Morphisms of 2-dimensional structures with applications

By

Branko Nikolić

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Summary

This thesis consists of three chapters providing solutions to three problems. All of them involve morphisms of bicategories: lax functors, enriched categories, and categories enriched on two sides.

The first problem was to obtain explicit constructions for various 2-categories which represent 2-categorical concepts involving monads and comonads. We considered lax functors (these are the morphisms of bicategories in the sense of Bénabou) between 2-categories $\mathcal{C}$ and $\mathcal{D}$ and define strictification tensor product for them. Let $\mathrm{Lax}(\mathcal{C}, \mathcal{D})$ denote the 2-category of lax functors, lax natural transformations and modifications, and $[\mathcal{C}, \mathcal{D}]_{\mathrm{lnt}}$ its full sub-2-category of (strict) 2-functors. Since monads can be seen as lax functors from 1 (the terminal category), the bicategory of monads in $\mathcal{D}$, denoted $\mathrm{Mnd}(\mathcal{D})$, is isomorphic to $\mathrm{Lax}(1, \mathcal{D})$. A concise way of defining distributive laws is as monads in $\mathrm{Mnd}(\mathcal{D})$. We give a construction of a 2-category $\mathcal{C} \boxtimes \mathcal{D}$ satisfying $\mathrm{Lax}(\mathcal{C}, \mathrm{Lax}(\mathcal{D}, \mathcal{E})) \cong [\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E}]_{\mathrm{lnt}}$, thus generalizing the case of the free distributive law $1 \boxtimes 1$. We also analyse dual constructions.

The second problem involves enriching in a monoidal category similar to the one used by Lawvere to obtain (generalized) metric spaces. He expressed Cauchy completeness in purely categorical terms which led to the possibility of applying it to an arbitrary base; for example, an ordinary category is Cauchy complete when all its idempotents split. What we do is to obtain spaces of relativistic events as enriched categories and show that they are always Cauchy complete in the categorical sense. We then see this as a more general phenomenon by providing conditions on the base monoidal category which ensure Cauchy completeness of those enriched categories having all idempotents splitting in the underlying

category. The splitting condition was not seen in the case of our partially ordered base since the only idempotents are identities.

Finally, in order to analyse Cauchy modules for categories enriched in graded and differential graded Abelian groups (GAb and DGAb), we consider two-sided enriched categories between bicategories, forming a tricategory Caten. The construction of DGAb from Ab, which exists in the literature, can be factored via GAb, and we prove that it is an instance of semidirect product of Hopf bimonoids, applicable to an arbitrary base symmetric monoidal category. To extend this approach to the bicategories of modules, we considered a generalization from Hopf bimonoids in a symmetric monoidal category to Hopf comonads in Caten. The crucial property of such comonads is that the forgetful functor creates left Kan extensions, which generalizes creation of duals and cohoms in the monoidal category case, and adjoints in the bicategory case.

Statement of Originality

This work has not previously been submitted for a degree or diploma in any university. To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.

Sydney, 27th February 2018

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1

Introduction and overview

Higher (two-three) dimensional category theory can be sliced and served in different ways, depending on the taste and use. The quickest way to describe a (strict) n-category $\mathcal{C}$ is by considering a category enriched in $(n-1)$-Cat; a cartesian closed and (co)complete $\mathcal{V}$ produces a cartesian closed and (co)complete $\mathcal{V}$-Cat [19], and the base of induction 0-Cat = Set is cartesian closed and (co)complete [25], so for every natural number n the category of n-categories (n-Cat) is cartesian closed and (co)complete. All compositions and identities in $\mathcal{C}$ are strictly associative and unital, enriched functors $F : \mathcal{C} \to \mathcal{D}$ preserve compositions and identities, and enriched transformations $\alpha : F \Rightarrow G$ are (strictly) natural.

Enrichment does not access 2-cells of $\mathcal{V}$, even if $\mathcal{V}$ has them. In two dimensions ($\mathcal{V}$ = Cat) existence of 2-cells allows weakening the laws for $\mathcal{C}$ leading to a list of progressively weaker structures: 2-category, bicategory, left (right) skew bicategory, lax bicategory. Weakening the laws for F and α leads to weaker morphisms: (strict) 2-functor, pseudofunctor, lax (or oplax) functor. And similarly, the list for natural transformations is: strict natural transformation, pseudonatural transformation, lax (or oplax) natural transformation. With 2-cells in $\mathcal{V}$ we can form modifications between (weak) natural transformations $m : \alpha \to \beta$. The totality of strict/pseudo/lax functors, strict/pseudo/lax natural transformations and modifications forms a different (strict) 2-category for each of the cases. In Chapter 2 we construct a tensor product of 2-categories such that homing out of it with strict functors, lax natural transformations and modifications corresponds to taking the lax hom twice.

A different, perhaps more immediate, way to generalise functors is to consider enriched modules (aka profunctors, distributors). Lawvere showed that metric spaces can be viewed as enriched categories [24], and Cauchy completeness can be expressed as representability of left-adjoint modules. The notion of Cauchy completeness can then be generalized to an arbitrary base, or even proarrow equipment [37, 38]. For example, a usual category ($\mathcal{V} = \mathrm{Set}$) is Cauchy complete if its idempotents split. A preadditive category ($\mathcal{V} = \mathrm{Ab}$) is Cauchy complete if idempotents split and it has direct sums; rings (one-object Ab-categories) are Morita equivalent if their Cauchy completions are equivalent [19]. In Chapter 3 we give a quick review of Lawvere's argument, and modify the base of enrichment to give a description of relativistic causal spaces. All such spaces are Cauchy complete, and we provide sufficient conditions on the base $\mathcal{V}$ that ensure that a $\mathcal{V}$-category is Cauchy complete if and only if idempotents split in its underlying category. In particular, n-Cat satisfies the conditions.

A $\mathcal{V}$-enriched category is a many-object version of a monoid in $\mathcal{V}$. A monoid and a comonoid on the same object can be compatible in two important and distinct ways; they can form a Frobenious monoid or a bimonoid (if $\mathcal{V}$ is braided). The latter can have a property of being Hopf, if it has an antipode. Tensoring with a bimonoid A induces a monoidal comonad on $\mathcal{V}$, whose category of coalgebras, denoted A-CoAlg or $\mathcal{V}^{A\otimes -}$, has a monoidal structure. If $\mathcal{V}$ is symmetric, and A has a braiding element [16] then the monoidal structure on A-CoAlg becomes braided. In the first half of Chapter 4 we show how graded abelian groups and chain complexes of abelian groups can be viewed as coalgebras for particular Hopf monoids, which are related via (a generalization of) the semidirect product.

Every monoidal category can equivalently be thought of as a one-object bicategory. Also, enrichment in $\mathcal{V}$ can be extended to the case when $\mathcal{V}$ is a bicategory [34]. A pseudofunctor $\mathcal{U} : \mathcal{W} \to \mathcal{V}$ whose functors on hom-categories $\mathcal{U}(W, W') : \mathcal{W}(W, W') \to \mathcal{V}(FW, FW')$ have right adjoints, always induces a change of base functor $\mathcal{U}' : \mathcal{W}\text{-Cat} \to \mathcal{V}\text{-Cat}$ which has a right adjoint in[1] 2-CAT. The main motivation for introducing new morphisms between bicategories [20], called 2-sided enriched categories (forming a tricategory Caten), was to introduce a right adjoint of $\mathcal{U}$, call it $\mathcal{R}$, such that $\mathcal{U}' \dashv \mathcal{R}'$. In the second part of Chapter 4 we will review 2-sided enrichment, characterise comonads $\mathcal{G}$ in Caten (generalizing monoidal comonads of [8]), construct the bicategory of coalgebras $\mathcal{V}^{\mathcal{G}}$, show that when the comonad is (left) Hopf the underlying functor creates left Kan extensions. This then applies to the underlying functor $\mathcal{U} : \mathcal{V}^{\mathcal{G}} \to \mathcal{V}$, as well as change of base functors $\mathcal{U}' : \mathcal{V}^{\mathcal{G}}\text{-Cat} \to \mathcal{V}\text{-Cat}$ and $\widetilde{\mathcal{U}} : \mathcal{V}^{\mathcal{G}}\text{-Mod} \to \mathcal{V}\text{-Mod}$, under certain conditions. The general theory then applies to the (differential) graded abelian groups of the first part of the Chapter 4.

This thesis, although worked out and written by me, grew on the fertile soil of the Centre of Australian Category Theory, and in the rest of the introduction I will outline the origins of different concepts presented in the thesis. The motivation for the strictification tensor product (Chapter 2) comes from my Supervisor Ross Street's earlier work on free (mixed) distributive laws [36], and he suggested using computads (defined in [30] and reviewed in Section 2.3.3) and introduced me to Bénabou's path construction, described in Section 2.2.1. During my talk to the Australian Category Seminar on the topic, James Dolan and Richard Garner suggested that the strictification tensor product would be a 2-step construction via the Bénabou construction and lax Gray tensor product, see Section 2.2.2.

The construction of causal spaces via enriched categories (Section 3.3) comes from my physics background, while searching for the conditions on the monoidal base that simplify Cauchy completeness (Section 3.4) was Ross' suggestion.

Finally, Ross noticed that DGAb is a category of coalgebras for a Hopf monoid in GAb (see Section 4.2), that Pareigis' Hopf monoid is a semidirect product, and that a Hopf-comonadic forgetful functor creates left Kan extensions (Theorem 4.5.1). Fitting everything into the

[1]Note that $\mathcal{V}$-Cat has a large set (proper class) of objects, but it is locally small.

2-sided enriched setting was my suggestion and realization. Other parts of that work came after discussions with Ross, followed by my own calculations.

Chapters 2 and 3 are my own original work while Chapter 4 is joint work with Ross Street and will be published accordingly.

2

Strictification tensor product of 2-categories

2.1 Introduction

Monads (aka triples, standard constructions) are given by a category C, an endofunctor $F : C \to C$ and two natural transformations $\eta : 1_C \Rightarrow F$ and $\mu : F^2 \Rightarrow F$, satisfying unit and associativity axioms [25]. Their use is ubiquitous and the most common one is describing a (possibly complicated) algebraic structure as Eilenberg-Moore (EM) algebras [25] on a category of simpler ones. An EM algebra is given by a map $TX \to X$ compatible with μ and η. With algebra morphisms, they form a category $\mathrm{EM}(T)$. The full subcategory of $\mathrm{EM}(T)$ consisting of free algebras is (up to equivalence) usually denoted $\mathrm{KL}(T)$. A typical example is the Abelian group monad on the category of sets taking a set S to the set of elements of the free Abelian group on S.

A distributive law [3] consists of two different monads on the same category satisfying a compatibility condition. Then their composite is a new monad. A typical example is the Abelian group monad together with the monoid monad producing the ring monad, hence the name.

Monads are in fact definable in an arbitrary bicategory $\mathcal{E}$ [29], just by replacing words "functor" with arrow and natural transformation by 2-cell. For example, in a bicategory of spans, monads are precisely (small) categories [4]. A morphism between a monad T on X and S on Y, consists of an arrow $X \xrightarrow{F} Y$ and a "crossing" 2-cell $S \circ F \overset{\sigma}{\Rightarrow} F \circ T$ which is compatible with unit and multiplication for both monads. A morphism between monad morphisms F and G, consists of a 2-cell $F \overset{\alpha}{\Rightarrow} G$ compatible with crossing 2-cells. These form the 2-category of monads in $\mathcal{E}$, called $\mathrm{Mnd}(\mathcal{E})$. Now, a distributive law in $\mathcal{E}$ has a short description as a monad in $\mathrm{Mnd}(\mathcal{E})$. Various duals are expressible using dualities of 2-categories, for instance, the 2-category of comonads is defined as $\mathrm{Cmd}(\mathcal{E}) = \mathrm{Mnd}(\mathcal{E}^{\mathrm{co}})^{\mathrm{co}}$, mixed distributive laws as $\mathrm{Cmd}(\mathrm{Mnd}(\mathcal{E}))$. Since objects of $\mathcal{E}$ are no longer categories, we have no access to their elements, and cannot form an EM-category; but we can use the 2-dimensional universal property of lax limit to obtain, if exists, an EM-object $\mathrm{EM}(T)$, also denoted C^T. The main topic of [22] is completion of $\mathcal{E}$ under these limits. Dually, lax colimits give KL(T), also denoted C_T.

The free monad [23] is a 2-category FM which classifies monads; that is, the 2-category of strict functors, lax natural transformations and modifications $[\mathrm{FM}, \mathcal{E}]_{\mathrm{lnt}}$ is isomorphic to $\mathrm{Mnd}(\mathcal{E})$. It is given by the suspension of the opposite of the algebraist's category of simplices, $\Delta_{\mathrm{a}}^{\mathrm{op}}$ with ordinal sum as the monoidal structure. We will use it a lot, so we review its definition and some properties in Appendix A.1. The free mixed distributive law (FMDL) was constructed by Street [36], and is a special case of the construction presented here.

A lax functor [4] (aka morphism) between bicategories generalises the notion of a (strict) 2-functor, by relaxing the conditions of preservation of the unit and composition of arrows. Instead, a lax functor $F : \mathcal{D} \to \mathcal{E}$ is equipped with comparison maps

$$\eta_D : 1_{FD} \Rightarrow F(1_D) \text{ and } \mu_{dd'} : F(d') \circ F(d) \Rightarrow F(d' \circ d)$$

for each object D of $\mathcal{D}$, and composable pair (d, d') of arrows in $\mathcal{D}$. These are required to satisfy unit and associativity laws, and μ is required to be natural in c and c'. The special case of $\mathcal{D} = 1$, that is, if $\mathcal{D}$ has only one 0/1/2-cell, then giving a lax functor exactly corresponds to giving a monad in $\mathcal{E}$. A lax functor from the chaotic category[1] on a set X corresponds to a category enriched in $\mathcal{E}$. Another example, lax functors from $\mathbb{I}(:= 0 \to 1)$ into Span correspond to choosing two categories and a module (aka profunctor, distributor) between them. Lax natural transformations $F \overset{\sigma}{\Rightarrow} G$ between two such functors consist of arrows $FD \xrightarrow{\sigma_D} GD$, for each $D \in \mathcal{D}$, and $Gd \circ \sigma_D \xrightarrow{\sigma_d} \sigma'_D \circ Fd$, for each $D \xrightarrow{d} D'$ in $\mathcal{D}$, natural in d and compatible with η and μ. Finally a modification $\sigma \xrightarrow{m} \tau$ consists of 2-cells $\sigma_D \overset{m_D}{\Longrightarrow} \tau_D$, for each D, compatible with σ. These form a 2-category $\mathrm{Lax}(\mathcal{D}, \mathcal{E})$. The choice of directions gives an isomorphism of 2-categories $\mathrm{Lax}(1, \mathcal{E}) \cong \mathrm{Mnd}(\mathcal{E})$, and by the definition of (free) distributive law (FDL) we have $\mathrm{Lax}(1, \mathrm{Lax}(1, \mathcal{E})) \cong [\mathrm{FDL}, \mathcal{E}]_{\mathrm{lnt}}$.

Our goal is, given 2-categories $\mathcal{C}$ and $\mathcal{D}$, to construct a 2-category $\mathcal{C} \boxtimes \mathcal{D}$ that is "free", in the sense that it strictifies the lax functors, so that

$$\mathrm{Lax}(\mathcal{C}, \mathrm{Lax}(\mathcal{D}, \mathcal{E})) \cong [\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E}]_{\mathrm{lnt}}. \tag{2.1.1}$$

The variables C, c, γ used to describe cells in $\mathcal{C}$ (similarly for D, d and δ in $\mathcal{D}$), have sources and targets according to the diagram 2.1.2.

$$\begin{tikzcd} C \arrow[r, bend left=40, "c"] \arrow[r, "\bar{c}" description] \arrow[r, bend right=40, "\bar{\bar{c}}"'] \arrow[r, phantom, shift left=2, "\gamma\Downarrow"] \arrow[r, phantom, shift right=2, "\bar{\gamma}\Downarrow"] & C' \arrow[r, bend left=40, "c'"] \arrow[r, "\bar{c}'" description] \arrow[r, bend right=40, "\bar{\bar{c}}'"'] \arrow[r, phantom, shift left=2, "\gamma'\Downarrow"] \arrow[r, phantom, shift right=2, "\bar{\gamma}'\Downarrow"] & C'' \end{tikzcd} \tag{2.1.2}$$

Horizontal composition is denoted by $\circ$ and vertical by $\bullet$.

2.2 Connection with lax Gray tensor product

In this section we recall a construction due to Bénabou which can be interpreted as a 2-category of paths in a 2-category. We use it, together with the lax Gray tensor product, to express our strictification tensor product.

[1] That is, the category having exactly one arrow in each hom.

2.2.1 Bénabou construction of the 2-category of paths

Let $\mathcal{C}$ be a 2-category and $\mathcal{C}^\dagger$ the 2-category of "paths" in $\mathcal{C}$, consisting of the same objects as $\mathcal{C}$, and arrows between C and C' are strict 2-functors p representing paths in $\mathcal{C}$ between C and C'; that is,

$$[n] \xrightarrow{p} \mathcal{C}, \;\; p(0) = C, \;\; p(n) = C', \tag{2.2.1}$$

where $[n]$ is an object of $\Delta_{\perp\top}$, for details see Appendix A.1. Denote by[2] $(p)_i$ the i^{th} component in the path

$$(p)_i = p\left((i-1) \to i\right) . \tag{2.2.2}$$

The identity is a path of zero length on C:

$$[0] \to \mathcal{C} \tag{2.2.3}$$

$$0 \mapsto C \tag{2.2.4}$$

and composition is given by "concatenation",

$$(n', p') \circ (n, p) = (n + n', p + p') \tag{2.2.5}$$

where $(p+p')_i = (p)_i$ if $i \leqslant n$ and $(p+p')_i = (p')_{i-n}$ otherwise. This composition is strictly associative and unital.

Finally, 2-cells between (n,p) and $(\bar{n},\bar{p})$, are pairs (ξ,α) where $\xi : [\bar{n}] \to [n]$ is a morphism in $\Delta_{\perp\top}$ and α is an identity on components, oplax-natural transformation, shortly icon, introduced in [21]:

$$\alpha : p \circ \xi \Rightarrow \bar{p}, \text{ with } \alpha_{1_i} = 1_{1_{\bar{p}(i)}} . \tag{2.2.6}$$

So, α is determined by $\bar{n}$ components on non-identity arrows:

$$\alpha_i := \alpha_{(i-1)\to i} : (p \circ \xi)\left((i-1) \to i\right) \Rightarrow (\bar{p})_i\,. \tag{2.2.7}$$

Note that if $\xi_i = 0$, meaning $\xi(i) = \xi(i-1)$ (see Appendix A.1 for details), then the source of the corresponding component of α is the identity, $\alpha_i : 1_{p\xi(i)} \Rightarrow (\bar{p})_i$. The identity is given

[2] We reserve p_i, without brackets, to mean the length of the image as in (A.1.14).

by $1_{(n,p)} = (1_{[n]}, 1_p)$. The vertical composite of (ξ, α) and $(\bar{\xi}, \bar{\alpha})$ is obtained by pasting, as in the diagram 2.2.8.

$$\begin{tikzcd}
{[n]} \arrow[dr, "p"] \arrow[d, phantom, shift left=4ex, "\alpha \Downarrow"] & \\
{[\bar{n}]} \arrow[u, "\xi"] \arrow[r] \arrow[d, phantom, shift left=4ex, "\bar{\alpha} \Downarrow"] & \mathcal{C} \\
{[\bar{\bar{n}}]} \arrow[u, "\bar{\xi}"] \arrow[ur, "\bar{\bar{p}}"'] &
\end{tikzcd} \tag{2.2.8}$$

The horizontal composition is concatenation, analogous to the one for path (1-cells), $(\xi', \alpha') \circ (\xi, \alpha) = (\xi + \xi', \alpha + \alpha')$, where $(\alpha + \alpha')_i = \alpha_i$ if $i \leqslant n$, and $(\alpha + \alpha')_i = \alpha'_{i-n}$ otherwise.

Lax functors out of $\mathcal{C}$ correspond to strict 2-functors out of $\mathcal{C}^\dagger$. In fact, there is an isomorphism of 2-categories

$$\mathrm{Lax}(\mathcal{C}, \mathcal{E}) \cong [\mathcal{C}^\dagger, \mathcal{E}]_{\mathrm{lnt}} \,. \tag{2.2.9}$$

2.2.2 Lax Gray tensor product

Lax Gray tensor product [15], $\otimes_l$: 2-Cat $\times$ 2-Cat $\to$ 2-Cat, is a tensor product for the internal hom $[-,-]_{\mathrm{lnt}}$, that is

$$[\mathcal{C}, [\mathcal{D}, \mathcal{E}]_{\mathrm{lnt}}]_{\mathrm{lnt}} \cong [\mathcal{C} \otimes_l \mathcal{D}, \mathcal{E}]_{\mathrm{lnt}} \,. \tag{2.2.10}$$

The left hand side of Eq. (2.1.1) can be transformed

$$\begin{aligned}
\mathrm{Lax}(\mathcal{C}, \mathrm{Lax}(\mathcal{D}, \mathcal{E})) &\overset{(2.2.9)}{\cong} [\mathcal{C}^\dagger, [\mathcal{D}^\dagger, \mathcal{E}]_{\mathrm{lnt}}]_{\mathrm{lnt}} & (2.2.11)\\
&\overset{(2.2.10)}{\cong} [\mathcal{C}^\dagger \otimes_l \mathcal{D}^\dagger, \mathcal{E}]_{\mathrm{lnt}} & (2.2.12)
\end{aligned}$$

leading to a characterization

$$\mathcal{C} \boxtimes \mathcal{D} \cong \mathcal{C}^\dagger \otimes_l \mathcal{D}^\dagger \,. \tag{2.2.13}$$

The lax Gray product $\otimes_l$ is defined via its universal property, and the explicit description involves relations and quotienting. Our direct description, explained in Section 2.4, involves no quotienting.

2.3 Tensor product via computads

We begin by fully unpacking the LHS of (2.1.1), which involves familiar, but numerous axioms - there are eighteen axioms for an object (lax functor) B, five axioms for an arrow (lax natural transformation) $b : B \to B'$, and two axioms for a 2-cell (modification) $\beta : b \Rightarrow \bar{b}$. Then we review the definition of computads [30] which play the same role for 2-categories as graphs do for usual categories - they are part of a monadic adjunction. We then proceed to construct a computad $\mathcal{G}$ to give a convenient generator-relation description of the tensor product.

2.3.1 Unpacking

An object B of $\mathrm{Lax}(\mathcal{C}, \mathrm{Lax}(\mathcal{D}, \mathcal{E}))$ assigns to each $C \in \mathcal{C}$ a lax functor $BC : \mathcal{D} \to \mathcal{E}$, which amounts to giving the following data in $\mathcal{E}$:

- for each D an object $BCD \in \mathcal{E}$
- for each d an arrow $BCd : BCD \to BCD'$
- for each δ a 2-cell $BC\delta : BCd \Rightarrow BC\bar{d}$, functorially

$$BC1_d = 1_{BCd} \tag{2.3.1}$$

$$BC(\bar{\delta} \bullet \delta) = BC\bar{\delta} \bullet BC\delta \tag{2.3.2}$$

- (f1) for each D a unit comparison 2-cell $\eta_{BC1_D} : 1_{BCD} \Rightarrow BC1_D$
- (f1) for each composable pair (d, d') a composition comparison 2-cell $\mu_{BCdd'} : (BCd') \circ (BCd) \Rightarrow (BCd' \circ d)$,

  satisfying unit and associativity axioms,

$$\mu \bullet (1 \circ \eta) = 1 = \mu \bullet (\eta \circ 1) \tag{2.3.3}$$

$$\mu \bullet (1 \circ \mu) = \mu \bullet (\mu \circ 1) \tag{2.3.4}$$

  together with a naturality condition,

$$\mu_{BC\bar{d}\bar{d}'} \bullet (BC\delta' \circ BC\delta) = BC(\delta' \circ \delta) \bullet \mu_{Cdd'}. \tag{2.3.5}$$

Also, B assigns to each $c : C \to C'$ a lax natural transformation $Bc : BC \to BC'$ consisting of:

- arrows $BcD : BCD \to BC'D$
- **(t1)** 2-cells $\sigma_{Bcd} : BC'd \circ BcD \Rightarrow BcD' \circ BCd$,

  with the two axioms expressing compatibility with unit and composition,

$$\sigma \bullet (\eta \circ 1) = 1 \circ \eta \tag{2.3.6}$$

$$\sigma \bullet (\mu \circ 1) = (1 \circ \mu) \bullet (\sigma \circ 1) \bullet (1 \circ \sigma) \tag{2.3.7}$$

  and one expressing naturality,

$$\sigma_{Bc\bar{d}} \bullet (BC'\delta \circ 1_{BcD}) = (1_{BcD'} \circ BC\delta) \bullet \sigma_{Bcd}\,. \tag{2.3.8}$$

Finally, B assigns (functorially) to each 2-cell $\gamma : c \to \bar{c}$ a modification $B\gamma : Bc \Rightarrow B\bar{c}$, which in $\mathcal{E}$ means:

- 2-cells $B\gamma D : BcD \Rightarrow B\bar{c}D$,

  satisfying the modification axiom,

$$\sigma_{B\bar{c}d} \bullet (1_{BC'd} \circ B\gamma D) = (B\gamma D' \circ 1_{BCd}) \bullet \sigma_{Bcd} \tag{2.3.9}$$

  and the functoriality condition

$$B1_c D = 1_{BcD} \tag{2.3.10}$$

$$B(\bar{\gamma} \bullet \gamma)D = B\bar{\gamma}D \bullet B\gamma D\,. \tag{2.3.11}$$

Being a lax functor, B has to provide the unit and composition comparison modifications given by data:

- **(f2)** unit 2-cells $\eta_{B1_C D} : 1_{BCD} \Rightarrow B1_C D$
- **(f2)** composition 2-cells $\mu_{Bcc'D} : (Bc'D) \circ (BcD) \Rightarrow (Bc' \circ cD)$

  which, in addition to the naturality condition

$$\mu_{B\bar{c}\bar{c}'D} \bullet (B\gamma' D \circ B\gamma D) = B(\gamma' \circ \gamma)D \bullet \mu_{Bcc'D} \tag{2.3.12}$$

and modification axiom,

$$\sigma \bullet (1 \circ \eta) = \eta \circ 1 \tag{2.3.13}$$

$$\sigma \bullet (1 \circ \mu) = (\mu \circ 1) \bullet (1 \circ \sigma) \bullet (\sigma \circ 1) \tag{2.3.14}$$

satisfy the unit and associativity axioms (2.3.3)-(2.3.4).

An arrow $b : B \to B'$, being a lax transformation between lax functors B and B', assigns to each $C \in \mathcal{C}$ a lax transformation $bC : BC \to B'C$ and to each $c : C \to C'$ a modification $\sigma_{bc} : B'c \circ bC \Rightarrow bC' \circ Bc$, which means the following data in $\mathcal{E}$:

- 1-cells $bCD : BCD \to B'CD$

$\bullet_{(\mathbf{t1})}$ 2-cells $\sigma_{bCd} : B'Cd \circ bCD \Rightarrow bCD' \circ BCd$

$\bullet_{(\mathbf{t2})}$ 2-cells $\sigma_{bcD} : B'cD \circ bCD \Rightarrow bC'D \circ BcD$,

subject to naturality

$$\sigma_{b\bar{c}D} \bullet (B'\gamma D \circ 1_{bCD}) = (1_{bC'D} \circ B\gamma D) \bullet \sigma_{bcD} \tag{2.3.15}$$

$$\sigma_{bC\bar{d}} \bullet (B'C\delta \circ 1_{bCD}) = (1_{bCD'} \circ BC\delta) \bullet \sigma_{bCd} \tag{2.3.16}$$

lax transformation

$$\sigma \bullet (\eta \circ 1) = 1 \circ \eta \tag{2.3.17}$$

$$\sigma \bullet (\mu \circ 1) = (1 \circ \mu) \bullet (\sigma \circ 1) \bullet (1 \circ \sigma) \tag{2.3.18}$$

and a modification

$$(1 \circ \sigma) \bullet (\sigma \circ 1) \bullet (1 \circ \sigma) = (\sigma \circ 1) \bullet (1 \circ \sigma) \bullet (\sigma \circ 1) \tag{2.3.19}$$

axioms.

A 2-cell $\beta : b \to \bar{b}$ in $\mathrm{Lax}(\mathcal{C}, \mathrm{Lax}(\mathcal{D}, \mathcal{E}))$, being a modification, assigns to each $C \in \mathcal{C}$ a modification $\beta C : bC \Rightarrow \bar{b}C$, which in $\mathcal{E}$ means

- 2-cells $\beta CD : bCD \Rightarrow \bar{b}CD$, with modification axioms,

$$\sigma_{\bar{b}cD} \bullet (1_{B'cD} \circ \beta CD) = (\beta C'D \circ 1_{BcD}) \bullet \sigma_{bcD} \tag{2.3.20}$$

$$\sigma_{\bar{b}Cd} \bullet (1_{B'Cd} \circ \beta CD) = (\beta CD' \circ 1_{BCd}) \bullet \sigma_{bCd}\,. \tag{2.3.21}$$

2.3.2 Symmetries

Denote by $(\mathrm{Op})\mathrm{Lax}_{(\mathrm{op})}(\mathcal{D},\mathcal{E})$ the 2-category of (op)lax functors (first op), (op)lax natural transformations (subscript op) and modifications.

Proposition 2.3.1. *There are isomorphisms:*

$$Lax_{op}(\mathcal{D},\mathcal{E}) \cong Lax(\mathcal{D}^{op},\mathcal{E}^{op})^{op} \tag{2.3.22}$$

$$OpLax_{op}(\mathcal{D},\mathcal{E}) \cong Lax(\mathcal{D}^{co},\mathcal{E}^{co})^{co} \tag{2.3.23}$$

$$Lax(\mathcal{C}, Lax_{op}(\mathcal{D},\mathcal{E})) \cong Lax_{op}(\mathcal{D}, Lax(\mathcal{C},\mathcal{E})) \tag{2.3.24}$$

$$Lax(\mathcal{C}, OpLax_{op}(\mathcal{D},\mathcal{E})) \cong OpLax_{op}(\mathcal{D}, Lax(\mathcal{C},\mathcal{E}))\,. \tag{2.3.25}$$

Proof. Data and axioms for the LHS of (2.3.22) (resp. (2.3.23)) are obtained from the beginning of Section 2.3.1 until the equation (2.3.11), by ignoring the letter B in all the names, and reversing the direction of 2-cells for data marked by **(t1)** (resp. **(f1)** or **(t1)**). On the other hand, the data and axioms for the RHS of (2.3.22) (resp. (2.3.23)) have reversed sources and targets of arrows (resp. 2-cells), compared to the diagram (2.1.2), but they also live in $\mathcal{E}^{\mathrm{op}}$ (resp. $\mathcal{E}^{\mathrm{co}}$), rather than $\mathcal{E}$; interpreted in $\mathcal{E}$, they have reversed 2-cells marked by (**t1**) (resp. (**f1**) or (**t1**)). A possibly easier way to see this is to draw string diagrams in $\mathcal{E}^{\mathrm{op}}$ (resp. $\mathcal{E}^{\mathrm{co}}$), and then flip them horizontally (resp. vertically).

To prove (2.3.24), observe that the data and axioms in Section 2.3.1, with **(t1)** 2-cells reversed (LHS), and second and third letter in all labels formally swapped, corresponds to the same data and axioms when C (resp. c, γ) is substituted for D (resp. d, δ), and vice versa, and then **(t2)** 2-cells are reversed (RHS).

Similarly, in (2.3.25) reversing **(f1)** and **(t1)** 2-cells, followed by swapping positions in labels, leads the same result as swapping variables and then reversing 2-cells marked by **(f2)** and **(t2)**.

Once the directions for data are fixed, all axioms are determined in a unique way, and there is no need to analyse them separately. □

Corollary 2.3.1. *There are isomorphism:*

$$OpLax(\mathcal{D},\mathcal{E}) \cong Lax(\mathcal{D}^{co\,op},\mathcal{E}^{co\,op})^{co\,op} \tag{2.3.26}$$

$$OpLax(\mathcal{C}, Lax_{op}(\mathcal{D}, \mathcal{E})) \cong Lax_{op}(\mathcal{D}, OpLax(\mathcal{C}, \mathcal{E}))\,. \tag{2.3.27}$$

Corollary 2.3.2. *There are isomorphism:*

$$[\mathcal{D}, \mathcal{E}]_{ont} \cong [\mathcal{D}^{op}, \mathcal{E}^{op}]_{lnt}^{op} \tag{2.3.28}$$

$$[\mathcal{D}, \mathcal{E}]_{ont} \cong [\mathcal{D}^{co}, \mathcal{E}^{co}]_{lnt}^{co} \tag{2.3.29}$$

$$[\mathcal{C}, [\mathcal{D}, \mathcal{E}]_{ont}]_{lnt} \cong [\mathcal{D}, [\mathcal{C}, \mathcal{E}]_{lnt}]_{ont}\,. \tag{2.3.30}$$

2.3.3 Reviewing computads

The content of this part is taken from [30]. We describe the major ideas and leave out the details.

