# Children's quantitative sense of fractions

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# Contents

# ABSTRACT

Learning the meaning of common fractions and how to operate with them is a traditionally difficult aspect of learning mathematics. The symbol system used to represent fractions, one whole number written above another whole number, is not transparent to the meaning of fractions. This difficulty of interpreting fractions contributes to traditional practice in teaching fractions emphasising the syntax of fractions over their semantics. Without a sense of the size of fractions students must rely solely on learning the fraction syntax, as there is no feasible way of checking the reasonableness of an answer.

To investigate how well students had developed a sense of the size of fractions, a large cross section of students from Years 4–8 (over 300 students in each grade) completed 37 tasks designed to draw upon a quantitative sense of fractions. The students' answers provided data for both a conceptual analysis and a related Rasch item analysis. The conceptual analysis found that although some students have a strong sense of the size of fractions, many students have developed pseudo fraction concepts. Rather than seeing fractions as relational numbers, more than 10% of the students responded as if fractions corresponded to a count of the number of parts in a fraction representation. Questions involving the fraction notation increased the variety of incorrect interpretations of fractions. The Rasch analysis confirmed that the items related to the same trait and provided an ordering of the difficulty of the questions. The Rasch item map provided backing for a description of how students develop a sense of the size of fractions.

The current methods of developing students' understanding of fractions have resulted in a large number of pseudo concepts being formed. The basis of making meaning from models used to introduce fractions, needs to be the focus of teaching fractions. As well as using counter-examples to limit the number of unintended features of models students associate with fractions, comparison of length rather than area should be used to introduce fractions.

The work described in this thesis is original and has not been submitted in any form for a higher degree at any other university or institution. All of the work presented in this thesis is my own and was undertaken during my PhD candidature.

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# Chapter 1 INTRODUCTION

*The concept of a fraction as a mathematical object has developed over time, as has the notation system for fractions. A historical perspective on the development of the fraction concept is important in understanding the evolution of the approaches used with the teaching of fractions and the corresponding research into the acquisition of rational number knowledge.*

*Common fractions have multiple embodiments and representations. Also, the symbol system used with fractions is very powerful but is not transparently linked to the concept. Traditional practice in teaching fractions has been to emphasise the syntax of fractions over their semantics. However, neither the procedural knowledge of operating with fractions nor the conceptual knowledge associated with estimating the size of fractions as mathematical objects, appear widespread in school students.*

**Preview**

In this chapter an examination of the historical origins of fractions highlights the relationship between the need to solve division problems and the teaching of algorithms, both of which are needed to address the demands placed on the use of fractions in society. Two different forms of fractions, partitioned fractions and quantity fractions, are described to underscore the role of a quantitative sense of fractions in connecting the two forms.

The teaching of fractions may not have kept pace with the shift of focus from procedural knowledge to the conceptual knowledge needed to appreciate fractions as mathematical objects. This mismatch may contribute to fundamental difficulties students have in learning fractions.

The decimal fraction is more properly an extension of the base-ten place-value system, than a fraction as a mathematical object. As such, decimal fractions are not specifically addressed in this study.

## 1.1 The origins of common fractions

Fractions originated from practical problems of division. Accurate calculations with fractions, as Joseph (1991) notes, were required in societies that did not use money and where transactions were carried out in kind. In very early examples of writing, such as the Ahmes Papyrus (c. 1650 B.C.), the practical problems of division of food and mixing different ingredients for beer or bread illustrate the applications of fractions to the needs of society and the associated arithmetical procedures (Joseph, 1991). Remuneration for temple workers in ancient Egypt was distributed in portions of bread and beer according to the status of the worker. The practical nature of the division and the resulting fractional portions of loaves of bread gave rise to a unique system of fractions. As an illustration of the practical nature of fraction use in ancient Egypt, one of the problems on the Ahmes Papyrus deals with the division of 9 loaves among 10 men. Using current systems of notation we would state that each man would receive nine-tenths of a loaf. Our current system of fractions overlooks the practical problem of division. Namely, the tenth worker may not consider the division just if he receives 9 pieces, each one-tenth of a loaf whereas the other men receive a single piece of bread, nine-tenths of a loaf. The Egyptian system of fractional division dealt with a form of equivalent fraction decomposition, based upon fractions as parts of one whole. The practical division problems addressed by the Egyptians meant that their use of fractioning was quite different from our modern interpretation of fractions as mathematical objects.

To represent a fraction, the Egyptians used the symbol  meaning ‘part’ with the divisor underneath. Thus a “fifth part” would appear as . Although Egyptian fractions have often been described as unitary fractions, this is misleading as a unitary fraction implies the existence of a numerator as well as a denominator (Benoit, Chemla, & Ritter, 1992). Egyptian fractions were initially developed as practical solutions to problems, not as mathematical objects.

### *1.1.1 The development of our current fraction notation*

> When above any number a line is drawn, and above that is written any other number, the superior number stands for the part or parts of the inferior number; the inferior number is called the denominator, the superior the numerator. Thus, if above the two a line is drawn, and above that unity is written, this unity stands for one part of two parts of an integer, i.e., for a half, thus $\frac{1}{2}$.
>
> Translated from *Il Liber abbaci di Leonardo Pisano* (ed. B. Boncompagni; Roma, 1857), pp 23–24, quoted in Cajori (1974, p. 269).

Indian mathematics provided both our current place-value symbol system including zero, as well as decimal fractions. Our method of writing common fractions is due essentially to the Hindus, although they did not use the fraction bar (D. E. Smith, 1958, p. 215 ). Common fractions are represented in the Bakhshali Manuscript as two whole numbers with the denominator placed below the numerator without using a fraction bar or vinculum (Joseph, 1991). The Arabs introduced the vinculum, although it was not used by all of their writers. Al-Khwarizmi introduced the Hindu notion of fraction to the Arabic world and the word “algorithm”, a corruption of the name al- Khwarizmi, is still associated with any structured scheme involving the Hindu numerals, especially the arithmetical operations on fractions. He also showed how operations on these fractions could be achieved through largely mechanical methods around 800 AD.

The cultural practices associated with the religion of Islam, such as the Islamic laws of inheritance, make substantial demands on proportional reasoning. For example, when a woman dies her husband receives one-quarter of her estate, and the rest is divided among the children such that a son receives twice as much as a daughter. However, if a legacy is left to a stranger, the division gets more complicated as a stranger cannot receive more than one-third of the estate without the permission of the natural heirs. If only some of the heirs endorse the legacy those who do must between them pay, pro rata, out of their own shares the amount by which the stranger’s legacy exceeds one-third (Joseph, 1991). Consequently, reducing the challenge of operating with ratios to a mechanical procedure of manipulating pairs of whole numbers, eventually written as $\frac{a}{b}$, was a significant cultural achievement. Fibonacci (Leonardo of Pisa) in his *Liber abbaci* was perhaps the first European mathematician to use the fraction bar as it is used today. However, he followed the Arab practice of placing the fraction to the left of the integer (Cajori, 1974, p. 89). It is through his *Liber abbaci* that the current representation of a fraction as a mathematical object was communicated and popularised.

The representational system for common fractions has developed over time and with influences from many cultures (Benoit et al., 1992). Ratio and proportional reasoning were formalised as a mode of reasoning in Euclid’s Elements (c. 300 B.C.), yet fractions as objects were not well understood for another 1500 years (Davis, 2003).

### 1.1.2 The changing needs of society

The history of the development of mathematical ideas provides a useful insight into the problems students encounter in attempting to learn to use fractions. Fractions, which arose from the practical problem of managing division of quantities, developed over time into abstract mathematical objects, numbers. The fraction notation system has formed in response to the need for manageable techniques to deal with proportional reasoning and measures of rates of change (Thompson & Saldanha, 2003). However, the multiple representations of the one entity, a fraction, provide both the power and the challenge of fractions. A fraction such as $\frac{2}{3}$ can be used to describe the operation of dividing 2 by 3 as well as representing the answer to the problem of sharing 14 loaves of bread equally among 21 people. The very same fraction is also a mathematical entity, a unique number corresponding to a position on the number line and is itself a number that can be involved in division, multiplication, addition and subtraction.

In addition to the multifaceted complexity of the fraction concept, the symbol system used to record fractions is counter-intuitive. The symbol system shows "two whole numbers", the numerator and the denominator, one over the other separated by a bar. The fractional quantity is a single entity representing the relationship between the two whole numbers. Students may interpret the symbol as whole numbers, rather than as a relationship. Operating with fractions also reduces to operating with whole numbers, the numerator and denominator. Further, many students learn to interpret "$\frac{a}{b}$" as denoting a part-whole relationship, for example that "$\frac{3}{7}$" means "three out of seven" (Brown, 1993). This interpretation encourages a perception of fractions as the result of a double count; one count, the numerator, over a second count, the denominator.

The idea of ratio and proportional reasoning described in Euclid's *Elements* portray a fraction as a geometric comparison. In the geometric sense of a fraction as the ratio of two lengths, we have a robust feel for fractions as comparisons; however, any reasoning involving fractions using this geometric sense is complex and this complexity may itself have given rise to the need for fraction thinking as process or "algorithm". In the arithmetic sense of a fraction as an object, the representational system is extremely important. The power of al-Khwarizmi's notion of fractions is that the representational system allows us to be assured of a correct answer based on following the rules alone.

Viewed from the stance of history, the algorithmic approach to fractions was a necessary and significant advance to thinking, which naturally led to teaching fractions as algorithmic or procedural knowledge. The expansion of business and commerce during the Industrial Revolution led to computation of fractions assuming an important role in the school mathematics curriculum (National Council of Teachers of Mathematics, 1970). A time-efficient path to formal algorithmic computation involving fractions served the needs of society at that time. Clerical tasks involving calculations with measures and money relied upon the correct manipulation of fractions. However, the computational role played by fractions in society changed as the day-to-day manipulation of "common" fractions became less common. Nevertheless, an emphasis on formal symbolic computation involving fractions has persisted within schools (Behr, Wachsmuth, Post, & Lesh, 1984; Ellerton & Clements, 1994). The business and social needs for fractions moved from algorithmic procedures to a greater need for proportional reasoning and measures of rates of change (Litwiller, 2002; Thompson & Saldanha, 2003). Nevertheless, as the social need for algorithmic fraction knowledge changed, the approach to teaching fractions changed only superficially. Computation involving fractions was identified as a common component of the curriculum for the 41 countries involved in the Third International Mathematics and Science Study (TIMSS) (Lokan, Ford, & Greenwood, 1996).

## 1.2 Teaching fractions: Algorithm or concept?

One of the long-standing debates in mathematics education concerns the relative importance of conceptual knowledge versus procedural knowledge, or understanding versus skill (Brownell, 1935; Bruner, 1960). In his 1978 seminal paper on what it means to understand mathematics, Richard Skemp described two meanings of understanding mathematics: relational and instrumental. Relational understanding is described as knowing both what to do and why, whereas instrumental understanding can be thought of as rules without reasons (Skemp, 1978).

The two types of understanding, conceptual knowledge and procedural knowledge, are evident in the various approaches to teaching and learning fractions (Heibert, 1986). The rules for operating with fractions are clearly examples of procedural knowledge. The history of the use of fractions provides a rationale for the initial emphasis on the procedural knowledge of methods of manipulating fractions.

When the manipulation of fractional quantities in day-to-day calculations was a common need of society, being able to operate correctly with fractions was a clear educational objective. The teaching of fraction algorithms was oriented towards developing an instrumental understanding as described in the following excerpt.

> In work on operations with fractions, the algorithms frequently are developed as an extension of whole number algorithms… an extension of counting (adding with common denominators)… Because of this, the curriculum and instruction prematurely emphasise technical symbolic operating rules (lining up decimal points, finding least common denominator, etc.). These extensions usually are not built on the intuitive mathematics of fractional numbers… Children get the form but not the substance of the system. This may result in temporary achievements with fragments of knowledge but not in lasting, useful, powerful, personal knowledge.
>
> (Kieren, 1988, p. 177)

Although computers and calculators have reduced the need for efficient formal symbolic computation with fractions, algorithmic computation with fractions remains a component of current curricula due to the algebraic efficiency of combining the process of division with the answer to that division in fraction form. Internationally, computation with fractions is an area of poor performance (Lokan et al., 1996) and fraction learning is a serious obstacle in the mathematical development of children (Behr, Harel, Post, & Lesh, 1992; Kieren, 1976, 1988; Mack, 1990; Pitkethly & Hunting, 1996).

## 1.3 What is meant by a fraction?

Fractions are sometimes described formally by the notation $\frac{a}{b}$ where $a$ and $b$ are integers and $b \neq 0$, or even more technically as elements of a quotient field consisting of equivalence classes. These formal approaches to defining fractions rely upon a pre-existing sense of what is meant by a fraction. In practice, fractions exist in essentially two forms; embodied representations of comparisons, sometimes called partitioned fractions, and mathematical objects, also known as quantity fractions (K. Yoshida, 2004). An embodied representation of a comparison could be 3 parts out of 8 equal parts, such as three-eighths of a pizza. As with any fraction describing an embodied representation of a comparison, $\frac{3}{8}$ as a partitioned fraction is always $\frac{3}{8}$ of something. When dealing with a partitioned fraction, $\frac{3}{8}$ of something cannot reasonably be compared to $\frac{1}{2}$ of something else, a different partitioned fraction, because $\frac{3}{8}$ of something is not always less than $\frac{1}{2}$ of something else. By contrast, fractions as quantities are more than parts of a whole — they are abstract measurement units that make use of a notional 'one whole' with meaning independent of specific contexts. Consequently, $\frac{3}{8}$ considered as a mathematical object is less than $\frac{1}{2}$, as both fractions are like measurement units referencing the same 'one'. However, fractions as mathematical objects are in effect dimensionless. One unit of length divided by two units of length produces one-half as a number in the same way that one kilogram divided by two kilograms produces the number one-half.

### 1.3.1 What is a quantitative sense of fractions?

Teaching fractions usually starts from models of partitioned fractions (Watanabe, 2002) but to compare and order fractions students have to make the transition from partitioned fractions to abstract quantity fractions and to develop a sense of the size of fractions. A sense of the size of fractions is what Saenz-Ludlow (1994) refers to as conceptualising fractions as quantities. She argues that children need to conceptualise fractions as quantities before they are introduced to standard symbolic computation algorithms.

Similarly, in trying to determine which fraction form should play a central role in the development of the basic rational number concept, Behr, Lesh, Post, & Silver (1983) refer to "a strand of data" arising from their teaching experiments that bears on this question. They describe this data strand as concerning children's development of a *quantitative notion of rational number* by which they mean children's ability to demonstrate the size of rational numbers. I describe this quantitative notion of rational number as a *sense of the size of fractions*. The importance of this quantitative sense of fractions in enabling students to move from partitioned fractions to quantity fractions is highlighted by Behr et al. (1983, p. 120) who observed that a quantitative notion of fractions is "… fundamental in children's development of rational-number concepts, relations, and operations. It apparently underlies children's ability to order rational numbers, to internalize the concept of equivalent fractions, and to have a meaningful grasp of addition and multiplication of fractions. What meaning does the addition of $\frac{3}{8}$ and $\frac{4}{8}$ have for a child without an internalized notion of the 'bigness' of each addend and the sum?"

Despite the fundamental value of a sense of the size of fractions, it does not appear to be specifically taught or learnt. Studies in several countries (T. P. Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Hart, 1989; Kerslake, 1986; Ni & Zhou, 2005; H. Yoshida & Kuriyama, 1995) suggest that the underpinning knowledge of fractions as mathematical objects is frequently absent from students' concepts.

### *1.3.2 The fundamental difficulties students have in learning fractions*

Treating the fraction as being composed of two separate whole numbers often influences the difficulties that students experience in learning fractions. Sometimes these whole numbers are loosely related by division, as was the case in Mack's (1995) study where some children interpreted $\frac{1}{3}$ as "one whole divided into three parts". This incomplete interpretation allowed the children to rationalise the incorrect addition "$\frac{1}{3}+\frac{1}{3}=\frac{2}{6}$" by thinking that combining one whole divided into three parts with another whole divided into three parts would result in two wholes and six parts altogether.

Some children appear to see fractions as two whole numbers (Streefland, 1993). When viewed this way, it is not uncommon for children to apply whole number strategies to fraction problems (Lamon, 1999; Mack, 1995). This application of whole number strategies can result in the belief that $\frac{1}{5}$ is bigger than $\frac{1}{3}$ because 5 is bigger than 3 (Kerslake, 1986), whereas having a quantitative sense of fractions involves treating each fraction as a single numeric quantity. Lack of a quantitative sense of fractions is evident when children treat a fraction as two unrelated or loosely related whole numbers.

In addition to the problem of the application of whole number strategies to fraction problems, the dual function of the fraction notation creates problems for students. The fraction notation is subject to a paradox that is central to mathematical thinking (Lehrer & Lesh, 2003). On one hand, a fraction such as $\frac{2}{3}$ takes its meaning from the situations to which it refers (partitioned fraction); on the other hand, it derives its mathematical power by divorcing itself from those situations (abstract quantity fraction). The parts-of-a-whole model gives meaning to "$\frac{3}{7}$" as "three out of seven" from working with partitioned embodiments of this fraction, but unless the notation is divorced from this context "$\frac{7}{3}$" does not make sense.

## 1.4 The purpose of this study

This study seeks to identify:

- a quantitative sense of fractions (i.e. what it means to understand the size of a fraction) as a construct that is helpful in establishing fractions as mathematical objects, and

- any significant misconceptions associated with this quantitative sense of fractions among children.

Non-examples of a quantitative sense of fractions help to create the boundaries and clarify the construct. Although the need to carry out computations with fractions has reduced, the need for a quantitative sense of fractions has not diminished in society. Fractional quantities remain embedded in many measurement tasks required each day.

Taking up Saenz-Ludlow's phrase of *conceptualising fractions as quantities* we need to ask, "What does it mean to conceptualise fractions as quantities?" Fractions are relational numbers and it is the process of forming the relation, part-to-whole and whole-to-part, which creates the sense of the absolute size of fractions.

# Chapter 2 FRACTIONS AS MATHEMATICAL OBJECTS: AN OVERVIEW OF THE RESEARCH

*Research into the nature and acquisition of rational number knowledge and, more specifically, fractions as mathematical objects, is very extensive. The focus of this review of research is on how children come to understand fractions as quantities.*

*Although the terms 'rational number' and 'fraction' are sometimes used interchangeably in the research, the focus of the review is common fractions. Decimal fractions and percentages are distinct areas of research, shaped by their own distinctive representational systems.*

**Preview**

This chapter provides an overview of research related to investigating a quantitative sense of fractions by addressing two interrelated themes concerning fractions: structure and meaning. The theme of structure includes the fraction notation and syntax, as well as associated misconceptions related to interpreting the symbols with which one operates. The syntax of fractions is complex and cannot be totally separated from the semantic problem of extracting meaning. However, meaning — the symbolisation, language and representations of fractions — is viewed in this chapter from the stance of its contribution to quantitative ideas of fractions.

Starting with the theme of 'structure', the evolution of the theoretical analyses of rational numbers is described, along with the notion of various *subconstructs* (ratio number, operator, quotient, measure and part-whole) that contribute to the domain of rational numbers. Two interpretations of part-whole number are discussed; additive part-whole and multiplicative part-whole, followed by an examination of the critical role played by partitioning in the formation of fractions.

The common fraction notation is then described as an algebraic notation system and the influence of the fraction symbols on students' answers to tasks is outlined. In particular, the role of single-unit counting schemes, used to interpret the notation, is outlined as contributing to the development of 'whole number bias' in some students. The fraction notation and the language used to name fractions are considered as cultural tools.

A brief semantic analysis of the multiplicative domain structure associated with fractions involves an elaboration of part-whole relations and partitioning, which leads into the analysis of the action basis of scheme development, and in particular, unitising. The various subconstructs may all be understood as compositions and recompositions of units (halving, measuring, partitioning-sharing, compensation).

For fractions to make the transition from embodied partitions to mathematical objects they need to draw upon a universal unit-whole or 'one' that remains the same size in all contexts and is similar to a standard unit of measure. The various embodiments of the fraction concept and the related action-based schemes contribute to or inhibit the development of fractions as mathematical objects which reference a universal unit-whole (abstract quantity fractions).

The *evoked concept image* is described at the individual fraction level as a way of interpreting the subconstructs and action-based schemes in stMackudents' responses and to investigate both positive and negative instances of growth in a quantitative sense of fractions.

## 2.1 Introduction

Fractions have a long history as a topic of educational research (e.g. McLellan & Dewey, 1895/2007). The complexity of the fraction concept is reflected in the extensive analyses of rational number meaning mapped out by Kieren (1976, 1980b, 1988, 1993) and the Rational Number Project (Behr et al., 1992; Behr, Harel, Post, & Lesh, 1993; Lesh, Behr, & Post, 1987).

As Kieren (1994) has pointed out, researchers who have undertaken theoretical analyses in the domain of rational numbers have adopted two general stances: epistemological and psychological. The emphasis of epistemological research has been to clarify the nature of rational numbers as mathematical constructs and the subconstructs which comprise rational numbers (e.g. ratio, quotient, measure and operator). By contrast, the aim of psychological research has been to identify the knowledge structures that children bring to the domain of rational numbers and the way in which this knowledge develops as the children are introduced to the domain in a more formal fashion. Mack (1990, 1995, 2001) refers to this development as building on informal knowledge through instruction in a complex content domain.

## 2.2 The nature of the fraction construct

The analyses of rational number meaning, sometimes referred to as *interpretations* of rational numbers (Kieren, 1976), *subconstructs* (Behr et al., 1983) or *personalities* (Behr et al., 1992) have evolved over time. The history and critique of the analyses of rational number as captured by Ohlsson (1988), Olive (1999) and in the detailed semantic analysis of rational number provided by Behr et al. (1992) are briefly described below.

Although Behr et al. (1992) argue that five subconstructs of rational number — part-whole, quotient, ratio number, operator, and measure — have stood the test of time, it is the interpretation and place of part-whole that is particularly germane to the development of fractions as quantities in teaching and learning. Part-whole models or relationships can be troublesome. The problem of the simplistic use of part-whole models is well described in the following passage.

> *Because part-whole models of fractions conveniently help produce fractional language, the school mathematics fraction language of teacher and texts alike tend to orient a student to a static double count image and knowledge of fractions. The child, while being able to produce "correct" answers to questions, develops a mental model which is inappropriately inclusive (parts of a whole), rather than a powerful measure of inclusion (comparison to a unit) suggested by Vergnaud (1983). Further, the language of fractions frequently orients the child to a static outcome (a double count of parts) rather than to the act of "dividing into 'n' parts" and its representation mathematically "/n".* (Kieren, 1988, p. 177)

Using simple part-whole models as the primary means of defining fractions has clear limitations (Freudenthal, 1983; Wachsmuth, Behr, & Post, 1983) because a student may focus on whole numbers within these models, rather than the equal-partitioning and the relationship between the parts and the whole. A focus on the parts that make up the whole (part + part = whole) can lead to an additive relationship, rather than a multiplicative comparison of the part to the whole. Many of the problems associated with learning fractions have been attributed to teaching efforts that have focussed almost exclusively on the part-whole construct (Kerslake, 1986; Streefland, 1991). Consequently, this teaching practice has been called into question (Behr et al., 1992; Kieren, 1993). Despite the limitations of part-whole models, many curricular offerings emphasise part-whole models almost exclusively in the teaching of fractions in the primary years (Middleton, Toluk, deSilva, & Mitchell, 2001).

The idea that rational number consists of several interpretations was first introduced by Kieren who argued that from the point of view of curriculum, "…it has been common to implicitly assume that rationals had some single interpretation, and ideas were then developed within that one interpretation" (1976, p. 127). Kieren's initial rational number interpretations were:

a) Fractions which can be compared, added, subtracted, etc.
b) Equivalence classes of fractions
c) Ratio numbers of the form $\frac{p}{q}$, where $p$, $q$ are integers and $q \neq 0$
d) Multiplicative operators or mappings
e) Elements of a quotient field
f) Measures or points on a number line
g) Decimal fractions.

The major thesis of Kieren's 1976 paper was that rational numbers, from the point of view of instruction, must be considered in all of the above interpretations.

Given the historical evolution of ratio, proportion and, fractions as objects, describing (c) *ratio numbers of the form* $\frac{p}{q}$ was perhaps an attempt to describe fractions as *relational numbers*. A ratio, such as 2 : 3 or 2 : 3 : 1, is a comparison of quantities and is not a single number. An interval such as the number line interval from 0 to 1 may be divided into a given ratio. However, the relational numbers or fractions (e.g. $\frac{2}{5}$ or $\frac{1}{3}$) are uniquely located on the number line only by reference to the unit "1". Although the term "ratio number" is seldom used now and has been taken to correspond to the ratio subconstruct, confounding ratios (part : part) with fractions (part : whole) can be misleading when considering fractions as quantities. Dividing the interval from 0 to 1 in the ratio 2 : 3 corresponds to the number $\frac{2}{5}$, not $\frac{2}{3}$.

In 1983 Behr, Lesh, Post and Silver proposed the following seven interpretations of fractions, which they called *subconstructs*:

a) Fractional measure (how much there is of a quantity relative to a specified unit of a quantity). This subconstruct was proposed as a reformulation of the part-whole concept.
b) Ratio
c) Rate (a relationship between two quantities)[1]
d) Quotient
e) Linear coordinate (point on a number line)
f) Decimal
g) Operator

These seven subconstructs were a redefinition of some of Kieren's 1976 categories and a subdivision of others.

Ohlsson (1988) identifies a third analysis, an analysis of *fractions* proposed by Nesher in 1985 (cited in Ohlsson, 1988, pp. 55-56). Nesher's analysis identified five concepts:

a) A part-whole relationship (describing the partitioning of an object into parts).
b) Result of a division between two whole numbers.
c) Ratio (a multiplicative comparison between two quantities)
d) Operator
e) Probability

The three analyses agree that quotient, ratio, operator and *some version of the part-whole relation* are central concepts. However, the analyses are difficult to reconcile, as the criteria used to make distinctions within each analysis were not specified. What these analyses do show is the complexity of the rational number concept, and the difficulty of agreeing on its component subconcepts.

Kieren's initial rational number *interpretations* have evolved into descriptions of rational number *subconstructs*. Thus in 1988, Kieren maintained that a fully developed rational

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[1] Two views on the nature of rates were described by Lesh, Post and Behr (1988). In the first, rate defines a new quantity as a relationship between two different types of quantities in the tradition established by the early Greeks. For example, speed is defined as a relationship between distance and time. The second interpretation of rates arises from distinguishing between two basic types of quantities, extensive quantities and intensive quantities. Extensive quantities are amounts that are derived from the environment and quantified by counting or measuring. They tell how much (i.e. the extent) of a quantity is associated with an object. Intensive quantities are usually not measured or counted but rather are generated through the act of division and express relationships between one quantity and one unit of another quantity. Using these two types of quantities, a rate can be considered to be a single intensive quantity whereas a ratio is a relationship between two quantities.

number construct comprised four subconstructs (measure, quotient, ratio number and multiplicative operator). Within these four subconstructs, measure and quotient provide a version of the part-whole relation. Both Kieren (1988) and Freudenthal (1983) identified the part-whole relationship as the foundation of rational number knowledge and fundamental to all later interpretations. Given Kieren's statements on the dangers of part-whole models of fractions, the challenge of capturing the appropriate nature of the part-whole relation in descriptions of rational numbers is a significant one.

### *2.2.1 Elaborating the fraction subconstructs*

The Rational Number Project was a multi-university cooperative research effort in the USA that commenced in 1979 and developed instructional and evaluation materials concerned with the learning of rational number (Post, Behr, & Lesh, 1986). A great deal of attention was paid to the various embodiments of fractions, such as continuous and discrete embodiments based on identified subconstructs, as part of the project (Behr, Wachsmuth, & Post, 1984; Post, Wachsmuth, Lesh, & Behr, 1985).

The Rational Number Project developed and piloted instructional materials that reflected the four rational number subconstructs identified by Kieren (measure, quotient, ratio number and multiplicative operator) as well as a fifth subconstruct: part-whole relations (Behr et al., 1992; Behr, Post, Silver, & Mierkiewicz, 1980; Behr, Wachsmuth, Post, & Lesh, 1984). Olive (1999) cites Nesher's (1985) analysis as the basis of the added fifth subconstruct, part-whole relations. Indeed, Kieren initially (1976) used the term whole-part relationships as a description of all rational numbers. Later (1993), Kieren subsumed part-whole relations under his measure and quotient subconstructs. The place of part-whole within the various descriptions of the facets of rational number is troublesome. In describing the theoretical foundations for the Rational Number Project, Behr et al. (1980, p. 62) incorrectly attribute five distinct subconstructs of rational number (part-whole, ratio, quotient, operator and measure) to Kieren's 1976 paper.

The organisation of the rational number subconstructs evolved over the span of the Rational Number Project. In particular, the Rational Number project sought to determine which of the subconstructs might best develop in children the basic fraction concept. Partitioning and part-whole were considered as basic to learning other constructs of rational number. This relationship is captured in the depiction of the conceptual scheme for rational number shown in Figure 2.1 (Behr et al., 1983; Behr et al., 1980).

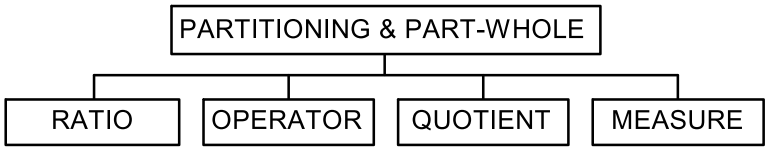



**Figure 2.1** Conceptual scheme for instruction on rational numbers

Although the subconstruct of fractional measure was originally proposed as a reformulation of the part-whole concept, both the measure subconstruct and part-whole appear in the above conceptual scheme. Figure 2.1 suggests that part-whole is viewed somewhat differently from the four subconstructs of ratio, operator, quotient and measure. The part-whole interpretation of rational number depends on partitioning either a continuous quantity or a set of discrete objects into equal-sized subparts or sets (Behr et al., 1983). In Figure 2.1,the part-whole interpretation is represented as being expressed through the other subconstructs. Structural equation modelling has been used (Charalambous & Pitta-Pantazi, 2005) to provide empirical support to the fundamental role of the part-whole interpretation of fractions in building understanding in the remaining subconstructs.

**Ratio**

Within the Rational Number Project the ratio subconstruct expressed a relationship between two quantities, for example, a relationship between the number of boys and girls in a room. If there are 20 girls and 10 boys in a class, the ratio of girls to boys is 2:1, while the fraction of girls in the class is $\frac{2}{3}$. Although both ratios and fractions can refer to the same data, they are conceptually different. The conventions of representing the "whole" are different. In a ratio, the whole is not explicitly present, only the parts, and so knowing the total is not essential. Thus, conceptually ratios are part:part or even part:part:part. By contrast, the fraction (as a quantity) cannot be determined without knowing the total or whole. Reasoning based on relations of one part compared to another is different from reasoning based on the relations of a part to the whole.

The ratio subconstruct attempts to capture the relationship between two numbers. However, in elaborating on rational number as ratio, Behr et al. (1983) recognise that a ratio may not be a rational number. Even the numbers that we associate with well-known ratios, such as the golden ratio or the ratio of the circumference to the diameter of a circle, are irrational numbers. Ratio echoes the relationship notion of fractions but because of its different representational and epistemological nature, ratio as a separate subconstruct may not be critical to the development of a quantitative sense of fractions. The essence of comparison may be better captured in other subconstructs, such as the fractional measure subconstruct.

**Operator**

The operator subconstruct treats a fraction as a type of function that transforms things into similar things that are $\frac{a}{b}$ times as big. The construct of operator is founded in the mathematical concept of functions and field structure. The operator subconstruct was elaborated by Behr et al. (1993) who identified five potential, hypothetical "personalities" of rational number as operator. The two main personalities were the duplicator/partition-reducer (DPR) and the stretcher/shrinker (SS).

The duplicator/partition-reducer has the effect of partitive division of the unit operated upon, into parts equal in number to the denominator and equal in size, a reduction of this number of parts to one, and a duplication (iteration) of this one part to a number of parts equal to the numerator. For example, in finding $\frac{3}{4}$ of eight bundles of four sticks, the partitive division by four results in two bundles reduced to a single unit (composed of two bundles) which is iterated to create three of this new unit.

The "stretcher-shrinker" notion of operator is really a composition of two inverse operations, multiplication and division. For example, using $\frac{3}{4}$ as an operator, the numerator (3) acts as a stretcher, making whatever it acts upon three times as big. The denominator (4) acts as a shrinker, making whatever it acts upon one-quarter of the size. The operator subconstruct can be thought of as similar to a scale factor.

In a study of preservice teachers' understanding of the operator construct (Behr, Khoury, Harel, Post, & Lesh, 1997) both personalities were observed, with the stretcher/shrinker used less often than the duplicator/partition-reducer.

**Quotient**

The quotient subconstruct interprets a rational number as an indicated division. That is, $\frac{a}{b}$ is interpreted as *a* divided by *b*. Division can be considered to be quotitive (sometimes termed measurement division, reflecting its conceptual links with the operations of measurement) or partitive. Partitive division arises from the notion of sharing, or distributing equally into a specified number of parts. For example, 15 ÷ 3 means to share 15 things equally among 3 positions. The quotitive interpretation of division determines, for example, how many composite units of 3 are contained in the composite unit of 15, a measurement notion.

The relationship between whole number division and fair sharing in the formation of a fraction-as-division scheme has been investigated through a series of teaching experiments (Middleton & Toluk, 2004; Middleton et al., 2001; Toluk, 2004). Analysis of the teaching experiments suggested that children developed in order, a whole number quotient scheme, fractional quotient schemes, a division as fraction scheme and finally a fraction as division scheme.

**Measure**

The *measure* subconstruct refers to the amount of a quantity relative to a specified unit of that quantity, in effect a quotitive division. For example, a measure could result from successively partitioning a number line to determine the number of a specified unit. The fractional measure subconstruct (Behr et al., 1983) represents a re-conceptualisation of the part-whole relationship notion of fraction.

In a study of 205 fifth-graders and 208 sixth-graders from China (Ni, 2001) the sixth-graders showed significantly improved performance on equivalent fraction items representing the part-whole relation but not on those representing the measure aspect of rational number. The results suggest that the measure subconstruct may be a source of difficulty in children constructing the concept of fraction equivalence.

