Pricing of European Options using Empirical Characteristic Functions

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Abstract

Pricing problems of financial derivatives are among the most important ones in Quantitative Finance. Since 1973 when a Nobel prize winning model was introduced by Black, Merton and Scholes the Brownian Motion (BM) process gained huge attention of professionals. It is now known, however, that stock market log-returns do not follow the very popular BM process. Derivative pricing models which are based on more general Lévy processes tend to perform better.

Carr & Madan (1999) and Lewis (2001) (CML) developed a method for vanilla options valuation based on a characteristic function of asset log-returns assuming that they follow a Lévy process. Assuming that at least part of the problem is in adequate modeling of the distribution of log-returns of the underlying price process, we use instead a nonparametric approach in the CML formula and replaced the unknown characteristic function with its empirical version, the Empirical Characteristic Functions (ECF). We consider four modifications of this model based on the ECF. The first modification requires only historical log-returns of the underlying price process. The other three modifications of the model need, in addition, a calibration based on historical option prices. We compare their performance based on the historical data of the DAX index and on ODAX options written on the index between the 1st of June 2006 and the 17th of May 2007. The resulting pricing errors show that one of our models performs, at least in the cases considered in the project, better than the Carr & Madan (1999) model based on calibration of a parametric Lévy model, called a VG model.

Our study seems to confirm a necessity of using implied parameters, apart from an adequate modeling of the probability distribution of the asset log-returns. It indicates that to precisely reproduce behaviour of the real option prices yet other factors like stochastic volatility need to be included in the option pricing model. Fortunately the discrepancies between our model and real option prices are reduced by introducing the implied

parameters which seem to be easily modeled and forecasted using a mixture of regression and time series models. Such approach is computationaly less expensive than the explicit modeling of the stochastic volatility like in the Heston (1993) model and its modifications.

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Certificate

I certify that the work in this thesis is original and has not been submitted for a higher degree to any other university or institution.

Karol Binkowski
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$(\Omega, \mathcal{F}, \mathbf{P})$ - probability space,

$\mathbf{P}, \mathbf{Q}$ - probability measures,

$\mathcal{F}$ - σ-field,

$(\mathbf{F}_t)_{t \in [0,T]}$ - an increasing family of sub σ-fields of $\mathcal{F}$,

$\mathbb{N}$ - set of positive integers,

$E^{\mathbf{P}} X$ - expected value of r.v. X with respect to measure $\mathbf{P}$,

EX - expected value of r.v. X,

$E(X|Y)$ - expected value of r.v. X, given r.v. Y,

$\mathbf{L}_X(dx)$ - Lévy measure of r.v. X,

$l_X(x)$ - Lévy density function of r.v. X,

X_t - stochastic process,

$\phi^{\mathbf{P}}_{X_t}(u)$ - characteristic function of X_t, under measure $\mathbf{P}$,

$\phi_{X_t}(u)$ - characteristic function of X_t,

$\psi_{X_t}(u)$ - cumulant function of X_t,

A_X - analyticity strip of a characteristic function of r.v. X,

$\hat{\phi}_n(u)$ - Empirical Characteristic Function (ECF),

$C(t, T, K)$ - model price of Call option at time t, maturity at time T and strike price K,

$\bar{C}(0, T, K)$ - historical price of Call option at time zero, maturity T and strike price K,

$\hat{C}_n(t, T, K; w, p)$ - ECF model price of Call option at time t, maturity at time T, strike price K and parameters w, p,

$H(x)$ - payoff function of European option,

w - Mean Martingale Correcting Term (MMCT),

$\hat{w}_n$ - empirical MMCT,

w^*_n - implied MMCT,

w^f_n - forecasted implied w^*_n,

p^*_n - implied number of days to option expiry,

p^f_n - forecasted implied p^*_n,

Δ^*_n - implied time length of increments,

$o(\cdot)$ - little o,

$\mathbf{1}_D(x)$ - indicator function of a set D.

Chapter 1

Introduction

In this project we are concerned with accurate and numerically efficient methods for pricing of European options. The modern mathematical theory of option pricing started from the Nobel Prize winning model of Black & Scholes (1973) (BS model) and Merton (1973). The BS model assumes that the logarithmic returns of the underlying stock follow a Brownian Motion with a constant variance $\sigma^2 > 0$. It has been, however, observed that the real option prices did not exactly trade according to the BS model. The differences have been conveniently reported by calibrating the BS model to the real option prices and reporting the resulting parameter σ^* as implied volatility. The implied volatility has had over years a persisting U-shape as a function of strikes and this phenomena has been labeled as a 'volatility smile'. The smile has changed over years and became quite flat on its right hand side, yet the label 'volatility smile' is still being used referring to deviations between the model and real option prices. Even more, by including also time to option expiration, the volatility curve evolved into a volatility surface. Accurate modelling of the volatility surface became one of the central problems in the theory of option pricing.

The BS model evolved in various directions, including modelling the stochastic behaviour of volatility (e.g. Heston (1993)) and more sophisticated modelling of assets log-returns via Lévy processes. Among the most popular generalizations of the log-normal assumption of returns are the Normal Inverse Gaussian (NIG) (Barndorff-Nielsen, Kent & Sørensen 1982), Variance Gamma (VG) (Madan & Seneta 1990), (Carr, Madan & Chang 1998) and Hyperbolic (Barndorff-Nielsen & Halgreen 1977), (Eberlein & Keller 1995) distributions. These classes of distributions are included in the class of Generalized Hyperbolic (GH) distributions, introduced by Barndorff-Nielsen & Halgreen (1977).

Prause (1999), Eberlein & Prause (2002) modelled the log-returns with a Generalized Hyperbolic (GH) distribution, which resulted in a slight improvement of the shape of the implied volatility surface. However, this improvement is not yet quite satisfactory.

Some other models are based on other specific (non-GH) distributions such as the Jump-Diffusion model (Merton 1976), Double Exponential Jump-Diffusion model (Kou 2002), Stochastic Volatility (SV) model (Heston 1993), or more sophisticated Lévy SV models used by Barndorff-Nielsen, Nicolato & Shephard (2002) and Carr, Geman, Madan & Yor (2003).

Carr & Madan (1999) and Lewis (2001), which will be referred to as CML, derived a general option pricing model based on a characteristic function of logarithmic returns driven by a fairly general Lévy process. The CML model has been used by Carr & Madan (1999) in the case of a parametric sub-family of Lévy processes, called VG processes, introduced by Madan & Seneta (1990). It has also been used assuming full knowledge of the characteristic function by Lewis (2001). We suggest in our project to use in the CML the empirical characteristic function, capturing in this way possibly other than the VG infinitely divisible distributions corresponding to the real asset log-returns. We show that the CML method of option pricing based on the empirical characteristic function is strongly convergent to the CML option value based on the characteristic function. Hence it can be considered as a strongly consistent estimator of the CML option price. In this way our method is nonparametric and does not require perfect knowledge of the characteristic function of the log-returns.

In fact we introduce and consider four nonparametric models. In Model 1 we just replace the characteristic function in the CML formula with empirical characteristic function based on historical data. This model does not reproduce very precisely real option prices. Let us note that another alternative approach consisting in estimation of parameters of distributions of log-returns and, next, using the estimated parameters in option pricing, is not providing satisfactory results (cf. Carr & Madan (1999), Ait-Sahalia, Wang & Yared (2001)). These and our results for Model 1 suggest that other market factors should in fact also be included into option pricing models. One of such a factor is stochastic volatility.

To achieve a better option pricing we introduced some implied parameters p_n^* and w_n^* which will be given proper interpretation in Chapter 5. Our Model 2 is based on the ECF and has p_n^* as an implied parameter. Model 3 is based on the ECF and uses w_n^* as an implied parameter. Model 4, apart from being based on the ECF, uses two implied parameters: p_n^* and w_n^*. Not surprisingly, Model 4, with the two implied parameters performs best, even better than the 3-parameter model of Carr & Madan (1999).

We tested our model on historical data of DAX index and ODAX options written on the index and we obtained several series of the implied parameters p_n^* and w_n^*. Like in the case of the classical implied volatility σ^* referring to the BS model it is plausible that the dynamic of our implied parameters p_n^* and w_n^* can be described by a simple model. In our project we explore modeling of p_n^* and w_n^* by a mixture of regression and time series model. This allows a simple forecast of the implied parameters for the next day. We present results of our preliminary exploration of the performance of option pricing with the use of the forecasted parameters and we obtained the best results for Model 4.

In our case it is evident that the fitted parameters are showing some dynamics. Therefore, testing performance of the option pricing on out-of-sample data does not seem appropriate. Instead, it is desirable to suggest some time-series models for the implied parameters. By forecasting the parameters we are going even beyond the scope of the available so far papers, by trying to achieve good pricing environment for a near future, e.g. for the next day (cf. Remark 11 in Section 5.5).

In Chapter 2 we briefly introduce Lévy processes and include a few examples of Lévy processes. In particular we describe in more detail the VG distribution, its properties, and the corresponding VG process.

In Chapter 3 we present a brief introduction to the European option pricing. We show two methods of pricing European options: by conditional integration and by Fourier Transformation technique. We present the Mean Martingale Correcting Term (MMCT) w as a convenient parameter describing the risk-neutral probability measure. For the sake of completeness we include in Chapters 2 and 3 some classical results and provided their proofs in Appendix A.

In Chapter 4 we present our new nonparametric method to price options with the use of the ECF. This section includes a detailed description of our four option pricing models. The first model requires only historical log-returns of the underlying price process. The other three modifications of the model need, in addition, real option prices for calibration of implied parameters.

In Chapter 5 we present performance of our model based on historical data of DAX index and ODAX options written on the index between the 1st of June 2006 and the 17th of May 2007. The resulting pricing errors show that our Model 4 performs better, than that of the CML.

In Chapter 6 we present concluding remarks about our project and we discuss directions for expansion of the present research.

For the clarity of exposition we collected proofs and technical references in Appendices A and B, respectively. In Appendix C we included the MATLAB® code used in our project.

Chapter 2

Lévy processes used in option pricing

It has been known since long that the Brownian Motion (BM) is not a perfect model for market stock log-returns. Many authors have extended the model to more general processes. We are choosing to work in this project with Lévy processes which are known to have many nice features. Our working example is a 3-parameter[1] Variance Gamma (VG) process introduced by Madan & Seneta (1990) and Carr et al. (1998). This process is a particular case of a class of a 5-parameter Generalized Hyperbolic (GH) processes introduced by Barndorff-Nielsen & Halgreen (1977). Prause (1999) remarks that the GH distributions tend to overfit and are computationally demanding. Since the VG distributions are closed under convolution, it is also convienient to use it for modelling Value at Risk (VaR). Therefore we choose to focus on the VG process, which is one of the most popular tractable semi heavy-tailed Lévy process.

In this chapter we present a definition and some examples of Lévy processes. Next we introduce the VG process and discuss its distributional properties.

Let $\big(\Omega, \mathcal{F}, \mathbf{P}\big)$ be a probability space.

2.1 Introduction to Lévy processes

Definition 1 *A real valued stochastic process* $(X_t)_{t\geq 0}$*, defined on* $\big(\Omega, \mathcal{F}, \mathbf{P}\big)$*, is called a Lévy process if the following conditions are satisfied:*

[1]The fourth parameter can be included as an additional drift.

a. X_t *is right continuous with left hand side limits,*
b. $X_0 = 0$ *with probability* 1,
c. X_t *has independent increments, i.e for any* $0 \leq r < s < t$ *increment* $X_s - X_r$ *is independent of* $X_t - X_s$,
d. X_t *has stationary increments, i.e. the distribution of* $X_{t+h} - X_t$ *is the same as of* $X_h - X_0$,
e. X_t *is stochasticaly continuous, i.e. for any* $\epsilon > 0$,

$$\lim_{h \to 0} \mathbf{P}\Big(|X_{t+h} - X_t| \geq \epsilon\Big) = 0.$$

We shall refer to the parameter $t > 0$ as to time.

Definition 2 *A characteristic function of a Lévy process* X_t *is given by*

$$\phi_{X_t}(u) = Ee^{iuX_t},$$

where $u \in A \subset \mathbb{C}$ *and* A *is a strip of the form* $\{z \in \mathbb{C} | Im(z) \in (a,b)\}$*, where* $a, b \in \mathbb{R}$ *are specific for a given process.*

Definition 3 *A cumulant function of a Lévy process* X_t *is given by*

$$\psi_{X_t}(u) = \log \phi_{X_t}(u)$$

where $u \in A \subset \mathbb{C}$ *and* A *is a strip.*

Definition 4 *A probability distribution of a random variable* X *is called infinitely divisible if for any positive integer* n *there exist independent and identically distributed random variables* $X_1, \ldots, X_n$*, such that r.v.* $X_1 + \ldots + X_n$ *has the same distribution as* X*.*

Since the increments of a Lévy process are independent and stationary, its probability distribution is infinitely divisible, i.e. for any $t > 0$ and for any $n \in \mathbb{N}$, we have the following decomposition of X_t

$$X_t = \big(X_t - X_{\frac{n-1}{n}t}\big) + \ldots + \big(X_{\frac{2}{n}t} - X_{\frac{1}{n}t}\big) + \big(X_{\frac{1}{n}t} - X_0\big),$$

where a size of an increment in the decomposition is $\frac{t}{n}$. This decomposition gives the following relation

$$\phi_{X_t}(u) = \phi_{\sum_{j=1}^{n}\left(X_{\frac{j}{n}t} - X_{\frac{j-1}{n}t}\right)}(u) = \prod_{j=1}^{n} \phi_{\left(X_{\frac{j}{n}t} - X_{\frac{j-1}{n}t}\right)}(u) = \left(\phi_{X_{\frac{t}{n}}}(u)\right)^n, \tag{2.1.1}$$

where the second and third equalities come from independence and stationarity properties of increments of X_t. Also for any $t > 0$ we have

$$\phi_{X_t}(u) = \left(\phi_{X_1}(u)\right)^t. \tag{2.1.2}$$

We refer to Appendix A for a derivation of this relation.

In Chapter 4 we develop a method for pricing of European options, which relies on (2.1.1) and (2.1.2).

Lemma 1 *(Sato (1999)) If $\phi_{X_t}(u)$ ia characteristic function of a Lévy process X_t, then there exists a unique and continuous cumulant function $\psi(u)$ such that*

$$\phi_{X_t}(u) = e^{t\psi(u)}. \tag{2.1.3}$$

A complete characterization of cumulant functions for Lévy processes is given by the following Lévy-Khintchine representation.

Theorem 1 *(Lévy-Khintchine formula). A probability distribution of a random variable X is infinitely divisible if and only if there exist $\sigma^2 > 0, m \in \mathbb{R}$ and a measure $\mathbf{L}$ satisfying conditions*

$$\mathbf{L}(\{0\}) = 0$$

and

$$\int_{-\infty}^{\infty} (1 \wedge |x|^2)\mathbf{L}(dx) < \infty,$$

such that the characteristic function of X has the following representation

$$Ee^{iuX} = e^{\psi_X(u)} = \exp\left(imu - \frac{\sigma^2 u^2}{2} + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux1_{|x|\leq 1}(x))\mathbf{L}(dx)\right). \tag{2.1.4}$$

Definition 5 *We call $\mathbf{L}(dx)$ in formula (2.1.4) a Lévy measure, σ^2 a Gaussian coefficient, m a drift of the Lévy process, and $(\sigma^2, \mathbf{L}, m)$ a generating triplet. If the Lévy measure is absolutely continuous with respect to the Lebesgue measure, such that $\mathbf{L}(dx) = l(x)dx$, then $l(x)$ is called a Lévy density function.*

By (2.1.3) we have the following representation for characteristic functions of Lévy processes.

Corollary 1 *A characteristic function of a Lévy process X_t takes form*

$$Ee^{iuX_t} = e^{t\psi(u)} = \exp\Big(t\big[imu - \frac{\sigma^2 u^2}{2} + \int_{-\infty}^{\infty}(e^{iux} - 1 - iux1_{|x|\leq 1}(x))\mathbf{L}(dx)\big]\Big).$$

Below we present several examples of characteristic functions of simple Lévy processes and their Lévy-Khintchine representations.

Example 1 *A characteristic function of a linear drift $X_t = \mu t$ has representation*

$$\phi_{X_t}(u) = \exp(ti\mu u).$$

Both the Gaussian coefficient σ^2 and the Lévy measure $\mathbf{L}$ here equal to zero.

Example 2 *A characteristic function of a Brownian Motion $X_t = \sigma W_t$ has representation*

$$\phi_{X_t}(u) = \exp(-\frac{u^2\sigma^2}{2}t). \tag{2.1.5}$$

In this case the Lévy measure $\mathbf{L}$ and the drift m are equal to zero.

Example 3 *A characteristic function of a Poisson process with a probability function of increments over time interval of length $t > 0$ given by*

$$\mathbf{P}(X_{t+h} - X_t = k) = \frac{e^{-\lambda h}(\lambda h)^k}{k!},\ \textit{for } \lambda > 0 \textit{ and } k = 0, 1, \ldots \tag{2.1.6}$$

has the representation

$$\phi_{X_t}(u) = \exp\left(\lambda t(e^{iu} - 1)\right) = \exp\left(\lambda t\int_{-\infty}^{\infty}(e^{iux} - 1)\delta_1(x)dx\right),$$

where δ_1 is the Dirac measure centered at 1. In this case the Lévy measure $\mathbf{L}(x) = \delta_1(x)$. The Gaussian coefficient σ^2 equals zero.

Example 4 *A characteristic function of a Laplace distribution with a probability function given by*

$$f(x) = \frac{1}{2s}e^{-\frac{|x-\theta|}{s}},$$

where $\theta \in \mathbb{R}$ and $s > 0$ are location and scale parameters, respectively, has representation (cf. Kotz, Kozubowski & Podgórski (2001))

$$\phi(u) = \exp\left(iu\theta + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - iux1_{|x|\leq 1}(x)\right) e^{-\frac{|x|}{s}} |x|^{-1} dx \right).$$

Example 5 *A Gamma process γ_t with a probability density function $f_{\gamma_t}(y)$ of increments over interval of length $t > 0$, is given by*

$$f_{\gamma_t}(y) = g_{\frac{t}{\nu},\frac{1}{\nu}}(y), \quad \textit{for } \nu > 0, \tag{2.1.7}$$

where

$$g_{\alpha,\beta}(y) = \frac{\beta^\alpha y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)} \tag{2.1.8}$$

is a density function of Gamma distribution with a shape parameter α and scale parameter β. The moment generating function (mgf) of X_t is given by

$$mgf(u) = (1 + u\nu)^{-\frac{t}{\nu}}$$

where $u \in \mathbb{R}$. The cumulant function $\psi_{X_t}(u)$ is given by

$$\psi_{X_t}(u) = -\frac{t}{\nu} \log(1 + u\nu) = -\frac{t}{\nu} \log\left(\frac{\frac{1}{\nu} + u}{\frac{1}{\nu}} \right)$$

and by Frullani equality (cf. Spiegel (1968) or (B.1.1) *in Appendix B.1) we have*

$$\psi_{X_t}(u) = -\frac{t}{\nu} \frac{\int_0^\infty \frac{e^{-\frac{x}{\nu}} - e^{-(\frac{1}{\nu}+u)x}}{x} dx}{\lim_{y\to 0} e^y - \lim_{y\to\infty} e^y} = t \int_0^\infty (e^{-ux} - 1) \frac{e^{-\frac{x}{\nu}}}{\nu x} dx.$$

Let us note that $\psi_{X_t}(u)$ allows a unique analytic extension onto complex plane. In the following we will consider $\psi_{X_t}(u)$ as a function of a complex argument $u \in A$. In particular for a transformation $u \mapsto -iu$ we get

$$\psi_{X_t}(-iu) = t \int_{-\infty}^\infty (e^{iux} - 1) \frac{e^{-\frac{x}{\nu}}}{\nu x} dx.$$

This implies that the Lévy measure is of the form

$$\mathbf{L}_{X_t}(x) = \frac{e^{-\frac{x}{\nu}}}{\nu x} \tag{2.1.9}$$

and that the drift m equals to zero. There is no Gaussian component here and the generating triplet satisfies conditions of finite variation which will be discussed in Theorem 2.1.11.

Remark 1 *Another way to derive the Lévy measure of the Gamma process relies on formula*

$$\lim_{t\to 0} t^{-1} f_{X_t}(x) = l(x),$$

which can be found in Barndorff-Nielsen (2000)(formula 3.22). Indeed, we have

$$t^{-1} f_{X_t}(x) = \frac{x^{\frac{t}{\nu}-1} e^{-\frac{x}{\nu}}}{\nu^{\frac{t}{\nu}+1}\Gamma(\frac{t}{\nu}+1)} \to \frac{e^{-\frac{x}{\nu}}}{\nu x}, \text{ as } t \to 0.$$

In the next section we will discuss a VG process, which is a particular case of the general Lévy process. The VG process has finite variation, where the finite variation is defined in the following way.

Definition 6 *Stochastic process X_t has finite variation if with probability 1 its trajectories are functions of finite variation, i.e.*

$$\mathbf{P}\left(\sup \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}| < \infty\right) = 1, \tag{2.1.10}$$

where the supremum is taken over all partitions $(t_i)_{i=1,\dots,n}$ of any closed interval $[a,b]$.

We have the following characterization for Lévy processes with finite variation.

Theorem 2 *(Cont & Tankov (2004), Proposition 3.9). A Lévy process is of finite variation if and only if its generating triplet $(\sigma^2, \mathbf{L}, m)$ satisfies the following conditions*

$$\sigma^2 = 0$$

and

$$\int_{|x|\le 1} |x|\mathbf{L}(dx) < \infty. \tag{2.1.11}$$

2.2 Review of Lévy processes used for option pricing

2.2.1 Variance gamma process

A VG process has been first introduced in its symmetric form by Madan & Seneta (1990) with application to model behavior stock returns. Carr et al. (1998) extended the VG process onto non-symmetric distributions. The VG process can be constructed in one of the following three ways:

a. by subordinating a BM to a Gamma process in time parameter,

b. by specifying the density function of the process increments to have VG distribution, or

c. by specifying the Lévy measure.

In the following subsections we will discuss each of these methods in detail.

Construction of the VG process by subordination

We can obtain the VG process as a Brownian motion with a randomly changed time, as shown in Madan & Seneta (1990) and Carr et al. (1998)

$$X_t = \theta\gamma_t + \sigma W_{\gamma_t}, \tag{2.2.1}$$

where W_t is a standard Brownian Motion independent of a Gamma process γ_t with pdf (2.1.7) and parameter $\nu > 0$. Subordination of the time parameter of the BM allows the following interpretation. We assume that time between transactions of assets are random. The lengths of such random increments can be modelled by a positive random process called *operational time*[2] or *chronometer*[3], and in the VG case it is a Gamma process.

The resulting process has three parameters θ, σ and ν which correspond to the drift and volatility of the BM, and to the variance rate of the Gamma process, respectively. The VG process has no continuous component and hence it is called a pure-jump process. These properties can be summarized in the following proposition.

Proposition 1 *The VG process X_t with three parameters $\theta \in \mathbb{R}$, $\sigma, \nu > 0$ has a generating triplet $(0, L_X, 0)$, where*

$$\mathbf{L}_X(dx) = \frac{1}{\nu}\left[\mathbf{1}_{\{x<0\}} \exp\left(\frac{\theta + \sqrt{2\frac{\sigma^2}{\nu} + \theta^2}}{\sigma^2}x\right) + \mathbf{1}_{\{x>0\}} \exp\left(\frac{\theta - \sqrt{2\frac{\sigma^2}{\nu} + \theta^2}}{\sigma^2}x\right)\right]|x|^{-1}dx,$$

and hence its Lévy-Khintchine representation for characteristic function reduces to

$$\phi_{X_t}(u) = \exp\left(t[iu\mu + \int_{-\infty}^{\infty}(e^{iux} - 1)\mathbf{L}_X(dx)]\right),$$

[2] Feller (1966) (p. 347)

[3] Barndorff-Nielsen, Maejima & Sato (2006) (p. 437)

where $\mu = -\int_{|x|\leq 1} x\mathbf{L}_X(dx)$.
Moreover, X_t *is a process with finite variation.*

We refer to Appendix A for a sketch of the proof.

This approach to construction of VG processes implies an easy way to simulate its paths. First we simulate a subordinator, which is a random time Gamma process. Then based on the simulated subordinator we simulate a BM process. The simulation becomes even simpler if we use following decomposition. Any function of finite variation can be decomposed onto difference of two increasing functions. Carr et al. (1998)[4] showed that we can represent the VG process as

$$X_t = \gamma_t^1 - \gamma_t^2,$$

i.e. as a difference of two independent increasing Gamma processes γ_t^1 and γ_t^2 with the density functions for increments given by

$$g_{\frac{\mu_1^2 t}{\nu_1}, \frac{\mu_1}{\nu_1}}(y)$$

and

$$g_{\frac{\mu_2^2 t}{\nu_2}, \frac{\mu_2}{\nu_2}}(y),$$

(cf. 2.1.8), respectively. The characteristic functions of these processes are given by

$$\phi_{\gamma_t^k(u)} = \left(1 - iu\frac{\nu_k}{\mu_k}\right)^{-\frac{\mu_k^2 t}{\nu_k}} \text{ for } k = 1, 2,$$

and the characteristic function of the process X_t is given by

$$\phi_{X_t}(u) = \left(1 - iu\left(\frac{\nu_1}{\mu_1} - \frac{\nu_2}{\mu_2}\right) + u^2\frac{\nu_1}{\mu_1}\frac{\nu_2}{\mu_2}\right)^{-\frac{t}{\nu}},$$

[4]It is noted in Madan & Seneta (1990) (p. 518) that the symmetric VG process can be approximated as a difference of two independent compound Poisson processes.

where

$$\mu_1 = \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} + \frac{\theta}{2},$$
$$\mu_2 = \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} - \frac{\theta}{2},$$
$$\nu_1 = \mu_1^2\nu,$$
and
$$\nu_2 = \mu_2^2\nu.$$

Hence, one can simulate the VG process by taking a difference of paths of two simulated Gamma processes with proper parameters.

Construction of the VG process by specifying the density function of increments

The marginal probability distributions of the VG process are variance-mean mixtures of Normal distributions[5]. They are special cases of the Normal Variance-Mean distributions (Barndorff-Nielsen et al. 1982) and General Normal Variance-Mean distributions (Seneta & Tjetjep 2006). Kotz et al. (2001) classified VG distribution as a special case of class of *Asymmetric Generalized Laplace* distributions, indicating a relation to classic Laplace distributions. Assuming that the conditional probability distribution of a random variable X given Y is $N(a(b+Y), c^2Y + d^2)$ and that Y is a positive random variable, where a, b, c, d are real numbers, we obtain a marginal probability distribution, referred to as the GNVM probability distribution. In the case of probability distribution associated with the VG process we have $a = \theta$, $c = \sigma > 0$, $d = 0$, and the mixing distribution Y is a Gamma distribution with parameter $\nu > 0$, from Example 5.

Proposition 2 *If X_t is a VG process then the pdf of increments $X_{t+h} - X_t$ over time of*

[5] Teichroew (1957) considered such symmetric density function expressed in terms of Modified Bessel function of the second kind.

length $h > 0$ *is given by*

$$f_{X_{t+h}-X_t}(x) = \frac{2e^{\frac{\theta x}{\sigma^2}}}{\nu^{\frac{h}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{h}{\nu})}\left(\frac{x^2}{2\frac{\sigma^2}{\nu}+\theta^2}\right)^{\frac{h}{2\nu}-\frac{1}{4}} K_{\frac{h}{\nu}-\frac{1}{2}}\left(\frac{\sqrt{x^2(2\frac{\sigma^2}{\nu}+\theta^2)}}{\sigma^2}\right), \qquad (2.2.2)$$

where $K_a(\cdot)$ *is modified Bessel function of the second kind*[6]*,*
The density function $f_{X_{t+h}-X_t}(x)$ *is decreasing for large* x *like a power-modified exponential function, i.e.*

$$f_{X_{t+h}-X_t}(x) = const(\theta,\sigma,\nu,h)|x|^{\frac{h}{\nu}-1}e^{(\beta\mp\alpha)x} + o(1), \quad \text{as } x \to \pm\infty, \qquad (2.2.3)$$

where $\alpha = \sqrt{\frac{\theta^2\nu+2\sigma^2}{\nu\sigma^4}}$, $\beta = \frac{\theta}{\sigma^2}$ *and* $const(\theta,\sigma,\nu,h) = \frac{\left(2\frac{\sigma^2}{\nu}+\theta^2\right)^{-\frac{h}{2\nu}}}{\nu^{\frac{h}{\nu}}\Gamma(\frac{h}{\nu})}$.

