Singular Integrals With Rough Kernels and BMO Spaces Associated To Operators

By

Georges Nader

A thesis submitted to Macquarie University
for the degree of Master of Research
Department of Mathematics and Statistics
January 2020

MACQUARIE
UNIVERSITY
FACULTY OF SCIENCE

This thesis entitled:

Singular Integrals With Rough Kernels and BMO Spaces Associated To Operators

written by Georges Nader
has been approved for the Department of Mathematics and Statistics

---

Xuan Thinh Duong

Ji Li

Date:________________

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

Typeset in $\LaTeX\,2_\varepsilon$.

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree to any other university or institution other than Macquarie University.

I also certify that the thesis is an original piece of research and it has been written by me. Any help and assistance that I have received in my research work and the preparation of the thesis itself has been appropriately acknowledged.

In addition, I certify that all information sources and literature used are indicated in the thesis.

Georges Nader

Abstract

The purpose of this thesis is to study the new criterion for boundedness of some singular integrals. The main results of this thesis are presented in four parts.

1. Recall the Hörmander condition for boundedness of singular integrals which has been an important result of the Calderón-Zygmund theory.
2. Discuss a new criterion for singular integral operators to be bounded on $L^p(X)$, $1 < p < \infty$, where X is a space of homogeneous type. This criterion is an improvement of the Hörmander condition and it has had many applications in recent research of singular integrals in the last 20 years.
3. Discuss new function spaces which suit these operators such as BMO_A spaces associated to operators.
4. Use these results to study the functional calculus of operators satisfying certain kernel estimates.

Acknowledgements

Firstly, I would like to express the deepest appreciation to my supervisor Prof. Xuan Thinh Duong for giving me a opportunity to study harmonic analysis.

My sincere thanks also goes to Dr The Anh Bui and Dr Ji Li for their helpful discussions, guidance and support.

I also would like to thank the staff in the Department of Mathematics and Statistics for giving me a great environment of research.

Last but not the least, I would like to thank my family: my parents, my sisters and my relatives for their support and encouragement throughout my studies.

Contents

1
Introduction

Harmonic analysis is a mathematical discipline that is originated with the fundamental problem of representing functions as sum of sine and cosine functions. Nowadays, harmonic analysis has been developed extensively which has had important links to other fields of mathematics, especially complex analysis, number theory and partial differential equations. Applied harmonic analysis has been instrumental in a number of engineering and industrial mathematics, such as signal processing.

A central part of modern harmonic analysis is the Calderón-Zygmund theory which was developed by many famous mathematicians since 1960's. A main aim of this theory is to study the boundedness of singular integral operators. A typical example of singular integral operator is the Hilbert transform. It is given by

$$Hf(x) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{f(x-y)}{y} dy$$

where p.v. is abbreviation for the term "principal value". Historically, the study of Hilbert transform on the real line $\mathbb{R}$ relied on complex analysis. The extension of the Hilbert transform to higher dimension spaces, namely $\mathbb{R}^n$, gives us the Riesz transform $R_j = \frac{\partial}{\partial x_j} \Delta^{-1/2}$ (where Δ denotes the Laplace operator). The boundedness of the Riesz transform R_j gives one tool to compare the norm of the partial derivatives $\frac{\partial}{\partial x_j}$ and the square root of the Laplace operator. Let us remind that the Calderón-Zygmund theory asserts a sufficient condition so-called the Hörmander integral condition for a singular integral to be of weak type $(1,1)$. Recall that an integral operator T with the associated kernel $k(x,y)$ satisfies the Hörmander integral condition if there exist $C > 0$ and $\delta > 0$ such that

$$\int_{d(x,y) \geq \delta d(x,y_1)} |k(x,y) - k(x,y_1)| d\nu(x) \leq C$$

for all $y, y_1 \in X$.

However, in practice, there are a number of operators which do not fall within the scope of the Calderón–Zygmund theory, i.e. their associated kernels do not satisfy the Hörmander integral condition. For example, in this thesis we consider the functional calculus $f(L)$ of an general operator L which enjoys only the suitable upper bound where

f is a bounded holomorphic function. One of the most typical examples for a such operator is the Schrödinger operator $L = -\Delta + V$ on $\mathbb{R}^n$ where $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. For the details, we refer to Chapter 5.

Therefore, in order to treat these operators, we require new approaches. In [4], Duong and McIntosh introduced a sufficient condition which is weaker than the Hörmander integral condition for a class of singular integrals to be of weak-type $(1,1)$. We now briefly describe the main result in [4]. Let T be a linear/sublinear operator which is bounded on $L^2(X)$ and with associated kernel $k(x,y)$. Suppose that there are operators A_t with associated kernels $a_t(x,y)$ satisfying suitable upper bound estimates. If the operators TA_t have the associated kernels K_t such that there is a constant $\delta, C > 0$ so

$$\int_{d(x,y) \geq \delta t^{\frac{1}{m}}} |k(x,y) - K_t(x,y)| d\nu(x) \leq C, \text{ for all } y \in X, \tag{1.1}$$

then the operator is of weak type $(1,1)$.

Under the particular choice of the family A_t, the condition (1.1) turns out to be the Hörmander integral condition. The flexibility of the family A_t allows us to prove the weak type estimates of singular integrals beyond the Calderón-Zygmund theory such as the functional calculus of a general operator and the generalized Riesz transforms in various settings. See for example [4].

Motivated by this problem, the main aim of this thesis is to discuss the main results in [4] and [5]. More precisely, we present the proof of the weak type $(1,1)$ estimate for singular integrals satisfying the condition (1.1). Then we also review the main results in [5] which considered a new BMO space associated to the family of operators A_t. It is interesting to note that the new BMO space in [5] is similar to the classical BMO space in the sense that it pertains a number of important properties of the classical BMO spaces such as the interpolation property with the Lebesgue spaces and the endpoint estimates in the study of the boundedness of singular integrals.

The structure of the thesis is organized as follows. In Chapter 2, we recall some backgrounds in harmonic analysis related to interpolation theorem, singular integrals and BMO spaces in the classical case $\mathbb{R}^n$ with simple settings. Chapter 3 will discuss the main results in [4]. A discussion on the theory of the new BMO space associated to a general family of operator A_t in [5] will be given in Chapter 4. We first recall the definition of the new BMO space and then reprove important properties of the new spaces such as John–Nirenberg's inequality and the interpolation property for the new BMO space with the L^p space. In Chapter 5, we consider a case study of the functional calculus of a general operator satisfying a suitable upper bounds for its heat kernel. We will show that the functional calculus fits nicely into the settings in [4] and [5], hence we obtain the weak type estimate and the endpoint estimate on the new BMO space for this operator.

2

Calderón-Zygmund Theory and BMO($\mathbb{R}^n$)

2.1 Calderón-Zygmund Theory on Singular Integrals in $\mathbb{R}^n$

We will first discuss some results for real and complex valued functions on $\mathbb{R}^n$. For $a \in \mathbb{R}^n$ and $r > 0$, we denote $B(a, r)$ the ball of center a and radius r. In addition, for every measurable sets $B, C \subset \mathbb{R}^n$ we denote $B - C$ to be $B \cap C^c$ and $|B|$ to be the measure of B. Throughout this paper, the letters c and c' will denote (possibly different) constants that are independent of the essential variables.

2.1.1 Maximal Function

For a locally integrable function f, for any $r > 0$ and for any open ball $B \subset \mathbb{R}^n$, we define the centered Hardy-Littlewood maximal function of f as

$$\mathcal{M}^c f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)| dy \tag{2.1}$$

and the uncentered Hardy-Littlewood maximal function of f as

$$Mf(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)| dy \tag{2.2}$$

where the supremum in (2.2) is taken over all open balls containing x.

Lemma 2.1.1 *The uncentered Hardy-Littlewood function and the centered Hardy-Littlewood function are equivalent in the sense:*

$$\mathcal{M}^c f(x) \leq Mf(x) \leq c\mathcal{M}^c f(x)$$

for every locally integrable function f.

Proof. The first inequality is clear. On the other hand, if $x \in B(x_0, r)$, then $B(x_0, r) \subset B(x, 2r)$ and

$$\begin{aligned}\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}|f(y)|dy &\leq \frac{|B(x,2r)|}{|B(x_0,r)|}\frac{1}{|B(x,2r)|}\int_{B(x,2r)}|f(y)|dy\\ &\leq c\mathcal{M}^c f(x).\end{aligned}$$

Hence, the second inequality is obtained by taking the supremum over all the balls containing x.

The following lemma and theorem illustrate some properties of the Hardy-Littlewood maximal functions.

Lemma 2.1.2 *Let f be a locally integrable function. Then $E_\lambda = \{x \in \mathbb{R}^n : Mf(x) > \lambda\}$ is an open set.*

Proof. Let $x \in E_\lambda$. There exists a ball B containing x such that:

$$\frac{1}{|B|}\int_B |f(y)|dy > \lambda.$$

Hence, for every $z \in B$ we have $Mf(z) > \lambda$, and

$$B \subset E_\lambda.$$

Therefore, E_λ is open in $\mathbb{R}^n$.

Lemma 2.1.3 *Let X be a measurable subset of $\mathbb{R}^n$ covered by a family $\{B_\alpha\}_{\alpha\in I}$ of balls of bounded diameters. Then there exists a disjoint sequence B_{α_i} of these balls such that*

$$|X| \leq 5^n \sum_i |B_{\alpha_i}|. \tag{2.3}$$

For the proof of the lemma see chapter 1, page 9 of [9].

Theorem 2.1.4 *Let f be a function defined on $\mathbb{R}^n$.*

a. If $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, then Mf is finite almost everywhere.

b. If $f \in L^1(\mathbb{R}^n)$, then $|\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| \leq \frac{c}{\alpha}\|f\|_1$.

c. If $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, then $\|M(f)\|_p \leq c_p\|f\|_p$.
In the above estimates c and c_p are constants depend on the dimension n.

Proof. For the case where $f = 0$ the proof is direct. Now if $f \neq 0$ and $\alpha > 0$, let $E_\alpha = \{x \in \mathbb{R}^n, Mf(x) > \alpha\}$. Thus, for each $x \in E_\alpha$, there is $r_x > 0$ such that

$$|B(x, r_x)|^{-1}\int_{B(x,r_x)}|f(x)|dx > \alpha.$$

Therefore,

$$|B(x, r_x)| < \frac{1}{\alpha}\int_{B(x,r_x)}|f(x)|dx.$$

In addition, we have

$$E_\alpha \subseteq \underset{x\in E_\alpha}{\cup} B(x, r_x).$$

Hence, by using Lemma 2.1.3 there is a mutually disjoint family of balls $B(x_i, r_{x_i})$ such that

$$5^n \sum_i |B(x, r_{x_i})| \geq |E_\alpha|.$$

Thus,

$$|E_\alpha| \leq c \sum_i |B(x, r_{x_i})| \leq \frac{c}{\alpha} \int_{\cup B(x, r_{x_i})} |f(x)| dx < \frac{c}{\alpha} \parallel f \parallel_1$$

which proves (b).

We now prove (c).
For $p = \infty$ we have

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(x)| dx \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} \|f\|_\infty dx = \|f\|_\infty.$$

Thus, $\|M(f)\|_\infty \leq \|f\|_\infty$.
Suppose that $1 < p < \infty$ and let $f_1 = 1_{\{x, |f(x)| \geq \frac{\alpha}{2}\}} f(x)$. Note that if $f \in L^p(\mathbb{R}^n)$, then $|\{x \in \mathbb{R}^n, |f(x)| \geq \frac{\alpha}{2}|\} < \infty$. Thus, by using Hölder's inequality we have

$$\int_{\mathbb{R}^n} |f_1(x)| dx = \int_{\mathbb{R}^n} |f(x)| 1_{\{x\in\mathbb{R}^n : |f(x)| \geq \frac{\alpha}{2}\}} dx \leq \|f\|_p |\{x \in \mathbb{R}^n \; |f(x)| \geq \frac{\alpha}{2}\}|^{\frac{1}{q}}.$$

Hence, $f_1 \in L^1(\mathbb{R}^n)$. In addition, we have $M(f) < M(f_1) + \frac{\alpha}{2}$, so

$$|E_\alpha| \leq |\{x \in \mathbb{R}^n : M(f_1)(x) > \tfrac{\alpha}{2}\}| \leq \tfrac{c}{\alpha} \|f_1\|_1.$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^n} M(f)(x)^p dx &= p \int_0^\infty \alpha^{p-1} |\{x \in \mathbb{R}^n ; M(f(x)) > \alpha\}| d\alpha \\
&\leq c \int_0^\infty \alpha^{p-2} \int_{\{x\in\mathbb{R}^n ; |f(x)| > \frac{\alpha}{2}\}} |f(x)| dx d\alpha \\
&= c \int_{\mathbb{R}^n} |f(x)| \int_0^{2|f(x)|} \alpha^{p-2} d\alpha dx \qquad \text{(due to Fubbini's theorem)} \\
&= c \int_{\mathbb{R}^n} |f(x)| |2f(x)|^{p-1} dx = c \|f\|_p
\end{aligned}$$

which proves (c) and then (a).

Definition 2.1.5 *Let T be a mapping from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. Then we say*

(i) T is of type (p, q) if T is bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

(ii) T is of weak-type (p, q), for $q = \infty$, if T is bounded operator from $L^p(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

(iii) T is of weak-type (p, q) , for $q < \infty$, if for every $\alpha > 0$ and $f \in L^p(\mathbb{R}^n)$ we have

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \alpha\}| \leq (\frac{c\|f\|_p}{\alpha})^q.$$

Lemma 2.1.6 *If T is of type (p,q), then T is of weak-type (p,q).*

Proof. For $q<\infty$, we have

$$|\{x\in\mathbb{R}^n;|Tf(x)|>\alpha\}|\le\int_{|\{x\in\mathbb{R}^n;|Tf(x)|>\alpha\}|}\frac{|Tf(y)|^q}{\alpha^q}dy\le c\frac{\|Tf\|_q^q}{\alpha^q}\le c\frac{\|f\|_p^q}{\alpha^q}.$$

2.1.2 Calderón-Zygmund Decomposition

In Fourier analysis, harmonic analysis and singular integrals, the Calderón-Zygmund decomposition is a fundamental result. The idea is partitioning $\mathbb{R}^n$ into two sets: one where the function is essentially small and the other where the function is essentially large but with some control.

In order to prove the Calderón-Zygmund decomposition, we first state the two lemmas below.

Lemma 2.1.7 *Given a non empty close subset E of $\mathbb{R}^n$, then its complement is a union of countable cubes Q_i, whose sides are parallel to the x-axis, whose interiors are mutually disjoint, and whose diameters are approximately proportional to their distances from E. More explicitly:*

(*a*) $\cup_i Q_i=E^c$.

(*b*) *The interiors of Q_i are mutually disjoint.*

(*c*) *There exists positive constants c_1 and c_2 so that*

$$c_1 diam(Q_i)\le d(Q_i,E)\le c_2 diam(Q_i).$$

Where $d(Q_i,E)$ is the distance between Q_i and the set E and $diam(Q_i)$ is the length of its diameter.

For the proof of the lemma see page 16 of [9].

Lemma 2.1.8 *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a non negative integrable function and α be a positive constant. Then there is an open set Ω such that:*

(*i*) *Ω is a union of cubes Q_i whose interiors are mutually disjoint.*

(*ii*) $|\Omega|\le\frac{c}{\alpha}\|f\|_1$.

(*iii*) $\frac{1}{|(Q_i)|}\int_{Q_i}f(x)dx\le c\alpha$.

(*iv*) *$f(x)\le\alpha$ almost everywhere in $F=\Omega^c$.*

Proof. Let $F=\{x\in\mathbb{R}^n: Mf(x)\le\alpha\}$, F is a closed set. Thus, by using Lemma 2.1.7, we have that $\Omega=F^c$ is equal to union of countable cubes Q_i that verifies (a), (b) and (c). Furthermore, Theorem 2.1.4 (b) shows that $|\Omega|\le\frac{c}{\alpha}\|f\|_1$.
Let Q_i be one of these cubes and let $\overline{B}_i$ be a closed sphere that contains Q_i and intersects F. Let $\{x_n\}$ be a sequence of points in $\overline{B}_i\cap F$ such that $d(x_n,Q_i)$ converges to $d(Q_i,F)$ where $d(x_n,Q_i)$ is the distance between the point x_n and the set Q_i and where $d(Q_i,F)$ is the distance between Q_i and F. Since $\overline{B}_i$ is a compact set, there is a subsequence

$(x_{n_k})_k$ that converges to a point p_i. F is a closed set, thus $p_i \in F$ and the distance $d(p_i, Q_i) = d(F, Q_i)$. Let $B(p_i, r_i)$ be the smallest ball which contains Q_i. Furthermore, Lemma 2.1.7 and some elementary geometry show that the ratio $\sigma_i = \frac{|B(p_i,r_i)|}{|Q_i|}$ is bounded by a constant for any Q_i. As $p_i \in F$, $Mf(p_i) \leq \alpha$. Therefore,

$$\frac{1}{|Q_i|}\int_{Q_i} f(x)dx = \frac{1}{\frac{|B(p_i,r_i)||Q_i|}{|B(p_i,r_i)|}}\int_{Q_i} f(x)dx \leq \sigma_i M(f)(p_i) \leq c\alpha.$$

Thus (iii) is proved. In addition, by using the Lebesgue differentiation theorem $f(x) \leq \alpha$ for $x \in F$. Hence, (iv) is proved.

Lemma 2.1.9 *Suppose $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$. Then there is a decomposition of f to a "good" function g and a "bad" function b, where*

$$f = g + b \text{ and where } b = \sum_i b_i$$

such that:

(i) $|g(x)| \leq c\alpha$ *for almost all* $x \in \mathbb{R}^n$.

(ii) *There is a sequence of cubes $\{Q_i\}$ with mutually disjoint interiors, such that the support of each b_i is contained in Q_i.*

(iii) $\int_{\mathbb{R}^n} b_i(x)dx = 0$ *and* $f(x) \leq \alpha$, *for* $x \in (\bigcup_i Q_i)^c$.

(iv) $\int_{Q_i} |b_i(x)|dx \leq c\alpha|Q_i|$.

(v) $\Sigma_i|Q_i| \leq \frac{c}{\alpha}\int |f(x)|dx$.