### 2.2.2 Interpretations of part-whole

Different interpretations of part-whole can be analysed by examining the relationship between the component parts and the whole. Any description of part-whole will always refer to two of three pieces of information; the part, the complement and the whole. The concept of a ratio and its related notation[2] foregrounds *part : complement* (or part : part) whereas the concept of a common fraction as a relational number and its associated notation brings to the fore $\frac{\text{part}}{\text{whole}}$. The two different interpretations of the parts and the whole are shown diagrammatically in Figure 2.2.

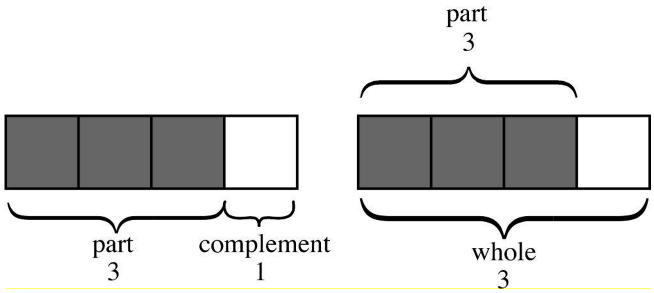



**Figure 2.2** Different relationships interpreted as part-whole

Both interpretations are referred to as part-whole; the first more appropriately described as exclusive while the second is inclusive part-whole. With the first interpretation, the whole is formed as a result of combining the equal parts in a given ratio (three of one type and one of the other) whereas the second interpretation starts with the whole and uses equi-partitioning of the whole. The equal parts formed in the second diagram can be thought of as being multiplicatively related to the whole through the division process.

**Additive part-whole**

Within research related to whole number acquisition, part-whole knowledge relates to the principle of additive composition by which whole numbers are combined to form other whole numbers (Meron & Peled, 2004; Resnick, 1989; Riley & Greeno, 1988; Sophian & McCorgray, 1994). For this whole number interpretation of part-whole relations, the whole is the sum of the parts.

One possible interpretation of the rectangles in Figure 2.2 is that 3 + 1 = 4, additive part-whole. Although divided rectangles similar to those in Figure 2.2 are frequently used to teach fractions, the additive interpretation of part-whole is more likely to arise from the diagram than a multiplicative sense of part-whole.

---

[2] Although some curriculum materials have suggested that ratios may be recorded in fraction form, presumably to use the same process of removing common factors to simplify both forms, this practice confounds the fraction and ratio notations. The confusion between ratios and fraction notation appears to be a consequence of the evolution of notation for division. Liebniz introduced the colon as the symbol for division in Europe and frequently used a colon when writing fractions, leading to recording 1:3 as meaning $\frac{1}{3}$ (Cajori, 1974).

Dealing with part-whole using a discrete set model produces a representation that appears strongly additive. In Figure 2.3 the part-whole additive interpretation 2 + 6 = 8 is more readily gained than any sense of "one of four" because both the part and the whole need to be "constructed" or unitised before they can be referenced.

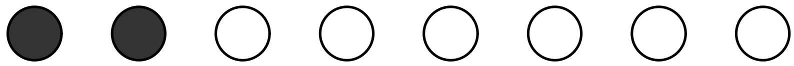


**Figure 2.3** Part-whole with discrete objects

Naming a fraction (part-whole discrete) is different from finding a fraction of a discrete quantity. In naming a fraction of a number of discrete objects, one works part-part-whole. That is, two counters out of eight counters may be viewed as two (part), six (other part), and eight (whole). This part-part-whole (2, 6, 8) is an additive interpretation, as the whole is the *sum of* the parts. However, in finding a fraction the relationship between the part and the whole requires a multiplicative interpretation.

**Multiplicative part-whole**

A second, separate meaning of part-whole relationships is used when considering part-whole relations with fractions. The part and the whole are related by a multiplicative interpretation. For this part-whole relation, the whole is a *product* of the one part. Consequently, eight is two plus six (additive part-whole) but four times one unit of two makes a whole (multiplicative part-whole). Due to the nature of discrete units, it is difficult to represent them in a strongly multiplicative way; that is, in a way that precludes an additive (counting) interpretation. Figure 2.4 is intended to suggest a multiplicative relationship between a unit of two and four duplicates of this unit of two.

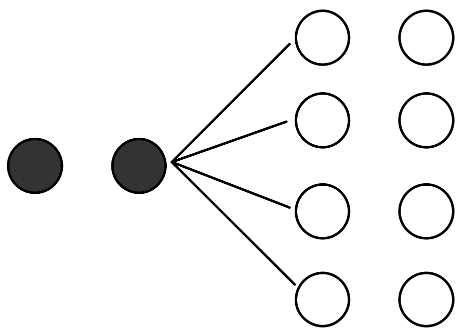


**Figure 2.4** Multiplicative part-whole

Additive part-whole can be seen as forming a composite unit as an *a posteriori* entity based on an *a priori* conceived abstract entity, the counting unit. That is, the whole (composite unit) is dependent on the counting unit, and is formed part-to-whole due to the precedence of the part. When a 'whole' is given *a priori*, an equal partitioning of this whole can generate a subunit that is dependent upon the whole in a multiplicative sense, whole-to-part, that can be quantified fractionally (Sáenz-Ludlow, 1994).

**Children's development of part-whole knowledge**

The traditional part-whole interpretation of fractions derives from partitioning either a continuous quantity or a set of discrete objects into equal-sized subparts or subsets. An example of partitioning a continuous quantity would be to divide the area of a rectangle into five equal parts, so that the five equal parts constituted the whole. An example of a physical context for partitioning a set of discrete objects is provided in a practical fraction task used to assess 11 year-old students in England, reported in Dickson, Brown, & Gibson (1984, p. 280). Students were presented with 4 square tiles, 3 yellow and 1 red, and asked, "What fraction of these squares are red?" Only 64% of the students assessed were successful. Many of those who were incorrect gave the answer one-third, presumably focusing on the parts and losing the sense of the whole.

A similar difficulty that can arise with partitioning continuous quantities is that some children focus on the *number of parts*, rather than the *equality of those parts*, which may be an artefact of a whole number interpretation of fractions. Many researchers consider children's difficulties with fractions are associated with interference from their whole number knowledge (Lamon, 1999; Post, Cramer, Behr, Lesh, & Harel, 1993). This whole number interpretation of fractions has been described as whole number dominance (Behr, Wachsmuth, Post et al., 1984) or whole number bias (Ni & Zhou, 2005).

Primary classrooms often include tasks where students associate models or pictures of partitioned and shaded areas with fraction symbols. However, little is known about how the properties of the models themselves facilitate or interfere with the students' ability to associate rational number ideas with them. The strategies students use to partition an area provide insight into their understanding of area as well as their notions of part-whole relationships.

A study of children in fourth-, sixth-, and eighth-grade designed to identify the strategies these students used to compare the areas of partitioned rectangles (Armstrong & Novillis Larson, 1995) found that most students used a direct comparison strategy, ignoring the part-whole relationships inherent in the tasks. The students were selected from those who had scored at or above the 25th percentile on the Iowa Test of Basic Skills. Each of the 21 tasks began by asking students to compare the shaded areas of two partitioned rectangular regions with one or more shaded parts. The cardboard regions were referred to as cakes cut into pieces and the shaded parts were referred to as frosted parts of the cakes. The tasks were designed so that:

- the number of parts shaded in each region could be the same or different,
- the total number of parts in each region could be the same or different,
- the size of the two regions could be the same or different, and
- the partitioning direction could be the same or different.

For example, the students were asked to indicate if the two cakes were the same size. If a student did not know how to directly compare the area of the two rectangular regions by overlapping them, the researcher demonstrated overlapping for the student. Then each student was asked if he or she ate the frosted part of the cakes whether one would give you more cake or if they would both give the same amount of cake. The images in Figure 2.5 show the cake comparisons for the first three questions.

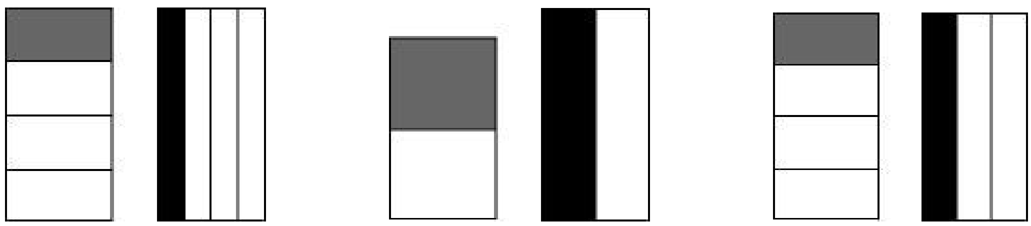


**Figure 2.5** Comparing $\frac{1}{4}$ with $\frac{1}{4}$, $\frac{1}{2}$ with $\frac{1}{2}$ (non-equal wholes), and $\frac{1}{3}$ with $\frac{1}{4}$

From Question 4 onwards non-unitary fraction comparisons were introduced. Also, fractional terms and symbols were introduced in the last eight tasks of the interview. Two broad groups of comparison strategies emerged in student responses: part-whole and direct comparison. The use of the part-whole strategy increased with the introduction of fraction terms and symbols, especially at the eighth-grade.

The broad category of part-whole was further subdivided into four types of strategies: rational number reasoning, partial rational number reasoning, area part-whole and partial area part-whole. Rational number reasoning responses included a comparison of the size of the wholes in relationship to a comparison of the size of the parts, which were expressed with fractional terms. An example of rational number reasoning in response to the first question would be that you get the same amount because they both take up one-quarter of the cake and the cakes are the same size.

The direct comparison strategies included those students who indicated that they were using decomposition followed by recomposition to compare areas. Other students compared a single dimension, length, or focused on the number of parts shaded. Some responses used a combination of strategies. The study did not notice any new strategies with the introduction of symbols into the questions. Moreover, the study found that "…it appears that students do not attend to the size of the wholes from whence parts come" (Armstrong & Novillis Larson, 1995, p. 16).

### *Looking back*

The development of the descriptions of the various subconstructs of rational number has contributed to an appreciation of the breadth of the rational number concept. However, the reorganisation of the nature and number of the subconstructs, as well as the changing interpretations of the individual subconstructs (Ohlsson, 1988; Olive, 1999) creates challenges in operationalising the subconstructs. In particular, the subconstructs frequently overlap. For example, the quotient subconstruct as the result of division is virtually identical to the subconstruct of a "ratio number $\frac{p}{q}$". Broader interpretations of the ratio subconstruct fall outside the domain of a fraction as an object and indeed outside of the domain of rational numbers. Mathematical definitions of rational numbers use the terms ratio and quotient interchangeably.

Thus, although Behr et al. (1992) argue that five subconstructs of rational number — part-whole, quotient, ratio number, operator, and measure — suffice to clarify the meaning of rational number, the discrete nature of part-whole as a subconstruct is questionable. The fractional measure subconstruct (Behr et al., 1983) was a re-conceptualisation of the part-whole relationship notion of fraction. Also, the measure subconstruct does not differ significantly from partitioning a continuous quantity, which was used by Behr et al. (1983) in describing the part-whole subconstruct.

Further, if ratio number, Kieren's (1976) initial description, is used to describe a subconstruct rather than ratio, then a ratio number $\frac{p}{q}$ would be largely indistinguishable from an indicated quotient $\frac{a}{b}$, used to describe the quotient subconstruct. The various descriptions of the different interpretations of rational numbers are not discrete and may obscure as much as they illuminate. However, it may be helpful to distinguish the contexts in which fractions are used (see for example Niemi, 2001; Vergnaud, 1983, 1988).

### *2.2.3 The role of partitioning*

The importance of partitioning in the understanding of fractional numbers has been acknowledged by Behr et al. (1983, 1985), Kieren (1976, 1980b, 1988), Piaget, Inhelder, and Szeminska (1960) and Pothier and Sawada (1983), among others. The ability to divide an object or a set of objects into equal parts appears critical to the logical development of part-part and part-whole relationships, as well as notions of equality and inequality. Kieren (1980b) theorized that partitioning experiences for students may be as important to the development of rational number concepts as counting experiences are to the development of whole number concepts.

Partitioning is a constructive mechanism that can facilitate the development of mature fraction concepts (Behr et al., 1983; Hunting, 1983, 1984; Vergnaud, 1983). The act of partitioning takes different forms depending upon the type of quantity involved. A continuous quantity is partitioned by being cut into separate pieces whereas a discrete quantity is partitioned by being sorted into separate collections (Ohlsson, 1988). The continuous expressions of fractional units formed by partitioning are effectively "measurable" and the discrete expressions are "countable" (Pitkethly & Hunting, 1996).

As early as first grade, children may use their understanding of partitioning to recognise the inverse relation between the denominator and the value of the fraction—they know that if a quantity is divided into many pieces then each piece will be smaller than if just a few pieces are formed (Empson, 1999). This inverse principle, the greater the number of equal pieces the smaller the size of the pieces, may develop unrelated to the size of the denominator and value of the fraction. It has been well documented that students often discover and use mathematical principles that they cannot express with words or symbols (Mack, 1993, 1995; Mix, Levine, & Huttenlocher, 1999).

**Levels of partitioning**

A description of the process of partitioning or subdividing a continuous whole into equal parts has been given by Pothier and Sawada (1983). Based on their clinical interviews of 43 children from Kindergarten to Grade 3, they proposed four levels of partitioning: sharing, algorithmic halving, evenness and oddness.

The first level, sharing, involves breaking things into parts and allocating the pieces. No attention is given to the specific size of the pieces. At the sharing level the child begins to learn a halving mechanism, that is, how to construct a line through the middle of a region.

At the second level, algorithmic halving, the child is able to partition rectangular and circular regions, not only into halves and quarters, but also into eighths and sixteenths. The child can double the number of parts by halving the size of the part. Algorithmic halving is applied without specific attention to the equality of the area of the parts formed. The child uses the tool of halving or splitting in an algorithmic manner, without concern for equality of area; for example, vertical parallel lines that work in a rectangular region may also be used in a circular region to produce "fourths".

At the third level equality and *evenness* becomes a critical characteristic in designating fair shares. The greater attention to the equality of the pieces comes closer to what Yoshida and Sawano (2002) call equal-partitioning. Pothier and Sawada characterise the evenness level as an ability to partition fractions with even denominators. The third level uses algorithmic halving followed by adjustment to make the parts "even". Of the five partitioning tasks devised for the study, one task, the cake problem, was considered to be highly effective, both in enabling children to demonstrate their partitioning capabilities and techniques, and in providing the basis for key insights into the partitioning process. The cake problem used one circular and four rectangular cakes of different sizes, a supply of sticks and a set of miniature dolls. The sticks, which were made available in three different lengths, were used to show cuts on the cakes. The children were asked to show how they would cut the cake so that each person (doll) at the birthday party would have the same amount. For example, at the third level to obtain sixths, "the child typically partitions a circular region into fourths, bisects two sections, and then rotates two sticks to obtain six equal parts" (p. 313).

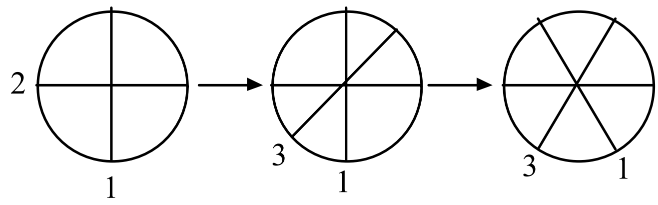



**Figure 2.6** Level III partitioning evenness (adapted from Pothier & Sawada, 1983)

The use of sticks with the "cake problem" may have influenced the behaviour described at the third level (see Figure 2.6).

The fourth level, oddness, is described as developing from the recognition of the limitation of the halving algorithm to create odd partitions. The child "…uses a *counting algorithm* to guide the partitioning; that is, pieces are produced one by one, and counting is used to keep track of results" (p. 315). The child comes to realise that a different initial cut from the halving algorithm is needed to make an odd number of parts. Frequent readjustment of the partitioning lines is needed to obtain equal shares.

As a theoretical extension of the levels, Pothier and Sawada proposed composition as a fifth level. Within the fifth level, a child would realise that to obtain ninths, a region can be partitioned into thirds and then each part divided into thirds. This composition of partitions employs a multiplicative algorithm. None of the children in Pothier and Sawada's study used a multiplicative algorithm when partitioning the regions.

Paper folding activities can provide models for compositions (Lamon, 1999, p. 104) and capture the sense that a fraction operator may be the result of an action performed on the result of a previous action. Pothier and Sawada's proposed fifth level has been elaborated in descriptions of recursive partitioning (Steffe, 2003) and unitising (Empson, 1999; Lamon, 1996). Recursive partitioning is described as follows: "When the result of a partition is given, recursive partitioning occurs when children can produce the result using two other partitions" (Steffe, 2003, p. 240). Both composition and recursive partitioning are unit fraction multiplying schemes.

### A taxonomy of partitioning strategies

A taxonomy for classifying young children's partitioning strategies in terms of how well such strategies facilitate the abstraction of the partitive quotient construct from the activity of partitioning objects or sets of objects was developed by Charles and Nason (2000). The taxonomy was based on the qualitative evaluation of clinical interviews with twelve Year 3 students. A complementary aim of the study was to identify new partitioning strategies. The twelve students were selected to produce maximum variation in cognitive functioning within the Year 3 students at a primary school. This process of selection relied upon the teacher's categorisation of the students into low, middle and high achievers.

Analysis of the interviews identified twelve strategies, which were then classified into three categories:

- Partitive quotient construct strategies
- Multiplicative strategies, and
- Iterative sharing strategies.

Of the twelve strategies identified, Charles and Nason identified half of them as having been previously reported and half of them as new strategies. Within the partitive quotient construct strategies, five strategies were identified. All of the strategies in this category had a common characteristic in that the relationship between the number of people sharing and the fractional name was used to generate the denomination of each share. For example, if the task involved sharing among five people then all of the shares were expressed as fifths.

Charles and Nason identified five partitive quotient construct strategies. Each strategy was described by a number of postulated steps in the children's reasoning. For example, in what Charles and Nason describe as the horizontal partitioning strategy, the described steps include:

1) recognise the number of people (y)
2) generate the fraction name from the number of people (y*ths*)
3) recognise the relationship between the fraction name and the number of pieces in each whole object (y)
4) partition each circular object into pieces horizontally (y pieces)
5) quantify each share ($\frac{1}{y}$ piece)
6) recognise that the shares are unequal.

The description of the horizontal partitioning strategy and the related steps appears to be overly context specific. The segmenting without specific attention to the equality of the area, that is, an emphasis on forming the correct number of parts, may not be orientation-dependent as the name "horizontal partitioning" suggests. Further, although the student who demonstrated this strategy, Claudia, may have recognised that the shares are unequal, recognition of equality or inequality was not a characteristic of the algorithmic halving level in the Pothier and Sawada (1983) study.

The only described multiplicative strategy is similar to Pothier and Sawada's (1983) proposed fifth level, composition. Although Charles and Nason indicate (p. 203) that none of the children in their study used the composition sharing strategy, one child, Thomas, used something akin to a multiplicative strategy when partitioning two pizzas among three people. His method was described as a "people by objects" strategy. Thomas appeared to find partitioning half a circle into thirds easier than partitioning a whole circle into thirds. In sharing the two pizzas among three people he first cut one pizza in half and then each half into three pieces. He then shared two-sixths to each person before repeating the process with the second pizza. Thomas quantified each person's share as $\frac{4}{6}$. He was the only child in the study who used a multiplicative strategy.

The four iterative sharing strategies included variations of repeated halving, as well as what has been labelled repeated sizing (p. 206). The repeated sizing strategy involved partitioning each whole into an even number of unequal pieces and sharing the pieces by attending to area rather than number. The use of an even number of unequal pieces is similar to the description in Pothier and Sawada's Level 2, of using halving in an algorithmic manner without concern for equality of area. The final two strategies were considered to be extensions of the "half the objects between half the people" strategy.

A taxonomy was developed according to the degree to which the 12 identified strategies contributed to 3 requisite conditions for the abstraction of the partitive quotient fraction construct. The three conditions were that children needed to be able to generate shares that are equal, accurately quantify and directly map the number of people and the number of shares. Four classes of strategies were described with Class 1 strategies meeting all three conditions (partitive quotient foundational strategy and proceduralised partitive quotient strategy), Class 2 strategies met the first two conditions (regrouping strategy, people by objects strategy and half to each person then quarter to each person strategy), Class 3 strategies met only the condition of generating equal shares (partition and quantify by part-whole notion strategy, halving the objects between half the people strategy, and whole to each person then half the remaining objects between half the people strategy) and Class 4 strategies met none of the conditions (horizontal partitioning strategy, repeated sizing strategy, and repeated halving/repeated sizing strategy). Children applying Class 3 strategies were reported as regularly suffering "loss of whole" (Charles & Nason, 2000, p. 214).

Charles and Nason's study confirms that young children's selection of partitioning strategies depends not only on their prior knowledge and experiences but also on the context of the task, the type and number of objects being shared, and the number of shares. One of the limitations of the description of partitioning strategies and the subsequent taxonomy arising from their study is the constraints upon its generalising power. With only 12 students in the study and 12 strategies being identified, the descriptions of the partitioning strategies may be overly specific to the students in the study.

**Partitioning and multiplicative structures**

The investigation of how children's approaches to partitioning support the construction of fractions as multiplicative structures has been taken up by Empson et al. (2005). They looked at how children coordinate quantities by manipulating the number of items to be shared, in conjunction with the number of people sharing them, to produce an exhaustive and equal sharing of the items. In particular, Empson et al. investigated how children would coordinate $bx$ people sharing $ax$ things to produce $\frac{a}{b}$. Two broad categories of coordination strategies were identified. The first was described as coordinating parts, where the goal was to produce a number of parts equal to (or occasionally a multiple of) the number of people. For example, a child might divide each pancake into 8 pieces if 8 children share 6 pancakes. The second category, ratio strategies, involved creating associated sets of discrete items, which avoided partitions of items. For example, a child might separate the 8 children into 2 groups of 4 and allocate 3 pancakes to each group resulting in a 4 to 3 ratio or three-quarters each. The main feature of ratio strategies appeared to be working on both groups (for example, the number of children and the number of pancakes) simultaneously.

### 2.2.4 Multiplicative domain structure

Recent epistemological approaches to the study of fractions include looking at the overall domain structure. Using concepts from pure mathematics, such as fields and vector spaces, is evident in Kieren's (1976) description of rational numbers as "elements of a quotient field" (p.118) and Vergnaud's "multiplicative structures" (1983, 1988, 1997). The description of fractions in pure mathematical terms can also be seen in Kieren's early work on rational numbers. For example, Kieren (1979) states "…fractional or rational numbers are by nature two-sided; they are simultaneously additive vectors and functions which can be composed" (p. 235).

The use of concepts borrowed from pure mathematics to describe mathematical thinking is described by Vergnaud (1988, p. 159) as follows, "…I use concepts borrowed from vector-space theory and from dimensional analysis to analyze what students do at the elementary and secondary levels". This approach leads to viewing fractions as part of a much broader conceptual field of multiplicative structures. Mathematically, rational numbers can be considered to be elements of a quotient field consisting of equivalence classes where the elements of the equivalence classes are fractions.

In clarifying the nature of rational numbers as mathematical constructs, Vergnaud treats rational numbers from the standpoint of mathematical structural properties. That is, a fraction represents an element of a *multiplicative* domain not an *additive* domain[3], which is why operating with the fraction notation can be counterintuitive. Children's understanding of additive structure "3 + 2 = 5" can interfere with development of the fraction concept when they attempt to apply this to fractions and wrongly obtain $\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$. Therefore children need to come to understand that the numerator and denominator of a common fraction are related through multiplication and division, not addition. Instances of multiplicative reasoning are also evident in knowing that given $\frac{1}{3}$, we recognise that there are three-times as many in the whole, not simply two more, and that the equivalence of fractions is multiplicative: $\frac{1}{3}=\frac{2}{6}=\frac{7}{21}=\cdots$

[3] The counting numbers (e.g. 2 = 1 + 1, 3 = 1 + 1 + 1, 5 = 1 + 1 + 1 + 1 + 1) are elements of an additive domain whereas fractions (e.g. $\frac{1}{3}=1\div 3$) are elements of a multiplicative domain.

The conceptual field of multiplicative structures is a network of distinct but interconnected concepts such as multiplication, division, fractions, and linear and nonlinear functions. The theory of conceptual fields stresses that situations and problems that substantiate the different aspects of mathematical concepts shape the acquisition of multiplicative structures (Vergnaud, 1994).

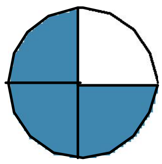
### *2.2.5 A semantic analysis of the multiplicative conceptual field*

A semantic analysis of the subconstructs of rational numbers with an associated notational system, designed to capture unit formation, re-unitisation and principles of unit conversion, is described in detail by Behr et al. (1992). In particular, the units analysis elaborates the part-whole subconstruct and the quotient subconstruct, taking both partitive and quotitive division into account. By addressing the role of conceptual units and the measures of quantity in the semantic analysis of part-whole, the semantic analysis attempts to enable recording of the similarities and differences between discrete and continuous representations of part-whole.

In the notational system of the structure of conceptual units proposed by Behr et al. (1992), part-whole for continuous quantities is quite distinct from part-whole for discrete quantities. For example, three-quarters of a continuous quantity as parts of a whole can be thought of as three "one-quarter units", recorded as 3 ($\frac{1}{4}$-units) in the notational system. By comparison, representing three-quarters for discrete quantity, say three-quarters of eight tiles, is more complex. Due to the "countable" nature of the discrete items, the notational system attempts to capture the re-unitising numerically. The eight tiles, forming the whole, are recorded as being equivalent to four units of two. Pictorially, the eight tiles could be represented as $((\square\ \square)(\square\ \square)(\square\ \square)(\square\ \square))$. Behr et al. (1992, p. 310) describe the resulting "four lots of two lots of one" as a "unit-of-units-of-units". Three-quarters of 8 is then recorded as three "one-quarter units", with the formation of the discrete sub-units characterised in the notational system as 3 ($\frac{1}{4}$(4(2-unit)s)-unit)s. Re-unitising a discrete quantity within this notational system appears far more complex than partitioning a continuous quantity.

The formal microanalysis proposed by Behr at al. (1992) did not take into account whether these operations have any basis in real problems or the processes that students actually use, and “the semantic analysis and notational system has not proved useful in ensuing research” (Lamon, 2007, p. 630). However, the distinction between part-whole for continuous quantity (which more readily captures the multiplicative sense of part-whole) and part-whole for discrete quantity (which may be more susceptible to an additive interpretation of part-whole) is important in understanding the changing role of part-whole and the associated teaching implications, as these often rely on part-whole.

The quotient construct describes interpretations of a rational number resulting from partitive and quotitive (measurement) division. Partitive division is used to solve tasks where a given quantity is divided into a number of equal shares. For example, the task of dividing three pikelets equally among four people involves partitive division or partitioning. Quotition, or measurement division, is used to determine the number of equal shares contained in a given quantity. For example, measurement division is used to determine how many quarter-chicken servings are possible from three complete chickens. The interpretations vary according to the identification of the nature of the composite unit. From a semantic analysis viewpoint, one interpretation of three-quarters resulting from partitive division is three one-quarter parts or, using the Behr et al. notation, 3 ($\frac{1}{4}$-units).



3 ($\frac{1}{4}$-units)

Another interpretation, also resulting from partitive division, is $\frac{1}{4}$ of a 3-unit, where a 3-unit is a composite unit. However, the problem context is likely to influence the way that the partitive (or quotitive) division is characterised. The semantic analysis described by Behr et al. (1992) represents idealised models of division. Children’s actual partitioning activity often involves other details (Charles & Nason, 2000; Empson et al., 2005; Lamon, 1996). For example, to divide three objects among four people, some children cut two of the 1-units into two parts and the other 1-unit into four parts before distributing the pieces. The practical process of sharing to create equal partitions can override theoretical units analyses of fractions. This type of sharing suggests a two-part interpretation to the quotient $\frac{3}{4}$, where the partitioning actions and the results of partitioning are separate entities.

### 2.2.6 Limits of epistemological approaches

The epistemological approaches tend to start from a unique conception of fractions as object—our current modern formulation of fractions. These approaches do not take into consideration the geographic and social distribution of the various working traditions in which fractions have emerged and their role in the maturation of the mathematical object itself.

The power of the symbol system used to describe fractions is that it can be applied to a range of different problems characterised by structural, rather than contextual similarities. Discrete and continuous contexts are contextually dissimilar yet yield to reasoning represented by a single symbol system. The fraction symbol system with its associated syntax has carried over into irrational numbers (e.g. $\frac{\pi}{2}$, $\frac{\sqrt{3}}{2}$). Indeed, the fraction symbol system contributes to an algebra of forms.

Fractions, as part-whole relationships, are intensive quantities. Indeed the fraction is the relationship. A fraction is not two whole number counts but rather the relationship between two units. The procedures (algorithms) for operating with these intensive quantities revolve around treating the components of the intensive quantity as extensive quantity replacements. Developing computational proficiency with fractions generally ignores the specific nature of the relationship between the numerator and the denominator that creates fractions as intensive quantities. For example, finding the lowest common denominator when adding or subtracting fractions focuses on the denominators only.

It is also unclear which, if any, of the subconstructs carries the equal-whole sense of fraction as number (i.e. the universal whole and its invariance). That is, which interpretation most closely aligns with a fraction as a mathematical object? Treating fractions as the sum total of the various subconstructs brings with it an added challenge in teaching fractions. Developing the equivalence of fraction units would need to operate across all of the subconstructs of fractions, and equivalent fractions would then need to be recursively developed (Kieren, 1993; Ni, 2001).

A major limitation of the research developed from the epistemological analysis of fractions has been its propensity to report in terms of success on tasks without an accompanying analysis of the knowledge that supported success and explained failure

(J. P. Smith, 1995). The nature of successful reasoning involving fractions and its evolution can remain largely a mystery.

## 2.3 The common fraction notation

Students' problems with written fraction notation are widely acknowledged (Hiebert, 1989). In particular, a child's representation of fractions reflects the coordination of knowledge of notational conventions and particular kinds of part-whole relations. However, a child's understanding of either fraction notation or part-whole relations does not ensure the other (Saxe, Taylor, McIntosh, & Gearhart, 2005). The analysis of 384 elementary students' notations for fractional parts of area suggested that notation and reference were acquired somewhat independently (Saxe et al., 2005). Indeed, for some students whose knowledge of fraction concepts is rooted in whole number, a "correct" notation may belie immature understandings of fractions. That is, even a student who records the correct notation in response to a continuous part-whole model may not appreciate the fraction as a relational entity (H. Yoshida & Kuriyama, 1995).

The fraction notation can be seen as a culturally developed tool (Cobb, 1995). Certain tasks are made easier because such culturally developed tools are available to perform the task. The primary task influencing the design of fraction notation is computation with fractions. However, the symbolic representation of fractions poses a "semantic" problem—each symbol representation means something. Not only does the symbol mean something, but also the symbol itself can be shaped by society's needs and be dependent upon an understanding of that society for its interpretation. The early Egyptian representation of fractions is sometimes described as using unitary fractions. This is because their representation system had no way of simply describing non-unitary fractions such as three-quarters. However, fractions were initially developed as practical solutions to problems, not as mathematical objects, and the fraction notation of ancient Egypt served the needs of the time.

The fraction notation, in signifying a single relational number, creates an affordance[4] for achieving results that is in essence an algebraic notational system. Multiplicative

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[4] The term affordance was used by the perceptual psychologist J. J. Gibson (1977) to refer to the actionable properties between the world and an actor. Put more simply, a perceived affordance typically describes whether the user perceives that some action is possible.

reasoning becomes encoded in syntactically defined rules of a symbol system. The notational system is *algebraic* in that quantitative relationships can be expressed in a way that assists in finding an unknown quantity. Bruner (1973) refers to such use of symbols as an "opaque" use of the symbols rather than a "transparent" use: the former implies attention to actions on the inscriptions, while the latter implies that actions are guided by reasoning about the entities to which the inscriptions are assumed to refer. Whitehead referred to the opaque use of symbols in 1911 by stating that "by relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems" (in Cajori, 1974, p. 332).

The opaque nature of the symbols used with fractions can contribute to the problems that students have in giving meaning to fractions. Students may have difficulty in understanding the meaning of a written symbol if the referents do not well represent the mathematical meaning or if the connection between the referent and the written symbol is not appropriate (Hiebert, 1989). Teaching students the operations with fraction symbols seeks to develop procedural knowledge. Adopting an information-processing approach, Hecht (1998) divided knowledge about rational numbers into three strands: procedural knowledge, factual knowledge and conceptual knowledge. His study isolated the contribution of these types of knowledge to children's competencies in working with fractions. He made two major conclusions: (a) conceptual knowledge and procedural knowledge uniquely explained variability in fraction computation solving and fraction word problem set up accuracy, and (b) conceptual knowledge uniquely explained individual differences in fraction estimation skills. In a later study, Hecht, Close, and Santisi (2003) found that structural equation modelling provided consistent evidence that conceptual knowledge independently contributes to individual differences in fraction computation, estimation and word problems. Consequently, teaching only the manipulation of the opaque symbolism of fractions appears to be inadequate to developing effective use of fractions for computation, estimation and problem solving.

### *2.3.1 Errors arising from the notation*

Many children appear to see fractions as two unrelated whole numbers—three-quarters is the whole number three written over the whole number four (Hart, 1989)—and apply whole number strategies to fraction problems (Lamon, 1999; Mack, 1995; Streefland, 1993). Behr, Wachsmuth, Post, and Lesh (1984) use the term whole-number-dominance to describe such strategies; Streefland (1993) refers to such errors as misapplication of whole-number arithmetic to a fraction situation; while Mack (1995) describes them as confounding whole-number and fraction concepts. The tendency of students to use the single-unit counting scheme to interpret instructional data on fractions has also been described as a whole number bias (Ni & Zhou, 2005). Instances of whole number bias include confounding the number of pieces in a partition with the size of each piece, (e.g. $\frac{1}{6}$ is bigger than $\frac{1}{3}$ because 6 is bigger than 3), adding across numerators and denominators to add fractions, and counting noncongruent parts to name a fraction. This whole-number interpretation of fractions may be exacerbated by focusing on algorithmic proficiency with fractions, disconnected from any concept of a fraction.