Definition 2.3.1. *([30], with a technical modification[3]) A computad $\mathcal{G}$ consists of a graph $|\mathcal{G}|$ (providing a set of objects $|\mathcal{G}|_0$ and a set of generating arrows $|\mathcal{G}|_1$), and for each pair of objects $G, G' \in |\mathcal{G}|_0$ a graph $\mathcal{G}(G, G')$ with a set nodes[4] $\mathcal{G}(G, G')_0 = (\mathcal{F}|\mathcal{G}|)(G, G')$ and a set of edges denoted $\mathcal{G}(G, G')_1$ (providing generating 2-cells).*

A computad morphism assigns all the data, respecting sources and targets, forming a category Cmp.

There is a free 2-category $\mathcal{F}\mathcal{G}$ on the computad $\mathcal{G}$ that has the same objects as $\mathcal{G}$. Arrows between G and G' are "paths" between G and G'; that is, elements of $\mathcal{G}(G, G')_0$. To define 2-cells, it is not enough to take the free category on $\mathcal{G}(G, G')$ since it does not take whiskering into account. Instead, consider the set of whiskered generating 2-cells

$$\begin{aligned}\mathcal{G}^1(G, G') = \{(p, \alpha, p')|p &\in \mathcal{G}(G, X)_0,\\ \alpha &\in \mathcal{G}(X, X')_1,\\ p' &\in \mathcal{G}(X', G')_0\}\,.\end{aligned}$$

Finally, to impose the middle of four interchange, take the set of whiskered pairs

$$\mathcal{G}^2(G, G') = \{(p, \alpha, p', \alpha', p'')|p \in \mathcal{G}(G, X)_0,$$

[3]We take all paths between two objects to be the nodes of $\mathcal{G}(G, G')$; that is, $\mathcal{G}(G, G')_0 = (\mathcal{F}|\mathcal{G}|)(G, G')$.

[4]$\mathcal{F}|\mathcal{G}|$ is the free category on a graph $|\mathcal{G}|$

$$\begin{aligned}&\alpha \in \mathcal{G}(X, X')_1,\\ &p' \in \mathcal{G}(X', X'')_0\\ &\alpha' \in \mathcal{G}(X'', X''')_1,\\ &p'' \in \mathcal{G}(X''', G')_0\}\end{aligned}$$

and form a coequalizer in Cat to obtain the hom $(\mathcal{F}\mathcal{G})(G, G')$

$$\mathcal{F}\mathcal{G}^2(G, G') \rightrightarrows \mathcal{F}\mathcal{G}^1(G, G') \to (\mathcal{F}\mathcal{G})(G, G') \tag{2.3.31}$$

where the two parallel arrows are the two obvious ways to compose whiskered α with whiskered α'; see [30] for details and the rest of the construction.

Given a 2-category $\mathcal{E}$, the underlying computad $\mathcal{U}\mathcal{E}$ has the underlying graph obtained from the underlying category of $\mathcal{E}$; that is, $|\mathcal{U}\mathcal{E}| = \mathcal{U}|\mathcal{E}|$, and the hom graphs have edges $(\mathcal{U}\mathcal{E})(E, E')(p, p') = \mathcal{E}(E, E')(\circ p, \circ \bar{p})$, where $\circ p$ denotes the arrow in $\mathcal{E}$ obtained by composing the path p in $\mathcal{E}$. Assignments $\mathcal{F}$ and $\mathcal{U}$ extend to morphisms and form an adjunction, giving a bijection between arrows in Cmp and 2-Cat

$$T : \mathcal{G} \to \mathcal{U}\mathcal{E} \;\;\longleftrightarrow\;\; \hat{T} : \mathcal{F}\mathcal{G} \to \mathcal{E}\,. \tag{2.3.32}$$

Intuitively, the 2-category $\mathcal{F}\mathcal{U}\mathcal{E}$ is the 2-category of pasting diagrams in $\mathcal{E}$, and the counit of the adjunction is the operation of actual pasting to obtain a (2-)cell in $\mathcal{E}$.

2.3.4 The tensor product computad

The goal is to construct a computad $\mathcal{G}$ which has data analogous to the one in Section 2.3.1, and then to impose further identification of 2-cells in $\mathcal{F}\mathcal{G}$, analogous to the axioms (2.3.1)-(2.3.14). Consider the computad $\mathcal{G}$ defined by the following data:

- a set $|\mathcal{G}|_0 = \mathrm{Ob}\mathcal{C} \times \mathrm{Ob}\mathcal{D}$ of nodes, whose elements are denoted $C \boxtimes D$
- the set $|\mathcal{G}|_1((C, D), (C', D'))$ of edges consists of arrows in $\mathcal{C}(C, C')$ if $D = D'$, denoted $c \boxtimes D$, and arrows in $\mathcal{D}(D, D')$ if $C = C'$, denoted $C \boxtimes d$, otherwise it is empty. The concatenation of $c \boxtimes D$ and $C' \boxtimes d$, as an arrow in the free category on $|\mathcal{G}|$, will be denoted

by $\{C \boxtimes D \xrightarrow{c\boxtimes D} C' \boxtimes D \xrightarrow{C'\boxtimes d} C' \boxtimes D'\}$, and the empty path on $C \boxtimes D$ by $\{C \boxtimes D\}$. When the meaning is clear from the context we omit the tensor product character. A concise way of expressing the collection of edges is as a disjoint union

$$|\mathcal{G}|_1((C,D),(C',D')) = \mathcal{C}(C,C') \times \delta_{DD'} + \delta_{CC'} \times \mathcal{D}(D,D'), \tag{2.3.33}$$

with δ_{XY} being an empty set when $X \neq Y$ and singleton $\{X\}$ when $X = Y$.

- 2-cells
  - for each object C of $\mathcal{C}$ and 2-cell $\delta : d \Rightarrow \bar{d}$ in $\mathcal{D}$,

$$C \boxtimes \delta : \{CD \xrightarrow{Cd} CD'\} \Rightarrow \{CD \xrightarrow{C\bar{d}} CD'\} \tag{2.3.34}$$

  - for each object D of $\mathcal{D}$ and 2-cell $\gamma : c \Rightarrow \bar{c}$ in $\mathcal{C}$,

$$\gamma \boxtimes D : \{CD \xrightarrow{cD} C'D\} \Rightarrow \{CD \xrightarrow{\bar{c}D} C'D\} \tag{2.3.35}$$

  -(f1) for each $(C,D) \in |\mathcal{G}|_0$, the unit comparisons

$$\mathbf{id}_{C1_D} : \{CD\} \Rightarrow \{CD \xrightarrow{C1_D} CD\} \tag{2.3.36}$$

  -(f2) for each $(C,D) \in |\mathcal{G}|_0$, the unit comparisons

$$\mathbf{id}_{1_C D} : \{CD\} \Rightarrow \{CD \xrightarrow{1_C D} CD\} \tag{2.3.37}$$

  -(f1) for each $C \in \mathcal{C}$ and composable pair (d,d') in $\mathcal{D}$, a composition comparison

$$\mathbf{comp}_{Cdd'} : \{CD \xrightarrow{Cd} CD' \xrightarrow{Cd'} CD''\} \Rightarrow \{CD \xrightarrow{C\boxtimes(d'\circ d)} CD''\} \tag{2.3.38}$$

  -(f2) for each $D \in \mathcal{D}$ and composable pair (c,c') in $\mathcal{C}$, a composition comparison

$$\mathbf{comp}_{cc'D} : \{CD \xrightarrow{cD} C'D \xrightarrow{c'D} C''D\} \Rightarrow \{CD \xrightarrow{(c'\circ c)\boxtimes D} C''D\} \tag{2.3.39}$$

  -(t1) for each pair of 1-cells (c,d),

$$\mathbf{swap}_{cd} : \{CD \xrightarrow{cD} C'D \xrightarrow{C'd} C'D'\} \Rightarrow \{CD \xrightarrow{Cd} CD' \xrightarrow{cD'} C'D'\}. \tag{2.3.40}$$

The 2-category $\mathcal{C} \boxtimes_{cmp} \mathcal{D}$ is obtained from $\mathcal{FG}$, the free 2-category on the computad $\mathcal{G}$, by imposing identifications:

- preservation of identity 2-cells

$$C \boxtimes 1_d = 1_{C \boxtimes d} \tag{2.3.41}$$

$$1_c \boxtimes D = 1_{c \boxtimes D} \tag{2.3.42}$$

- distributivity of $\boxtimes$ over vertical composition

$$(C \boxtimes \delta') \bullet (C \boxtimes \delta) = C \boxtimes (\delta' \bullet \delta) \tag{2.3.43}$$

$$(\gamma' \boxtimes D) \bullet (\gamma \boxtimes D) = (\gamma' \bullet \gamma) \boxtimes D \tag{2.3.44}$$

- compatibility with the composition comparison 2-cells

$$\mathbf{comp}_{C\bar{d}\bar{d}'} \bullet (C \boxtimes \delta' \circ C \boxtimes \delta) = C \boxtimes (\delta' \circ \delta) \bullet \mathbf{comp}_{Cdd'} \tag{2.3.45}$$

$$\mathbf{comp}_{\bar{c}\bar{c}'D} \bullet (\gamma' \boxtimes D \circ \gamma \boxtimes D) = (\gamma' \circ \gamma) \boxtimes D \bullet \mathbf{comp}_{cc'D} \tag{2.3.46}$$

- compatibility with the swapping 2-cells

$$\mathbf{swap}_{\bar{c}\bar{d}} \bullet (C' \boxtimes \delta \circ \gamma \boxtimes D) = (\gamma \boxtimes D' \circ C \boxtimes \delta) \bullet \mathbf{swap}_{cd} \tag{2.3.47}$$

- unit and associativity laws

$$\mathbf{comp} \bullet (1 \circ \mathbf{id}) = 1 \ \& \ \mathbf{comp} \bullet (\mathbf{id} \circ 1) = 1 \tag{2.3.48}$$

$$\mathbf{comp} \bullet (\mathbf{comp} \circ 1) = \mathbf{comp} \bullet (1 \circ \mathbf{comp}) \tag{2.3.49}$$

- compatibility of swapping with unit and composition

$$\mathbf{swap} \bullet (1 \circ \mathbf{id}) = \mathbf{id} \circ 1 \tag{2.3.50}$$

$$\mathbf{swap} \bullet (\mathbf{id} \circ 1) = 1 \circ \mathbf{id} \tag{2.3.51}$$

$$\mathbf{swap} \bullet (1 \circ \mathbf{comp}) = (\mathbf{comp} \circ 1) \bullet (1 \circ \mathbf{swap}) \bullet (\mathbf{swap} \circ 1) \tag{2.3.52}$$

$$\mathbf{swap} \bullet (\mathbf{comp} \circ 1) = (1 \circ \mathbf{comp}) \bullet (\mathbf{swap} \circ 1) \bullet (1 \circ \mathbf{swap})\,. \tag{2.3.53}$$

Proposition 2.3.2. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be 2-categories, $\mathcal{C} \boxtimes_{cmp} \mathcal{D}$ the 2-category defined above, then there is an isomorphism*

$$Lax(\mathcal{C}, Lax(\mathcal{D}, \mathcal{E})) \cong [\mathcal{C} \boxtimes_{cmp} \mathcal{D}, \mathcal{E}] \,. \tag{2.3.54}$$

Proof. The data for $\mathcal{G}$ and identifications when forming $\mathcal{C} \boxtimes_{cmp} \mathcal{D}$ correspond exactly to data and laws (2.3.1)-(2.3.14) for $B \in \mathrm{Lax}(\mathcal{C}, \mathrm{Lax}(\mathcal{D}, \mathcal{E}))$ in the Section 2.3.1. So, giving B corresponds to giving a computad map $B_{\mathrm{cmp}} : \mathcal{G} \to \mathcal{U}\mathcal{E}$ such that the strict 2-functor $\hat{B}_{\mathrm{cmp}} : \mathcal{F}\mathcal{G} \to \mathcal{E}$ respects the identifications (2.3.41)-(2.3.53), which corresponds to giving a strict 2-functor $\hat{B} : \mathcal{C} \boxtimes_{\mathrm{cmp}} \mathcal{D} \to \mathcal{E}$.

Define $\mathcal{E}^{\mathcal{D}} := [\mathcal{D}, \mathcal{E}]_{\mathrm{ont}}$. From (2.3.30) we get the following isomorphism

$$[\mathcal{F}\mathcal{G}, \mathcal{E}]^{\mathcal{J}}_{\mathrm{lnt}} \cong [\mathcal{F}\mathcal{G}, \mathcal{E}^{\mathcal{J}}]_{\mathrm{lnt}} \,. \tag{2.3.55}$$

In particular, we have a bijection on objects, so for a free arrow $\mathcal{J} = \mathbb{I}(:= 0 \to 1)$, (resp. free 2-cell $\mathcal{J} = \mathbb{D}(:= 0 \overset{\to}{\underset{\to}{\Downarrow}} 1)$), we get a bijection between arrows (resp. 2-cells) of $[\mathcal{F}\mathcal{G}, \mathcal{E}]_{\mathrm{lnt}}$ and 2-functors $\mathcal{F}\mathcal{G} \to \mathcal{E}^{\mathcal{I}}$ (resp. $\mathcal{F}\mathcal{G} \to \mathcal{E}^{\mathcal{D}}$).

Consider a lax natural transformation between 2-functors respecting identifications (2.3.41)-(2.3.53) (as above)

$$\hat{b}_{\mathrm{cmp}} : \hat{B}_{\mathrm{cmp}} \Rightarrow \hat{B}'_{\mathrm{cmp}} : \mathcal{F}\mathcal{G} \to \mathcal{E} \,. \tag{2.3.56}$$

It corresponds to a 2-functor

$$\hat{b}^{\mathrm{curry}}_{\mathrm{cmp}} : \mathcal{F}\mathcal{G} \to \mathcal{E}^{\mathbb{I}} \tag{2.3.57}$$

which corresponds to a lax natural transformation $b : B \Rightarrow B'$ - the correspondence goes as follows

$$\mathcal{G} \xrightarrow{b^{\mathrm{curry}}_{\mathrm{cmp}}} \mathcal{U}\mathcal{E}^{\mathbb{I}} \tag{2.3.58}$$

$$C \boxtimes D \mapsto bCD \tag{2.3.59}$$

$$c \boxtimes D,\, C \boxtimes d \mapsto \sigma_{bcD},\, \sigma_{bCd} \tag{2.3.60}$$

$$\gamma \boxtimes D,\, C \boxtimes \delta \mapsto (B\gamma D, B'\gamma D),\, (BC\delta, B'C\delta) \tag{2.3.61}$$

$$\mathbf{id}_{-},\, \mathbf{comp}_{-},\, \mathbf{swap}_{-} \mapsto (\eta_{B-}, \eta_{B'-}),\, (\mu_{B-}, \mu_{B'-}),\, (\sigma_{B-}, \sigma_{B'-}) \,. \tag{2.3.62}$$

The RHS of (2.3.61) (resp. (2.3.62)) being 2-cells of $\mathcal{E}^{\mathbb{I}}$ is equivalent to (2.3.15) and (2.3.16) (resp. (2.3.17), (2.3.18) and (2.3.19)). The 2-functor $\hat{b}^{\text{curry}}_{\text{cmp}}$ respects identifications (2.3.41)-(2.3.53) because its source and target do, and so it also corresponds to a 2-functor

$$\hat{b}^{\text{curry}} : \mathcal{C} \boxtimes_{\text{cmp}} \mathcal{D} \to \mathcal{E}^{\mathbb{I}} \tag{2.3.63}$$

which is equivalently a lax natural transformation

$$\hat{b} : \hat{B} \Rightarrow \hat{B}' : \mathcal{C} \boxtimes_{\text{cmp}} \mathcal{D} \to \mathcal{E}\,. \tag{2.3.64}$$

Similarly, a modification

$$\hat{\beta}_{\text{cmp}} : \hat{b}_{\text{cmp}} \to \hat{\bar{b}}_{\text{cmp}} : \hat{B}_{\text{cmp}} \Rightarrow \hat{B}'_{\text{cmp}} : \mathcal{F}\mathcal{G} \to \mathcal{E} \tag{2.3.65}$$

corresponds to a 2-functor

$$\hat{\beta}^{\text{curry}}_{\text{cmp}} : \mathcal{F}\mathcal{G} \to \mathcal{E}^{\mathbb{D}} \tag{2.3.66}$$

which corresponds to a modification $\beta : b \to \bar{b}$ via

$$\mathcal{G} \xrightarrow{\beta^{\text{curry}}_{\text{cmp}}} \mathcal{U}\mathcal{E}^{\mathbb{D}} \tag{2.3.67}$$

$$C \boxtimes D \mapsto \beta CD \tag{2.3.68}$$

$$c \boxtimes D,\, C \boxtimes d \mapsto (\sigma_{bcD}, \sigma_{\bar{b}cD}),\, (\sigma_{bCd}, \sigma_{\bar{b}Cd}) \tag{2.3.69}$$

$$\gamma \boxtimes D,\, C \boxtimes \delta \mapsto (B\gamma D, B'\gamma D),\, (BC\delta, B'C\delta) \tag{2.3.70}$$

$$\mathbf{id}_{-},\, \mathbf{comp}_{-},\, \mathbf{swap}_{-} \mapsto (\eta_{B-}, \eta_{B'-}),\, (\mu_{B-}, \mu_{B'-}),\, (\sigma_{B-}, \sigma_{B'-})\,. \tag{2.3.71}$$

The RHS of (2.3.69) being 1-cells of $\mathcal{E}^{\mathbb{D}}$ is equivalent to modification axioms (2.3.20) and (2.3.21). The RHS of (2.3.70) and (2.3.71) being 2-cells of $\mathcal{E}^{\mathbb{D}}$, and $\hat{\beta}^{\text{curry}}_{\text{cmp}}$ respecting identifications (2.3.41)-(2.3.53), are just componentwise properties of $\hat{b}^{\text{curry}}_{\text{cmp}}$ and $\hat{\bar{b}}^{\text{curry}}_{\text{cmp}}$. $\square$

2.3.5 Dual strictifications

Notice that all the data and identifications for $\mathcal{G}(=: \mathcal{G}^{\mathcal{C}\mathcal{D}}_{\text{lax}})$, apart from those involving **swap**, are invariant (up to relabelling) with respect to exchanging $\mathcal{C}$ and $\mathcal{D}$. However, if we exchange $\mathcal{C}$ and $\mathcal{D}$ and consider oplax natural transformations at the same time, we arrive at an isomorphic computad $\mathcal{G}^{\mathcal{DC}}_{oplax} \cong \mathcal{G}^{\mathcal{CD}}_{\mathrm{lax}}$, the isomorphism consisting of exchanging the two positions in all the labels. All the identifications are isomorphic as well. This directly leads us to observe

Corollary 2.3.3. *There is an isomorphism*

$$\mathcal{C} \boxtimes \mathcal{D} \cong (\mathcal{D}^{op} \boxtimes \mathcal{C}^{op})^{op} . \tag{2.3.72}$$

Proof. The computad $\mathcal{G}^{\mathcal{DC}}_{oplax}$, with its identifications, generates a 2-category strictifying $\mathrm{Lax}_{\mathrm{op}}(\mathcal{D}, \mathrm{Lax}_{\mathrm{op}}(\mathcal{C}, \mathcal{E}))$. On the other hand,

$$\mathrm{Lax}_{\mathrm{op}}(\mathcal{D}, \mathrm{Lax}_{\mathrm{op}}(\mathcal{C}, \mathcal{E})) \overset{(2.3.22)}{\cong} \mathrm{Lax}(\mathcal{D}^{\mathrm{op}}, \mathrm{Lax}(\mathcal{C}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}}))^{\mathrm{op}} \tag{2.3.73}$$

$$\overset{(2.3.54)}{\cong} [\mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{C}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}}]^{\mathrm{op}}_{\mathrm{lnt}} \tag{2.3.74}$$

$$\overset{(2.3.28)}{\cong} [(\mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \mathcal{E}]_{\mathrm{ont}} . \tag{2.3.75}$$

□

Corollary 2.3.4. *Given 2-categories $\mathcal{C}$ and $\mathcal{D}$ there are isomorphisms*

$$Lax_{op}(\mathcal{C}, Lax_{op}(\mathcal{D}, \mathcal{E})) \cong [\mathcal{D} \boxtimes \mathcal{C}, \mathcal{E}]_{ont} \tag{2.3.76}$$

$$OpLax_{op}(\mathcal{C}, OpLax_{op}(\mathcal{D}, \mathcal{E})) \cong [(\mathcal{C}^{co} \boxtimes \mathcal{D}^{co})^{co}, \mathcal{E}]_{ont} \tag{2.3.77}$$

$$OpLax(\mathcal{C}, OpLax(\mathcal{D}, \mathcal{E})) \cong [(\mathcal{D}^{co} \boxtimes \mathcal{C}^{co})^{co}, \mathcal{E}]_{lnt} . \tag{2.3.78}$$

When $\mathcal{C} = \mathcal{D} = 1$, we get free distributive laws between monads with opmorphisms (opfunctors in [29]), between comonads with opmorphisms and between comonads with morphisms, respectively.

Now we consider strictification for the case when one of the homs has oplax functors - $\mathrm{Lax}(\mathcal{C}, \mathrm{OpLax}(\mathcal{D}, \mathcal{E}))$. Consider a computad $\mathcal{G}_m$, obtained from $\mathcal{G}$ by reversing 2-cells marked by (**f1**) and changing identifications accordingly. It generates a mixed tensor product $\mathcal{C} \boxtimes^{\mathrm{m}}_{\mathrm{cmp}} \mathcal{D}$, which analogously to Proposition 2.3.2 and Corollary 2.3.3 satisfies Corollary 2.3.5.

Corollary 2.3.5. *There are isomorphisms:*

$$Lax(\mathcal{C}, OpLax(\mathcal{D}, \mathcal{E})) \cong [\mathcal{C} \boxtimes^{m}_{cmp} \mathcal{D}, \mathcal{E}]_{lnt} \tag{2.3.79}$$

$$\mathcal{C} \boxtimes^{m}_{cmp} \mathcal{D} \cong (\mathcal{D}^{co} \boxtimes^{m}_{cmp} \mathcal{C}^{co})^{co} . \tag{2.3.80}$$

The cases based on this one are:

$$\mathrm{OpLax}_{\mathrm{op}}(\mathcal{C}, \mathrm{Lax}_{\mathrm{op}}(\mathcal{D}, \mathcal{E})) \cong [\mathcal{D} \boxtimes^{\mathrm{m}}_{\mathrm{cmp}} \mathcal{C}, \mathcal{E}]_{\mathrm{ont}} \tag{2.3.81}$$

$$\mathrm{Lax}_{\mathrm{op}}(\mathcal{C}, \mathrm{OpLax}_{\mathrm{op}}(\mathcal{D}, \mathcal{E})) \cong [(\mathcal{C}^{\mathrm{op}} \boxtimes^{\mathrm{m}}_{cmp} \mathcal{D}^{\mathrm{op}})^{\mathrm{op}}, \mathcal{E}]_{\mathrm{ont}} \tag{2.3.82}$$

$$\mathrm{OpLax}(\mathcal{C}, \mathrm{Lax}(\mathcal{D}, \mathcal{E})) \cong [(\mathcal{D}^{\mathrm{op}} \boxtimes^{\mathrm{m}}_{cmp} \mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \mathcal{E}]_{\mathrm{lnt}}\,. \tag{2.3.83}$$

Finally, when the two homs have different choice for the direction of natural transformations, there is no strictification tensor product, mainly because we have to choose a type of natural transformation for the strict hom. For example, note that the objects $B \in \mathrm{Lax}(\mathcal{C}, \mathrm{Lax}_{\mathrm{op}}(\mathcal{D}, \mathcal{E}))$ correspond to the objects $B \in [\mathcal{D} \boxtimes \mathcal{C}, \mathcal{E}]_{(\mathrm{l})(\mathrm{o})\mathrm{nt}}$ but crossings in the former allow[5] $c \circ b \circ d \Rightarrow d \circ b \circ c$ while crossings of the latter allow $c \circ d \circ b \Rightarrow b \circ d \circ c$ for lax and $b \circ c \circ d \Rightarrow d \circ c \circ b$ for oplax natural transformations, suggesting that this case cannot be strictified. In a similar way, $\mathrm{Lax}(\mathcal{C}, \mathrm{OpLax}_{\mathrm{op}}(\mathcal{D}, \mathcal{E}))$ does not permit strictifications.

2.3.6 The n-fold product

Here we generalize the computad construction of the (binary) tensor product to the n-fold case. Then we show how it can be organized into a lax monoidal structure [11] on the category of 2-categories and lax functors, denoted Lax. A list of data will be denoted by an arrow on top of the typical letter, $\vec{x} := (x_1, \ldots, x_n)$. The same list of data with x_i substituted with y_i will be denoted by $\vec{x}[y_i]$. When emphasizing that the substitution is at place i, or when it is not clear from the context, we use $\vec{x}[y_i]_i$. More than one argument means multiple substitutions.

For a list of 2-categories $\vec{\mathcal{C}}$ define a computad $\mathcal{G}_{\vec{\mathcal{C}}}$ with a set of nodes $\{\vec{C} | C_i \in \mathcal{C}_i\}$. For each $\vec{C}$, i, and $C'_i \in \mathcal{C}_i$ define a set of edges[6] $|\mathcal{G}_{\vec{\mathcal{C}}}|(\vec{C}, \vec{C}[C'_i]) = \mathcal{C}(C_i, C'_i)$, whose elements are denoted by $\vec{C}[c_i]$. Similarly, the two cells in the computad, coming from the 2-cells of $\mathcal{C}_i$ will be denoted $\vec{C}[\gamma_i]$. Denote by $p.p'$ the concatenation of paths in $\mathcal{F}|\mathcal{G}_{\vec{\mathcal{C}}}|$. For each $\vec{C}$, i and $C_i \in \mathcal{C}_i$, there is a unit comparison

$$\mathbf{id}^{(i)}_{\vec{C}[1_{C_i}]} : \vec{C} \Rightarrow \vec{C}[1_{C_i}]\,. \tag{2.3.84}$$

[5] which is a shorter notation for $B'cD' \circ bCD' \circ BCd \Rightarrow B'C'd \circ bC'D \circ BcD$

[6] When $C_i = C'_i$ the the set of edges is defined twice, but the definitions coincide.

For each $\vec{C}$, i and $c_i, c_i' \in \mathcal{C}_i$, a composable pair in $\mathcal{C}_i$, as in (2.1.2), there is a composition comparison computad 2-cell

$$\mathbf{comp}^{(i)}_{\vec{C}[(c_i,c_i')_i]} : \vec{C}[c_i].\vec{C}[c_i'] \Rightarrow \vec{C}[(c_i' \circ c_i)_i] \,. \tag{2.3.85}$$

For each $\vec{C}$, i, $j > i$ and $c_i \in \mathcal{C}_i$, and $c_j \in \mathcal{C}_j$, there is a computad 2-cell

$$\mathbf{swap}^{(i<j)}_{\vec{C}[c_i,c_j]} : \vec{C}[c_i].\vec{C}[C_i', c_j] \Rightarrow \vec{C}[c_j].\vec{C}[c_i, C_j'] \,. \tag{2.3.86}$$

When forming the tensor product 2-category, denoted $\boxtimes_n \vec{\mathcal{C}}$ we need to impose identifications analogous to (2.3.41), (2.3.43), (2.3.45), (2.3.48) and (2.3.49) for each i, and (2.3.47), (2.3.50)-(2.3.53), for each i and $j > i$.

Let $F_i : \mathcal{C}_i \to \mathcal{D}_i$ be a list of lax functors. Denote by $F^{(0)}_{C_i}$ the unit and by $F^{(2)}_{c_i,c_i'}$ the composition comparison maps, and by $F\vec{\mathcal{C}} := (F_1C_1, \ldots, F_nC_n)$ the assignment on list of objects. Form the computad morphism

$$\mathcal{G}_{\vec{\mathcal{C}}} \xrightarrow{(\boxtimes_n \vec{F})^{\mathrm{cmp}}} \mathcal{U}\mathcal{F}\mathcal{G}_{\vec{\mathcal{D}}} \tag{2.3.87}$$

$$\vec{C}, \vec{C}[c_i], \vec{C}[\gamma_i] \mapsto F\vec{C}, (F\vec{C})[F_ic_i], (F\vec{C})[F_i\gamma_i] \tag{2.3.88}$$

$$\mathbf{id}^{(i)}_{\vec{C}[1_{C_i}]} \mapsto (F\vec{C})[F^{(0)}_{C_i}] \bullet \mathbf{id}^{(i)}_{F\vec{C}[1_{F_iC_i}]} \tag{2.3.89}$$

$$\mathbf{comp}^{(i)}_{\vec{C}[(c_i,c_i')_i]} \mapsto (F\vec{C})[F^{(2)}_{c_i,c_i'}] \bullet \mathbf{comp}^{(i)}_{F\vec{C}[(F_ic_i,F_ic_i')_i]} \tag{2.3.90}$$

$$\mathbf{swap}^{(i<j)}_{\vec{C}[c_i,c_j]} \mapsto \mathbf{swap}^{(i<j)}_{F\vec{C}[F_ic_i,F_jc_j]} \,. \tag{2.3.91}$$

The induced functor $\mathcal{F}\mathcal{G}_{\vec{\mathcal{C}}} \to \boxtimes_n \vec{\mathcal{D}}$ respects identifications imposed on $\mathcal{F}\mathcal{G}_{\vec{\mathcal{C}}}$, which follows from the axioms for a lax functor and the fact that the quotienting map $Q_D : \mathcal{F}\mathcal{G}_{\vec{\mathcal{D}}} \to \boxtimes_n \vec{\mathcal{D}}$ respects them. Hence, there is a 2-functor

$$\boxtimes_n \vec{F} : \boxtimes_n \vec{\mathcal{C}} \to \boxtimes_n \vec{\mathcal{D}} \,. \tag{2.3.92}$$

When each component is the identity on $\mathcal{C}_i$ the induced functor is the identity on $\boxtimes_n \vec{\mathcal{D}}$. To see that the composition of lax functors is preserved by tensoring, note that both ways lead to the following assignments

$$\mathbf{id}^{(i)}_{\vec{C}[1_{C_i}]} \mapsto (GF\vec{C})[G_iF^{(0)}_{F_iC_i}] \bullet (GF\vec{C})[G^{(0)}_{F_iC_i}] \bullet \mathbf{id}^{(i)}_{GF\vec{C}[1_{G_iF_iC_i}]} \tag{2.3.93}$$

$$\begin{aligned}\mathbf{comp}^{(i)}_{\vec{C}[(c_i,c'_i)_i]} \mapsto &(GF\vec{C})[G_iF^{(2)}_{c_i,c'_i}] \\ &\bullet (GF\vec{C})[G^{(2)}_{F_ic_i,F_ic'_i}] \\ &\bullet \mathbf{comp}^{(i)}_{GF\vec{C}[(G_iF_ic_i,G_iF_ic'_i)_i]}\end{aligned} \tag{2.3.94}$$

following from the definition of comparison cells for the composite lax functor on one hand, and the fact that $\boxtimes_n\vec{G}$ is functorial on homs. The rest of the assignments are trivially the same.

This proves that the assignment

$$\boxtimes_n : \mathrm{Lax}^n \to \mathrm{Lax} \tag{2.3.95}$$

is a functor. Note that the unary tensor product $\boxtimes_1\mathcal{C}$ is just the Bénabou construction of the 2-category of paths $\mathcal{C}^\dagger$, described in section 2.2.1.

Until the end of this part we informally discuss some further properties and generalizations without proofs.