Proof. Using Ω, F and the sequence of cubes $\{Q_i\}$ defined in Lemma 2.1.8 for the function $|f|$, let

$$g(x) = \begin{cases} f(x), & x \in F \\ \frac{1}{|Q_i|}\int_{Q_i} f(y)dy, & x \in Q_i \end{cases}$$

and

$$b_i(x) = \begin{cases} 0, & x \in F \\ f(x) - \frac{1}{|Q_i|}\int_{Q_i} f(y)dy, & x \in Q_i. \end{cases}$$

Property (iii) is obtained by integration. However, $(i), (ii)$ and (v) are obtained directly from Lemma 2.1.8.
Furthermore,

$$\int_{\mathbb{R}^n} |b_i|dx \leq 2\int_{Q_i} |f(x)|dx \leq c\alpha|Q_i|.$$

Hence, (iv) is proved. In addition, by using (iv) and (v) we have $\int |b(x)|dx \leq c\int |f(x)|dx$.

2.1.3 Marcinkiewicz Interpolation Theorem

The interpolation theorem is used to prove the boundedness of an operator. In order to prove this theorem, we need to discuss some properties of $L^p(\mathbb{R}^n)+L^q(\mathbb{R}^n)$, for $1 \le p \le \infty$, $1 \le q \le \infty$. We define $L^p(\mathbb{R}^n)+L^q(\mathbb{R}^n)$ to be the space of all functions, so that $f = f_1+f_2$, with $f_1 \in L^p(\mathbb{R}^n)$ and $f_2 \in L^q(\mathbb{R}^n)$.

Lemma 2.1.10 *For all r such that $p \le r \le q$, we have $L^r(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n) + L^q(\mathbb{R}^n)$.*

Proof. For $f \in L^r(\mathbb{R}^n)$ and $\alpha > 0$, let

$$f_1(x) = \begin{cases} f(x), & |f(x)| > \alpha \\ 0, & |f(x)| \le \alpha \end{cases}$$

and

$$f_2(x) = \begin{cases} f(x), & |f(x)| \le \alpha \\ 0, & |f(x)| > \alpha. \end{cases}$$

So

$$\int |f_1(x)|^p = \int_{\{x\in\mathbb{R}^n, |f(x)|>\alpha\}} |f_1(x)|^r |f_1(x)|^{p-r} dx \le \alpha^{p-r} \int |f(x)|^r dx < \infty$$

and

$$\int |f_2(x)|^q dx = \int |f_2(x)|^r |f_2(x)|^{q-r} dx \le \alpha^{q-r} \int |f(x)|^r dx < \infty.$$

Hence, $f_1 \in L^p(\mathbb{R}^n)$, $f_2 \in L^q(\mathbb{R}^n)$ and $f(x) = f_1(x) + f_2(x)$.

Theorem 2.1.11 *Suppose that $1 < r \le \infty$. Let T be a sublinear operator which is of weak-type $(1,1)$ and of weak-type (r,r). Then T is of type (p,p) for all p such that $1 < p < r$. In another way, for every $1 < p < r$, if $f \in L^p(\mathbb{R}^n)$, then*

$$\|T(f)\|_p \le A_p \|f\|_p.$$

Proof. First consider the case $r < \infty$. Let $f \in L^p(\mathbb{R}^n)$ and $\alpha > 0$. Then by using the decomposition in Lemma 2.1.10, there are $f_1 \in L^1(\mathbb{R}^n)$ and $f_2 \in L^r(\mathbb{R}^n)$ such that $f(x) = f_1(x) + f_2(x)$.
We have

$$|T(f)(x)| \le |T(f_1)(x)| + |T(f_2)(x)|.$$

Therefore,

$$\{x \in \mathbb{R}^n; |T(f)(x)| \ge \alpha\} \subseteq \{x \in \mathbb{R}^n; |T(f_1)(x)| \ge \frac{\alpha}{2}\} \cup \{x \in \mathbb{R}^n; |T(f_2)(x)| \ge \frac{\alpha}{2}\}.$$

Hence,

$$\int_{\mathbb{R}^n} |T(f)(x)|^p dx = p \int_0^\infty \alpha^{p-1} |\{x \in \mathbb{R}^n, |T(f)| \ge \alpha\}| d\alpha$$
$$\le p \int_0^\infty \alpha^{p-1} (|\{x \in \mathbb{R}^n; |T(f_1)(x)| \ge \frac{\alpha}{2}\}| + |\{x \in \mathbb{R}^n; |T(f_2)(x)| \ge \frac{\alpha}{2}\}|) d\alpha.$$

In addition, using the weak type $(1,1)$ and the weak type (r,r) of T we have

$$\begin{aligned}\int_0^\infty \alpha^{p-1}(|\{x\in\mathbb{R}^n; |T(f_1)(x)|\geq \frac{\alpha}{2}\}|)d\alpha &\leq c\int_0^\infty \alpha^{p-2}\int_{\mathbb{R}^n}|f_1(x)|dxd\alpha\\ &\leq c\int_0^\infty \alpha^{p-2}\int_{\{x\in\mathbb{R}^n;|f(x)|>\alpha\}}|f(x)|dxd\alpha\\ &\leq c\int_{\mathbb{R}^n}|f(x)|\int_0^{|f(x)|}\alpha^{p-2}d\alpha dx\\ &\leq c\int_{\mathbb{R}^n}|f(x)|^pdx\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \alpha^{p-1}(|\{x\in\mathbb{R}^n; |T(f_2)(x)|\geq \frac{\alpha}{2}\}|)d\alpha &\leq c\int_0^\infty \alpha^{p-r-1}\int_{\mathbb{R}^n}|f_2(x)|^rdxd\alpha\\ &\leq c\int_0^\infty \alpha^{p-r-1}\int_{\{x\in\mathbb{R}^n;|f(x)|\leq\alpha\}}|f(x)|^rdxd\alpha\\ &\leq c\int_{\mathbb{R}^n}|f(x)|^r\int_{|f(x)|}^{\infty}\alpha^{p-1-r}d\alpha dx\\ &\leq c\int_{\mathbb{R}^n}|f(x)|^pdx.\end{aligned}$$

Thus, $\|T(f)\|_p\leq c\|f\|_p$.

Now consider the case $r=\infty$. There is $a>0$ such that for every $g\in L^\infty$, $\|T(g)\|_\infty\leq a\|g\|_\infty$. Let $f\in L^p$. Write $f=f_1^\alpha+f_2^\alpha$ such that

$$f_1^\alpha(x)=\begin{cases} f(x) & ,|f(x)|>\frac{\alpha}{2a}\\ 0 & ,|f(x)|\leq\frac{\alpha}{2a}\end{cases}$$

and

$$f_2^\alpha(x)=\begin{cases} f(x) & ,|f(x)|\leq\frac{\alpha}{2a}\\ 0 & ,|f(x)|>\frac{\alpha}{2a}.\end{cases}$$

We have $f_2^\alpha\in L^\infty(\mathbb{R}^n)$. Hence, $\|Tf_2^\alpha\|_\infty\leq a\frac{\alpha}{2a}=\frac{\alpha}{2}$. Thus,

$$|\{x\in\mathbb{R}^n:|T(f_2^\alpha)(x)|>\frac{\alpha}{2}\}|=0$$

and

$$|\{x\in\mathbb{R}^n:|T(f)(x)|>\alpha|\}|\leq|\{x\in\mathbb{R}^n:|T(f_1^\alpha)(x)|>\frac{\alpha}{2}\}|\leq\frac{c}{\alpha}\int_{|f(x)|>\frac{\alpha}{2a}}|f(x)|dx.$$

Therefore,

$$\begin{aligned}\|T(f)\|_p^p=&p\int_0^\infty\alpha^{p-1}|\{x\in\mathbb{R}^n:|T(f)(x)|>\alpha\}|d\alpha\\ \leq& c\int_0^\infty\alpha^{p-2}\int_{|f(x)|>\frac{\alpha}{2a}}|f(x)|dxd\alpha\\ \leq& c\int_{\mathbb{R}^n}|f(x)|\int_0^{2af(x)}\alpha^{p-2}d\alpha\\ \leq& c\int_{\mathbb{R}^n}|f(x)|^pdx.\end{aligned}$$

Thus the theorem is proved.

2.1.4 Singular Integral Operators with The Hörmander Condition

The following theorem studies the boundedness of some singular operators where their kernels have some specific conditions such as the condition (3) below.

Theorem 2.1.12 *Let $K \in L^2(\mathbb{R}^n)$ and T be an operator such that*

1. *for all $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, we have*

$$(Tf)(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy,$$

2. *$\hat{K} \in L^\infty(\mathbb{R}^n)$ where $\hat{K}$ is the Fourier transform of K,*

3. *there exists a constant $C > 0$, such that*

$$\int_{\{x\in\mathbb{R}^n:|x|\geqslant 2|y|\}} |K(x-y) - K(x)|dx \leq C.$$

Then T is of weak-type (1,1). Furthermore, T can be extended to be a bounded operator on $L^p(\mathbb{R}^n)$, with $1 < p < \infty$.

Proof. First, we will prove that T is bounded on $L^2(\mathbb{R}^n)$, then we will prove that T is of weak-type $(1,1)$ in order to use the interpolation theorem.
Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we have $T(f)(x) = K * f(x)$ and $(Tf)^\wedge(y) = \hat{K}(y)\hat{f}(y)$. Thus, by using Plancherel theorem and the boundedness of $\hat{K}$, we have

$$\|T(f)\|_2 \leq C\|f\|_2$$

In addition, $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. Therefore, T can be extended to be a unique bounded operator on $L^2(\mathbb{R}^n)$. Hence, T is weak-type $(2,2)$.

We now prove that T is weak-type $(1,1)$. Let $f(x) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Fix α for a moment. Then by using Lemma 2.1.8 we have $\mathbb{R}^n = F \cup \Omega$, $F \cap \Omega = \emptyset$; $|f(x)| < \alpha$, $x \in F$ and $\Omega = \underset{i}{\cup} Q_i$. Using g, b and b_i defined in the proof of Lemma 2.1.9 we have $f = g + b = g + \sum\limits_i b_i$ and

$$\begin{aligned}
\|g(x)\|_2^2 &= \int_{\Omega^c} |g(x)|^2 dx + \sum_i \int_{Q_i} |g(x)|^2 dx \\
&= \int_{\Omega^c} |f(x)|^2 dx + \sum_i (\frac{1}{|Q_i|} \int_{Q_i} |f(x)|dx)^2 \int_{Q_i} dx \\
&\leq \alpha \int_{\mathbb{R}^n} |f(x)|dx + \sum_i \frac{1}{|Q_i|} (\int_{Q_i} |f(x)|dx)^2 \\
&\leq \alpha \int_{\mathbb{R}^n} |f(x)|dx + c\alpha(\sum_i \int_{Q_i} |f(x)|dx) \leq c\alpha\|f\|_1
\end{aligned}$$

Furthermore, T is bounded on $L^2(\mathbb{R}^n)$. Thus,

$$|\{x \in \mathbb{R}^n, |T(g)(x)| > \frac{\alpha}{2}\}| \leq \frac{c\|g\|^2}{\alpha^2} \leq \frac{c\alpha\|f\|_1}{\alpha^2} = \frac{c}{\alpha}\|f\|_1.$$

Set $Q_i^* = 2n^{\frac{1}{2}} Q_i$. Let $\Omega^* = \cap Q_i^*$ and $F^* = \Omega^{*c}$. Thus,

1. $Q_i \subset Q_i^*$.
2. $|\Omega^*| \leq (2n^{\frac{1}{2}})^n |\Omega|$.
3. If $x \notin Q_i^*$, then $|x - y_i| \geq 2|y - y_i|$ for all $y \in Q_i$, with y_i is the center of Q_i.

Given that $\int_{Q_i} b_i dx = 0$, we have

$$Tb_i(x) = \int_{Q_i} (K(x-y) - K(x-y_i))b_i(y)dy.$$

and

$$\begin{aligned}
\int_{F^*} |Tb(x)|dx &\leq \sum_i \int_{\cap Q_j^{*c}} \int_{Q_i} |K(x-y) - K(x-y_i)||b(y)|dydx \\
&\leq \sum_i \int_{Q_i^{*c}} \int_{Q_i} |K(x-y) - K(x-y_i)||b(y)|dydx \\
&\leq \sum_i \int_{Q_i} \left(\int_{|x-y_i| \geq 2|y-y_i|} |K(x-y) - K(x-y_i)|dx \right) |b(y)|dy \\
&\leq c \sum_i \int_{Q_i} |b(y)|dy \\
&\leq c\|f\|_1.
\end{aligned}$$

In addition,

$$\begin{aligned}
|\{x \in \mathbb{R}^n, |Tb(x)| > \frac{\alpha}{2}\}| &= |\{x \in F^*, |Tb(x)| > \frac{\alpha}{2}\}| + |\{x \in \Omega^*, |Tb(x)| > \frac{\alpha}{2}\}| \\
&\leq \frac{c}{\alpha}\|f\|_1 + |\Omega^*| \\
&\leq \frac{c}{\alpha}\|f\|_1 + (2n^{\frac{1}{2}})^n|\Omega| \\
&\leq \frac{c}{\alpha}\|f\|_1.
\end{aligned}$$

Hence,

$$\begin{aligned}
|\{x \in \mathbb{R}^n, |Tf(x)| > \alpha| &\leq |\{x \in \mathbb{R}^n, |Tb(x)| > \frac{\alpha}{2}\}| + |\{x \in \mathbb{R}^n, |T(g)(x)| > \frac{\alpha}{2}\}| \\
&\leq \frac{c}{\alpha}\|f\|_1.
\end{aligned}$$

It follows that T is of weak-type (1,1). Hence, by using interpolation theorem, we have T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < 2$. In another way, for every $f \in L^p(\mathbb{R}^n)$ there is $c_p > 0$, such that

$$\|T(f)\|_p \leq c_p\|f\|_p$$

We now prove the boundedness in case $p > 2$. Let $C_0(\mathbb{R}^n)$ be the set of continuous functions with compact a support and q be a positive real number such that $\frac{1}{q} + \frac{1}{p} = 1$. Let

$$S_q = \{\phi \in C_0(\mathbb{R}^n) : \|\phi\|_q \leq 1\}$$

If f is locally integrable and

$$\sup_{\phi \in S_q} |\langle f, \phi \rangle| = \sup_{\phi \in S_q} |\int_{\mathbb{R}^n} f(x)\phi(x)dx| < \infty,$$

then $f \in L_p(\mathbb{R}^n)$ and $\|f\|_p = \sup\limits_{\phi \in S_q} |\langle f, \phi \rangle|$.

Let $g \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. We have $K \in L^2(\mathbb{R}^n)$, so $K * g \in L^2(\mathbb{R}^n)$. Hence, for every $\phi \in S_q$

$$\int_{\mathbb{R}^n} (K * g)(x)\phi(x) dx dy$$

converge absolutely and by using Fubini's theorem, we have

$$\langle Tg, \phi \rangle = \int_{\mathbb{R}^n} (K * g)(x)\phi(x) dx = \int_{\mathbb{R}^n} g(y) \int_{\mathbb{R}^n} K(x - y)\phi(x) dx dy.$$

Let $K^*(x) = K(-x)$ and T^* be an operator such that for all $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$,

$$(T^* f)(x) = \int_{\mathbb{R}^n} K^*(x - y) f(y) dy.$$

Let $\phi \in S_q$. Given that $p > 2$ then $q \in (1, 2)$, so by using the previous result we have T^* is bounded on $L^q(\mathbb{R}^n)$ and

$$\|T^*(\phi)\|_q \leq c_q \|\phi\|_q \leq c_q.$$

Hence,

$$\begin{aligned}
|\langle Tg, \phi \rangle| &= |\int_{\mathbb{R}^n} T(g)(x)\phi(x) dx| \\
&= |\int_{\mathbb{R}^n} g(y) \int_{\mathbb{R}^n} K(x - y)\phi(x) dx dy| \\
&= |\int_{\mathbb{R}^n} g(y) T^*(\phi)(y) dy| \\
&\leq \|g(y)\|_p \|T^*\phi\|_q \\
&\leq c_q \|g\|_p.
\end{aligned}$$

Thus, $\|T(g)\|_p \leq c_q \|g\|_p$ and this completes our proof.

Theorem 2.1.12 can be generalized as follow.

Theorem 2.1.13 *Let T be a bounded operator on $L^2(\mathbb{R}^n)$ such that:*

1. *T has an associated kernel $K(x, y)$, i.e*

$$(Tf)(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy$$

for each continuous function of compact support f and for almost all x not in the support of f.

2. *There are positive constants a and C so that :*

$$\int_{|x-y| \geq a|y-y_1|} |K(x, y) - K(x, y_1)| dx < C$$

and

$$\int_{|x-y| \geq a|x-x_1|} |K(x, y) - K(x_1, y)| dy < C.$$

Then T is of weak-type $(1, 1)$. Furthermore, T can be extended to be a bounded operator on $L^p(\mathbb{R}^n)$, with $1 < p < \infty$.

The proof of Theorem 2.1.13 is similar to the proof of Theorem 2.1.12 with some modifications and we omit the details.

2.2 BMO Spaces

Definition 2.2.1 *For a complex-valued locally integrable function f on $\mathbb{R}^n$ and for a measurable set $Q \subset \mathbb{R}^n$, we define the mean of f over Q as:*

$$f_Q = \frac{1}{|Q|} \int_Q f(x)dx$$

Definition 2.2.2 *For a complex-valued locally integrable function f on $\mathbb{R}^n$, we define*

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \tag{2.4}$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$. Let $BMO(\mathbb{R}^n)$ be the set of all locally integrable functions f on $\mathbb{R}^n$ with $\|f\|_{BMO} < \infty$.

Note that if f is a constant function then $\|f\|_{BMO} = 0$, therefore $\|.\|_{BMO}$ is not a norm. However, if $f, g \in BMO(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$, then $f + g \in BMO$, $\lambda f \in BMO$ and

$$\|\lambda f + g\|_{BMO} \leq |\lambda| \|f\|_{BMO} + \|g\|_{BMO}$$

In addition, $L^\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ and if $f \in L^\infty(\mathbb{R}^n)$ then $\|f\|_{BMO} \leq 2\|f\|_\infty$.