In a National Assessment of Educational Progress assessment (T. P. Carpenter et al., 1981), more than half of U.S. eighth graders appeared to believe that fractions were a form of recording whole numbers and chose 19 or 21 as the best estimate of $\frac{12}{13}+\frac{7}{8}$. Their responses suggest that whole-number dominance may overpower an individual's quantitative sense of fractions. Saenz-Ludlow (1994) argues that the results indicate that students need to "…conceptualize fractions as quantities before they are introduced to standard symbolic computation algorithms" (p. 50).

With common fractions, students may reason that $\frac{1}{8}$ is larger than $\frac{1}{7}$ because 8 is larger than 7 or they may believe that $\frac{3}{4}$ equals $\frac{4}{5}$ because in both fractions the difference between numerator and denominator is 1 (Behr, Wachsmuth, Post, & Lesh, 1984). The Behr at al. analysis of 12 Grade 4 students' responses to ordering fractions tasks identified several distinct strategies for comparing fractions. For fractions with the same numerators, they described five distinct strategies—denominator only, numerator and denominator, reference point, manipulative and whole number dominance. The denominator only strategy, which dominated the explanations, referred only to the denominators of the fractions. For example, an explanation that referred only to the denominator, such as "the bigger the number is, the smaller the pieces get" was recorded in their study as denominator only. The "numerator and denominator" strategy was identified by explanations that referred to both the numerators and the denominators, indicating that the same number of parts was present but the fraction with the larger denominator had the smaller sized parts. The reference point strategy made use of a third number, such as one-half, in comparing the fractions. The manipulative strategy identified explanations that made use of pictures or manipulative materials. Whole number dominance was used to describe an ordering consistent with whole number arithmetic applied to the denominators. Some of these strategies (reference point, manipulative and whole number dominance) were also evident in comparing fractions with different numerators and denominators. The whole number dominance reasoning appears to reflect students continuing to use properties they learned from operating with whole numbers. "For all children, their previous whole-number schemas[5] have influenced their ability to reason about the order relation for fractions" (Post & Cramer, 1987, p. 33).

The notation for fractions can evoke a range of fraction images in addition to incomplete part-whole relationships. Mack (1995) studied seven third and fourth graders over the course of a 3-week period to determine how their informal knowledge of fractions linked with the written notation. She found that the students tended to interpret written notation in whole number terms. For example, one child stated that the notation for five-eighths of an area could be written as either "5" or "$\frac{5}{8}$", explaining that "It doesn't matter. It's the same thing" (p. 437). The student's focus may have been on the five parts as discrete pieces, without linking the notation to a description of the size of the pieces.

In studying eight sixth-graders, Mack (1990) found that students' informal knowledge related to comparing fractional quantities was initially disconnected from their knowledge of fraction symbols. For example, four of five of the students unsuccessfully compared fractions represented symbolically immediately after successfully comparing the same fractions presented in the context of real-world problems. Further, when asked a question like "Tell me which fraction is bigger, $\frac{1}{6}$ or $\frac{1}{8}$", seven of the eight students asked responded that one-eighth is bigger because eight is bigger than six.

This focus on the size of the denominator in interpreting the relative size of fractions appears common across countries. In a study of number sense in Australia, the United States of America and Sweden (Bana, Farrell, & McIntosh, 1997), when students were presented with four fractions with the same numerator and asked to determine which was the largest number, many students selected the fraction with the greatest denominator.

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[5] The term "schema" is the Latin form of "scheme".

Vinner (1997) described fraction comparison strategies that focused on the size of the numbers in fraction notation, such as “the bigger the denominator the smaller the fraction”, as pseudo-analytical behaviour. The application of these incomplete strategies can result in correct answers for the wrong reasons. A question from the Third International Mathematics and Science Study (Beaton et al., 1996), “Which of the following numbers is the smallest? (a) $\frac{1}{6}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$” could be correctly answered using “the bigger the denominator the smaller the fraction”. However, the application of a similar strategy, “the smaller the denominator the greater the fraction”, to another TIMSS question, “Which number is the greatest? (a) $\frac{4}{5}$ (b) $\frac{3}{4}$ (c) $\frac{5}{8}$ (d) $\frac{7}{10}$” would lead to the selection of option (b). Vinner reported that 42.1% fewer $8^{th}$ grade Israeli students correctly answered the second question than the first question. Approximately 39% of the students selected option (b), which may have been influenced by the application of a strategy such as “the smaller the denominator the greater the fraction”. The difference in the percentage of correct responses to these two similar fraction comparison tasks suggests that the expected method of using equivalent fractions may not have been the only substantial coherent method applied.

A premature emphasis on using fraction symbols and the associated algorithmic manipulation of whole numbers to achieve a symbolic answer to fraction operations may account for many of the common errors with fractions, such as adding numerators and denominators, that can be characterised as whole number interpretations of fractions (Mack, 1990, 1995). Connecting meaning to different representation systems is one of the most significant aspects of learning about mathematics, particularly when children are learning about fractions. For otherwise similar fraction tasks, children show contrasting performance between those involving symbols and those not involving symbols (Hart et al., 1981).

### *2.3.2 Diagrams using the number line*

Another representational tool associated with fractions is the use of the measurement analogue: the structured number line. Fractions, as mathematical objects, are represented by locations on a structured number line, which is a very sophisticated analogue as it carries a sense of the density of fractions, as well as of their ordering. The number line is also a many-to-one analogue as unique locations on the number line are associated with equivalence classes of fractions. For example, $\left\{\frac{1}{3},\frac{2}{6},\frac{3}{9},\cdots\right\}$ would map to a single point on the number line.

Not all tools are equally easy to use. Novillis (1976) found that associating a fraction less than one with a point on a number line was more difficult for children than associating the same fraction with a part-whole model, discrete or regional. Different semantic meanings of fractions embodied in graphical representations pose different challenges to children (Dufour-Janvier, Bednarz, & Belanger, 1987; Kieren, 1993; Lesh et al., 1987). When interpreting an area model of a fraction the unit can be taken for granted. Recognising equivalent fractions on the number line relies upon recognising a unit of measure can always be divided up into finer and finer subunits with different fraction names resulting when different subunits cover the same distance. The number line, as an analogue of a fraction, is not necessarily less abstract than the notation.

A study involving 400 fifth and sixth graders from China investigated the influence of semantic meanings of fractions embodied in graphical representations on children's construction of fraction equivalence (Ni, 2001). The students performed best on the area items, less well on the discrete set items and the poorest on the number line items. This study confirmed earlier results that tasks involving the number line are often poor estimators of children's understanding of the measurement aspect of rational number (Ni, 2000).

### *2.3.3 Fraction words as symbols*

Considering the language of mathematics also as a culturally developed tool, the English language naming of fractions invites confusion with ordinals. "One-third", a term composed of a cardinal (one) and an ordinal (third), is used as a composite term to refer to a fractional part ($\frac{1}{3}$). Rather than having the language assist the acquisition of the concept, the English language appears to work against the fraction concept. In Japanese, one-third is spoken as *san bun no ichi*, which translates literally as "of three parts, one" and in Korean, one third is spoken as *sam bun ui il*, which also translates literally as "of three parts, one" (Miura, Okamoto, Vlahovic-Stetic, Kim, & Han, 1999).

Unlike the English language, the Japanese and Korean languages directly support the concept of the whole divided into three parts. The linguistic tag connects the fraction to its part-whole meaning rather than miscuing an ordinal interpretation. The linguistic system for naming fractions may influence the learning of fractions and the possible misconceptions that students form. A study of Croatian, Korean and American children's knowledge of fractions prior to school instruction supported the view that the Korean vocabulary of fractions may influence the part-whole meaning 6- to 7-year-old children ascribed to numerical fractions (Miura et al., 1999). The Croatian and American children had numerical language characteristics that are similar to each other and different from Korean. The Korean children were better able to associate numerical fractions with corresponding pictorial representations prior to formal fraction education. This study's results are consistent with the hypothesis that East Asian mathematical languages may influence the understanding of part-whole fraction concepts. However, focusing on both of the numerals presented in the fraction, for example, the 2 and the 5 for two-fifths, led to similar errors (selecting 2 parts shaded and 5 unshaded as a representation for $\frac{2}{5}$) across countries.

## 2.4 Psychological approaches

The patterns of thinking (schemes) that have been traditionally identified as important for learning about fractions include halving, measuring, partitioning-sharing, qualitative and quantitative compensation, and the integration of partitioning and measuring (Hunting, 1989). These schemes have one thing in common; they are based on actions. The *mental actions* are taken to reveal themselves explicitly in the behaviour of the child or else are inferred from discussions with children as they solve fraction problems. Schemes, connected groups of ideas, can be constructed in three ways: by building from physical experience and testing to see whether predictions are confirmed, via communication and testing through discussion, and through mental creativity and testing of new ideas for consistency with what is already known (Skemp, 1989).

According to von Glasersfeld (1980), a scheme consists of three parts:

a) the experiential situation as perceived by the child that he or she has experienced before,

b) the child's specific activity or procedures the child associates with the situation, and
c) the result that the child has come to expect of the activity in the given situation.

Schemes are described in the psychological literature as "networks of connected concepts" (Olive & Steffe, 2002; Skemp, 1979). A complex scheme can be characterised as having a large network of ideas that are built around one or more core concepts.

The link between fractions and the action-based schemes derived from the process of division clarify the origins of many terms associated with fractions as quantities. Davydov and Tsvetkovich (1991) in investigating the origin of the concept of fractions, point out that the two principal sources of fractions are measurement and division. Although division is given some attention in teaching fractions they argue that there are significant advantages to introducing students to fractions from the viewpoint of measuring quantities. In developing the notion that a fraction is a number, natural and rational numbers can both be derived from measurement, and the relationships between these numbers can be developed.

### *2.4.1 Fraction schemes*

A number of fraction schemes have been identified as important for learning about fractions (Hunting, 1986; Sáenz-Ludlow, 1994; Streefland, 1993) and are described briefly below. The schemes describe the processes needed for equal partitioning and segmenting, the action of subdividing an amount into parts of a specified size.

**Halving**

The act of halving by subdividing at approximately the midpoint of continuous quantities is a scheme which Pothier and Sawada (1983) called algorithmic halving. This powerful strategy can inhibit the development of other partitions such as the 3-partition. Halving and doubling have their roots in a primitive scheme that Confrey calls splitting (1994). In splitting, the primitive action is creating simultaneous multiple versions of an original through dividing symmetrically, growing, magnifying and folding. Confrey defines splitting as "an action of creating equal parts or copies of an original… an operation that requires only recognition of the type of split and the requirement that the parts are equal" (1994, p. 300). For Confrey, splitting is an action of duplicating and constructing simultaneous splits of an original.

## Measuring

Piaget defined measurement of length as a synthesis of change of position and subdivision (Piaget et al., 1960). Measurement of length for Piaget involved (a) understanding space or the length of an object as being partitionable, that is as being able to be subdivided, and (b) partitioning off a unit from an object and iterating that unit without overlap or empty intervals (change of position). Consequently, schemes related to partitioning are important in measuring. Sharing or partitioning is the action of distributing an amount of something. The idea of partitioning "develops from informal sharing situations as well as from halving activities" (Kieren, 1980a, p. 76). Hunting (1989) describes three distinct qualities of partitioning knowledge for determining fractional units: action-based partitioning, operation-based partitioning (using internal representations of external objects), and operation-based partitioning using whole number relationships.

A more general scheme, called a fair-share scheme, was proposed by Streefland (1991, 1993) as an important scheme for teaching fractions. A fair-share scheme means that when *N* persons share *M* objects, they should share the objects equally. This scheme has also been called equal-partitioning (H. Yoshida & Sawano, 2002).

## Compensation

To create fair-shares or equal partitions, adjustments need to be made to the amount in each share. This is where compensatory schemes are employed. There are two forms of compensation, qualitative and quantitative. Qualitative compensation is the coordination of number and size of fractional units across transformations of partitions or in comparisons of equivalent quantities each partitioned differently (Hunting, 1981). The logic of the reasoning is that a larger number of units implies units of smaller size. This reasoning has also been identified in children's early measurement learning as an inverse relationship between the size of units and the number needed (T. P. Carpenter & Lewis, 1976).

When children are able to tell by how much the size or number of units will vary across various partitions they are said to have quantitative compensation (Hunting, 1989). That is, quantitative compensation enables children to recognise that four times the number of partitions of a referent unit will result in parts that are one quarter of the original size. Quantitative compensation may be considered as a precursor of reforming units or re-unitising. Re-unitising builds on Lamon's (1996) notion of unitising, taken to be the process of conceptualising the amount of a given share before, during and after the sharing process.

**The equal-whole**

Building on the idea of the invariance of the whole (Piaget et al., 1960), the equal-whole scheme relies on recognising that magnitudes of one as a whole should be the same in all fractions (H. Yoshida & Shinmachi, 1999). When asked to order fractions, many Grade 4 students drew representations (rectangles) in which the size of the whole that each fraction represented was in direct proportion to the size of the denominator, that is, each representation of one was a different size (H. Yoshida & Kuriyama, 1995). Using an equal-whole scheme appears to underpin understanding fractions as mathematical objects as it enables ordering fractions as relational numbers. Consequently, it may be an important component of the psychological processes contributing to a student's sense of the size of fractions.

The role played by the equal-whole scheme is evident in a study of U.S. students' strategies used to compare partitioned rectangles.

> Even when some of the students used Part-Whole strategies, they ignored the size of the whole when they made their comparisons. Comparison-of-fraction problems are usually presented to students where the wholes are the same size. From the results of the study it appears that students do not attend to the size of the wholes from whence the parts come.
>
> (Armstrong & Novillis Larson, 1995, p. 16)

The equal-whole scheme may also be over-generalised. One of the problems from the research program *Concepts in Secondary Mathematics and Science* (CSMS) that operated out of the University of London in the years 1974–1979 (Hart et al., 1981) appears to describe an over generalisation of the equal-whole scheme. In response to a problem that asked children if one-quarter of an amount of pocket money could be greater than one-half of an amount of pocket money, a large number responded in the negative reasoning that $\frac{1}{2}$ is (always) bigger than $\frac{1}{4}$. As this response accounted for 41.5% of 12 year-olds, 34.3% of 13 year-olds, 27.6% of 14 year-olds and 19.1% of 15 year-olds, an absolute "equal-whole" appears to apply for many children even when it is not warranted.

So students have to learn to consider an equal whole when comparing fractions as numbers and to recognise that when comparing fractions of different quantities, the size of the quantities (implicit wholes) influences the size of the fractional parts.

### 2.4.2 *The construction of conceptual units*

The phenomenological foundation of fractions involves a “fractioning” (Freudenthal, 1983) of some physical or mental object—a unit. Constructing a unit is an act of abstraction. Technically, von Glasersfeld and Richards take the construction of units to be two acts of abstraction: the first act of abstraction is to produce units from sensory-motor material; the second act of abstraction “…takes these units as the material for the construction of a unit that comprises them” (von Glaserfeld & Richards, 1983).

Composite units are a collection of units sharing a common attribute (Steffe & Olive, 1990). For example, if a plate of 12 pikelets had 3 plain pikelets, 3 with strawberry jam, 3 with honey and 3 with butter, each topping could be considered as a unit and the plate of pikelets could be considered as one unit composed of four subunits. Similarly, three-quarters of a strip of paper could be considered as one unit composed of three units of one-quarter each.

$\frac{3}{4}$

Units-of-units are composite units that are re-conceptualised into a new, encompassing unit (Steffe, Cobb, & von Glasersfeld, 1988). For example, the plate of 12 pikelets, or four units of three, could be re-conceptualised as one unit of 12, or one dozen. Similarly, four-sixths of a single pikelet could be re-conceptualised as two units of two-sixths or as one unit of two-thirds.

Unitising can be thought of as the cognitive assignment of a unit of measurement to a given quantity (Lamon, 1996, p. 170). In particular, Lamon distinguishes between partitioning — an operation that generates quantity — and unitising, which she takes to be the process of conceptualising the amount of a given share before, during and after the sharing process. Using a cross-sectional study, Lamon analysed the partitioning strategies of 346 children in the U.S. from grades four through eight to infer their unitising processes. As grade levels increased, a greater percentage of children used economical partitioning strategies indicating a shift away from the distribution of singleton units toward the use of composite units. This shift towards the increased use of composite units in partitioning is analogous to Pothier and Sawada’s postulated fifth level (see Section 2.2.3).

Lamon coded the relative sophistication of the partitioning according to the level of redundant marking and cutting carried out in the partitioning tasks. For example, in sharing four pizzas among three people, a response that marked subdivisions on all four pizzas would be considered overly-marked as three whole pizzas would not need to be partitioned. Lamon reported that some students tended to be aware of the context and used different strategies when the same task used different types of foods. For example, some students considered sharing several pizzas all with the same topping to be a different context than sharing pizzas with different toppings.

Lamon's study identified at least four dimensions for differentiating student's strategies for fair sharing:

- Preservation of pieces that do not require cutting when each person receives more than one whole discrete quantity,
- Economy of the marking (e.g. not using sixths when thirds would suffice),
- Economy of cutting (not making more cuts than necessary)
- Social practices related to the objects being shared (such as providing variety when sharing three pizzas with different toppings among three people).

The influence of social practices on children's fair sharing may affect their solution strategies even when the context is not explicitly stated.

The construction of conceptual units through unitising provides a mechanism for understanding the many subconstructs of rational number outlined in section 2.2.1, which children must conceptually coordinate. The subconstructs of rational number "… may all be understood as compositions and recompositions of units" (Lamon, 1996, p. 192).

### *2.4.3 Clarifying fractions as quantities*

Taking up Saenz-Ludlow's phrase of *conceptualising fractions as quantities* we need to ask, "What does it mean to conceptualise fractions as quantities?" Fractions are relational numbers and it is the process of forming the relation, part-to-whole and whole-to-part, which creates the sense of the absolute size of fractions. Part-to-whole and whole-to-part are inverse operations which, when fully established, can result in a better understanding of a fraction as a relational quantity (Sáenz-Ludlow, 1994). Reversibility is a key component of understanding for Piaget, Inhelder and Szeminska (1960) who argued over

30 years earlier that fraction understanding is incomplete if the learner is not able to reconstruct the whole from the parts.

Traditionally, understanding equal-partitioning as well as the order and equivalence of fractions is considered basic to understanding fractions in general (Behr, Wachsmuth, Post et al., 1984; Post & Cramer, 1987; Post et al., 1985; J. P. Smith, 1995; Streefland, 1991, 1993). Cramer and Post (1995) simply state that to think quantitatively about fractions, "students should know something about the relative size of fractions and be able to estimate reasonable answers when fractions are operated on" (p. 377). It is not possible to estimate reasonable answers when fractions are operated on without having a quantitative sense of fractions. Rounding common fractions (e.g. treating $\frac{6}{11}$ as $\frac{1}{2}$ to estimate an answer) only makes sense when there is a corresponding sense of location, such as determining whether a fraction is greater or less than one-half (Behr, Wachsmuth, Post, & Lesh, 1984).

For comparisons of fractions, the equal-whole scheme proposed by Yoshida and Shinmachi (1999) as an extension of the idea of the invariance of the whole (Piaget et al., 1960) can aid the design of tasks to map children's sense of the size of fractions. The scheme addresses part of the perceived problem of the negative interaction between whole number knowledge and fraction knowledge. If we consider whole numbers as composite units (Steffe, 1994) the iteration of a unit to form a composite whole can support the development of the iterative fraction scheme (Tzur, 1999) provided that iterating unit fractions can be followed by a reorganisation of an appropriate number of unit fractions into wholes. This reorganisation of iterated fraction units, employing the equal-whole scheme, is necessary to make the transition from an additive iteration of units to a multiplicative association between parts of an equal-whole.

Thinking quantitatively about fractions also relies significantly upon equal-partitioning (Lamon, 1996) and the invariance of the whole (H. Yoshida & Sawano, 2002). In representing a number less than one, the whole should be of a fixed size in order to allow comparison of fractions. The invariance of the whole is essential in comparing quantity fractions, that is, those that reference a universal unit (K. Yoshida, 2004). Although the equal-whole is a critically important concept in understanding the multiplicative structure of fractions, it appears that the equal-whole concept is difficult to acquire (Hart, 1988).

Without the equal-whole concept, ordering fractions as quantities draws upon significantly different mental models of fractions compared to ordering fractions based on numeric rules.

The development of fractions as quantities is also influenced by the different interpretations of the fraction notation. The use of $\frac{a}{b}$ to record *a* parts out of *b* parts draws upon the universal whole in a similar way to transforming *a* divided by *b* into the mathematical object (number) $\frac{a}{b}$. The power of the notation for dual representation — fraction as a number and an indicated division — means that the creation of the equal whole remains invisible for many students. That is, for many students abstract quantity fractions may be indistinguishable from partitioned fractions.

### 2.4.4 From partitioning to iterating

Children's fractional schemes can emerge as accommodations in their numerical counting schemes (Steffe, 2002). Steffe and Olive (cited in Steffe, 2002) hypothesised that through abstracting the foundational activity of partitioning combined with the activity of iterating a unit, the child would base his or her first conception of fractions, the *equi-partitioning scheme*, on the equality of all parts and on the number of times that the unit was iterated to produce the partitioned whole. The learner anticipates that partitioning a given whole brings about a unique-size quantity relative to the size of the whole and the inverse relationship between the number and size of parts. Steffe (2002) further suggested that children can use their number knowledge, in conjunction with the equi-partitioning scheme, to reorganise their fraction knowledge through iterating unit fractions to produce composite fractions, those with a numerator greater than one such as $\frac{3}{5}$ and $\frac{7}{5}$. He termed this more advanced conception the *iterative fraction scheme.* This scheme underpins addition and subtraction of like-denominator fractions (Tzur, 1999, 2004).

Tzur (1996) also investigated what he termed the *partitive fraction scheme*, characterised by the child's ability to iterate and operate numerically on unitary and composite fractions, as long as those fractions do not exceed the partitioned whole. The purpose of the partitive fraction scheme is to partition a unit into so many equal parts, take one out of those parts, and establish a one-to-many relation between the part and the partitioned whole. The partitive fraction scheme is taken to be an intermediate conception on the way to the *iterative fraction scheme*. The iterative fraction scheme involves whole number knowledge, in conjunction with the equi-partitioning scheme, leading children to reorganise their fraction knowledge through iterating unit fractions to produce non-unit fractions.

One obstacle to the creation of the iterative fraction scheme is that some children consistently regard an improper fraction produced via iteration of a unit fraction as a new whole (Tzur, 1999). That is, they think of $\frac{1}{8}$ iterated nine times as $\frac{9}{9}$ and each part was considered $\frac{1}{9}$. This belief could be attributed to children failing to reform the iterated eight-eighths into the equivalent unit whole. Empson (1999, 2001) describes reunitising associated with equivalent fractions as "chunking". The task Tzur used to overcome this obstacle involved doubling fractions such as $\frac{3}{11}$ and then $\frac{6}{11}$ rather than iterating the unit fraction, $\frac{1}{11}$. This approach suggests that doubling may form a better link to the whole than iteration of a unit to form a composite whole. More recently, Tzur (2004) described the *reversible fraction conception* as a transformation in children's iteration-based fraction concepts. Tzur describes the reversible fraction conception as "…the learner's partitioning of a non-unit fraction ($\frac{n}{m}$) into *n* parts to produce the unit fraction ($\frac{1}{m}$) from which the non-unit fraction was composed in the first place. This allows, for example, for producing the whole ($\frac{m}{m}$) of which the unit fraction is part." (p. 93).

Mentally reconceptualizing a quantity according to some unit of measure that is convenient for thinking about and operating on the quantity (constructing different-sized chunks) can be described as *unitising* (Lamon, 1996). Earlier, von Glasersfeld (1981) described a model for the operation of unitising which he considered to be the conceptual act of conferring unity on a collection of elements. The process of unitising may be thought of as an elaboration of the schemes involved in partitioning. In particular, splitting (Confrey, 1994) and the capacity to *unpartition* (Behr et al., 1983) are constituent components of Lamon's *reconceptualizing* involved in unitising. This shift towards the use of more economical composite units or chunks in partitioning described by Lamon is analogous to Pothier and Sawada's (1983) postulated fifth level (see section 2.2.3 The role of partitioning).

Unit iterations and subsequent modifications for conceptual understanding of fractions have also been considered by Sáenz-Ludlow (1994). In particular, she sees the link between the mental acts of iteration and measuring as important to the conceptual development of fractions. A whole number is viewed as a composite unit formed from the iteration of the counting unit through a part-to-whole process. Partitioning a composite whole can result in a whole-to-part operation where the resulting unit is dependent upon the whole and the part can be quantified fractionally as a part of the whole. Sáenz-Ludlow argues that fractional quantifications seem to be the result of the concomitant establishment of these two operations; part-to-whole and whole-to-part (Sáenz-Ludlow, 1994). Tzur's (2004) *reversible fraction conception* encompasses both the whole-to-part and part-to-whole operations described by Sáenz-Ludlow. The approaches of Olive (1999), Sáenz-Ludlow (1994), Tzur (1999, 2004) and Steffe (2002) to children's construction of fraction schemes are similar in that each views iteration of a unit as basic to the fraction concept.

Consequently, iterating and partitioning operations appear to be parts of the same psychological structure (Steffe, 2002) with reversibility needing to become a key component of equi-partitioning in forming a reversible fraction conception. Any single part of a partitioning should be able to be used to reconstitute the unpartitioned whole by iterating the part. Reforming the whole from the iterated parts requires a history of the partitioning and a capacity to chunk or reunitise.

### *2.4.5 Common errors of interpretation*

Results from the Rational Number Project (Behr, Wachsmuth, & Post, 1984, 1985; Post et al., 1986), the National Assessment of Educational Progress in the USA (T. P. Carpenter et al., 1981; Kouba et al., 1988) and Strategies and Errors in Secondary Mathematics (Kerslake, 1986) documented numerous common errors and suggested that many students' understanding of fractions is characterised by a knowledge of rote procedures which are often incorrect. Moreover, some students interpret the fraction by relying solely on the size of the denominator or treating the size of the fraction as the difference between the numerator and the denominator (see Section 2.3.1).

In an Australian study of children in Grades 2, 3, 4, and 5, Clements and Del Campo (1987) investigated how, and to what extent, children's meanings and conceptions of the fractions one-half, one-quarter, and one-third differed across different kinds of continuous quantities and different displays of discrete sets of objects. They used a 36-item pen and paper test of fraction knowledge and then interviewed about one-quarter of the 1024 children in the study. Of the 36 items, 24 involved the recognition of one-half, one-quarter or one-third in either discrete or continuous pictorial embodiments. That is, given a fraction picture, children were asked to indicate whether it represented one-half, one-quarter, one-third or none of these. Six items involved equal sharing of a number of lollies among a specified number of people by drawing a ring around the number of lollies each person would receive, and the remaining six items required the child to indicate which of a set of four pictures illustrated a specific fraction.

During the interviews the children were presented with a range of materials, all possible fraction embodiments, and challenged to use these materials to construct representations of one-half, one-quarter and one-third. Ten different possible fraction embodiments, such as a 10 cm length of plasticine and an equal arm balance, circular paper discs, pieces of string and jars with water, were used for the interviews with two of the ten being discrete embodiments (12 unifix cubes, 8 identical squares and 2 identical circles). The interviews showed that many children simply could not think of a set of discrete objects as a unit that could be partitioned. One Grade 4 child is reported to have said, “There are 12 blocks here, so how can you get a half? What do you mean? Half a block?” (1987, p. 103). Not only were the questions involving discrete embodiments of $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{3}$ found to be much more difficult than those involving continuous embodiments, but only about one-half of the fifth-grade interview sample could cope with “discrete” $\frac{1}{4}$ questions.

Another error identified in the Clements and Del Campo study was that a drawing showing one-quarter shaded was often interpreted as one-third because “one section was shaded and three were not” (p. 104). They described this kind of response as relying on the word association of the numbers “one” and “three” with the expression “one-third”. When children were asked to make one-quarter, they often divided the material into pieces that were not equal in size, despite being able to correctly identify drawings of one-quarter in the receptive mode. The need for identical subunits appeared to develop across the grades.

Clements and Del Campo also reported that on paper and pencil items requiring children in Grades 2 through 5 to associate the fraction one-third with a given picture, they performed little better than what might have been expected from random responses. Children's identification of one-third with one-quarter was reported as very common in the interviews. As well as linking one-third with a representation of one-quarter, possibly see as one part shaded and three not, it was not uncommon for the three quarter-sections to be associated with one-third. Children in this sample had even less idea of one-third in a discrete context than they did in a continuous context. The concept of one-third did not appear well formed at any of the grades, and several children appeared to have developed a different form of equivalence of fractions when they commented, "That's about three-quarters, and three-quarters is a third, isn't it?" (p. 108).

The misconceptions identified in the Clements and Del Campo study often related to the nature of the mental images the children accessed in answering the question. For example, when asked to identify which of four pictures showed one-half, over 10% of children in each of Grades 2, 3, 4, and 5 gave an incorrect response when the line of symmetry identifying half of an equilateral triangle was at an oblique angle. Partitioning often appeared to be a visual matter, sometimes with no attention to the numerosity of the subsets.

## 2.5 Non-symbolic representation of fractions

The capacity to develop a non-symbolic representation of fractions appears to be essential for the development of children's quantitative sense of fractions. Without adequate conceptual benchmarks, comparison of fractions reduces to a numeric rule basis. For example, a numeric rule of "the bigger the number, the smaller the fraction" might be formed by some children. Thinking quantitatively about fractions depends upon the concept image children have of fractions. The research into fraction learning has not resulted in the looked-for advances in children's fraction concepts themselves. Possible reasons for the lack of progress may be the multiple interpretations of fractions and related descriptions of subconstructs, and even "personalities" of the subconstructs (Behr et al., 1993; Lamon, 1996), the non-transparent symbol system and the impact of traditional part-whole teaching of double counts. Many children do not appear to have coherent fraction concepts (Hart et al., 1981; Mack, 1995). Yet conflicting interpretations of fractions do not by themselves appear to produce constructive conflict in the child. Perhaps the very breadth of the fraction concept has meant that it is frequently compartmentalized.

### 2.5.1 The fraction concept image

A concept image is described as all of the cognitive structure in the individual's mind that is associated with a given concept, “which includes all of the mental pictures and associated properties and processes” (Tall & Vinner, 1981, p. 152). The evoked concept image is referred to as the portion of the concept image activated at a particular time. In this way, seemingly conflicting images may be evoked at different times without necessarily producing any sense of conflict in a child.

A child’s fraction concept image may then include models of fractions, acting as prototypic images[6], as well as continuous or discrete models for increasingly abstract reasoning about fractions. It may also include the algorithmic manipulation of whole numbers used in operating with fraction symbols. Further, a child’s concept image may include the linguistic tags associated with fractions, such as “half”, “quarters” and “thirds”. The symbol system $\frac{a}{b}$ is another significant component of a fraction concept image. Over time the evoked concept image may display many different facets of fraction knowledge, not all of which need be logically consistent.

Children’s prototypical visual images for fractions are likely to be influenced by the dominance of the horizontal and vertical orientations in visual perception (Howard & Templeton, 1966). Studies of the dependence of perception of form on orientation (Rock, 1973) suggest that determining an axis of symmetry, a process associated with halving of regional models of fractions, is dependent upon the orientation of the axis of symmetry. Thus the visual prototypic images associated with fraction concept images are likely to be orientation dependent.

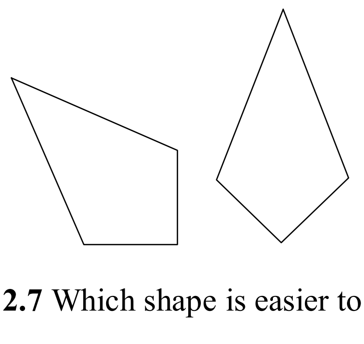


**Figure 2.7** Which shape is easier to halve?

[6] Prototype theory is a model of graded categorisation in cognitive science, where all members of a category do not have equal status (Rosch, 1983).

In Figure 2.7, identifying half of the shape is easier for the shape on the right with a vertical axis of symmetry, although the two shapes are congruent.

Piaget's notion of assimilation was used by Skemp used when he described understanding as assimilating "into an appropriate scheme," (1986, p. 43) by which he meant attaching appropriate meanings and imagery to the utterances or inscriptions that a person interprets. The fraction concept image may contain both procedural knowledge and conceptual understanding of fractions and either or both of these types of knowledge may be evoked in response to a specific task.

The multiple subconstructs and schemes associated with fractions suggest that the field of fractions is conceptually rich. Indeed, the wealth of concepts involved with fraction learning makes it a challenging area for both students and researchers. Utilising the idea of the evoked concept image of various fractions provides a way of recognising the multiple research perspectives associated with fraction learning without being overwhelmed by the different insights provided by different research traditions.

A concept, when considered as a mental construct, can be taken to consist of "a person's organized information about an item or a class of items that enables the person to discriminate the item or the class of items from other items and also to relate it to other items and classes of items" (Klausmeier, 1990, p. 94). If the multiple representations, subconstructs and instances of the scheme for a particular fraction class are taken to form a concept image for that fraction, we have a way of thinking about each fraction as both a partitioned fraction and a quantity fraction. For example, a concept image for one-quarter could contain the notation, the linguistic tag, one-quarter as a half of a half for a continuous or a discrete representation, as well as one quarter as the result of a division of two whole numbers, such as one and four or six and twenty-four.

A concept image of a particular fraction contains sufficient representations and scripts to be a concept, as well as being a subordinate to the larger concept images of partitioned fractions or elements of multiplicative fields. The multiple representations of the one entity, a fraction, provide both the power and the challenge of fractions. Referencing the concept image of a fraction, such as a quarter, provides a manageable way of dealing with the multiple representations and schemes evoked in response to situations dealing with a quarter.

Another pertinent idea from classical category and concept research is the role played by positive and negative instances of a concept (Shumway, White, Wilson, & Brombacher, 1983). As categories and concepts are based upon defined boundaries, clear or fuzzy, negative instances play a role in forming concepts. Specifically, coming to know what one-quarter is can be assisted by knowing what one-quarter is not. Traditional misconceptions could be described as negative instances of a concept that have been unintentionally construed as positive instances of the concept.