It is known that bicategories, lax functors and lax natural transformation do not form a bicategory because whiskering on the right cannot be defined. If we restrict to strict functors, adding ((op)lax)) natural transformations as 2-cells is not compatible with the tensor $\boxtimes$. However, if we take 2-categories, lax functors, and icons (defined in [21], and here in Eq. (2.2.6)), we get a nice 2-category Licon, and the n-fold tensor product extends to a 2-functor

$$\boxtimes_n : \mathrm{Licon}^n \to \mathrm{Licon}\,. \tag{2.3.96}$$

The need to define n-fold product already suggests that 2-fold does not determine the higher ones, in the way it does for monoidal categories. However, we give components of the unit

$$\eta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^\dagger \tag{2.3.97}$$

$$C, c, \gamma \mapsto C, c, \gamma \tag{2.3.98}$$

$$\{C\} \xRightarrow{\mathbf{id}_C} \eta_{\mathcal{C}}(1_C) \tag{2.3.99}$$

$$\eta_{\mathcal{C}}(c).\eta_{\mathcal{C}}(c') \xRightarrow{\mathbf{comp_{c,c'}}} \eta_{\mathcal{C}}(c' \circ c) \tag{2.3.100}$$

as well as for each partition $\xi : \langle m \rangle \to \langle n \rangle$ of m a comparison functor μ_ξ which "flattens out" the structure. Given a list of lists of categories $\vec{\vec{\mathcal{C}}}$, form a 2-functor

$$\mu_\xi : \boxtimes_n \overrightarrow{\boxtimes_{m_i} \vec{\mathcal{C}}_i} \to \boxtimes_m \vec{\mathcal{C}} \tag{2.3.101}$$

from the computad morphism

$$\mathcal{G}_{\overrightarrow{\boxtimes \vec{\mathcal{C}}}} \to \mathcal{U} \boxtimes_m \vec{\mathcal{C}} \tag{2.3.102}$$

$$\vec{\vec{C}} \mapsto \vec{C} \tag{2.3.103}$$

$$\vec{\vec{C}}[p_i] \mapsto \cdot_{j=1}^{l(p_i)} \vec{C}[(p_i)_j]_i =: \mathrm{fl}(p_i) \tag{2.3.104}$$

$$\vec{\vec{C}}[\pi_i] \mapsto \bullet_{j=1}^{l(\pi_i)} \vec{C}[(\pi_i)_j]_i \tag{2.3.105}$$

$$\mathbf{id}^{(i)}_{\vec{\vec{C}}[1_{\vec{C}_i}]} \mapsto 1_{1_{\vec{C}}} \tag{2.3.106}$$

$$\mathbf{comp}^{(i)}_{\vec{C}[(p_i, p_i')]_i} \mapsto 1_{\mathrm{fl}(p_i + p_i')} \tag{2.3.107}$$

$$\mathbf{swap}^{(i<j)}_{\vec{\vec{C}}[p_i, p_j]} \mapsto ! : \mathrm{fl}(p_i).\mathrm{fl}(p_j) \Rightarrow \mathrm{fl}(p_j).\mathrm{fl}(p_i)\,. \tag{2.3.108}$$

We omit proving that it preserves identifications, naturality in $\mathcal{C}_{ij}$, axioms for lax monoidal structure, how it is defined on icons, and just state the following proposition.

Proposition 2.3.3. *Functors $\boxtimes_n$, together with natural transformations η and μ_ξ, form a lax monoidal structure on Licon.*

2.4 Simplicial approach

We proceed to describe a model $\mathcal{C} \boxtimes_{sim} \mathcal{D}$ for the strictification tensor product and then show that it is isomorphic to $\mathcal{C} \boxtimes_{\mathrm{cmp}} \mathcal{D}$.

Objects of $\mathcal{C} \boxtimes_{sim} \mathcal{D}$ are pairs $(C; D)$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

An arrow in $\mathcal{C} \boxtimes_{sim} \mathcal{D}$ is a sextuple $(n, p, r; m, q, s)$. It consists of a path in $\mathcal{C}$ of length n, a path in $\mathcal{D}$ (of length m)

$$p : [n] \to \mathcal{C}, \quad q : [m] \to \mathcal{D} \tag{2.4.1}$$

and a way to combine them into a string of length $n + m$; that is, a shuffle

$$[n] \xleftarrow{r} [n+m] \xrightarrow{s} [m] \tag{2.4.2}$$

where r and s satisfy a compatibility condition (A.1.28) saying that one increases if and only if the other one does not.

The identity (empty path) on $(C; D)$ is defined by taking $m = n = 0$, $r = s = 1_{[0]}$, and p and q pick the objects C and D. Composition is defined by path concatenation, formally expressed as tensor product of shuffles.

Below is an example of a 1-cell $\{c_1, d_1, c_2, c_3, d_2\} : (C_1, D_1) \to (C_4, D_3)$ in $\mathcal{C} \boxtimes_{\mathrm{dir}} \mathcal{D}$. Here, $n = 3$, $m = 2$, $r : [5] \to [3]$ and $s : [5] \to [2]$ give the coordinates of the corresponding node in the path, and $p : [3] \to \mathcal{C}$ and $q : [2] \to \mathcal{D}$ are the obvious functors producing the paths $\{c_i\}_{i=1}^3$ and $\{d_i\}_{i=1}^2$ in $\mathcal{C}$ and $\mathcal{D}$.

$$\begin{tikzcd}
 & C_1 \arrow[r, "c_1"] & C_2 \arrow[r, "c_2"] & C_3 \arrow[r, "c_3"] & C_4 \\
D_1 \arrow[d, "d_1"'] & C_1D_1 \arrow[r, "c_1"] & C_2D_1 \arrow[d, "d_1"'] & & \\
D_2 \arrow[d, "d_2"'] & & C_2D_2 \arrow[r, "c_2"] & C_3D_2 \arrow[r, "c_3"] & C_4D_2 \arrow[d, "d_2"'] \\
D_3 & & & & C_4D_3
\end{tikzcd} \tag{2.4.3}$$

A 2-cell

$$(\xi, \alpha; \rho, \beta) : (n, p, r; m, q, s) \to (\bar{n}, \bar{p}, \bar{r}; \bar{m}, \bar{q}, \bar{s}) \tag{2.4.4}$$

consists of:

- a shuffle morphism, that is functors $\xi : [\bar{n}] \to [n]$, $\rho : [\bar{m}] \to [m]$ preserving the first and the last element and satisfying, for all $\bar{i} \leqslant \bar{n} + \bar{m}$,

$$\min r^{-1}(\xi \bar{r} \bar{i}) \leqslant \max s^{-1}(\rho \bar{s} \bar{i}) \tag{2.4.5}$$

  a condition ensuring that there are no swaps of arrows from $\mathcal{C}$ and $\mathcal{D}$ in the wrong direction. The condition (2.4.5) is an explicitly written condition for the existence of the natural transformation (A.1.31).

- path 2-cells, that is, icons $\alpha : p \circ \xi \Rightarrow \bar{p}$ and $\beta : q \circ \rho \Rightarrow \bar{q}$, as defined in section 2.2.1

Below is an example of a 2-cell.

$$\begin{tikzcd}[column sep=small]
C_1D_1 \arrow[r, "c_1"'] & C_2D_1 \arrow[r, "d_1"'] & C_2D_2 \arrow[r, "c_2"'] & C_3D_2 \arrow[r, "c_3"'] & C_4D_2 \arrow[r, "d_2"'] & C_4D_3 \\
C_1D_1 \arrow[r, "\bar{c}_1"] & C_2D_1 \arrow[r, "\bar{d}_1"] & C_2D_3 \arrow[r, "\bar{c}_2"] & C_2D_3 \arrow[r, "\bar{c}_3"] & C_4D_3 &
\end{tikzcd} \tag{2.4.6}$$

The above graph represents two 1-cells and data of ξ and ρ, and what remains is to specify icon components $\alpha_1 : c_1 \Rightarrow \bar{c}_1$, $\alpha_2 : 1_{C_2} \Rightarrow \bar{c}_2$, $\alpha_3 : c_3 \circ c_2 \Rightarrow \bar{c}_3$ in $\mathcal{C}$ and $\beta_1 : d_2 \circ d_1 \Rightarrow \bar{d}_1$ in $\mathcal{D}$.

Vertical composition and whiskerings are defined componentwise as in **Shuff**, $\mathcal{C}^\dagger$ and $\mathcal{D}^\dagger$.

2.4.1 As a limit

The category $\mathcal{C} \boxtimes_{sim} \mathcal{D}$ is a limit of the following diagram in 2-Cat.

$$\begin{array}{ccccccccc}
\mathcal{C}^\dagger & \to & \Sigma\Delta_{\perp\top} & \leftarrow & \mathrm{FDL} & \to & \Sigma\Delta_{\perp\top} & \leftarrow & \mathcal{D}^\dagger \\
C & \mapsto & * & \leftarrow\!\shortmid & * & \mapsto * & & \leftarrow\!\shortmid & D \\
(n,p) & \mapsto & [n] & \leftarrow\!\shortmid & (n,m,s,r) & \mapsto [m] & & \leftarrow\!\shortmid & (m,q) \\
(\xi,\alpha) & \mapsto & \xi & \leftarrow\!\shortmid & (\xi,\rho,\gamma) & \mapsto \rho & & \leftarrow\!\shortmid & (\rho,\beta)
\end{array}$$

2.4.2 Isomorphism between two constructions

This part is about proving the following proposition.

Proposition 2.4.1. *There is an isomorphism*

$$\mathcal{C} \boxtimes_{sim} \mathcal{D} \cong \mathcal{C} \boxtimes_{cmp} \mathcal{D}\,. \tag{2.4.7}$$

We shall define a computad morphism $T : \mathcal{G} \to \mathcal{U}(\mathcal{C} \boxtimes_{sim} \mathcal{D})$, show that the induced strict 2-functor $\hat{T} : \mathcal{FG} \to \mathcal{C} \boxtimes_{sim} \mathcal{D}$ respects the identifications (2.3.41)-(2.3.53), and that any other 2-functor $\hat{V} : \mathcal{FG} \to \mathcal{E}$ respecting them factors uniquely through $\hat{T}$. Then, from the universal property of $\mathcal{C} \boxtimes_{cmp} \mathcal{D}$ it will follow that $\mathcal{C} \boxtimes_{cmp} \mathcal{D} \cong \mathcal{C} \boxtimes_{sim} \mathcal{D}$.

The computad morphism $T : \mathcal{G} \to \mathcal{U}(\mathcal{C} \boxtimes_{sim} \mathcal{D})$ is defined on nodes by

$$T(C \boxtimes D) = (C; D)\,, \tag{2.4.8}$$

on edges by

$$T(C \boxtimes d) = (0, \{C\}, \sigma_0^1; 1, \{D \xrightarrow{d} D'\}, 1_{[1]}) \tag{2.4.9}$$

$$T(c \boxtimes D) = (1, \{C \xrightarrow{c} C'\}, 1_{[1]}; 0, \{D\}, \sigma_0^1)\,, \tag{2.4.10}$$

on 2-cells inherited from $\mathcal{C}$ and $\mathcal{D}$ by

$$\begin{aligned} T(C \boxtimes \delta) = (1_{[0]}, \{\}; 1_{[1]}, \{\delta\}) &: (0, \{C\}, \sigma_0^1; 1, \{D \xrightarrow{d} D'\}, 1_{[1]}) \\ &\Rightarrow (0, \{C\}, \sigma_0^1; 1, \{D \xrightarrow{\bar{d}} D'\}, 1_{[1]}) \end{aligned} \tag{2.4.11}$$

$$\begin{aligned} T(\gamma \boxtimes D) = (1_{[1]}, \{\gamma\}; 1_{[0]}, \{\}) &: (1, \{C \xrightarrow{c} C'\}, 1_{[1]}; 0, \{D\}, \sigma_0^1) \\ &\Rightarrow (1, \{C \xrightarrow{\bar{c}} C'\}, 1_{[1]}; 0, \{D\}, \sigma_0^1) \end{aligned} \tag{2.4.12}$$

and on the comparison and swapping 2-cells by

$$\begin{aligned} T(\mathbf{id}_{1_C D}) = (\sigma_0^1, \{1_{1_C}\}; 1_{[0]}, \{\}) &: (0, \{C\}, 1_{[0]}; 0, \{D\}, 1_{[0]}) \\ &\Rightarrow (1, \{C \xrightarrow{1_C} C\}, 1_{[1]}; 0, \{D\}, \sigma_0^1) \end{aligned} \tag{2.4.13}$$

$$\begin{aligned} T(\mathbf{id}_{C 1_D}) = (1_{[0]}, \{\}; \sigma_0^1, \{1_{1_D}\}) &: (0, \{C\}, 1_{[0]}; 0, \{D\}, 1_{[0]}) \\ &\Rightarrow (0, \{C\}, \sigma_0^1; 1, \{D \xrightarrow{1_D} D\}, 1_{[1]}) \end{aligned} \tag{2.4.14}$$

$$\begin{aligned} T(\mathbf{comp}_{c,c',D}) =& (\partial_1^2, \{1_{c' \circ c}\}; 1_{[0]}, \{\}) : \\ & (2, \{C \xrightarrow{c} C' \xrightarrow{c'} C''\}, 1_{[2]}; 0, \{D\}, !_{[2] \to [0]}) \\ & \Rightarrow (1, \{C \xrightarrow{c' \circ c} C''\}, 1_{[2]}; 0, \{D\}, \sigma_0^1) \end{aligned} \tag{2.4.15}$$

$$\begin{aligned} T(\mathbf{comp}_{C,d,d'}) =& (1_{[0]}, \{\}; \partial_1^2, \{1_{d' \circ d}\}) : \\ & (0, \{C\}, !_{[2] \to [0]}; 2, \{D \xrightarrow{d} D' \xrightarrow{d'} D''\}, 1_{[2]}) \\ & \Rightarrow (0, \{C\}, \sigma_0^1; 1, \{D \xrightarrow{d' \circ d} D''\}, 1_{[1]}) \end{aligned} \tag{2.4.16}$$

$$\begin{aligned} T(\mathbf{swap}_{c,d}) =& (1_{[1]}, \alpha = \{1_c\}; 1_{[1]}, \beta = \{1_d\}) : \\ & (1, \{C \xrightarrow{c} C'\}, \sigma_1^2; 1, \{D \xrightarrow{d} D'\}, \sigma_0^2) \\ & \Rightarrow (1, \{C \xrightarrow{c} C'\}, \sigma_0^2; 1, \{D \xrightarrow{d} D'\}, \sigma_1^2)\,. \end{aligned} \tag{2.4.17}$$

To check that the last 2-cell is the valid one, write equation (A.1.31) as

$$L\sigma_1^2 \circ 1 \circ \sigma_0^2 = \partial_2^2 \circ \sigma_0^2 \Rightarrow \partial_0^2 \circ \sigma_1^2 = R\sigma_0^2 \circ 1 \circ \sigma_1^2 \,. \tag{2.4.18}$$

The cells on the RHS of (2.4.11)-(2.4.17) will be called *elementary 2-cells.*

To see that the induced strict 2-functor respects identifications (2.3.41)-(2.3.53), note that $T(\mathbf{id})$, $T(\mathbf{comp})$, and $T(\mathbf{swap})$ have trivial icon components, while the definition of T on other parts of the computad have trivial components in **Shuff**, and that the composition of 2-cells in $\mathcal{C} \boxtimes_{sim} \mathcal{D}$ is done independently in each of the components.

Given a computad map $V : \mathcal{G} \to \mathcal{UE}$, such that $\hat{V} : \mathcal{FG} \to \mathcal{E}$ respects the identifications (2.3.41)-(2.3.53), form the following assignments $W : \mathcal{C} \boxtimes_{sim} \mathcal{D} \to \mathcal{E}$ on objects

$$W(C; D) = V(C \boxtimes D) \tag{2.4.19}$$

and on elementary arrows

$$W(0, \{C\}, \sigma_0^1; 1, \{D \xrightarrow{d} D'\}, 1_{[1]}) = W(T(C \boxtimes d)) = V(C \boxtimes d) \tag{2.4.20}$$

$$W(1, \{C \xrightarrow{c} C'\}, 1_{[1]}; 0, \{D\}, \sigma_0^1) = W(T(c \boxtimes D)) = V(c \boxtimes D)\,. \tag{2.4.21}$$

Since every shuffle can be written uniquely as a sum of shuffles of unit length, the above assignment determines assignment on all 1-cells; given $(n, p, r; m, q, s)$, assign to it the composite given by (2.4.22).

$$W(n, p, r; m, q, s) = \circ_{i=n+m}^{1} \begin{cases} V((p)_i \boxtimes qsi), \text{ if } s_i = 0 \\ V(pri \boxtimes (q)_i), \text{ if } r_i = 0 \end{cases} \tag{2.4.22}$$

When $n = m = 0$ we get that W preserves identities; that is,

$$W(1_{(C;D)}) = 1_{W(C;D)}\,. \tag{2.4.23}$$

Also, W preserves composition

$$W(n', p', r'; m', q', s') \circ W(n, p, r; m, q, s) =$$

$$\circ_{i'=n'+m'}^{1} \begin{cases} V((p')_{i'} \boxtimes q's'i'), \text{ if } s'_{i'} = 0 \\ V(p'r'i' \boxtimes (q')_{i'}), \text{ if } r'_{i'} = 0 \end{cases} \quad \circ_{i=n+m}^{1} \begin{cases} V((p)_i \boxtimes qsi), \text{ if } s_i = 0 \\ V(pri \boxtimes (q)_i), \text{ if } r_i = 0 \end{cases} \tag{2.4.24}$$

$$= \circ^{1}_{i=n'+m'+n+m} \begin{cases} V((p')_i \boxtimes q's'i), \text{ if } s'_i = 0, \text{ and } i > n+m \\ V(p'r'i \boxtimes (q')_i), \text{ if } r'_i = 0, \text{ and } i > n+m \\ V((p)_i \boxtimes qsi), \text{ if } s_i = 0, \text{ and } i \leqslant n+m \\ V(pri \boxtimes (q)_i), \text{ if } r_i = 0, \text{ and } i \leqslant n+m \end{cases} \tag{2.4.25}$$

$$= W(n'+n, p'+p, r'+r; m'+m, q'+q, s'+s) \tag{2.4.26}$$

$$= W((n', p', r'; m', q', s') \circ (n, p, r; m, q, s)) \,. \tag{2.4.27}$$

Hence, it is a functor on the underlying categories.

The requirement that $WT = V$ determines the assignment on identities

$$T(1_g) = 1_{Tg} \tag{2.4.28}$$

on elementary 2-cells $T\pi$

$$W(T\pi) = V(\pi) \tag{2.4.29}$$

and similarly on whiskered elementary 2-cells

$$W(Tg'' \circ T\pi \circ Tg) := Vg'' \circ V\pi \circ Vg = V(g'' \circ \pi \circ g) \tag{2.4.30}$$

where $T\pi$ is an elementary 2-cell and Tg and Tg' are 1-cells.

Given any 2-cell $(\xi, \alpha; \rho, \beta)$, as in (2.4.4), choose a decomposition into whiskered elementary 2-cells in the following order, starting from the target 1-cell,

- elementary β, $j = \bar{m}, .., 1$

$$J_j = 1 \circ T(\bar{p}\bar{r}j \boxtimes \beta_j) \circ 1 \tag{2.4.31}$$

$$= (1_{[\bar{n}]}, \{1_{p_1}, .., 1_{p_{\bar{n}}}\}; 1_{[\bar{m}]}, \{1_{q_1}, .., \beta_j, .., 1_{q_{\bar{n}}}\}) \tag{2.4.32}$$

- elementary α, $i = \bar{n}, .., 1$

$$I_i = 1 \circ T(\alpha_i \boxtimes \bar{q}\bar{s}i) \circ 1 \tag{2.4.33}$$

$$= (1_{[\bar{n}]}, \{1_{p_1}, .., \alpha_i, .., 1_{p_{\bar{n}}}\}; 1_{[\bar{m}]}, \{1_{q_1}, .., 1_{q_{\bar{n}}}\}) \tag{2.4.34}$$

- comparisons in $\mathcal{D}$, $j = \bar{m}, .., 1$
  - if $\bar{\rho}_j = 0$ then

$$L_{j,1} = 1 \circ T(\mathbf{id}) \circ 1 =: L_j^{(\mathbf{id})} \tag{2.4.35}$$

  - if $\bar{\rho}_j \geqslant 2$, $k = \bar{\rho}_j - 1, ..., 1$

$$L_{j,k} = 1 \circ T(\mathbf{comp}) \circ 1 =: L_{j,k}^{(\mathbf{comp})} \tag{2.4.36}$$

    This order corresponds to left bracketing.
  - if $\bar{\rho}_j = 1$ then $L_{j,1} = 1$, and can be ignored.
- comparisons in $\mathcal{C}$, $i = \bar{n}, .., 1$
  - if $\bar{\xi}_i = 0$ then

$$K_{i,1} = 1 \circ T(\mathbf{id}) \circ 1 =: K_j^{(\mathbf{id})} \tag{2.4.37}$$

  - if $\bar{\xi}_j \geqslant 2$, $k = \bar{\xi}_j - 1, ..., 1$

$$K_{i,k} = 1 \circ T(\mathbf{comp}) \circ 1 =: K_{j,k}^{(\mathbf{comp})} \tag{2.4.38}$$

    This order corresponds to left bracketing.
  - if $\bar{\xi}_j = 1$ then $K_{j,1} = 1$, and can be ignored.
- crossings - the remaining 2-cell to decompose has trivial icon components as well as trivial ξ and ρ. In the relation tables - which define the two shuffles - elementary crossings correspond to switching ones to zeros, or, going backwards, switching zeros to ones. Let (x, y) be the coordinates of the corresponding crossings, order them by $x - y$ and then (if the $x - y$ value is the same) by $x + y$. Our backward decomposition starts with the last crossing in the table. Denote them by S_i.

Now, define

$$W(\xi, \alpha; \rho, \beta) = \circ_i W(J_i) \circ_i W(I_i) \circ_{i,j} W(L_{i,j}) \circ_{i,j} W(K_{i,j}) \circ_i W(S_i) \tag{2.4.39}$$

Given a composable pair of 2-cells, the composite of their their images under W, $W(\bar{\xi}, \bar{\alpha}; \bar{\rho}, \bar{\beta}) \circ W(\xi, \alpha; \rho, \beta)$, is equal to

$$\begin{aligned} &\circ_i W(\bar{J}_i) \circ_i W(\bar{I}_i) \circ_{i,j} W(\bar{L}_{i,j}) \circ_{i,j} W(\bar{K}_{i,j}) \circ_i W(\bar{S}_i) \\ &\circ_i W(J_i) \circ_i W(I_i) \circ_{i,j} W(L_{i,j}) \circ_{i,j} W(K_{i,j}) \circ_i W(S_i) \end{aligned} \tag{2.4.40}$$

which need not be in the canonical form. The assignment on the composite 2-cell

$$(\xi \circ \bar{\xi}, \bar{\alpha} \bullet (\alpha \circ \bar{\xi}); \rho \circ \bar{\rho}, \bar{\beta} \bullet (\beta \circ \bar{\rho})) \tag{2.4.41}$$

is in the canonical form, and the two are equal which we show by "bubble-sorting" the decomposition (2.4.40). In each step one of two cases can happen:

- the output (target of the elementary part) of the first 2-cell does not overlap with the input (source of the elementary part) of the second 2-cell. Then we can write the vertical composite of their images as

$$\begin{aligned} &W(Tg_5 \circ T\bar{g}_4 \circ Tg_3 \circ T\pi_2 \circ Tg_1) \\ &\bullet W(Tg_5 \circ T\pi_1 \circ Tg_3 \circ Tg_2 \circ Tg_1) \\ &= V(g_5 \circ \pi_1 \circ g_3 \circ \pi_2 \circ g_1) = \\ &W(Tg_5 \circ T\pi_1 \circ Tg_3 \circ T\bar{g}_2 \circ Tg_1) \\ &\bullet W(Tg_5 \circ Tg_4 \circ Tg_3 \circ T\pi_2 \circ Tg_1) \end{aligned} \tag{2.4.42}$$

  meaning that we can change the order of their composition after suitably changing the whiskering 1-cells.

- the output of the first 2-cell overlaps with the input of the second 2-cell. Depending on which elementary 2-cells meet, do an operation according to the following table.

$1^{st}\backslash 2^{nd}$	$\bar{J}$	$\bar{I}$	$\bar{L}^{(\mathbf{id})}$	$\bar{L}^{(\mathbf{comp})}$	$\bar{K}^{(\mathbf{id})}$	$\bar{K}^{(\mathbf{comp})}$	$\bar{S}$
J	(2.3.43)	$\perp$	$\perp$	(2.3.45)	$\perp$	$\perp$	$\perp$/(2.3.47)
I	$\perp$	(2.3.44)	$\perp$	$\perp$	$\perp$	(2.3.46)	(2.3.47)/$\perp$
$L^{(\mathbf{id})}$	R	$\perp$	$\perp$	(2.3.48)	$\perp$	$\perp$	$\perp$/(2.3.51)
$L^{(\mathbf{comp})}$	R	$\perp$	$\perp$	R/(2.3.49)	$\perp$	$\perp$	$\perp$/(2.3.53)
$K^{(\mathbf{id})}$	$\perp$	R	$\perp$	$\perp$	$\perp$	(2.3.48)	(2.3.50)/$\perp$
$K^{(\mathbf{comp})}$	$\perp$	R	$\perp$	$\perp$	$\perp$	R/(2.3.49)	(2.3.52)/$\perp$
S	$\perp/R$	$R/\perp$	$\perp$	$\perp/\perp/R$	$\perp$	$R/\perp/\perp$	$R/\perp/\perp$

(2.4.43)

If the first 2-cell has n outputs and the second 2-cell has m inputs, there are $n+m-1$ ways to match them. When different, these cases are separated by "/". The symbol $\perp$ denotes that matching is not possible for that case, and R denotes that the matching is possible, but the order is already correct (lower triangle). Finally, an equation number tells us to apply apply $\hat{T}$ to both sides, and substitute the LHS, which appears in the composition, with the RHS. Each step changes the decomposition of the 2-cell, and the fact that $\hat{V}$ preserves relations ensures that the composite in $\mathcal{E}$ does not change.

This proves that W is functorial on homs.

A 2-cell in $\mathcal{C} \boxtimes \mathcal{D}$, obtained by whiskering, has the same elementary 2-cells in its decomposition as the original 2-cell. Hence, the two different composites

$$(WT\bar{g}' \circ W(\xi, \alpha; \rho, \beta)) \bullet (W(\xi', \alpha'; \rho', \beta') \circ WTg) \tag{2.4.44}$$

and

$$(W(\xi', \alpha'; \rho', \beta') \circ WT\bar{g}) \bullet (WTg' \circ W(\xi, \alpha; \rho, \beta)) \tag{2.4.45}$$

necessarily bubble-sort to $W((\xi', \alpha'; \rho', \beta') \circ (\xi, \alpha; \rho, \beta))$. This completes the proof that W is a 2-functor.

The functor $\hat{T}$ is bijective on objects and arrows, and surjective on 2-cells, so W is the unique 2-functor satisfying $\hat{V} = W\hat{T}$.

2.4.3 Mixed tensor product

The case covering the free mixed distributive law, strictifying $\mathrm{Lax}(\mathcal{C}, \mathrm{OpLax}(\mathcal{D}, \mathcal{E}))$, produces $\mathcal{C} \boxtimes^{\mathrm{m}}_{\mathrm{dir}} \mathcal{D}$ that has the same objects and arrows as $\mathcal{C} \boxtimes \mathcal{D}$, and 2-cells differ by changing the direction of $\rho : [m] \to [\bar{m}]$ to accommodate comultiplication and counit, change in icon $\beta : q \Rightarrow \bar{q}\rho : [m] \to \mathcal{D}$, with the restriction for crossings taking a slightly different form

$$Lr \circ \xi \circ \bar{r} \Rightarrow Rs \circ R\rho \circ \bar{s}\,. \tag{2.4.46}$$

With a proof following the same steps as the non-mixed case, we state the following proposition.

Proposition 2.4.2. *There is an isomorphism*

$$\mathcal{C} \boxtimes^{m}_{dir} \mathcal{D} \cong \mathcal{C} \boxtimes^{m}_{cmp} \mathcal{D}\,. \tag{2.4.47}$$

2.5 Properties and an example

There is an obvious 2-functor $L : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{C} \times \mathcal{D}$ that forgets shuffles and composes paths. It has a right adjoint R in the 2-category of 2-categories, lax functors and icons:

$$\mathcal{C} \times \mathcal{D} \xrightarrow{R} \mathcal{C} \boxtimes \mathcal{D} \tag{2.5.1}$$

$$(C, D) \mapsto C \boxtimes D \tag{2.5.2}$$

$$(c, d) \mapsto CD \xrightarrow{Cd} CD' \xrightarrow{cD'} C'D' \tag{2.5.3}$$

$$(\gamma, \delta) \mapsto (\gamma \boxtimes D') \circ (C \boxtimes \delta) \tag{2.5.4}$$

with identity and composition comparison maps

$$! : 1_{C \boxtimes D} \Rightarrow CD \xrightarrow{C1_D} CD \xrightarrow{1_C D} CD \tag{2.5.5}$$

$$(\partial^2_1, 1; \partial^2_1, 1) : CD \xrightarrow{Cd} CD' \xrightarrow{cD'} C'D' \xrightarrow{C'd'} C'D'' \xrightarrow{c'D''} C''D'' \tag{2.5.6}$$

$$\Rightarrow CD \xrightarrow{C(d' \circ d)} CD'' \xrightarrow{(c' \circ c)D''} C''D''\,. \tag{2.5.7}$$

The composite $L \circ R$ is just the identity functor $1_{\mathcal{C} \times \mathcal{D}}$, while the unit of the adjunction is an icon

$$\eta : 1_{\mathcal{C} \boxtimes \mathcal{D}} \Rightarrow R \circ L \tag{2.5.8}$$

assigning to each arrow $(n, p, r; m, q, s)$ in $\mathcal{C} \boxtimes \mathcal{D}$ a 2-cell

$$(!_{[1]\to[n]}, 1_{\circ p}; !_{[1]\to[m]}, 1_{\circ q}) : (n, p, r; m, q, s) \Rightarrow (1, \circ p, \sigma_0^2; 1, \circ q, \sigma_1^2)\,. \tag{2.5.9}$$

Whiskering η on the left (resp. right) by L (resp. R) gives the identity on L (resp. R), proving the adjunction axioms.

Any strict functor $\hat{B} : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{E}$ can be precomposed with R to give a lax functor

$$\hat{B} \circ R : \mathcal{C} \times \mathcal{D} \to \mathcal{E}\,. \tag{2.5.10}$$

This generalizes the notion of a composite monad induced by a distributive law.

2.5.1 Parametrizing parametrization of categories

Take $\mathcal{C}$ and $\mathcal{D}$ to be just categories (seen as locally discrete 2-categories), and[7] $\mathcal{E} = \mathrm{Span}$.

The bicategory of spans is equivalent to the bicategory of matrices, which is in turn a full subcategory of[8] Mod. Each strict functor $\hat{B} : \mathcal{C} \boxtimes \mathcal{D} \to \mathrm{Span}$ is, in particular, a normal lax functor, so we can use the Bénabou construction [33] (after forgetting 2-cells) to obtain a category $\tilde{B}_{\mathrm{nerve}}$ parametrised over $\mathcal{C}\boxtimes\mathcal{D}$. Explicitly, $\tilde{B}_{\mathrm{nerve}}$ has objects over $C\boxtimes D$ given by the set BCD. Arrows over $C\boxtimes d$ and $c\boxtimes D$ are elements of spans BCd and BcD respectively, and they generate arrows over arbitrary paths, which are, due to composition in Span, composable tuples.

The 2-cells that we have temporarily forgotten are mapped to span morphisms. In particular, the images $\hat{B}\eta_p$ of the unit of the adjunction (2.5.8) give a unique way of "composing" arbitrary arrows in $\tilde{B}_{\mathrm{nerve}}$, resulting in an arrow over a path in $\mathcal{C} \boxtimes \mathcal{D}$ of the form $CD \xrightarrow{Cd} CD' \xrightarrow{cD} C'D'$. The image of this assignment forms a category $\tilde{B}$ whose composition is concatenation in $\tilde{B}_{\mathrm{nerve}}$ followed by applying (the unique) appropriate $B\eta$. Uniqueness guarantees the identity and associativity laws.

Explicitly, $\tilde{B}$ with the same objects as $\tilde{B}_{\mathrm{nerve}}$, and arrows between $X \in B(C \boxtimes D)$ and $X' \in B(C' \boxtimes D')$ are elements of $B(CD \xrightarrow{Cd} CD' \xrightarrow{cD} C'D')$, denoted by pairs (g, f),. The

[7]Instead of Span one can take a strict version with objects sets X, Y... and arrows cocontinuous functors $\mathrm{Set}/X \to \mathrm{Set}/Y$ which are determined by the assignment of singletons.

[8]Consisting of categories and modules (aka profunctors or distributors)

identity is

$$1_X = (1_X^D, 1_X^C),\ \text{with} \tag{2.5.11}$$

$$1_X^D := (B\mathbf{id}_{C1_D})(X) \tag{2.5.12}$$

$$1_X^C := (B\mathbf{id}_{1_C D})(X) \tag{2.5.13}$$

and composition is given by

$$(g', f') \circ (g, f) = B((\mathbf{comp} \circ \mathbf{comp}) \bullet (1 \circ \mathbf{swap} \circ 1))(g', f', g, f)\,. \tag{2.5.14}$$

For each object $D \in \mathcal{D}$ we get a subcategory $\pi_D \tilde{B}$ parametrized by $\mathcal{C}$ - an object X over C is an element of BCD, and arrow $f : X \to X'$ over c is an element of BcD, which can be identified with an arrow $(1_X^D, f)$ of $\tilde{B}$. Similarly, each object $C \in \mathcal{C}$ gives a subcategory $\pi_C \tilde{B}$, parametrized by $\mathcal{D}$. Furthermore, each arrow (g, f) in $\tilde{B}$ can be decomposed as

$$(1_{D'}, f) \circ (g, 1_C) \tag{2.5.15}$$

or as

$$(g, 1_{C'}) \circ (1_D, f)\,. \tag{2.5.16}$$

3

Cauchy completeness and causal spaces

3.1 Introduction

Considering causal structures as fundamental and space-time as emerging was considered in [7]. We provide a novel way to construct causal preordered sets, together with maximal intervals between events, as categories enriched in a particular monoidal category we called $\mathcal{R}_{\perp}$. All $\mathcal{R}_{\perp}$-enriched categories turn out to be Cauchy complete, and we address the question of which bases share this property. This abstract approach to general relativity might shine new light on already existing categorical approaches to quantum foundations [1] or quantum gravity [2].