Definition 2.2.3 *For a complex-valued locally integrable function f on $\mathbb{R}^n$, set*

$$\|f\|_{BMO_{ball}} = \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| dx \tag{2.5}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Note that there are $a_n, b_n > 0$ such that

$$a_n \|f\|_{BMO} \leq \|f\|_{BMO_{ball}} \leq c_n \|f\|_{BMO}.$$

Proposition 2.2.4 *Let $f \in BMO(\mathbb{R}^n)$. We have the following properties:*

(1) If a cube Q_1 is contained in a cube Q_2, then

$$|f_{Q_2} - f_{Q_1}| \leq \frac{|Q_2|}{|Q_1|} \|f\|_{BMO}.$$

(2) Given a ball B and a positive integer m, we have

$$|f_B - f_{2^m B}| \leq m 2^n \|f\|_{BMO}.$$

Proof. For the inequality (1), we have

$$\begin{aligned} |f_{Q_2} - f_{Q_1}| &\leq \frac{1}{|Q_1|} \int_{Q_1} |f_{Q_2} - f(x)| dx \\ &\leq \frac{1}{|Q_1|} \int_{Q_2} |f_{Q_2} - f(x)| dx \\ &\leq \frac{|Q_2|}{|Q_1|} \|f\|_{BMO}. \end{aligned}$$

For the inequality (2), we have

$$\begin{aligned}|f_B - f_{2^m B}| &\leq \sum_{i=1}^{m} |f_{2^{i-1}B} - f_{2^i B}| \\ &\leq \sum_{i=1}^{m} \frac{|2^i B|}{|2^{i-1}B|} \|f\|_{BMO} \\ &\leq \sum_{i=1}^{m} 2^n \|f\|_{BMO} \\ &= m2^n \|f\|_{BMO}\end{aligned}$$

2.2.1 John-Nirenberg Inequality

One of the features of BMO functions is their exponential integrability. This is the main aim of the following theorem.

Theorem 2.2.5 *For $f \in BMO(\mathbb{R}^n)$, for all cubes Q, and all positive α we have:*

$$|\{x \in Q : |f(x) - f_Q| > \alpha\}| \leq c|Q|e^{\frac{-a\alpha}{\|f\|_{BMO}}}$$

with a and c are positive constants that depend on n only.

Proof. It is sufficient to prove the case when $\|f\|_{BMO} = 1$.
Let Q be a fix cube in $\mathbb{R}^n$ and a constant $b > 1$. We will set up a criterion in order to have a collection of cubes Q_i^k such that there is some control on their measures and

$$\{x \in \mathbb{R}^n : |f(x) - f_Q| > 2^n kb\} \subseteq \underset{i}{\cup} Q_i^k.$$

We will set the stopping time for a cube P:

$$\frac{1}{|P|} \int_P |f(x) - f_Q| dx > b. \tag{2.6}$$

Given $\|f\|_{BMO} = 1$, then we have that Q doesn't verify the property (2.6). Subdivide Q into 2^n equal closed subcubes with disjoint interiors. We choose the subcubes with property (2.6) and then for the subcubes that are not chosen, we subdivide them to 2^n equal closed subcubes and choose again the ones that verify (2.6). By repeating the process, we will obtain a countable set of cubes $\{Q_i^1\}$ that satisfy:

(1) $b < \frac{1}{|Q_j^1|} \int_{Q_j^1} |f(x) - f_Q| dx \leq 2^n b$.

(2) $|f_Q - f_{Q_i^1}| \leq 2^n b$.

(3) $\sum_i |Q_i^1| \leq \frac{1}{b}|Q|$.

(4) $|f - f_Q| \leq b$ almost everywhere on the set $Q - \cup_i Q_i$.

The first inequality in (1) can be obtained from property (2.6). For second inequality in (1), we have

$$\frac{1}{|Q_i^1|} \int_{Q_i^1} |f(x) - f_Q| dx \leq \frac{|Q|}{|Q||Q_i^1|} \int_{Q_i^1} |f(x) - f_Q| \leq 2^n \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq 2^n b.$$

For (2), we have $|f_Q - f_{Q_i^1}| \le \frac{1}{|Q_i^1|}\int_{Q_i^1}|f_Q - f(y)|dy \le 2^n b$.
Using (1) and the fact that Q_i^1 have disjoints interiors, we have:

$$\sum_i |Q_i^1| \le \frac{1}{b}\sum_i \int_{Q_j^1} |f(x)-f_Q|dx \le \frac{1}{b}\int_Q |f(x)-f_Q|dx \le \frac{1}{b}\|f\|_{BMO}|Q| \le \frac{1}{b}|Q|.$$

Thus, the proof of (3). Finally (4) is obtained by using the Lebesgue differentiation theorem.

We repeat the same process for each Q_i^1, but with the property

$$\frac{1}{|P|}\int_P |f(x)-f_{Q_i^1}|dx > b. \tag{2.7}$$

Hence, we obtain a countable set of cubes of $\{Q_j^2\}$. We repeat the process for all Q_i^2 to get a collection of cubes $\{Q_s^3\}$. By iteration, we will get a collection of cubes Q_i^s that verifies:

(a) The interior of each Q_j^s is included in a unique Q_k^{s-1}

(b) $b < \frac{1}{|Q_j^s|}\int_{Q_j^s}|f(x)-f_{Q_k^{s-1}}|dx \le 2^n b$.

(c) $|f_{Q_k^{s-1}} - f_{Q_j^s}| \le 2^n b$.

(d) $\sum_j |Q_j^s| \le \frac{1}{b^s}|Q|$.

(e) $|f - f_{Q_j^{s-1}}| \le b$ for almost everywhere on the set $Q_k^{s-1} - \cup_j Q_j^s$.

The proof of these properties is similar to the one above.
Furthermore, we have $|f - f_{Q_i^1}| \le b$ for a.e on $Q_i^1 - \underset{j}{\cup} Q_j^2$ and $|f_Q - f_{Q_i^1}| \le 2^n b$.
Thus

$$|f - f_Q| \le \sup\{b, b + 2^n b\} = b + 2^n b \le 2^n 2b$$

a.e on $Q - \underset{j}{\cup} Q_j^2$.
With a similar argument with some modifications, we obtain:

$$|f - f_Q| \le 2^n sb$$

almost everywhere on $Q - \underset{j}{\cup} Q_j^s$. Hence,

$$|\{x \in Q : |f(x) - f_Q| > 2^n sb\}| \le |\underset{j}{\cup} Q_j^s| \le b^{-s}|Q|.$$

Let $\alpha > 0$. If $\alpha \le 2^n b$, then

$$|\{x \in Q : |f(x)-f_Q| > \alpha\}| \le |Q| \le |Q|e^{2^n b}e^{-\alpha} \le |Q|e^{2^n b}e^{-\frac{\log b}{2^n b}\alpha}$$

If $\alpha > 2^n b$, then there is a positive integer s such that

$$2^n sb < \alpha \le 2^n(s+1)b.$$

Thus,

$$\begin{aligned}|\{x \in Q : |f(x)-f_Q| > \alpha\}| &\le |\{x \in Q : |f(x)-f_Q| > 2^n sb\}| \\ &\le b^{-s}|Q| \\ &\le |Q|be^{-\frac{\log b}{2^n b}\alpha} \\ &\le |Q|e^{2^n b}e^{-\frac{\log b}{2^n b}\alpha}.\end{aligned}$$

Thus the proof of Theorem 2.2.5 is complete.

Corollary 2.2.6 *For all $1 < p < \infty$, we have*

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx\Big)^{\frac{1}{p}} \approx \|f\|_{BMO}$$

Proof. We first prove that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx\right)^{\frac{1}{p}} \leq c\|f\|_{BMO}.$$

We have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx &= \frac{p}{|Q|}\int_0^\infty \alpha^{p-1}|\{x \in Q : |f(x) - f_Q| > \alpha\}|d\alpha \\
&\leq \frac{pc|Q|}{|Q|}\int_0^\infty \alpha^{p-1}e^{-\frac{a\alpha}{\|f\|_{BMO}}}d\alpha \\
&= c\|f\|^p_{BMO}.
\end{aligned}$$

By taking the supremum over all cubes we get the inequality.
In addition,

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq \frac{|Q|^{\frac{1}{q}}}{|Q|}\left(\int_Q |f(x) - f_Q|^p dx\right)^{\frac{1}{p}} = \frac{1}{|Q|^{\frac{1}{p}}}\left(\int_Q |f(x) - f_Q|^p dx\right)^{\frac{1}{p}}.$$

By taking the supremum of both sides over all cubes we get

$$\|f\|_{BMO} \leq \mathop{sup}_Q \left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx\right)^{\frac{1}{p}}.$$

This completes the proof of Corollary 2.2.6.

2.2.2 Interpolation of BMO Spaces

Theorem 2.2.7 *Let $1 \leq p_0 < \infty$. Let T be a bounded linear operator from $L^{p_0}(\mathbb{R}^n)$ into $L^{p_0}(\mathbb{R}^n)$ and from $L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$. Then for all p with $p_0 < p < \infty$ there is a constant c such that for all $f \in L^p(\mathbb{R}^n)$ we have*

$$\|T(f)\|_p \leq c\|f\|_p.$$

For the proof of Theorem 2.2.7 see Theorem 7.4.7 of [7]. This proof is similar to the proof of Theorem 4.3.6 in chapter 4.

2.3 Hardy Spaces

Definition 2.3.1 *A complex-valued function a is called 2-atom if there is a cube Q such that*

(a) a is supported by Q

(b) $\int_{\mathbb{R}^n} a(x)dx = 0$

(c) $\|a\|_2 \leq \frac{1}{|Q|^{\frac{1}{2}}}$

Definition 2.3.2 *We define $H^1(\mathbb{R}^n)$ as*

$$H^1(\mathbb{R}^n) = \{\sum_i \lambda_i a_i : a_i \text{ 2-atom}, \lambda_i \in \mathbb{C}, \sum_i |\lambda_i| < \infty\}$$

and the norm on $H^1(\mathbb{R}^n)$ as

$$\|f\|_{H^{1,2}} = \inf\{\sum_i |\lambda_i| : f = \lambda_i a_i, \lambda_i \in \mathbb{C}, a_i \text{ 2-atom}\}.$$

Theorem 2.3.3 *The dual of $H^1(\mathbb{R}^n)$ is isomorphic to $BMO(\mathbb{R}^n)$ with equivalent norms.*

For the proof of this theorem see [7], chapter 7.

3

Singular Integrals with Rough Kernels

In this section, we will discuss a new criterion for singular integral operators to be bounded on $L^p(X)$, $1 < p < \infty$, where X is a space of homogeneous type. This criterion is an improvement of the Hörmander condition and it has had many applications. The main reference of this chapter is [4].

3.1 Preliminaries

Definition 3.1.1 *A quasi metric d on a set X is a function from $X \times X$ to $[0, \infty)$ such that :*

1. *$d(x, y) = 0$ if and only if $x = y$.*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$.*
3. *There is a constant $C_1 \in [1, \infty)$ such that for all $x, y, z \in X$,*

$$d(x, y) \leq C_1(d(x, z) + d(z, y)).$$

Let X be a topological space equipped with a measure ν and a quasi metric d which is a measurable function on $X \times X$. We define (X, d, ν) to be of homogeneous type if the doubling property is verified uniformly for all $x \in X$ and $r > 0$. That is,

$$\nu(B(x; 2r)) \leq c_1 \nu(B(x, r)) < \infty \tag{3.1}$$

for some $c_1 \geq 1$ uniformly for all $x \in X$ and $r > 0$. Note that the doubling property implies that there are $c_2, n > 0$ such that for all $x \in X$ and $\lambda \geq 1$

$$\nu(B(x, \lambda r)) \leq c_2 \lambda^n \nu(B(x, r)) \tag{3.2}$$

and there are $c_3 > 0$ and N, $0 \leq N \leq n$, such that for all $x, y \in X$ and $r \geq 0$

$$\nu(B(y, r)) \leq c_3 (1 + \frac{d(x, y)}{r})^N \nu(B(x, r)). \tag{3.3}$$

In the following, we will state some useful theorems for space (X, d, v) of homogeneous type that are similar to Theorem 2.1.4 and Theorem 2.1.9 in classical case.

Definition 3.1.2 *Let $f \in L^p(X)$, $1 \leq p \leq \infty$. We define the maximal function of f as*

$$Mf(x) = \sup_{x \in B} \frac{1}{B} \int_B |f(x)| d\nu(x), \text{ where } B \text{ is an open ball in } X.$$

Theorem 3.1.3 *Let f be a measurable function on X;*

1. *If $f \in L^1(X)$, then $\nu\{x \in X, (Mf)(x) > \alpha\} \leq \frac{c}{\alpha} \|f\|_1$.*

2. *If $f \in L^p$, with $1 < p \leq \infty$, then there is $C_p > 0$, such that for all $f \in L^p(X)$ we have*
$$\|Mf\|_p \leq C_p \|f\|_p.$$

For the proof of part (1), see [2]. For the part (2), we mimic the proof of (c) in Theorem 2.1.4 by replacing $\mathbb{R}^n$ by X.

Theorem 3.1.4 *Let $O \subsetneq X$ be an open set. Then there is a collection of balls $\{B(x_i, r_i)\}_I$ such that:*

1. *$\underset{i}{\cup} B(x_i, r_i) = O$,*

2. *each point of O is contained in at most a finite number K of balls $B(x_i, r_i)$,*

3. *there is $c > 1$ such that $B(x_i, cr_i) \cap O^c \neq \emptyset$.*

For the proof see chapter 3 of [2].

Theorem 3.1.5 *Suppose that $f \in L^1(X)$ and $\alpha > \frac{\|f\|_1}{\nu(X)}$. Then there exist functions g and b such that:*

(a) $f = g + b$,

(b) $|g(x)| \leq c\alpha$ *for almost all* $x \in X$,

(c) There is a sequence of functions b_i and balls B_i so that the support of each b_i is contained in B_i and $b = \sum_i b_i$,

(d) $\int_{B_i} b_i(x) d\nu(x) = 0$,

(e) $\int_{B_i} |b_i(x)| d\nu(x) \leq c\alpha\nu(B_i)$,

(f) $\sum_i \nu(B_i) \leq \frac{c}{\alpha} \int |f(x)| d\nu(x)$,

(g) $\sum_i 1_{B_i} \leq N$.

For the proof see Corollary 2.3 in Chapter III of [2].

Remark 3.1.6 *If $\nu(x) = \infty$, then we take any $\alpha > 0$. In addition, using the properties (e) and (f) of Theorem 3.1.5 we have*

$$\|b\|_1 \leq \sum_i \int |b_i(x)| d\nu(x) \leq c\alpha \sum_i \nu(B_i) \leq c\|f\|_1.$$

Thus, $\|g\|_1 \leq (1 + c)\|f\|_1$.

Definition 3.1.7 *Let T be a mapping from $L^p(X)$ to $L^q(X)$, $1 \leq p, q \leq \infty$. Then we say*

1. *T is of type (p, q) if T is bounded operator.*

2. *T is of weak-type (p, q), for $q = \infty$, if T is bounded operator.*

3. *T is of weak-type (p, q) , for $q < \infty$, if for every $\alpha > 0$ and $f \in L^p(X)$ we have*

$$\nu(\{x \in X : |Tf(x)| > \alpha\}) \leq (\frac{c\|f\|_p}{\alpha})^q. \tag{3.4}$$

Theorem 3.1.8 *Suppose that $p < r \leq \infty$. Let T be a sublinear operator of weak-type (p, p) and of weak-type (r, r), then T is of type (q, q) for all q such that $p < q < r$. In another way, there is $A_q > 0$, such that if $f \in L^q(X)$ then*

$$\|T(f)\|_q \leq A_q\|f\|_q.$$

For the proof see Theorem 1.3.2 of [7].

The following theorem gives an analogue of the dyadic cubes in $\mathbb{R}^n$. For the proof see Theorem 11 of [3].

Theorem 3.1.9 *Let (X, d, ν) be a space of homogeneous type where d is a quasi metric. There exist a collection of open sets $\{Q^k_\alpha \subset X : k \in Z, \alpha \in I_k\}$, and constants $\delta \in (0, 1)$, $a_0 > 0$, $\eta > 0$ and $0 < D < \infty$, where I_k denotes some index set depending on k, such that*

(i) $\nu(X - \underset{\alpha}{\cup} Q^k_\alpha) \to 0$ *as* $k \to 0$.

(ii) If $l \geq k$ *then either* $Q^l_\beta \subset Q^k_\alpha$ *or* $Q^k_\alpha \cap Q^l_\beta = \emptyset$.

(iii) For each (k, α) *and each* $l < k$, *there is a unique* β *such that* $Q^k_\alpha \subset Q^l_\beta$.

(iv) Diameter $(Q^k_\alpha) \leq D\delta^k$.

(v) Each Q^k_α *contains some ball* $B(z^k_\alpha, a_0\delta^k)$.

3.2 Singular Integral Operators with Rough Kernels

In this section, (X, d, ν) is a space of homogeneous type where d is a metric.

Definition 3.2.1 *A family of operators $\{A_t, t > 0\}$ is said to be a generalized approximation of the identity if, for every $t > 0$, A_t is represented by a kernel $a_t(x, y)$ in the following sense: for every function $f \in L^p(X), 1 \leq p \leq \infty$,*

$$A_tf(x) = \int_X a_t(x, y)f(y)d\nu(y) \tag{3.5}$$

and

$$|a_t(x, y)| \leq h_t(x, y), \tag{3.6}$$

for all $x, y \in X$ where $h_t(x, y)$ is given by

$$h_t(x, y) = \frac{1}{\nu(B(x, t^{\frac{1}{m}}))} g(d(x, y)^m t^{-1}) \tag{3.7}$$

in which $m > 0$ and g is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+N+\epsilon} g(r^m) = 0 \tag{3.8}$$

for some $\epsilon > 0$, where n and N are the constants that appeared previously in (3.2) and (3.3).

Now we list some lemmas related to the functions h_t.

Lemma 3.2.2 *Given $k > 0$ and the function $h_t(x, y)$ defined in (3.7), there are $c, k > 0$ such that*

$$\sup_{z\in B(y,r)} h_t(x, z) \le c \inf_{z\in B(y,r)} h_{kt}(x, z) \tag{3.9}$$

uniformly for all $x, y \in X$, and $r, t > 0$ with $r^m \le kt$.

