Modern Harmonic Analysis: Singular Integral Operators, Function Spaces and Applications.

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Summary

In this thesis, we aim to study Besov spaces associated with operators and our work has 2 new parts: one part is to extend certain results of [BDY] to different type of heat kernels not included in [BDY] and another part is to obtain molecular and atomic decompositions for Besov spaces for a larger range of indices.

Recently, in [BDY] the authors investigated the theory of Besov spaces associated to operators whose heat kernel satisfies an upper bound of Poisson type on the space of polynomial upper bound on volume growth. They also carried out that by different choices of operators L, they can recover most of the classical Besov spaces. Moreover, in some particular choices of L, they obtain new Besov spaces. In the first new part of this thesis, we aim to extend certain results in [BDY] to a more general setting when the underlying space can have different dimensions at 0 and infinity.

In the second new part of this thesis, the main aim is to lay out the theory of Besov spaces associated to operators L whose heat kernel satisfies the Gaussian upper bounds on spaces of homogeneous type. Adapting some ideas in [BDY], we construct the Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ associated to the operators. The main contribution is to investigate the atomic and molecular decompositions of functions in the new Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$. We also carry out the study that depending on the choice of L, our new Besov spaces may coincide with or may be properly larger than the classical Besov spaces in the space of homogeneous type. Finally, the behaviour of fractional integrals and spectral multipliers on the new Besov spaces is also investigated.

Declaration

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree to any other university or institution other than Macquarie University.

I also certify that the thesis is an original piece of research and it has been written by me. Any help and assistance that I have received in my research work and the preparation of the thesis itself has been appropriately acknowledged.

In addition, I certify that all information sources and literature used are indicated in the thesis.

Anthony Wong

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Contents

Chapter 1

Introduction

The main theme of this thesis is to study Besov spaces associated with operators.

This thesis is organised as follows.

In Chapter 1 we give some background of modern harmonic analysis, some preliminaries, and our main results.

In Chapter 2 we give some background of classical Besov spaces and Triebel–Lizorkin spaces.

In Chapters 3 and 4 we develop a theory of Besov spaces associated with an operator L under the assumption that L generates an analytic semigroup e^{-tL} with Gaussian kernel bounds on $L^2(\mathcal{X})$, where $\mathcal{X}$ is a quasi-metric space of polynomial upper bound on volume growth.

In Chapter 5 we derive atomic and molecular decompositions of Besov spaces associated to operators on spaces of homogeneous type.

1.1 Background

1.1.1 Modern harmonic analysis

The standard Calderón–Zygmund theory of singular integrals was one of the main achievements in modern harmonic analysis in the 60s and 70s (see [CZ, SW, Hö, CM]). Furthermore, the theory of Hardy spaces developed in the 70s (see [SW1, FS, Co1, La, TW1, St2]) has been very successful with many applications; in particular with proving boundedness of singular in-

tegrals and with regularity of solutions to partial differential equations. The development of the theory of Hardy spaces in $\mathbb{R}^n$ was initiated by Stein and Weiss [SW1], and was originally tied to the theory of harmonic functions. Real variable methods were introduced into this subject in the seminal paper of Fefferman and Stein [FS], the evolution of whose ideas led eventually to characterizations of Hardy spaces via the atomic or molecular decomposition. This enabled the extension of Hardy spaces to a far more general setting, that of a "space of homogeneous type" in the sense of Coifman and Weiss [CW].

The Calderón–Zygmund theory and Hardy space theory have formed the framework for the study of singular integrals that are bounded on L^2 whose kernels satisfy the standard Hörmander condition. To study L^p-boundedness ($1 < p < \infty$), one first studies the weak type $(1,1)$ estimate of the singular integrals and then the L^p-boundedness follows from interpolation and duality. Alternatively, one can study the $H^1 \to L^1$ estimate of the singular integrals in place of the weak type $(1,1)$ estimate.

In practical applications, there is a need to study singular integrals with non-smooth kernels which do not fall in the scope of Calderón-Zygmund operators. One of the breakthroughs in this direction was the work [DM^{c}] of X.T. Duong and A. McIntosh in the late 1990s. They considered the singular integral operators with non-smooth kernels on irregular domains, and obtained the weak type $(1,1)$ estimate, hence L^p estimate ($1 < p \leq 2$), for operators which are bounded on L^2 and whose kernels satisfy an "average" estimate which is strictly weaker than the usual Hörmander condition ([Hö]).

Now let (X, d, μ) be a space of homogeneous type in the sense of Coifman and Weiss. Note that spaces of homogeneous type were first introduced by R. Coifman and G. Weiss ([CW]) in the 1970s in order to extend the theory of Calderón–Zygmund singular integrals to a more general setting. In this setting, there are no translations or dilations, no analogue of the Fourier transform or convolution operation and no group structure. Let T be a bounded linear operator

on $L^2(X)$ with an associated kernel $k(x,y)$ in the sense that

$$Tf(x) = \int_X k(x,y)f(y)\,d\mu(y), \tag{1.1}$$

where $k(x,y)$ is a measurable function, and the above formula holds for each continuous function f with compact support, and for almost all x not in the support of f.

The well-known Hörmander condition on the kernel $k(x,y)$ states that an L^2 bounded operator T satisfies weak type $(1,1)$ estimates if there exist constants C and $\delta > 1$ so that

$$\int_{d(x,y)\geq \delta d(y_1,y)} |k(x,y) - k(x,y_1)|\,d\mu(x) \leq C \tag{1.2}$$

for all $y, y_1 \in X$.

In [DM^{c}], the authors assume that there exists a class of integral operators $\{A_t\}_{t>0}$, which plays the role of approximations to the identity as follows: Assume that A_t can be represented by kernels $a_t(x,y)$ in the sense that

$$A_t u(x) = \int_X a_t(x,y)u(y)\,d\mu(y), \tag{1.3}$$

and the kernels $a_t(x,y)$ satisfy

$$|a_t(x,y)| \leq h_t(x,y) = \frac{1}{\mu(B(x,t^{1/s}))} h(d(x,y)^s/t), \tag{1.4}$$

where s is a positive constant and h is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\eta} h(r^s) = 0 \tag{1.5}$$

for some $\eta > 0$.

The main result of [DM^{c}] is the following:

Theorem 1.1 (Duong–McIntosh)**.** *Let T be a bounded operator from $L^2(X)$ to $L^2(X)$ with an associated kernel $k(x,y)$. Assume that there exists a class of approximations to the identity $\{A_t\}_{t>0}$ so that TA_t have kernels $k_t(x,y)$ in the sense of (1.1) and there exist constants $C, c > 0$ so that*

$$\int_{d(x,y)\geq ct^{1/s}} |k(x,y) - k_t(x,y)|\,d\mu(x) \leq C \tag{1.6}$$

for all $y \in X$. Then the operator T is of weak type $(1,1)$. Hence, T can be extended from $L^2(X) \cap L^p(X)$ to a bounded operator on $L^p(X)$ for all $1 < p \leq 2$.

It is not difficult to check that condition (1.6) is strictly weaker than the Hörmander condition (1.2) by choosing an appropriate $\{A_t\}_{t>0}$.

1.1.2 Hardy space theory

Let us now consider the following inequalities for $f \in C_0^\infty(\mathbb{R}^n)$.

$$\|\nabla f\|_{L^p(\mathbb{R}^n)} \leq C_p \|\Delta^{1/2} f\|_{L^p(\mathbb{R}^n)} \tag{1.7}$$

$$\|\nabla^2 f\|_{L^p(\mathbb{R}^n)} \leq C_p \|\Delta f\|_{L^p(\mathbb{R}^n)} \tag{1.8}$$

The second is commonly known as the *Calderón–Zygmund inequality*. Here the constant C_p may depend on p and the dimension n, but not on f. Both are valid for all $1 < p < \infty$.

Inequalities such as these, often referred to as 'L^p-estimates', along with their analogues (when the space $L^p(\mathbb{R}^n)$ is replaced by other function spaces) have been thoroughly studied in the harmonic analysis literature, motivated in part by their connections with partial differential equations.

When one considers p below 1, inequalities (1.7) and (1.8) are valid for $p \leq 1$ once we replace the $L^p(\mathbb{R}^n)$ spaces and their norms by the Hardy spaces $H^p(\mathbb{R}^n)$ and their respective norms.

For $0 < p < \infty$ the tempered distribution f is said to belong to the Hardy space $H^p(\mathbb{R}^n)$ if the so-called "square function"

$$Sf(x) = \Big(\int_0^\infty \int_{|x-y|<t} \big| t^2 \Delta e^{t^2 \Delta} f(y) \big|^2 dy \frac{dt}{t^{n+1}} \Big)^{1/2}, \qquad x \in \mathbb{R}^n \tag{1.9}$$

satisfies $Sf \in L^p(\mathbb{R}^n)$. The study of these spaces began in [SW1] in the early 1960s. Real variable methods were introduced in [FS], and since then, the theory of Hardy spaces has undergone a rich development. We refer the reader to the monograph [St2] for an exposition on this subject.

For p below 1 these spaces are the natural continuation of the $L^p(\mathbb{R}^n)$ spaces because

firstly it can be shown that L^p coincides with H^p for $p > 1$, and secondly, on replacing L^p by H^p then (1.7) and (1.8) holds for all $0 < p < \infty$.

Part of the interest in the H^p spaces stems from their role in partial differential equations and in harmonic analysis. However it is known that there are many situations in which these classical spaces are not directly applicable. For instance the classical Riesz transforms $\nabla(-\Delta)^{-1/2}$ are bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ (and even $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$). In fact, $\nabla(-\Delta)^{-1/2}f \in L^p$ is one criterion for membership of f in H^p. See [St2] and [Gr]. Unfortunately given an arbitrary differential operator L, its associated Riesz transform $\nabla L^{-1/2}$ may not necessarily be bounded from H^1 to L^1. This may happen, for example, when L is an elliptic operator in divergence form with complex coefficients (see the discussion in [HMa] and also [Au, BK, HM] for results on the intervals of boundedness of $\nabla L^{-1/2}$ on $L^p(\mathbb{R}^n)$).

The notion of a *Hardy space adapted to an operator* was introduced to address some of these deficiencies. Given an operator L and in analogy with (1.9) we say that $f \in H^p_L(\mathbb{R}^n)$ provided the associated square function

$$S_L f(x) = \Big(\int_0^\infty \int_{|x-y|<t} \big|t^2 L e^{-t^2 L} f(y)\big|^2 dy \frac{dt}{t^{n+1}}\Big)^{1/2}, \qquad x \in \mathbb{R}^n$$

satisfies $S_L f \in L^p(\mathbb{R}^n)$. Depending on L, these spaces may or may not coincide with the classical Hardy spaces. Nevertheless under suitable conditions on L, the spaces $H^p_L(\mathbb{R}^n)$ may still interpolate with $L^p(\mathbb{R}^n)$. This is useful in applications. For instance the proof of some theorems given in [DOY] takes advantage of this fact. These spaces were initially introduced (for operators whose heat kernels satisfy suitable pointwise bounds) in [ADM^c, DY, DY1], and were further developed (for more general classes operators) in [AM^{c}R, HLMMY, HMa]. We refer the reader to these articles for the details and relevant references as well as some historical notes on the evolution of these ideas. For some recent applications of these H^p_L spaces to partial differential equations we refer the reader to [DHMMY].

In recent years, P. Auscher, X.T. Duong and A. McIntosh ([ADM^c]) first introduced the Hardy space $H^1_L(\mathbb{R}^n)$ associated with an operator L, and obtained a molecular decomposition,

assuming that L has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$ and the kernel of the heat semigroup e^{-tL} generated by L enjoys a pointwise Poisson upper bound. Later, with the same assumptions on L, X.T. Duong and L.X. Yan introduced the space $\mathrm{BMO}_L(\mathbb{R}^n)$ associated with L in [DY1] and established the duality of $H^1_L(\mathbb{R}^n)$ and $\mathrm{BMO}_{L^*}(\mathbb{R}^n)$ in [DY], where L^* denotes the adjoint operator of L in $L^2(\mathbb{R}^n)$.

In their approach, modeled on the work of X.T. Duong and A. McIntosh [DM^{c}] (see also X.T. Duong and D.W. Robinson [DR]) on weak-type $(1,1)$ bounds for generalized singular integrals, the heat semigroup or resolvent replaces the usual averaging operator over cubes or balls (in this connection, see also the work of J.M. Martell [Ma] on adapted sharp functions), and in place of a standard vanishing moment condition, "cancellation" becomes a matter of membership in the range of L. Subsequent work on this subject has been based on these two cornerstones.

More specifically, let L be a linear operator of type ω on $L^2(\mathbb{R}^n)$ with $\omega < \pi/2$; hence L generates a holomorphic semigroup e^{-zL}, $0 \le |\arg(z)| < \pi/2 - \omega$. Assume that

(i) The operator L has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$. That is, there exists $c_{\nu,2} > 0$ such that $b(L) \in \mathcal{L}(L^2, L^2)$, and for $b \in H_\infty(S^0_\nu)$:

$$\|b(L)g\|_2 \le c_{\nu,2}\|b\|_\infty \|g\|_2$$

for any $g \in L^2(\mathbb{R}^n)$, where $\nu > 0$ is related to holomorphic functional calculus. For the precise definition of ν, we refer to [M^{c}].

(ii) The holomorphic semigroup e^{-zL}, $0 \le |\arg(z)| < \pi/2 - \omega$ is represented by the kernel $p_z(x,y)$ which satisfies the upper bound

$$|p_z(x,y)| \le C_\theta h_{|z|}(x,y) \tag{1.10}$$

for $x, y \in \mathbb{R}^n$, $|\arg(z)| < \pi/2 - \theta$ for $\theta > \omega$, and $h_t(x,y)$ is the same as that in (1.4). (For the notation of operator of type ω and the H_∞-calculus, we refer to [M^{c}].)

For a detailed study of operators which have holomorphic functional calculi, we refer the reader to [M$^{\rm c}$].

Under the assumptions (i) and (ii), the Hardy space associated with L is defined as

$$H^1_L(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : S_L(f) \in L^1(\mathbb{R}^n)\},$$

where

$$S_L(f)(x) = \Big(\int_{\Gamma(x)} |Q_{t^s} f(y)|^2 \, \frac{dy\, dt}{t^{n+1}}\Big)^{1/2}.$$

Here s is the constant in (1.4) and $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$ is the standard cone of aperture 1 with vertex $x \in \mathbb{R}^n$ and $Q_t = tLe^{-tL} = -t\frac{d}{dt}e^{-tL}$.

For any $\beta > 0$, a function $f \in L^2_{loc}(\mathbb{R}^n)$ is said to be a function of β-type if f satisfies

$$\Big(\int_{\mathbb{R}^n} \frac{|f(x)|^2}{1+|x|^{n+\beta}}\, dx\Big)^{1/2} \le c < \infty. \tag{1.11}$$

Denote by $\mathcal{M}_\beta$ the collection of all functions of β-type. For $f \in \mathcal{M}_\beta$, the norm of f in $\mathcal{M}_\beta$ is defined by

$$\|f\|_{\mathcal{M}_\beta} = \Big(\int_{\mathbb{R}^n} \frac{|f(x)|^2}{1+|x|^{n+\beta}}\, dx\Big)^{1/2}.$$

It is shown in [DY] that $\mathcal{M}_\beta$ is a Banach space with respect to the norm $\|f\|_{\mathcal{M}_\beta}$. For any given operator L, we let

$$\Theta(L) = \sup\{\eta : (1.5) \text{ holds for } (1.10)\}.$$

Then we define $\mathcal{M} = \mathcal{M}_{\Theta(L)}$ if $\Theta(L) < \infty$; $\mathcal{M} = \bigcup_{0<\beta<\infty} \mathcal{M}_\beta$ if $\Theta(L) = \infty$. The BMO space associated to L is defined as

$$\mathrm{BMO}_L(\mathbb{R}^n) = \{f \in \mathcal{M} : \sup_B \frac{1}{|B|}\int_B |f(x) - e^{-r_B^s L} f(x)|\, dx < \infty\},$$

where the *sup* is taken over all balls in $\mathbb{R}^n$, r_B is the radius of the ball B and s is the constant in (1.4).

The main result of [DY] is the following:

Theorem 1.2 (Duong–Yan)**.** *Assume that L satisfies (i) and (ii). Then,*

$$(H^1_L(\mathbb{R}^n))' = \mathrm{BMO}_{L^*}(\mathbb{R}^n).$$

In subsequent work, L.X. Yan [Ya] generalized those to the Hardy space $H^p_L(\mathbb{R}^n)$ for all $0 < p \leq 1$.

Recently, P. Auscher, A. McIntosh and E. Russ [AM^{c}R], and Hofmann and Mayboroda [HMa], treated Hardy spaces H^p_L, $p \geq 1$, (and in the latter paper, BMO spaces) adapted, respectively, to the Hodge Laplacian on a Riemann manifold with doubling measure, and to a second order divergence form elliptic operator on $\mathbb{R}^n$ with complex coefficients, in which settings pointwise heat kernel bounds may fail. By making use of a notion of "L-cancellation" of molecules, they studied properties of the Hardy space H^1_L including a molecular decomposition, a square function characterization, its dual space and many others. Furthermore, in [HLMMY], S. Hofmann, G. Lu, D. Mitrea, M. Mitrea and L.X. Yan developed the theory of H^1 and BMO spaces associated with an operator L on spaces of homogeneous type X. The assumptions of L are the following:

(a) L is a non-negative self-adjoint operator on $L^2(X)$;

(b) The analytic semigroup $\{e^{-tL}\}_{t>0}$ satisfies the Davies–Gaffney condition. That is, there exist constants C, $c > 0$ such that for any open subsets U_1, $U_2 \subset X$,

$$|\langle e^{-tL} f_1, f_2\rangle| \leq C \exp\Big(-\frac{\operatorname{dist}(U_1, U_2)^2}{ct}\Big)\|f_1\|_{L^2(X)}\|f_2\|_{L^2(X)}, \quad \forall t > 0, \tag{1.12}$$

for every $f_i \in L^2(X)$ with $\operatorname{supp} f_i \subset U_i$, $i = 1, 2$, where

$$\operatorname{dist}(U_1, U_2) = \inf_{x \in U_1, y \in U_2} d(x, y).$$

For this Hardy space H^1_L, they obtained an atomic decomposition, that is, any function $f \in H^1_L$ can be represented as sum of atoms which are compactly supported.

1.2 Preliminaries

1.2.1 Notation

We collect here some standard notation we shall employ throughout this thesis.

All functions appearing in this thesis shall be measurable, have domain $\mathbb{R}^n$, and be complex-valued unless stated otherwise. We will reserve the letter n to denote the dimension of the Euclidean space $\mathbb{R}^n$. We use the usual definitions $\mathbb{N} = \{0, 1, 2, \ldots\}$ for the natural numbers and $\mathbb{Z}$ for the integers.

For a multi index $\kappa = (\kappa_1, \kappa_2, \ldots \kappa_n)$, $\kappa_j \in \mathbb{N}$, we define $|\kappa| = \kappa_1 + \kappa_2 + \ldots \kappa_n$ and

$$D^\kappa = \partial_{x_1}^{\kappa_1} \partial_{x_2}^{\kappa_2} \ldots \partial_{x_n}^{\kappa_n}$$

where ∂_{x_j} is the partial derivative with respect to x_j. We also use the obvious notation

$$x^\kappa = x_1^{\kappa_1} x_2^{\kappa_2} \ldots x_n^{\kappa_n}$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

We use $\mathcal{D}$ to denote the family of all dyadic cubes in $\mathbb{R}^n$ and $\mathcal{D}_j$ to be the family of dyadic cubes of the jth resolution. The cubes of $\mathcal{D}_j$ have sidelength equal to 2^{-j}. Let us explain briefly how to obtain $\mathcal{D}$. Consider firstly the set of cubes of form $[k_1, k_1+1) \times [k_2, k_2+1) \times \cdots \times [k_n, k_n+1)$ where $k_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, n$, which all have sidelength 1. We call this set $\mathcal{D}_0$. Next we bisect each cube of $\mathcal{D}_0$ to obtain a new collection of cubes. We call this new collection $\mathcal{D}_1$, each cube of this collection having sidelength $1/2$. We could also double each cube of $\mathcal{D}_0$ and obtain a collection of cubes of sidelength 2. We call this collection $\mathcal{D}_{-1}$. We see that

$$\mathcal{D} = \bigcup_{j \in \mathbb{Z}} \mathcal{D}_j$$

For $\alpha \in \mathbb{R}$ we use the notation $[\alpha]$ to denote the integer part of α, i.e.

$$[\alpha] = \max\{k \in \mathbb{Z} : k \leq \alpha\}$$

We will also make use of the usual definitions,

$$\widetilde{\eta}(y) = \eta(-y),$$

and

$$\tau_y \eta(x) = \eta(x - y).$$

Our underlying measure space, unless otherwise noted, will be $\mathbb{R}^n$ with the Lebesgue measure. Given a measurable set $E \subset \mathbb{R}^n$ we write $|E|$ to mean the Lebesgue measure of E. The notation $\int_E f(x)\,dx$ denotes the Lebesgue integral of f over E. At times we often drop the dx to simplify notation. We also use the notation

$$\fint_E f \;=\; \frac{1}{|E|}\int_E f$$

to mean the average of f over the measurable set E.

Given a measure space (X,μ) and $1 \le p < \infty$, we denote by $L^p(X,\mu)$ the Banach space of complex valued functions on X that are p-integrable. That is, we say that $f \in L^p(X,\mu)$ if the $L^p(X,\mu)$-norm of f,

$$\|f\|_{L^p(X)} = \Big(\int_X |f|^p \, d\mu\Big)^{1/p}$$

is finite. When $X = \mathbb{R}^n$ and $d\mu = dx$ and we will often write L^p in place of $L^p(\mathbb{R}^n)$. If $d\mu = w\,dx$ for some locally-integrable function w, then we write $L^p(w)$ instead. When we use the expressions *almost everywhere* or *almost every* x (abbreviated "a.e." or "a.e.x") we mean that the properties to which they refer hold except on a set of measure zero. The scalar product in $L^2(X)$ is denoted by $\langle\cdot,\cdot\rangle$.

Given normed spaces $\big(X, \|\cdot\|_X\big)$ and $\big(Y, \|\cdot\|_Y\big)$, the expression

$$T : X \to Y$$

will mean that T is a bounded mapping or operator (or admits a bounded extension) from X into Y. In this case we write $\|T\|_{X\to Y}$ to mean the operator norm of T, defined as

$$\|T\|_{X\to Y} = \inf\{C > 0 : \|Tx\|_Y \le C\,\|x\|_X\}.$$

When we refer to a ball centred at $x \in \mathbb{R}^n$ with radius $r > 0$, we mean the open set

$$B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}\,.$$

When we mention 'a ball B' we mean that a ball with a designated centre x_B and radius r_B has been chosen and fixed. By a cube $Q = Q(x_Q, l_Q)$ in $\mathbb{R}^n$ we mean a cube centred at x_Q

with sidelength l_Q, and with sides parallel to the coordinate axes. If $\lambda > 0$ then we write $\lambda B = B(x_B, \lambda r_B)$ (respectively $\lambda Q = Q(x_Q, \lambda l_Q)$) to mean the ball with the same centre as B but with radius dilated by a factor of λ (respectively a cube with the same centre as Q but with sidelength dilated by a factor of λ).

We define the *distance* between two subsets $E, F \subset \mathbb{R}^n$ as

$$\operatorname{dist}(E, F) = \inf\{|x - y| : x \in E, y \in F\}.$$

The notation $\mathbf{1}_E$ will be used to denote the indicator or characteristic function of the set E: $\mathbf{1}_E(x) = 1$ if $x \in E$ and 0 if $x \notin E$.

Given a function $\gamma : \mathbb{R}^n \to (0, \infty)$, we define balls associated to γ by $B(x, \gamma(x))$. We shall use the notation $\mathfrak{B}^\gamma(x) = B(x, \gamma(x))$. When we mention a ball $\mathfrak{B}^\gamma$ we mean that a ball with a designated centre x_B and radius $\gamma(x_B)$ has been fixed. That is, $\mathfrak{B}^\gamma = B(x_B, \gamma(x_B))$.

We will often discretise the space $\mathbb{R}^n$ into concentric annuli centred at a fixed ball B as follows:

$$U_j(B) = \begin{cases} B & j = 0 \\ 2^j B \backslash 2^{j-1} B & j \geq 1 \end{cases}$$

We can replace B by the balls $\mathfrak{B}^\gamma$ or a cube Q, with the obvious modifications.

Given a number $p \in [1, \infty]$ we shall use the notation p' to denote the conjugate exponent of p. That is, p and p' satisfy the relationship $1/p + 1/p' = 1$. We also write p^* to denote the Sobolev exponent of p. This is defined as

$$p^* = \begin{cases} \dfrac{np}{n-p} & p < n \\ \infty & p \geq n \end{cases}$$

We will also make use of the l^p spaces. We say the sequence $(a_k)_{k \in \mathbb{Z}}$ of complex numbers belongs to l^p if

$$\|(a_k)_{k \in \mathbb{Z}}\|_{l^p} = \Big(\sum_{k \in \mathbb{Z}} |a_k|^p \Big)^{1/p} < \infty.$$

We note that the l^p spaces are monotone in the sense that $l^p \subseteq l^q$ whenever $p \leq q$. This is a

consequence of the easily proven fact that

$$\|(a_k)_{k\in\mathbb{Z}}\|_{l^q} \leq \|(a_k)_{k\in\mathbb{Z}}\|_{l^p}$$

for any $p \leq q$.

For an open subset X of $\mathbb{R}^n$ and $k \in \mathbb{N}$ we denote by $C^k(X)$ the space of all continuous functions on X which have continuous partial derivatives up to order k. As remarked above when $X = \mathbb{R}^n$ we will use the short hand $C^k = C^k(\mathbb{R}^n)$. We denote by $\mathcal{P}$ the set of all polynomials.

A function f is in the Schwartz class, $\mathcal{S}(\mathbb{R}^n)$ if it is infinitely differentiable and if all of its derivatives decrease rapidly at infinity; that is, if for all multiindices α and β there exist positive constants $C_{\alpha,\beta}$ such that

$$\sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial^\beta f(x)\right| = C_{\alpha,\beta} < \infty.$$

The dual of $\mathcal{S}$, that is, the space of bounded linear functionals on $\mathcal{S}$, $\mathcal{S}'$, is called the space of tempered distributions. A linear map T from $\mathcal{S}$ to $\mathbb{C}$ is in $\mathcal{S}'$ if

$$\lim_{k\to\infty} T(\phi_k) = 0 \quad \text{whenever} \quad \lim_{k\to\infty} \phi_k = 0 \quad \text{in } \mathcal{S}.$$

The Fourier transform will initially be defined for any $\psi \in L^1$ by

$$\widehat{\psi}(\xi) = \int_{\mathbb{R}^n} \psi(x) e^{-2\pi i x\cdot\xi}\, dx.$$

We then extend the Fourier Transform to L^2 and $\mathcal{S}'$ in the usual manner. Throughout this thesis the notation $\widehat{\psi}$ will always denote the Fourier transform of ψ.

For measurable functions ψ and η, we define the convolution $\psi * \eta$ by

$$\psi * \eta(x) = \int_{\mathbb{R}^n} \psi(x-y)\eta(y)\, dy,$$

whenever the last integral is defined. We note that for $p \geq 1$ we have Young's Inequality

$$\|\psi * \eta\|_p \leq \|\psi\|_1 \|\eta\|_p.$$

The convolution can also be extended extended to $\mathcal{S}'$. For $f \in \mathcal{S}'$ and $\phi \in \mathcal{S}$ we define

$$\phi * f(x) = f(\tau_x \widetilde{\phi}).$$

It is well known that the function $\phi * f \in C^\infty$, and moreover, that $\phi * f$ is *not increasing too quickly* in the sense that there exists $k \in \mathbb{N}$ such that, for every $x \in \mathbb{R}^n$,

$$|\phi * f(x)| \leq C(1 + |x|)^k.$$

Let $0 < t < \infty$. We define the dilation ψ_t by

$$\psi_t(x) = t^{-n}\psi(x/t).$$

For a discrete parameter $j \in \mathbb{Z}$ we make the slight abuse of notation and write

$$\psi_j(x) = \psi_{2^{-j}}(x) = 2^{jn}\psi(2^j x).$$

We have the following result. Let $\psi \in L^1$ with $\widehat{\psi}(0) = 1$. If $g \in L^p \cap C^0$ for $1 \leq p \leq \infty$ we have, for every $x \in \mathbb{R}^n$,

$$\lim_{t \to 0} \psi_t * g(x) = g(x). \tag{1.13}$$

A similar result holds when $(1 + |\cdot|)^\lambda \psi(\cdot) \in L^1$ and $(1 + |\cdot|)^{-\lambda} g(\cdot) \in L^\infty \cap C^0$, since then the convolution $\psi * g$ is well-defined. We remark that (1.13) also applies when $\psi \in \mathcal{S}$ and $g \in \mathcal{S}'$, since then again the convolution $\psi * g$ is well-defined. However the convergence is now in $\mathcal{S}'$.

Let $k \in \mathbb{Z}$. We say $\psi \in L^1$ has k *vanishing moments* if for every $|\kappa| \leq k$ we have

$$\int_{\mathbb{R}^n} x^\kappa \psi(x)\, dx = 0. \tag{1.14}$$

If $k < 0$ we make the convention that no moment condition is required. It is easy to see that if $\psi \in \mathcal{S}$ then $D^\kappa \psi$ has $|\kappa| - 1$ vanishing moments, thus taking derivatives gives vanishing moments. The following theorem shows that in fact the converse is also true.

Theorem 1.3 ([BT2])**.** *Suppose $\mu \in \mathcal{S}$ has k vanishing moments. Then for every $|\kappa| = k + 1$ there exists $\mu^\kappa \in \mathcal{S}$ such that*

$$\mu = \sum_{|\kappa| = k+1} D^\kappa \mu^\kappa.$$

We remark that if $\widehat{\psi}$ vanishes in a neighbourhood of the origin then ψ has infinite vanishing moments. This in turn implies

$$\int_{\mathbb{R}^n} \rho(x)\psi(x)\, dx = 0$$

for any polynomial ρ.

Let $\psi \in L^1$. We say ψ satisfies the *Tauberian condition* if for every $|\xi| = 1$ there exists a $c > 0$ and $0 < 2\sigma \leq \varrho < \infty$ such that for each $\sigma < t < \varrho$

$$|\widehat{\psi}(t\xi)|^2 \geq c > 0. \tag{1.15}$$

The Tauberian condition implies that the family of functions $(\widehat{\psi}(2^{-j}\xi))_{j\in\mathbb{Z}}$ do not simultaneously vanish for every $|\xi| > 0$. This observation allows one to construct a function η such that for every $|\xi| > 0$ we have the equality

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi)\widehat{\eta}(2^{-j}\xi) = 1.$$

Formally, by multiplying both sides by $\widehat{\mu}$ and taking the inverse transform, this equality would then give the representation, the Calderón reproducing formula

$$\mu(x) = \sum_{j\in\mathbb{Z}} \psi_j * \eta_j * \mu(x). \tag{1.16}$$

The Poisson kernel on $\mathbb{R}^n$ is given by

$$P(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}}$$

where $\Gamma(z)$ is the gamma function given by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ for complex number z with Re $z > 0$.

On $\mathbb{R}^n$ the Riesz transforms associated to L are $\partial_j L^{-1/2}$ for $j \in \{1, ..., n\}$. We set $\nabla L^{-1/2} = (\partial_1 L^{-1/2}, \ldots, \partial_n L^{-1/2})$ and hence

$$\left|\nabla L^{-1/2} f\right| = \Big(\sum_{j=1}^n \left|\partial_j L^{-1/2} f\right|^2\Big)^{1/2}.$$

The second-order Riesz transforms associated to L are $\partial_j\partial_k L^{-1}$ for $j, k \in \{1, ..., n\}$. We take $\nabla^2 L^{-1}$ to mean the $n \times n$ matrix $(\partial_j\partial_k L^{-1})_{j,k}$, and also

$$\left|\nabla^2 L^{-1} f\right| = \Big(\sum_{j,k=1}^n \left|\partial_j\partial_k L^{1-} f\right|^2\Big)^{1/2}.$$

The following are well known representation formulae.

$$L^{-\alpha/2} = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} \frac{dt}{t^{1-\alpha/2}}, \qquad \alpha > 0,$$

$$\nabla L^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \nabla e^{-tL} \frac{dt}{\sqrt{t}},$$
$$\nabla^2 L^{-1} = \int_0^\infty \nabla^2 e^{-tL}\, dt.$$

One can arrive at these via functional calculus or spectral theory (see [Ha3]).

Throughout the thesis, we always use C and c to denote positive constants that are independent of the main parameters involved but whose values may differ from line to line. We shall write $A \lesssim B$ if there is a universal constant C so that $A \leq CB$ and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

1.2.2 Schrödinger operators

In this section we give the definition of the Schrödinger operator via forms and introduce the semigroup associated to this operator. For more on forms, operators and semigroups we refer the reader to [Da1, Ou, Ha3].

Let $n \geq 1$ and V be a non-negative locally-integrable function on $\mathbb{R}^n$. We define the form $\mathcal{Q}_V$ by

$$\mathcal{Q}_V(u,v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v + \int_{\mathbb{R}^n} V u v$$

with domain

$$\mathcal{D}(\mathcal{Q}_V) = \Big\{ u \in W^{1,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V\, |u|^2 < \infty \Big\}.$$

It is well known that this symmetric form is closed. It was also shown by Simon [Si2] that this form coincides with the minimal closure of the form given by the same expression but defined on $C_0^\infty(\mathbb{R}^n)$. In other words, $C_0^\infty(\mathbb{R}^n)$ is a *core* of the form $\mathcal{Q}_V$.

Let us denote by L the self-adjoint operator associated with $\mathcal{Q}_V$. Its domain is

$$\mathcal{D}(L) = \Big\{ u \in \mathcal{D}(\mathcal{Q}_V) : \exists\, v \in L^2(\mathbb{R}^n) \quad \text{with} \quad \mathcal{Q}_V(u,\varphi) = \int v\varphi, \quad \forall \varphi \in \mathcal{D}(\mathcal{Q}_V) \Big\}.$$

We write formally $L = -\Delta + V$.

We now introduce the heat kernel associated to L. Consider the following parabolic

equation

$$\Big(\frac{\partial}{\partial t}+L\Big)u(x,t)=0, \qquad (x,t)\in\mathbb{R}^n\times(0,\infty).$$

We are interested in the fundamental solution $\Gamma(x,y,t)$ of this equation. That is, Γ satisfies, for each $y\in\mathbb{R}^n$

$$\Big(\frac{\partial}{\partial t}+L\Big)\Gamma(x,y,t)=0, \qquad \forall\, x\in\mathbb{R}^n,\ x\neq y,\ t>0,$$

$$\lim_{t\to 0}\Gamma(x,y,t)=\delta(x-y).$$

This fundamental solution is called the *heat kernel* of L. We use the notation $p_t(x,y)$ in place of $\Gamma(x,y,t)$. The heat kernel generates a semigroup family of integral operators associated to L, which we shall denote by $\{e^{-tL}\}_{t>0}$ and refer to as the *heat semigroup associated to* L. That is, $p_t(x,y)$ is the integral kernel associated to e^{-tL} in the sense that

$$e^{-tL}f(x)=\int_{\mathbb{R}^n}p_t(x,y)\,f(y)\,dy, \qquad \text{for any } f\in L^2.$$

We denote by $h_t(x,y)$ the heat kernel of $-\Delta$ in $\mathbb{R}^n$. When $n\geq 3$ for each $x,y\in\mathbb{R}^n$ and $t>0$ it is well known that

$$h_t(x,y)=\frac{1}{(4\pi t)^{n/2}}e^{-|x-y|^2/4t}. \tag{1.17}$$

This is the integral kernel of the semigroup generated by $-\Delta$. That is,

$$e^{t\Delta}f(x)=\int_{\mathbb{R}^n}h_t(x,y)f(x)\,dy.$$

We also record the following fact.

$$\Big(\frac{\partial}{\partial t}+L\Big)^{-1}f(x,t)=\int_0^t e^{-(t-s)L}f(x,s)\,ds=\int_{\mathbb{R}^n}\int_0^t p_{t-s}(x,y)f(y,s)\,ds\,dy. \tag{1.18}$$

That is, the integral kernel of $\Big(\frac{\partial}{\partial t}+L\Big)^{-1}$ is $p_{t-s}(x,y)\mathbf{1}_{(0,t)}(s)$.

A useful formulation of the semigroup property is:

$$p_{2t}(x,y)=\int_{\mathbb{R}^n}p_t(x,u)\,p_t(u,y)\,du=e^{-tL}p_t(\cdot,y)(x). \tag{1.19}$$

for any $x, y \in \mathbb{R}^n$ and $t > 0$. Indeed,

$$\begin{aligned}\int p_{2t}(x,y)f(y)\,dy &= e^{-2tL}f(x) = e^{-tL}e^{-tL}f(x)\\ &= \int p_t(x,u)e^{-tL}f(u)\,du\\ &= \int p_t(x,u)\Big(\int p_t(u,y)f(y)\,dy\Big)du\\ &= \int \Big(\int p_t(x,u)\,p_t(u,y)\,du\Big)f(y)\,dy\\ &= \int e^{-tL}p_t(x,y)f(y)\,dy.\end{aligned}$$

The following *perturbation formula* holds as a consequence of perturbation for semigroups of operators (see for example [Pa]).

$$e^{t\Delta} - e^{-tL} = \int_0^t e^{(t-s)\Delta}Ve^{-sL}ds = \int_0^t e^{s\Delta}Ve^{-(t-s)L}ds. \tag{1.20}$$

This gives

$$\begin{aligned}h_t(x,y) - p_t(x,y) &= \int_0^t \int_{\mathbb{R}^n} h_{t-s}(x,z)V(z)p_s(z,y)\,dz\,ds\\ &= \int_0^t \int_{\mathbb{R}^n} h_s(x,z)V(z)p_{t-s}(z,y)\,dz\,ds.\end{aligned}$$

We remark that we can interchange the role of $-\Delta$ and L in (1.20).

In Section 4.2 we study decomposition of Besov spaces associated with Schrödinger operators.

1.2.3 The reverse Hölder class

In this section we define a class of potentials, and give a list of their known properties. These properties originated in [Sh].

Definition 1.4 (Reverse Hölder class)**.** *Let $1 < q < \infty$. We say that a non-negative and locally integrable function V belongs to the reverse Hölder class of order q if there exists $C > 0$ such that*

$$\Big(\fint_B V^q\Big)^{1/q} \le C \fint_B V$$

for all balls B. In this case we write $V \in \mathcal{B}_q$. We say that $V \in \mathcal{B}_\infty$ if there exists $C > 0$ such that for all balls B

$$V(x) \le C \fint_B V \qquad a.e.\ x \in B.$$

For all $1 < s < q$, it is easily seen that $\mathcal{B}_s \supset \mathcal{B}_q$. Furthermore, $V(x)dx$ is a doubling measure. That is, there is a constant $C_0 > 1$ such that

$$\int_{2B} V(x)\,dx \ \le\ C_0 \int_B V(x)\,dx.$$

It follows that for each $\lambda \ge 1$ there exists $n_0 > 0$ and $C > 0$ such that

$$\int_{\lambda B} V(x)\,dx \ \le\ C\lambda^{n_0} \int_B V(x)\,dx. \tag{1.21}$$

In fact we can take $n_0 = \log_2 C_0$.

Definition 1.5 (Critical radius)**.** *For $V \ge 0$ we define the* critical radius *associated to V at x by the following expression.*

$$\gamma(x) = \gamma(x, V) = \sup\Big\{r > 0 :\ r^2 \fint_{B(x,r)} V \le 1\Big\}. \tag{1.22}$$

Lemma 1.6 ([Sh] Lemmas 1.2 and 1.8)**.** *If $n \ge 1$ and $V \in \mathcal{B}_q$ for some $q > 1$ then there exists $C > 0$ such that the following holds:*

(a) *for each $\lambda > 1$ and all balls B,*

$$r_B^2 \fint_B V \ \le\ C\lambda^{n/q-2}(\lambda r_B)^2 \fint_{\lambda B} V\,,$$

(b) *for all balls B satisfying $r_B \ge \gamma(x_B)$,*

$$r_B^2 \fint_B V \ \le\ C\Big(\frac{r_B}{\gamma(x_B)}\Big)^\sigma$$

where $\sigma = n_0 - n + 2$.

Lemma 1.7 ([Sh] estimates 1.6 and 1.7)**.** *Let $V \in \mathcal{B}_q$. Then the following holds.*

(a) *If $q > n/2$ then there exists $C = C(n, q, V)$ such that for any ball B,*

$$\int_B \frac{V(x)}{|x_B - x|^{n-2}}\,dx \le \frac{C}{r_B^{n-2}} \int_B V(x)\,dx.$$

(b) *If $q \geq n$ then there exists $C > 0$ such that for any ball B,*

$$\int_B \frac{V(x)}{|x_B - x|^{n-1}} dx \leq \frac{C}{r_B^{n-1}} \int_B V(x)\, dx.$$

The next property states that the function γ is slowly varying.

Lemma 1.8 ([Sh] Lemma 1.4)**.** *Let $V \in \mathcal{B}_q$ with $q \geq n/2$. Then there exists $C_0 > 0$ and $\kappa_0 \geq 1$ with*

$$C_0^{-1}\, \gamma(x) \left(1 + \frac{|x-y|}{\gamma(x)}\right)^{-\kappa_0} \leq \gamma(y) \leq C_0\, \gamma(x) \left(1 + \frac{|x-y|}{\gamma(x)}\right)^{\frac{\kappa_0}{\kappa_0+1}}. \tag{1.23}$$

In particular if $x, y \in B(x_B, \lambda\gamma(x_B))$ for some $\lambda > 0$, then

$$\gamma(x) \leq C_\lambda \gamma(y) \tag{1.24}$$

where $C_\lambda = C_0^2 (1+\lambda)^{\frac{2\kappa_0+1}{\kappa_0+1}}$.

A consequence of (1.24) is that $\mathbb{R}^n$ admits a covering with 'critical balls' that has bounded overlap.

Lemma 1.9 ([DZ1])**.** *Let $V \in \mathcal{B}_q$ with $q \geq n/2$. Let $\gamma : \mathbb{R}^n \to (0, \infty)$ be as defined in (1.22). Then there exists a countable collection of critical balls $\{\mathfrak{B}_j^\gamma\}_j = \{B(x_{B_j}, \gamma(x_{B_j}))\}_j$ satisfying the following properties.*

(i) $\bigcup_j \mathfrak{B}_j^\gamma = \mathbb{R}^n$.

(ii) *For every $\sigma \geq 1$ there exists constants C and N such that $\sum_j \mathbf{1}_{\sigma\mathfrak{B}_j^\gamma} \leq C\sigma^N$.*

Remark 1.10. *We note the following dilation. Set $\sigma = C2^{\kappa/(\kappa+1)}$ where C and κ are from (1.23). Then there exists C and $\widetilde{N}$ such that $\sum_j \mathbf{1}_{\sigma\mathfrak{B}_j^\gamma} \leq C\sigma^{\widetilde{N}}$ and it follows from (1.23) that for each j,*

$$\bigcup_{x \in \mathfrak{B}_j^\gamma} \mathfrak{B}^\gamma(x) \subseteq \widetilde{\mathfrak{B}_j^\gamma}$$

where $\widetilde{\mathfrak{B}_j^\gamma} = \sigma\mathfrak{B}_j^\gamma$.

1.3 Main results

In Chapter 2 we give some background of classical Besov spaces and Triebel–Lizorkin spaces. We look at the main stages of development of classical Besov spaces and Triebel–Lizorkin spaces. We then give definitions and some properties (Theorem 2.1) of classical Besov spaces and Triebel–Lizorkin spaces. We also give further properties of classical Besov spaces, including decomposition (Theorem 2.10), potential theory and singular integrals on Besov spaces.

The main theme of this thesis is to study Besov spaces associated with operators. Our study of Besov spaces associated with operators is divided over three chapters, Chapters 3, 4 and 5. We study Besov spaces associated with an operator L under the assumption that L generates an analytic semigroup e^{-tL} with Gaussian kernel bounds on $L^2(\mathcal{X})$, where $\mathcal{X}$ is a quasi-metric space of polynomial upper bound on volume growth. We extend certain results in [BDY] to a more general setting when the underlying space can have different dimensions at 0 and infinity.

In Section 3.2, we give definitions of quasi-metric spaces of polynomial upper bounds on volume growth, then some assumptions on the operator L, and define Besov norms associated with L (Definition 3.4). We also give an upper bound estimate of the Besov norm of the heat kernels (Proposition 3.6).

In Section 3.3, we introduce the space of test functions associated with L (Definition 3.7). We then define the Besov norms for linear functionals (on space of test functions) and Besov spaces associated with L (Definition 3.9). In order to study properties of these Besov spaces, we prove several versions of Calderón reproducing formulas for linear functionals (Theorems 3.3, 3.13 and 3.14). Furthermore, we show that the classical Besov spaces $B^{\alpha}_{p,q}(\Omega)$ and ${}_zB^{\alpha}_{p,q}(\Omega)$ are special cases in this theory (Proposition 3.15).

In Section 4.1, we study an embedding theorem (Theorem 4.1) for the Besov spaces and give discrete characterizations of the Besov norms associated to operators (Proposition 4.3). We also study the equivalence of the Besov norms with respect to different functions of L (Proposition 4.4). We extend the Besov norm equivalence to more general class of functions

$\Psi_t(L)$ with suitable decay at 0 and infinity, and to non-integer $k \geq 1$ (Propositions 4.5 and 4.6). Then we study the behaviour of fractional integrals on the Besov spaces (Theorem 4.9).

In Section 4.2, we study decomposition of Besov spaces associated with Schrödinger operators with non-negative potentials satisfying reverse Hölder estimates on $\mathbb{R}^n$ (Theorem 4.13). We also show that, in some special cases, the classical Besov spaces are proper subspaces of these spaces (Theorem 4.14). We also extend the decomposition of Besov spaces associated with Schrödinger operators to more general values α, p, q (Theorem 4.19).

The main aim of chapter 5 is to lay out the theory of Besov spaces associated to operators L whose heat kernel satisfies the Gaussian upper bounds on spaces of homogeneous type. Adapting some ideas in [BDY], we construct the Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ associated to the operators. Note that since the assumption of polynomial upper bound on volume growth in [BDY] do not include the spaces of homogeneous type, some refinements and improvements would be required. The main contribution of chapter 5 is to investigate the atomic and molecular decompositions of functions in new Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$. (Note that there are no results on atomic and molecular decompositions for the general Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ in [BDY]). Precisely, we are able to prove the following results:

(i) Under the Gaussian upper bound assumption only, we prove that each function in our Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$, with $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$, admits a linear molecular decomposition. We would like to emphasize that there are no smoothness conditions on the molecules. Conversely, each linear molecular decomposition belongs to the Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$, with $-1 < \alpha < 0$ and $1 \leq p, q \leq \infty$. See Theorem 5.20 and Theorem 5.22.

(ii) Under the Gaussian upper bound, Hölder continuity and conservation assumptions, we prove the theory of molecular decomposition on Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$, with $-1 < \alpha < \delta$ and $1 \leq p, q \leq \infty$ where δ is a positive constant depending on the smoothness order of the heat kernel of the operator L. See Theorem 5.27.

(iii) We study the theory of molecular decomposition on Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$, with $-1 <$

$\alpha < \delta_0$ and $1 \leq p, q \leq \infty$ for some δ_0, associated with operators of Schrödinger type. It is worth noting that the conservation property is not assumed here. See Theorem 5.30. Note that our findings have applications in various settings such as Schrödinger operators, degenerate Schrödinger operators on $\mathbb{R}^n$ and Schrödinger operators on Heisenberg groups and connected and simply connected nilpotent Lie groups.

(iv) In the particular case $p = q$, the atomic decomposition of Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ is also obtained in Theorem 5.36.

We also carry out the relationship between our Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ and the Besov spaces $\dot{B}^{\alpha}_{p,q}(X)$ of Han and Sawyer in [HS]. See Section 5.5. We prove the following results:

(i) Under the Gaussian upper bound, Hölder continuity and conservation assumptions, we show the coincidence between our Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ and the Besov spaces $\dot{B}^{\alpha}_{p,q}(X)$ for all indices α being close to zero.

(ii) When the operator L is an operator of Schrödinger type, we show the inclusion $\dot{B}^{\alpha}_{p,q}(X) \subset \dot{B}^{\alpha,L}_{p,q}(X)$ for some suitable values of p, q and α.

Note that for the investigation on the atomic decomposition, the approach in [HS] was based on a construction of a family of approximation to the identity and a Calderón reproducing formula. Roughly speaking, these kernels associated to this family satisfy the Gaussian upper bound, Hölder continuity and conservation properties. In most parts of our work, we do not need the conservation assumption. Even if L satisfies the Gaussian upper bound, Hölder continuity and conservation properties, our obtained results are still new as we do not assume either the polynomial growth nor the reverse doubling property on the volume of the balls on the underlying spaces. Moreover, when the order of the family of approximation to the identity is less than 1, the results on Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ in chapter 5 are new for values α being close to -1.

The organisation of chapter 5 is as follows. In Section 5.1, we recall some basic properties on the regularity of the time derivative of the heat kernels and the covering lemma of Christ in

[Ch1]. The theory of Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ and their basic properties is addressed in Section 5.2. The molecular and atomic decompositions on Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ are investigated in Section 5.3 and Section 5.4, respectively. In Section 5.5, we compare our Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ and Besov spaces $\dot{B}^{\alpha}_{p,q}(X)$. As applications, in Section 5.6 we study the behaviour of fractional integrals and spectral multipliers on new Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$.

Chapter 2

Classical Besov spaces and Triebel–Lizorkin spaces

In this chapter we give some background of classical Besov spaces and Triebel–Lizorkin spaces.

In Section 2.1 we look at the main stages of development of classical Besov spaces and Triebel–Lizorkin spaces.

In Section 2.2 we give definitions and some properties of classical Besov spaces and Triebel–Lizorkin spaces.

In Section 2.3 we give further properties of classical Besov spaces, including decomposition, potential theory and singular integrals on Besov spaces.

2.1 Main stages of development

We now turn to the background of the theory of Besov spaces and Triebel–Lizorkin spaces. The theory of Besov spaces has a long history and plays a central part in harmonic analysis and partial differential equations. The theory of Besov spaces has been an active area of research in the last few decades because of its important role in the study of approximation of functions and regularity of solutions to partial differential equations.

Classical theory of Besov spaces, for example, can be found in [Be, Ta, Pe1, Pe, He, Tr1, St1, FJ1, BPT]. Some of more recent results on Besov spaces are [SW, Tr2, GHL, DHY]. The studies on Besov spaces on the Euclidean spaces $\mathbb{R}^n$, for example, can be found in [Tr1, Tr2, Be, BPT2, BPT, BPT1, FJ1]. The theory of Besov spaces on a domain Ω of $\mathbb{R}^n$ was investigated in [Ry1, Ry2, TW]. The recent theory on Besov spaces on the spaces of homogeneous type, for

example, is developed in [HS, HMY, MY]. A historical account of the subject is beyond the scope of this thesis, and we refer the reader to [Pe, Tr1, Tr2] for that. We shall, however, briefly discuss the main stages of its development which are relevant to this thesis.

The Besov spaces $B^{\alpha}_{p,q}(\mathbb{R}^n) = B^{\alpha}_{p,q}$ (also called Lipschitz spaces or Besov–Lipschitz spaces) were introduced by O.V. Besov [Be, Be1] for $\alpha > 0$, via the modulus of continuity method. The works of O.V. Besov, S.M. Nikol'skii, V.P. Il'in, P.J. Lizorkin and their collaborators were partially motivated by the theory of approximation of functions, see e.g. [BIN, Ni, Tr1].

A short time after the appearance of Besov's works, M.H. Taibleson [Ta] characterized Besov spaces by using the Hardy–Littlewood method via the Poisson kernel and the Gaussian kernel. As a consequence, he also extended the definition of $B^{\alpha}_{p,q}$ to $\alpha \leq 0$. Further works in this direction were done by T.M. Flett [Fl1, Fl2], R. Johnson [Jo], among others.

The third development came from the works by J. Peetre [Pe1, Pe2, Pe] where the Fourier analytic method and the real variable techniques (developed by C.L. Fefferman and E.M. Stein [FS]) were used to obtain some new characterizations. Many contributions in this direction were also made by H. Triebel and others (see [Tr1, FJ1, FJ2]). J. Peetre also succeeded in extending the theory to the case where either $0 < p < 1$ or $0 < q < 1$. After some preliminary results by Peetre and Triebel, the Hardy–Littlewood type characterization of Besov and Triebel–Lizorkin spaces was completed by H.-Q. Bui, M. Paluszýnski and M.H. Taibleson [BPT, BPT1], by using a crucial reproducing formula of A.P. Calderón [JT]. Recent works connected with the Peetre's approach include [BZ, DP] where the authors defined certain classes of Besov spaces associated with the Schrödinger operators.

The classical Besov spaces $B^{\alpha}_{p,q}(\Omega)$ on a domain Ω of $\mathbb{R}^n$ are usually defined by either the restriction method or certain intrinsic characterization. These characterizations of the Besov spaces have been investigated extensively, see [Tr, Tr1, Mu, Ry1, Ry2, TW]. Another existing classical Besov space on Ω is the space $\widetilde{B}^{\alpha}_{p,q}(\Omega) = {}_zB^{\alpha}_{p,q}(\Omega)$ which is defined by "zero extension" outside $\overline{\Omega}$ (see [Tr]). When Ω is bounded and smooth, H. Sikić and M.H. Taibleson [ST] obtained a Hardy–Littlewood type characterization of ${}_zB^{\alpha}_{p,q}(\Omega)$ and $B^{\alpha}_{p,q}(\Omega)$ using the kernels of "killed

Brownian motion" and "reflecting Brownian motion".

The existence of two different kinds of Besov spaces on a domain is similar to the situation for the Hardy spaces in the paper [CKS] by D.-C. Chang, S.G. Krantz and E.M. Stein. In all these works on a general domain Ω, some regularity on the boundary of Ω is assumed. Moreover, the classical theory of the homogeneous Besov spaces $\dot{B}^{\alpha}_{p,q}(\Omega)$ was not fully developed, due mainly to the difficulty with the invariance under diffeomorphisms.