In Section 3.2 we review the work of Lawvere who viewed positive real numbers as a monoidal category, denoted $\mathcal{R}$, to obtain generalized metric spaces as enriched categories.

In Section 3.3 we give a modification of the base category $\mathcal{R}$, call it $\mathcal{R}_\perp$, which gives causal spaces as $\mathcal{R}_\perp$-enriched categories, and explain how black holes and wormholes (see 3.3.3) can be described using enriched modules. We also prove a surprising fact that all causal spaces are Cauchy complete, in the sense of enriched category theory.

In Section 3.4 we give conditions on a monoidal category $\mathcal{V}$ which ensure that a $\mathcal{V}$-category $\mathcal{C}$ is Cauchy complete if and only if the underlying (Set-enriched) category $\mathcal{C}_0$ is Cauchy complete, which for Set-enrichment means that idempotents in $\mathcal{C}_0$ split. As a corollary we add a few more conditions on $\mathcal{V}$ ensuring that all $\mathcal{V}$-enriched categories are Cauchy complete, generalizing the case of $\mathcal{R}_\perp$.

3.2 Metric spaces as enriched categories

A generalized metric space X consists of a set of points and, for each pair of points P and Q, a distance $d(P,Q) \in [0,\infty]$ from P to Q such that, for all points P, Q and R,

$$d(P,P) = 0 \tag{3.2.1}$$

$$d(P,Q) + d(Q,R) \geqslant d(P,R)\,. \tag{3.2.2}$$

"Generalized" comes from dropping conditions of finiteness (allowing infinite distance), symmetry (allowing $d(P,Q) \neq d(Q,P)$), and distinguishability (allowing $d(P,Q) = 0$ without $P = Q$). Those spaces correspond precisely [24] to categories enriched in $\mathcal{R}$ - a monoidal category (more concretely, a totally ordered set) with positive reals and infinity as objects, an arrow between a and b if and only if $b \leqslant a$, and monoidal structure given by sum. $\mathcal{R}$ is also closed, with internal hom given by truncated subtraction, uniquely defined right adjoint to summation. To see the correspondence, recall [19] that a category $\mathcal{X}$ enriched in a monoidal category $\mathcal{V}$ consists of a set of objects (points in this case), for each pair of objects a hom, that is, an object in $\mathcal{V}$ (a number providing distance in this case), and unit and composition arrows of $\mathcal{V}$ (providing (in)equalities (3.2.1)-(3.2.2), in this case) satisfying unit and associativity laws (trivially true in this case because $\mathcal{R}$ is a poset).

Denote by $\mathcal{I}$ the space having only one point $*$. An enriched module (aka profunctor, distributor) $\mathcal{I} \overset{M}{\nrightarrow} \mathcal{X}$, alternatively expressed as an enriched presheaf $M : \mathcal{X}^{\mathrm{op}} \to \mathcal{R}$, assigns to each point P in $\mathcal{X}$ a distance from P to $*$, $M(P,*)$, with an action ensuring triangle inequality for the newly introduced distances

$$\mathcal{X}(P,Q) + M(Q,*) \geqslant M(P,*)\,. \tag{3.2.3}$$

For example, each point $P \in \mathcal{X}$ defines a module $M_P(Q,*) = \mathcal{X}(Q,P)$ - this motivates a general definition A.2.2 for convergent modules. Dually, an enriched module $\mathcal{X} \overset{N}{\nrightarrow} \mathcal{I}$ assigns to each point P in $\mathcal{X}$ a distance from $*$ to P, with actions

$$N(*,P) + \mathcal{X}(P,Q) \geqslant N(*,Q)\,. \tag{3.2.4}$$

Asking for M and N to form an adjunction in $\mathcal{R}$-Mod imposes existence of a counit

$$M(P,*) + N(*,Q) \geqslant \mathcal{X}(P,Q) \tag{3.2.5}$$

expressing that the newly introduced distances do not violate the triangular inequality via $*$, enabling us to consider a new space $\mathcal{X}_*$, with an added point $*$. Finally, the unit of the adjunction[1]

$$0 \geqslant \inf_{P\in\mathcal{X}} (N(*,P) + M(P,*)) \tag{3.2.6}$$

forces the newly adjoined point to have zero distance from (and to) the rest of the space, providing a Cauchy condition analogous to the one for Cauchy sequences. This motivates general definitions A.2.1 of Cauchy modules, and of Cauchy completeness of enriched categories A.2.3.

An important base is the monoidal category Ab of Abelian groups, where one-object Ab-categories are rings, and they are Morita equivalent (have equivalent categories of (left) modules) if and only if their Cauchy completions are equivalent [19]. We review definitions and some results related to general Cauchy completeness in Appendix A.2.

3.3 Causal spaces as enriched categories

Given a space-time E one can assign to each time-like path p in E its proper time $T(p)$. Maximizing the proper time $T(p)$ over all time-like paths between two events gives an interval

[1]The coend involved in the module composition reduces to inf when the base of enrichment is $\mathcal{R}$.

or “distance” between them. This is not distance in the sense of a metric space, mainly because the triangle inequality is inverted. The maximal time will usually (in physical situations) correspond to time measured by an inertial observer, while any accelerated reference frame would measure a shorter time, with a photon bouncing from appropriately set up mirrors would “measure” a zero time. However, we used maximizing over all time-like paths, rather than an inertial path, because of possible existence of Lorentzian manifolds where there are causally related points which do not have a (unique) inertial path between them. This is analogous to minimizing path length over all paths on a Riemannian manifold to obtain a metric; for example, antipodal points on a sphere have multiple shortest paths, or two points in a plane on the opposite side of a cut out (closed) disc have no path with a minimal length between them.

To get the inverted triangular inequality one could just invert the arrows of $\mathcal{R}$. On one hand, such a category could no longer be closed because the object 0 would be the monoidal identity and the initial object at the same time, which would mean that tensoring (summing) does not preserve colimits (in particular, the initial object), since, for example

$$1 = 1 + 0 \neq 0\,. \tag{3.3.1}$$

On the other hand, physically, there would be no object in the monoidal category that could be assigned to space-like separated events. Both of the problems are solved by freely adding an initial object which we denote by $\perp$. So, the correct base for enrichment is formally given by

Definition 3.3.1. *A symmetric closed monoidal category $\mathcal{R}_\perp$ is defined to have*

- *objects the real positive numbers $[0, \infty)$ with infinity ∞ and the additional object $\perp$*

- *arrows $a \to b$ existing uniquely if $a = \perp$, $b = \infty$ or $a \leqslant b$, forming a total order*

- *tensor product* $+ : \mathcal{R}_\perp \times \mathcal{R}_\perp \to \mathcal{R}_\perp$ *given by*

$$\begin{array}{c|c|c|c} + & \perp & b & \infty \\ \hline \perp & \perp & \perp & \perp \\ \hline a & \perp & a+b & \infty \\ \hline \infty & \perp & \infty & \infty \end{array} \tag{3.3.2}$$

- *internal hom* $- : \mathcal{R}_\perp{}^{\mathrm{op}} \times \mathcal{R}_\perp \to \mathcal{R}_\perp$ *given by*

$$\begin{array}{c|c|c|c} - & \perp & a & \infty \\ \hline \perp & \infty & \perp & \perp \\ \hline b & \infty & \begin{cases} b-a, & a \leqslant b \\ \perp, & a > b \end{cases} & \perp \\ \hline \infty & \infty & \infty & \infty \end{array} \tag{3.3.3}$$

With this direction of arrows, all the colimits are suprema, and limits are infima.

A category $\mathcal{E}$ enriched in $\mathcal{R}_\perp$ has objects X, Y, ... interpreted as events, and homs $\mathcal{E}(X,Y) \in \mathcal{R}_\perp$ interpreted as "distances" or intervals. If $\mathcal{E}(X,Y) = \perp$ then Y is not in the future of X, equivalently said, X cannot cause Y. The composition of homs witnesses that the chosen time between the two events is the largest,

$$\mathcal{E}(X,Y) + \mathcal{E}(Y,Z) \leqslant \mathcal{E}(X,Z) \tag{3.3.4}$$

and the unit

$$0 \leqslant \mathcal{E}(X,X) \tag{3.3.5}$$

prevents endohoms from being $\perp$. The associativity and unit axioms are trivially satisfied because $\mathcal{R}_\perp$ is a poset.

Example 3.3.1. *In a Minkowski 2D space-time objects are points in* $(t,x) \in \mathbb{R}^2$ *and homs are*

$$\mathcal{E}((t,x),(t',x')) = \begin{cases} \sqrt{(t'-t)^2 - (x'-x)^2}, & \textit{if } t'-t \geqslant |x'-x| \\ \perp, & \text{otherwise}. \end{cases} \tag{3.3.6}$$

Proposition 3.3.1. *Properties of homs of $\mathcal{E}$ include*

(i). endohoms are monoidal idempotents

$$\mathcal{E}(X,X) + \mathcal{E}(X,X) = \mathcal{E}(X,X) \tag{3.3.7}$$

(ii). the action of endohoms on other homs is given by equalities

$$\mathcal{E}(Y,X) + \mathcal{E}(X,X) = \mathcal{E}(Y,X) \tag{3.3.8}$$

$$\mathcal{E}(X,X) + \mathcal{E}(X,Y) = \mathcal{E}(X,Y) \tag{3.3.9}$$

(iii). possible endohoms are

$$\mathcal{E}(X,X) = 0 \text{ or } \mathcal{E}(X,X) = \infty \tag{3.3.10}$$

(a) if $\mathcal{E}(X,X) = \infty$, all the homs $\mathcal{E}(Y,X)$ and $\mathcal{E}(X,Y)$ are either $\perp$ or ∞

(b) if $\mathcal{E}(X,X) = 0$, either both $\mathcal{E}(X,Y)$ and $\mathcal{E}(Y,X)$ equal 0 or at least one equals $\perp$

Proof. (i). Adding $\mathcal{E}(X,X)$ to the unit (3.3.5) gives

$$\mathcal{E}(X,X) \leqslant \mathcal{E}(X,X) + \mathcal{E}(X,X) \tag{3.3.11}$$

On the other hand, the composition (3.3.4) for Y and Z equal X gives

$$\mathcal{E}(X,X) + \mathcal{E}(X,X) \leqslant \mathcal{E}(X,X) \tag{3.3.12}$$

(ii). Adding $\mathcal{E}(X,Y)$ to the unit, and the compositions

$$\mathcal{E}(X,Y) + \mathcal{E}(X,X) \leqslant \mathcal{E}(X,Y) \tag{3.3.13}$$

$$\mathcal{E}(X,X) + \mathcal{E}(Y,X) \leqslant \mathcal{E}(Y,X) \tag{3.3.14}$$

give the required result.

(iii). By part (i) of the proposition, noting that objects $\perp$, 0 and ∞ are the only monoidal idempotents in $\mathcal{R}_\perp$, and using the unit (3.3.5), restricts possible endohoms to 0 and ∞.

(a) Case analysis on (3.3.8)-(3.3.9)

(b) Case analysis on $\mathcal{E}(Y,X) + \mathcal{E}(X,Y) \leqslant \mathcal{E}(X,X) = 0$

□

Call an event with the infinite endohom (situation (3a)) *irregular*. Although unphysical, these are needed to keep $\mathcal{R}_\perp$ closed. For instance, $\perp$ is irregular, since $\perp - \perp = \infty$. However, part (3a) of Proposition 3.3.1 ensures that such points in space are either causally unrelated to, or at an infinite temporal distance from, the rest of the (physical) space. Part (3b) of Proposition 3.3.1 prevents the grandfather paradox in the physical part of the space - given two regular (endohom being 0) events X and Y, it is not possible for both of them to cause each other, unless they happen simultaneously.

A program for formulating quantum gravity using discrete partial orders, started in [7] and reviewed in, for example, [12], has a notion of *causal set* as a basic mathematical structure. If we take the underlying category $\mathcal{E}_0$ of a causal space $\mathcal{E}$, we get a general preordered set without requirements for antisymmetry and local finiteness - the information about local time-like intervals is contained in homs, and allows different events to happen at the same point in space-time. On the other hand, each causal set has a corresponding causal space, where homs come from the local finiteness condition - if A causes B, then $\mathcal{E}(A,B)$ is the (integer) length of the longest (necessarily finite) path between A and B.

3.3.1 Enrichment in $[-\infty, \infty]$

A possible generalization of both metric and event spaces, would be enrichment in $[-\infty, \infty]$, with an arrows from A to B, if $B \leqslant A$. Then positive length would denote space-like intervals, with triangle inequality (3.2.2), while negative numbers would be interpreted as time-like intervals. However, the triangle inequality with mixed entries is too restrictive, so the Minkowski 2D space-time is not enriched in $[-\infty, \infty]$. For example,

$$A = (0,0) \tag{3.3.15}$$

$$B = (-1,0) \tag{3.3.16}$$

$$C = (0,1) \tag{3.3.17}$$

gives

$$\mathcal{E}(A,B) + \mathcal{E}(B,C) = -1 + 0 = -1 \tag{3.3.18}$$

$$\mathcal{E}(A,C) = 1\,. \tag{3.3.19}$$

3.3.2 $\mathcal{R}_\perp$-Cat

An $\mathcal{R}_\perp$-functor $F : \mathcal{D} \to \mathcal{E}$ maps events in $\mathcal{D}$ to events in $\mathcal{E}$ such that the distances increase

$$\mathcal{D}(A,B) \leqslant \mathcal{E}(FA,FB)\,. \tag{3.3.20}$$

In particular, space-like intervals (given by $\perp$) can map to time-like intervals.

Natural transformations $\eta : F \to G$ indicate that for all $A \in D$ the event GA is in the future of FA.

Since $\mathcal{R}_\perp$ is symmetric, closed and (co)complete, so is $\mathcal{R}_\perp$-Cat [19]. Explicitly, the tensor product $\mathcal{D} + \mathcal{E}$ of $\mathcal{D}$ and $\mathcal{E}$ has

- objects pairs (A,X)
- homs $(\mathcal{D}+\mathcal{E})((A,X),(B,Y)) = \mathcal{D}(A,B) + \mathcal{E}(X,Y)$

and $[\mathcal{D},\mathcal{E}]$ has

- objects $\mathcal{R}_\perp$-functors F, G...
- homs
$$[\mathcal{D},\mathcal{E}](F,G) = \int_{A\in\mathcal{D}} \mathcal{E}(FA,GA) = \inf_{A\in\mathcal{D}} \mathcal{E}(FA,GA)\,. \tag{3.3.21}$$

Finally, given a causal space $\mathcal{E}$, using symmetry of $\mathcal{R}_\perp$ we can form the opposite $\mathcal{E}^{\mathrm{op}}$ by taking the same set of objects and

$$\mathcal{E}^{\mathrm{op}}(X,Y) = \mathcal{E}(Y,X) \tag{3.3.22}$$

for homs.

3.3.3 Modules, black holes and wormholes

A (2-sided) module $M : \mathcal{D} \twoheadrightarrow \mathcal{E}$ is defined as an $\mathcal{R}_\perp$-functor

$$M : \mathcal{E}^{\mathrm{op}} + \mathcal{D} \to \mathcal{R}_\perp \tag{3.3.23}$$

and can be equivalently given by actions

$$\mathcal{E}(Y,X) + M(X,A) \leqslant M(Y,A) \tag{3.3.24}$$

$$M(X,A) + \mathcal{D}(A,B) \leqslant M(X,B)\,. \tag{3.3.25}$$

These inequalities enable us to "glue" the two causal spaces with homs between objects of $\mathcal{E}$ and $\mathcal{D}$ given by M, and all homs from $\mathcal{D}$ to $\mathcal{E}$ being $\perp$, a process known as a lax colimit or collage [33].

Remark 3.3.1. *Physically, such a module can be interpreted as a wormhole going from $\mathcal{E}$ to $\mathcal{D}$. In particular, when $\mathcal{D} = \mathcal{I}$ the module M is a black hole in $\mathcal{E}$.*

Composition of modules $N : \mathcal{C} \twoheadrightarrow \mathcal{D}$ and $M : \mathcal{D} \twoheadrightarrow \mathcal{E}$ is given by

$$(M \circ N)(X,P) = \int^{A \in \mathcal{D}} M(X,A) + N(A,P) \tag{3.3.26}$$

$$= \sup_{A \in \mathcal{D}} (M(X,A) + N(A,P)) \tag{3.3.27}$$

for all $P \in \mathcal{C}$ and $X \in \mathcal{E}$.

3.3.4 Cauchy completeness

To give a pair of adjoined modules $(M \dashv N) : \mathcal{I} \twoheadrightarrow \mathcal{E}$ is the same as to give a pair of $\mathcal{R}_\perp$-functors

$$M : \mathcal{E}^{\mathrm{op}} \to \mathcal{R}_\perp \tag{3.3.28}$$

$$N : \mathcal{E} \to \mathcal{R}_\perp \tag{3.3.29}$$

which, in addition to the actions (3.3.24)-(3.3.25)

$$\mathcal{E}(Y,X) + M(X) \leqslant M(Y) \tag{3.3.30}$$

$$N(X) + \mathcal{E}(X,Y) \leqslant N(Y) \tag{3.3.31}$$

satisfy (existence of the unit and counit of the adjunction)

$$0 \leqslant \sup_X (N(X) + M(X)) \tag{3.3.32}$$

$$\mathcal{E}(X,Y) \geqslant M(X) + N(Y)\,. \tag{3.3.33}$$

Proposition 3.3.2. *Any $\mathcal{R}_\perp$ enriched category $\mathcal{E}$ is Cauchy complete.*

Proof. First, consider the case when $\mathcal{E}$ is empty. Then M and N are unique empty functors, but they cannot be adjoint as the RHS of (3.3.32) equals $\perp$. Since there are no Cauchy modules, $\mathcal{E}$ is Cauchy complete.

Now, assume $\mathcal{E}$ is non-empty and M is a Cauchy module, that is there is N such that (3.3.30)-(3.3.33) hold. In particular, since $\perp$ is the only element smaller than 0, equation (3.3.32) implies that there is $Z \in \mathcal{E}$ such that

$$0 \leqslant N(Z) + M(Z)\,. \tag{3.3.34}$$

If either $N(Z)$ or $M(Z)$ was equal to $\perp$ the sum would equal $\perp$ as well, so we have that both terms are greater or equal than 0,

$$0 \leqslant N(Z) \;\text{ and }\; 0 \leqslant M(Z)\,. \tag{3.3.35}$$

Now we have

$$M(Y) \leqslant M(Y) + N(Z) \tag{3.3.36}$$

$$\leqslant \mathcal{E}(Y,Z) \tag{3.3.37}$$

$$\leqslant \mathcal{E}(Y,Z) + M(Z) \tag{3.3.38}$$

$$\leqslant M(Y) \tag{3.3.39}$$

proving that $M(Y) = \mathcal{E}(Y,Z)$, and showing that Z represents M. $\square$

3.4 Cauchy completeness via idempotent splitting

Here we consider which monoidal categories $\mathcal{V}$ produce enriched categories whose Cauchy completeness is determined by idempotent splitting in the corresponding underlying category. We begin with an easy direction.

Proposition 3.4.1. *Let $\mathcal{V}$ be a locally small, cocomplete symmetric monoidal closed category. If a small $\mathcal{V}$-category $\mathcal{E}$ is Cauchy complete then idempotents split in the underlying category*[2] *$\mathcal{E}_0$.*

Proof. Let $I \xrightarrow{e} \mathcal{E}(E, E)$ be an idempotent in $\mathcal{E}_0$. Let $E_* : \mathcal{I} \nrightarrow \mathcal{E}$ and $E^* : \mathcal{E} \nrightarrow \mathcal{I}$ denote the modules induced by the $\mathcal{V}$-functor picking the object E. That is

$$E_*(X) = \mathcal{E}(X, E) \tag{3.4.1}$$

$$E^*(X) = \mathcal{E}(E, X) \tag{3.4.2}$$

with actions given by composition in $\mathcal{E}$. The induced module endomorphisms $e_* : E_* \Rightarrow E_*$ and $e^* : E^* \Rightarrow E^*$ are idempotent because e is. Since in the corresponding presheaf category idempotents split, there is a module $M : \mathcal{I} \nrightarrow \mathcal{E}$, and module morphisms $f : E_* \Rightarrow M$, $g : M \Rightarrow E_*$ splitting e_*. Similarly, there is a module $N : \mathcal{E} \nrightarrow \mathcal{I}$, and module morphisms $k : E^* \Rightarrow N$, $l : N \Rightarrow E^*$ splitting e^*. Using the fact that e^* and e_* are mates under the adjunction $E_* \dashv E^*$, it is easy to show that[3] $(k \otimes f) \circ \eta$ and $\epsilon \circ (g \otimes l)$ are unit and a counit of the adjunction $M \dashv N$. Since $\mathcal{E}$ is Cauchy complete, M is represented by an object, say $D \in \mathcal{E}$, and so, using the weak Yoneda lemma, e splits through it. □

Proposition 3.4.2. *Consider the following properties of a cocomplete, locally small, symmetric monoidal closed category $\mathcal{V}$:*

(i) the underlying functor

$$\mathcal{V}(I, -) : \mathcal{V} \to \mathbf{Set} \tag{3.4.3}$$

takes regular epi families to epi families (joint surjections),

[2]The underlying (Set-enriched) category [19] of a $\mathcal{V}$-category $\mathcal{E}$ has the same objects as $\mathcal{E}$, and homsets defined by mapping out of the unit of $\mathcal{V}$.

[3]Here $\otimes$ denotes the horizontal composition, and $\circ$ the vertical composition of module morphisms.

(ii) the function

$$\mathcal{V}(I,A) \times \mathcal{V}(I,B) \xrightarrow{\otimes} \mathcal{V}(I \otimes I, A \otimes B) \xrightarrow{-\circ\rho_I} \mathcal{V}(I, A \otimes B) \tag{3.4.4}$$

is a bijection,

then a small $\mathcal{V}$-category $\mathcal{E}$ is Cauchy complete if idempotents split in the underlying category $\mathcal{E}_0$.

Proof. Let $M : \mathcal{I} \nrightarrow \mathcal{E}$ be a Cauchy module with a right adjoint N which amounts to giving actions

$$M(X) \otimes \mathcal{E}(Y,X) \xrightarrow{\alpha_{Y,X}} M(Y) \tag{3.4.5}$$

$$\mathcal{E}(X,Y) \otimes N(X) \xrightarrow{\beta_{X,Y}} N(Y) \tag{3.4.6}$$

compatible with unit and composition in $\mathcal{E}$, and unit and counit for the adjunction

$$\eta : I \to \int^{Y} M(Y) \otimes N(Y)\,, \tag{3.4.7}$$

$$\epsilon_{X,Y} : N(Y) \otimes M(X) \to \mathcal{E}(X,Y)\,. \tag{3.4.8}$$

The coend cowedge components

$$M(X) \otimes N(X) \xrightarrow{w_X} \int^{Y} M(Y) \otimes N(Y) \tag{3.4.9}$$

form a jointly regular epic family, see section A.3 example A.3.3. By condition (i), the functor $\mathcal{V}(I,-)$ takes them to a jointly surjective family of functions $\mathcal{V}(I, w_X)$. This in particular means that the unit of the adjunction is in the image of a function $\mathcal{V}(I, w_Z)$, for some Z. So, the unit decomposes as $\eta = w_Z \circ z$. From condition (ii) we get that z can be further decomposed as $m \otimes n$ for a unique pair of maps $m : I \to M(Z)$ and $n : I \to N(Z)$, to give a final decomposition of the unit

$$\eta = w_Z \circ (m \otimes n) \tag{3.4.10}$$

One of the adjunction axioms, together with (3.4.10) gives a commutative diagram shown in (3.4.11).

$$
\begin{tikzcd}
M(Y) \arrow[d, "\cong"'] \arrow[dddd, bend right=60, "1"'] & \\
I \otimes M(Y) \arrow[d, "\eta \otimes 1"'] \arrow[dr, "m \otimes n \otimes 1"] & \\
\int^C M(C) \otimes N(C) \otimes M(Y) \arrow[d, "\int^C 1 \otimes \epsilon_{Y,C}"'] & M(Z) \otimes N(Z) \otimes M(Y) \arrow[l, "w_Z \otimes 1"'] \arrow[d, "1 \otimes \epsilon_{Y,Z}"] \\
\int^C M(C) \otimes \mathcal{E}(Y,C) \arrow[d, "\cong"'] & M(Z) \otimes \mathcal{E}(Y,Z) \arrow[l, "w_Z"'] \\
M(Y) &
\end{tikzcd}
\tag{3.4.11}
$$

From the outside of the diagram (3.4.11) it follows that the identity on $M(Y)$ decomposes into the following two maps

$$M(Y) \xrightarrow{n \otimes 1} N(Z) \otimes M(Y) \xrightarrow{\epsilon_{Y,Z}} \mathcal{E}(Y,Z) \tag{3.4.12}$$

$$\mathcal{E}(Y,Z) \xrightarrow{m \otimes 1} M(Z) \otimes \mathcal{E}(Y,Z) \xrightarrow{\alpha_{Y,Z}} M(Y)\,. \tag{3.4.13}$$

Both of these sets of arrows are $\mathcal{V}$-natural in Y, following from $\mathcal{V}$-naturality of ϵ and compatibility of action α with composition in $\mathcal{E}$. Composing them the other way around we get an idempotent $\mathcal{V}$-natural transformation on $\mathcal{E}(-,Z)$, which is represented by an idempotent arrow $Z \xrightarrow{e} Z$ in $\mathcal{E}_0$. Since idempotents split, there is Z' through which e splits, hence Z' is a representing object for M. $\square$

Remark 3.4.1. *The only place we used symmetry and closedness of $\mathcal{V}$ was the definition of module compositions using coends, and the definition of the category of enriched presheaves. Both of these notions are definable for non-symmetric $\mathcal{V}$, or even when the base of enrichment is a bicategory [34], so we expect the above theorems to work at that level of generality as well.*

Corollary 3.4.1. *A cocomplete quantale $\mathcal{Q}$ such that any collection of its objects $\{A_i\}$ with an arrow*

$$I \to \bigvee_i A_i \tag{3.4.14}$$

contains an object $Z \in \{A_i\}$ *with an arrow*

$$I \to Z \tag{3.4.15}$$

has the property that all small $\mathcal{Q}$*-categories* $\mathcal{E}$ *are Cauchy complete.*

Example 3.4.1. *The motivating example* $\mathcal{R}_\perp$ *has this property.*

Corollary 3.4.2. *If a cocomplete category* $\mathcal{V}$ *is Cartesian closed and*

$$\mathcal{V}(1, -) : \mathcal{V} \to \mathrm{Set} \tag{3.4.16}$$

has a right adjoint, then $\mathcal{V}$ *satisfies the requirements of proposition 3.4.2.*

Denoting by G the right adjoint we need a (natural) bijection

$$\mathcal{V}(A, GS) \cong \mathbf{Set}(\mathcal{V}(I, A), S)\,. \tag{3.4.17}$$

Example 3.4.2. *For* $\mathcal{V} = \mathrm{Set}$, $G = 1_{\mathrm{Set}}$. *More generally, if* $\mathcal{V} = [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ *and* $\mathcal{C}$ *has a terminal object* 1 *then*

$$(GS)C = \mathrm{Set}(\mathcal{C}(1, C), S) \tag{3.4.18}$$

functorially in C. *The isomorphism (3.4.17) follows from*

$$\begin{aligned}
&[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}](A, \mathrm{Set}(\mathcal{C}(1, -), S)) && (3.4.19)\\
&\cong \int_{C \in \mathcal{C}} \mathrm{Set}(AC, \mathrm{Set}(\mathcal{C}(1, C), S)) && (3.4.20)\\
&\cong \mathrm{Set}\left(\int^{C \in \mathcal{C}} AC \times \mathcal{C}(1, C), S\right) && (3.4.21)\\
&\cong \mathrm{Set}(A1, S) && (3.4.22)\\
&\cong \mathrm{Set}\left([\mathcal{C}^{\mathrm{op}}, \mathrm{Set}](\mathcal{C}(-, 1), A), S\right) && (3.4.23)\\
&\cong \mathrm{Set}\left([\mathcal{C}^{\mathrm{op}}, \mathrm{Set}](1, A), S\right) && (3.4.24)
\end{aligned}$$

where 1 *in the last line denotes the terminal presheaf which is the monoidal unit in* $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$.

Example 3.4.3. *For $\mathcal{V}$ = Cat, GS is the chaotic category on the set S, because mapping into it is uniquely determined by the assignment on objects. More generally, for $\mathcal{V} = n$-Cat, GS is a the chaotic category seen as a locally discrete n-category (each hom is the terminal $(n-1)$-category).*

In some cases condition (ii) holds when the product is not Cartesian.

Example 3.4.4. $\mathrm{Gray}_{(l)}$ *has the same objects and arrows as* 2-Cat*, but (lax) Gray tensor product, rather than the Cartesian one for the monoidal structure. Strict functors* $1 \to \mathcal{A} \otimes_{(l)} \mathcal{B}$ *detect (pick) objects, which are pairs consisting of an object in $\mathcal{A}$ and an object in $\mathcal{B}$, hence satisfying condition (ii).*

Proposition 3.4.3. *Let $\mathcal{V}$ be a monoidal category. The following are equivalent:*

(i). every $\mathcal{V}$-category $\mathcal{C}$ has a Cauchy complete underlying category $\mathcal{C}_0$,

(ii). every monoid (T, μ, η) in $\mathcal{V}$ induces an idempotent-splitting monoid on the hom-set $\mathcal{V}(I, T)$.

Proof. $(1 \Rightarrow 2)$ Consider a one-object category $\mathcal{C}$ with the endohom, multiplication and unit given by (T, μ, η). The underlying category is precisely the suspension of the monoid $\mathcal{V}(I, T)$, so idempotent-splitting in $\mathcal{C}_0$ is the same as idempotent-splitting in $\mathcal{V}(I, T)$.
$(2 \Rightarrow 1)$ Let $I \xrightarrow{e} \mathcal{C}(A, A)$ be an idempotent in $\mathcal{C}_0$. Since $\mathcal{C}(A, A)$ is a monoid in $\mathcal{V}$, e is also an idempotent in the induced monoid on $\mathcal{V}(I, \mathcal{C}(A, A))$, and, by condition 2, it splits. □

Remark 3.4.2. *Under condition 2, all idempotents in $\mathcal{C}_0$ split through the same object they live on. As a consequence, if an array of maps composes to the identity on an object A, then all intermediate objects are isomorphic to A.*

Corollary 3.4.3. *A monoidal category $\mathcal{V}$ satisfying conditions of the proposition 3.4.2, and the second of 3.4.3, has all small $\mathcal{V}$-categories Cauchy complete.*

4

Comonadic base change

4.1 Introduction

Characterising Cauchy completeness of differential graded (DG) categories provided motivation for this chapter. We showed that a DGAb-module is Cauchy if and only if its underlying GAb-module (or Ab-module) is: an enriched module is Cauchy (by definition) if it has a right adjoint in the bicategory of modules, adjoints can be expressed using Kan extensions, and our main theorem states that underlying 2-functors for certain comonads create Kan extensions (Theorem 4.5.1).

Chain complexes of abelian groups (also called differential graded Abelian groups) form a symmetric monoidal closed category DGAb (explained in detail in Section 4.2) which can be obtained as a category of coalgebras for a Hopf ring[1] in Ab, the symmetric monoidal closed category of abelian groups [28]. DGAb has a full symmetric monoidal closed subcategory GAb, consisting of graded abelian groups seen as complexes with trivial differential, which can also be obtained as a category of coalgebras of a different Hopf ring in Ab.

In Section 4.3 we consider Hopf monoids A in an arbitrary symmetric monoidal category $\mathcal{V}$. We need a braiding in the category of (co)algebras in order to consider Hopf monoids H there, which is obtained using a braiding (co)element [16], and symmetry in $\mathcal{V}$. Their semidirect product $H \rtimes A$, called bosonization in [26], is also a Hopf monoid, and its category of (co)algebras is isomorphic to the category of H-(co)algebras. A particular example, when $\mathcal{V}$ is additive, resembles the $\mathcal{V} = \text{Ab}$ case, and Pareigis' ring is an instance of it, see Section 4.4.