Proof. If $x \in B(y, 3r)$, then $d(x, z) \le 4r$ for every $z \in B(y, r)$. Hence

$$g(d(x, z)^m t^{-1}) \ge g(4^m r^m t^{-1}).$$

Therefore, for all $z_1, z_2 \in B(y, r)$ we have

$$\begin{aligned} h_t(x, z_2) &\le \frac{g(0)}{\nu(B(x, t^{\frac{1}{m}}))} \\ &\le \frac{g(0)}{\nu(B(x, t^{\frac{1}{m}}))} \times \frac{g(d(x, z_1)^m t^{-1})}{g(4^m r^m t^{-1})} \\ &\le g(0)(g(4^m r^m t^{-1}))^{-1} h_t(x, z_1). \end{aligned}$$

Thus,

$$\sup_{z\in B(y,r)} h_t(x, z) \le g(0)(g(4^m r^m t^{-1}))^{-1} h_t(x, z_1).$$

In addition,

$$\begin{aligned} h_t(x, z_1) &= \frac{g(d(x, z_1)^m t^{-1})}{\nu(B(x, t^{\frac{1}{m}}))} \times \frac{\nu(B(x, 2t^{\frac{1}{m}}))}{\nu(B(x, 2t^{\frac{1}{m}}))} \\ &\le c2^m h_{2^m t}(x, z_1). \end{aligned}$$

Therefore,

$$\sup_{z\in B(y,r)} h_t(x, z) \le c2^m h_{2^m t}(x, z_1).$$

Thus, the Lemma is valid for $x \in B(y, 3r)$. Now for $x \notin B(y, 3r)$, we have for all $z_1, z_2 \in B(y, r)$

$$d(x, z_1) \le d(x, z_2) + d(z_2, z_1) \le 2d(x, z_2). \qquad (d \text{ is a metric})$$

Thus,

$$h_t(x, z_2) \le \frac{B(x, 2t^{\frac{1}{m}})}{B(x, t^{\frac{1}{m}})} h_{2^m t}(x, z_1) \le c' h_{2^m t}(x, z_1).$$

This leads to the conclusion that

$$\sup_{z\in B(y,r)} h_t(x, z) \le c' \inf_{z\in B(y,r)} h_{kt}(x, z).$$

Hence, the proof is complete.

Lemma 3.2.3 *There are constant $a, b > 0$ such that*

$$a \leq \int_X h_t(x,y)d\nu(x) \leq b. \tag{3.10}$$

Proof. Note that

$$\nu(B(y,t^{\frac{1}{m}})) \leq c(1+\frac{d(x,y)}{t^{\frac{1}{m}}})^N \nu(B(x,t^{\frac{1}{m}})).$$

Hence,

$$\begin{aligned}\int_X h_t(x,y)d\nu(x) &= \int_{d(x,y)\leq t^{\frac{1}{m}}} \frac{g(\frac{d(x,y)^m}{t})}{\nu(B(x,t^{\frac{1}{m}}))}d\nu(x) + \sum_{i=0}^{\infty}\int_{2^i t^{\frac{1}{m}}<d(x,y)\leq 2^{i+1}t^{\frac{1}{m}}} \frac{g(\frac{d(x,y)^m}{t})}{\nu(B(x,t^{\frac{1}{m}}))}d\nu(x)\\ &\leq \frac{cg(0)}{\nu(B(y,t^{\frac{1}{m}}))}\int_{d(x,y)\leq t^{\frac{1}{m}}} d\nu(x) + c\sum_{i=0}^{\infty}\int_{B(y,2^{i+1}t^{\frac{1}{m}})} \frac{2^{(i+1)N}g(2^{mi})}{\nu(B(y,t^{\frac{1}{m}}))}d\nu(x)\\ &\leq c' + c\sum_{i}^{\infty} 2^{(i+1)(N+n)}g(2^{im}) = b.\end{aligned}$$

It is enough to take $a = \frac{g(1)}{2}$ to complete the lemma.

Lemma 3.2.4 *For any $f \in L^p(X)$, $1 \leq p \leq \infty$, we have*

$$\int_X |f(y)h_t(x,y)|d\nu(y) \leq cMf(x).$$

The proof of this lemma is similar to the proof of lemma 3.2.3, and we omit the details.

Definition 3.2.5 *Let T be a bounded linear operator in $L^2(X)$. We say T has an associated kernel $k(x,y)$ if*

$$(Tf)(x) = \int_X k(x,y)f(y)d\nu(y), \tag{3.11}$$

where $k(x,y)$ is a measurable function, and the above formula holds for all continuous functions with compact support, and for almost all x not in the support of f.

The following theorem discusses a new criterion for singular integral operators to be bounded on $L^p(X)$, $1 < p < \infty$, where X is a space of homogeneous type. This criterion is an improvement of the Hörmander condition (2.1.12).

Theorem 3.2.6 *Let T be a bounded linear operator on $L^2(X)$, we suppose:*

(A-1) T has an associated kernel $k(x,y)$.

(A-2) There is integral operators A_t, $t > 0$, which plays the role of approximation of the identity with kernel $a_t(x,y)$ satisfying the conditions (3.5) – (3.7).

(A-3) The operators TA_t have associated kernels k_t, in the sense of (3.11), such that there are constants $\delta, C > 0$, so

$$\int_{d(x,y)\geq \delta t^{\frac{1}{m}}} |k(x,y)-k_t(x,y)| \leq C, \text{ for all } y \in X. \tag{3.12}$$

Then T is of weak type $(1,1)$. Therefore, T can be extended to a bounded operator on $L^p(X)$, for all $1 < p \leq 2$.

Proof. Let $f \in L^2(X) \cap L^1(X)$ and $\alpha > \frac{\|f\|_1}{\nu(X)}$. By applying Theorem 3.1.5 there are functions f, b and b_i that verify the properties of this theorem.

We have $Tf = Tg + Tb$, thus

$$\nu(\{x \in X, |Tf(x)| > \alpha\}) \leq \nu(\{x \in X, |Tg(x)| > \frac{\alpha}{2}\}) + \nu(\{x \in X, |Tb(x)| > \frac{\alpha}{2}\})$$

First, we examine $\nu(\{x \in X, |Tg(x)| > \frac{\alpha}{2}\})$. By using (b) of Theorem 3.1.5, we have

$$\|g\|_2^2 = \int_X |g(x)|^2 dx \leq c\alpha \int_X |f(x)| d\nu(X).$$

Thus, $g \in L^2(X)$. Additionally, T is bounded on $L^2(X)$. Hence, T is of weak-type $(2,2)$. Therefore, we have

$$\nu(\{x \in X, |Tg(x)| > \frac{\alpha}{2}\}) \leq \frac{c}{\alpha^2}\|g\|^2 \leq \frac{c}{\alpha}\|f\|_1. \tag{3.13}$$

Now we examine $\nu(\{x \in X, |Tb(x)| > \frac{\alpha}{2}\})$. Let r_i be the radius of the ball B_i mentioned in Theorem 3.1.5. We have

$$Tb = \sum_i Tb_i = \sum_i (Tb_i + (TA_{r_i^m} - TA_{r_i^m})b_i) = \sum_i (TA_{r_i^m} b_i + (T - TA_{r_i^m})b_i)$$

where m is the constant for h_t that appeared in (3.7). Thus,

$$\begin{aligned}
&\nu(\{x \in X, |Tb(x)| > \frac{\alpha}{2}\}) \\
&\leq \nu(\{x \in X : |\sum_i TA_{r_i^m} b_i(x)| > \frac{\alpha}{4}\}) + \nu(\{x \in X : |\sum_i (T - TA_{r_i^m})b_i(x)| > \frac{\alpha}{4}\}).
\end{aligned}$$

We now analyze:

(i) $\nu(\{x \in X : |\sum_i (TA_{r_i^m} b_i(x))| > \frac{\alpha}{4}\})$,

(ii) $\nu(\{x \in X : |\sum_i (T - TA_{r_i^m})b_i(x)| > \frac{\alpha}{4}\})$.

For (i), we first prove that $\sum_i A_{r_i^m} b_i \in L^2(X)$.

$$\begin{aligned}
|A_{r_i^m} b_i(x)| &\leq \int_{B_i} |h_{r_i^m}(x,y) b_i(y)| d\nu(y) \\
&\leq \int_{B_i} \sup_{z \in B_i} |h_{r_i^m}(x,z) b_i(y)| d\nu(y) \\
&\leq c \int_{B_i} \inf_{z \in B_i} |h_{kr_i^m}(x,z) b_i(y)| d\nu(y) \qquad \text{(due to lemma 3.2.2).}
\end{aligned}$$

Hence,

$$\begin{aligned}
|A_{r_i^m} b_i(x)| &\leq c \int_{B_i} \inf_{z \in B_i} |h_{kr_i^m}(x,z) b_i(y)| d\nu(y) \\
&= c \inf_{z \in B_i} h_{kr_i^m}(x,z) \|b_i\|_1 \\
&\leq c\alpha\nu(B_i) \inf_{z \in B_i} h_{kr_i^m}(x,z) \qquad \text{(due to theorem 3.1.5 (e))} \\
&= c\alpha \int_X h_{kr_i^m}(x,y) 1_{B_i}(y) d\nu(y).
\end{aligned}$$

Thus, for every $\psi \in L^2(X)$, we have:

$$\begin{aligned}
|\langle \psi, A_{r_i^m} b_i \rangle| &\leq c\alpha \int_X \int_X |\psi(x)| h_{kr_m^i}(x,y) 1_{B_i}(y) d\nu(y) d\nu(x) \\
&\leq c\alpha \int_X 1_{B_i}(y) (\int_X |\psi(x)| h_{kr_m^i}(x,y) d\nu(x)) d\nu(y) \\
&\leq c\alpha \langle M\psi, 1_{B_i} \rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
|\langle \psi, \sum_i A_{r_i^m} b_i \rangle| &\leq \sum_i |\langle \psi, A_{r_i^m} b_i \rangle| \\
&\leq c\alpha \sum_i \langle M\psi, 1_{B_i} \rangle \\
&= c\alpha \langle M\psi, \sum_i 1_{B_i} \rangle \\
&\leq c\alpha \|M\psi\|_2 \| \sum_i 1_{B_i} \|_2 \\
&\leq c\alpha \|\psi\|_2 \| \sum_i 1_{B_i} \|_2.
\end{aligned}$$

In addition, by using (g) of Theorem 3.1.5, we have $\sum_i 1_{B_i} \leq N 1_{\cup B_i}$. Thus,

$$\begin{aligned}
|\langle \psi, \sum_i A_{r_i^m} b_i \rangle| &\leq c\alpha \|\psi\|_2 \|N 1_{(\cup B_i)}\|_2 \\
&\leq c\alpha \|\psi\|_2 (\sum_i \nu(B_i))^{\frac{1}{2}} \\
&\leq c\alpha^{\frac{1}{2}} \|\psi\|_2 \|f\|_1^{\frac{1}{2}}. && \text{(due to theorem 3.1.5)}
\end{aligned}$$

Therefore,

$$\| \sum_i A_{r_i^m} b_i \|_2 \leq c\alpha^{\frac{1}{2}} \|f\|_1^{\frac{1}{2}}.$$

Using the fact that T is of weak-type $(2,2)$, we have

$$\nu(\{x \in X : |T(\sum_i (A_{r_i^m} b_i))| > \frac{\alpha}{4}\}) \leq \frac{c}{\alpha^2} \alpha \|f\|_1 \leq \frac{c}{\alpha} \|f\|_1. \tag{3.14}$$

Now for (ii), let $D_i = (1+\delta)B_i$, so if $y \in B_i$ and $x \in D_i^c$ then $d(x,y) \geq \delta r_i$. Let $\Omega^* = \cup_i D_i$

and $F^* = \Omega^{*c} = \cap_i D_i^c$. Hence

$$\begin{aligned}
&\int_{F^*} |\sum_i (T - TA_{r_i^m}) b_i(x)| d\nu(x) \\
&\qquad \leq \sum_i \int_{F^*} |(T - TA_{r_i^m}) b_i(x)| d\nu(x) \\
&\qquad \leq \sum_i \int_{D_i^c} |(T - TA_{r_i^m}) b_i(x)| d\nu(x) \\
&\qquad \leq \sum_i \int_{D_i^c} |\int_{B_i} (k(x,y) - k_{r_i^m}(x,y)) b_i(y) d\nu(y)| d\nu(x) \\
&\qquad \leq \sum_i \int_{B_i} |b_i(y)| (\int_{d(x,y)\geq \delta r_i} |k(x,y) - k_{r_i^m}(x,y)| d\nu(x)) d\nu(y) \\
&\qquad \leq c \sum_i \int_{B_i} |b_i(y)| d\nu(y) \\
&\qquad \leq c\|f\|_1.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\nu(\{x \in X : |\sum_i (T - TA_{r_i^m}) b_i| > \frac{\alpha}{4}\}) \\
&\leq \sum_i \nu(D_i) + \nu(\{x \in F^* : |\sum_i (T - TA_{r_i^m}) b_i| > \frac{\alpha}{4}\}) \\
&\leq c_2 (1+\delta)^n \sum_i \nu(B_i) + \frac{4}{\alpha} \int_{F^*} |\sum_i (T - TA_{r_i^m}) b_i(x)| d\nu(x) \\
&\leq \frac{c}{\alpha} \|f\|_1.
\end{aligned}$$

Thus,

$$\nu(\{x \in X : |\sum_i (T - TA_{r_i^m}) b_i| > \frac{\alpha}{4}\}) \leq \frac{c}{\alpha} \|f\|_1. \tag{3.15}$$

Combining the results in (3.13), (3.14) and (3.15) implies that T is of weak-type $(1,1)$. Thus, T can be extended to be a bounded on $L^p(X)$ for all $1 < p \leq 2$.

Note that if there exits a class of operators B_t whose kernel satisfy the conditions (3.5) – (3.7) so that $B_t T$ have associated kernels $K_t(x,y)$, and there are constants δ' and C', such that

$$\int_{d(x,y)\geq \delta' t^{\frac{1}{m}}} |k(x,y) - K_t(x,y)| d\nu(y) \leq C' \tag{3.16}$$

for all $x \in X$. Then the adjoint operator T^* can be extended to be bounded on $L^p(X)$ for $1 < p \leq 2$. Therefore, T can be extended to a bounded operator on $L^q(X)$ for $2 \leq q < \infty$.

4

Spaces BMO_A Associated with the Generalized Approximation to the Identity

In this chapter, we will discuss a new function spaces $BMO_A(X)$. We demonstrate that the John-Nirenberg holds in these spaces and they interpolate with $L^p(X)$. In this chapter, (X, d, ν) is a space of homogeneous type where d is a quasi metric and the condition (3.8) of $h_t(x, y)$ is replaced by

$$\lim_{r\to\infty} r^{n+2N+\epsilon} g(r^m) = 0 \tag{4.1}$$

for some $\epsilon > 0$, where n and N are the constant that appeared previously in (3.2) and (3.3). In addition, if B is a set in X, we denote $2^{-1}B$ to be the empty set. The main reference of this chapter is [5].

4.1 BMO_A Spaces.

Definition 4.1.1 *Let $\beta \in (0, \epsilon)$. A function $f \in L^1_{loc}(X)$ is said to be a function of type (x_0, β) if f satisfies*

$$\int_X \frac{|f(x)|}{(1 + d(x_0, x))^{2N+\beta}\nu(B(x_0, 1 + d(x_0, x)))} d\nu(x) \le c < \infty. \tag{4.2}$$

We denote by $M_{(x_0,\beta)}$ the collection of all functions of type (x_0, β). For $f \in M_{(x_0,\beta)}$, we denote

$$\|f\|_{M(x_0,\beta)} = \inf\{c \ge 0 : (4.2) \text{ holds}\}.$$

In addition, $\|.\|_{M(x_0,\beta)}$ defines a norm on $M(x_0, \beta)$ and $(M_{(x_0,\beta)}, \|.\|_{M(x_0,\beta)})$ is a Banach space. In fact let f_n be a Cauchy sequence in $(M_{(x_0,\beta)}, \|.\|_{M(x_0,\beta)})$. Then

$$\frac{|f_n(x)|}{(1 + d(x_0, x))^{2N+\beta}\nu(B(x_0, 1 + d(x_0, x)))}$$

is a Cauchy sequence in $L^1(X)$. Therefore,

$$\frac{|f_n(x)|}{(1 + d(x_0, x))^{2N+\beta}\nu(B(x_0, 1 + d(x_0, x)))}$$

converges to a function g in $L^1(X)$. Let $f = (1 + d(x_0, x))^{2N+\beta}\nu(B(x_0, 1 + d(x_0, x)))g$. It is clear that $f \in M_{(x_0,\beta)}$ and that f_n converges to f in $(M_{(x_0,\beta)}, \|.\|_{M_{(x_0,\beta)}})$.
We denote

$$\mathcal{M} = \bigcup_{x\in X} \bigcup_{0<\beta<\epsilon} M_{(x,\beta)}$$

where ϵ is the constant in (4.1). In this section, $\{A_t, t > 0\}$ is a family of operators having associated kernels $a_t(x, y)$ in the following sense: for every function $f \in \mathcal{M}, 1 \le p \le \infty$,

$$A_t f(x) = \int_X a_t(x, y) f(y) d\nu(y) \tag{4.3}$$

where

$$|a_t(x, y)| \le h_t(x, y).$$

Before we give the definition of $BMO_A(X)$, we will list some of the properties of set $\mathcal{M}$.