## 2.6 Summary

Various researchers and projects have used five subconstructs of rational number (part-whole, quotient, ratio number, operator, and measure) to clarify the meaning of rational number. Given the reorganisation of the various subconstructs over time, dealing with the subconstructs as discrete entities is unlikely to form a productive framework for a quantitative sense of fractions.

The approach of viewing fractions as part of a much broader conceptual field of multiplicative structures provides an alternate organization of fraction tasks, and contributes to the focus on the role of managing different types of units. What has emerged from the research (Pitkethly & Hunting, 1996) is the view that the fraction concept develops from partitioning discrete and continuous quantities leading to identification of the unit and then building up amounts by iterating the unit to form units-of-units. The relationship between the unit and the whole is emphasised by iterating the unit to form the whole or re-organising the iterated units when the whole has been exceeded. That is, children's initial reorganisation of fraction conceptions may be considered to fall into four strands:

(a) equidivision of wholes into parts,
(b) recursive partitioning of parts,
(c) reconstruction of the unit (i.e. the whole), and
(d) iterating beyond the whole and reforming the unit.

The idea of the division and reconstruction of the unit as a central feature of fraction knowledge is not new. It was described by Piaget, Inhelder and Szeminska (1960) as well as McLellan and Dewey (1895/2007).

What needs to be added to this overview of the development of fractions as quantities is the function of the equal whole. Fractions only exist as apparently unit-less mathematical objects when they reference an immutable universal whole or “one”.

The conceptual field of fractions is very large. Dealing with evoked concept images for individual fractions as mathematical objects provides a way to bring together the various subconstructs and action-based schemes associated with fractions. Both positive and negative instances of the fraction concept images for various common fractions can, and should, contribute to mapping the development of a quantitative sense of fractions in students. The specific individual concept images also include the range of traditional fraction perspectives, such as discrete, continuous, part-whole and measure. Finally, using misconceptions as negative instances of the evoked concept images for specific fractions can strengthen what it means to have a quantitative sense of fractions.

# Chapter 3 METHOD

*The detailed analysis of students' thinking associated with in-depth descriptions of the formation of fraction concepts is usually dependent upon relatively small numbers of students being involved. The method employed to investigate how children come to understand fractions as quantities needed to address both the conceptual analysis and provide a robust empirically founded description of the development of the quantitative sense of fractions.*

**Preview**

To investigate students' quantitative sense of fractions, tasks used to assess fraction understanding arising from measurement, partitioning continuous quantities, and making comparisons between fractions expressed in words and numbers were identified or developed for this study. The tasks were given to a large cross-section of students from Years 4–8 to obtain data for both a conceptual analysis and a related Rasch item analysis. The sample for this study contained over 300 students in each year group to obtain robust subgroups for analysis.

Provisional coding categories were formed and then reviewed, looking for any overlap or redundancy. The coding categories were refined over several iterations of conceptual analysis by testing whether the codes could be interpreted in terms of an emerging theory. The process of coding and data reduction was shaped by an emerging theory of student intent, arising from the simultaneous process of coding and analysis.

Having developed a rich description of the development of students' quantitative sense of fractions through the conceptual analysis, Rasch analysis was used with the items to investigate and describe a hypothesised developmental continuum within students' sense of the size of fractions.

## 3.1 Introduction

Investigating the nature and acquisition of a quantitative sense of fractions, that is, fractions as mathematical objects, makes use of a large sample cross-sectional item analysis. Items were developed to measure the data strand identified by Behr, Lesh, Post, & Silver (1983) as concerning children's development of a quantitative notion of rational number. The items were designed to reflect the most relevant subconstructs and schemes identified in the research literature as contributing to children understanding fractions as quantities.

A sufficiently large sample size with more than 300 students in each year group was used to produce a robust Rasch item analysis in addition to a conceptual analysis. Follow-up interviews were used to clarify any categories of responses where students' recordings did not appear to provide sufficient insight into their reasoning.

## 3.2 Task development

In general, the criteria adopted for selection and development of tasks were:

a) that each task should produce responses that are significant in terms of their psychological importance as described in the research literature;
b) that the tasks should have validity with respect to a quantitative sense of fractions, rather than a procedural emphasis;
c) that, wherever possible, the tasks should be in the "expressive mode" involving students in demonstrating their understanding by constructing fractions across a range of contexts.

In seeking to capture the essence of a quantitative sense of fractions, the tasks needed to address the underlying invariance of the fraction across multiple representations. The comparison of the part to a whole in evoking a fraction as a relational number across a range of contexts was central to identifying tasks that will contribute to the conceptual analysis.

Starting from the description that to think quantitatively about fractions, "students should know something about the relative size of fractions and be able to estimate reasonable answers when fractions are operated on" (Cramer & Post, 1995) tasks were identified or developed to address a range of related concepts and different contexts. (The mathematical object that constitutes a fraction remains unchanged regardless of the context or embodiment.) The form of fraction knowledge and contexts selected include:

- fractions as area,
- fractions as part-whole comparisons of collections of discrete items,
- fractions arising from measurement,
- fractions arising from partitioning continuous quantities,

- comparisons between fractions expressed in words and numbers, and
- the relationship between partitioning, unitising and iterating unit fractions.

The range of contexts designed to elicit students' quantitative sense of fractions also provided perceptual variability for the concept. Dienes (1964) argued that many mathematical concepts are essentially multi-dimensional and particular representations illustrate specific aspects of a concept. The tasks selected for this study emphasise the multiplicative part-whole relationship associated with a universal whole.

Tasks can be considered to be either in the "expressive" or the "receptive" mode of communication. Tasks in the expressive mode involve students in constructing, drawing, making and doing. Traditional assessment tasks normally emphasise the receptive mode of communication, that is, students are passively engaged in processing material generated by someone else. Clements and Del Campo (1987) found that many students who could correctly respond to fraction items in the "receptive mode", such as correctly naming a shaded quarter of a circle, could not construct one-quarter of a circle in an individual interview. Expressive mode tasks are more likely to indicate understanding of concepts than receptive mode tasks.

### *3.2.1 Criteria for item selection or development*

Thinking quantitatively about fractions relies significantly upon equal-partitioning (Lamon, 1996) and the invariance of the whole (H. Yoshida & Sawano, 2002). In representing a number less than one, magnitudes of "one" need to be of the same size if fractions are to be compared, because the invariance of the whole is crucial in comparing quantity fractions (K. Yoshida, 2004).

The importance of partitioning in the understanding of fractional numbers has been widely acknowledged (Behr et al., 1983; Kieren, 1976; Piaget et al., 1960; Pothier & Sawada, 1983; Streefland, 1991; Vergnaud, 1983). However, it is the process of partitioning that is more important to a quantitative sense of fractions than the allocation of numbers to pre-partitioned parts (Armstrong & Novillis Larson, 1995; Empson, 1999; Kieren, 1988). Consequently, the questions involving comparisons of the size of fractions employed in this study did not use fully partitioned entities. For example, the first three questions required students to name one-half and one-quarter as parts of a strip of paper, however, the paper was divided into three parts (a half and two quarters), not four parts.

The relationship between the part and the whole is the pertinent feature of a quantitative sense of fractions in this study. When students were asked to name fractional parts of a whole, the tasks had to allow opportunities to determine what the student attended to in naming the fraction. For example, a focus on the number of parts should be readily distinguishable from a regional multiplicative part-whole interpretation of the parts.

Similarly, the next three questions asked students to name what fraction of the area of a hexagon was formed by a trapezium (half), an equilateral triangle (one-sixth) and a rhombus (one-third). As with the paper strip, the number of pieces forming the whole did not indicate the fractional parts. The hexagon was formed from three, not six pieces so students could not simply count the number of partitioned pieces to name the fraction.

Questions involving locating fractions on number lines were not used to measure students' conceptual measurement knowledge of fractions, as number line test items appear to be poor estimators of the measurement aspect of fractions (Ni, 2000). Thus, although questions involving locating fractions on number lines have been used in some international comparisons of number knowledge, they were not used in attempting to identify students' quantitative sense of fractions.

The capacity to develop a non-symbolic representation of fractions appears to be essential for children's quantitative sense of fractions. Concerns have been raised about the relationship between students' use of notation and the related fraction concepts. These concerns were identified in qualitative studies such as those of Mack (1995) and Ball (1993), and confirmed in the elementary grades in the quantitative study by Saxe et al. (2005). Therefore, none of the first 23 of 29 questions used the $\frac{a}{b}$ notation. To keep the focus on a quantitative sense of fractions and to avoid as much as possible algorithmic manipulation of fraction notation, students were asked to draw their answers to questions 15 through 18.

Having a sense of the size of fractions requires students to conceptualise fractions as quantities (quantity fractions). That these quantities can be compared, ordered and operated upon was the basis for items used to measure students' quantitative sense of fractions.

### 3.2.2 Purpose and organization of the tasks

The fraction concept can develop from partitioning discrete and continuous quantities (Pitkethly & Hunting, 1996). Consequently, some tasks used in this study involve continuous contexts, and others involve discrete items. As the discrete expressions of fractional units are effectively "countable" which could contribute to an additive interpretation, fewer discrete item tasks were used than continuous item tasks.

The most commonly used continuous quantity models are the "regional part-whole models" of fractions. Many curricular offerings emphasise part-whole models almost exclusively in the teaching of fractions in the primary years (Middleton et al., 2001). The common regional part-whole models of fractions often involve students in colouring in a number of subdivisions of a region to represent a fraction or naming a fraction represented in this way. To determine whether students had developed a mental model which is inappropriately inclusive (parts-of-a-whole), rather than a powerful measure of inclusion (comparison to a unit) (Kieren, 1988; Vergnaud, 1983), tasks were carefully designed to allow incorrect interpretations of the regional model to be evident in students' responses. Indeed, both positive and negative instances of the fraction concept images for various common fractions contribute to identifying a quantitative sense of fractions in students, and are essential in a conceptual analysis.

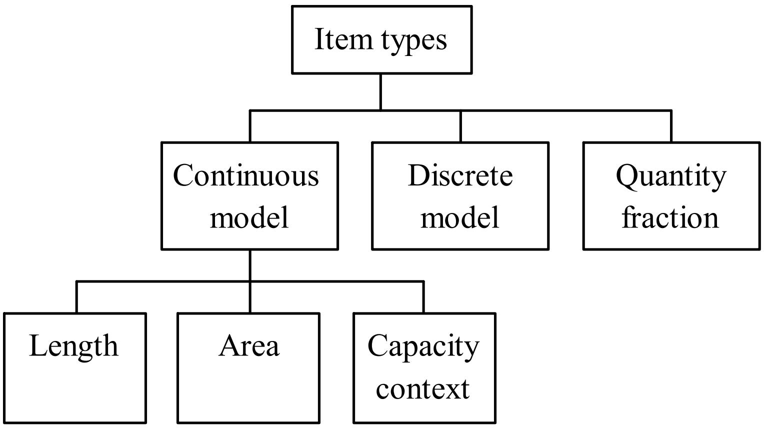



**Figure 3.1** Organization of tasks

Three main types of items were used, those relying on continuous, discrete and quantity fraction interpretations. The tasks involving a continuous part-whole model included part-whole comparison of length or area. A word problem encapsulating a capacity context — cups of milk used in a recipe — was used to tap informal knowledge of operations involving fractions. Students' representations of their evoked fraction concept images for this problem often involved drawings based on area, as they were asked to draw their answers. The capacity context allowed imagination to influence both process and recording. Items using discrete representations referred to plastic counters and quantity fractions were presented without a specific material model.

To assist with the management of the tasks, they were grouped according to the context or materials being used. Consequently continuous model tasks were interspersed with discrete model tasks and quantity fraction tasks. The order of presentation was constant across grades and classes.

## 3.3 The tasks

In total, 37 items were used across 2 separate tests. The responses to the first 30 items were recorded on a single A3 test sheet formatted as four A4 pages. Each classroom teacher also used a strip of paper one-quarter of the width of an A4 sheet (269 mm x 53 mm), three transparent coloured pattern block tiles for use on an overhead projector and transparent coloured counters (2 yellow, 6 red and 12 of one colour) to administer the tasks on the first test.

The final seven tasks and the associated materials made up the second test. As well as an A4 answer sheet, each student was provided with three strips of paper, one blue (200 x 15 mm), one yellow (150 x 15 mm) and one green (80 x 15 mm), to use in measurement comparisons. Each student also received a half of a coloured paper circle (diameter 12 cm) and a coloured square piece of paper (edge length 10 cm). The entire test materials were provided in A4 sized envelopes.

The school coordinators for the fraction assessment tasks were provided with a briefing on 16 June 2004 to gain consistency in the delivery of the tasks. The two series of tasks were given to classes on separate days within one week, selected by the school to fit with other commitments, during the three weeks from 26 July 2004 to 13 August 2004. The teachers checked the date of birth recorded for each student and the coordinator returned the completed tasks in the envelopes provided.

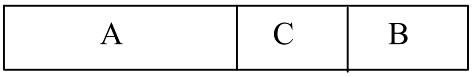
### 3.3.1 The questions and instructions

The teacher presented the first 11 questions to the whole class. The oral instructions are presented in italics and a copy of the answer sheet is included in Appendix A. The purpose statements were not included on the answer sheets.

*Write the name of the school, your name, class and date of birth on the answer sheet.*
*Write your answers to the following questions on the answer sheet.*

***Purpose: Interpret divided length***

*Instructions:*
*Watch carefully.*
In front of your students, fold a strip of paper (269 mm x 53 mm) in half as below, unfold, fold one-half in half then unfold.

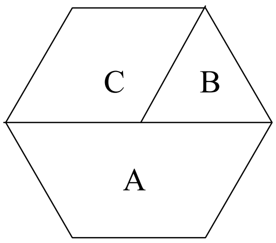



*I want you to think about your answers and then write them down on the answer sheet next to the question number.*

1. Point to one-half of the strip (A) and move your hand across it so that students are clear as to which part you are referring to. *What fraction of the strip is this piece of paper?*
2. Point to the outside quarter of the strip (B) and move your hand across it so that students are clear as to which part you are referring to. *What fraction is this piece?*
3. Point to the inside quarter (C) of the strip and move your hand across it so that students are clear as to which part you are referring to. *What fraction is this piece?*

***Purpose: Interpret regional model, non-equal parts***

*Instructions:* Place the following three pattern block transparencies on the overhead projector to form a hexagon.




4. Point to the trapezium (A). *What fraction of the whole shape is this piece?*
5. Point to the equilateral triangle (B). *What fraction of the whole shape is this piece?*

6. Point to the rhombus (C). *What fraction of the whole shape is this piece?*

### *Purpose: Interpret discrete model*

*Instructions:* Place 2 yellow transparent counters and 6 red transparent counters in a line on the overhead projector. Point to the 2 yellow counters.

7. *What fraction of all the counters is yellow?*
8. *Can you write this fraction another way?*

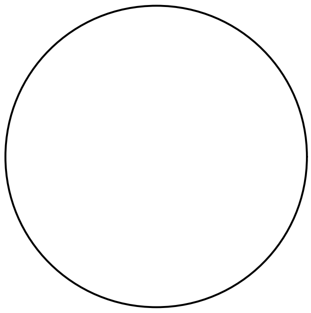
### *Purpose: Fractional quantity of discrete items*

*Instructions:* Place 12 transparent counters of the same colour in two equal columns on the overhead projector.

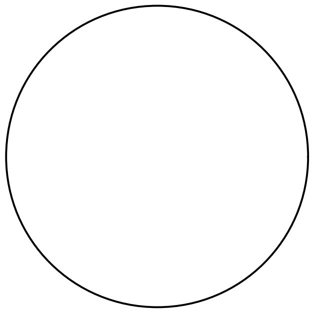
9. *How many of these counters would you have if I gave you 1/4 of them?*
10. *How many of these counters would you have if I gave you 3/4 of them?*
11. *Would you have more counters if I gave you 1/4 or 1/6 of them? How do you know?*

### *Purpose: Partition to construct a regional model*

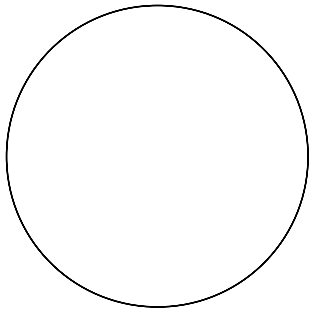
12. Shade one-half of this circle.



13. Shade one-third of this circle.



14. Shade one-sixth of this circle.



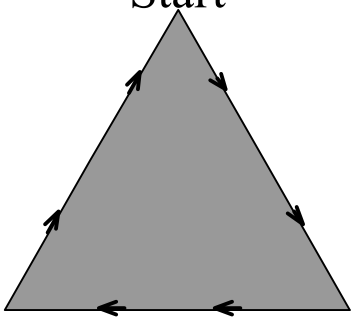
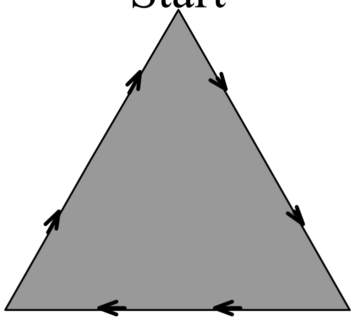
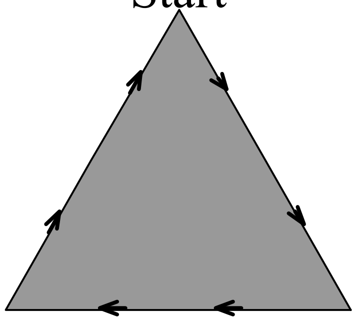
### *Purpose: Representing multiples of non-unit fractions*

15. A recipe uses three-quarters of a cup of milk. If I make double the recipe, how many cups of milk do I use? Draw your answer.
16. A recipe uses two-thirds of a cup of milk. If I make three lots of the recipe, how many cups of milk do I use? Draw your answer.

### *Purpose: Representing quotitive (measurement) division by fractions*

17. I have 6 cups of milk. A recipe needs one-third of a cup of milk. How many times can I make the recipe before I run out of milk? Draw your answer.
18. I have 6 cups of milk. A recipe needs three-quarters of a cup of milk. How many times can I make the recipe before I run out of milk? Draw your answer.

***Purpose: Locate fractional position on a continuous model***

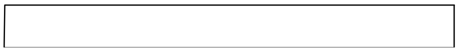
| | | |
|---|---|---|
| 19. An ant crawls around the outside of this triangle following the direction of the arrows from the Start. Mark a X on the triangle where it will be when it is half of the way around. | 20. An ant crawls around the outside of this triangle following the direction of the arrows from the Start. Mark a X on the triangle where it will be when it is one-third of the way around. | 21. An ant crawls around the outside of this triangle following the direction of the arrows from the Start. Mark a X on the triangle where it will be when it is one-quarter of the way around. |
| Start  | Start  | Start  |

***Purpose: Justify quantitative comparison of fractions as mathematical objects***

| | Which is the bigger number? | How do you know? |
|---|---|---|
| 22 | One-half or one-quarter | |
| 23 | One-third or one-half | |
| 24 | $\frac{1}{3}$ or $\frac{1}{4}$ | |
| 25 | $\frac{1}{3}$ or $\frac{1}{6}$ | |
| 26 | $\frac{2}{3}$ or $\frac{5}{6}$ | |
| 27 | $\frac{9}{10}$ or $\frac{12}{13}$ | |

***Purpose: Reconstructing the unit from a continuous part***

28. If this drawing shows three-quarters ($\frac{3}{4}$) of a piece of paper, mark on it where one-half ($\frac{1}{2}$) of the **whole piece of paper** would be.



29. If this drawing shows three-quarters ($\frac{3}{4}$) of a piece of paper, draw where the whole piece of paper would end.

***Purpose: Estimating the size of the sum of two fractions***

30. What would be the best estimate of the answer to $\frac{4}{5} + \frac{11}{12}$?

A 17

B 15

C 2

D 1

## Test 2: Tasks with paper units

The following questions were presented on a single answer sheet (see Appendix A). The teachers read the questions aloud on request or if reading the questions was part of the normal class routine.

*Materials:* Three strips of paper, one blue (200 x 15 mm) one yellow (150 x 15 mm) and one green (80 x 15 mm).

***Purpose: Measurement comparison of two quantities (fraction remainder)***

31. Exactly how many of the short green strip of paper would be needed to be equal to the length of the blue strip of paper? How do you know?

***Purpose: Halving the unit of measure***

32. If you had a strip of paper that was half as long as the green strip of paper, exactly how many of these would you need to equal the blue strip of paper?

***Purpose: Reversing the measurement comparison***

33. Exactly how many of the yellow strip of paper would be needed to be equal to the length of the blue strip of paper? How do you know?

34. What fraction of the length of the blue strip of paper is the yellow strip of paper?

35. What fraction of the length of the yellow strip of paper is the blue strip of paper?

***Purpose: Composition of partitioning***

*Material:* Half of a circular disk (diameter 12 cm).

36. Fold the half of a paper circle to show what one sixth of the **full circle** would be. Shade your answer so that it is clear.

*Material:* A square piece of paper (edge length 10 cm).

37. Fold the square piece of paper to show one-ninth of the square. Shade your answer so that it is clear where the ninth of the square is.

## 3.4 Summary of the tasks

The tasks have been developed from the existing research literature. Some of the questions are minor variations of questions used in other studies. The remaining tasks designed specifically for this study have been developed to investigate schemes, extensions of partitioning levels or understandings of the unit-whole proposed by experimental curricula. Many conventional curricula introduce common fractions as standing for part of a whole but little attention is given to the whole from which the fraction extracts its meaning (Kilpatrick, Swafford, & Findell, 2001). The tasks seek to determine what students have abstracted from the relationship between the parts and the whole, which contributes to a sense of fractions as quantities that can be compared.

### 3.4.1 Continuous length

With the partitioning process visible in the first three questions, the naming of common fractional parts (one-half and one-quarter) through interpreting a divided length depends on what the student attends to. That is, students' answers can refer to the number of parts, the area of the parts, the equality of the parts or other elements of their evoked concept images.

In treating a perimeter as an example of a continuous length, students were asked to locate a position corresponding to a fraction of the total perimeter of an equilateral triangle in Questions 19 to 21. Clements and Del Campo (1987) claim that this task has both continuous and discrete features. The discrete feature they refer to is that the triangle has three sides. However, very few students in their study made use of this discrete feature. In the Clements and Del Campo study, the journey around the equilateral triangle started at the bottom left vertex. For Questions 19 to 21 the journey started from the apex of the equilateral triangle. This variation in the starting point for the tasks has implications for how the specific schemes, such as splitting or halving, are deployed by students.

Two questions, 28 and 29, required students to reconstruct the unit-whole from a non-unit part. These two questions were designed to look for evidence of students using the reversible fraction conception. The reversible fraction conception has been described as "…the learner's partitioning of a non-unit fraction ($\frac{n}{m}$) into $n$ parts to produce the unit fraction ($\frac{1}{m}$) from which the non-unit fraction was composed in the first place. This allows, for example, for producing the whole ($\frac{m}{m}$) of which the unit fraction is part." (Tzur, 2004, p. 93). These two questions developed from a suggested problem designed to require middle years students to think flexibly about rational numbers (National Council of Teachers of Mathematics, 2000, p. 215).

Davydov and Tsvetkovich (1991) suggested introducing the idea of fractions by starting from comparison of units in measurement. They argue that fractions appear quite naturally from measuring when a quantity is not an exact multiple of the unit of measure. With this approach, the size of the unit is always in the foreground, making the part-whole multiplicative relationship more obvious. This process provides a measurement-based comparison of two quantities and a natural context for fractional remainders. The comparisons of the lengths of paper strips (Questions 31 to 35) developed out of the description of Davydov and Tsvetkovich's approach to teaching fractions from measurement (Morris, 2000). Viewing fractional parts as arising from acts of measurement enables students to consider a composite measurement unit as a referent whole and to partition that unit by splitting quantities in ways that reflect the multiplicative nature of the problem.

### 3.4.2 Continuous area

Pattern blocks were used in Questions 4 to 6 to engage students in interpreting area as essential feature of a continuous embodiment of a fraction. A hexagon was formed from three different coloured pattern blocks; a trapezium (half), an equilateral triangle (one-sixth) and a rhombus (one-third). Three different pattern blocks were used so that the number of pieces forming the whole did not indicate the fractional parts. These questions required students to interpret a regional model composed of related but non-equal parts of a hexagon.

Students were also asked to partition circles to construct their own regional models to represent three common fractions, $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$. These questions were designed to capture the process of partitioning rather than colouring in pre-partitioned circles. The sequence $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$ was chosen to examine composition of partitioning. That is, constructing half a circle and one-third of a circle should assist in constructing one-sixth of a circle if composition of partitioning is an established component of students' fraction concepts.

The idea of composition of fractions, recursive partitioning and unitising through paper folding is explored further in Questions 36 and 37. In Question 36, students were provided with half of a circular disc of paper and asked to fold it to determine one-sixth of the whole circle. Conceptually, this task again addresses the sequence $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$ through composition of partitioning. In Question 37, students were asked to fold a square piece of paper to show one-ninth of the area. This question was designed to address the composition level of partitioning and the idea of recursive partitioning. Halving and quartering relate to the notion of splitting (Confrey, 1994) or repeated halving (Pothier & Sawada, 1983) whereas finding one-ninth of a square by folding is more likely to draw on a multiplicative composition of fractions in determining a third of one-third. This idea of taking a part of a part of a whole is also associated with understanding fractions as operators (Behr et al., 1993).

### 3.4.3 Continuous capacity context

In exploring students' informal knowledge of fractions Mack (2001) suggests embedding fraction operations in a realistic problem. The idea for Questions 15 to 18 came from an example of using a recipe to determine the fractional unit (e.g. $\frac{2}{3}$ of a cup of sugar) as the basis of fraction calculations (National Research Council, 2001, p. 239). The first two questions deal with representing multiples of non-unit fractions and the next two questions require students to interpret, represent and carry out non-symbolic measurement division involving fractions. Conventional instruction on fraction computation is commonly rule-based (Hiebert, 1989), so the request to draw the answers was an attempt to move the explanation towards a sense-making process, rather than a rule-based manipulation of whole numbers.

### 3.4.4 Discrete model

Actions used to subdivide continuous quantities are different from those used to subdivide discrete quantities (Hunting, 1984). Consequently, students' conceptions of fractions may be quite different if restricted to either context. Questions 7 and 8 require students to name a fraction by interpreting 2 red counters and 6 yellow counters in a line as forming units composed of discrete items, and then determine an equivalent fraction name. Question 7 is an elaboration of a simpler discrete item fraction task involving four coloured tiles used to assess 11 year-old students in England, reported in Dickson, Brown and Gibson (1984, p. 280). Question 8 deals with the relationship between two-eighths and one-quarter as fractions. In Questions 9 and 10, students are asked to find fractional amounts (one-quarter and three-quarters) of 12 counters. The counters are presented in two equal columns of 6 counters, acting as sub-unit halves, to make the process easier. The fractions in these questions act as operators to compare fractions of a composite unit of 12. The process by which students systematically allocate items resulting in equal shares is known as the dealing procedure (Davis & Pitkethly, 1990). Having access to the images of the counters rather than the actual items reduces the opportunity for one-to-one dealing as a process for constructing quarters. Finally, Question 11 introduces the fraction one-sixth and asks the students to compare fractional quantities of 12 discrete items, and to justify their answer.

### 3.4.5 *Abstract quantity fractions*

In Questions 22 to 27, students were asked to justify their comparison of the size of fractions as mathematical objects. Appreciating the magnitude of each fraction as a single entity depends upon being able to locate it as a quantity fraction. Students' explanations may equally capture the lack of a universal unit in their reasoning about quantity fractions.

Questions 26 and 27 used fractions that were near the universal 'one'. These fractions have been used in these comparison tasks because these fractions are only near one if you can conceptualise them as quantities. The fractions used in Question 26 ($\frac{2}{3}$ and $\frac{5}{6}$) and Question 27 ($\frac{9}{10}$ and $\frac{12}{13}$), were also designed to have a ready additive interpretation. As well as the numerators being one less than the denominators, the numerators and denominators of each pair of fraction are 3 apart. The denominators of the first pair of fractions are also linked by a multiplicative relationship. That is, the denominators 3 and 6 can be thought of as three apart or that one is twice the other.

Question 30, estimating the size of the sum of two fractions near one, was based on an item (determining the best estimate for $\frac{12}{13} + \frac{7}{8}$) from the Second Mathematics Assessment of the National Assessment of Educational Progress (T. P. Carpenter et al., 1981). Again the two fractions are only near one if they are conceptualised as quantities.

## 3.5 The sample

Nine high schools, one central school and eleven primary schools were selected from across New South Wales' government schools. At least two complete classes in each of the larger schools were selected from Years 4, 5 and 6 in the primary schools and Years 7 and 8 in the high schools and the central school. Although item response modelling is not dependent upon using a representative sample (Wright & Stone, 1979), the schools were selected to provide a balance across metropolitan, regional and remote rural areas. Schools close to a major regional area, such as Wagga Wagga or Tamworth, were considered regional rather than remote rural. Twelve regional schools formed the bulk of the sample. Three of the schools, one high school, one central school and one primary school, were described as remote rural in that they were part of the Country Area Program, which services schools that are educationally disadvantaged by geographic isolation. Six metropolitan schools completed the geographic mix of schools.

The classes within the schools were a mixture, including composite multi-age classes, ungraded, middle-band streamed classes as well as opportunity classes (OC). Students with identified intellectual disabilities were, at the teacher's discretion, able to take part in the assessment but their results were not coded.

The total cross-sectional sample of 1676 (832 F, 844 M) students whose responses were coded comprised 331 Year 4 students (164 F, 167 M), 330 Year 5 students (164 F, 166 M), 331 Year 6 students (157 F, 174 M), 342 Year 7 students (170 F, 172 M) and 342 Year 8 students (177 F, 165 M).

## 3.6 Analysis scheme

The analysis uses the process of analytic induction and draws on ideas derived from a method suggested by Glaser and Strauss (1967). The analysis is largely a "conceptual analysis", seeking examples, counterexamples and any evidence of transition between the two. Analytical induction calls for finding commonalities in the data which first lead to a description, and then commonly to an explanation of that regularity. In contrast to the constant comparison process, which develops its theory during the observational process, analytic induction focuses on theory development during the analytic process with already gathered data (Krathwohl, 1998).

In line with the idea of theoretical sampling proposed by Glaser and Strauss (1967), the analysis will guide any further data collection deemed necessary to have a rich description of the sense of the size of fractions. "Theoretical sampling is the process of data collection for generating theory whereby the analyst jointly collects, codes and analyzes his data and decides what data to collect next and where to find them, in order to develop his theory as it emerges. This process of data collection is controlled by the emerging theory, whether substantive or formal." (Glaser & Strauss, 1967, p. 45) Analytic induction calls for "finding commonalities in the data which lead first to a description and then to an explanation of that regularity" (Krathwohl, 1998, p. 260) and will be augmented by theoretical sensitivity formed through the analysis of the literature review.

Examining the relative location of categories of responses within the data by locating the responses on a common scale using item response theory will further enhance or challenge the explanation of the regularity in students' responses.

### *3.6.1 Cross-case analysis*

A cross-case analysis is used to look for commonalities in the pattern of responses. As all bar one of the tasks are free response, the categories will emerge from the data, and in particular attempts to reduce the data, which is the essence of inductive analysis (Patton, 1990).

Categorising is always a crucial element in the process of analysis. It begins with initial attempts to record the information students have provided in their explanations, drawings, symbolising and records of actions on materials, in responding to the tasks. These data records are then brought together to form temporary categories by looking across cases, organised around those data that apparently relate to the same content. The temporary nature of these categories is important as there is a need to be prepared to extend, change and discard categories, as well as to consider alternative ways of categorising and interpreting the data. As Dey (1993) notes, flexibility is required to accommodate fresh observations and new directions in the analysis.

During the course of the analysis, the criteria for including and excluding observations, rather vague in the beginning, become more precise. The analysis seeks to define and, as necessary, redefine categories by specifying and changing the criteria used for labelling data to specific categories.

## 3.7 The Rasch analysis

The statistical analysis of the data includes an item analysis using the Rasch item response model or one parameter logistic model. The Rasch item response model is the modern development of a method set out by Louis Guttman in the 1940s in a less statistically advanced formulation (Andrich, 1985). The Guttman scaling model was used in the CSMS studies reported by Hart et al. (1981). The Rasch model is based on the conjoint measurement of student performance and item difficulty.

The Rasch model provides measures of student performance on a scale that is independent of both the items employed and the persons who respond (Keeves, Johnson, & Afrassa, 2000; Wright & Stone, 1979). This model is used for several reasons. To begin with, the assumption that a quantitative sense of fractions as a construct is unidimensional aligns with the basic assumption of the Rasch one parameter logistic model. What the model requires is the presence of one dominant component or factor that influences test performance. The essence of the Rasch model is that person-item interaction can be modelled by independent parameters for items, which are assumed to be independent and separable (Bond & Fox, 2001).

The feature of parameter separation enables direct comparisons of person ability and item difficulty estimates independently of the distribution of those abilities and difficulties in the particular sample of persons and items under examination. Additionally, where data do not conform to the expectations of the Rasch model, rather than seeking to find a model that better accounts for the data, this study treats statistical misfit as a substantive anomaly that needs to be understood. If one task does not accord with all the other items it raises the question, why not? In this way the Rasch model is used to understand the data. That is, the intent of the use of the Rasch model is not as a model-fitting activity but rather as an investigative tool.

The Rasch model is a probabilistic unidimensional model premised on the principle that the easier the question the more likely the student will respond correctly to it, and the more “able” the student, the more likely he or she will correctly answer the question. The model assumes that the probability that a student will correctly answer a question is a logistic function of the difference between the student’s ability [θ] and the difficulty of the question [δ] (i.e. the ability required to answer the question correctly) (Wright & Stone, 1979).

$$p\left(X_{ni} = 1 \middle| \theta_n, \delta_i\right) = \frac{e^{(\theta_n - \delta_i)}}{1 + e^{(\theta_n - \delta_i)}}$$

The Rasch model provides a precise measurement of the difficulty of each task or item and a way to determine the association of each item with the construct being measured. The ability and difficulty scale arising from the Rasch one parameter logistic model have a common log-odds measurement unit called the *logit*[7]. This common unit creates the opportunity to plot the items on a scale according to their difficulty (as determined by the students' responses) and to locate the students according to their measured "sense of fractions" on the same scale. The logits form the basis of an interval scale and can be averaged on a "like with like" basis to determine the average score of a "sense of fractions" of students making a particular response.

