dense in $\mathscr{F}$, then we show that the lax centre of $\mathscr{F}$ is equal to the centre of $\mathscr{F}$. For a Hopf algebra H we find two cases where the lax centre and centre becomes equal: the lax centre of the monoidal category **Comod**H of left H-comodules is equal to its centre and if H is finite dimensional then the lax centre of the monoidal category **Mod**H of left H-modules is equal to its centre.

We also calculate the centre and lax centre for cartesian monoidal categories; here $\mathscr{V}$ = **Set** with cartesian monoidal structure. The lax centre $\mathcal{Z}_l\mathscr{C}$ of a cartesian monoidal category $\mathscr{C}$ has objects pairs (A,ϕ) with A in $\mathscr{C}$ and ϕ is a family of functions $\phi_X : \mathscr{C}(A,X) \longrightarrow \mathscr{C}(X,X)$ satisfying a commutative diagram. A morphism $g : (A,\phi) \longrightarrow (A',\phi')$ in $\mathcal{Z}_l\mathscr{C}$ is a morphism $g : A \longrightarrow A'$ in $\mathscr{C}$ such that $\phi_X(vg) = \phi'_X(v)$ for all $v : A' \longrightarrow X$. We identify the objects of $\mathcal{Z}_l\mathscr{C}$ with pairs (A,θ) where A is an object of $\mathscr{C}$ and $\theta_X : A \times X \longrightarrow X$ is a family of morphisms. The *core* $C_{\mathscr{X}}$ of the category $\mathscr{X}$ with finite products in the sense of Freyd [Fr2] is precisely a terminal object in $\mathcal{Z}_l\mathscr{X}$. If the core exists, we identify the lax centre of $\mathscr{X}$ with a slice category:

$$\mathcal{Z}_l\mathscr{X} \cong \mathscr{X}/C_{\mathscr{X}}.$$

If $\mathscr{C}$ is a category in which every endomorphism is invertible then we show that the lax centre $\mathcal{Z}_l[\mathscr{C},\mathbf{Set}]$ of the cartesian monoidal category $[\mathscr{C},\mathbf{Set}]$ is equal to the centre $\mathcal{Z}[\mathscr{C},\mathbf{Set}]$. If $\mathscr{C}$ is a small category equipped with the promonoidal structure such that the convolution structure on $[\mathscr{C},\mathbf{Set}]$ is cartesian monoidal, then we show that

$$[\mathcal{Z}_l\mathscr{C},\mathbf{Set}] \xrightarrow{\simeq} [\mathscr{C},\mathbf{Set}]/C_{[\mathscr{C},\mathbf{Set}]} \xrightarrow{\cong} \mathcal{Z}_l[\mathscr{C},\mathbf{Set}].$$

If $\mathscr{C}$ is also a groupoid then we prove the following equalities

$$\mathcal{Z}_l\mathscr{C} = \mathcal{Z}\mathscr{C} = \mathscr{C}^{\mathbf{Z}}, \quad \mathcal{Z}_l[\mathscr{C},\mathbf{Set}] = \mathcal{Z}[\mathscr{C},\mathbf{Set}], \quad C_{[\mathscr{C},\mathbf{Set}]} = \mathrm{Aut}_{\mathscr{C}},$$

and show that the following functor

$$\mathcal{Z}[\mathscr{C},\mathbf{Set}] \xrightarrow{\sim} [\mathscr{C}^{\mathbf{Z}},\mathbf{Set}]$$

is a braided monoidal equivalence. We examine when the centre of $[\mathscr{C},\mathscr{V}]$ with a convolution monoidal structure (in the sense of [Da1]) is again a functor category $[\mathscr{D},\mathscr{V}]$. If $\mathscr{C}$ is a small category equipped with the promonoidal structure such that the convolution structure on $[\mathscr{C}, \mathbf{Set}]$ is cartesian then the adjunction $\hat{\Psi} \dashv \tilde{\Psi}$ defines an equivalence of lax-braided monoidal categories

$$\mathcal{Z}_l[\mathscr{C}, \mathbf{Set}] \simeq [\mathcal{Z}_l\mathscr{C}, \mathbf{Set}]$$

which restricts to a braided monoidal equivalence

$$\mathcal{Z}[\mathscr{C}, \mathbf{Set}] \simeq [\mathcal{Z}\mathscr{C}, \mathbf{Set}].$$

If $\mathscr{C}$ is a promonoidal k-linear category with finite-dimensional homs and the category $\mathscr{F} = [\mathscr{C}, \mathscr{V}]$ has the convolution monoidal structure, where $\mathscr{V} = \mathbf{Vect}_k$, then we obtain the following result

$$\mathcal{Z}\mathscr{F} = \mathcal{Z}_l\mathscr{F} \cong \mathscr{F}^M \simeq [\mathscr{C}_M, \mathscr{V}],$$

where $\mathscr{C}_M$ is the Kleisli category for the promonad M on $\mathscr{C}$.