Lemma 4.1.2 *Let $\{A_t, t > 0\}$ be a family of operators as mentioned above. Then*

(1) $BMO(X) \subset \mathcal{M}$

(2) *If $f \in \mathcal{M}$ then $|A_t f(x)| < \infty$, for all $x \in X$ and $t > 0$.*

(3) *If $f \in \mathcal{M}$ then $|A_t(A_s f)(x)| < \infty$, for all $x \in X$ and $t, s > 0$.*

For property (1). Let $f \in BMO(X)$. Then we have

$$\begin{aligned}
&\int_X \frac{|f(x)|}{(1 + d(x_0, x))^{2N+\beta}\nu(B(x_0, 1 + d(x_0, x)))} d\nu(x) \\
&\le \sum_{i=0} \int_{2^i B(x_0,1) - 2^{i-1}B(x_0,1)} \frac{|f(x) - f_{2^i B(x_0,1)}| + |f_{2^i B(x_0,1)}|}{(1 + d(x_0, x))^{2N+\beta}\nu(B(x_0, 1 + d(x_0, x)))} d\nu(x) \\
&\le \|f\|_{BMO} + |f_{B(x_0,1)}| + \sum_{i=1} \int_{2^i B(x_0,1)} \frac{|f(x) - f_{2^i B(x_0,1)}| + |f_{2^i B(x_0,1)}|}{(1 + 2^{i-1})^{2N+\beta}\nu(B(x_0, 1 + 2^{i-1}))} d\nu(x).
\end{aligned}$$

Furthermore, we have $|f_{2^k B} - f_B| \le c(1 + k)\|f\|_{BMO}$ (for the proof see [10]). Thus,

$$|f_{2^k B}| \le c(1 + k)\|f\|_{BMO} + f_B.$$

Hence,

$$\begin{aligned}
&\int_X \frac{|f(x)|}{(1 + d(x_0, x))^{2N+\beta}\nu(B(x_0, 1 + d(x_0, x)))} d\nu(x) \\
&\le \|f\|_{BMO} + |f_{B(x_0,1)}| + c'' \sum_{i=1} 2^{-(i-1)(2N+\beta)}(c'\|f\|_{BMO} + c(1 + i)\|f\|_{BMO} + f_{B(x_0,1)}) < \infty.
\end{aligned}$$

Thus, $BMO(X) \subset \mathcal{M}$.
For property (2), let $f \in \mathcal{M}$. Then there are $x_0 \in M$ and $\beta > 0$ such that $f \in M_{(x_0,\beta)}$. Fix $x_0 \in X$ and let $x \in X$. Then

$$|A_t f(x)| \le \int_X |a_t(x, y) f(y)| d\nu(y) \le \int_X h_t(x, y)|f(y)| d\nu(y).$$

Let $X_1 = \{y \in X : d(x,y) > \max(1 + d(x_0,x), t^{\frac{1}{m}})\}$ and $X_2 = X_1^c$. Given $f \in L^1_{loc}(X)$, then

$$\int_{X_2} h_t(x,y)|f(y)|d\nu(y) \le \frac{g(0)}{\nu(B(x,t^{\frac{1}{m}}))}\int_{X_2}|f(y)|d\nu(y) < \infty.$$

For X_1 we have

$$\begin{aligned}\int_{X_1} h_t(x,y)|f(y)|d\nu(y) &= \int_{X_1}\frac{g(\frac{d(x,y)^m}{t})|f(y)|}{\nu(B(x,t^{\frac{1}{m}}))}d\nu(y)\\ &\le \sup_{y\in X_1}(\frac{g(\frac{d(x,y)^m}{t})(1+d(x_0,x))^{2N+\beta}\nu(B(x_0,1+d(x_0,y)))}{\nu(B(x,t^{\frac{1}{m}}))})\|f\|_{M_{(x_0,\beta)}}\\ &\le \sup_{y\in X_1}(\frac{g(\frac{d(x,y)^m}{t})(d(x,y))^{2N+\beta}\nu(B(x_0,1+d(x_0,y)))}{\nu(B(x,t^{\frac{1}{m}}))})\|f\|_{M_{(x_0,\beta)}}.\end{aligned}$$

Note that if $y \in X_1$, then

$$\begin{aligned}1 + d(x_0,y) &\le 1 + C_1 d(x_0,x) + C_1 d(x,y)\\ &\le C_1 + C_1 d(x_0,x) + C_1 d(x,y)\\ &\le C_1(1 + d(x_0,x) + d(x,y))\\ &\le 2C_1 d(x,y).\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{X_1} h_t(x,y)|f(y)|d\nu(y)\\ &\le c\sup_{y\in X_1}(\frac{g(\frac{d(x,y)^m}{t})(d(x,y))^{2N+\beta}\nu(B(x_0,d(x,y)))}{\nu(B(x,t^{\frac{1}{m}}))})\|f\|_{M_{(x_0,\beta)}}\\ &\le c\sup_{y\in X_1}(\frac{g(\frac{d(x,y)^m}{t})(d(x,y))^{2N+\beta}(1+\frac{d(x_0,x)}{d(x,y)})^N\nu(B(x,d(x,y)))}{\nu(B(x,t^{\frac{1}{m}}))})\|f\|_{M_{(x_0,\beta)}}\\ &\le c\sup_{y\in X_1}(\frac{g(\frac{d(x,y)^m}{t})(d(x,y))^{2N+\beta}(2)^N\nu(B(x,d(x,y)))}{\nu(B(x,t^{\frac{1}{m}}))})\|f\|_{M_{(x_0,\beta)}}\\ &\le 2^N c\sup_{y\in X_1}(\frac{g(\frac{d(x,y)^m}{t})(d(x,y))^{2N+\beta+n}\nu(B(x,t^{\frac{1}{m}}))}{t^{\frac{n}{m}}\nu(B(x,t^{\frac{1}{m}}))})\|f\|_{M_{(x_0,\beta)}}\\ &\le ct^{\frac{2N+\beta}{m}}\sup_{y\in X_1}(g(\frac{d(x,y)^m}{t})(\frac{d(x,y)}{t^{\frac{1}{m}}})^{2N+\beta+n})\|f\|_{M_{(x_0,\beta)}}\\ &< \infty.\end{aligned}$$

Thus,

$$|A_t f(x)| \le \int_{X_1} h_t(x,y)d\nu(y) + \int_{X_2} h_t(x,y)d\nu(y) < \infty.$$

For (3), we have

$$\begin{aligned}|A_t(A_s f)(x)| &\le \int_X\int_X h_t(x,z)h_s(z,y)f(y)d\nu(y)d\nu(z)\\ &\le \int_X f(y)\int_X h_t(x,z)h_s(z,y)d\nu(z)d\nu(y).\end{aligned}$$

Now we examine the last term of the inequality. For any $x, y \in X$, let

$$F_1 = \{z \in X, d(x,z) \geq (3C_1)^{-1} d(x,y)\}$$

and

$$F_2 = \{z \in X, d(z,y) \geq (3C_1)^{-1} d(x,y)\}.$$

If $z \notin F_1$, then

$$d(x,y) \leq C_1 d(x,z) + C_1 d(y,z) \leq \frac{d(x,y)}{3} + C_1 d(y,z).$$

Hence,

$$d(y,z) \geq \frac{2d(x,y)}{3C_1}.$$

Therefore, $z \in F_2$ which implies that $X = F_1 \cup F_2$. In addition,

$$\begin{aligned}
\int_{F_1} h_t(x,y) h_s(y,z) d\nu(z) &= \int_{F_1} \frac{1}{\nu(B(x, t^{\frac{1}{m}}))} g(\frac{d(x,y)^m}{t}) h_s(z,y) d\nu(z) \\
&\leq c \frac{1}{\nu(B(x, t^{\frac{1}{m}}))} g(\frac{d(x,y)^m}{(3C_1)^m t}) \int_{F_1} h_s(z,y) d\nu(z) \\
&\leq c(1 + \frac{s}{t})^{\frac{n}{m}} \frac{1}{\nu(B(x, (t+s)^{\frac{1}{m}}))} g(\frac{d(x,y)^m}{(3C_1)^m (t+s)}) \\
&\leq c(1 + \frac{s}{t})^{\frac{n}{m}} h_{(3C_1)^m (t+s)}(x,y)
\end{aligned}$$

where in the second step we used the fact that g is decreasing and in the third step we used Lemma 3.2.3.

Similarly, we get

$$\int_{F_2} h_t(x,y) h_s(y,z) d\nu(z) \leq c(1 + \frac{t}{s})^{\frac{n+N}{m}} h_{(3C_1)^m (t+s)}(x,y).$$

Hence,

$$\int_X h_t(x,y) h_s(y,z) d\nu(z) \leq c_{(s,t)} h_{(3C_1)^m (t+s)}(x,y).$$

Thus, we have

$$|A_t(A_s f)(x)| \leq c_{(s,t)} \int_X h_{(3C_1)^m (t+s)}(x,y) |f(y)| d\nu(y) < \infty.$$

Definition 4.1.3 *For $f \in M$, we say that $f \in BMO_A(X)$ if there is a constant c such that for every open ball B in X we have:*

$$\frac{1}{\nu(B)} \int_B |f(x) - A_{t_B} f(x)| d\nu(x) \leq c \tag{4.4}$$

where $t_B = r_B^m$ and r_B is the radius of B. We denote

$$\|f\|_{BMO_A} = \inf\{c \in \mathbb{R} : c \text{ verifies } (4.4)\}.$$

Let $K_A = \{f \in M : (A_t f)(x) = f(x)\}$, note that $\|f\|_{BMO} = 0$ if $f \in K_A$. Hence, $BMO_A(X)$ modulo K_A is a normed vector space. In the rest of this section, $BMO_A(X)$ is understood to be K_A modulo and the operators $\{A_t\}$ form a semigroup where A_0 is the identity operator.

The following proposition is similar to the proposition 2.2.4

Theorem 4.1.4 *Let $f \in BMO_A(X)$. Then for any $k > 1$ and $t > 0$ we have*

$$|A_t f(x) - A_{kt} f(x)| \leq c(1 + \log k)\|f\|_{BMO_A} \tag{4.5}$$

for almost all $x \in X$, where $c > 0$ is a constant independent of x and k.

In order to prove the theorem, we will use the following lemma;

Lemma 4.1.5 *Let $f \in BMO_A(X)$. Then for any $t > 0$ we have*

$$|A_t f(x) - A_{t+s} f(x)| \leq c\|f\|_{BMO_A} \tag{4.6}$$

for almost all $x \in X$, where $\frac{t}{4} \leq s \leq t$ and $c > 0$ is independent of x.

Proof.

Let $x \in X$, $t > 0$ and $s \in [\frac{t}{4}, t]$. We have

$$\begin{aligned}
|A_t f(x) - A_{t+s} f(x)| &\leq \int_X h_t(x,y)|f(y) - A_s f(y)|d\nu(y) \\
&= \frac{1}{\nu(B(x, t^{\frac{1}{m}}))} \int_X g(\frac{d(x,y)^m}{t})|f(y) - A_s f(y)|d\nu(y) \\
&\leq \frac{g(0)}{\nu(B(x, s^{\frac{1}{m}}))} \int_{B(x,s^{\frac{1}{m}})} |f(y) - A_s f(y)|d\nu(y) \\
&+ \frac{1}{\nu(B(x, s^{\frac{1}{m}}))} \int_{B(x,s^{\frac{1}{m}})^c} g(\frac{d(x,y)^m}{t})|f(y) - A_s f(y)|d\nu(y) \\
&\leq c\|f\|_{BMO_A} + \frac{1}{\nu(B(x, s^{\frac{1}{m}}))} \int_{B(x,s^{\frac{1}{m}})^c} g(\frac{d(x,y)^m}{t})|f(y) - A_s f(y)|d\nu(y).
\end{aligned}$$

Therefore, in order to prove Lemma 4.1.5 it is suffices to prove that

$$\frac{1}{\nu(B(x, s^{\frac{1}{m}}))} \int_{B(x,s^{\frac{1}{m}})^c} g(\frac{d(x,y)^m}{t})|f(y) - A_s f(y)|d\nu(y) \leq c\|f\|_{BMO_A}.$$

By using Theorem 3.1.9, there exists a collection of open sets $\{Q^k_\alpha \subset X : k \in Z, \alpha \in I_k\}$, and constants $\delta \in (0,1)$, $a_0 > 0$, $\eta > 0$ and $0 < D < \infty$, where I_k denotes some index set depending on k, such that

(i) $\nu(X - \underset{\alpha}{\cup} Q^k_\alpha) = 0$.

(ii) If $l \geq k$, then either $Q^l_\beta \subset Q^k_\alpha$ or $Q^k_\alpha \cap Q^l_\beta = \emptyset$.

(iii) For each (k, α) and each $l < k$, there is a unique β such that $Q^k_\alpha \subset Q^l_\beta$.

(iv) Diameter $(Q^k_\alpha) \leq D\delta^k$.

(v) Each Q_α^k contains some ball $B(z_\alpha^k, a_0\delta^k)$.

Let $l_0 \in \mathbb{Z}$ such that $D\delta^{l_0} \leq s^{\frac{1}{m}} < D\delta^{l_0-1}$. There is $Q_{\alpha_0}^{l_0}$ such that $x \in Q_{\alpha_0}^{l_0}$. Hence, by using (iv) of Theorem 3.1.9 we have $Q_{\alpha_0}^{l_0} \subset B(x, D\delta^{l_0})$. Let

$$M_k = \{\beta \in I_{l_0}, Q_\beta^{l_0} \cap B(x, D\delta^{l_0-k}) \neq \emptyset\}$$

and k_0 be an integer such that $\delta^{-k_0} > 2C_1$.
For $\beta \in M_k$, $y' \in Q_\beta^{l_0}$ and $x' \in Q_\beta^{l_0} \cap B(x, D\delta^{l_0-k})$, we have

$$d(x,y') \leq C_1(d(x,x') + d(x',y')) \leq 2C_1 D\delta^{l_0-k} \leq D\delta^{l_0-k-k_0}.$$

Thus $Q_\beta^{l_0} \subset B(x, D\delta^{l_0-k-k_0})$. Hence,

$$B(x, D\delta^{l_0-k}) \subset \bigcup_{\beta\in M_k} Q_\beta^{l_0} \subset B(x, D\delta^{l_0-(k+k_0)}).$$

Using z_α^k defined in (v) of the Theorem 3.1.9 we have for any $\beta_1, \beta_2 \in M_k$,

$$d(z_{\beta 1}^{l_0}, z_{\beta_2}^{l_0}) \leq C_1(d(z_{\beta_1}^{l_0}, x) + d(x, z_{\beta_2}^{l_0})) \leq 2C_1 D\delta^{l_0-(k+k_0)}.$$

We now have

$$\begin{aligned}
\nu(Q_{\beta_1}^{l_0}) &\leq \nu(B(z_{\beta_1}^{l_0}, D\delta^{l_0}) \\
&\leq c(1 + \frac{d(z_{\beta 1}^{l_0}, z_{\beta_2}^{l_0})}{D\delta^{l_0}})^N \nu(B(z_{\beta_2}^{l_0}, D\delta^{l_0})) \\
&\leq c(1 + \frac{(2C_1 D\delta^{l_0-(k+k_0)})}{D\delta^{l_0}})^N \nu(B(z_{\beta_2}^{l_0}, D\delta^{l_0})) \\
&\leq c\delta^{-kN}\nu(B(z_{\beta_2}^{l_0}, D\delta^{l_0})) \\
&\leq c\delta^{-kN}\nu(B(z_{\beta_2}^{l_0}, a_0\delta^{l_0})) \\
&\leq c\delta^{-kN}\nu(Q_{\beta_2}^{l_0}).
\end{aligned}$$

Hence,

$$\nu(Q_{\alpha_0}^{l_0}) \leq c\delta^{-kN} \inf_{\beta\in M_k} \nu(Q_\beta^{l_0}).$$

Let m_k be the cardinal number of M_k. We have

$$\begin{aligned}
m_k\nu(B(x, D\delta^{l_0})) &\leq cm_k\nu(B(z_{\alpha_0}^{l_0}, D\delta^{l_0})) && \text{(due to (3.3))} \\
&\leq cm_k\nu(B(z_{\alpha_0}^{l_0}, a_0\delta^{l_0})) \\
&\leq c\delta^{-kN} m_k \inf_{\beta\in M_k} \nu(Q_\beta^{l_0}) \\
&\leq c\delta^{-kN}\nu(B(x, D\delta^{l_0-(k+k_0)})) && (Q_\beta^{l_0} \subset B(x, D\delta^{l_0-(k+k_0)}) \text{ for } \beta \in M_k) \\
&\leq c\delta^{-k(n+N)}\nu(B(x, D\delta^{l_0})).
\end{aligned}$$

Thus, there is $c > 0$ such that

$$m_k \leq c\delta^{-k(n+N)} \tag{4.7}$$

We have $s \in [\frac{t}{4}, t]$ and $D\delta^{l_0} \le s^{\frac{1}{m}} < D\delta^{l_0-1}$. Hence,

$$\begin{aligned}
&\frac{1}{\nu(B(x,s^{\frac{1}{m}}))}\int_{B(x,s^{\frac{1}{m}})^c} g(\frac{d(x,y)^m}{t})|f(y)-A_sf(y)|d\nu(y)\\
&\le \frac{1}{\nu(B(x,s^{\frac{1}{m}}))}\int_{B(x,D\delta^{l_0})^c} g(\frac{d(x,y)^m}{t})|f(y)-A_sf(y)|d\nu(y)\\
&\le \frac{1}{\nu(B(x,s^{\frac{1}{m}}))}\sum_{k=0}^{\infty}\int_{B(x,D\delta^{l_0-(k+1)})-B(x,D\delta^{l_0-k})} g(\frac{d(x,y)^m}{t})|f(y)-A_sf(y)|d\nu(y))\\
&\le \frac{1}{\nu(B(x,s^{\frac{1}{m}}))}\sum_{k=0}^{\infty} g(\frac{(D\delta^{l_0-k})^m}{t})\int_{B(x,D\delta^{l_0-(k+1)})-B(x,D\delta^{l_0-k})} |f(y)-A_sf(y)|d\nu(y)\\
&\le \frac{1}{\nu(B(x,s^{\frac{1}{m}}))}\sum_{k=0}^{\infty} g(\frac{\delta^{(1-k)m}(s^{\frac{1}{m}})^m}{4s})\int_{B(x,D\delta^{l_0-(k+1)})-B(x,D\delta^{l_0-k})} |f(y)-A_sf(y)|d\nu(y)\\
&\le \frac{1}{\nu(B(x,s^{\frac{1}{m}}))}\sum_{k=0}^{\infty} g(4^{-1}\delta^{-(k-1)m})\int_{B(x,D\delta^{l_0-(k+1)})} |f(y)-A_sf(y)|d\nu(y)\\
&\le \frac{1}{\nu(B(x,s^{\frac{1}{m}}))}\sum_{k=0}^{\infty}\sum_{\beta\in M_{k+1}} g(4^{-1}\delta^{-(k-1)m})\int_{Q_\beta^{l_0}} |f(y)-A_sf(y)|d\nu(y).
\end{aligned}$$

In addition, we have

$$\begin{aligned}
\nu(B(z_\beta^{l_0}, s^{\frac{1}{m}}) &\le (1+\frac{d(x,z_\beta^{l_0})}{s^{\frac{1}{m}}})^N\nu(B(x,s^{\frac{1}{m}}))\\
&\le (1+\frac{D\delta^{l_0-k-k_0}}{D\delta^{l_0}})^N\nu(B(x,s^{\frac{1}{m}}))\\
&\le c\delta^{-kN}\nu(B(x,s^{\frac{1}{m}})),
\end{aligned}$$

and $Q_\beta^{l_0} \subset B(z_\beta^{l_0}, D\delta_0) \subset B(z_\beta^{l_0}, s^{\frac{1}{m}})$. Hence,

$$\begin{aligned}
&\frac{1}{\nu(B(x,s^{\frac{1}{m}}))}\int_{B(x,s^{\frac{1}{m}})^c} g(\frac{d(x,y)^m}{t})|f(y)-A_sf(y)|d\nu(y)\\
&\le c\sum_{k=0}^{\infty}\sum_{\beta\in M_{k+1}} \delta^{-kN} g(4^{-1}\delta^{-(k-1)m})\frac{1}{\nu(B(z_\beta^{l_0},s^{\frac{1}{m}}))}\int_{Q_\beta^{l_0}} |f(y)-A_sf(y)|d\nu(y))\\
&\le c\sum_{k=0}^{\infty}\sum_{\beta\in M_{k+1}} \delta^{-kN} g(4^{-1}\delta^{-(k-1)m})\|f\|_{BMO_A}\\
&\le c\sum_{k=0}^{\infty} m_{k+1}\delta^{-kN} g(4^{-1}\delta^{-(k-1)m})\|f\|_{BMO_A}\\
&\le c\sum_{k=0}^{\infty} \delta^{-k(n+2N)} g(4^{-1}\delta^{-(k-1)m})\|f\|_{BMO_A} \qquad \text{(due to (4.7))}\\
&\le c\|f\|_{BMO_A}.
\end{aligned}$$

Thus, Lemma 4.1.5 is proved.