If the above assumptions hold true then the relationship between the performance of students on an individual question and the underlying trait (a quantitative sense of fractions) can be described by an S-shaped curve known as the item response function. The steepness of the curve indicates the rapidity with which the probability that a student responding to the question correctly changes as a function of this trait.

The Rasch model forms a basis for maximum likelihood estimation of the location of items to be measured on a continuum, based on collections of categorical data. This model will allow examination of the relative difficulty of various representations of the same fractions on a common scale of a quantitative sense of fractions.

### 3.7.1 Model fit

The coded categories from the data were explored using Quest, a Rasch modelling tool developed by Adams & Khoo (1999). To estimate individual question difficulty using the Rasch model, all of the incorrect response categories were treated together as one incorrect option. Treating the responses as dichotomous allows the individual item characteristics to be determined. To determine how well each item fits the model, and so contributes to a single trait, a set of fit statistics are used which test the extent to which the observed data match those expected by the model. The method of evaluating the goodness of fit test draws on the chi-square statistic.

Adams and Khoo (1999) suggest that in practice using the mean-square residual summary statistics can be a useful way of considering the compatibility of the model and the data. The goodness of fit of an item is expressed by a mean square fit statistic, or the weighted average of the squared item residuals, which are the differences between the observed score for the task and the corresponding expected score according to the Rasch model.

[7] The logit is the natural logarithm of the odds ratio of success. If a person has a 0.5 probability of answering a question correctly the odds ratio is 1:1 and the natural logarithm of this ratio is zero.

Two types of fit statistics can be computed for each item, the ‘infit’ and ‘outfit’ statistics. The infit statistic is more robust than the outfit statistic as a test of item-model fit because the outfit statistic is particularly sensitive to a few individual responses misfitting. INFIT MNSQ values that range from 0.8 to 1.2 are generally deemed to be acceptable INFIT statistics for high stakes assessment (Wright & Linacre, 1994). Values of the INFIT MNSQ less than one indicate that the observations are too predictable, that is, there may be redundancy in the data. Values of the INFIT MNSQ greater than one indicate that the observations are too unpredictable and contain unmodeled noise.

# Chapter 4 CODING THE RESPONSES

*The coding of open responses to the fraction tasks begins with making provisional categories before reviewing the results, and looking for overlap and redundancy. The initial ordering of categories in terms of the number of common responses is combined with a search for the salient features of the responses.*

*The coding categories were refined over several iterations of conceptual analysis. Subcategories were created for some items to explore the use of potentially idiosyncratic solution strategies. In addition to carrying out an initial recording of students' responses, observational notes arising from the initial coding were kept to assist in determining possible alternate ways of categorising the data. The process of coding and data reduction was shaped by an emerging theory of student intent, arising from the simultaneous process of coding and analysis.*

*The central concern in categorising responses was to capture students' understandings of fractions as mathematical objects. Correct reasoning was afforded prominence over correct answers without reasons. The aggregating of data across cases required making provisional categories before reviewing the results.*

**Preview**

Creating provisional coding categories for the items used requires different methods of coding. This chapter provides a description of the different types of coding used with free response items, coding that combined reasoning for a choice with whether the answer was correct or incorrect, forming categories of explanations, coding regions depicted as parts of circles, interpreting explanations conveyed through drawings and finally looking at the limitations of coding simple responses.

## 4.1 Coding open responses

Coding tasks designed to evoke students' conceptual images of fractions brings with it the challenge of coding in a way that allows the student's intent to be captured. Coding responses from the perspective of an adult's relatively sophisticated understanding of fractions can produce an artificial and inaccurate portrait of students' thinking.

Coding always begins by attempting to describe what the student has recorded. In essence, coding is making decisions about what things mean (Krathwohl, 1998). The coded responses are then tentatively categorised to begin to reduce the data. The creation of categories relies on the coded responses, which in turn are a reflection of the way the coder interprets the responses.

Only one of the 37 items (see Appendix A) required students to select the correct answer from a fixed number of choices. Consequently, beyond capturing what the student actually recorded, the aggregating of data across cases required making provisional categories before reviewing the results, and examining the potential interpretive value of the codes. The design of the items gave precedence to what Bogdan and Biklen (1992) describe as strategy codes. Strategy codes capture the ways people accomplish things through methods, techniques or tactics. These strategy codes attempt to summarise students' evoked concept images of the very idea of a fraction.

### 4.1.1 Creating category codes

Each student in the study was allocated a unique four-digit code with the first digit identifying the enrolled school year of the student. The date of birth and gender of each student were recorded, as well as the responses provided to the questions. Whatever students wrote down in response to a question was initially recorded as close as possible to verbatim in a spreadsheet against the student's unique four-digit code. The results were then arranged in a frequency distribution table which aided in the formation of the initial response categories.

For example, in response to Question 1, the vast majority of students (1472 in total) recorded the answer as $\frac{1}{2}$. The answer form $\frac{2}{4}$ accounted for another 45 responses. Both answer forms had initially been recorded as separate entries, as the primary coding was as close to verbatim as an entry in a spreadsheet will allow. Other forms of $\frac{1}{2}$, such as $\frac{5}{10}$ or $\frac{50}{100}$, were also present. All of the answers equivalent to $\frac{1}{2}$ were then assigned to the same category and given the numeric code of 1. The next distinct most frequent response was $\frac{1}{3}$ with 50 responses, and this was assigned the numeric code of 2. Continuing down the list of frequencies the next highest response was $\frac{2}{3}$, with 8 students recording this answer. With the small number of responses recording this answer, an initial category was formed and assigned the numeric code 3, recognising that it could readily be combined with the category of “other”, and recoded as 7. Non-attempts were assigned the code 0. For Question 1, 14 students had left the response blank. The proportion of students who omitted a particular question acted as a “rule-of-thumb” in forming categories. That is, responses corresponding to a percentage lower than the omit rate were usually included in the “other” category. Subsequently, all other responses to Question 1 with frequencies less than 8 were included in the “other” category.

A two-way table was then formed showing the category of response by school year, as in Table 4.1. Both the response description (e.g. $\frac{1}{3}$) and the code are recorded in the table. Examining responses across the school years determines if different interpretations of fractions are more common in some school years than others.

Table 4.1

*Number of responses in each category (Q1–3)*

| No. | Code | Response | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|---|---|
| Q1 | 1 | $\frac{1}{2}$ | 273 | 304 | 318 | 317 | 324 |
| | 2 | $\frac{1}{3}$ | 18 | 8 | 5 | 10 | 9 |
| | 3 | $\frac{2}{3}$ | 3 | 1 | 1 | 2 | 1 |
| | 7 | Other | 31 | 17 | 5 | 11 | 4 |
| | 0 | Omit | 6 | 0 | 2 | 2 | 4 |
| Q2 | 1 | $\frac{1}{4}$ | 236 | 249 | 269 | 271 | 294 |
| | 2 | $\frac{1}{3}$ | 42 | 50 | 38 | 40 | 33 |
| | 3 | $\frac{1}{2}$ (or $\frac{2}{4}$) | 11 | 9 | 6 | 4 | 4 |
| | 7 | Other | 35 | 20 | 16 | 24 | 7 |
| | 0 | Omit | 7 | 2 | 2 | 3 | 4 |
| Q3 | 1 | $\frac{1}{4}$ | 217 | 226 | 263 | 262 | 288 |
| | 2 | $\frac{1}{3}$ | 34 | 42 | 35 | 44 | 35 |
| | 3 | $\frac{1}{2}$ | 14 | 18 | 14 | 9 | 4 |
| | 4 | $\frac{2}{3}$ | 10 | 16 | 8 | 1 | 0 |
| | 7 | Other | 44 | 24 | 7 | 22 | 9 |
| | 0 | Omit | 12 | 4 | 4 | 4 | 6 |

The "other" category in each question contained some interesting responses. For example, one student answered "$\frac{1}{3}$ or $\frac{1}{4}$" to Question 2 while another answered "25% $\frac{1}{3}$ $\frac{25}{100}$". However, as these were very low frequency responses, they were added to the "other" category.

Some categories were more pronounced in the earlier school years and disappeared in the later years. For example, the response $\frac{2}{3}$ for Question 3 was relatively popular in Years 4 and 5 although it had disappeared completely by Year 8.

### *4.1.2 Coding related answers*

The coding for Question 8 was a little more involved as the answer depended on the response provided to Question 7. In Question 7, students were shown 2 yellow counters and 6 red counters in a line and asked what fraction of all the counters were yellow. In Question 8, students were asked if they could write this fraction another way. As the relationship between what was considered a correct answer to Question 7 had an impact on the way Question 8 was coded, it is helpful to understand the decision process involved in determining a correct answer for Question 7.

In Question 7, the expected correct answers were $\frac{1}{4}$ or $\frac{2}{8}$. Students' actual answers included responses such as 25%, 2 8ths, 2 eighths, 2 eights, 2 of 8, 20/80, two over eight and 2 yellow/6 red. Initially alternate correct expressions for $\frac{1}{4}$ or $\frac{2}{8}$, such as 25% or $\frac{20}{80}$ were given the code 2. The response "2 out of 8" was coded 3, based on the decision that a quantity fraction, such as $\frac{2}{8}$, is a single entity whereas "2 out of 8" is not one thing — although "2 out of 8" is consistent with the way that students are introduced to fractions through the early units of the syllabus (NSW Department of Education, 1989). In seeking to identify a quantitative sense of fractions and indeed equivalent fractions, "2 out of 8" alone did not contribute to the construct. Answers that were not common fractions were plentiful. Indeed, 16 students recorded the fraction as the whole number 2.

Writing the fraction in words, including the one Year 4 student who used the ordinal name "8ths", was accepted as correct. The three students who answered "2 eights" were considered case by case as it is possible that the answers could be considered to be spelling errors. Two of the three answered $\frac{2}{8}$ to the next question and the third answered 2 8ths, resulting in the three answers to Question 7 being treated as spelling errors and coded as correct.

The coding for Question 8 developed as follows. In looking at the actual responses provided, some students managed a translation of the correct fraction used in response to Question 7. That is, some students who wrote $\frac{2}{8}$ for Question 7 responded with $\frac{1}{4}$, 0.25, 25%, two eighth, or 2 out of 8 to Question 8. Although "2 out of 8" was not initially coded as a correct answer to Question 7, "2 out of 8" was accepted as another way to write $\frac{2}{8}$. If a student wrote one-quarter as the answer to Question 7 and 2 out of 8 to Question 8, this was considered a translation of the form of a correct answer. To guarantee that the items were independent of the order in which they were presented, an answer of 2 out of 8 to Question 7 was coded as an alternate correct representation if the answer to Question 8 was a correct translation of the form. All of these responses corresponding to a translation of the form of a correct answer to Question 7 were combined into a single category and given the code of 1. Answers that were a simple repetition of the answer to Question 7 without transformation formed two more categories, depending upon whether the answer to Question 7 was correct.

There was also a group of students who "inverted" their answer to Question 7, presumably believing that a reciprocal is simply a different way of writing a fraction. These responses were given the numeric code of '5'.

Table 4.2

*Initial coding categories for Question 8*

| **Code** | **Description** |
|---|---|
| 0 | Non attempt |
| 1 | Equivalent correct fraction (symbols, words, decimal or percentage, "2 out of 8") |
| 2 | Same fraction (correct) |
| 3 | Same fraction (incorrect) |
| 4 | Equivalent incorrect fraction |
| 5 | Inverted |
| 7 | Other |

As questions need local independence within a Rasch analysis, those students whose answers fell into the category of "equivalent incorrect fraction" were also considered to be correct, and recoded as 1. Otherwise an incorrect response to Question 7 would necessarily result in an incorrect result for Question 8. That is, if a student incorrectly answered $\frac{2}{6}$ to Question 7 and then answered $\frac{1}{3}$ for Question 8, an equivalent incorrect fraction, Question 8 was coded as a correct answer to avoid having an incorrect answer to Question 7 influence the coding of Question 8.

### 4.1.3 Coding explanations

In Question 11, students were for the first time asked to explain their response. The initial coding of the explanations was a literal capture. The first clear category to appear out of the coding was the category of "fact". A reason was coded as "fact" if the reasoning was simply a statement of fact, such as "a quarter is bigger", "1 sixth is smaller" or "there's just more". The "fact" category also included belief statements such as "I just know" or "it is common sense". A similar category was "guess", which included statements such as "I took a guess" and "I don't know".

Categories were formed by considering recurring elements in students' explanations. Reading across the cases, a number of responses included statements to the effect that "one sixth is 6" in the reasoning. For example, a Year 6 student wrote "cause sixth is 6 and quarter is a four" as the reasoning behind her response to Question 11 and a Year 5 student wrote "one sixth is 6 out of 12". Similarly, another Year 5 student wrote, "$\frac{1}{4}$ is 3 but one sixth is 6 so one sixth is more". Any explanations that included statements to the effect that "one sixth is 6" were considered as part of the category "$\frac{1}{6}$ is 6 or $\frac{1}{4}$ is 4".

Other categories arose by examining the students' responses closely. For example, the strategy code "bigger is smaller" (BIS) arose in response to explanations such as "the bigger the bottom the smaller the amount". This descriptive strategy code develops its label from the student's language whilst being applicable to a range of responses using the same mode of reasoning. For example, "because it a rule if the denominator is bigger and the top is 1 then it's a smaller number" was allocated the descriptive strategy code BIS.

Another descriptive strategy code that emerged from students' explanations in Question 11 was a focus on the size of the whole number in the denominator. Explanations such as $\frac{1}{6}$ is larger "because it has a higher denomenator" or $\frac{1}{6}$ is larger "because 6 is bigger than 4" appeared to equate the size of the fraction with the size of the "whole number denominator" (WND). There is some similarity with the reasoning that "one sixth is 6" and eventually both strategies were allocated the same numeric code.

Coding the students' explanations in Question 11 required making provisional categories before reviewing the results, and looking for overlap and redundancy. The three strategy codes described above were provisional categories that were reviewed when allocating numeric codes necessary for the Rasch analysis.

In seeking to produce a single coded response for Question 11, the reasons students offered could not be considered independently of their choice of which fraction corresponded to the larger number of counters. The question involving the 12 counters, "*Would you have more counters if I gave you $\frac{1}{4}$ or $\frac{1}{6}$ of them?*" could have four different responses: omitted, $\frac{1}{4}$, $\frac{1}{6}$ or other. If a student chose the correct answer and provided a correct reason, the response was coded A. The most common correct reason involved determining the number of counters equivalent to one-sixth and one-quarter, such as "$\frac{1}{6}$ that will only equal to 2 but if I had a $\frac{1}{4}$ that will equal to 3". A correct answer with an incorrect reason was coded B. A reason such as "$\frac{1}{6}$ is 1 out of 6 and a quater is a section of them" was coded as an incorrect reason, even though the answer of $\frac{1}{4}$ was correct. An incorrect response such as "$\frac{1}{3}$" and correct reason "because in a 6th you would only have 2 and 1/3 you would have 4" was coded C and an incorrect response and incorrect reason such as "because I like six more than four" was coded as D. Alphabetical coding was used initially rather than numeric coding to avoid confusing numeric categories.

The provisional categories for the reasons students provided in Question 11 included fact, guess, BIS, WND and "$\frac{1}{6}$ is 6 or $\frac{1}{4}$ is 4". The major categories are summarised in Table 4.3 together with an example of responses typical of the category.

Table 4.3

*Initial coding categories for Question 11*

| **Category** | **Example** |
|---|---|
| Fact | Reason: Becds a quater is Biger |
| Guess | Reason: I guest |
| Bigger is Smaller (BIS) | Reason: because the bigger the number the smaller it is. |
| Whole Number Denominator (WND) | Reason: Because 6 is bigger than 4 |
| $\frac{1}{6}$ is 6 or $\frac{1}{4}$ is 4 | Reason: onequater is 3 but one sixth is6 so one sixth is more |

In reviewing the provisional categories and looking for overlap and redundancy, the guess category was coded as B if the reason was associated with a correct answer and D if associated with an incorrect answer. Further, the explanations "$\frac{1}{6}$ is 6 or $\frac{1}{4}$ is 4" appeared very similar to reasoning solely from the whole number denominator (WND) and these two provisional categories were both coded the same. The provisional category of BIS was included in correct reasoning whereas an omitted reason was treated as an incorrect reason.

A small number of students (19) indicated that "$\frac{1}{6}$ was 1" while correctly determining that a quarter is 3. This response may have been influenced by interpreting $\frac{1}{6}$ as 1 out of 6. Consequently, there were occasions where a single strategy was not consistently applied to both fractions, one-sixth and one-quarter. As the reasoning component of Question 11 was more important than simply identifying the larger of two given fractions, the reasoning component was given greater emphasis when combining the selected fraction with the reason.

### 4.1.4 Estimating sector size

In coding the drawings of one-third and one-sixth of a circle for Questions 13 and 14, it was necessary to comprehend the physical limits of hand drawn sectors as well as the difficulty of angle judgement. It has been recognised for over 100 years that there are difficulties associated with making judgements of angle size (Wundt, 1902, p. 137) because we tend to overestimate acute angles and underestimate obtuse ones (R. H. Carpenter & Blakemore, 1973). There also appears to be a spatial norm effect, where we generally see better in the horizontal and vertical orientations (Howard & Templeton, 1966). Consequently, the accuracy of students' estimates of angle sizes depend upon the orientation of the angles drawn.

Given the range of ages and experiences of children in this study, error limits on the angle estimates for one-third and one-sixth were set at ±15° which meant that a sector up to 135°, corresponding to $\frac{3}{8}$, was coded as $\frac{1}{3}$ unless the construction lines indicated that the intent was to draw $\frac{3}{8}$. To assist in coding the sectors drawn to represent the two fractions, a transparency was created showing the required sectors along with the accepted range.

### 4.1.5 Coding tolerances for length

Questions 19 to 21 required students to mark a location on the boundary of an equilateral triangle with a cross. The first two of these three questions were relatively easy to code as the first was the midpoint of a side and the second corresponded to a vertex of the triangle. Question 21 was more difficult to code. Using composition of fractions, three-quarters of one-third of the perimeter corresponded to one-quarter of the perimeter. Three-quarters of a side was usually found by successive halving. The length of each side of the equilateral triangle was 5.7 cm and answers between 3.8 cm and 4.8 cm from the vertex labelled 'Start' on the right-hand side of the triangle were accepted as correct. This range, centred on a distance of 4.3 cm from the apex, excluded the next closest unitary fraction estimate of one-fifth.

In Questions 28 and 29, given a drawing of three-quarters of the length of a rectangle, students needed to mark where half of a whole rectangle would be, and then show where a whole rectangular piece of paper would end. For Question 28, the responses were coded as either correct or incorrect. Two-thirds of 6.0 cm (that is, 4.0 cm) corresponded to half of the whole rectangle, with a tolerance of ±0.4 cm was used as the range of acceptable answers. In Question 29, the 5.6 cm long rectangle corresponding to three-quarters of the total length meant that the end of the whole rectangle was accepted as correct if it was marked within a range of 1.5 cm to 2.3 cm from the end of the initial rectangle. That is, one-third of the 5.6 cm length was taken as 1.9 cm, to be added to the existing rectangle for a total length of (7.5 ± 0.4) cm.

## 4.2 Interpreting the coded data

The initial coding of Questions 12 to 14 (shade $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$ of a circle) viewed the responses through a traditional area model interpretation. Although it is possible to consistently record the approximate area of a circle shaded, this area did not necessarily code the intent of the student. For example, the response in Figure 4.1 was initially coded as $\frac{1}{4}$. That is, the approximate area shaded was initially recorded. From the perspective of an adult with well-developed fraction concepts, this response looks like one-quarter.

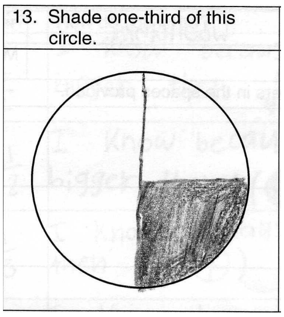



**Figure 4.1** Area coded as $\frac{1}{4}$

The provisional categories were then considered as to whether the codes could be interpreted in terms of an emerging theory of student intent. The initial percentage distribution of area categories drawn to represent one-third is outlined in Table 4.4.

Table 4.4

*Initial percentage distribution of the shaded area of a circle to represent one-third*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{1}{3}$ | 30 | 43 | 62 | 43 | 54 |
| $\frac{1}{4}$ | 16 | 11 | 10 | 14 | 12 |
| Segmented | 12 | 15 | 7 | 10 | 6 |
| $\frac{3}{4}$ | 10 | 7 | 3 | 4 | 3 |
| $\frac{3}{8}$ | 2 | 1 | 1 | 4 | 1 |
| Other | 28 | 22 | 17 | 25 | 23 |
| Omitted | 2 | 0 | 0 | 0 | 0 |

Two factors led to recoding the provisional category of one-quarter for Question 13. The first was that the coded response of "one-quarter" was the next largest category after the correct answer of one-third, constituting over 12% of all responses. What made the size of this category surprising was the lack of a satisfactory interpretation of why students would draw one-quarter when asked to shade one-third. Why would so many students confuse one-third with one-quarter? This response (one-quarter) also appeared to be relatively insensitive to grade level with about 40 responses in each grade being coded as one-quarter. The second, and perhaps more telling, observation was that representing one-third as one-quarter did not appear to be consistently applied by students. That is, students answering questions designed to elicit different fractions used essentially the same area representation. Not only could one-third be represented as one-quarter of a circle but so too could one-sixth (see Figure 4.2 for an example).

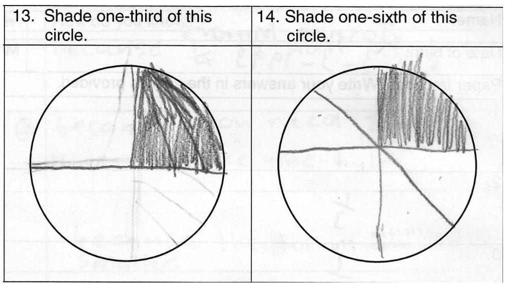



**Figure 4.2** Two different fractions represented by one-quarter

From an emergent perspective of sense making, the student's intent must have been to represent one-third in response to Question 13 and one-sixth in response to Question 14. The interpretation of this student's responses becomes consistent if the responses are coded from the perspective of the number of parts rather than the area of those parts. Both responses in Figure 4.2 represent the given fraction in this way. This observation that students might be representing the number of parts indicated by a fraction rather than the area led to all of the responses to Questions 13 and 14 initially coded as $\frac{1}{4}$ being re-examined to determine how frequently the intent to display one-third as "one part out of three parts" or one-sixth as "one part out of six parts" was evident in the responses. In Question 14, this entailed re-examining the initial response categories of $\frac{1}{4}$ and $\frac{1}{8}$. The re-examination of the responses categorised as $\frac{1}{8}$ resulted in distinguishing between $\frac{1}{8}$ constructed and 6 parts constructed (4 eighths and 2 quarters) with $\frac{1}{8}$ shaded.

### 4.2.1 Interpreting drawings

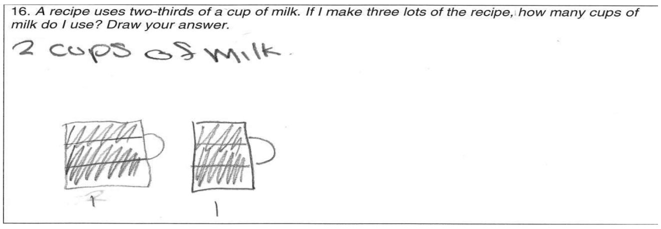
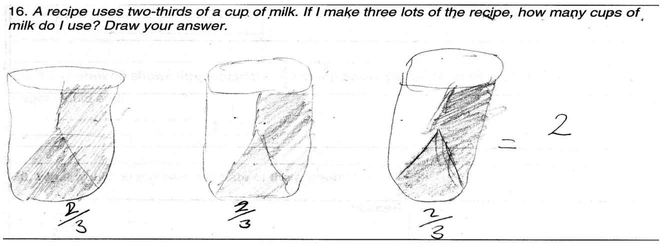
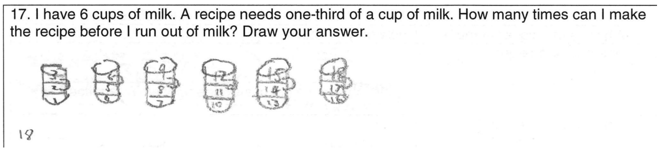
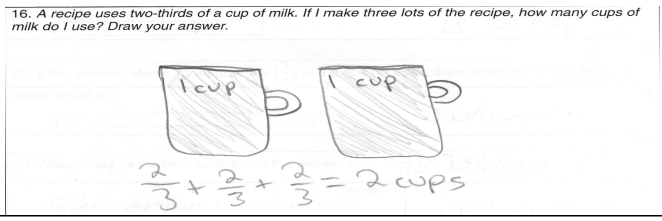
Questions 15 to 18, determining the number of cups of milk required by multiples of a recipe or the number of times a recipe can be followed when 6 cups of milk are available, relied heavily upon students' drawings to provide explanations. In a similar way to dealing with the reasons provided in Question 11, the coding of students' drawings evolved from four initial categories, A to D, to include:

- A Correct response and correct reason
- B Correct response and incorrect reason
- C Incorrect response and correct reason
- D Incorrect response and incorrect reason
- E Correct response and no reason
- F Incorrect response and no reason
- M Multiple different responses

Out of these initial categories, subcategories were formed and allocated individual codes, derived from the parent code, to record the use of units in students' reasoning. The additional information on the use of units was added to assist in the identification of strategy codes that capture the ways students accomplish the tasks. The extended categories are summarised in Table 4.5.

Table 4.5

*Extended categories for Questions 15 to 18*

| Code | Meaning |
|---|---|
| A1 | Correct answer, accumulated units |
| |  |
| A2 | Correct answer, separate units |
| |  |
| A3 | Correct answer, accumulated units (numbered) |
| |  |
| A4 | Correct answer, diagrammatic number sentence |
| |  |
| B1 | Correct answer, incorrect units |
| C1 | Incorrect, incorrect unit, correct number of units |
| C2 | Incorrect, correct unit, correct number of units (not accumulated) |
| C3 | Incorrect, correct unit, correct number of units (error accumulating) |
| C4 | Incorrect, disaggregated units, incorrect total |
| C5 | Incorrect, correct expression, incorrect answer |
| D1 | Incorrect, correct unit, incorrect number of units |
| D2 | Incorrect, incorrect unit, incorrect number of units |
| D3 | Incorrect, incorrect expression |

In Question 15, the above coding subcategories were also supplemented by the “fraction answer” where the response was incorrect, for example, C3 ($\frac{6}{8}$). This additional annotation to the coding, identifying the final fraction answer, enabled a simple frequency analysis to be carried out on the resulting responses. Subsequently, it became apparent that $\frac{6}{8}$ was referred to so frequently in the incorrect responses for Question 15 that this answer was analysed in more detail (see Section 5.6).

### *4.2.2 Allocating numerical codes to subcategories*

A collapsible second-order numerical coding (e.g. 1, 11, 12, 13) was initially used to allow further analyses to be carried out on various categories of responses. Using exploratory data analysis methods, the decision to code an answer in a particular way should leave as much flexibility as possible to regain information if needed. That is, coding a large data set using exploratory methods should seek to capture as much detail as possible whilst maintaining a manageable data set. Early decisions on categories of responses can result in data reduction methods that become a one-way mapping of data towards an emerging theory. The decision to collapse response categories was made as late as possible and only after looking across the data for identifiable patterns of responses. Data reduction was necessary to make use of item analysis techniques. Item response theory typically deals with dichotomous or ordered-category data.

The data entry for the Rasch analysis requires that responses are characterised by a single digit, with the digit “1” allocated to the correct response and “0” reserved for omitted responses. The initial alphabetic coding of answers and reasons (A, B, C, D, E, F, M and 0) supplemented with additional information, such as using a regional model in an explanation, resulted in more categories than available digits. A correct response with an incorrect reason would be in category B and initially allocated the digit “2” for the Rasch analysis data entry. For example, when comparing the size of two fractions, a correct answer that used a regional model with an inaccurate representation of the fraction, a specific subcategory of incorrect reasoning, was initially recorded as “21”, and considered as a potential 2. Similarly a correct answer with no reasoning (originally category E) ultimately became part of code 2. The emphasis in coding was on the communicated reasoning of the student.

The initial coding of the reasoning students described in answering Questions 22 to 27 was based on two smaller studies (Behr, Wachsmuth, Post et al., 1984; Gould, 2005). When comparing the size of fractions with the same numerators, students may use an ordering consistent with whole number arithmetic applied to the denominators and reason that $\frac{1}{8}$ is larger than $\frac{1}{7}$ because 8 is larger than 7. This type of reasoning was categorised as comparing the whole numbers in the denominator (WND). When students stated that two different fractions were equal because in both fractions the difference between the numerator and the denominator was one, their reasoning was categorised as considering the size of the fraction to be determined by the difference between the numerator and the denominator (FID = Fraction In Difference). Some of the descriptive categories such as "bigger is smaller" (BIS) and "whole number denominator" (WND) were the same as those used in coding the reasoning in Question 11. Other than the first three codes in Table 4.6, the remaining codes emerged when considering student responses to questions that asked them to compare the size of fractions.

Table 4.6

*Categories of reasoning about the size of fractions*

| Code | Description | Explanation |
|---|---|---|
| WND | Whole Number Denominator | The size of the fraction is determined by the size of the denominator. |
| BIS | Bigger Is Smaller | There is an inverse relationship between the size of the denominator and the size of the fraction. |
| NEW | No Equal Whole | The lack of recognition of the need for equality of the whole when comparing fractions. |
| BLN | Both Larger Numbers | The size of a fraction is determined by the size of the whole numbers in both the numerator and the denominator e.g. 5/6 is bigger than 2/3 because 5 and 6 are larger numbers than 2 and 3. |
| FID | Fraction In Difference | The size of a fraction is articulated in the difference between the numerator and the denominator. |
| SOC | Size Of Complement | Determining which fraction is the bigger number by reasoning about how much remains to make the whole number. |
| EQV | Equivalent fraction | Determining which fraction is the bigger number by converting to equivalent fractions. |
| WNN | Whole Number Numerator | The size of a non-unit fraction is determined by the size of the numerator only. |
| ENSD | Equal numerator size of denominator | If two fractions have equal numerators the size of the denominator determines the fraction. |
| SDO | Size of denominator only | The size of a non-unit fraction is determined by the size of the denominator only. |

Many of the categories described in Table 4.6 only became apparent when students were asked to compare the size of non-unit fractions. For example, using the size of the complement of a fraction is only useful in special cases such as when comparing $\frac{9}{10}$ with $\frac{12}{13}$. Similarly, arguing about the size of both the numerator and the denominator (BLN) more readily appears in response to comparing $\frac{5}{6}$ and $\frac{2}{3}$ than comparing $\frac{1}{3}$ to $\frac{1}{4}$.

Simple dichotomous coding of items as correct or incorrect has some quite significant limitations. With dichotomous coding, reasoning is frequently overlooked. Students can achieve a correct answer to a question involving fractions without having a multiplicative conceptual base to fractions (see Figure 4.3). Question 30 was the only multiple-choice item used in the assessment and the issue of a correct answer being obtained from spurious reasoning is discussed in greater depth later (Section 5.9).

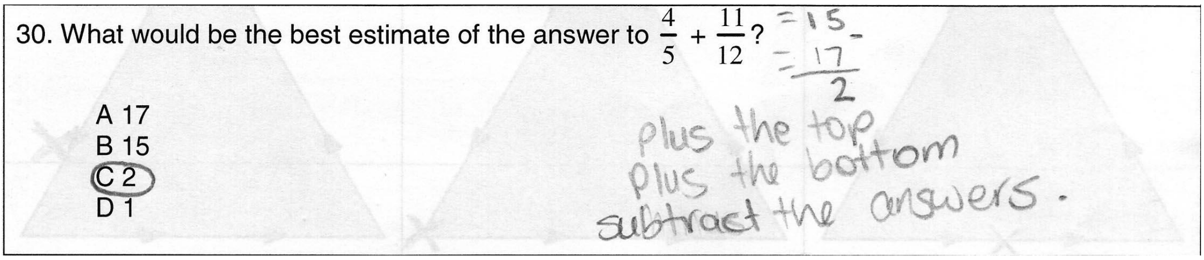



**Figure 4.3** A correct answer from an incorrect method

The above example is provided as a rationale for the emphasis on students' reasoning in the coding process and the need to maintain as much information in the coding as possible for as long as possible.

### *4.2.3 Coding folding*

The final two questions required students to partition half a circular disc to show one-sixth of the whole circle and a square piece of paper to show one-ninth. Creating one-sixth of the whole circle, given half of the circle, produced some inventive folding. One Year 8 student created one-quarter of the circle and then folded the quarter into thirds before shading two-thirds of one-quarter (see Figure 4.4).

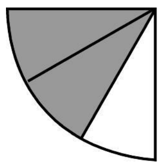


**Figure 4.4** Creating $\frac{1}{6}$ as $\frac{2}{3}$ of $\frac{1}{4}$

Although this solution produces one-sixth, the sequence of folds suggests a different conceptual pathway from folding the semi-circle into thirds and the coding was designed to describe the different folding method. Creating one-sixth as two-thirds of one-quarter when you are presented with one-half appeared to be quite different from creating one-sixth as one-third of one-half.

As well as coding the resulting area (e.g. $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{16}$, etc), a brief summary of the folding sequence was included for unusual folds in the coding notes. Some students created three parts, not all equal, of the semi-circle and shaded one of those parts. For example, folding one-eighth (half of one-quarter) creates one part of the half circle. Folding the bottom corner up (see Figure 4.5) creates three parts of the half circle with two of them equal.