We refer to Appendix A for a proof.

Remark 2 *Let us note that the VG process can be obtained as a limiting process of truncated* α*-Stable processes (cf. Cont & Tankov (2004)). The Lévy density of the* α*-Stable process is given by*

$$l_S(x) = \frac{A}{|x|^{1+\alpha}}1_{(-\infty,0)}(x) + \frac{B}{x^{1+\alpha}}1_{(0,\infty)}(x), \qquad (2.2.4)$$

where $A, B > 0$ *and* $\alpha \in (0,2]$*, while Lévy density of truncated* α*-Stable process is given by*

$$l_{X_t}(x) = \frac{A}{|x|^{1+\alpha}}1_{(-\infty,0)}(x)e^{C_1x} + \frac{B}{x^{1+\alpha}}1_{(0,\infty)}(x)e^{C_2x},$$

where $C_1 > 0$ *and* $C_2 < 0$*. If we pass with* $\alpha \to \infty$ *we get a Lévy density of the VG process* X_t *given by*

$$l_{X_t}(x) = \frac{A}{|x|}1_{(-\infty,0)}(x)e^{C_1x} + \frac{B}{x}1_{(0,\infty)}(x)e^{C_2x},$$

where $A = B = \frac{1}{\nu}$, $C_1 = \frac{\theta+\sqrt{2\frac{\sigma^2}{\nu}+\theta^2}}{\sigma^2}$ *and* $C_2 = \frac{\theta-\sqrt{2\frac{\sigma^2}{\nu}+\theta^2}}{\sigma^2}$ *(cf. Proposition 1). Consequently, the tails of the Lévy density of VG process have a power-modified exponential decay at infinity.*

[6] *Seneta (2004) remarked about some ambiguity of terminology, and he refers after Erdélyi et al. (1953) to the same function as a modified Bessel function of the third kind.*

Remark 3 *The VG process is known as a so called pure-jump process. It can be also interpreted as a Stochastic Volatility (SV) process. In this case the random volatility equals* $\sigma\sqrt{\gamma_t}$. *A probability distribution of a subordinated BM with a drift is given by*

$$X_t = \theta\gamma_t + \sigma W_{\gamma_t},$$

which is equivalent to the distribution of

$$X_t = \theta\gamma_t + \sigma\sqrt{\gamma_t}W_t.$$

In the following Proposition we present an explicit form of the characteristic function of the VG process X_t.

Proposition 3 *A characteristic function of the VG process is given by*

$$\phi_{X_t}(u) = \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-\frac{t}{\nu}}, \tag{2.2.5}$$

and is single-valued and analytical in the strip $A_X = \{z \in \mathbb{C} | Imz \in (a,b)\}$, *where* $a = \frac{\theta}{\sigma^2} - \sqrt{\frac{\theta^2\nu + 2\sigma^2}{\nu\sigma^4}}$ *and* $b = \frac{\theta}{\sigma^2} + \sqrt{\frac{\theta^2\nu + 2\sigma^2}{\nu\sigma^4}}$.

We refer to Appendix A for a proof.

Corollary 2 *The moment generating function of the VG process is given by*

$$mgf_{X_t}(u) = \phi_{X_t}(-iu) = \left(1 - u\theta\nu - \frac{1}{2}\sigma^2\nu u^2\right)^{-\frac{t}{\nu}},$$

and is defined for $Re(u) \in (a,b)$.

Since all moments of the VG process exist (cf. Madan & Seneta (1990), Carr et al. (1998)) the mean, variance, skewness and kurtosis are finite. We have

Proposition 4 *The mean, variance, skewness and kurtosis of a length* t *increment of the VG process are given by:*

$$\begin{aligned}
EX_t &= \theta t, \\
VX_t &= (\theta^2\nu + \sigma^2)t, \\
Skewness(X_t) &= \frac{2\theta^3\nu^2 + 3\sigma^2\theta\nu}{(\theta^2\nu + \sigma^2)^{\frac{3}{2}}}t^{-\frac{1}{2}}, \\
Kurtosis(X_t) &= \frac{(3\theta^4\nu^2 + 6\theta^2\nu\sigma^2)(t + 2\nu) + 3\sigma^4(t + \nu)}{(\theta^2\nu + \sigma^2)^2 t}.
\end{aligned}$$

In the case of the VG process we can find a and b of the strip A_X from the explicit form of the characteristic function. Alternatively we can derive them from the Lévy measure of X_t. This comes with the help of the following theorem on exponential moments for Lévy processes.

Theorem 3 *(Sato (1999), Theorem 25.17). Let X_t be a Lévy process with the generating triplet $(\sigma^2, \mathbf{L}, m)$. Let*

$$C = \{c \in \mathbb{R} | \int_{|x|>1} e^{cx} \mathbf{L}(dx) < \infty\}.$$

Then the set C is convex and contains the origin.
Moreover, $c \in C$ if and only if $Ee^{cX_t} < \infty$ for some $t > 0$ or, equivalently, for every $t > 0$.

2.2.2 Other Lévy processes used for option pricing

Normal Inverse Gaussian (NIG) processes

Distribution of the NIG process increments is a GNVM distribution where mixing distribution is Inverse Gaussian (IG). This distribution was used for pricing options by Prause (1999).

If X_t is a NIG process then pdf of increments over time length $h > 0$ is given by

$$f_{X_{t+h}-X_t}(x) = \frac{\alpha}{\pi} \exp(h\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - h\mu)) \frac{h\delta K_1(\alpha\sqrt{(h\delta)^2 + (x - h\mu)^2})}{\sqrt{(h\delta)^2 + (x - h\mu)^2}} \tag{2.2.6}$$

where $\alpha, \delta > 0$, $|\beta| \leq \alpha$ and $K_1(\cdot)$ is modified Bessel function of the second kind.
Rates of decrease of the pdf of increments of X_t in tails are power-modified exponential, i.e.

$$f_{X_{t+h}-X_t}(x) = \text{const}(\alpha, \beta, \delta, \mu, h) \left((h\delta)^2 + (x - h\mu)^2\right)^{-\frac{3}{4}} e^{\beta(x-h\mu)-\alpha\sqrt{(h\delta)^2+(x-h\mu)^2}}$$
$$+o(1), \quad \text{as } x \to \pm\infty,$$

where $\text{const}(\alpha, \beta, \delta, \mu, h) = \frac{\sqrt{\alpha}h\delta\exp((h\delta)^2\sqrt{\alpha^2-\beta^2})}{\sqrt{2\pi}}$.
The characteristic function of the NIG process increments is given by

$$\phi_{X_t}(u) = \frac{\exp(iut\mu + t\delta\sqrt{\alpha^2 - \beta^2})}{exp(t\delta\sqrt{\alpha^2 - (\beta + iu)^2})}$$

and is regular (single-valued and analytical) in the strip $A_X = \{z \in \mathbf{C} | \mathrm{Im} z \in (\beta - \alpha, \beta + \alpha)\}$.
In contrast to VG process, NIG process in a process of infinite variation.

Generalized Hyperbolic processes

Distribution of increments of GH process are generalization of VG and NIG distributions. The distribution has been used for pticing options by Prause (1999). This class also contains Hyperbolic distribution used for option pricing by Eberlein & Keller (1995). GH distribution is a GNVM distribution where mixing distribution is Generalized Inverse Gaussian (GIG) distribution. The density of the Generalized Hyperbolic distribution is given by

$$f(x) = \frac{(\alpha^2 - \beta^2)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\alpha^{\lambda - \frac{1}{2}}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}(\delta^2 + (x-\mu)^2)^{(\lambda - \frac{1}{2})/2} e^{\beta(x-\mu)} K_{\lambda - \frac{1}{2}}(\alpha\sqrt{\delta^2 + (x-\mu)^2}),$$

where $\mu \in \mathbb{R}$ and

$$\begin{aligned} &\delta \geq 0, \quad |\beta| < \alpha \quad \text{if } \lambda > 0, \\ &\delta > 0, \quad |\beta| < \alpha \quad \text{if } \lambda = 0, \\ &\delta > 0, \quad |\beta| \leq \alpha \quad \text{if } \lambda < 0, \end{aligned}$$

and K denotes a modified Bessel function of the second kind. The characteristic function of the GH process increments of size t is given by

$$\phi_{X_t}(u) = e^{iu\mu t} \frac{(\delta\sqrt{\alpha^2 - \beta^2})^{\lambda t} K_\lambda(\delta\sqrt{\alpha^2 - (\beta + iu)^2})^t}{K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})^t (\delta\sqrt{\alpha^2 - (\beta + iu)^2})^{\lambda t}},$$

and is analytical in the strip $A_X = \{z \in \mathbf{C} | \mathrm{Im} z \in (\beta - \alpha, \beta + \alpha)\}$. GH process has infinite variation, unless it is degenerated to a process of finite variation, for example VG process.

Finite Moment LogStable processes (FMLS)

Carr & Wu (2000) modified α-stable process for purpose of option pricing. They introduced Finite Moment LogStable processes. The FMLS process does not have the density function in a closed form. If X_t is a FMLS process then the characteristic function of increments over time length $h > 0$ is given by

$$\phi_{X_{t+h} - X_t}(u) = \exp\left(iu\lambda h - (iu\sigma)^\alpha h \sec\left(\frac{\pi\alpha}{2}\right)\right),$$

where $\lambda \in \mathbb{R}$, $\sigma \geq 0$, $\alpha \in (1, 2)$, and is analytical in the strip $A_X = \{z \in \mathbf{C} | \mathrm{Im} z < 0\}$. The FMLS processes have infinite variation.

Chapter 3

Option pricing for Lévy processes

In this chapter we present a framework for pricing of European options.

3.1 Risk-neutral market model for option pricing

We consider a riskless bond $(B_t)_{t\in[0,T]}$ and a risky asset with a price process $(S_t)_{t\in[0,T]}$. We denote a probability space by $\big(\Omega, \mathcal{F}, \mathbf{P}\big)$ and by $(\mathbf{F}_t)_{t\in[0,T]}$ an increasing family of sub σ-fields of $\mathcal{F}$, representing the history of the asset S_t. We shall refer to (B_t, S_t) as a market model.

To price derivatives in such market models we need a definition of *absence of arbitrage*. The absence of arbitrage means that one cannot make riskless profits, or, in other words, that it is a *fair* market. The lack of arbitrage guarantees existence of the so called *risk-neutral* or *martingale* measure. This is illustrated by Theorem 4 below, called a Fundamental Theorem of Asset Pricing (cf. Harrison & Pliska (1981), Delbaen & Schachermayer (1998), or Cont & Tankov (2004) (Proposition 9.2)).

Definition 7 *A stochastic process M_t is a martingale if*

$$E(M_t|\mathbf{F}_s) = M_s, \quad 0 \leq s \leq t.$$

Theorem 4 *The market model defined by a probability space $\big(\Omega, \mathcal{F}, \mathbf{P}\big)$, a riskless bond $(B_t)_{t\in[0,T]}$ and asset prices $(S_t)_{t\in[0,T]}$ is arbitrage-free if and only if there exists an equivalent probability measure $\mathbf{Q}$ ($\mathbf{Q} \approx \mathbf{P}$) such that the discounted asset price process $\left(\frac{S_t}{B_t}\right)_{t\in[0,T]}$ is a martingale with respect to the measure $\mathbf{Q}$.*

In an arbitrage free market, prices of all financial instruments can be computed as discounted expectations of their terminal payoffs with respect to some risk-neutral measure $\mathbf{Q}$, for example for the price process we have (cf. Cont & Tankov (2004))

$$\frac{S_t}{B_t} = E^{\mathbf{Q}}\Big(\frac{S_T}{B_T}|\mathbf{F}_t\Big) \quad \text{for } t \le T. \tag{3.1.1}$$

Since we do not assume that the price process S_t has log-normal distribution the market model can be *incomplete*. Let us recall that in incomplete markets there exist more than one risk-neutral measure.

3.1.1 Exponential Lévy market model

We shall assume that the bond price process is of the form $B_t = e^{rt}$ where the riskless rate $r > 0$ is constant. We shall assume that X_t follows a Lévy process and call

$$S_t = S_0 e^{rt+X_t} \tag{3.1.2}$$

a geometric or exponential Lévy process. We shall refer to (B_t, S_t) as an exponential Lévy market model.
We assume that our underlying asset pays no dividends, or that the dividends are already included into its price[1].

The following theorem shows that the exponential Lévy model is arbitrage free.

Theorem 5 *(Cont & Tankov (2004), Proposition 9.9). If the trajectories of a Lévy process X_t are neither increasing nor decreasing with probability 1, then the model given by $S_t = e^{rt+X_t}$ is arbitrage free; i.e. there exists a probability measure $\mathbf{Q}$ equivalent to $\mathbf{P}$ such that $(e^{-rt}S_t)_{t\in[0,T]}$ is a martingale with respect to $\mathbf{Q}$.*

The processes which are of our interest satisfy the assumptions of Theorem 5. For example the VG processes, described in Chapter 2, are neither increasing nor decreasing with probability 1. This comes, for example, from the decomposition of the VG process into a difference of two increasing Gamma processes (Carr et al. 1998).

[1] For example Lewis (2001) considers $S_t = S_0 e^{rt-qt+X_t}$, where q denotes the rate of the dividend. We assume $q = 0$, because we work with options, for which dividends are included in the price.

Remark 4 *In similar settings Eberlein & Jacod (1997), showed for some processes that by choosing different equivalent risk-neutral measures one obtains prices of European options between* $(S_0 - e^{-rt}K)^+$ *and* S_0*, which lie in the interval of all possible European option pricess.*

3.1.2 Esscher transform and the Mean Martingale Correcting Term

There are several methods existing in the literature for choosing the risk-neutral measure for pricing options in incomplete markets. Different choices of the risk-neutral measure may result in different financial instrument prices. The most popular are the Esscher Martingale Measure (ESSMM) (Eberlein & Keller (1995), Prause (1999), Boyarchenko & Levendorskiĭ (2002)), and the Mean Martingale Correcting Term (MMCT) method (Madan & Seneta (1990), Madan & Milne (1991), Carr et al. (1998), Lewis (2001), Schoutens (2003)).

We chose the MMCT as the method for obtaining a risk-neutral measure. Let us note that Miyahara (2005) showed that the MMCT is a special case of the ESSMM. To illustrate how the measure is introduced into the model, we first describe the ESSMM method. For this purpose we start with the Radon-Nikodym Theorem.

Theorem 6 *Radon-Nikodym Theorem. A probability measure* $\mathbf{P}$ *is absolutely continuous with respect to a probability measure* $\mathbf{Q}$*, if and only if there exists a nonnegative random variable* ξ*, such that for any* $A \in \mathcal{F}$*,*

$$\mathbf{P}(A) = \int_A \xi(\omega)\mathbf{Q}(d\omega).$$

The random variable ξ *is called a Radon-Nikodym derivative and it is denoted by* $\xi = \frac{d\mathbf{P}}{d\mathbf{Q}}$*.*

The Esscher transform

Let X_t be a Lévy process such that for some $w \in \mathbb{R}$ we have $\mathrm{E}e^{wX_t} < \infty$ and let

$$Z_t = \frac{e^{wX_t}}{\mathrm{E}e^{wX_t}}. \tag{3.1.3}$$

We note that $Z_t \geq 0$, $EZ_t = 1$ and that Z_t is a martingale with respect to the probability measure $\mathbf{P}$. The Essher transformed measure $\mathbf{Q}$ on $\mathcal{F}$ is defined by

$$\mathbf{Q}(A) = E(1_A Z_t), \tag{3.1.4}$$

for $A \in \mathcal{F}$. The measure $\mathbf{Q}$ is equivalent to $\mathbf{P}$ and Z_t is the Radon-Nikodym derivative $\frac{dP}{dQ}$. To price financial instruments we need w, such that $(e^{-rt}S_t)_{t\in[0,T]}$ is a martingale with respect to the measure $\mathbf{Q}$ (cf. Theorems 4 and 5). The martingale condition implies

$$E^{\mathbf{Q}}(e^{-rt}S_t|\mathbf{F}_0) = S_0, \tag{3.1.5}$$

which by (3.1.2) is equivalent to the condition

$$E^{\mathbf{Q}}(e^{X_t}|\mathbf{F}_0) = E^{\mathbf{Q}}(e^{X_t}) = 1$$

and can be expressed in terms of a characteristic function of X_t as

$$\phi^{\mathbf{Q}}_{X_t}(-i) = 1. \tag{3.1.6}$$

The characteristic function of X_t with respect to the measure $\mathbf{Q}$ is given by

$$\phi^{\mathbf{Q}}_{X_t}(u) = E^{\mathbf{Q}}(e^{iuX_t}) = E(e^{iuX_t}\frac{d\mathbf{P}}{d\mathbf{Q}}) = \frac{E(e^{(iu+w)X_t})}{Ee^{wX_t}} = \frac{\phi_{X_t}(u-iw)}{\phi_{X_t}(-iw)}. \tag{3.1.7}$$

The following proposition gives the generating triplet for the Esscher transformed process.

Proposition 5 *(cf. Miyahara (2004)) If X_t has a generating triplet $(\sigma^2, \mathbf{L}, m)$, then the Esscher transformed process has a generating triplet given by $((\sigma^{\mathbf{Q}})^2, \mathbf{L}^{\mathbf{Q}}, m^{\mathbf{Q}})$, where*

$$(\sigma^{\mathbf{Q}})^2 = \sigma^2,$$
$$m^{\mathbf{Q}} = m + w\sigma^2 + \int_R (e^{wx}-1)iux\mathbf{1}_{|x|\leq 1}(x)\mathbf{L}(dx),$$
$$\mathbf{L}^{\mathbf{Q}}(dx) = e^{wx}\mathbf{L}(dx).$$

We refer to Appendix A for a proof.

The Mean Martingale Correcting Term

Miyahara (2005) noticed that the MMCT coincides with the ESSMM in the case when X_t is a Wiener process. The generating triplet for the Wiener process W_t is $(1, 0, 0)$. The

Esscher transformed triplet is given by $(1, 0, w)$. The characteristic function of W_t with respect to the transformed measure $\mathbf{Q}$ is given by

$$\phi^{\mathbf{Q}}_{W_t}(u) = \phi_{W_t}(u)e^{wt}. \tag{3.1.8}$$

We calculate w from the martingale condition (3.1.6), i.e.

$$w = -\frac{\log\big(\phi_{W_t}(-i)\big)}{t} = -\frac{1}{2},$$

which coincides with the classical Black-Scholes theory. If we apply transformation (3.1.8) to any Lévy process X_t, then we get

$$\phi^{\mathbf{Q}}_{X_t}(u) = \phi_{X_t}(u)e^{wt} = Ee^{iuX_t}e^{wt} \tag{3.1.9}$$

and we obtain the MMCT change of measure. The martingale measure has been obtained by shifting the process X_t to $X_t + wt$. Hence we can get exponential Lévy price process

$$S_t = S_0 e^{rt+X_t+wt} \tag{3.1.10}$$

and the martingale condition

$$w = -\frac{\log\big(\phi_{X_t}(-i)\big)}{t}. \tag{3.1.11}$$

In the following we shall refer to (3.1.11) as to the MMCT.

3.2 Pricing of European options

In this section we discuss pricing of European options. As we noted in Section 3.1, under the martingale measure $\mathbf{Q}$ the value of a financial instrument is given by the discounted expectation of its terminal payoff. For an European call option $C(t_0, T, K)$ issued at time t_0 with strike price K and maturity time T, the terminal payoff is $(S_T - K)^+$, and the price is given by

$$C(t_0, T, K) = e^{-r(T-t_0)}E^{\mathbf{Q}}((S_T - K)^+ | F_{t_0}). \tag{3.2.1}$$

Hence, the pricing of the option can be done in two steps, by determining the distribution of S_T and integrating $(S_T - K)^+$ with respect to this distribution. An alternative consists in using the Fourier Transform if the characteristic function of X_T is known, as was first indirectly noted by Merton (1973)[2]. We shall discuss this method in the next section. For simplicity of notation we shall assume $t_0 = 0$.

In the Black-Scholes model, the risk-neutral model of an asset price was described by the exponential of a Brownian motion with drift

$$S_t = S_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W_t}.$$

We replace the Brownian motion with drift by a Lévy process X_t.

Proposition 6 *If X_t is a Lévy process and*

$$S_t = S_0 e^{rt+X_t+wt}, \tag{3.2.2}$$

where

$$w = -\frac{\log(\phi_{X_t}(-i))}{t}, \tag{3.2.3}$$

then a discounted pricing process $e^{-rt}S_t$ is a martingale under measure $\mathbf{Q}$ given by (3.1.4).

We refer to Appendix A for a proof.

The simplest method to compute the European option price, where the log-return process is modelled by a subordinated BM is to integrate the Black-Scholes price conditioned on the random time increments. For example for the VG process (2.2.1) we integrate the conditioned BS price formula with respect to the probability distribution of the Gamma process γ_T, as it has been proposed by Madan & Seneta (1990). For the VG case with parameters θ, σ, ν we have the following proposition.

Proposition 7 *Assumed that S_t is given by* (3.2.2)*, where X_t is a VG process with parameters (θ, σ, ν). Let $C(0,T,K)$ be the current price of an European call option with a spot price S_0, a time to maturity T, a strike price K and a risk neutral rate r. Then*

$$C(0,T,K) = \int_0^\infty \left(S_0 e^{wT+(\theta+\frac{1}{2}\sigma^2)y} F_N(d_1|y) - e^{-rT} K F_N(d_2|y) \right) f_{\gamma_T}(y)dy,$$

[2] Merton (1973) p.167 and footnote 49.

where $F_N(d|y) = E(1_{\{Z<d\}}|Y_T = y)$ *is the conditional Normal cdf with variance* y*,* Z *is a standard Normal variable,* $f_{\gamma_T}(y)$ *is a density function of the distribution of the Gamma increments* Y_T *over time of length* T *(cf.* (2.1.7)*),*

$$d_2 = \frac{\log\frac{S_0}{K} + (r+w)T + \theta y}{\sigma\sqrt{y}},$$

$$d_1 = d_2 + \sigma\sqrt{y},$$

and

$$w = \frac{1}{\nu}\log(1 - \theta\nu - \frac{1}{2}\sigma^2\nu).$$

We refer to Appendix A for a proof.

3.3 Option pricing based on characteristic functions

Carr et al. (1998) derived a formula for the European Call option in a classical form for the VG process. Bakshi & Madan (2000) generalized the formula and showed that the value of the European Call option with strike price K and maturity T at time 0 is given by

$$C(0,T,K) = \Pi_1 S_0 - Ke^{-rT}\Pi_2,$$

where

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\frac{\exp(-iu\log K)\phi(u-i)}{iu\phi(-i)}du$$

and

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\frac{\exp(-iu\log K)\phi(u)}{iu}du$$

and where $\phi(u)$ is the characteristic function of the log of the stock price process $\log\big(S_t\big)$.

Carr & Madan (1999) pointed out some numerical drawbacks, related to the singularities at zero in the above formula. They derived a new pricing method based on the characteristic function[3] of the log of the stock price process $\log\big(S_t\big)$ which was next generalized by Lewis (2001). Let us note that based on a similar Laplace transform method and

[3] Heston (1993) was the first one who derived formula for price of an European option based on characteristic function. However, he assumed a different stochastic process to model log-returns. His model does not imply i.i.d. logarithmic increments.

Fourier transform method Raible (2000) and Borovkov & Novikov (2002), respectively, derived other formulas for the European option prices. Here we present the Lewis method.

Let us recall that the characteristic function of X_T for $u \in \mathbb{C}$ and $a < \mathrm{Im}u < b$ is defined as $\phi_{X_T}(u) = E(e^{iuX_T})$. Since

$$\phi_{X_T}(0) = 1$$

by (3.2.2) and by the martingale condition (3.1.5) we get

$$S_0 = E^{\mathbf{Q}}(e^{-rT} S_T | \mathbf{F}_0) = S_0 \phi_{X_T}(-i)$$

and we have

$$\phi_{X_T}(-i) = 1.$$

Hence the characteristic function exists at both points $u = 0$ and $u = -i$.

By Theorem 9 in Appendix B.2, since $\phi_{X_T}(u)$ is analytical in the neighborhood of $u = 0$, it is also analytical in a horizontal strip, which is either a whole complex plane or it has two horizontal boundary lines, and is of the form $A = \{z \in \mathbb{C} | \mathrm{Im}(z) \in (a, b)\}$. Since the characteristic function exists at point $-i$ the strip has to include this point. Hence, $a \leq -1$ and $b \geq 0$.

For example for the VG process the horizontal boundary lines are given by (cf. Proposition 3),

$$a = \frac{\theta}{\sigma^2} - \sqrt{\frac{\theta^2 \nu + 2\sigma^2}{\nu \sigma^4}}$$

and

$$b = \frac{\theta}{\sigma^2} + \sqrt{\frac{\theta^2 \nu + 2\sigma^2}{\nu \sigma^4}}.$$

Moreover, Theorem 9 in Appendix B.2 shows that the characteristic function $\phi_{X_T}(u)$ is analytic in the strip A, i.e. between these two horizontal boundary lines.

The following theorem gives a price of an European option with a payoff function $H(x)$. The resulting formula (3.3.1) is based on the characteristic function $\phi_{X_T}(u)$ of X_T.

Theorem 7 *Lewis (2001) (Theorem 3.2)*
Let $C(0,T,K)$ be an arbitrage free price of a European option with a payoff function $H(\log S_T) \geq 0$, where S_T is the value of the stock at expiry T. Assume that:
a. a Fourier transform $\hat{H}(z)$ of $H(x)$ exists in the strip A_w,
b. $S_t = S_0 e^{rt+Y_t}$, where Y_t is a Lévy process and e^{Y_t} is a martingale,
c. Y_T has a characteristic function $\phi_{Y_T}(z)$ analytic and one-valued in a strip $A_{Y_T} = \{z \in \mathbb{C} : Im(z) \in (a,b)\}$, where $a < -1$ and $b > 0$.
Under assumptions a-c, if $x + i\nu \in A_C = A_H \cap A^*_{Y_T}$, for some $\nu \in \mathbb{R}$ then the option price is given by

$$C(0,T,K) = \frac{e^{-rT}}{2\pi} \int_{-\infty}^{\infty} e^{-i(x+i\nu)(\log S_0 + rT)} \phi_{Y_T}(-(x+i\nu)) \hat{H}(x+i\nu) dx. \tag{3.3.1}$$

We refer to Appendix A for a sketch of the proof of Theorem 7.

The following corollary gives a price of a European Call option.

Corollary 3 *Under the assumptions of Theorem 7, and assuming that:*
a. $H(x) = (e^x - K)^+$ is a payoff function of a Call option,
b. we can integrate along a real line in a complex plane $\{x - \frac{i}{2} | x > 0\}$,
c. $Y_t = X_t + wt$, where w is determined by the martingale condition,
the option price is given by

$$C(0,T,K) = S_0 - \frac{\sqrt{S_0 K}}{\pi} e^{-\frac{rT}{2} + \frac{wT}{2}} \int_0^{\infty} Re\left[e^{-iu(\log \frac{S_0}{K} + rT + wT)} \phi_{X_T}(-u - \frac{i}{2}) \right] \frac{du}{u^2 + \frac{1}{4}}, \tag{3.3.2}$$

We refer to Appendix A for a sketch of the proof of Corollary 3.
We shall refer to formula (3.3.2) as to the CML formula. Let us note that to calculate price of an European Put option $P(0,T,K)$ we use the put-call parity relation

$$P(0,T,K) = C(0,T,K) - S_0 + Ke^{-rT}.$$

Chapter 4

Option pricing based on empirical characteristic functions

In this chapter we depart from the typical parametric approach in modelling the distribution of the underlying price process. We use instead a nonparametric approach in the CML formula by using the Empirical Characteristic Function (ECF) to price European options. We consider several modifications of this model based on the ECF. In particular, we introduce models with implied parameters p_n^* (or Δ_n^*) and w_n^* and compare results with those obtained by applying the CML method in the case of a parametric VG distribution of log-returns.