2.2 Classical Besov and Triebel–Lizorkin spaces

The Besov and Triebel–Lizorkin spaces have a long history and their properties have been well studied. As with many function spaces, the Besov and Triebel–Lizorkin spaces come in two varieties, the homogeneous versions, denoted by $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$ respectively, and the inhomogeneous versions, denoted by $B^{\alpha}_{p,q}$ and $F^{\alpha}_{p,q}$ These types of spaces arise naturally when one considers the problem of measuring the smoothness of a distribution. There are various ways of measuring the smoothness of a distribution, one way is to ask for a generalisation of the classical homogeneous Sobolev spaces $\dot{W}^{p,k}$ where, for $k \in \mathbb{N}$ with $k > 0$, we define

$$\dot{W}^{p,k} = \{f \in \mathcal{S}' : D^{\kappa} f \in L^p \quad \forall\, |\kappa| = k\}.$$

The homogeneous Triebel–Lizorkin space, $\dot{F}^{\alpha}_{p,q}$, satisfies this requirement in the sense that $\dot{F}^{k}_{p,2} = \dot{W}^{p,k}$ whenever $k \in \mathbb{N}$ with $k > 0$, and $1 < p < \infty$. We also remark that the restriction of $f \in \dot{W}^{p,k}$ on $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ belongs to a certain Besov space; see [AF]. These results hint at the important applications of Triebel–Lizorkin and Besov spaces to the study of partial differential equations (PDEs), as the Sobolev spaces are widely used in the theory of PDEs.

The scale of spaces, $\dot{F}^{\alpha}_{p,q}$, also includes other important function spaces such as the Hardy space, H^p, where we define, using the characterisation of Fefferman and Stein (see [St2], pg 91),

$$H^p = \left\{ f \in \mathcal{S}' : \|f\|_{H^p} = \left\| \sup_{t>0} |\phi_t * f| \right\|_p < \infty \right\}$$

with $\phi \in \mathcal{S}$ such that $\widehat{\phi}(0) > 0$. In this case it can be proven that we have the equivalence $\dot{F}^{0}_{p,2} = H^p$; see [Bu1]. Throughout this thesis we restrict our attention to the homogeneous

Besov and Triebel–Lizorkin spaces. We refer the interested reader to [Tr1] and [Pe] for more on the history and origins of the Besov and Triebel–Lizorkin spaces.

2.2.1 Definitions

The definition of Besov and Triebel–Lizorkin spaces will require a kernel $\varphi \in \mathcal{S}$ satisfying, for any $\xi \in \mathbb{R}^n \backslash \{0\}$,

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{j}\xi) = 1 \tag{2.1}$$

and moreover

$$\operatorname{supp} \widehat{\varphi} \subseteq \{1/2 \leq |\xi| \leq 2\}. \tag{2.2}$$

Note that $\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j}\xi)$ where $\varphi_j(x) = 2^{jn}\varphi(2^j x)$, thus

$$\operatorname{supp} \widehat{\varphi}_j(\xi) \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$$

and so $(\widehat{\varphi}_j)_{j\in\mathbb{Z}}$ decomposes $\mathbb{R}^n$ into dyadic annuli. Let $\widehat{\phi}_0 = \sum_{j\leq 0} \widehat{\varphi}(2^{-j}\xi)$.

We can now define the Besov space, $B^{\alpha}_{p,q}$, as follows. For $0 < p, q \leq \infty$, $\alpha \in \mathbb{R}$, and any distribution $f \in \mathcal{S}'$, we define

$$\|f\|_{B^{\alpha}_{p,q}} = \|\phi_0 * f\|_p + \left(\sum_{j\geq 1} (2^{j\alpha}\|\varphi_j * f\|_p)^q\right)^{1/q}$$

and

$$B^{\alpha}_{p,q} = \{f \in \mathcal{S}' : \|f\|_{B^{\alpha}_{p,q}} < \infty\}.$$

Similarly, for $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, and $0 < p < \infty$, we let

$$\|f\|_{F^{\alpha}_{p,q}} = \|\phi_0 * f\|_p + \left\| \left(\sum_{j\geq 1} (2^{j\alpha}|\varphi_j * f|)^q\right)^{1/q} \right\|_p$$

and define the Triebel–Lizorkin space, $F^{\alpha}_{p,q}$, by

$$F^{\alpha}_{p,q} = \{f \in \mathcal{S}' : \|f\|_{F^{\alpha}_{p,q}} < \infty\}.$$

We define the homogeneous Besov space, $\dot{B}^{\alpha}_{p,q}$, as follows. For $0 < p, q \leq \infty$, $\alpha \in \mathbb{R}$, and any distribution $f \in \mathcal{S}'$, we define

$$\|f\|_{\dot{B}^{\alpha}_{p,q}} = \left(\sum_{j\geq 1} (2^{j\alpha}\|\varphi_j * f\|_p)^q\right)^{1/q}$$

and

$$\dot{B}^{\alpha}_{p,q} = \{f \in \mathcal{S}' : \|f\|_{\dot{B}^{\alpha}_{p,q}} < \infty\}.$$

Similarly, for $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, we let

$$\|f\|_{\dot{F}^{\alpha}_{p,q}} = \left\| \left(\sum_{j \geq 1} (2^{j\alpha} |\varphi_j * f|)^q \right)^{1/q} \right\|_p$$

and define the homogeneous Triebel–Lizorkin space, $\dot{F}^{\alpha}_{p,q}$, by

$$\dot{F}^{\alpha}_{p,q} = \{f \in \mathcal{S}' : \|f\|_{\dot{F}^{\alpha}_{p,q}} < \infty\}.$$

The following Poisson integral definition is due to Taibleson [Ta]: Fix $k \in \mathbb{N}$ and for $0 < \alpha < k$, $1 \leq p, q \leq \infty$ we define

$$\dot{B}^{\alpha}_{p,q} = \{f \in L^p : \left(\int_0^{\infty} (t^{-\alpha} \|t^k \frac{\partial^k}{\partial t^k} e^{-t\sqrt{\Delta}}\|_{L^p})^q \, \frac{dt}{t} \right)^{1/q} < \infty\}.$$

Note that the assumption (2.2) on φ implies that φ has infinite vanishing moments. Thus, for any polynomial ρ, we have $\varphi * \rho = 0$ and as a result

$$\|\rho\|_{\dot{B}^{\alpha}_{p,q}} = 0,$$

hence $\|\cdot\|_{\dot{B}^{\alpha}_{p,q}}$ is not a norm. However, if we consider elements of $\dot{B}^{\alpha}_{p,q}$ modulo polynomials, then $\|\cdot\|_{\dot{B}^{\alpha}_{p,q}}$ does form a norm (quasi-norm if $0 < \min\{p, q\} < 1$). Moreover, if we regard $\dot{B}^{\alpha}_{p,q}$ as a subset of $\mathcal{S}'/\mathcal{P}$, then $\dot{B}^{\alpha}_{p,q}$ is a Banach space (quasi-Banach space if $0 < \min\{p, q\} < 1$); see [Bu1]. A similar comment applies for the Triebel–Lizorkin space $\dot{F}^{\alpha}_{p,q}$. For the remainder of this thesis we will slightly abuse notation by refering to the quasi-norms, $\|\cdot\|_{\dot{B}^{\alpha}_{p,q}}$ and $\|\cdot\|_{\dot{F}^{\alpha}_{p,q}}$, as norms.

We also remark that different choices of φ satisfying (2.1) and (2.2) lead to the same spaces $\dot{B}^{\alpha}_{p,q}$ and $\dot{F}^{\alpha}_{p,q}$ with equivalent norms. We refer the reader to the work of J. Peetre, [Pe2], for a proof of this fact.

2.2.2 Properties of Besov and Triebel–Lizorkin spaces

We note the following elementary properties of the (homogeneous) Besov and Triebel–Lizorkin spaces.

Theorem 2.1. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $\alpha \in \mathbb{R}$.*

(i) The spaces $B^\alpha_{p,q}$ and $F^\alpha_{p,q}$ are (semi) Banach spaces and are independent of the generating function φ. Moreover $\mathcal{S}$ forms a dense subspace if $p, q < \infty$.

(ii) The homogeneous spaces $\dot{B}^\alpha_{p,q}$ and $\dot{F}^\alpha_{p,q}$ are (semi) Banach spaces when considered modulo polynomials. Moreover the collection of all $\varphi \in \mathcal{S}$ with infinite vanishing moments (see (1.14)) forms a dense subspace if $p, q < \infty$.

(iii) For any $f \in \mathcal{S}'$ we have

$$\|f\|_{B^\alpha_{p,\max\{p,q\}}} \leq \|f\|_{F^\alpha_{p,q}} \leq \|f\|_{B^\alpha_{p,\min\{p,q\}}},$$

$$\|f\|_{\dot{B}^\alpha_{p,\max\{p,q\}}} \leq \|f\|_{\dot{F}^\alpha_{p,q}} \leq \|f\|_{\dot{B}^\alpha_{p,\min\{p,q\}}},$$

and, if $0 < q_1 < q_2 \leq \infty$,

$$\|f\|_{B^\alpha_{p,q_2}} \leq \|f\|_{B^\alpha_{p,q_1}}, \qquad \|f\|_{F^\alpha_{p,q_2}} \leq \|f\|_{F^\alpha_{p,q_1}},$$

$$\|f\|_{\dot{B}^\alpha_{p,q_2}} \leq \|f\|_{\dot{B}^\alpha_{p,q_1}}, \qquad \|f\|_{\dot{F}^\alpha_{p,q_2}} \leq \|f\|_{\dot{F}^\alpha_{p,q_1}}.$$

Moreover,

$$\|f\|_{\dot{B}^{\alpha-n/p}_{\infty,q}} \leq C\|f\|_{\dot{B}^\alpha_{p,q}}.$$

(iv) If $\alpha > n/p$ then $f \in \dot{B}^\alpha_{p,q}$ implies that f is a slowly increasing, continuous function.

(v) For $0 < p_1 < p_2 \leq \infty$ we have the Sobolev type embedding

$$\|f\|_{B^\alpha_{p_2,q}} \leq C\|f\|_{B^{\alpha+\frac{n}{p_1}-\frac{n}{p_2}}_{p_1,q}}$$

(vi) Convergence in $B^\alpha_{p,q}$, $F^\alpha_{p,q}$, ($\dot{B}^\alpha_{p,q}$ or $\dot{F}^\alpha_{p,q}$) implies convergence in $\mathcal{S}'$ ($\mathcal{S}'/\mathcal{P}$).

Proof. The proof of properties (i), (ii) and (vi) can be found in [Tr1], (iii) is a straightforward application of the Minkowski inequality and the l^p estimate

$$\|(a_j)\|_{l^p} \leq \|(a_j)\|_{l^r}$$

which holds for $r \leq p$. See [Bu1].

Property (v) follows from an application of the following lemma.

Lemma 2.2. *Assume* $1 \le p \le r \le \infty$. *Then*

$$\|\varphi_j * f\|_{L^r} \le C 2^{j(\frac{n}{p}-\frac{n}{r})} \|\varphi_j * f\|_{L^p}.$$

Proof. Let $\phi \in \mathcal{S}$ satisfy $\widehat{\phi}(\xi) = 1$ for $2^{-1} \le |\xi| \le 2$. Then $\widehat{\varphi} = \widehat{\varphi}\widehat{\phi}$ and so

$$\varphi_j * f = \phi_j * \varphi_j * f.$$

Recalling Young's inequality for convolutions

$$\|g * h\|_{L^r} \le \|g\|_{L^q} \|h\|_{L^p}$$

where $1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{p}$ we have

$$\begin{aligned} &\|\varphi_j * f\|_{L^r} \\ &\quad = \|\phi_j * \varphi_j * f\|_{L^r} \\ &\quad \le \|\phi_j\|_{L^q} \|\varphi_j * f\|_{L^p}. \end{aligned}$$

Thus as

$$\begin{aligned} &\|\phi_j\|_{L^q} \\ &\quad = 2^{j(n-\frac{n}{q})} \|\phi\|_{L^q} \\ &\quad \le C 2^{j(\frac{n}{p}-\frac{n}{r})} \end{aligned}$$

the result follows. □

To prove (iv), suppose $f \in \dot{B}^{\alpha}_{p,q}$ and $\alpha > n/p$. We begin by defining, for $|\xi| > 0$,

$$\widehat{\phi}(\xi) = \sum_{j=-\infty}^{0} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi)$$

where, by our convention, $\varphi \in \mathcal{S}$ satisfies (2.1) and (2.2). If we take $\widehat{\phi}(0) = 1$ then ϕ satisfies

$$\widehat{\phi}(\xi) = \begin{cases} 1, & |\xi| \le 1 \\ 0, & |\xi| \ge 2. \end{cases}$$

and hence $\phi \in \mathcal{S}$. Moreover, for $m, M \in \mathbb{Z}$ with $M > m$,

$$\begin{aligned}
&\sum_{j=m+1}^{M} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi) \\
&\quad = \sum_{j=-\infty}^{M} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi) - \sum_{j=-\infty}^{m} \widehat{\varphi}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi) \\
&\quad = \widehat{\phi}(2^{-M}\xi)\widehat{\phi}(2^{-m}\xi)
\end{aligned}$$

and hence

$$\sum_{j=m+1}^{M} \varphi_j * \varphi_j(x) = \phi_M(x) - \phi_m(x). \tag{2.3}$$

Take $\mu \in \mathcal{S}$ such that

$$\widehat{\mu}(\xi) = \begin{cases} 1, & |\xi| \leq 2 \\ 0, & |\xi| \geq 3 \end{cases}$$

and let $g = f - \mu * f$. Since $\widehat{\mu} = 1$ on the support of $\widehat{\phi}$ we have, for any $\xi \in \mathbb{R}^n$, $\widehat{\phi}(\xi)\widehat{\mu}(\xi) = \widehat{\phi}(\xi)$.

Therefore

$$\phi * g = \phi * f - \phi * \eta * f = \phi * f - \phi * f = 0. \tag{2.4}$$

Similarly, as $\operatorname{supp} \widehat{\varphi} = \{1/2 \leq |\xi| \leq 2\}$, we have $\varphi_j * \mu = 0$ whenever $j \geq 3$. Thus

$$\varphi_j * g = \varphi_j * f \tag{2.5}$$

for any $j \geq 3$. Combining (2.4) and (2.5) with (2.3) we obtain, for any $3 \leq m < M$,

$$\phi_M * g - \phi_m * g = \sum_{j=m+1}^{M} \varphi_j * \varphi_j * f.$$

Therefore, using the assumption $f \in \dot{B}^{\alpha}_{p,q}$ together with property (iii), we see that

$$\begin{aligned}
&\|\phi_M * g - \phi_m * g\|_\infty \\
&\quad \leq C \sum_{j=m+1}^{M} \|\varphi_j * f\|_\infty \\
&\quad \leq C \sum_{j=m+1}^{M} 2^{-j(\alpha - n/p)}
\end{aligned}$$

and hence the family of functions $(\phi_M * g)_{M \in \mathbb{N}}$ forms a Cauchy sequence in L^∞. Thus, as L^∞ forms a Banach space, there exists an $h \in L^\infty$ such that $\phi_M * g$ converges to h uniformly.

Moreover, the uniform convergence implies that h is also continuous. Finally, since $\phi_M * g$ converges to g in $\mathcal{S}'$, we must have $g = h \in L^\infty$ and therefore, as $\mu * f$ is a slowly increasing, smooth function,

$$f = g + \mu * f$$

is a slowly increasing, continuous function. □

Remark. The important exponents are α and p. The exponent q is used more to fine tune the function spaces (one should think of q as measuring the difference between say weak-L^1 and L^1). The α exponent measures the smoothness of f. To see this pretend for the moment we know how to differentiate α times for any $\alpha \in \mathbb{R}$. Then since we expect $\partial^\alpha \varphi_j \sim 2^{\alpha j}(\partial^\alpha \varphi)_j$ and $\partial^\alpha \varphi \sim \varphi$ we have

$$2^{\alpha j}\varphi_j * f \sim 2^{j\alpha}(\partial^\alpha \varphi)_j * f \sim \partial^\alpha(\varphi_j * f) \sim \varphi_j * \partial^\alpha f.$$

Thus $f \in \dot{B}^\alpha_{p,q}$ is roughly the same as $\partial^\alpha f \in \dot{B}^0_{p,q}$ and so a large α implies that we have control over a number of derivatives of f. In other words a large α implies that f should be relatively smooth.

Property (v) in the above proposition is known as a Sobolev type embedding because it allows one to trade regularity for integrability. Thus we can make p larger at the expense of making α smaller.

Another property of the Besov and Triebel–Lizorkin spaces is the following characterisation obtained in [BPT] and [BPT1]. Before we state the theorem we briefly define the Peetre maximal function, $\mu_k^* f$, by

$$\mu_k^* f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\mu_k * f(y)|}{(1 + |x - y|)^\lambda}$$

where $\lambda > 0$ is some fixed constant.

Theorem 2.3. *Fix $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$. Suppose $\mu \in \mathcal{S}$ has $[\alpha]$ vanishing moments (see (1.14)) and $\eta \in \mathcal{S}$ satisfies the Tauberian condition (see (1.15)). Then if $\lambda > n/p$ and $f \in \mathcal{S}'$,*

there exists a polynomial ρ depending only on f such that

$$\begin{aligned}&\left(\int_0^\infty \left(t^{-\alpha}\|\mu_t * (f-\rho)\|_{H^p}\right)^q \frac{dt}{t}\right)^{1/q}\\ &\quad\le C_1\|f\|_{\dot{B}^\alpha_{p,q}}\\ &\quad\le C_2\left(\int_0^\infty \left(t^{-\alpha}\|\eta_t * f\|_p\right)^q \frac{dt}{t}\right)^{1/q}.\end{aligned} \tag{2.6}$$

Similarly, if $p<\infty$ and $\lambda > \max\{n/p, n/q\}$, then for any $f\in\mathcal{S}'$ there exists a polynomial ρ depending only on f such that

$$\begin{aligned}&\left\|\left(\int_0^\infty \left(t^{-\alpha}\mu_t^*(f-\rho)\right)^q \frac{dt}{t}\right)^{1/q}\right\|_p\\ &\quad\le C_1\|f\|_{\dot{F}^\alpha_{p,q}}\\ &\quad\le C_2\left\|\left(\int_0^\infty \left(t^{-\alpha}|\eta_t * f|\right)^q \frac{dt}{t}\right)^{1/q}\right\|_p.\end{aligned} \tag{2.7}$$

Moreover, the discrete version of both inequalities also holds.

Proof. For a proof, we refer the reader to [BPT] and [BPT1]. □

We make the following remarks. Firstly, the polynomial term in (2.6) and (2.7) cannot be removed. To see this note that the infinite vanishing moments of φ implies that the convolution $\varphi_j * f$ annihilates all polynomials, i.e. $\varphi * \rho = 0$ for any polynomial ρ. On the other hand, as we only require μ to have $[\alpha]$ vanishing moments, the convolution $\mu * \rho$ will not vanish for polynomials with degree higher than $[\alpha]$. Thus taking $f=\rho$ with ρ some polynomial of degree higher than $[\alpha]$ gives $\|f\|_{\dot{B}^\alpha_{p,q}} = 0$ but

$$\left(\int_0^\infty \left(t^{-\alpha}\|\mu_t * f\|_{H^p}\right)^q \frac{dt}{t}\right)^{1/q} > 0.$$

Thus the polynomial term in (2.6) cannot be removed. A similar comment applies to the Triebel–Lizorkin version, (2.7).

Secondly, since $\|\mu_t * f\|_p \le \|\mu_t * f\|_{H^p}$ for all $0<p,t<\infty$ and any $\mu\in\mathcal{S}$, the above theorem implies that

$$\left(\int_0^\infty \left(t^{-\alpha}\|\mu_t * (f-\rho)\|_p\right)^q \frac{dt}{t}\right)^{1/q} \le C\|f\|_{\dot{B}^\alpha_{p,q}},$$

assuming $f \in \mathcal{S}'$ and μ has $[\alpha]$ vanishing moments. In particular, when $p = q = \infty$, we have

$$\sup_{t>0} t^{-\alpha} \|\mu_t * (f - \rho)\|_\infty \leq C \|f\|_{\dot{B}^\alpha_{\infty,\infty}}.$$

Moreover, as remarked in [BPT1], we may replace $\|\mu_t * (f - \rho)\|_{H^p}$ with $\|\mu_t^*(f - \rho)\|_p$ in (2.6).

The major reason why these spaces are useful is that the spaces $B^\alpha_{p,q}$ and $F^\alpha_{p,q}$, as well as their homogeneous versions, include a large number of the classical function spaces which appear in analysis.

Theorem 2.4. *The following function spaces are equivalent.*

$$\begin{aligned}
&\dot{F}^0_{p,2} \approx F^0_{p,2} \approx L^p && 1 < p < \infty \\
&\dot{F}^0_{p,2} \approx H^p && 0 < p \leq 1 \\
&F^0_{p,2} \approx H^p_{loc} && 0 < p \leq 1 \\
&F^s_{p,2} \approx W^{s,p} \approx L^p_s && 1 < p < \infty \\
&\dot{F}^s_{p,2} \approx \dot{W}^{s,p} \approx \dot{L}^p_s && 1 < p < \infty \\
&\dot{F}^0_{\infty,2} \approx \mathrm{BMO} \\
&F^0_{\infty,2} \approx \mathrm{bmo} \\
&B^\alpha_{\infty,\infty} \approx \Lambda^\alpha && \alpha > 0 \\
&\dot{B}^\alpha_{\infty,\infty} \approx \dot{\Lambda}^\alpha && \alpha > 0
\end{aligned}$$

Proof. See the first chapter in [Gr2]. The equivalence between H^p_{loc} and $F^0_{p,2}$ is proven in [Bu1]. □

Remark. We should mention that although the Besov spaces $B^\alpha_{p,q}$ do not appear so often on the above list, they are still important. One reason for this is they occur when one tries to restrict say functions in $W^{s,p}$ to a hyperplane or more generally to some lower dimensional submanifold, see [AF]. They are also much easier to work with than the Triebel–Lizorkin variants as one takes the L^p norm first, thus they often appear in nonlinear PDEs as a replacement for the Sobolev spaces $W^{s,p}$ when some endpoint embedding fails.

We will briefly sketch one application of the theory of Besov and Triebel–Lizorkin spaces. This concerns the embedding $L^\infty \subset W^{s,p}$ which holds for every $s > \frac{n}{p}$. This embedding is a simple application of the characterisation of $W^{s,p}$ proved earlier as

$$\begin{aligned}
&\|f\|_{L^\infty} \\
&\quad \leq \sum_{j\geq 0} \|\varphi_j * f\|_{L^\infty} \\
&\quad \leq C \sum_{j\geq 0} 2^{j\frac{n}{p}} \|\varphi_j * f\|_{L^p} \\
&\quad = \|f\|_{B^{\frac{n}{p}}_{p,1}}.
\end{aligned}$$

The embedding $F^s_{p,2} \subset B^{\frac{n}{p}}_{p,1}$ which holds for every $s > \frac{n}{p}$ then completes the proof. Note that we have to spend slightly more derivatives than we would like. Scaling would suggest that we should have the embedding $L^\infty \subset W^{\frac{n}{p},p}$, however what we get instead is that L^∞ is contained in the slightly larger space $B^{\frac{n}{p}}_{p,1}$. This is useful as it allows us to take advantage of the endpoint embedding, often in nonlinear PDEs we cant afford to waste derivatives so using the Besov space is a useful way to make the most of any regularity our functions have.

We note that the embedding $L^\infty \subset W^{s,p}$ is known to fail at the endpoint $s = \frac{n}{p}$. We illustrate this in the (easier) case $p = 2$. We need to show there exists f such that

$$\int_{\mathbb{R}^n} (1 + |\xi|)^n |\widehat{f}(\xi)|^2 \, d\xi < \infty$$

but $f \notin L^\infty$. We define f via its Fourier transform as

$$\widehat{f} = \begin{cases} \frac{1}{|\xi|^n \log|\xi|}, & |\xi| > 2 \\ 1, & |\xi| \leq 2. \end{cases}$$

It is easy to check $\widehat{f} \in L^2$ and so the inverse Fourier transform is well-defined. Similarly we can check that $f \in W^{\frac{n}{2},2}$. On the other hand

$$f(0) = 2^n + \int_{|\xi|\geq 2} \frac{1}{|\xi|^n \log|\xi|} \, d\xi = \infty$$

thus an approximation argument can show $f \notin L^\infty$ as required. This example illustrates a useful

method for finding counter examples, namely using the fact that for $p > 1$ we have

$$\int_{|x|>2} \frac{1}{|x|^n (\log |x|)^p}\, dx < \infty$$

while for $p = 1$ the integral diverges. This is easily proved by noting that

$$\int \frac{1}{r(\log r)^p}\, dr = \begin{cases} \log\log r, & p = 1 \\ (\log r)^{1-p}, & p > 1. \end{cases}$$

2.3 Further properties of classical Besov spaces

2.3.1 Atomic decomposition

We now give an atomic characterization of the Besov spaces. We first define the notion of smooth atoms.

Definition 2.5. *A function $a_Q \in C_c^\infty(\mathbb{R}^n)$ is a smooth N-atom for a cube Q if and only if*

(i) $\operatorname{supp} a_Q \subset 3Q$

(ii) $\int x^\gamma a_Q(x)\, dx = 0$ *for all* $|\gamma| \le N$

(iii) $|\partial^\gamma a_Q(x)| \le C_\gamma |Q|^{-\frac{\gamma}{n} - \frac{1}{2}}$ *for all multi-indices* $\gamma \in \mathbb{Z}_+^n$.

Next we define some sequence spaces.

Definition 2.6. *For $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, we define*

$$\dot{b}^\alpha_{p,q} = \Big\{ \{s_Q\}_{Q \in \mathcal{D}} : \|\{s_Q\}\|_{\dot{b}^\alpha_{p,q}} < \infty \Big\}$$

where $\{s_Q\}$ is a sequence of complex numbers indexed by the family of all dyadic cubes in $\mathbb{R}^n$, and

$$\|\{s_Q\}\|_{\dot{b}^\alpha_{p,q}} = \Bigg(\sum_{j \in \mathbb{Z}} \Bigg(\sum_{Q \in \mathcal{D}_j} (|Q|^{-\frac{\alpha}{n} - \frac{1}{2} + \frac{1}{p}} |s_Q|)^p \Bigg)^{q/p} \Bigg)^{1/q}.$$

The space $\dot{b}^{\alpha}_{p,q}$ plays the same role for $\dot{B}^{\alpha}_{p,q}$ as $l^p(\mathbb{Z})$ does for $L^p([0,1])$.

Now we give some discrete Calderón reproducing formula that we will use to decompose our Besov spaces.

Lemma 2.7 ([FJ1], Lemma 2.1)**.** *Let $f \in \mathcal{S}'/\mathcal{P}$ and φ, ψ be functions such that*

(i) $\varphi, \psi \in \mathcal{S}$,

(ii) $\operatorname{supp} \widehat{\varphi} \subseteq \{\xi : |\xi| < \pi\}$ *and* $\operatorname{supp} \widehat{\psi} \subseteq \{\xi : |\xi| < \pi\}$,

(iii) $\sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^j\xi)\widehat{\psi}(2^j\xi) = 1$ *for* $\xi \neq 0$.

Then

$$f(x) = \sum_{j\in\mathbb{Z}} 2^{-jn} \sum_{k\in\mathbb{Z}} \varphi_j * f(2^{-j}k)\psi_j(x - 2^{-j}k).$$

Here both the convergence of the right hand side and the equality is in $\mathcal{S}'/\mathcal{P}$.

Lemma 2.8 ([FJW], Lemma 5.12)**.** *Let $f \in \mathcal{S}'/\mathcal{P}$ and φ, ψ be functions such that*

(i) $\operatorname{supp} \theta \subset \{x : |x| \leq 1\}$,

(ii) $\int x^\gamma \theta(x)\, dx = 0$ *if* $|\gamma| \leq N$,

(iii) $\operatorname{supp} \widehat{\varphi} \subset \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$,

(iv) $|\widehat{\varphi}(\xi)| \geq c > 0$ *if* $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$,

(v) $\sum_{j\in\mathbb{Z}} \widehat{\theta}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi) = 1$ *for all* $\xi \neq 0$.

We then have

$$f = \sum_{j\in\mathbb{Z}} \varphi_j * \theta_j * f$$

in $\mathcal{S}'/\mathcal{P}$.

Next we give one more lemma that will be used to prove (2.9). Many proofs of this lemma exist in the literature and has a connection with Hardy spaces.

Lemma 2.9 ([FJ1], Lemma 2.4)**.** *Let* $0 < p \leq \infty$, $j \in \mathbb{Z}$ *and* $g \in \mathcal{S}'$ *with* $\operatorname{supp} \widehat{g} \subset \{\xi : |\xi| \leq 2^{j+1}\}$. *Then*

$$\left(\sum_{Q \in \mathcal{D}_j} \sup_{x \in Q} |g(x)|^p \right)^{1/p} \leq C_{n,p} 2^{jn/p} \|g\|_2^p.$$

Now we present the atomic decomposition. It was first proved in [FJ1].

Theorem 2.10. *Let* $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, *and* $N \in \mathbb{Z}_+$. *For each* $f \in \dot{B}^{\alpha}_{p,q}$ *there exists a sequence* $\{s_Q\}_{Q \in \mathcal{D}} \in \dot{b}^{\alpha}_{p,q}$ *and smooth* N*-atoms* $\{a_Q\}_{Q \in \mathcal{D}}$ *such that*

$$f = \sum_{Q \in \mathcal{D}} s_Q a_Q \tag{2.8}$$

where the convergence is in $\mathcal{S}'/\mathcal{P}$. *Furthermore we also have*

$$\|\{s_Q\}\|_{\dot{b}^{\alpha}_{p,q}} \leq C \|f\|_{\dot{B}^{\alpha}_{p,q}}. \tag{2.9}$$

Proof. The idea is to decompose our Besov objects in the time and space variables. The first step is to write our Besov distribution as a discrete sum over the integers. This is roughly a decomposition in the time variable. Then for each integer in this sum, we further decompose over the underlying domain over dyadic cubes. Then we may select the atoms from the resulting expression. We now proceed with the details.

Firstly we use Lemma 2.8 to write

$$f = \sum_{j \in \mathbb{Z}} \varphi_j * \theta_j * f$$

which is valid in $\dot{B}^{\alpha}_{p,q}$ because $\dot{B}^{\alpha}_{p,q}$ is a subspace of $\mathcal{S}'/\mathcal{P}$.

Next for each j we further decompose $\varphi_j * \theta_j * f$ over the dyadic cubes of resolution j.

$$\varphi_j * \theta_j * f(x) = \sum_{Q \in \mathcal{D}_j} \int_Q \theta_j(x-y)(\varphi_j * f)(y)\, dy.$$

In other words, we split up the θ_j convolution over dyadic cubes.

Now we define our coeffcients

$$s_Q = |Q|^{1/2} \sup_{y \in Q} |(\varphi_j * f)(y)| \tag{2.10}$$

and our atoms

$$a_Q(x) = \frac{1}{s_Q} \int_Q \theta_j(x-y)(\varphi_j * f)(y)\, dy. \tag{2.11}$$

We must first check that these are indeed smooth N-atoms.

(i) $\operatorname{supp} a_Q \subset 2Q$.

Since $\operatorname{supp} \theta \subset \{x : |x| \leq 1\}$ then $\operatorname{supp} \theta_j \subset \{x : |x| \leq 2^{-j}\}$. Hence in the definition for a_Q we see that $\theta_j(x-y)$ is non-zero precisely when $|x-y| \leq 2^{-j}$. Since $y \in Q$ this is equivalent to x being at most a distance of 2^{-j} away from the cube Q. Finally noting that $l(Q) = 2^{-j}$ tells us $x \in 3Q$.

(ii) $\int x^\gamma a_Q(x)\, dx = 0$ for $|\gamma| \leq N$.

Note that

$$\int x^\gamma a_Q(x)\, dx = \int \frac{x^\gamma}{s_Q} \int_Q \theta_j(x-y)(\varphi_j * f)(y)\, dy\, dx = 0$$

follows by Fubini's theorem and the properties of θ.

(iii) $|\partial^\gamma a_Q(x)| \leq |Q|^{-\frac{|\gamma|}{n} - \frac{1}{2}}$.

We have firstly that a_Q is smooth because $\theta \in \mathcal{S} \subset C^\infty$. Next

$$\begin{aligned}
&|\partial^\gamma a_Q(x)| \\
&\quad \leq \frac{1}{s_Q} \int_Q |\partial_x^\gamma \theta_j(x-y)||\varphi_j * f(y)|\, dy \\
&\quad \leq \frac{1}{s_Q} \left(\int_Q |\partial_x^\gamma \theta_j(x-y)|^2\, dy \right)^{1/2} \left(\int_Q |\varphi_j * f(y)|^2\, dy \right)^{1/2} \\
&\quad \leq \frac{1}{s_Q} \left(\int_Q |\partial_x^\gamma \theta_j(x-y)|^2\, dy \right)^{1/2} \|\varphi_j * f\|_\infty |Q|^{1/2} \\
&\quad = \left(\int_Q |\partial_x^\gamma \theta_j(x-y)|^2\, dy \right)^{1/2} \\
&\quad = \left(\int_Q 2^{2j(n+|\gamma|)} |\partial_x^\gamma \theta(x-y)|^2\, dy \right)^{1/2} \\
&\quad = 2^{j(n+|\gamma|)} \|\partial^\gamma \theta\|_2 \\
&\quad \leq 2^{j(n+|\gamma|)} \|\partial^\gamma \theta\|_\infty |Q|^{1/2} \\
&\quad = l(Q)^{-n-|\gamma|} \|\partial^\gamma \theta\|_\infty |Q|^{1/2}
\end{aligned}$$

$$\leq C_\gamma |Q|^{-|\gamma|-\frac{1}{2}}$$

In the fourth equality we have used that $|Q| = l(Q)^n = 2^{-jn}$ and in the second we have used that $\partial^\gamma \theta(2^j x) = 2^{j|\gamma|}\partial^\gamma \theta(x)$. In the final inequality we use that $\|\partial^\gamma \theta\|_\infty \leq C_\gamma$ with the constant chosen to be independent of θ.

Therefore the definitions (2.11) and (2.10) give a decomposition of f into smooth N-atoms in $\mathcal{S}'/\mathcal{P}$, as in the expression (2.8).

Now we prove the inequality (2.9). We do this by applying Lemma 2.9 to $\varphi_j * f$ in the $\dot{b}^\alpha_{p,q}$ norm of $\{s_Q\}_Q$. We have

$$\begin{aligned}
&\left(\sum_{Q\in\mathcal{D}_j} (|Q|^{-\frac{\alpha}{n}-\frac{1}{2}+\frac{1}{p}}|s_Q|)^p\right)^{1/p} \\
&= \left(\sum_{Q\in\mathcal{D}_j} (|Q|^{-\frac{\alpha}{n}+\frac{1}{p}} \sup_{y\in Q} |\varphi_j * f(y)|)^p\right)^{1/p} \\
&= 2^{j\alpha - j\frac{n}{p}} \left(\sum_{Q\in\mathcal{D}_j} \sup_{y\in Q} |\varphi_j * f(y)|^p\right)^{1/p} \\
&\leq 2^{j\alpha - j\frac{n}{p}} C_{n,p} 2^{j\frac{n}{p}} \|\varphi_j * f\|_p \\
&= C_{n,p} 2^{j\alpha} \|\varphi_j * f\|_p
\end{aligned}$$

Hence (2.9) follows after taking $l^q(\mathbb{Z})$ norms. □

2.3.2 Potential theory

Next we look at some potential theory.

We define the Riesz potentials for $0 < \alpha < n$ by

$$\mathcal{I}_\alpha f(x) = (-\Delta)^{-\alpha/2} f(x) = \frac{1}{\gamma_1(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,$$

with

$$\gamma_1(\alpha) = \pi^{n/2} 2^\alpha \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n}{2} - \frac{\alpha}{2})}$$

They act as smoothing operators. We have the following fact:

Theorem 2.11 ([FJW]). *For $0 < \alpha < n$, $1 \leq p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, we have*

$$\|\mathcal{I}_\alpha f\|_q \leq C\|f\|_p.$$

The Riesz potentials are nice locally ($|x| \to 0$) but not globally ($|x| \to \infty$). As an alternative we consider replacing the non-negative operator $-\Delta$ by the strictly positive operator $I-\Delta$. The resulting operators are called Bessel potentials. The following are some representation formulae.

$$\mathcal{J}_\alpha f(x) = (I-\Delta)^{-\alpha/2} f(x) := G_\alpha * f(x), \qquad \alpha > 0$$

where G_α is defined by

$$G_\alpha(x) = \frac{1}{\gamma_2(\alpha)} \int_0^\infty e^{-\pi|x|^2/t} e^{-\frac{t}{4\pi}} t^{\frac{-n+\alpha}{2}} \frac{dt}{t}, \qquad \alpha > 0$$

with $\gamma_2(\alpha) = (4\pi)^{-\alpha/2}\Gamma(\alpha/2)^{-1}$.

If we take $\mathcal{J}_0 = I$ then the family of operators $\{\mathcal{J}_\alpha\}_{\alpha \geq 0}$ form a semigroup.

Next note that $\mathcal{J}_\alpha$ is well defined on L^p. In fact we have

$$\|\mathcal{J}_\alpha f\|_p \leq \|f\|_p, \qquad 1 \leq p \leq \infty.$$

Hence for each α it makes sense to define the following *potential space* or *generalized Sobolev space*

$$\begin{aligned}
&\mathcal{L}^p_\alpha(\mathbb{R}^n) \\
&\quad = (I-\Delta)^{-\alpha/2} L^p(\mathbb{R}^n) \\
&\quad = \left\{ (I-\Delta)^{-\alpha/2} g : g \in L^p \right\} \\
&\quad = \left\{ ((1+|\cdot|^2)^{\alpha/2})^\vee \widehat{f} \in L^p : f \in L^p \right\} \\
&\quad = \left\{ f \in L^p : \left\| (I-\Delta)^{\alpha/2} f \right\|_p < \infty \right\}
\end{aligned}$$

with norm

$$\|f\|_{\mathcal{L}^p_\alpha} = \left\| (I-\Delta)^{\alpha/2} f \right\|_p.$$

They are defined to be the images of L^p under the Bessel potentials. More generally they can be defined as spaces of tempered distributions. Homogeneous versions can also be defined. They are called generalized Sobolev spaces because when k is an integer and $1 < p < \infty$,

$$\mathcal{L}^p_k = W^p_k.$$

The following indicates the relationship of these spaces to Besov spaces.

Theorem 2.12 ([FJW]). *For $\alpha > 0$, and $2 \leq p < \infty$,*

$$\dot{B}^{\alpha}_{p,2} \subset \mathcal{L}^p_\alpha \subset \dot{B}^{\alpha}_{p,p}$$

For $\alpha > 0$, and $1 \leq p \leq 2$,

$$\dot{B}^{\alpha}_{p,p} \subset \mathcal{L}^p_\alpha \subset \dot{B}^{\alpha}_{p,2}$$

From this we see that $\mathcal{L}^p_\alpha = \dot{B}^{\alpha}_{2,2}$.

2.3.3 Singular integrals on Besov spaces

We next look at the action of singular integrals on Besov spaces.

Let $K(x)$ be a function defined away from the origin on $\mathbb{R}^n$ that satisfies the size estimate

$$\sup_{0<R<\infty} \frac{1}{R} \int_{|x|\leq R} |K(x)||x|\, dx \leq A_1, \tag{2.12}$$

Hörmander's condition

$$\sup_{y\in\mathbb{R}^n\setminus\{0\}} \int_{|x|\geq 2|y|} |K(x-y) - K(x)|\, dx \leq A_2, \tag{2.13}$$

and the cancellation condition

$$\sup_{0<R_1<R_2<\infty} \left| \int_{R_1<|x|<R_2} K(x)\, dx \right| \leq A_3, \tag{2.14}$$

and $A_1, A_2, A_3 < \infty$. Condition (2.14) implies that there exists a sequence $\varepsilon_j \downarrow 0$ as $j \to \infty$ such that the limit exists:

$$\lim_{j\to\infty} \int_{\varepsilon_j \leq |x| \leq 1} K(x)\, dx = L_0.$$

This gives that for a smooth and compactly supported function f on $\mathbb{R}^n$, the limit

$$\lim_{j\to\infty} \int_{|x-y|>\varepsilon_j} K(x-y) f(y)\, dy = T(f)(x) \tag{2.15}$$

exists and defines a linear operator T. This operator T is given by convolution with a tempered distribution W that coincides with the function K on $\mathbb{R}^n\setminus\{0\}$.

We know by singular integral theory that such a T, which is initially defined on $C_0^\infty(\mathbb{R}^n)$, admits an extension that is L^p bounded for all $1 < p < \infty$ and also of weak type $(1,1)$. All these norms are bounded above by dimensional multiples of the quantity $A_1 + A_2 + A_3$. Therefore, such a T is well defined on $L^1(\mathbb{R}^n)$. We have the following result concerning Besov spaces.

Theorem 2.13 ([Gr])**.** *Let K satisfy (2.12), (2.13), and (2.14), and let T be defined as in (2.15). Let $1 \le p \le \infty$, $0 < q \le \infty$ and $\alpha \in \mathbb{R}$. Then there is a constant $C_{n,p,q,\alpha}$ such that for all f in $\mathcal{S}(\mathbb{R}^n)$ we have*

$$\|T(f)\|_{\dot{B}^\alpha_{p,q}} \le C_{n,p,q,\alpha}(A_1 + A_2 + A_3)\|f\|_{\dot{B}^\alpha_{p,q}}. \tag{2.16}$$

Therefore, T admits a bounded extension on all homogeneous Besov spaces $\dot{B}^\alpha_{p,q}$ with $p \ge 1$.

Chapter 3

Besov spaces associated with operators I

Some of the content in this chapter are contained in [Wo1]. In this chapter we develop a theory of Besov spaces associated with an operator L under the assumption that L generates an analytic semigroup e^{-tL} with Gaussian kernel bounds on $L^2(\mathcal{X})$, where $\mathcal{X}$ is a quasi-metric space of polynomial upper bound on volume growth. We extend certain results in [BDY] to a more general setting when the underlying space can have different dimensions at 0 and infinity.

In Section 3.1, we give some preliminaries.

In Section 3.2, we give definitions of quasi-metric spaces of polynomial upper bounds on volume growth, then some assumptions on the operator L, and define Besov norms associated with L. We also give an upper bound estimate of the Besov norm of the heat kernels.

In Section 3.3, we introduce the space of test functions associated with L. We then define the Besov norms for linear functionals (on space of test functions) and Besov spaces associated with L. In order to study properties of these Besov spaces, we prove several versions of Calderón reproducing formulas for linear functionals. Furthermore, we show that the classical Besov spaces $B^{\alpha}_{p,q}(\Omega)$ and ${}_zB^{\alpha}_{p,q}(\Omega)$ are special cases in this theory.

3.1 Preliminaries

The theory of Besov spaces has been an active area of research in the last few decades because of its important role in the study of approximation of functions and regularity of solutions to partial differential equations.

Classical theory of Besov spaces, for example, can be found in [Pe], [St1], [FJ1].

Let us give an equivalent definition of the classical Besov spaces on the Euclidean spaces. Suppose that φ is a function satisfying:

$$\varphi \in C_0^\infty(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} \varphi(x)\, dx = 0,$$

and the standard Tauberian condition, that is

$$\forall \xi \neq 0,\, \exists t = t_\xi > 0 \quad \text{such that} \quad \hat{\varphi}(t\xi) \neq 0.$$

We shall use $\varphi_t, t > 0$, to denote the dilation of φ:

$$\varphi_t(x) = t^{-n}\varphi(x/t), \quad x \in \mathbb{R}^n.$$

Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. The classical (homogeneous) Besov space $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ on the Euclidean space $\mathbb{R}^n$ can be defined as follows:

$$\dot{B}^\alpha_{p,q} = \left\{ f \in \mathcal{S}' : \left[\int_0^\infty (t^{-\alpha}\|\varphi_t * (f)\|_p)^q \, \frac{dt}{t} \right]^{1/q} = \|f\|_{\dot{B}^\alpha_{p,q}} < \infty \right\}.$$

where $\mathcal{S}'$ is the space of tempered distributions.

It is well known that in this definition, if the function φ_t with compact support is replaced by the time derivative of the heat kernel,

$$t\frac{d}{dt}h_t(x) = t\frac{d}{dt}\left(\frac{c}{t^{n/2}} e^{-|x|^2/4t} \right)$$

(and $t^{-\alpha}$ is replaced by $t^{-\alpha/2}$), or the time derivative of the Poisson kernel

$$t\frac{d}{dt}p_t(x) = t\frac{d}{dt}\left(\frac{c}{t^n} \times \frac{1}{(1 + |x|^2/t^2)^{(n+1)/2}} \right)$$

(and the convolution is suitably defined), then one obtains equivalent Besov spaces with equivalent norms. See, for example, [BPT2, BPT, JT].

From this observation, we can say that the classical Besov spaces are associated with the Laplace operator $-\Delta$ (or its square root $\sqrt{-\Delta}$). When one studies operators with non-smooth coefficients, function spaces associated with the standard Laplacian might not be the

most suitable ones. The classical approach also has complications when one considers function spaces on domains which are more general than the Euclidean spaces $\mathbb{R}^n$ and is not applicable when the domain has no regularity on its boundary.

We study the theory of Besov spaces associated with a certain operator L under the weak assumption that L generates an analytic semigroup e^{-tL} with Gaussian kernel bounds on $L^2(\mathcal{X})$, where $\mathcal{X}$ is a quasi-metric space of polynomial upper bounds on volume growth. We will develop the theme in the works [DY1, DY, AM^{c}R, AR] and define the Besov norm $\dot{B}^{\alpha,L}_{p,q}$ via quadratic norms of L. We now give a brief overview of the important features of the theory of the Besov spaces $\dot{B}^{\alpha,L}_{p,q}$.

(i) By choosing different operators L, we can recover most of the classical Besov spaces. More specifically, we can prove the following:

- When the space $\mathcal{X} = \mathbb{R}^n$ and if the chosen operator L and its adjoint L^* possess Hölder continuity on their heat kernels as well as conservative property $e^{-tL}1 = e^{-tL^*}1 = 1$ (for example, L is the Laplace operator $-\Delta$ or its square root $\sqrt{-\Delta}$ or an elliptic divergence form operator with bounded, real coefficients on the Euclidean space $\mathbb{R}^n$), then the Besov space $\dot{B}^{\alpha,L}_{p,q}$ is equivalent to the classical Besov spaces $\dot{B}^{\alpha}_{p,q}$ (see Theorem 5.1 in [BDY]).
- When the space $\mathcal{X} = \Omega$ where Ω is a domain of $\mathbb{R}^n$ with smooth boundary and L is chosen as the Laplace operator $-\Delta_N$ with Neumann boundary conditions on Ω, then we obtain that $L^p(\Omega) \cap \dot{B}^{\alpha,-\Delta_N}_{p,q}$ is equivalent to the classical Besov space $B^{\alpha}_{p,q}(\Omega)$ (Proposition 3.15).
- When the space $\mathcal{X} = \Omega$, where Ω is a smooth domain of $\mathbb{R}^n$, and L is chosen as the Laplace operator $-\Delta_D$ with Dirichlet boundary conditions on Ω, then we obtain that $L^p(\Omega) \cap \dot{B}^{\alpha,-\Delta_D}_{p,q}$ is equivalent to the Besov space ${}_zB^{\alpha}_{p,q}(\Omega)$ (Proposition 3.15).

We note that, in addition to recovering the classical Besov spaces, this theory might also give new characterizations of the classical Besov norms through the quadratic norms of L.

(ii) The underlying space $\mathcal{X}$ is assumed to be a quasi-metric space of polynomial upper bounds on volume growth (see definition in Section 3.2), hence $\mathcal{X}$ might not satisfy the doubling volume property. This allows us to treat the case $\mathcal{X} = \Omega$ where Ω is any subset of Euclidean spaces. Indeed, while the standard theory of Besov spaces always requires smoothness on the boundary of the domain, this theory goes beyond the classical spaces and gives definitions to natural Besov spaces when L possesses Gaussian heat kernel bounds without any regularity assumption on the boundary of the domain. Hence we can take L as the Laplace operator with Dirichlet boundary conditions on a general open domain Ω in $\mathbb{R}^n$ and obtain the Besov space $\dot{B}^{\alpha,-\Delta_D}_{p,q}(\Omega)$.

(iii) In the general setting of L and $\mathcal{X}$, we can prove embedding properties and discrete characterizations for the family of Besov spaces $\dot{B}^{\alpha,L}_{p,q}$, similarly to the classical Besov spaces (Theorem 4.1 and Proposition 4.3).

(iv) The Besov spaces $\dot{B}^{\alpha,L}_{p,q}$ are natural settings for estimates of certain singular integrals associated with L such as the fractional powers L^γ for real value γ. See Theorem 4.9. For other choices of the operator L such as the Schrödinger operator on $\mathbb{R}^n$ or a divergence form operator on $\mathbb{R}^n$ or a domain Ω of $\mathbb{R}^n$, we obtain new Besov spaces. While we study Besov spaces associated with a general operator L, additional information from specific operators L such as the Schrödinger operator or a divergence form operator would certainly give further important properties for the new Besov spaces in those cases.

(v) Due to the lacking of smoothness on the heat kernels of L (we assume heat kernel upper bound for L but there is no assumption on regularity of the space variables of heat kernels) as well as the possible rough boundary and non-doubling volume growth of the underlying space $\mathcal{X}$, there are substantial technical difficulties to be overcome. Quite a few of the estimates rely on several key Calderón reproducing formulas in Sections 3.2 and 3.3 (Theorems 3.3, 3.13 and 3.14). These formulas are proved not only for functions in Lebesgue spaces but also for continuous linear functionals on certain spaces of test functions, which are

defined via the operator L together with an appropriate decay condition at infinity. In this theory, these test functions play an important role, similarly to the role of the Schwartz class in the classical theory. These new reproducing formulas are of independent interest and they should be useful for research in harmonic analysis related to the operator L.

For an investigation of a class of Besov spaces on spaces with non-doubling measures and polynomial growth, we refer the reader to [DHY]. The approach in [DHY] was based on a construction of a family of approximation to the identity with compactly supported kernels and a Calderón reproducing formula. Note that these kernels are constructed so that they satisfy the conservation property and Hölder continuity estimates. In contrast, we make no assumption on the supports of the kernels, noting that the classical Poisson and Gaussian kernels do not have compact supports. In most part of this work, we require neither a regularity condition nor the conservation property on the kernels. The flexibility of choosing L in this approach also gives rise to different Besov spaces on the same domain $\mathcal{X}$.

The study of the classical Besov spaces on a smooth domain Ω was, to some extent, motivated by the application to partial differential equations (see, for example, [Tr]). When Ω is a Lipschitz domain, the authors in [MM1] had investigated properties of both the spaces $B^{\alpha}_{p,q}(\Omega)$ and ${}_zB^{\alpha}_{p,q}(\Omega)$ to obtain sharp estimates for the Green potentials. Some further results in this direction can be found in [MM2, MMS]. The Besov spaces we study could serve as useful tools in the investigation of properties of solutions to partial differential equations on non-smooth domains or with rough coefficients.

We note that there are a number of recent papers which defined and characterized Hardy spaces associated with operators under various assumptions on heat kernel bounds by using the area integral estimates and atomic decompositions on doubling domains. These function spaces retained a number of important properties of the classical spaces and played a positive role in the study of the boundedness of singular integral operators with non-smooth kernels. See, for example, [AM^{c}R, AR, DY, HLMMY, HMa] and the references therein. A study of Hardy spaces

associated with operators under the same assumptions as this work (on non-doubling domains) would be interesting.

Recent work of Bui, Duong and Yan in [BDY] defined Besov spaces associated with a certain operator L under the weak assumption that L generates an analytic semigroup e^{-tL} with Poisson kernel bounds on $L^2(\mathcal{X})$ where $\mathcal{X}$ is a (possibly non-doubling) quasi-metric space of polynomial upper bound on volume growth. When L is the Laplace operator $-\Delta$ or its square root $\sqrt{-\Delta}$ acting on the Euclidean space $\mathbb{R}^n$, this class of Besov spaces associated with the operator L are equivalent to the classical Besov spaces. Depending on the choice of L, the Besov spaces are natural settings for generic estimates for certain singular integral operators such as the fractional powers L^α.

We aim to extend certain results in [BDY] to a more general setting when the underlying space can have different dimensions at 0 and infinity, that is, for some $n > 0$, $N \geq 0$, and $C > 0$,

$$\mu(B(x,r)) \leq \begin{cases} Cr^n, & 0 < r \leq 1 \\ Cr^N, & 1 < r < \infty \end{cases}$$

for all balls B. Here n is the local dimension and N is the global dimension or the dimension at infinity.

An example of this case is in Lie groups of polynomial growth (see, for example, [Al] and [Ro]). Consider when L is the Laplace operator Δ_N with Neumann boundary conditions on a bounded Lipschitz domain Ω of $\mathbb{R}^n$. See, for example, [Da1]. The heat kernel $p_t(x,y)$ in this case satisfies

$$\begin{aligned} 0 \leq p_t(x,y) &\leq \frac{C}{V(x,\sqrt{t})} e^{-\alpha|x-y|^2/t} \\ &= C \max\left\{\frac{1}{t^{n/2}}, 1\right\} e^{-\alpha|x-y|^2/t} \\ &= \begin{cases} \dfrac{C}{t^{n/2}} e^{-\alpha|x-y|^2/t}, & 0 < t \leq 1 \\ Ce^{-\alpha|x-y|^2/t}, & 1 < t < \infty \end{cases} \end{aligned}$$

for some positive constants C and α, where $V(x,\sqrt{t})$ denotes the volume of the ball with centre

x and radius $\sqrt{t}$ in $\mathbb{R}^n$. In this case N can be chosen to be 0, so that $V(x, \sqrt{t})$ is bounded by a constant.

While many results in [BDY] carry over, there are some difficulties with the change in dimension. Instead of using Poisson kernel bounds (polynomial type), which posed some technical difficulties, we use Gaussian kernel bounds (exponential type), which is a stronger assumption.

3.2 Besov norms associated with operators

3.2.1 Spaces of polynomial upper bounds on volume growth

We shall first give some standard definitions. A quasi-metric d on a set $\mathcal{X}$ is a function from $\mathcal{X} \times \mathcal{X}$ to $[0, \infty)$ satisfying the following:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;

(iii) There exists a constant $C \in [1, \infty)$ such that for all x, y and $z \in \mathcal{X}$,

$$d(x, y) \leq C(d(x, z) + d(z, y)).$$

Any quasi-metric defines a topology, for which the balls $B(x, r) = \{y \in \mathcal{X} : d(y, x) < r\}$ form a basis. However, the balls themselves need not be open when $C > 1$ (see [CW]).

We let μ be a non-negative Borel measure on $\mathcal{X}$ which satisfies the following conditions:

(iv) For some $n > 0$, $N \geq 0$, and $C > 0$,

$$\mu(B(x, r)) \leq \begin{cases} Cr^n, & 0 < r \leq 1 \\ Cr^N, & 1 < r < \infty \end{cases}$$

for all balls B and $x \in \mathcal{X}$. Here n is the local dimension and N is the global dimension or the dimension at infinity. Note that n and N need not be integers (although in many examples n and N are positive integers).

We note that properties (i)–(iv) would not imply that μ is doubling. An example of a possibly non-doubling space $(\mathcal{X}, d, \mu)$ of polynomial growth is given when $\mathcal{X}$ is a subset of $\mathbb{R}^n$ equipped with the Euclidean distance and the Lebesgue measure. Without regularity assumption on the boundary of $\mathcal{X}$, μ can be non-doubling.