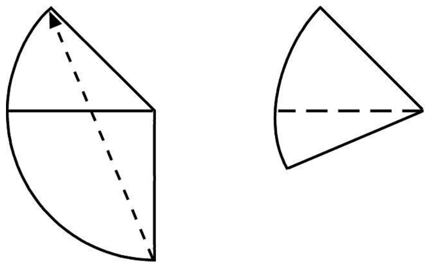


**Figure 4.5** Folding half a circle into $\frac{1}{8}$ and two $\frac{3}{16}$ pieces

The student could interpret this as "creating $\frac{1}{6}$" by dividing the half circle into three parts (two $\frac{3}{16}$ and one $\frac{1}{8}$). What students attend to in creating $\frac{1}{6}$ through composition of partitioning is dependent upon their sense of "parts-of-a-whole". The same process of creating three parts of the half with two of them equal could also result in answers such as $\frac{1}{8}$. Instead of folding the bottom corner up as in Figure 4.5, it is possible to fold the top corner over to produce one-quarter and two one-eighth sections (see Figure 5.142). The two one-eighth parts are the equal parts with the remaining one-quarter acting as the third part. In addition to the folds resulting in sectors, some students used parallel partitioning as in Figure 4.6.

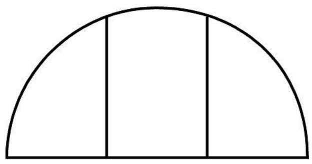


**Figure 4.6** Parallel partitioning

As well as categories based on the shaded area, answers formed by processes similar to parallel partitioning led to the creation of a response category of "segmented".

Coding the types of folds used in partitioning the square fell into three broad categories: 1-direction, 2-direction and 4-direction folds. Folding in one direction, usually vertically or horizontally, is uni-directional or 1-directional folding. Within the category of 1-direction folds, students could use repeated halving, a fan fold or a roll fold (Figure 4.7).

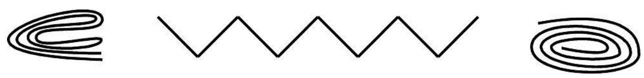


**Figure 4.7** End view of repeated halving, fan fold and roll fold

The uni-directional roll fold invariably resulted in the pieces increasing in size due to the accumulated width. The area of the specific part shaded within a 1-direction roll fold could vary markedly. Bi-directional or 2-directional folding is typically achieved by folding in two perpendicular directions aligned to the sides. 'Quilting' is used to describe 4-directional folding as it results in a pattern with many half-square triangular units similar to a quilt form, as in Figure 4.8.

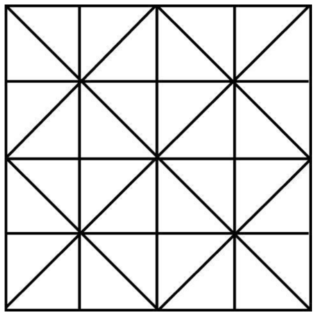


**Figure 4.8** An example of 4-directional folding

Although we imagine that students see the area as the determining characteristic of the area model, many students may not. For example, uni-directional folds could be used to create eighths before halving one of the eighths to make nine unequal pieces. Coding that focuses only on the shaded area may miss the intent of the students. The process of subdividing into parts might be more important than the size of the parts. Consequently the coding was developed to record the number of parts formed and how they are formed, as well as the area of the shaded part of the square.

The coding of students' folding of the square recorded the final area shaded (e.g. $\frac{1}{9}$, $\frac{1}{16}$, $\frac{9}{16}$, or $\frac{1}{8}$) and subcategories were used to capture the type of folding (uni-directional, bi-directional or quilting). As bi-directional folding to form sixteenths could be achieved in different ways, the resulting array structure was also recorded (i.e. 4 x 4 or 2 x 8).

## 4.3 Summary

The categories used in coding attempted to retain as much information as possible whilst focusing on students' understanding of fractions as mathematical objects. Coding students' explanations drew upon common features such as referring directly to the size of the whole number in the denominator of the fraction or the inverse relationship between the size of the denominator and the size of the fraction. Coding of regional models of fractions focused on trying to understand student intent rather than simply coding the relative size of the area shaded.

When students' drawings or paper folding were used to represent reasoning the coding system linked correct or incorrect answers with correct or incorrect reasoning. These overarching categories were further subdivided to record how students created and managed partitions of the whole. For example, creating one-ninth of a square of paper was coded according to the folding technique (1-directional, 2-directional or 4-directional) as well as the portion shaded. Again, within these broad categories further detail was provided as to the method of "folding".

# Chapter 5 ANALYSIS OF STUDENTS' RESPONSES

*When interpreting regional models of fractions, some students appear to focus on the number of parts while others utilise prototypical images for common fractions, such as a quarter or a third. The evoked concept image, as the portion of the concept image activated at a particular time, localises the child's rational action in response to a particular item. Rather than integrating the various "personalities" of fractions, the child can hold conflicting views with little need to accommodate these views across items. For example, when comparing the size of different pairs of fractions, a child can access an image, use a heuristic based on the size of the denominators if the numerators are the same, and also base judgements on the difference between the numerator and the denominator.*

*Additionally, phonemic similarity between fraction names and integers appears to have led some students to believe that "a sixth is six". Rather than developing an inverse relationship between the size of the denominator and the size of the fraction (bigger is smaller), some students have developed a direct relationship. Those who have developed a direct relationship with the denominator make quantitative arguments about fractions using the size of the whole number from the denominator.*

*Even when equality of parts becomes a salient feature of students' evoked fraction concepts, representations of the number of equal parts can result in significant mismatches, such as three-quarters of a circle being used to stand in place of one-third or sixth-eighths to represent one-sixth. These idiosyncratic interpretations of regional models lead some students to believe that one-third is made up of three quarters. When three-eighths is used with this interpretation of a regional model to represent one-third (three equal parts), the proximity of the resulting area to the prototypical image of one-third can entrench the misinterpretation of fraction as quantity.*

*Some commonly used fraction questions can mean that even a correct response can mask an inaccurate or incomplete quantitative sense of fractions. For 10% or more of students in this sample from the middle years of schooling the 'number of parts' (only sometimes equal parts) can be the sole salient feature of fraction contexts.*
*The introduction of the standard fraction notation into solution methods appears to increase the likelihood of inappropriate quantitative reasoning.*

**Preview**

This chapter begins by distinguishing between fractions as they were presented in tasks and the way that students construed the fraction tasks. Student responses to the tasks are taken as evidence of the component of the fraction concept activated or evoked, that is, the evoked fraction concept image. The analysis of the responses to the tasks follows a pathway from describing the fraction models in the tasks as presented, to determining the evoked concept images evident in the students' responses. The tasks are analysed in groups, with each group identified according to the intent of the design of the tasks. The analysis is carried out in the same order the tasks were presented in order to look for possible instances of one answer or method influencing the next.

The tasks are presented as:

- *interpreting* divided length, a regional model with non-equal parts, and a discrete model (Questions 1–3, 4–6, and 7–8),
- *determining a fractional quantity of discrete units* (Questions 9–11),
- *constructing* regional models for parts of a circle (Questions 12–14),
- *representing* multiples of non-unit fractions and measurement division by fractions (Questions 15–18),
- *locating* fractional position on a continuous model (Questions 19–21),
- *justifying* quantitative comparisons of fractions as numbers (Questions 22–27),
- *reconstructing the unit* from a continuous part (Questions 28–29),
- *estimating* the size of the sum of two fractions (Question 30),
- *comparing measurements of two quantities* with fraction remainders, using half the unit of measure, and reversing the comparison (Questions 31–35), and
- *composition of partitioning* (Questions 36–37).

## Introduction

In general, the models implicit in the fraction test items can be considered to be of two types, discrete and continuous.

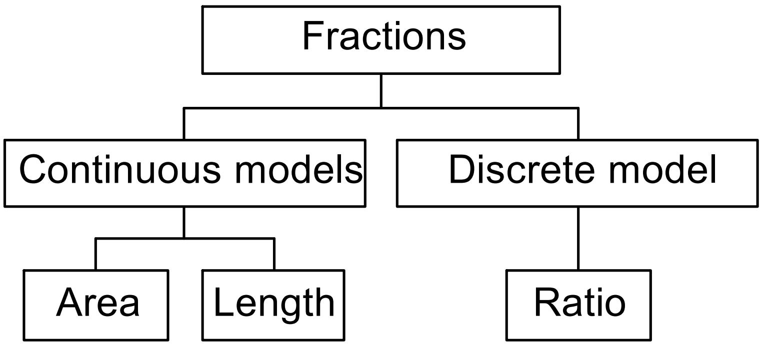



**Figure 5.1** Organisation chart of fraction models

This simplified organisation of the general categories of the tasks as they were presented does not seek to capture the physical aspects of the task materials. Some tasks used concrete materials that the students could manipulate, others used drawings, while others used concrete materials that students could see but not manipulate. Questions also used the description of common fraction contexts, such as working with quantities in a recipe.

The part-whole model emphasises the language used with fractions, and is the dominant method of fraction instruction referred to in NSW curriculum documents (Mathematics K-6 syllabus, pp 263-266; Mathematics Syllabus Years 7 and 8, pp 91–93). The NSW Primary Mathematics Syllabus develops the concept of a fraction by dividing objects or groups of objects into equal parts and relating one part or parts of a group to the whole. However, fractions as presented need not correspond to the way that students construe the tasks. For example, when asked to draw answers to doubling three-quarters of a cup of milk, the task context as presented was considered to draw on a continuous model of capacity. Although the context of fractional parts of a cup appears to deal with capacities, in drawing the answers, the dimension addressed in practice is length.

## 5.1 Interpreting divided length: Q1–3

The first three questions asked students to identify fractions of a rectangular strip of paper folded into halves and then one half folded into halves (Figure 5.2). The dimension of length was the attribute of the rectangle being partitioned, with the number of parts resulting from the partition designed not to cue the naming of the fractions.

| | | |
|---|---|---|
| A | C | B |

**Figure 5.2** Folded strip for Questions 1–3.

Having watched the strip of paper being folded, each part in turn (A, B and C) was identified and students were asked to determine what fraction that piece of paper was.

### *Question 1*

Students were asked to identify part A of the rectangle as one-half and their responses are summarised in Table 1.

Table 1

*Percentage distribution of responses to Question 1, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{1}{2}$ [a] | 83 | 92 | 96 | 93 | 95 |
| $\frac{1}{3}$ | 5 | 2 | 2 | 3 | 3 |
| Other | 10 | 6 | 2 | 4 | 1 |
| Omitted | 2 | 0 | 1 | 1 | 1 |

*Note.* [a] The category $\frac{1}{2}$ includes those who responded $\frac{2}{4}$.

Most students identified part A as one-half (more than 90% of each of Years 5 to 8). The slight dip in the percentage correct between Year 6 and Year 7 in this question and many of the questions may be an artefact of the sample. A small percentage of Years 4 and Year 5 (10% and 6% respectively) gave answers other than $\frac{1}{2}$ or $\frac{1}{3}$. These other responses included $\frac{2}{3}$, $\frac{1}{4}$, $\frac{2}{2}$, and descriptive text such as "the first part", "left hand side", "right hand side" or "rectangle".

A few students (3.0% overall[8]) appear to have interpreted the area of one-half as one part of three, leading to the answer of one-third. For these students, the number of parts has become a more important feature of the fraction than the comparison to the whole.

### *Question 2*

Students were next asked to identify what fraction part B was of the whole strip of paper. The results are summarised in Table 2.

[8] Percentages of the total population of students are reported to two significant figures as 0.1% of the population corresponds to more than one student.

Table 2

*Percentage distribution of responses to Question 2, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{1}{4}$ | 71 | 75 | 81 | 79 | 86 |
| $\frac{1}{3}$ | 13 | 15 | 11 | 12 | 10 |
| $\frac{1}{2}$ | 3 | 3 | 2 | 1 | 1 |
| Other | 11 | 6 | 5 | 7 | 2 |
| Omitted | 2 | 1 | 1 | 1 | 1 |

About three-quarters of the sample correctly identified part B of the whole strip of paper as one-quarter. The next most common response to Question 2 was one-third, with 12% overall providing this answer. These students appear to have interpreted part B as one part of three, indicating that for them the number of parts was the more salient feature of the representation than a comparison of the area to the whole. The traditional regional model of a fraction does not appear to have resulted in a robust sense of *one-quarter as a relational number* for nearly one-fifth of this sample.

Apart from those students who answered the question correctly or appeared to interpret the one part of three as one-third, about 2% of the sample answered one-half. This response could have been influenced by the common use of one-half to refer to a fractional piece or the visibility of both one part and two parts (for $\frac{1}{2}$). The next most frequent response after one-half was three-quarters (11 students) and this response has been included in the 'other' category, as the number of students who answered three-quarters was lower than the number who omitted the question. It is possible that this small group of students combined a positional (ordinal) interpretation with "parts-of-a-whole" reasoning. The right-hand quarter (B) could be described as the third quarter or piece.

### *Question 3*

In Question 3, students were asked to identify what fraction part C was of the whole strip of paper. The results are summarised in Table 3.

Table 3

*Percentage distribution of responses to Question 3, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{1}{4}$ | 66 | 69 | 79 | 77 | 84 |
| $\frac{1}{3}$ | 10 | 13 | 11 | 13 | 10 |
| $\frac{1}{2}$ | 4 | 5 | 4 | 3 | 1 |
| $\frac{2}{3}$ | 3 | 5 | 2 | 0 | 0 |
| Other | 13 | 7 | 2 | 6 | 3 |
| Omitted | 4 | 1 | 1 | 1 | 2 |

Overall, just under three-quarters of students answered Question 3 correctly, slightly less than those who correctly answered Question 2 correctly. The number of students who responded $\frac{1}{3}$ was 11% overall. The slight reduction in the percentage answering $\frac{1}{3}$ for Question 3 compared to Question 2 may be balanced by the appearance of the category of $\frac{2}{3}$ for Question 3 in the primary grades. As with the possible reasoning associated with the answer three-quarters in Question 2, there appears to be a blending in the earlier grades of the part-whole component "the number of parts" and "the position of parts". That is, over 3% of the students in primary grades 4–6 recorded the ***second part of the three parts*** as $\frac{2}{3}$.

Two patterns emerged related to this possible blending of position and part name in the interpretation of the second part of three parts. The first is a consistent blend of the number of parts and the position of parts resulting in $\frac{1}{3}$, $\frac{3}{3}$, $\frac{2}{3}$ as the sequence of answers to the first three questions (7 students). The second pattern of answers to the first three questions was X, $\frac{1}{3}$, $\frac{2}{3}$ (24 students) where X was usually $\frac{1}{2}$ (20 students) but also took values of $\frac{2}{3}$ (2 students), $\frac{3}{3}$ (1 student) and $\frac{3}{4}$ (1 student). This second pattern of responses could correspond to a positional interpretation, with part C the second part of three parts (from either end), or an accumulation of thirds (summing from right to left). Although the identification of $\frac{1}{2}$ is a strong response to the first question, the students' introduction of thirds in the answers to Questions 2 and 3 suggests that the representation of one-quarter is subject to other interpretations beyond one half of a half.

Students appear to focus on the number of parts independently of the equality of parts. In Question 3 over 10% of each year indicated that one-quarter of a strip of paper was one-third when the strip was divided into three unequal parts. Although this error might be expected in younger students, it did not diminish across the grades with its occurrence in Year 8 at a similar level to Year 4.

The response $\frac{1}{2}$ corresponded to about 4% of the entire sample of answers to Question 3. The only difference between Question 2 and Question 3 was the location of the part identified. The increased percentage in the $\frac{1}{2}$ category from Question 2 appears to be influenced by the rise of the answer $\frac{2}{4}$. Over 2% of the answers in this category for Question 3 answered $\frac{2}{4}$ compared to less than 1% in Question 2. Perhaps more telling is that none of the 2% who answered $\frac{2}{4}$ for Question 3 provided the answer "$\frac{1}{2}$" for Question 2.

It is clear that the context in which the basic concept of one-quarter is assessed has a substantial bearing on the responses students make. If the strip of paper had been fully partitioned into quarters, a focus only on the number of parts would have resulted in an answer that would have been indistinguishable from a part-whole relational interpretation.

**Looking back**

How common was a focus on the number of parts in the given context? One way to examine this question is to look for evidence of consistent reference to the number of parts. As noted for Question 1, 3.0% overall (50 students out of 1676) described one-half as one-third. As one-half is the first fraction that students learn and is encountered in so many different contexts, the existence of one-third as a separate category of response to Question 1 is important. In total, 22 students answered $\frac{1}{3}$ to each of the first three questions.

Of the 50 students who described one-half as one-third in answering Question 1, the pattern of responses to the first three questions was revealing. Twenty-two labelled each of the first three answers as one-third. Seven of the Year 4 and 5 students answered the first three questions with $\frac{1}{3}$, $\frac{3}{3}$ and $\frac{2}{3}$. This pattern of answers is indicative of a number of parts focus combined with ‘positional’ labelling. The number of parts determines the denominator and the position of the parts identifies the numerator, as in Figure 5.3.

| | | |
|---|---|---|
| $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{3}{3}$ |

**Figure 5.3** Left-to-right position and number of parts

For another group of students, the process of halving to form the final two quarters probably influenced the answers to Questions 2 and 3. That is, although they answered $\frac{1}{3}$ to Question 1, a further 5 students answered $\frac{1}{4}$ to Questions 2 and 3.

The responses generated by the context used with the first three questions made it clear that many students (more than 10% in each grade) displayed a *number of parts* concept image of fractions rather than a relational number interpretation. Additionally, their image of a fraction can be a blend of the position of the part with the number of parts. When interpreting fractions depicted by a divided length, there is a clear positional order to be read from the context. The use of the ordinal names for fractions creates unfortunate support for blending positional descriptions with a number-of-parts reading of the context.

## 5.2 Interpreting a regional model with non-equal parts: Q4–6

In Questions 4 to 6 students were asked to indicate what fraction of the hexagon each of the individual pattern blocks was (Figure 5.4). While partitioning the dimension of length could be used with the first context, Questions 4 to 6 required a visual comparison of area to interpret the regional model of fractions.

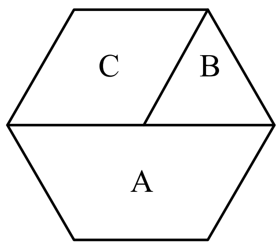



**Figure 5.4** Pattern block figure for Questions 4 to 6

The pattern block arrangement in Figure 5.4 was displayed to the students using an overhead projector and coloured transparent pattern blocks.

***Question 4***

Students were asked to identify what fraction of the whole shape the trapezoidal piece (part A) corresponded to. The students' responses are summarised in Table 4.

Table 4

*Percentage distribution of responses to Question 4, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{1}{2}$ | 79 | 91 | 92 | 94 | 94 |
| $\frac{1}{3}$ | 7 | 5 | 4 | 1 | 3 |
| $\frac{1}{4}$ | 2 | 0 | 1 | 1 | 0 |
| Other | 10 | 4 | 2 | 4 | 1 |
| Omitted | 3 | 0 | 1 | 1 | 2 |

About 90% of students correctly identified the trapezoidal shape as corresponding to one-half. An alternate interpretation would be to see the trapezoidal shape as one part out of three parts. The percentage of students apparently using this interpretation was higher in this question for Years 4, 5 and 6 than in Question 1. This result suggests that although both questions would usually be described as regional models, the area model used in this question may be less familiar or it may emphasise area to a greater extent than a divided rectangle. Decisions about fractional parts represented by a divided rectangle are perhaps more likely to be based on linear subdivision rather than area.

Although the number of students who provided the answer $\frac{1}{4}$ to Question 4 was quite low and would normally be included with the "other" category, it has been included for comparison with the same incorrect answer to Questions 5 and 6. The students appeared not to confuse one-half with one-quarter in Question 4, as the percentages answering $\frac{1}{4}$ are similar to the omit rate.

***Question 5***

Students were asked to identify what fraction of the whole shape was represented by the equilateral triangle (part B). The students' responses are summarised in Table 5.

Table 5

*Percentage distribution of responses to Question 5, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{1}{6}$ | 18 | 45 | 59 | 25 | 43 |
| $\frac{1}{3}$ | 25 | 19 | 14 | 27 | 31 |
| $\frac{1}{4}$ | 34 | 22 | 17 | 27 | 16 |
| Other | 20 | 13 | 9 | 19 | 7 |
| Omitted | 2 | 2 | 1 | 3 | 4 |

The percentage of students who answered one-third or one-quarter was frequently greater than the percentage of those who answered one-sixth. Indeed, these categories are so large when compared to other answers that they are strongly suggestive of significantly different evoked fraction concepts.

As the regular triangular pattern block was clearly one part out of three parts, this interpretation provided a ready explanation for the number of students who described the pattern block as one-third of the hexagon. The percentage of students who described the equilateral triangle as one-third of the hexagon is much higher than the percentage of students who described one-quarter of a strip of paper as one-third in Question 3. The difference is most pronounced in Years 4, 7 and 8. With the removal of *equality* of the pieces apparent in the first three questions, more students appeared to rely on a 'number of parts' interpretation of fractions.

An explanation for the high percentage of students who described the triangular pattern block as one-quarter of the hexagon was less obvious. The percentage of students describing the equilateral triangle as one-quarter of the hexagon was at least as great as those describing it as one-third in all of the years except Year 8. Indeed, $\frac{1}{4}$ was the most popular answer in Year 4 but for some reason dipped quite sharply in Year 8. This dip is unlikely to be influenced by differences in curriculum access, as the focus of the high school fraction curriculum in Years 7 and 8 was algorithmic operations with fractions.

The high percentage of students who described the triangle as one quarter of the hexagon may arise because halves and quarters are more readily modelled than one-third. Consider a student who learns that one-quarter can be represented as in Figure 5.5.

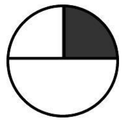


**Figure 5.5** One-quarter of a circle

Applying a similar prototypical image to the regular hexagon could result in a possible confusion between the images in Figure 5.6.

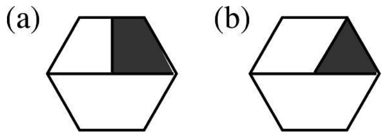



**Figure 5.6** Hexagon images

Simple "quartering" could occur in two distinct ways. What I describe as uni-directional quartering (see Question 37) occurs when all of the cuts are parallel. Bi-directional quartering occurs when the cuts are perpendicular to each other. The preference for vertical and horizontal partitioning may be influenced by our greater visual acuity in the horizontal and vertical orientations (Howard & Templeton, 1966). The first image (a) in Figure 5.6 results from bi-directional quartering.

By looking across the items it is likely that the high percentage of students describing the equilateral triangle as one-quarter may also be influenced by a second interpretation of one-quarter. In addition to a "near" vertical and horizontal quartering of the object, some students appear to use the term "a quarter" to refer to a general fraction part that is not one-half. As noted earlier, there are students for whom the term "quarter" meant "fraction part". The explanations in Figure 5.7 use the term "quarter" as an equivalent to "fractional part".

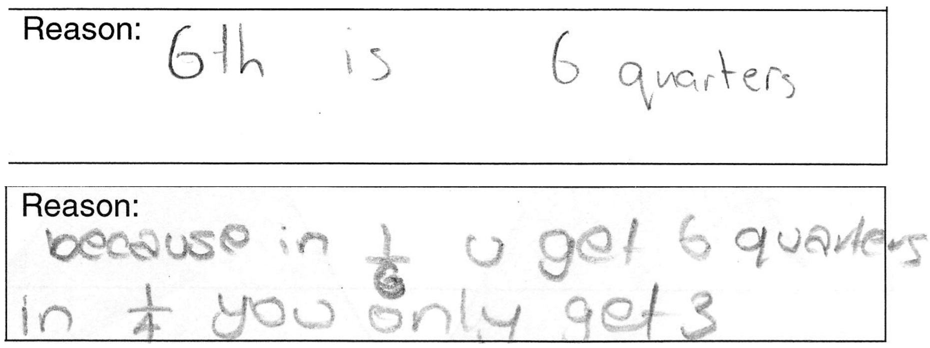



**Figure 5.7** "Quarters" as fractional parts

The answer $\frac{1}{5}$ was initially treated as a separate category of answer for Question 5, despite the low percentages, because it is plausible that a student could treat it as one part compared to five parts. However, as $\frac{1}{5}$ persisted at a similar level in Question 6 without a plausible explanation, the answer $\frac{1}{5}$ was recoded and added to the 'other' category for both Questions 5 and 6. In Questions 5 and 6, the response $\frac{1}{5}$ was higher in Years 7 and 8 (3% and 5% respectively) than in the other years, perhaps because rather than interpreting $\frac{1}{5}$ as one part compared to five parts, these students were influenced by the size of the denominator and saw $\frac{1}{5}$ as larger than $\frac{1}{4}$.

Unlike the first few questions where there was a slight dip in the percentage correct between Year 6 and Year 7, there was a profound reduction in the percentage correct between Year 6 and Year 7 on Question 5. This difference in performance was investigated using Rasch analysis. The students' responses were treated as belonging to one of three school stages, used to describe the curriculum. Students in Years 7 and 8, the first two years of high school, are taught Stage 4 curriculum whereas students in Years 5 and 6, the final two years of primary school, are taught Stage 3 curriculum and, students in Year 4 the Stage 2 mathematics curriculum. Using the number of questions students answered correctly as an estimate of their quantitative sense of fractions, each school stage was subdivided into four ability groups. The probability of correctly answering Question 5 for the average ability estimate corresponding to each of the four ability groups within a school stage created a point on the item characteristic curve for that stage. Each stage curve is formed from four points. Figure 5.8 shows the performance of each stage on Question 5 as well as the theoretical item characteristic curve.

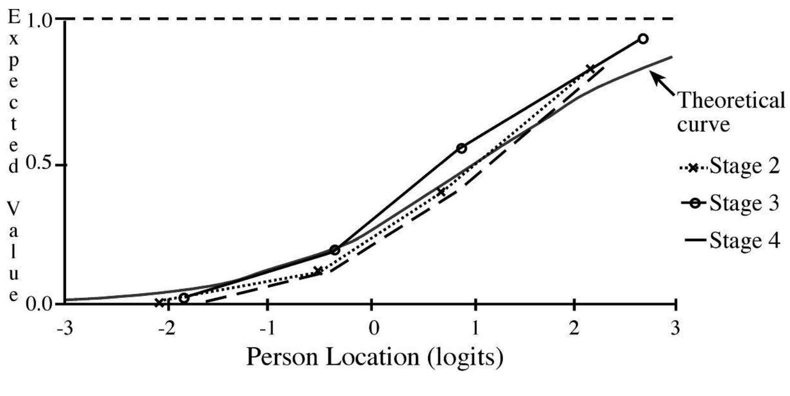



**Figure 5.8** Like-ability comparison of performance on Q5 by school stage.

The curves map the probability that someone with a known level of ability as measured by the total assessment will answer the question correctly. As you move from left to right on the curve, the higher the ability of the person the greater the probability of answering the question correctly. Although the behaviour of Stage 2 and Stage 4 are very similar on this graph, the Stage 3 curve is generally above the Stage 4 curve. That is, for students of similar ability as measured by the total assessment, Stage 3 students were more likely to correctly answer Question 5 than Stage 4 students. Thus the higher percentage of correct answers to this question provided by Years 5 and 6 compared to Years 7 and 8 could not be attributed to Years 5 and 6 having a larger number of able students, as the less able Stage 3 students were also more likely to correctly answer this question than students of comparable ability estimates in Stage 4.

### *Question 6*

Students were asked to identify the fraction of the whole shape represented by the rhombus (part C). The students' responses are summarised in Table 6.

Table 6

*Percentage distribution of responses to Question 6, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{1}{3}$ [a] | 36 | 53 | 67 | 44 | 61 |
| $\frac{1}{4}$ | 30 | 26 | 14 | 23 | 17 |
| $\frac{1}{2}$ | 7 | 6 | 4 | 6 | 2 |
| $\frac{2}{3}$ | 5 | 4 | 3 | 4 | 7 |
| $\frac{3}{4}$ | 1 | 1 | 3 | 4 | 2 |
| Other | 17 | 9 | 8 | 17 | 7 |
| Omitted | 5 | 1 | 1 | 3 | 4 |

[a] Or equivalent.

Across the entire sample, 52% of students answered $\frac{1}{3}$ or an equivalent fraction to Question 6. However, the percentage of students who answered $\frac{1}{3}$ to Question 6 may well include students who saw the rhombus as one part out of three parts. That is, an incorrect interpretation of a regional "parts-of-a-whole" model of fractions could result in a correct answer to this question. The percentage of each year group answering $\frac{1}{4}$ to this question is comparable with the percentage of students who answered $\frac{1}{4}$ in Question 5 to name the fraction for the equilateral triangle. The responses exemplify a lack of consistency in the evoked concept images. What appears to be particularly germane to this lack of consistency of the evoked concept image of one-quarter for these students is that 19% (62) of Year 4 students, 10% (33) of Year 5 students and 8% of each of Year 6, 7 and 8 indicated that *both* the equilateral triangle and the rhombus represented one-quarter of the hexagon despite the triangle and the rhombus being different shapes, different colours, and one twice the size of the other.

Although the robust nature of the knowledge of one-half as an area model has been reported (Hart, 1989), 85 students (5%) described the rhombus as half of the hexagon. This response might have been influenced in part by the visual relationship between the equilateral triangle and the adjacent rhombus.

### *Who understands one-half, one-quarter or one-third?*

For the purpose of analysis, I define a strong form of a student having a quantitative sense of say, one-half, as requiring a student to answer one-half to questions where the answer is one-half and not answer one-half to questions where the answer is not one-half. Using these criteria, who understands one-half? In Year 4 the percentage of students who appear to understand one-half changes from 79% to 73% when those who call the rhombus one-half of the hexagon are removed. This is the greatest change of any of the year groups. Across the population of 1676 students, applying this criterion reduces the percentage of students who appear to understand the regional model of one-half from 90% to 86%. This stronger form of a quantitative sense of a fraction could provide a more robust measure of those who *have* the concept.

In Year 4 the percentage of students who appear to understand one-quarter in Question 2 (one-quarter of a strip of paper) changes dramatically from 71% to 47% when those who incorrectly call the equilateral triangle one-quarter of the hexagon are removed. Again this is the greatest change of any of the year groups. Across the entire population of 1676 students, applying this criterion reduces the percentage of students who appear to understand the regional model of one-quarter from 79% to 61%.

Similarly, the percentage of students who appear to understand the regional model of one-third drops from 52% to 43% when those who incorrectly call the equilateral triangle one-third of the hexagon are removed. The greatest change of any year group was again in Year 4, where the 36% of students who appeared to be able to correctly identify the rhombus as one-third of the hexagon dropped to 22% when those who also named the equilateral triangle one-third were not counted.

**Looking back**

The number of parts interpretation of fractions is not the only incorrect interpretation of the regional model when equality of the parts is removed from the model. The bilateral symmetry usually associated with quarters could influence the prototypical image a student forms of one quarter. That is, the "best image" or prototype that students hold of a quarter could be influenced by horizontal and vertical visual bias. Shapes that are near the prototype (generic example) for a quarter are considered by some students to be a quarter. There also appears to be a number of students who use the term "a quarter" to refer to a general fraction part other than one-half.

## 5.3 Interpreting the discrete model: Q7–8

Two yellow transparent counters and six red transparent counters were placed in a line on an overhead projector to act as a discrete item fraction model.

### *Question 7*

Students were asked what fraction of all the counters was yellow.

Table 7

*Percentage distribution of responses to Question 7, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{1}{4}$ [a] | 67 | 85 | 90 | 85 | 93 |
| 2 out of 8 | 9 | 3 | 1 | 2 | 1 |
| $\frac{2}{6}$ or $\frac{1}{3}$ | 5 | 3 | 3 | 2 | 3 |
| Other | 19 | 9 | 5 | 9 | 1 |
| Omitted | 1 | 0 | 0 | 1 | 2 |

[a] Including $\frac{2}{8}$ or equivalent.

Overall 84% of students correctly identified the 2 yellow counters as corresponding to two-eighths of the counters. The response ‘2 out of 8’ was not coded as a correct answer to the request to identify the fraction of the counters that are yellow for Question 7 because the description ‘2 out of 8’ corresponded more to a double count interpretation than a quantitative sense of fractions.

The strong visual influence of the two distinct colours on the sense of association or grouping may have influenced the number of students who interpreted the two out of eight discrete counters as $\frac{2}{6}$ or $\frac{1}{3}$. With discrete models of fractions, “units formed from the parts” may take the foreground while the whole remains in the background. That is, when looking at 2 yellow counters and 6 red counters it is easier to see the “2 and 6” than the “8” (see Piaget et al., 1960). Overall, 3.2% of students used this “part-part” interpretation of the discrete model of fractions. In Year 4, the other category included a range of “2 things” responses such as 2 wholes, 2 quarters, 2 yellows, $\frac{2}{7}$, $\frac{2}{9}$ and $\frac{2}{12}$.

### *Question 8*

Students were asked if they could write the fraction answer to Question 7 another way. The results are summarised in Table 8.

Table 8

*Percentage distribution of responses to Question 8, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Equivalent representation | 28 | 49 | 70 | 44 | 71 |
| Same fraction | 36 | 23 | 15 | 32 | 11 |
| Inverted fraction | 4 | 6 | 3 | 2 | 2 |
| Other | 27 | 16 | 10 | 18 | 13 |
| Omitted | 5 | 6 | 2 | 4 | 4 |

In total 52% of students could write an equivalent fraction for their answer to Question 7. Although ‘2 out of 8’ was not accepted as a correct answer for Question 7, it was accepted as another way of *representing* two-eighths or one-quarter. In spite of the fact that this question sought to elicit an equivalent fraction, such as one-quarter, as another way to write two-eighths, many students wrote exactly the same fraction they had provided to Question 7 for Question 8. Overall 21% of students wrote the same (correct) answer to Question 8 that they gave for Question 7. In addition, 19% of all students answered $\frac{1}{4}$ to Question 7 and responded with $\frac{2}{8}$ for Question 8. A number of students (about 1%) who answered $\frac{2}{8}$ to Question 7 answered $\frac{6}{8}$ to Question 8.

In responding to Question 8, 55 students or a little over 3% of the total indicated that they believed that another way to write a fraction was to invert the fraction. Both the complement of the fraction ($\frac{6}{8}$) and the multiplicative inverse of the fraction ($\frac{8}{2}$) were considered by students to be alternate ways of writing a fraction.