Nonparametric approaches have already been used in option pricing. Approximation of risk-neutral density has been done through a tree-based method, cf. Cox & Rubinstein (1979), Rubinstein (1994), Jackwerth (1999). Spline method has been proposed by Shimko (1993) as an extension of Breeden & Litzenberger (1978) approximation. This approach later has been generalized in Ait-Sahalia & Lo (1998) where the nonparametric kernel regression has been used. Another approach is based on Edgeworth expansion by Jarrow & Rudd (1982), and approximation of risk-neutral density by Hermite polynomials, cf. Madan & Milne (1994) and Schlogl (2007).

The chapter is organized in the following way. In Section 4.1 we recall the definition of the Empirical Characteristic Function and its properties. Next, we extend a result of Csörgő & Totik (1983) on a uniform consistency of the ECF on the real line onto a strip in the complex plane. We introduce our nonparametric model and show convergence with probability 1 of our ECF pricing formula to the original one. In Section 4.2 we specify the five models considered in the project, in the cases with and without the implied parameters.

4.1 Approximate option pricing using empirical characteristic functions

Let us assume that $X_1, \dots, X_n$ represent independent identically distributed random variables with a cumulative distribution function (CDF) $F(x)$ and a characteristic function $\phi(u)$, where $u \in A \subset \mathbb{C}$ and where A is a strip of the form $A = \{z \in \mathbb{C} | \mathrm{Im}(z) \in (a, b)\}$ for some $a, b \in \mathbb{R}$.
Moreover, we assume that the characteristic function exists at imaginary points $-y - \frac{i}{2}$ and $-i$, where $y > 0$, i.e. $\phi(-y - \frac{i}{2}) = Ee^{-iyX + \frac{X}{2}} < \infty$ and $\phi(-i) = Ee^{X} < \infty$.

The ECF $\hat{\phi}_n(u)$ is given by

$$\hat{\phi}_n(u) = \frac{1}{n} \sum_{j=1}^{n} e^{iuX_j},$$

where $u \in A$.

By the Strong Law of Large Numbers, the ECF is a consistent estimator of the characteristic function at each point $u \in A \subset \mathbb{C}$. We note this in the following lemma.

Lemma 2 *For any $x \in \mathbb{R}$ and $\nu \in A$, where A is a strip of analyticity of the characteristic function we have*

$$P\left(\lim_{n\to\infty} \hat{\phi}_n(x + i\nu) = \phi(x + i\nu)\right) = 1, \tag{4.1.1}$$

We refer to Appendix A for a proof.

This extends to a uniform consistency on closed intervals $[-U, U]$ on the real line:

$$\mathbf{P}\Big(\lim_{n\to\infty} \sup_{|u|\le U} |\hat{\phi}_n(u) - \phi(u)| = 0\Big) = 1,$$

cf. Feuerverger & Mureika (1977), or even to a uniform consistency on an increasing intervals of the form $[-U_n, U_n]$, where $U_n = \exp(n/G_n)$, where $G_n \to \infty$, cf. Csörgő & Totik (1983),

$$\mathbf{P}\Big(\lim_{n\to\infty} \sup_{|u|\le U_n} |\hat{\phi}_n(u) - \phi(u)| = 0\Big) = 1.$$

The following proposition extends the result of Csörgő & Totik (1983) for uniform consistency on increasing sequence of intervals on the real line. We show that the strong consistency is valid in the strip of analyticity of the characteristic function.

Proposition 8 *A characteristic function $\phi(u)$ is analytical in a strip*

$$A = \{z \in \mathbb{C} | Im(z) \in (a, b)\}.$$

If

$$\lim_{n\to\infty} \frac{\log U_n}{n} = 0, \tag{4.1.2}$$

then

$$\mathbf{P}\Big(\lim_{n\to\infty} \sup_{|x|\leq U_n} \big|\hat{\phi}_n(x+i\nu) - \phi(x+i\nu)\big| = 0\Big) = 1, \tag{4.1.3}$$

where $\nu \in (a, b)$.

We refer to Appendix A for a proof.

Before we introduce our pricing formula we have some preliminary remarks. We assume that time increment between our observations is $\Delta = \frac{1}{365}$, i.e. we have daily log-returns[1]. Size of time increment between the time when option is issued, and the time when option expiries is $T = p \times \Delta$, where p is the number of days to expiration. Assuming that log-returns are i.i.d. random variables we get, by the property of characteristic function that

$$\phi_{X_T}(u) = (\phi_{X_\Delta}(u))^p. \tag{4.1.4}$$

Hence we can get the characteristic function of log-returns on long interval T by knowing the characteristic function of log-returns on a shorter interval Δ. This is particularly useful in working with historical data. We can estimate an ECF for log-returns by applying (4.1.4) to an ECF of log-returns on short time intervals.

$$\hat{\phi}_{T,n}(u) = \left(\hat{\phi}_n(u)\right)^p, \tag{4.1.5}$$

[1] The most popular conventions are 365 or 252 working days in a year. We chose the first convention.

where $p = \frac{T}{\Delta}$ and $\hat{\phi}_n(u)$ is an ECF of log-returns on interval of length Δ. Hence $\hat{\phi}_{T,n}(u)$ is an estimator of $\phi_{X_T}(u)$. We consider pricing of an European option, with time to maturity T and strike price K, by replacing formula (3.3.2) with its empirical version

$$\hat{C}_n(0,T,K;\hat{w}_n,p) = \\ S_0 - \frac{\sqrt{S_0 K}}{\pi} e^{-\frac{rT}{2}+\frac{\hat{w}_n T}{2}} \int_0^\infty \mathrm{Re}\left[e^{-iu(\log\frac{S_0}{K}+rT+\hat{w}_n T)} \hat{\phi}_{T,n}(-u-\frac{i}{2}) \right] \frac{du}{u^2+\frac{1}{4}}, \quad (4.1.6)$$

where $\hat{w}_n$ is the empirical version of the MMCT w, given by

$$\hat{w}_n = -\frac{\log\left(\hat{\phi}_n(-i)\right)}{\Delta}, \quad (4.1.7)$$

and where $\hat{\phi}_{T,n}(u)$ is given by (4.1.5).

In Lemma 3, in the following remarks and in Proposition 9 we provide a formal justification of our method.

Lemma 3 *The integral in formula (4.1.6) is finite.*

We refer to Appendix A for a proof.

Remark 5 *In formula (4.1.6) we have to take into consideration a convergence of a sequence $\hat{\phi}_n(-i)$ to the value of the true characteristic function at point $-i$. The condition*

$$\phi(-i) < \infty \quad (4.1.8)$$

holds for most probability distributions considered in the financial literature concerned with option pricing. In particular, it is met for probability distributions listed in Section 2.2.2.

For example for the VG distribution of log-returns the strip of analyticity of the characteristic function is given by

$$A_X = \left\{ z \in \mathbb{C} | Im(z) \in \left(\frac{\theta}{\sigma^2} - \sqrt{\frac{\theta^2\nu + 2\sigma^2}{\nu\sigma^4}}, \frac{\theta}{\sigma^2} + \sqrt{\frac{\theta^2\nu + 2\sigma^2}{\nu\sigma^4}} \right) \right\},$$

(cf. Proposition 3). The strip does not depend on size of the time increment between log returns. Carr et al. (1998) obtained the following estimates of parameters of the VG distribution of log-returns of S&P500 index, $\theta = 0.0591, \sigma = 0.1172, \nu = 0.002$, for which the strip is $A_X = \{z \in \mathbb{C} | Im(z) \in (-27.3242, 35.9294)\}$. Condition (4.1.8) does

not hold for all values of the four parameters $\alpha-$Stable distributions. Carr & Wu (2000) considered a subclass of that class of distributions by keeping some parameters fixed, hence achieving that condition (4.1.8) *was met. They call their class Finite Moment LogStable*[2] *distributions.*

As we already noted in Section 3.3, the existence and analyticity of a characteristic function at points 0 *and* $-i$ *implies the existence and analyticity of the characteristic function in a horizontal strip*

$$A_X = \{z \in \mathbb{C} | Im(z) \in (a, b)\},$$

where $a < -1$ *and* $b > 0$*. To verify if that assumption is met for probability distributions considered in the present research, we estimated the strip for the VG process based on options data. Using the Maximum Likelihood Estimation method we obtained triplets* (θ, σ, ν) *of parameters for each day from our data set, i.e.* $\{(\theta_k, \sigma_k, \nu_k)\}_{k=1,\dots,243}$[3]*. Then we calculated the horizontal boundaries for the strip, i.e.* $\{(a_k, b_k)\}_{k=1,\dots,243}$*. Figure 4.1 illustrates how the strip changes over time. From the plot we observed that points* 0 *and* $-i$ *are always included in the strip of the VG process. Although we are not tied to the VG process, the above example justifies our confidence that the assumption on finitness of the exponential moments are fulfilled for probability distributions of the DAX log-returns.*

Remark 6 *Pricing options using formula (4.1.6) allows the following interpretation related to the classical method of pricing options. Assume that the probability distribution of log-returns of the price process* S_t *belongs to a certain parametric family. We can first estimate parameters of the risk-neutral distribution using some method based on the ECF*[4]*.Then, we can use the estimated parameters in formula (3.3.2) to obtain prices of options. This procedure consists of two steps. Our procedure consists of only one step, we do not estimate parameters of the distribution, but instead we use estimator of the characteristic function. Both approaches have a common drawback. Because we do not use historical option data, but only historical data of the underlying price process, it is difficult*

[2] *Cf. Section 2.2.2.*

[3] *We consider* 243 *days in our options database, between the 1st of June 2006 and the 17th of May 2007.*

[4] *For example, we can estimate parameters of the distribution by solving a set of equations which involved ECF, cf. Feuerverger & McDunnough (1981)*

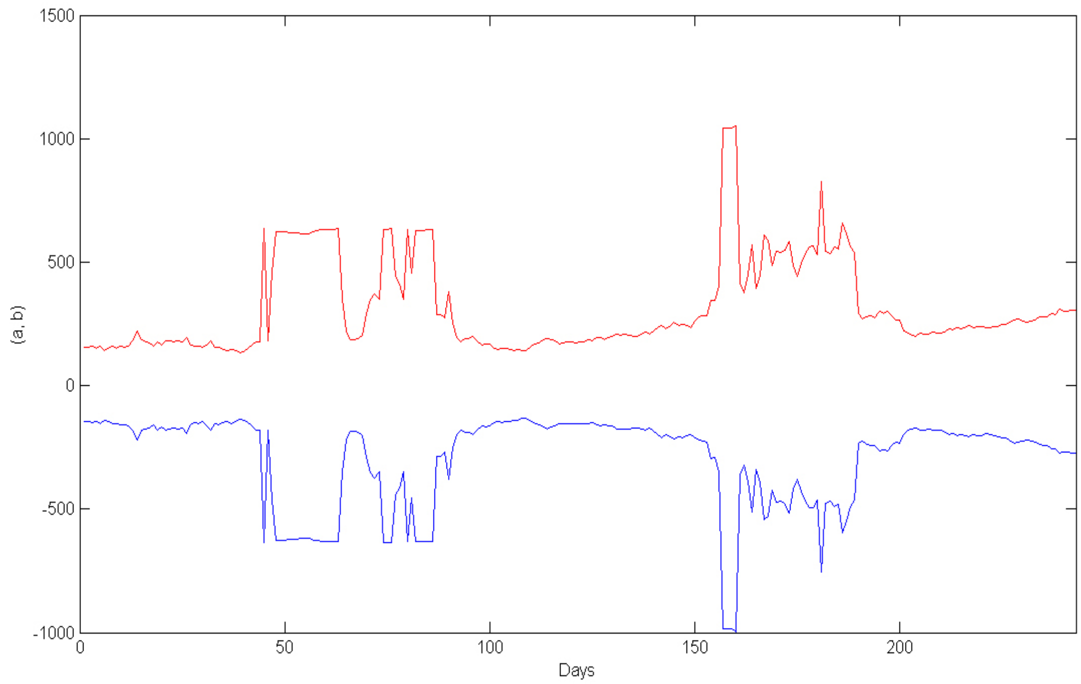

Figure 4.1: Estimated strip for VG process, using MLE, for ODAX options between the 1st of June 2006 and the 17th of May 2007, (a_k - blue line, b_k - red line).

to obtain option prices close to the real ones. This also shows discrepancy between the risk-neutral measure implied by real option prices and the one implied by the underlying price process.

Remark 7 *Instead of using the ECF in formula* (3.3.2) *we could alternatively choose to integrate the payoff against the empirical cumulative distribution function*

$$e^{-rT}\int_R (S_T - K)^+ dF_n^T(x),$$

where $F_n^T(x) = \frac{1}{n}\sum_{j=1}^n \mathbf{1}_{\{X_j \leq x\}}(x)$*. However, this approach has two serious drawbacks. First, such an integral is harder to evaluate numerically. For example, if we were to approximate the cdf* $F_n^T(x)$ *of the distribution of the terminal payoff, we would need to take a p-fold convolution of the empirical cdfs* $F_n^\Delta(x)$*, where p is the number of days to option expiration. In the case of the ECF we just take it to the power p, using properties of characteristic functions. Secondly, it is also not clear how to incorporate change of measure through MMCT in such a case.*

Remark 8 *The assumption on independent identically distributed log-returns is crucial for our approach. For not i.i.d. log-returns,* $\hat{\phi}_{T,n}(u)$ *in the formula* (4.1.6) *is not longer an estimator of* $\phi_{X_T}(u)$ *in the formula* (3.3.2). *There exist option pricing models based on semimartingales for which increments of the stochastic process are not i.i.d. The simplest examples of such models are the SV model (Heston 1993), where the price process is modelled as a CIR*[5] *process, or its extension like a jump-diffusion SV (Bates 1996). More advanced processes of this type for option pricing were used by Barndorff-Nielsen et al. (2002) and Carr et al. (2003). They include pure jump processes. These models do not imply independent logarithmic increments.*

We also mention exponential additive processes used for option pricing in Carr, Geman, Madan & Yor (2007). These processes satisfy the same assumption as Lévy processes, except the one on stationary increments. Hence, in such models the logarithmic increments can be not identically distributed. Our assumption exludes all of these processes and our method is not applicable in such models.

Remark 9 *Approximations obtained from our approach can be compared to the classical Black-Scholes model, where deviations from the model are described by a so called implied volatility. In the present case the implied parameters* w_n^*, p_n^* *and* (w_n^*, p_n^*) *play similar role as the implied volatility in the Black-Scholes model. Clearly, this shows that some assumptions of our model, like i.i.d., may be violated and that more sophisticated, like Heston (1993), models are needed to precisely describe the market behaviour. However, in such cases only very time consuming Monte-Carlo simulation technics are available to date to calibrate the models. Hence, our approach seems to be a reasonable compromise between numerical sophistication and perfect accuracy of modeling of market behaviour. In particular, in Chapter 5 we observe that the parameters, obtained through daily calibration to real option prices, are changing with time. In Section 5 of Chapter 5 we model the implied parameters from our model using a simple regression-time-series approach, a process which is much faster than Monte-Carlo simulations.*

The following proposition shows convergence of our formula (4.1.6) to the original formula (3.3.2).

[5] *Cox-Ingersoll-Ross.*

Proposition 9 *We have*

$$\mathbf{P}\Big(\lim_{n\to\infty}\hat{C}_n(0,T,K;\hat{w}_n,p)=C(0,T,K)\Big)=1. \tag{4.1.9}$$

We refer to Appendix A for a proof.

Formula (4.1.6) includes parameters w and p, which in the case of perfect modelling are completely specified, i.e. w is given by (3.2.3) and p is the number of days to option expiry. In our case it is natural to estimate w by $\hat{w}_n$ given by (4.1.7). However, though model (4.1.6) relying on modelling real log-returns by Lévy processes captures the main features of the behaviour of real options, yet it is not perfect. We suggest to consider w and/or p as parameters and fit them to the historical option prices. The obtained parameters w_n^* and p_n^*, respectively, are called implied because the real option prices are used in fitting their optimal values. Let us note that the implied parameters are functions of both historical log-returns and of historical option prices, while the empirical $\hat{w}_n$ depends only on historical log-returns of the underlying price process.

Accepting replacement of the characteristic function in (3.3.2) with its empirical version we consider the following cases. In the first model we estimate $w=\hat{w}_n$ by Nonlinear Least Squares method using real option prices, and denote it by w_n^*. We call w_n^* an *implied* MMCT (or *implied* w). In another model we allow p to be a parameter which can be fitted to the historical option prices. We denote the fitted parameter p_n^*, and we call it *implied* p. We refer to the number of days to option expiration as *the true* p. Since time to expiration T is related to p by

$$p=\frac{T}{\Delta},$$

it is natural to keep T fixed and to have implied the time length of increments Δ_n^*. Hence, we have the following relation

$$p_n^*=\frac{T}{\Delta_n^*}, \tag{4.1.10}$$

used in the project. Let us note that one can also consider a model, where T is implied and Δ_n^* fixed, however this is beyond the scope of the present project. In the next section we discuss the resulting models in details.

4.2 Five models for option pricing

As we discussed in Section 4.1 one can allow w and p to become implied parameters of the model. We use the Nonlinear Least Squares method to obtain the best fitting values

w_n^* and p_n^* of w and p, respectively. We consider the following four models using the ECF formula (4.1.6), and the fifth model where like Carr & Madan (1999), we use the CML formula (3.3.2). By $\bar{C}(0,T,K)$ we denote the real option price at time zero with expiration at time T, and strike price K.

4.2.1 Model 1: one estimated parameter $\hat{w}_n$

Our first model can be labelled as Empirical ECF pricing.
We price options by formula (4.1.6), where we use the empirical version of MMCT, i.e.

$$\hat{w}_n = -\frac{\log\left(\phi_n(-i)\right)}{\Delta},$$

and the true value p of the number of days to option expiry.

4.2.2 Model 2: estimated parameter $\hat{w}_n$ and implied Δ_n^*

The second of our model is the ECF pricing model (4.1.6) with empirical $\hat{w}_n$ given by (4.1.7) and implied p_n^*, where

$$p_n^* = \frac{T}{\Delta_n^*} = \mathrm{argmin}_{\{p\}} \sum_l \left\{\hat{C}_n(0,T,K_l;\hat{w}_n,p) - \bar{C}(0,T,K_l)\right\}^2,$$

and where l indexes the set of the considered strikes.

Model 2 loses the interpretation of p as a number of days to option expiry. Change of p affects the ECF in $\hat{\phi}_{T,n}(u)$ but is not directly related to the maturity, because we do not change T in the formula 4.1.6. Discrepancy between p and p_n^* can be interpreted as a change of speed of time due to other factors than the historical log-returns and not included explicitely in the model.

4.2.3 Model 3: implied w_n^*

The third model considered in the project is the ECF pricing model 4.1.6 with implied w_n^* (i.e. implied MMCT w) and and the true expiry p (in days), where

$$w_n^* = \mathrm{argmin}_{\{w\}} \sum_l \left\{\hat{C}_n(0,T,K_l;w,p) - \bar{C}(0,T,K_l)\right\}^2,$$

and where l indexes the set of the considered strikes.

In Model 3 we lose interpretation of finding empirically the MMCT w. We do not need to assume that log-returns follow specific subclass of Lévy processes to satisfy the assumption $\phi(-i) < \infty$. However we still need the assumption that characteristic function exists at points $-y - \frac{i}{2}$, where $y > 0$. Discrepancy between w_n^* and $\hat{w}_n$ can be also interpreted as adjusting approximate model 4.1.6 to the real process of option pricing.

4.2.4 Model 4: implied w_n^* and implied Δ_n^*

The fourth model is the ECF pricing model with both implied w_n^* (i.e. implied MMCT w) and p_n^* (i.e. implied p), where

$$(w_n^*, p_n^* = \frac{T}{\Delta_n^*}) = \text{argmin}_{\{w,p\}} \sum_l \left\{ \hat{C}_n(0, T, K_l; w, p) - \bar{C}(0, T, K_l) \right\}^2,$$

and where l indexes the set of the considered strikes.
In Model 4 we have two parameters to bring model 4.1.6 closer to real process of option pricing. Similarly, as for Model 3 we still need the assumption that characteristic function exists at points $-y - \frac{i}{2}$, where $y > 0$.
Models 3 and 4 give us a class of curves which fit option prices, parameterized by w_n^* and by w_n^* and p_n^*, respectively.

4.2.5 Model 5: a Variance Gamma model with estimated parameters

The CML pricing with implied parameters of the VG distribution. For a comparison with a parametric model we also calibrate the CML pricing model, like Carr & Madan (1999), in the case of three parameters of the VG distribution.

$$(\theta^*, \sigma^*, \nu^*) = \text{argmin}_{\{\theta,\sigma,\nu\}} \sum_l \left\{ C(0, T, K_l) - \bar{C}(0, T, K_l) \right\}^2,$$

where l indexes over strikes.

To obtain prices of options we used adaptive quadrature, and numerical nonlinear Least Square minimization functions from MATLAB® Optimization Toolbox.

Remark 10 *It is common to use weighting factors, which depend on liquidity, or transformations such as logarithm or implied volatilities of prices using the non-linear Least Squares minimization in the calibration of option prices. We used non weighted non linear Least Squares minimization. Our approach may result in larger percentage errors especially for far out-of-the-money options, but those options are of less interest for investors because of their smaller liquidity. In fitting the implied parameters we used only the most liquid options.*

Chapter 5

Performance of the five models on historical data

In the present chapter we report results of fitting and calibrating of the five models introduced in Chapter 4. Let us recall that we use terms fitting or estimating when parameters of the model are chosen in the best way, according to the chosen criterion, to comply with the historical DAX log-retuns. Whenever the historical option prices are used we use the term calibration. If both historical DAX log-returns and historical option prices are used to determine the model parameters we refer to estimation (fitting) and calibration.

In Section 5.1 we describe our data sets. In Section 5.2 we present our verification of the numerical accuracy of evaluation of integrals in (4.1.6) in the project. In Section 5.3 we present results of estimation and calibration of our models. In Section 5.4 we report on pricing less liquid options using the parameters obtained from the liquid ones. In Section 5.5 we model the time behaviour of the obtained parameters by simple times series methods. We forecast the parameters of the models and price the options using the forecasted parameters. In this way we explore if the fitted parameters can be useful, eg. for market makers.

We use the following format for presenting numbers. For numbers between -0.0001 and 0.0001 we use floating point format, with four digits after the decimal point, e.g. 7.5477e-08. For other numbers we use scaled fixed point format, with four digits after the decimal point, e.g. 485.3104.

5.1 The underlying DAX index and ODAX options data

We test performance of our models on historical data consisting of Deutsche Boerse AG DAX index (XETRA: GDAXI, ISIN[1]: DE0008469008) and European Call Options ODAX (ISIN: DE0008469495) written on the index, and traded on Eurex. The options data include daily close price, strike price, and the time to maturity. The strike prices are set at 50 points space intervals. The options have been recorded on Eurex exchange between the 1st of June 2006 and the 17th of May 2007 (243 days). The number of maturities change over time and range from 1 to 6 of the closest ones to expiration. We used the data obtained from the Securities Industry Research Centre of Asia-Pacific (SIRCA Ltd., http://www.sirca.org.au). Interest rates for this period have been taken from the European Central Bank web site (http://www.ecb.int/). From the options data set we chose only the most traded options, i.e. the 3 or 4 strikes nearest to the spot price. There were 2985 such options in our data set.

5.2 Precision of the numerical integration

Prior to reporting on performance of our models we present our check of the precision of the numerical integration in (4.1.6) which we use in our project. The error of numerical integration has two components. The first error results from integrating the integrand in the formula (4.1.6) over a finite interval $[0, U]$ while in the formula the the integration region is $[0, \infty)$. The second error comes from a choice of a numerical method for integration on interval $[0, U]$. We consider these two errors separately. They do not exceed 4.0e-03 and 3.0e-08, respectively.

[1]International Securities Identification Number.

5.2.1 Integration cut-off error

By evaluation of the last integral in (A.0.3) from the cutoff point U to the infinity we can estimate the integration error $\mathrm{er}(U)$.

$$\mathrm{er}(U) \leq 2\hat{M} \int_U^\infty \frac{du}{u^2 + \frac{1}{4}} = 4\hat{M} \left(\lim_{u \to \infty} \arctan(u) - \arctan(2T) \right) = 2\hat{M}\pi - 4\hat{M} \arctan(2U),$$

where $\hat{M}$ is between 0.9998 and 1.0009 for the whole considered period.

In our numerical procedure we set $U = 512$. This gives $\mathrm{er}(512) < 0.004$ for any value of $\hat{M}$ for the whole year. This error is much larger than the error coming from the integration between zero and U. For example from Table 5.1 we can read that for Gauss-Lobatto quadrature the error does not exceed 3.0e-07. We found that for all considered options the magnitude of errors was of similar order.

5.2.2 Error of the numerical integration method

From Figure 5.2 we can spot some "outliers" among the implied p_n^*. We chosed the one the most away from the regression line. Then we checked the numerical precision of the integration method used to get this value. We chosed one option with the smallest strike, for which the price is available. For integration we used three different methods available in MATLAB®[2]: the trapezoidal rule (with spacing 0.1), the adaptive Simpson quadrature, and the adaptive Gauss-Lobatto quadrature (for calibration we used the third one). We report the results in Table 5.1. We integrate the integrand in formula (4.1.6) from 0 to the cutoff point which is one of the 32 multiples of 16, i.e. ranging from $16, 32, \ldots, 512$. The date when the option price is chosen is the 30th of June 2006, the spot price is $S_0 = 5683.31$, the strike price is $K = 5600$, the interest rate is $r = 0.0375$, the maturity is 262 days, the empirical $\hat{w}_n = -0.1251$ and the implied $p_n^* = 920.12$. The first column contains the right hand side cutoffs. The columns 2nd, 4th and 6th show prices computed with the use of the three methods for numerical integration, respectively. The columns 3rd, 5th and 7th show differences between consecutive prices.
We observe that differences between the maximum price and the minimum price do not exceed 2.0e-06 for all methods of integration, what gives a satisfactory accuracy for our

[2] trapz.m, quad.m and quadl.m, respectively.

method. Variations of errors for different levels of cutoffs are caused by the oscillatory behaviour of the ECF, and they do not exceed 1.0e-05.
For the estimation procedure for one year of options data we used the right hand side cutoff $U = 512$.

5.3 Estimation and calibration of the models

5.3.1 Measures of accuracy of option pricing for Models 1-5

In Tables 5.2-5.4 we report the following model pricing errors.

The Mean Absolute Error (MAE):

$$\frac{1}{N}\sum_{l=1}^{N}\left|\bar{C}(0,T,K_l) - \hat{C}_n(0,T,K_l;w,p)\right|,$$

the relative MAE:

$$\frac{1}{N}\sum_{l=1}^{N}\left|\bar{C}(0,T,K_l) - \hat{C}_n(0,T,K_l;w,p)\right| / |\bar{C}(0,T,K_l)|,$$

the Root Mean Square Error (RMSE):

$$\sqrt{\frac{1}{N}\sum_{l=1}^{N}\left(\bar{C}(0,T,K_l) - \hat{C}_n(0,T,K_l;w,p)\right)^2},$$

the relative RMSE:

$$\sqrt{\frac{1}{N}\sum_{l=1}^{N}\left[\left(\bar{C}(0,T,K_l) - \hat{C}_n(0,T,K_l;w,p)\right)/\bar{C}(0,T,K_l)\right]^2},$$

where N is the number of option prices depending on the considered case. For interpretation of other symbols we refer to Sections 4.1 and 4.2.