Other examples of $(\mathcal{X}, d, \mu)$ include smooth n-dimensional submanifolds of $\mathbb{R}^m$, $n \leq m$, μ the volume (area) measure on $\mathcal{X}$, and d the Euclidean distance. Another class of examples is the class of "regular" subsets of $\mathbb{R}^m$ of Hausdorff dimension n, where $n \leq m$ may not be an integer, and $\mu = \mathcal{H}^n$, the n-dimensional Hausdorff measure (see [EG]).

We will assume that $(\mathcal{X}, d, \mu)$ satisfies properties (i)–(iv).

The following estimate will be frequently used in the thesis.

Lemma 3.1. *Let $1 \leq p \leq \infty$. For every $\alpha > 0$, there exists $C > 0$ such that*

$$\int_{\mathcal{X}} \left[e^{-\alpha d(x,y)^2/t}\right]^p \, d\mu(x) \leq \begin{cases} Ct^{n/2}, & 0 < t \leq 1 \\ Ct^{N/2}, & 1 < t < \infty \end{cases}$$

for $y \in \mathcal{X}$.

Proof. Fix $y \in \mathcal{X}$. For $p = \infty$, we clearly have

$$\sup_x e^{-\alpha d(x,y)^2/t} \leq C, \quad 0 < t \leq 1$$
$$\sup_x e^{-\alpha d(x,y)^2/t} \leq C, \quad 1 < t < \infty$$

Next suppose $1 \leq p < \infty$. For $0 < t \leq 1$ we have

$$\int_{\mathcal{X}} \left[e^{-\alpha d(x,y)^2/t}\right]^p \, d\mu(x)$$
$$\leq \int_{B(y,\sqrt{t})} \left[e^{-\alpha d(x,y)^2/t}\right]^p \, d\mu(x)$$

$$
\begin{aligned}
&+\sum_{k=1}^{\infty}\int_{2^{k-1}\sqrt{t}\le d(x,y)<2^k\sqrt{t}}\left[e^{-\alpha d(x,y)^2/t}\right]^p\,d\mu(x)\\
&\le C\left\{t^{n/2}+\sum_{k=1}^{\infty}\left[e^{-\alpha(2^{2k})t}\right]^p(2^k\sqrt{t})^N\right\}\\
&\le C\left\{t^{n/2}+\sum_{k=1}^{\infty}\left[e^{-\alpha(2^{2k})t}\right]^p 2^{kN}t^{n/2}\right\}\quad\text{for } n<N\\
&\le Ct^{n/2}.
\end{aligned}
$$

For $1<t<\infty$ we have

$$
\begin{aligned}
&\int_{\mathcal{X}}\left[e^{-\alpha d(x,y)^2/t}\right]^p\,d\mu(x)\\
&\le\int_{B(y,\sqrt{t})}\left[e^{-\alpha d(x,y)^2/t}\right]^p\,d\mu(x)\\
&\quad+\sum_{k=1}^{\infty}\int_{2^{k-1}\sqrt{t}\le d(x,y)<2^k\sqrt{t}}\left[e^{-\alpha d(x,y)^2/t}\right]^p\,d\mu(x)\\
&\le C\left\{t^{N/2}+\sum_{k=1}^{\infty}\left[e^{-\alpha(2^{2k})t}\right]^p(2^k\sqrt{t})^N\right\}\\
&\le Ct^{N/2}.
\end{aligned}
$$

Hence the inequalities follow. □

3.2.2 Assumptions on operators

Assume L is densely-defined on $L^2(\mathcal{X})$ and satisfies

(S) L generates a holomorphic semigroup e^{-zL} for $z=t+is$ with $t>0$ and $|\arg z|<\rho$ for some $\rho>0$,

(K) the heat kernel of L satisfies bounds of Gaussian type, i.e. the kernel $p_t(x,y)$ of e^{-tL} satisfies

$$
|p_t(x,y)|\le\begin{cases}\dfrac{C}{t^{n/2}}e^{-\alpha d(x,y)^2/t}, & 0<t\le 1\\[2ex] \dfrac{C}{t^{N/2}}e^{-\alpha d(x,y)^2/t}, & 1<t<\infty\end{cases}
$$

for some $C>0$ and for all $x,y\in\mathcal{X}$.

(H) the kernels $p_t(x,y)$ of e^{-tL} satisfy the Hölder continuity estimates

$$|p_t(x,y) - p_t(x,y')| \leq \begin{cases} \dfrac{Cd(y,y')}{t^{n/2+1}} e^{-\alpha d(x,y)^2/t}, & 0 < t \leq 1 \\ \dfrac{Cd(y,y')}{t^{N/2+1}} e^{-\alpha d(x,y)^2/t}, & 1 < t < \infty \end{cases}$$

whenever $d(y,y') \leq d(x,y)/2$.

(C) L satisfies the conservation property $e^{-tL}1 = 1$. This is equivalent to

$$\int_{\mathcal{X}} p_t(x,y)\, d\mu(y) = 1.$$

The following are some well-known examples of operators which satisfy some or all the assumptions above.

(i) The Laplace operator $-\Delta$ and its square root $\sqrt{-\Delta}$ on $\mathbb{R}^n$ satisfy the assumptions (S), (K), (H) and (C). So do elliptic divergence form operators with bounded, real coefficients on $\mathbb{R}^n$. However, if the coefficients of the elliptic divergence form operators are bounded and complex, (K) is known to fail in dimension $n \geq 5$.

(ii) The Schrdinger operator $-\Delta + V$ on $\mathbb{R}^n$ where the potential $0 \leq V \in L^1_{\text{loc}}$ satisfies (S) and (K) but need not satisfy (H) and (C). However, if V belongs to some reverse Hölder class, then $-\Delta + V$ satisfies (H).

(iii) The Laplace operator Δ_D with Dirichlet boundary conditions on an open set Ω of $\mathbb{R}^n$ satisfies (S) and (K), but not (C). If Ω is a bounded Lipschitz domain, then Δ_D also satisfies (H) but (H) does not hold for general open sets.

(iv) The Laplace operator Δ_N with Neumann boundary conditions on a bounded domain Ω of $\mathbb{R}^n$ with extension property satisfies (S) and (C) for all $t > 0$, satisfies (K) and (H) for $0 < t \leq 1$. See [Da1]. More specifically, the heat kernel $p_t(x,y)$ in this case satisfies

$$\begin{aligned} 0 \leq p_t(x,y) &\leq \frac{C}{V(x,\sqrt{t})} e^{-\alpha|x-y|^2/t} \\ &= C \max\left\{\frac{1}{t^{n/2}}, 1\right\} e^{-\alpha|x-y|^2/t} \end{aligned}$$

for some positive constants C and α, where $V(x, \sqrt{t})$ denotes the volume of the ball with centre x and radius $\sqrt{t}$ in $\mathbb{R}^n$.

We also note that a Lipschitz domain satisfies the extension property.

(v) The Laplace Beltrami operator on a non-compact complete Riemann manifold satisfies (S) and (C), but in general not (K) and (H). If the manifold has non-negative Ricci curvature and the measure of the ball radius r is equivalent to r^n, then the Laplace Beltrami operator also satisfies (K) and (H). If one considers the example of the manifold of two copies of $\mathbb{R}^n$ smoothly glued together by a cylinder of length 1, then the Laplace Beltrami operator satisfies (S), (K) and (C) but not (H).

We note that condition (K) implies that the semigroup e^{-tL}, initially defined on $L^2(\mathcal{X})$, can be extended to $L^p(\mathcal{X})$, $1 \leq p \leq \infty$. Furthermore, $e^{-tL}f$ makes sense for certain functions f which satisfy appropriate growth condition but might not belong to $L^p(\mathcal{X})$. Combining (S) and (K) would imply that the time derivatives of the semigroups also possess Gaussian bounds as in the next proposition which states some useful properties related to our assumptions.

The following are some useful properties related to our assumptions.

Proposition 3.2. *For $k = 1, 2, \ldots,$ let $p_{k,t}(x, y)$ denote the kernel of the operator $t^k L^k e^{-tL}$.*

(a) Suppose L satisfies (S) and (K). Then $p_{k,t}(x, y)$ satisfies the size estimate (DK), i.e. for every $k \in \mathbb{N}$, there is a constant c_k satisfying

$$|p_{k,t}(x, y)| \leq \begin{cases} \dfrac{c_k}{t^{n/2}} e^{-\alpha_k d(x,y)^2/t}, & 0 < t \leq 1 \\ \dfrac{c_k}{t^{N/2}} e^{-\alpha_k d(x,y)^2/t}, & 1 < t < \infty \end{cases}$$

for all $x, y \in \mathcal{X}$.

(b) Suppose L satisfies (S), (K) and (H). Then $p_{k,t}(x, y)$ satisfies the Hölder estimate (DH),

i.e. there is a constant c_k satisfying

$$|p_{k,t}(x,y) - p_{k,t}(x,y')| \le \begin{cases} \dfrac{c_k d(y,y')}{t^{n/2+1}} e^{-\alpha_k d(x,y)^2/t}, & 0 < t \le 1 \\ \dfrac{c_k d(y,y')}{t^{N/2+1}} e^{-\alpha_k d(x,y)^2/t}, & 1 < t < \infty \end{cases}$$

whenever $d(y,y') \le d(x,y)/2$.

(c) Suppose L satisfies (C). Then we have

$$\int_{\mathcal{X}} p_{k,t}(x,y)\, d\mu(y) = 0$$

for every $x \in \mathcal{X}$.

Proof. For a proof of part (a), we refer the reader to Theorem 3 in [Da2] and Theorem 6.17 in [Ou].

To show part (b), we first observe that

$$t^k L^k e^{-tL} = (-2)^k \left(\frac{d^k}{dt^k} e^{-\frac{t}{2}L} \right) e^{-\frac{t}{2}L}.$$

Next, by using assumption (H) and (DK), we obtain, for $0 < t \le 1$,

$$\begin{aligned} &|p_{k,t}(x,y) - p_{k,t}(x,y')| \\ &\quad = 2^k \left| \int_{\mathcal{X}} p_{k,\frac{t}{2}}(x,z) \big(p_{\frac{t}{2}}(z,y) - p_{\frac{t}{2}}(z,y') \big) d\mu(z) \right| \\ &\quad \le c_k \int_{\mathcal{X}} \frac{1}{t^{n/2}} e^{-\alpha_k d(x,z)^2/t} \frac{d(y,y')}{t^{n/2+1}} e^{-\alpha_k d(z,y)^2/t}\, d\mu(z) \\ &\quad \le \frac{c_k d(y,y')}{t^{n/2+1}} e^{-\alpha_k d(x,y)^2/t}. \end{aligned}$$

Similarly, for $1 < t < \infty$, we have

$$\begin{aligned} &|p_{k,t}(x,y) - p_{k,t}(x,y')| \\ &\quad = 2^k \left| \int_{\mathcal{X}} p_{k,\frac{t}{2}}(x,z) \big(p_{\frac{t}{2}}(z,y) - p_{\frac{t}{2}}(z,y') \big) d\mu(z) \right| \\ &\quad \le c_k \int_{\mathcal{X}} \frac{1}{t^{N/2}} e^{-\alpha_k d(x,z)^2/t} \frac{d(y,y')}{t^{N/2+1}} e^{-\alpha_k d(z,y)^2/t}\, d\mu(z) \\ &\quad \le \frac{c_k d(y,y')}{t^{N/2+1}} e^{-\alpha_k d(x,y)^2/t}. \end{aligned}$$

Hence we have shown (b).

To show part (c), we just use assumption (C) and also

$$t^k L^k e^{-tL} = (-1)^k \left(\frac{d}{dt}\right)^k e^{-tL}.$$

Thus the proof of the proposition is finished. □

Theorem 3.3 (Calderón reproducing formula I)**.** *Suppose L is a densely-defined operator in $L^2(\mathcal{X})$ and satisfies (S) and (K). Assume that $f \in L^p(\mathcal{X})$, $1 < p < \infty$. Then we have*

$$f(x) = \frac{1}{(k-1)!} \int_0^\infty t^k L^k e^{-tL} f(x) \, \frac{dt}{t}, \quad k = 1, 2, \ldots, \tag{3.1}$$

where the integral converges strongly in $L^p(\mathcal{X})$.

Proof. Let $f \in L^p(\mathcal{X})$, $1 < p < \infty$. Assume $\epsilon > 0$ is given. There exists a $C_0(\mathcal{X})$-function g with compact support with $\|f - g\|_p \le \epsilon$. Suppose that $\operatorname{supp} g \subset B(x_0, r)$ for $r > 0$ and some $x_0 \in \mathcal{X}$. Applying Minkowski's inequality, Proposition 3.2 and Lemma 3.1, it follows that for every $m = 0, 1, 2, \ldots,$

$$\begin{aligned}
&\|t^m L^m e^{-tL} g\|_p \\
&\quad \le \Big(\int_{\mathcal{X}} \Big| \int_{B(x_0,r)} p_{m,t}(x,y) g(y) \, d\mu(y) \Big|^p \, d\mu(x) \Big)^{1/p} \\
&\quad \le \|g\|_1 \Big(\sup_{y \in \mathcal{X}} \int_{\mathcal{X}} |p_{m,t}(x,y)|^p \, d\mu(x) \Big)^{1/p} \\
&\quad \le C t^{-N/2p'} \|g\|_1 \quad \text{for } t \text{ large enough,}
\end{aligned} \tag{3.2}$$

which tends to 0 as $t \to \infty$. Thus it follows that

$$\begin{aligned}
&\lim_{t\to\infty} \|t^m L^m e^{-tL} f\|_p \\
&\quad \le \lim_{t\to\infty} \|t^m L^m e^{-tL}(f-g)\|_p + \lim_{t\to\infty} \|t^m L^m e^{-tL} g\|_p \\
&\quad \le C\epsilon.
\end{aligned}$$

By letting $\epsilon \to 0$, we obtain

$$\lim_{t\to\infty} \|t^m L^m e^{-tL} f\|_p = 0.$$

Using integration by parts, we have

$$\frac{1}{(k-1)!} \int_0^\infty t^k L^k e^{-tL} f \, \frac{dt}{t}$$

$$= (-1)^k \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \frac{\partial^k}{\partial t^k} e^{-tL} f \, dt$$
$$= \lim_{t\to 0} \Big(e^{-tL} f + \sum_{m=1}^{k-1} c_m t^m L^m e^{-tL} f \Big) \quad \text{in } L^p(\mathcal{X})$$

for some constants $\{c_m\}_{m=1}^{k-1}$. To complete the proof of Theorem 3.3, it is enough to prove that

(i) $\lim_{t\to 0} e^{-tL} f = f$ in $L^p(\mathcal{X})$;

(ii) $\lim_{t\to 0} t^m L^m e^{-tL} f = 0$ in $L^p(\mathcal{X})$, $m = 1, 2, \ldots$

Let us show (i). As above, for any given $\epsilon > 0$, we can find a $C_0(\mathcal{X})$-function g with compact support with $\|f - g\|_p \le \epsilon$, such that $\operatorname{supp} g \subset B(x_0, r)$ for $r > 0$ and some $x_0 \in \mathcal{X}$. We then have

$$\|e^{-tL} g - g\|_p \le \|e^{-tL} g\|_{L^p(B(x_0,2r)^c)} + \|e^{-tL} g - g\|_{L^p(B(x_0,2r))}.$$

By using condition (K), we obtain

$$\|e^{-tL} g\|_{L^p(B(x_0,2r)^c)} \le Ct \|g\|_1,$$

which implies that

$$\lim_{t\to 0} \|e^{-tL} g\|_{L^p(B(x_0,2r)^c)} = 0.$$

Because $\lim_{t\to 0} e^{-tL} g = g$ in $L^2(\mathcal{X})$, and

$$\|e^{-tL} g - g\|_{L^p(B(x_0,2r))} \le \begin{cases} \mu(B(x_0,2r))^{1-\frac{p}{2}} \|e^{-tL} g - g\|_2, & 1 < p \le 2 \\ C\Big(\sup_x |g(x)|^{\frac{p-2}{p}} \Big) \|e^{-tL} g - g\|_2^{2/p}, & 2 < p < \infty, \end{cases}$$

it follows that

$$\lim_{t\to 0} \|e^{-tL} g - g\|_{L^p(B(x_0,2r))} = 0.$$

These estimates together prove that $\lim_{t\to 0} \|e^{-tL} g - g\|_p = 0$. Thus using condition (K), we have

$$\begin{aligned} &\limsup_{t\to 0} \|e^{-tL} f - f\|_p \\ &\quad \le \limsup_{t\to 0} \|e^{-tL}(f-g)\|_p + \|f - g\|_p + \lim_{t\to 0} \|e^{-tL} g - g\|_p \\ &\quad \le C\epsilon. \end{aligned}$$

By letting $\epsilon \to 0$, it follows that $e^{-tL}f \to f$ in L^p as $t \to 0$. Thus (i) is proved.

All that remains is to show (ii). Notice that L is a densely-defined operator in $L^2(\mathcal{X})$. For any $f \in \mathcal{D}(L)$, $g = Lf \in L^2(\mathcal{X})$, which implies that for every $m = 1, 2, \ldots$,

$$\begin{aligned}
&\lim_{t\to 0} \|t^m L^m e^{-tL} f\|_2 \\
&\quad \le \lim_{t\to 0} t\|t^{m-1} L^{m-1} e^{-tL}(Lf)\|_2 \\
&\quad \le \lim_{t\to 0} t\|g\|_2 \\
&\quad = 0.
\end{aligned}$$

Applying a density argument, we have that $\lim_{t\to 0} t^m L^m e^{-tL} f = 0$ in $L^2(\mathcal{X})$. Let $f \in L^p(\mathcal{X})$. By the same arguments we used in (i), it follows that $t^m L^m e^{-tL} f \to 0$ in L^p as $t \to 0$. Thus (ii) is proved. We have now finished proving Theorem 3.3. □

3.2.3 Besov norms associated with operators

Assume L satisfies (S) and (K). Let $k_t(x, y) = p_{1,t}(x, y)$ be the kernel of $\Psi_t(L) = tLe^{-tL}$. By Proposition 3.2, $k_t(x, y)$ satisfies

$$|k_t(x, y)| \le \begin{cases} \dfrac{c}{t^{n/2}} e^{-\alpha d(x,y)^2/t}, & 0 < t \le 1 \\ \dfrac{c}{t^{N/2}} e^{-\alpha d(x,y)^2/t}, & 1 < t < \infty. \end{cases}$$

Let f be a complex valued measurable function on $\mathcal{X}$ satisfying the following growth condition (G):

$$\int_{\mathcal{X}} |f(x)| e^{-\alpha d(x,y_0)^2}\, d\mu(x) < \infty$$

for some $y_0 \in \mathcal{X}$. Then we have that

$$\Psi_t(L)f(x) = \int_{\mathcal{X}} k_t(x, y) f(y)\, d\mu(y)$$

is defined for all $x \in \mathcal{X}$.

Definition 3.4. *Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. For any f satisfying (G), we define its $\dot{B}^{\alpha,L}_{p,q}$-norm by*

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \left\{ \int_0^\infty (t^{-\alpha}\|\Psi_t(L)f\|_p)^q \, \frac{dt}{t} \right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \sup_{t>0} t^{-\alpha}\|\Psi_t(L)f\|_p$$

for $q = \infty$, whenever these are finite.

Although we consider the Besov spaces in the range $-1 < \alpha < 1$, the above definition of the norm for functions satisfying the growth condition (G) remains valid when $\alpha \leq -1$ (in analogy with the classical situation). But we will not define the Besov spaces in the range $\alpha \leq -1$, as we expect these spaces to contain not only functions but also certain "distributions". Another complication in this case is finding an appropriate space of test functions.

When $\alpha \geq 1$, the Besov norm of a function satisfying (G) can be defined by replacing the kernel $k_t = p_{1,t}$ by $p_{k,t}$, where $k > \alpha$. However, as the classical case shows us, the homogeneous Besov spaces can contain functions of order $O(|x|^\alpha)$ at infinity. These functions may not satisfy the growth condition (G), and therefore the weak decay of $p_{k,t}$ (see Proposition 3.2) would make it unsuitable to be used for investigating these spaces. By duality, the case $\alpha \leq -1$ also presents a challenge.

The difficulty discussed above is a main reason why we restrict our study to the case $-1 < \alpha < 1$. On the other hand, when the kernel has more smoothness in the spatial variable or possess stronger decay, such as the case for some specific operators L, it would be feasible to carry out the investigation for a larger range of α. We do not carry out a study in this direction here.

We postpone the formal definition of the Besov spaces until Section 3.3, after investigating properties of the $\dot{B}^{\alpha,L}_{p,q}$–norm. We first prove a simple property of the $\dot{B}^{\alpha,L}_{p,q}$-norm.

Proposition 3.5. *Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. Suppose that f satisfies (G) and $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = 0$. Then for every $t > 0$, we have $\Psi_t(L)f = 0$ almost everywhere.*

Proof. Clearly the result is true for $q = \infty$ (by the definition of the norm). Suppose that $q < \infty$.

Then $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = 0$ implies $\|\Psi_t(L)f\|_p = 0$ for almost everywhere $t \in (0,\infty)$. Notice that for all $s, t > 0$,

$$\begin{aligned}
&\|\Psi_{t+s}(L)f\|_p \\
&\quad = \left\| \frac{t+s}{t} e^{-sL} \Psi_t(L) f \right\|_p \\
&\quad \le C \frac{t+s}{t} \|\Psi_t(L)f\|_p, \qquad (3.3)
\end{aligned}$$

where $C = \sup_{s>0} \|e^{-sL}\|_{p\to p} < \infty$. Then we have $\|\Psi_t(L)f\|_p = 0$ for all $t \in (0,\infty)$. Thus we have finished the proof of the proposition. □

Remark. From (3.3) above, it follows that for every measurable function f and $t > 0$,

$$\|\Psi_s(L)f\|_p \le 2C\|\Psi_t(L)f\|_p \qquad t \le s \le 2t. \qquad (3.4)$$

There exists functions with finite Besov norm but not necessarily smooth. In the following proposition we give an upper bound estimate of the Besov norm of the heat kernels. For any $k \in \mathbb{N}$, we denote $\Psi_{k,t}(L) = t^k L^k e^{-tL}$ to be the operator whose kernel is $p_{k,t}$; so $\Psi_{1,t}(L) = \Psi_t(L)$.

Proposition 3.6. *Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. Suppose that f satisfies (S) and (K). Then for $k \in \mathbb{N}$ and $z \in \mathcal{X}$,*

$$\|p_{k,s}(\cdot, z)\|_{\dot{B}^{\alpha,L}_{p,q}} \le \begin{cases} C_n s^{-\alpha - n/2p'}, & 0 < s \le 1 \\ C_N s^{-\alpha - N/2p'}, & 1 < s < \infty \end{cases}$$

where $C_n > 0$ depends on α, n, k, p and q, and $C_N > 0$ depends on α, N, k, p and q.

Proof. Fix $k \in \mathbb{N}$. Using (DK) in Proposition 3.2, the kernel of the operator $\Psi_t(L)\Psi_{k,s}(L)$ is

$$\mathcal{K}_{t,s}(x,z) = \int_{\mathcal{X}} k_t(x,y) p_{k,s}(y,z)\, d\mu(y).$$

Let $\tilde{K}_t$ be a kernel satisfying

$$\left|\tilde{K}_t(x,z)\right| \le \begin{cases} \dfrac{c_k}{t^{n/2}} e^{-\alpha_k d(x,z)^2/t}, & 0 < t \le 1 \\ \dfrac{c_k}{t^{N/2}} e^{-\alpha_k d(x,z)^2/t}, & 1 < t < \infty \end{cases}$$

for all $x, z \in \mathcal{X}$. Then $\mathcal{K}_{t,s}$ satisfies the size estimate

$$|\mathcal{K}_{t,s}(x,z)| \le C \min\left\{\frac{s}{t}, \frac{t}{s}\right\} |\tilde{K}_{t+s}(x,z)|.$$

Put $\phi(y) = p_{k,s}(y,z)$, $y \in \mathcal{X}$. We then have that

$$\begin{aligned}
&|\Psi_t(L)\phi(x)| \\
&\quad = |\mathcal{K}_{t,s}(x,z)| \\
&\quad \le \begin{cases} C \min\left\{\frac{s}{t}, \frac{t}{s}\right\} \frac{e^{-\alpha_k d(x,z)^2/(t+s)}}{(t+s)^{n/2}}, & 0 < t+s \le 1 \\ C \min\left\{\frac{s}{t}, \frac{t}{s}\right\} \frac{e^{-\alpha_k d(x,z)^2/(t+s)}}{(t+s)^{N/2}}, & 1 < t+s < \infty \end{cases}
\end{aligned}$$

Hence, using Lemma 3.1,

$$\|\Psi_t(L)\phi\|_p \le \begin{cases} C \min\left\{\frac{s}{t}, \frac{t}{s}\right\} (t+s)^{-n/2p'}, & 0 < t+s \le 1 \\ C \min\left\{\frac{s}{t}, \frac{t}{s}\right\} (t+s)^{-N/2p'}, & 1 < t+s < \infty \end{cases}$$

Therefore, for $0 < s \le 1$,

$$\begin{aligned}
&\|\phi\|_{\dot{B}^{\alpha,L}_{p,q}} \\
&\quad = \left\{ \int_0^\infty (t^{-\alpha} \|\Psi_t(L)(\phi)\|_p)^q \, \frac{dt}{t} \right\}^{1/q} \\
&\quad \le C \left\{ \int_0^s \left(\frac{t^{1-\alpha}}{s(t+s)^{n/2p'}} \right)^q \frac{dt}{t} + \int_s^\infty \left(\frac{t^{-1-\alpha} s}{(t+s)^{n/2p'}} \right)^q \frac{dt}{t} \right\}^{1/q} \\
&\quad \le C s^{-\alpha - n/2p'} \left\{ \int_0^1 \left(\frac{t^{1-\alpha}}{(1+t)^{n/2p'}} \right)^q \frac{dt}{t} + \int_1^\infty \left(\frac{t^{-1-\alpha}}{(1+t)^{n/2p'}} \right)^q \frac{dt}{t} \right\}^{1/q} \\
&\quad \le C s^{-\alpha - n/2p'},
\end{aligned}$$

where the constant C in the final inequality depends on α, n, k, p and q.

For $1 < s < \infty$,

$$\begin{aligned}
&\|\phi\|_{\dot{B}^{\alpha,L}_{p,q}} \\
&\quad = \left\{ \int_0^\infty (t^{-\alpha} \|\Psi_t(L)(\phi)\|_p)^q \, \frac{dt}{t} \right\}^{1/q} \\
&\quad \le C \left\{ \int_0^s \left(\frac{t^{1-\alpha}}{s(t+s)^{N/2p'}} \right)^q \frac{dt}{t} + \int_s^\infty \left(\frac{t^{-1-\alpha} s}{(t+s)^{N/2p'}} \right)^q \frac{dt}{t} \right\}^{1/q} \\
&\quad \le C s^{-\alpha - N/2p'} \left\{ \int_0^1 \left(\frac{t^{1-\alpha}}{(1+t)^{N/2p'}} \right)^q \frac{dt}{t} + \int_1^\infty \left(\frac{t^{-1-\alpha}}{(1+t)^{N/2p'}} \right)^q \frac{dt}{t} \right\}^{1/q}
\end{aligned}$$

$$\leq Cs^{-\alpha-N/2p'},$$

where the constant C in the final inequality depends on α, N, k, p and q. □

3.3 Besov spaces associated with operators

3.3.1 Definitions of Besov spaces

Firstly, we use a similar approach as in [BDY] to define a "space of test functions".

Definition 3.7. *Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. A function f is in the space of test functions $\mathcal{M}^{\alpha,L}_{p,q}$ if $f = Lg$ for some g, and the following are satisfied:*

(i) $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} < \infty$;

(ii) There is a $C > 0$ such that

$$|f(x)| + |g(x)| \leq Ce^{-\alpha d(x,x_0)^2} \tag{3.5}$$

for some $x_0 \in \mathcal{X}$, and for every $x \in \mathcal{X}$.

For $q = \infty$, we assume, in addition, that

$$\|t^{-\alpha}\Psi_t(L)f\|_p \to 0 \text{ as } t \to 0 \text{ or } t \to \infty,$$

and when $p = \infty$, we assume that

$$e^{-sL}f \to f \text{ in } \dot{B}^{\alpha,L}_{\infty,q} \text{ as } s \to 0.$$

Remark. There are two main features of the space of test functions $\mathcal{M}^{\alpha,L}_{p,q}$. The first is an "$L$-cancellation" property, which is expressed by the condition $f = Lg$. The second is the finiteness of the norm $\|f\|_{\dot{B}^{\alpha,L}_{p,q}}$. This space of test functions plays the role of $\mathcal{S}_0$ in the classical theory of the Besov spaces $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$, $-1 < \alpha < 1$, where $\mathcal{S}_0$ is the subspace of those $\varphi \in \mathcal{S}$ such that

$\int_{\mathbb{R}^n} \varphi(x)\,dx = 0$. Observe that $\|\varphi\|_{\dot{B}^{\alpha}_{p,q}} < \infty$ for all $\varphi \in \mathcal{S}_0$ and $-1 < \alpha < 1$. On the one hand, the dependence of the space of test functions on p, q and α seems undesirable, but, on the other hand, it hints at a duality result. This situation is similar to the theory of the Hardy spaces $H^p(\mathcal{X})$ on a space of homogeneous type $\mathcal{X}$ (see pp. 591–592 in [CW1]), where the test functions satisfy a Hölder condition whose exponent depends on p.

Using Proposition 3.5 $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = 0$ if and only if, for every $t > 0$,

$$\Psi_t(L)(f) = tLe^{-tL}f = 0$$

almost everywhere. Thus the space $\mathcal{M}^{\alpha,L}_{p,q}$ is a normed linear space equipped with the norm

$$\|f\|_{\mathcal{M}^{\alpha,L}_{p,q}} = \|f\|_{\dot{B}^{\alpha,L}_{p,q}}$$

when we identify with the zero element all those f's satisfying $Le^{-tL}f = 0$ (almost everywhere) for every $t > 0$.

Proposition 3.8. *Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. Then for every $t > 0$, $k \in \mathbb{N}$, and $y \in \mathcal{X}$, $p_{k,t}(\cdot, y) \in \mathcal{M}^{\alpha,L}_{p,q}$, where $p_{k,t}(\cdot, y)$ is the kernel of $\Psi_{k,t}(L) = t^k L^k e^{-tL}$.*

Proof. Put $\phi = p_{k,t}(\cdot, y)$. From Proposition 3.6, the $\|\cdot\|_{\dot{B}^{\alpha,L}_{p,q}}$-norm of ϕ is finite. Using the semigroup property,

$$p_{k,t}(\cdot, y) = 2^k e^{-t/2L} p_{k,t/2}(\cdot, y),$$

so that $p_{k,t}(\cdot, y) \in D(L)$ (see [Ou], page 37). By a standard argument it follows that

$$\phi = p_{k,t}(\cdot, y) = tL(p_{k-1,t}(\cdot, y)).$$

Thus we have the result for $1 \le p, q < \infty$.

Next suppose $p = \infty$. We need to show that $e^{-sL}\phi \to \phi$ in $\dot{B}^{\alpha,L}_{\infty,q}$ as $s \to 0$. When $s, u > 0$,

$$e^{-sL}\Psi_u(L)\phi - \Psi_u(L)\phi$$

$$= 2e^{-uL/2}(e^{-sL}\Psi_{u/2}(L)\phi - \Psi_{u/2}(L)\phi)$$

$$= 2e^{-uL/2}g_{t,u,s},$$

where $g_{t,u,s} = e^{-sL}\Psi_{u/2}(L)\phi - \Psi_{u/2}(L)\phi \in L^2$. Thus for every $x \in \mathcal{X}$, using Schwarz's inequality,

$$|e^{-sL}\Psi_u(L)\phi(x) - \Psi_u(L)\phi(x)| \leq 2\|p_{u/2}(x,\cdot)\|_2\|g_{t,u,s}\|_2.$$

Because $g_{t,u,s} = e^{-sL}\Psi_{u/2}(L)\phi - \Psi_{u/2}(L)\phi \to 0$ in L^2 as $s \to 0$, using (S), we have that

$$\|\Psi_u(L)(e^{-sL}\phi - \phi)\|_\infty = \|e^{-sL}\Psi_u(L)\phi - \Psi_u(L)\phi\|_\infty \to 0, \quad \text{as } s \to 0$$

for every $u > 0$. Furthermore,

$$\|e^{-sL}\Psi_u(L)\phi - \Psi_u(L)\phi\|_\infty \leq C\|\Psi_u(L)\phi\|_\infty.$$

Applying the Dominated Convergence theorem, it follows that

$$\|e^{-sL}\phi - \phi\|_{\dot{B}^{\alpha,L}_{\infty,q}} = \left(\int_0^\infty (u^{-\alpha}\|\Psi_u(L)(e^{-sL}\phi - \phi)\|_\infty)^q \frac{du}{u}\right)^{1/q} \to 0$$

for $q < \infty$. For $q = \infty$, by (3.4), it follows that

$$\|e^{-sL}\phi - \phi\|_{\dot{B}^{\alpha,L}_{\infty,\infty}} \leq C\|e^{-sL}\phi - \phi\|_{\dot{B}^{\alpha,L}_{\infty,1}} \to 0$$

as $s \to 0$.

Finally, assume $q = \infty$. Because $\|\phi\|_{\dot{B}^{\alpha,L}_{p,1}} < \infty$, from (3.4) it follows that

$$\|u^{-\alpha}\Psi_u(L)\phi\|_p \to 0 \quad \text{as } u \to 0 \text{ or } u \to \infty.$$

Thus we have finished the proof of the proposition. □

Proposition 3.8 implies that for any $t > 0$ and $x \in \mathcal{X}$,

$$k_t(x,\cdot) = k_t^*(\cdot,x) = p_{1,t}^*(\cdot,x) \in \mathcal{M}_{p,q}^{\alpha,L^*},$$

$-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. Thus for any $f \in \left(\mathcal{M}_{p',q'}^{-\alpha,L^*}\right)'$, and for each $t > 0$ and $x \in \mathcal{X}$,

$$\Psi_t(L)f(x) = (f, k_t(x,\cdot)) = \int_{\mathcal{X}} f(x)k_t(x,\cdot)\,d\mu(x)$$

is well-defined, where $(\cdot,\cdot)$ denotes the pairing between a linear functional and a test function.

Definition 3.9. *Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. We define the Besov space $\dot{B}^{\alpha,L}_{p,q}$ associated to an operator L by*

$$\dot{B}^{\alpha,L}_{p,q} = \left\{ f \in \left(\mathcal{M}^{-\alpha,L^*}_{p',q'}\right)' : \|f\|_{\dot{B}^{\alpha,L}_{p,q}} < \infty \right\},$$

where

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \left\{ \int_0^\infty (t^{-\alpha}\|\Psi_t(L)(f)\|_p)^q \, \frac{dt}{t} \right\}^{1/q}.$$

Similarly to the space of test functions $\mathcal{M}^{\alpha,L}_{p,q}$, the space $\dot{B}^{\alpha,L}_{p,q}$ is a normed linear space if the zero element is identified with all f satisfying $Le^{-tL}f = 0$ for all $t > 0$ almost everywhere.

Definition 3.10. *Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$, $1 \le p, q \le \infty$ and $s > 0$. Let $f \in \left(\mathcal{M}^{-\alpha,L^*}_{p',q'}\right)'$. Define a linear functional $e^{-sL}f$ on $\mathcal{M}^{-\alpha,L^*}_{p',q'}$ by*

$$(e^{-sL}f, \phi) = (f, e^{-sL^*}\phi), \quad \forall \phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}. \tag{3.6}$$

In the following proposition we will show that $e^{-sL}f \in \left(\mathcal{M}^{-\alpha,L^*}_{p',q'}\right)'$.

Proposition 3.11. *Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. Let $f \in \dot{B}^{\alpha,L}_{p,q}$. Then there exists a constant $C > 0$ such that for all $s > 0$, $e^{-sL}f \in \dot{B}^{\alpha,L}_{p,q}$ and*

$$\|e^{-sL}f\|_{\dot{B}^{\alpha,L}_{p,q}} \le C\|f\|_{\dot{B}^{\alpha,L}_{p,q}}.$$

Proof. Firstly we show that $e^{-sL}f \in \left(\mathcal{M}^{-\alpha,L^*}_{p',q'}\right)'$ for all $f \in \left(\mathcal{M}^{-\alpha,L^*}_{p',q'}\right)'$ and $s > 0$. To show this claim, we note that for any test function $\phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$ and $s, t > 0$,

$$\Psi_t(L^*)e^{-sL^*}\phi = e^{-sL^*}\Psi_t(L^*)\phi.$$

Thus, using the continuity of the semigroup $\{e^{-sL}\}_{s>0}$, we obtain

$$\|\Psi_t(L^*)e^{-sL^*}\phi\|_p \le C\|\Psi_t(L^*)\phi\|_p.$$

We then have

$$\|e^{-sL^*}\phi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}} \le C\|\phi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}}$$

and thus the claim is proved.

To show the proposition, for all $s, t > 0$, we let $K_{\Psi_t(L)e^{-sL}}(x, y)$ be the kernel of the operator $\Psi_t(L)e^{-sL}$, and $K_{e^{-sL}\Psi_t(L)}(x, y)$ be the kernel of the operator $e^{-sL}\Psi_t(L)$. Note that

$$K_{\Psi_t(L)e^{-sL}}(x, y) = K_{e^{-sL}\Psi_t(L)}(x, y)$$

for all $x, y \in \mathcal{X}$. Using a similar argument to the proofs of Proposition 3.6 and Proposition 3.8, it follows that $K_{\Psi_t(L)e^{-sL}}(x, \cdot) \in \mathcal{M}^{-\alpha, L^*}_{p', q'}$ for all $s, t > 0$ and $x \in \mathcal{X}$. Thus for any $f \in \dot{B}^{\alpha, L}_{p,q}$ and $x \in \mathcal{X}$,

$$\begin{aligned}
&\Psi_t(L)e^{-sL}f(x) \\
&\quad = (f, K_{\Psi_t(L)e^{-sL}}(x, \cdot)) \\
&\quad = (f, K_{e^{-sL}\Psi_t(L)}(x, \cdot)) \\
&\quad = e^{-sL}\Psi_t(L)f(x).
\end{aligned} \tag{3.7}$$

Hence we have

$$\begin{aligned}
&\|e^{-sL}f\|_{\dot{B}^{\alpha,L}_{p,q}} \\
&\quad = \left\{ \int_0^\infty (t^{-\alpha}\|\Psi_t(L)e^{-sL}f\|_p)^q \, \frac{dt}{t} \right\}^{1/q} \\
&\quad = \left\{ \int_0^\infty (t^{-\alpha}\|e^{-sL}\Psi_t(L)f\|_p)^q \, \frac{dt}{t} \right\}^{1/q} \\
&\quad \le C\left\{ \int_0^\infty (t^{-\alpha}\|\Psi_t(L)f\|_p)^q \, \frac{dt}{t} \right\}^{1/q} \\
&\quad = C\|f\|_{\dot{B}^{\alpha,L}_{p,q}}.
\end{aligned}$$

Thus the proof of the proposition is complete. □

The next subsection will require the following *approximation to the identity* result for the Besov spaces.

Proposition 3.12. *Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$.*

(i) Assume $1 \le p, q < \infty$ *and* $f \in \dot{B}^{\alpha,L}_{p,q}$. *Then*

$$\lim_{s\to 0} e^{-sL} f = f \quad \text{in } \dot{B}^{\alpha,L}_{p,q}.$$

(ii) Assume $\phi \in \mathcal{M}^{\alpha,L}_{p,q}$. *Then*

$$\lim_{s\to 0} e^{-sL} \phi = \phi \quad \text{in } \mathcal{M}^{\alpha,L}_{p,q}.$$

Proof. The proofs of (i) and (ii) when $1 \le p < \infty$ use arguments similar to the proof of Proposition 3.8, by applying (3.7), the continuity of the semigroup $\{e^{-sL}\}_{s>0}$, and the Dominated Convergence Theorem.

The result (ii) for $p = \infty$ follows from the definition of test functions. Thus it remains to show (ii) for $p < \infty$ and $q = \infty$. Let $s > 0$ and $\phi \in \mathcal{M}^{\alpha,L}_{p,\infty}$. Using the finiteness of $C = \sup_{s>0} \|e^{-sL}\|_{L^p \to L^p}$ and the definition of test functions, it follows that

$$\begin{aligned} &\|t^{-\alpha}\Psi_t(L)e^{-sL}\phi\|_p \\ &\quad = \|t^{-\alpha}e^{-sL}\Psi_t(L)\|_p \\ &\quad \le C\|t^{-\alpha}\Psi_t(L)\|_p \to 0 \end{aligned}$$

as $t \to \infty$ or $t \to 0$. Fix $0 < r < R < \infty$. We then have

$$\begin{aligned} &\sup_{r\le t\le R} t^{-\alpha}\|\Psi_t(L)(e^{-sL}\phi - \phi)\|_p \\ &\quad \le C\left(\sup_{t>0} \|\Psi_t(L)\|_{L^p\to L^p}\right)\|e^{-sL}\phi - \phi\|_p \to 0 \end{aligned}$$

as $s \to 0$, using the continuity property of $\{e^{-sL}\}_{s>0}$. We therefore have (ii) for $q = \infty$. Thus we have finished the proof of the Proposition. □

3.3.2 The Calderón reproducing formulas

Two forms of the Calderón reproducing formula will be developed to investigate properties of the Besov spaces. These formulas together with the other Calderón reproducing formulas in the thesis are important in our study, and they are of independent interest.

Theorem 3.13 (Calderón reproducing formula II)**.** *Suppose L satisfies (S) and (K), and L^* its adjoint operator. Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. Let p' and q' be the conjugate exponents of p and q respectively. Then for $\Psi_t(L) = tLe^{-tL}$, we have*

$$(f, \phi) = 4 \int_0^\infty \int_{\mathcal{X}} \Psi_t(L) f(x) \Psi_t(L^*) \phi(x) \, d\mu(x) \, \frac{dt}{t} \tag{3.8}$$

for every $f \in \dot{B}^{\alpha,L}_{p,q}$ and $\phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$.

Proof. Let $f \in \dot{B}^{\alpha,L}_{p,q}$ and $\phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$. The double integral in (3.8) converges absolutely since

$$\begin{aligned}
&\int_0^\infty \int_{\mathcal{X}} |\Psi_t(L) f(x) \Psi_t(L^*) \phi(x)| \, d\mu(x) \, \frac{dt}{t} \\
&\quad \le \int_0^\infty t^{-\alpha} \|\Psi_t(L) f\|_p \, t^{\alpha} \|\Psi_t(L^*) \phi\|_{p'} \, \frac{dt}{t} \\
&\quad \le \|f\|_{\dot{B}^{\alpha,L}_{p,q}} \|\phi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}} \\
&\quad < \infty.
\end{aligned} \tag{3.9}$$

We then have

$$\begin{aligned}
&\int_0^\infty \int_{\mathcal{X}} \Psi_t(L) f(x) \Psi_t(L^*) \phi(x) \, d\mu(x) \, \frac{dt}{t} \\
&\quad = \lim_{M \to \infty} \int_{1/M}^M \Big(f, (\Psi_t(L^*))^2 \phi \Big) \, \frac{dt}{t} \\
&\quad = \lim_{M \to \infty} \Big(f, \int_{1/M}^M (\Psi_t(L^*))^2 \phi \, \frac{dt}{t} \Big).
\end{aligned}$$

We note that in the above we have used

$$\int_{1/M}^M (\Psi_t(L^*))^2 \phi \, \frac{dt}{t} \in \mathcal{M}^{-\alpha,L^*}_{p',q'}.$$

This fact follows from (3.11) and Proposition 3.6. To finish the proof we need to show that

$$\lim_{M \to \infty} \int_{1/M}^M (\Psi_t(L^*))^2 \phi \, \frac{dt}{t} = \frac{1}{4} \phi \quad \text{in } \mathcal{M}^{-\alpha,L^*}_{p',q'} \tag{3.10}$$

By using the fact that $(\Psi_t(L^*))^2 = (tL^*)^2 e^{-2tL^*}$ and by using integration by parts, it follows that

$$\int_{1/M}^M (\Psi_t(L^*))^2 \phi \, \frac{dt}{t}$$

$$
\begin{aligned}
&= \frac{1}{4}\int_{1/M}^{M} t\frac{d^2}{dt^2}(e^{-2ML^*})\phi\,dt \\
&= \frac{1}{4}(-2ML^*e^{-2ML^*}\phi + \frac{2}{M}L^*e^{-\frac{2}{M}L^*}\phi - e^{-2ML^*}\phi + e^{-\frac{2}{M}L^*}\phi).
\end{aligned}
\tag{3.11}
$$

Using Proposition 3.12 (ii),

$$
\lim_{M\to\infty} e^{-\frac{2}{M}L^*}\phi = \phi \quad \text{in } \mathcal{M}^{-\alpha,L^*}_{p',q'} \tag{3.12}
$$

Thus to show (3.10), it is enough to verify the convergence in the $\dot{B}^{-\alpha,L^*}_{p',q'}$-norm of the following three expressions:

(i) $\lim_{M\to\infty} 2ML^*e^{-2ML^*}\phi = 0.$

(ii) $\lim_{M\to\infty} \frac{2}{M}L^*e^{-\frac{2}{M}L^*}\phi = 0.$

(iii) $\lim_{M\to\infty} e^{-2ML^*}\phi = 0.$

By using $\phi = L^*g$, we obtain

$$
\begin{aligned}
&\|2ML^*e^{-2ML^*}\phi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}} \\
&\quad = \left\{\int_0^\infty t^\alpha \|\Psi_t(L^*)(2ML^*e^{-2ML^*}\phi)\|_{p'})^{q'}\frac{dt}{t}\right\}^{1/q'} \\
&\quad \le \left\{\int_0^1 (t^\alpha \|Mt(L^*)^2e^{-(t+2M)L^*}\phi)\|_{p'})^{q'}\frac{dt}{t}\right\}^{1/q'} \\
&\qquad + \left\{\int_1^\infty (t^\alpha \|Mt(L^*)^3e^{-(t+2M)L^*}g)\|_{p'})^{q'}\frac{dt}{t}\right\}^{1/q'} \\
&\quad = I + II.
\end{aligned}
$$

Note that

$$
Mt(L^*)^2e^{-(t+2M)L^*} = \frac{Mt}{(t+2M)^2}\times((t+2M)L^*)^2e^{-(t+2M)L^*}.
$$

By the $L^{p'}$-boundedness of $((t+2M)L^*)^2e^{-(t+2M)L^*}$ we have

$$
\begin{aligned}
I &\le C\|\phi\|_{p'}\left\{\int_0^1 \left(t^\alpha \times \frac{Mt}{(t+2M)^2}\right)^{q'}\frac{dt}{t}\right\}^{1/q'} \\
&\le \frac{C}{M}\|\phi\|_{p'}\left\{\int_0^1 t^{(\alpha+1)q'}\frac{dt}{t}\right\}^{1/q'} \\
&\le \frac{C}{M}\|\phi\|_{p'}
\end{aligned}
$$

tends to zero as $M \to \infty$.

To estimate term II, we select α_0 satisfying $|\alpha| < \alpha_0 < 1$. Because the operator

$$((t+2M)L^*)^3 e^{-(t+2M)L^*}$$

is bounded on $L^{p'}(\mathcal{X})$, it follows that

$$\begin{aligned} II &\leq C\|g\|_{p'} \left\{ \int_1^\infty \left(t^\alpha \times \frac{Mt}{(t+2M)^3} \right)^{q'} \frac{dt}{t} \right\}^{1/q'} \\ &\leq \frac{C}{M^{1-\alpha_0}} \|g\|_{p'} \left\{ \int_1^\infty t^{(\alpha-\alpha_0)q'} \frac{dt}{t} \right\}^{1/q'} \\ &\leq \frac{C}{M^{1-\alpha_0}} \|g\|_{p'} \end{aligned}$$

tends to zero as $M \to \infty$. Thus the proof of (i) is complete.

An argument similar to the above will give (iii). Let us now show (ii). For each $t > 0$,

$$\|\Psi_t(L^*)(sL^* e^{-sL^*}\phi)\|_{p'} = \frac{ts}{(t+s)^2} \|(t+s)^2 e^{-(t+s)L^*}\phi\|_{p'} \to 0$$

as $s \to 0$. Furthermore, by the $L^{p'}$-boundedness of $sL^* e^{-sL^*}$ we have

$$\|\Psi_t(L^*)(sL^* e^{-sL^*}\phi)\|_{p'} \leq C\|\Psi_t(L^*)\phi)\|_{p'}.$$

Because

$$\int_0^\infty (t^\alpha \|\Psi_t(L^*)\phi)\|_{p'})^{q'} \frac{dt}{t} < \infty,$$

by the Dominated Convergence Theorem it follows that

$$\begin{aligned} &\lim_{s\to 0} \|sL^* e^{-sL^*}\phi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}} \\ &\quad = \lim_{s\to 0} \int_0^\infty (t^\alpha \|\Psi_t(L^*)(sL^* e^{-sL^*}\phi)\|_{p'})^{q'} \frac{dt}{t} \\ &\quad = 0. \end{aligned}$$

Hence the proof of (ii) is complete. Thus we have finished proving (3.10). Thus the proof of the theorem is complete. □

The following Calderón reproducing formula will be needed to show that the class of functions

of finite Besov norm is contained in $\dot{B}^{\alpha,L}_{p,q}$. Even though its proof follows similar steps to the proof of Theorem 3.13, Theorem 3.14 is not a corollary of Theorem 3.13.

Theorem 3.14 (Calderón reproducing formula III)**.** *Suppose L satisfies (S) and (K), and L^* its adjoint operator. Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. Let p' and q' be the conjugate exponents of p and q respectively. Also suppose*

$$\int_{\mathcal{X}} |f(x)| e^{-\alpha d(x,x_0)^2}\, d\mu(x) < \infty$$

for some $x_0 \in \mathcal{X}$, and $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} < \infty$. Then for $\Psi_t(L) = tLe^{-tL}$, we have

$$(f,\phi) = \int_{\mathcal{X}} f(x)\phi(x)\, d\mu(x) = 4\int_0^\infty \int_{\mathcal{X}} \Psi_t(L) f(x) \Psi_t(L^*)\phi(x)\, d\mu(x)\, \frac{dt}{t} \tag{3.13}$$

for every $\phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$.

Proof. Let $\phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$. Then we have that $\phi \in L^r(\mathcal{X})$ for any $r > 1$. Suppose f is a function that satisfies

$$\int_{\mathcal{X}} |f(x)| e^{-\alpha d(x,x_0)^2}\, d\mu(x) < \infty \tag{3.14}$$

for some $x_0 \in \mathcal{X}$, and $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} < \infty$. As in the proof of Theorem 3.13, the double integral in (3.13) converges absolutely. Moreover, we have

$$\begin{aligned}&\int_0^\infty \int_{\mathcal{X}} \Psi_t(L) f(x) \Psi_t(L^*)\phi(x)\, d\mu(x)\, \frac{dt}{t} \\ &\quad = \lim_{M\to\infty} \int_{\mathcal{X}} \int_{1/M}^{M} f(x)(\Psi_t(L^*))^2 \phi(x)\, \frac{dt}{t}\, d\mu(x).\end{aligned}$$

Because the semigroup e^{-tL^*} is continuous and differentiable with $\lim_{t\to 0} e^{-tL^*} = I$ in L^r, using Theorem 3.3 it follows that

$$\lim_{M\to\infty} \int_{1/M}^{M} (\Psi_t(L^*))^2 \phi\, \frac{dt}{t} = \frac{1}{4}\phi \quad \text{in } L^r.$$

By the L^r convergence, there exists a sub-sequence $\{M_j\}$ of integers {M} satisfying

$$\lim_{j\to\infty} \int_{1/M_j}^{M_j} (\Psi_t(L^*))^2 \phi\, \frac{dt}{t} = \frac{1}{4}\phi \tag{3.15}$$

almost everywhere. Using (3.11), it follows that

$$\int_{1/M}^{M} (\Psi_t(L^*))^2 \phi \, \frac{dt}{t} \\ = \frac{1}{4}(-2ML^* e^{-2ML^*}\phi + \frac{2}{M} L^* e^{-\frac{2}{M}L^*}\phi - e^{-2ML^*}\phi + e^{-\frac{2}{M}L^*}\phi). \tag{3.16}$$

Because $-2ML^* e^{-2ML^*} = \Psi_{2M}(L^*)$ satisfies the kernel bound (K), from the equality $\phi = L^* g$, and condition (3.5), it follows that for every $x \in \mathcal{X}$,

$$\begin{aligned} &\big| -2ML^* e^{-2ML^*}\phi(x)\big| \\ &= \Big| -\frac{1}{2M}(2ML^*)^2 e^{-2ML^*} g(x)\Big| \\ &\le \frac{C}{M}\int_{\mathcal{X}} \frac{e^{-\alpha d(x,y)^2/2M}}{(2M)^{\min(n,N)/2}} e^{-\alpha d(x_0,y)^2}\, d\mu(y) \\ &\le C\frac{M^{-1}}{(2M)^{\min(n,N)/2}} e^{-\alpha d(x,x_0)^2/2M} \\ &\le C e^{-\alpha d(x,x_0)^2} \end{aligned}$$

for $M > 1$. In a similar way, the other three terms in (3.16) satisfy the same estimates. Thus we obtain that for every x,

$$\left| \int_{1/M_j}^{M_j} (\Psi_t(L^*))^2 \phi(x) \, \frac{dt}{t} \right| \le C e^{-\alpha d(x,x_0)^2} \tag{3.17}$$

By (3.17), it follows that for all M_j and all x

$$\left| f(x) \int_{1/M_j}^{M_j} (\Psi_t(L^*))^2 \phi(x) \, \frac{dt}{t} \right| \le C|f(x)| e^{-\alpha d(x,x_0)^2}$$

which is integrable, from the growth assumption on f. Hence using (3.15) and the Dominated Convergence Theorem,

$$\lim_{j\to\infty} \int_{\mathcal{X}} f(x) \int_{1/M_j}^{M_j} (\Psi_t(L^*))^2 \phi(x) \, \frac{dt}{t} \, d\mu(x) = \frac{1}{4}\int_{\mathcal{X}} f(x)\phi(x)\, d\mu(x).$$

Thus the proof of the theorem is complete. □

We end this subsection by a brief discussion of two Besov spaces on domains existing in current literature (see, for example, [Tr, Tr1, ST, Mu]).

Let Ω be a smooth, bounded domain in $\mathbb{R}^n$. Let $0 < \alpha < 1$ and $1 \leq p, q \leq \infty$. In this case, the inhomogeneous Besov space $B^{\alpha}_{p,q}(\Omega)$ is defined as follows:

$$B^{\alpha}_{p,q}(\Omega) = \{f \in L^p(\Omega) : \exists g \in B^{\alpha}_{p,q}(\mathbb{R}^n), \quad f = g|_{\Omega}\},$$

where $g|_{\Omega}$ is the restriction of g to Ω. See [Tr, Tr1] for properties of this space as well as the definition for all $\alpha \in \mathbb{R}$.