Semidirect product works even when $\mathcal{V}$ has no braiding if we consider Hopf monoidal comonads [10], or dually Hopf opmonoidal monads [8], not necessarily induced by tensoring with a Hopf monoid. These are comonads in the 2-category of monoidal categories, monoidal functors and monoidal natural transformations, in the sense of [29], with a Hopf condition: the fusion maps are invertible. In order to study categories, and modules between them, enriched in a category of coalgebras, in Section 4.5, we generalize further, by considering comonads in Caten [20], whose objects are bicategories, arrows are categories enriched on 2-sides, and 2-cells are enriched functors. Then, a change of base along the forgetful functor $U : \mathcal{V}^{\mathcal{G}} \to \mathcal{V}$ induces comonadic arrows in Caten $\bar{U} : \mathcal{V}^{\mathcal{G}}\text{-Cat} \to \mathcal{V}\text{-Cat}$ and $\tilde{U} : \mathcal{V}^{\mathcal{G}}\text{-Mod} \to \mathcal{V}\text{-Mod}$. The main theorem (4.5.1) states that when a comonad in Caten is Hopf, the comonadic forgetful functor creates left Kan extensions. Left extensions (dually liftings) are a generalization of left (dually right) cohoms from monoidal to bicategorical setting, so if $\mathcal{V}$ is (a suspension of) a monoidal category, U creates cohoms and duals. Adjunctions in a bicategory can also be expressed using left extensions, so $\tilde{U}$ creates Cauchy modules.

[1] The word "ring" will denote a monoid in an additive monoidal category. We save the word "(co)algebra" for (co)algebras for a (co)monad.

4.2 (Differential) graded abelian groups

The category DGAb (differential graded abelian groups) has chain complexes A, B, as objects. They are defined by diagrams in Ab

$$\dots \xrightarrow{d} A_{n+1} \xrightarrow{d} A_n \xrightarrow{d} A_{n-1} \xrightarrow{d} \dots \tag{4.2.1}$$

with group homomorphisms d (full notation would be d_n^A) satisfying $d \circ d = 0$. An arrow $f : A \to B$, called chain map, consists of group homomorphisms $f_n : A_n \to B_n$, indexed by integers, satisfying

$$f_n \circ d = d \circ f_{n-1} \,. \tag{4.2.2}$$

DGAb is monoidal with tensor product defined by

$$(A \otimes B)_n = \Sigma_{i+j=n} A_i \otimes B_j \tag{4.2.3}$$

$$d(a \otimes b) = da \otimes b + (-1)^i a \otimes db, \text{ for } a \in A_i \text{ and } b \in B_j \,. \tag{4.2.4}$$

The unit is given by $I_n = \delta_{n0}\mathbb{Z}$. There is a symmetry

$$\sigma(a \otimes b) = (-1)^{ij} b \otimes a \tag{4.2.5}$$

and a closed structure

$$[B, C]_n = \prod_j \mathrm{Ab}(B_j, C_{j+n}) \tag{4.2.6}$$

$$(df)_j b = d(f_j(b)) - (-1)^n f_{j-1}(db), \text{ for } f \in [B, C]_n \text{ and } b \in B_j \,. \tag{4.2.7}$$

The category DGAb can be obtained as the EM-category of P-coalgebras, for a Hopf ring P [28]

$$P = \mathbb{Z}\langle \xi, \xi^{-1}, \psi \rangle / (\xi\psi + \psi\xi, \psi^2) \tag{4.2.8}$$

$$\Delta(\xi) = \xi \otimes \xi, \ \epsilon(\xi) = 1 \tag{4.2.9}$$

$$\Delta(\psi) = \psi \otimes 1 + \xi^{-1} \otimes \psi, \ \epsilon(\psi) = 0 \tag{4.2.10}$$

$$s(\xi) = \xi^{-1} \tag{4.2.11}$$

$$s(\psi) = \psi\xi \tag{4.2.12}$$

where Δ is the comultiplication, s is the antipode, and corner brackets denote non-commutativity.

4.2.1 GAb

The category of graded abelian groups, denoted GAb, can be seen as a full subcategory of DGAb consisting of chain complexes with all $d = 0$. GAb inherits symmetric monoidal closed structure, which follows from (4.2.4) and (4.2.7). On the other hand, there is a forgetful functor $U : \mathrm{DGAb} \to \mathrm{GAb}$, with adjoints $L \dashv U \dashv R$ given by

$$L(C)_n = C_{n+1} \oplus C_n \tag{4.2.13}$$

$$R(C)_n = C_n \oplus C_{n-1} \tag{4.2.14}$$

$$d = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{4.2.15}$$

U reflects isomorphisms, since f^{-1} satisfies (4.2.2) if and only if f does. The functor U, having both adjoints, preserves all limits and colimits, in particular U-split equalizers and coequalizers. Hence, U is both comonadic and monadic.

There is a functor $\Sigma : \mathrm{GAb} \to \mathrm{Ab}$ that takes the coproduct (sum) of all components. It has a right adjoint which creates $\mathbb{Z}$ copies of an abelian group. The diagram below summarises all relevant adjunctions.

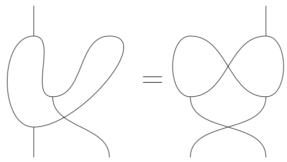
$$\begin{tikzcd} \mathrm{DGAb} \arrow[r, "U" description] & \mathrm{GAb} \arrow[l, bend right, "L"', "\perp"] \arrow[l, bend left, "R", "\perp"'] \arrow[r, "\Sigma"] & \mathrm{Ab} \arrow[l, bend left, "C", "\perp"'] \end{tikzcd} \tag{4.2.16}$$

Both $\Sigma \circ C$ and $R \circ U$ are comonads isomorphic to tensoring with a certain Hopf ring in Ab and GAb respectively. In section 4.3 we discuss the semidirect product construction in general, and then in section 4.4 we show that the Pareigis biring is the semidirect product of the two birings generating $\Sigma \circ C$ and $R \circ U$.

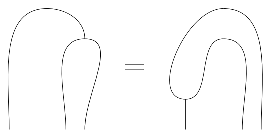
4.3 Semidirect product

Let $\mathcal{W} = \mathcal{V}^{A\otimes -}$ be the category of algebras for a (Hopf) bimonoid A in a symmetric monoidal (closed) $\mathcal{V}$. For $\mathcal{W}$ to be braided we need A to have a braiding element $\gamma : I \to A \otimes A$

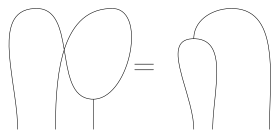
satisfying the three axioms at page 58 of [16], which we quote here in the form we are going to use later (half-turned compared to the ones in [16]; that is, we read from top to bottom):



(4.3.1)



(4.3.2)



(4.3.3)

Explicitly, the braiding of (X, α_X) and (Y, α_Y) is[2]

$$s_{XY} = XY \xrightarrow{\gamma 11} AAXY \xrightarrow{1\sigma 1} AXAY \xrightarrow{\alpha_X \alpha_Y} XY \xrightarrow{\sigma} YX \tag{4.3.4}$$

For an A-algebra (X, α_X) define

$$\tau_X = AX \xrightarrow{\delta 1} AAX \xrightarrow{1\sigma_{AX}} AXA \xrightarrow{\alpha_X 1} XA\,. \tag{4.3.5}$$

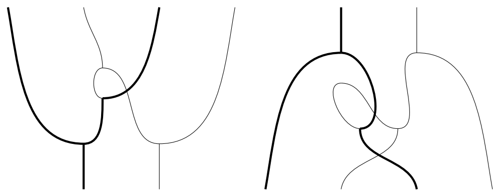
Proposition 4.3.1. *If X is a monoid in $\mathcal{W}$, then τ_X is a distributive law in $\mathcal{V}$.*

Proof. There are 4 axioms to check. The two involving unit and multiplication for A use the compatibility of unit with the comultiplication of A, and the bimonoid axiom, respectively. The two involving unit and multiplication for X follow from the fact that they are A-algebra morphisms. □

Let H be a (Hopf) bimonoid in $\mathcal{W}$. It automatically inherits a (co)monoid structure in $\mathcal{V}$ by forgetting that (co)unit and (co)multiplication maps are A-algebra morphisms. Note that, unless $\gamma = \eta \otimes \eta$, H need not be a bimonoid in $\mathcal{V}$.

[2]We sometimes omit writing $\otimes$.

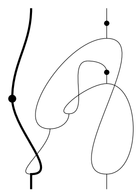
Definition 4.3.1. *The semidirect product of a bimonoid A in $\mathcal{V}$ and a bimonoid (H, α_H) in A-Mod, denoted $H \rtimes A$, is given by the object HA, with monoid structure given via the distributive law τ_H, and comonoid structure via the codistributive law s_{AH}. Using a thick line for H, thin line for A, the multiplication and the comultiplication of $H \rtimes A$ are given by the following string diagrams*



(4.3.6)

where all relevant arrows in $\mathcal{V}$ are uniquely determined by their source and target, so there is no need for labelling.

Proposition 4.3.2. *The semidirect product, $H \rtimes A$, is a bimonoid in $\mathcal{V}$. If H and A are Hopf, with antipodes graphically represented by dots, then so is $H \rtimes A$, with the antipode given by diagram (4.3.7).*

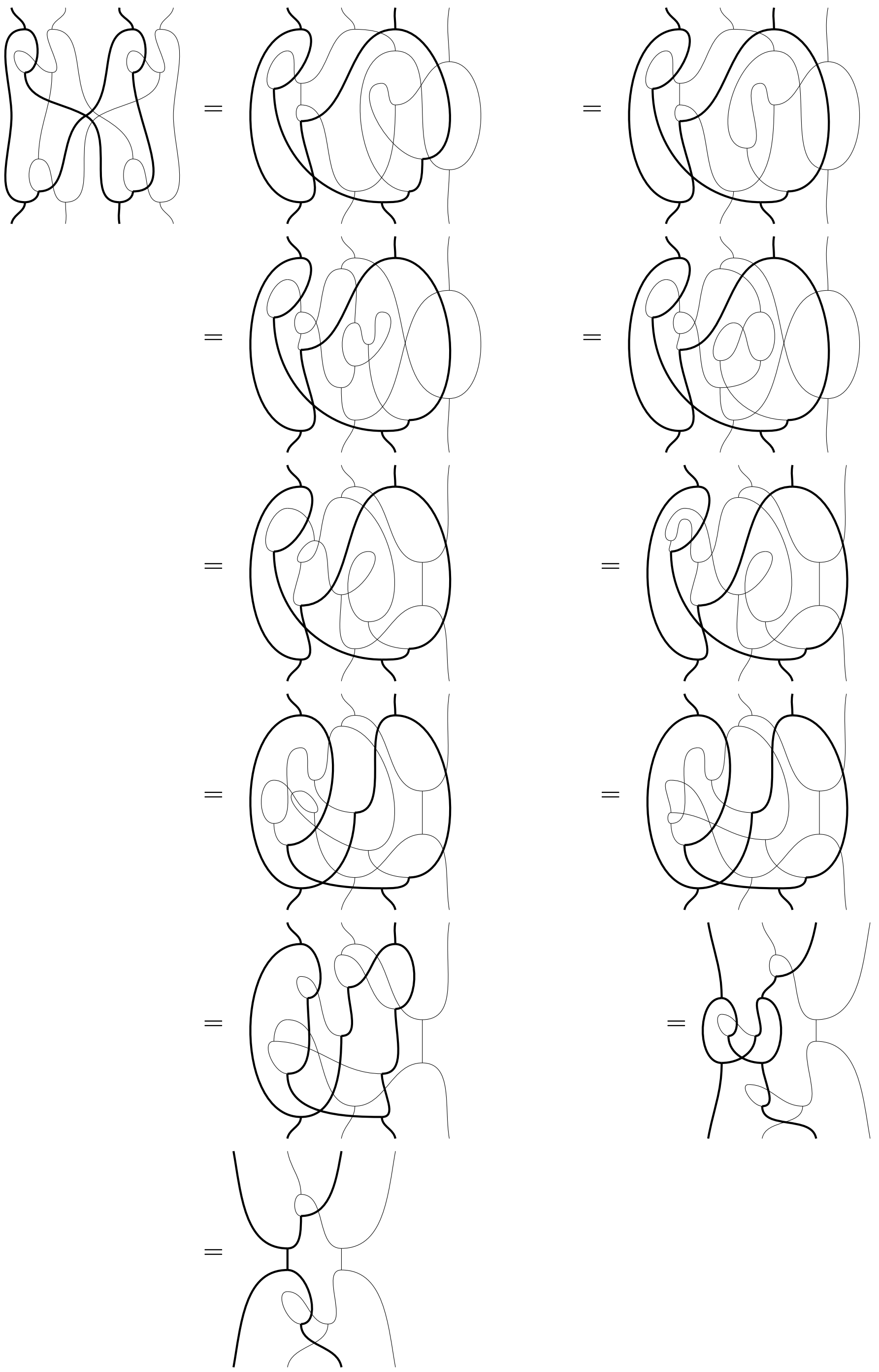


(4.3.7)

Proof. The defined (co)multiplication is already part of a (co)monoid structure. The compatibility of counit with unit, counit with multiplication and unit with comultiplication follows directly. What remains to show is the bimonoid axiom, which we have done using the manipulation of string diagrams shown in Figure 4.3 and described below.

In line (4.3.8), after rearrangement we used the compatibility of multiplication of A with action of A on H, in the bottom right corner of the middle diagram.

Going from line (4.3.8) to line (4.3.9) we used the bimonoid axiom for A on the top-left part of the diagram, followed by the (co)associativity for A. In the line (4.3.9) we used the element axiom (4.3.1). When passing from line (4.3.9) to line (4.3.10) we used the (co)associativity for A, together with the bimonoid axiom for A, in the right side of the diagram. In the line (4.3.10) we used the element axiom (4.3.2) on the top-left. Line (4.3.10) to (4.3.11) involves



(4.3.8)

(4.3.9)

(4.3.10)

(4.3.11)

(4.3.12)

(4.3.13)

Figure 4.1: Diagrams used in the proof of Proposition 4.3.2.

just a rearrangement, followed by the element axiom (4.3.3), on the left of the diagram, in line (4.3.11).

Passing from line (4.3.11) to line (4.3.12) uses the compatibility of multiplication of A with action of A on H at three different places. In line (4.3.12) we used that the (co)multiplication of H is an A-algebra morphism. Finally, going from line (4.3.12) to line (4.3.13) uses the bimonoid axiom for H.

That (4.3.7) is indeed an antipode follows in a similar way. The strategy to show the "right inverse" axiom is to use the compatibility of α_H with μ_A, and bimonoid axioms to get all comultiplications to the top, and multiplications to the bottom of the diagram, and then use the right inverse axiom for A multiple times, followed by the right inverse axiom for H. The strategy to show the "left inverse" axiom is to bring all actions α_H below μ_H, using the definition of action on the product of algebras, followed by the left inverse axiom for H, followed by the compatibility of ϵ_A with μ_A, and the left inverse axiom for A. □

Proposition 4.3.3. *The comparison functor*

$$\mathcal{W}^{H\otimes -} \xrightarrow{F} \mathcal{V}^{H\rtimes A\otimes -} \tag{4.3.14}$$

$$((B,\alpha_B),\chi_B) \mapsto (B, \chi_B \circ 1_H\alpha_B) \tag{4.3.15}$$

$$(f : B \to C) \mapsto (f : B \to C) \tag{4.3.16}$$

is strict monoidal and has a strict monoidal inverse

$$\mathcal{V}^{H\rtimes A\otimes -} \xrightarrow{F^{-1}} \mathcal{W}^{H\otimes -} \tag{4.3.17}$$

$$(B,\beta) \mapsto ((B, \beta\circ\eta_H 1_A \circ \lambda_A^{-1}), \beta \circ 1_H\eta_A \circ \rho_H^{-1}) \tag{4.3.18}$$

$$(f : B \to C) \mapsto (f : B \to C)\,. \tag{4.3.19}$$

Proof. Using Beck's monadicity theorem, we show that the forgetful functor

$$\mathcal{W}^{H\otimes -} \xrightarrow{\mathcal{U}} \mathcal{V} \tag{4.3.20}$$

$$((B,\alpha_B),\chi_B) \mapsto B \tag{4.3.21}$$

$$(f : B \to C) \mapsto (f : B \to C) \tag{4.3.22}$$

is monadic. Since $\mathcal{U}$ is the composite of two monadic functors, $\mathcal{U}_H$ and $\mathcal{U}_A$, it has a left adjoint and it reflects isomorphisms. The third criterion, not necessarily preserved by composition, is the existence and preservation of $\mathcal{U}$-split coequalizers. So, assume the parallel pair

$$f, g : ((B, \alpha_B), \chi_B) \to ((C, \alpha_C), \chi_C) \tag{4.3.23}$$

in $\mathcal{W}^{H\otimes -}$ has a split coequalizer $h : C \to E$ in $\mathcal{V}$. That is, there are maps

$$E \xrightarrow{s} C \xrightarrow{t} B \tag{4.3.24}$$

satisfying

$$h \circ s = 1_E \tag{4.3.25}$$

$$f \circ t = 1_C \tag{4.3.26}$$

$$s \circ h = g \circ t\,. \tag{4.3.27}$$

Monadicity of $\mathcal{U}_A$ implies that E is an A-algebra, with action

$$\alpha_E = (AE \xrightarrow{1s} AC \xrightarrow{\alpha_C} C \xrightarrow{h} E) \tag{4.3.28}$$

and that h is a coequalizer in $\mathcal{W}$, but not necessarily split. The proof involves the following identities

$$h \circ \alpha_C \circ 1s \circ 1h = h \circ \alpha_C \circ 1g \circ 1t \tag{4.3.29}$$

$$= h \circ \alpha_C \circ 1f \circ 1t \tag{4.3.30}$$

$$= h \circ \alpha_C \tag{4.3.31}$$

where the first equality follows from (4.3.27), the second from $h \circ f = h \circ g$ and the fact that f and g are A-algebra morphisms, and the third comes from (4.3.26). Exactly the same equalities hold with A replaced by H, for the same reasons:

$$h \circ \chi_C \circ 1s \circ 1h = h \circ \chi_C \circ 1g \circ 1t \tag{4.3.32}$$

$$= h \circ \chi_C \circ 1f \circ 1t \tag{4.3.33}$$

$$= h \circ \chi_C\,. \tag{4.3.34}$$

Now, the map

$$\chi_E = (HE \xrightarrow{1s} HC \xrightarrow{\chi_C} C \xrightarrow{h} E) \tag{4.3.35}$$

is an A-algebra morphism

$$\begin{aligned}
\chi_E \circ \alpha_H \alpha_E \circ 1\sigma_{AH}1 \circ \delta_A 11 &\stackrel{\text{def.}}{=} h \circ \chi_C \circ 1s \circ 1h \circ \alpha_H \alpha_C \circ 1\sigma_{AH}1 \circ \delta_A 1s && (4.3.36)\\
&\stackrel{(4.3.34)}{=} h \circ \chi_C \circ \alpha_H \alpha_C \circ 1\sigma_{AH}1 \circ \delta_A 1s && (4.3.37)\\
&\stackrel{\chi_C \in \mathcal{W}}{=} h \circ \alpha_C \circ 1\chi_C \circ 11s && (4.3.38)\\
&\stackrel{(4.3.31)}{=} h \circ \alpha_C \circ 1s \circ 1h \circ 1\chi_C \circ 11s && (4.3.39)\\
&\stackrel{\text{def.}}{=} \alpha_E \circ 1\chi_E && (4.3.40)
\end{aligned}$$

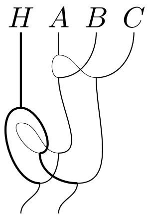
compatible with unit and multiplication on H, which follows from the compatibility of χ_C with unit and multiplication and equations (4.3.25) and (4.3.34). Therefore $((E, \alpha_E), \chi_E)$ is an object of $\mathcal{V}^{H\otimes -}$.

The arrow $h : C \to E$ is an H-algebra morphism, which follows directly from (4.3.34). To show that it coequalizes f and g, take $((X, \alpha_X), \chi_X)$ to be an H-algebra in $\mathcal{W}$ and $m : C \to X$ an H-algebra morphism satisfying $m \circ f = m \circ g$. In $\mathcal{V}$, $m \circ s : E \to X$ is the unique comparison map, since h is the coequalizer of f and g. But $m \circ s$ is an H-algebra

$$\begin{aligned}
m \circ s \circ \chi_E &\stackrel{\text{def.}}{=} m \circ s \circ h \circ \chi_C \circ 1s && (4.3.41)\\
&\stackrel{(4.3.27)}{=} m \circ g \circ t \circ \chi_C \circ 1s && (4.3.42)\\
&= m \circ f \circ t \circ \chi_C \circ 1s && (4.3.43)\\
&\stackrel{(4.3.26)}{=} m \circ \chi_C \circ 1s && (4.3.44)\\
&\stackrel{m \in \mathcal{V}^{H\otimes -}}{=} \chi_X \circ 1m \circ 1s && (4.3.45)
\end{aligned}$$

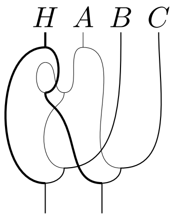
completing the proof that $\mathcal{U}$ is monadic.

The comparison functor F is strict monoidal: $F(((B, \alpha_B), \chi_B) \otimes ((C, \alpha_C), \chi_C))$ is given by the following action



(4.3.46)

while $F((B,\alpha_B),\chi_B)\otimes F((C,\alpha_C),\chi_C)$ is given by the following action



(4.3.47)

which are equal because α_B is compatible with multiplication on A. □

Example 4.3.1. *When $\mathcal{V} = (Set, \times)$, the comultiplication is forced to be the diagonal map, A is a monoid - with identity e_A, and the only possible braiding element is (e_A, e_A). An A-algebra bimonoid H is the same as a monoid morphism $\phi : A \to End(H)$, and $H \rtimes A$ is precisely the semidirect product for monoids generalizing the one for groups.*

4.4 Birings

In this section we consider a particular choice for bimonoids in a braided monoidal additive category with direct sums preserved by tensoring. Braiding, being a natural transformation between additive functors, is compatible with direct sums - for $H = A \oplus B$ and $H' = A' \oplus B'$

$$\sigma_{HH'} = \begin{bmatrix} \sigma_{AA'} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{BA'} & 0 \\ 0 & \sigma_{AB'} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{BB'} \end{bmatrix} \tag{4.4.1}$$

which can be concisely written by specifying non-zero components

$$\begin{tikzcd}[row sep=tiny] AA' \arrow[r, "\sigma_{AA'}"] & A'A \\ AB' \arrow[dr, "\sigma_{AB'}"] & A'B \\ BA' \arrow[ur, "\sigma_{BA'}"'] & B'A \\ BB' \arrow[r, "\sigma_{BB'}"'] & B'B \end{tikzcd} \tag{4.4.2}$$

where concatenation is the tensor product and vertical empty space is the direct sum.

4.4.1 Grading Hopf ring

Let $\mathcal{W}$ be a symmetric monoidal additive category with direct sums preserved by tensoring. We also assume $\mathcal{W}$ has countable coproducts, and denote by $\mathbb{Z} \cdot C$ the copower of the object $C \in \mathcal{W}$ by the set of integers $\mathbb{Z}$. In particular, there is an object

$$Z := \mathbb{Z} \cdot I\,. \tag{4.4.3}$$

The addition of integers gives $\mathbb{Z}$ a group (Hopf monoid) structure in $(\mathrm{Set}, \times)$, and induces a Hopf ring structure on Z, given by

$$I \cong \{*\} \cdot I \underset{0 \cdot I}{\overset{! \cdot I}{\leftrightarrows}} \mathbb{Z} \cdot I \underset{+ \cdot I}{\overset{\Delta \cdot I}{\rightleftarrows}} (\mathbb{Z} \times \mathbb{Z}) \cdot I \cong Z \otimes Z\,. \tag{4.4.4}$$

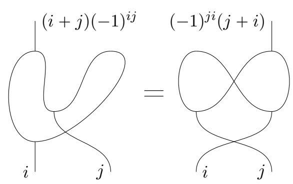
Tensoring with Z gives a functor isomorphic to taking a copower by $\mathbb{Z}$

$$\begin{aligned} Z \otimes C &= (\mathbb{Z} \cdot I) \otimes C && (4.4.5)\\ &\cong \mathbb{Z} \cdot (I \otimes C) && (4.4.6)\\ &\cong \mathbb{Z} \cdot C\,. && (4.4.7) \end{aligned}$$

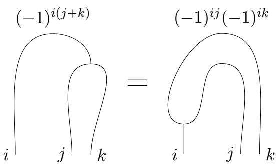
Since $\mathcal{W}$ is symmetric monoidal, the category Z-CoAlg of Z-coalgebras is monoidal, and with a braiding coelement given by

$$\begin{tikzcd} (\mathbb{Z} \times \mathbb{Z}) \cdot I \arrow[d, "\gamma"'] & I \arrow[l, "c_{ij}"'] \arrow[dl, "(-1)^{ij}"] \\ I & \end{tikzcd} \tag{4.4.8}$$

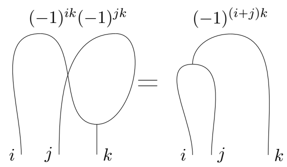
Z-CoAlg becomes braided. Arrows c_{ij} denote coproduct coprojections, and γ satisfies the coelement axioms, dual to the element axioms drown in (4.3.1)-(4.3.3). Because of duality, we need to invert the direction (read the diagrams from bottom to top):



(4.4.9)



(4.4.10)



(4.4.11)

Z-CoAlg inherits direct sums: if $B \xrightarrow{\beta} \mathbb{Z} \cdot B$ and $C \xrightarrow{\gamma} \mathbb{Z} \cdot C$ are Z- coalgebras, then

$$\begin{array}{c} B \xrightarrow{\ \ \beta\ \ } \mathbb{Z} \cdot B \\ C \xrightarrow{\ \ \gamma\ \ } \mathbb{Z} \cdot C \end{array} \tag{4.4.12}$$

is a Z-coalgebra as well, and the braiding induced from the cobraiding element γ is automatically compatible with direct sums.

Example 4.4.1. *When $\mathcal{W} = \mathrm{Ab}$, the biring $Z = \mathbb{Z}[x, x^{-1}]$ is the Laurent polynomial ring with integer coefficients. The coring structure is given by $1 \leftarrow\!\shortmid x \mapsto x \otimes x$. Then*

$$\mathrm{GAb} \to Z\text{-CoAlg} \tag{4.4.13}$$

$$C \mapsto \Sigma C_n \xrightarrow{\xi} Z \otimes \Sigma C_n \tag{4.4.14}$$

$$c \in C_n \mapsto x^n \otimes c \tag{4.4.15}$$

is an equivalence of categories. Consider a Z-coalgebra

$$B \xrightarrow{\beta} Z \otimes B \tag{4.4.16}$$

$$b \mapsto \Sigma_i x^i \otimes \beta_i^{(b)} \tag{4.4.17}$$

β being a group homomorphism ensures that

$$\beta_i^{(b)} + \beta_i^{(b')} = \beta_i^{(b+b')} \text{ and } \beta_i^{(0)} = 0 \tag{4.4.18}$$

which enable us to define abelian subgroups

$$B_i = \{\beta_i^{(b)} | b \in B\} \tag{4.4.19}$$

while the compatibility with counit and comultiplication give

$$\Sigma_i \beta_i^{(b)} = b \tag{4.4.20}$$

$$\beta_j^{(\beta_i^{(b)})} = \delta_{i,j}\beta_i^{(b)} \tag{4.4.21}$$

which ensure that

$$B = \Sigma_i B_i \tag{4.4.22}$$

$\delta_{i,j}$ *denotes the Kronecker delta, it equals* 1 *when* $i = j$ *and* 0 *otherwise.*

The braiding coelement (4.4.8) corresponds to the group homomorphism

$$\mathbb{Z}[x, x^{-1}] \otimes \mathbb{Z}[x, x^{-1}] \xrightarrow{\gamma} \mathbb{Z} \tag{4.4.23}$$

$$x^i \otimes x^j \mapsto (-1)^{ij} \tag{4.4.24}$$

and gives a braiding (symmetry, in fact) in GAb.

4.4.2 Differential Hopf ring

Let $\mathcal{V}$ be a braided monoidal additive category with direct sums preserved by tensoring.

Proposition 4.4.1. *An object* D *with braiding* $\sigma_{DD} = -1_{DD}$ *induces a Hopf ring* $H = D \oplus I$*, whose monoid structure* $HH \xrightarrow{\mu} H \xleftarrow{\eta} I$ *has non-zero components*

$$\begin{array}{l} DD \\ DI \xrightarrow{\rho := \rho_D} D \\ ID \xrightarrow{\lambda := \lambda_D} D \\ II \xrightarrow{i := \rho_I = \lambda_I} I \xleftarrow{1} I\,. \end{array} \tag{4.4.25}$$

the comonoid structure (Δ, ϵ) *has inverses of (4.4.25) as non-zero components, and the antipode is*

$$S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proof. The (co)associativity and (co)unit axioms follow from coherence for monoidal categories, after noting that a component is non-zero if and only if it contains either one D in its source and target, or none.

The compatibility of unit with counit and comultiplication is obvious.

The bimonoid axiom

$$HH \xrightarrow{\Delta\Delta} HHHH \xrightarrow{1\sigma 1} HHHH \xrightarrow{\mu\mu} HH \tag{4.4.26}$$

$$= HH \xrightarrow{\mu} H \xrightarrow{\Delta} HH \tag{4.4.27}$$

imposes that

$$\begin{tikzcd}
DD \arrow[r, "\rho^{-1}\lambda^{-1}"] \arrow[dr, "\lambda^{-1}\rho^{-1}"'] & DIID \arrow[r, "1\sigma_{II}1"] & DIID \arrow[r, "\rho\lambda"] & DD \\
& IDDI \arrow[r, "1\sigma_{DD}1"] & IDDI \arrow[ur, "\lambda\rho"'] & \\
DI \arrow[r, "\rho^{-1}i^{-1}"] \arrow[dr, "\lambda^{-1}i^{-1}"'] & DIII \arrow[r, "1\sigma_{II}1"] & DIII \arrow[r, "\rho i"] & DI \\
& IDII \arrow[dr, "1\sigma_{DI}1"] & IDII \arrow[ur, "\lambda i"'] & \\
ID \arrow[r, "i^{-1}\rho^{-1}"] \arrow[dr, "i^{-1}\lambda^{-1}"'] & IIDI \arrow[ur, "1\sigma_{ID}1"'] & IIDI \arrow[r, "i\rho"] & ID \\
& IIID \arrow[r, "1\sigma_{II}1"] & IIID \arrow[ur, "i\lambda"'] & \\
II \arrow[r, "i^{-1}i^{-1}"] & IIII \arrow[r, "1\sigma_{II}1"] & IIII \arrow[r, "i\lambda"] & II
\end{tikzcd} \tag{4.4.28}$$

equals

$$\begin{tikzcd}
DD & & DD \\
DI \arrow[r, "\rho"] & D \arrow[r, "\rho^{-1}"] \arrow[dr, "\lambda^{-1}"'] & DI \\
ID \arrow[ur, "\lambda"'] & & ID \\
II \arrow[r, "i"'] & I \arrow[r, "i^{-1}"'] & II
\end{tikzcd} \tag{4.4.29}$$

which follows from $\sigma_{DD} = -1$, braiding coherences [17]: $\sigma_{DI} = \lambda^{-1} \circ \rho$, $\sigma_{ID} = \rho^{-1} \circ \lambda$ and $\sigma_{II} = 1$, and coherences for unit and associator.

Finally, the Hopf axioms hold, for example the left inverse part gives

$$\begin{tikzcd}
& DI \arrow[r, "-1"] & DI \arrow[dr, "\rho"] & \\
D \arrow[ur, "\rho^{-1}"] \arrow[r, "\lambda^{-1}"'] & ID \arrow[r, "1"] & ID \arrow[r, "\lambda"'] & D \\
I \arrow[r, "i^{-1}"'] & II \arrow[r, "1"] & II \arrow[r, "i"'] & I
\end{tikzcd} \tag{4.4.30}$$

equals

$$\begin{tikzcd}
D & & D \\
I \arrow[r, "1"] & I \arrow[r, "1"] & I
\end{tikzcd} \tag{4.4.31}$$

$\square$

$\mathcal{V}$ as a category of coalgebras

Let $\mathcal{W}$ be a symmetric monoidal additive category with direct sums preserved by tensoring, and A a biring there, with a braiding coelement $A \otimes A \xrightarrow{\gamma} I$. Take $\mathcal{V} = A$-CoAlg. Now D as an A-coalgebra is an object of $\mathcal{W}$, together with a coaction $d : D \to A \otimes D$ satisfying

$$(\gamma 11) \circ (1\sigma_{AD}1) \circ (dd) \circ \sigma_{DD} = -1_{DD} \tag{4.4.32}$$

where on the left we have the braiding in $\mathcal{V}$ and σ is the symmetry in $\mathcal{W}$. From Proposition 4.4.1, we have that $H = D \oplus I$ with the coaction

$$\begin{array}{ccc} D & \xrightarrow{d} & AD \\ I & \xrightarrow{\eta} & A \end{array} \tag{4.4.33}$$

is a Hopf ring in $\mathcal{V}$. Hence, by (the dual of) Proposition 4.3.2, there is a semidirect product $H \rtimes A = DA \oplus A$, with the Hopf ring structure in $\mathcal{W}$ having components

$$\begin{tikzcd}
DADA \\
DAIA \arrow[r, "1\mu"] & DA \\
IADA \arrow[ur, "1\mu \circ s_{AD}1"'] & IA & I \arrow[l, "\eta"'] \\
IAIA \arrow[ur, "\mu"']
\end{tikzcd} \tag{4.4.34}$$

$$\begin{tikzcd}
DADA \\
DAIA & DA \arrow[l, "1\delta"'] \arrow[dl, "\tau_D 1 \circ 1\delta"] \\
IADA & IA \arrow[dl, "\delta"] \arrow[r, "\epsilon"] & I\,. \\
IAIA
\end{tikzcd} \tag{4.4.35}$$

Pareigis' example

Finally, take $\mathcal{W} = \mathrm{Ab}$, and $A = Z(= \mathbb{Z}[x, x^{-1}])$. The coalgebra (D, d) can be thought of as a graded abelian group, see Example 4.4.1. Let $d_i \in D_i$ and $d'_j \in D_j$. Condition (4.4.32) gives

$$(-1)^{ij} d'_j \otimes d_i = -d_i \otimes d'_j \tag{4.4.36}$$

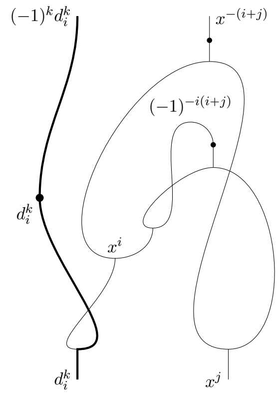
which forces all $D_j = 0$, except for $j = i$ for a fixed odd i. In addition, D_i can have only one generator, call it d.