5.3.2 Examples of calibration based on one-day data

First, for illustration, we present in Figure 5.1 results of option pricing using models 1-4, for the 1st of June 2006. We used 120 prior days to calculate the ECF, i.e. we take

Cutoff	Trapez.	Diff.	Simpson	Diff.	Lob.-Gauss	Diff.
16	485.3104	-2.1702E-07	485.3104	7.5477E-08	485.3104	-2.1702E-07
32	485.3104	-1.8117E-09	485.3104	-2.3979E-07	485.3104	-1.8117E-09
48	485.3104	1.3642E-11	485.3104	-1.4278E-07	485.3104	1.3642E-11
64	485.3104	1.2551E-10	485.3104	1.9693E-07	485.3104	1.2551E-10
80	485.3104	-8.9130E-11	485.3104	-6.8918E-08	485.3104	-8.9130E-11
96	485.3104	4.6384E-11	485.3104	1.5937E-06	485.3104	4.6384E-11
112	485.3104	-1.3097E-10	485.3104	-1.6275E-06	485.3104	-1.3097E-10
128	485.3104	1.0914E-10	485.3104	1.0227E-07	485.3104	1.0914E-10
144	485.3104	1.8190E-11	485.3104	4.3383E-10	485.3104	1.8190E-11
160	485.3104	2.2944E-07	485.3104	3.7653E-09	485.3104	2.2944E-07
176	485.3104	-2.5509E-07	485.3104	-7.2694E-08	485.3104	-2.5509E-07
192	485.3104	2.2766E-08	485.3104	4.9352E-08	485.3104	2.2766E-08
208	485.3104	2.6603E-09	485.3104	1.5444E-06	485.3104	2.6603E-09
224	485.3104	2.0373E-10	485.3104	-1.4318E-06	485.3104	2.0373E-10
240	485.3104	-1.8736E-10	485.3104	-1.9573E-07	485.3104	-1.8736E-10
256	485.3104	2.0009E-11	485.3104	8.6593E-08	485.3104	2.0009E-11
272	485.3104	5.2751E-11	485.3104	1.5679E-08	485.3104	5.2751E-11
288	485.3104	1.6007E-10	485.3104	3.5028E-07	485.3104	1.6007E-10
304	485.3104	-2.8740E-10	485.3104	-3.4985E-07	485.3104	-2.8740E-10
320	485.3104	3.5780E-09	485.3104	-2.2185E-08	485.3104	3.5780E-09
336	485.3104	-2.0100E-09	485.3104	2.5951E-08	485.3104	-2.0100E-09
352	485.3104	-5.8481E-10	485.3104	-1.8847E-08	485.3104	-5.8481E-10
368	485.3104	-5.5115E-10	485.3104	-5.3848E-08	485.3104	-5.5115E-10
384	485.3104	-8.8221E-11	485.3104	6.9225E-07	485.3104	-8.8221E-11
400	485.3104	-3.7744E-10	485.3104	-6.4289E-07	485.3104	-3.7744E-10
416	485.3104	1.5916E-10	485.3104	-5.8090E-08	485.3104	1.5916E-10
432	485.3104	4.5475E-11	485.3104	1.6025E-06	485.3104	4.5475E-11
448	485.3104	9.6406E-11	485.3104	-3.1568E-07	485.3104	9.6406E-11
464	485.3104	1.1823E-11	485.3104	-1.1161E-06	485.3104	1.1823E-11
480	485.3104	-2.4374E-10	485.3104	-1.8792E-07	485.3104	-2.4374E-10
496	485.3104	2.1646E-10	485.3104	-7.8171E-09	485.3104	2.1646E-10
512	485.3104		485.3104		485.3104	

Table 5.1: Numerical integration errors and prices of option for the 30th of June 2006, where $p_n^* = 920.12$.

$\hat{\phi}_{120}(u)$ and $\hat{w}_{120}$ given by (4.1.7). The spot price was $S_0 = 5707.59$, the interest rate was $r = 0.035$. We used four maturities with 4 strikes for each maturity. Strikes are presented on the horizontal axes and prices of options are on the vertical axis. The black line is the option payoff. Rectangles denote the real ODAX option prices and circles denote the model prices. We considered the following maturities: $18, 53, 109$ and 200 days to options expirations.

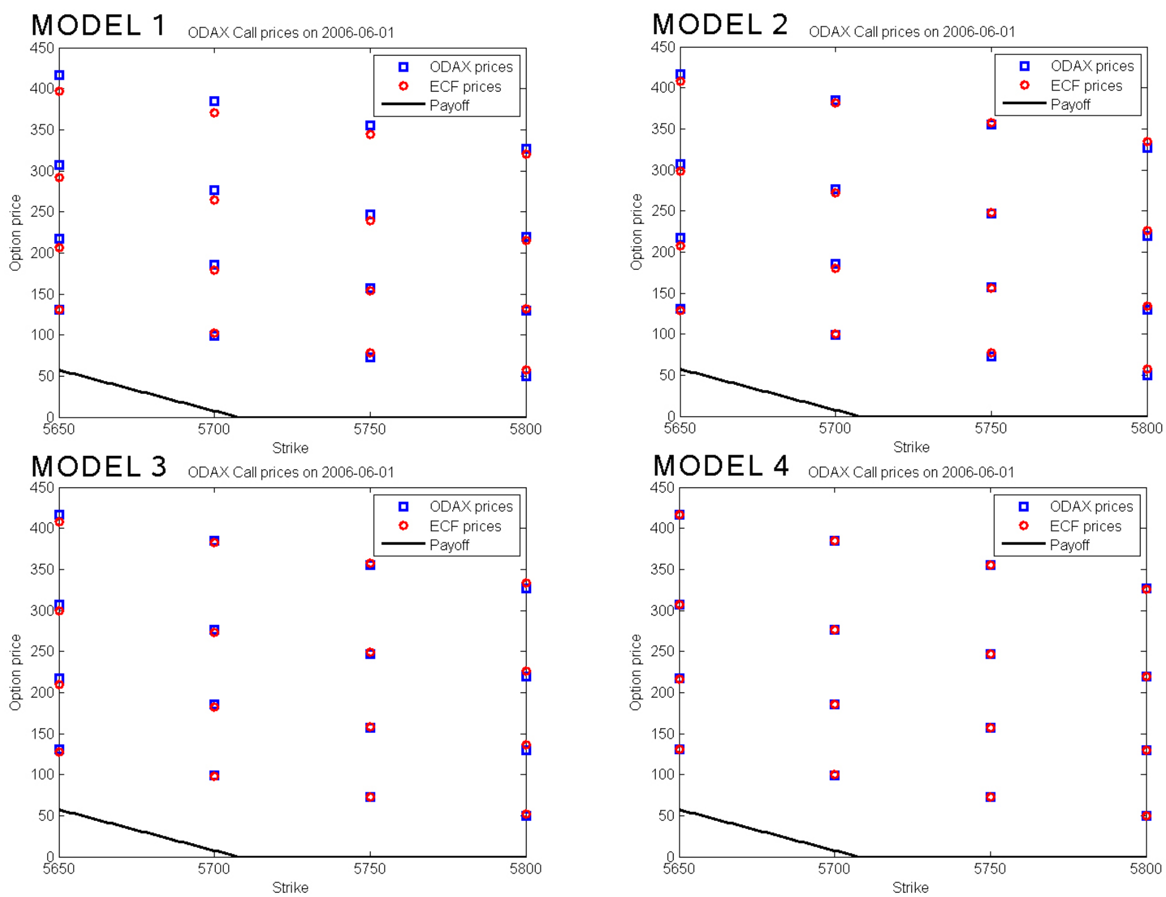

Figure 5.1: Prices of options obtained by Models 1-4, for the 1st June 2006.

In Table 5.2 we present the results of ODAX Call option pricing, the implied parameters and errors. We consider four models based on the ECF. Days to expiry p are presented in the 2nd column, the 3rd column contains values p_n^* of the implied days to expiry, whenever it is applicable. The 4th column is filled by the same value of the empirical MMCT $\hat{w}_n$, and the 5th column contains values of the implied MMCTs w_n^*. The last two columns report the relative errors for each maturity: the 6th column contains the relative MAE between the historical option prices and the ones obtained from our modelling while the 7th column contains the relative RMSE of the historical and modelled option prices. We observe decreasing errors as we calibrate more parameters.

We observe that for Model 2 the implied p_n^* are smaller than the true ones. This is not the case for Model 4, where the implied p_n^* are larger than the true ones. For Model 3 the implied w_n^* are closer to the empirical $\hat{w}_n$ than for Model 4.

5.3.3 Calibration based on one-year data

We consider all five models and one year of the ODAX Call options data. We price Call options for Model 1 for one year, calibrate and price the Call options using models 2-5. The calibration was done for each set of 3 to 4 strikes with the same maturity, between the 1st of June 2006 and the 17th of May 2007.

Table 5.3 contains measurements of all errors. In rows we present the MAE, the relative MAE, the RMSE and the relative RMSE, respectively. The number of considered Call options equals 2985. We observe the largest errors for Model 1 which is based only on a nonparametric estimation of the characteristic function of log-returns of DAX index and is not using any historical Call options for calibration. For Models 2 and 3, where we calibrate one parameter in each model, the errors of pricing are, not surpricingly, smaller than for Model 1. The difference is clearly seen in the case of the relative MAE, where the error was reduced from almost 40 percent to about 10 percent. In Model 4 the errors of Call option pricing are even smaller, the value is around 0.55 percent for the relative MAE. This precision is quite satisfactory. It is interesting to note that Model 5 of Carr & Madan (1999) with three parameters of the VG distribution calibrated to option prices does not perform better than Model 4 with two parameters. The relative MAE for Model 5 is satisfactory, but higher than the one resulting from pricing by our Model 4.

Let us recall that Lévy processes are often interpreted as subordinated Brownian Motions. It means that the time flow in the Brownian Motion can be interpreted as random with a varying speed. Behaviour of our implied parameter p_n^* given by (4.1.10) shows that there may be a component of the time flow which is not yet included in the subordinating process.

Referring to Figure 5.3 we can interpret the implied parameter w_n^* as an indicator of changes occuring to the real market environment. In particular the linear relation between implied w_n^* and empirical MMCT may be of some interest and even subject of some further study.

Figure 5.2 shows results of calibration for Model 2. Each point with coordinates (p, p_n^*) refers to one maturity: p denotes the true number of days to option expiration while p_n^* denotes the value obtained from calibration. There are 774 calibrated p_n^*. The straight line represents regression $p_n^* = \alpha_0 + \alpha_1 p$, with coefficients $\alpha_0 = 3.1510$ and $\alpha_1 = 0.8866$. It shows a linear relationship between those p that are true and those that are implied, however the figure shows heteroscedasticity of the data and outliers.

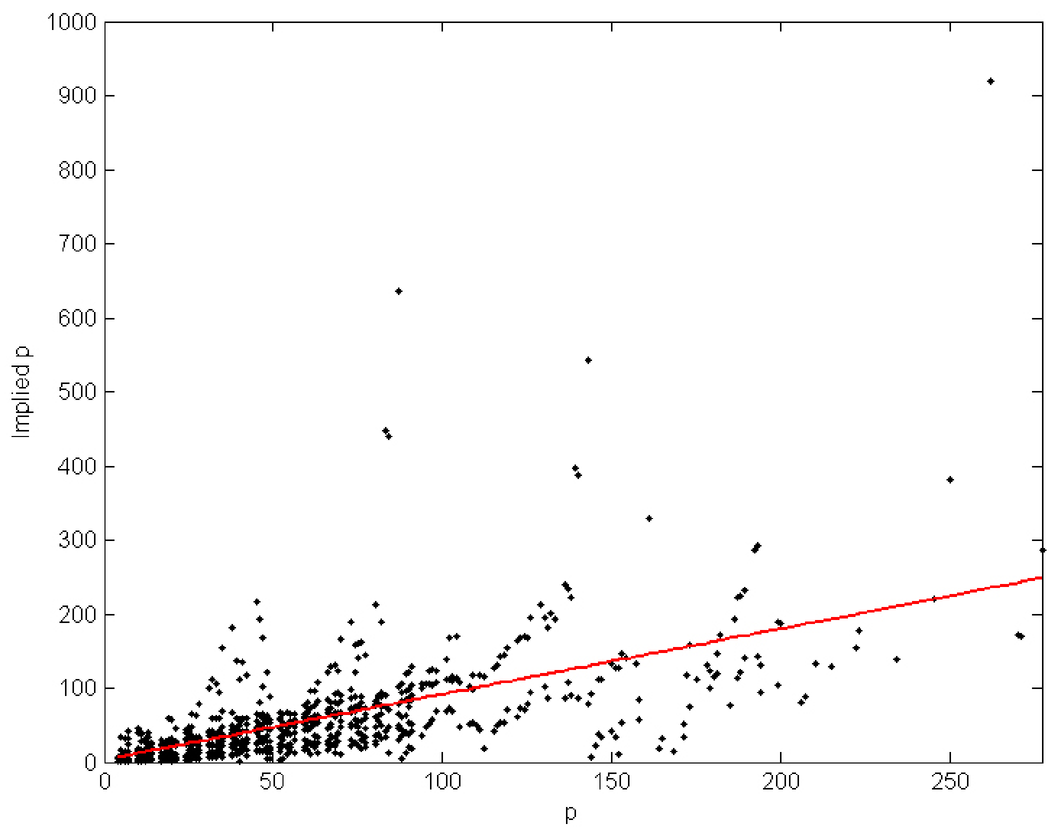

Figure 5.2: Number of days to expiration p vs. implied p_n^*, based on Model 2 for one year of pricing options.

Figure 5.3 shows results of calibration of parameter w_n^* in Model 3. It contains a plot of w_n^* versus $\hat{w}_n$. Each point represents one maturity, altogether the 774 calibrated w_n^*. The straight line represents regression $w_n^* = \alpha_0 + \alpha_1 \hat{w}_n$, with coefficients $\alpha_0 = 0.1590$ and $\alpha_1 = 1.1417$. Like in Figure 5.2 we observe heteroscedasticity in the data and outliers, as well as skewness.

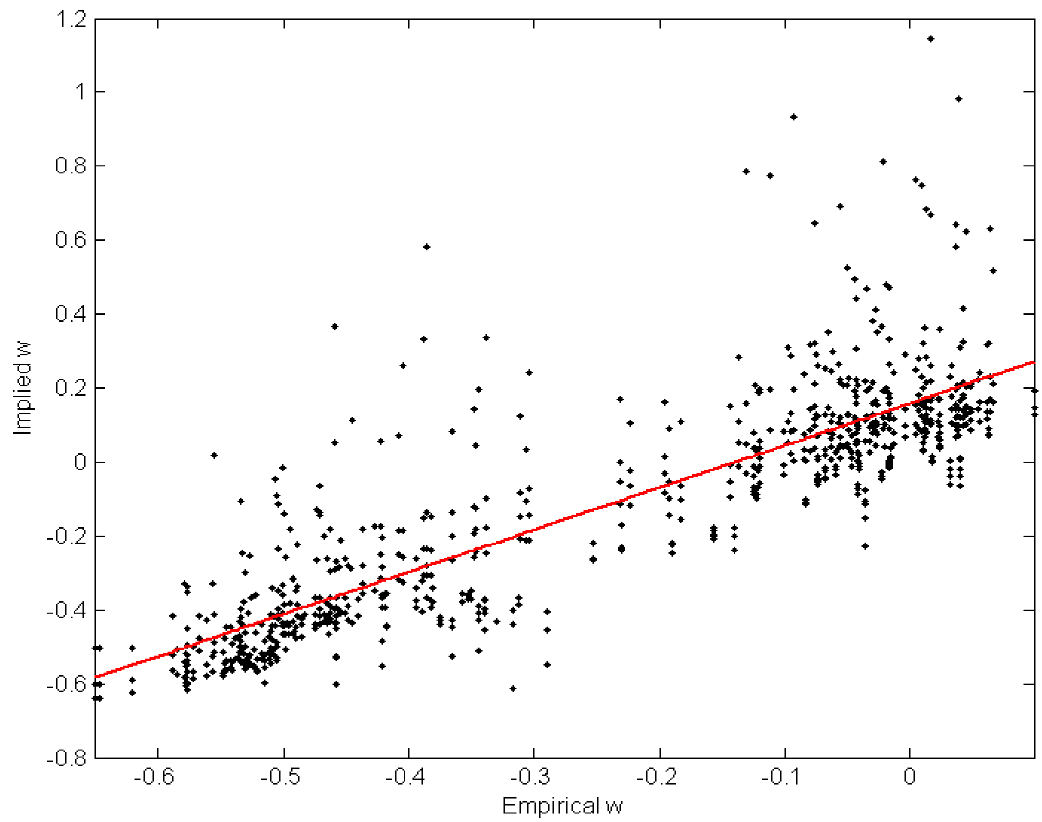

Figure 5.3: Empirical $\hat{w}_n$ vs. implied w_n^*, based on Model 3 for one year of pricing options.

We are not presenting similar plots for Model 4, for which both parameters are calibrated jointly. The relation between the empirical MMCT, the number of days to expiration and the implied parameters, is not so straightforward.

Figure 5.4 presents how the empirical $\hat{w}_{120}$ (the black bold line), the maximum (red line) and the minimum (blue line) of the implied w_n^* from Model 3 change over the year. The maximum and minimum are taken over different implied parameters related to different maturites for each day, respectively. The empirical MMCT has been used in Models 1 and 2.

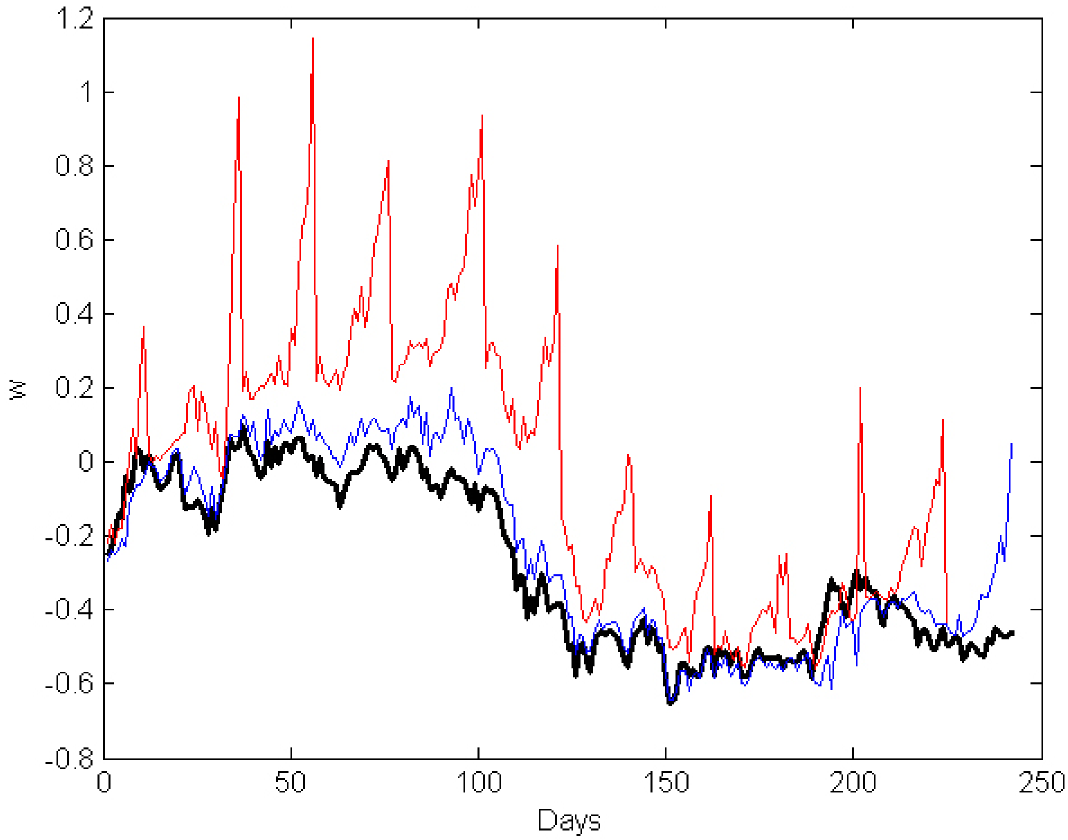

Figure 5.4: Empirical $\hat{w}_{120}(u)$ based on 120 historical log-returns between the 1st of June 2006 and the 17th of May 2007 for each day (black line) and the implied minimum (blue line) and maximum (red line) w_n^* for Model 3.

Similarly, Figure 5.5 shows the empirical $\hat{w}_{120}$ (black bold line), and the maximum (red line) and the minimum (blue line) of the implied w_n^* from Model 4. The maximum and minimum are taken over different implied parameters related to different maturites for each day, respectively.

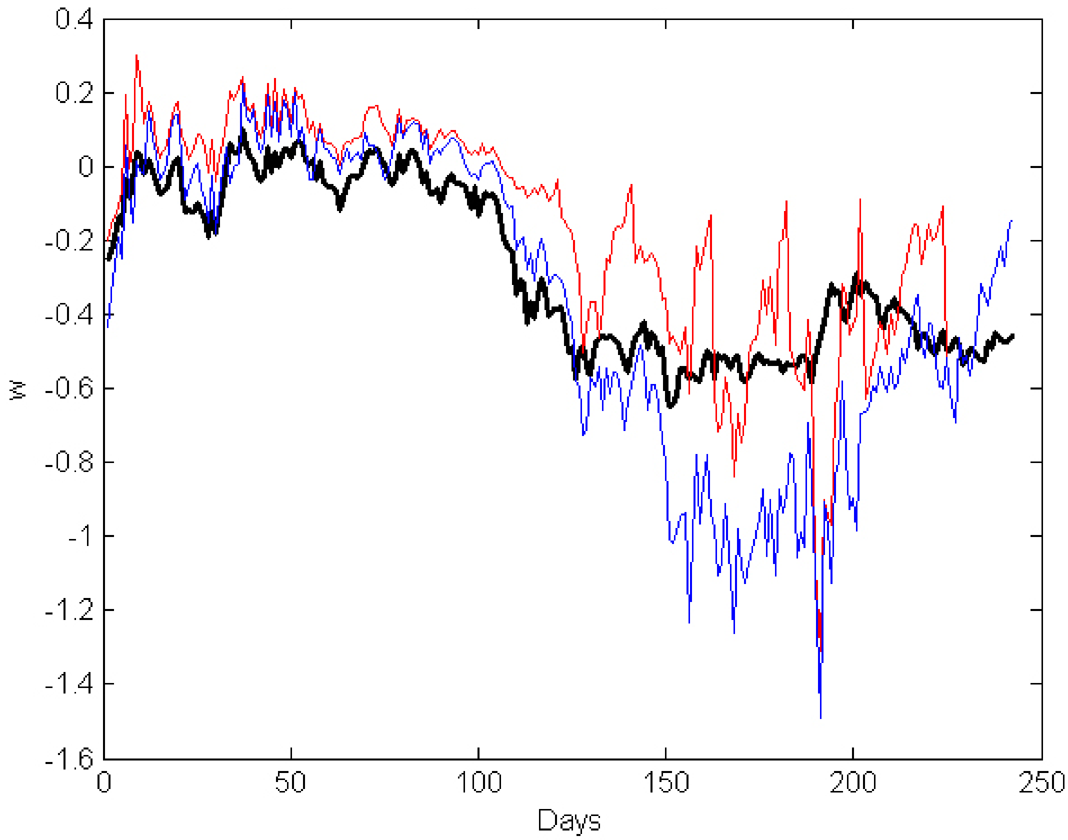

Figure 5.5: Empirical $\hat{w}_{120}(u)$ calculated from 120 historical log-returns between the 1st of June 2006 and 17th of May 2007 for each day (black line) and the minimum (blue line) and maximum (red line) of the implied parameter w_n^* for Model 4.

Figure 5.6 shows close prices of DAX between 120 working days prior to the 1st June 2006 (marked by a vertical red line) and the 17th of May 2007.

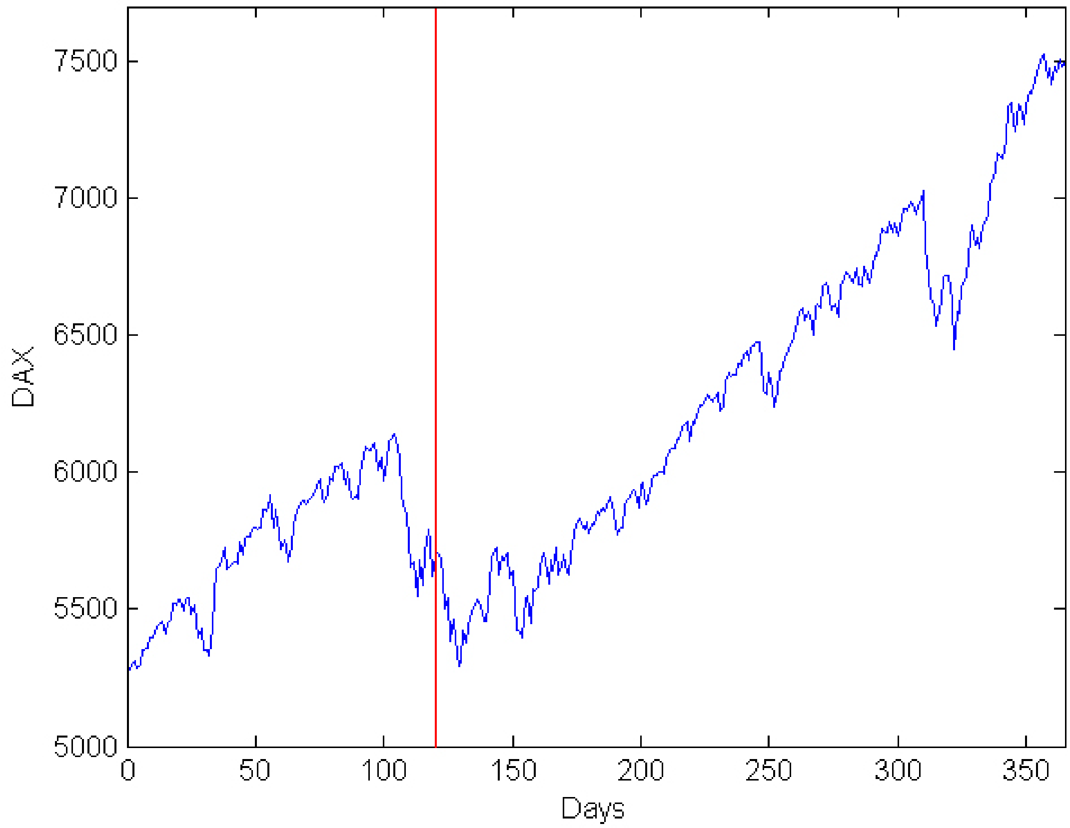

Figure 5.6: Close prices of DAX between 120 working days prior to the 1st June 2006, and 17th May 2007 (blue line). Red line indicates the 1st June 2006.

5.4 Pricing of less liquid options using implied w_n^* and p_n^* obtained from liquid cases

In this subsection we present the results of pricing options for a range of strike prices between $S_0 - 200$ and $S_0 + 200$. We used the implied parameters w_n^* and p_n^* obtained in Model 4. In table 5.4 we report the errors of option pricing. The relative error measurements increased to almost 1.9 percent for the relative MAE and to 6 percent for the relative RMSE.

Carr et al. (1998) have shown superiority of their CML model over the classical Black-Scholes approach. Our results have shown that the Model 4 outperforms slightly Model 5 based on the CML formula, cf. Table 5.3. Hence, Model 4 also outperforms the classical Black-Scholes models.

	p	p_n^*	$\hat{w}_n$	w_n^*	rel. MAE	rel. RMSE
Model 1	18	-	-0.2528	-	0.0651	0.0871
($\hat{w}_n$,p)	53	-	-0.2528	-	0.0271	0.0307
	109	-	-0.2528	-	0.0370	0.0391
	200	-	-0.2528	-	0.0328	0.0344
Model 2	18	15.4649	-0.2528	-	0.0598	0.0821
($\hat{w}_n$,p_n^*)	53	49.3644	-0.2528	-	0.0275	0.0310
	109	97.1398	-0.2528	-	0.0210	0.0238
	200	187.394	-0.2528	-	0.0151	0.0169
Model 3	18	-	-0.2528	-0.2206	0.0228	0.0263
(w_n^*,p)	53	-	-0.2528	-0.2626	0.0256	0.0292
	109	-	-0.2528	-0.2653	0.0182	0.0206
	200	-	-0.2528	-0.2618	0.0139	0.0156
Model 4	18	23.2664	-0.2528	-0.2014	0.0062	0.0065
(w_n^*,p_n^*)	53	96.9268	-0.2528	-0.3170	0.0018	0.0018
	109	215.4613	-0.2528	-0.3808	0.0007	0.0007
	200	433.6934	-0.2528	-0.4367	0.0005	0.0005

Table 5.2: Empirical and implied parameters and error measurements for Models 1-4, for pricing options on the 1st of June 2006.