In [ST], H. Sikić and M.H. Taibleson considered the Besov–Lipschitz space ${}_zB^{\alpha}_{p,q}(\Omega)$, which is defined as the space of all $f \in L^p(\overline{\Omega})$ such that its zero extension to $\mathbb{R}^n$, $\widetilde{f} \in B^{\alpha}_{p,q}(\mathbb{R}^n)$, where

$$\widetilde{f} = \begin{cases} f(x), & x \in \overline{\Omega}, \\ 0, & x \notin \overline{\Omega}. \end{cases}$$

As noted in [ST], ${}_zB^{\alpha}_{p,q}(\Omega)$ is the same as the space $\widetilde{B}^{\alpha}_{p,q}(\Omega)$ in [Tr]; we refer the reader to Remark 4, p. 320 in [Tr] for a discussion on the origin of this space. By (2.8) in [ST], we obtain

$${}_zB^{\alpha}_{p,q}(\Omega) \subseteq B^{\alpha}_{p,q}(\Omega).$$

Furthermore, the relationship between these two spaces and the space $\mathring{B}^{\alpha}_{p,q}(\Omega)$, the closure of $C^{\infty}_c(\Omega)$ in $B^{\alpha}_{p,q}(\Omega)$, has been elaborated in Chapter 4 of [Tr].

By p. 144 in [ST], the kernel $Q(t; x, y)$ in that paper is the heat kernel associated with the Laplacian on Ω with Dirichlet boundary condition, and by Section 5.6 in [ST], the kernel $R(t; x, y)$ is the heat kernel associated with the Laplacian on Ω with Neumann boundary condition. Thus it follows from Theorem 4.1 and (5.20) in [ST] that we have the following result.

Proposition 3.15. *Suppose that Ω is a smooth and bounded domain in $\mathbb{R}^n$. Assume that $0 < \alpha < 1$ and $1 \leq p, q \leq \infty$. Let $-\Delta_N$ and $-\Delta_D$ denote the Laplace operators on Ω with Neumann boundary condition and Dirichlet boundary condition, respectively. Then we have*

$$\dot{B}^{\alpha,-\Delta_N}_{p,q}(\Omega) \cap L^p(\Omega) = B^{\alpha}_{p,q}(\Omega),$$

$$\dot{B}^{\alpha,-\Delta_D}_{p,q}(\Omega) \cap L^p(\Omega) = {}_zB^{\alpha}_{p,q}(\Omega).$$

We note that in all the existing works in the (inhomogeneous) Besov spaces, such as the classical Besov spaces $B^{\alpha}_{p,q}(\Omega)$ and ${}_zB^{\alpha}_{p,q}(\Omega)$, some regularity (smoothness) of the boundary of Ω is assumed. In contrast, we made no assumptions on the boundary of Ω, but impose instead a heat kernel bound on the kernel $p_t(x,y)$. As the example (iii) in Section 3.2.2 shows, this heat kernel bound condition can be satisfied when the domain Ω has no regularity condition on its boundary. Therefore the Besov spaces $\dot{B}^{\alpha,L}_{p,q}(\Omega) \cap L^p(\Omega)$ go beyond the classical case and give new spaces when Ω possesses no regularity condition on its boundary.

Because of the usefulness of Besov spaces in the study of solutions of the Dirichlet and Neumann problems on Lipschitz domains (see, for example, [Ke, MM1, MMS]), it would be of considerable interest to extend the results in Proposition 3.15 to the case where the boundary of Ω has minimal regularity.

Chapter 4

Besov spaces associated with operators II

Some of the content in this chapter are contained in [Wo1] and [Wo2]. In this chapter we study some properties and decomposition of Besov spaces associated with operators.

In Section 4.1 we study an embedding theorem for the Besov spaces and give discrete characterizations of the Besov norms associated to operators. We also study the equivalence of the Besov norms with respect to different functions of L. We extend the Besov norm equivalence to more general class of functions $\Psi_t(L)$ with suitable decay at 0 and infinity, and to non-integer $k \geq 1$. Then we study the behaviour of fractional integrals on the Besov spaces.

In Section 4.2 we study decomposition of Besov spaces associated with Schrödinger operators with non-negative potentials satisfying reverse Hölder estimates on $\mathbb{R}^n$. We also show that, in some special cases, the classical Besov spaces are proper subspaces of these spaces. We also extend the decomposition of Besov spaces associated with Schrödinger operators to more general values α, p, q.

4.1 Properties of Besov spaces associated with operators

4.1.1 Embedding theorem

Theorem 4.1. *Suppose that L satisfies (S) and (K). Let $-1 < \alpha < 1$, $1 \leq p \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$. Then the following statements are true:*

(i) $\dot{B}^{\alpha,L}_{p,q} \subseteq (\mathcal{M}^{-\alpha,L^*}_{p',q'})'$ *(continuous embedding).*

(ii) $\dot{B}^{\alpha,L}_{p,q_1} \subseteq \dot{B}^{\alpha,L}_{p,q_2}$.

(iii) $\dot{B}^{\alpha,L}_{p,q}$ *is complete.*

(iv) If $1 \le p_1 \le p_2 \le \infty$, $-1 < \alpha_2 \le \alpha_1 < 1$ *and* $\alpha_1 - \frac{\min(n,N)}{2p_1} = \alpha_2 - \frac{\min(n,N)}{2p_2}$, *then*

$$\dot{B}^{\alpha_1,L}_{p_1,q} \subseteq \dot{B}^{\alpha_2,L}_{p_2,q},$$

for every $1 \le q \le \infty$.

Proof. Firstly, (i) follows from Theorem 3.13.

Let us now show (ii). Let $f \in \dot{B}^{\alpha,L}_{p,q_1}$. Using the same arguments as in the proof of Proposition 3.5, it follows that (3.3) and (3.4) are true for f. Then for all $t > 0$, we have

$$\|\Psi_s(L)f\|_p \ge c\|\Psi_t(L)f\|_p, \quad t/2 \le s \le t, \tag{4.1}$$

so then we obtain

$$\begin{aligned}
&\|f\|_{\dot{B}^{\alpha,L}_{p,q_1}} \\
&\quad \ge \left\{ \int_{t/2}^{t} (s^{-\alpha}\|\Psi_s(L)f\|_p)^{q_1} \, \frac{ds}{s} \right\}^{1/q_1} \\
&\quad \ge c_1 t^{-\alpha}\|\Psi_t(L)f\|_p.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\|f\|_{\dot{B}^{\alpha,L}_{p,\infty}} \\
&\quad = \sup_{t>0} t^{-\alpha}\|\Psi_t(L)\phi\|_p \\
&\quad \le \frac{1}{c_1}\|f\|_{\dot{B}^{\alpha,L}_{p,q_1}}.
\end{aligned}$$

Hence (ii) is true for the case $q_2 = \infty$. The embedding for the case $q_2 < \infty$ easily follows from the case $q_2 = \infty$.

Let us now show (iii). Suppose that $\{f_n\}$ is a Cauchy sequence in $\dot{B}^{\alpha,L}_{p,q}$. From (i), we have that $\{f_n\}$ is a Cauchy sequence in the Banach space $\left(\mathcal{M}^{-\alpha,L^*}_{p',q'}\right)'$. Hence there exists

$f \in \left(\mathcal{M}_{p',q'}^{-\alpha,L^*}\right)'$ satisfying

$$f_n \to f \quad \text{in} \quad \left(\mathcal{M}_{p',q'}^{-\alpha,L^*}\right)'.$$

It follows that for every $t > 0$ and $x \in \mathcal{X}$,

$$(f_n, k_t(x,\cdot)) = \Psi_t(L)f_n(x) \to (f, k_t(x,\cdot)) = \Psi_t(L)f(x).$$

Using (ii) and the completeness of $L^p(\mathcal{X})$ we obtain $\Psi_t(L)f_n \to \Psi_t(L)f$ in $L^p(\mathcal{X})$.

Let us now prove that f_n converges to f in $\dot{B}_{p,q}^{\alpha,L}$. Let $\epsilon > 0$. Because $\{f_n\}$ is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$\left\{\int_0^\infty (t^{-\alpha}\|\Psi_t(L)(f_m - f_n)\|_p)^q \frac{dt}{t}\right\}^{1/q} < \epsilon.$$

Fix $n \geq N$, and let $m \to \infty$ in the above. It follows that

$$\left\{\int_0^\infty (t^{-\alpha}\|\Psi_t(L)(f_n - f)\|_p)^q \frac{dt}{t}\right\}^{1/q} < \epsilon.$$

Thus we have $f_n \to f$ in $\dot{B}_{p,q}^{\alpha,L}$. Thus the proof of (iii) is finished.

Let us now prove (iv). Suppose $f \in \dot{B}_{p,q}^{\alpha_1,L}$. Because

$$\Psi_{2t}(L)f = 2e^{-tL}\Psi_t(L)f, \quad t > 0,$$

from the kernel bound condition (K), it follows that for all $x \in \mathcal{X}$,

$$\Psi_{2t}(L)f(x) \leq C\int_{\mathcal{X}} \frac{e^{-\alpha d(x,y)^2/t}}{t^{\min(n,N)/2}}|\Psi_t(L)f(y)|\,d\mu(y).$$

Let $r \geq 0$, where $\frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{r} - 1$. By applying a similar argument as in the proof of Young's inequality (see e.g., Theorem 1.2.12 in [Gr]) and Lemma 3.1, it follows that

$$\begin{aligned}
&\|\Psi_{2t}(L)\|_{p_2} \\
&\quad \leq C\|\Psi_t(L)f\|_{p_1}\left(\sup_y \left\|\frac{e^{-\alpha d(\cdot,y)^2/t}}{t^{\min(n,N)/2}}\right\|_r\right) \\
&\quad \leq C\|\Psi_t(L)f\|_{p_1} t^{\min(n,N)(\frac{1}{p_2}-\frac{1}{p_1})/2}.
\end{aligned}$$

Then we have

$$\|f\|_{\dot{B}_{p_2,q}^{\alpha_2,L}}$$

$$
\begin{aligned}
&\le C\Big\{\int_0^\infty (t^{-\alpha_2}\|\Psi_{2t}(L)f\|_{p_2})^q\,\frac{dt}{t}\Big\}^{1/q}\\
&\le C\Big\{\int_0^\infty \Big(t^{-\alpha_2+\min(n,N)(\frac{1}{p_2}-\frac{1}{p_1})/2}\|\Psi_t(L)f\|_{p_1}\Big)^q\,\frac{dt}{t}\Big\}^{1/q}\\
&= C\|f\|_{\dot{B}^{\alpha_1,L}_{p_1,q}}.
\end{aligned}
$$

Thus we have finished proving the Theorem. □

In the following we show that $\dot{B}^{\alpha,L}_{p,q}$ contains functions satisfying some growth condition.

Proposition 4.2. *Suppose L satisfies (S) and (K). Let $-1<\alpha<1$ and $1\le p,q\le\infty$. Suppose that f satisfies the growth condition (G) and $\|f\|_{\dot{B}^{\alpha,L}_{p,q}}<\infty$. Then $f\in\dot{B}^{\alpha,L}_{p,q}$.*

Proof. It is enough to show that $f\in\big(\mathcal{M}^{-\alpha,L^*}_{p',q'}\big)'$. For all $\phi\in\mathcal{M}^{-\alpha,L^*}_{p',q'}$, we obtain by Theorem 3.14 that

$$
(f,\phi)=\int_{\mathcal{X}} f(x)\phi(x)\,d\mu(x)=\int_0^\infty\int_{\mathcal{X}}\Psi_t(L)f(x)\Psi_t(L^*)\phi(x)\,d\mu(x)\,\frac{dt}{t}.
$$

From Hölder's inequality, we have

$$
\begin{aligned}
&\Big|\int_0^\infty\int_{\mathcal{X}}\Psi_t(L)f(x)\Psi_t(L^*)\phi(x)\,d\mu(x)\,\frac{dt}{t}\Big|\\
&\quad\le\int_0^\infty t^{-\alpha}\|\Psi_t(L)f\|_p\,t^{\alpha}\|\Psi_t(L^*)\phi\|_{p'}\,\frac{dt}{t}\\
&\quad\le\|f\|_{\dot{B}^{\alpha,L}_{p,q}}\|\phi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}}\\
&\quad<\infty,
\end{aligned}
$$

hence $f\in\big(\mathcal{M}^{-\alpha,L^*}_{p',q'}\big)'$. Thus the proof of the proposition is complete. □

4.1.2 Norm equivalence

In the following proposition, we present discrete characterizations of Besov norms associated with operators.

Proposition 4.3. *Suppose L satisfies (S) and (K). Let $-1<\alpha<1$ and $1\le p\le\infty$. The following three statements are equivalent for $f\in\big(\mathcal{M}^{-\alpha,L^*}_{p',q'}\big)'$.*

(a) $f \in \dot{B}^{\alpha,L}_{p,q}$

(b) f *satisfies* $\Big\{ \sum_{j=-\infty}^{\infty} (2^{j\alpha} \|\Psi_{2^{-j}}(L)f\|_p)^q \Big\}^{1/q} < \infty$

(c) f *satisfies* $\Big\{ \sum_{j=-\infty}^{\infty} (2^{j\alpha} \|\Delta_j(L)f\|_p)^q \Big\}^{1/q} < \infty$

where $\Delta_j(L)f = e^{-2^{-j}L}f - e^{-2^{-j-1}L}f$.

Furthermore, each infinite sum in (b) and (c) are equivalent to $\|f\|_{\dot{B}^{\alpha,L}_{p,q}}$.

Proof. Let us begin by showing that (a) is equivalent to (b). For each $j \in \mathbb{Z}$, applying (4.1) we obtain

$$\|\Psi_t(L)f\|_p \le C\|\Psi_{2^{-j-1}}(L)f\|_p, \quad 2^{-j-1} \le t \le 2^{-j}.$$

Then we have

$$\begin{aligned}
&\|f\|_{\dot{B}^{\alpha,L}_{p,q}} \\
&\quad = \left\{ \int_0^\infty (t^{-\alpha}\|\Psi_t(L)f\|_p)^q \, \frac{dt}{t} \right\}^{1/q} \\
&\quad = \left\{ \sum_{j=-\infty}^{\infty} \int_{2^{-j-1}}^{2^{-j}} (t^{-\alpha}\|\Psi_t(L)f\|_p)^q \, \frac{dt}{t} \right\}^{1/q} \\
&\quad \le C_1 \left(\sum_{j=-\infty}^{\infty} \left(2^{(j+1)\alpha}\|\Psi_{2^{-j-1}}(L)f\|_p\right)^q \right)^{1/q} \\
&\quad = C_1 \left(\sum_{j=-\infty}^{\infty} (2^{j\alpha}\|\Psi_{2^{-j}}(L)f\|_p)^q \right)^{1/q}.
\end{aligned}$$

Using (4.1), it follows that

$$\|\Psi_t(L)f\|_p \ge c\|\Psi_{2^{-j}}(L)f\|_p, \quad 2^{-j-1} \le t \le 2^{-j}.$$

Using a similar argument, we obtain

$$\left(\sum_{j=-\infty}^{\infty} (2^{j\alpha}\|\Psi_{2^{-j}}(L)f\|_p)^q \right)^{1/q} \le c_2\|f\|_{\dot{B}^{\alpha,L}_{p,q}}.$$

Thus we have shown that (a) is equivalent to (b).

Next we show that (a) implies (c). Observe that for every j,

$$\Delta_j(L)f = (e^{-2^{-j}L} - e^{-2^{-j-1}L})f = -\int_{2^{-j-1}}^{2^{-j}} \Psi_t(L)f \, \frac{dt}{t}.$$

By applying Hölder's and Minkowski's inequalities we obtain

$$\begin{aligned}
&2^{j\alpha}\|\Delta_j(L)f\|_p \\
&\quad \le 2^{j\alpha}\int_{2^{-j-1}}^{2^{-j}} \|\Psi_t(L)f\|_p \,\frac{dt}{t} \\
&\quad \le C2^{j\alpha}\bigg(\int_{2^{-j-1}}^{2^{-j}} \|\Psi_t(L)f\|_p^q \,\frac{dt}{t}\bigg)^{1/q} \\
&\quad \le C\bigg(\int_{2^{-j-1}}^{2^{-j}} (t^{-\alpha}\|\Psi_t(L)f\|_p)^q \,\frac{dt}{t}\bigg)^{1/q}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\bigg(\sum_{j=-\infty}^{\infty} (2^{j\alpha}\|\Delta_j(L)f\|_p)^q\bigg)^{1/q} \\
&\quad \le C\bigg(\int_0^{\infty} (t^{-\alpha}\|\Psi_t(L)f\|_p)^q \,\frac{dt}{t}\bigg)^{1/q} \\
&\quad = C\|f\|_{\dot{B}^{\alpha,L}_{p,q}}.
\end{aligned}$$

Lastly, we will show that (c) implies (a). For every $j \in \mathbb{Z}$, using the mean value theorem, there exists some $\eta_j \in [2^{-j-1}, 2^{-j}]$ satisfying

$$\Delta_j(L)f = (e^{-2^{-j}L} - e^{-2^{-j-1}L})f = -2^{-j-1}Le^{-\eta_j L}f.$$

Applying the L^p-continuity of the semigroup $\{e^{-sL}\}_{s>0}$ gives

$$\begin{aligned}
&\|\Psi_{2^{-j}}(L)f\|_p \\
&\quad = 2\|e^{-(2^{-j}-\eta_j)L}(2^{-j-1}Le^{-\eta_j L}f)\|_p \\
&\quad \le C\|2^{-j-1}Le^{-\eta_j L}f\|_p \\
&\quad = C\|\Delta_j(L)f\|_p,
\end{aligned}$$

then by using the equivalence of (a) and (b) it follows that

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} \le C\bigg(\sum_{j=-\infty}^{\infty} (2^{j\alpha}\|\Delta_j(L)f\|_p)^q\bigg)^{1/q}.$$

Hence we have shown that (a) is equivalent to (c). Thus we have finished the proof of the proposition. $\square$

The following proposition gives the result that the Besov norms defined by $t^k L^k e^{-tL}$ are equivalent to each other for positive $k \geq 1$.

Proposition 4.4. *Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. For any $f \in \left(\mathcal{M}_{p',q'}^{-\alpha,L^*}\right)'$, and $k = 1, 2, \ldots$, we define a family of Besov norms by*

$$\|f\|_{\dot{B}_{p,q}^{\alpha,L,k}} = \left\{ \int_0^\infty (t^{-\alpha} \|t^k L^k e^{-tL} f\|_p)^q \, \frac{dt}{t} \right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}_{p,q}^{\alpha,L,k}} = \sup_{t>0} t^{-\alpha} \|t^k L^k e^{-tL} f\|_p$$

for $q = \infty$, where $t^k L^k e^{-tL} f(x) = (f, p_{k,t}(x, \cdot))$. Then these norms for different values of k are equivalent to each other.

Proof. We will prove that the Besov norms are equivalent for the choices of $t^k L^k e^{-tL}$ and $t^{k+1} L^{k+1} e^{-tL}$ for all $k \in \mathbb{N}$. Firstly, it can be seen that

$$\begin{aligned}
&\|t^{k+1} L^{k+1} e^{-tL} f\|_p \\
&\quad = \|tLe^{-tL/2} t^k L^k e^{-tL/2} f\|_p \\
&\quad \leq \|tLe^{-tL/2}\|_{p\to p} \|t^k L^k e^{-tL/2} f\|_p \\
&\quad \leq C \|t^k L^k e^{-tL/2} f\|_p,
\end{aligned}$$

where the final inequality is true because the operator norm $\|tLe^{-tL/2}\|_{p\to p}$ is uniformly bounded as a consequence of its kernel bound. Therefore we have

$$\|f\|_{\dot{B}_{p,q}^{\alpha,L,k+1}} \leq C \|f\|_{\dot{B}_{p,q}^{\alpha,L,k}}$$

for any positive value $k \geq 1$.

In order to prove the reverse inequality, suppose that $1 \leq q < \infty$. Let us recall Hardy's inequality: For $0 < r < \infty$ and non-negative measurable function g,

$$\left(\int_0^\infty t^{r-1} \left[\int_t^\infty g(s)\, ds \right]^q dt \right)^{1/q} \leq \frac{q}{r} \left(\int_0^\infty t^{r-1} [tg(t)]^q \, dt \right)^{1/q}.$$

(See for example, Lemma 3.14, Chapter V in [SW].)

Next we see that, for every $\phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$,

$$\begin{aligned}\frac{d}{ds}(L^k e^{-sL} f, \phi) &= (-L^{k+1} e^{-sL} f, \phi),\\ \int_t^u (-L^{k+1} e^{-sL} f, \phi)\, ds &= (L^k e^{-uL} f, \phi) - (L^k e^{-tL} f, \phi)\\ &= (f, (L^*)^k e^{-uL^*} \phi) - (L^k e^{-tL} f, \phi).\end{aligned}$$

Using an argument similar to the proof of Theorem 3.13, we observe that $(L^*)^k e^{-uL^*}\phi \to 0$ in $\mathcal{M}^{-\alpha,L^*}_{p',q'}$ norm as $u \to \infty$. Then we have

$$(t^k L^k e^{-tL} f, \phi) = t^k \int_t^\infty (L^{k+1} e^{-sL} f, \phi)\, ds. \tag{4.2}$$

Applying this and Hardy's inequality with $g(s) = \|L^{k+1} e^{-sL} f\|_p$ we obtain

$$\begin{aligned}&\left\{ \int_0^\infty (t^{-\alpha} \|t^k L^k e^{-tL} f\|_p)^q \frac{dt}{t} \right\}^{1/q}\\ &\quad \le \left\{ \int_0^\infty t^{-\alpha q} t^{kq} \left(\int_t^\infty \|L^{k+1} e^{-sL} f\|_p\, ds \right)^q \frac{dt}{t} \right\}^{1/q}\\ &\quad \le \frac{q}{r} \left\{ \int_0^\infty t^{-\alpha q} t^{kq} t^q \|L^{k+1} e^{-tL} f\|_p^q \frac{dt}{t} \right\}^{1/q}\\ &\quad = \frac{q}{r} \left\{ \int_0^\infty \left(t^{-\alpha} \|t^{k+1} L^{k+1} e^{-tL} f\|_p \right)^q \frac{dt}{t} \right\}^{1/q},\end{aligned}$$

where $r = q(k-\alpha) > 0$; that is,

$$\|f\|_{\dot{B}^{\alpha,L,k}_{p,q}} \le \frac{q}{r} \|f\|_{\dot{B}^{\alpha,L,k+1}_{p,q}}.$$

Lastly, consider the case $q = \infty$. Using (4.2) and Minkowski's inequality it follows that

$$\begin{aligned}&t^{-\alpha} \|t^k L^k e^{-tL} f\|_p\\ &\quad \le t^{k-\alpha} \int_t^\infty s^{-\alpha} \|s^{k+1} L^{k+1} e^{-sL} f\|_p s^{\alpha-k-1}\, ds\\ &\quad \le \left(\sup_{s>0} s^{-\alpha} \|s^{k+1} L^{k+1} e^{-sL} f\|_p \right) t^{k-\alpha} \int_t^\infty s^{\alpha-k-1}\, ds\\ &\quad = \left(\sup_{s>0} s^{-\alpha} \|s^{k+1} L^{k+1} e^{-sL} f\|_p \right) t^{k-\alpha} \left[\frac{s^{\alpha-k}}{\alpha-k} \right]_t^\infty\\ &\quad = \left(\sup_{s>0} s^{-\alpha} \|s^{k+1} L^{k+1} e^{-sL} f\|_p \right) t^{k-\alpha} \left(0 + \frac{t^{k-\alpha}}{k-\alpha} \right)\\ &\quad = \frac{1}{k-\alpha} \|f\|_{\dot{B}^{\alpha,L,k+1}_{p,\infty}}.\end{aligned}$$

Thus we obtain the reverse inequality for $q = \infty$. Hence the proof of the proposition is complete.

□

In the next proposition we show the equivalence of Besov norms of more general class of functions $\Psi_t(L)$ with suitable decay at 0 and infinity.

Proposition 4.5. *Suppose L satisfies (S) and (K). Let $0 < \alpha < 1$ and $1 \le p, q \le \infty$. For any $f \in \left(\mathcal{M}_{p',q'}^{-\alpha,L^*}\right)'$, we define a family of Besov norms by*

$$\|f\|_{\dot{B}_{p,q}^{\alpha,\Psi_t(L)}} = \left\{ \int_0^\infty (t^{-\alpha}\|\Psi_t(L)f\|_p)^q \, \frac{dt}{t} \right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}_{p,q}^{\alpha,\Psi_t(L)}} = \sup_{t>0} t^{-\alpha}\|\Psi_t(L)f\|_p$$

for $q = \infty$. Assume that $\Psi_t(L)$ and $\tilde{\Psi}_t(L)$ are two classes of functions of L which satisfy the following conditions:

(i) $\Psi(\xi)$ and $\tilde{\Psi}(\xi)$ are holomorphic functions on the positive x-axis such that $\Psi(\xi)$ and $\tilde{\Psi}(\xi)$ tend to 0 as ξ tends to 0 and as ξ tends to infinity.

(ii) The operators $\Psi_t(L)$ and $\tilde{\Psi}_t(L)$ have kernel bounds (K).

(iii) There exists $\tilde{\tilde{\Psi}}_t(L)$ with kernel bounds (K) such that

$$\tilde{\Psi}_t(L) = \Psi_t(L)\tilde{\tilde{\Psi}}_t(L).$$

(iv) The functions $\Psi(\xi)$ and $\tilde{\Psi}(\xi)$ satisfy for some constant C

$$\tilde{\Psi}_t(L) = Ct\frac{d}{dt}(\Psi_t(L)).$$

Then the Besov norms with respect to $\Psi_t(L)$ and $\tilde{\Psi}_t(L)$ are equivalent to each other.

Proof. First, it follows from condition (iii) that there exists a constant C such that

$$\begin{aligned}
&\|\tilde{\Psi}_t(L)f\|_p \\
&\quad = \|\tilde{\tilde{\Psi}}_t(L)\Psi_t(L)f\|_p \\
&\quad \le \|\tilde{\tilde{\Psi}}_t(L)\|_{p\to p}\|\Psi_t(L)f\|_p \\
&\quad \le C\|\Psi_t(L)f\|_p,
\end{aligned}$$

This then gives

$$\|f\|_{\dot{B}^{\alpha,\tilde{\Psi}_t(L)}_{p,q}} \le C\|f\|_{\dot{B}^{\alpha,\Psi_t(L)}_{p,q}}.$$

To obtain the reverse inequality, first assume $1 \le q < \infty$. Recall Hardy's inequality: For $0 < r < \infty$ and non-negative measurable function g,

$$\left(\int_0^\infty t^{-r-1}\left[\int_0^t g(s)\,ds\right]^q dt\right)^{1/q} \le \frac{q}{r}\left(\int_0^\infty t^{-r-1}[tg(t)]^q\,dt\right)^{1/q}.$$

(See for example, Lemma 3.14, Chapter V in [SW].)

Next, it follows from condition (iv) that, for every $\phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$,

$$\begin{aligned}\frac{d}{ds}(\Psi_s(L)f,\phi) &= (\frac{1}{s}\tilde{\Psi}_s(L)f,\phi),\\ \int_u^t (\tilde{\Psi}_s(L)f,\phi)\,\frac{ds}{s} &= (\Psi_t(L)f,\phi) - (\Psi_u(L)f,\phi)\\ &= (\Psi_t(L)f,\phi) - (f,\Psi_u(L^*)\phi).\end{aligned}$$

By condition (i) and an argument similar to the proof of Theorem 3.13, we observe that $\Psi_u(L^*)\phi \to 0$ in $\mathcal{M}^{-\alpha,L^*}_{p',q'}$ norm as $u \to 0$. It follows that

$$(\Psi_t(L)f,\phi) = \int_0^t (\tilde{\Psi}_s(L)f,\phi)\,\frac{ds}{s}. \tag{4.3}$$

This and Hardy's inequality with $g(s) = \dfrac{1}{s}\|\tilde{\Psi}_s(L)f\|_p$ gives

$$\begin{aligned}&\left\{\int_0^\infty (t^{-\alpha}\|\Psi_t(L)f\|_p)^q\,\frac{dt}{t}\right\}^{1/q}\\ &\quad\le \left\{\int_0^\infty t^{-\alpha q}\left(\int_0^t \|\tilde{\Psi}_s(L)f\|_p\,\frac{ds}{s}\right)^q\frac{dt}{t}\right\}^{1/q}\\ &\quad\le \frac{q}{r}\left\{\int_0^\infty t^{-\alpha q}\left(\frac{t}{t}\|\tilde{\Psi}_t(L)f\|_p\right)^q\frac{dt}{t}\right\}^{1/q}\\ &\quad= \frac{q}{r}\left\{\int_0^\infty \left(t^{-\alpha}\|\tilde{\Psi}_t(L)f\|_p\right)^q\frac{dt}{t}\right\}^{1/q},\end{aligned}$$

where $r = \alpha q > 0$; that is,

$$\|f\|_{\dot{B}^{\alpha,\Psi_t(L)}_{p,q}} \le \frac{q}{r}\|f\|_{\dot{B}^{\alpha,\tilde{\Psi}_t(L)}_{p,q}}.$$

Finally, assume $q = \infty$. Then by (4.3) and Minkowski's inequality, it follows that

$$t^{-\alpha}\|\Psi_t(L)f\|_p$$

$$
\begin{aligned}
&\le t^{-\alpha}\int_0^t s^{-\alpha}\|\tilde{\Psi}_s(L)f\|_p s^{\alpha}\,\frac{ds}{s}\\
&\le \Big(\sup_{s>0} s^{-\alpha}\|\tilde{\Psi}_s(L)f\|_p\Big)t^{-\alpha}\int_0^t s^{\alpha-1}\,ds\\
&= \Big(\sup_{s>0} s^{-\alpha}\|\tilde{\Psi}_s(L)f\|_p\Big)t^{-\alpha}\left[\frac{s^{\alpha}}{\alpha}\right]_0^t\\
&= \Big(\sup_{s>0} s^{-\alpha}\|\tilde{\Psi}_s(L)f\|_p\Big)t^{-\alpha}\left(\frac{t^{\alpha}}{\alpha}-0\right)\\
&= \frac{1}{\alpha}\|f\|_{\dot{B}^{\alpha,\tilde{\Psi}_t(L)}_{p,\infty}}.
\end{aligned}
$$

Hence the reverse inequality for $q=\infty$ follows. Thus the proof of the proposition is complete. $\square$

Next we look more at the equivalence of Besov norms of more general class of functions. Let $0<\alpha<1$ and $f\in$ domain of L^{α}. Assume L has a bounded holomorphic functional calculus on L^2. We have

$$
\begin{aligned}
&\|f\|_{\dot{B}^{\alpha,L}_{p,q}}\\
&\quad=\left\{\int_0^{\infty}\left(t^{-\alpha}\|\Psi_t(L)f\|_p\right)^q\frac{dt}{t}\right\}^{1/q}\\
&\quad=\left\{\int_0^{\infty}\left(t^{-\alpha}\|tLe^{-tL}f\|_p\right)^q\frac{dt}{t}\right\}^{1/q}\\
&\quad=\left\{\int_0^{\infty}\left(t^{-\alpha}\Big(\int_{\mathcal{X}}\left|tLe^{-tL}f\right|^p dx\Big)^{1/p}\right)^q\frac{dt}{t}\right\}^{1/q}\\
&\quad=\left\{\int_0^{\infty}\left(\int_{\mathcal{X}}\left|t^{-\alpha}tLe^{-tL}f\right|^p dx\right)^{q/p}\frac{dt}{t}\right\}^{1/q},
\end{aligned}
$$

If we replace tLe^{-tL} by $t^kL^ke^{-tL}$ for $k\ge 1>\alpha$, put $\tilde{\Psi}_t(z)=(tz)^{-\alpha}\Psi_t(z)$ and $g=L^{\alpha}f$, with $g\in L^p$, then it follows that

$$
\begin{aligned}
&\|f\|_{\dot{B}^{\alpha,L}_{p,q}}\\
&\quad=\left\{\int_0^{\infty}\left(\int_{\mathcal{X}}\left|t^{-\alpha}t^kL^ke^{-tL}f\right|^p dx\right)^{q/p}\frac{dt}{t}\right\}^{1/q}\\
&\quad=\left\{\int_0^{\infty}\left(\int_{\mathcal{X}}\left|t^{k-\alpha}L^{k-\alpha}L^{\alpha}e^{-tL}f\right|^p dx\right)^{q/p}\frac{dt}{t}\right\}^{1/q}\\
&\quad=\left\{\int_0^{\infty}\left(\int_{\mathcal{X}}\left|t^{k-\alpha}L^{k-\alpha}e^{-tL}L^{\alpha}f\right|^p dx\right)^{q/p}\frac{dt}{t}\right\}^{1/q}\\
&\quad=\left\{\int_0^{\infty}\left(\int_{\mathcal{X}}\left|\tilde{\Psi}_t(L)g\right|^p dx\right)^{q/p}\frac{dt}{t}\right\}^{1/q}\\
&\quad=\left\{\int_0^{\infty}\|\tilde{\Psi}_t(L)g\|_p^q\,\frac{dt}{t}\right\}^{1/q}.
\end{aligned}
$$

Let $\tilde{\Psi}_t(L) = t^{-\alpha}L^{-\alpha}\Psi_t(L)$ and $\tilde{\beta}_t(L) = t^{-\alpha}L^{-\alpha}\beta_t(L)$. Then [M^{c}Y] gives us that the Besov norms with respect to $\tilde{\Psi}_t(L)$ and $\tilde{\beta}_t(L)$ are equivalent to each other for the case $p = q = 2$. That is,

$$\|f\|_{\dot{B}^{\alpha,\Psi_t(L)}_{2,2}} = \left\{ \int_0^\infty \|\tilde{\Psi}_t(L)g\|_2^2 \, \frac{dt}{t} \right\}^{1/2}$$

is equivalent to

$$\|f\|_{\dot{B}^{\alpha,\beta_t(L)}_{2,2}} = \left\{ \int_0^\infty \|\tilde{\beta}_t(L)g\|_2^2 \, \frac{dt}{t} \right\}^{1/2}.$$

Furthermore, in [CDM^{c}Y] we have that, for $1 < p < \infty$,

$$\left\| \left\{ \int_0^\infty |\tilde{\Psi}_t(L)g|^2 \, \frac{dt}{t} \right\}^{1/2} \right\|_p$$

is equivalent to

$$\left\| \left\{ \int_0^\infty |\tilde{\beta}_t(L)g|^2 \, \frac{dt}{t} \right\}^{1/2} \right\|_p.$$

When $p = 2$, it follows that

$$\begin{aligned}
&\left\| \left\{ \int_0^\infty |\tilde{\Psi}_t(L)g|^2 \, \frac{dt}{t} \right\}^{1/2} \right\|_p \\
&\quad = \left\{ \int_{\mathcal{X}} \left(\int_0^\infty \left|\tilde{\Psi}_t(L)g\right|^2 \frac{dt}{t} \right)^{p/2} dx \right\}^{1/p} \\
&\quad = \left\{ \int_{\mathcal{X}} \left(\int_0^\infty \left|\tilde{\Psi}_t(L)g\right|^2 \frac{dt}{t} \right)^{2/2} dx \right\}^{1/2} \\
&\quad = \left\{ \int_{\mathcal{X}} \left(\int_0^\infty \left|\tilde{\Psi}_t(L)g\right|^2 \frac{dt}{t} \right) dx \right\}^{1/2} \\
&\quad = \left\{ \int_0^\infty \left(\int_{\mathcal{X}} \left|\tilde{\Psi}_t(L)g\right|^2 dx \right) \frac{dt}{t} \right\}^{1/2} \\
&\quad = \left\{ \int_0^\infty \left(\int_{\mathcal{X}} \left|\tilde{\Psi}_t(L)g\right|^2 dx \right)^{2/2} \frac{dt}{t} \right\}^{1/2} \\
&\quad = \left\{ \int_0^\infty \|\tilde{\Psi}_t(L)g\|_2^2 \, \frac{dt}{t} \right\}^{1/2} \\
&\quad = \|f\|_{\dot{B}^{\alpha,\Psi_t(L)}_{2,2}}.
\end{aligned}$$

Therefore [CDM^{c}Y] also gives us that the Besov norms with respect to $\tilde{\Psi}_t(L)$ and $\tilde{\beta}_t(L)$ are equivalent to each other for the case $p = q = 2$. That is,

$$\|f\|_{\dot{B}^{\alpha,\Psi_t(L)}_{2,2}} = \left\{ \int_0^\infty \|\tilde{\Psi}_t(L)g\|_2^2 \, \frac{dt}{t} \right\}^{1/2}$$

is equivalent to

$$\|f\|_{\dot{B}^{\alpha,\beta_t(L)}_{2,2}} = \left\{ \int_0^\infty \|\tilde{\beta}_t(L)g\|_2^2 \, \frac{dt}{t} \right\}^{1/2}.$$

The following proposition extends the Besov norm equivalence result of Proposition 4.4 to non-integer $k \geq 1$.

Proposition 4.6. *Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. For any $f \in \left(\mathcal{M}^{-\alpha,L^*}_{p',q'}\right)'$, and $k = 1, 2, \ldots$, we define a family of Besov norms by*

$$\|f\|_{\dot{B}^{\alpha,L,k}_{p,q}} = \left\{ \int_0^\infty (t^{-\alpha}\|t^k L^k e^{-tL} f\|_p)^q \, \frac{dt}{t} \right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}^{\alpha,L,k}_{p,q}} = \sup_{t>0} t^{-\alpha}\|t^k L^k e^{-tL} f\|_p$$

for $q = \infty$, where $t^k L^k e^{-tL} f(x) = (f, p_{k,t}(x,\cdot))$. Then these norms for different values of non-integer w, for $k < w < k+1$, are equivalent to each other.

Proof. Let $w = k + \alpha$, where $0 < \alpha < 1$. Firstly, it can be seen that

$$\begin{aligned}
&\|t^w L^w e^{-tL} f\|_p \\
&\quad = \|t^\alpha L^\alpha e^{-tL/2} t^k L^k e^{-tL/2} f\|_p \\
&\quad \leq \|t^\alpha L^\alpha e^{-tL/2}\|_{p\to p} \|t^k L^k e^{-tL/2} f\|_p \\
&\quad \leq C\|t^k L^k e^{-tL/2} f\|_p,
\end{aligned}$$

where the final inequality is true because the operator norm

$$\|t^\alpha L^\alpha e^{-tL/2}\|_{p\to p}$$

is uniformly bounded, which follows from interpolation between $\alpha = 0$ and $\alpha = 1$. Therefore we have

$$\|f\|_{\dot{B}^{\alpha,L,w}_{p,q}} \leq C\|f\|_{\dot{B}^{\alpha,L,k}_{p,q}}$$

for any non-integer value w, where $k < w < k+1$.

To prove the reverse inequality, observe that

$$\begin{aligned}
&\|t^{k+1}L^{k+1}e^{-tL}f\|_p \\
&\quad = \|t^{1-\alpha}L^{1-\alpha}e^{-tL/2}t^{k+\alpha}L^{k+\alpha}e^{-tL/2}f\|_p \\
&\quad \le \|t^{1-\alpha}L^{1-\alpha}e^{-tL/2}\|_{p\to p}\|t^{k+\alpha}L^{k+\alpha}e^{-tL/2}f\|_p \\
&\quad \le C\|t^{k+\alpha}L^{k+\alpha}e^{-tL/2}f\|_p, \\
&\quad = C\|t^{w}L^{w}e^{-tL/2}f\|_p,
\end{aligned}$$

where the final inequality is true because the operator norm

$$\|t^{1-\alpha}L^{1-\alpha}e^{-tL/2}\|_{p\to p}$$

is uniformly bounded, which follows from interpolation between $\alpha = 0$ and $\alpha = 1$. Therefore we have

$$\|f\|_{\dot{B}^{\alpha,L,k+1}_{p,q}} \le C\|f\|_{\dot{B}^{\alpha,L,w}_{p,q}}$$

for any non-integer value w, where $k < w < k+1$. Hence the proof of the proposition is complete. □

Next we see if we have norm equivalence when we replace the semigroup $t^kL^ke^{-tL}$ by the resolvent $t^kL^k(tL+1)^{-m}$, for $k < m$.

Let $\lambda > 0$. The Laplace transform gives

$$(\lambda I + L)^{-m} = \frac{1}{m!}\int_0^\infty s^{m-1}e^{-\lambda s}e^{-sL}\,ds.$$

We have

$$\begin{aligned}
&(tL+1)^{-m} \\
&\quad = \Big[t\big(L+\frac{1}{t}\big)\Big]^{-m} \\
&\quad = t^{-m}\big(L+\frac{1}{t}\big)^{-m} \\
&\quad = \frac{t^{-m}}{m!}\int_0^\infty s^{m-1}e^{-s/t}e^{-sL}\,ds.
\end{aligned}$$

Then for $k < m$, it follows that

$$\begin{aligned}
&t^k L^k (tL+1)^{-m} \\
&\quad = \frac{t^{k-m}}{m!} \int_0^\infty s^{m-1} e^{-s/t} L^k e^{-sL}\, ds \\
&\quad = \frac{t^{-m}}{m!} \int_0^\infty s^{m-1} e^{-s/t} t^k L^k e^{-sL}\, ds \\
&\quad = \frac{t^{-m}}{m!} \int_0^\infty s^{m} e^{-s/t} \left(\frac{t}{s}\right)^k s^k L^k e^{-sL}\, \frac{ds}{s}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\|t^k L^k (tL+1)^{-m}\|_p \\
&\quad \le \frac{t^{-m}}{m!} \int_0^\infty s^{m} e^{-s/t} \left(\frac{t}{s}\right)^k \|s^k L^k e^{-sL}\|_p\, \frac{ds}{s} \\
&\quad \le C t^{-m} \int_0^\infty s^{m} e^{-s/t} \left(\frac{t}{s}\right)^k \frac{ds}{s}.
\end{aligned}$$

By change of variables $s/t \to w$ it then follows that

$$\begin{aligned}
&\|t^k L^k (tL+1)^{-m}\|_p \\
&\quad \le C \int_0^\infty w^{m-k} e^{-w}\, \frac{ds}{w} \\
&\quad \le C \int_0^\infty w^{m-k-1} e^{-w}\, ds \\
&\quad \le C.
\end{aligned}$$

By using the above and also observing that

$$\begin{aligned}
&\|t^k L^k e^{-tL}\|_p \\
&\quad \le \|t^k L^k (tL+1)^{-m}\|_p \|(tL+1)^m e^{-tL}\|_p \\
&\quad \le C \|t^k L^k (tL+1)^{-m}\|_p \\
&\quad \le C,
\end{aligned}$$

this gives a simpler proof for Proposition 4.4 than by using Hardy's inequality.

To conclude this section we present a result that gives norm equivalence for the Besov spaces with positive α.

Proposition 4.7. *Suppose L satisfies (S) and (K). Let $0 < \alpha < 1$ and $1 \le p, q \le \infty$. A functional f belongs to $\dot{B}^{\alpha,L}_{p,q}$ if and only if f satisfies*

$$\left\{ \int_0^\infty (t^{-\alpha} \|(I - e^{-tL})f\|_p)^q \, \frac{dt}{t} \right\}^{1/q} < \infty. \tag{4.4}$$

Furthermore, the above expression is equivalent to $\|f\|_{\dot{B}^{\alpha,L}_{p,q}}$.

Proof. Let $f \in \dot{B}^{\alpha,L}_{p,q}$. We shall show that f satisfies (4.4). Observe that by (S) and (K) we have

$$(I - e^{-tL})f = \int_0^t Le^{-sL} f \, ds \quad \text{in } (\mathcal{M}^{-\alpha,L^*}_{p',q'})'.$$

Observe that the right hand side of the above equality is a function. By Hardy's inequality, it follows that

$$\begin{aligned}
&\left\{ \int_0^\infty (t^{-\alpha} \|(I - e^{-tL})f\|_p)^q \, \frac{dt}{t} \right\}^{1/q} \\
&\quad = \left\{ \int_0^\infty \left(t^{-\alpha} \| \int_0^t Le^{-sL} f \, ds \|_p \right)^q \frac{dt}{t} \right\}^{1/q} \\
&\quad \le \left\{ \int_0^\infty t^{-\alpha q - 1} \left(\int_0^t \|Le^{-sL} f\|_p \, ds \right)^q dt \right\}^{1/q} \\
&\quad \le \frac{1}{\alpha} \left\{ \int_0^\infty s^{-\alpha q - 1} \|sLe^{-sL} f\|_p^q \, ds \right\}^{1/q} \\
&\quad = \frac{1}{\alpha} \|f\|_{\dot{B}^{\alpha,L}_{p,q}},
\end{aligned}$$

which gives (4.4).

Assume that f satisfies (4.4). We shall prove that $f \in \dot{B}^{\alpha,L}_{p,q}$. Following [HMa], for every $t > 0$ we shall write

$$\begin{aligned}
I &= t^{-1} \int_t^{2t} ds \cdot I \\
&= t^{-1} \int_t^{2t} (I - e^{-sL}) \, ds + t^{-1} \int_t^{2t} e^{-sL} \, ds.
\end{aligned} \tag{4.5}$$

But we have $\frac{d}{ds} e^{-sL} = -Le^{-sL}$ so it follows that

$$L \int_t^{2t} e^{-sL} \, ds = e^{-tL} - e^{-2tL} = e^{-tL}(I - e^{-tL}). \tag{4.6}$$

By substituting (4.6) into (4.5), it follows that

$$tLe^{-tL} f = Le^{-tL} \int_t^{2t} (I - e^{-sL}) \, ds + e^{-2tL}(I - e^{-tL}). \tag{4.7}$$

Using condition (K) we have that tLe^{-tL} and e^{-tL} are bounded on L^p for all $1 \le p \le \infty$. This, together with (4.7) gives

$$\begin{aligned}
&\left\{ \int_0^\infty (t^{-\alpha}\|tLe^{-tL}f\|_p)^q \, \frac{dt}{t} \right\}^{1/q} \\
&\quad \le \left\{ \int_0^\infty \left(t^{-\alpha} \left\| Le^{-tL} \int_t^{2t} (I - e^{-sL})\, ds \right\|_p \right)^q \frac{dt}{t} \right\}^{1/q} \\
&\qquad + \left\{ \int_0^\infty \left(t^{-\alpha} \left\| e^{-2tL}(I - e^{-tL}) \right\|_p \right)^q \frac{dt}{t} \right\}^{1/q} \\
&\quad \le C \left\{ \int_0^\infty \left(t^{-\alpha} \int_t^{2t} \left\| (I - e^{-sL}) \right\|_p \frac{ds}{s} \right)^q \frac{dt}{t} \right\}^{1/q} \\
&\qquad + C \left\{ \int_0^\infty \left(t^{-\alpha} \left\| (I - e^{-tL}) \right\|_p \right)^q \frac{dt}{t} \right\}^{1/q} \\
&\quad \le C' \left\{ \int_0^\infty \left(t^{-\alpha} \left\| (I - e^{-tL}) \right\|_p \right)^q \frac{dt}{t} \right\}^{1/q}.
\end{aligned}$$

Therefore we have $f \in \dot{B}^{\alpha,L}_{p,q}$. Thus we have finished proving the proposition. □

4.1.3 Applications: fractional integrals

Let $0 < \alpha < 1, 0 < \gamma < 1$ and $\alpha + \gamma < 1$. For any $f \in \dot{B}^{\alpha,L}_{p,q}$, we define fractional integrals $L^{-\gamma} f$ associated with an operator L by

$$(L^{-\gamma} f, \phi) = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} (e^{-tL} f, \phi)\, dt \tag{4.8}$$

for every $\phi \in \mathcal{M}^{-(\alpha+\gamma),L^*}_{p',q'}$, where $\Gamma(\gamma)$ is an appropriate constant.

Lemma 4.8. *Suppose L satisfies (S) and (K). Let $0 < \alpha < 1$, $0 < \gamma < 1$ and $\alpha + \gamma < 1$. If $f \in \dot{B}^{\alpha,L}_{p,q}$, then $L^{-\gamma} f$ is well-defined.*

Proof. Let $\phi \in \mathcal{M}^{-(\alpha+\gamma),L^*}_{p',q'}$ and $f \in \dot{B}^{\alpha,L}_{p,q}$. To show that the above integral converges, we first consider $0 < t \le 1$. For every $0 < \gamma < 1$,

$$\begin{aligned}
&\|\phi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}} \\
&\quad = \left\{ \int_0^\infty \left(s^\alpha \|sL^* e^{-sL^*} \phi\|_{p'} \right)^{q'} \frac{ds}{s} \right\}^{1/q'} \\
&\quad \le \|\phi\|_{p'} \left\{ \int_0^1 s^{\alpha q'} \frac{ds}{s} \right\}^{1/q'} + \left\{ \int_1^\infty \left(s^{\alpha+\gamma} \|sL^* e^{-sL^*} \phi\|_{p'} \right)^{q'} \frac{ds}{s} \right\}^{1/q'} \\
&\quad \le C\|\phi\|_{p'} + \|\phi\|_{\dot{B}^{-(\alpha+\gamma),L^*}_{p',q'}}.
\end{aligned}$$

We also have $\phi = L^*g$ for some g satisfying condition (3.5). Thus we have $\phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$. Applying Theorem 3.13, we obtain

$$\begin{aligned}
&|(e^{-tL}f,\phi)| \\
&\quad \le \|e^{-tL}f\|_{\dot{B}^{\alpha,L}_{p,q}}\|\phi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}} \\
&\quad \le C\|f\|_{\dot{B}^{\alpha,L}_{p,q}}\|\phi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}} \\
&\quad \le C\|f\|_{\dot{B}^{\alpha,L}_{p,q}}(\|\phi\|_{p'} + \|\phi\|_{\dot{B}^{-(\alpha+\gamma),L^*}_{p',q'}}).
\end{aligned} \tag{4.9}$$

We shall now consider the case $t \ge 1$. Using the proof of Theorem 4.1, it follows that $f \in \dot{B}^{\alpha,L}_{p,\infty}$, so we have $\|tLe^{-tL}f\|_p \le Ct^{\alpha}\|f\|_{\dot{B}^{\alpha,L}_{p,q}}$. Furthermore, $\phi = L^*g$ for some $g \in L^{p'}(\mathcal{X})$, hence we obtain

$$\begin{aligned}
&|(e^{-tL}f,\phi)| \\
&\quad \le \frac{1}{t}|(tLe^{-tL}f,g)| \\
&\quad \le \frac{1}{t}\|tLe^{-tL}f\|_p\|g\|_{p'} \\
&\quad \le Ct^{\alpha-1}\|f\|_{\dot{B}^{\alpha,L}_{p,q}}\|g\|_{p'}.
\end{aligned} \tag{4.10}$$

Then it can be easily shown that the integral on the right hand side of (4.8) converges absolutely. Thus the proof of the lemma is complete. □

Observe that if we let L be the Laplacian $-\Delta$ on $\mathbb{R}^n$, then $L^{-\gamma}$ will be the classical fractional integral. See, for instance, [SW].

Suppose that α, γ, p, q and f are as given in Lemma 4.8. From this lemma we have that for each $k \in \mathbb{N}$, $(tL)^k e^{-tL}f$ is well-defined. Then we define the norm

$$\|L^{-\gamma}f\|_{\dot{B}^{\alpha+\gamma,L}_{p,q}} = \left\{ \int_0^\infty (t^{-(\alpha+\gamma)}\|tLe^{-tL}(L^{-\gamma}f)\|_p)^q \, \frac{dt}{t} \right\}^{1/q}$$

when the last integral is finite.

Theorem 4.9. *Suppose L satisfies (S) and (K). Let $0 < \alpha < 1$, $0 < \gamma < 1$, $\alpha + \gamma < 1$ and $1 \le p, q \le \infty$. Then there exists a positive constant C such that for all $f \in \dot{B}^{\alpha,L}_{p,q}$,*

$$C^{-1}\|f\|_{\dot{B}^{\alpha,L}_{p,q}} \le \|L^{-\gamma}f\|_{\dot{B}^{\alpha+\gamma,L}_{p,q}} \le C\|f\|_{\dot{B}^{\alpha,L}_{p,q}}. \tag{4.11}$$

Proof. Let us first show the right hand inequality of (4.11). Let $\Psi_t(L) = (tL)^2 e^{-tL}$. Using Proposition 4.4 we obtain

$$\begin{aligned}
&\|L^{-\gamma} f\|_{\dot{B}^{\alpha+\gamma,L}_{p,q}} \\
&\quad \le \left\{ \int_0^\infty (t^{-(\alpha+\gamma)} \|\Psi_t(L)(L^{-\gamma} f)\|_p)^q \, \frac{dt}{t} \right\}^{1/q} \\
&\quad = \left\{ \int_0^\infty (t^{-\alpha} \|(tL)^{-\gamma}(t^2 L^2 e^{-tL} f)\|_p)^q \, \frac{dt}{t} \right\}^{1/q}.
\end{aligned} \tag{4.12}$$

We shall now estimate the term $\|(tL)^{-\gamma}(t^2 L^2 e^{-tL} f)\|_p$. Applying (4.8) we have that

$$(tL)^{-\gamma}(t^2 L^2 e^{-tL} f) = C_\gamma \int_0^\infty \left(\frac{s}{t}\right)^\gamma \left(\frac{t}{t+s}\right) \left((t+s) L e^{-(s+\frac{t}{2})}\right) tL e^{-\frac{t}{2}L} f \, \frac{ds}{s}.$$

Then it follows that

$$\begin{aligned}
&\|(tL)^{-\gamma}(t^2 L^2 e^{-tL} f)\|_p \\
&\quad \le C \int_0^\infty \left(\frac{s}{t}\right)^\gamma \left(\frac{t}{t+s}\right) \left\| \left((t+s) L e^{-\frac{t+s}{2}}\right) \right\|_{p,p} \frac{ds}{s} \, \|tL e^{-\frac{t}{2}L} f\|_p \\
&\quad \le C \int_0^\infty \left(\frac{s}{t}\right)^\gamma \left(\frac{t}{t+s}\right) \frac{ds}{s} \, \|tL e^{-\frac{t}{2}L} f\|_p \\
&\quad \le C \|tL e^{-\frac{t}{2}L} f\|_p.
\end{aligned} \tag{4.13}$$

By putting (4.13) back into (4.12), we obtain $\|L^{-\gamma} f\|_{\dot{B}^{\alpha+\gamma,L}_{p,q}} \le C \|f\|_{\dot{B}^{\alpha,L}_{p,q}}$.