The biring $(D \oplus \mathbb{Z}) \rtimes \mathbb{Z}[x, x^{-1}]$ has underlying abelian group $Q := \mathbb{Z}[x, x^{-1}] \oplus D \otimes \mathbb{Z}[x, x^{-1}]$, with (co)unit and (co)multiplication determined using (4.4.34) and (4.4.35):

$$\begin{aligned} Q &\xrightarrow{\epsilon} \mathbb{Z} \\ d \otimes x^j &\mapsto 0 \\ x^j &\mapsto 1 \end{aligned} \tag{4.4.37}$$

$$\begin{aligned} \mathbb{Z} &\xrightarrow{\eta} Q && (4.4.38)\\ 1 &\mapsto x^0 \\ Q &\xrightarrow{\delta} Q\otimes Q && (4.4.39)\\ d\otimes x^j &\mapsto d\otimes x^j\otimes x^j + x^{i+j}\otimes d\otimes x^j \\ x^k &\mapsto x^k\otimes x^k \\ Q\otimes Q &\xrightarrow{\mu} Q && (4.4.40)\\ d\otimes x^j\otimes d\otimes x^k &\mapsto 0 \\ d\otimes x^j\otimes x^k &\mapsto d\otimes x^{j+k} \\ x^j\otimes d\otimes x^k &\mapsto (-1)^j d\otimes x^{j+k} \\ x^j\otimes x^k &\mapsto x^{j+k}\,. \end{aligned}$$

To see what the antipode is, consider the general antipode diagram (4.3.7), and label the edges



(4.4.41)

where either $k = 1$ and i is odd, or $i = k = 0$. So we have

$$\begin{aligned} Q &\xrightarrow{s} Q && (4.4.42)\\ d\otimes x^j &\mapsto (-1)^j d\otimes x^{-(i+j)} \\ x^j &\mapsto x^{-j}\,. \end{aligned}$$

When $i = -1$ and $D_{-1} = \mathbb{Z}$, we get exactly the Pareigis Hopf ring P, by identifying

$$\xi = x \tag{4.4.43}$$

$$\psi = d\otimes x^0\,. \tag{4.4.44}$$

4.5 Comonadic base change via 2-sided enrichment

The forgetful functor $\mathcal{V}^{\mathcal{G}} \xrightarrow{\mathcal{U}} \mathcal{V}$ induces change of base functors

$$\mathcal{V}^{\mathcal{G}}\text{-Cat} \xrightarrow{\mathcal{U}'} \mathcal{V}\text{-Cat} \tag{4.5.1}$$

$$\mathcal{V}^{\mathcal{G}}\text{-Mod} \xrightarrow{\widetilde{\mathcal{U}}} \mathcal{V}\text{-Mod}\,. \tag{4.5.2}$$

In this section we introduce the context in which $\mathcal{U}'$ and $\widetilde{\mathcal{U}}$ have right adjoints. It is given by the tricategory Caten [20] whose objects are bicategories and homs are 2-categories of categories enriched on 2-sides. First we give a short review of [20], where we also introduce the notation for the rest of the section, and then generalize the notion of (Hopf) monoidal comonad [8] to that level.

4.5.1 2-sided enrichment

Objects of Caten are bicategories $\mathcal{V}$, $\mathcal{W}$, etc. Their hom categories $\mathcal{V}(V, V')$ are sometimes denoted $\mathcal{V}_V^{V'}$ to shorten the notation. The horizontal composition is denoted by tensor product $\otimes$.

Arrows $\mathcal{A} : \mathcal{W} \to \mathcal{V}$ are called 2-sided enriched categories. They consist of:

- a set of objects Ob$\mathcal{A}$ whose elements are denoted A, A', etc. together with a span

$$\begin{tikzcd} & \mathrm{Ob}\mathcal{A} \arrow[dl, "(-)_-"'] \arrow[dr, "(-)_+"] & \\ \mathrm{Ob}\mathcal{W} & & \mathrm{Ob}\mathcal{V} \end{tikzcd} \tag{4.5.3}$$

  assigning to each object A an object A_- in $\mathcal{W}$ and A_+ in $\mathcal{V}$

- homs $\mathcal{A}(A, A')$, also denoted $\mathcal{A}_A^{A'}$, defined to be functors

$$\mathcal{A}_A^{A'} : \mathcal{W}_{A_-}^{A'_-} \to \mathcal{V}_{A_+}^{A'_+} \tag{4.5.4}$$

- unit and composition natural transformations

$$\begin{tikzcd} 1 \arrow[r, "1_{A_-}"] \arrow[dr, bend right, "1_{A_+}"'] & \mathcal{W}_{A_-}^{A_-} \arrow[d, "\mathcal{A}_A^A"] \arrow[dl, phantom, "\eta_A\Rightarrow"] \\ & \mathcal{V}_{A_+}^{A_+} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{W}_{A'_-}^{A''_-} \times \mathcal{W}_{A_-}^{A'_-} \arrow[r, "\otimes"] \arrow[d, "\mathcal{A}_{A'}^{A''} \times \mathcal{A}_A^{A'}"'] \arrow[dr, phantom, "\mu_{AA''}^{A'}\Rightarrow"] & \mathcal{W}_{A_-}^{A''_-} \arrow[d, "\mathcal{A}_A^{A''}"] \\ \mathcal{V}_{A'_+}^{A''_+} \times \mathcal{V}_{A_+}^{A'_+} \arrow[r, "\otimes"'] & \mathcal{V}_{A_+}^{A''_+} \end{tikzcd} \tag{4.5.5}$$

satisfying unit and associativity laws.

Composition of 2-sided enriched categories is given by composition of spans (pullback), composition of functors defining homs, and pasting unit and multiplication natural transformations.

A 2-cell $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is an (enriched) functor consisting of

- a map of spans $\mathrm{Ob}\mathcal{F} =: F$
$$F : \mathrm{Ob}\mathcal{A} \to \mathrm{Ob}\mathcal{B} \tag{4.5.6}$$
which means
$$(FA)_- = A_-, \text{ and } (FA)_+ = A_+ \tag{4.5.7}$$
- natural transformations
$$\mathcal{F}_A^{A'} : \mathcal{A}_A^{A'} \Rightarrow \mathcal{B}_{FA}^{FA'} \tag{4.5.8}$$
which are compatible with unit and multiplication of $\mathcal{A}$ and $\mathcal{B}$.

A 3-cell $\psi : \mathcal{F} \to \mathcal{E}$ is an (enriched) natural transformation consisting of components
$$\psi_A : 1_{A_+} \Rightarrow \mathcal{B}_{FA}^{EA} 1_{A_-} \tag{4.5.9}$$
satisfying an enriched naturality condition (the filled in coherence 2-cells in $\mathcal{V}$ can be found in [20])
$$\begin{tikzcd}
\mathcal{A}_A^{A'}(w) \arrow[r, "\psi_{A'} \otimes (\mathcal{F}_A^{A'})_w"] \arrow[d, "(\mathcal{E}_A^{A'})_w \otimes \psi_A"'] & \mathcal{B}_{FA'}^{EA'}(1_{A'_-}) \otimes \mathcal{B}_{FA}^{FA'}(w) \arrow[d, "\mu"] \\
\mathcal{B}_{EA}^{EA'}(w) \otimes \mathcal{B}_{FA}^{EA}(1_{A_-}) \arrow[r, "\mu"'] & \mathcal{B}_{FA}^{EA'}(w)\,.
\end{tikzcd} \tag{4.5.10}$$

All axioms, compositions, whiskerings, and the fact that Caten is a tricategory are explained in detail in [20].

Example 4.5.1. *When $\mathcal{W} = 1$, $\mathcal{A}$ is precisely a category enriched in the bicategory $\mathcal{V}$.*

Example 4.5.2. *When $\mathrm{Ob}\mathcal{A} = \mathrm{Ob}\mathcal{W}$ and $(-)_- = 1$, $\mathcal{A}$ is precisely a lax functor from $\mathcal{W}$ to $\mathcal{V}$, and 2-cells are icons [21].*

Modules

Instead of (enriched) functors we could have chosen enriched modules $M : \mathcal{A} \nrightarrow \mathcal{B}$ as 2-cells. They consist of

- functors

$$M^A_B : \mathcal{W}^{A_-}_{B_-} \to \mathcal{V}^{A_+}_{B_+} \tag{4.5.11}$$

- action natural transformations

$$\begin{tikzcd} \mathcal{W}^{A'_-}_{A_-} \times \mathcal{W}^{A_-}_{B_-} \arrow[r, "\otimes"] \arrow[d, "\mathcal{A}^{A'}_{A} \times M^A_B"'] \arrow[dr, phantom, "\lambda^A_{BA'}\Rightarrow"] & \mathcal{W}^{A'_-}_{B_-} \arrow[d, "M^{A'}_B"] \\ \mathcal{V}^{A'_+}_{A_+} \times \mathcal{V}^{A_+}_{B_+} \arrow[r, "\otimes"'] & \mathcal{V}^{A'_+}_{B_+} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{W}^{A_-}_{B'_-} \times \mathcal{W}^{B'_-}_{B_-} \arrow[r, "\otimes"] \arrow[d, "M^A_{B'} \times \mathcal{B}^{B'}_B"'] \arrow[dr, phantom, "\rho^{B'}_{BA}\Rightarrow"] & \mathcal{W}^{A_-}_{B_-} \arrow[d, "M^A_B"] \\ \mathcal{V}^{A_+}_{B'_+} \times \mathcal{V}^{B'_+}_{B_+} \arrow[r, "\otimes"'] & \mathcal{V}^{A_+}_{B_+} \end{tikzcd} \tag{4.5.12}$$

  compatible with each other, and units and compositions in $\mathcal{A}$ and $\mathcal{B}$.

A module morphism $\sigma : M \Rightarrow N$ consists of natural transformations

$$\sigma^A_B : M^A_B \Rightarrow N^A_B \tag{4.5.13}$$

compatible with actions (4.5.12).

Module morphisms compose, and we get a category of modules between $\mathcal{A}$ and $\mathcal{B}$, which we call $\mathrm{Moden}(\mathcal{W}, \mathcal{V})(\mathcal{A}, \mathcal{B})$. When $\mathcal{V}$ is locally cocomplete $\mathrm{Moden}(\mathcal{W}, \mathcal{V})$ becomes a bicategory equivalent to the bicategory of enriched modules $\mathrm{Conv}(\mathcal{W}, \mathcal{V})$-Mod, where Conv denotes internal hom in Caten for the usual product of bicategories [20].

Each functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ defines a module $\mathcal{F}_* : \mathcal{A} \nrightarrow \mathcal{B}$ by taking (this is properly typed because of (4.5.7))

$$(\mathcal{F}_*)^A_B = \mathcal{B}^{FA}_B : \mathcal{W}^{A_-}_{B_-} \to \mathcal{V}^{A_+}_{B_+} \tag{4.5.14}$$

$$\lambda^A_{BA'} = \otimes\,(\mathcal{A}^{A'}_A \times \mathcal{B}^{FA}_B) \xRightarrow{1(\mathcal{F}^{A'}_A \times 1)} \otimes(\mathcal{B}^{FA'}_{FA} \times \mathcal{B}^{FA}_B) \xRightarrow{\mu^{FA}_{B,FA'}} \mathcal{B}^{FA'}_B \otimes \tag{4.5.15}$$

$$\rho^{B'}_{BA} = \otimes\,(\mathcal{B}^{FA}_{B'} \times \mathcal{B}^{B'}_B) \xRightarrow{\mu^{B'}_{B,FA}} \mathcal{B}^{FA}_B \otimes\ . \tag{4.5.16}$$

Compatibility of ρ with $\mu^{(\mathcal{B})}$ and $\eta^{(\mathcal{B})}$ are just the unit and associativity axioms for $\mu^{(\mathcal{B})}$ and $\eta^{(\mathcal{B})}$. Compatibility of λ with $\mu^{(\mathcal{A})}$ and $\eta^{(\mathcal{A})}$ follows by applying compatibility of the functor $\mathcal{F}$ with $\mu^{(\mathcal{A})}$ and $\eta^{(\mathcal{A})}$, followed by unit associativity laws for $\mu^{(\mathcal{A})}$ and $\eta^{(\mathcal{A})}$.

Similarly, each natural transformation $\psi : \mathcal{F} \to \mathcal{E}$ has an induced module morphism,

$$\psi_* : \mathcal{F}_* \to \mathcal{E}_* \tag{4.5.17}$$

$$(\psi_*)^A_B := \mathcal{B}^{FA}_B \Rightarrow \mathcal{B}^{EA}_B \tag{4.5.18}$$

$$\begin{aligned}((\psi_*)^A_B)_w = \mathcal{B}^{FA}_B(w) &\xRightarrow{\psi_A \otimes 1} \mathcal{B}^{EA}_{FA}(1_{A_-}) \otimes \mathcal{B}^{FA}_B(w) \\ &\xRightarrow{\mu^{FA}_{B,EA}} \mathcal{B}^{EA}_B(w)\,.\end{aligned} \tag{4.5.19}$$

To see that ψ_* is compatible with λ, tensor diagram (4.5.10) by $\mathcal{B}^{FA}_B(w')$, whisker the resulting square with $\mu^{(\mathcal{B})}$ on the right, and add obvious commutative squares to get the required compatibility. Compatibility with ρ follows from associativity of $\mu^{(\mathcal{B})}$. Also, every module morphism between modules induced by functors gives rise to a natural transformation. Given

$$\sigma^A_B : \mathcal{B}^{FA}_B \Rightarrow \mathcal{B}^{EA}_B \tag{4.5.20}$$

we can form

$$\sigma_A : 1_{A_+} \xRightarrow{\eta_{FA}} \mathcal{B}^{FA}_{FA}(1_{A_-}) \xRightarrow{(\sigma^A_{FA})_{1_{A_-}}} \mathcal{B}^{EA}_{FA}(1_{A_-}) \tag{4.5.21}$$

and the natural transformation axiom (4.5.10) is witnessed by commutativity of

$$\begin{tikzcd}
\mathcal{A}^{A'}_A(w) \arrow[rr, "\eta_{FA'} \otimes 1"] \arrow[dr, "(\mathcal{F}^{A'}_A)_w"] \arrow[dd, "1 \otimes \eta_{FA}"'] & & \mathcal{B}^{FA'}_{FA'}(1_{A'_-}) \otimes \mathcal{A}^{A'}_A(w) \arrow[rr, "(\sigma^{A'}_{FA'})_{1_{A'_-}} \otimes 1"] \arrow[d, "1 \otimes (\mathcal{F}^{A'}_A)_w"] & & \mathcal{B}^{EA'}_{FA'}(1_{A'_-}) \otimes \mathcal{A}^{A'}_A(w) \arrow[ddd, "1 \otimes (\mathcal{F}^{A'}_A)_w"] \\
& \mathcal{B}^{FA'}_{FA}(w) \arrow[r, "\eta_{FA'} \otimes 1"] \arrow[ddd, "1 \otimes \eta_{FA}"'] \arrow[dddrr, "1"] & \mathcal{B}^{FA'}_{FA'}(1_{A'_-}) \otimes \mathcal{B}^{FA'}_{FA}(w) \arrow[dddr, "\mu"'] \arrow[ddrr, "(\sigma^{A'}_{FA'})_{1_{A'_-}} \otimes 1"] & & \\
\mathcal{A}^{A'}_A(w) \otimes \mathcal{B}^{FA}_{FA}(1_{A_-}) \arrow[ddd, "1 \otimes (\sigma^A_{FA})_{1_{A_-}}"'] \arrow[ddr, "(\mathcal{F}^{A'}_A)_w \otimes 1"] & & & & \\
& & & & \mathcal{B}^{EA'}_{FA'}(1_{A'_-}) \otimes \mathcal{B}^{FA'}_{FA}(w) \arrow[dd, "\mu"] \\
& \mathcal{B}^{FA'}_{FA}(w) \otimes \mathcal{B}^{FA}_{FA}(1_{A_-}) \arrow[rr, "\mu"'] & & \mathcal{B}^{FA'}_{FA}(w) \arrow[dr, "(\sigma^{A'}_{FA})_w"] & \\
\mathcal{A}^{A'}_A(w) \otimes \mathcal{B}^{EA}_{FA}(1_{A_-}) \arrow[rr, "(\mathcal{E}^{A'}_A)_w \otimes 1"'] & & \mathcal{B}^{EA'}_{EA}(w) \otimes \mathcal{B}^{EA'}_{FA}(1_{A_-}) \arrow[rr, "\mu"'] & & \mathcal{B}^{EA'}_{FA}(w)
\end{tikzcd} \tag{4.5.22}$$

where the hexagon and the bottom right square are compatibility conditions between module morphism σ and actions (4.5.15) and (4.5.16) respectively.

Proposition 4.5.1. *The functor*

$$(-)_* : \mathrm{Caten}(\mathcal{W}, \mathcal{V})(\mathcal{A}, \mathcal{B}) \to \mathrm{Moden}(\mathcal{W}, \mathcal{V})(\mathcal{A}, \mathcal{B}) \tag{4.5.23}$$

is full and faithful.

Proof. The processes of turning a natural transformation into a module morphism given by (4.5.19) and the one turning module morphism of convergent modules into a natural transformation, given by (4.5.21), are inverse to each other, as witnessed by the commuting diagrams (4.5.24).

$$\begin{tikzcd} 1_{A_+} \arrow[rr, "\eta_{FA}"] \arrow[d, "\psi_A"'] & & \mathcal{B}_{FA}^{FA}(1_{A_-}) \arrow[d, "\psi_A \otimes 1"] \\ \mathcal{B}_{FA}^{EA}(1_{A_-}) \arrow[rr, "1 \otimes \eta_{FA}"] \arrow[drr, "1"'] & & \mathcal{B}_{FA}^{EA}(1_{A_-}) \otimes \mathcal{B}_{FA}^{FA}(1_{A_-}) \arrow[d, "\mu"] \\ & & \mathcal{B}_{FA}^{EA}(1_{A_-}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}_{B}^{FA}(w) \arrow[d, "\eta_{FA} \otimes 1"'] \arrow[dr, "1"] & \\ \mathcal{B}_{FA}^{FA}(1_{A_-}) \otimes \mathcal{B}_{B}^{FA}(w) \arrow[r, "\mu"'] \arrow[d, "(\sigma_{FA}^{A})_{1_{A_-}} \otimes 1"'] & \mathcal{B}_{B}^{FA}(w) \arrow[d, "(\sigma_{B}^{A})_w"] \\ \mathcal{B}_{FA}^{EA}(1_{A_-}) \otimes \mathcal{B}_{B}^{FA}(w) \arrow[r, "\mu"'] & \mathcal{B}_{B}^{EA}(w) \end{tikzcd} \tag{4.5.24}$$

□

4.5.2 Comonads in Caten

Let $\mathcal{G} : \mathcal{V} \to \mathcal{V}$ be a comonad in Caten, that is, a 2-sided enriched category with enriched functors

$$1_{\mathcal{V}} \xleftarrow{\epsilon} \mathcal{G} \xrightarrow{\delta} \mathcal{G}^2 \tag{4.5.25}$$

satisfying the three comonoid axioms. The existence of the span morphism $\mathrm{Ob}\mathcal{G} \xrightarrow{\mathrm{Ob}\epsilon} \mathrm{Ob}\mathcal{V}$ forces $G_+ = G_- = (\mathrm{Ob}\epsilon)(G) =: G_0$, for all G. The two counit axioms give $(\mathrm{Ob}\delta)(G) = (G, G)$. With these simplifications, the remaining data for $\mathcal{G}$ is given by endofunctors

$$\mathcal{G}_G^{G'} : \mathcal{V}_{G_0}^{G'_0} \to \mathcal{V}_{G_0}^{G'_0} \tag{4.5.26}$$

and natural transformations with components

$$(\mu_{GG''}^{G'})_{v',v} : \mathcal{G}_{G'}^{G''}(v') \otimes \mathcal{G}_G^{G'}(v) \Rightarrow \mathcal{G}_G^{G''}(v' \otimes v) \tag{4.5.27}$$

$$\eta_G : 1_{G_0} \Rightarrow \mathcal{G}_G^G(1_{G_0}) \tag{4.5.28}$$

$$(\delta_G^{G'})_v : \mathcal{G}_G^{G'}(v) \Rightarrow (\mathcal{G}_G^{G'})^2(v) \tag{4.5.29}$$

$$(\epsilon_G^{G'})_v : \mathcal{G}_G^{G'}(v) \Rightarrow v \tag{4.5.30}$$

satisfying enriched functor compatibility axioms, which together with the comonad axioms, correspond exactly to the monoidal comonad axioms dual to opmonoidal monad ones appearing in [8].

The bicategory of $\mathcal{G}$-coalgebras

Each hom $\mathcal{G}_G^{G'}$ becomes a comonad in the usual sense (in Cat). Let $\mathcal{V}^{\mathcal{G}}$ denote a (soon to become) bicategory with the same objects as $\mathcal{G}$ and with homs the categories of EM-coalgebras

$$\mathcal{V}^{\mathcal{G}}(G,G') := \mathcal{V}(G_0,G'_0)^{\mathcal{G}(G,G')} . \tag{4.5.31}$$

The identity coalgebra is $(1_{G_0}, \eta_G)$ and composition is given on coalgebras by

$$\begin{aligned} &\mathcal{V}^{\mathcal{G}}(G',G'') \times \mathcal{V}^{\mathcal{G}}(G,G') \to \mathcal{V}^{\mathcal{G}}(G,G'') \\ &(v',\gamma_{v'}),(v,\gamma_v) \mapsto (v'\otimes v, (\mu_{GG''}^{G'})_{v',v} \circ (\gamma_{v'}\otimes\gamma_v)) . \end{aligned} \tag{4.5.32}$$

The assigned map is a coalgebra: compatibility with δ is witnessed by commutativity of[3]

$$\begin{tikzcd} v'\otimes v \arrow[r, "\gamma\otimes\gamma"] \arrow[d, "\gamma\otimes\gamma"'] & \mathcal{G}v'\otimes\mathcal{G}v \arrow[r, "\mu"] \arrow[d, "\delta\otimes\delta"'] & \mathcal{G}(v'\otimes v) \arrow[dd, "\delta"] \\ \mathcal{G}v'\otimes\mathcal{G}v \arrow[r, "\mathcal{G}\gamma\otimes\mathcal{G}\gamma"'] \arrow[d, "\mu"'] & \mathcal{G}^2v'\otimes\mathcal{G}^2v \arrow[d, "\mu"'] \arrow[dr, "\mu^{(\mathcal{G}^2)}"] & \\ \mathcal{G}(v'\otimes v) \arrow[r, "\mathcal{G}(\gamma\otimes\gamma)"'] & \mathcal{G}(\mathcal{G}v'\otimes\mathcal{G}v) \arrow[r, "\mathcal{G}\mu"'] & \mathcal{G}^2(v'\otimes v) \end{tikzcd} \tag{4.5.33}$$

where the upper left square is a componentwise compatibility of local coalgebras γ with comultiplication, the bottom left square is naturality of μ, the triangle is the definition of composition for the composite category, and the remaining square is compatibility of the enriched functor δ with compositions in its source and target, which one can also identify as a typical bimonoid (bialgebra) axiom. Similarly, ϵ being an enriched functor implies compatibility of (4.5.32) with ϵ. The assignment extends to coalgebra morphisms, which follows directly from naturality of μ. The unitors and associators are inherited from $\mathcal{V}$, they are coalgebra morphisms, and satisfy the usual monoidale axioms as they do in $\mathcal{V}$.

[3]When indices are omitted they can be deduced from the context. For example, $\mathcal{G}_G^{G'}(v)$ is the full notation.

There is an underlying (strict) functor $\mathcal{U} : \mathcal{V}^{\mathcal{G}} \to \mathcal{V}$ sending G to the underlying object G_0 in $\mathcal{V}$, and disregarding the colagebra structure on homs. By construction, each $\mathcal{U}_V^{V'}$ has a right adjoint $\mathcal{R}_V^{V'}$, and by Theorem 2.7 of [20] the right adjoints are part of a 2-sided enriched category $\mathcal{R} : \mathcal{V} \to \mathcal{V}^{\mathcal{G}}$ which has the same objects as $\mathcal{G}$, with span legs given by $G_- = G_0$ and $G_+ = G$, with unit and multiplication given by

$$(\mu^{(\mathcal{R})G'}_{GG''})_{v',v} : \mathcal{R}^{G''}_{G'}(v') \otimes \mathcal{R}^{G'}_{G}(v) \Rightarrow \mathcal{R}^{G''}_{G}(v' \otimes v) \tag{4.5.34}$$

$$= (\mathcal{G}v' \otimes \mathcal{G}v, \mu_{v',v} \circ (\delta_{v'} \otimes \delta_v)) \xrightarrow{\mu_{v',v}} (\mathcal{G}(v' \otimes v), \delta_{v' \otimes v}) \tag{4.5.35}$$

$$\eta^{(\mathcal{R})}_{G} : 1_G \Rightarrow \mathcal{R}^G_G(1_{G_0}) \tag{4.5.36}$$

$$= (1_{G_0}, \eta_G) \xrightarrow{\eta_G} (\mathcal{G}1_{G_0}, \delta_{1_{G_0}}) \,. \tag{4.5.37}$$

Now we have an adjunction in Caten.

$$\begin{tikzcd} \mathcal{V}^{\mathcal{G}} \arrow[r, bend left, "\mathcal{U}"] \arrow[r, phantom, "\perp"] & \mathcal{V} \arrow[l, bend left, "\mathcal{R}"] \end{tikzcd} \tag{4.5.38}$$

The counit and the unit of the adjunction are given by the enriched functors

$$\mathcal{U} \circ \mathcal{R} = \mathcal{G} \xrightarrow{\epsilon} 1_{\mathcal{V}} \tag{4.5.39}$$

$$1_{\mathcal{V}^{\mathcal{G}}} \xrightarrow{\gamma} \mathcal{R} \circ \mathcal{U} \tag{4.5.40}$$

$$(\mathrm{ob}\gamma)(G) = (G, G)$$

$$(v, \gamma_v : v \to \mathcal{G}^{G'}_{G} v) \xrightarrow{\gamma_v} (\mathcal{G}^{G'}_{G} v, \delta_v) \,.$$

Now we present a version of Beck's theorem that we are going to use in the rest of the chapter.

Proposition 4.5.2. *Any 2-sided enriched category $\mathcal{L} : \mathcal{W} \to \mathcal{V}$ such that*

- *$\mathcal{L}$ has a right adjoint $\mathcal{R}$ in Caten*
- *$\mathcal{L}$ is locally conservative*
- *$\mathcal{W}$ has, and $\mathcal{L}$ preserves, local $\mathcal{L}$-split equalizers*

gives rise to an equivalence to $\mathcal{W} \simeq \mathcal{V}^{\mathcal{G}}$, where $\mathcal{G}$ is the generated comonad $\mathcal{G} = \mathcal{L} \circ \mathcal{R}$.

Proof. As was shown in [20], $\mathcal{L}$ has a right adjoint if and only if it is a pseudo-functor and each functor $\mathcal{L}(L, L')$ has a right adjoint, call it $\mathcal{R}(L, L')$. Then, the right adjoint $\mathcal{R}$ has the same objects as $\mathcal{L}$ (and $\mathcal{W}$, since $\mathcal{L}$ is a pseudo-functor), and homs are precisely $\mathcal{R}(L, L')$. From the usual Beck (co-)monadicity theorem it follows that $\mathcal{W} \simeq \mathcal{V}^{\mathcal{G}}$: they have the same objects and equivalent homs. □

The category Caten$(\mathcal{X}, \mathcal{V})$ has an induced comonad Caten$(\mathcal{X}, \mathcal{G})$ on it. In particular, when $\mathcal{X} = \mathcal{V}^{\mathcal{G}}$ there is a natural coalgebra structure on $\mathcal{U}$ given by an enriched functor

$$\mathcal{U} \xrightarrow{\mathcal{U}\circ\gamma} \mathcal{G} \circ \mathcal{U} \tag{4.5.41}$$

whose components are exactly $\gamma_v : v \Rightarrow \mathcal{G}v$.

Lemma 4.5.1. *Let $\mathcal{X}$ be a bicategory. Whiskering with $\mathcal{U}$*

$$\text{Moden}(\mathcal{X}, \mathcal{V}^{\mathcal{G}})(\mathcal{A}, \mathcal{B}) \xrightarrow{\mathcal{U}\circ -} \text{Moden}(\mathcal{X}, \mathcal{V})(\mathcal{U} \circ \mathcal{A}, \mathcal{U} \circ \mathcal{B}) \tag{4.5.42}$$

is conservative and the source has, and $(\mathcal{U} \circ -)$ preserves, $(\mathcal{U} \circ -)$-split equalizers.

Proof. Let $M, N : \mathcal{A} \twoheadrightarrow \mathcal{B}$ be modules, and $\sigma : M \Rightarrow N$ a module morphism, with components

$$(\sigma^A_B)_x : M^A_B(x) \Rightarrow N^A_B(x) \tag{4.5.43}$$

which are 2-cells in $\mathcal{V}^{\mathcal{G}}$, natural in $x \in \mathcal{X}^{A-}_{B-}$.

Let $\psi : \mathcal{U} \circ M \Rightarrow \mathcal{U} \circ N$ be an inverse of $\mathcal{U} \circ \sigma$. This precisely means that the component

$$(\psi^A_B)_x : \mathcal{U}N^A_B(x) \Rightarrow \mathcal{U}M^A_B(x) \tag{4.5.44}$$

is an inverse of the component 2-cell $(\sigma^A_B)_x$ in $\mathcal{V}$. Since $\mathcal{U}$ is locally conservative, $(\psi^A_B)_x$ is also a coalgebra morphism. Hence, naturality squares for ψ^A_B consist of the same arrows regardless of whether it is seen as a morphism from $\mathcal{U}N^A_B$ to $\mathcal{U}M^A_B$, or from N^A_B to M^A_B. Compatibility of ψ with actions for M and N follows from the same compatibility conditions for σ and the fact that they are inverse of each other.