	Model 1	**Model 2**	**Model 3**	**Model 4**	**Model 5**
MAE	36.7306	2.8594	5.2871	0.3306	2.3868
Relative MAE	0.3908	0.0883	0.1172	0.0055	0.0156
RMSE	44.9672	4.0897	8.7297	0.4680	3.3061
Relative RMSE	1.2103	0.9575	0.9893	0.0269	0.0307

Table 5.3: Measurement of errors for Models 1-5 for one year of pricing options.

	Model 4	ITM and OTM with w_n^* and p_n^* from Model 4
MAE	0.3306	1.2263
Relative MAE	0.0055	0.0188
RMSE	0.4680	1.872
Relative RMSE	0.0269	0.0598

Table 5.4: Measurement of errors for pricing of 3-8 options per maturity with use of w_n^* and p_n^* from Model 4, for one year of pricing options.

5.5 Pricing options with the use of forecasted implied parameters

The CML model allows using general Lévy processes to model behaviour of the asset log-returns, however is still leaving not accounted for other driving market factors like stochastic volatility. The implied parameters should capture these deviations between the CML model and market behaviour, possibly in a similar way as the implied volatility captures deviations between the BS model and the real market. Figures 5.7 and 5.8 show the behaviour of implied parameters p_n^* and w_n^* over time as obtained from calibration of Models 2 and 3, respectively.

Remark 11 *Most of the papers on option pricing concentrate only on fitting models to real option prices, cf. Carr et al. (1998), Schoutens & Tistaert (2004), without veryfying the procedures on out-of-sample data. In our case it is evident that the fitted parameters are showing some dynamics. Therefore, testing performance of the option pricing on out-of-sample data does not seem appropriate. Instead, it is desirable to suggest some time-series models for the implied parameters. By forecasting the parameters we are going even beyond the scope of the available so far papers, by trying to achieve good pricing environment for a near future, e.g. for the next day. In other words, our model performance is done in-sample but instead we suggest a model for a dynamic of parameters. We believe that this is justified by the fact that the economy changes with time and the behaviour of market is dictated by permanently incoming news and events. Hence, prediction of the future seems to be more suitable than expectation of stability, where out-of-the-sample testing should be recommended.*

In this section we explore the behaviour of the obtained implied parameters by fitting mixture of regression and Autoregressive (AR) or Vector Autoregressive (VAR) time series models. We fit these models to series of implied p_n^* (Model 2), implied w_n^* (Model 3) and pairs of implied (w_n^*, p_n^*) (Model 4). There are two reasons for exploring this modelling. Let us recall that we used only 3 or 4 of option prices to fit one (in Model 2 and 3) or two (in Model 4) parameters. By fitting the implied parameters to a time series model we can test if the obtained implied parameters have not been overfitted. Secondly, by fitting the implied parameters to a regression-time-series model we can forecast values of the model parameters one step ahead and check if the next day implied parameters are producing reasonable option pricing. This may have practical value, for example for market makers. We present results of this exploration in the following subsections.

To fit the regression-time-series models we used a statistical package EViews®. We fit series of implied p_n^* and w_n^* to the following model (cf. Quantitative-Micro-Software (2007), Chapter 26)

$$y_m = x_m'\beta + u_m, \tag{5.5.1}$$

$$u_m = \alpha_1 u_{m-1} + \alpha_2 u_{m-2} + \epsilon_m, \tag{5.5.2}$$

where $\beta' = [\beta_1, \beta_2]$ are regression parameters, α_1, α_2 are parameters of a hidden AR model driving the regression noise and x_t is a vector of explanatory variables. Let us note that model (5.5.1)-(5.5.2) can be also presented equivalently without the hidden AR componentin the following way

$$y_m = x_m'\beta + \alpha_1(y_{m-1} - x_{m-1}'\beta) + \alpha_2(y_{m-2} - x_{m-2}'\beta) + \epsilon_m,$$

however, representation (5.5.1)-(5.5.2) allows a clear interpretation. We take y_m to be either p_n^* or w_n^*, respectively. In the case of implied p_n^* we include the number of days to option expiration as an explanatory variable, and in the case of w_n^* we include the empirical MMCT as an explanatory variable.

Let us note that equation (4.1.10) shows that the implied parameter p_n^* depends on the time to option expiry

$$p_n^* = \frac{T}{\Delta_n^*}.$$

Hence, in the case of an ideal model, where Δ_n^* is constant the p_n^* is a linear function of the time to the expiry T. This justifies our use of p as a regressor in the time-series modeling. We have observed that values of obtained implied parameters w_n^* are near values of the empirical MMCT, cf. Figure 5.3. This suggest that in modeling series of implied w_n^* we should regress on the empirical MMCT.

5.5.1 Regression-time-series model for implied p_n^*

We obtained 12 series of the implied p_n^* as a result of calibration of Model 2 to one year of options data. For a given day and for a given series, each element of the series relates to 3 or 4 options with the same maturity and different strike prices. Since only the most liquid options are chosen, the set of strike prices varies over time. Each series of the implied p_n^* has different length. For example, the first series consists of 11 of the implied p_n^*, which were obtained between the 1st of June 2006 and the 15th of June 2006. At the 1st of June 2006, the time to maturity of this group of 4 options is 18 days. For the second series the time to maturity is 53 days. The latter series has been obtained for a group of 4 options between the 1st of June 2006 and the 20th of July 2006 and consists of 36 of the implied p_n^*. Some of our series of the implied p_n^* start after the 1st of June 2006. These have been obtained from calibration of Model 2 to the sets of options which replaced previously expired options. This is illustrated in Figure 5.7.

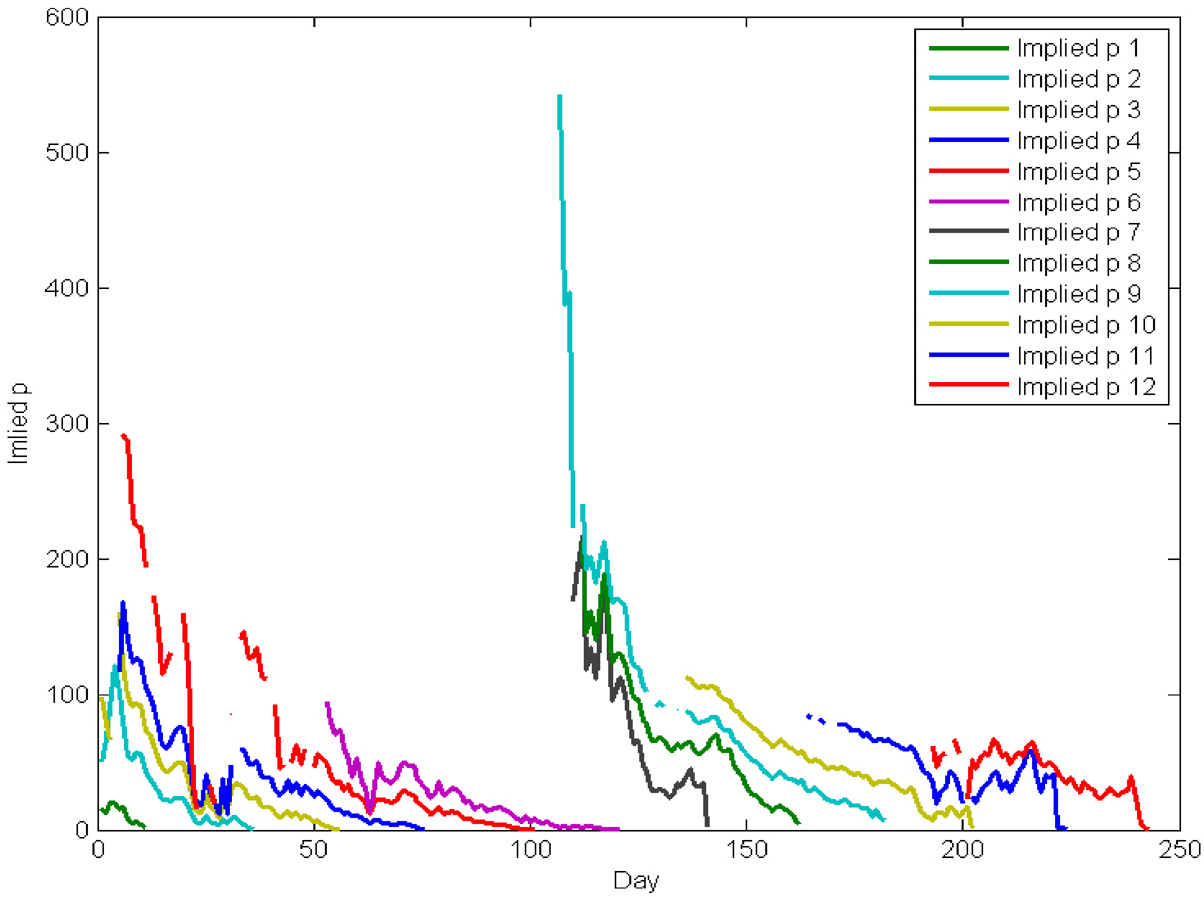

Figure 5.7: The 12 series of the implied p_n^*, obtained for Model 2 between the 1st of June 2006, and the 17th of May 2007.

For the modelling we chose the 5th series, which is the longest one, and covers the period between 2nd of June and 19th of October 2006. Let us note that series 5 seems to be the least regular in the first half, see Figure 5.7. We forecast the implied parameter p_n^f for the 2nd, 3rd and 4th of October 2006, respectively. To estimate the parameters of the regression-time-series model we used the historical data ranging from the 2nd of June 2006 until the day preceeding the forecast. We also removed 2 evident "outliers" from the series. For each of the fitted series we report in Table 5.5 the values of β (column 2) and α_1 (column 5) of the fitted coefficients of the regression part and of the time series part of the model. In columns 3-4 and 6-7 we report obtained from the EViews® package, values of the corresponding t-statistic and the p-value, for the test of significance of the coefficients β and α_1, respectively. The first column in Table 5.5 contains the number of the implied p_n^* used for estimation. The columns 8th and 9th contain values of the R^2 and the Durbin-Watson statistics for the fitted time series models, respectively. The 10th column contains values of Schwarz information criterion (BIC). The 11th column shows values of the implied p_n^* observed on a day following the estimated period. We forecast the implied parameter p_n^f for that day and report in the 12th column.

For the regression-time-series fitted models we report in Tables 5.6-5.8 values of the autocorrelations (AC), partial autocorrelations (PAC), Q-statistic and the related p-values, respectively. The first column shows the lags, the 2nd and 3rd columns show the corresponding values of AC and PAC, respectively. Column 4 shows values of Ljung-Box Q-statistic which test the null hypothesis of no serial correlation up to the given lag for residuals of the model. Column 5 shows the related P-values. Let us note that the behaviour of the autocorrelations and partial autocorrelations remained similar for lags 7-16.

5.5.2 Regression-time-series model for implied w_n^*

Similarly as for the Model 2, we obtained 12 series of the implied w_n^* as a result of calibration of Model 3 to one year of options data. For a given day and for a given series, each element of the series relates to 3 or 4 options with the same maturity and different strike prices. Since only the most liquid options are chosen, the set of strike prices varies in time. Each series of the implied w_n^* has different length. This is illustrated in Figure 5.7.

As in the case of the implied parameters from Model 2, for the modelling we chose

Series length	β	t-Stat	P-val	α_1	t-Stat	P-val	R^2	D-W	BIC	p_n^*	p_n^f
72	0.7548	6.9496	0	0.6988	8.1665	0	0.8160	1.8745	9.7528	6.2927	7.2894
73	0.7547	6.9978	0	0.6989	8.2319	0	0.8180	1.8749	9.7375	6.3992	8.4154
74	0.7546	7.0434	0	0.6991	8.2955	0	0.8200	1.8751	9.7226	4.8722	8.2604

Table 5.5: Coefficients and values of statistics of regression-time-series modelling of the 5th series of the implied p_n^* in Model 2, obtained between the 2nd of June 2006, and the 29th of September, 2nd and 3rd of October 2006, respectively. Forecasted values p_n^f for the 3 following days after these periods of time, between the 2nd and 4th of October 2006. Times to option expirations on the 2nd, 3rd and 4th of October 2006 were 21, 20 and 19 days, respectively.

Lags	AC	PAC	Q-Stat	P-value
1	0.0100	0.0100	0.0081	
2	-0.0700	-0.0700	0.3756	0.5400
3	0.0540	0.0560	0.5960	0.7420
4	-0.0400	-0.0470	0.7195	0.8690
5	-0.1240	-0.1160	1.9204	0.7500
6	-0.0050	-0.0110	1.9225	0.8600

Table 5.6: Values of autocorrelations, partial autocorrelations, Q-statistic and p-values for residuals from fitted regression-time-series model to the 5th series of the implied p_n^* in Model 2, obtained between the 2nd of June 2006 and the 29th of September 2006.

Lags	AC	PAC	Q-Stat	P-value
1	0.0100	0.0100	0.0080	
2	-0.0700	-0.0700	0.3808	0.5370
3	0.0540	0.0560	0.6039	0.7390
4	-0.0400	-0.0470	0.7290	0.8660
5	-0.1240	-0.1160	1.9446	0.7460
6	-0.0050	-0.0110	1.9467	0.8560

Table 5.7: Values of autocorrelations, partial autocorrelations, Q-statistic and p-values for residuals from fitted regression-time-series model to the 5th series of the implied p_n^* in Model 2, obtained between the 2nd of June 2006 and the 2nd of October 2006.

Lags	AC	PAC	Q-Stat	P-value
1	0.0100	0.0100	0.0079	
2	-0.0700	-0.0700	0.3866	0.5340
3	0.0540	0.0550	0.6122	0.7360
4	-0.0400	-0.0470	0.7392	0.8640
5	-0.1240	-0.1160	1.9702	0.7410
6	-0.0050	-0.0110	1.9723	0.8530

Table 5.8: Values of autocorrelations, partial autocorrelations, Q-statistic and p-values for residuals from fitted regression-time-series model to the 5th series of the implied p_n^* in Model 2, obtained between the 2nd of June 2006 and the 3rd of October 2006.

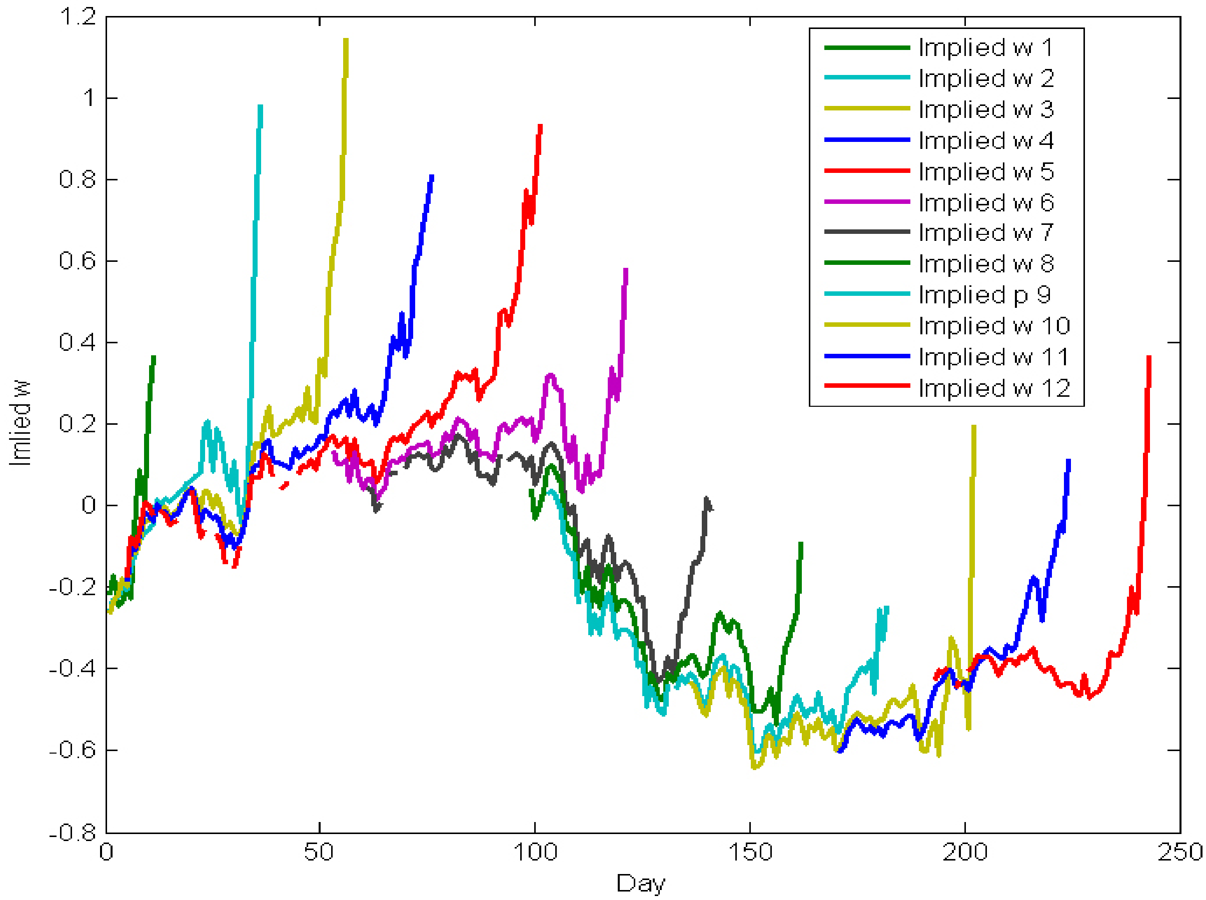

Figure 5.8: The 12 series of implied w_n^*, obtained for Model 3 between the 1st of June 2006 and the 17th of May 2007.

the 5th series, which is the longest one, and covers period between the 2nd of June and the 19th of October 2006. Let us note that series 5 seems to be the least regular in the first half, see Figure 5.8. We forecast the implied parameter w_n^f for the 2nd, 3rd and 4th of October 2006, respectively. To estimate the parameters of the regression-time-series models we used the historical data ranging from 2nd of June 2006 until the day preceeding the forecast. We removed 2 values of w_n^* which correspond to the "outliers" in the 5th series of the implied p_n^* from Model 2. For each of the fitted series we report in Table 5.9 values of β (column 2) and α_1, α_2 (columns 5 and 8) of the fitted coefficients of the regression part and the time series part of the model. In columns 3-4, 6-7 and 9-10 we report, obtained from the EViews® package, values of the corresponding t-statistic and the p-value, for the test of significance of the coefficients β and α_1, α_2, respectively. The first column in Table 5.9 contains number of the implied w_n^* used for estimation. The columns 11th and 12th contain values of the R^2 and the Durbin-Watson statistics for the fitted time series models, respectively. The 13th column contains values of the Schwarz information criterion (BIC). The 14th column shows values of the implied w_n^* observed on the day following the estimated period. We forecast the implied parameter w_n^f for that day and report them in the 15th column.

In Tables 5.10-5.12 we report values of the autocorrelation (AC), partial autocorrelation (PAC), Q-statistic and related P-value, for the regression-time-series fitted models. The first column shows the lags, the 2nd and 3rd columns show the corresponding values of AC and PAC, respectively. The 4th column shows values of Ljung-Box Q-statistic which test the null hypothesis of no serial correlation up to the given lag for residuals of the model. The 5th column shows the corresponding p-values.

Series length	β	t-Stat	P-val	α_1	t-Stat	P-val	α_2	t-Stat	P-val	R^2	D-W	BIC	w_n^*	w_n^f
72	0.7514	16.0261	0	0.7395	6.0319	0	0.2966	2.3301	0.0228	0.9776	1.9279	-5.1220	0.2924	0.2796
73	0.7518	16.2531	0	0.7164	6.1093	0	0.3229	2.6762	0.0093	0.9783	2.0149	-5.1314	0.2991	0.3048
74	0.7511	16.4075	0	0.7104	6.1919	0	0.3280	2.7654	0.0073	0.9790	2.0132	-5.1458	0.3080	0.2865

Table 5.9: Coefficients and values of statistics of regression-time-series modelling of the 5th series of the implied w_n^* in Model 3, obtained between the the 2nd of June 2006, and the 29th of September, 2nd and 3rd of October 2006, respectively. Forecasted values w_n^f for the 3 following days after these periods of time, between the 2nd and 4th of October 2006. Times to option expirations on the 2nd, 3rd and 4th of October 2006 were 21, 20 and 19 days, respectively.

Lags	AC	PAC	Q-Stat	P-value
1	-0.05	-0.05	0.1855	
2	-0.076	-0.079	0.6173	
3	-0.058	-0.067	0.8735	0.35
4	-0.057	-0.071	1.1189	0.572
5	-0.027	-0.046	1.1775	0.758
6	-0.015	-0.035	1.1944	0.879

Table 5.10: Values of autocorrelations, partial autocorrelations, Q-statistic and p-values for residuals from fitted regression-time-series model to the 5th series of the implied w_n^* in Model 2, obtained between the 2nd of June 2006 and the 29th of September 2006.

Lags	AC	PAC	Q-Stat	P-value
1	-0.0450	-0.0450	0.1468	
2	-0.0820	-0.0840	0.6513	
3	-0.0460	-0.0550	0.8151	0.3670
4	-0.0490	-0.0610	0.9992	0.6070
5	-0.0370	-0.0520	1.1043	0.7760
6	-0.0130	-0.0300	1.1170	0.8920

Table 5.11: Values of autocorrelations, partial autocorrelations, Q-statistic and p-values for residuals from fitted regression-time-series model to the 5th series of the implied w_n^* in Model 2, obtained between the 2nd of June 2006 and the 2nd of October 2006.

Lags	AC	PAC	Q-Stat	P-value
1	-0.0450	-0.0450	0.1492	
2	-0.0660	-0.0680	0.4803	
3	-0.0470	-0.0540	0.6531	0.4190
4	-0.0590	-0.0690	0.9266	0.6290
5	-0.0450	-0.0590	1.0845	0.7810
6	-0.0040	-0.0220	1.0860	0.8960

Table 5.12: Values of autocorrelations, partial autocorrelations, Q-statistic and p-values for residuals from fitted regression-time-series model to the 5th series of the implied w_n^* in Model 2, obtained between the 2nd of June 2006 and the 3rd of October 2006.

5.5.3 Regression-time-series model for implied (p_n^*, w_n^*)

Similarly, as for Models 2 and 3, we obtained 12 series of pairs of the implied (p_n^*, w_n^*) as a result of calibration of Model 4 to one year of options data. For a given day and for a given series, each element of the series relates to 3 or 4 options with the same maturity and different strike prices. Since only the most liquid options are chosen, the set of strike prices varies in time. Each series of the implied (p_n^*, w_n^*) has different length.

We fit the following mixture of the regression and 2nd order Vector Autoregressive (VAR) models (cf. Quantitative-Micro-Software (2007), Chapter 34) to the series of implied pairs (p_n^*, w_n^*) from Model 4, i.e.

$$y_m = \beta x'_m + u_m,$$
$$u_m = c + A_1 u_{m-1} + A_2 u_{m-2} + \epsilon_m,$$

where

$$A_j = \begin{pmatrix} \alpha_{1,1}^j & \alpha_{1,2}^j \\ \alpha_{2,1}^j & \alpha_{2,2}^j \end{pmatrix}, j = 1, 2,$$
$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, x'_m = \begin{pmatrix} x_{1,m} \\ x_{2,m} \end{pmatrix}, u_m = \begin{pmatrix} u_{1,m} \\ u_{2,m} \end{pmatrix} \text{ and } c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Matrices $A_j, j = 1, 2$, and β contain the VAR parameters, x_m is a vector of explanatory variables and c is a vector of constants. We take $y_{1,m}$ to be p_n^*, and $y_{2,m}$ to be w_n^*. We included the empirical MMCT $\hat{w}_n$ as an explanatory variable with coefficients (β_1, β_2).

As in the case of the implied parameters from Model 2 and 3, for the modelling we chose the 5th series, which is the longest one, and covers the period between the 2nd of

June and the 19th of October 2006. We forecast the implied parameter w_n^f for the 2nd, 3rd and 4th of October 2006, respectively. To estimate the parameters of the regression-time-series model we used the historical data ranging from the 2nd of June 2006 until the day preceeding the forecast. We removed 2 pairs of (p_n^*, w_n^*) which correspond to "outliers" in the 5th series of the implied p_n^* from Model 2.

For each of the fitted series we report in Table 5.13 values of β_1, β_2 (column 2), c (column 3), and A_1, A_2 (columns 4-7, respectively) of the fitted coefficients of the regression part and the time series part of the model. We report values of the R^2, the Durbin-Watson statistics and the Schwarz information criterion (BIC). The first column in Table 5.13 shows number of the implied pairs (p_n^*, w_n^*) used for estimation. The 8th column shows values of the R^2 statistics for the fitted time series models. The 9th column contains values of Schwarz information criterion (BIC). The 10th column shows values of the implied (p_n^*, w_n^*) observed on the day following the estimated period. We forecasted the implied parameters (p_n^f, w_n^f) for that day and reported in the 11th column.

In Tables 5.14-5.16 we report results of the Portmanteau Autocorrelation (Q-Statistic) and the Autocorrelation Lagrange Multiplier (LM) tests for the residuals. The first column shows the lags, the 2nd and the 3rd columns show values of the multivariate Ljung-Box Q-statistic and the corresponding p-value, respectively, which test the null hypothesis of no serial correlation up to the given lag for residuals of the model. The 4th and the 5th columns shows values of the multivariate LM statistic and the corresponding p-values for residual serial correlation up to the lag indicated in column 1, respectively.

In Table 5.17 we present error measurements for prices between the historical ODAX prices, the model prices, and the model prices based on the forecasted implied parameters from Models 2-4. By ODAX-MODEL we denote thr errors between the historical option prices and the model prices. By ODAX-FORECAST we denote errors between the historical option prices and the model prices with the use of the forecasted parameters. MODEL-FORECAST denotes the errors between the model prices and the model prices with the use of the forecasted parameters. The errors have been calculated for the next three days, following the last day used in the regression-time-series estimation. The 2nd row contains dates of pricing of 4 options. Rows 3-5 contain the Relative MAE and rows 6-8 contain the Relative RMSE. Row 9 shows a number of the the implied parameters used to estimate the regression-time-series model.

The errors in option pricing are the smallest for Model 4. Hence, the obtained results

Series length	β_1 β_2	c_1 c_2	$\alpha^1_{1,1}$ $\alpha^1_{1,2}$	$\alpha^1_{2,1}$ $\alpha^1_{2,2}$	$\alpha^2_{1,1}$ $\alpha^2_{1,2}$	$\alpha^2_{2,1}$ $\alpha^2_{2,2}$	R^2	BIC	p^*_n w^*_n	p^f_n w^f_n
72	314.3965	37.3477	0.8510	-0.0001	0.0529	1.3700e-05	0.9719	4.8168	11.1751	9.419111
	1.2560	0.1128	-306.6136	0.0104	-4.1301	-0.0141	0.8989		0.0636	0.0175161
73	313.8365	37.3805	0.8515	-4.57e-05	0.0522	-4.7500e-06	0.9722	4.8437	11.2898	8.314149
	1.2413	0.1137	-306.9024	0.0028	-3.7519	-0.0042	0.8935		0.083	0.0360027
74	254.4107	41.9829	-	-	0.8359	-0.0001	0.9534	5.2667	11.0082	4.975881
	1.2257	0.1141	-	-	-337.5252	0.0050	0.8873		0.1008	-0.006075

Table 5.13: Coefficients and values of statistiscs of regression-time-series modelling of the 5th series of the implied (p^*_n, w^*_n) in Model 4, obtained between the 2nd of June 2006, and the 29th of September, 2nd and 3rd of October 2006, respectively. Forecasted values (p^f_n, w^f_n) for the 3 following days after these periods of time, between the 2nd and 4th of October 2006. Times to option expirations on the 2nd, 3rd and 4th of October 2006 were 21, 20 and 19 days, respectively.