We will now prove Theorem 4.1.4.

Proof. In case $0 < s < \frac{t}{4}$, we have $\frac{t+s}{4} \le t-s < t+s$. Thus, by using Lemma 4.1.5 we have

$$(A_t f(x) - A_{t+s}(f(x)) = (A_t f(x) - A_{t+t}(f(x)) - A_{t+s}(f - A_{t-s}f)(x) \leq c\|f\|_{BMO_A}.$$

So we proved Theorem 4.1.4 for the case $1 < k \leq 2$. Now for the case $k > 2$, let l be an integer such that $2^l \leq k < 2^{l+1}$. Then

$$\begin{aligned}|A_t f(x) - A_{kt}f(x)| &\leq \sum_{k=0}^{l-1} |A_{2^k t}f(x) - A_{2^{k+1}t}f(x)| + |A_{2^l t}f(x) - A_{kt}f(x)| \\ &\leq c(l+1)\|f\|_{BMO_A} \\ &\leq c(1+\log k)\|f\|_{BMO_A}.\end{aligned}$$

Hence, the proof of the theorem is complete.

4.2 The John-Nirenberg Inequality on $BMO_A(X)$

Theorem 4.2.1 *If $f \in BMO_A(X)$, then there are constants c_1 and c_2 such that for every ball $B = B(x_0, r)$ and every $\alpha > 0$, we have:*

$$\nu\{x \in B(x_0, r) : |f(x) - A_{t_B}f(x)| > \alpha\} \leq c_1\nu(B(x_0, r))e^{-\frac{c_2\alpha}{\|f\|_{BMO_A}}} \tag{4.8}$$

where $x_0 \in X$ and $r > 0$ and $t_B = r^m$.

Proof. Similarly to Theorem 2.2.5, it is enough to prove the theorem for the case when $\|f\|_{BMO_A} = 1$. The case $\alpha < 1$ is obvious if we take $c_1 > e$ and $c_2 < 1$, so we only study the case where $\alpha \geq 1$.
Let $\beta > 1$ and B be a fixed ball of center $x_0 \in X$ and radius $r_B > 0$. We will set up a criterion in order to have a collection of balls $B_{i,k} = B(x_{B_{i,k}}, r_{B_{i,k}})$ that verifies:

(i) For any $B_{k,m} \cap B_{k+1,j} \neq \emptyset$, we have $|A_{t_{B_{k,m}}}f(x) - A_{t_{B_{k+1,j}}}f(x)| \leq c\beta$, $x \in B_{k+1,j}$.

(ii) $\{x \in B : |f(x) - A_{t_B}f(x)| > ck\beta\} \subseteq \underset{i}{\cup} B_{k,i}$.

(iii) $\sum_{i=1}^{\infty} \nu(B_{k,i}) \leq (\frac{c}{\beta})^k \nu(B)$, where in this inequality we have $\frac{c}{\beta} < 1$.

Let $f_0 = (f - A_{t_B})1_{10C_1^4 B(x_0, r_B)}$, where $C_1 > 1$ is the constant in Definition 3.1.1.

$$\|f_0\|_1 \leq \int_{10C_1^4 B} |f(x) - A_r f(x)| d\nu(x) \leq \|f\|_{BMO_A}\nu(10C_1^4 B(x_0, r)) = c_3\nu(B) < \infty.$$

Let $\beta > 0$, $\Omega = \{x \in X : M(f_0)(x) > \beta\}$ and $F = \Omega^c$. We have Ω is open set. Using Theorem 3.1.4, there exists a collection of open balls $B_{1,i}$ such that $\Omega = \cup_i B_{1,i}$, each point of Ω is contained in no more then K balls $B_{1,i}$ and there is $c > 1$ such that $cB_{1,i} \cap F \neq \emptyset$ for each i. Using (i) of the Theorem 3.1.3, we have

$$\sum_i \nu(B_{1,i}) \leq K\nu(\Omega) \leq \frac{c}{\beta}\|f_0\|_1 \leq \frac{c_4}{\beta}\nu(B).$$

We now prove that if $B_{1,i} \cap B \neq \emptyset$, then there is $c_5 > 0$ such that

$$|A_{t_{B_{1,i}}}f(x) - A_{t_B}f(x)| \leq c_5\beta, \text{ for all } x \in B_{1,i}.$$

Let $B_{1,i} \cap B \neq \emptyset$ and assume that $r_{B_{1,i}} \geq r_B$. Then

$$\begin{aligned}
\nu(B(x_0, r_B)) &\leq c(\frac{r_B}{r_{B_{1,i}}})^n \nu(B(x_0, r_{B_{1,i}})) \\
&\leq c(\frac{r_B}{r_{B_{1,i}}})^n (1 + \frac{C_1(r_B + r_{B_{1,i}})}{r_{B_{1,i}}})^N \nu(B(x_{B_{1,i}}, r_{B_{1,i}})) \\
&\leq c(1 + 2C_1)^N \nu(B_{1,i}) \\
&\leq \frac{c_6}{\beta} \nu(B(x_0, r_B)).
\end{aligned}$$

Hence, if we take $\beta > c_6$, we get a contradiction. Therefore, in case $\beta > c_6$ we have $r_{B_{1,i}} < r_B$ and

$$\begin{aligned}
\nu(B(x_0, r_B)) &\leq c(\frac{r_B}{r_{B_{1,i}}})^n (1 + \frac{C_1(r_B + r_{B_{1,i}})}{r_{B_{1,i}}})^N \nu(B(x_{B_{1,i}}, r_{B_{1,i}})) \\
&\leq \frac{c_7}{\beta} (\frac{r_B}{r_{B_{1,i}}})^{n+N} \nu(B(x_0, r_B)).
\end{aligned}$$

We choose β such that $\beta > \max\{c_7(10C_1)^{n+N}, c_4^2, c_6\}$. Then $r_B > 10C_1 r_{B_{1,i}}$.
Let $B_{1,i} \cap B \neq \emptyset$. We have

$$A_{t_{B_{1,i}}} f(x) - A_{t_B} f(x) = A_{t_{B_{1,i}}} (f - A_{t_B} f)(x) + (A_{(t_{B_{1,i}} + t_B)} f(x) - A_{t_B} f(x)).$$

Additionally, we have $t_{B_{1,i}} + t_B \leq 2t_B$. Thus, by using Theorem 4.1.4 we have

$$|A_{(t_{B_{1,i}} + t_B)} f(x) - A_{t_B} f(x)| \leq c.$$

We now prove that

$$|A_{t_{B_{1,i}}} (f - A_{t_B} f)(x))| < c\beta, \text{ for } x \in B_{1,i}.$$

Let $x \in B_{1,i}$, q_i be the smallest integer such that $2C_1^2 B \subset 2^{q_i+1} B_{1,i}$ and $2C_1^2 B \cap (2^{q_i} B_{1,i})^c \neq \emptyset$. Then for $y \in 2C_1^2 B \cap (2^{q_i} B_{1,i})^c$, we have

$$2^{q_i} r_{B_{1,i}} < d(y, x_{B_{1,i}}) \leq C_1(d(y, x_0) + d(x_0, x_{B_{1,i}})) \leq C_1(2C_1^2 r_B + C_1(r_B + r_{B_{1,i}}))$$

For $x' \in 2^{q_i+1} B_{1,i}$, we have

$$\begin{aligned}
d(x, x_0) &\leq C_1(d(x', x_{B_{1,i}}) + d(x_{B_{1,i}}, x_0)) \\
&\leq C_1(2^{q_i+1} r_{B_{1,i}} + C_1(r_B + r_{B_{1,i}})) \\
&\leq C_1 \left(2C_1(2C_1^2 r_B + C_1(r_B + r_{B_{1,i}})) + C_1(r_B + r_{B_{1,i}})\right) \\
&\leq 10C_1^4 r_B.
\end{aligned}$$

Thus $2^{q_i+1} B_{1,i} \subset 10C_1^4 B$.
Furthermore, we have $B_{1,i} \cap B \neq \emptyset$ and $r_B > 10C_1 r_{B_{1,i}}$. Let $y' \in B_{1,i}$, then

$$d(y', x_0) \leq C_1(d(y', x_{B_{1,i}}) + d(x_{B_{1,i}}, x_0)) \leq C_1(r_{B_{1,i}} + C_1(r_B + r_{B_{1,i}})) \leq 2C_1^2 r_B.$$

Hence, $B_{1,i} \subset 2C_1^2 B$. We write

$$
\begin{aligned}
|A_{t_{B_{1,i}}}(f - A_{t_B} f)(x)| &\leq c\frac{1}{\nu(B_{1,i})}\int_X g(\frac{d^m(x,y)}{t_{B_{1,i}}})|f(y) - A_{t_B} f(y)|d\nu(y) \\
&\leq c\sum_{k=0}^{q_i+1} \frac{1}{\nu(B_{1,i})}\int_{2^k B_{1,i} - 2^{k-1}B_{1,i}} g(\frac{d^m(x,y)}{t_{B_{1,i}}})|f(y) - A_{t_B} f(y)|d\nu(y) \\
&+ c\frac{1}{\nu(B_{1,i})}\int_{X - 2^{q_i+1}B_{1,i}} g(\frac{d^m(x,y)}{t_{B_{1,i}}})|f(y) - A_{t_B} f(y)|d\nu(y) \\
&\leq I + II.
\end{aligned}
$$

We now study I.
We have $2^{q_i+1}B_{1,i} \subset 10C_1^4 B$. Hence, for $0 \leq k \leq q_i + 1$ we have

$$
\begin{aligned}
\frac{1}{\nu(2^k B_{1,i})}\int_{2^k B_{1,i}} |f(y) - A_{t_B} f(y)|d\nu(y) &= \frac{1}{\nu(2^k B_{1,i})}\int_{2^k B_{1,i}} |f_0(y)|d\nu(y) \\
&\leq \frac{c^n}{\nu(c2^k B_{1,i})}\int_{c2^k B_{1,i}} |f_0(y)|d\nu(y) \\
&\leq c'\beta \qquad\qquad \text{(due to } (Mf_0)(y) \leq \beta \text{ in } F)
\end{aligned}
$$

where c is the constant in Theorem 3.1.4.

Note that if an integer $k > [log_2 C_1] + 1$ (where $[log_2 C_1]$ denotes the integer part of $log_2 C_1$) then for all $x' \in B_{1,i}$ and $y \in 2^k B_{1,i} - 2^{k-1}B_{1,i}$ we have

$$
2^{k-1}r_{B_{1,i}} < d(y, x_{B_{1,i}}) \leq C_1 d(x', y) + C_1 d(x', x_{B_{1,i}}).
$$

Thus,

$$
d(x', y) \geq \frac{1}{C_1}(2^{k-1} - C_1)r_{B_{1,i}} \geq c_8 2^{k-1} r_{B_{1,i}}.
$$

Therefore,

$$
\begin{aligned}
I &\leq c\sum_{k=0}^{[log_2 C_1]+1} 2^{kn} g(0)\frac{1}{\nu(2^k B_{1,i})}\int_{2^k B_{1,i}} |f(y) - A_{t_B} f(y)|d\nu(y) \\
&+ \sum_{k=[log_2 C_1]+2}^{q_i+1} 2^{kn} g((c_8 2^{k-1})^m)\frac{1}{\nu(2^k B_{1,i})}\int_{2^k B_{1,i}} |f(y) - A_{t_B} f(y)|d\nu(y) \\
&\leq c'\beta + c''\beta \sum_{k=[log_2 C_1]+2}^{q_i+1} 2^{kn} g((c_8 2^{k-1})^m) \leq c\beta.
\end{aligned}
$$

Now we study II.
Let p_i be an integer such that $2^{p_i} r_{B_{1,i}} \leq r_B < 2^{p_i+1} r_{B_{1,i}}$. Then

$$
\nu(B(x_0, r_{B_{1,i}})) \leq c(1 + \frac{C_1(r_B + r_{B_{1,i}})}{r_{B_{1,i}}})^N \nu(B_{1,i}) \leq c2^{p_i N}\nu(B_{1,i}).
$$

Since $2C_1^2 B \subset 2^{q_i+1}B_{1,i}$ we have

$$
\begin{aligned}
II &\leq c\sum_{k=[2log_2 C_1]+1}^{\infty} \frac{1}{\nu(B_{1,i})}\int_{2^{k+1}B - 2^k B} g(\frac{d^m(x,y)}{t_{B_{1,i}}})|f(y) - A_{t_B} f(y)|d\nu(y) \\
&\leq c\sum_{k=[2log_2 C_1]+1}^{\infty} \frac{2^{p_i N} 2^{(k+1)n}}{\nu(2^{k+1}B)}\int_{2^{k+1}B - 2^k B} g(\frac{d^m(x,y)}{t_{B_{1,i}}})|f(y) - A_{t_B} f(y)|d\nu(y).
\end{aligned}
$$

In addition, if $k > [2log_2 C_1] + 1$, $x' \in B_{1,i}$ and $y \in 2^{k+1}B - 2^k B$ then there is $c_9 > 0$ such that $d(x', y) \geq c_9 2^{k+p_i} r_{B_{1,i}}$. Hence

$$
\begin{aligned}
II &\leq c \sum_{k=[2log_2 C_1]+1}^{\infty} \frac{2^{p_i N} 2^{(k+1)n} g(c_9^m 2^{(k+p_i)m})}{\nu(2^{k+1}B)} \int_{2^{k+1}B} |f(y) - A_{t_B} f(y)| d\nu(y) \\
&\leq c \sum_{k=[log_2 C_1]+1}^{\infty} 2^{(k+p_i)(n+N)} g(c_9^m 2^{(k+p_i)m}) \|f\|_{BMO_A} \leq c\beta.
\end{aligned}
$$

Combining the results for I and II, we have $|A_{t_{B_{1,i}}}(f - A_{t_B} f)(x))| < c\beta$ for all $x \in B_{1,i}$. Hence, $|A_{t_{B_{1,i}}} f(x) - A_{t_B} f(x)| \leq c_5 \beta$, for all $x \in B_{1,i}$.

By replacing B by $B_{1,i}$ and f_0 by $f_{B_{1,i}} = (f - A_{t_{B_{1,i}}} f) X_{10C_1^4 B_{1,i}}$, we get a sequence of balls $\{B_{2,j}\}_j$ such that:

1. $\sum_j \nu(B_{2,j}) \leq \frac{c_4}{\beta} \nu(B_{1,i})$,
2. for any $x \in B_{1,i} - \cup_j B_{2,j}$ we have $|f(x) - A_{t_{B_{1,i}}} f(x)| \leq \beta$,
3. for any $B_{2,j} \cap B_{1,i} \neq \emptyset$ we have $|A_{t_{B_{2,j}}} f(x) - A_{t_{B_{1,i}}} f(x)| \leq c_5 \beta$, for all $x \in B_{2,j}$.

Doing this for all $B_{1,i}$, we get countable open balls $\{B_{2,i}\}_J$. Hence if $x \in B - \cup_m B_{2,m}$, we have

$$|f(x) - A_{t_B} f(x)| \leq |f(x) - A_{t_{B_{1,i}}} f(x)| + |A_{t_{B_{1,i}}} f(x) - A_{t_B} f(x)| \leq 2c_5 \beta.$$

In addition,

$$\sum_{j=1} \nu(B_{2,j}) \leq \frac{c_4}{\beta} \sum_i \nu(B_{1,i}) \leq (\frac{c_4}{\beta})^2 \nu(B).$$

By repeating the procedure we get a collection of balls $B_{k,i} = B(x_{B_{i,k}}, r_{B_{i,k}})$ which verifies

(i) For any $B_{k,m} \cap B_{k+1,j} \neq \emptyset$, we have $|A_{t_{B_{k,m}}} f(x) - A_{t_{B_{k+1,j}}} f(x)| \leq c_5 \beta$, for all $x \in B_{k+1,j}$.

(ii) $\{x \in B : |f(x) - A_{t_B} f(x)| > c_5 k\beta\} \subseteq \bigcup_i B_{k,i}$.

(iii) $\sum_{i=1}^{\infty} \nu(B_{k,i}) \leq (\frac{c_4}{\beta})^k \nu(B)$.