Let us now show the left hand inequality of (4.11). We shall write

$$\begin{aligned}
&\|f\|_{\dot{B}^{\alpha,L}_{p,q}} \\
&\quad = \left\{ \int_0^\infty (t^{-\alpha} \|tL e^{-tL} f\|_p)^q \, \frac{dt}{t} \right\}^{1/q} \\
&\quad = \left\{ \int_0^\infty (t^{-(\alpha+\gamma)} \|(tL)^{\gamma-1}(t^2 L^2 e^{-tL} L^{-\gamma} f)\|_p)^q \, \frac{dt}{t} \right\}^{1/q}.
\end{aligned}$$

Because $0 < 1-\gamma < 1$, by a similar argument to (4.13) it can be proved that for every $1 \le p \le \infty$,

$$\|(tL)^{\gamma-1}(t^2 L^2 e^{-tL} L^{-\gamma} f)\|_p \le C \|tL e^{-\frac{t}{2}L}(L^{-\gamma} f)\|_p,$$

and hence we have $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} \le C \|L^{-\gamma} f\|_{\dot{B}^{\alpha+\gamma,L}_{p,q}}$. Thus the proof of the left hand inequality of (4.11) is complete. Therefore we have finished the proof of the theorem. □

4.2 Molecular decomposition of Besov spaces associated with Schrödinger operators

Suppose that V is a fixed non-negative function on $\mathbb{R}^n$, $n \geq 3$, satisfying a *reverse Hölder inequality* $RH_S(\mathbb{R}^n)$ for some $s > \frac{n}{2}$; that is, there is a $C = C(s, V) > 0$ with the property that

$$\Big(\frac{1}{|B|}\int_B V(x)^s\,dx\Big)^{1/s} \leq \frac{C}{|B|}\int_B V(x)\,dx \tag{4.14}$$

for all balls $B \subset \mathbb{R}^n$. Let us consider the time independent Schrödinger operator with the potential V on $L^2(\mathbb{R}^n)$:

$$L = -\Delta + V(x). \tag{4.15}$$

We note that the operator L is non-negative self-adjoint on $L^2(\mathbb{R}^n)$ and it generates a semigroup

$$e^{-tL}f(x) = \int_{\mathbb{R}^n} p_t(x,y)f(y)\,dy, \quad f \in L^2(\mathbb{R}^n), \quad t > 0,$$

where the kernel $p_t(x,y)$ is dominated by the heat kernel of the Laplacian on $\mathbb{R}^n$, thus $p_t(x,y)$ has a Gaussian upper bound.

Let us recall some estimates for the heat kernel of e^{-tL}. In the same way as in [Sh], we shall define a function $\rho(x;V) = \rho(x)$ by

$$\rho(x) = \sup\Big\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \leq 1\Big\}. \tag{4.16}$$

In this section we make the assumption that $V \not\equiv 0$, hence $0 < \rho(x) < \infty$. Using a result in [Sh], there exist $k_0 \geq 1$ and $c > 0$ such that for every $x, y \in \mathbb{R}^n$,

$$c^{-1}\rho(x)\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-k_0} \leq \rho(y) \leq c\rho(x)\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{k_0+1}}. \tag{4.17}$$

In particular, we have $\rho(x) \sim \rho(y)$ when $r \leq \rho(x)$ and $y \in B(x,r)$. Furthermore, when $r = \rho(x)$, we have

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \leq 1.$$

In the case where $V = P(x)$ is a non-negative polynomial of degree k, it can be shown that

$$\rho(x)^{-1} \sim \sum_{|\alpha|\leq k} |\partial_x^\alpha P(x)|^{1/(|\alpha|+2)}.$$

See pp. 516–517 in [Sh].

Lemma 4.10. *Suppose that $V \in RH_S(\mathbb{R}^n), s > \frac{n}{2}$. Then for every N there exists a constant C_N such that the kernel $p_t(x,y)$ of the semigroup e^{-tL} satisfies*

$$0 \le p_t(x,y) \le C_N t^{-\frac{n}{2}} \exp\Big(-\frac{|x-y|^2}{5t}\Big)\Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-N}. \tag{4.18}$$

Proof. For a proof, we refer the reader to p. 332, Proposition 2 in [DGMTZ]. See also [DZ] and [Ku]. □

We will require estimates for the kernel of the operator $t^2 L e^{-t^2 L}$,

$$q_t(x,y) = t^2 \frac{\partial p_s(x,y)}{\partial s}\Big|_{s=t^2}, \tag{4.19}$$

as follows.

Proposition 4.11. *There are constants $c, \delta > 0$ such that for every N there exists a constant $C_N > 0$ so that*

(i) $|q_t(x,y)| \le C_N t^{-n} \exp\Big(-\frac{|x-y|^2}{ct^2}\Big)\Big(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\Big)^{-N}$;

(ii) $|q_t(x+h,y) - q_t(x,y)|$
$\le C_N \Big(\frac{|h|}{t}\Big)^{\delta} t^{-n} \exp\Big(-\frac{|x-y|^2}{ct^2}\Big)\Big(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\Big)^{-N}$ *for all* $|h| \le t$;

(iii) $\Big|\int_{\mathbb{R}^n} q_t(x,y)\,dy\Big| \le C_N \Big(\frac{t}{\rho(x)}\Big)^{\delta}\Big(1+\frac{t}{\rho(x)}\Big)^{-N}$.

Proof. For a proof, we refer the reader to p. 332, Proposition 4 in [DGMTZ]. □

4.2.1 Molecular decomposition of $\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$

Let us define the notion of molecules.

In the following, the definition of a molecule associated with a cube

$$Q = \{x \in \mathbb{R}^n : a_i \le x_i \le b_i, i = 1, 2, \ldots, n\}$$

involves the "lower left corner of Q", $x_Q = a = (a_1, a_2, \ldots, a_n)$, and $\ell(Q)$, the side length of Q.

Definition 4.12. *Let $\epsilon \in (0,1]$. A function m_Q is called an (ϵ, L, Q)-molecule if $m_Q = Lg_Q$ for some g_Q, and the following conditions hold:*

$$|m_Q(x)| + \ell(Q)^{-2}|g_Q(x)| \le |Q|^{-1}\left\{1 + \frac{|x - x_Q|}{\ell(Q)}\right\}^{-n-\epsilon} \quad \text{for } x \in \mathbb{R}^n; \tag{4.20}$$

$$\int_{|y|\le \ell(Q)} \|m_Q(x+y) - m_Q(x)\|_{L^1(dx)} \frac{dy}{|y|^n} \le 1. \tag{4.21}$$

The following result is a molecular characterization of $\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$.

Theorem 4.13. *Suppose that $L = -\Delta + V$, where $V \not\equiv 0$ is a non-negative potential in $RH_s(\mathbb{R}^n)$ for some $s > \frac{n}{2}$. Assume that $f \in L^1(\mathbb{R}^n)$. The following are equivalent properties of f:*

(i) $f \in \dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$.

(ii) For any $0 < \epsilon \le 1$, there exist a sequence of coefficients $\{s_Q\}$, $0 \le s_Q < \infty$, where Q ranges over the dyadic cubes, and a sequence $\{m_Q\}$ of (ϵ, L, Q)-molecules such that

$$f = \sum_Q s_Q m_Q \qquad \text{in} \quad \dot{B}^{0,L}_{1,1}(\mathbb{R}^n), \tag{4.22}$$

and

$$\sum_Q |s_Q| \le C\|f\|_{\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)}.$$

Proof. Assume that $f \in \dot{B}^{0,L}_{1,1}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. By using Theorem 3.3 (Calderón reproducing formula I) for f we have

$$f(x) = 8\int_0^\infty \left(t^2 L\right)^2 e^{-2t^2 L} f(x)\, \frac{dt}{t}$$

which converges in $L^2(\mathbb{R}^n)$ and almost everywhere.

Next we "discretize" the right-hand side as follows: For a dyadic cube $Q \subset \mathbb{R}^n$, let

$$T(Q) = Q \times [\frac{\ell(Q)}{2}, \ell(Q)].$$

Then the set $\{T(Q)\}$, Q dyadic, is a collection of half cubes covering

$$\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t > 0\}$$

whose interiors are pairwise disjoint. Recall that $p_t(x,y)$ denotes the kernel of e^{-tL} and $q_t(x,y)$ denotes the kernel of $t^2Le^{-t^2L}$ in (4.19). Therefore,

$$\begin{aligned} f(x) &= 8\iint_{\mathbb{R}^{n+1}_+} q_t(x,y)t^2Le^{-t^2L}f(y)\,dy\,\frac{dt}{t} \\ &= 8\sum_Q \iint_{T(Q)} q_t(x,y)t^2Le^{-t^2L}f(y)\,dy\,\frac{dt}{t}. \end{aligned}$$

Put

$$s_Q = \iint_{T(Q)} \left|t^2Le^{-t^2L}f(y)\right|\,dy\,\frac{dt}{t},$$

and, when $s_Q \neq 0$,

$$\begin{aligned} &m_Q(x) \\ &\quad= \frac{8}{s_Q}\iint_{T(Q)} q_t(x,y)t^2Le^{-t^2L}f(y)\,dy\,\frac{dt}{t} \\ &\quad= Lg_Q(x), \end{aligned}$$

where

$$g_Q(x) =: \frac{8}{s_Q}\iint_{T(Q)} t^2p_{t^2}(x,y)t^2Le^{-t^2L}f(y)\,dy\,\frac{dt}{t}.$$

It is clear that

$$\sum_Q s_Q \leq C\|f\|_{\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)},$$

thus (4.22) is true in L^1 (and pointwisely). Furthermore, using (i) of Proposition 4.11 we obtain

$$\begin{aligned} &|m_Q(x)| \\ &\quad\leq C\sup_{(y,t)\in T(Q)} t^{-n}\exp\Big(-\frac{|x-y|^2}{ct^2}\Big)\Big\{\frac{1}{s_Q}\iint_{T(Q)}\left|t^2Le^{-t^2L}f(y)\right|\,dy\,\frac{dt}{t}\Big\} \\ &\quad\leq C|Q|^{-1}\Big\{1+\frac{|x-x_Q|}{\ell(Q)}\Big\}^{-n-1}. \end{aligned}$$

By a similar argument using (4.18) it follows that (4.20) is true for g_Q. Next we check that (4.21) holds. Applying property (ii) of Proposition 4.11 we obtain

$$\|q_t(x+z,y)-q_t(x,y)\|_{L^1(dx)} \leq C\big(\frac{|z|}{t}\big)^\delta.$$

Then we have

$$\int_{|z|\leq\ell(Q)} \|m_Q(x+z)-m_Q(x)\|_{L^1(dx)}|z|^{-n}\,dz$$

$$
\begin{aligned}
&\leq \frac{C}{s_Q} \iint_{T(Q)} \left|t^2 L e^{-t^2 L} f(y)\right| \int_{|z|\leq \ell(Q)} \|q_t(x+z) - q_t(x)\|_{L^1(dx)} |z|^{-n}\, dz\, dy\, \frac{dt}{t} \\
&\leq \frac{C}{s_Q} \iint_{T(Q)} \left|t^2 L e^{-t^2 L} f(y)\right| \left\{ \int_{|z|\leq \ell(Q)} \left(\frac{|z|}{t}\right)^{\delta} |z|^{-n}\, dz \right\} dy\, \frac{dt}{t} \\
&\leq C
\end{aligned}
\tag{4.23}
$$

since $t \in [\frac{\ell(Q)}{2}, \ell(Q)]$. Hence m_Q/C is an (ϵ, L, Q)-molecule. We will show below that

$$
\|m_Q\|_{\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)} \leq A,
$$

where A is an absolute constant that does not depend on Q. Then it follows that the convergence in (4.22) holds in $\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$.

Let $f \in \dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$. Note that $e^{-tL} f \to f$ in $\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$ as $t \to 0$. Moreover, we also have

$$
e^{-tL} f \in \dot{B}^{0,L}_{1,1}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),
$$

for every $t > 0$. By applying a standard argument it follows that f has an (ϵ, L, Q)-molecule decomposition as in (4.22).

For the converse, let f be a function as in (4.22), where the m_Q's are (ϵ, L, Q)-molecules. We need to show that $f \in \dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$. It is enough to prove that for every (ϵ, L, Q)-molecule m_Q, there is a constant $A > 0$, that does not depend on m_Q, such that $\|m_Q\|_{\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)} \leq A$. To show this, let us write

$$
\|m_Q\|_{\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)} = \Big(\int_0^{\ell(Q)} + \int_{\ell(Q)}^{\infty} \Big) \int_{\mathbb{R}^n} \left| t^2 L e^{-t^2 L}(m_Q)(x) \right| \frac{dx\, dt}{t} =: I + II.
$$

By applying the condition $m_Q = L g_Q$ for some function g_Q that satisfies (4.20), it follows that

$$
\begin{aligned}
II &\leq \int_{\ell(Q)}^{\infty} \int_{\mathbb{R}^n} \left| (t^2 L)^2 e^{-t^2 L}(g_Q)(x) \right| \frac{dx\, dt}{t^3} \\
&\leq C \|g_Q\|_1 \int_{\ell(Q)}^{\infty} \frac{dt}{t^3} \\
&= C \|g_Q\|_1 \left[-\frac{t^{-2}}{2} \right]_{\ell(Q)}^{\infty} \\
&= C \|g_Q\|_1 \left(0 + \frac{\ell(Q)^{-2}}{2} \right) \\
&\leq C.
\end{aligned}
$$

To estimate term I, let us rewrite

$$\begin{aligned}
&\int_0^{\ell(Q)} \int_{\mathbb{R}^n} \left| t^2 L e^{-t^2 L}(m_Q)(x) \right| \frac{dx\,dt}{t} \\
&= \int_0^{\ell(Q)} \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} q_t(x,y)(m_Q)(y)\,dy \Big| \frac{dx\,dt}{t} \\
&\le \int_0^{\ell(Q)} \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} q_t(x,y)\big(m_Q(y) - m_Q(x)\big)\,dy \Big| \frac{dx\,dt}{t} \\
&\quad + \int_0^{\ell(Q)} \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} q_t(x,y)\,dy \Big| \big|m_Q(x)\big| \frac{dx\,dt}{t} \\
&= I_1 + I_2.
\end{aligned}$$

Using (iii) of Proposition 4.11 we obtain

$$\begin{aligned}
I_2 &\le C_N \int_0^{\ell(Q)} \int_{\mathbb{R}^n} \frac{\big(t/\rho(x)\big)^\delta}{\big(1 + t/\rho(x)\big)^N} \big|m_Q(x)\big| \frac{dx\,dt}{t} \\
&\le C_N \int_{\mathbb{R}^n} \Big\{ \int_0^\infty \frac{\big(t/\rho(x)\big)^\delta}{\big(1 + t/\rho(x)\big)^N} \frac{dt}{t} \Big\} \big|m_Q(x)\big|\,dx \\
&\le C_{N,\delta}.
\end{aligned}$$

To estimate term I_1, we apply (i) of Proposition 4.11 to obtain

$$I_1 \le C \int_0^{\ell(Q)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} t^{-n} \exp\Big(-\frac{|x-y|^2}{ct^2} \Big) \big|m_Q(y) - m_Q(x)\big|\,dy\,\frac{dx\,dt}{t}.$$

Thus we have

$$I \le C + C \int_0^{\ell(Q)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} t^{-n} \exp\Big(-\frac{|x-y|^2}{ct^2} \Big) \big|m_Q(y) - m_Q(x)\big|\,dy\,\frac{dx\,dt}{t}. \tag{4.24}$$

For estimating the right hand side, consider two cases in the y-integral: $|y-x| \le \ell(Q)$ and $|y-x| \ge \ell(Q)$. In the first case, we apply condition (4.21) to obtain

$$\begin{aligned}
&\int_0^{\ell(Q)} \int_{\mathbb{R}^n} \int_{|y-x|\le \ell(Q)} t^{-n} \exp\Big(-\frac{|x-y|^2}{ct^2} \Big) \big|m_Q(y) - m_Q(x)\big|\,dy\,\frac{dx\,dt}{t} \\
&\le \int_{|w|\le \ell(Q)} \big\|m_Q(x+w) - m_Q(x)\big\|_{L^1(dx)} \Big\{ \int_0^{\ell(Q)} t^{-n} \exp\Big(-\frac{|w|^2}{ct^2} \Big) \frac{dt}{t} \Big\}\,dw \\
&\le C \int_{|w|\le \ell(Q)} \big\|m_Q(x+w) - m_Q(x)\big\|_{L^1(dx)} \frac{dw}{|w|^n} \\
&\le C.
\end{aligned}$$

In the case $|y-x| \ge \ell(Q)$, using (4.20) and an elementary integration we have

$$\int_0^{\ell(Q)} \int_{\mathbb{R}^n} \int_{|y-x|\ge \ell(Q)} t^{-n} \exp\Big(-\frac{|x-y|^2}{ct^2} \Big) \Big(\big|m_Q(y)\big| + \big|m_Q(x)\big| \Big)\,dy\,\frac{dx\,dt}{t}$$

$$
\begin{aligned}
&\leq C \int_0^{\ell(Q)} \Big(\frac{t}{\ell(Q)}\Big)^2 \frac{dt}{t} \\
&= \frac{C}{\ell(Q)^2} \int_0^{\ell(Q)} t\, dt \\
&= \frac{C}{\ell(Q)^2} \left[\frac{t^2}{2}\right]_0^{\ell(Q)} \\
&= \frac{C}{\ell(Q)^2} \left(\frac{\ell(Q)^2}{2} - 0\right) \\
&= \frac{C}{2} \\
&\leq C,
\end{aligned}
$$

and hence $I \leq C$. Thus we have shown that $\|m_Q\|_{\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)} \leq C$. Therefore we have finished proving the theorem. □

4.2.2 The inclusion $\dot{B}^0_{1,1}(\mathbb{R}^n) \subsetneqq \dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$

In this section we aim to prove the following result.

Theorem 4.14. *Suppose that $L = -\Delta + V$, where $V \not\equiv 0$ is a non-negative potential in $RH_s(\mathbb{R}^n)$ for some $s > \frac{n}{2}$. Then the following inclusion is true*

$$\dot{B}^0_{1,1}(\mathbb{R}^n) \subsetneqq \dot{B}^{0,L}_{1,1}(\mathbb{R}^n). \tag{4.25}$$

That is, the classical space $\dot{B}^0_{1,1}(\mathbb{R}^n)$ is a proper subspace of $\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$.

To prove Theorem 4.14, we will require the following atomic decomposition of $\dot{B}^0_{1,1}(\mathbb{R}^n)$. In the following, $\mathcal{F}$ is the set of C^1-functions supported by the unit ball of $\mathbb{R}^n$ which satisfy the following conditions:

(i) $\|a\|_\infty \leq 1$;

(ii) $\left\|\dfrac{\partial a}{\partial x_j}\right\|_\infty \leq 1$ for all $1 \leq j \leq n$;

(iii) $\displaystyle\int_{\mathbb{R}^n} a(x)\, dx = 0$.

Then we let $\mathcal{A} \subseteq L^1(\mathbb{R}^n)$ be the set of functions $b(x) = t^{-n} a(\frac{x-x_0}{t})$ where $a \in \mathcal{F}$, $x_0 \in \mathbb{R}^n$ and $t > 0$. The functions $b \in \mathcal{A}$ are known as *very special atom*. We have the following decomposition of the classical Besov space $\dot{B}^0_{1,1}(\mathbb{R}^n)$ (see [Me]).

Lemma 4.15. *Suppose that $f \in L^1(\mathbb{R}^n)$. The following properties of f are equivalent.*

(i) $f \in \dot{B}^0_{1,1}(\mathbb{R}^n)$.

(ii) There exist a sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ of scalars and a sequence of very special atoms $\{a_j\}_{j\in\mathbb{N}}$ satisfying

$$f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x), \quad \text{and} \quad \sum_{j=0}^{\infty} |\lambda_j| < \infty.$$

Remark. By Lemma 4.15 we have that $\int_{\mathbb{R}^n} f(x)\,dx = 0$ for every $f \in \dot{B}^0_{1,1}(\mathbb{R}^n)$. We note that this fact also follows from the well-known inclusion $\dot{B}^0_{1,1} \subseteq H^1$ (see [Pe2]).

Definition 4.16. *The function b is called a* very special L-atom *associated with a ball $B(x_0, r)$ when $b(x) = r^{-n}a(\frac{x-x_0}{r})$ where $a \in C^1(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$ and $r > 0$ such that*

$$\|a\|_\infty \le 1, \quad \|\nabla a\|_\infty \le 1, \quad \text{and} \quad \operatorname{supp} a \subset B(0,1), \tag{4.26}$$

and also,

$$\int_{\mathbb{R}^n} b(x)\,dx = 0, \quad \text{when} \quad 0 < r < \rho(x_0). \tag{4.27}$$

From Definition 4.16, it is clear that every very special atom is a very special L-atom. The following is a main result of this section.

Theorem 4.17. *Assume that $f \in L^1(\mathbb{R}^n)$. Suppose that $f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x)$, where $a_j(x)$, $j \in \mathbb{N}$, is a sequence of very special L-atoms and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$. Then the series $\sum_{j=0}^{\infty} \lambda_j a_j$ converges in $\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$, and there exists a constant $C > 0$ satisfying*

$$\left\| \sum_{j=0}^{\infty} \lambda_j a_j \right\|_{\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)} \le C \sum_{j=0}^{\infty} |\lambda_j|.$$

As a consequence, the following inclusion is true

$$\dot{B}^0_{1,1}(\mathbb{R}^n) \subseteq \dot{B}^{0,L}_{1,1}(\mathbb{R}^n).$$

Proof. In order to show Theorem 4.17, it is enough to prove that for each very special L-atom $b(x) = r^{-n}a(\frac{x-x_0}{r})$ associated with a ball $B(x_0, r)$ for some $r > 0$, $x_0 \in \mathbb{R}^n$, there is a positive constant $C > 0$, that does not depend on b, such that

$$\|b\|_{\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)}$$

$$
\begin{aligned}
&= \int_0^\infty \int_{\mathbb{R}^n} \left| t^2 L e^{-t^2 L}(b)(x) \right| \frac{dx\,dt}{t} \\
&\leq C. \qquad (4.28)
\end{aligned}
$$

We shall now show (4.28). Firstly, we have that

$$
\begin{aligned}
&\|b\|_{\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)} \\
&= \int_0^\infty \int_{\mathbb{R}^n} \left| t^2 L e^{-t^2 L}(b)(x) \right| \frac{dx\,dt}{t} \\
&= \int_0^r \int_{\mathbb{R}^n} \left| t^2 L e^{-t^2 L}(b)(x) \right| \frac{dx\,dt}{t} \\
&\quad + \int_r^\infty \int_{\mathbb{R}^n} \left| t^2 L e^{-t^2 L}(b)(x) \right| \frac{dx\,dt}{t} \\
&= I + II.
\end{aligned}
$$

To estimate term I, let us rewrite

$$
\begin{aligned}
&\int_0^r \int_{\mathbb{R}^n} \left| t^2 L e^{-t^2 L}(b)(x) \right| \frac{dx\,dt}{t} \\
&= \int_0^r \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} q_t(x,y) b(y)\,dy \Big| \frac{dx\,dt}{t} \\
&\leq \int_0^r \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} q_t(x,y) \big(b(y) - b(x)\big)\,dy \Big| \frac{dx\,dt}{t} \\
&\quad + \int_0^r \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} q_t(x,y)\,dy \Big| \big|b(x)\big| \frac{dx\,dt}{t} \\
&= I_1 + I_2.
\end{aligned}
$$

Using property (4.26) of very special L-atoms and (iii) of Proposition 4.11 we obtain

$$
\begin{aligned}
I_2 &\leq C_N \int_0^r \int_{\mathbb{R}^n} \frac{\big(t/\rho(x)\big)^\delta}{\big(1 + t/\rho(x)\big)^N} \big|b(x)\big| \frac{dx\,dt}{t} \\
&\leq C_N \int_{\mathbb{R}^n} \Big\{ \int_0^\infty \frac{\big(t/\rho(x)\big)^\delta}{\big(1 + t/\rho(x)\big)^N} \frac{dt}{t} \Big\} \big|b(x)\big|\,dx \\
&\leq C_{N,\delta}.
\end{aligned}
$$

To estimate term I_1, we apply (i) of Proposition 4.11 and the fact that $b(x) = r^{-n} a(\frac{x-x_0}{r})$ to get

$$
I_1 \leq C \int_0^r \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} t^{-n} \exp\Big(-\frac{|x-y|^2}{ct^2} \Big) r^{-n} \Big| a\Big(\frac{y-x_0}{r}\Big) - a\Big(\frac{x-x_0}{r}\Big) \Big|\,dy\,\frac{dx\,dt}{t}.
$$

By making the change of variables $x - y \to z$ and $\frac{x-x_0}{r} \to w$, it follows that

$$
I_1 \leq C \int_0^r \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} t^{-n} \exp\Big(-\frac{|z|^2}{ct^2} \Big) r^{-n} \Big| a\Big(\frac{x-x_0-z}{r}\Big) - a\Big(\frac{x-x_0}{r}\Big) \Big|\,dz\,\frac{dx\,dt}{t}
$$

$$\leq C\int_0^r\int_{\mathbb{R}^n} t^{-n}\exp\Big(-\frac{|z|^2}{ct^2}\Big)\Big\{\int_{\mathbb{R}^n}\Big|a\Big(w-\frac{z}{r}\Big)-a(w)\Big|\,dw\Big\}\,\frac{dz\,dt}{t}.$$

By using property (4.26) of very special L-atoms, we then have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\Big|a\Big(w-\frac{z}{r}\Big)-a(w)\Big|\,dw\\
&\quad\leq \frac{|z|}{r}\sum_{j=1}^n\Big\|\frac{\partial a}{\partial w_j}\Big\|_{L^1(\mathbb{R}^n)}\\
&\quad\leq C\frac{|z|}{r}.
\end{aligned}$$

It then follows that

$$\begin{aligned}
I_1 &\leq C\int_0^r\int_{\mathbb{R}^n} t^{-n}\exp\Big(-\frac{|z|^2}{2ct^2}\Big)\Big(\frac{|z|}{t}\Big)\Big(\frac{t}{r}\Big)\,\frac{dz\,dt}{t}\\
&\leq C\int_0^r r^{-1}\,dt\\
&= \frac{C}{r}\int_0^r dt\\
&= \frac{C}{r}\big[t\big]_0^r\\
&= \frac{C}{r}\Big(r-0\Big)\\
&= C.
\end{aligned}$$

Combining this with the estimate for I_2, we obtain $I\leq C$.

Let us now estimate term II. For any given very special L-atom $b(y)=r^{-n}a(\frac{y-x_0}{r})$, consider two cases:

Case 1 $(0<r<\rho(x_0))$. For this case, we have the fact that $\int_{\mathbb{R}^n} b(x)\,dx=0$. By using (ii) of Proposition 4.11 and the symmetry of q_t we obtain

$$\begin{aligned}
&\int_r^\infty\int_{\mathbb{R}^n}\Big|t^2Le^{-t^2L}(b)(x)\Big|\,\frac{dx\,dt}{t}\\
&\quad\leq \int_r^\infty\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}\big(q_t(x,y)-q_t(x,x_0)\big)r^{-n}a\Big(\frac{y-x_0}{r}\Big)\,dy\Big|\,\frac{dx\,dt}{t}\\
&\quad\leq \int_r^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big(\frac{|y-x_0|}{t}\Big)^\delta t^{-n}\exp\Big(-\frac{|x-y|^2}{ct^2}\Big)r^{-n}\Big|a\Big(\frac{y-x_0}{r}\Big)\Big|\,dy\,\frac{dx\,dt}{t}\\
&\quad\leq C\int_r^\infty\Big(\frac{r}{t}\Big)^\delta\,\frac{dt}{t}\\
&\quad= Cr^\delta\int_r^\infty\frac{1}{t^{\delta+1}}\,dt\\
&\quad= Cr^\delta\Big[-\frac{t^{-\delta}}{\delta}\Big]_r^\infty
\end{aligned}$$

$$
\begin{aligned}
&= Cr^{\delta}\left(0+\frac{1}{\delta r^{\delta}}\right)\\
&= \frac{C}{\delta}\\
&= C
\end{aligned}
$$

because $\operatorname{supp} a \subset B(0,1)$, and then $|y-x_0| \le r \le t$.

Case 2 ($r \ge \rho(x_0)$). For this case, using (i) of Proposition 4.11 it follows that

$$
\begin{aligned}
&\int_r^{\infty}\int_{\mathbb{R}^n}\left|t^2 Le^{-t^2 L}(b)(x)\right|\frac{dx\,dt}{t}\\
&\le C\int_r^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\right)^{-1} t^{-n}\exp\Big(-\frac{|x-y|^2}{ct^2}\Big)\big|b(y)\big|\,dy\,\frac{dx\,dt}{t}\\
&\le C\int_r^{\infty}\int_{\{|y-x_0|\le r\}}\left(\frac{\rho(y)}{t}\right)r^{-n}\left|a\Big(\frac{y-x_0}{r}\Big)\right|\frac{dy\,dt}{t}. \qquad (4.29)
\end{aligned}
$$

By using (4.17), we obtain

$$
\begin{aligned}
\rho(y) &\le C\rho(x_0)\left(1+\frac{|y-x_0|}{\rho(x_0)}\right)^{\frac{k_0}{k_0+1}}\\
&\le C\rho(x_0)\left(1+\frac{r}{\rho(x_0)}\right)\\
&\le Cr
\end{aligned}
$$

for all $|y-x_0| \le r$. Then it follows that

$$
\begin{aligned}
&\text{RHS of (4.29)}\\
&\le Cr\int_r^{\infty}\int_{\mathbb{R}^n} t^{-1}r^{-n}\left|a\Big(\frac{y-x_0}{r}\Big)\right|\frac{dy\,dt}{t}\\
&\le Cr\int_r^{\infty} t^{-1}\,\frac{dt}{t}\\
&= Cr\int_r^{\infty}\frac{1}{t^2}\,dt\\
&= Cr\left[-\frac{1}{t}\right]_r^{\infty}\\
&= Cr\left(0+\frac{1}{r}\right)\\
&= C.
\end{aligned}
$$

Thus we have proved that $II \le C$, and hence we have estimate (4.28). Therefore we have finished proving the theorem. □

Let us now turn to the proof of Theorem 4.14.

Proof of Theorem 4.14. Firstly, the inclusion "$\dot{B}^0_{1,1}(\mathbb{R}^n) \subseteq \dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$" follows readily from Theorem 4.17.

Next, we shall show that "$\dot{B}^0_{1,1}(\mathbb{R}^n) \subsetneqq \dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$". Let $x_0 \in \mathbb{R}^n$ and $r \geq \rho(x_0)$. Take a function $b(x) = r^{-n}a(\frac{x-x_0}{r})$, where $a \in C^1(\mathbb{R}^n)$ and $\operatorname{supp} a \subset B(0,1)$ such that

(i) $\|a\|_\infty \leq 1$;

(ii) $\|\frac{\partial a}{\partial x_j}\|_\infty \leq 1$ for all $1 \leq j \leq n$;

(iii) $\int_{\mathbb{R}^n} b(x)\,dx \neq 0$.

On the one hand, from Theorem 4.17 we have that $b \in \dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$ and $\|b\|_{\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)} \leq C$. On the other hand, because $\int_{\mathbb{R}^n} b(x)\,dx \neq 0$, it follows that $b \notin \dot{B}^0_{1,1}(\mathbb{R}^n)$ by using the remark after Lemma 4.15. Therefore, the proof of "$\dot{B}^0_{1,1}(\mathbb{R}^n) \subseteq \dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$" is finished. □

4.2.3 Molecular decomposition of $\dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)$

In this subsection we extend the decomposition of Besov spaces associated with Schrödinger operators (Theorem 4.13) to more general values α, p, q.

Let us define the notion of molecules.

In the following, the definition of a molecule associated with a cube

$$Q = \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i, i = 1, 2, \ldots, n\}$$

involves the "lower left corner of Q", $x_Q = a = (a_1, a_2, \ldots, a_n)$, and $\ell(Q)$, the side length of Q.

Definition 4.18. *Let $\epsilon \in (0,1]$, $\alpha \in (-1,1)$ and $p \geq 1$. A function m_Q is called an (ϵ, α, p)-molecule for L associated to the cube Q if $m_Q = Lg_Q$ for some g_Q, and the following conditions hold:*

$$|m_Q(x)| + \ell(Q)^{-2}|g_Q(x)| \leq \ell(Q)^{\alpha - n/p}\left\{1 + \frac{|x - x_Q|}{\ell(Q)}\right\}^{-n-\epsilon} \quad \text{for } x \in \mathbb{R}^n; \tag{4.30}$$

$$\int_{|y| \leq \ell(Q)} \|m_Q(x+y) - m_Q(x)\|_{L^p(dx)} \frac{dy}{|y|^{n+\alpha}} \leq 1. \tag{4.31}$$

The following result is a molecular characterization of $\dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)$. In the following, given $j \in \mathbb{Z}$, we use $\mathbb{D}_j$ to denote the set of all dyadic cubes of sidelength 2^{-j}.

Theorem 4.19. *Suppose that $L = -\Delta + V$, where $V \not\equiv 0$ is a non-negative potential in $RH_s(\mathbb{R}^n)$ for some $s > \frac{n}{2}$. Assume that $f \in L^1(\mathbb{R}^n)$ and let σ be the constant δ from Proposition 4.11. Let $-1 < \alpha < \min\{1, \sigma\}$ and $1 \le p \le q < \infty$. Then in the following we have (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c):*

(a) $f \in \dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

(b) *For any $0 < \epsilon \le 1$, there exist a sequence of coefficients $\{s_Q\}$, $0 \le s_Q < \infty$, where Q ranges over the dyadic cubes, and a sequence $\{m_Q\}$ of (ϵ, α, p)-molecules for L, such that*

$$f = \sum_Q s_Q m_Q \qquad \text{in} \quad \dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n), \tag{4.32}$$

and

$$\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{Q\in\mathbb{D}_j} |s_Q|^p\Big)^{q/p}\Big)^{1/q} \approx \|f\|_{\dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)}. \tag{4.33}$$

(c) $f \in \dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)$.

Proof of Theorem 4.19. We shall show that (b) $\Rightarrow$ (c).

Let m_Q be an (ϵ, α, p)-molecule for L associated to a cube Q. We will prove that

$$\|m_Q\|_{\dot{B}^{\alpha,L}_{p,q}} \le C.$$

We first split

$$\begin{aligned}\|m_Q\|_{\dot{B}^{\alpha,L}_{p,q}} &= \Big\{\Big(\int_0^{\ell(Q)} + \int_{\ell(Q)}^{\infty}\Big)\big\|t^2 L e^{-t^2 L} m_Q\big\|^q_{L^p} \frac{dt}{t^{1+\alpha q}}\Big\}^{1/q} \\ &\le I + II\end{aligned}$$

where

$$I = \Big\{\int_0^{\ell(Q)} \big\|t^2 L e^{-t^2 L} m_Q\big\|^q_{L^p} \frac{dt}{t^{1+\alpha q}}\Big\}^{1/q},$$

$$II = \Big\{ \int_{\ell(Q)}^{\infty} \big\| t^2 L e^{-t^2 L} m_Q \big\|_{L^p}^q \frac{dt}{t^{1+\alpha q}} \Big\}^{1/q}.$$

Let us estimate the second term. Firstly the bounds for g_Q in (4.30) allow us to obtain

$$\|g_Q\|_{L^1} \leq C\ell(Q)^{\alpha+2+n(1-1/p)}.$$

Next using that $m_Q = Lg_Q$ for some g_Q, the kernel bounds in Proposition 4.11 (i), and Minkowski's inequality, we have

$$\begin{aligned}
II &= \Big\{ \int_{\ell(Q)}^{\infty} \Big(\int_{\mathbb{R}^n} \big| (t^2 L)^2 e^{-t^2 L} g_Q(x) \big|^p dx \Big)^{q/p} \frac{dt}{t^{1+q(\alpha+2)}} \Big\}^{1/q} \\
&\leq C \Big\{ \int_{\ell(Q)}^{\infty} \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} e^{-p|x-y|^2/ct^2} dx \Big)^{1/p} |g_Q(y)| \, dy \Big)^q \frac{dt}{t^{1+q(\alpha+2+n)}} \Big\}^{1/q} \\
&\leq C \|g_Q\|_{L^1} \Big\{ \int_{\ell(Q)}^{\infty} \frac{dt}{t^{1+q(\alpha+2+n(1-1/p))}} \Big\}^{1/q} \\
&\leq C.
\end{aligned}$$

To estimate the first term we write

$$\begin{aligned}
I &= \Big\{ \int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} q_t(x,y) \, m_Q(y) \, dy \Big|^p dx \Big)^{q/p} \frac{dt}{t^{1+\alpha q}} \Big\}^{1/q} \\
&= \Big\{ \int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} q_t(x,y) \, [m_Q(y) + m_Q(x) - m_Q(x)] \, dy \Big|^p dx \Big)^{q/p} \frac{dt}{t^{1+\alpha q}} \Big\}^{1/q} \\
&\leq I_1 + I_2
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \Big\{ \int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} q_t(x,y) \, m_Q(x) \, dy \Big|^p dx \Big)^{q/p} \frac{dt}{t^{1+\alpha q}} \Big\}^{1/q}, \\
I_2 &= \Big\{ \int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} q_t(x,y) \, [m_Q(y) - m_Q(x)] \, dy \Big|^p dx \Big)^{q/p} \frac{dt}{t^{1+\alpha q}} \Big\}^{1/q}.
\end{aligned}$$

Let us estimate I_1. By using (iii) of Proposition 4.11, Minkowski's inequality and the assumption that $p \leq q$ we obtain

$$\begin{aligned}
I_1 &\leq \Big\{ \int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} q_t(x,y) \, dy \Big|^p |m_Q(x)|^p \, dx \Big)^{q/p} \frac{dt}{t^{1+\alpha q}} \Big\}^{1/q} \\
&\leq C_N \Big\{ \int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} \Big| \frac{(t/\rho(x))^\sigma}{(1+t/\rho(x))^N} \Big|^p |m_Q(x)|^p \, dx \Big)^{q/p} \frac{dt}{t^{1+\alpha q}} \Big\}^{1/q} \\
&= C_N \Bigg(\Big\{ \int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} \Big| \frac{(t/\rho(x))^\sigma}{(1+t/\rho(x))^N} \Big|^p |m_Q(x)|^p \, dx \Big)^{q/p} \frac{dt}{t^{1+\alpha q}} \Big\}^{p/q} \Bigg)^{1/p}
\end{aligned}$$

$$\leq C_N \left(\int_{\mathbb{R}^n} \Big\{ \int_0^{\ell(Q)} \Big| \frac{(t/\rho(x))^\sigma}{(1+t/\rho(x))^N} \Big|^q \frac{dt}{t^{1+\alpha q}} \Big\}^{p/q} |m_Q(x)|\, dx \right)^{1/p}$$

$$\leq C_{N,\sigma}.$$

We estimate the second term by splitting the region of integration in the y variable into two regions: $|x-y| \geq \ell(Q)$ and $|x-y| < \ell(Q)$. That is,

$$I_2 \leq \Big\{ \int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} e^{-|x-y|^2/ct^2} |m_Q(y) - m_Q(x)|\, dy \Big)^p dx \Big)^{q/p} \frac{dt}{t^{1+q(n+\alpha)}} \Big\}^{1/q}$$

$$\leq I_{2.1} + I_{2.2}$$

where

$$I_{2.1} = \Big\{ \int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} \Big(\int_{|x-y|\geq \ell(Q)} e^{-|x-y|^2/ct^2} |m_Q(y) - m_Q(x)|\, dy \Big)^p dx \Big)^{q/p} \frac{dt}{t^{1+q(n+\alpha)}} \Big\}^{1/q},$$

$$I_{2.2} = \Big\{ \int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} \Big(\int_{|x-y|<\ell(Q)} e^{-|x-y|^2/ct^2} |m_Q(y) - m_Q(x)|\, dy \Big)^p dx \Big)^{q/p} \frac{dt}{t^{1+q(n+\alpha)}} \Big\}^{1/q}$$

For the first case we integrate

$$I_{2.1} \leq I_{2.1.1} + I_{2.1.2}$$

where

$$I_{2.1.1} = \Big\{ \int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} \Big(\int_{|x-y|\geq \ell(Q)} e^{-|x-y|^2/ct^2} |m_Q(y)|\, dy \Big)^p dx \Big)^{q/p} \frac{dt}{t^{1+q(n+\alpha)}} \Big\}^{1/q},$$

$$I_{2.1.2} = \Big\{ \int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} \Big(\int_{|x-y|\geq \ell(Q)} e^{-|x-y|^2/ct^2} |m_Q(x)|\, dy \Big)^p dx \Big)^{q/p} \frac{dt}{t^{1+q(n+\alpha)}} \Big\}^{1/q}$$

Then for any $\delta > \frac{n}{2}(1-1/p) + \alpha$, Minkowski's inequality gives

$$I_{2.1.1} \leq \Big\{ \int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} \Big(\int_{|x-y|\geq \ell(Q)} e^{-p|x-y|^2/ct^2} dx \Big)^{1/p} |m_Q(y)|\, dy \Big)^q \frac{dt}{t^{1+q(n+\alpha)}} \Big\}^{1/q}$$

$$\leq C\ell(Q)^{-2\delta/p} \, \|m_Q\|_{L^1} \Big\{ \int_0^{\ell(Q)} \frac{dt}{t^{1+q(\alpha+n(1-1/p)-2\delta/p)}} \Big\}^{1/q}$$

$$\leq C.$$

In the last step we used the estimate

$$\|m_Q\|_{L^1} \leq C\ell(Q)^{n+\alpha-n/p}$$

which holds via the bounds in (4.30).

Next, for any $\delta > 0$, $x \in \mathbb{R}^n$ and cube Q we have

$$\int_{|x-y|\geq \ell(Q)} e^{-|x-y|^2/ct^2} dy \;\leq\; Ct^{n+2\delta}\ell(Q)^{-2\delta}.$$

Applying this with some $\delta > \alpha/2$ gives

$$\begin{aligned} I_{2.1.2} &= \Big\{\int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} |m_Q(x)|^p \Big(\int_{|x-y|\geq \ell(Q)} e^{-|x-y|^2/ct^2} dy\Big)^p dx\Big)^{q/p} \frac{dt}{t^{1+q(n+\alpha)}}\Big\}^{1/q} \\ &\leq C\ell(Q)^{2\delta}\, \|m_Q\|_{L^p} \Big\{\int_0^{\ell(Q)} \frac{dt}{t^{1+q(\alpha-2\delta)}}\Big\}^{1/q} \\ &\leq C. \end{aligned}$$

In the last step we used the estimate

$$\|m_Q\|_{L^p} \;\leq\; C\ell(Q)^{\alpha}$$

which holds via again the bounds in (4.30).

Next, with a change of variable $y = x + w$, and applying Minkowski's inequality twice, we obtain

$$\begin{aligned} I_{2.2} &= \Big\{\int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n} \Big(\int_{|w|\leq \ell(Q)} e^{-|w|^2/ct^2} |m_Q(x+w) - m_Q(x)|\, dw\Big)^p dx\Big)^{q/p} \frac{dt}{t^{1+q(n+\alpha)}}\Big\}^{1/q} \\ &\leq \Big\{\int_0^{\ell(Q)} \Big(\int_{|w|\leq \ell(Q)} e^{-|w|^2/ct^2} \|m_Q(\cdot + w) - m_Q(\cdot)\|_{L^p}\, dw\Big)^q \frac{dt}{t^{1+q(n+\alpha)}}\Big\}^{1/q} \\ &\leq \int_{|w|\leq \ell(Q)} \|m_Q(\cdot + w) - m_Q(\cdot)\|_{L^p} \Big\{\int_0^{\ell(Q)} e^{-q|w|^2/ct^2} \frac{dt}{t^{1+q(n+\alpha)}}\Big\}^{1/q} dw \\ &\leq C \int_{|w|\leq \ell(Q)} \|m_Q(\cdot + w) - m_Q(\cdot)\|_{L^p} \frac{dw}{|w|^{n+\alpha}} \\ &\leq C \end{aligned}$$

In the last step we applied (4.31).

We show (a) $\Rightarrow$ (b).

Let $f \in \dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Applying the Calderón reproducing formula I to f we obtain

$$f = \frac{1}{8}\int_0^\infty (t^2L)^2 e^{-2t^2L} f \frac{dt}{t} = \frac{1}{8}\int_0^\infty \int_{\mathbb{R}^n} q_t(x,y)\big(t^2Le^{-t^2L}f\big)(y)\frac{dy\,dt}{t}$$

We then "discretize" the right hand side by splitting $\mathbb{R}^n$ into dyadic cubes. Let Q be a dyadic cube. We define

$$\mathcal{T}(Q) = \big\{(x,t) \in \mathbb{R}^{n+1}_+ : x \in Q,\ \ell(Q)/2 < t \leq \ell(Q)\big\}$$

to be the "half-cube" in $\mathbb{R}^{n+1}_+$ over Q.

We then have

$$\begin{aligned} f(x) &= \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathbb{D}_j} \frac{1}{8} \iint_{\mathcal{T}(Q)} q_t(x,y)\big(t^2 L e^{-t^2 L} f\big)(y) \frac{dy\,dt}{t} \\ &= \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathbb{D}_j} s_Q\ m_Q(x) \end{aligned}$$

where

$$\begin{aligned} s_Q &= \frac{1}{\ell(Q)^{\alpha+n(1-1/p)}} \iint_{\mathcal{T}(Q)} \big|t^2 L e^{-t^2 L} f(y)\big| \frac{dy\,dt}{t} \\ m_Q(x) &= \frac{1}{8 s_Q} \iint_{\mathcal{T}(Q)} q_t(x,y)\big(t^2 L e^{-t^2 L} f\big)(y) \frac{dy\,dt}{t}. \end{aligned}$$

We now show that m_Q satisfies (4.30) and (4.31).

We first check (4.31). By using estimate (ii) from Proposition 4.11, we have

$$\begin{aligned} &\int_{|z|\leq \ell(Q)} \|m_Q(\cdot + z) - m_Q(\cdot)\|_{L^p} \frac{dz}{|z|^{n+\alpha}} \\ &\leq \frac{1}{8 s_Q} \iint_{\mathcal{T}(Q)} \big|t^2 L e^{-t^2 L} f(y)\big| \Big(\int_{|z|\leq \ell(Q)} \|q_t(\cdot + z, y) - q_y(\cdot, y)\|_{L^p} \frac{dz}{|z|^{n+\alpha}} \Big) \frac{dy\,dt}{t} \\ &\leq C \frac{1}{s_Q} \iint_{\mathcal{T}(Q)} \big|t^2 L e^{-t^2 L} f(y)\big| \frac{dy\,dt}{t^{1+\sigma+n(1-1/p)}} \int_{|z|\leq \ell(Q)} \frac{dz}{|z|^{n+\alpha-\sigma}} \\ &\leq C \frac{1}{\ell(Q)^{\alpha+n(1-1/p)}\ s_Q} \iint_{\mathcal{T}(Q)} \big|t^2 L e^{-t^2 L} f(y)\big| \frac{dy\,dt}{t} \\ &\leq C \end{aligned}$$

In the next to last step we used the condition that $\sigma > \alpha$ in the second integral. We also used that $(y,t) \in \mathcal{T}(Q)$ implies $t \approx \ell(Q)$.

We now check (4.30). For each $x \in \mathbb{R}^n$, and any $\epsilon > 0$

$$|m_Q(x)| \leq C \frac{1}{s_Q} \iint_{\mathcal{T}(Q)} |q_t(x,y)| \big|t^2 L e^{-t^2 L} f(y)\big| \frac{dy\,dt}{t}$$

$$\leq C\ell(Q)^{\alpha+n(1-1/p)} \sup_{(y,t)\in\mathcal{T}(Q)} |q_t(x,y)|$$

$$\leq C\ell(Q)^{\alpha-n/p} \sup_{(y,t)\in\mathcal{T}(Q)} e^{-|x-y|^2/ct^2}$$

$$\leq C\ell(Q)^{\alpha-n/p}\Big(1+\frac{|x-x_Q|}{\ell(Q)}\Big)^{-n-\epsilon}.$$

By a similar argument using (4.18) it follows that (4.30) is true for g_Q. $\square$

Chapter 5

Besov spaces associated with operators III: Atomic and molecular decompositions of Besov spaces associated to operators on spaces of homogeneous type

The content of this chapter is essentially a joint work with T.A. Bui, X.T. Duong and F.K. Ly [BDLW], in which my contribution is fair and reasonable.

The main aim of this chapter is to lay out the theory of Besov spaces associated to operators L whose heat kernel satisfies the Gaussian upper bounds on spaces of homogeneous type. The organisation of the chapter is as follows. In Section 5.1, we recall some basic properties on the regularity of the time derivative of the heat kernels and the covering lemma of Christ in [Ch1]. The theory of Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ and their basic properties is addressed in Section 5.2. The molecular and atomic decompositions on Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ are investigated in Section 5.3 and Section 5.4, respectively. In Section 5.5, we compare our Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ and Besov spaces $\dot{B}^{\alpha}_{p,q}(X)$. As applications, in Section 5.6 we study the behaviour of fractional integrals and spectral multipliers on new Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$.

5.1 Preliminaries

Recently, in [BDY] the authors investigated the theory of Besov spaces associated to operators whose heat kernel satisfies an upper bound of Poisson type on the space of polynomial upper bound on volume growth. They also carried out that by different choices of operators L, they can recover most of the classical Besov spaces. Moreover, in some particular choices of L, they

obtain new Besov spaces.

The main aim of this chapter is to lay out the theory of Besov spaces associated to operators L whose heat kernel satisfies the Gaussian upper bounds on spaces of homogeneous type. Adapting some ideas in [BDY], we construct the Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ associated to the operators. Note that since the assumption of polynomial upper bound on volume growth in [BDY] do not include the spaces of homogeneous type, some refinements and improvements would be required. The main contribution of this chapter is to investigate the atomic and molecular decompositions of functions in the new Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$. (Note that there are no results on atomic and molecular decompositions for the general Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ in [BDY]). Precisely, we are able to prove the following results:

(i) Under the Gaussian upper bound assumption only, we prove that each function in our Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$, with $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$, admits a linear molecular decomposition. We would like to emphasize that there are no smoothness conditions on the molecules. Conversely, each linear molecular decomposition belongs to the Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$, with $-1 < \alpha < 0$ and $1 \leq p, q \leq \infty$. See Theorem 5.20 and Theorem 5.22.

(ii) Under the Gaussian upper bound, Hölder continuity and conservation assumptions, we prove the theory of molecular decomposition on Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$, with $-1 < \alpha < \delta$ and $1 \leq p, q \leq \infty$ where δ is a positive constant depending on the smoothness order of the heat kernel of the operator L. See Theorem 5.27.

(iii) We study the theory of molecular decomposition on Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$, with $-1 < \alpha < \delta_0$ and $1 \leq p, q \leq \infty$ for some δ_0, asscociated with operators of Schrödinger type. It is worth noting that the conservation property is not assumed here. See Theorem 5.30. Note that our findings have applications in various settings such as Schrödinger operators, degenerate Schrödinger operators on $\mathbb{R}^n$ and Schrödinger operators on Heisenberg groups and connected and simply connected nilpotent Lie groups.

(iv) In the particular case $p = q$, the atomic decomposition of Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ is also

obtained in Theorem 5.36.

We also carry out the relationship between our Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ and the Besov spaces $\dot{B}^{\alpha}_{p,q}(X)$ of Han and Sawyer in [HS]. See Section 5.5. We prove the following results:

(i) Under the Gaussian upper bound, Hölder continuity and conservation assumptions, we show the coincidence between our Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ and the Besov spaces $\dot{B}^{\alpha}_{p,q}(X)$ for all indices α being close to zero.

(ii) When the operator L is an operator of Schrödinger type, we show the inclusion $\dot{B}^{\alpha}_{p,q}(X) \subset \dot{B}^{\alpha,L}_{p,q}(X)$ for some suitable values of p, q and α.

Note that for the investigation on the atomic decomposition, the approach in [HS] was based on a construction of a family of approximation to the identity and a Calderón reproducing formula. Roughly speaking, these kernels associated to this family satisfy the Gaussian upper bound, Hölder continuity and conservation properties. In most parts of our work, we do not need the conservation assumption. Even if L satisfies the Gaussian upper bound, Hölder continuity and conservation properties, our obtained results are still new as we do not assume either the polynomial growth nor the reverse doubling property on the volume of the balls on the underlying spaces. Moreover, when the order of the family of approximation to the identity is less than 1, the results on Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ in this chapter are new for values α being close to -1.

Let (X, d) be a metric space, with the distance d. Assume that X is equipped with a nonnegative, Borel, doubling measure μ. Throughout this paper, we assume that $\mu(X) = \infty$. Denote by $B(x, r)$ the open ball of radius $r > 0$ and centre $x \in X$. The doubling property of μ provides us with the fact that there exists a constant $C > 0$ so that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \tag{5.1}$$

for all $x \in X$ and $r > 0$.

Notice that the doubling property (5.1) implies that

$$\mu(B(x, \lambda r)) \leq C\lambda^n \mu(B(x, r)), \tag{5.2}$$

for some positive constant n uniformly for all $\lambda \geq 1, x \in X$ and $r > 0$. There also exists a constant $0 \leq N \leq n$ such that

$$\mu(B(x,r)) \leq C\Big(1 + \frac{d(x,y)}{r}\Big)^N \mu(B(y,r)), \tag{5.3}$$

uniformly for all $x, y \in X$ and $r > 0$.

In the rest of the chapter, unless specified, we always assume that L is a linear operator of type θ with $\theta < \pi/2$, has dense range in $L^2(X)$ and has a bounded H_∞-calculus on $L^2(X)$. The main assumption is the following:

(G) *Gaussian upper bound:* There exist constants C, $c > 0$ such that

$$|p_t(x,y)| \leq \frac{C}{\mu(B(x,\sqrt{t}))} \exp\Big(-\frac{d(x,y)^2}{ct}\Big), \tag{5.4}$$

for all $x, y \in X$ and $t > 0$, where $p_t(x,y)$ is the associated kernel of e^{-tL}.

In certain situations, we shall assume one or both of the following additional conditions:

(C) *Conservation property:* $\int_X p_t(x,y)d\mu(y) = \int_X p_t(y,x)d\mu(y) = 1$ for all $x \in X$ and $t > 0$.

(H) *Hölder continuity:* There exists $\delta \in (0,1]$ so that

$$|p_t(x,y) - p_t(x',y)| + |p_t(y,x) - p_t(y,x')| \leq \Big(\frac{d(x,x')}{\sqrt{t}}\Big)^\delta \frac{C}{\mu(B(x,\sqrt{t}))} \exp\Big(-c\frac{d(x,y)^2}{t}\Big)$$

for all $t > 0$ and $d(x,x') < \sqrt{t}/2$.

For $k \in \mathbb{N}$, denote by $p_{t,k}(x,y)$ the heat kernel of $(tL)^k e^{-tL}$. We have the following result.