In Question 8 there again appears to be a marked drop in performance between Year 6 and Year 7. However, the stage-based Rasch analysis represented in Figure 5.9 suggests that if this dip in performance occurs it must be very short in duration as the matched performance of Stage 4 students was as good or better than the performance of Stage 3 students on this question. The graph of the expected value or the probability of obtaining a correct answer for Stage 4 is generally above the graph of Stage 3 in Figure 5.9. That is, for students of similar ability as measured by their performance on the fraction assessment, Stage 4 students were more likely to correctly answer Question 8 than Stage 3 students.

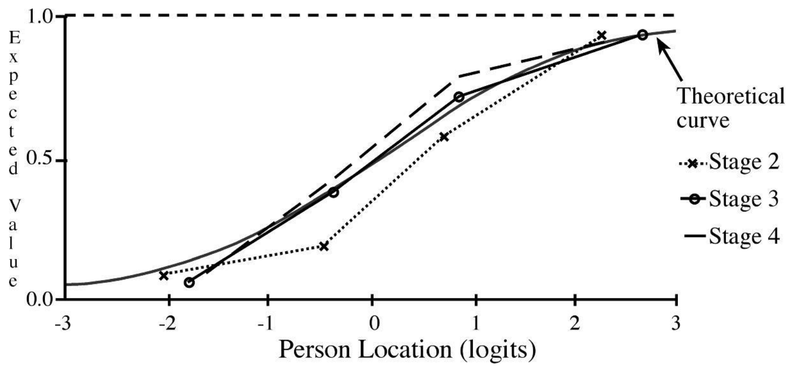



**Figure 5.9** Rasch comparison of performance by school stage on Q8.

## 5.4 Fractional quantity of discrete items: Q9–11

As well as investigating students' ability to interpret discrete models of fractions, students were asked to determine fractions of a collection of discrete identical items.

### *Question 9*

Students were asked to determine one-quarter of twelve counters, displayed in two columns of 6 counters. The results are summarised in Table 9.

Table 9

*Percentage distribution of responses to Question 9, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| 3 | 66 | 74 | 86 | 78 | 83 |
| 4 | 7 | 10 | 5 | 10 | 8 |
| 6 | 5 | 2 | 1 | 3 | 1 |
| Other | 20 | 13 | 7 | 8 | 6 |
| Omitted | 2 | 1 | 1 | 1 | 3 |

Finding one-quarter of twelve counters required students to apply their fraction knowledge to discrete items by constructing units (four groups of three) that were not present. The percentage of correct answers (77%) was comparable to the identification of one-quarter in Question 2 (79%) and in Question 3 (75%). That is, determining one-quarter of 12 counters appears to be of similar challenge to identifying one-quarter of a strip of paper. It is likely that the arrangement of the counters into two columns of six counters assisted students in partitioning into four groups. With the vertical partition effectively done, the horizontal halving partition would readily create a group of three counters. Similarly, the answer "6" could have been influenced by the arrangement of the 12 counters into two columns of 6. The proportion of students who answered "4" is quite large (5–10%) and may have been influenced by the link between the fraction name "one-quarter" and the number "4".

### *Question 10*

After determining one-quarter of 12 counters arranged in two columns of 6, students were then asked to determine three-quarters of the twelve counters. The results are summarised in Table 10.

Table 10

*Percentage distribution of responses to Question 10, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| 9 | 63 | 69 | 83 | 71 | 78 |
| 4 | 5 | 5 | 2 | 7 | 6 |
| 6 | 6 | 6 | 2 | 3 | 1 |
| 3 | 4 | 3 | 3 | 3 | 2 |
| Other | 18 | 16 | 10 | 14 | 11 |
| Omitted | 3 | 2 | 0 | 2 | 2 |

Determining three-quarters of twelve counters usually depends upon being able to first find one-quarter. Consequently, it was expected that the percentage of correct answers in each school year would be less than in Question 9. However, the reduction in the proportion of correct answers is only slightly lower, suggesting that once a quarter of 12 items has been determined, finding three-quarters may not be much more difficult.

Again, the introduction of the term three-quarters in the question may have contributed to the percentage of responses falling into the answer categories of "3" and "4", as simple "translations" of the terms "three" and "quarters". Although there is a decrease in the proportion of students who answered "4" to Question 10 compared with Question 9, the total of those answering "3" and "4" to Question 10 is comparable to those answering "4" to Question 9. The comparability of the proportion of students answering "3" or "4" when asked to find three-quarters of 12 to the proportion of students answering "4" to finding one-quarter of 12 suggests that translation of the fraction name may have operated as a method of solution for both Question 9 and 10. Overall, 57% of Year 4, 65% of Year 5, 81% of Year 6, 68% of Year 7 and 75% of Year 8 obtained the correct answer to both Question 9 and Question 10. This suggests that a reasonably robust sense of quarters with respect to discrete (structured) items exists for most students in this sample across Years 4 to 8.

### *Question 11*

Question 11 asked students if they would have more counters if they were given $\frac{1}{4}$ or $\frac{1}{6}$ of the 12 counters they could see, and why. The first part of the response called for a simple choice between $\frac{1}{4}$ and $\frac{1}{6}$ of the 12 counters. Where an answer was provided, it was coded as $\frac{1}{4}$, $\frac{1}{6}$ or other. Sixty-seven responses (4.0%) were coded as "other". For the second part of the question, students' reasons were recorded verbatim.

In looking across the explanations a number of categories were formed. The most common correct reason was to simply calculate $\frac{1}{4}$ and $\frac{1}{6}$ of 12 and to state that $\frac{1}{4} = 3$ and $\frac{1}{6} = 2$. Calculation-based explanations were recorded in the "calculated response" category. Explanations that involved reasoning about the inverse relationship between the size of the denominator and the size of the fraction were categorised as "bigger is smaller" (BIS). For example, responses such as "because it a rule if the denominator is bigger and the top is 1 then it's a smaller number" or "the lesser the fraction the higher" were included in this category, as was the response in Figure 5.10.

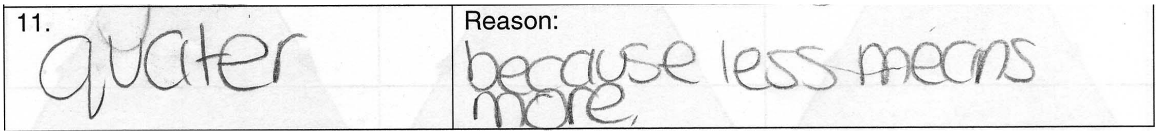



**Figure 5.10** A Year 8 student's explanation that less means more

Another response category that emerged from students' explanations in Question 11 was a focus on the size of the whole number in the denominator. Explanations such as $\frac{1}{6}$ is larger "because it has a higher denomenator" or $\frac{1}{6}$ is larger "because 6 is bigger than 4" appeared to equate the size of the fraction with the size of the "whole number denominator" (WND). In effect, taking the size of the fraction to be directly proportional to the size of the denominator (WND) is the opposite of "bigger is smaller" (BIS).

Statements of fact, such as "because it's more bigger" were recorded in the "fact" category irrespective of whether they were correct or incorrect statements. Answers such as "because I like six more than four" or "they would both be the same because $\frac{1}{6}$ is the same as $\frac{1}{4}$" were included in the "other incorrect" category.

Table 11

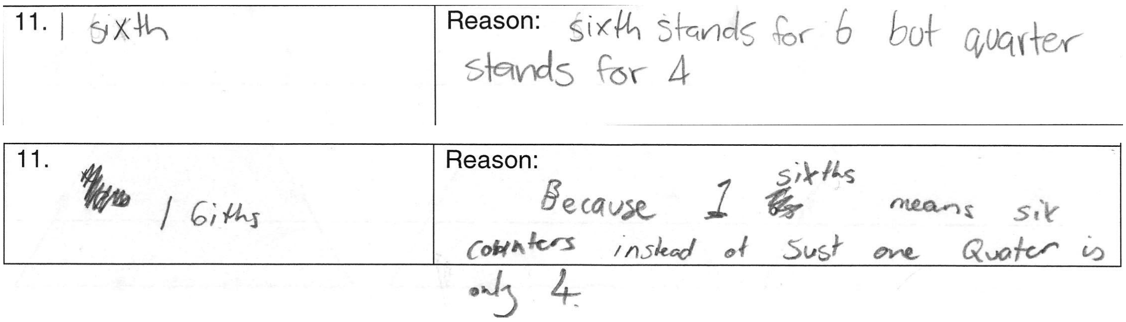
*Percentage distribution of responses to Question 11, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Right answer/right reason | | | | | |
| Calculated response | 18 | 32 | 54 | 26 | 39 |
| Bigger is smaller | 3 | 6 | 4 | 4 | 7 |
| Right answer/wrong reason | | | | | |
| Fact | 17 | 19 | 12 | 27 | 27 |
| Guess | 7 | 5 | 3 | 3 | 1 |
| Other (includes no reason) | 10 | 7 | 7 | 13 | 10 |
| Wrong answer/wrong reason | | | | | |
| $\frac{1}{6}$ is 6 or $\frac{1}{4}$ is 4 | 11 | 8 | 4 | 4 | 4 |
| Fact | 6 | 5 | 5 | 4 | 5 |
| Guess | 4 | 3 | 1 | 2 | 1 |
| Whole No. denominator | 4 | 2 | 2 | 2 | 1 |
| Other (includes no reason) | 17 | 10 | 6 | 11 | 2 |
| Wrong answer/right reason | 0 | 1 | 0 | 1 | 1 |
| Omitted | 3 | 2 | 1 | 2 | 3 |

Although the majority of students (72%) could correctly identify the larger fraction, far fewer students (39%) could provide a satisfactory reason for the answer. The two largest identified categories of reasoning, “calculated response” and “fact”, may reflect student expectations of mathematics questions. That is, students are most frequently asked to carry out calculations in mathematics classes and they are infrequently asked to justify their reasoning.

**Phonemic links**

There were a number of responses that included statements to the effect that “one sixth is 6” in the reasoning. Similarly, some students argued that a quarter was four and overall, 6.0% of students argued that a sixth stands for six or a quarter stands for four.


11. 1 sixth

Reason: sixth stands for 6 but quarter stands for 4

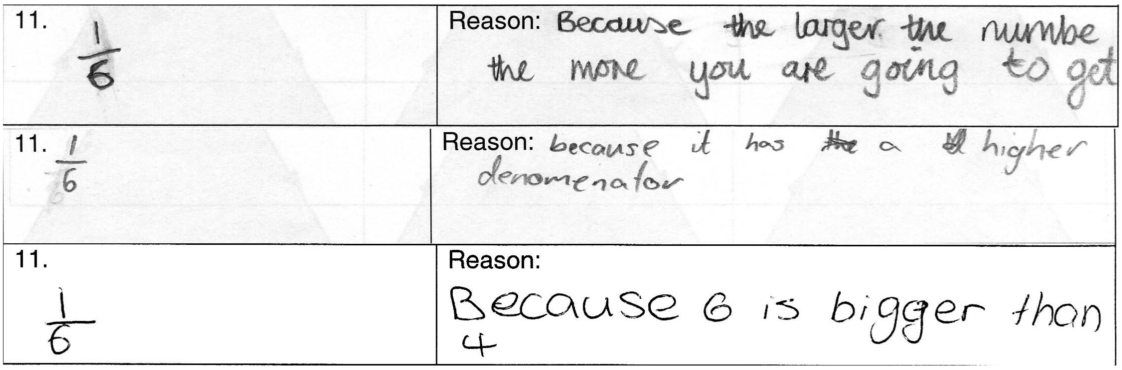
11. 1 6iths

Reason: Because 1 sixths means six counters instead of just one Quater is only 4.

**Figure 5.11** Responses showing equating fractions with cardinals

The use of the numeral in abbreviations of cardinals (e.g. the "6" in 6$^{th}$) may also reinforce the association of the integer with the fraction. The association between the ordinal fraction name and the cardinal may be absolute, as when a Year 6 student wrote "cause sixth is 6 and quarter is a four" or it may blend with traditional lessons leading to recording "one sixth is 6 out of 12". Similarly, a Year 5 student wrote, "1/4 is 3 but one sixth is 6 so one sixth is more". These explanations suggest that the confusion between phonemically similar fractional and integer values may be contributing to the interpretation of fractions as being equivalent to the *whole number denominator.* Cross-cultural research (e.g. Muira et al., 1999) has identified the language support provided to the fraction concept by various Asian languages but has not identified the specific limitations of the fraction and ordinal naming system in English. In particular, the prevalence of the belief in the equivalence between the fraction name and the cardinal corresponding to the denominator is likely to be influenced by the phonemic similarity of the terms.

It is also plausible that the direct association between the size of the whole number in the denominator and the size of the denominator, described in the results as the response category of "whole number denominator" (WND), is another manifestation of the phonemically influenced beliefs that leads students to state that "one sixth is six".


11. $\frac{1}{6}$ Reason: Because the larger the numbe the more you are going to get

11. $\frac{1}{6}$ Reason: because it has ~~the~~ a higher denomenator

11. $\frac{1}{6}$ Reason: Because 6 is bigger than 4

**Figure 5.12** Responses showing reasoning directly from the size of the denominator

A focus on the whole number in the denominator led to simple direct reasoning, as illustrated in Figure 5.12. Rather than an inverse relationship between the size of the denominator and the size of the fraction, this reasoning suggests that the fraction notation may not be supporting the fraction concept for 5% or more of the students in each grade in this study. The two categories ("$\frac{1}{6}$ is 6 or $\frac{1}{4}$ is 4" and "whole number denominator") account for 15% of the errors in Year 4 and 10% of the errors in Year 5.

Of course, some students appear to have mixed fraction concepts. The Year 8 student whose response to Question 11 is shown in Figure 5.13 has correctly determined that one-quarter of twelve counters is three counters but then incorrectly states that $\frac{1}{6}$ is six counters.

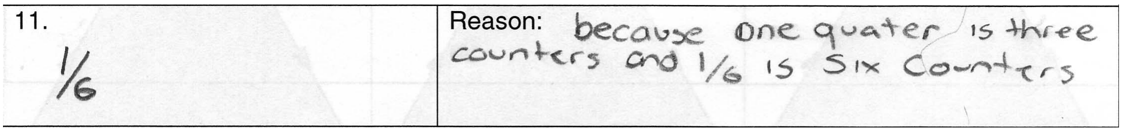

11. 1/6 Reason: because one quater is three counters and 1/6 is six counters

**Figure 5.13** Response of a Year 8 student with mixed meanings attributed to fractions

Even though the standard symbolism for $\frac{1}{6}$ is not used, the Year 8 student's response in Figure 5.14 suggests that the naming system for fractions has led to confusion between ordinal names and the phonemically similar cardinal. Referring to a sixth as half of the twelve counters suggests that for this student, the "sixth" did not correspond to naming a fraction.

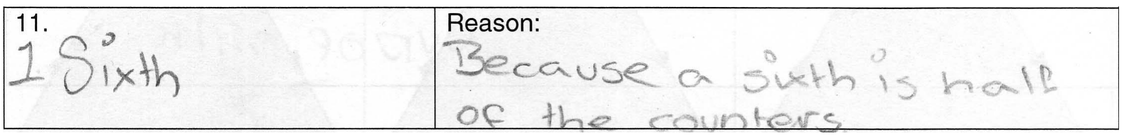

11. 1 Sixth Reason: Because a sixth is half of the counters

**Figure 5.14** Response indicating a "sixth is half of twelve"

A student’s use of fraction notation can appear to be additive even when he or she has an essentially multiplicative sense of fractions. In Figure 5.15, the Year 7 student explained “you would only have two if it is one sixth because $\frac{1}{6}+\frac{1}{6}=\frac{2}{12}$ and $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{3}{12}$”. As all of this student’s responses to the questions preceding this one were correct, she appears to possess quite a robust sense of the size of fractions. Looking beyond the conventions of symbolic addition of fractions, this use of the fraction notation is suggestive of forming groups. The reason states that “you would only have two if it is one sixth” and in the preceding questions the student has correctly determined that one-quarter of twelve is three and three-quarters of twelve is nine. Although this is not the conventional use of the fraction notation, it has an internal logic of additive groups. For example, this use of the symbol system could equally demonstrate that one-sixth of eighteen is three, $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{3}{18}$. Perhaps the most pertinent observation from this student’s response is the possibility that an additive use of the fraction notation system may not reflect absence of a multiplicative understanding of fractions.

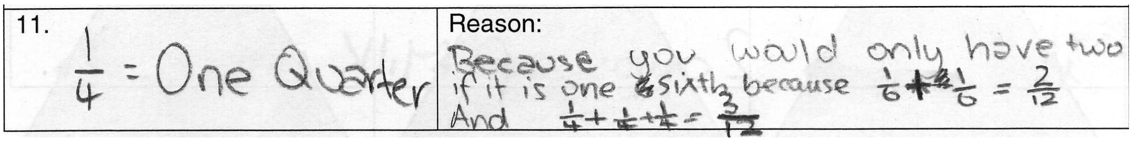



**Figure 5.15** Response showing an apparent additive use of fraction notation[9]

The response in Figure 5.16 is typical of the explanation that $\frac{1}{6}$ was 1 while correctly determining that a quarter of twelve is three (a correct answer resulting from using incorrect reasoning). This response was more common amongst high school students with 15 of the 19 students who reasoned in this manor being in Years 7 and 8.

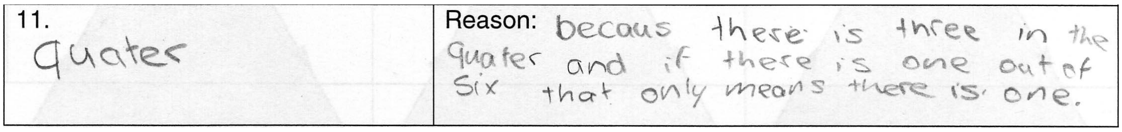



**Figure 5.16** A response indicating “one out of six is one”

It is also possible that students had developed a more robust appreciation of halves and quarters and a superficial interpretation of one-sixth.

Incorrect regional models also featured in some explanations. For example, the Year 4 student’s answer in Figure 5.17 reasons that one-quarter is more than one-sixth from a

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[9] In later questions, such as Questions 15 and 16, this student’s fraction notation supported by diagrams is used conventionally with addition.

regional model that represents one-sixth as one-eighth of a circle. The explanation also describes a multiplicative relationship between the area representations for one-sixth and one-quarter. As appears later in response to Question 14 (shading one-sixth of a circle), this Year 4 student was not alone in believing that one-sixth could be represented by one-eighth of a circle.

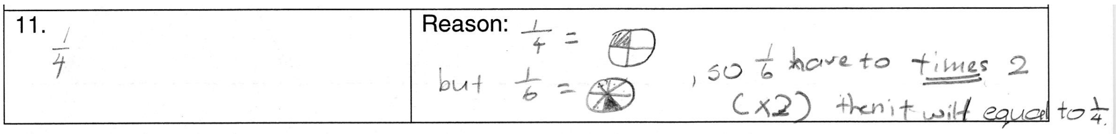


**Figure 5.17** Representing one-sixth as one-eighth

**Looking back**

Though 84% of students could name a number of discrete items as a fraction of the whole, equivalent fractions, the conceptual underpinning of all operations with fractions, were not well known in this sample. While a fraction such as $\frac{2}{8}$ has infinitely many equivalent fractions, understanding how to produce an equivalent fraction was not evident for more than one-half of the students in Year 4, 5 and 7 involved in the study. Expressing equivalent fractions is not an explicit expectation of the Mathematics syllabus for the primary years, but it is for Years 7 and 8. Despite the absence of operating with fractions in the primary syllabus, 18 % of Year 4, 33% of Year 5 and 54% of Year 6 calculated $\frac{1}{6}$ and $\frac{1}{4}$ of 12 to demonstrate which was larger.

The percentage of students who could correctly find one-quarter of 12 discrete counters is comparable to the percentage of students identifying one-quarter of a strip of paper folded into one-half and two quarters. Students' reasoning strategies came to the fore in Question 11. The interpretation of fractions as being equivalent to the *whole number denominator* (that is, reasoning from the direct relationship between the size of the denominator and the size of the fraction) aligned with phonemic miscuing (such as "one sixth is 6"). Overall 8.2% of students used a direct relationship between the size of the denominator and the size of a fraction rather than an inverse relationship.

## 5.5 Constructing parts of a circle regional models: Q12-14

Using a regional model of a fraction can be considered to require subdividing an area into a number of pieces, then counting and shading a number of pieces. When shading half of a circular region this process can be achieved more simply using basic "algorithmic halving".

### *Question 12*

In Question 12, students were asked to shade the area of an unmarked circle to represent one-half. The results are summarised in Table 12.

Table 12

*Percentage distribution of responses to Question 12, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{1}{2}$ | 98 | 100 | 99 | 100 | 100 |
| Other | 1 | 0 | 0 | 0 | 0 |
| Omitted | 1 | 0 | 1 | 0 | 0 |

The percentage of students correctly shading one half of a circle was very high (100% to 2 significant figures) with every Year 5, 7 and 8 student correctly answering the question. Students' knowledge of one-half of a circle tended to confirm Hart's (1989) observation that one-half, in this context, is not a typical fraction but "…seems to be an honorary whole number" (p. 216).

### *Question 13*

In Question 13, students were asked to shade the area of an unmarked circle to represent one-third. The responses to shading one-third of a circle were much more varied and contributed to a more complex coding process. Application of a "number of pieces" interpretation of the regional model led to attempts to create thirds as segments of a circle (see Figure 5.18).

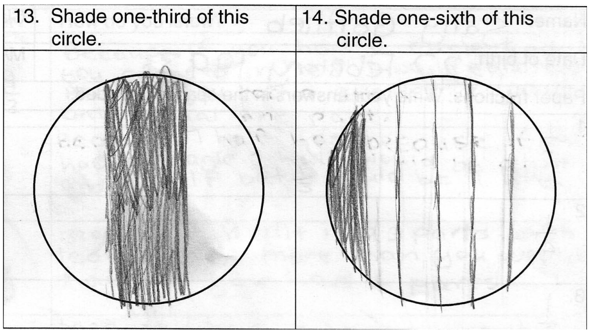


**Figure 5.18** A year 6 student's response coded as segments[10]

This form of partitioning of a circle appears as equi-distant cutting with attention usually given to the distance between the cuts rather than the resulting area of the pieces. In the present study, this form of parallel partitioning strategy persisted well past the ages of students involved in other studies that report horizontal partitioning methods (Charles & Nason, 2000; Pothier & Sawada, 1983; Streefland, 1991).

An initial coding and classification of the responses according to the area shaded was made but the preliminary analysis identified several apparent anomalies that prompted a more thorough conceptual analysis of the evoked concept images. The first was that the coded response of "one-quarter" was surprisingly large, constituting over 12% in total of all responses. After the correct answer of one-third, one-quarter was the next largest category. This "unusual error" also appeared to be relatively insensitive to grade level, with about 40 responses in each grade being coded as one-quarter. The second, and perhaps more telling observation was that representing one-third as one-quarter did not appear to be a consistent error in the way that the region was represented.

In the Year 7 student's response shown in Figure 5.19, the area of a circle model of one-third was initially coded one-quarter using an area interpretation. Similarly, the attempt to create one-sixth of a circle was also initially coded as one-quarter. Clearly the student either believes that the same areas represent one-third and one-sixth, or else area is not the intended focus of the student's response.

[10] Attempting to represent a fraction by segmenting a circle was not always consistently used. The example provided in Figure 5.19 is meant only to clarify the category of "segmented" and not to imply that students consistently segmented both circles.

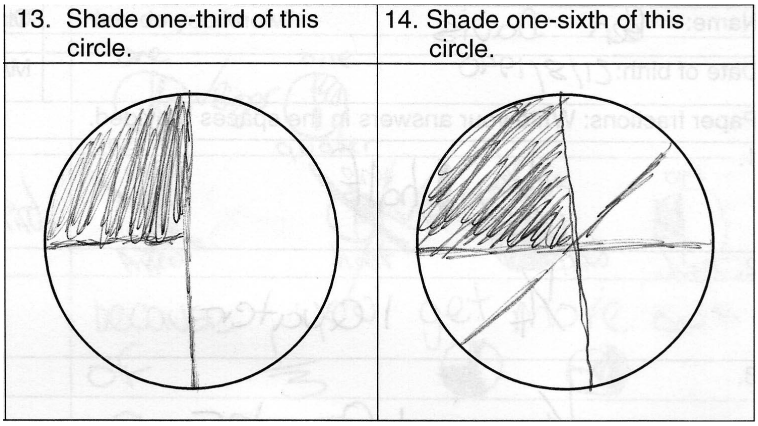


**Figure 5.19** A year 7 student's response initially coded as one-quarter

From an emergent perspective of sense making, it must be assumed that the student's intent was to represent one-third in response to Question 13 and one-sixth in response to Question 14. The interpretations of this student's responses become consistent if the responses are coded from the perspective of the number of parts rather than the area of those parts. This led to all of the responses initially coded as $\frac{1}{4}$ being recoded to determine how frequently the intent to display one-third as "one part out of three parts" was evident in the responses. The results of this revised coding are shown in Table 13.

Table 13

*Percentage distribution of responses to Question 13, by year (recoded)*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{1}{3}$ | 30 | 43 | 62 | 43 | 54 |
| Segmented | 12 | 15 | 7 | 10 | 6 |
| $\frac{1}{4}$ | 11 | 7 | 8 | 9 | 10 |
| $\frac{3}{4}$ | 10 | 7 | 3 | 4 | 3 |
| 3 parts ($\frac{1}{2}$ & 2 $\frac{1}{4}$s) | 5 | 4 | 3 | 5 | 3 |
| $\frac{3}{8}$ constructed | 2 | 1 | 1 | 4 | 1 |
| Other | 28 | 22 | 17 | 25 | 23 |
| Omitted | 2 | 0 | 0 | 0 | 0 |

Overall, only 46% of students could create an adequate representation of one-third of a circle's area. As 10% of students used parallel partitioning to attempt to create thirds, the area basis for the representation of fractions appears suspect in students' concept images.

The responses coded as "segmented" arising from parallel partitioning, align more with a "number of parts" interpretation than an area-based interpretation of fractions.

The recoding resulted in fewer responses being coded $\frac{1}{4}$ than "segmented", but the total is still quite high. The category one-quarter contains constructions of two-eighths as in Figure 5.20, and other creations of one-quarter, presumably drawn to align with the common prototypical image of one-third. In Figure 5.20 the construction lines used to represent one-third by shading two eighths are still evident, although somewhat faint from being erased.

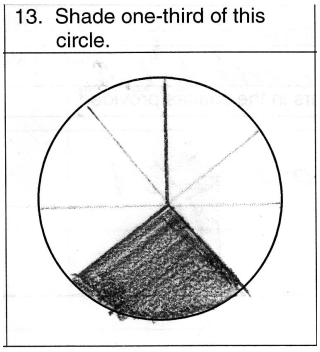


**Figure 5.20** A Year 6 student's drawing coded as one-quarter

The answers coded as three-quarters were an unexpected response to the instruction to shade one-third of a circle. If one-half is the first fraction that students learn, one-quarter is usually the next, and shading three-quarters to represent one-third suggests evoked concept images unrelated to traditional regional models. A further investigation of this category will be undertaken below in connection with students' representations of one-sixth. The newly formed category of one part out of three parts (one half and two quarters) accounted for 3.9% of all responses. Most students shaded one of the quarters, but a small number shaded the one-half.

The visual similarity between three-eighths and one-third of a circle may be sufficient for some students to equate the two regional models. Overall, 1.7% of students clearly constructed $\frac{3}{8}$ of a circle to represent $\frac{1}{3}$. The actual percentage of students who drew $\frac{3}{8}$ of a circle to represent $\frac{1}{3}$ may have been a little higher than this, as the range of acceptable drawings for $\frac{1}{3}$ had an upper sector angle boundary of 135°, corresponding to $\frac{3}{8}$, which meant that a sector corresponding to $\frac{3}{8}$ would have been recorded as $\frac{1}{3}$ unless the construction lines indicated that the intent was to draw $\frac{3}{8}$. Consequently, the category was described as "$\frac{3}{8}$ constructed".

### *Question 14*

In Question 14, students were asked to shade the area of an unmarked circle to represent one-sixth of the circle. The interpretations of responses to shading one-sixth of a circle were similar to those arising from shading in one-third of a circle. As with Question 13, the responses to Question 14 were coded twice to reflect the evolving explanatory categories.

The construction lines students left in creating their answers helped to distinguish categories. For example, a response needed to have the construction lines evident to be included in the category of "$\frac{1}{8}$ constructed" and the construction lines also distinguished these responses from the category of "6 parts constructed and one-eighth shaded". In addition to the emergence of the category of 6 parts (2 quarters and 4 eighths) in the recoding, a new type of response was evident in the segmented category. As well as the usual vertical cuts to create segments, some students introduced horizontal cuts to create half-segments as in Figure 5.21. This partitioning through vertical and horizontal lines is likely to be influenced by the dominance of the horizontal and vertical orientations in visual perception (Howard & Templeton, 1966) which creates a visual bias in these two directions.

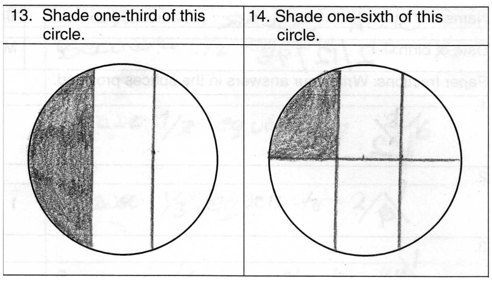


**Figure 5.21** Response showing horizontal cuts used to create "sixths"

In Figure 5.21, although the area is incorrectly represented, the answer to Question 14 suggests that for this student one-sixth may well be half of one-third. That is, a multiplicative relationship (e.g. creating sixths by halving thirds) may exist between some fractions even though the regional representation is incorrect. Moreover, this vertical and horizontal partitioning would be correct for a rectangle, emphasising the need for non-examples in developing regional fraction concepts. The responses to Question 14 are summarised in Table 14.

Table 14

*Percentage distribution of responses to Question 14, by year (recoded)*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{1}{6}$ | 29 | 43 | 62 | 42 | 59 |
| Segmented or parts | 15 | 13 | 11 | 10 | 5 |
| 6 parts (2 $\frac{1}{4}$s & 4 $\frac{1}{8}$s) | 16 | 10 | 7 | 8 | 10 |
| $\frac{1}{8}$ constructed | 5 | 3 | 2 | 7 | 5 |
| $\frac{6}{8}$ | 2 | 2 | 1 | 1 | 1 |
| Other | 31 | 28 | 18 | 31 | 19 |
| Omitted | 3 | 1 | 0 | 2 | 1 |

Overall, 47% of the students created an acceptable representation of one-sixth of a circle. This result is comparable to the 46% of students who achieved an acceptable representation of one-third of a circular region. Based on these results, shading one-sixth of a circle appears to be of the same order of difficulty for students as shading one-third of a circle. Just as finding three-quarters of 12 counters presented a similar level of challenge to finding one-quarter of 12 counters (cf. Questions 9 and 10), representing one-sixth (half of one-third) was no more challenging than representing one-third of the area of a circle. In total, 8.1% attempted to represent one-sixth using segments and another 2.6% used parts of segments (see Figure 5.21).

In Question 14, an unexpected high response category was described initially as "$\frac{1}{8}$". As in dealing with the unusually high response category of $\frac{1}{4}$ in Question 13, the construction lines students left in creating their answers helped to clarify their intent in producing $\frac{1}{8}$. This resulted in the creation of the category of "6 parts (2 $\frac{1}{4}$s & 4 $\frac{1}{8}$s)" as well as "$\frac{1}{8}$ constructed". In total, 7.1% of students created 2 quarters and 4 eighths before shading $\frac{1}{8}$ and another 3.1% shaded $\frac{1}{4}$. The category of one part out of six parts (two quarters and four eighths), which included responses such as Figure 5.22, accounted for 10% of all responses.

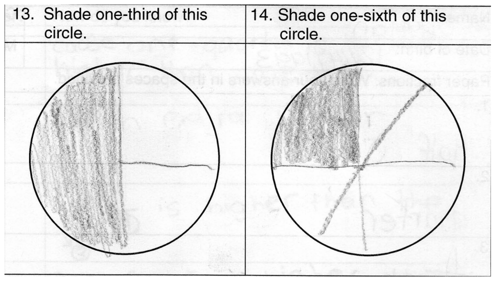


**Figure 5.22** Shading one part of three (Q13) and one of six (Q14)

This mode of partitioning the circle into six parts, some of which are equal, was quite popular. As well as accounting for 10% of all responses to shading one-sixth of a circle, this "equal parts" representation did not readily diminish across the school grades.

As well as creating 6 parts (2 quarters and 4 eighths) and then shading $\frac{1}{8}$, some of the responses, such as the Year 7 student's response shown in Figure 5.23, constructed $\frac{1}{8}$ as one part out of eight.

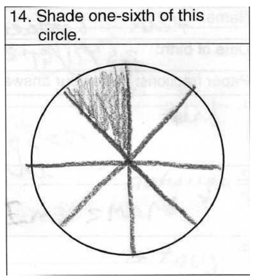


**Figure 5.23** Response showing "$\frac{1}{8}$ constructed"

Not all responses coded as "$\frac{1}{8}$" were constructed in this way for Question 14. Some students, possibly influenced by their construction of one-third, simply bisected a quarter. If a student holds a prototypical image of one-third that is close to one-quarter of the area of a circle, creating one-sixth by halving this ambiguous prototypical image can result in constructing one-eighth for one-sixth, as in Figure 5.24.

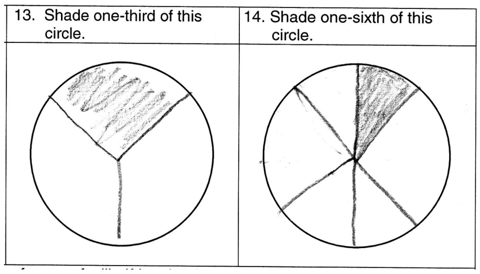


**Figure 5.24** Halving an image of one-third

Thus, this example of the category of "$\frac{1}{8}$ constructed" may be seen as arising from halving a prototypical image of a one-third that is indistinguishable from one-quarter.