We now study $\nu(\{x \in B : |f(x) - A_{t_B} f(x)| > \alpha\})$ where $\alpha > 0$.
The case if $\alpha < c_5 \beta$ we have

$$\nu(\{x \in B : |f(x) - A_{t_B} f(x)| > \alpha\}) \leq e^{1 - \frac{\alpha}{c_5 \beta}} \nu(B).$$

For $kc_5\beta \leq \alpha < (k+1)c_5\beta$, where $k \geq 1$ is an integer and $\beta > c_4^2$, we have

$$
\begin{aligned}
\nu(\{x \in B : |f(x) - A_{t_B} f(x)| > \alpha\}) &\leq \sum_i \nu(B_{k,i}) \\
&\leq (\frac{c_4}{\beta})^k \nu(B) \\
&\leq e^{\frac{-k \log \beta}{2}} \nu(B) \\
&\leq \beta^{\frac{1}{2}} e^{\frac{-\alpha log\beta}{2c_5 \beta}} \nu(B).
\end{aligned}
$$

Combining these two estimates demonstrates

$$\nu\{x \in B(x_0, r) : |f(x) - A_{t_B} f(x) > \alpha\} \leq c_1 \nu(B(x_0, r)) e^{-c_2 \alpha}.$$

Hence the proof of Theorem 4.2.1 is complete.

Definition 4.2.2 *Given $p \in [1, \infty)$, we define the space $BMO_A^p(X)$ as follows: We say that $f \in M$ is in $BMO_A^p(X)$ if there exists some constant c such that for any ball B,*

$$\left(\frac{1}{\nu(B)} \int_B |f(x) - A_{t_B} f(x)|^p d\nu(x) \right)^{\frac{1}{p}} \leq c \tag{4.9}$$

where $t_B = r_B^m$ and r_B is the radius of the ball B. The smallest c which (4.9) is satisfied is taken to be the norm of f in this space and is denoted by $\|f\|_{BMO_A^p}$.

Theorem 4.2.3 *For $p \in [1, \infty)$, the spaces BMO_A^p coincide, and the norms are equivalent with respect to different values of p.*
The proof is similar to Corollary 2.2.6, where the John-Nirenberg Inequality was an essential tool to prove it.

4.3 The Spaces BMO_A(X) and L^p Interpolation

Similar to the classical case, we have interpolation between BMO_A and $L^p(X)$. One of the main tools to prove this interpolation is the good-λ inequality.

Definition 4.3.1 *Let f be a locally integrable function and $1 \leq s < \infty$. The Hardy-Littlewood maximal function $M_s f$ is defined by*

$$M_s f(x) = \left(\sup_{x \in B} \frac{1}{\nu(B)} \int_B |f(y)|^s d\nu(y) \right)^{\frac{1}{s}}.$$

Note that by using $M_s(f) = M(|f|^s)^{\frac{1}{s}}$, M_s is of weak-type (s, s) and bounded from L^p to itself for $s < p < \infty$.

Definition 4.3.2 *Let $f \in M$ and $1 \leq s < \infty$. The sharp maximal function $M_{A,s}^{\#}$ associated with the generalized approximation to the identity $\{A_t\}_{t>0}$ is defined by*

$$M_{A,s}^{\#} f(x) = \sup_{x \in B} \left(\frac{1}{\nu(B)} \int_B |f(y) - A_{r_B^m} f(y)|^s d\nu(y) \right)^{\frac{1}{s}}$$

where r_B is the radius of the ball B. We denote $M_{A,1}^{\#}$ by $M_A^{\#}$.

Remark: for $f \in L^p(X)$, $M_{A,s}^{\#}$ is pointwise bounded by $M_s f$. Therefore we have

$$\|M_{A,s}^{\#} f\|_p \leq c \|M_s f\|_p \leq c \|f\|_p.$$

Lemma 4.3.3 *For every ball $B_0 \subset X$ and every function $f \in L^p(X)$, $1 \leq p < \infty$, such that there exists $x_0 \in B_0$ with $Mf(x_0) \leq \lambda$. Then there is $c_0 > 0$ such that*

$$A_{2^m r_0^m}(f)(x) \leq c_0 \lambda$$

for any $x \in B_0$, where r_0 is the radius of B_0.

Proof. Let $B_0 = B(x_{B_0}, r_0)$ and let $x \in B_0$. Hence,

$$\nu(B(x_{B_0}, 3r_0)) \leq c \left(1 + \frac{d(x, x_{B_0})}{3r_0} \right)^N \nu\left(B(x, 3r_0)\right) \leq c\nu \left(B(x, \frac{3}{2} 2r_0) \right) \leq c\nu\left(B(x, 2r_0)\right).$$

Hence,

$$\begin{aligned}
|A_{2^m r_0^m} f(x)| &\leq c\nu\left(B(x,2r_0)\right)^{-1}\int_X g\left(\frac{d^m(x,y)}{2^m r_0^m}\right)|f(y)|d\nu(y)\\
&\leq c\nu(B(x_{B_0},3r_0))^{-1}\sum_{k=0}^{\infty}\int_{3^k B_0 - 3^{k-1}B_0} g\left(\frac{d^m(x,y)}{2^m r_0^m}\right)|f(y)|d\nu(y)\\
&\leq c\nu(B(x_{B_0},3r_0))^{-1}\sum_{k=0}^{\infty} g(3^{(k-2)m})\int_{3^k B_0}|f(y)|d\nu(y)\\
&\leq c\sum_{k=0}^{\infty} g(3^{(k-2)m})\nu(3B_0)^{-1}\nu(3^k B_0)\lambda\\
&\leq c\lambda\sum_{k=0}^{\infty} 3^{kn} g(3^{(k-2)m}) \leq c_0\lambda.
\end{aligned}$$

Thus, the proof of Lemma 4.3.3 is complete.

The following lemma is called good-λ inequality.

Lemma 4.3.4 *There exist $K_0 > 1$ and $c > 0$, such that for every $\lambda > 0$ and every $K > K_0$ and $\gamma > 0$, for every ball B_0 in X and every function $f \in L^p(X)$, $1 \leq p < \infty$, such that there is $x_0 \in B_0$ with $Mf(x_0) \leq \lambda$, we have*

$$\nu\{x \in B_0 : |f(x)| > K\lambda, M^{\#}_{A,s} f(x) \leq \gamma\lambda\} \leq c\gamma^s \nu(B_0). \tag{4.10}$$

Proof. Let $B_0 = B(x_0, r_0)$ be an open ball in X and let $K_0 = c_0 + 1$, where c_0 is the constant defined in Lemma 4.3.3. Set

$$U_{B_0} = \{x \in B_0 : |f(x)| > K\lambda, M^{\#}_{A,s} f(x) \leq \gamma\lambda\}.$$

If $U_{B_0} = \emptyset$, then (4.10) is obvious. Assume that $U_{B_0} \neq \emptyset$. Using Lemma 4.3.3 we have for every $x \in B_0$,

$$A_{2^m r_0^m}(f)(x) \leq c_0\lambda.$$

Hence, when $x \in U_{B_0}$,

$$|f(x) - A_{2^m r_0^m} f(x)| \geq |f(x)| - |A_{2^m r_0^m} f(x)| \geq K\lambda - c_0\lambda \geq (c_0+1)\lambda - c_0\lambda \geq \lambda.$$

Therefore, for any $x \in U_{B_0}$ we have

$$\begin{aligned}
\nu(U_{B_0}) = \int_{U_{B_0}} 1 d\nu(y) &\leq \lambda^{-s}\int_{U_{B_0}} |f(y) - A_{2^m r_0^m} f(y)|^s d\nu(y)\\
&\leq \lambda^{-s}\nu(B(x,2r_0))(M^{\#}_{A,s} f(x))^s\\
&\leq \lambda^{-s}\nu(3B_0)(\lambda\gamma)^s\\
&\leq c\gamma^s\nu(B_0).
\end{aligned}$$

Hence, Lemma 4.3.4 is proved.

The following lemma is a consequence of good-λ inequality and it is essential to the proof of interpolation theorem using BMO_A.

Lemma 4.3.5 *For every $f \in L^1(X)$, for $< p < \infty$, there is c_p such that*

$$\|f\|_{L^p} \leq c_p(\|M^{\#}_{A,s} f\|_p + \|f\|_1)$$

where the last term on the right-hand side can be canceled if $\nu(X) = \infty$.

Proof. For $\lambda > 0$, let $E_\lambda = \{x \in X : Mf(x) > \lambda\}$. We take $\lambda_0 = 0$ if $\nu(X) = \infty$ and $\lambda_0 = c\|f\|_1(\nu(X))^{-1}$ if $\nu(X) < \infty$, where c is the constant appearing in the Theorem 3.1.3. By using Lemma 3.1.4 and assuming $\lambda > \lambda_0$ we have a collection of balls B_i such that:

1. $\underset{i}{\cup} B_i = E_\lambda$,
2. each point of E_λ is contained in at most a finite number K of balls B_i,
3. there is $c > 1$ such that $cB_i \cap E_\lambda^c \neq \emptyset$.

Given that $cB_i \cap E_\lambda^c \neq \emptyset$, there is $x_i \in cB_i$ such that $Mf(x_i) < \lambda$. Hence, by using Lemma 4.3.4 we have for every $\gamma > 0$ and $K > K_0$ (note that $K_0 > 1$),

$$\nu\{x \in cB_i : |f(x)| > K\lambda, M^{\#}_{A,s}f(x) \leq \gamma\lambda\} \leq c\gamma^s \nu(cB_i) \leq c\gamma^s \nu(B_i).$$

In addition, we have

$$\{x \in X : |f(x)| > K\lambda, M^{\#}_{A,s}f(x) \leq \gamma\lambda\} \subset E_\lambda.$$

Thus,

$$\begin{aligned}
&\nu\Big(\{x \in X : |f(x)| > K\lambda, M^{\#}_{A,s}f(x) \leq \gamma\lambda\}\Big) \\
&\leq \nu\Big(\underset{i}{\cup}\{x \in cB_i : |f(x)| > K\lambda, M^{\#}_{A,s}f(x) \leq \gamma\lambda\}\Big) \qquad (c > 1 \text{ and } cB_i \cap E_\lambda^c \neq \emptyset) \\
&\leq \sum_i \nu\Big(\{x \in cB_i : |f(x)| > K\lambda, M^{\#}_{A,s}f(x) \leq \gamma\lambda\}\Big) \\
&\leq c\gamma^s \sum_i \nu(B_i) \leq c\gamma^s \nu(E_\lambda).
\end{aligned}$$

We will prove first the lemma for the case where X is unbounded. Using good-λ inequality we have

$$\begin{aligned}
\|f\|_p^p &= pK^p \int_0^\infty \lambda^{p-1} \nu(\{x \in X : |f(x)| > K\lambda\}) d\lambda \\
&\leq pK^p \int_0^\infty \lambda^{p-1} \Big(\nu(\{x \in X : |f(x)| > K\lambda, M^{\#}_{A,s}f(x) \leq \gamma\lambda\}) \\
&\qquad + \nu(\{x \in X : M^{\#}_{A,s}f(x) \geq \gamma\lambda\})\Big) d\lambda \\
&\leq cpK^p\gamma^s \int_0^\infty \lambda^{p-1}\nu(E_\lambda)d\lambda + \frac{pK^p}{\gamma^p}\|M^{\#}_{A,s}f\|_p^p \\
&\leq cpK^p\gamma^s\|Mf\|_p^p + \frac{pK^p}{\gamma^p}\|M^{\#}_{A,s}f\|_p^p \\
&\leq c'pK^p\gamma^s\|f\|_p^p + \frac{pK^p}{\gamma^p}\|M^{\#}_{A,s}f\|_p^p.
\end{aligned}$$

Therefore,

$$\|f\|_p^p \leq c'pK^p\gamma^s\|f\|_p^p + \frac{pK^p}{\gamma^p}\|M^{\#}_{A,s}f\|_p^p.$$

Hence, by choosing $\gamma = (2c'pK^p)^{-\frac{1}{s}}$ and moving the first part of the right side of the inequality to other side, we obtain

$$\|f\|_p^p \leq c\|M^{\#}f\|_p^p.$$

If X is bounded, we have:

$$\begin{aligned}
\|f\|_p^p &= pK^p \int_0^\infty \lambda^{p-1} \nu(\{x \in X : |f(x)| > K\lambda\}) d\lambda \\
&\le pK^p \left(\int_0^{\lambda_0} \lambda^{p-1} \nu(\{x \in X : |f(x)| > K\lambda\}) d\lambda + \int_{\lambda_0}^\infty \lambda^{p-1} \nu(\{x \in X : |f(x)| > K\lambda\}) d\lambda \right).
\end{aligned}$$

Additionally,

$$\begin{aligned}
&\int_{\lambda_0}^\infty \lambda^{p-1} \nu(\{x \in X : |f(x)| > K\lambda\}) d\lambda \\
&\le pK^p \int_{\lambda_0}^\infty \lambda^{p-1} (\nu(\{x \in X : |f(x)| > K\lambda, M_{A,s}^{\#} f(x) \le \gamma\lambda\}) + \nu(\{x \in X : M_{A,s}^{\#} f(x) \ge \gamma\lambda\})) d\lambda \\
&\le cpK^p \gamma^s \int_{\lambda_0}^\infty \lambda^{p-1} \nu(E_\lambda) d\lambda + \frac{pK^p}{\gamma^p} \|M_{A,s}^{\#} f\|_p^p \\
&\le cpK^p \gamma^s \|Mf\|_p^p + \frac{pK^p}{\gamma^p} \|M_{A,s}^{\#} f\|_p^p \\
&\le c' pK^p \gamma^s \|f\|_p^p + \frac{pK^p}{\gamma^p} \|M_{A,s}^{\#} f\|_p^p.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|f\|_p^p &\le K^p \lambda_0^p \nu(X) + c' pK^p \gamma^s \|f\|_p^p + \frac{pK^p}{\gamma^p} \|M_{A,s}^{\#} f\|_p^p \\
&\le c\|f\|_1^p + c' pK^p \gamma^s \|f\|_p^p + \frac{pK^p}{\gamma^p} \|M_{A,s}^{\#} f\|_p^p. \qquad (\lambda_0 = c\|f\|_1 (\nu(X))^{-1})
\end{aligned}$$

Thus,

$$\|f\|_p \le c(\|M_{A,s}^{\#} f\|_p + \|f\|_1).$$

We now state the main result of this section on the interpolation of the space BMO_A.

Theorem 4.3.6 *Let $1 \le s \le q$. Assume that T is a sublinear operator that is bounded on $L^q(X)$, $1 \le q < \infty$ and*

$$\|M_{A,s}^{\#} Tf\|_\infty \le c\|f\|_\infty.$$

Then T is bounded on $L^p(X)$ for all $q < p < \infty$.

Proof. Let $f \in L^p(X)$ and $1 \le s \le q$. We define the operator $M_{T,A,s}^{\#}$ as

$$M_{T,A,s}^{\#} f(x) = M_{A,s}^{\#} Tf(x).$$

The aim is to show that $M_{T,A,s}^{\#}$ is bounded from $L^p(X)$ to it self for all $q < p < \infty$.
First, we prove that $M_{T,A,s}^{\#} f(x) \le cM_s(Tf)(x)$. We have

$$\begin{aligned}
\frac{1}{\nu(B)} \int_B |Tf(y) - A_{t_B} Tf(y)|^s d\nu(y) &\le c\frac{1}{\nu(B)} \int_B (|Tf(y)|^s + |A_{t_B} Tf(y)|^s) d\nu(y) \\
&\le c\frac{1}{\nu(B)} \int_B |Tf(y)|^s d\nu(y) + c\frac{1}{\nu(B)} \int_B |A_{t_B} Tf(y)|^s d\nu(y).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sup_{x\in B}\frac{1}{\nu(B)}\int_B |Tf(y)-A_{t_B}Tf(y)|^s d\nu(y)\\
&\le \sup_{x\in B}\frac{c}{\nu(B)}\int_B |Tf(y)|^s d\nu(y)+\sup_{x\in B}\frac{c}{\nu(B)}\int_B |A_{t_B}Tf(y)|^s d\nu(y)\\
&\le c(M_s(Tf)(x))^s.
\end{aligned}$$

Hence, $M^{\#}_{T,A,s}f(x)\le cM_s(Tf)(x)$.
Therefore, for $1\le s\le q$ and $\lambda>0$ we have

$$\begin{aligned}
\nu\{x\in X : M^{\#}_{T,A,s}f(x)>\lambda\} &\le \nu\{x\in X : M_s(Tf)(x)>c\lambda\}\\
&\le \frac{c}{\lambda^q}\|Tf\|_q \le \frac{c}{\lambda^q}\|f\|_q,
\end{aligned}$$

so $M^{\#}_{T,A,s}$ is of weak-type (q,q). In addition, we have

$$\|M^{\#}_{T,A,s}f\|_\infty = \|M^{\#}_{A,s}Tf\|_\infty \le c\|f\|_\infty.$$

Thus, $M^{\#}_{T,A,s}$ is bounded on $L^\infty(X)$. Hence, by using interpolation theorem we have $M^{\#}_{T,A,s}$ is bounded from L^p to its self for $q<p<\infty$.

If $\nu(X)=\infty$, we have

$$\begin{aligned}
\|Tf\|_p &\le c\|M^{\#}_{A,s}Tf\|_p && \text{due to Lemma 4.3.5}\\
&= c\|M^{\#}_{T,A,s}f\|_p\\
&\le c\|f\|_p. && \text{due to the boundedness of } M^{\#}_{T,A,s}
\end{aligned}$$

Therefore, T is bounded on L^p for all $q<p<\infty$.

For the case $\nu(X)<\infty$ we have

$$\begin{aligned}
\|Tf\|_p &\le c(\|M^{\#}_{A,s}Tf\|_p+\|Tf\|_1) && (\text{ to Lemma 4.3.5 })\\
&\le c(\|M^{\#}_{T,A,s}f\|_p)+c_X\|Tf\|_q && (\text{Holder's inequality})\\
&\le c\|f\|_p+c_X\|f\|_q && (\text{due to boundedness of } M^{\#}_{T,A,s} \text{ on } L^p \text{ and of } T \text{ on } L^q\ )\\
&\le c\|f\|_p. && (\text{Holder's inequality and } \nu(x)<\infty)
\end{aligned}$$

Thus, the theorem is proved.