Lemma 5.1. *Assume that L satisfies* (G) *and* (H)*. Then for each $k \in \mathbb{N}$, there exist $C, c > 0$ so that*

(i) $|p_{t,k}(x,y)| \leq \dfrac{C}{\mu(B(x,\sqrt{t}))} \exp\Big(-c\dfrac{d(x,y)^2}{t}\Big), \quad \forall t > 0, x, y \in X;$

(ii) $\|p_{t,k}(\cdot,y)\|_p + \|p_{t,k}(y,\cdot)\|_p \leq \dfrac{C}{\mu(B(y,\sqrt{t}))^{1/p'}}, \quad \forall t > 0, x, y \in X$ *and* $1 \leq p \leq \infty;$

(iii) $|p_{t,k}(x,y) - p_{t,k}(x',y)| + |p_t(y,x) - p_t(y,x')| \le \Big(\frac{d(x,x')}{t}\Big)^{\delta} \frac{C}{\mu(B(x,\sqrt{t}))} \exp\Big(-c\frac{d(x,y)^2}{t}\Big)$ *for all* $t>0$ *and* $d(x,x') < \sqrt{t}/4$.

Proof. For the proof of (i) we refer to [CD2]. The assertion (ii) follows immediately from (i).

To prove (iii), we write $(tL)^k e^{-tL} = 2^k e^{-\frac{t}{2}L}(\frac{t}{2}L)^k e^{-\frac{t}{2}L}$. Hence

$$\begin{aligned}
&|p_{t,k}(x,y) - p_{t,k}(x',y)| \\
&\quad \le 2^k \int_X \Big|(p_{t/2}(x,z) - p_{t/2}(x',z))p_{t/2,k}(z,y)\Big| d\mu(z) \\
&\quad \lesssim \int_X \Big|\Big(\frac{d(x,x')}{\sqrt{t}}\Big)^{\delta} \frac{1}{\mu(B(x,\sqrt{t}))} \exp\Big(-c\frac{d(x,z)^2}{t}\Big) \frac{1}{\mu(B(y,\sqrt{t}))} \exp\Big(-c\frac{d(y,z)^2}{t}\Big)\Big| d\mu(z).
\end{aligned} \tag{5.5}$$

Note that

$$\begin{aligned}
\exp\Big(-c\frac{d(x,z)^2}{t}\Big)\exp\Big(-c\frac{d(y,z)^2}{t}\Big) &\lesssim \exp\Big(-c_1\frac{d(x,z)^2+d(z,y)^2}{t}\Big)\exp\Big(-c_2\frac{d(y,z)^2}{t}\Big) \\
&\lesssim \exp\Big(-c_1'\frac{d(x,y)^2}{t}\Big)\exp\Big(-c_2\frac{d(y,z)^2}{t}\Big).
\end{aligned}$$

This together with (5.5) gives

$$\begin{aligned}
&|p_{t,k}(x,y) - p_{t,k}(x',y)| \\
&\quad \lesssim \Big(\frac{d(x,x')}{\sqrt{t}}\Big)^{\delta} \frac{1}{\mu(B(x,\sqrt{t}))} \exp\Big(-c\frac{d(x,y)^2}{t}\Big) \int_X \Big|\frac{1}{\mu(B(y,\sqrt{t}))} \exp\Big(-c\frac{d(y,z)^2}{t}\Big)\Big| d\mu(z) \\
&\quad \lesssim \Big(\frac{d(x,x')}{\sqrt{t}}\Big)^{\delta} \frac{1}{\mu(B(x,\sqrt{t}))} \exp\Big(-c\frac{d(x,y)^2}{t}\Big).
\end{aligned}$$

This completes our proof.

We will recall an important result on covering lemma in [Ch1]. □

Lemma 5.2. *There exists a collection of open sets* $\{Q_\tau^k \subset X : k \in \mathbb{Z}, \tau \in I_k\}$, *where* I_k *denotes certain (possibly finite) index set depending on* k, *and constants* $\rho \in (0,1), a_0 \in (0,1)$ *and* $C_1 \in (0,\infty)$ *such that*

(i) $\mu(X \setminus \cup_\tau Q_\tau^k) = 0$ *for all* $k \in \mathbb{Z}$;

(ii) if $i \ge k$, *then either* $Q_\tau^i \subset Q_\beta^k$ *or* $Q_\tau^i \cap Q_\beta^k = \emptyset$;

(iii) for (k,α) *and each* $i < k$, *there exists a unique* τ' *such* $Q_\tau^k \subset Q_{\tau'}^i$;

(iv) the diameter $\operatorname{diam}(Q_\tau^k) \le C_1\rho^k$;

(v) each Q_τ^k contains certain ball $B(x_{Q_\tau^k}, a_0\rho^k)$.

Remark 5.3. *(i) Without loss of generality, we may assume that $\rho = 1/2$ and $a_0 = 1$. See for example [HS]. We then fix a collection of open sets in Lemma 5.2 and denote this collection by $\mathcal{D}$. We also call these open sets the dyadic cubes in X and $x_{Q_\tau^k}$ the centre of the cube Q_τ^k. We also denote $\mathcal{D}_k := \{Q \in \mathcal{D} : \operatorname{diam} Q \sim 2^k\}$ for each $k \in \mathbb{Z}$.*

(ii) If x_Q is the centre of the cube $Q \in \mathcal{D}_k$ then there exists $C > 0$ so that for any $\lambda > 1$,

$$\sharp\{S : S \in \mathcal{D}_k, S \cap \lambda Q \neq \emptyset\} \leq \lambda^n$$

where $\lambda Q = B(x_Q, \lambda 2^k)$.

(iii) From the doubling property, it is easy to see that for each k, the set of indices I_k is at most countable.

Throughout the chapter, we set $\Psi_t(L) = t^2 L e^{-t^2 L}$. The following simple lemma will be useful in the sequel.

Lemma 5.4. *Assume that L satisfies* (G). *Let $p \in [1, \infty]$. There exist $C_1, C_2 > 0$ so that*

$$C_1 \Big\| \Psi_{2^k}(L) f \Big\|_p \leq \Big\| \Psi_t(L) f \Big\|_p \leq C_2 \Big\| \Psi_{2^{k-1}}(L) f \Big\|_p$$

for all $f \in L^p(X), k \in \mathbb{Z}$ and $2^{k-1} \leq t \leq 2^k$.

5.2 Besov spaces associated to operators

5.2.1 Definition of Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ and basic properties

Assume that L satisfies (G). Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. For an appropriate function f, we define the Besov norm $\| \cdot \|_{\dot{B}^{\alpha,L}_{p,q}(X)}$ by

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} = \Big(\int_0^\infty (t^{-\alpha} \|\Psi_t(L) f\|_p)^q \frac{dt}{t} \Big)^{1/q}, \ q < \infty,$$

and

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} = \sup_{t>0} t^{-\alpha}\|\Psi_t(L)f\|_p,\ \ q=\infty.$$

In the definition above, it should be understood that "an appropriate function f" may be any function so that $\Psi_t(L)f$ is well-defined. An example of such functions is the class of functions satisfying the *growth condition*. A function $f \in L^1_{\mathrm{loc}}(X)$ is said to satisfy the *growth condition* if

$$\int_X \frac{|f(x)|}{(1+d(x_0,x))^{N+\beta}\mu(B(x_0,1+d(x_0,x)))}d\mu(x) < \infty \tag{5.6}$$

for some $\beta > 0$ and some $x_0 \in X$. We then denote by $\mathcal{M}$ the class of all functions satisfying the growth condition (5.6). Note that if (5.6) holds for some $x_0 \in X$ then it holds for any choice of $x_0 \in X$. For $f \in \mathcal{M}$, it can be verified that $|\Psi_t(L)f(x)| < \infty$ for all $t > 0, x \in X$. See for example [DY1].

We will show that the Besov norm of the heat kernels $p_{t,k}(\cdot, y)$ is finite for $t > 0, y \in X$ and $k \in \mathbb{N}_+$. This statement plays a crucial role in the construction of the Besov spaces associated to the operator L.

Lemma 5.5. *Assume that L satisfies* (G). *Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. Then for all $y \in X, s > 0$ and $k \in \mathbb{N}_+$, we have*

$$\|p_{s,k}(\cdot,y)\|_{\dot{B}^{\alpha,L}_{p,q}(X)} \lesssim s^{-\alpha/2}V(y,\sqrt{s})^{-1/p'}.$$

Proof. Note that $p_{s,k}(\cdot,y) \in \mathcal{M}$. Moreover, we have

$$\Psi_t(L)[p_{s,k}(\cdot,y)](x) = \frac{t^2 s^{k+1}}{(t^2+s)^k}p_{t^2+s,k}(x,y).$$

Hence,

$$\begin{aligned}\|p_{s,k}(\cdot,y)\|^q_{\dot{B}^{\alpha,L}_{p,q}(X)} &= \int_0^\infty \Big(t^{-\alpha}\frac{t^2 s^k}{(t^2+s)^{k+1}}\|p_{t^2+s,k}(\cdot,y)\|_p\Big)^q\frac{dt}{t}\\ &= \int_0^{\sqrt{s}} \ldots + \int_{\sqrt{s}}^\infty \ldots := A_1 + A_2.\end{aligned}$$

Note that

$$\|p_{t^2+s,k}(\cdot,y)\|_p \lesssim \frac{1}{V(y,\sqrt{t^2+s})^{1/p'}}.$$

So, we have

$$A_1 \lesssim \int_0^{\sqrt{s}} \Big(t^{-\alpha} \frac{t^2 s^k}{(t^2+s)^{k+1}} \frac{1}{V(y,\sqrt{t^2+s})^{1/p'}}\Big)^q \frac{dt}{t} \quad \lesssim \int_0^{\sqrt{s}} \Big(t^{-\alpha} \frac{t^2}{s} \frac{1}{V(y,\sqrt{s})^{1/p'}}\Big)^q \frac{dt}{t}$$
$$\lesssim s^{-\alpha q/2} V(y,\sqrt{s})^{-q/p'}.$$

Likewise,

$$A_2 \lesssim \int_{\sqrt{s}}^{\infty} \Big(t^{-\alpha} \frac{t^2 s^k}{(t^2+s)^{k+1}} \frac{1}{V(y,\sqrt{t^2+s})^{1/p'}}\Big)^q \frac{dt}{t} \quad \lesssim \int_{\sqrt{s}}^{\infty} \Big(t^{-\alpha} \frac{s^k}{t^{2k}} \frac{1}{V(y,\sqrt{s})^{1/p'}}\Big)^q \frac{dt}{t}$$
$$\lesssim s^{-\alpha q/2} V(y,\sqrt{s})^{-q/p'}.$$

□

To be able to define the Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$, we adapt some ideas in [BDY] to define the space of test functions.

Definition 5.6. *Assume that L satisfies* (G). *Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. A function f is said to be in the space of test functions $\mathcal{M}^{\alpha,L}_{p,q}$ if there holds:*

(i) $f = Lg$;

(ii) *There exist $\epsilon > 0$ and $C > 0$ so that*

$$|f(x)| + |g(x)| \le \frac{C}{(1+d(x_0,x))^{N+\epsilon}\mu(B(x_0, 1+d(x_0,x)))}$$

for some $x_0 \in X$.

(iii) $\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} < \infty$;

When $q = \infty$, we assume additionally that

$$t^{-\alpha}\|\Psi_t(L)f\|_p \to 0, \text{ as } t \to 0 \text{ or } t \to \infty, \tag{5.7}$$

and when $p = \infty$, that $\lim_{s\to 0} \|e^{-sL}f - f\|_{\dot{B}^{\alpha,L}_{\infty,q}} = 0$.

For $f \in \mathcal{M}^{\alpha,L}_{p,q}$, we define $\|f\|_{\mathcal{M}^{\alpha,L}_{p,q}} = \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}$.

We also define $(\mathcal{M}^{\alpha,L}_{p,q})'$ as the dual space of the space of the test functions $\mathcal{M}^{\alpha,L}_{p,q}$ equipped with the weak* topology. In particular, a sequence (f_n) in $(\mathcal{M}^{\alpha,L}_{p,q})'$ converges to $f \in (\mathcal{M}^{\alpha,L}_{p,q})'$ if

and only if

$$\lim_{n\to\infty}\langle f_n,\varphi\rangle=\langle f,\varphi\rangle,\quad \text{for all } \varphi\in\mathcal{M}^{\alpha,L}_{p,q},$$

where $\langle\cdot,\cdot\rangle$ is the pairing between a linear functional in $(\mathcal{M}^{\alpha,L}_{p,q})'$ and the test function in $\mathcal{M}^{\alpha,L}_{p,q}$.

From the definition of the spaces of test functions, some basic properties can be withdrawn.

Proposition 5.7. *Assume that L satisfies* (G). *Let $-1<\alpha<1$ and $1\le p\le\infty$ and $1\le q_1\le q_2\le\infty$. Then we have*

(i) $\mathcal{M}^{\alpha,L}_{p,q_1}\subset\mathcal{M}^{\alpha,L}_{p,q_2}$ *with continuous embedding;*

(ii) *Assume that X has volume growth at least polynomial, that is, $\mu(B(x,r))\gtrsim r^\nu$ for some $\nu>0$. If $1\le p_1\le p_2\le\infty, -1<\alpha_2\le\alpha_1<1$ and $\alpha_1-\nu/p_1=\alpha_2-\nu/p_2$ then*

$$\mathcal{M}^{\alpha_1,L}_{p_1,q}\subset\mathcal{M}^{\alpha_2,L}_{p_2,q}\quad \textit{with continuous embedding.}$$

Proof. (i) We first prove for $-1<\alpha<1$, $1\le p\le\infty$, $1\le q_1<\infty$ and $q_2=\infty$. Let $f\in\mathcal{M}^{\alpha,L}_{p,q_1}$. Since $\|f\|_{\dot B^{\alpha,L}_{p,q}(X)}<\infty$, $\lim_{t\to0}t^{-\alpha}\|\Psi_t(L)f\|_p=\lim_{t\to\infty}t^{-\alpha}\|\Psi_t(L)f\|_p=0$ and $\|f\|_{\dot B^{\alpha,L}_{p,\infty}(X)}=\sup_{t>0}t^{-\alpha}\|\Psi_t(L)f\|_p<\infty$. This implies $f\in\mathcal{M}^{\alpha,L}_{p,\infty}$. It remains to show that

$$\|f\|_{\dot B^{\alpha,L}_{p,\infty}(X)}\lesssim\|f\|_{\dot B^{\alpha,L}_{p,q_1}(X)}. \tag{5.8}$$

If $\|f\|_{\dot B^{\alpha,L}_{p,\infty}(X)}>0$ there exists t_0 so that $t_0^{-\alpha}\|\Psi_{t_0}(L)f\|_p>\frac12\|f\|_{\dot B^{\alpha,L}_{p,\infty}(X)}$. Let $k\in\mathbb{Z}$ so that $2^k<t_0\le2^{k+1}$. By Lemma 5.4, we have

$$\begin{aligned}\|f\|^q_{\dot B^{\alpha,L}_{p,q_1}(X)}&\ge\int_{2^{k-1}}^{2^k}\left(t^{-\alpha}\|\Psi_t(L)f\|_p\right)^q\frac{dt}{t}\gtrsim\left(2^{-k\alpha}\|\Psi_{2^k}(L)f\|_p\right)^q\\&\gtrsim\left(t_0^{-\alpha}\|\Psi_{t_0}(L)f\|_p\right)^q\gtrsim\|f\|_{\dot B^{\alpha,L}_{p,\infty}(X)}.\end{aligned}$$

When $1\le q_2<\infty$, for $f\in\mathcal{M}^{\alpha,L}_{p,q_1}$, by (5.8) we have

$$\begin{aligned}\|f\|^{q_2}_{\dot B^{\alpha,L}_{p,q_2}(X)}&=\int_0^\infty\left(t^{-\alpha}\|\Psi_t(L)f\|_p\right)^{q_2}\frac{dt}{t}\\&\lesssim\|f\|^{q_2-q_1}_{\dot B^{\alpha,L}_{p,\infty}(X)}\int_0^\infty\left(t^{-\alpha}\|\Psi_t(L)f\|_p\right)^{q_1}\frac{dt}{t}\\&\lesssim\|f\|^{q_2-q_1}_{\dot B^{\alpha,L}_{p,q_1}(X)}\|f\|^{q_1}_{\dot B^{\alpha,L}_{p,q_1}(X)}:=\|f\|^{q_2}_{\dot B^{\alpha,L}_{p,q_1}(X)}.\end{aligned}$$

This completes the proof of (i).

(ii) We first claim that for $f \in L^{p_1}(X)$, we have

$$\|e^{-t^2 L} f\|_{p_2} \lesssim t^{-\nu(1/p_1 - 1/p_2)} \|f\|_{p_1}. \tag{5.9}$$

Indeed, by Minkowski's inequality we have

$$\begin{aligned} \|e^{-t^2 L} f\|_{p_2} &= \Big\| \int_X p_{t^2}(x,y) f(y) d\mu(y) \Big\|_{p_2} \\ &\lesssim \int_X \|p_{t^2}(x,\cdot)\|_{p_2} |f(y)| d\mu(y). \end{aligned}$$

Using (ii) of Lemma 5.1 gives

$$\|p_{t^2}(x,\cdot)\|_{p_2} \lesssim \mu(B(x,t))^{-(1-1/p_2)} \lesssim t^{-\nu(1-1/p_2)}.$$

Hence,

$$\|e^{-t^2 L} f\|_{p_2} \lesssim t^{-\nu(1-1/p_2)} \|f\|_1. \tag{5.10}$$

Moreover, $\|e^{-t^2 L} f\|_{p_2} \lesssim \|f\|_{p_2}$. This together with (5.10) and interpolation implies (5.9).

We now return to the proof of (ii). Let $f \in \mathcal{M}^{\alpha_1, L}_{p_1, q}$. We first observe that

$$2t^2 L e^{-2t^2 L} f = 2e^{-t^2 L}[\Psi_t(L) f].$$

Hence, by (5.9) we have

$$\|2t^2 L e^{-2t^2 L} f\|_{p_2} \lesssim t^{-\nu(1-1/p_2)} \|\Psi_t(L) f\|_{p_1}$$

which implies

$$\begin{aligned} \|f\|^q_{\dot{B}^{\alpha_2, L}_{p_2, q}} &\approx \int_0^\infty (t^{-\alpha_2} \|2t^2 L e^{-2t^2 L} f\|_{p_2})^q \frac{dt}{t} \\ &\lesssim \int_0^\infty (t^{-\alpha_2 - \nu(1/p_1 - 1/p_2)} \|\Psi_t(L) f\|_{p_1})^q \frac{dt}{t} \\ &\lesssim \int_0^\infty (t^{-\alpha_2 - \nu(1/p_1 - 1/p_2)} \|\Psi_t(L) f\|_{p_1})^q \frac{dt}{t} \\ &\lesssim \|f\|^q_{\dot{B}^{\alpha_1, L}_{p_1, q}}. \end{aligned}$$

This completes the proof of (ii). □

Proposition 5.8. *Assume that L satisfies* (G). *Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. Then for each $k \in \mathbb{N}_+$, $p_{s,k}(\cdot, y) \in \mathcal{M}^{\alpha, L}_{p,q}$ and $p_{s,k}(y, \cdot) \in \mathcal{M}^{\alpha, L^*}_{p,q}$ for all $y \in X, s > 0$.*

Proof. We need to prove $p_{s,k}(\cdot,y)\in\mathcal{M}^{\alpha,L}_{p,q}$. The assertion $p_{s,k}(y,\cdot)\in\mathcal{M}^{\alpha,L^*}_{p,q}$ follows immediately by duality.

Note that $p_{s,k}(x,y)=L(tp_{s,k-1}(\cdot,y))(x)$. Since the heat kernels $p_{s,k}(x,y)$ and $p_{s,k-1}(x,y)$ satisfy Gaussian upper bounds, $p_{s,k}(\cdot,y)$ and $p_{s,k-1}(\cdot,y)$ satisfy (ii) in Definition 5.6. Moreover, by Lemma 5.5, $\|p_{s,k}(\cdot,y)\|_{\dot{B}^{\alpha,L}_{p,q}(X)}<\infty$. Hence, $p_{s,k}(\cdot,y)\in\mathcal{M}^{\alpha,L}_{p,q}$ for $1\le p,q<\infty$.

If $q=\infty$, using the fact that

$$\Psi_t(L)p_{s,k}(\cdot,y)=\frac{t^2s^k}{(t^2+s)^{k+1}}p_{t^2+s,k+1}(\cdot,y),$$

we arrive at

$$\|\Psi_t(L)p_{s,k}(\cdot,y)\|_p=\frac{t^2s^k}{(t^2+s)^{k+1}}\|p_{t^2+s,k+1}(\cdot,y)\|_p\lesssim\frac{t^2s^k}{(t^2+s)^{k+1}}V(y,\sqrt{t^2+s})^{-1/p'}$$

which implies that $\|\Psi_t(L)p_{s,k}(\cdot,y)\|_p\to 0$ as $t\to 0$ or $t\to\infty$. Hence, $p_{s,k}(\cdot,y)\in\mathcal{M}^{\alpha,L}_{p,q}$ for $1\le p<\infty$ and $q=\infty$.

If $p=\infty$, we need to verify that $\lim_{\tau\to 0}\|(I-e^{-\tau L})p_{s,k}(\cdot,y)\|_{\dot{B}^{\alpha,L}_{\infty,q}(X)}=0$. Indeed, we have

$$\begin{aligned}\Psi_t(L)(I-e^{-\tau L})p_{s,k}(\cdot,y)(x)&=\int_0^\tau\Psi_t(L)Le^{-uL}p_{s,k}(\cdot,y)du\\&=\int_0^\tau\frac{t^2s^k}{(u+s+t^2)^{k+1}}p_{u+s+t^2,k+2}(x,y)du.\end{aligned}\tag{5.11}$$

Therefore,

$$|\Psi_t(L)(I-e^{-\tau L})p_{s,k}(\cdot,y)(x)|\le C\int_0^\tau\frac{t^2s^k}{(s+t^2)^{k+1}}\frac{1}{V(y,\sqrt{s+t^2})}du=C\tau\frac{t^2s^k}{(s+t^2)^{k+1}}\frac{1}{V(y,\sqrt{s+t^2})}$$

as long as $\tau<s$. This implies

$$\lim_{\tau\to 0}\|\Psi_t(L)(I-e^{-\tau L})p_{s,k}(\cdot,y)\|_\infty=0.$$

Moreover, $\|\Psi_t(L)(I-e^{-\tau L})p_{s,k}(\cdot,y)\|_\infty\lesssim\|\Psi_t(L)p_{s,k}(\cdot,y)\|_\infty$. Hence, the dominated convergence theorem yields that

$$\lim_{\tau\to 0}\|(I-e^{-\tau L})p_{s,k}(\cdot,y)\|_{\dot{B}^{\alpha,L}_{\infty,q}(X)}=0,\ \text{for } 1\le q<\infty.$$

When $q=\infty$, we will show that

$$\lim_{\tau\to 0}\sup_{t>0}t^{-\alpha}\|\Psi_t(L)(I-e^{-\tau L})p_{s,k}(\cdot,y)\|_\infty=0.$$

Indeed, we have

$$\lim_{\tau\to 0}\sup_{t>0} t^{-\alpha}\|\Psi_t(L)(I-e^{-\tau L})p_{s,k}(\cdot,y)\|_\infty \le \lim_{\tau\to 0}\sup_{0<t<\sqrt{s}} \ldots + \lim_{\tau\to 0}\sup_{t\ge\sqrt{s}} \ldots.$$

Using (5.5), we have

$$\begin{aligned} t^{-\alpha}|\Psi_t(L)(I-e^{-\tau L})p_{s,k}(\cdot,y)(x)| &\le C\int_0^\tau \frac{t^{2-\alpha}s^k}{(u+s+t^2)^{k+1}}\frac{1}{V(y,\sqrt{u+s+t^2})}du \\ &\lesssim C\tau s^{1-\alpha}\frac{1}{V(y,\sqrt{s})} \end{aligned}$$

provided $t<\sqrt{s}$ and $\tau<s$. This implies

$$\lim_{\tau\to 0}\sup_{0<t<\sqrt{s}} t^{-\alpha}\|\Psi_t(L)(I-e^{-\tau L})p_{s,k}(\cdot,y)\|_\infty = 0.$$

Similarly,

$$\lim_{\tau\to 0}\sup_{t\ge\sqrt{s}} t^{-\alpha}\|\Psi_t(L)(I-e^{-\tau L})p_{s,k}(\cdot,y)\|_\infty = 0.$$

This completes our proof. □

As a direct consequence of Lemma 5.8, for $t>0$, $k\in\mathbb{N}_+$ and $f\in(\mathcal{M}_{p',q'}^{-\alpha,L^*})'$ we can define

$$(tL)^k e^{-tL}f(x) = \langle f, p_{t,k}(x,\cdot)\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the pairing between a linear functional in $(\mathcal{M}_{p',q'}^{-\alpha,L^*})'$ and the test function in $\mathcal{M}_{p',q'}^{-\alpha,L^*}$. This leads us to define the Besov spaces $\dot{B}_{p,q}^{\alpha,L}(X)$ as follows:

Definition 5.9. *Assume that L satisfies* (G). *Let $-1<\alpha<1$ and $1\le p,q\le\infty$. The Besov spaces $\dot{B}_{p,q}^{\alpha,L}(X)$ associated to the operator L are defined by*

$$\dot{B}_{p,q}^{\alpha,L}(X) = \{f\in(\mathcal{M}_{p',q'}^{-\alpha,L^*})' : \|f\|_{\dot{B}_{p,q}^{\alpha,L}(X)}<\infty\}.$$

We have the following simple result on Besov spaces $\dot{B}_{p,q}^{\alpha,L}(X)$.

Lemma 5.10. *Assume that L satisfies* (G). *Let $-1<\alpha<1$ and $1\le p,q\le\infty$. If $f\in\cup_{1<r<\infty}L^r(X)$ and $\|f\|_{\dot{B}_{p,q}^{\alpha,L}(X)}<\infty$ then $f\in\dot{B}_{p,q}^{\alpha,L}(X)$.*

Proof. Assume that $f \in L^r(X)$ for some $1 < r < \infty$. By spectral theory, it can be verified that

$$f(x) = C \int_0^\infty (t^2 L)^2 e^{-2t^2 L} f(x) \frac{dt}{t}$$

in $L^r(X)$.

Therefore, for any $\varphi \in \mathcal{M}_{p',q'}^{-\alpha,L^*}$, by Hölder's inequality we have

$$\begin{aligned}\langle f, \varphi \rangle &= \int_X f(x)\varphi(x)dx = C \int_X \int_0^\infty (t^2 L)^2 e^{-2t^2 L} f(x)\varphi(x) \frac{dt}{t} dx \\ &= C \int_X \int_0^\infty \Psi_t(L) f(x) \Psi_t(L^*)\varphi(x) \frac{dt}{t} dx \\ &\lesssim \|f\|_{\dot{B}_{p,q}^{\alpha,L}(X)} \|\varphi\|_{\dot{B}_{p',q'}^{-\alpha,L^*}}\end{aligned}$$

which implies that $f \in (\mathcal{M}_{p',q'}^{-\alpha,L^*})'$. □

It is worth to note that the result of Lemma 5.10 can be extended to the class of functions in $\mathcal{M}$.

Lemma 5.11. *Assume that L satisfies* (G)*. Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. If $f \in \mathcal{M}$ and $\|f\|_{\dot{B}_{p,q}^{\alpha,L}(X)} < \infty$ then $f \in \dot{B}_{p,q}^{\alpha,L}(X)$.*

Proof. The proof of this lemma is similar to that of [BDY, Proposition 4.2]. □

We now give the discrete characterization of the Besov spaces $\dot{B}_{p,q}^{\alpha,L}(X)$.

Proposition 5.12. *Assume that L satisfies* (G)*. Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. If $f \in (\mathcal{M}_{p',q'}^{-\alpha,L^*})'$ then the following statements are equivalent:*

(i) $f \in \dot{B}_{p,q}^{\alpha,L}(X)$;

(ii) $\Big(\sum_{j \in \mathbb{Z}} (2^{-j\alpha} \|\Psi_{2^j}(L) f\|_p)^q \Big)^{1/q} < \infty$.

Moreover, the sum in (ii) *and $\|f\|_{\dot{B}_{p,q}^{\alpha,L}(X)}$ are equivalent.*

Proof. The proof of this proposition is similar to that of [BDY, Proposition 4.3]. □

Note that for $t > 0$ and $k \in \mathbb{N}$, it can be verified that $(tL)^k e^{-tL}\varphi \in \mathcal{M}_{p,q}^{\alpha,L}$ whenever $\varphi \in \mathcal{M}_{p,q}^{\alpha,L}$. Moreover,

$$\|(tL)^k e^{-tL}\varphi\|_{\dot{B}_{p,q}^{\alpha,L}(X)} \lesssim \|\varphi\|_{\dot{B}_{p,q}^{\alpha,L}(X)}.$$

This leads us to define $(tL)^k e^{-tL}$ as a linear functional on $(\mathcal{M}^{\alpha,L}_{p,q})'$.

Definition 5.13. *Assume that L satisfies* (G). *Let $-1 < \alpha < 1$, $1 \le p, q \le \infty$, $s > 0$ and $k \in \mathbb{N}$. For each $f \in (\mathcal{M}^{-\alpha,L^*}_{p',q'})'$, we define $(sL)^k e^{-sL} f$ as a linear functional in $(\mathcal{M}^{-\alpha,L^*}_{p',q'})'$ by setting*

$$\langle (sL)^k e^{-sL} f, \varphi \rangle = \langle f, (sL^*)^k e^{-sL^*} \varphi \rangle$$

for all $\varphi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$.

We will show that our new Besov spaces still retain some basic properties as for the classical Besov spaces.

Proposition 5.14. *Assume that L satisfies* (G). *Let $-1 < \alpha < 1$, $1 \le p \le \infty$ and $1 \le q_1 \le q_2 \le \infty$. Then we have*

(i) $\dot{B}^{\alpha,L}_{p,q_1}(X) \subset \dot{B}^{\alpha,L}_{p,q_2}(X)$ *with continuous embedding;*

(ii) *Assume that X has volume growth at least polynomial, that is, $\mu(B(x,r)) \gtrsim r^\nu$ for some $\nu > 0$. If $1 \le p_1 \le p_2 \le \infty, -1 < \alpha_2 \le \alpha_1 < 1$ and $\alpha_1 - \nu/p_1 = \alpha_2 - \nu/p_2$ then*

$$\dot{B}^{\alpha_1,L}_{p_1,q}(X) \subset \dot{B}^{\alpha_2,L}_{p_2,q}(X) \text{ with continuous embedding.}$$

Proof. (i) Let $f \in \dot{B}^{\alpha,L}_{p,q_1}(X)$. We then have $f \in (\mathcal{M}^{-\alpha,L^*}_{p',q_1'})'$. Due to Proposition 5.7, $(\mathcal{M}^{-\alpha,L^*}_{p',q_1'})' \subset (\mathcal{M}^{-\alpha,L^*}_{p',q_2'})'$ and hence $f \in (\mathcal{M}^{-\alpha,L^*}_{p',q_2'})'$. Therefore, it suffices to show that

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q_2}(X)} \lesssim \|f\|_{\dot{B}^{\alpha,L}_{p,q_1}(X)}.$$

To do this we write

$$\Psi_t(L) f = 4 e^{-\frac{3t^2}{4} L} \Psi_{t/2}(L) f.$$

At this stage, we can use the same argument done in Proposition 5.7 of which we replace Lemma 5.4 by the following inequality

$$\|e^{-sL} g\|_p \lesssim \|e^{-tL} g\|_p \lesssim \|e^{-4sL} g\|_p, \quad s \le t \le 4s \tag{5.12}$$

for all $p \in [1, \infty]$ and $g \in L^p(X)$. This completes the proof of (i).

(ii) Thanks to Proposition 5.7, $(\mathcal{M}^{-\alpha_2,L^*}_{p_1',q'})' \subset (\mathcal{M}^{-\alpha_2,L^*}_{p_2',q'})'$. Hence, it suffices to show that if $f \in \dot{B}^{\alpha_1,L}_{p_1,q}(X)$ then we have

$$\|f\|_{\dot{B}^{\alpha_1,L}_{p_2,q}(X)} \lesssim \|f\|_{\dot{B}^{\alpha_1,L}_{p_1,q}(X)}$$

To do this, we write $\Psi_t(L)f = 4e^{-\frac{3t^2}{4}L}\Psi_{t/2}(L)f$ and then using the similar argument done in Proposition 5.7 together with (5.12) we get the desired estimate. □

5.2.2 Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ and Lipschitz spaces

In this section, we investigate the relationship between the Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ and the spaces of Lipschitz type.

Let $1 \le p, q \le \infty$ and $\alpha > 0$. A function $f \in L^p_{\mathrm{loc}}(X)$ is said to be in the Lipschitz-type space $L^s_{p,q}(X)$ if and only if

$$\|f\|_{L^s_{p,q}(X)} := \left[\sum_{k=-\infty}^{\infty} 2^{-ksq}\Big(\int_X \frac{1}{\mu(B(x,2^k))}\int_{B(x,2^k)} |f(x)-f(y)|^p d\mu(y)d\mu(x)\Big)^{p/q}\right]^{1/q} < \infty.$$

When X is an appropriate subset of $\mathbb{R}^n$, the spaces of Lipschitz type $L^s_{p,q}(X)$ was introduced in [Jo1, JW]. When X is a space of homogeneous type satisfying the reverse doubling property, the Lipschitz spaces was introduced in [MY].

Proposition 5.15. *Assume that L satisfies* (G), (C) *and* (H). *Let $1 \le p, q \le \infty$ and $0 < \alpha < \infty$. Then $L^\alpha_{p,q}(X) \cap L^r(X) \subset \dot{B}^{\alpha,L}_{p,q}(X)$ for any $1 < r < \infty$.*

Proof. By Lemma 5.10, it suffices to show that $\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} < \infty$. Indeed, for each $k \in \mathbb{Z}$, we have

$$\begin{aligned}
\|2^{2k}Le^{-2^{2k}L}f\|_p &= \Big\|\int_X p_{2^{2k},1}(x,y)(f(y)-f(x))d\mu(x)\Big\|_p \\
&\lesssim \Big(\int_X \Big|\int_X \frac{1}{\mu(B(x,2^k))}\exp\Big(-c\frac{d(x,y)^2}{2^{2k}}\Big)|f(x)-f(y)|d\mu(y)\Big|^p d\mu(x)\Big)^{1/p} \\
&\lesssim \sum_{j\ge 0} 2^{-j(\alpha+\epsilon)}\Big(\int_X \Big|\frac{1}{\mu(B(x,2^{k+j}))}\int_{B(x,2^{k+j})} |f(x)-f(y)|d\mu(y)\Big|^p d\mu(x)\Big)^{1/p}.
\end{aligned}$$

Hence, by Young's inequality we arrive at

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}}(2^{-k\alpha}\|2^{2k}Le^{-2^{2k}L}f\|_p)^q\Big)^{1/q}\\
&\quad\lesssim\left\{\sum_{k\in\mathbb{Z}}\Big[\sum_{j\geq 0}2^{-j\epsilon}2^{-(k+j)\alpha}\Big(\int_X\Big|\frac{1}{\mu(B(x,2^{k+j}))}\int_{B(x,2^{k+j})}|f(x)-f(y)|d\mu(y)\Big|^p d\mu(x)\Big)\Big]^{q/p}\right\}^{1/q}\\
&\quad\lesssim\Big(\sum_{j\geq 0}2^{-j\epsilon}\Big)\left(\sum_{k\in\mathbb{Z}}2^{-(k+j)\alpha q}\Big(\int_X\Big|\frac{1}{\mu(B(x,2^{k+j}))}\int_{B(x,2^{k+j})}|f(x)-f(y)|d\mu(y)\Big|^p d\mu(x)\Big)^{q/p}\right)^{1/q}\\
&\quad\lesssim\|f\|_{L^{\alpha}_{p,q}(X)}.
\end{aligned}$$

This completes our proof. □

Let $\beta\in(0,1)$. The Lipschitz space $L_\beta(X)$ is defined as the set of all continuous functions f satisfying

$$\|f\|_{L_\beta(X)}=\sup_{x\neq y}\frac{|f(x)-f(y)|}{d(x,y)^\beta}.$$

Note that if $f\in L_\beta(X)$ with compact support then $f\in L^p(X)$ for all $p\in[1,\infty]$.

We recall that the measure μ is said to satisfy the reverse doubling property if there exist positive constants κ and C such that

$$\lambda^\kappa\mu(B(x,r))\leq C\mu(B(x,\lambda r)) \tag{5.13}$$

for all $x\in X$, $r>0$ and $\lambda>1$.

Proposition 5.16. *Assume that L satisfies* (G), (C) *and* (H), *and (X,μ) satisfies (5.13). Let $f\in L_\beta(X)$ with compact support for some $\beta\in(0,1)$. Then $f\in\dot{B}^{\alpha,L}_{p,q}(X)$ for all $1\leq p,q\leq\infty$ and $-\kappa(1-1/p)<\alpha<\beta$.*

Proof. By Lemma 5.10, it suffices to show that $\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}<\infty$. Indeed, assume that $\operatorname{supp}\varphi\subset B:=B(x_0,r)$ for some $x_0\in X$ and $r>0$. Since $f\in L_\beta(X)$ with compact support, $f\in L^2(X)$. We will show that $\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}<\infty$ provided $1\leq p\leq\infty, 0<q\leq\infty$ and $-\kappa(1-1/p)<\alpha<\beta$.

Indeed, we have

$$\|f\|^q_{\dot{B}^{\alpha,L}_{p,q}(X)}=\int_0^r(t^{-\alpha}\|t^2Le^{-t^2L}f\|_p)^q\frac{dt}{t}+\int_r^\infty(t^{-\alpha}\|t^2Le^{-t^2L}f\|_p)^q\frac{dt}{t}:=A+B.$$

By Minkowski's inequality and (5.13), for $t \geq r$ we have

$$\begin{aligned}\|t^2 Le^{-t^2 L} f\|_p &\lesssim \int_{B(x_0,r)} \|p_{t^2,1}(\cdot,y)\|_p |f(y)| d\mu(y) \\ &\lesssim \frac{\|f\|_1}{\mu(B(y,t))^{1-1/p}} \lesssim \frac{\|f\|_1}{\mu(B(y,r))^{1-1/p}} \Big(\frac{r}{t}\Big)^{\kappa(1-1/p)}.\end{aligned}$$

Since $y \in B(x_0,r)$, $\mu(B(y,r)) \approx \mu(B)$. Hence,

$$\|t^2 Le^{-t^2 L} f\|_p \lesssim t^{-\kappa(1-1/p)}$$

which implies that $B < \infty$ whenever $-\kappa(1-1/p) < \alpha$.

To estimate A, we write

$$\begin{aligned}\|t^2 Le^{-t^2 L} f\|_p^p &= \int_X \Big| \int_{B(x_0,r)} p_{t^2,1}(x,y) f(y) d\mu(y) \Big|^p d\mu(x) \\ &= \int_{B(x_0,4r)} \Big| \int_{B(x_0,r)} p_{t^2,1}(x,y) f(y) d\mu(y) \Big|^p d\mu(x) \\ &\quad + \int_{X\setminus B(x_0,4r)} \Big| \int_{B(x_0,r)} p_{t^2,1}(x,y) f(y) d\mu(y) \Big|^p d\mu(x) \\ &:= I_1 + I_2.\end{aligned}$$

Since L satisfies (C), we have

$$\begin{aligned}I_1 &= \int_{B(x_0,4r)} \Big| \int_X p_{t^2,1}(x,y)(f(y)-f(x)) d\mu(y) \Big|^p d\mu(x) \\ &\lesssim \int_{B(x_0,4r)} \Big| \int_X \frac{1}{\mu(B(x,t))} \exp\Big(-c\frac{d(x,y)^2}{t^2}\Big) |f(y)-f(x)| d\mu(y) \Big|^p d\mu(x) \\ &\lesssim \int_{B(x_0,4r)} \Big| \int_X \frac{1}{\mu(B(x,t))} \exp\Big(-c\frac{d(x,y)^2}{t^2}\Big) d(x,y)^\beta \|f\|_{L_\beta(X)} d\mu(y) \Big|^p d\mu(x) \\ &\lesssim \int_{B(x_0,4r)} \Big| \int_X \frac{1}{\mu(B(x,t))} \exp\Big(-c'\frac{d(x,y)^2}{t^2}\Big) t^\beta \|f\|_{L_\beta(X)} d\mu(y) \Big|^p d\mu(x) \\ &\lesssim t^{p\beta}.\end{aligned}$$

We next consider the term I_2. We write

$$\begin{aligned}I_2 &\lesssim \int_{X\setminus B(x_0,4r)} \Big| \int_{B(x_0,4r)} \frac{1}{\mu(B(x,t))} \exp\Big(-c\frac{d(x,y)^2}{t^2}\Big) |f(y)| d\mu(y) \Big|^p d\mu(x) \\ &\lesssim \sum_{j=2}^\infty \int_{2^j r \leq d(x,x_0) \leq 2^{j+1} r} \Big| \int_{B(x_0,4r)} \frac{1}{\mu(B(x,d(x,y)))} \exp\Big(-c\frac{d(x,y)^2}{t^2}\Big) |f(y)| d\mu(y) \Big|^p d\mu(x) \\ &\lesssim \sum_{j=2}^\infty \frac{\|f\|_1^p}{\mu(x_0,2^j r)^{p-1}} \Big(\frac{t}{2^j r}\Big)^{p\beta} \\ &\lesssim t^{p\beta}.\end{aligned}$$

From the estimates of I_1 and I_2, we have $\|t^2 Le^{-t^2 L} f\|_p \lesssim t^\beta$ which implies that $A < \infty$. This completes our proof. $\square$

5.3 Molecular decompositions of Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$

5.3.1 Molecular decompositions

We establish the following Calderón reproducing formula in Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ which plays an essential role in the sequel.

Theorem 5.17. *Assume that L satisfies* (G). *Let $1 \le p, q \le \infty$, $-1 < \alpha < 1$ and $M \ge 1$. Then for $f \in \dot{B}^{\alpha,L}_{p,q}(X)$, we have*

$$f(x) = \frac{1}{(M-1)!} \int_0^\infty (tL)^M e^{-tL} f(x) \frac{dt}{t} \tag{5.14}$$

in $(\mathcal{M}^{-\alpha,L^*}_{p',q'})'$.

Proof. We will prove this theorem by using induction on M.

We temporarily assume that (5.14) is true for $M = 1$. We will provide the proof later.

Assume that (5.14) holds for $M = k$. We need to prove (5.14) for $M = k+1$. In this situation, by using integration by parts we obtain that

$$\int_\epsilon^K (tL)^{k+1} e^{-tL} f(x) \frac{dt}{t} = -(KL)^k e^{-KL} f(x) + (\epsilon L)^k e^{-\epsilon L} f(x) + k \int_\epsilon^K (tL)^k e^{-tL} f(x) \frac{dt}{t}. \tag{5.15}$$

We need the following auxiliary lemma whose proof will be given later.

Lemma 5.18. *Assume that L satisfies* (G). *Let $1 \le p, q \le \infty$, $-1 < \alpha < 1$ and $k \in \mathbb{N}_+$. For $f \in \dot{B}^{\alpha,L}_{p,q}(X)$, we have*

(i) $\lim_{\epsilon \to 0} (\epsilon L)^k e^{-\epsilon L} f(x) = 0$ *in* $(\mathcal{M}^{-\alpha,L^*}_{p',q'})'$;

(ii) $\lim_{K \to \infty} (KL)^k e^{-KL} f(x) = 0$ *in* $(\mathcal{M}^{-\alpha,L^*}_{p',q'})'$.

Letting $\epsilon \to 0$ and $K \to \infty$ in (5.15), thanks to Lemma 5.18, we get that

$$\lim_{\epsilon\to 0}\lim_{K\to\infty}\int_{\epsilon}^{K}(tL)^{k+1}e^{-tL}f(x)\frac{dt}{t} = k\lim_{\epsilon\to 0}\lim_{K\to\infty}\int_{\epsilon}^{K}(tL)^{k}e^{-tL}f(x)\frac{dt}{t}$$
$$= k!f(x)$$

in $(\mathcal{M}^{-\alpha,L^*}_{p',q'})'$. □

Proof of Lemma 5.18: (i)

Since

$$\langle(\epsilon L)^k e^{-\epsilon L}f,\varphi\rangle = \langle f,(\epsilon L^*)^k e^{-\epsilon L^*}\varphi\rangle, \quad \text{for all } \varphi\in\mathcal{M}^{-\alpha,L^*}_{p',q'},$$

we need to show that $\lim_{\epsilon\to 0}\|(\epsilon L^*)^k e^{-\epsilon L^*}\varphi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}(X)} = 0$. Observe that

$$\Psi_t(L^*)(\epsilon L^*)^k e^{-\epsilon L^*}\varphi = \frac{\epsilon^k t^2}{(\epsilon+t^2)^{k+1}}((\epsilon+t^2)L^*)^k e^{-(\epsilon+t^2)L^*}\varphi.$$

Hence

$$\|\Psi_t(L^*)(\epsilon L^*)^k e^{-\epsilon L^*}\varphi\|_{p'} \le C\frac{\epsilon^k t^2}{(\epsilon+t^2)^{k+1}}\|\varphi\|_{p'}$$

which implies

$$\lim_{\epsilon\to 0}\|\Psi_t(L^*)(\epsilon L^*)^k e^{-\epsilon L}\varphi\|_{p'} = 0.$$

On the other hand,

$$\|\Psi_t(L^*)(\epsilon L^*)^k e^{-\epsilon L^*}\varphi\|_{p'} = \|(\epsilon L^*)^k e^{-\epsilon L^*}\varphi\|_{p'} \le C\|\Psi_t(L^*)\varphi\|_{p'}.$$

At this stage, by using the dominated convergence theorem we arrive at

$$\lim_{\epsilon\to 0}\|(\epsilon L^*)^k e^{-\epsilon L^*}\varphi\|_{\dot{B}^{-\epsilon,L^*}_{p',q'}(X)} = 0.$$

(ii) We will claim that

$$\lim_{K\to\infty}\|\Psi_t(L^*)(KL^*)^k e^{-KL^*}\varphi\|_{p'} = 0, \varphi\in\mathcal{M}^{-\alpha,L^*}_{p',q'}. \tag{5.16}$$

Once (5.16) is proved, $\lim_{K\to\infty}\|(KL^*)^k e^{-KL^*}\varphi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}(X)} = 0$ follows immediately by using the fact that

$$\|\Psi_t(L^*)(KL^*)^k e^{-KL^*}\varphi\|_{p'} = \|(KL^*)^k e^{-KL^*}\varphi\|_{p'} \le C\|\Psi_t(L^*)\varphi\|_{p'}$$

and the same argument done in (i).

To prove (5.16), we write

$$\|\Psi_t(L^*)(KL^*)^k e^{-KL^*}\varphi\|_{p'} = \frac{t^2K^k}{(t^2+K)^{k+1}}\|((t^2+K)L^*)^{k+1}e^{-(t^2+K)L^*}\phi\|_{p'} \le \frac{C}{K}\|\phi\|_{p'}$$

which implies (5.16). This completes our proof.

□

Complete the proof of Theorem 5.17: To complete the proof of Theorem 5.17, we now show that (5.14) is true for $M = 1$. Indeed, in this situation we have

$$\int_\epsilon^K tLe^{-tL}f(x)\frac{dt}{t} = -e^{-KL}f(x) + e^{-\epsilon L}f(x).$$

By the same argument done in Lemma 5.18, we can show that

$$\lim_{K\to\infty} e^{-KL}f = 0 \ \text{ in } (\mathcal{M}^{-\alpha,L^*}_{p',q'})'.$$

It remains to prove that

$$\lim_{\epsilon\to 0} e^{-\epsilon L}f = f \ \text{ in } (\mathcal{M}^{-\alpha,L^*}_{p',q'})'.$$

The proof of this assertion is close to that of [BDY, Proposition 3.3]. Since

$$\langle f - e^{-\epsilon L}f, \varphi\rangle = \langle f, (I - e^{-\epsilon L^*})\varphi\rangle, \quad \varphi \in \mathcal{M}^{-\alpha,L^*}_{p',q'},$$

it suffices to show that $\lim_{\epsilon\to 0}\|(I - e^{-\epsilon L^*})\varphi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}(X)} = 0$.

If $p' = \infty$, the assertion $\lim_{\epsilon\to 0}\|(I - e^{-\epsilon L^*})\varphi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}(X)} = 0$ follows from the definition of the test function.

If $1 \le p' < \infty$ and $1 \le q' < \infty$, it is easy to see that

$$\begin{aligned}
\|\Psi_t(L^*)(I - e^{-\epsilon L^*})\varphi\|_{p'} &= \Big\|\int_0^\epsilon \Psi_t(L^*)Le^{-sL}\varphi ds\Big\|_{p'} \\
&= \Big\|\int_0^\epsilon \frac{t^2}{(t^2+s)^2}((s+t^2)L)^2 e^{-(t^2+s)L}\varphi ds\Big\|_{p'} \\
&\le \int_0^\epsilon \Big\|\frac{t^2}{(t^2+s)^2}((s+t^2)L)^2 e^{-(t^2+s)L}\varphi\Big\|_{p'} ds \\
&\lesssim \int_0^\epsilon \|\varphi\|_{p'} ds \lesssim \epsilon\|\varphi\|_{p'}
\end{aligned}$$

which implies $\lim_{\epsilon\to 0}\|\Psi_t(L^*)(I-e^{-\epsilon L^*})\varphi\|_{p'}=0$. On the other hand,

$$\|\Psi_t(L^*)(I-e^{-\epsilon L^*})\varphi\|_{p'}\lesssim\|\Psi_t(L^*)\varphi\|_{p'}.$$

This in combination with the dominated convergence theorem implies that

$$\lim_{\epsilon\to 0}\|(I-e^{-\epsilon L^*})\varphi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}(X)}=0.$$

If $1\le p'<\infty$ and $q'=\infty$, we have

$$\|(I-e^{-\epsilon L^*})\varphi\|_{\dot{B}^{-\alpha,L^*}_{p',\infty}(X)}=\sup_{t>0}t^{-\alpha}\|\Psi_t(L^*)(I-e^{-\epsilon L^*})\varphi\|_{p'}.$$

Using (5.7), we have

$$t^{-\alpha}\|\Psi_t(L^*)(I-e^{-\epsilon L^*})\varphi\|_{p'}\lesssim t^{-\alpha}\|\Psi_t(L^*)\varphi\|_{p'}\to 0$$

as $t\to 0$ or $t\to\infty$. Hence, we can pick two positive constants $c_1>c_2>0$ independent of ϵ so that

$$\sup_{t>0}t^{-\alpha}\|\Psi_t(L^*)(I-e^{-\epsilon L^*})\varphi\|_{p'}\lesssim\sup_{c_1<t<c_2}t^{-\alpha}\|\Psi_t(L^*)(I-e^{-\epsilon L^*})\varphi\|_{p'}\lesssim\|(I-e^{-\epsilon L^*})\varphi\|_{p'}$$

which implies

$$\lim_{\epsilon\to 0}\|(I-e^{-\epsilon L^*})\varphi\|_{\dot{B}^{-\alpha,L^*}_{p',\infty}(X)}=0.$$

This completes our proof.

□

We are now in the position to establish the molecular decomposition of the functions in $\dot{B}^{\alpha,L}_{p,q}(X)$. We first describe the notion of a $(L,M,\alpha,p,,\epsilon)$ molecule.

Definition 5.19. *Let $1\le p\le\infty$, $-1<\alpha<1$, $\epsilon>0$ and $M\in\mathbb{N}_+$. A function m is said to be a (L,M,α,p,ϵ) molecule if there exists a dyadic cube $Q\in\mathcal{D}$ so that*

(i) $m=L^Mb$;

(ii) $|L^kb(x)|\le\dfrac{\ell(Q)^{2(M-k)+\alpha}}{\mu(Q)^{1/p}}\Big(1+\dfrac{d(x,x_Q)}{\ell(Q)}\Big)^{-(N+n+\epsilon)}$, $k=0,1,\dots,M$, *for all* $x\in X$.

Theorem 5.20. *Assume that L satisfies* (G). *Let $1 \le p, q \le \infty$, $-1 < \alpha < 1$, $\epsilon > 0$ and $M \in \mathbb{N}$. Then for each $f \in \dot{B}^{\alpha,L}_{p,q}(X)$, there exist a sequence of coefficients $0 \le s_Q < \infty$, and a sequence m_Q of $(L, M, \alpha, p, \epsilon)$ molecules, where Q ranges over the dyadic cubes, such that*

$$f = \sum_{Q \in \mathcal{D}} s_Q m_Q \quad \text{in } (\mathcal{M}^{-\alpha,L^*}_{p',q'})',$$

and

$$\Big[\sum_{k\in\mathbb{Z}}\Big[\sum_{Q\in\mathcal{D}_k}|s_Q|^p\Big]^{q/p}\Big]^{1/q} \approx \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}. \tag{5.17}$$

Proof. Let $f \in \dot{B}^{\alpha,L}_{p,q}(X)$. By Theorem 5.17 we have

$$f(x) = C_M \int_0^\infty (t^2L)^M e^{-t^2L}\Psi_t(L)f(x)\frac{dt}{t}$$

which converges in $(\mathcal{M}^{-\alpha,L^*}_{p',q'})'$.