Consider a pair $\sigma, \chi : M \Rightarrow N$ with a split equalizer

$$\begin{tikzcd} E \arrow[r, "\xi"] & \mathcal{U} \circ M \arrow[l, bend left, "\phi"] \arrow[r, shift left, "\mathcal{U}\circ\sigma"] \arrow[r, shift right, "\mathcal{U}\circ\chi"'] & \mathcal{U} \circ N \arrow[l, bend left=50, "\psi"] \end{tikzcd} \tag{4.5.45}$$

meaning that we have the following componentwise formulas:

$$(\phi^A_B)_x \bullet (\xi^A_B)_x = 1_{E^A_B(x)} \tag{4.5.46}$$

$$(\psi^A_B)_x \bullet (\sigma^A_B)_x = 1_{\mathcal{U}M^A_B(x)} \tag{4.5.47}$$

$$(\psi^A_B)_x \bullet (\chi^A_B)_x = (\xi^A_B)_x \bullet (\phi^A_B)_x\,. \tag{4.5.48}$$

This in particular means that the pair $(\sigma^A_B)_x, (\chi^A_B)_x : M^A_B(x) \rightrightarrows N^A_B(x)$ has a $\mathcal{U}^{A+}_{B_+}$-split equalizer in $\mathcal{V}^{A+0}_{B+0}$. Since $\mathcal{U}^{A+}_{B_+}$ is comonadic, $(\xi^A_B)_x$ is an equalizer of $(\sigma^A_B)_x$ and $(\chi^A_B)_x$ in $\mathcal{V}^{A+}_{B_+}$, with an algebra structure on its source

$$\begin{aligned}\gamma_{E^A_B(x)} := E^A_B(x) &\xRightarrow{(\xi^A_B)_x} \mathcal{U}M^A_B(x) \\ &\xRightarrow{\gamma_{M^A_B(x)}} \mathcal{G}^{A+}_{B_+}\mathcal{U}M^A_B(x) \\ &\xRightarrow{\mathcal{G}^{A+}_{B_+}(\phi^A_B)_x} \mathcal{G}^{A+}_{B_+}E^A_B(x)\,.\end{aligned} \tag{4.5.49}$$

The action components for the module E are coalgebra morphisms, the proof for λ (dually for ρ) comes from the diagram (4.5.50) (all indices can be deduced from the top left term).

$$\begin{tikzcd}
\mathcal{A}^{A'}_A(x')\otimes E^A_B(x) \arrow[rr, "\lambda"] \arrow[d, "\gamma\otimes\xi"'] \arrow[dr, "1\otimes\xi"] & & E \arrow[d, "\xi"] \\
\mathcal{G}\mathcal{A}\otimes M \arrow[d, "1\otimes\gamma"'] & \mathcal{A}\otimes M \arrow[r, "\lambda"] \arrow[dl, "\gamma\otimes\gamma"'] \arrow[d, "\gamma"'] & M \arrow[d, "\gamma"] \\
\mathcal{G}\mathcal{A}\otimes\mathcal{G}M \arrow[r, "\mu"'] \arrow[d, "1\otimes\mathcal{G}\phi"'] & \mathcal{G}(\mathcal{A}\otimes M) \arrow[r, "\mathcal{G}\lambda"] \arrow[d, "\mathcal{G}(1\otimes\phi)"'] & \mathcal{G}M \arrow[d, "\mathcal{G}\phi"] \\
\mathcal{G}\mathcal{A}\otimes\mathcal{G}E \arrow[r, "\mu"'] & G(\mathcal{A}\otimes E) \arrow[r, "\mathcal{G}\lambda"'] & \mathcal{G}E
\end{tikzcd} \tag{4.5.50}$$

Diagrams for compatibility of actions of E with units and multiplications in $\mathcal{A}$ and $\mathcal{B}$ are the same as the ones for $\mathcal{U}\circ\mathcal{A}$ and $\mathcal{U}\circ\mathcal{B}$. This proves that $E : \mathcal{A} \nrightarrow \mathcal{B}$ is a module. Components $(\xi^A_B)_x$ are natural in x, and compatible with actions of E as coalgebra morphisms because they are natural and compatible as usual arrows. This proves that ξ is a module morphism between E (with coalgebra structure) and M.

It remains to show that ξ is an equalizer of σ and χ, so assume $L \xrightarrow{\omega} M$ is another $\mathcal{V}^{\mathcal{G}}$-module morphism satisfying $\sigma \bullet \omega = \chi \bullet \omega$. The components of $\phi \bullet \omega$, obtained by composing components of ϕ and ω, are coalgebra maps since $\mathcal{U}$ is locally comonadic, and naturality in x and compatibility with actions follows as for ξ. □

Corollary 4.5.1. *Whiskering with* $\mathcal{U}$

$$\mathrm{Caten}(\mathcal{X}, \mathcal{V}^{\mathcal{G}})(\mathcal{A}, \mathcal{B}) \xrightarrow{\mathcal{U}\circ -} \mathrm{Caten}(\mathcal{X}, \mathcal{V})(\mathcal{U} \circ \mathcal{A}, \mathcal{U} \circ \mathcal{B}) \tag{4.5.51}$$

is conservative and the source has, and $(\mathcal{U} \circ -)$ *preserves,* $(\mathcal{U} \circ -)$*-split equalizers.*

Proof. Direct consequence of Proposition 4.5.1, Lemma 4.5.1, and commutativity of $(-)_*$ with $\mathcal{U} \circ -$. □

Proposition 4.5.3. *The bicategory* $\mathcal{V}^{\mathcal{G}}$ *is an* EM*-object for the comonad* $\mathcal{G}$ *in* Caten.

Proof. Mapping out of $\mathcal{X}$,

$$\mathrm{Caten}(\mathcal{X}, -) : \mathrm{Caten} \to \text{2-CAT} \tag{4.5.52}$$

is a pseudo-functor, therefore preserves adjunctions. In particular, applying it to (4.5.38) gives

$$\begin{tikzcd} \mathrm{Caten}(\mathcal{X}, \mathcal{V}^{\mathcal{G}}) \arrow[rr, bend left, "\mathcal{U}' := \mathrm{Caten}(\mathcal{X}, \mathcal{U})"] \arrow[rr, phantom, "\perp"] & & \mathrm{Caten}(\mathcal{X}, \mathcal{V}) \arrow[ll, bend left, "\mathcal{R}' := \mathrm{Caten}(\mathcal{X}, \mathcal{R})"] \end{tikzcd} . \tag{4.5.53}$$

The composite is isomorphic to $\mathrm{Caten}(\mathcal{X}, \mathcal{G})$, and what remains to show is that $\mathcal{U}'$ is comonadic in the sense of Proposition 4.5.2. It has a right adjoint $\mathcal{R}'$, and the rest follows from Corollary 4.5.1. □

4.5.3 Hopf comonads

Definition 4.5.1. *A comonad* $\mathcal{G}$ *is left Hopf if, for all* G, G', G'', $v \in \mathcal{V}^{\mathcal{G}}(G, G')$ *and* $v' \in \mathcal{V}(G'_0, G''_0)$*, the fusion map*

$$\mathrm{v}_{v',v} : \mathcal{G}v' \otimes \mathcal{U}v \xRightarrow{1 \otimes \gamma_v} \mathcal{G}v' \otimes \mathcal{G}\mathcal{U}v \xRightarrow{(\mu^{G'}_{G,G''})_{v',\mathcal{U}v}} \mathcal{G}(v' \otimes \mathcal{U}v) \tag{4.5.54}$$

is invertible. This is equivalent to (left) Hopf maps

$$\mathrm{h}_{v',v} : \mathcal{R}v' \otimes v \xRightarrow{1\otimes\gamma_v} \mathcal{R}v' \otimes \mathcal{R}\mathcal{U}v \xRightarrow{(\mu^{G'}_{G,G''})_{v',v}} \mathcal{R}(v' \otimes \mathcal{U}v) \tag{4.5.55}$$

being invertible.

Proposition 4.5.4. *The inverse fusion maps are $\mathcal{G}$-compatible in the first variable, meaning*

$$\begin{tikzcd}
\mathcal{G}(v'\otimes v) \arrow[d, "\epsilon_{v'\otimes v}"'] \arrow[dr, "\mathrm{v}^{-1}_{v',v}"] & \\
v'\otimes v & \mathcal{G}v'\otimes v \arrow[l, "\epsilon_{v'}\otimes 1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{G}(v'\otimes v) \arrow[r, "\mathrm{v}^{-1}_{v',v}"] \arrow[d, "\delta_{v'\otimes v}"'] & \mathcal{G}v'\otimes v \arrow[r, "\delta_{v'}\otimes 1"] & \mathcal{G}^2v'\otimes v \\
\mathcal{G}^2(v'\otimes v) \arrow[rr, "\mathcal{G}\mathrm{v}^{-1}_{v',v}"] & & \mathcal{G}(\mathcal{G}v'\otimes v) \arrow[u, "\mathrm{v}^{-1}_{\mathcal{G}v',v}"']
\end{tikzcd} \tag{4.5.56}$$

as well as compatible with any coalgebra structure existing on v', in the sense

$$\begin{tikzcd}
\mathcal{G}(v'\otimes v) \arrow[dr, "\mathrm{v}^{-1}_{v',v}"] & \\
v'\otimes v \arrow[u, "\gamma_{v'\otimes v}"] \arrow[r, "\gamma_{v'}\otimes 1"'] & \mathcal{G}v'\otimes v\ .
\end{tikzcd} \tag{4.5.57}$$

Proof. Follows directly from the commuting diagrams:

$$\begin{tikzcd}
\mathcal{G}(v'\otimes v) \arrow[d, "\epsilon_{v'\otimes v}"'] & \mathcal{G}v'\otimes\mathcal{G}v \arrow[l, "\mu_{v',v}"'] \arrow[dl, "\epsilon_{v'}\otimes\epsilon_v"] \\
v'\otimes v & \mathcal{G}v'\otimes v \arrow[l, "\epsilon_{v'}\otimes 1"] \arrow[u, "1\otimes\gamma_v"']
\end{tikzcd} \tag{4.5.58}$$

$$\begin{tikzcd}
 & \mathcal{G}v'\otimes v \arrow[d, "1\otimes\gamma"] \arrow[dr, "\delta\otimes 1"] \arrow[ddr, "\delta\otimes\gamma"] & \\
\mathcal{G}(v'\otimes v) \arrow[dd, "\delta"'] & \mathcal{G}v'\otimes\mathcal{G}v \arrow[l, "\mu"'] \arrow[d, "\delta\otimes\delta"'] & \mathcal{G}^2v'\otimes v \arrow[d, "1\otimes\gamma"] \\
 & \mathcal{G}^2v'\otimes\mathcal{G}^2v \arrow[d, "\mu"'] & \mathcal{G}^2v'\otimes\mathcal{G}v \arrow[l, "1\otimes\mathcal{G}\gamma"] \arrow[d, "\mu"] \\
\mathcal{G}^2(v'\otimes v) & \mathcal{G}(\mathcal{G}v'\otimes\mathcal{G}v) \arrow[l, "\mathcal{G}\mu"] & \mathcal{G}(\mathcal{G}v'\otimes v) \arrow[l, "\mathcal{G}(1\otimes\gamma)"]
\end{tikzcd} \tag{4.5.59}$$

$$\begin{tikzcd}
\mathcal{G}(v'\otimes v) & \mathcal{G}v'\otimes\mathcal{G}v \arrow[l, "\mu_{v',v}"'] \\
v'\otimes v \arrow[u, "\gamma_{v'\otimes v}"] \arrow[ur, "\gamma_{v'}\otimes\gamma_v"'] \arrow[r, "\gamma_{v'}\otimes 1"'] & \mathcal{G}v'\otimes v \arrow[u, "1\otimes\gamma_v"']
\end{tikzcd} \tag{4.5.60}$$

□

Theorem 4.5.1. *If the comonad $\mathcal{G}$ is left Hopf, then the underlying (pseudo-)functor $\mathcal{U}$: $\mathcal{V}^{\mathcal{G}} \to \mathcal{V}$ creates left Kan extensions.*

Proof. Consider two coalgebras (u, γ_u) and (v, γ_v) whose underlying arrows have a left extension $k = \mathrm{lan}_v u$ as shown

$$\begin{tikzcd} & G'' \\ G \arrow[ur, "{(u,\gamma_u)}"] \arrow[r, "{(v,\gamma_v)}"'] & G' \arrow[u, dashed, "{(k,\gamma_k)}"'] \end{tikzcd}$$
$$\mathcal{U}\downarrow$$
$$\begin{tikzcd} & G_0'' \\ G_0 \arrow[ur, "u"] \arrow[r, "v"'] & G_0' \arrow[u, dashed, "k"'] \arrow[ul, phantom, "\kappa \Rightarrow"] \end{tikzcd} \quad . \tag{4.5.61}$$

The universal property of left Kan extensions says there is a bijection

$$\phi : u \Rightarrow l \otimes v \tag{4.5.62}$$

$$\overline{\phantom{\bar{\phi} : k \Rightarrow l \otimes v}}$$

$$\bar{\phi} : k \Rightarrow l \tag{4.5.63}$$

such that $\phi = (\bar{\phi} \otimes 1) \bullet \kappa$. In particular, there is a map $\gamma_k : k \to \mathcal{G}k$ corresponding to $u \overset{\gamma_u}{\Longrightarrow} \mathcal{G}u \overset{\mathcal{G}\kappa}{\Longrightarrow} \mathcal{G}(k \otimes v) \overset{\mathrm{v}_{k,v}^{-1}}{\Longrightarrow} \mathcal{G}k \otimes v$ such that the diagram below commutes.

$$\begin{tikzcd} u \arrow[r, "\gamma_u"] \arrow[d, "\kappa"'] & \mathcal{G}u \arrow[r, "\mathcal{G}\kappa"] & \mathcal{G}(k \otimes v) \arrow[d, "\mathrm{v}_{k,v}^{-1}"] \\ k \otimes v \arrow[rr, "\gamma_k \otimes 1"] & & \mathcal{G}(k) \otimes v \end{tikzcd} \tag{4.5.64}$$

The obtained arrow, γ_k, defines an coalgebra structure on k, the compatibility with δ and ϵ

follows from

$$
\begin{tikzcd}
 & u \arrow[dr, "\gamma"] \arrow[drrr, "\kappa"] \arrow[ddl, "\kappa"'] \arrow[dd, "\gamma"] & & & \\
 & & \mathcal{G}u \arrow[d, "\mathcal{G}\gamma"'] \arrow[ddr, "\mathcal{G}\kappa"] & & k\otimes v \arrow[ddd, "\gamma\otimes 1"] \\
k\otimes v \arrow[ddr, "\gamma\otimes 1"'] & \mathcal{G}u \arrow[r, "\delta"] \arrow[d, "\mathcal{G}\kappa"] & \mathcal{G}^2 u \arrow[d, "\mathcal{G}^2\kappa"'] & & \\
 & \mathcal{G}(k\otimes v) \arrow[r, "\delta"] \arrow[d, "\mathrm{v}^{-1}"] & \mathcal{G}^2(k\otimes v) \arrow[d, "\mathcal{G}\mathrm{v}^{-1}"'] & \mathcal{G}(k\otimes v) \arrow[dl, "\mathcal{G}(\gamma\otimes 1)"] \arrow[dr, "\mathrm{v}^{-1}"] & \\
 & \mathcal{G}k\otimes v \arrow[dr, "\delta\otimes 1"'] & \mathcal{G}(\mathcal{G}k\otimes v) \arrow[d, "\mathrm{v}^{-1}"] & & \mathcal{G}k\otimes v \arrow[dll, "\mathcal{G}\gamma\otimes 1"] \\
 & & \mathcal{G}^2 k\otimes v & &
\end{tikzcd}
\tag{4.5.65}
$$

$$
\begin{tikzcd}
u \arrow[rrr, "\kappa"] \arrow[dr, "1"] \arrow[ddr, "\gamma"'] \arrow[ddd, "\kappa"'] & & & k\otimes v \\
 & u \arrow[urr, "\kappa"] & & \\
 & \mathcal{G}u \arrow[u, "\epsilon"'] \arrow[r, "\mathcal{G}\kappa"] & \mathcal{G}(k\otimes v) \arrow[uur, "\epsilon"'] \arrow[dr, "\mathrm{v}^{-1}"'] & \\
k\otimes v \arrow[rrr, "\gamma\otimes 1"'] & & & \mathcal{G}k\otimes v \arrow[uuu, "\epsilon\otimes 1"']
\end{tikzcd}
\tag{4.5.66}
$$

The 2-cell κ is a coalgebra morphism, which is obvious after substituting v^{-1} in (4.5.64).

To see that κ exhibits (k, γ_k) as a left extension of (u, γ_u) through (v, γ_v), consider a coalgebra $(l, \gamma_l) : G' \to G''$, and a coalgebra morphism $\phi : u \Rightarrow l \otimes v$. In $\mathcal{V}$, the Kan extension universal property gives $\bar{\phi} : k \Rightarrow l$. From the following commuting diagram it follows that $\bar{\phi}$ is a coalgebra morphism.

$$
\begin{tikzcd}
 & u \arrow[r, "\kappa"] \arrow[dl, "\kappa"'] \arrow[d, "\gamma"] \arrow[drr, "\phi"'] & k\otimes v \arrow[dr, "\bar{\phi}\otimes 1"] & \\
k\otimes v \arrow[ddr, "\gamma\otimes 1"'] & \mathcal{G}u \arrow[d, "\mathcal{G}\kappa"] \arrow[dr, "\mathcal{G}\phi"] & & l\otimes v \arrow[dl, "\gamma"] \arrow[ddl, "\gamma\otimes 1"] \\
 & \mathcal{G}(k\otimes v) \arrow[r, "\mathcal{G}(\bar{\phi}\otimes 1)"'] \arrow[d, "\mathrm{v}^{-1}"] & \mathcal{G}(l\otimes v) \arrow[d, "\mathrm{v}^{-1}"'] & \\
 & \mathcal{G}k\otimes v \arrow[r, "\mathcal{G}\bar{\phi}\otimes 1"'] & \mathcal{G}l\otimes v &
\end{tikzcd}
\tag{4.5.67}
$$

□

Recall that m having an adjoint is equivalent to existence of a left Kan extension $\mathrm{lan}_m 1_M$ which is respected by m; that is, $m \circ \mathrm{lan}_m 1_M = \mathrm{lan}_m m$.

Corollary 4.5.2. *With a Hopf-comonadic $\mathcal{U} : \mathcal{N} \to \mathcal{M}$, an arrow $n \in \mathcal{N}(N, N')$ has a right adjoint if and only if $\mathcal{U}n$ does.*

Proof. $\mathcal{U}$, being a pseudo functor, preserves adjoints.

The other way around, assume Un has a right adjoint, that is both $\mathrm{lan}_{Un} 1_{UN}$ and $\mathrm{lan}_{Un} Un$ exist. From the previous theorem, $\mathrm{lan}_n 1_N$ exists and $n \circ \mathrm{lan}_n 1_N$ is taken to $U(n \circ \mathrm{lan}_n 1_N) \cong Un \circ \mathrm{lan}_{Un} 1_{UN} \cong \mathrm{lan}_{Un} Un$ which creates $\mathrm{lan}_n n$. □

Theorem 4.5.2. *If $\mathcal{V}$ is locally cocomplete then the induced underlying functor is well-defined*

$$\mathcal{N} := \mathit{Moden}(\mathcal{X}, \mathcal{V}^{\mathcal{G}}) \xrightarrow{\widetilde{\mathcal{U}} := \mathit{Moden}(\mathcal{X}, \mathcal{U})} \mathcal{M} := \mathit{Moden}(\mathcal{X}, \mathcal{V}) \tag{4.5.68}$$

and it is comonadic in CATEN. Denote its right adjoint by $\widetilde{\mathcal{R}}$. If $\mathcal{R}$ preserves local colimits, and $\mathcal{G}$ is Hopf, then the induced comonad $\widetilde{\mathcal{G}} := \widetilde{\mathcal{U}} \circ \widetilde{\mathcal{R}}$ is also Hopf.

Proof. We will consider the case when $\mathcal{X}$ is the terminal bicategory: then $\mathcal{N} = \mathcal{V}^{\mathcal{G}}$-Mod, and $\mathcal{M} = \mathcal{V}$-Mod. By Proposition 7.5 of [20], $\widetilde{\mathcal{U}}$ is a lax functor. First we show that it has local right adjoints $\widetilde{\mathcal{R}}_{\mathcal{A}}^{\mathcal{B}}$ given by

$$\mathcal{M}(\mathcal{U} \circ \mathcal{A}, \mathcal{U} \circ \mathcal{B}) \xrightarrow{\widetilde{\mathcal{R}}_{\mathcal{A}}^{\mathcal{B}}} \mathcal{N}(\mathcal{A}, \mathcal{B}) \tag{4.5.69}$$

$$(\mathcal{U} \circ \mathcal{A} \overset{M}{\nrightarrow} \mathcal{U} \circ \mathcal{B}, \alpha) \mapsto (\mathcal{A} \overset{\widetilde{\mathcal{R}}M}{\nrightarrow} \mathcal{B}, \widetilde{\mathcal{R}}\alpha) \tag{4.5.70}$$

$$(\sigma : M \Rightarrow N) \mapsto (\widetilde{\mathcal{R}}\sigma : \widetilde{\mathcal{R}}M \Rightarrow \widetilde{\mathcal{R}}N) \tag{4.5.71}$$

where α denotes a 2-sided action (the analogous 1-sided ones are denoted by λ and ρ) and the assignments are defined by

$$(\widetilde{\mathcal{R}}M)_B^A := \mathcal{R}M_B^A \tag{4.5.72}$$

$$\begin{aligned}(\widetilde{\mathcal{R}}\alpha)_{B,A'}^{B',A} &:= \mathcal{A}_A^{A'} \otimes \mathcal{R}M_{B'}^A \otimes \mathcal{B}_B^{B'} \\ &\xrightarrow{\gamma \otimes 1 \otimes \gamma} \mathcal{R}\mathcal{U}\mathcal{A}_A^{A'} \otimes \mathcal{R}M_{B'}^A \otimes \mathcal{R}\mathcal{U}\mathcal{B}_B^{B'}\end{aligned} \tag{4.5.73}$$

$$\xRightarrow{\mu^{(\mathcal{R})}} \mathcal{R}(\mathcal{U}\mathcal{A}_A^{A'} \otimes M_{B'}^A \otimes \mathcal{U}\mathcal{B}_B^{B'})$$
$$\xRightarrow{\mathcal{R}(\alpha)} \mathcal{R}M_B^{A'}$$
$$(\widetilde{\mathcal{R}}\sigma)_B^A := \mathcal{R}\sigma_B^A = \sigma_B^A\,. \tag{4.5.74}$$

Actions $\widetilde{\mathcal{R}}\alpha$, (or separately $\widetilde{\mathcal{R}}\lambda$ and $\widetilde{\mathcal{R}}\rho$) are compatible with unit and composition in $\mathcal{A}$ and $\mathcal{B}$. For example, compatibility of ρ with composition is witnessed by commutativity of diagram (4.5.75).

$$\begin{tikzcd}
\mathcal{R}M_{B''}^A \otimes \mathcal{B}_{B'}^{B''} \otimes \mathcal{B}_B^{B'} \arrow[rrrr, "1\otimes\mu^{(\mathcal{B})}"] \arrow[dr, "1\otimes\gamma\otimes\gamma"] \arrow[dd, "1\otimes\gamma\otimes 1"'] & & & & \mathcal{R}M \otimes \mathcal{B} \arrow[dd, "1\otimes\gamma"] \\
& \mathcal{R}M \otimes \mathcal{R}\mathcal{U}\mathcal{B} \otimes \mathcal{R}\mathcal{U}\mathcal{B} \arrow[rr, "1\otimes\mu^{(\mathcal{R}\circ\mathcal{U})}"] \arrow[dr, "1\otimes\mu^{(\mathcal{R})}"] \arrow[dd, "\mu^{(\mathcal{R})}\otimes 1"'] & & \mathcal{R}M \otimes \mathcal{R}\mathcal{U}(\mathcal{B}\otimes\mathcal{B}) \arrow[dr, "1\otimes\mathcal{R}\mathcal{U}\mu^{(\mathcal{B})}"] \arrow[dd, "\mu^{(\mathcal{R})}"] & \\
\mathcal{R}M \otimes \mathcal{R}\mathcal{U}\mathcal{B} \otimes \mathcal{B} \arrow[ur, "1\otimes 1\otimes\gamma"] \arrow[dd, "\mu^{(\mathcal{R})}\otimes 1"'] & & \mathcal{R}M \otimes \mathcal{R}(\mathcal{U}\mathcal{B}\otimes\mathcal{U}\mathcal{B}) \arrow[ur, "1\otimes\mathcal{R}\mu^{(\mathcal{U})}"] \arrow[ddd, "\mu^{(\mathcal{R})}"'] & & \mathcal{R}M \otimes \mathcal{R}\mathcal{U}\mathcal{B} \arrow[dd, "\mu^{(\mathcal{R})}"] \\
& \mathcal{R}(M\otimes\mathcal{U}\mathcal{B}) \otimes \mathcal{R}\mathcal{U}\mathcal{B} \arrow[ddr, "\mu^{(\mathcal{R})}"] \arrow[ddd, "\mathcal{R}\rho\otimes 1"'] & & \mathcal{R}(M\otimes\mathcal{U}(\mathcal{B}\otimes\mathcal{B})) \arrow[dr, "\mathcal{R}(1\otimes\mathcal{U}\mu^{\mathcal{B}})"] & \\
\mathcal{R}(M\otimes\mathcal{U}\mathcal{B}) \otimes \mathcal{B} \arrow[ur, "1\otimes\gamma"] \arrow[dd, "\mathcal{R}\rho\otimes 1"'] & & & & \mathcal{R}(M\otimes\mathcal{U}\mathcal{B}) \arrow[dd, "\mathcal{R}\rho"] \\
& & \mathcal{R}(M\otimes\mathcal{U}\mathcal{B}\otimes\mathcal{U}\mathcal{B}) \arrow[uur, "\mathcal{R}(1\otimes\mu^{(\mathcal{U})})"] \arrow[urr, "\mathcal{R}(1\otimes\mu^{(\mathcal{U}\circ\mathcal{B})})"'] \arrow[dr, "\mathcal{R}(\rho\otimes 1)"] & & \\
\mathcal{R}M \otimes \mathcal{B} \arrow[r, "1\otimes\gamma"'] & \mathcal{R}M \otimes \mathcal{R}\mathcal{U}\mathcal{B} \arrow[rr, "\mu^{(\mathcal{R})}"'] & & \mathcal{R}(M\otimes\mathcal{U}\mathcal{B}) \arrow[r, "\mathcal{R}\rho"'] & \mathcal{R}\mathcal{B}
\end{tikzcd} \tag{4.5.75}$$

In this, the non-obvious equalities might be the top pentagon, which is just stating that components of $\mu^{(\mathcal{B})}$ are coalgebra morphisms, naturality of μ squares, and the bottom right square obtained by applying $\mathcal{R}$ to compatibility of ρ with $\mu^{(\mathcal{U}\circ\mathcal{B})}$. Similarly, components of $\eta^{(\mathcal{B})}$ being coalgebra morphisms leads to compatibility of $\widetilde{\mathcal{R}}\rho$ with $\eta^{(\mathcal{B})}$. Compatibility of $\widetilde{\mathcal{R}}\sigma$ with $\widetilde{\mathcal{R}}\rho$ (and $\widetilde{\mathcal{R}}\lambda$) follows directly from the compatibility of σ with ρ (and λ).

The components of the unit and counit of the local adjunctions are given by components of γ and ϵ:

$$\tilde{\eta}_{\mathcal{A}}^{\mathcal{B}} : 1_{\mathcal{N}(\mathcal{A},\mathcal{B})} \Rightarrow \widetilde{\mathcal{R}}_{\mathcal{A}}^{\mathcal{B}}(\mathcal{U}\circ -) \tag{4.5.76}$$
$$((\tilde{\eta}_{\mathcal{A}}^{\mathcal{B}})_N)_B^A = \gamma_{N_B^A} : N_B^A \Rightarrow \mathcal{R}\mathcal{U}N_B^A \tag{4.5.77}$$
$$\tilde{\epsilon}_{\mathcal{A}}^{\mathcal{B}} : \mathcal{U}\circ\widetilde{\mathcal{R}}_{\mathcal{A}}^{\mathcal{B}}(-) \Rightarrow 1_{\mathcal{M}(\mathcal{U}\circ\mathcal{A},\mathcal{U}\circ\mathcal{B})} \tag{4.5.78}$$
$$((\tilde{\epsilon}_{\mathcal{A}}^{\mathcal{B}})_M)_B^A = \epsilon_{M_B^A} : \mathcal{U}\mathcal{R}M_B^A \Rightarrow M_B^A\,. \tag{4.5.79}$$

They form module morphisms, as witnessed by diagrams

$$\begin{tikzcd}
N_B^A \otimes \mathcal{B}_B^{B'} \arrow[rr, "\gamma\otimes 1"] \arrow[drr, "\gamma\otimes\gamma"] \arrow[ddr, "\gamma"] \arrow[dddd, "\rho^{(N)}"'] & & \mathcal{RU}N_B^A \otimes \mathcal{B}_{B'}^B \arrow[d, "1\otimes\gamma"] \\
 & & \mathcal{RU}N_B^A \otimes \mathcal{RU}\mathcal{B}_{B'}^B \arrow[dl, "\mu^{(\mathcal{RU})}"'] \arrow[dd, "\mu^{(\mathcal{R})}"] \\
 & \mathcal{RU}(N_B^A \otimes \mathcal{B}_B^{B'}) \arrow[ddr, "\mathcal{RU}\rho^{(N)}"'] & \\
 & & \mathcal{R}(\mathcal{U}N_B^A \otimes \mathcal{U}\mathcal{B}_{B'}^B) \arrow[ul, "\mathcal{R}\mu^{(\mathcal{U})}"] \arrow[d, "\mathcal{R}\rho^{(\mathcal{U}N)}"] \\
N_{B'}^A \arrow[rr, "\gamma"'] & & \mathcal{RU}N_{B'}^A
\end{tikzcd} \tag{4.5.80}$$

$$\begin{tikzcd}
\mathcal{UR}M_B^A \otimes \mathcal{U}\mathcal{B}_{B'}^B \arrow[d, "\mu^{(\mathcal{U})}"'] \arrow[dr, "1\otimes\mathcal{U}\gamma"] \arrow[drr, "\epsilon\otimes 1"] & & \\
\mathcal{U}(\mathcal{R}M_B^A \otimes \mathcal{B}_{B'}^B) \arrow[d, "\mathcal{U}(1\otimes\gamma)"'] & \mathcal{UR}M_B^A \otimes \mathcal{URU}\mathcal{B}_{B'}^B \arrow[r, "\epsilon\otimes\epsilon"] \arrow[dl, "\mu^{(\mathcal{U})}"'] \arrow[ddl, "\mu^{(\mathcal{UR})}"] & M_B^A \otimes \mathcal{U}\mathcal{B}_{B'}^B \arrow[d, "\rho^{(M)}"] \\
\mathcal{U}(\mathcal{R}M_B^A \otimes \mathcal{RU}\mathcal{B}_{B'}^B) \arrow[d, "\mathcal{U}\mu^{(\mathcal{R})}"'] & & M_{B'}^A \\
\mathcal{UR}(M_B^A \otimes \mathcal{U}\mathcal{B}_{B'}^B) \arrow[d, "\mathcal{UR}\rho^{(M)}"'] \arrow[uurr, "\epsilon"'] & & \\
\mathcal{UR}M_{B'}^A \arrow[uurr, "\epsilon"'] & &
\end{tikzcd} \tag{4.5.81}$$

and they satisfy the adjunction axioms because γ and ϵ do. Since $\widetilde{\mathcal{U}}$ has local right adjoints, it preserves local colimits, which, together with pseudofunctoriality of $\mathcal{U}$, gives sufficient conditions for pseudofunctoriality of $\widetilde{\mathcal{U}}$,

$$\begin{tikzcd}
\sum \mathcal{U}M_{B'}^A \otimes \mathcal{U}\mathcal{B}_B^{B'} \otimes \mathcal{U}N_{C'}^B \arrow[r, shift left, "\mathcal{U}\rho\otimes 1"] \arrow[r, shift right, "1\otimes\mathcal{U}\lambda"'] \arrow[d, "\sum\mu^{(\mathcal{U})}"'] & \sum \mathcal{U}M_B^A \otimes \mathcal{U}N_{C'}^B \arrow[r, "\mathrm{coeq}"] \arrow[d, "\sum\mu^{(\mathcal{U})}"] & (\mathcal{U}N \circ_{\mathcal{UB}} \mathcal{U}M)_{C'}^A \arrow[d, dashed, "\mu^{(\widetilde{\mathcal{U}})}"] \\
\sum \mathcal{U}(M_{B'}^A \otimes \mathcal{B}_B^{B'} \otimes N_{C'}^B) \arrow[r, shift left, "\mathcal{U}(\rho\otimes 1)"] \arrow[r, shift right, "\mathcal{U}(1\otimes\lambda)"'] & \sum \mathcal{U}(M_B^A \otimes N_{C'}^B) \arrow[r, "\mathrm{coeq}", "\mathcal{U}(\mathrm{coeq})"'] & \mathcal{U}(N \circ_{\mathcal{B}} M)_{C'}^A\,.
\end{tikzcd} \tag{4.5.82}$$

Since $\widetilde{\mathcal{U}}$ is a pseudofunctor and has local right adjoints, by Proposition 2.7 of [20], $\widetilde{\mathcal{R}}$ extends to a 2-sided enriched category which is a right adjoint to $\widetilde{\mathcal{U}}$. $\widetilde{\mathcal{U}}$ also satisfies the other two conditions of Proposition 4.5.2 as stated in Lemma 4.5.1. This completes the proof that $\widetilde{\mathcal{U}}$ is comonadic.