We consider the monoids in the lax centre of a monoidal $\mathscr{V}$-category $\mathscr{C}$. A monoid (A, u) in the lax centre $\mathcal{Z}_l\mathscr{C}$ determines a canonical enrichment of the $\mathscr{V}$-functor

$$- \otimes A : \mathscr{C} \longrightarrow \mathscr{C}$$

to a monoidal functor:

$$X \otimes A \otimes Y \otimes A \xrightarrow{1 \otimes u_Y \otimes 1} X \otimes Y \otimes A \otimes A \xrightarrow{1 \otimes 1 \otimes \mu} X \otimes Y \otimes A$$

$$U \xrightarrow{\eta} A \cong U \otimes A.$$

If M is a Mackey functor on a finite group H then by Lindner [Li1] the functor $M : \mathbf{Spn}(H\text{-}\mathbf{set}_{fin}) \longrightarrow \mathbf{Mod}_k$ preserves coproducts, where H-$\mathbf{set}_{fin}$ is the category of finite left H-sets (or *permutation representation*) for a finite group H. If $F : G\text{-}\mathbf{set}_{fin} \longrightarrow H\text{-}\mathbf{set}_{fin}$ is a pullback and finite coproduct preserving functor (where G is finite) then the functor

$$M \circ \mathbf{Spn}(F) : \mathbf{Spn}(G\text{-}\mathbf{set}_{fin}) \longrightarrow \mathbf{Mod}_k$$

is a Mackey functor on G. In the last paper entitled "Pullback and finite coproduct preserving functors between categories of permutation representations" which is in Chapter 4, we study the functors which preserves finite coproducts and pullbacks between categories of permutation representations of finite groups and give a categorical explanation for the work of Bouc [Bo1]. For finite groups K, G and H Bouc [Bo1] shows that, every functor $F : G\text{-}\mathbf{set}_{fin} \longrightarrow H\text{-}\mathbf{set}_{fin}$ which preserves finite coproducts and pullbacks is isomorphic to one of the form $A \circ_G -$ where A is a $(G^{\mathrm{op}} \times H)$-set ($G^{\mathrm{op}}$ is G with opposite multiplication). The set $A \circ_G B$ is defined by the following equation

$$A \circ_G B = (A \wedge_G B)/G$$

which is a $(K^{\mathrm{op}} \times H)$-set for each $(K^{\mathrm{op}} \times G)$-set B, where

$$A \wedge_G B = \{(a, b) \in A \times B \mid g \in G, ag = a \Rightarrow \text{there exists } k \in K \text{ with } gb = bk\}.$$

We provide a direct proof of the well-known representability theorem for the case where "small" means "finite". We show that there is an equivalence of categories

$$(G^{\mathrm{op}} \times H)\text{-}\mathbf{set} \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{fin}, H\text{-}\mathbf{set}_{fin}), \quad A \longmapsto A \circ_G -$$

where $\mathbf{CopPb}(\mathscr{A}, \mathscr{B})$ is the category of finite coproduct and pullback preserving functors from $\mathscr{A}$ to $\mathscr{B}$. Let **Bouc** be the bicategory whose objects are finite groups and hom-categories are

$$\mathbf{Bouc}(G, H) = (G^{\mathrm{op}} \times H)\text{-}\mathbf{set},$$

then we obtain a homomorphism of bicategories

$$\mathbf{Bouc} \longrightarrow \mathbf{CopPb}.$$

If the functor $F : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$ preserves pullback and finite coproducts then we can obtain an exact functor

$$\bar{F} : \mathbf{Mky}_{\mathit{fin}}(H) \longrightarrow \mathbf{Mky}_{\mathit{fin}}(G)$$

defined by $\bar{F}(N) = N \circ \mathbf{Spn}(F)$ for all $N \in \mathbf{Mky}_{\mathit{fin}}(H)$, where $\mathbf{Mky}_{\mathit{fin}}$ is the category of finite-dimensional Mackey functors for a finite group G. The functor $\bar{F}$ has a left adjoint

$$\mathbf{Mky}_{\mathit{fin}}(F) : \mathbf{Mky}_{\mathit{fin}}(G) \longrightarrow \mathbf{Mky}_{\mathit{fin}}(H)$$

defined by

$$\mathbf{Mky}_{\mathit{fin}}(F)(M) = \int^{C} \mathbf{Spn}(H\text{-}\mathbf{set}_{\mathit{fin}})(F(C), -) \otimes MC$$

where C runs over the connected G-sets as objects of $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ and the tensor product is that of additive (commutative) monoids. Let $\mathbf{AbCat}_k$ denote the 2-category of abelian k-linear categories, k-linear functors with right exact right adjoints, and natural transformations. Then we obtain a homomorphism of bicategories

$$\mathbf{Mky}_{\mathit{fin}} : \mathbf{Bouc} \longrightarrow \mathbf{AbCat}_k$$

given by $(A : G \longrightarrow H) \longmapsto (\mathbf{Mky}_{\mathit{fin}}(A \circ_G -) : \mathbf{Mky}_{\mathit{fin}}(G) \longrightarrow \mathbf{Mky}_{\mathit{fin}}(H))$. Actually $\mathbf{Mky}_{\mathit{fin}}(G)$ is much more than an abelian k-linear category; we show that it is $*$-autonomous category in Chapter 1. Let $*$-$\mathbf{AbCat}_k$ denote the 2-category of $*$-autonomous monoidal abelian k-linear categories, $*$-preserving strong-monoidal k-linear functors with right exact right adjoints, and natural transformations. Many of our results are summed up in the statement that we have the following homomorphism (or "pseudofunctor") between bicategories

$$\mathbf{Mky}_{\mathit{fin}} : \mathbf{Boucc} \longrightarrow *\text{-}\mathbf{AbCat}_k.$$

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