We then have

$$\begin{aligned} f(x) &= C_M \sum_{k\in\mathbb{Z}} \int_{2^{k-1}}^{2^k} (t^2L)^M e^{-t^2L}\Psi_t(L)f(x)\frac{dt}{t} \\ &= C_M \sum_{k\in\mathbb{Z}} \int_{2^{k-1}}^{2^k} 2^{-2(k-1)}t^2(t^2L)^M e^{-(2t^2-2^{2(k-1)})L}\Psi_{2^{k-1}}(L)f(x)\frac{dt}{t} \\ &= C_M \sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_k} \int_{2^{k-1}}^{2^k} 2^{-2(k-1)}t^2(t^2L)^M e^{-(2t^2-2^{2(k-1)})L}[\Psi_{2^{k-1}}(L)f.\chi_Q](x)\frac{dt}{t}. \end{aligned}$$

For each $k \in \mathbb{Z}$ and $Q \in \mathcal{D}_k$, we set

$$s_Q = 2^{-(k-1)\alpha}\Big(\int_Q |\Psi_{2^{k-1}}(L)f(y)|^p d\mu(y)\Big)^{1/p}$$

and

$$m_Q = \frac{1}{s_Q}\int_{2^{k-1}}^{2^k} 2^{-2(k-1)}t^2(t^2L)^M e^{-(2t^2-2^{2(k-1)})L}[\Psi_{2^{k-1}}(L)f.\chi_Q](x)\frac{dt}{t}.$$

It is clear that $f = \sum_{Q\in\mathcal{D}} s_Q m_Q$ in $(\mathcal{M}^{-\alpha,L^*}_{p',q'})'$. We now check (5.17). Indeed, we have

$$\begin{aligned} \sum_{Q\in\mathcal{D}_k}|s_Q|^p &= \sum_{Q\in\mathcal{D}_k} 2^{-(k-1)\alpha p}\int_Q |\Psi_{2^{k-1}}(L)f(y)|^p d\mu(y) \\ &= 2^{-(k-1)\alpha p}\int_X |\Psi_{2^{k-1}}(L)f(y)|^p d\mu(y). \end{aligned}$$

This implies that

$$\sum_{Q\in\mathcal{D}_k}|s_Q|^p \lesssim \int_{2^{k-2}}^{2^{k-1}} t^{-\alpha p}\int_X |\Psi_{2^{k-1}}(L)f(y)|^p d\mu(y)\frac{dt}{t}.$$

Hence

$$\Big[\sum_{Q\in\mathcal{D}_k}|s_Q|^p\Big]^{q/p}\lesssim\int_{2^{k-2}}^{2^{k-1}}\Big[t^{-\alpha}\Big(\int_X|\Psi_{2^{k-1}}(L)f(y)|^p d\mu(y)\Big)^{1/p}\Big]^q\frac{dt}{t}. \tag{5.18}$$

Note that for $2^{k-2}\le t\le 2^{k-1}$, by Lemma 5.4 we have

$$\|\Psi_{2^{k-1}}(L)f\|_p\lesssim\|\Psi_t(L)f\|_p.$$

This together with (5.18) implies that

$$\Big[\sum_{k\in\mathbb{Z}}\Big[\sum_{Q\in\mathcal{D}_k}|s_Q|^p\Big]^{q/p}\Big]^{1/q}\lesssim\Big[\sum_{k\in\mathbb{Z}}\int_{2^{k-2}}^{2^{k-1}}\Big(t^{-\alpha}\|\Psi_t(L)f(y)\|_p\Big)^q\frac{dt}{t}\Big]^{1/q}:=\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}.$$

Moreover, for $2^{k-1}\le t\le 2^k$, by Lemma 5.4 we have

$$\|\Psi_t(L)f\|_p\lesssim\|\Psi_{2^{k-1}}(L)f\|_p.$$

Hence,

$$\begin{aligned}\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}&=\Big[\sum_{k\in\mathbb{Z}}\int_{2^{k-1}}^{2^k}\Big(t^{-\alpha}\|\Psi_{2^{k-1}}(L)f(y)\|_p\Big)^q\frac{dt}{t}\Big]^{1/q}\\&\lesssim\Big[\sum_{k\in\mathbb{Z}}\Big(2^{-k\alpha}\|\Psi_{2^{k-1}}(L)f(y)\|_p\Big)^q\Big]^{1/q}\\&\lesssim\Big[\sum_{k\in\mathbb{Z}}\Big(2^{-k\alpha p}\sum_{Q\in\mathcal{D}_k}\int_Q\Big|\Psi_{2^{k-1}}(L)f(y)\Big|^p d\mu(y)\Big)^{q/p}\Big]^{1/q}\\&\lesssim\Big[\sum_{k\in\mathbb{Z}}\Big[\sum_{Q\in\mathcal{D}_k}|s_Q|^p\Big]^{q/p}\Big]^{1/q}.\end{aligned}$$

This proves (5.17).

We now claim that for each $Q\in\mathcal{D}_k, k\in\mathbb{Z}$, m_Q is a multiple of a (L,M,α,p,ϵ) molecule with a harmless multiple constant. Indeed, we have $m_Q=L^M b_Q$ with

$$b_Q=\frac{1}{s_Q}\int_{2^{k-1}}^{2^k}2^{-2(k-1)}t^2(t^2L)^M e^{-(2t^2-2^{2(k-1)})L}[\Psi_{2^{k-1}}(L)f.\chi_Q](x)\frac{dt}{t}.$$

For each $m=0,1,\dots,M$, we have

$$L^m b_Q(x)=\frac{1}{s_Q}\int_{2^{k-1}}^{2^k}\int_Q\frac{t^{2(M-m+1)}}{2^{-2(k-1)}}\Big(\frac{t^2}{2t^2-2^{2(k-1)}}\Big)^m p_{2t^2-2^{2(k-1)},m}(x,y)\Psi_{2^{k-1}}(L)f(y)d\mu(y)\frac{dt}{t}.$$

Using the Gaussian upper bounds for $p_{2t^2-2^{2(k-1)},m}(x,y)$, the doubling property and the fact

that $t^2 \approx 2^{-2(k-1)} \approx 2t^2 - 2^{2(k-1)}$ whenever $t \in [2^{k-1}, 2^k]$, we obtain that for all $y \in Q$,

$$\begin{aligned}\frac{t^{2(M-m+1)}}{2^{-2(k-1)}}\Big(\frac{t^2}{2t^2-2^{2(k-1)}}\Big)^m p_{2t^2-2^{2(k-1)},m}(x,y) &\lesssim 2^{2k(M-m)}\frac{1}{\mu(B(y,t))}\Big(1+\frac{d(x,y)}{t}\Big)^{-N-n-\epsilon}\\ &\lesssim 2^{2k(M-m)}\frac{1}{\mu(Q)}\Big(1+\frac{d(x,x_Q)}{\ell(Q)}\Big)^{-N-n-\epsilon}.\end{aligned}$$

This together with Hölder's inequality implies

$$\begin{aligned}|L^m b_Q(x)| &\lesssim \frac{2^{2k(M-m)}}{s_Q}\frac{1}{\mu(Q)}\Big(1+\frac{d(x,x_Q)}{\ell(Q)}\Big)^{-N-n-\epsilon}\int_{2^{k-1}}^{2^k}\int_Q |\Psi_{2^{k-1}}(L)f(y)|d\mu(y)\frac{dt}{t}\\ &\lesssim \frac{2^{2k(M-m)}}{s_Q}\frac{1}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{\ell(Q)}\Big)^{-N-n-\epsilon}\Big(\int_Q |\Psi_{2^{k-1}}(L)f(y)d\mu(y)|^p\Big)^{1/p}\\ &\lesssim \frac{2^{2k(M-m)}}{s_Q}\frac{1}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{\ell(Q)}\Big)^{-N-n-\epsilon}2^{(k-1)\alpha}s_Q\\ &\lesssim \frac{\ell(Q)^{2(M-m)+\alpha}}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{\ell(Q)}\Big)^{-N-n-\epsilon}.\end{aligned} \tag{5.19}$$

This completes our proof. □

Note that in the definition of $(L, M, \alpha, p, \epsilon)$ molecules, we do not require the cancellation property. However, this property will hold if L satisfies the conservation property (C).

Lemma 5.21. *Assume that L satisfies* (G) *and* (C). *If m is a $(L, M, \alpha, p, \epsilon)$ molecule for $1 \le p \le \infty$, $-1 < \alpha < 1$, $\epsilon > 0$ and $M \ge 1$ then*

$$\int_X m(x)d\mu(x) = 0.$$

Proof. The proof of this lemma is similar to that of [HLMMY, Lemma 9.1]. For the sake of completeness, we provide the proof here.

Note that since m is a $(L, M, \alpha, p, \epsilon)$ molecule, $m \in L^1(X)$ and hence $e^{-tL}m \in L^1(X)$ for all $t > 0$. Observe that $(I+L)^{-1}m = \int_0^\infty e^{-t}e^{-tL}m(x)dt$. We then have

$$\begin{aligned}\int_X (I+L)^{-1}m(x)d\mu(x) &= \int_X\int_0^\infty e^{-t}e^{-tL}m(x)dtd\mu(x)\\ &= \int_X\int_0^\infty\int_X e^{-t}p_t(x,y)m(y)d\mu(y)dtd\mu(x)\\ &= \int_X m(y)d\mu(y).\end{aligned} \tag{5.20}$$

By the definition of $(L, M, \alpha, p, \epsilon)$ molecules, $m = L^M b$ with b satisfying (ii) in Definition 5.19. Set $m_1 = L^{M-1}b$. We then have $m = Lm_1$. Using (5.20), we obtain that

$$\int_X m(x)d\mu(x) = \int_X (I+L)^{-1}m(x)d\mu(x) = \int_X (I+L)^{-1}Lm_1(x)d\mu(x)$$
$$= \int_X m_1(x)d\mu(x) - \int_X (I+L)^{-1}m_1(x)d\mu(x).$$

By the argument used in (5.20), we arrive at

$$\int_X m_1(x)d\mu(x) = \int_X (I+L)^{-1}m_1(x)d\mu(x)$$

which implies $\int_X m(x)d\mu(x) = 0$. □

In Theorem 5.20, we proved that each function in Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$, $1 \le p, q \le \infty$ and $-1 < \alpha < 1$, can be decomposed into a linear combination of molecules. Conversely, we will claim that any molecular decomposition belongs to the Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$. In this section, we consider the case $1 \le p, q \le \infty$ and $-1 < \alpha < 0$. The case $1 \le p, q \le \infty$ and $\alpha \ge 0$ should require some extra conditions and will be investigated in the next subsection.

Theorem 5.22. *Assume that L satisfies* (G). *Let $-1 < \alpha < 0$, $1 \le p, q \le \infty$ and $\epsilon > 0$ and let $M > N + n + \epsilon$. Assume that*

$$f = \sum_{Q \in \mathcal{D}} s_Q m_Q \quad \text{in } (\mathcal{M}^{-\alpha,L^*}_{p',q'})',$$

where $\{s_Q\}_Q$ is a sequence of nonnegative numbers and $\{m_Q\}$ is a sequence of $(L, M, \alpha, p, \epsilon)$ molecules. Then we have

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} \lesssim \Big[\sum_{k\in\mathbb{Z}}\Big[\sum_{Q\in\mathcal{D}_k}|s_Q|^p\Big]^{q/p}\Big]^{1/q} \tag{5.21}$$

To prove Theorem 5.22, we need the following auxiliary results.

Proposition 5.23. *Assume that L satisfies* (G). *Let $-1 < \alpha < 0$, $1 \le p, q \le \infty$ and $M \in \mathbb{N}_+$. For each $(L, M, \alpha, p, \epsilon)$ molecule m_Q associated to the dyadic cube $Q \in \mathcal{D}_\eta$, we have*

(i) $|\Psi_{2^k}(L)m_Q(x)| \lesssim \dfrac{2^{\eta\alpha}}{\mu(Q)^{1/p}}\Big(1 + \dfrac{d(x, x_Q)}{2^\eta}\Big)^{-(N+n+\epsilon)}$, *for all $k \le \eta$;*

(ii) $|\Psi_{2^k}(L)m_Q(x)| \lesssim \dfrac{2^{2M(\eta-k)}2^{\eta\alpha}}{\mu(Q)^{1/p}}\Big(1 + \dfrac{d(x, x_Q)}{2^k}\Big)^{-(N+n+\epsilon)}$, *for all $\eta < k$.*

Proof. (i) Using the Gaussian upper bounds for the heat kernels of $\Psi_{2^k}(L)$, we have

$$\begin{aligned}
|\Psi_{2^k}(L)m_Q(x)| &\lesssim \int_X \frac{1}{\mu(B(x,2^k))}\exp\Big(-c\frac{d(x,y)^2}{2^{2k}}\Big)|m_Q(y)|d\mu(y)\\
&\lesssim \frac{2^{\eta\alpha}}{\mu(Q)^{1/p}}\int_X \frac{1}{\mu(B(x,2^k))}\exp\Big(-c\frac{d(x,y)^2}{2^{2k}}\Big)\Big(1+\frac{d(y,x_Q)}{2^\eta}\Big)^{-(N+n+\epsilon)}d\mu(y)\\
&\lesssim \frac{2^{\eta\alpha}}{\mu(Q)^{1/p}}\int_{d(x,y)\le d(x,x_Q)/2}\cdots+\frac{2^{\eta\alpha}}{\mu(Q)^{1/p}}\int_{d(x,y)>d(x,x_Q)/2}\cdots\\
&:= I_1+I_2.
\end{aligned}$$

Note that $d(y,x_Q)\approx d(x,x_Q)$ whenever $d(x,y)\le d(x,x_Q)/2$. Hence,

$$I_1 \lesssim \frac{2^{\eta\alpha}}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{2^\eta}\Big)^{-(N+n+\epsilon)}.$$

On the other hand, when $d(x,y)>d(x,x_Q)/2$, we obtain that

$$\begin{aligned}
\exp\Big(-c\frac{d(x,y)^2}{2^{2k}}\Big) &\lesssim \exp\Big(-c'\frac{d(x,y)^2}{2^{2k}}\Big)\Big(1+\frac{d(x,x_Q)}{2^k}\Big)^{-(N+n+\epsilon)}\\
&\lesssim \exp\Big(-c'\frac{d(x,y)^2}{2^{2k}}\Big)\Big(1+\frac{d(x,x_Q)}{2^\eta}\Big)^{-(N+n+\epsilon)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
I_2 &\lesssim \frac{2^{\eta\alpha}}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{2^\eta}\Big)^{-(N+n+\epsilon)}\int_X \frac{1}{\mu(B(x,2^k))}\exp\Big(-c'\frac{d(x,y)^2}{2^{2k}}\Big)d\mu(y)\\
&\lesssim \frac{2^{\eta\alpha}}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{2^\eta}\Big)^{-(N+n+\epsilon)}.
\end{aligned}$$

This completes the proof of (i).

(ii) Using $m_Q = L^M b_Q$ to give

$$\begin{aligned}
|\Psi_{2^k}(L)m_Q(x)| = |2^{2k}L^{M+1}e^{-2^{2k}L}b_Q(x)| &\lesssim \int_X \frac{2^{-2Mk}}{\mu(B(x,2^k))}\exp\Big(-c\frac{d(x,y)^2}{2^{2k}}\Big)|b_Q(y)|d\mu(y)\\
&\lesssim \frac{2^{2M(\eta-k)}2^{\eta\alpha}}{\mu(Q)^{1/p}}\int_X \frac{1}{\mu(B(x,2^k))}\exp\Big(-c\frac{d(x,y)^2}{2^{2k}}\Big)\Big(1+\frac{d(y,x_Q)}{2^\eta}\Big)^{-(N+n+1)}d\mu(y)\\
&\lesssim \frac{2^{2M(\eta-k)}2^{\eta\alpha}}{\mu(Q)^{1/p}}\int_{d(x,y)\le d(x,x_Q)/2}\cdots+\frac{2^{2M(\eta-k)}2^{\eta\alpha}}{\mu(Q)^{1/p}}\int_{d(x,y)>d(x,x_Q)/2}\cdots.
\end{aligned}$$

At this stage, we can apply the same argument used in (i) to obtain that

$$|\Psi_{2^k}(L)m_Q(x)| \lesssim \frac{2^{2M(\eta-k)}2^{\eta\alpha}}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{2^k}\Big)^{-(N+n+\epsilon)}.$$

This completes the proof of (ii). □

The following proposition is inspired from [FJ1]. However, since we are working on a more general setting, some significant modifications are required.

Proposition 5.24. (i) *For $\eta \le k$, we have*

$$\Big\| \sum_{Q\in\mathcal{D}_\eta} s_Q \Psi_{2^k}(L) m_Q \Big\|_p \lesssim 2^{\eta\alpha} 2^{(2M-N-n-\epsilon)(\eta-k)} \Big(\sum_{Q\in\mathcal{D}_\eta} |s_Q|^p \Big)^{1/p}.$$

(ii) *For $k < \eta$, we have*

$$\Big\| \sum_{Q\in\mathcal{D}_\eta} s_Q \Psi_{2^k}(L) m_Q \Big\|_p \lesssim 2^{\eta\alpha} \Big(\sum_{Q\in\mathcal{D}_\eta} |s_Q|^p \Big)^{1/p}.$$

Proof. By Proposition 5.23, we have

$$\begin{aligned}\Big\| \sum_{Q\in\mathcal{D}_\eta} s_Q \Psi_{2^k}(L) m_Q \Big\|_p^p &= \sum_{B\in\mathcal{D}_\eta} \int_B \Big| \sum_{Q\in\mathcal{D}_\eta} s_Q \Psi_{2^k}(L) m_Q(x) \Big|^p d\mu(x) \\ &\lesssim \sum_{B\in\mathcal{D}_\eta} \int_B \Big| \sum_{Q\in\mathcal{D}_\eta} s_Q \frac{2^{2M(\eta-k)} 2^{\eta\alpha}}{\mu(Q)^{1/p}} \Big(1 + \frac{d(x,x_Q)}{2^k}\Big)^{-(N+n+\epsilon)} \Big|^p d\mu(x).\end{aligned}$$

Observe that for $x \in B$ we have

$$1 + \frac{d(x,x_Q)}{2^k} \approx 1 + \frac{d(x_B,x_Q)}{2^k}$$

which implies that

$$\Big\| \sum_{Q\in\mathcal{D}_\eta} s_Q \Psi_{2^k}(L) m_Q \Big\|_p^p \lesssim \sum_{B\in\mathcal{D}_\eta} \int_B \Big| \sum_{Q\in\mathcal{D}_\eta} s_Q \frac{2^{2M(\eta-k)} 2^{\eta\alpha}}{\mu(Q)^{1/p}} \Big(1 + \frac{d(x,x_Q)}{2^k}\Big)^{-(N+n+\epsilon)} \Big|^p d\mu(x).$$

Using the fact that

$$\mu(B) \lesssim \mu(Q) \Big(1 + \frac{d(x_B,x_Q)}{2^\eta}\Big)^N \lesssim 2^{N(k-\eta)} \mu(Q) \Big(1 + \frac{d(x_B,x_Q)}{2^k}\Big)^N,$$

we conclude that

$$\begin{aligned}\Big\| \sum_{Q\in\mathcal{D}_\eta} s_Q \Psi_{2^k}(L) m_Q \Big\|_p^p &\lesssim \sum_{B\in\mathcal{D}_\eta} \Big| \sum_{Q\in\mathcal{D}_\eta} s_Q 2^{(2M-N)(\eta-k)} 2^{\eta\alpha} \Big(1 + \frac{d(x_B,x_Q)}{2^k}\Big)^{-(n+\epsilon)} \Big|^p \\ &\lesssim \sum_{j\in\mathbb{Z}} \sum_{\substack{B\in\mathcal{D}_\eta \\ B\cap S_j(Q)\neq\emptyset}} \Big| \sum_{Q\in\mathcal{D}_\eta} s_Q 2^{(2M-N)(\eta-k)} 2^{\eta\alpha} \Big(1 + \frac{d(x_B,x_Q)}{2^k}\Big)^{-(n+\epsilon)} \Big|^p,\end{aligned}$$

where $S_j(Q) = 2^{j+1}Q \backslash 2^j Q$ if $j > 0$ and $S_j(Q) = Q$ if $j = 0$.

This together with Remark 5.3 (ii) gives

$$\Big\| \sum_{Q\in\mathcal{D}_\eta} s_Q \Psi_{2^k}(L) m_Q \Big\|_p^p \lesssim \sum_{j\in\mathbb{Z}} 2^{jn} \Big| \sum_{Q\in\mathcal{D}_\eta} s_Q 2^{(2M-N)(\eta-k)} 2^{\eta\alpha} \Big(1 + \frac{2^{j+\eta}}{2^k}\Big)^{-(n+\epsilon)} \Big|^p.$$

By Remark 5.3 (iii), we can assume that the set $\mathcal{D}_\eta$ is countable. Hence we can write

$$\Big\| \sum_{Q\in\mathcal{D}_\eta} s_Q \Psi_{2^k}(L) m_Q \Big\|_p^p \lesssim \sum_{j\in\mathbb{Z}} \Big| \sum_i s_i 2^{(2M-N)(\eta-k)} 2^{\eta\alpha} (1+2^{\eta-k})^{-(n+\epsilon)} 2^{-\epsilon j} \Big|^p$$
$$\lesssim \sum_{j\in\mathbb{Z}} \Big| \sum_i s_i 2^{(2M-N-n-\epsilon)(\eta-k)} 2^{\eta\alpha} 2^{-\epsilon|j-i|} \Big|^p.$$

This together with Young's inequality for the discrete convolution gives

$$\Big\| \sum_{Q\in\mathcal{D}_\eta} s_Q \Psi_{2^k}(L) m_Q \Big\|_p^p \lesssim 2^{p(2M-N-n-\epsilon)(\eta-k)} 2^{\eta\alpha p} \Big(\sum_{j\in\mathbb{Z}} |s_j|^p \Big) \Big(\sum_{j\in\mathbb{Z}} 2^{-\epsilon|j|} \Big)^p$$
$$\lesssim 2^{p(2M-N-n-\epsilon)(\eta-k)} 2^{\eta\alpha p} \Big(\sum_{Q\in D_\eta} |s_Q|^p \Big).$$

This gives (i).

The proof of (ii) is similar to that of (i) and hence we omit the details here. □

We are now in the position to give the proof of Theorem 5.22.

Proof of Theorem 5.27: We will adapt the arguments in [FJ1] to our situation.

We have

$$\|f\|^q_{\dot{B}^{\alpha,L}_{p,q}(X)} = \sum_{k\in\mathbb{Z}} \Big(2^{-k\alpha} \Big\| \sum_{\eta\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_\eta} s_Q \Psi_{2^k}(L) m_Q \Big\|_p \Big)^q$$
$$\lesssim \sum_{k\in\mathbb{Z}} \Big(2^{-k\alpha} \Big\| \sum_{\eta\le k} \sum_{Q\in\mathcal{D}_\eta} s_Q \Psi_{2^k}(L) m_Q \Big\|_p \Big)^q$$
$$+ \sum_{k\in\mathbb{Z}} \Big(2^{-k\alpha} \Big\| \sum_{\eta> k} \sum_{Q\in\mathcal{D}_\eta} s_Q \Psi_{2^k}(L) m_Q \Big\|_p \Big)^q.$$

This in combination with Proposition 5.24 implies that

$$\|f\|^q_{\dot{B}^{\alpha,L}_{p,q}(X)} \lesssim \sum_{k\in\mathbb{Z}} \Big(\sum_{\eta\le k} 2^{(\eta-k)\alpha} \Big(\sum_{Q\in\mathcal{D}_\eta} |s_Q|^p \Big)^{1/p} \Big)^q + \sum_{k\in\mathbb{Z}} \Big(\sum_{\eta\le k} 2^{(\eta-k)(2M-N/p-n-\epsilon+\alpha)} \Big(\sum_{Q\in\mathcal{D}_\eta} |s_Q|^p \Big)^{1/p} \Big)^q.$$

Due to $\alpha < 0$ and $2M - N/p - n - \epsilon + \alpha > 0$, by using Young's inequality we arrive at

$$\|f\|^q_{\dot{B}^{\alpha,L}_{p,q}(X)} \lesssim \sum_{\eta\in\mathbb{Z}} \Big(\sum_{Q\in\mathcal{D}_\eta} |s_Q|^p \Big)^{q/p}.$$

This completes our proof.

□

By careful examination of the proof of Theorem 5.27 we have the following important remark.

Remark 5.25. *Assume that L satisfies* (G). *Let $1 \le p, q \le \infty$, $-1 < \alpha < 1$ and $M > N+n+\epsilon$. Assume that*

$$f = \sum_{Q \in \mathcal{D}} s_Q m_Q \quad \text{in } (\mathcal{M}_{p',q'}^{-\alpha,L^*})',$$

where $\{s_Q\}$ is a sequence of nonnegative numbers and $\{m_Q\}$ is a sequence of $(L, M, \alpha, p, \epsilon)$ molecules. In addition, if there holds

$$|\Psi_{2^k}(L) m_Q(x)| \lesssim \frac{2^{\eta\alpha} 2^{\epsilon'(k-\eta)}}{\mu(Q)^{1/p}} \Big(1 + \frac{d(x, x_Q)}{2^\eta}\Big)^{-(N+n+\epsilon)}$$

for all $k < \eta$ and for some $\epsilon' \in (0, 1]$, then we have

$$\|f\|_{\dot{B}_{p,q}^{\alpha,L}(X)} \lesssim \Big[\sum_{k \in \mathbb{Z}} \Big[\sum_{Q \in \mathcal{D}_k} |s_Q|^p\Big]^{q/p}\Big]^{1/q}$$

for $-1 < \alpha < \epsilon'$.

5.3.2 Smooth molecular decompositions

We now describe the concept of ϵ-smooth (L, M, α, p, β) molecules.

Definition 5.26. *Let $1 \le p \le \infty$, $-1 < \alpha < 1$, $\epsilon, \beta > 0$ and $M \in \mathbb{N}_+$. A function m is said to be an ϵ-smooth (L, M, α, p, β) molecule if there exists a dyadic cube $Q \in \mathcal{D}$ so that*

(i) $m = L^M b$;

(ii) $|L^k b(x)| \le \dfrac{\ell(Q)^{2(M-k)+\alpha}}{\mu(Q)^{1/p}} \Big(1 + \dfrac{|x - x_Q|}{\ell(Q)}\Big)^{-(N+n+\beta)}, k = 0, 1, \ldots, M$;

(iii) $|L^k b(x) - L^k b(y)| \le \Big(\dfrac{d(x,y)}{\ell(Q)}\Big)^\epsilon \dfrac{\ell(Q)^{2(M-k)+\alpha}}{\mu(Q)^{1/p}} \Big(1 + \dfrac{d(x, x_Q)}{\ell(Q)}\Big)^{-(N+n+\beta)}, k = 0, 1, \ldots, M$, *whenever $d(x, y) < \ell(Q)/8$.*

Theorem 5.27. *Assume that L satisfies* (G), (C) *and* (H). *Let $1 \le p, q \le \infty$, $-1 < \alpha < 1$ and $M \in \mathbb{N}_+$.*

(a) *Then for each $f \in \dot{B}_{p,q}^{\alpha,L}(X)$, there exist a sequence of coefficients $0 \le s_Q < \infty$, where Q ranges over the dyadic cubes, and a sequence m_Q of ϵ-smooth (L, M, α, p, β) molecules with $\epsilon \in (0, \delta]$ and $\beta > 0$, such that*

$$f = \sum_{Q \in \mathcal{D}} s_Q m_Q \quad \text{in } (\mathcal{M}_{p',q'}^{-\alpha,L^*})',$$

and

$$\Big[\sum_{k\in\mathbb{Z}}\Big[\sum_{Q\in\mathcal{D}_k}|s_Q|^p\Big]^{q/p}\Big]^{1/q}\approx\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}. \tag{5.22}$$

(b) *Conversely, suppose*

$$f=\sum_{Q\in\mathcal{D}}s_Qm_Q\quad\text{in }(\mathcal{M}^{-\alpha,L^*}_{p',q'})',$$

where $\{s_Q\}_Q$ *is a sequence of nonnegative numbers and* $\{m_Q\}$ *is a sequence of* ϵ*-smooth* (L,M,α,p,β) *molecules with* $\epsilon\in(0,\delta]$, $\beta>0$ *and* $M>N+n+\beta$. *Then we have*

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}\lesssim\Big[\sum_{k\in\mathbb{Z}}\Big[\sum_{Q\in\mathcal{D}_k}|s_Q|^p\Big]^{q/p}\Big]^{1/q} \tag{5.23}$$

for $-1<\alpha<\epsilon$.

Proof. (a) In the proof of Theorem 5.20, we proved that

$$f=\sum_{Q\in\mathcal{D}}s_Qm_Q\quad\text{in }(\mathcal{M}^{-\alpha,L^*}_{p',q'})',$$

where

$$s_Q=2^{-(k-1)\alpha}\Big(\int_Q|\Psi_{2^{k-1}}(L)f(y)|^pd\mu(y)\Big)^{1/p}$$

and

$$m_Q=\frac{1}{s_Q}\int_{2^{k-1}}^{2^k}2^{-2(k-1)}t^2(t^2L)^Me^{-(2t^2-2^{2(k-1)})L}[\Psi_{2^{k-1}}(L)f.\chi_Q](x)\frac{dt}{t}.$$

Moreover, it was also proved that (5.22) holds and m_Q satisfies (i)–(ii) in Definition 5.26. Hence, to complete the proof we need only to verify the condition (iii) in Definition 5.26. Indeed, we have $m_Q=L^Mb_Q$ where

$$b_Q=\frac{1}{s_Q}\int_{2^{k-1}}^{2^k}2^{-2(k-1)}t^2(t^2L)^Me^{-(2t^2-2^{2(k-1)})L}[\Psi_{2^{k-1}}(L)f.\chi_Q](x)\frac{dt}{t}.$$

Hence, for each $m=0,1,\dots,M$, we have

$$L^mb_Q(x)=\frac{1}{s_Q}\int_{2^{k-1}}^{2^k}\int_Q\frac{t^{2(M-m+1)}}{2^{-2(k-1)}}\Big(\frac{t^2}{2t^2-2^{2(k-1)}}\Big)^mp_{2t^2-2^{2(k-1)},m}(x,y)\Psi_{2^{k-1}}(L)f(y)d\mu(y)\frac{dt}{t}.$$

which implies

$$\begin{aligned}|L^mb_Q(x)-L^mb_Q(y)|\le\frac{1}{s_Q}\int_{2^{k-1}}^{2^k}\int_Q\frac{t^{2(M-m+1)}}{2^{-2(k-1)}}\Big(\frac{t^2}{2t^2-2^{2(k-1)}}\Big)^m\\ \times\Big[p_{2t^2-2^{2(k-1)},m}(x,z)-p_{2t^2-2^{2(k-1)},m}(y,z)\Big]\Psi_{2^{k-1}}(L)f(y)d\mu(z)\frac{dt}{t}.\end{aligned}$$

Thanks to Lemma 5.1, the doubling property and the fact that $2t^2 - 2^{2(k-1)} \approx t^2 \approx 2^{2(k-1)}$ whenever $t \in [2^{k-1}, 2^k]$, for $d(x,y) < \ell(Q)/8 < t/2$ we obtain that

$$\begin{aligned}
|L^m b_Q(x) - L^m b_Q(y)| &\lesssim \frac{2^{2k(M-m)}}{s_Q.\mu(Q)} \int_{2^{k-1}}^{2^k} \int_Q \Big[\frac{d(x,y)}{t}\Big]^\delta \Big(1 + \frac{d(x,x_Q)}{t}\Big)^{-(N+n+\beta)} |\Psi_{2^{k-1}}(L)f(z)| d\mu(z) \frac{dt}{t} \\
&\lesssim \frac{2^{2k(M-m)}}{s_Q.\mu(Q)} \Big[\frac{d(x,y)}{\ell(Q)}\Big]^\delta \Big(1 + \frac{d(x,x_Q)}{\ell(Q)}\Big)^{-(N+n+\beta)} \int_Q |\Psi_{2^{k-1}}(L)f(z)| d\mu(z) \\
&\lesssim \frac{2^{2k(M-m)}}{s_Q.\mu(Q)^{1/p}} \Big[\frac{d(x,y)}{\ell(Q)}\Big]^\delta \Big(1 + \frac{d(x,x_Q)}{\ell(Q)}\Big)^{-(N+n+\beta)} \Big(\int_Q |\Psi_{2^{k-1}}(L)f(z)|^p d\mu(z)\Big)^{1/p} \\
&\lesssim \frac{\ell(Q)^{2(M-k)+\alpha}}{\mu(Q)^{1/p}} \Big[\frac{d(x,y)}{\ell(Q)}\Big]^\delta \Big(1 + \frac{d(x,x_Q)}{\ell(Q)}\Big)^{-(N+n+\beta)}.
\end{aligned}$$

This completes the proof of (a).

(b) Assume that m_Q is an ϵ-smooth (L, M, α, p, β) molecule associated to the dyadic cube $Q \in \mathcal{D}_\eta$ with $\epsilon \in (0, \delta]$ and $\beta > 0$. Since an ϵ-smooth (L, M, α, p, β) molecule is also an (L, M, α, p, β) molecule, in the light of Remark 5.25 it suffices to claim that

$$|\Psi_{2^k}(L) m_Q(x)| \lesssim \frac{2^{(k-\eta)\epsilon} 2^{\eta\alpha}}{\mu(Q)^{1/p}} \Big(1 + \frac{d(x,x_Q)}{2^\eta}\Big)^{-(N+n+\beta)}, k \le \eta. \tag{5.24}$$

Indeed, due to (C) we have

$$\begin{aligned}
|\Psi_{2^k}(L) m_Q(x)| &= \Big| \int_X p_{2^{2k},1}(x,y)(m_Q(y) - m_Q(x)) d\mu(y) \Big| \\
&\lesssim \Big| \int_{d(x,y)<2^\eta/8} \cdots \Big| + \Big| \int_{d(x,y)\ge 2^\eta/8} \cdots \Big| \\
&= E_1 + E_2.
\end{aligned}$$

Since m_Q satisfies (iii) in Definition 5.26, we have

$$\begin{aligned}
E_1 &\lesssim \frac{2^{\eta\alpha}}{\mu(Q)^{1/p}} \int_X \frac{1}{V(x,2^k)} \exp\Big(-c\frac{d(x,y)^2}{2^{2k}}\Big) \Big(1 + \frac{d(x,x_Q)}{2^\eta}\Big)^{-(N+n+\beta)} \Big(\frac{d(x,y)}{2^\eta}\Big)^\epsilon d\mu(y) \\
&\lesssim \frac{2^{(k-\eta)\epsilon} 2^{\eta\alpha}}{\mu(Q)^{1/p}} \int_X \frac{1}{V(x,2^k)} \exp\Big(-c'\frac{d(x,y)^2}{2^{2k}}\Big) \Big(1 + \frac{d(x,x_Q)}{2^\eta}\Big)^{-(N+n+\beta)} d\mu(y) \\
&\lesssim \frac{2^{(k-\eta)\epsilon} 2^{\eta\alpha}}{\mu(Q)^{1/p}} \Big(1 + \frac{d(x,x_Q)}{2^\eta}\Big)^{-(N+n+\beta)}.
\end{aligned}$$

For the term E_2, we have

$$\begin{aligned}
E_2 &\lesssim \int_{d(x,y)\ge 2^\eta/8} \frac{1}{V(x,2^k)} \exp\Big(-c\frac{d(x,y)^2}{2^{2k}}\Big) |m_Q(x) - m_Q(y)| d\mu(y) \\
&\lesssim \int_{d(x,y)\ge 2^\eta/8} \Big(\frac{2^k}{d(x,y)}\Big)^\epsilon \frac{1}{V(x,2^k)} \exp\Big(-c'\frac{d(x,y)^2}{2^{2k}}\Big) |m_Q(x) - m_Q(y)| d\mu(y) \\
&\lesssim \int_{d(x,y)\ge 2^\eta/8} \frac{1}{V(x,2^k)} \exp\Big(-c'\frac{d(x,y)^2}{2^{2k}}\Big) |m_Q(x) - m_Q(y)| d\mu(y).
\end{aligned}$$

By (ii) in Definition 5.26 it is easy to see that

$$\int_{d(x,y)\geq 2^{\eta}/8} \frac{1}{V(x,2^k)} \exp\Big(-c'\frac{d(x,y)^2}{2^{2k}}\Big)|m_Q(x)|d\mu(y) \lesssim \frac{2^{(k-\eta)\epsilon}2^{\eta\alpha}}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{2^{\eta}}\Big)^{-(N+n+\beta)}.$$

On the other hand, by using the same argument done in Proposition 5.23 (i), we arrive at

$$\int_{d(x,y)\geq 2^{\eta}/8} \frac{1}{V(x,2^k)} \exp\Big(-c'\frac{d(x,y)^2}{2^{2k}}\Big)|m_Q(y)|d\mu(y) \lesssim \frac{2^{(k-\eta)\epsilon}2^{\eta\alpha}}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{2^{\eta}}\Big)^{-(N+n+\beta)}.$$

Taking these two estimates into account, we conclude that

$$E_2 \lesssim \frac{2^{(k-\eta)\epsilon}2^{\eta\alpha}}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{2^{\eta}}\Big)^{-(N+n+\beta)}.$$

This completes our proof. □

5.3.3 Critical molecular decompositions

In the previous section, in order to establish the reverse molecular theorem for Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ with $\alpha \geq 0$, the conservation property (C) was necessary. The aim of this section is to remove the conservation property (C). To do this, we need a different kind of molecules.

Let γ be a positive function on X. The function γ is called a critical function if there exist positive constants C and k_0 so that

$$\gamma(y) \leq C[\gamma(x)]^{\frac{1}{1+k_0}}[\gamma(x)+d(x,y)]^{\frac{k_0}{k_0+1}} \tag{5.25}$$

for all $x, y \in X$.

Note that the concept of critical functions was introduced first to the setting of Schrödinger operators on $\mathbb{R}^n$ in [Fe] (see also [Sh]) and then was extended to the spaces of homogeneous type in [YZ]. We have some basic properties on the critical function in [YZ].

Lemma 5.28. *Let γ be a critical function on X. Then*

(i) *For any $\widetilde{C} > 0$, there exists a positive constant C, depending on $\widetilde{C}$, such that if $y \in B(x, \widetilde{C}\gamma(x))$, then $C^{-1}\gamma(x) \leq \gamma(y) \leq C\gamma(x)$.*

(ii) *There exists a positive constant C such that for all $x, y \in X$,*

$$\gamma(y) \geq C[\gamma(x)]^{1+k_0}[\gamma(x) + d(x,y)]^{-k_0}.$$

One of the most important examples of critical functions is the class of critical functions associated to the weights satisfying the reverse Hölder's inequality. If $w \in RH_q$ with $q > \max\{1, n/2\}$ then the function defined by

$$\gamma(x) = \sup\Big\{r : \frac{r^2}{\mu(B(x,r))}\int_{B(x,r)} w(y)d\mu(y) \leq 1\Big\}$$

is a critical function. See for example [YZ].

Let γ be a critical function on X. In this section, we assume that L is a nonnegative self-adjoint operator on $L^2(X)$ satisfying the following conditions:

(I) For all $K > 0$, there exist positive constants c and C so that

$$|p_t(x,y)| \leq \frac{C}{V(x,\sqrt{t})}\exp\Big(-c\frac{d(x,y)^2}{t}\Big)\Big(1 + \frac{\sqrt{t}}{\gamma(x)} + \frac{\sqrt{t}}{\gamma(y)}\Big)^{-K}$$

for all $x, y \in X$ and $t > 0$;

(II) There is a positive constant $\delta_0 > 0$ so that for all $K > 0$, there exist positive constants c and C:

$$|p_{t,1}(x,y) - p_{t,1}(\overline{x},y)| \leq \frac{C}{V(x,\sqrt{t})}\Big[\frac{d(x,\overline{x})}{\sqrt{t}}\Big]^{\delta_0}\exp\Big(-c\frac{d(x,y)^2}{t}\Big)\Big(1 + \frac{\sqrt{t}}{\gamma(x)} + \frac{\sqrt{t}}{\gamma(y)}\Big)^{-K}$$

whenever $d(x,\overline{x}) \leq \sqrt{t}/2$ and $t > 0$; and

$$\Big|\int_X p_t(x,y)d\mu(y) - 1\Big| + \Big|\int_X p_{t,1}(x,y)d\mu(y)\Big| \leq C\Big(\frac{\sqrt{t}}{\sqrt{t} + \gamma(x)}\Big)^{\delta_0}\Big(1 + \frac{\sqrt{t}}{\gamma(x)}\Big)^{-N}$$

for all $x \in X$ and $t > 0$.

Examples of operators satisfying (I)–(II) include:

(i) the Schrödinger operators on $\mathbb{R}^n, n \geq 3$ with nonnegative potentials V in the reverse Hölder class $RH_{n/2}$;

(ii) the degenerate Schrödinger operators on $\mathbb{R}^n, n \geq 3$ with nonnegative potentials V satisfying certain reverse Hölder inequalities;

(iii) the sub-Laplace Schrödinger operator on Heisenberg groups with nonnegative potentials V satisfying certain reverse Hölder inequalities;

(iv) the sub-Laplace Schrödinger operator on connected and simply connected nilpotent Lie groups with nonnegative potentials V satisfying certain reverse Hölder inequalities.

See for example [YZ] and the references therein.

Definition 5.29. *Let γ be a critical function in X. Assume that L is a nonnegative self-adjoint operator on $L^2(X)$ satisfying* (I)–(II). *Let $1 \leq p \leq \infty$, $-1 < \alpha < 1$, $\epsilon, \beta > 0$ and $M \in \mathbb{N}_+$. A function m is said to be a (γ, ϵ)-critical (L, M, α, p, β) molecule if there exists a dyadic cube $Q \in \mathcal{D}$ so that*

(i) $m = L^M b$;

(ii) $|L^k b(x)| \leq \dfrac{\ell(Q)^{2(M-k)+\alpha}}{\mu(Q)^{1/p}} \Big(1 + \dfrac{|x - x_Q|}{\ell(Q)}\Big)^{-(N+n+\beta)} \Big(1 + \dfrac{\ell(Q)}{\gamma(x)}\Big)^{-\epsilon}, k = 0, 1, \ldots, M$;

(iii) $|L^k b(x) - L^k b(y)| \leq \Big(\dfrac{d(x,y)}{\ell(Q)}\Big)^{\epsilon} \dfrac{\ell(Q)^{2(M-k)+\alpha}}{\mu(Q)^{1/p}} \Big(1 + \dfrac{d(x, x_Q)}{\ell(Q)}\Big)^{-(N+n+\beta)}, k = 0, 1, \ldots, M$, *whenever $d(x,y) < \ell(Q)/8$.*

Theorem 5.30. *Assume that L is a nonnegative self-adjoint operator on $L^2(X)$ satisfying* (I)–(II) *and γ is a critical function in X. Let $1 \leq p, q \leq \infty$, $-1 < \alpha < 1$ and $M \in \mathbb{N}_+$.*

(a) *Then for each $f \in \dot{B}^{\alpha,L}_{p,q}(X)$, there exist a sequence of coefficients $0 \leq s_Q < \infty$, where Q ranges over the dyadic cubes, and a sequence m_Q of (γ, ϵ)-critical (L, M, α, p, β) molecules with $\epsilon \in (0, \delta_0]$ and $\beta > 0$, such that*

$$f = \sum_{Q \in \mathcal{D}} s_Q m_Q \quad \text{in } (\mathcal{M}^{-\alpha,L}_{p',q'})',$$

and

$$\Big[\sum_{k \in \mathbb{Z}} \Big[\sum_{Q \in \mathcal{D}_k} |s_Q|^p\Big]^{q/p}\Big]^{1/q} \sim \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}. \tag{5.26}$$

(b) *Conversely, suppose*

$$f = \sum_{Q\in\mathcal{D}} s_Q m_Q \quad \text{in } (\mathcal{M}^{-\alpha,L}_{p',q'})',$$

where $\{s_Q\}_Q$ *is a sequence of nonnegative numbers and* $\{m_Q\}$ *is a sequence of* (γ,ϵ)*-critical* (L,M,α,p,β) *molecules with* $\epsilon\in(0,\delta_0]$ *and* $M>N+n+\beta$*. Then we have*

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} \lesssim \Big[\sum_{k\in\mathbb{Z}}\Big[\sum_{Q\in\mathcal{D}_k}|s_Q|^p\Big]^{q/p}\Big]^{1/q} \tag{5.27}$$

for $-1<\alpha<\epsilon$*.*

To prove Theorem 5.30 we need the following auxiliary lemma.

Lemma 5.31. *Assume that* L *is a nonnegative self-adjoint operator on* $L^2(X)$ *satisfying* (I)–(II)*. Then for each* $k\in\mathbb{N}_+$ *and* $K>0$*, there exist positive constants* c *and* C *so that*

(i) $|p_{t,k}(x,y)|\le \dfrac{C}{V(x,\sqrt{t})}\exp\Big(-c\dfrac{d(x,y)^2}{t}\Big)\Big(1+\dfrac{\sqrt{t}}{\gamma(x)}+\dfrac{\sqrt{t}}{\gamma(y)}\Big)^{-K}$ *for all* $x,y\in X$ *and* $t>0$*; and*

(ii) $|p_{t,k}(x,y)-p_{t,k}(\overline{x},y)| \le \dfrac{C}{V(x,\sqrt{t})}\Big[\dfrac{d(x,\overline{x})}{d(x,y)}\Big]^{\delta_0}\exp\Big(-c\dfrac{d(x,y)^2}{t}\Big)\Big(1+\dfrac{\sqrt{t}}{\gamma(x)}+\dfrac{\sqrt{t}}{\gamma(y)}\Big)^{-K}$ *whenever* $d(x,\overline{x})\le\sqrt{t}/4$ *and* $t>0$*;*

Proof. The proof of (i) is very standard and we omit the details here. The proof of (ii) is similar to that of Lemma 5.1. □

We are now in the position to give the proof for Theorem 5.30.

Proof of Theorem 5.30: (a) We can proceed with the arguments used in the proof of Theorem 5.27 (a) of which we use Lemma 5.31 in place of Lemma 5.1.

(b) Due to Remark 5.25, it suffices to show that for any (γ,ϵ)-critical (L,M,α,p,β) molecule m_Q associated to the dyadic cube $Q\in\mathcal{D}_\eta$ there holds

$$|\Psi_{2^k}(L)m_Q(x)| \lesssim \frac{2^{(k-\eta)\epsilon}2^{\eta\alpha}}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{2^\eta}\Big)^{-(N+n+\beta)}, k\le\eta.$$

Indeed, we have

$$\begin{aligned}|\Psi_{2^k}(L)m_Q(x)| &\le \Big|\int_X p_{2^{2k},1}(x,y)(m_Q(y)-m_Q(x))d\mu(y)\Big| + \Big|\int_X p_{2^{2k},1}(x,y)m_Q(x)d\mu(y)\Big|\\ &= A_1+A_2.\end{aligned}$$

Using the similar argument to that of (5.24) we can show that

$$A_1 \lesssim \frac{2^{(k-\eta)\epsilon}2^{\eta\alpha}}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{2^\eta}\Big)^{-(N+n+\beta)}$$

Since m_Q satisfies (iii) in Definition 5.29, we have

$$\begin{aligned} A_2 &\lesssim \frac{2^{\eta\alpha}}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{2^\eta}\Big)^{-(N+n+\beta)}\Big(1+\frac{2^\eta}{\gamma(x)}\Big)^{-\epsilon}\Big|\int_X p_{2^{2k},1}(x,y)d\mu(y)\Big| \\ &\lesssim \frac{2^{\eta\alpha}}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{2^\eta}\Big)^{-(N+n+\beta)}\Big(1+\frac{2^\eta}{\gamma(x)}\Big)^{-\epsilon}\Big(\frac{2^k}{\gamma(x)}\Big)^{\epsilon} \\ &\lesssim \frac{2^{(k-\eta)\epsilon}2^{\eta\alpha}}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{2^\eta}\Big)^{-(N+n+\beta)}. \end{aligned}$$

This completes our proof.

Remark 5.32. *It is interesting to note that in Theorems 5.20, 5.22, 5.27 and 5.30 if $f \in L^r(X)\cap \dot{B}^{\alpha,L}_{p,q}(X)$ for some $1<r<\infty$ then the series $f=\sum_{Q\in\mathcal{D}} s_Q m_Q$ converges in $L^r(X)$. We leave the details to the interested reader.*

5.4 Atomic decompositions of Besov spaces $\dot{B}^{\alpha,L}_{p,p}(X)$

Let L be a nonnegative and self-adjoint operator on $L^2(X)$ satisfying (G). In what follows, given a $L^2(X)$ bounded linear operator, we shall denote by $K_T(x,y)$ the kernel of the operator T. Then there exists a constant $c_0>0$ such that

$$\operatorname{supp} K_{\cos(t\sqrt{L})} \subset \{(x,y)\in X\times X : d(x,y)\le c_0 t\}. \tag{5.28}$$

See for example [Si].

Lemma 5.33. *Let L be a nonnegative and self-adjoint operator on $L^2(X)$ and let γ be a critical function on X. Let $\varphi\in C_0^\infty(\mathbb{R})$ be even and $\operatorname{supp}\varphi\subset(-c_0^{-1},c_0^{-1})$, with c_0 as in (5.28). Let Φ denote the Fourier transform of φ. For every $k\in\mathbb{N}$ and $t>0$, denote by $K_{(t^2L)^k\Phi(t\sqrt{L})}$ the kernel of $(t^2L)^k\Phi(t\sqrt{L})$.*

(i) *If L satisfies* (G) *then we have*

$$\operatorname{supp} K_{(t^2L)^k\Phi(t\sqrt{L})} \subset \{(x,y)\in X\times X : d(x,y)\le t\}, \tag{5.29}$$

and

$$|K_{(t^2L)^k\Phi(t\sqrt{L})}(x,y)| \leq \frac{C}{\mu(B(x,t))} \tag{5.30}$$

for all $x, y \in X$ *and* $t > 0$.

(ii) *If* L *satisfies* (G) *and* (H) *then we have*

$$\Big|K_{(t^2L)^k\Phi(t\sqrt{L})}(x,y) - K_{(t^2L)^k\Phi(t\sqrt{L})}(x',y)\Big| \leq \frac{C}{\mu(B(x,t))}\Big(\frac{d(x,x')}{t}\Big)^{\delta}$$

for all $t > 0$ *and* $d(x,x') < t/2$.

(iii) *If* L *satisfies* (I) *and* (II) *then for each* $K > 0$ *there exist* $C, c > 0$ *so that*

$$|K_{(t^2L)^k\Phi(t\sqrt{L})}(x,y)| \leq \frac{C}{\mu(B(x,t))}\Big(1 + \frac{t}{\gamma(x)} + \frac{t}{\gamma(y)}\Big)^{-K}$$

and

$$\Big|K_{(t^2L)^k\Phi(t\sqrt{L})}(x,y) - K_{(t^2L)^k\Phi(t\sqrt{L})}(x',y)\Big| \leq \frac{C}{\mu(B(x,t))}\Big(\frac{d(x,x')}{t}\Big)^{\delta_0}\Big(1 + \frac{t}{\gamma(x)} + \frac{t}{\gamma(y)}\Big)^{-K}$$

for all $t > 0$ *and* $d(x,x') < t/2$.

Proof. For the proof of (5.29) we refer the reader to Lemma 3.5 in [HLMMY]. The proof of (5.30) is similar to that of (ii) but easier. Hence we provide the proof of (ii) only. Note that in the particular case when X satisfies the condition $\mu(B(x,r)) \approx r^n$, the proof of (5.30) was given in [Si].

Observe that for $m \in \mathbb{N}$ we have

$$(I + t^2L)^{-m} = c_m \int_0^\infty e^{-\lambda t^2 L} e^{-\lambda} \lambda^{m-1} d\lambda$$

which implies

$$K_{(I+t^2L)^{-m}}(x,y) = c_m \int_0^\infty p_{\lambda t^2}(x,y) e^{-\lambda} \lambda^{m-1} d\lambda. \tag{5.31}$$

Hence for $d(x,x') \le t/2$ we have

$$\begin{aligned}|K_{(I+t^2L)^{-m}}(x,y) - K_{(I+t^2L)^{-m}}(x',y)| &= c_m \int_0^\infty |p_{\lambda t^2}(x,y) - p_{\lambda t^2}(x',y)| e^{-\lambda}\lambda^{m-1} d\lambda \\ &= c_m \int_0^{2d(x,x')^2/t^2} \ldots + c_m \int_{2d(x,x')^2/t^2}^\infty \ldots \\ &= I_1(x,x',y) + I_2(x,x',y).\end{aligned}$$

Using (H) we have

$$I_2(x,x',y) \lesssim \int_{2d(x,x')^2/t^2}^\infty \Big(\frac{d(x,x')}{t\sqrt{\lambda}}\Big)^\delta \frac{1}{V(x,t\sqrt{\lambda})} \exp\Big(-c\frac{d(x,y)^2}{\lambda t^2}\Big) e^{-\lambda}\lambda^{m-1} d\lambda$$

which implies

$$\begin{aligned}\|I_2(x,x',\cdot)\|_2 &\lesssim \int_{2d(x,x')^2/t^2}^\infty \Big(\frac{d(x,x')}{t\sqrt{\lambda}}\Big)^\delta \frac{1}{\mu(B(x,t\sqrt{\lambda}))^{1/2}} e^{-\lambda}\lambda^{m-1} d\lambda \\ &\lesssim \Big(\frac{d(x,x')}{t}\Big)^\delta \frac{1}{\mu(B(x,t))^{1/2}} \int_{2d(x,x')^2/t^2}^\infty \lambda^{-\delta/2}(1+1/\sqrt{\lambda})^{n/2} e^{-\lambda}\lambda^{m-1} d\lambda \\ &\lesssim \Big(\frac{d(x,x')}{t}\Big)^\delta \frac{1}{\mu(B(x,t))^{1/2}}\end{aligned}$$

provided $m > n/4 + 1$.

To estimate $I_1(x,x',y)$, we break

$$\begin{aligned}I_1(x,x',y) &\lesssim c_m \int_0^{2d(x,x')^2/t^2} |p_{\lambda t^2}(x,y)| e^{-\lambda}\lambda^{m-1} d\lambda + c_m \int_0^{2d(x,x')^2/t^2} |p_{\lambda t^2}(x',y)| e^{-\lambda}\lambda^{m-1} d\lambda \\ &\lesssim I_{11}(x,x',y) + I_{12}(x,x',y).\end{aligned}$$

By Minkowski's inequality, we have

$$\begin{aligned}\|I_{11}(x,x',\cdot)\|_2 &\lesssim \int_0^{2d(x,x')^2/t^2} \frac{1}{\mu(B(x,t\sqrt{\lambda}))^{1/2}} e^{-\lambda}\lambda^{m-1} d\lambda \\ &\lesssim \int_0^{2d(x,x')^2/t^2} \frac{1}{\mu(B(x,t))^{1/2}} (1+1/\sqrt{\lambda})^{n/2} e^{-\lambda}\lambda^{m-1} d\lambda \\ &\lesssim \int_0^{2d(x,x')^2/t^2} \Big(\frac{d(x,x')}{t}\Big)^\delta \frac{1}{\mu(B(x,t))^{1/2}} (1+1/\sqrt{\lambda})^{n/2} e^{-\lambda}\lambda^{m-1-\delta/2} d\lambda \\ &\lesssim \Big(\frac{d(x,x')}{t}\Big)^\delta \frac{1}{\mu(B(x,t))^{1/2}}\end{aligned}$$

provided $m > n/4 + 1$.

Likewise,

$$\|I_{12}(x,x',\cdot)\|_2 \lesssim \Big(\frac{d(x,x')}{t}\Big)^\delta \frac{1}{\mu(B(x',t))^{1/2}} \lesssim \Big(\frac{d(x,x')}{t}\Big)^\delta \frac{1}{\mu(B(x,t))^{1/2}},$$

where in the last inequality we use the fact that $\mu(B(x,t)) \approx \mu(B(x',t))$ since $d(x,x') < t/2$.