Explicitly, the unit for $\widetilde{\mathcal{R}}$ is a module morphism defined using the enriched functor (4.5.40)

$$\eta_{\mathcal{A}}^{(\widetilde{\mathcal{R}})} := (\gamma \circ \mathcal{A})_* \tag{4.5.83}$$

and the multiplication components

$$\widetilde{\mathcal{R}}(N) \circ_{\mathcal{B}} \widetilde{\mathcal{R}}(M) \overset{\mu^{(\widetilde{\mathcal{R}})}}{\Longrightarrow} \widetilde{\mathcal{R}}(N \circ_{\mathcal{U}\mathcal{B}} M) \tag{4.5.84}$$

are given by the right column of

$$\begin{tikzcd}
\sum \mathcal{R}M^A_{B'} \otimes \mathcal{B}^{B'}_B \otimes \mathcal{R}N^B_{C'} \arrow[r, shift left, "\widetilde{\mathcal{R}}\rho \otimes 1"] \arrow[r, shift right, "1 \otimes \widetilde{\mathcal{R}}\lambda"'] \arrow[d, "(\mathcal{R}i_{BB'} \bullet \mu^{(\mathcal{R})} \bullet 1 \otimes \gamma \otimes 1)_{BB'}"'] & \sum \mathcal{R}M^A_B \otimes \mathcal{R}N^B_{C'} \arrow[r, "\mathrm{coeq}"] \arrow[d, "(\mathcal{R}i_B \bullet \mu^{(\mathcal{R})})_B"] & (\mathcal{R}N \circ_{\mathcal{B}} \mathcal{R}M)^A_{C'} \arrow[d, dashed, "\mu^{(\widetilde{\mathcal{R}})}"] \\
\mathcal{R} \sum M^A_{B'} \otimes \mathcal{U}\mathcal{B}^{B'}_B \otimes N^B_{C'} \arrow[r, shift left, "\mathcal{R}(\rho \otimes 1)"] \arrow[r, shift right, "\mathcal{R}(1 \otimes \lambda)"'] & \mathcal{R} \sum M^A_B \otimes N^B_{C'} \arrow[r, "\mathcal{R}(\mathrm{coeq})"'] & \mathcal{R}(N \circ_{\mathcal{U}\mathcal{B}} M)^A_{C'}
\end{tikzcd} \tag{4.5.85}$$

where the top line is defining composition of modules in $\mathcal{N}$, the bottom line is $\mathcal{R}$ applied to the defining composition of modules in $\mathcal{M}$, i_B and $i_{BB'}$ are the coproduct inclusions, and $(-)_B$ denotes the induced map for mapping out of a coproduct.

For modules $M \in \mathcal{N}(\mathcal{A}, \mathcal{B})$ and $N \in \mathcal{M}(\mathcal{U} \circ \mathcal{B}, \mathcal{U} \circ \mathcal{C})$, the (left) Hopf map, given by the right column of

$$\begin{tikzcd}
\sum M^A_{B'} \otimes \mathcal{B}^{B'}_B \otimes \mathcal{R}N^B_{C'} \arrow[r, shift left, "\widetilde{\mathcal{R}}\rho \otimes 1"] \arrow[r, shift right, "1 \otimes \widetilde{\mathcal{R}}\lambda"'] \arrow[d, "\sum \gamma \otimes 1 \otimes 1"'] & \sum M^A_B \otimes \mathcal{R}N^B_{C'} \arrow[r, "\mathrm{coeq}"] \arrow[d, "\sum \gamma \otimes 1"] & (\mathcal{R}N \circ_{\mathcal{B}} M)^A_{C'} \arrow[d, "(1 \otimes \gamma)^A_{C'}"'] \\
\sum \mathcal{R}\mathcal{U}M^A_{B'} \otimes \mathcal{B}^{B'}_B \otimes \mathcal{R}N^B_{C'} \arrow[r, shift left, "\widetilde{\mathcal{R}}\rho \otimes 1"] \arrow[r, shift right, "1 \otimes \widetilde{\mathcal{R}}\lambda"'] \arrow[d, "\sum \mu^{(\mathcal{R})} \bullet 1 \otimes \gamma \otimes 1"'] & \sum \mathcal{R}\mathcal{U}M^A_B \otimes \mathcal{R}N^B_{C'} \arrow[r, "\mathrm{coeq}"] \arrow[d, "\sum \mu^{(\mathcal{R})}"] & (\mathcal{R}N \circ_{\mathcal{B}} \mathcal{R}\mathcal{U}M)^A_{C'} \arrow[d, "\mu^{(\widetilde{\mathcal{R}})}"'] \\
\sum \mathcal{R}(\mathcal{U}M^A_{B'} \otimes \mathcal{B}^{B'}_B \otimes N^B_{C'}) \arrow[r, shift left, "\mathcal{R}(\rho \otimes 1)"] \arrow[r, shift right, "\mathcal{R}(1 \otimes \lambda)"'] & \sum \mathcal{R}(\mathcal{U}M^A_B \otimes N^B_{C'}) \arrow[r, "\mathrm{coeq}", "\mathcal{R}(\mathrm{coeq})"'] & \mathcal{R}(N \circ_{\mathcal{U}\mathcal{B}} \mathcal{U}M)^A_{C'}
\end{tikzcd} \tag{4.5.86}$$

is invertible because the other two columns are invertible: they are determined by Hopf maps for $\mathcal{G}$. $\square$

Example 4.5.3. *Functors U and Σ from diagram (4.2.16) create duals and cohoms. An abelian group A has a dual if and only if it is finitely generated and projective [35]. As a consequence of Σ being Hopf-comonadic, a graded abelian group A has a dual if and only if it has finitely many non-zero components each of which is finitely generated and projective. As a consequence of U being Hopf-comonadic, a chain complex A has a dual if and only if its underlying graded abelian group does.*

Example 4.5.4. *The change of base functors $\widetilde{U}$ and $\widetilde{\Sigma}$, induced from U and Σ, create Cauchy modules.*

5

Conclusion and outlook

To finalise, we slice and serve low-dimensional categories in a somewhat different way, compared to the introduction.

We have used, or at least mentioned, various structures that satisfy strict associativity and unit laws: monoids (in Set, or any monoidal $\mathcal{V}$), monads, categories, lax functors, enrichment, 2-sided enrichment. There are numerous interesting statements connecting them that arise from this fact: monoids are one-object categories; monads are monoids in endohoms; categories are monads in Span(Set); monads are one-object categories (enriched in a bicategory); mapping out of 1 gives monads from lax functors, and enrichment (in a bicategory) from 2-sided enrichment; 2-sided enrichment is a lax functor into the matrix construction on the codomain bicategory [5].

Structures whose data is not strictly associative and unital, but only up to (invertible) associators and unitors satisfying 2-dimensional axioms, include: monoidal categories, monoidales (pseudomonoids), pseudomonads, bicategories, bienrichment... Similarly to the above, a bicategory $\mathcal{V}$, bienriched[1] [14] in a monoidal bicategory $\mathcal{M}$, is a many-object version of a monoidale in $\mathcal{M}$. The tricategory $\mathcal{M}$-Caten of $\mathcal{M}$-bicategories and 2-sided enriched categories was mentioned in [20]. Our analysis of Hopf comonads in Caten, Chapter 4, corresponds to the $\mathcal{M} = \text{Cat}$ case. However, it extends to any $\mathcal{M}$ which has EM-coalgebra objects, generalizing Hopf monoidal comonads on monoidales [10]. Various Hopf concepts obtainable from monoidal comonads on monoidales are summarised in [6]. Furthermore, duoidal categories, a differently generalized context for bimonoids, are monoidales in MonCat. With addition of 2-cells, one could consider (Hopf) bimonoidales, or Frobenious monoidales.

There are structures whose associators and unitors do not satisfy axioms strictly, but have one dimension higher (invertible) cells satisfying 3-dimensional axioms. We already mentioned monoidal bicategories, which can be seen as one-object tricategories. The iterated enrichment from the beginning of the introduction does not directly generalize to bienrichment since Bicat is a genuine tricategory, but one can use icons [9] instead of pseudonatural transformations, or by generalizing 1-cells, use 2-sided enrichment ($\mathcal{M}$-Caten is a monoidal bicategory). Finally, one could as well imagine 2-sided bienrichment as a convenient morphism of tricategories, and a zoo of structures that could live there.

[1] weakly enriched, or just enriched

A

Appendices

A.1 Simplices, intervals and shuffles

The *algebraist's delta*, denoted by Δ_a, is the full subcategory of **Cat** consisting of categories $\langle n \rangle$ whose objects are numbers $0, ..., n-1$ and 1-cells are unique $i \to j$ when $i \leqslant j$. The empty category is denoted $\langle 0 \rangle$. Arrows between $\langle n \rangle$ and $\langle n' \rangle$ are functors; that is, order preserving functions, generated by face and degeneracy maps

$$\sigma_i^n : \langle n+1 \rangle \to \langle n \rangle,\ i = 0, \ldots, n-1 \tag{A.1.1}$$

$$\partial_i^n : \langle n \rangle \to \langle n+1 \rangle,\ i = 0, \ldots, n \tag{A.1.2}$$

which can be presented in a diagram

$$
\langle 0 \rangle \xrightarrow{\partial^0_0} \langle 1 \rangle
\begin{array}{c} \xrightarrow{\partial^1_1} \\ \xleftarrow{\sigma^1_0} \\ \xrightarrow{\partial^1_0} \end{array}
\langle 2 \rangle
\begin{array}{c} \xrightarrow{\partial^2_2} \\ \xleftarrow{\sigma^2_1} \\ \xrightarrow{\partial^2_1} \\ \xleftarrow{\sigma^2_0} \\ \xrightarrow{\partial^2_0} \end{array}
\langle 3 \rangle \quad \cdots \tag{A.1.3}
$$

A natural transformation between f and $\bar{f}$, if one exists, is unique and witnesses that $fi \leqslant \bar{f}i$ for all i, turning $\Delta_a[\langle n \rangle, \langle n' \rangle]$ into a poset. The 2-category Δ_a is equipped with a strict monoidal structure, the ordinal sum $\oplus$.

A.1.1 Intervals - free monoid

Denote by $\Delta_{\perp\top}$ the subcategory of Δ_a, called the category of intervals, consisting of relabelled objects

$$
[n] := \langle n+1 \rangle,\ n = 0, 1, ... \tag{A.1.4}
$$

and 1-cells that preserve the first and the last element; it is generated by the arrows from the inside of the diagram (A.1.3), represented by the bold part of

$$
\cdot \longrightarrow \mathbf{[0]}
\begin{array}{c} \longrightarrow \\ \xleftarrow{\sigma^1_0} \\ \longrightarrow \end{array}
\mathbf{[1]}
\begin{array}{c} \longrightarrow \\ \xleftarrow{\sigma^2_1} \\ \xrightarrow{\partial^2_1} \\ \xleftarrow{\sigma^2_0} \\ \longrightarrow \end{array}
\mathbf{[2]} \quad \cdots \tag{A.1.5}
$$

It is clear that suspension (moving nodes to the left) gives an isomorphism

$$
\Delta^{op}_{\perp\top} \cong \Delta_a \tag{A.1.6}
$$

$$
[n] = \langle n+1 \rangle \mapsto \langle n \rangle \tag{A.1.7}
$$

$$
\sigma^n_i \mapsto \partial^{n-1}_i,\ i = 0, \ldots, n-1 \tag{A.1.8}
$$

$$
\partial^n_i \mapsto \sigma^{n-1}_{i-1},\ i = 1, \ldots, n-1 \tag{A.1.9}
$$

The tensor product on $\Delta_{\perp\top}$ is inherited from the ordinal sum under the isomorphism (A.1.6), and has the interpretation of path concatination;

$$
\xi : [n] \to [m] \tag{A.1.10}
$$

$$\xi' : [n'] \to [m'] \tag{A.1.11}$$

concatinate to

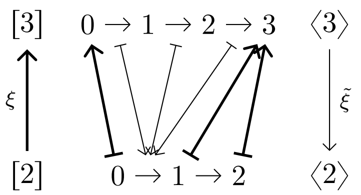
$$\xi + \xi' : [n + n'] \to [m + m'] \tag{A.1.12}$$

$$i \mapsto \begin{cases} \xi(i), \text{ if } i \leqslant n \\ \\ \xi'(i - n), \text{ otherwise.} \end{cases} \tag{A.1.13}$$

In particular, every such 1-cell ξ can be decomposed

$$\xi = \sum_{i=1}^{n} ! : [1] \to [\xi_i], \text{ with } \sum_{i=1}^{n} \xi_i = m. \tag{A.1.14}$$

The image of ξ under the isomorphism is an order preserving function that takes ξ_i points in $\langle m \rangle$ to $i \in \langle n \rangle$. An example of the isomorphism, for $n = 2$ and $m = 3$ can be visualized as



(A.1.15)

The embedding $\Delta_{\perp\top} \hookrightarrow \Delta_a$ is a monoidal functor with comparison maps representing

$$\langle 0 \rangle \xrightarrow{\partial_0^0} \langle 1 \rangle = [0] \tag{A.1.16}$$

$$[n] \oplus [n'] = \langle n + n' + 2 \rangle \xrightarrow{z_{n,n'} := \sigma_n^{n+n'+1}} \langle n + n' + 1 \rangle = [n] + [n'] \tag{A.1.17}$$

There is a functor

$$\Delta_{\perp\top}^{op} \xrightarrow{L} \Delta_a \tag{A.1.18}$$

$$[n] = \langle n + 1 \rangle \mapsto \langle n + 1 \rangle \tag{A.1.19}$$

$$\sigma_i^n \mapsto \partial_{i+1}^n, \, i = 0, \ldots, n - 1 \tag{A.1.20}$$

$$\partial_i^n \mapsto \sigma_i^n, \, i = 1, \ldots, n - 1 \tag{A.1.21}$$

assigning to each 1-cell in $\Delta_{\perp\top}$ its left adjoint (Galois connection) in Δ_a. Explicitly, for $\xi : [n] \to [m]$,

$$L(\xi) : \langle m + 1 \rangle \to \langle n + 1 \rangle \tag{A.1.22}$$

$$i \mapsto \min\{j | i \leqslant \xi(j)\}. \tag{A.1.23}$$

The functor L is oplax monoidal, with the same comparison maps (A.1.16)-(A.1.17), but the naturality holds up to a 2-cell

$$L(\xi + \xi') \circ z_{m,m'} \Rightarrow z_{n,n'} \circ (L\xi \oplus L\xi'). \tag{A.1.24}$$

Dually, there is a lax monoidal functor $\Delta_{\perp\top}^{op} \xrightarrow{R} \Delta_a$ assigning right adjoints, with a 2-cell

$$R(\xi + \xi') \circ z_{m,m'} \Leftarrow z_{n,n'} \circ (R\xi \oplus R\xi'). \tag{A.1.25}$$

The free 2-category containing a monad [23] is obtained as the suspension of the monoidal category of intervals,

$$\mathrm{FM} := \Sigma\Delta_{\perp\top}. \tag{A.1.26}$$

A.1.2 Shuffles - free distributive law

A shuffle of $\langle n \rangle$ and $\langle m \rangle$ in Δ_a is defined to be a pair of complement embeddings $\langle n \rangle \to \langle n+m \rangle \leftarrow \langle m \rangle$. Shuffles in $\Delta_{\perp\top}$ are inherited via the isomorphism (A.1.6) and have the following explicit description:

$$[n] \xleftarrow{r} [n+m] \xrightarrow{s} [m] \tag{A.1.27}$$

with the constraint

$$r_i + s_i = 1\,. \tag{A.1.28}$$

The numbers r_i and s_i are lengths (either 0 or 1 in this case) of the image of the i^{th} subinterval of $[n+m]$, as in (A.1.14). The condition (A.1.28) states that each subinterval maps to an interval of length 1 either in $[n]$ or in $[m]$.

An equivalent description of a shuffle is given by a relation of "appearing before in the shuffle"

$$\langle m \rangle^{\text{op}} \times \langle n \rangle \xrightarrow{l} \langle 2 \rangle\,. \tag{A.1.29}$$

The same relation can be interpreted as a shuffle of segments $[n]$ and $[m]$, for example

$$\begin{array}{c|cccccc}
[2]\backslash[3] & 0 \longrightarrow & 1 & \longrightarrow & 2 & \longrightarrow & 3 \\
0 & \longrightarrow & & & & & \\
\downarrow & 0 & \downarrow & 1 & & 1 & \\
1 & & & \longrightarrow & & \longrightarrow & \\
\downarrow & 0 & & 0 & & 0 & \downarrow \\
2 & & & & & &
\end{array} \tag{A.1.30}$$

A shuffle morphism $(\xi, \rho) : (n, m, s, r) \to (\bar{n}, \bar{m}, \bar{s}, \bar{r})$ consists of 1-cells $\xi : [\bar{n}] \to [n]$ and $\rho : [\bar{m}] \to [m]$ in $\Delta_{\perp\top}$, such that the following 2-cell in Δ_a exists

$$Lr \circ \xi \circ \bar{r} \Rightarrow Rs \circ \rho \circ \bar{s}\ . \tag{A.1.31}$$

When $\xi = 1_{[n]}$ and $\rho = 1_{[m]}$, the condition (A.1.31) is equivalent to the fact that the induced relations $l, \bar{l} : \langle m \rangle^{\mathrm{op}} \times \langle n \rangle \to \langle 2 \rangle$ satisfy $l \leqslant \bar{l}$, or that the $\bar{l}$ path in the table (A.1.30) appears to the down-left of the l path.

Shuffles and their morphisms form a category **Shuff** with the identity morphism $(1_{[n]}, 1_{[m]})$ and composition $(\xi \circ \bar{\xi}, \rho \circ \bar{\rho})$ for which the condition (A.1.31) is obtained by pasting

$$\begin{tikzcd}
& {[\bar{\bar{n}}]} \arrow[rr, "\bar{\xi}"] & & {[\bar{n}]} \arrow[rr, "1"] \arrow[dr, "L\bar{r}"'] \arrow[rr, phantom, shift right=3, "\eta \Downarrow"] & & {[\bar{n}]} \arrow[rr, "\xi"] & & {[n]} \arrow[dr, bend left, "Lr"] & \\
{[\bar{\bar{n}} + \bar{\bar{m}}]} \arrow[ur, bend left, "\bar{\bar{r}}"] \arrow[dr, bend right, "\bar{\bar{s}}"'] & & \Downarrow & & {[\bar{n} + \bar{m}]} \arrow[ur, "\bar{r}"'] \arrow[dr, "\bar{s}"] & & \Downarrow & & {[n+m]} \\
& {[\bar{\bar{m}}]} \arrow[rr, "\bar{\rho}"'] & & {[\bar{m}]} \arrow[rr, "1"'] \arrow[ur, "R\bar{s}"] \arrow[rr, phantom, shift left=3, "\epsilon \Downarrow"] & & {[\bar{m}]} \arrow[rr, "\rho"'] & & {[m]} \arrow[ur, bend right, "Rs"'] &
\end{tikzcd} \tag{A.1.32}$$

Shuff inherits a tensor product from $\Delta_{\perp\top}$ which (algebraically) follows from

$$L(r + r') \circ (\xi + \xi') \circ (\bar{r} + \bar{r}') \circ z \overset{(A.1.17)}{=} L(r + r') \circ z \circ (\xi \oplus \xi') \circ (\bar{r} \oplus \bar{r}') \tag{A.1.33}$$

$$\overset{(A.1.24)}{\Rightarrow} z \circ (Lr \oplus Lr') \circ (\xi \oplus \xi') \circ (\bar{r} \oplus \bar{r}') \tag{A.1.34}$$

$$\overset{(A.1.31)}{\Rightarrow} z \circ (Rs \oplus Rs') \circ (\rho \oplus \rho') \circ (\bar{s} \oplus \bar{s}') \tag{A.1.35}$$

$$\overset{(A.1.25)}{\Rightarrow} R(s + s') \circ z \circ (\rho \oplus \rho') \circ (\bar{s} \oplus \bar{s}') \tag{A.1.36}$$

$$\overset{(A.1.17)}{=} R(s + s') \circ (\rho + \rho') \circ (\bar{s} + \bar{s}') \circ z \tag{A.1.37}$$

but can also be seen as "direct summing"[1] the relation tables, for example the shuffle (A.1.30) can be interpreted as $([2] \xleftarrow{\sigma_1^3} [3] \xrightarrow{\sigma_0^2 \circ \sigma_2^3} [1]) + ([1] \xleftarrow{\sigma_1^2} [2] \xrightarrow{\sigma_0^2} [1])$.

The free 2-category containing a distributive law is obtained as the suspension of the monoidal category of shuffles,

$$\mathrm{FDL} := \Sigma \mathbf{Shuff}. \tag{A.1.38}$$

A.1.3 Mixed shuffle morphisms - free mixed distributive law

The category of mixed shuffles **MShuff** can be obtained by slightly modifying the construction of **Shuff**; the ρ component of the mixed shuffle morphism has the opposite direction $\rho : [m] \to [\bar{m}]$, and the existence condition (A.1.31) becomes

$$Lr \circ \xi \circ \bar{r} \Rightarrow Rs \circ R\rho \circ \bar{s}. \tag{A.1.39}$$

The 2-category containing a free mixed distributive law (FMDL) is obtained as the suspension of the monoidal category of mixed shuffles,

$$\mathrm{FMDL} := \Sigma \mathbf{MShuff}. \tag{A.1.40}$$

A.2 Cauchy completeness

Here we summarize basic definitions and results related to the general theory of Cauchy completeness. The motivating example is in the introduction.

Definition A.2.1. *A $\mathcal{V}$-module $M : \mathcal{B} \nrightarrow \mathcal{C}$ is called Cauchy if it has a right adjoint in* $\mathcal{V}$-Mod.

Proposition A.2.1. *[31] A $\mathcal{V}$-module M is Cauchy if and only if all M-weighted colimits are absolute.*

More on absolute colimits in (Set-)categories can be found in [27]. Absolute weights for enrichment in a bicategory were further examined in [13].

[1] As one would direct sum k-matrices between finite-dimensional k-vector spaces

Proposition A.2.2. *[19] For symmetric closed complete and cocomplete $\mathcal{V}$, a $\mathcal{V}$-module $M : \mathcal{I} \nrightarrow \mathcal{C}$ is Cauchy if and only if it is small-projective; that is, the representable functor*

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{V}](M, -) : [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \to \mathcal{V} \tag{A.2.1}$$

preserves small colimits.

Definition A.2.2. *A right $\mathcal{C}$-module $M : \mathcal{B} \nrightarrow \mathcal{C}$ is called convergent if there is a $\mathcal{V}$-functor $F : \mathcal{B} \to \mathcal{C}$ such that $M \cong F_* := \mathcal{C}(-, F-)$.*

When $\mathcal{B} = \mathcal{I}$, M being convergent is equivalent to M being representable in the usual sense.

Definition A.2.3. *A $\mathcal{V}$-category $\mathcal{C}$ is Cauchy complete if all Cauchy modules into $\mathcal{C}$ are representable.*

Proposition A.2.3. *A $\mathcal{V}$-category $\mathcal{C}$ is Cauchy complete if and only if it has all absolute-weighted colimits.*

A.3 Familial epiness

In this section we explore the notion of jointly epi families and how it can be extended to extremal, strong and regular epi families. The letter $\mathcal{V}$ denotes an ordinary category. Most of the concepts here are taken from [32].

Definition A.3.1. *A family of maps $\{A_i \xrightarrow{w_i} B\}_{i \in I}$ in $\mathcal{V}$ is jointly epi if any two maps $B \xrightarrow{f} C$ and $B \xrightarrow{g} C$ satisfying, for all i, $f \circ w_i = g \circ w_i$ implies $f = g$.*

Definition A.3.2. *A family of maps $\{A_i \xrightarrow{w_i} B\}_{i \in I}$ in $\mathcal{V}$ is jointly extremal epi if it is jointly epi and satisfies the invertible mono condition: namely, that any mono m through which all w_i factor is necessarily an isomorphism.*

Definition A.3.3. *A family of maps $\{A_i \xrightarrow{w_i} B\}_{i \in I}$ in $\mathcal{V}$ is jointly strong epi if it is jointly epi and satisfies the diagonal fill in condition: namely, that for any map $B \xrightarrow{g} D$, any mono $C \xrightarrow{m} D$, and any family of maps $\{A_i \xrightarrow{f_i} C\}_{i \in I}$ such that $m \circ f_i = g \circ w_i$, there is a unique diagonal filler $B \xrightarrow{d} C$ such that all triangles commute.*

Remark A.3.1. *As in the single epi case, if equalizers exist in $\mathcal{V}$, the condition of being jointly epi in order to be jointly extremal/strong, follows from the invertible-mono/diagonal-fill-in condition.*

Remark A.3.2. *As in the single epi case, any jointly strong epi family is jointly extremal epi, and in the presence of pullbacks, every jointly extremal epi family is a jointly strong epi family.*

Definition A.3.4. *A relation R on a family $\{A_i\}_{i\in I}$ of objects in $\mathcal{V}$ is given by a set $R_{i,j}$ of spans between A_i and A_j, for each i and j. We use R to denote the (disjoint) union of all $R_{i,j}$. A quotient of R is a family $\{A_i \xrightarrow{w_i} B\}_{i\in I}$ that is (part of) a colimit cone for the diagram consisting of objects $\{A_i\}_{i\in I}$ and spans in R between them. Explicitly, for each span*

$$A_i \xleftarrow{x} D \xrightarrow{y} A_j \tag{A.3.1}$$

in $R_{i,j}$, the square

$$\begin{tikzcd} D \arrow[r, "x"] \arrow[d, "y"'] & A_i \arrow[d, "w_i"] \\ A_j \arrow[r, "w_j"'] & B \end{tikzcd} \tag{A.3.2}$$

commutes, and the quotient is a universal family with this property. A kernel of an arbitrary family $\{A_i \xrightarrow{w_i} B\}_{i\in I}$, denoted $\mathrm{Ker}(\{w_i\})$, is the relation containing all spans of the form (A.3.1) satisfying (A.3.2).

If a family $\{A_i \xrightarrow{w_i} B\}_{i\in I}$ quotients some relation, then it quotients its kernel. That is because adding more spans (such that (A.3.2) commutes) to the colimit diagram does not change the colimit.

Definition A.3.5. *A family of maps $\{A_i \xrightarrow{w_i} B\}_{i\in I}$ in $\mathcal{V}$ is jointly regular epi if it is a quotient for some relation.*

Example A.3.1. *Cowedge components of a coend form a regular epi family. An (ordinary) functor $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{V}$ has a coend if and and only if the relation on $\{T(C,C)\}_{C\in\mathcal{C}}$ formed by spans*

$$T(C,C) \xleftarrow{T(f,C)} T(C',C) \xrightarrow{T(C',f)} T(C',C') \tag{A.3.3}$$

for each $f : C \to C'$, has a quotient, and they are the same (up to isomorphism). This is a reformulation of obtaining a coend [25] via a colimit.

Example A.3.2. *The same is true for an enriched coend. Let $\mathcal{V}$ be a locally small symmetric monoidal closed category, and $\mathcal{C}$ a $\mathcal{V}$-category. An enriched functor $T : \mathcal{C}^{op} \otimes \mathcal{C} \to \mathcal{V}$ can equivalently be seen as an endomodule on $\mathcal{C}$, given by actions*

$$\mathcal{C}(C', C'') \otimes T(C, C') \xrightarrow{\lambda^{C'}_{CC''}} T(C, C'') \tag{A.3.4}$$

$$T(C', C'') \otimes \mathcal{C}(C, C') \xrightarrow{\rho^{C'}_{CC''}} T(C, C'')\,. \tag{A.3.5}$$

It has a coend, defined as the quotient of the relation on $\{T(C,C)\}_{C \in \mathcal{C}}$ formed by spans

$$T(C, C) \xleftarrow{\rho^{C'}_{CC} \circ \sigma} \mathcal{C}(C, C') \otimes T(C', C) \xrightarrow{\lambda^{C}_{C'C'}} T(C', C') \tag{A.3.6}$$

for each pair of objects C, C'. Note that this quotient is isomorphic to the one quotienting the relation formed from

$$T(C, C) \xleftarrow{\rho^{C'}_{CC}} T(C', C) \otimes \mathcal{C}(C, C') \xrightarrow{\lambda^{C}_{C'C'} \circ \sigma} T(C', C') \tag{A.3.7}$$

since σ is an isomorphism of spans constituting the colimit diagrams.

Example A.3.3. *Module composition cocone components form a regular epi family. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be small $\mathcal{V}$-enriched categories for a cocomplete monoidal category $\mathcal{V}$, together with a pair of modules $\mathcal{C} \overset{M}{\nrightarrow} \mathcal{D} \overset{N}{\nrightarrow} \mathcal{E}$. Fix objects $C \in \mathcal{C}$, $E \in \mathcal{E}$, and consider the relation on $\{M(D,C) \otimes N(E,D)\}_{D \in \mathcal{D}}$ consisting of spans as in (A.3.8).*

$$\begin{tikzcd}[column sep=small] & M(D', C) \otimes \mathcal{D}(D, D') \otimes N(E, D) \arrow[dl, "\rho^{(M)}_{CDD'} \otimes 1"'] \arrow[dr, "1 \otimes \lambda^{(N)}_{DD'E}"] & \\ M(D, C) \otimes N(E, D) & & M(D', C) \otimes N(E, D') \end{tikzcd} \tag{A.3.8}$$

Its quotient is precisely the definition of the composite module, with quotient maps

$$M(D, C) \otimes N(E, D) \xrightarrow{w^{CE}_{D}} (N \circ_{\mathcal{D}} M)(E, C)\,. \tag{A.3.9}$$

In particular, when $\mathcal{V}$ is symmetric closed, this is isomorphic to the enriched coend

$$\int^{D \in \mathcal{D}} M(D, C) \otimes N(E, D)\,. \tag{A.3.10}$$

Remark A.3.3. *If $J = 1$ the above definitions reduce to the definitions of (extremal, strong or regular [18]) epimorphisms. Furthermore, if $\mathcal{V}$ has coproducts, the induced map $\sum_i A_i \xrightarrow{w} B$ is extremal/strong epi if and only if $\{w_i\}_{i\in I}$ is a jointly extremal/strong epi family.*

The regular case is examined in

Proposition A.3.1. *In the category $\mathcal{V}$ with coproducts, a jointly regular epi family $\{w_i\}_{i\in I}$ induces a regular epi map $\sum_i A_i \xrightarrow{w} B$. The converse is true if for all parallel pairs $x, y : D \to \sum_i A_i$ the family*

$$F_{xy} = \{p : P \to D | \exists i, j, p_i : P \to A_i, p_j : P \to A_j, \ \textit{such that} \tag{A.3.11}$$

$$x \circ p = \theta_i \circ p_i \ \textit{and} \ y \circ p = \theta_i \circ p_j\} \tag{A.3.12}$$

is jointly epi.

Proof. Considering the diagram

$$\begin{tikzcd} & & & B \\ D \arrow[r, "x"] & A_i \arrow[r, "\theta_i"] \arrow[urr, bend left, "w_i"] \arrow[drr, bend right, "f_i"'] & \sum A_i \arrow[ur, "w"'] \arrow[dr, "f"] & \\ & & & C \end{tikzcd} \tag{A.3.13}$$

it is easy to see that

$$\mathrm{Ker}(w) \subset \mathrm{Ker}(f) \implies \mathrm{Ker}\{w_i\} \subset \mathrm{Ker}\{f_i\} \tag{A.3.14}$$

so given f satisfying $\mathrm{Ker}(w) \subset \mathrm{Ker}(f)$, and using that $\{w_i\}$ is joint regular epi we get a unique factorization of f through w, proving that w is regular epi.

Conversely, given a regular epi w, and f such that $\mathrm{Ker}\{w_i\} \subset \mathrm{Ker}\{f_i\}$, consider an arbitrary element of $\mathrm{Ker}(w)$, $x, y : D \to \sum_i A_i$, that is $w \circ x = w \circ y$, and an arrow $p \in F_{xy}$. Chasing diagrams gives

$$w_i \circ p_i = w \circ \theta_i \circ p_i \tag{A.3.15}$$

$$= w \circ x \circ p \tag{A.3.16}$$

$$= w \circ y \circ p \tag{A.3.17}$$

$$= w \circ \theta_j \circ p_j \tag{A.3.18}$$

$$= w_j \circ p_j \tag{A.3.19}$$

so $(p_i, p_j) \in \mathrm{Ker}(\{w_i\})$, and using the assumption for f, $(p_i, p_j) \in \mathrm{Ker}(\{f_i\})$. So we have

$$f_i \circ p_i = f_j \circ p_j \tag{A.3.20}$$

$$f \circ \theta_i \circ p_i = f \circ \theta_j \circ p_j \tag{A.3.21}$$

$$f \circ x \circ p = f \circ y \circ p\,. \tag{A.3.22}$$

Using joint epiness of F_{xy} we conclude that $(x, y) \in \mathrm{Ker}(f)$, and, because w is regular epi, f factors uniquely through it. □

Remark A.3.4. *As in the single epi case, any jointly regular epi family is automatically jointly strong epi. The converse is true when $\mathcal{V}$ is familialy regular, a proof of a stronger statement is given in [32].*

Example A.3.4. *In a preordered set $\mathcal{V}$, any family $\{A_i \xrightarrow{w_i} B\}_{i \in I}$ is jointly epi.*

Example A.3.5. *In a poset $\mathcal{V}$ with arbitrary joins, a family $\{A_i \xrightarrow{w_i} B\}_{i \in I}$ is jointly extremal/strong/regular if and only if $B = \bigvee_i A_i$.*

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