5

Applications: Holomorphic Functional Calculus of Elliptic Operators

The main references of this chapter are [4, 5].
In this chapter, (X, d, ν) is a space of homogeneous type where d is a metric and the condition (3.8) of $h_t(x, y)$ is replaced by

$$\lim_{r\to\infty} r^{n+2N+\epsilon} g(r^m) = 0. \tag{5.1}$$

We will first state the holomorphic functional calculus given by McIntoch [11].
Let $0 \le w < v < \pi$. We define the S_w as

$$S_w = \{z \in \mathbb{C} : |\arg z| \le w\} \cup \{0\}$$

Let S_w^0 be the interior of S_w, $H(S_v^0)$ be the space of all holomorphic functions on S_v^0 ,

$$H_\infty(S_v^0) = \{b \in H(S_v^0) : \|b\|_\infty < \infty\}$$

where

$$\|b\|_\infty = \sup\{|b(z)| : z \in S_v^0\}\}$$

and

$$\Psi(S_v^0) = \{\psi \in H(S_v^0) : \text{ there is } s > 0, |\psi(z)| \le c|z|^s(1 + |z|^{2s})^{-1}\}.$$

Let $0 \le w < \pi$. A closed operator L in $L^p(X)$ is said to be of type w if $\sigma(L) \subset S_w$, and for each $v > w$, there exists a constant c_v, such that

$$\|(L - \lambda I)^{-1}\|_{p,p} \le c_v|\lambda|^{-1}, \quad \lambda \notin S_v.$$

If L is of type w and $\psi \in \Psi(S_v^0)$, we define $\psi(L) \in L(L^p, L^p)$ by

$$\psi(L) = \frac{1}{2\pi i}\int_\Gamma (L - \lambda I)^{-1}\psi(\lambda)d\lambda,$$

where $\Gamma = \{z \in \mathbb{C} : z = re^{\pm i\theta}, r \ge 0\}$ is parameterized clockwise around S_w, and $w < \theta < v$. This definition is independent of the choice of $w < \theta < v$. Furthermore, if L is one-one and has dense range and if $b \in H_\infty(S_v^0)$, then $b(L)$ can be defined by

$$b(L) = [\psi(L)]^{-1}(b\psi)(L),$$

where $\psi(z) = z(1 + z)^{-2}$.

Definition 5.0.1 *The operator L has a bounded H_∞ functional calculus in L^p, $1 < p < \infty$, if there exists $c_{v,p} > 0$ such that $b(L) \in L(L^p, L^p)$, and*

$$\|b(L)\|_{p,p} \leq c_{v,p}\|b\|_\infty$$

for all $b \in H_\infty(S_v^0)$.

The following lemma helps to extend the boundedness of $b(L)$ for b in $\Psi(S_v^0)$ to $H_\infty(S_v^0)$. For the proof see [11].

Lemma 5.0.2 *Let $0 \leq w < v \leq \pi$ and $1 < p < \infty$. Let L be an operator of type w which is one-one with dense range. Let $\{b_\alpha\}_\alpha$ be a uniformly bounded net in $H_\infty(S_v^0)$. Let $b \in H_\infty(S_v^0)$, and suppose, for some $M < \infty$, that*

1. *$\|b_\alpha(L)\|_{p,p} \leq M$.*
2. *for each $0 < \delta < \beta < \infty$,*

$$\sup\{|b_\alpha(z) - b(z)| : z \in S_v^0 \text{ and } \delta \leq |z| \leq \beta\} \to 0.$$

Then $b(L) \in L(L^p, L^p)$, $b_\alpha(L)u \to b(L)u$ in $L^p(X)$ for all $u \in L^p(X)$ and

$$\|b(L)\|_{p,p} \leq sup_\alpha\|b_\alpha\|_{p,p}$$

Theorem 5.0.3 *Let L be a linear operator of type w on $L^2(X)$ with $w < \frac{\pi}{2}$, so that $(-L)$ generates a holomorphic semigroup e^{-zL}, $0 \leq |Arg(z)| < \frac{\pi}{2} - w$. If*

1. *The kernel $a_z(x, y)$ of the holomorphic semigroup e^{-zL}, $|Arg(z)| < \frac{\pi}{2} - w$, satisfy:*

$$|a_z(x, y)| \leq h_{|z|}(x, y)$$

   for $x, y \in X$, $|Arg(z)| < \frac{\pi}{2} - \theta$ for some $\theta > w$.

2. *The operator L has a bounded holmorphic functional calculus in $L^2(X)$.*

Then L has a bounded holmorphic function on L^p, $1 < p < \infty$.

Proof. We will first prove that for any $b \in \Psi(S_v^0)$, $b(L)$ satisfies the conditions of Theorem 3.2.6. Hence, the operator $b(L)$ is bounded on L^p, then by using the convergent lemma 5.0.2, we prove that L has a bounded H_∞ functional calculus in L^p, $1 < p < \infty$.
Let $b \in \Psi(S_v^0)$, then L^2 boundedness of $b(L)$ is direct result from (2) of 5.0.3. In addition, if we take A_t in Theorem 3.2.6 to be e^{-tL}, then (1) of Theorem 5.0.3 gives the condition (A_2) of Theorem 3.2.6. Now we will prove (A_1) and (A_3) of theorem 3.2.6 . Choose v, θ and ν such that $w < \theta < \nu < v < \frac{\pi}{2}$ and let $b \in \Psi(S_v^0)$. Note $\gamma = \gamma_+ + \gamma_-$, where $\gamma+ = \{te^{i\nu}, t \geq 0\}$ and $\gamma- = \{-te^{-i\nu}, t \leq 0\}$, we have

$$b(L) = \frac{1}{2\pi i}\int_\gamma (L - \lambda I)^{-1} b(\lambda) d\lambda.$$

If $\lambda \in \gamma_+$, then we have

$$(L - \lambda I)^{-1} = \int_{\sigma_+} e^{\lambda z} e^{-zL} dz$$

where $\sigma_+ = \{te^{i\frac{\pi}{2}-\theta}, t \geq 0\}$, and if $\lambda \in \gamma_-$ then we have

$$(L - \lambda I)^{-1} = \int_{\sigma_-} e^{\lambda z} e^{-zL} dz$$

where $\sigma_- = \{-te^{-i\frac{\pi}{2}+\theta}, t \leq 0\}$.
Note

$$b_+(L) = \frac{1}{2i\pi} \int_{\gamma_+} \int_{\sigma_+} e^{\lambda z} e^{-zL} b(\lambda) dz d\lambda = \frac{1}{2i\pi} \int_{\sigma_+} e^{-zL} \int_{\gamma_+} e^{\lambda z} b(\lambda) d\lambda dz = \int_{\sigma_+} e^{-zL} f_+(z) dz$$

where $f_+(z) = \frac{1}{2\pi i} \int_{\gamma_+} e^{\lambda z} b(\lambda) d\lambda$, and

$$b_-(L) = \frac{1}{2i\pi} \int_{\gamma_-} \int_{\sigma_-} e^{\lambda z} e^{-zL} b(\lambda) dz d\lambda = \frac{1}{2i\pi} \int_{\sigma_-} e^{-zL} \int_{\gamma_-} e^{\lambda z} b(\lambda) d\lambda dz = \int_{\sigma_-} e^{-zL} f_-(z) dz$$

where $f_-(z) = \frac{1}{2\pi i} \int_{\gamma_-} e^{\lambda z} b(\lambda) d\lambda$. We have $b(L) = b_+(L) + b_-(L)$. Hence, the kernel $K_b(x, y)$ of $b(L)$ is given by

$$K_b(x, y) = \int_{\sigma_+} a_z(x, y) f_+(z) dz + \int_{\sigma_-} a_z(x, y) f_-(z) dz.$$

Let $m_t = e^{-tz} b(z)$ and $g_t(z) = (1 - e^{-tz}) b(z)$ for $t > 0$. We now prove that:

When $d(x, y) \geq ct^{\frac{1}{m}}$, we have

$$K_{g_t}(x, y) \leq c \|b\|_\infty \frac{t^{\frac{\alpha}{m}}}{\nu(B(x, d(x, y))) d(x, y)^\alpha}. \tag{5.2}$$

for some $\alpha > 0$, and this we will lead to prove (A_3).

We have $|1 - e^{-t\lambda}| \leq c$ if $\Re(\lambda) \geq 0$ and $t \geq 0$ and $|1 - e^{-t\lambda}| \leq c|t\lambda| \leq c|t\lambda|^\beta$ for $0 < \beta < \min\{\epsilon, 1\}$ when $|t\lambda| \leq 1$ (ϵ is constant in 4.1). In addition, $e^{-s} \leq s^{-\beta}$, for $s > 0$ and $\beta < \min\{\epsilon, 1\}$.

Let $d(x, y) \geq ct^{\frac{1}{m}}$ and $\beta < \min\{1, \frac{\epsilon}{m}\}$. Setting $a = |cos(\nu + \beta)|$, we have

$$\begin{aligned}
|K_{g_t}(x, y)| &\leq c\|b\|_\infty \int_0^\infty |a_z(x, y)| \int_0^\infty |e^{z\lambda}(1 - e^{-t\lambda})| d|\lambda| d|z| \\
&\leq c\|b\|_\infty \int_0^\infty |a_z(x, y)| \left(\int_0^{t^{-1}} e^{-a|z||\lambda|} |t\lambda|^\beta d|\lambda| + \int_{\frac{1}{t}}^\infty e^{-a|\lambda||z|} d|\lambda| \right) d|z| \\
&\leq c\|b\|_\infty t^\beta \int_0^\infty |a_z(x, y)| |z|^{-1-\beta} d|z|.
\end{aligned}$$

We will find an upper bound for the last term

$$\begin{aligned}
&\int_0^\infty |a_z(x,y)||z|^{-1-\beta}d|z| \\
&\quad\le \int_0^\infty \frac{g(\frac{d(x,y)^m}{|z|})}{\nu(B(x,|z|^{\frac{1}{m}}))}|z|^{-1-\beta}d|z| \\
&\quad\le \int_0^{d(x,y)^m} \frac{g(\frac{d(x,y)^m}{|z|})}{\nu(B(x,|z|^{\frac{1}{m}}))}|z|^{-1-\beta}d|z| + \int_{d(x,y)^m}^\infty \frac{g(\frac{d(x,y)^m}{|z|})}{\nu(B(x,|z|^{\frac{1}{m}}))}|z|^{-1-\beta}d|z| \\
&\quad\le c\int_0^{d(x,y)^m} \frac{(\frac{d(x,y)}{|z|^{\frac{1}{m}}})^{-2N-\epsilon}}{\nu(B(x,d(x,y)))}|z|^{-1-\beta}d|z| + c\int_{d(x,y)^m}^\infty \frac{1}{\nu(B(x,d(x,y)))}|z|^{-1-\beta}d|z| \\
&\quad\le c\int_0^{d(x,y)^m} \frac{\left(\frac{d(x,y)}{|z|^{\frac{1}{m}}}\right)^{-2N-\beta m}}{\nu(B(x,d(x,y)))}|z|^{-1-\beta}d|z| + c\int_{d(x,y)^m}^\infty \frac{1}{\nu(B(x,d(x,y)))}|z|^{-1-\beta}d|z| \\
&\quad\le c\frac{1}{\nu(B(x,d(x,y))d(x,y)^{m\beta}}.
\end{aligned}$$

Therefore, by taking $\alpha = \beta m$, we have

$$K_{g_t}(x,y) \le c\|b\|_\infty \frac{t^{\frac{\alpha}{m}}}{\nu(B(x,d(x,y)))d(x,y)^\alpha}.$$

We will prove (A_3) of Theorem 3.2.6.

$$\begin{aligned}
\int_{d(x,y)>c't^{\frac{1}{m}}} |K_{g_t}(x,y)|d\nu(x) &\le c\int_{d(x,y)>c't^{\frac{1}{m}}} \frac{t^{\frac{\alpha}{m}}}{\nu(B(x,d(x,y)))d(x,y)^\alpha}d\nu(x) \\
&\le c\sum_{i=0}^\infty \int_{2^ic't^{\frac{1}{m}}<d(x,y)<2^{i+1}c't^{\frac{1}{m}}} \frac{t^{\frac{\alpha}{m}}}{\nu(B(x,2^ic't^{\frac{1}{m}}))(2^ic't^{\frac{1}{m}})^\alpha}d\nu(x) \\
&\le c\sum_{i=0}^\infty 2^{-i\alpha}\int_{B(y,2^{i+1}c't^{\frac{1}{m}})} \frac{1}{\nu(B(y,2^ic't^{\frac{1}{m}}))}d\nu(x) \le c.
\end{aligned}$$

Therefore, by taking $A_t = e^{-tL}$ we have that $b(L)$ verifies $(A_1), (A_2)$ and (A_3) of Theorem 3.2.6. Hence, $b(L)$ is bounded on $L^p(X)$, $1 < p \le 2$. In addition, $b(L)$ and e^{-tL} commute. Thus, by using duality argument we have $b(L)$ is bounded on $L^p(X)$, $1 < p < \infty$.

Theorem 5.0.4 *Let T be an operator satisfying the following conditions:*

1. *T is a bounded linear operator from $L^2(X)$ to $L^2(X)$ with the kernel k such that for every f in $L^\infty(X)$ with bounded support,*

$$Tf(x) = \int_X k(x,y)f(y)d\nu(y)$$

for ν-almost all $x \notin \operatorname{supp} f$.

2. *There exists a generalized approximation to the identity $\{A_t\}_{t>0}$ such that the operators $(T - A_tT)$ have associated kernels $k_t(x,y)$ and there exist positive constants c_1 and c_2 such that:*

$$\int_{d(x,y)>c_1t^{\frac{1}{m}}} |k_t(x,y)|d\nu(y) \le c_2 \qquad \textit{for all } x \in X.$$

Then

$$\|Tf\|_{BMO_A} \leq c\|f\|_\infty$$

for all $f \in L^2(X) \cap L^\infty(X)$. (Note, we can always assume that $c_1 > 1$).

Proof. To prove the theorem, we will show that

$$\frac{1}{\nu(B)}\int_B |Tf(x) - A_{t_B}Tf(x)|d\nu(x) \leq c\|f\|_\infty.$$

Let $f \in L^2(X) \cap L^\infty(X)$ and let $f_1 = 1_{4c_1B}f$ and $f_2 = 1_{(4c_1B)^c}f$.

$$\begin{aligned}\frac{1}{\nu(B)}\int_B |Tf(x) - A_{t_B}Tf(x)|d\nu(x) &\leq \frac{1}{\nu(B)}\int_B (|Tf_1(x) - A_{t_B}Tf_1(x)| \\ &\quad + |(T - A_{t_B}T)f_2(x)|)d\nu(x) \\ &\leq c\int_B M(|Tf_1|)(x)d\nu(x) + \int_B |(T - A_{t_B}T)f_2(x)|d\nu(x).\end{aligned}$$

Furthermore,

$$\begin{aligned}\int_B M(|Tf_1|)(x)d\nu(x) &\leq c(\nu(B))^{\frac{1}{2}}(\int_X |M(|Tf_1|)(x)|^2 d\nu(x))^{\frac{1}{2}} \\ &\leq c(\nu(B))^{\frac{1}{2}}(\int_{4c_1B} |f(x)|^2 d\nu(x))^{\frac{1}{2}} \qquad \text{M and T are bounded on } L^2(X) \\ &\leq c\nu(B)\|f\|_\infty.\end{aligned}$$

In addition,

$$\begin{aligned}\int_B |(T - A_{t_B}T)f_2(x)|d\nu(x) &\leq \int_B \int_{(4c_1B)^c} |k_{t_B}(x,y)||f(y)|d\nu(y)d\nu(x) \\ &\leq \|f\|_\infty \int_B \int_{d(x,y)>c_1t^{\frac{1}{m}}} |k_{t_B}(x,y)|d\nu(y)d\nu(x) \\ &\leq c\nu(B)\|f\|_\infty.\end{aligned}$$

Hence,

$$\frac{1}{\nu(B)}\int_B |Tf(x) - A_{t_B}Tf(x)|d\nu(x) \leq c\|f\|_\infty.$$

Thus,

$$\|Tf\|_{BMO_A} \leq c\|f\|_\infty$$

and the theorem is proved.

Note that by using Theorem 4.3.6 we can show that T can be extended to a bounded operator on $L^p(X)$, $2 < p < \infty$.

Remark: The approach in this thesis can be adapted to study the end-point estimates and L^p boundedness of other singular integrals whose kernels are rough and do not belong to the class of Calderón-Zygmund operators. Examples are the Riesz transforms of the Laplace Beltrami operators on certain doubling manifolds, the Riesz transforms associated with the Divergence form operators on $\mathbb{R}^n$ or associated to the Schrödinger operators with non-negative potentials, see [1]. One can also obtain weighted estimates for singular integrals with rough kernels, see [6]. However, we do not pursue these results within this Master thesis.

References

[1] P. Auscher, T. Coulhon, X. T. Duong and S. Hofmann, 2004. November. Riesz transform on manifolds and heat kernel regularity. In Annales scientifiques de l'École normale supérieure (Vol. 37, No. 6, pp. 911–957). No longer published by Elsevier. 47

[2] R. R. Coifman and G. Weiss, 2006. Analyse harmonique non-commutative sur certains espaces homogénes: Etude de certaines intégrales singulières (Vol. 242). Springer. 20

[3] M. Christ, 1990. A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. In Colloquium Mathematicum (Vol. 2, No. 60-61, pp. 601–628). 21

[4] X. T. Duong and A. G. McIntosh, 1999. Singular integral operators with non-smooth kernels on irregular domains. Rev. Mat. Iberoamericana, 15(2), pp.233–265. 2, 19, 43

[5] X. T. Duong and L. Yan, 2005. New function spaces of BMO type, the John-Nirenberg inequality, interpolation, and applications. Communications on Pure and Applied Mathematics, 58(10), pp.1375–420. 2, 27, 43

[6] X. T. Duong and L. Yan, 2005. Weighted inequalities for holomorphic functional calculi of operators with heat kernel bounds. Journal of the Mathematical Society of Japan, 57(4), pp.1129–1152. 47

[7] L. Grafakos, 2004. Classical and modern Fourier analysis. Pearsson Education. Inc., London. 16, 17, 21

[8] E. M. Stein and T. S. Murphy, 1993. Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals (Vol. 3). Princeton University Press.

[9] E. M. Stein, 1970. Singular integrals and differentiability properties of functions (Vol. 2). Princeton university press. 4, 6

[10] G. B. Folland, E. M. Stein and E. M. Mather, 1982. Hardy spaces on homogeneous groups (Vol. 28). Princeton University Press. 28

[11] A. G. McIntosh, 1986. Operators which have an H_∞ functional calculus. In Miniconference on operator theory and partial differential equations (pp. 210-231). Centre for Mathematics and its Applications, Mathematical Sciences Institute, The Australian National University. 43, 44