Lags	Q-Stat	P-value	LM-Stat	P-value
1	10.2372		32.2836	0
2	20.3541		10.6926	0.0302
3	22.9021	0.0001	2.6652	0.6153
4	25.9778	0.0011	2.8341	0.5860
5	28.8917	0.0041	2.9157	0.5720
6	33.1552	0.0070	4.4001	0.3546
7	34.7965	0.0212	1.8178	0.7692
8	37.1521	0.0423	2.4038	0.6619
9	39.3047	0.0761	2.1774	0.7032
10	42.9345	0.0938	4.2728	0.3703
11	45.6618	0.1298	2.7688	0.5972
12	52.1787	0.0940	7.0672	0.1324
13	55.7201	0.1107	4.1944	0.3803
14	57.5542	0.1625	1.8525	0.7629
15	62.1280	0.1588	5.5922	0.2317
16	65.3321	0.1842	3.6519	0.4552

Table 5.14: Values of Q-statistic, LM-statistic and corresponding p-values for residuals from fitted regression-time-series model to the 5th series of the implied (p_n^*, w_n^*) in Model 4, obtained between the 2nd of June 2006 and the 29th of September 2006.

Lags	Q-Stat	P-value	LM-Stat	P-value
1	11.1359		34.9831	0
2	21.5785		11.0854	0.0256
3	23.8842	0.0001	2.3922	0.6640
4	26.9155	0.0007	2.9121	0.5726
5	31.0960	0.0019	4.2606	0.3719
6	36.9686	0.0021	6.1176	0.1905
7	38.9240	0.0068	2.2664	0.6869
8	40.9088	0.0170	2.0633	0.7241
9	42.8641	0.0359	2.0889	0.7194
10	45.9702	0.0523	3.7482	0.4411
11	48.6845	0.0771	2.8051	0.5910
12	54.8023	0.0596	6.7306	0.1508
13	59.5763	0.0586	5.8057	0.2141
14	61.5690	0.0903	2.0645	0.7239
15	65.5183	0.0986	4.9670	0.2907
16	68.6158	0.1201	3.5447	0.4711

Table 5.15: Values of Q-statistic, LM-statistic and corresponding p-values for residuals from fitted regression-time-series model to the 5th series of the implied (p_n^*, w_n^*) in Model 4, obtained between the 2nd of June 2006 and the 2nd of October 2006.

Lags	Q-Stat	P-value	LM-Stat	P-value
1	18.1547		20.0245	0.0005
2	38.9111		24.2917	0.0001
3	42.6167	0	3.6755	0.4517
4	46.7368	0	4.2452	0.3738
5	51.0351	0	4.6293	0.3275
6	55.7837	0	5.2149	0.2659
7	56.2007	0	0.3883	0.9834
8	58.6200	0.0001	2.9294	0.5697
9	59.6707	0.0004	1.0245	0.9061
10	65.1094	0.0005	6.5523	0.1615
11	66.9862	0.0013	2.1225	0.7132
12	69.7115	0.0025	3.1951	0.5257
13	75.0488	0.0024	6.5871	0.1594
14	77.1866	0.0048	2.6404	0.6197
15	81.3218	0.0058	4.9179	0.2958
16	85.8270	0.0063	6.0232	0.1974

Table 5.16: Values of Q-statistic, LM-statistic and corresponding p-values for residuals from fitted regression-time-series model to the 5th series of the implied (p_n^*, w_n^*) in Model 4, obtained between the 2nd of June 2006 and the 3rd of October 2006.

	Model 2			Model 3			Model 4		
Forecasted date	2nd Oct	3rd Oct	4th Oct	2nd Oct	3rd Oct	4th Oct	2nd Oct	3rd Oct	4th Oct
R-MAE ODAX-FORECAST	0.0695	0.1248	0.2181	0.6371	0.7111	0.6593	0.0115	0.0585	0.1354
R-MAE ODAX-MODEL	0.0440	0.0477	0.0707	0.6424	0.7085	0.6695	0.0033	0.0046	0.0074
R-MAE MODEL-FORECAST	0.0486	0.1026	0.1781	0.0155	0.0066	0.0252	0.0113	0.0584	0.1350
R-RMSE ODAX-FORECAST	0.0977	0.1588	0.2698	0.7584	0.8512	0.8003	0.0132	0.0599	0.1386
R-RMSE ODAX-MODEL	0.0563	0.0610	0.0922	0.7649	0.8480	0.8131	0.0035	0.0049	0.0080
R-RMSE MODEL-FORECAST	0.0509	0.1079	0.1863	0.0211	0.0083	0.0328	0.0146	0.0595	0.1381
Estimated series length	72	73	74	72	73	74	72	73	74

Table 5.17: Relative MAE and relative RMSE for the 2nd, 3rd and 4th of October 2006 between historical ODAX prices, model prices and model prices with the use of forecasted parameters for Models 2-4. The forecast is based on the 5th series of pairs of the implied (p_n^*, w_n^*), obtained between the 2nd of June 2006 and the 29th of September, the 2nd and the 3rd of October 2006, respectively. Times to option expirations on the 2nd, 3rd and 4th of October 2006 were 21, 20 and 19 days, respectively. For the each day 4 options has been priced.

may indicate that the Model 4 is the best one, however the number of the forecasted days used in our preliminary study is too small to make this conclusion reliable. Definitely further study in this direction is needed, yet the approach looks very promising.

Chapter 6

Conclusions

We introduced four nonparametric models for pricing of European options. The first model, Model 1, requires only historical log-returns of the underlying price process. The other three models need, in addition, real option prices to calibrate implied parameters. In some of the cases (for some days) the first model gives prices with relatively small error (around 5 percent), but in most other cases the calibration of parameters w and p results in a more accurate option pricing. Models 2 and 3 with the implied parameters p_n^* and w_n^*, respectively, perform in general much better than Model 1, but still the relative errors remain on level between 8 to 12 percent. The Model 4, with two implied parameters p_n^* and w_n^*, shows the smallest errors which are not larger than 3 percent. Our Model 4 outperforms Model 5, which is based on the original Carr & Madan (1999) parametric approach.

So far our Model 4 with two implied parameters p_n^* and w_n^* results in the best option pricing. This shows that the ECF captures more information about the distribution of the underlying price process than the parametric model assuming a VG distribution of the asset log-returns.

The CML model and our derived Models 2-4 do not model explicitly the stochastic volatility and, possibly, other factors not included into the CML model. Therefore, it is of interest to try to understand the behaviour of the implied parameters p_n^* and w_n^*, respectively. We report on our preliminary exploration of this behaviour and on modelling it by regression-time-series models using package EViews®. We have fitted the series of the obtained implied parameters to a mixture of regression and Autoregressive (Vector Autoregressive in the case of Model 4) time series models and next we priced options based on the forecasted parameters. Clearly, the procedure does not reproduce the same errors as the original calibration. Our preliminary results using Model 4 look encouraging. Definitely further research and exploration in this direction is needed. With the results of pricing using forecasted implied parameters for 3 days only, it is impossible to come with firm conclusions. However, this exploratory research suggest that there is a good prospect for modelling of the implied p_n^* and w_n^* hence offering a quick and computationaly inexpensive method useful e.g. for market makers.

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Appendix A

Proofs

For the clarity of exposition we collected all proofs in the present Appendix.

Proof of equation (2.1.2)

For any positive integer $n, m \in \mathbb{N}$ we have $\phi_{X_1}(u) = \Big(\phi_{X_{\frac{1}{n}}}(u)\Big)^n$ and $\phi_{X_m}(u) = \Big(\phi_{X_1}(u)\Big)^m$, where we use increments of size $\frac{1}{n}$ and 1 respectively. Hence, $\phi_{X_{\frac{m}{n}}}(u) = \Big(\phi_{X_{\frac{1}{n}}}(u)\Big)^m = \Big(\phi_{X_1}(u)\Big)^{\frac{m}{n}}$. For any irrational number $t > 0$ we can find a sequence of rational numbers t_k, such that $t = \lim_{k\to\infty} t_k$, then

$$\phi_{X_t}(u) = \lim_{k\to\infty} \phi_{X_{t_k}}(u) = \lim_{k\to\infty} \Big(\phi_{X_1}(u)\Big)^{t_k} = \Big(\phi_{X_1}(u)\Big)^t.$$

The first equality holds under assumption of stochastic continuity on the process X_t.

Proof of Proposition 1.

We derive the drift and Lévy measure of the VG process using a subordination approach. The generating triplet of the subordinated Brownian motion with drift can be obtained from Theorem 8 in Appendix B.2 (Sato (1999), Theorem 30.1). Since drift of the Gamma process is zero, by (B.2.1) we have $\sigma_Z^2 = 0$. Since

$$\int_{-1}^{1} x e^{-\frac{x^2}{2t}} dx = t(e^{-\frac{1}{2t}} - e^{-\frac{1}{2t}}) = 0,$$

the drift of the VG process given by (B.2.3) equals

$$m = 0 \cdot \theta + \int_0^\infty \frac{e^{-\frac{t}{\nu}}}{\nu t} \left(\int_{-1}^1 x \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \right) dt = 0.$$

Let $l_{\gamma_t}(y)$ be the Lévy density of the subordinating Gamma process (2.1.9), and let $f_{BM}(x)$ be the probability density function of increments of the subordinated BM with drift, i.e. $\theta t + \sigma W_t$. The Lévy triplet of the BM with drift θ is given by $(\sigma^2, 0, \theta)$. The Lévy density $l_X(x)$ of the VG process is given by (B.2.2), i.e.

$$l_X(x) = \int_0^\infty f_{BM}(x) l_{\gamma_t}(y) dy = \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma\sqrt{y}} e^{-\frac{(x-\theta y)^2}{2\sigma^2 y}} \frac{y^{-1} e^{-\frac{y}{\nu}}}{\nu} dy.$$

Then by (B.1.2), for $a = -\frac{1}{2}$, $b = \frac{x^2}{2\sigma^2}$ and $c = \frac{\theta^2}{2\sigma^2} + \frac{1}{\nu}$, we get

$$l_X(x) = \frac{2e^{\frac{\theta x}{\sigma^2}}}{\nu\sqrt{2\pi}\sigma} \left(\frac{x^2}{2\frac{\sigma^2}{\nu} + \theta^2} \right)^{-\frac{1}{4}} K_{-\frac{1}{2}} \left(\frac{\sqrt{x^2(2\frac{\sigma^2}{\nu} + \theta^2)}}{\sigma^2} \right).$$

By (B.1.3) we have

$$l_X(x) = \frac{\exp\frac{\theta x}{\sigma^2}}{\nu|x|} \exp\left(-\frac{\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}}}{\sigma} |x| \right). \tag{A.0.1}$$

Since $\sigma_Z^2 = 0$, conditions 2.1.11 of Theorem 2 are satisfied we get

$$\int_{-1}^1 l_X(x) dx = \int_{-1}^1 \frac{\exp\frac{\theta x}{\sigma^2}}{\nu} \exp\left(-\frac{\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}}}{\sigma} |x| \right) dx < \infty,$$

hence we can conclude that the VG process has finite variation. We can write the cumulant function as

$$\psi_{X_t}(u) = \int_{-\infty}^\infty (e^{iux} - 1 - iux 1_{|x|\leq 1}(x)) l_X(dx) = iu\mu + \int_{-\infty}^\infty (e^{iux} - 1) l_X(dx).$$

Finally, we note that we can write the Lévy density of the VG process in the following way

$$l_X(x) = \frac{1}{\nu} \Big[\mathbf{1}_{\{x<0\}}(x) \exp\left(\frac{\theta + \sqrt{2\frac{\sigma^2}{\nu} + \theta^2}}{\sigma^2} x \right)$$
$$+ \mathbf{1}_{\{x>0\}}(x) \exp\left(\frac{\theta - \sqrt{2\frac{\sigma^2}{\nu} + \theta^2}}{\sigma^2} x \right) \Big] |x|^{-1}.$$

Proof of Proposition 2.

If the conditional distribution of X given $Y = y$ is $N(\theta y, \sigma^2 y)$ and r.v. Y has a Gamma pdf $g_{\frac{1}{\nu},\frac{1}{\nu}}(y)$ (cf. (2.1.8)), then the marginal probability distribution function of X is given by

$$f_X(x) = \int_0^\infty f_{X|Y}(x,y) f_Y(y) dy = \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma\sqrt{y}} e^{-\frac{(x-\theta y)^2}{2\sigma^2 y}} \frac{y^{\frac{1}{\nu}-1} e^{-\frac{y}{\nu}}}{\nu^{\frac{1}{\nu}}\Gamma(\frac{1}{\nu})} dy.$$

Following Carr et al. (1998) we use relation (B.1.2) in the Appendix with $a = \frac{1}{\nu} - \frac{1}{2}$, $b = \frac{x^2}{2\sigma^2}$ and $c = \frac{\theta^2}{2\sigma^2} + \frac{1}{\nu}$, to rewrite the density in terms of a modified Bessel function of the second kind $K_a(\cdot)$, i.e.

$$f_X(x) = \frac{2e^{\frac{\theta x}{\sigma^2}}}{\nu^{\frac{1}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{1}{\nu})} \left(\frac{x^2}{2\frac{\sigma^2}{\nu} + \theta^2} \right)^{\frac{1}{2\nu}-\frac{1}{4}} K_{\frac{1}{\nu}-\frac{1}{2}} \left(\frac{\sqrt{x^2(2\frac{\sigma^2}{\nu} + \theta^2)}}{\sigma^2} \right).$$

Similarly, for increments $X_{t+h} - X_t$, for any $h > 0$ and $t > 0$, the pdf is

$$\begin{aligned} f_{X_{t+h}-X_t}(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma\sqrt{y}} e^{-\frac{(x-\theta y)^2}{2\sigma^2 y}} \frac{y^{\frac{h}{\nu}-1} e^{-\frac{y}{\nu}}}{\nu^{\frac{h}{\nu}}\Gamma(\frac{h}{\nu})} dy \\ &= \frac{2e^{\frac{\theta x}{\sigma^2}}}{\nu^{\frac{h}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{h}{\nu})} \left(\frac{x^2}{2\frac{\sigma^2}{\nu} + \theta^2} \right)^{\frac{h}{2\nu}-\frac{1}{4}} K_{\frac{h}{\nu}-\frac{1}{2}} \left(\frac{\sqrt{x^2(2\frac{\sigma^2}{\nu} + \theta^2)}}{\sigma^2} \right). \end{aligned}$$

Using relation (B.1.4) we obtain tail behaviour of the density function

$$f_{X_{t+h}-X_t}(x) = \frac{\left(2\frac{\sigma^2}{\nu} + \theta^2\right)^{-\frac{h}{2\nu}}}{\nu^{\frac{h}{\nu}}\Gamma(\frac{h}{\nu})} |x|^{\frac{h}{\nu}-1} e^{\frac{\theta \mp \sqrt{2\frac{\sigma^2}{\nu}+\theta^2}}{\sigma^2} x} + o(1), \text{ as } x \to \pm\infty,$$

where $\alpha = \sqrt{\frac{\theta^2\nu + 2\sigma^2}{\nu\sigma^4}}$ and $\beta = \frac{\theta}{\sigma^2}$. From the above equation we see that the rates of decrease of the pdf are power-modified exponential.

The above result is stated in more general settings, for Normal Variance-Mean mixture distributions in Barndorff-Nielsen et al. (1982).

Proof of Proposition 3.

The characteristic function of the VG process can be calculated using properties of conditional expectations. For $t > 0$ we denote

$$X_t = \theta Y_t + \sigma\sqrt{Y_t} Z,$$

where Y_t is a random variable Gamma associated with a random time change independent of Z, and $Z \approx N(0,1)$. We have

$$\begin{aligned} E\left(e^{iuX_t}\right) &= E\left(e^{iu(\theta Y_t+\sigma\sqrt{Y_t}Z)}\right) = E\left(E\left(e^{iu(\theta Y_t+\sigma\sqrt{Y_t}Z)}|Y_t\right)\right) \\ &= E\left(e^{(iu\theta-\frac{\sigma^2u^2}{2})Y_t}\right) = \phi_{Y_t}\left(\theta u + i\frac{\sigma^2u^2}{2}\right) = \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-\frac{t}{\nu}}. \end{aligned}$$

The characteristic function exists in a complex strip $A_X = \{z \in \mathbf{C}|\mathrm{Im}(z) \in (a,b)\}$, such that $|\phi_{X_t}(u)| < \infty$ for $u \in A_X$. We have

$$\begin{aligned} \left|\phi_{X_t}(u)\right| &= \left|Ee^{iuX_t}\right| \le E\left|e^{iuX_t}\right| = Ee^{-\mathrm{Im}(u)X_t} = \phi_{X_t}(i\mathrm{Im}(u)) \\ &= \left(1 + \mathrm{Im}(u)\theta\nu - \frac{1}{2}\sigma^2\nu(\mathrm{Im}(u))^2\right)^{-\frac{t}{\nu}}. \end{aligned}$$

The last expression if finite if

$$1 + \mathrm{Im}(u)\theta\nu - \frac{1}{2}\sigma^2\nu(\mathrm{Im}(u))^2 > 0,$$

which is satisfied for

$$\frac{\theta}{\sigma^2} - \sqrt{\frac{\theta^2\nu + 2\sigma^2}{\nu\sigma^4}} < \mathrm{Im}(u) < \frac{\theta}{\sigma^2} + \sqrt{\frac{\theta^2\nu + 2\sigma^2}{\nu\sigma^4}}.$$

Both characteristic functions of Normal and Gamma distributions are analytical in a neighborhood of the origin, hence by Theorem 9 in Appendix B.2 the characteristic function of the VG process is analytical in the complex strip A_X.

Proof of Proposition 5.

By (3.1.7) we have

$$\phi^{\mathbf{Q}}_{X_T}(u) = \frac{\phi_{X_t}(u - i\theta)}{\phi_{X_t}(-i\theta)}$$

and by the Lévy-Khintchine representation this equals to

$$\begin{aligned} &\frac{\exp\left(t\left[im(u-i\theta) - \frac{\sigma^2(u-i\theta)^2}{2} + \int_{-\infty}^{\infty}(e^{i(u-i\theta)x} - 1 - i(u-i\theta)x1_{|x|\le1})L(dx)\right]\right)}{\exp\left(t\left[m\theta + \frac{\sigma^2\theta^2}{2} + \int_{-\infty}^{\infty}(e^{\theta x} - 1 - \theta x1_{|x|\le1})\mathbf{L}(dx)\right]\right)} \\ &= \exp\left(t\left[imu - \frac{\sigma^2u^2}{2} + i\theta\sigma^2u + \int_{-\infty}^{\infty}e^{\theta x}(e^{iux} - 1)\mathbf{L}(dx) - \int_{-\infty}^{\infty}iux1_{|x|\le1})\mathbf{L}(dx)\right]\right). \end{aligned}$$

Hence, we have

$$\phi^{\mathbf{Q}}_{X_T}(u) = \exp\Big(t\big[i(m + \theta\sigma^2 + \int_{-\infty}^{\infty}(e^{\theta x} - 1)iux1_{|x|\leq 1}\mathbf{L}(dx))u - \frac{\sigma^2 u^2}{2} + \int_{-\infty}^{\infty}(e^{iux} - 1 - iux1_{|x|\leq 1})e^{\theta x}\mathbf{L}(dx)\big]\Big).$$

and the required triplet shows from the above representation.

Proof of Proposition 6

Process $e^{-rt}S_t$ is a martingale if condition

$$S_0 = E^{\mathbf{Q}}(e^{-rt}S_t|\mathbf{F}_0)$$

is satisfied. Since

$$E^{\mathbf{Q}}(e^{-rt}S_t|\mathbf{F}_0) = S_0 e^{wt}E^{\mathbf{Q}}(e^{X_t}) = S_0 e^{wt}\phi_{X_t}(-i),$$

and by (3.1.5) we have

$$e^{-wt} = \phi_{X_t}(-i).$$

Proof of Lemma 4

We have

$$e^{\frac{1}{2}c^2}EH(Z+c) = e^{\frac{1}{2}c^2}\int_{-\infty}^{\infty}H(z+c)f(z)dz$$

$$= e^{\frac{1}{2}c^2}\int_{-\infty}^{\infty}H(y)f(y-c)dy = \int_{-\infty}^{\infty}e^{cy}H(y)f(y)dy = E(e^{cZ}H(Z)),$$

where $f(z)$ is a pdf of Z. We used have substitution $y = z + c$, and the following relation

$$f(y-c) = e^{-\frac{1}{2}c^2 + cy}f(y).$$

Proof of Proposition 7.

The price of the European call option can be derived in the following way.

$$C(0,T,K) = e^{-rT}E(S_{\gamma_T} - K)^+ = e^{-rT}E\left[E[(S_{\gamma_T} - K)^+|\gamma_T]\right]$$
$$= e^{-rT}E\left[E[(S_0 e^{rT + \theta\gamma_T + \sigma W_{\gamma_T} + wT} - K)\mathbf{1}_{\{S_{\gamma_T} - K > 0\}}|\gamma_T]\right].$$

Conditioning on γ_T, W_{γ_T} is $N(0,\gamma_T)$ and we have

$$
\begin{aligned}
&= e^{-rT}E\left[E[(S_0e^{rT+\theta\gamma_T+\sigma\sqrt{\gamma_T}Z+wT}-K)\mathbf{1}_{\{S_0e^{rT+\theta\gamma_T+\sigma\sqrt{\gamma_T}Z+wT}>K\}}|\gamma_T]\right]\\
&= e^{-rT}E\Bigg[E[S_0e^{rT+\theta\gamma_T+\sigma\sqrt{\gamma_T}Z+wT}\mathbf{1}_{\left\{Z>\frac{\log\frac{K}{S_0}-(r+w)T-\theta\gamma_T}{\sigma\sqrt{\gamma_T}}\right\}}|\gamma_T\Bigg]\\
&\qquad -E\Bigg[K\mathbf{1}_{\left\{Z>\frac{\log\frac{K}{S_0}-(r+w)T-\theta\gamma_T}{\sigma\sqrt{\gamma_T}}\right\}}|\gamma_T]\Bigg]\\
&= e^{-rT}E\left[E[S_0e^{rT+\theta\gamma_T+\sigma\sqrt{\gamma_T}Z+wT}\mathbf{1}_{\{Z>-d_2\}}|\gamma_T]-E[K\mathbf{1}_{\{Z>-d_2\}}|\gamma_T]\right],
\end{aligned}
$$

where

$$d_2=\frac{\log\frac{S_0}{K}+(r+w)T+\theta\gamma_T}{\sigma\sqrt{\gamma_T}}.$$

Applying B.2.4 in Lemma 4 to the first conditional expectation, we have

$$
\begin{aligned}
&= e^{-rT}E\left[S_0e^{rT+\theta\gamma_T+\frac{1}{2}\sigma^2\gamma_T+wT}E[\mathbf{1}_{\{Z+\sigma\sqrt{\gamma_T}>-d_2\}}|\gamma_T]-E[K\chi\{Z>-d_2\}|\gamma_T]\right]\\
&= e^{-rT}E\left[S_0e^{(r+w)T+\theta\gamma_T+\frac{1}{2}\sigma^2\gamma_T}E[\mathbf{1}_{\{Z\}>-d_1\}}|\gamma_T]-E[K\mathbf{1}_{\{Z>-d_2\}}|\gamma_T]\right]\\
&= e^{-rT}E\left[S_0e^{(r+w)T+(\theta+\frac{1}{2}\sigma^2)\gamma_T}F_N(d_1|\gamma_T)-KF_N(d_2|\gamma_T)\right]\\
&= \int_0^\infty\left(S_0e^{wT+(\theta+\frac{1}{2}\sigma^2)y}F_N(d_1|y)-e^{-rT}KF_N(d_2|y))\right)f_\gamma(y)dy,
\end{aligned}
$$

where

$$d_1=d_2+\sigma\sqrt{\gamma_T},$$

Z is a standard Normal variable, $F_N(d|y)=E(1_{Z<d}|Y_T=y)$ is the conditional cdf, which is a cdf of the Normal distribution with variance y and where $w=\frac{1}{\nu}\log(1-\theta\nu-\frac{1}{2}\sigma^2\nu)$.

Proof of Theorem 7.

Since the Fourier transform $\hat{H}(z)$ exists in some strip A_H (cf. Definition 2), by the inversion formula for $z\in A_H$ we have for every ν such that $u+i\nu\in A_H$ for every $y\in\mathbb{R}$

$$H(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i(y+i\nu)x}\hat{H}(y+i\nu)dy.$$

By (3.2.1) and since $\mathbf{F}_0$ is a trivial σ-field

$$C(0,T,K)=e^{-rT}E\left[(e^{\log(S_T)}-K)^+\right]=e^{-rT}E\left[H(\log(S_T))\right]$$

and by the inverse Fourier transform of $\hat{H}(z)$, we have

$$= \frac{e^{-rT}}{2\pi} E\left[\int_{-\infty}^{+\infty} e^{-i(y+i\nu)\log S_T} \hat{H}(y+i\nu)dy\right]$$
$$= \frac{e^{-rT}}{2\pi} \int_{-\infty}^{+\infty} e^{-i(y+i\nu)(\log S_0+rT)} E\left[e^{-i(y+i\nu)(X_T+wT)}\right] \hat{H}(y+i\nu)dy.$$

By the Fourier transform of the probability distribution of $Y_T = X_T + wT$ we get that this equals

$$= \frac{e^{-rT}}{2\pi} \int_{-\infty}^{\infty} e^{-i(y+i\nu)(\log S_0+rT)} \phi_{Y_T}(-(y+i\nu))\hat{H}(y+i\nu)dy,$$

where $y + i\nu \in A_H \cap A_Y^*$.

Proof of Corollary 3.

The payoff function of the European Call option is given by $H(x) = (e^x - K)^+$. The Fourier transform of $H(x)$ exists for $z \in A_H = \{z|\mathrm{Im}(z) > 1\}$ and is given by

$$\hat{H}(z) = \int_{-\infty}^{\infty} e^{izx}(e^x - K)^+ dx$$
$$= \lim_{x\to\infty}\left(\frac{e^{(iz+1)x}}{iz+1} - K\frac{e^{izx}}{iz}\right) - \left(\frac{e^{(iz+1)\log K}}{iz+1} - K\frac{e^{iz\log K}}{iz}\right) = -\frac{K^{iz+1}}{z^2 - iz}.$$

We assume that the price process is given by

$$S_t = S_0 e^{rt+X_t+wt},$$

where w is determined by the martingale condition, so that the discounted process $e^{-rt}S_t$ is a martingale. By Theorem 7 the European Call option price is given by

$$C(0,T,K) =$$
$$-\frac{Ke^{-rT}}{2\pi} \int_{-\infty}^{+\infty} e^{-i(x+i\nu)(\log\frac{S_0}{K}+rT+wT)} \phi_{X_T}(-(x+i\nu)) \frac{dx}{(x+i\nu)^2 - i(x+i\nu)},$$

where $x + i\nu \in A_H \cap A_X^* = \{z : \mathrm{Im}(z) \in (1, \alpha - \beta)\}$. The integration is along the real line in the complex plane, i.e. $\mathbb{R} + i\nu$. We write the integrand as

$$-\frac{Ke^{-rT}}{2\pi} e^{-iz(\log\frac{S_0}{K}+rT+wT)} \phi_{X_T}(-z) \frac{1}{z^2 - iz},$$

where $z \in A_H \cap A_X^*$. The integrand has simple poles at 0 and i. The residue at i is given by

$$\lim_{z\to i}\left[(z-i)(-\frac{Ke^{-rT}}{2\pi}e^{-iz(\log\frac{S_0}{K}+rT+wT)}\frac{\phi_{X_T}(-z)}{z^2-iz})\right] = -\frac{S_0}{2\pi i},$$

and by the Residue Theorem (cf. Karunakaran (2005)) for $\nu_1 \in (0,1)$ we have

$$C(0,T,K) =$$

$$-\frac{Ke^{-rT}}{2\pi}\int_{-\infty}^{+\infty} e^{-i(x+i\nu_1)(\log\frac{S_0}{K}+rT+wT)}\phi_{X_T}(-(x+i\nu_1))\frac{dx}{(x+i\nu_1)^2-i(x+i\nu_1)} - 2\pi i(-\frac{S_0}{2\pi i})$$

$$= S_0 - \frac{Ke^{-rT}}{2\pi}\int_{-\infty}^{+\infty} e^{-i(x+i\nu_1)(\log\frac{S_0}{K}+rT+wT)}\phi_{X_T}(-(x+i\nu_1))\frac{dx}{(x+i\nu_1)^2-i(x+i\nu_1)}.$$

Hence, for $\nu_1 = \frac{1}{2}$ the option price reduces to

$$C(0,T,K) = S_0 - \frac{\sqrt{S_0K}}{\pi}e^{-\frac{rT}{2}+\frac{wT}{2}}\int_0^{\infty}\mathrm{Re}\left[e^{-iu(\log\frac{S_0}{K}+rT+wT)}\phi_{X_T}(-u-\frac{i}{2})\right]\frac{du}{u^2+\frac{1}{4}}.$$

Proof of Lemma 2.