To sum up, we have shown that

$$\|K_{(I+t^2L)^{-m}}(x,\cdot) - K_{(I+t^2L)^{-m}}(x',\cdot)\|_2 \lesssim \Big(\frac{d(x,x')}{t}\Big)^{\delta} \frac{1}{\mu(B(x,t))^{1/2}}. \tag{5.32}$$

We now write

$$(t^2L)^k\Phi(t\sqrt{L}) = (I+t^2L)^{-m}\Upsilon_t(L)(I+t^2L)^{-m}$$

where $\Upsilon_t(L) = (I+t^2L)^{2m}(t^2L)^k\Phi(t\sqrt{L})$. Hence, using Hölder's inequality and (5.32) gives

$$\begin{aligned}
&\Big|K_{(t^2L)^k\Phi(t\sqrt{L})}(x,y) - K_{(t^2L)^k\Phi(t\sqrt{L})}(x',y)\Big| \\
&\qquad = \Big|\int_X \Big(K_{(I+t^2L)^{-m}}(x,z) - K_{(I+t^2L)^{-m}}(x',z)\Big) K_{\Upsilon_t(L)(I+t^2L)^{-m}}(z,y)d\mu(z)\Big| \\
&\qquad \lesssim \Big\|K_{(I+t^2L)^{-m}}(x,\cdot) - K_{(I+t^2L)^{-m}}(x',\cdot)\|_2\Big\|K_{\Upsilon_t(L)(I+t^2L)^{-m}}(\cdot,y)\Big\|_2 \\
&\qquad \lesssim \Big(\frac{d(x,x')}{t}\Big)^{\delta}\frac{1}{\mu(B(x,t))^{1/2}}\Big\|K_{\Upsilon_t(L)(I+t^2L)^{-m}}(\cdot,y)\Big\|_2.
\end{aligned}$$

On the other hand, we have

$$K_{\Upsilon_t(L)(I+t^2L)^{-m}}(z,y) = \Upsilon_t(L)[K_{(I+t^2L)^{-m}}(\cdot,y)](z)$$

which implies

$$\begin{aligned}
\Big\|K_{\Upsilon_t(L)(I+t^2L)^{-m}}(\cdot,y)\Big\|_2 &\lesssim \|\Upsilon_t(L)\|_{2\to 2}\Big\|K_{(I+t^2L)^{-m}}(\cdot,y)\Big\|_2 \\
&\lesssim \|\Upsilon_t\|_\infty\Big\|K_{(I+t^2L)^{-m}}(\cdot,y)\Big\|_2 \lesssim \Big\|K_{(I+t^2L)^{-m}}(\cdot,y)\Big\|_2.
\end{aligned}$$

Using formula (5.31) and the Gaussian upper bound of $p_t(x,y)$, it can be verified that

$$\Big\|K_{(I+t^2L)^{-m}}(\cdot,y)\Big\|_2 \lesssim \frac{1}{\mu(B(y,t))^{1/2}} \approx \frac{1}{\mu(B(x,t))^{1/2}}$$

as long as $d(x,y) < t$. Therefore,

$$\Big|K_{(t^2L)^k\Phi(t\sqrt{L})}(x,y) - K_{(t^2L)^k\Phi(t\sqrt{L})}(x',y)\Big| \le \frac{C}{\mu(B(x,t))}\Big(\frac{d(x,x')}{t}\Big)^{\delta}$$

for all $t > 0$ and $d(x,x') < t/2$.

The proofs of (iii) and (iv) are similar to that of (ii) and hence we leave them to the interested reader. □

Definition 5.34. *Assume that L is a nonnegative and self-adjoint operator on $L^2(X)$ satisfying* (G). *Let $1 \le p \le \infty$, $-1 < \alpha < 1$, ϵ and $M \in \mathbb{N}$. A function a is said to be an (L, M, α, p) atom if there exists a dyadic cube $Q \in \mathcal{D}$ so that*

(i) $a = L^M b$;

(ii) $\operatorname{supp} L^k b \subset 3Q, k = 0, 1, \ldots, M$;

(iii) $\|L^k b\|_\infty \le \frac{\ell(Q)^{2(M-k)+\alpha}}{\mu(Q)^{1/p}}, k = 0, 1, \ldots, M.$

Moreover, a function a is said to be an ϵ-smooth (L, M, α, p) atom if a satisfies additionally the following condition

$$|L^k b(x) - L^k b(y)| \le \Big(\frac{d(x,y)}{\ell(Q)}\Big)^\epsilon \frac{\ell(Q)^{2(M-k)+\alpha}}{\mu(Q)^{1/p}}, k = 0, 1, \ldots, M.$$

Definition 5.35. *Let γ be a critical function on X. Assume that L is a nonnegative and self-adjoint operator on $L^2(X)$ satisfying* (I) *and* (II). *Let $1 \le p \le \infty$, $-1 < \alpha < 1$, $\epsilon > 0$ and $M \in \mathbb{N}$. A function a is said to be a (γ, ϵ)-critical (L, M, α, p) atom if there exists a dyadic cube $Q \in \mathcal{D}$ so that*

(i) $a = L^M b$;

(ii) $\operatorname{supp} L^k b \subset 3Q, k = 0, 1, \ldots, M$;

(iii) $\|L^k b\|_\infty \le \frac{\ell(Q)^{2(M-k)+\alpha}}{\mu(Q)^{1/p}} \Big(1 + \frac{\ell(Q)}{\gamma(x)}\Big)^{-\epsilon}, k = 0, 1, \ldots, M.$

(iv) $|L^k b(x) - L^k b(y)| \le \Big(\frac{d(x,y)}{\ell(Q)}\Big)^\epsilon \frac{\ell(Q)^{2(M-k)+\alpha}}{\mu(Q)^{1/p}}, k = 0, 1, \ldots, M.$

Obviously, each (L, M, α, p) atom (ϵ-smooth (L, M, α, p) atom, (γ, ϵ)-critical (L, M, α, p) atom, respectively) is also a (L, M, α, p, β) molecule (ϵ-smooth (L, M, α, p, β) molecule, (γ, ϵ)-critical (L, M, α, p, β) molecule, respectively).

It is interesting that similar to the classical case, the function in Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ admits an atomic decomposition.

Theorem 5.36. *Let γ be a critical function on X. Let L be a nonnegative and self-adjoint operator on $L^2(X)$. Let $-1 < \alpha < 1$, $1 \le p \le \infty$, $\epsilon > 0$ and $M > N + n$. Assume that $f \in L^2(X)$.*

(a) *If L satisfies* (G) *then the following statements are equivalent:*

(i) $f \in \dot{B}^{\alpha,L}_{p,p}(X)$;

(ii) *there exist a sequence of coefficients $0 \le s_Q < \infty$, where Q ranges over the dyadic cubes, and a sequence a_Q of (L, M, α, p)-atoms, such that*

$$f = \sum_{Q \in \mathcal{D}} s_Q a_Q \quad \text{in } L^2(X);$$

and

$$\Big[\sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_k} |s_Q|^p\Big]^{1/p} < \infty. \tag{5.33}$$

(b) *If L satisfies* (G), (C) *and* (H) *then the following statements are equivalent:*

(i) $f \in \dot{B}^{\alpha,L}_{p,p}(X)$ *with $-1 < \alpha < \epsilon$;*

(ii) *there exist a sequence of coefficients $0 \le s_Q < \infty$, where Q ranges over the dyadic cubes, and a sequence a_Q of ϵ-smooth (L, M, α, p)-atoms with $\epsilon \in (0, \delta]$, such that*

$$f = \sum_{Q \in \mathcal{D}} s_Q a_Q \quad \text{in } L^2(X);$$

and

$$\Big[\sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_k} |s_Q|^p\Big]^{1/p} < \infty. \tag{5.34}$$

(c) *If L satisfies* (I)–(II) *then the following statements are equivalent:*

(i) $f \in \dot{B}^{\alpha,L}_{p,p}(X)$ *with $-1 < \alpha < \epsilon$;*

(ii) *there exist a sequence of coefficients $0 \le s_Q < \infty$, where Q ranges over the dyadic cubes, and a sequence a_Q of (γ, ϵ)-critical (L, M, α, p)-atoms with $\epsilon \in (0, \delta_0]$, such that*

$$f = \sum_{Q \in \mathcal{D}} s_Q a_Q \quad \text{in } L^2(X);$$

and

$$\Big[\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_k}|s_Q|^p\Big]^{1/p}<\infty. \tag{5.35}$$

Proof. (a) $(ii)\to(i)$: The proof is similar to that of Theorem 5.22.

$(i)\to(ii)$: Let $f\in L^2(X)$. By spectral theory, we have

$$f(x)=C_M\int_0^\infty (t^2L)^M\Phi(t\sqrt{L})\Psi_t(L)f(x)\frac{dt}{t}$$

which converges in $L^2(X)$ almost everywhere, where Φ is as in Lemma 5.33.

For each $Q\in\mathcal{D}_k$, we set $\widehat{Q}=Q\times[2^{k-1},2^k]$. We then have

$$\begin{aligned}f(x)&=C_M\sum_{Q\in\mathcal{D}}\int_0^\infty (t^2L)^M\Phi(t\sqrt{L})[\Psi_t(L)f.\chi_{\widehat{Q}}](x)\frac{dt}{t}\\&=C_M\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_k}\int_0^\infty (t^2L)^M\Phi(t\sqrt{L})[\Psi_t(L)f.\chi_{\widehat{Q}}](x)\frac{dt}{t}.\end{aligned}$$

For each $k\in\mathbb{Z}$ and $Q\in\mathcal{D}_k$, we set

$$s_Q=\Big(\int_{2^{k-1}}^{2^k}t^{-\alpha p}\int_Q|\Psi_t(L)f(y)|^p d\mu(y)\frac{dt}{t}\Big)^{1/p}$$

and

$$a_Q=\frac{1}{s_Q}\int_0^\infty (t^2L)^M\Phi(t\sqrt{L})[\Psi_t(L)f.\chi_{\widehat{Q}}](x)\frac{dt}{t}.$$

It is clear that $f=\sum_{Q\in\mathcal{D}}s_Qa_Q$ in $L^2(X)$. Moreover, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_k}|s_Q|^p&=\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_k}\int_{2^{k-1}}^{2^k}t^{-\alpha p}\int_Q|\Psi_t(L)f(y)|^p d\mu(y)\frac{dt}{t}\\&=\int_0^\infty t^{-\alpha p}\int_X|\Psi_t(L)f(y)|^p d\mu(y)\frac{dt}{t}\\&:=\|f\|^p_{\dot{B}^{\alpha,L}_{p,p}(X)}.\end{aligned}$$

We now claim that for each $Q\in\mathcal{D}_k, k\in\mathbb{Z}$, a_Q is a multiple of an (L,M,α,p)-atom with a harmless multiple constant. Indeed, we have $a_Q=L^Mb_Q$ with

$$b_Q=\frac{1}{s_Q}\int_0^\infty t^{2M}\Phi(t\sqrt{L})[\Psi_t(L)f.\chi_{\widehat{Q}}](x)\frac{dt}{t}.$$

For each $m=0,1,\dots,M$, we have

$$L^mb_Q(x)=\frac{1}{s_Q}\int_{2^{k-1}}^{2^k}\int_Q t^{2(M-m)}K_{(t^2L)^m\Phi(t\sqrt{L})}(x,y)\Psi_t(L)f(y)d\mu(y)\frac{dt}{t}$$

which together with Lemma 5.33 implies that $\operatorname{supp} L^m b_Q \subset 3Q$. Moreover, from Lemma 5.33 and the doubling property, we obtain that

$$K_{(t^2L)^m\Phi(t\sqrt{L})}(x,y) \lesssim \frac{1}{\mu(B(y,t))} \lesssim \frac{1}{\mu(Q)}.$$

This together with Hölder's inequality implies

$$\begin{aligned}
|L^m b_Q(x)| &\lesssim \frac{2^{k(M-m)}}{s_Q}\frac{2^{k\alpha}}{\mu(Q)}\int_{2^{k-1}}^{2^k} t^{-\alpha}\int_Q |\Psi_t(L)f(y)|d\mu(y)\frac{dt}{t} \\
&\lesssim \frac{2^{k(M-m)}}{s_Q}\frac{2^{k\alpha}}{\mu(Q)^{1/p}}\Big(\int_{2^{k-1}}^{2^k} t^{-\alpha p}\int_Q |\Psi_t(L)f(y)d\mu(y)|^p\frac{dt}{t}\Big)^{1/p} \\
&\lesssim \frac{2^{k(M-m)}2^{k\alpha}s_Q}{s_Q\mu(Q)^{1/p}} \\
&\lesssim \frac{\ell(Q)^{2(M-m)+\alpha}}{\mu(Q)^{1/p}}.
\end{aligned} \tag{5.36}$$

This completes the proof of (a).

The proofs of (b) and (c) are similar to that in (a) of which we use (iii) and (iv) in place of (ii) in Lemma 5.33 and hence we omit the details here. □

We are unable to address the atomic decomposition for Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ with $p \neq q$. This problem would be an interesting open problem and we leave it to the interested reader.

5.5 Relationship between the classical Besov spaces and $\dot{B}^{\alpha,L}_{p,q}(X)$ spaces

5.5.1 Coincidence of $\dot{B}^{\alpha}_{p,q}(X)$ and $\dot{B}^{\alpha,L}_{p,q}(X)$

In this subsection we assume that

$$\mu(B(x,r)) \approx r^n \tag{5.37}$$

for all $x \in X$ and $r > 0$. We now recall the definition of Besov spaces $\dot{B}^{\alpha}_{p,q}(X)$ on spaces of homogeneous type in [HS, HMY].

Definition 5.37. *Let $\epsilon > 0$. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of bounded linear operators on $L^2(X)$ is said to be an approximation of the identity of order ϵ if there exists a constant $C > 0$ such that for all $k \in Z$ and all $x, x', y, y' \in X$, the integral kernel $S_k(x, y)$ of S_k satisfies*

(i) $|S_k(x,y)| \le C\dfrac{2^k\epsilon}{(2^k + d(x,y))^{n+\epsilon}}$;

(ii) $|S_k(x,y) - S_k(x',y)| + |S_k(y,x) - S_k(y,x')| \le C\Big(\dfrac{d(x,x')}{2^k + d(x,y)}\Big)^{\epsilon}\dfrac{2^{k\epsilon}}{(2^k + d(x,y))^{n+\epsilon}}$

whenever $d(x,x') \le \frac{1}{2}(2^k + d(x,y))$;

(iii) *If* $\max\{d(x,x'), d(y,y')\} \le \frac{1}{2}(2^k + d(x,y))$ *then*

$$|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]| \\ \le C\Big(\frac{d(x,x')}{2^k + d(x,y)}\Big)^{\epsilon}\Big(\frac{d(y,y')}{2^k + d(x,y)}\Big)^{\epsilon}\frac{2^{k\epsilon}}{(2^k + d(x,y))^{n+\epsilon}};$$

(iv) $\displaystyle\int_X S_k(x,y)d\mu(x) = \int_X S_k(x,y)d\mu(y) = 1.$

Note that if $\{S_k\}_k$ satisfies (i), (ii) and (iv) then $\{S_k^2\}$ satisfies (i)–(iv). See for example [HMY]. It was proved in [HMY] that under condition (5.37) we always construct an approximation of the identity of order $\theta > 0$ and we fixed the constant θ.

Definition 5.38 ([HMY])**.** *Fix $\epsilon > 0$ and $0 < \beta \le \theta$. A function f defined on X is said to be a test function of type (x_0, r, β, γ) with $x_0 \in X$ and $r > 0$ if the following holds:*

(i) $|f(x)| \le C\dfrac{r^{\epsilon}}{(r + d(x,x_0))^{n+\epsilon}}$;

(ii) $|f(x) - f(y)| \le C\Big(\dfrac{d(x,y)}{r + d(x,x_0)}\Big)^{\beta}\dfrac{r^{\epsilon}}{(r + d(x,x_0))^{n+\epsilon}}$ *for* $d(x,y) \le (r + d(x,x_0))/2$.

If f is a test function of type (x_0, r, β, γ), we write $f \in \mathcal{G}(x_0, r, \beta, \epsilon)$ and we set

$$\|f\|_{\mathcal{G}(x_0,r,\beta,\epsilon)} = \inf\{C : \text{(i) and (ii) hold}\}.$$

Now fix $x_0 \in X$ and let $\mathcal{G}(\beta,\epsilon) = \mathcal{G}(x_0, 1, \beta, \epsilon)$. It is easy to see that $\mathcal{G}(x_1, r, \beta, \epsilon) = \mathcal{G}(\beta, \epsilon)$ with equivalent norms for all $x_1 \in X$ and $r > 0$. Moreover, $\mathcal{G}(\beta,\epsilon)$ is a Banach space. We let $\overset{\circ}{\mathcal{G}}(\beta,\epsilon)$ be a completion of the space $\mathcal{G}(\theta,\theta)$ when $0 < \beta, \epsilon < \theta$ and let $(\overset{\circ}{\mathcal{G}}(\beta,\epsilon))'$ be the dual space of all continuous linear functionals on $\overset{\circ}{\mathcal{G}}(\beta,\epsilon)$. See for example [HS, HMY].

Definition 5.39. *Let $-\theta < \alpha < \theta$ and $1 \le p, q \le \infty$. Let $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation of the identity of order $\theta > 0$. The (homogeneous) Besov spaces $\dot{B}^{\alpha}_{p,q}(X)$ are defined as the set of all functions $f \in (\overset{\circ}{\mathcal{G}}(\beta,\epsilon))'$ with $0 < \beta, \epsilon < \theta$ so that*

$$\|f\|_{\dot{B}^{\alpha}_{p,q}(X)} = \Big(\sum_k (2^{-k\alpha}\|D_k f\|_p)^{1/q}\Big)^{1/q},$$

where $D_k = S_k - S_{k-1}$.

It is important to note that the Besov spaces $\dot{B}^{\alpha}_{p,q}(X)$ are independent of the choice of the approximation of the identity $\{S_k\}_{k\in\mathbb{Z}}$. See for example [HMY].

Definition 5.40. *Let $-\theta < \alpha < \theta$, $1 \le p \le \infty$ and $\epsilon > 0$. A function a is said to be an (ϵ, p)-smooth atom if there exist a dyadic cube Q and the constant $\kappa > 0$ is independent of a and Q so that*

(i) $\operatorname{supp} a \subset 3\kappa Q$;

(ii) $\displaystyle\int_X a(x)d\mu(x) = 0$;

(iii) $|a(x)| \le \ell(Q)^{\alpha}\mu(Q)^{-1/p}$;

(iv) $|a(x) - a(y)| \le \ell(Q)^{\alpha}\mu(Q)^{-1/p}\Big(\dfrac{d(x,y)}{\ell(Q)}\Big)^{\epsilon}$.

Without loss of generality, we may assume that $\kappa = 1$.

Definition 5.41. *Let $-\theta < \alpha < \theta$, $1 \le p \le \infty$ and $\epsilon, \beta > 0$. A function m is said to be a (β, ϵ, p)-smooth molecule if there exists a dyadic cube Q so that*

(i) $\displaystyle\int_X m(x)d\mu(x) = 0$;

(ii) $|m(x)| \le \ell(Q)^{\alpha}\mu(Q)^{-1/p}\Big(1 + \frac{d(x,x_Q)}{\ell(Q)}\Big)^{-(n+\epsilon)}$;

(iii) $|m(x)-m(y)| \le \ell(Q)^{\alpha}\mu(Q)^{-1/p}\Big(\dfrac{d(x,y)}{\ell(Q)}\Big)^{\beta}\Big[\Big(1+\dfrac{d(x,x_Q)}{\ell(Q)}\Big)^{-(n+\epsilon)} + \Big(1+\dfrac{d(y,x_Q)}{\ell(Q)}\Big)^{-(n+\epsilon)}\Big]$.

Theorem 5.42. *[HS, HMY] Let $-\theta < \alpha < \theta$ and $1 \le p, q \le \infty$. For $f \in \dot{B}^{\alpha}_{p,q}(X) \cap (\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2))'$, $0 < \beta_1, \beta_2 < \theta$ there exist a sequence of nonnegative numbers $\{s_Q\}$ and a sequence of (ϵ, p)-smooth atoms $\{a_Q\}$ with $|\alpha| < \epsilon$ so that*

$$f = \sum_{Q \in \mathcal{D}} s_Q a_Q \quad \text{in } \dot{B}^{\alpha}_{p,q}(X) \text{ and } (\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2))'$$

and

$$\Big[\sum_{k \in \mathbb{Z}} \Big[\sum_{Q \in \mathcal{D}_k} |s_Q|^p\Big]^{q/p}\Big]^{1/q} \lesssim \|f\|_{\dot{B}^{\alpha}_{p,q}(X)}$$

Conversely, if $f = \sum_{Q \in \mathcal{D}} s_Q m_Q$ where $\{s_Q\}$ is a sequence of nonnegative numbers and $\{m_Q\}$ is a sequence of (ϵ, β, p)-smooth molecules with $|\alpha| < \epsilon, \beta < \theta$ satisfying

$$\Big[\sum_{k \in \mathbb{Z}} \Big[\sum_{Q \in \mathcal{D}_k} |s_Q|^p\Big]^{q/p}\Big]^{1/q} < \infty,$$

then $f \in \dot{B}^{\alpha}_{p,q}(X)$ and

$$\|f\|_{\dot{B}^{\alpha}_{p,q}(X)} \lesssim \Big[\sum_{k \in \mathbb{Z}} \Big[\sum_{Q \in \mathcal{D}_k} |s_Q|^p\Big]^{q/p}\Big]^{1/q}.$$

Remark 5.43. *By a careful examination of the proof of Theorem 5.42, it can be verified that if $f \in L^p(X) \cap \dot{B}^{\alpha}_{p,q}(X)$ for some $1 < p < \infty$ then the series*

$$f = \sum_{Q \in \mathcal{D}} s_Q a_Q$$

converges in $L^p(X)$.[1]

We have the following results.

Theorem 5.44. *Assume that L satisfies* (G), (C) *and* (H). *Let $|\alpha| < \min\{\delta, \theta\}$ and $1 \le p, q \le \infty$. If $f \in L^1_{\mathrm{loc}}(X)$ satisfying*

$$\int_X \frac{|f(x)|}{(1 + d(x_0, x))^{n+\epsilon}} d\mu(x) < \infty$$

for some $\epsilon < \min\{\delta, \theta\}$ and some $x_0 \in X$, then the following statements are equivalent:

(a) $f \in \dot{B}^{\alpha}_{p,q}(X)$;

[1] We thank D. Yang and W. Yuan for their discussions on this point.

(b) $f \in \dot{B}^{\alpha,L}_{p,q}(X)$.

Proof. $(b) \to (a)$: Since f satisfies the growth condition, $f \in (\overset{\circ}{\mathcal{G}}(\epsilon,\epsilon))'$. It suffices to show that $\|f\|_{\dot{B}^{\alpha}_{p,q}(X)} < \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}$. Indeed, since L satisfies (G), (C) and (H), we can choose $D_k = e^{-2^{2k}L} - e^{-2^{2(k-1)}L}$. Hence,

$$\|f\|^q_{\dot{B}^{\alpha}_{p,q}(X)} = \sum_{k\in\mathbb{Z}} (2^{-k\alpha}\|(e^{-2^{2k}L} - e^{-2^{2(k-1)}L})f\|_p)^q.$$

By Minkowski's inequality and Lemma 5.4, we have

$$\begin{aligned}\|(e^{-2^{2k}L} - e^{-2^{2(k-1)}L})f\|_p &\lesssim \Big\| \int_{2^{k-1}}^{2^k} \Psi_t(L)f\frac{dt}{t}\Big\|_p \lesssim \int_{2^{k-1}}^{2^k} \|\Psi_t(L)f\|_p \frac{dt}{t} \\ &\lesssim \|\Psi_{2^{k-1}}(L)f\|_p\end{aligned}$$

which together with Proposition 5.12 implies that

$$\|f\|^q_{\dot{B}^{\alpha}_{p,q}(X)} \lesssim \sum_{k\in\mathbb{Z}} (2^{k-1}\|\Psi_{2^{k-1}}(L)f\|_p)^q \lesssim \|f\|^q_{\dot{B}^{\alpha,L}_{p,q}(X)}.$$

$(a) \to (b)$: Let $f \in \dot{B}^{\alpha}_{p,q}(X)$. We then have $f \in \dot{B}^{\alpha}_{p,q}(X) \cap (\overset{\circ}{\mathcal{G}}(\epsilon,\epsilon))'$. By Theorem 5.42, there exist a sequence of nonnegative numbers $\{s_Q\}$ and a sequence of (ϵ,p)-smooth atoms $\{a_Q\}$ with $|\alpha| < \epsilon$ so that

$$f = \sum_{Q\in\mathcal{D}} s_Q a_Q \text{ in } (\overset{\circ}{\mathcal{G}}(\epsilon,\epsilon))'.$$

On the other hand, since L satisfies (G) and (H), $p_{t,k}(x,\cdot) \in \mathcal{G}(\epsilon,\epsilon)$. Hence,

$$\Psi_t(L)f(x) = \sum_{Q\in\mathcal{D}} s_Q \Psi_t(L)a_Q(x), \quad \text{a.e. } x \in X.$$

We need only to show that $\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} < \infty$. Indeed, the similar argument used in Theorem 5.27 tells us that

$$|\Psi_k(L)a_Q(x)| \lesssim \frac{2^{(k-\eta)\epsilon}}{\mu(Q)^{1/p}}\Big(1 + \frac{d(x,x_Q)}{2^\eta}\Big)^{-(n+\beta)}$$

for all $x \in X$ and $Q \in \mathcal{D}_\eta$ with $k \le \eta$.

For dyadic cubes $Q \in \mathcal{D}_\eta$ with $\eta \le k$, by the cancellation property of a_Q and (iii) of

Lemma 5.1 we have

$$\begin{aligned}|\Psi_k(L)a_Q(x)| &= \Big|\int_{3Q}[p_{2^{2k},1}(x,y)-p_{2^k,1}(x,x_Q)]a_Q(y)d\mu(y)\Big|\\ &\lesssim \int_{3Q}\Big(\frac{d(x,x_Q)}{2^k}\Big)^{\delta}\frac{1}{2^{kn}}\exp\Big(-c\frac{d(x,x_Q)^2}{2^{2k}}\Big)|a_Q(y)|d\mu(y)\\ &\lesssim \Big(\frac{\ell(Q)}{2^k}\Big)^{\delta}\frac{\mu(Q)}{2^{kn}}\Big(1+\frac{d(x,x_Q)}{2^k}\Big)^{-(n+\beta)}\frac{\ell(Q)^{\alpha}}{\mu(Q)^{1/p}}\\ &\lesssim 2^{(\eta-k)(n+\delta)}\frac{\ell(Q)^{\alpha}}{\mu(Q)^{1/p}}\Big(1+\frac{d(x,x_Q)}{2^k}\Big)^{-(n+\beta)}.\end{aligned}$$

At this stage, repeating the arguments used in the proofs of Proposition 5.24 and Theorem 5.27 gives $f\in\dot{B}^{\alpha,L}_{p,q}(X)$.

This completes our proof. □

It is worth to note that although in this situation our new Besov spaces $\dot{B}^{\alpha,L}_{p,q}(X)$ and the Besov spaces $\dot{B}^{\alpha}_{p,q}(X)$ coincide, the results on the atomic and molecular decompositions on $\dot{B}^{\alpha,L}_{p,q}(X)$ for $-1<\alpha<-\theta$ are new.

5.5.2 The inclusion $\dot{B}^{\alpha}_{p,q}(X)\subset\dot{B}^{\alpha,L}_{p,q}(X)$

Proposition 5.45. *Let γ be a critical function in X. Assume that L satisfies* (I)–(II). *Let $|\alpha|<\min\{\delta_0,\theta\}$ and $1\le p,q\le\infty$. If a is an (ϵ,p)-smooth atom with $\alpha<\epsilon<\min\{\delta_0,\theta\}$ then $a\in\dot{B}^{\alpha,L}_{p,q}(X)$. Moreover, if $1\le p\le q\le\infty$ and $-\min\{\delta_0,\theta\}<\alpha\le 0$ then there exists a constant $C>0$ so that*

$$\|a\|_{\dot{B}^{\alpha,L}_{p,q}(X)}\le C$$

for all (ϵ,p)-smooth atoms with $0<\epsilon<\min\{\delta_0,\theta\}$.

Proof. Assume that a is an (ϵ,p)-smooth atom associated to a dyadic cube Q. For $t>0$ we have

$$\begin{aligned}\|a\|^q_{\dot{B}^{\alpha,L}_{p,q}(X)} &= \int_0^{8\ell(Q)}(t^{-\alpha}\|\Psi_t(L)a\|_{L^p(6Q)})^q\frac{dt}{t}+\int_0^{8\ell(Q)}(t^{-\alpha}\|\Psi_t(L)a\|_{L^p(X\setminus 6Q)})^q\frac{dt}{t}\\ &\quad+\int_{8\ell(Q)}^{\infty}(t^{-\alpha}\|\Psi_t(L)a\|_p)^q\frac{dt}{t}:=A_1+A_2+A_3.\end{aligned}$$

Due to (II), the cancellation property of an (ϵ, p)-smooth atom a and Minkowski's inequality, we have

$$\begin{aligned}\|\Psi_t(L)a\|_p &= \Big(\int_X \Big|\int_{3Q}(p_{t^2,1}(x,y)-p_{t^2,1}(x,x_Q))a(y)d\mu(y)\Big|^p d\mu(x)\Big)^{1/p}\\ &\lesssim \Big(\int_X \Big|\int_{3Q}\Big(\frac{d(y,x_Q)}{t}\Big)^{\delta_0}\frac{1}{V(x,t)}\exp\Big(-c\frac{d(x,y)^2}{t^2}\Big)|a(y)|d\mu(y)\Big|^p d\mu(x)\Big)^{1/p}\\ &\lesssim \Big(\frac{\ell(Q)}{t}\Big)^{\delta_0}\frac{\ell(Q)^\alpha}{\mu(Q)^{-1/p'}}\sup_{y\in 3Q}\Big(\int_X\Big|\frac{1}{V(x,t)}\exp\Big(-c\frac{d(x,y)^2}{t^2}\Big)\Big|^p d\mu(x)\Big)^{1/p}.\end{aligned}$$

It can be verified that

$$\Big(\int_X\Big|\frac{1}{V(x,t)}\exp\Big(-c\frac{d(x,y)^2}{t^2}\Big)\Big|^p d\mu(x)\Big)^{1/p}\lesssim\frac{1}{V(y,t)^{1/p'}}.$$

This implies that

$$\|\Psi_t(L)a\|_p\lesssim\Big(\frac{\ell(Q)}{t}\Big)^{\delta_0}\frac{\ell(Q)^\alpha}{\mu(Q)^{-1/p'}}\sup_{y\in 3Q}\frac{1}{V(y,t)^{1/p'}}\lesssim\Big(\frac{\ell(Q)}{t}\Big)^{\delta_0}\ell(Q)^\alpha$$

which implies $A_3\le C$.

Using the Gaussian upper bound of the kernel of $\Psi_t(L)$ we have

$$\begin{aligned}\|\Psi_t(L)a\|_{L^p(X\setminus 6Q)}&\lesssim\exp\Big(-c\frac{d(3Q,X\setminus 6Q)^2}{t^2}\Big)\|a\|_p\\ &\lesssim\ell(Q)^\alpha\Big(\frac{t}{\ell(Q)}\Big)^{\delta_0}\end{aligned}$$

which implies $A_2\le C$.

To estimate the term A_1, observe that

$$\begin{aligned}\|\Psi_t(L)a\|_{L^p(6Q)}\le&\Big(\int_{6Q}\Big|\int_{3Q}p_{t^2,1}(x,y)(a(y)-a(x))d\mu(y)\Big|^p d\mu(x)\Big)\\ &+\Big(\int_{6Q}\Big|\int_{3Q}p_{t^2,1}(x,y)a(x)d\mu(y)\Big|^p d\mu(x)\Big)^{1/p}.\end{aligned}$$

By the smoothness condition of the atom a, we have

$$\begin{aligned}\Big|\int_{3Q}p_{t^2,1}(x,y)(a(y)-a(x))d\mu(y)\Big|&\lesssim\frac{\ell(Q)^\alpha}{\mu(Q)^{1/p}}\int_{3Q}\frac{1}{V(x,t)}\exp\Big(-c\frac{d(x,y)^2}{t^2}\Big)\Big(\frac{d(x,y)}{\ell(Q)}\Big)^\epsilon d\mu(y)\\ &\lesssim\frac{\ell(Q)^\alpha}{\mu(Q)^{1/p}}\Big(\frac{t}{\ell(Q)}\Big)^\epsilon.\end{aligned}\tag{5.38}$$

Invoking condition (II) gives

$$\Big|\int_{3Q}p_{t^2,1}(x,y)a(x)d\mu(y)\Big|\lesssim\frac{\ell(Q)^\alpha}{\mu(Q)^{1/p}}\Big(\frac{t}{\gamma(x)}\Big)^{\delta_0}.$$

If $\ell(Q) \leq \gamma(x_Q)$, then Lemma (5.28) implies that $\gamma(x_Q) \approx \gamma(x), x \in 6Q$. Hence, in this situation we have

$$\Big|\int_{3Q} p_{t^2,1}(x,y)a(x)d\mu(y)\Big| \lesssim \frac{\ell(Q)^\alpha}{\mu(Q)^{1/p}}\Big(\frac{t}{\gamma(x_Q)}\Big)^{\delta_0} \lesssim \frac{\ell(Q)^\alpha}{\mu(Q)^{1/p}}\Big(\frac{t}{\ell(Q)}\Big)^{\delta_0}. \tag{5.39}$$

Taking (5.38) and (5.39) into account leads to

$$\|\Psi_t(L)a\|_{L^p(6Q)} \lesssim \ell(Q)^\alpha\Big(\frac{t}{\ell(Q)}\Big)^{\epsilon}.$$

This implies that $A_1 \leq C$.

If $\ell(Q) > \gamma(x_Q)$, Lemma 5.28 deduces that $\gamma(x) \geq C_Q\gamma(x_Q)$. Hence,

$$\Big|\int_{3Q} p_{t^2,1}(x,y)a(x)d\mu(y)\Big| \leq c_Q\frac{\ell(Q)^\alpha}{\mu(Q)^{1/p}}\Big(\frac{t}{\gamma(x_Q)}\Big)^{\delta_0} \leq C'_Q\frac{\ell(Q)^\alpha}{\mu(Q)^{1/p}}\Big(\frac{t}{\ell(Q)}\Big)^{\delta_0}. \tag{5.40}$$

Taking (5.38) and (5.39) into account leads to

$$\|\Psi_t(L)a\|_{L^p(6Q)} \lesssim \ell(Q)^\alpha\Big(\frac{t}{\ell(Q)}\Big)^{\epsilon}.$$

This implies that $A_1 \leq \widetilde{C}_Q$.

In the particular case when $1 \leq p \leq q \leq \infty$, $-\min\{\delta_0, \theta\} < \alpha \leq 0$ and $\ell(Q) > \gamma(x_Q)$, by condition (II) we get that

$$\Big|\int_{3Q} p_{t^2,1}(x,y)a(x)d\mu(y)\Big| \lesssim \frac{\ell(Q)^\alpha}{\mu(Q)^{1/p}}\Big(\frac{t}{\gamma(x)}\Big)^{\delta_0}\Big(1+\frac{t}{\gamma(x)}\Big)^{-K}$$

for some $K > \delta_0$. By using Minkowski's inequality, (5.38) and the fact that

$$\int_0^\infty \Big|\frac{\ell(Q)^\alpha}{t^\alpha}\Big(\frac{t}{\gamma(x)}\Big)^{\delta_0}\Big(1+\frac{t}{\gamma(x)}\Big)^{-K}\Big|^{q/p}\frac{dt}{t} \leq C,$$

we arrive at $A_1 \leq C$. This completes our proof. □

As a direct consequence of the proposition above, we have the following result.

Corollary 5.46. *Assume that L satisfies* (I)–(II). *Let $1 \leq p \leq q \leq \infty$, $-\min\{\delta_0, \theta\} < \alpha \leq 0$ and $f \in \cup_{1<r<\infty} L^r(X)$. If $f \in \dot{B}^\alpha_{1,1}(X)$ then $f \in \dot{B}^{\alpha,L}_{1,1}(X)$. Moreover, we have*

$$\|f\|_{\dot{B}^{\alpha,L}_{1,1}(X)} \lesssim \|f\|_{\dot{B}^{\alpha}_{1,1}(X)}.$$

Definition 5.47. *Let $-\theta < \alpha < \theta$, $1 \le p \le \infty$ and $\epsilon > 0$. A function a is said to be a special (γ, ϵ, p)-smooth atom if there exists a dyadic cube Q so that*

(i) $\operatorname{supp} a \subset 3Q$;

(ii) $\displaystyle\int_X a(x)d\mu(x) = 0$ *if* $0 < \ell(Q) < \gamma(x_Q)$;

(iii) $|a(x)| \le \ell(Q)^\alpha \mu(Q)^{-1/p}$;

(iv) $|a(x) - a(y)| \le \ell(Q)^\alpha \mu(Q)^{-1/p} \Big(\dfrac{d(x,y)}{\ell(Q)}\Big)^\epsilon$.

Proposition 5.48. *Let γ be a critical function in X. Assume that L satisfies* (I)–(II). *Let $|\alpha| < \min\{\delta_0, \theta\}$ and $1 \le p, q \le \infty$. If a is a special (γ, ϵ, p)-smooth atom with $\alpha < \epsilon < \min\{\delta_0, \theta\}$ then $a \in \dot{B}^{\alpha,L}_{p,q}(X)$. Moreover, if $1 \le p \le q \le \infty$ and $-\min\{\delta_0, \theta\} < \alpha \le 0$ then there exists $C > 0$ independent of a so that $\|a\|_{\dot{B}^{\alpha,L}_{p,q}(X)} \le C$.*

Proof. Assume that a is a special (γ, ϵ, p)-smooth atom associated to a dyadic cube Q. If $0 < \ell(Q) < \gamma(x_Q)$ then Proposition 5.48 follows by Proposition 5.45.

If $\gamma(x_Q) \le \ell(Q)$, we write

$$\begin{aligned}\|a\|^q_{\dot{B}^{\alpha,L}_{p,q}(X)} &= \int_0^{8\ell(Q)} (t^{-\alpha}\|\Psi_t(L)a\|_{L^p(6Q)})^q \frac{dt}{t} + \int_0^{8\ell(Q)} (t^{-\alpha}\|\Psi_t(L)a\|_{L^p(X\setminus 6Q)})^q \frac{dt}{t} \\ &\quad + \int_{8\ell(Q)}^\infty (t^{-\alpha}\|\Psi_t(L)a\|_p)^q \frac{dt}{t} := B_1 + B_2 + B_3.\end{aligned}$$

The terms B_1 and B_2 can be done completely similarly to those in Proposition 5.45. It remains to estimate B_3. Observe that by Lemma 5.28 $\gamma(z) \lesssim \ell(Q)$ for all $z \in 3Q$. This together with (I) and Minkowski's inequality yields that

$$\begin{aligned}\|\Psi_t(L)a\|_p &= \Big(\int_X \Big|\int_{3Q} p_{t^2,1}(x,y)a(y)d\mu(y)\Big|^p d\mu(x)\Big)^{1/p} \\ &\lesssim \Big(\int_X \Big|\int_{3Q} \Big(\frac{\gamma(y)}{t}\Big)^{\delta_0} \frac{1}{V(x,t)} \exp\Big(-c\frac{d(x,y)^2}{t^2}\Big)|a(y)|d\mu(y)\Big|^p d\mu(x)\Big)^{1/p} \\ &\lesssim \Big(\int_X \Big|\int_{3Q} \Big(\frac{\ell(Q)}{t}\Big)^{\delta_0} \frac{1}{V(x,t)} \exp\Big(-c\frac{d(x,y)^2}{t^2}\Big)|a(y)|d\mu(y)\Big|^p d\mu(x)\Big)^{1/p} \\ &\lesssim \Big(\frac{\ell(Q)}{t}\Big)^{\delta_0} \frac{\ell(Q)^\alpha}{\mu(Q)^{-1/p'}} \sup_{y\in 3Q} \Big(\int_X \Big|\frac{1}{V(x,t)} \exp\Big(-c\frac{d(x,y)^2}{t^2}\Big)\Big|^p d\mu(x)\Big)^{1/p}.\end{aligned}$$

On the other hand, we have

$$\Big(\int_X \Big|\frac{1}{V(x,t)} \exp\Big(-c\frac{d(x,y)^2}{t^2}\Big)\Big|^p d\mu(x)\Big)^{1/p} \lesssim \frac{1}{V(y,t)^{1/p'}}$$

which implies that

$$\|\Psi_t(L)a\|_p \lesssim \Big(\frac{\ell(Q)}{t}\Big)^{\delta_0} \frac{\ell(Q)^\alpha}{\mu(Q)^{-1/p'}} \sup_{y\in 3Q} \frac{1}{V(y,t)^{1/p'}} \lesssim \Big(\frac{\ell(Q)}{t}\Big)^{\delta_0} \ell(Q)^\alpha.$$

Hence, $B_3 \le C$. This completes our proof. □

As a consequence of Corollary 5.46, Proposition 5.48 and the fact that $L^2(X)$ is dense in $\dot{B}^\alpha_{1,1}(X)$ for $|\alpha| < \theta$, we deduce the following result.

Corollary 5.49. *Let γ be a critical function in X. Assume that L satisfies* (I)–(II)*. Let $|\alpha| < \min\{\delta_0, \theta\}$. We then have that $\dot{B}^\alpha_{1,1}(X)$ is a proper subset of $\dot{B}^{\alpha,L}_{1,1}(X)$.*

5.6 Applications

5.6.1 Fractional integrals

In this section we study the behaviour of fractional integrals and fractional derivatives related to L on the Besov spaces.

Let L satisfy (G). Let $0 < \gamma < 1$, $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. For $f \in \dot{B}^{\alpha,L}_{p,q}(X)$ we define $L^\gamma f$ and $L^{-\gamma} f$ by setting

$$\langle L^\gamma f, \varphi\rangle = \frac{1}{\Gamma(1-\gamma} \int_0^\infty t^{-\gamma-1} \langle tLe^{-tL} f, \varphi\rangle dt, \quad \varphi \in \mathcal{M}^{-(\alpha-2\gamma),L^*}_{p',q'} \tag{5.41}$$

and

$$\langle L^{-\gamma} f, \varphi\rangle = \frac{1}{\Gamma(\gamma} \int_0^\infty t^{\gamma-1} \langle e^{-tL} f, \varphi\rangle dt, \quad \varphi \in \mathcal{M}^{-(\alpha+2\gamma),L^*}_{p',q'}. \tag{5.42}$$

Lemma 5.50. *Assume that L satisfies* (G)*. Let $0 < \alpha < 1$ and $1 \le p, q \le \infty$. Then there exists $C > 0$ so that*

(i) $C^{-1}\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} \le \|L^{-\gamma}\|_{\dot{B}^{\alpha+2\gamma,L}_{p,q}} \le C\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}$ *provided $\alpha + 2\gamma < 1$;*

(ii) $C^{-1}\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} \le \|L^{\gamma}\|_{\dot{B}^{\alpha-2\gamma,L}_{p,q}} \le C\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}$ *provided $\alpha - 2\gamma > 0$.*

To prove this lemma, we need the following result.

Lemma 5.51. *Assume that L satisfies* (G)*. Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. Then we have*

$$\langle f, \varphi \rangle \lesssim \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} \|\varphi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}(X)}$$

for all $f \in \dot{B}^{\alpha,L}_{p,q}(X)$ and $\varphi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$.

Proof. This lemma is a direct consequence of Theorem 5.17. □

We now return to the proof of Lemma 5.50.

Proof of Lemma 5.50:

The proof of (i) is similar to that of [BDY, Theorem 4.7].

(ii) We adapt some ideas in [BDY] to our situation. We first show that $\langle L^\gamma f, \varphi \rangle$ is well-defined whenever $\varphi \in \mathcal{M}^{-(\alpha-\gamma),L^*}_{p',q'}$. Indeed, let $\varphi \in \mathcal{M}^{-(\alpha-\gamma),L^*}_{p',q'}$. We then write $\varphi = L^*\phi$. Hence,

$$\begin{aligned}
\|\varphi\|^{q'}_{\dot{B}^{-\alpha,L^*}_{p',q'}(X)} &= \int_0^\infty \left(t^\alpha \|\Psi_t(L^*)\varphi\|_{p'}\right)^{q'} \frac{dt}{t} \\
&\le \int_0^1 \left(t^{\alpha-\gamma} \|\Psi_t(L^*)\varphi\|_{p'}\right)^{q'} \frac{dt}{t} + \int_1^\infty \left(t^{\alpha-2} \|\Psi_t(L^*)t^2 L^*\phi\|_{p'}\right)^{q'} \frac{dt}{t} \\
&\lesssim \|\varphi\|^{q'}_{\dot{B}^{-(\alpha-\gamma),L^*}_{p',q'}(X)} + \|\phi\|^{q'}_{p'}
\end{aligned}$$

which implies $\varphi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$.

We now write

$$\begin{aligned}
\left|\langle L^\gamma f, \varphi \rangle\right| &\lesssim \int_0^\infty t^{-\gamma-1} \left|\langle tLe^{-tL} f, \varphi \rangle\right| dt \\
&\lesssim \int_0^1 t^{-\gamma-1} \left|\langle tLe^{-tL} f, \varphi \rangle\right| dt + \int_1^\infty t^{-\gamma-1} \left|\langle tLe^{-tL} f, \varphi \rangle\right| dt.
\end{aligned}$$

Using Lemma 5.51 leads to

$$\begin{aligned}
\int_1^\infty t^{-\gamma-1} \left|\langle tLe^{-tL} f, \varphi \rangle\right| dt &\lesssim \int_1^\infty t^{-\gamma-1} \|tLe^{-tL} f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} \|\varphi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}} \\
&\lesssim \int_1^\infty t^{-\gamma-1} \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} \|\varphi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}} \lesssim \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} \|\varphi\|_{\dot{B}^{-\alpha,L^*}_{p',q'}}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_0^1 t^{-\gamma-1} \left|\langle tLe^{-tL} f, \varphi \rangle\right| dt &= \int_0^1 t^{-2\gamma} \left|\langle \Psi_t(L) f, \varphi \rangle\right| \frac{dt}{t} \\
&\lesssim \int_0^1 t^{-2\gamma} \|\Psi_t(L) f\|_p \|\varphi\|_{p'} \frac{dt}{t}.
\end{aligned}$$

Using the fact that $\|\Psi_t(L)f\|_p \lesssim t^\alpha \|f\|_{\dot{B}^{\alpha,L}_{p,\infty}} \lesssim t^\alpha \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}$, we obtain that

$$\int_0^1 t^{-\gamma-1}\Big|\langle tLe^{-tL}f,\varphi\rangle\Big|dt \lesssim \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}\|\varphi\|_{p'}\int_0^1 t^{\alpha-2\gamma}\frac{dt}{t}$$
$$\lesssim \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}\|\varphi\|_{p'}$$

as long as $\alpha - 2\gamma > 0$.

Taking these two estimates into account, we conclude that $\langle L^\gamma f,\varphi\rangle$ is well-defined whenever $\varphi \in \mathcal{M}^{-(\alpha-\gamma),L^*}_{p',q'}$.

We first prove that $\|L^\gamma f\|_{\dot{B}^{\alpha-2\gamma,L}_{p,q}} \lesssim \|f\|_{\dot{B}^{\alpha,L}_{p,q}}$. Indeed, observe that

$$\begin{aligned}(t^2L)^\gamma\Psi_t(L)f &= \frac{t^{2\gamma}}{\Gamma(1-\gamma)}\int_0^\infty s^{-\gamma}Le^{-sL}\Psi_t(L)f\,ds\\ &= \frac{1}{\Gamma(1-\gamma)}\int_0^\infty\Big(\frac{t^2}{s}\Big)^\gamma\frac{1}{(s+\frac{3}{4}t^2)}\big[(s+\frac{3}{4}t^2)Le^{-(s+\frac{3t^2}{4})L}\big]t^2Le^{-\frac{t^2}{4}L}f\,ds\end{aligned}$$

which implies that

$$\|(t^2L)^\gamma\Psi_t(L)f\|_p \lesssim \|\Psi_{t/2}(L)f\|_p$$

provided $0<\gamma<1$.

Therefore, we have

$$\begin{aligned}\|L^\gamma f\|_{\dot{B}^{\alpha-2\gamma,L}_{p,q}} &= \Big(\int_0^\infty (t^{-(\alpha-2\gamma)}\|\Psi_t(L)(L^\gamma f)\|_p)^q\frac{dt}{t}\Big)^{1/q}\\ &= \Big(\int_0^\infty (t^{-\alpha}\|(t^2L)^\gamma\Psi_t(L)f\|_p)^q\frac{dt}{t}\Big)^{1/q}\\ &\lesssim \Big(\int_0^\infty (t^{-\alpha}\|(t^2L)^\gamma\Psi_{t/2}(L)f\|_p)^q\frac{dt}{t}\Big)^{1/q}\\ &\lesssim \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}.\end{aligned}$$

It remains to show that $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} \lesssim \|L^\gamma f\|_{\dot{B}^{\alpha-2\gamma,L}_{p,q}}$. To do this, we write

$$\begin{aligned}\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} &= \Big(\int_0^\infty (t^{-\alpha}\|\Psi_t(L)f\|_p)^q\frac{dt}{t}\Big)^{1/q}\\ &= \Big(\int_0^\infty (t^{-\alpha+2\gamma}\|(t^2L)^{-\gamma}\Psi_t(L)(L^\gamma f)\|_p)^q\frac{dt}{t}\Big)^{1/q}.\end{aligned}$$

Using the argument above we can prove that

$$\|(t^2L)^{-\gamma}\Psi_t(L)(L^\gamma f)\|_p \lesssim \Psi_{t/2}(L)(L^\gamma f)\|_p$$

which implies $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} \lesssim \|L^\gamma f\|_{\dot{B}^{\alpha-2\gamma,L}_{p,q}}$. This completes our proof.

□

5.6.2 Spectral multipliers

Let L be a nonnegative self-adjoint operator on $L^2(X)$. By spectral theory, for any bounded Borel function $m : [0,\infty) \to \mathbb{C}$, one can define the operator

$$m(L) = \int_0^\infty m(\lambda) dE(\lambda),$$

which is bounded on $L^2(X)$, where $E(\lambda)$ is the spectral resolution of L.

Let $\phi \in L^\infty(\mathbb{R})$, we consider the function

$$m(\lambda) = \lambda \int_0^\infty \phi(t) e^{-t\lambda} dt. \tag{5.43}$$

For $f \in \dot{B}^{\alpha,L}_{p,q}(X)$, we define $m(L)f$ as follows:

$$\langle m(L)f, \varphi\rangle = \int_0^\infty \phi(t) \langle tLe^{-tL} f, \varphi\rangle \frac{dt}{t}$$

for all $\varphi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$.

The main aim of this section is to prove the following result.

Theorem 5.52. *Let L be a nonnegative self-adjoint operator on $L^2(X)$ satisfying* (G)*. Let $1 < p < \infty$, $1 \le q \le \infty$ and $0 < \alpha < 1$ and let m be a function defined as in (5.43). Then there exists a constant $C > 0$ so that*

$$\|m(L)f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} \le C\|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}$$

for all $f \in \dot{B}^{\alpha,L}_{p,q}(X)$.

Proof. We first prove that $\langle m(L)f, \varphi\rangle$ is well-defined for all $f \in \dot{B}^{\alpha,L}_{p,q}(X)$ and $\varphi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$. Indeed, we write

$$\begin{aligned}
\langle m(L)f, \varphi\rangle &= \int_0^\infty \phi(t) \langle tLe^{-tL} f, \varphi\rangle \frac{dt}{t} \\
&= \int_0^1 \phi(t) \langle tLe^{-tL} f, \varphi\rangle \frac{dt}{t} + \int_1^\infty \phi(t) \langle tLe^{-tL} f, \varphi\rangle \frac{dt}{t} \\
&= \int_0^1 \phi(t^2) \langle \Psi_t(L) f, \varphi\rangle \frac{dt}{t} + \int_1^\infty \phi(t^2) \langle \Psi_t(L) f, \varphi\rangle \frac{dt}{t} := I_1 + I_2.
\end{aligned}$$

By Hölder's inequality and the fact that $\|\Psi_t(L)\|_p \lesssim t^\alpha \|f\|_{\dot{B}^{\alpha,L}_{p,\infty}(X)} \lesssim t^\alpha \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}$, we get that

$$\begin{aligned} I_1 &\lesssim \int_0^1 \|\Psi_t(L)\|_p \|\varphi\|_{p'} \frac{dt}{t} \lesssim \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} \|\varphi\|_{p'} \int_0^1 t^\alpha \frac{dt}{t} \\ &\lesssim \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} \|\varphi\|_{p'}. \end{aligned}$$

To estimate I_2, we replace $\varphi = L^*\phi$ and then use Hölder's inequality to write

$$\begin{aligned} I_2 &= \int_1^\infty \phi(t^2) \langle \Psi_t(L) f, L^*\phi \rangle \frac{dt}{t} = \int_1^\infty \phi(t^2) t^{-2} \langle t^2 L \Psi_t(L) f, \phi \rangle \frac{dt}{t} \\ &\lesssim \int_1^\infty t^{-2} \|t^2 L \Psi_t(L) f\|_p \|\phi\|_{p'} \frac{dt}{t}. \end{aligned} \tag{5.44}$$

Note that $t^2 L \Psi_t(L) f = 4t^2 L e^{-\frac{3}{4}t^2 L} \Psi_{t/2}(L) f$. Hence

$$\|t^2 L \Psi_t(L) f\|_p \lesssim \|\Psi_{t/2}(L) f\|_p.$$

Inserting this into (5.44) we get that

$$I_2 \lesssim \|\phi\|_{p'} \int_1^\infty t^{-2} \|\Psi_{t/2}(L) f\|_p \frac{dt}{t} = \|\phi\|_{p'} \int_1^\infty t^{-2+\alpha} t^{-\alpha} \|\Psi_{t/2}(L) f\|_p \frac{dt}{t}.$$

This together with Hölder's inequality and the fact that $\alpha < 1$ shows us that

$$I_2 \lesssim \|\phi\|_{p'} \Big(\int_1^\infty (t^{-\alpha} \|\Psi_{t/2}(L) f\|_p)^q \frac{dt}{t} \Big)^{1/q} \lesssim \|\phi\|_{p'} \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}.$$

The estimates of I_1 and I_2 implies that $\langle m(L) f, \varphi \rangle$ is well-defined for all $f \in \dot{B}^{\alpha,L}_{p,q}(X)$ and $\varphi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$.

By the definition of the Besov norm, we have

$$\|m(L) f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} = \Big(\int_1^\infty (t^{-\alpha} \|\Psi_{t/2}(L) m(L) f\|_p)^q \frac{dt}{t} \Big)^{1/q} = \Big(\int_1^\infty (t^{-\alpha} \|m(L) \Psi_{t/2}(L) f\|_p)^q \frac{dt}{t} \Big)^{1/q}.$$

Since $m(L)$ is bounded on $L^p(X)$ (See for example [DOS]), we have

$$\|m(L) f\|_{\dot{B}^{\alpha,L}_{p,q}(X)} \lesssim \Big(\int_1^\infty (t^{-\alpha} \|\Psi_{t/2}(L) f\|_p)^q \frac{dt}{t} \Big)^{1/q} \lesssim \|f\|_{\dot{B}^{\alpha,L}_{p,q}(X)}.$$

This completes our proof. □

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