We have

$$E|e^{i(u+i\nu)X_j}| = Ee^{-\nu X_j}|e^{iuX_j}| = \phi(i\nu) < \infty$$

since the imaginary point $i\nu$ belongs to the strip of analyticity of the characteristic function. Then, by Strong Law of Large Numbers we have

$$P\left(\lim_{n\to\infty}\hat{\phi}_n(x+i\nu) = \phi(x+i\nu)\right) = 1,$$

for any $x \in \mathbb{R}$.

Proof of Lemma 3.

Since

$$\left|\hat{\phi}_{T,n}(-u-\frac{i}{2})\right| = \left|\frac{1}{n}\sum_{j=1}^{n}e^{i(-u-\frac{i}{2})X_j}\right|^p = \left|\frac{1}{n}\sum_{j=1}^{n}e^{-iuX_j}e^{\frac{1}{2}X_j}\right|^p$$

$$\leq \left(\frac{1}{n}\sum_{j=1}^{n}\left|e^{-iuX_j}\right|\left|e^{\frac{1}{2}X_j}\right|\right)^p \leq \left(\frac{1}{n}\sum_{j=1}^{n}e^{\frac{1}{2}X_j}\right)^p = \hat{\phi}_{T,n}(-\frac{i}{2}) = \hat{M}, \quad \text{(A.0.2)}$$

the left hand side expression is for all values of u bounded by the same random variable $\hat{M} > 0$. Finally, let us note that since $\mathrm{Re}(z_1 z_2) = \mathrm{Re}(z_1)\mathrm{Re}(z_2) - \mathrm{Im}(z_1)\mathrm{Im}(z_2)$ for $z_1, z_2 \in \mathbb{C}$ and $|\cos(u(b + \hat{w}_n T))|$ and $|\sin(u(b + \hat{w}_n T))|$ are bounded by 1, we have

$$\begin{aligned}
&\int_0^\infty \left| \mathrm{Re}\left[e^{-iu(\log \frac{S_0}{K} + rT + \hat{w}_n T)} \hat{\phi}_{T,n}(-u - \frac{i}{2}) \right] \frac{1}{u^2 + \frac{1}{4}} \right| du \\
&= \int_0^\infty \left| \cos(u(b + \hat{w}_n T)) \mathrm{Re}\left[\hat{\phi}_{T,n}(-u - \frac{i}{2}) \right] + \sin(u(b + \hat{w}_n T)) \mathrm{Im}\left[\hat{\phi}_{T,n}(-u - \frac{i}{2}) \right] \right| \frac{1}{u^2 + \frac{1}{4}} du \\
&\leq \int_0^\infty \left(\left| \mathrm{Re}\left[\hat{\phi}_{T,n}(-u - \frac{i}{2}) \right] \right| + \left| \mathrm{Im}\left[\hat{\phi}_{T,n}(-u - \frac{i}{2}) \right] \right| \right) \frac{1}{u^2 + \frac{1}{4}} du \\
&\leq 2 \int_0^\infty \left| \hat{\phi}_{T,n}(-u - \frac{i}{2}) \right| \frac{1}{u^2 + \frac{1}{4}} du \leq 2\hat{M} \int_0^\infty \frac{du}{u^2 + \frac{1}{4}} < +\infty, \quad \text{(A.0.3)}
\end{aligned}$$

where $b = \log \frac{S_0}{K} + rT$.

Proof of Proposition 8.

We adapt a proof of Csörgő & Totik (1983) to our case. For any $\epsilon > 0$ we choose $K > 0$ such that

$$\int_{|x|>K} e^{-\nu x} dF(x) < \frac{\epsilon}{8}.$$

Let

$$B(t + i\nu) = \int_{|x| \leq K} e^{i(t+i\nu)x} dF(x),$$

$$B_n(t + i\nu) = \int_{|x| \leq K} e^{i(t+i\nu)x} dF_n(x) = \frac{1}{n} \sum_{j=1}^n e^{i(t+i\nu)X_j} 1_{\{|X_j| \leq K\}},$$

and

$$D_n(t + i\nu) = B_n(t + i\nu) - B(t + i\nu).$$

We have

$$\sup_{|t| \leq U_n} \left| \hat{\phi}_n(t + i\nu) - \phi(t + i\nu) \right| \quad \text{(A.0.4)}$$

$$\leq \sup_{|t| \leq U_n} \left| B(t+i\nu) - \phi(t+i\nu) \right| + \sup_{|t| \leq U_n} \left| B_n(t+i\nu) - \hat{\phi}_n(t+i\nu) \right| + \sup_{|t| \leq U_n} \left| B_n(t+i\nu) - B(t+i\nu) \right|.$$

For the first term we have

$$\sup_{|t|\le U_n}\big|B(t+i\nu)-\phi(t+i\nu)\big| = \sup_{|t|\le U_n}\big|\int_{|x|>K} e^{i(t+i\nu)x}dF(x)\big| \le \int_{|x|>K} e^{-\nu x}dF(x) \le \frac{\epsilon}{8}.$$

For the second term we have

$$\sup_{|t|\le U_n}\big|B_n(t+i\nu)-\hat{\phi}_n(t+i\nu)\big| = \sup_{|t|\le U_n}\big|\int_{|x|>K} e^{i(t+i\nu)x}dF_n(x)\big| \le \int_{|x|>K} e^{-\nu x}dF_n(x).$$

Since by Lemma 2 the last expression converges to $\int_{|x|>K} e^{-\nu x}dF(x)$ with probability 1, we have that for sufficiently large n

$$\sup_{|t|\le U_n}\big|B_n(t+i\nu)-\hat{\phi}_n(t+i\nu)\big| < \frac{\epsilon}{8}.$$

Hence, for sufficiently large n we have

$$\sup_{|t|\le U_n}\big|\hat{\phi}_n(t+i\nu)-\phi(t+i\nu)\big| \le \sup_{|t|\le U_n}\big|D_n(t+i\nu)\big| + \frac{\epsilon}{4}. \tag{A.0.5}$$

We cover the interval $[-U_n, U_n]$ by $N_n = \big[\frac{16Ke^{\nu K}U_n}{\epsilon}\big]+1$ disjoint intervals $\Lambda_1,\ldots\Lambda_{N_n}$, each of length not exceeding $\frac{\epsilon}{8Ke^{\nu K}}$. Let $t_1,\ldots t_{N_n}$ denote the centers of these intervals.

For any $|t|\le U_n$ we can choose the closest t_k such that $t\in\Lambda_k$. Hence we have

$$\big|D_n(t+i\nu)\big| \le \big|D_n(t_k+i\nu)\big| + \sup_{t\in\Lambda_k}\big|D_n(t+i\nu)-D_n(t_k+i\nu)\big|,$$

which implies that

$$\sup_{|t|\le U_n}\big|D_n(t+i\nu)\big| \le \max_{1\le k\le N_n}\big|D_n(t_k+i\nu)\big| + \max_{1\le k\le N_n}\sup_{t\in\Lambda_k}\big|D_n(t+i\nu)-D_n(t_k+i\nu)\big|. \tag{A.0.6}$$

For any $s,t\in\mathcal{R}$ we have

$$\begin{aligned}\big|D_n(s+i\nu)-D_n(t+i\nu)\big| &\le \big|B_n(s+i\nu)-B_n(t+i\nu)\big| + \big|B(s+i\nu)-B(t+i\nu)\big| \\ &= \big|\int_{|x|\le K}\big(e^{i(s+i\nu)x}-e^{i(t+i\nu)x}\big)dF_n(x)\big| + \big|\int_{|x|\le K}\big(e^{i(s+i\nu)x}-e^{i(t+i\nu)x}\big)dF(x)\big|.\end{aligned}$$

By Taylor Theorem we have

$$\cos(sx) = \cos(tx) - x\sin(\xi_1 x)(s-t)$$

and

$$\sin(sx) = \sin(tx) + x\cos(\xi_2 x)(s-t),$$

for some $\xi_1, \xi_2 \in (s,t)$. Hence

$$e^{i(s+i\nu)x} - e^{i(t+i\nu)x} = -e^{-\nu x} x \sin(\xi_1 x)(s-t) + ie^{-\nu x} x \cos(\xi_2 x)(s-t).$$

We have

$$\begin{aligned} &\Big|\int_{|x|\le K} \big(e^{i(s+i\nu)x} - e^{i(t+i\nu)x}\big) dF_n(x)\Big| + \Big|\int_{|x|\le K} \big(e^{i(s+i\nu)x} - e^{i(t+i\nu)x}\big) dF(x)\Big| \\ &\le \Big|\int_{|x|\le K} 2|x||s-t|e^{-\nu x} dF_n(x)\Big| + \Big|\int_{|x|\le K} 2|x||s-t|e^{-\nu x} dF(x)\Big| \le 4Ke^{\nu K}|s-t| \end{aligned}$$

and

$$\max_{1\le k\le N_n} \sup_{t\in\Lambda_k} \big|D_n(t+i\nu) - D_n(t_k+i\nu)\big| \le \max_{1\le k\le N_n} \sup_{t\in\Lambda_k} \big\{4Ke^{\nu K}|t-t_k|\big\} \le \frac{\epsilon}{4}. \tag{A.0.7}$$

Now we find a bound for the first term in (A.0.6). Let

$$p_n = \mathbf{P}\Big(\max_{1\le k\le N_n} \big|D_n(t_k+i\nu)\big| > \frac{\epsilon}{2}\Big).$$

We will show that series $\sum_{n=1}^{\infty} p_n < \infty$. We have

$$\begin{aligned} p_n &= \mathbf{P}\Big(\max_{1\le k\le N_n} \big|D_n(t_k+i\nu)\big| > \frac{\epsilon}{2}\Big) \le \mathbf{P}\Big(\bigcup_{k=1}^{N_n} \big\{\big|D_n(t_k+i\nu)(\omega)\big| > \frac{\epsilon}{2}\big\}\Big) \\ &\le \sum_{k=1}^{N_n} \mathbf{P}\Big(\big|D_n(t_k+i\nu)\big| > \frac{\epsilon}{2}\Big) \le N_n \sup_{t\in\mathcal{R}} \mathbf{P}\Big(\big|D_n(t+i\nu)\big| > \frac{\epsilon}{2}\Big). \end{aligned}$$

We define random variables

$$R_j(t+i\nu) = \cos(tX_j)e^{-\nu X_j} 1_{\{|X_j|\le K\}} - \int_{|x|\le K} \cos(tx)e^{-\nu x} dF(x)$$

and

$$I_j(t+i\nu) = \sin(tX_j)e^{-\nu X_j} 1_{\{|X_j|\le K\}} - \int_{|x|\le K} \sin(tx)e^{-\nu x} dF(x),$$

for $j = 1, \ldots, n$.

We have the following bound

$$\begin{aligned}\mathbf{P}\Big(\big|D_n(t+i\nu)\big| > \frac{\epsilon}{2}\Big) &= \mathbf{P}\Big(\big|\frac{1}{n}\sum_{j=1}^{n} R_j(t+i\nu) + i\frac{1}{n}\sum_{j=1}^{n} I_j(t+i\nu)\big| > \frac{\epsilon}{2}\Big)\\ &\leq \mathbf{P}\Big(\frac{1}{n}\big|\sum_{j=1}^{n} R_j(t+i\nu)\big| + \frac{1}{n}\big|\sum_{j=1}^{n} I_j(t+i\nu)\big| > \frac{\epsilon}{2}\Big)\\ &\leq \mathbf{P}\Big(\frac{1}{n}\big|\sum_{j=1}^{n} R_j(t+i\nu)\big| > \frac{\epsilon}{4}\Big) + \mathbf{P}\Big(\frac{1}{n}\big|\sum_{j=1}^{n} I_j(t+i\nu)\big| > \frac{\epsilon}{4}\Big)\end{aligned}$$

Hence, with $N_n = MU_n$ for some constant $M > 0$, we get

$$\begin{aligned}p_n &\leq MU_n \sup_{t\in\mathcal{R}} \mathbf{P}\Big(\big|D_n(t+i\nu)\big| > \frac{\epsilon}{2}\Big)\\ &\leq MU_n \sup_{t\in\mathcal{R}}\Big\{\mathbf{P}\Big(\frac{1}{n}\big|\sum_{j=1}^{n} R_j(t+i\nu)\big| > \frac{\epsilon}{4}\Big) + \mathbf{P}\Big(\frac{1}{n}\big|\sum_{j=1}^{n} I_j(t+i\nu)\big| > \frac{\epsilon}{4}\Big)\Big\}.\end{aligned}$$

Random variables $R_j(t+i\nu)$ and $I_j(t+i\nu)$ are independent and identically distributed, and $ER_j(t+i\nu) = EI_j(t+i\nu) = 0$. We have

$$\begin{aligned}\big|R_j(t+i\nu)\big| &\leq \big|\cos(tX_j)e^{-\nu X_j}\big|1_{\{|X_j|\leq K\}} + \int_{|x|\leq K}\big|\cos(tx)e^{-\nu x}\big|dF(x)\\ &\leq e^{\nu K} + e^{\nu K}\int_{-\infty}^{\infty} dF(x) = 2e^{\nu K}\end{aligned}$$

and, similarly,

$$\big|I_j(t+i\nu)\big| \leq 2e^{\nu K}.$$

By the Hoeffding's inequality (cf. Theorem 11 in Appendix B.2) we have

$$\begin{aligned}\mathbf{P}\Big(\frac{1}{n}\big|\sum_{j=1}^{n} R_j(t+i\nu)\big| > \frac{\epsilon}{4}\Big) &\leq \mathbf{P}\Big(\frac{1}{n}\sum_{j=1}^{n} R_j(t+i\nu) > \frac{\epsilon}{4}\Big) + \mathbf{P}\Big(\frac{1}{n}\sum_{j=1}^{n} R_j(t+i\nu) \leq -\frac{\epsilon}{4}\Big)\\ &\leq 2e^{-\frac{\epsilon^2 n}{128e^{2\nu K}}}.\end{aligned}$$

Similarly,

$$\mathbf{P}\Big(\frac{1}{n}\big|\sum_{j=1}^{n} I_j(t+i\nu)\big| > \frac{\epsilon}{4}\Big) \leq 2e^{-\frac{\epsilon^2 n}{128e^{2\nu K}}}.$$

For $0 < \delta < \frac{\epsilon^2}{128e^{2\nu K}}$ and for sufficiently large n we have

$$U_n \leq e^{\delta n} < e^{\frac{\epsilon^2 n}{128e^{2\nu K}}}.$$

Hence,

$$p_n \leq 4MU_n e^{-\frac{\epsilon^2 n}{128e^{2\nu K}}} \leq 4Me^{n(\delta - \frac{\epsilon^2}{128e^{2\nu K}})},$$

and since $\delta - \frac{\epsilon^2}{128e^{2\nu K}} < 0$ we get

$$\sum_{n=1}^{\infty} p_n < \infty.$$

By the Borel-Canteli Lemma convergence of the sequence $\sum_{n=1}^{\infty} p_n$ implies that with probability 1

$$\max_{1 \leq k \leq N_n} \left| D_n(t_k + i\nu) \right| \leq \frac{\epsilon}{2}, \tag{A.0.8}$$

for n sufficiently large. Hence, by (A.0.5), (A.0.6), (A.0.7) and (A.0.8) with probability 1 for sufficiently large n we get that

$$\sup_{|t| \leq U_n} \left| \hat{\phi}_n(t + i\nu) - \phi(t + i\nu) \right| \leq \epsilon.$$

This completes the proof of Proposition 8.

Proof of Proposition 9.

We have

$$\begin{gathered}\left| \hat{C}_n(0, T, K; \hat{w}_n, p) - C(0, T, K) \right| = \\ \left| \frac{\sqrt{S_0 K}}{\pi} e^{-\frac{rT}{2} + \frac{\hat{w}_n T}{2}} \int_0^{\infty} \mathrm{Re}\left[e^{-iu(\log \frac{S_0}{K} + rT + \hat{w}_n T)} \left(\hat{\phi}_n(-u - \frac{i}{2}) \right)^p \right] \frac{du}{u^2 + \frac{1}{4}} \right. \\ \left. - \frac{\sqrt{S_0 K}}{\pi} e^{-\frac{rT}{2} + \frac{\omega T}{2}} \int_0^{\infty} \mathrm{Re}\left[e^{-iu(\log \frac{S_0}{K} + rT + \omega T)} \left(\phi_{X_\Delta}(-u - \frac{i}{2}) \right)^p \right] \frac{du}{u^2 + \frac{1}{4}} \right|\end{gathered}$$

Let

$$\begin{aligned} A &= \frac{\sqrt{S_0 K}}{\pi} e^{-\frac{rT}{2} + \frac{wT}{2}}, \\ \hat{A}_n &= \frac{\sqrt{S_0 K}}{\pi} e^{-\frac{rT}{2} + \frac{\hat{w}_n T}{2}}, \\ B &= \int_0^{\infty} \mathrm{Re}\left[e^{-iu(\log \frac{S_0}{K} + rT + wT)} \left(\phi_{X_\Delta}(-u - \frac{i}{2}) \right)^p \right] \frac{du}{u^2 + \frac{1}{4}} \end{aligned}$$

and

$$\hat{B}_n = \int_0^\infty \mathrm{Re}\left[e^{-iu(\log\frac{S_0}{K}+rT+\hat{w}_nT)}\left(\hat{\phi}_n(-u-\frac{i}{2})\right)^p\right]\frac{du}{u^2+\frac{1}{4}}.$$

We have

$$\left|\hat{C}_n(0,T,K;\hat{w}_n,p) - C(0,T,K)\right| = |\hat{A}_n\hat{B}_n - AB|$$
$$= |\hat{A}_n\hat{B}_n - \hat{A}_nB + \hat{A}_nB - AB| \leq |\hat{A}_n||\hat{B}_n - B| + |B||\hat{A}_n - A|$$

By Lemma 2, we have that

$$\hat{w}_n = -\frac{\log\left(\hat{\phi}_n(-i)\right)}{\Delta} \to -\frac{\log\left(\phi_{X_\Delta}(-i)\right)}{\Delta},$$

hence

$$\hat{A}_n - A = \frac{\sqrt{S_0K}}{\pi}e^{-\frac{rT}{2}}\left(e^{\frac{\hat{w}_nT}{2}} - e^{\frac{wT}{2}}\right) \to 0$$

with probability 1. Expression B is a part of formula (3.3.2) and is finite. Moreover, by Lemma 3 $\hat{A}_n$ is finite with probability 1. We will show that $\hat{B}_n - B$ converges to 0 with probability 1.

Let us denote

$$D = e^{-iu(\log\frac{S_0}{K}+rT+\omega T)}$$

and

$$\hat{D}_n = e^{-iu(\log\frac{S_0}{K}+rT+\hat{w}_nT)}.$$

We have

$$\begin{aligned}
|\hat{B}_n - B| &= \left|\int_0^\infty \mathrm{Re}\left[\hat{D}_n\left(\hat{\phi}_n(-u-\frac{i}{2})\right)^p - D\left(\phi_{X_\Delta}(-u-\frac{i}{2})\right)^p\right]\frac{du}{u^2+\frac{1}{4}}\right| \\
&= \left|\int_0^\infty \mathrm{Re}\big[\hat{D}_n\left(\hat{\phi}_n(-u-\frac{i}{2})\right)^p - \hat{D}_n\left(\phi_{X_\Delta}(-u-\frac{i}{2})\right)^p\right. \\
&\quad \left. +\hat{D}_n\left(\phi_{X_\Delta}(-u-\frac{i}{2})\right)^p - D\left(\phi_{X_\Delta}(-u-\frac{i}{2})\right)^p\big]\frac{du}{u^2+\frac{1}{4}}\right| \\
&\leq \int_0^\infty\left|\mathrm{Re}\big[\hat{D}_n\Big((\hat{\phi}_n(-u-\frac{i}{2}))^p - \left(\phi_{X_\Delta}(-u-\frac{i}{2})\right)^p\Big)\big]\right|\frac{du}{u^2+\frac{1}{4}} \\
&\quad + \int_0^\infty\left|\mathrm{Re}\left[\left(\hat{D}_n - D\right)\left(\phi_{X_\Delta}(-u-\frac{i}{2})\right)^p\right]\right|\frac{du}{u^2+\frac{1}{4}} \\
&= \int_0^{U_n}\left|\mathrm{Re}\left[\hat{D}_n\Big((\hat{\phi}_n(-u-\frac{i}{2}))^p - \left(\phi_{X_\Delta}(-u-\frac{i}{2})\right)^p\Big)\right]\right|\frac{du}{u^2+\frac{1}{4}} \\
&\quad + \int_{U_n}^\infty\left|\mathrm{Re}\left[\hat{D}_n\Big((\hat{\phi}_n(-u-\frac{i}{2}))^p - \left(\phi_{X_\Delta}(-u-\frac{i}{2})\right)^p\Big)\right]\right|\frac{du}{u^2+\frac{1}{4}} \\
&\quad \int_0^\infty\left|\mathrm{Re}\left[\left(\hat{D}_n - D\right)\left(\phi_{X_\Delta}(-u-\frac{i}{2})\right)^p\right]\right|\frac{du}{u^2+\frac{1}{4}} = I_1 + I_2 + I_3,
\end{aligned}$$

where we choose U_n satisfying assumptions of Proposition 8. Since $\hat{D}_n - D$ is bounded and converges to 0, we get by Lebesgue dominated convergence theorem that

$$I_3 \to 0$$

as $n \to \infty$.

Moreover, we have

$$|I_2| \leq \int_{U_n}^\infty\left|\mathrm{Re}\left[\hat{D}_n\Big(\hat{\phi}_n(-u-\frac{i}{2})\Big)^p\right]\right|\frac{du}{u^2+\frac{1}{4}} + \int_{U_n}^\infty\left|\mathrm{Re}\left[\hat{D}_n\left(\phi_{X_\Delta}(-u-\frac{i}{2})\right)^p\right]\right|\frac{du}{u^2+\frac{1}{4}}.$$

Both integrals are finite and converge to 0 as $U_n \to \infty$ with $n \to \infty$.

Since

$$|\hat{D}_n| = |\cos(u(\log\frac{S_0}{K} + rT + \hat{w}_nT)) + i\sin(u(\log\frac{S_0}{K} + rT + \hat{w}_nT)| \leq 2,$$

$$|I_1| \leq 2\int_0^{U_n}\left|(\hat{\phi}_n(-u-\frac{i}{2}))^p - \left(\phi_{X_\Delta}(-u-\frac{i}{2})\right)^p\right|\frac{du}{u^2+\frac{1}{4}}.$$

For any $u \in [0, U_n]$ by the Taylor's Theorem (cf. Theorem 10 in Appendix B.2) and for sufficiently large n

$$\left(\hat{\phi}_n(-u - \frac{i}{2})\right)^p - \left(\phi_{X_\Delta}(-u - \frac{i}{2})\right)^p =$$
$$\frac{1}{2\pi i}\int_{\mathbf{C}} \frac{\zeta^p}{(\zeta - \hat{\phi}_n(-u - \frac{i}{2}))(\zeta - \phi_{X_\Delta}(-u - \frac{i}{2}))} d\zeta \left(\hat{\phi}_n(-u - \frac{i}{2}) - \phi_{X_\Delta}(-u - \frac{i}{2})\right)$$

where we expand the complex power function z^p for $z = \hat{\phi}_n(-u - \frac{i}{2})$ around point $a = \phi_{X_\Delta}(-u - \frac{i}{2})$, and where C is a circle around point a with a radius $r \geq |\hat{\phi}_n(-u - \frac{i}{2}) - \phi_{X_\Delta}(-u - \frac{i}{2})|$.

By Proposition 8 for any $\epsilon \in (0, r)$ and for any $u \in [0, U_n]$ there exists integer $N > 0$, such that for any $n > N$ we have

$$\left|\hat{\phi}_n(-u - \frac{i}{2}) - \phi_{X_\Delta}(-u - \frac{i}{2})\right| < \epsilon.$$

Hence, for $\zeta \in C$ we have

$$|\zeta - \hat{\phi}_n(-u - \frac{i}{2})| \geq |\zeta - \phi_{X_\Delta}(-u - \frac{i}{2})| - |\hat{\phi}_n(-u - \frac{i}{2}) - \phi_{X_\Delta}(-u - \frac{i}{2})| \geq r - \epsilon$$

and this implies that for sufficiently large n the integral over C is well defined.

We have

$$\left|\frac{1}{2\pi i}\int_{\mathbf{C}} \frac{\zeta^p}{(\zeta - \hat{\phi}_n(-u - \frac{i}{2}))(\zeta - \phi_{X_\Delta}(-u - \frac{i}{2}))} d\zeta \left(\hat{\phi}_n(-u - \frac{i}{2}) - \phi_{X_\Delta}(-u - \frac{i}{2})\right)\right|$$
$$\leq \frac{1}{2\pi}\frac{M}{(r-\epsilon)r} 2\pi r\epsilon \leq \frac{2M\epsilon}{r}, \text{ for } \epsilon < \frac{r}{2}.$$

Hence, we have

$$|I_1| \leq 2\int_0^{U_n} \frac{2M\epsilon}{r}\frac{du}{u^2 + \frac{1}{4}} \leq \frac{4M\pi}{r}.$$

This concludes proof of the Proposition.

Appendix B

Supplements

B.1 Supplementary formulas

Frullani equality

Assume $f'(x)$ is continuous and the integral converges, then

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = [\lim_{x\to 0} f(x) - \lim_{x\to\infty} f(x)] \log\left(\frac{b}{a}\right), \tag{B.1.1}$$

for $a, b > 0$.

The modified Bessel function of the second kind $K_a(\cdot)$

We have

$$\int_0^\infty y^{a-1} e^{-\frac{b}{y} - cy} dy = 2\left(\frac{b}{c}\right)^{\frac{a}{2}} K_a\left(2\sqrt{bc}\right), \ (\mathrm{Re}(b) > 0, \mathrm{Re}(c) > 0), \tag{B.1.2}$$

cf. Gradshteyn & Ryzhik (1965) - formula 3.471.9,
and

$$K_{-\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2}} z^{-\frac{1}{2}} e^{-z} \tag{B.1.3}$$

cf. Prause (1999) - formula B.17.

Asymptotic behaviour of the modified Bessel function of the second kind $K_a(\cdot)$

$$K_a(x) = \sqrt{\frac{\pi}{2}} x^{-\frac{1}{2}} e^{-x} + o(1), \text{ as } x \to \infty, \tag{B.1.4}$$

Barndorff-Nielsen et al. (1982) - section 6.

B.2 Supplementary theorems

Theorem 8 *(Part of Theorem 30.1 in (Sato 1999)) Let $(X_t)_{t\geq 0}$ be a Lévy process with generating triplet $(\sigma^2, \mathbf{L}_X, m)$ and let $(Y_t)_{t\geq 0}$ be a subordinator with a generating triplet $(0, \rho, b)$. Then the process $(Z_t)_{t\geq 0}$ defined for each $\omega \in \Omega$ by $Z(t,\omega) := X(Y(t,\omega),\omega)$ is a Lévy process with a characteristic triplet $(\sigma_Z^2, \mathbf{L}_Z, m_Z)$, where*

$$\sigma_Z^2 = b\sigma^2, \tag{B.2.1}$$

$$\mathbf{L}_Z(dx) = b\mathbf{L}_X(dx) + \int_0^\infty f_{X_s}(x)\rho(ds), \tag{B.2.2}$$

$$m_Z = bm + \int_0^\infty \rho(ds) \int_{|x|\leq 1} x f_{X_s}(x)dx. \tag{B.2.3}$$

Theorem 9 *Lukacs (1960) (Theorem 7.1.1)*
If a characteristic function $\phi(z)$ is analytical and one-valued in a neighborhood of the origin, then it is also analytical and one-valued in a horizontal strip and can be represented in this strip by a Fourier integral. This strip is either the whole plane, or it has one or two horizontal boundary lines. The purely imaginary points on the boundary of the strip (if this strip is not the whole plane) are singular points of $\phi(z)$.

Lemma 4 *Uni-variate Gaussian Shift Theorem (GST) (cf. Workshop on Exotic Option Pricing (2006) by P.Buchen & O.Konstandatos). Let $Z \sim N(0,1)$, $c \in \mathbb{R}$ and H be any measurable function with a finite expectation. Then*

$$E(e^{cZ}H(Z)) = e^{\frac{1}{2}c^2} EH(Z+c). \tag{B.2.4}$$

Theorem 10 *(cf. Karunakaran (2005) (Theorem 4.4.12))*
Let $f(z)$ be analytic in a region A and $a \in A$. Then $f(z)$ can be expanded in the following form

$$f(z) = f(a) + f'(a)(z-a) + \ldots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-a)^n(\zeta-z)}(z-a)^n,$$

where C is any circle with centre at point a and radius r, such that disc $|z-a| \le r$ is contained in A. This expansion is valid for $z \in A$ and for $n = 1, 2, \ldots$.

Theorem 11 *Devroye & Lugosi (2001) (Chapter 2, Theorem 2.1, p. 6)*
Let $X_1, \ldots, X_n$ be independent bounded random variables such that $X_i \in [a_i, b_i]$, $i = 1, \ldots, n$ with probability 1. Then for any $t > 0$, we have

$$\mathbf{P}(S_n - ES_n \ge t) \le \exp\left(\frac{-2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$$

and

$$\mathbf{P}(S_n - ES_n \le -t) \le \exp\left(\frac{-2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right),$$

where $S_n = \sum_{i=1}^n X_i$.

Appendix C

Matlab programs

This Appendix contains MATLAB® programs used to price and calibrate Models 1 - 5, used in the project.

1. ecfvanilla.m

```
function y = ecfvanilla(S,K,r,q,t,tinc,data,w,p,aa,cutoff,u)
% ECF pricing, formula (4.1.6)
% S    - spot price
% K    - option strike
% r    - interest rate
% q    - divident yield
% t    - option maturity
% tinc - assumed time increments of log-rets, tinc * length(data) should be
     equal t
% data - data for DAX log-returns used in estimation of ecf
% w    - used in calibration of implied MMCT
% aa   - alpha
% u    - flag: 1 for call and 0 for put
rint=quadl('ecfintegrand',0,cutoff,[],[],K, S, r, t, data, w, p, aa);
R=S*(aa<=0) - exp(-r*t)*K*(aa<=-1) - (S*(aa==0) - exp(-r*t)*K*(aa==-1))/2;
y1=R + exp(-r*t) .* rint/pi;
y=y1+(1-u).*(K.*exp(-r.*t)-S.*exp(-q.*t)); %call or put
```

2. ecfintegrand.m

```
function y = ecfintegrand(v,K,S,r,t,data,w,p,aa)
% integrand in formula (4.1.6)
% v    - variable
```

```
% S    - spot price
% K    - option strike
% r    - interest rate
% t    - option maturity
% data - data for calculating ecf
% w    - implied w
% p    - implied p
% aa   - integration parameter
y1=exp(-i*(v-i*aa)*log(K));
vv=v-i*(aa+1);
y2=exp(i*vv*(log(S)+(r+w)*t)).*ecfcn(vv,data).^p;
y3=1./(-vv.*(v-i*aa));
y=real(y1.*y3).*real(y2) - imag(y1.*y3).*imag(y2);
```

3. ecfcalib.m

```
function [wecf, pecf, wdata, modelprices, resnorm, difplus, difminus,
    meandif, mediandif, rmserel] = ecfcalib(S0, data, ndata, r, strike,
    price, expdays, tinc, optiontype, wflag, pflag)
% Calibrates the Models 2-4 to ODAX option prices
% S0          - price of the underlying
% data        - underlying asset log returns
% ndata       - number of elements back of data to calculate ecf
% r           - riskfree rate
% strike      - strike price
% price       - prices of options
% expdays     - number of days to option expiration
% pp          - power of ECF
% tinc        - time increments between log returns
% optiontype  - 1 for call, 0 for put, vector of length same as strike and
    price
% wflag       - if 1 then calibrate w, if 0 then use wdata
% pflag       - if 1 then calibrate p, if 0 then use number of days to
    maturity
% output:
% wecf        - implied w
% implied p   - implied p
% wdata       - empirical MMCT
% modelprices - prices obtained with the use of the optimal parameters
% resnorm     - the value of the squared 2-norm of the residual
% difplus     - maximum of relative errors
```

```
% difminus     - minimum of relative errors
% meandif      - mean of relative errors
% mediandif    - median of relative errors
% rmserel      - relative resnorm
%
aa=-.5; %uses Lewis formula
cutoff=512;
tolfun = 10^(-6); tolx = 10^(-3);
opts = optimset('Display','iter','TolFun',tolfun,'TolX',tolx);

data1=data(length(data)-ndata+1:length(data));
wdata = -log(ecfcn(-i,data1))/tinc;

maturity = expdays*tinc;
S01 = ones(length(strike),1).*S0;
r1 = ones(length(strike),1).*r;

if (wflag==0)&(pflag==0) %use wdata and set p to number of days to
    expiration
  modelprices = zeros(length(strike),1);
  for j=1:length(strike)
    modelprices(j) = ecfvanilla(S01(j),strike(j),r1(j),0,maturity,tinc,
        data1,wdata,expdays,aa,cutoff,optiontype(j));
  end
  resnorm = norm(modelprices-price)^2;
  wecf=0;
  pecf=0;
elseif (wflag==1)&(pflag==0) %calibrate w and set p to number of days to
    expiration
  mats=ones(length(strike),1)*maturity;
  w0 = wdata;
  [wecf, resnorm] = lsqnonlin(@ecffuncpowfixed, w0, [], [], opts, expdays,
     aa, cutoff, mats, tinc, strike, price, r1, 0, S01, data1, optiontype)
  %pricing with calibrated w
  modelprices = zeros(length(strike),1);
  for j=1:length(strike)
    modelprices(j) = ecfvanilla(S01(j),strike(j),r1(j),0,maturity,tinc,
        data1,wecf,expdays,aa,cutoff,optiontype(j));
  end
  pecf=0;
```

```
elseif (wflag==0)&(pflag==1) %use wdata and calibrate p
  mats=ones(length(strike),1)*maturity;
  pL = 0;
  [pecf, resnorm] = lsqnonlin(@ecffuncpowcalib, expdays, pL, [], opts,
      wdata, aa, cutoff, mats, tinc, strike, price, r1, 0, S01, data1,
      optiontype)
  %pricing with calibrated w
  modelprices = zeros(length(strike),1);
  for j=1:length(strike)
    modelprices(j) = ecfvanilla(S01(j),strike(j),r1(j),0,maturity,tinc,
        data1,wdata,pecf,aa,cutoff,optiontype(j));
  end
  wecf=0;
else %calibrate w and p
  mats=ones(length(strike),1)*maturity;
  w0 = wdata;
  wp0=[w0,expdays];
  wpL = [-10 0];
  [wpecf, resnorm] = lsqnonlin(@ecffunc, wp0, wpL, [], opts, aa, cutoff,
      mats, tinc, strike, price, r1, 0, S01, data1, optiontype)
  %pricing with calibrated w
  modelprices = zeros(length(strike),1);
  for j=1:length(strike)
    modelprices(j) = ecfvanilla(S01(j),strike(j),r1(j),0,maturity,tinc,
        data1,wpecf(1),wpecf(2),aa,cutoff,optiontype(j));
  end
  wecf=wpecf(1);
  pecf=wpecf(2);
end
  difplus = max((modelprices-price)./price);
  difminus = min((modelprices-price)./price);
  meandif = mean((modelprices-price)./price);
  mediandif = median((modelprices-price)./price);
  rmserel = sqrt(norm((modelprices-price)./price)^2/length(price));

%end of the main function

function dd = ecffunc(par, aa, cutoff, maturity, tinc, strike, price,
    interest, dividend, underlprice, data, u)
% calculates vector of differences between real and model prices
```

```
% maturity, strike, optionprice - equal sized vectors for comparision
% underprice                    - price of the underlying asset
% u                             - 1 for call, 0 for put
%
w=par(1); p=par(2);
l=length(maturity);
dd=zeros(l,1);
for j=1:l
  dd(j) = ecfvanilla(underlprice(j), strike(j), interest(j), dividend,
      maturity(j),tinc, data, w, p, aa, cutoff, u(j)) - price(j);
end

function dd = ecffuncpowcalib(p, w, aa, cutoff, maturity, tinc, strike,
    price, interest, dividend, underlprice, data, u)
% calculates vector of differences between real and model prices
% maturity, strike, optionprice - equal sized vectors for comparision
% underprice                    - price of the underlying asset
% u                             - 1 for call, 0 for put
%
l=length(maturity);
dd=zeros(l,1);
for j=1:l
  dd(j) = ecfvanilla(underlprice(j), strike(j), interest(j), dividend,
      maturity(j),tinc, data, w, p, aa, cutoff, u(j)) - price(j);
end

function dd = ecffuncpowfixed(w, p, aa, cutoff, maturity, tinc, strike,
    price, interest, dividend, underlprice, data, u)
% calculates vector of differences between real and model prices
% maturity, strike, optionprice  - equal sized vectors for comparision
% underprice                     - price of the underlying asset
% u                              - 1 for call, 0 for put
%
l=length(maturity);
dd=zeros(l,1);
for j=1:l
  dd(j) = ecfvanilla(underlprice(j), strike(j), interest(j), dividend,
      maturity(j),tinc, data, w, p, aa, cutoff, u(j)) - price(j);
end
```

4. rvg.m

```
function y = rvg(S,K,r,q,sig,nu,th,t,aa,u)
% CML option price for Variance Gamma (VG) distribution
%
% S            - spot price
% K            - option strike
% r            - interest rate
% q            - divident yield
% sig, nu, th  - VG parameters
% t            - option maturity
% aa           - integration parameter
% u            - flag: 1 for call and 0 for put
%
% Refererences:
% Lewis, Alan L., 'A simple option formula for general jump-diffusion and
% other exponential Levy processes. Manuscript, Envision Financial Systems
% and OptionCity.net, 2001
% Roger Lord and Christian Kahl, 'Optimal Fourier Inversion in Semi
% -analytical Option Pricing', Tinbergen Institute Discussion Papers, 2006
%
w = 1/nu*log(1 - th*nu - .5*nu*sig^2);
rint=quadl('rintegrand',0,512,[],[],K, S, r, q, t, sig, nu, th, w, aa);
R=S*(aa<=0) - exp(-r*t)*K*(aa<=-1) - (S*(aa==0) - exp(-r*t)*K*(aa==-1))/2;
y1=R + exp(-r*t) .* rint/pi;
y=y1+(1-u).*(K.*exp(-r.*t)-S.*exp(-q.*t)); %call or put

function y = rintegrand(v,K,S,r,q,t,sig,nu,th,w,aa)
y1=exp(-i*(v-i*aa)*log(K));
y2=exp(i*(v-i*(aa+1))*(log(S)+(r+w)*t)).*rvgchf(v-i*(aa+1),th,sig,nu,t);
y3=1./(-(v-i*(aa+1)).*(v-i*aa));
y=real(y1.*y2.*y3);
```

5. cmlcalib.m

```
function [paramopt, modelprices, resnorm, difplus, difminus, meandif,
    mediandif, rmserel] = cmlcalib(S0, data, ndata, r, strike, price,
    expdays, tinc, params, optiontype)
% Calibrates the CML model to ODAX option prices
% used in Model 5
% S0           - price of the underlying,
% data         - timeseries of the underlying,
```

```
% ndata        - number of elements to calculate ecf,
% r            - riskfree rate,
% strike       - strike price,
% price        - price of option,
% expday       - number of days to option expiration,
% tinc         - time increments of log returns,
% optiontype   - 1 for call, 0 for put,
% params       - starting parameters for calibration
% output:
% paramopt     - optimal parameters
% modelprices  - prices obtained with the use of the optimal parameters
% resnorm      - the value of the squared 2-norm of the residual
% difplus      - maximum of relative errors
% difminus     - minimum of relative errors
% meandif      - mean of relative errors
% mediandif    - median of relative errors
% rmserel      - relative resnorm
%
aa=-.5; %use the CML formula
cutoff=512;
tolfun = 10^(-6); tolx = 10^(-3);
opts = optimset('Display','iter','TolFun',tolfun,'TolX',tolx);

data1=data(length(data)-ndata+1:length(data));
theta=params(1); sigma=params(2); nu=params(3);

maturity = expdays*tinc;
S01 = ones(length(strike),1).*S0;
r1 = ones(length(strike),1).*r;

mats=ones(length(strike),1)*maturity;
par0 = [params(1) params(2) params(3)];
parL = [-Inf 0 0];
opts = optimset('Display','iter','TolFun',10^(-9),'TolX', 10^(-9));
[paramopt resnorm] = lsqnonlin(@VG_model_price, par0, parL, [], opts, mats,
    strike, price, r1, 0, S0, optiontype);
sigma = paramopt(2);
nu = paramopt(3);
theta = paramopt(1);
%pricing with calibrated params
```

```
modelprices = zeros(length(strike),1);
for j=1:length(strike)
  modelprices(j) = rvg(S01(j),strike(j),r1(j),0,sigma,nu,theta,maturity,aa,
      optiontype(j));
end
resnorm = norm(modelprices-price)^2;

difplus = max((modelprices-price)./price);
difminus = min((modelprices-price)./price);
meandif = mean((modelprices-price)./price);
mediandif = median((modelprices-price)./price);
rmserel = sqrt(norm((modelprices-price)./price)^2/length(price));

%end of the main function


function v = VG_model_price(params, maturity, strike, price, interest,
    dividend, underlprice, u)
% calculates vector of differences between real and model prices
%
% params                           - Variance Gamma parameters
% maturity, strike, optionprice - equal sized vectors
% underprice                       - price of the underlying asset
% u                                - 1 for call, 0 for put
%
th=params(1); sg=params(2); nu=params(3);
l=length(maturity);
v=zeros(l,1);
underlprice1 = ones(length(strike),1).*underlprice;
for k=1:l
  v(k)=rvg(underlprice1(k), strike(k), interest(k), dividend, sg, nu, th,
      maturity(k), -.5, u(k)) - price(k);
end
```

6. Model1234.m

```
% Script for Models 1-4
%
% set Model:
% 1 - pflag = 0 and wflag = 0
% 2 - pflag = 1 and wflag = 0
```

```
% 3 - pflag = 0 and wflag = 1
% 4 - pflag = 1 and wflag = 1
%
pflag=0;
wflag=0;

%open connection to MySQL database using MySQL Database Connector
%http://sourceforge.net/projects/mym, http://www.mmf.utoronto.ca/resrchres/mysql/
mysql( 'open', 'localhost', 'karol', '');mym( 'open', 'localhost', 'karol', '');
mysql('use options');mym('use options');

q = 0; %dividend zero
daycount = 365; tinc=1/daycount;

colormap(hsv(daycount)); %set colors for plotting
Mcolor=hsv;
daxcloseseries = mysql('SELECT dadjclose FROM dax WHERE dax.daxdate >= "2006-06-01" AND dax.daxdate <= "2007-05-31"  ORDER BY daxdate');
[odate, daxprice, expdays, strike, closeprice, putcall]  = mysql('SELECT odate, daxprice, days, strike, closeprice, putcall FROM odax WHERE odate >= "2006-06-01" AND odate <= "2007-05-31" AND qualifiers <> "Missing" AND daxprice <> "NULL" ORDER BY odate, ric');
odaxdates = unique(odate);
numberofdays = length(odaxdates);
table=[]; table_LS=[]; tabledate=[];
table_price=[]; table_modelprice=[]; table_regress=[];

for j=1:numberofdays
   odaxdatestr = datestr(odaxdates(j),29);    %adjust odax date format
  mym('truncate dummydate'); mym('INSERT INTO dummydate(col1) VALUES("{S}")', odaxdatestr); %updating dummydate for the next line..
  ODAX = mym('SELECT daxprice, days, strike, closeprice, putcall, interestrate FROM odax, dummydate WHERE odax.odate = dummydate.col1 AND qualifiers <> "Missing" AND odax.volume>=1 ORDER BY days, strike'); %select current=j day
   daxprice = ODAX.daxprice; expdays = ODAX.days; ostrike = ODAX.strike; closeprice = ODAX.closeprice; putcall = ODAX.putcall; interestrates = ODAX.interestrate/100;
```

```
S0 = daxprice(1);
%use only four strikes around S0
srl=100;srr=100;
strikes_idx=and(ODAX.strike<=S0+srr,ODAX.strike>S0-srl);
ostrike=ostrike(strikes_idx); daxprice=daxprice(strikes_idx); expdays=
    expdays(strikes_idx); closeprice=closeprice(strikes_idx); putcall=
    putcall(strikes_idx); interestrates=interestrates(strikes_idx);

%reading DAX closeprices for calculating ECF
daxclose = mysql('SELECT dadjclose FROM dax, dummydate WHERE dax.daxdate
    <= dummydate.col1 ORDER BY daxdate');
daxlogrets=diff(log(daxclose));

[exps, m, n] = unique(expdays); %sort with respect to maturities
numberofmaturities = length(exps);
m1 = [0;m];
minprice=-10; %for plotting

for k=1:numberofmaturities
  S0 = daxprice(1);
  data=daxlogrets;
  maturity=expdays(m1(k)+1:m1(k+1))/daycount; %select same maturities
  expday=exps(k);
  ndata=120;
  strike=ostrike(m1(k)+1:m1(k+1));
  price=closeprice(m1(k)+1:m1(k+1));
  rate=interestrates(m1(k)+1:m1(k+1));
  optiontype=ones(length(putcall(m1(k)+1:m1(k+1))),1);

%if number of options is less than 3, skip it
if length(strike)>=3

  [wecf, pecf, wdata, modelprices, resnorm, difplus, difminus, meandif,
     mediandif, rmserel] = ecfcalib(S0, data', ndata, rate, strike, price
     , expday, tinc, optiontype, wflag, pflag);
  %saving calibrated parameters
  rmse = sqrt(resnorm / length(price));
  tableinput = [odaxdates(j), expday, ndata, rate(1), wdata, resnorm,
     rmse, wecf, pecf, difplus, difminus, meandif, mediandif,rmserel];
```

```
    table = [table; tableinput]; tabledateinput = [odaxdatestr];
    tabledate = [tabledate; tabledateinput];

    %saving ODAX prices and model prices
    tableinput = [price, modelprices, S0./strike, maturity, rate];
    table_regress = [table_regress; tableinput];

    ff=figure(j); lcolor=expday;
    plot(strike, price, 's','Color',Mcolor(lcolor,:),'LineWidth',1); hold
        on; %%
    plot(strike, modelprices, 'o','Color',Mcolor(lcolor,:),'LineWidth',2);
        hold on;
    K1 = [min(strike):1:max(strike)]; Y1 = max(S0-K1,0); plot(K1, Y1,'k','
        LineWidth',2); hold on;

    xlabel('Strike'); ylabel('Option price');
    legend('ODAX prices','ECF prices','Payoff');
    title(sprintf('ODAX Call prices on %s',odaxdatestr));

    minprice1=min([price;modelprices]); %for plotting
    minprice=min([minprice;price;modelprices]);

else end %if length(strike)<3 next maturity

  end
  h1=gca; x1=get(h1,'XLim'); y1=get(h1,'YLim');
  text(round((x1(2)-x1(1))/3+x1(1)),y1(2)-round(.1*(y1(2)-y1(1))),sprintf('
      S0 = %g',S0),'FontSize',12);

  for k1=1:numberofmaturities
    text(strike(1)+round(.1*(x1(2)-x1(1))),10+k1*round(.03*(y1(2)-y1(1))),
        num2str(exps(k1)),'Color',Mcolor(exps(k1),:),'FontSize',8,'
        HorizontalAlignment','right', 'FontWeight','bold');
  end
  text(strike(1)+round(.01*(x1(2)-x1(1))),10,'(days to expiry)','FontSize'
      ,8,'HorizontalAlignment','left', 'FontWeight','bold');

  saveas(ff,sprintf('ODAX%s_ECF%g',odaxdatestr, ndata),'fig');
  saveas(ff,sprintf('ODAX%s_ECF%g',odaxdatestr, ndata),'pdf');
  close(ff);
```

```
end
%Error measurements
price_err = table_regress(:,1)-table_regress(:,2);
price_err_rel = (table_regress(:,1)-table_regress(:,2))./table_regress(:,1)
    ;
price_err_abs = abs(price_err);
price_err_rel_abs = abs(price_err_rel);
table_errors = [mean(price_err); mean(price_err_rel); mean(price_err_abs);
    mean(price_err_rel_abs); sqrt(sum(price_err.^2)/length(price_err)); sqrt
    (sum(price_err_rel.^2)/length(price_err_rel))];
```

7. Model5.m

```
% Script for Model 5
%
%open connection to MySQL database using MySQL Database Connector
%http://sourceforge.net/projects/mym, http://www.mmf.utoronto.ca/resrchres/
    mysql/
mysql( 'open', 'localhost', 'karol', ''); mym( 'open', 'localhost', 'karol'
    , '')
mysql('use options'); mym('use options');

q = 0;
daycount = 365;
tinc=1/daycount;

colormap(hsv(daycount)); %set colors for plotting
Mcolor=hsv;
daxcloseseries = mysql('SELECT dadjclose FROM dax WHERE dax.daxdate >=
    "2006-06-01" AND dax.daxdate <= "2007-05-31"  ORDER BY daxdate');
%choose time interval and calculate number of days to analyse
[odate, daxprice, expdays, strike, closeprice, putcall] = mysql('SELECT
    odate, daxprice, days, strike, closeprice, putcall FROM odax WHERE odate
    >= "2006-06-01" AND odate <= "2007-05-31" AND qualifiers <> "Missing"
    AND daxprice <> "NULL" ORDER BY odate, ric');
odaxdates = unique(odate);
numberofdays = length(odaxdates);
%create table for results
table=[];
table_LS=[];
tabledate=[]; %table for dates
```

```
table_price=[];
table_modelprice=[];
table_regress=[];
for j=1:numberofdays
  odaxdatestr = datestr(odaxdates(j),29);   %adjust odax date format
  mym('truncate dummydate'); mym('INSERT INTO dummydate(col1) VALUES("{S}")
      ', odaxdatestr); %updating dummydate
  ODAX = mym('SELECT daxprice, days, strike, closeprice, putcall,
      interestrate FROM odax, dummydate WHERE odax.odate = dummydate.col1
      AND qualifiers <> "Missing" AND odax.volume>=1 ORDER BY days, strike')
      ; %select current=j day
  daxprice = ODAX.daxprice; expdays = ODAX.days; ostrike = ODAX.strike;
      closeprice = ODAX.closeprice; putcall = ODAX.putcall; interestrates =
      ODAX.interestrate/100;
  S0 = daxprice(1);
  %use only four strikes around S0
  srl=100;srr=100;
  strikes_idx=and(ODAX.strike<=S0+srr,ODAX.strike>S0-srl);
  ostrike=ostrike(strikes_idx);
  daxprice=daxprice(strikes_idx);
  expdays=expdays(strikes_idx);
  closeprice=closeprice(strikes_idx);
  putcall=putcall(strikes_idx);
  interestrates=interestrates(strikes_idx);
  %read DAX closeprices for calculating ECF, up to the date analysed in
      dummydate(col1)
  daxclose = mysql('SELECT dadjclose FROM dax, dummydate WHERE dax.daxdate
      <= dummydate.col1 ORDER BY daxdate');
  daxlogrets=diff(log(daxclose));
  [exps, m, n] = unique(expdays); %sort with respect to maturities
  numberofmaturities = length(exps);
  m1 = [0;m];
  minprice=-10; %for plotting
  ndata=250;
  % calculate starting parameters by Method of Moments, use 250 historical
  % log returns
  DATA=daxlogrets(length(daxlogrets)-ndata+1:length(daxlogrets)); DATAmc=
      DATA - mean(DATA);
  theta_mm = mean(DATA)/tinc; sigma_mm = std(DATAmc)/sqrt(tinc); nu_mm = (
      kurtosis(DATAmc)/3 - 1)*tinc;
```

```
numberofmaturities_done=0;
for k=1:numberofmaturities
  S0 = daxprice(1);
  data=daxlogrets;
  maturity=expdays(m1(k)+1:m1(k+1))/daycount; %select same maturities
  expday=exps(k);       %expdays(m(k));
  strike=ostrike(m1(k)+1:m1(k+1));
  price=closeprice(m1(k)+1:m1(k+1));
  rate=interestrates(m1(k)+1:m1(k+1));
  optiontype=ones(length(putcall(m1(k)+1:m1(k+1))),1);
% if number of available options is less than 3, skip it
if length(strike)>=3
  [par_LS, modelprices_LS, resnorm_LS, difplus_LS, difminus_LS,
     meandif_LS, mediandif_LS, rmserel_LS] = cmlcalib(S0, data', ndata,
     rate, strike, price, expday, tinc, [theta_mm sigma_mm nu_mm],
     optiontype);
  %saving calibrated parameters
  rmse_LS = sqrt(resnorm_LS / length(price));
  tableinput_LS = [odaxdates(j), expday, ndata, par_LS, rate(1),
     resnorm_LS, rmse_LS, difplus_LS, difminus_LS, meandif_LS,
     mediandif_LS, rmserel_LS];
  table_LS = [table_LS; tableinput_LS];
  tabledateinput = [odaxdatestr];
  tabledate = [tabledate; tabledateinput];

 %saving ODAX prices and model prices
  tableinput = [price, modelprices_LS, S0./strike, maturity, rate];
  table_regress = [table_regress; tableinput];

  ff=figure(j); lcolor=expday;
  plot(strike, price, 's','Color',Mcolor(lcolor,:),'LineWidth',1); hold
     on;
  plot(strike, modelprices_LS,'go','LineWidth',2); hold on;
  K1 = [min(strike):1:max(strike)]; Y1 = max(S0-K1,0); plot(K1, Y1,'k','
     LineWidth',2); hold on;

  xlabel('Strike'); ylabel('Option price');
  legend('ODAX prices','CML MLE prices','CML LS prices','Payoff');
  title(sprintf('ODAX Call prices on %s',odaxdatestr));
  minprice1=min([price;modelprices_LS]); %for plotting
```

```
    minprice=min([minprice;price;modelprices_LS]);

  numberofmaturities_done=numberofmaturities_done+1;
else end %if length(strike) < 3 then next maturity
  end
  axis([min(strike)-1 max(strike)+1 minprice-1 max([price; modelprices_LS])
      +1]);
  h1=gca; x1=get(h1,'XLim'); y1=get(h1,'YLim');
  text(round((x1(2)-x1(1))/3+x1(1)),y1(2)-round(.1*(y1(2)-y1(1))),sprintf('
      S0 = %g',S0),'FontSize',12);
  for k1=1:numberofmaturities_done
    text(strike(1)+round(.1*(x1(2)-x1(1))),10+k1*round(.03*(y1(2)-y1(1))),
        num2str(exps(k1)),'Color',Mcolor(exps(k1),:),'FontSize',8,'
        HorizontalAlignment','right', 'FontWeight','bold');
  end
  text(strike(1)+round(.01*(x1(2)-x1(1))),10,'(days to expiry)','FontSize'
      ,8,'HorizontalAlignment','left', 'FontWeight','bold');
  saveas(ff,sprintf('ODAX%s_CML%g',odaxdatestr, ndata),'fig');
  saveas(ff,sprintf('ODAX%s_CML%g',odaxdatestr, ndata),'pdf');
  close(ff);
end
%Error measurements
price_err = table_regress(:,1)-table_regress(:,2);
price_err_rel = (table_regress(:,1)-table_regress(:,2))./table_regress(:,1)
    ;
price_err_abs = abs(price_err);
price_err_rel_abs = abs(price_err_rel);
table_errors = [mean(price_err); mean(price_err_rel); mean(price_err_abs);
    mean(price_err_rel_abs); sqrt(sum(price_err.^2)/length(price_err)); sqrt
    (sum(price_err_rel.^2)/length(price_err_rel))];
```