The category $\frac{6}{8}$ appeared to correspond to a simple number of parts interpretation of the fraction. For example, the response to Questions 13 and 14 shown in Figure 5.25 depicts a fraction as an indicated number of equal parts corresponding to the denominator of the fraction to be shaded.

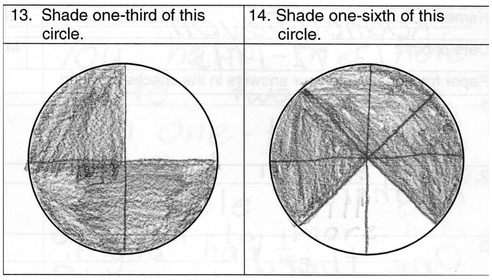


**Figure 5.25** Fractions indicated by the number of parts

That is, a "number of pieces" interpretation of the regional model that represents the number of pieces described by the denominator of a fraction could translate into drawing an area of six-eighths to represent one-sixth. Although the total area shaded is the same for the two fractions, from the student's perspective it indicates that the number of equal parts and not the area forms the fraction.

**Number of parts interpretations to Questions 13 and 14**

The responses to Question 13 included the category $\frac{3}{4}$ and similarly Question 14 had the category $\frac{6}{8}$. From the standpoint of the area shaded these two categories are identical. However, a closer analysis of the responses suggests a "number of parts" interpretation of these answers may better represent what the students intended. In Figure 5.26, the Year 5 student's numbered drawing represents an attempt to indicate one-third as three parts and one-sixth as six parts. Not only is this a "number of pieces" interpretation but it is also a "number of equal pieces" evoked image of a fraction.

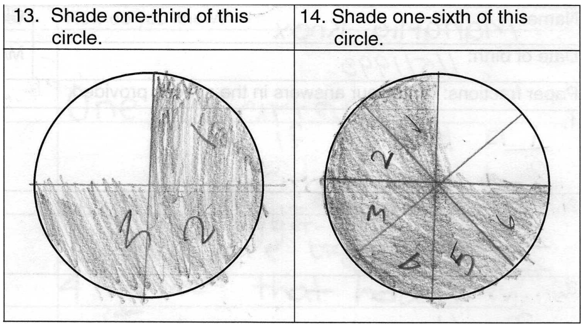


**Figure 5.26** A year 5 student's drawing identifying the number of parts

A "number of pieces" interpretation of the regional model that represents the number of pieces described by the denominator of a fraction appears to have been translated into drawing an area of three-quarters to represent one-third in Question 13 and shading six-eighths to represent one-sixth in Question 14. The Year 8 student's response in Figure 5.27 is also suggestive of this interpretation.

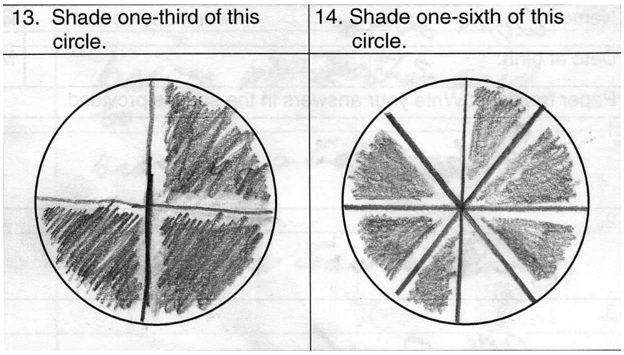


**Figure 5.27** A "number of equal pieces" representation of a unit fraction

There is a type of pattern logic behind creating sixths as part of a sequence of diameters. Figure 5.28 demonstrates how one-sixth fits into such a pattern.

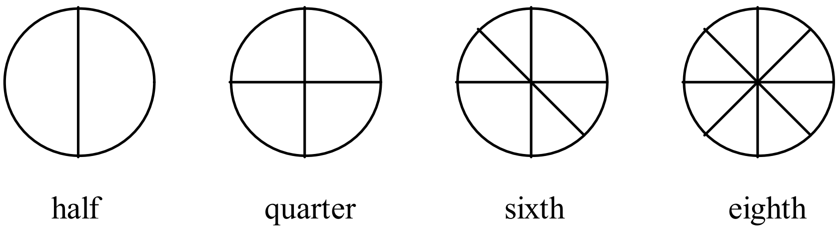


**Figure 5.28** A pattern sequence including "sixths"

This pattern logic resonates with Pothier and Sawada's (1983) third level of partitioning, *evenness* (see Section 2.2.3). Pothier and Sawada characterise the evenness level as an ability to partition fractions with even denominators. Their third level of partitioning uses algorithmic halving followed by adjustment to make the parts "even". The materials used with their partitioning tasks, such as the sticks used to show the cuts with the "cake problem", may have encouraged the adjustments they observed. Partitioning that has been formed from drawing may be more prone to algorithmic halving *without adjustment*.

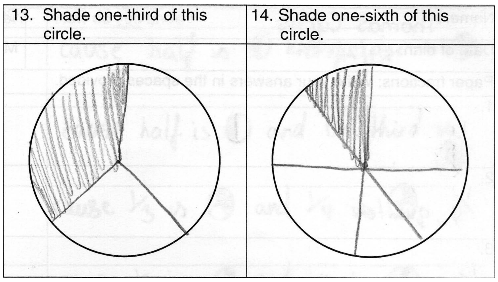


**Figure 5.29** One-sixth not represented as half of one-third

As can be seen in Figure 5.29, representing one-sixth as one part of six (eighths and quarters) persisted even in the presence of an evoked concept image of one-third similar to the common prototype (generic example). This Year 8 student did not construct one-sixth as one-half of one-third.

A "number of equal pieces" interpretation could also be applied to those responses where three-eighths was constructed in response to the request to shade one-third of a circle (Figure 5.30). The total number of students who constructed three-eighths in response to Question 13 was 28, a much smaller number than those constructing three quarter-parts to represent one-third. Although it is reasonable to consider that some students' constructions of three-eighths may be interpreted as a "number of equal pieces" evoked concept image of a fraction, the proximity of the resulting area to a prototypical image of one-third is also pertinent. A prototypical image of one-third (commonly ) is the image or prototype (generic example) that we use when we think of one-third. Visually, this creation of one-third as three countable units (see Figure 5.30) has resulted in an image that is quite close to the correct sub-division of the area of the circle for one-third.

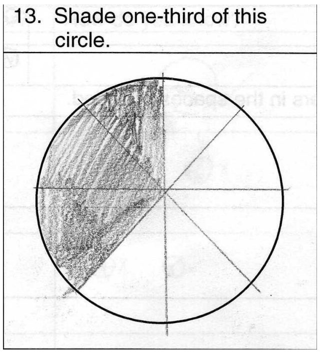


**Figure 5.30** A year 7 student's construction of $\frac{3}{8}$ for $\frac{1}{3}$

Occasionally the number of parts and the count corresponding to the denominator operated together, as in Figure 5.31 (Question 14). This Year 5 student has shaded three parts in Question 13 to demonstrate one-third and six parts in Question 14 to demonstrate one-sixth[11].

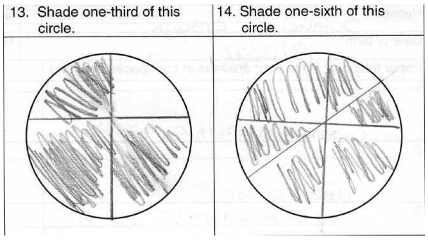


**Figure 5.31** Shading three parts (Q13) and shading six parts (Q14)

Other students created the six parts in the same way but shaded different numbers of parts.

**Looking back**

Area was often not the salient feature of regional model representations created by students. Consistent interpretation of students' diagrams was achieved when some responses were coded from the perspective of the number of parts rather than the area of those parts.

A "number of pieces" interpretation has three common variations. The first interpretation can lead to attempts to create thirds and sixths as segments of a circle, using parallel partitioning. The second interpretation is to construct a number of pieces corresponding to the denominator and then to shade in one of the pieces. A third interpretation is to shade in a number of pieces corresponding to the denominator. The proximity of some incorrect fraction representations to common prototypical images (e.g., $\frac{3}{8}$, $\frac{1}{4}$ and $\frac{1}{3}$) may reinforce and mask the misconception. Looking at the way that students have interpreted fractions from Questions 1 to 14, suggests that simply using part-whole models to represent fractions frequently does not result in students developing complete part-whole fraction concepts. Indeed, with 10% of the Year 8 students subdividing the circle into 4 eighths and 2 quarters, 5% dividing the circle into unequal parts and 5% constructing one-eighth for a sixth, the progressive development of the part-whole concept of a fraction, such as one-sixth or one-third, has failed for approximately 20% of students.

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[11] This response of forming six parts and then shading them all to indicate one-sixth was not common. Consequently, this response to Q14 and other similar types of responses were added to the 'other' category.

## 5.6 Representing multiples of non-unit fractions: Q15-18

Questions 15 to 18 presented a practical context for operating with fractional quantities. Fractions were written in words and students were asked to draw their answers in an attempt to discourage them from reducing the task to algorithmic manipulation of the fraction notation. That is, the task was designed to draw on students' quantitative sense of fractions rather than their knowledge of fraction algorithms.

### *Question 15*

In Question 15, students were told that a recipe uses three-quarters of a cup of milk and were then asked to determine how many cups of milk would be needed to make double the recipe. Although the fraction term three-quarters was written in words, this did not stop some students from introducing the fraction notation and subsequently being misled by the notation (Figure 5.32).

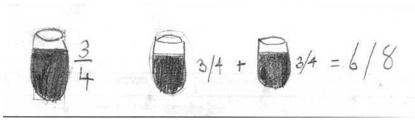



**Figure 5.32** Fraction notation introduced by a Year 4 student and used additively

A range of students' evoked fraction concept images was recorded. The drawings were broadly categorised as correct answers with correct reasoning (Category 1), incorrect answers with correct reasoning (Category 2), correct answers with incorrect or inadequate reasoning (Category 3), and incorrect answers with incorrect reasoning (Category 4). An answer without any reasoning was treated as inadequate reasoning and included in either Category 3 or 4. The percentage of each school year producing reasoned responses to Question 15 is shown in Table 15.

Table 15

*Percentage distribution of responses to Question 15, by year*

| | | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|---|
| Right answer/right reason | (1) | 43 | 57 | 72 | 58 | 64 |
| Wrong answer/right reason | (2) | 13 | 13 | 8 | 15 | 16 |
| Right answer/wrong reason | (3) | 1 | 1 | 1 | 1 | 1 |
| Wrong answer/wrong reason | (4) | 43 | 29 | 19 | 25 | 19 |
| Omitted | | 1 | 1 | 0 | 1 | 1 |

Correct answers with correct supporting reasoning (Category 1) included simple aggregation of fractional units, as well as multiplicative expressions (see Figure 5.33).

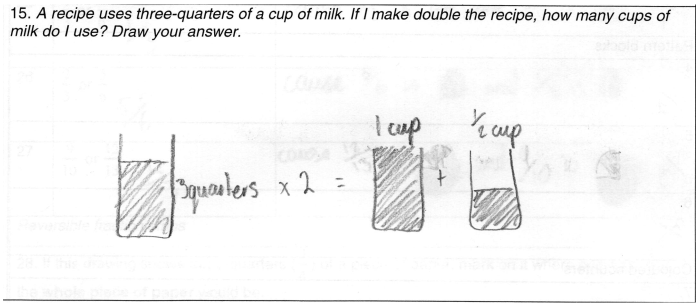



**Figure 5.33** A correct answer with correct reasoning

Overall, 59% of students achieved the correct answer through correct modes of representing the problem, either diagrammatically or symbolically. However, correct representation did not always result in a correct answer. To have an answer included in the category of incorrect answer with correct reasoning (Category 2) required that there was some element of the reasoning that should have contributed to a correct solution. The majority of the answers in Category 2 corresponded to students drawing the correct fractional unit and the correct number of units but then not accumulating the units, as in Figure 5.34.

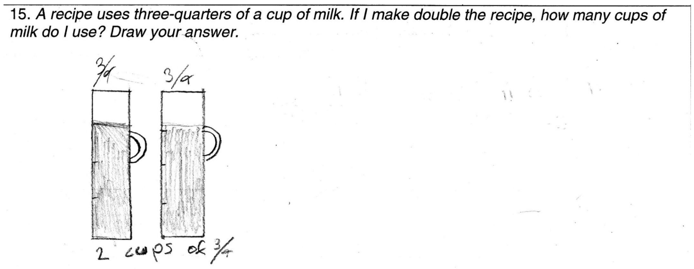



**Figure 5.34** A Year 5 response showing the correct unit and correct number of units

The initial categories had a number of subcategories (as described in Table 4.3) to refine the recording of the use of units in students' reasoning. The subcategories recorded features such as whether the correct unit and number of units were used, as well as whether the units were correctly accumulated.

If a student correctly transformed the problem by writing out a correct mathematical expression (see Figure 5.35) and then recorded an incorrect answer, the response was also included in Category 2.

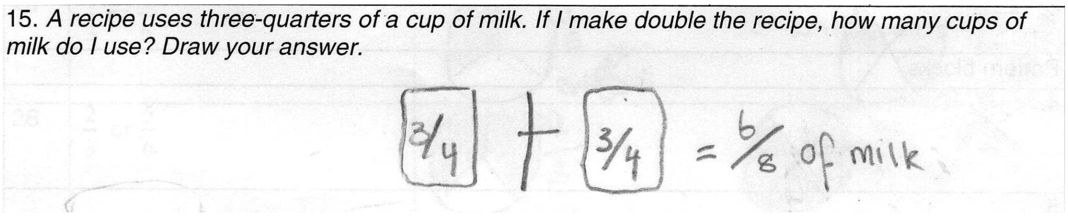



**Figure 5.35** A Year 5 student's correct expression with an incorrect answer

This additive interpretation of the fraction notation resulted in just over 3% of all students indicating that the answer was $\frac{6}{8}$ (Year 4, 2%; Year 5, 3%; Year 6, 1%; Year 7, 4%; Year 8, 7%). There was an increase in the percentage of the school grade answering $\frac{6}{8}$ in Year 8, perhaps because students have been taught algorithms for the four operations with fractions and mixed numbers. Most of the Year 8 students who answered $\frac{6}{8}$ to Question 15 made a similar error ($\frac{6}{9}$) on Question 16.

Overall 69% of students' answers were attributed to correct reasoning involving the fraction units (Categories 1 and 2). However, additive interpretations of the fraction units persisted despite the practical context of the question.

The small number of correct answers with incorrect or inadequate reasoning (Category 3) included responses that diagrammatically represented the accumulated units with incorrect notation, as in the Year 5 student's response in Figure 5.36. A correct answer with no supporting reasoning also fell into this category.

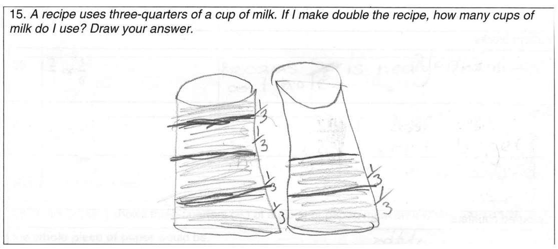



**Figure 5.36** Incorrect notation with correctly represented area units (Category 3)

If 7.5% of all the students in this study can construct two lots of three-quarters of a cup but then not determine how many cups of milk this corresponds to (as in Figure 5.34), is it because the question is in some way ambiguous or is it because addition and conversion of simple fractional parts is too challenging? The number of students in this subcategory does not change markedly across the school years (Yr 4, 8%; Yr 5, 9%; Yr 6, 6%; Yr 7, 8%; Yr 8, 7%). This lack of aggregation of fractional units may reflect the paucity of opportunities to aggregate fractional units beyond one-whole in school problem contexts.

Category 4 corresponded to an incorrect answer with no reasoning or incorrect reasoning. Overall, 26% of responses fell into this category with Year 4 having the highest proportion of incorrect responses. Equal proportions of Category 1 and Category 4 responses for Year 4 on this question suggests that although 63% could correctly determine three-quarters of 12 counters in Question 10, doubling three-quarters in a practical context appears to be more difficult. This may also be influenced by lack of opportunities to aggregate fractional units beyond one-whole in Year 4.

The request to draw the answer did appear to assist some students. The response by the Year 7 student shown in Figure 5.37 (Category 1) shows the answer in fraction notation originally recorded as $\frac{6}{8}$ with the denominator overwritten to become $\frac{6}{4}$. The use of the diagrams may have prompted the need to correct the notation for the sum.

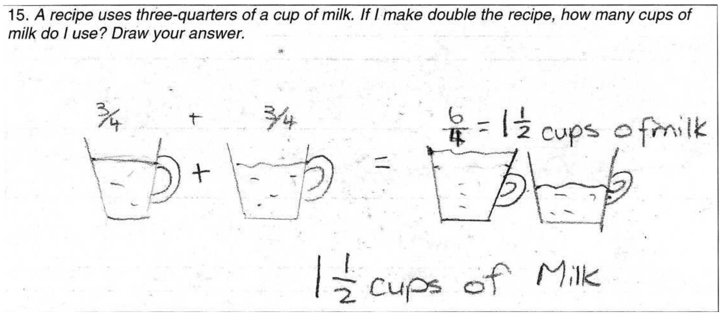



**Figure 5.37** A correct diagrammatic representation and adjusted notation

However, not all of the diagrams that students created were helpful. In the Year 4 student's response (Category 4) depicted in Figure 5.38, the student seems to have attempted to represent three individual "quarters" of a cup doubled, disaggregating the three-quarter cups, which was not a constructive a move towards the solution. The answer shown in Figure 5.38 was coded as incorrect as there are no supporting indicators (e.g. partition marks, symbols or words) on the diagram of that the drawings are meant to represent quarters.

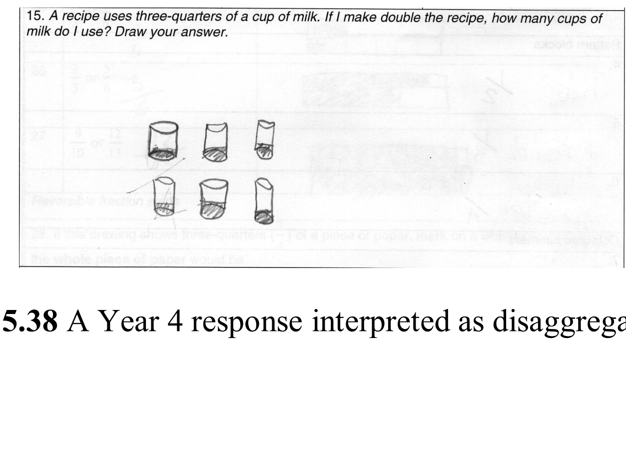



**Figure 5.38** A Year 4 response interpreted as disaggregated units

Question 15 was designed to investigate the capacity of students to double three-quarters of a common unit, using drawing to represent the problem solution. For some students, such as the Year 8 student whose response is shown in Figure 5.39, the invitation to draw the answer revealed unanticipated interpretations of the fraction notation. For this student, doubling three-quarters resulted in $\frac{6}{8}$.

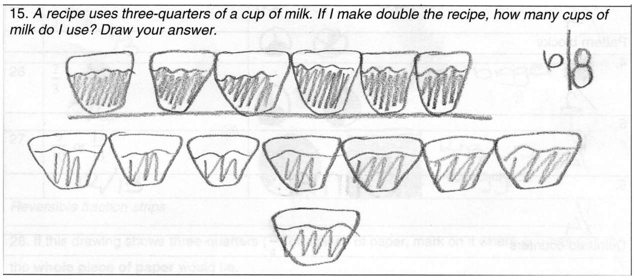



**Figure 5.39** A Year 8 response interpreted as doubling both parts

In Figure 5.39, the comparison of two discrete numbers in three-quarters, three is to four, appears to result in "doubling the ratio" to produce $\frac{6}{8}$. Oddly, by Year 8 students have been taught to simplify fractions and presumably, many of those who answered $\frac{6}{8}$ could have simplified $\frac{6}{8}$ to $\frac{3}{4}$. However, the evoked concept images associated with doubling a fraction were rarely accompanied by an appreciation of fraction equivalence.

When some students introduced the fraction notation with accompanying manipulation of the notation through a standard problem translation approach, several anomalies were revealed. For example, the Year 5 student whose answer is shown in Figure 5.40 multiplied all of the numbers involved.

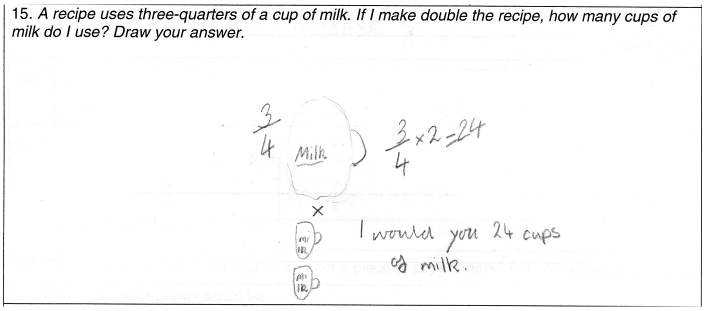



**Figure 5.40** Multiplying all of the numbers (3, 4 and 2)

This procedural response of transforming the problem to a multiplication expressed with fraction notation has resulted in an answer (24) devoid of any quantitative sense. Other responses from this student suggest that she did not have a quantitative sense of one-quarter. For example, the answers to Questions 2 and 3, both of which should have been

$\frac{1}{4}$, were $\frac{1}{3}$ and $\frac{2}{3}$. Operations on fractions without an appreciation of fractions as quantities can clearly result in answers disconnected from making sense of the context. Another example of "rules over reasons" is evident in the Year 8 student's response in Figure 5.41. The process of multiplication is confounded with the fraction division algorithm.

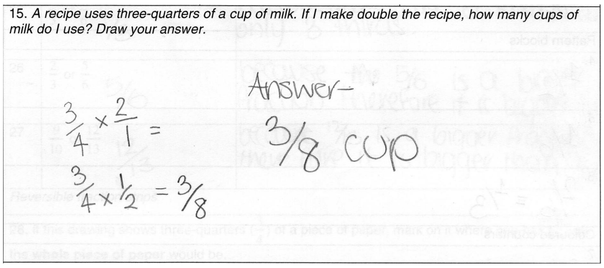



**Figure 5.41** A response treating multiplication as invert and multiply

The sequence of the steps in the explanation suggests that replacing a multiplication involving fractions with multiplying by the reciprocal of a fraction is a thoughtless application of a remembered rule.

As indicated earlier, the sample of students was not selected specifically to produce comparative ability between school years and it is likely that the Year 6 group was more capable than the high school groups. For example, comparing the performance of Year 6 and Year 8 on this question suggests that overall they are at a similar level (80% correct reasoning) with the Year 6 group being able to carry out or communicate the reasoning to achieve a correct answer more often then the Year 8 students.

### *Question 16*

In Question 16, students were told that a recipe uses two-thirds of a cup of milk and then asked to determine how many cups of milk would be needed to make three lots of the recipe. The same broad categories as Question 15 were used to organise the results and the percentage of each school year using these types of reasoned responses is shown in Table 16.

Table 16

*Percentage distribution of responses to Question 16, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Right answer/right reason (1) | 26 | 39 | 56 | 31 | 40 |
| Wrong answer/right reason (2) | 15 | 15 | 12 | 15 | 20 |
| Right answer/wrong reason (3) | 1 | 0 | 1 | 1 | 1 |
| Wrong answer/wrong reason (4) | 53 | 42 | 32 | 47 | 35 |
| Omitted | 5 | 4 | 0 | 6 | 5 |

The percentage of students who could correctly reason with multiples of fractions declined substantially between Question 15 and Question 16. While 59% of all students could correctly double three-quarters of a cup, only 38% could determine three lots of two-thirds of a cup. The greatest percentage decline occurred in Year 7 with an appreciably smaller decrease in Category 1 in the primary years (4–6) between questions.

Overall 51% of students' answers were attributed to correct reasoning involving the fraction units, including incorrect answers (Category 2). Of the 13% of students who did not answer the question correctly although their reasoning about units was correct, the majority used the correct unit and number of units but did not accumulate the units. The Year 8 student's response in Figure 5.42 is typical of drawings showing the correct unit and the correct number of units, without the units being accumulated.

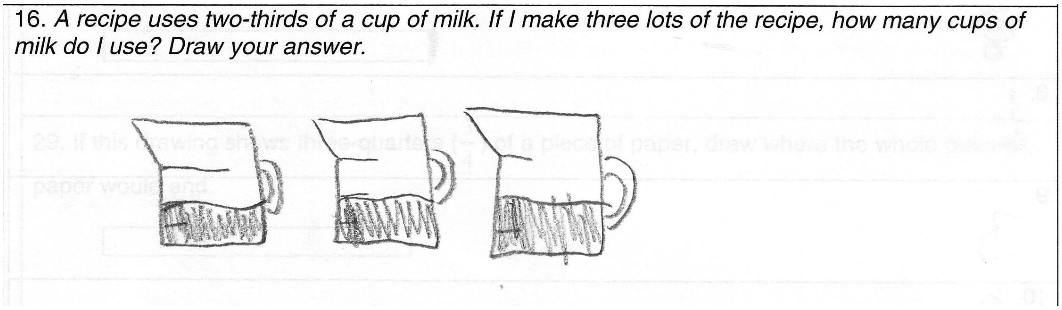



**Figure 5.42** Correct units and correct number of units

It could be argued that this Year 8 student has drawn quarters rather than thirds or, if representing two-thirds, has not recognised the need to accumulate the two-third units. However, the same student's response to the previous question (Figure 5.43) did accumulate the units in the drawing and helped to distinguish the pouring spout from the whole and the fractional parts.

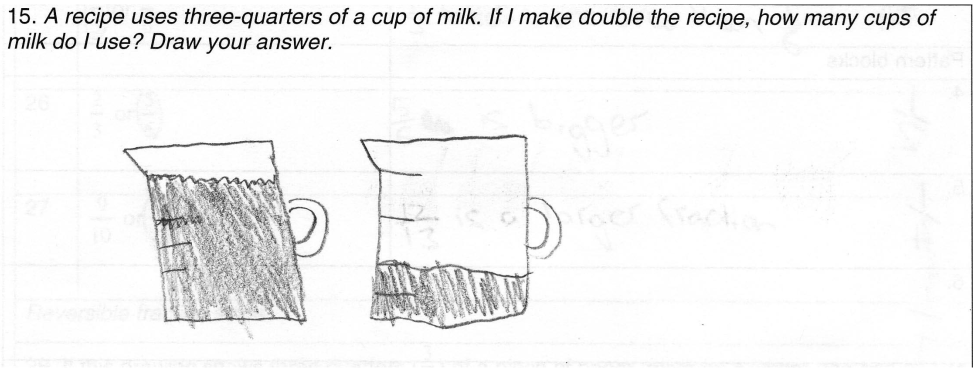



**Figure 5.43** Same Year 8 student's answer to Q15 (accumulated units and spout)

The decline in the Year 8 performance (compared to Year 6 and relative to the previous question) may be associated with the increased use of an erroneous fraction algorithm (3 lots of $\frac{2}{3}$) resulting in $\frac{6}{9}$. Just over 7% of Year 8 students answered $\frac{6}{9}$ compared to 1% of Year 6 students.

The overall decline in correct answers could also be influenced by the use of thirds rather than quarters in Question 16. If we compare the 38% of students who could correctly determine three lots of two-thirds of a cup (and communicate their reasoning) to the 46% of students who could correctly represent one-third of a circle in Question 13, it is clear that operating on two-thirds of a cup is more difficult. This reduction of the proportion of students who could correctly answer Question 13 compared to Question 16 was more pronounced in secondary schools. In Year 6 the gap between the percentages of correct answers to the two questions was 6% but in Year 7 the gap increased to 12% and 14% in Year 8. As the four operations with fractions are introduced in Years 7 and 8, it may be that learning fraction algorithms contributes to a decline in student performance on contextual fraction problems.

In total, 3.3% of students answered $\frac{6}{9}$ to Question 16 with 40 of these 55 responses coming from Years 7 and 8. In general, students' additive strategies were consistently applied and 73% of the students who answered $\frac{6}{9}$ to Question 16 had previously answered $\frac{6}{8}$ to Question 15. Both responses reflected a continued application of an additive solution method, as in Figure 5.44.

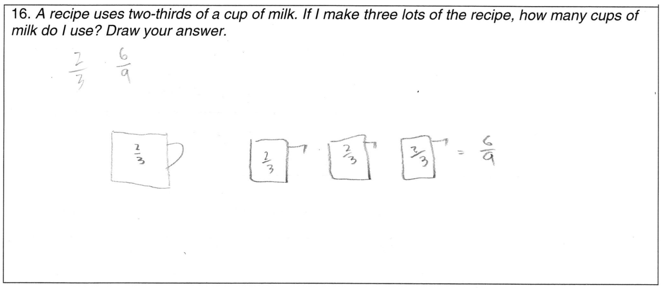



**Figure 5.44** An additive interpretation of the fraction notation

Even when students could correctly comprehend the question, the requisite understanding of how to iterate or accumulate the fraction units was frequently absent. The Year 8 student's response in Figure 5.45 suggests that creating three lots of the recipe does not involve repeated addition but rather a form of repeated "magic doubling"[12].

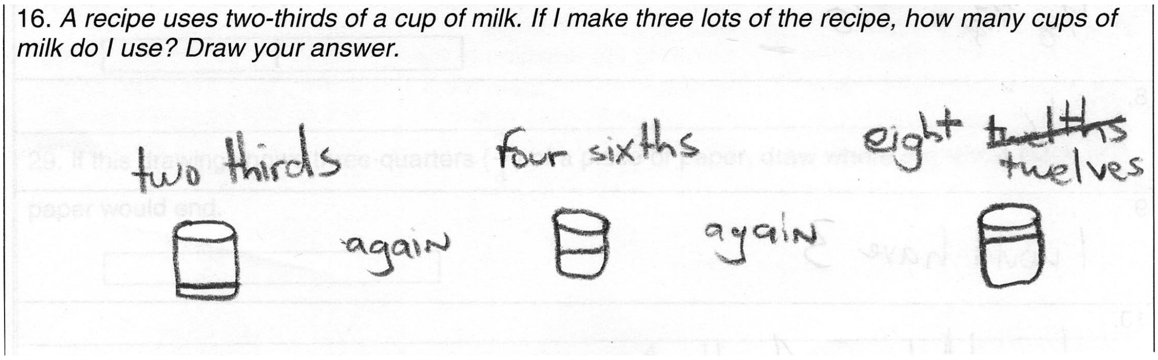



**Figure 5.45** Addition of fractions replaced by "magic doubling"

Perhaps influenced by an appreciation of the whole numbers used in recording fractions, the response in Figure 5.45 also records the decision to replace the fraction ordinal term "twelfths" with the cardinal term "twelves".

**Looking back**

In Questions 15 and 16, having a ratio interpretation of a fraction (3:4 in Question 15 and 2:3 in Question 16) gave rise to what would be considered to be equivalent fractions when doubling or finding three lots of a fraction ($\frac{6}{8}$ as double $\frac{3}{4}$ and $\frac{6}{9}$ as three lots of $\frac{2}{3}$). Even when the referent unit (a cup of milk) is well defined within the question, the evoked concept images included inappropriate additive interpretations of the problem context. The decline in performance in Years 7 and 8 suggests that learning fraction algorithms might contribute to poorer performance on contextual fraction problems. Aggregating fractions

[12] The term 'magic doubling' is sometimes applied to a significant type of erroneous strategy students employ with proportional reasoning (Misailidou & Williams, 2003). The term is only used in a descriptive sense here, as repeated doubling is clearly inappropriate.

beyond the whole also acted as a barrier with over 10% of Year 7 and 8 students failing to correctly aggregate individual fractions.

## 5.7 Representing measurement division by fractions: Q17-18

In a sense, a quantitative feel for fractions must rely upon the division process, as discrete and continuous embodiments of the fraction concept are the product of a process of division or subdivision. Question 17 explored students' capacity to use fractions with the division process.

### *Question 17*

In Question 17, students were given a fractional unit, one-third of a cup of milk, and asked to determine how many lots of that unit corresponded to six whole cups of milk, a measurement division process. The percentages of students who answered the question (correctly or incorrectly) with correct or incorrect reasoning are shown in Table 17.

Table 17

*Percentage distribution of responses to Question 17, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Right answer/right reason (1) | 24 | 43 | 63 | 41 | 55 |
| Wrong answer/right reason (2) | 8 | 3 | 6 | 2 | 6 |
| Right answer/wrong reason (3) | 2 | 2 | 2 | 4 | 2 |
| Wrong answer/wrong reason (4) | 57 | 49 | 27 | 47 | 31 |
| Omitted | 9 | 3 | 2 | 6 | 6 |

Overall, 50% of students' answers were attributed to correct reasoning involving the fraction units, including incomplete or incorrect answers (Category 2). Of the 4.9% of students who did not answer the question correctly although their reasoning about units was correct, the majority used the correct unit and number of units but either did not accumulate the units or made an error accumulating them. In determining the number of one-third cups in six cups some students subdivided the six cups into thirds, using either standard or non-standard partitioning of the cups. The common prototypical image of one-third of a circle emerged as an example of a non-standard partitioning in the Year 4 student's response to Question 17 (see Figure 5.46). The multiplicative link between the number of thirds in each cup and the fraction measurement division can be seen in the progressive sum of multiples of three.

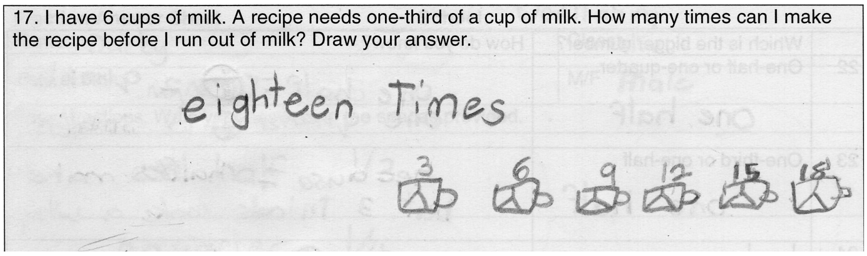



**Figure 5.46** A Year 4 response showing each cup holds three thirds

The application of the prototypical image for one-third of a circle is also evident in the Year 6 student's response in Figure 5.47. Unlike the Year 4 response in Figure 5.47 which records a multiple count, the Year 6 response shows each third enumerated.

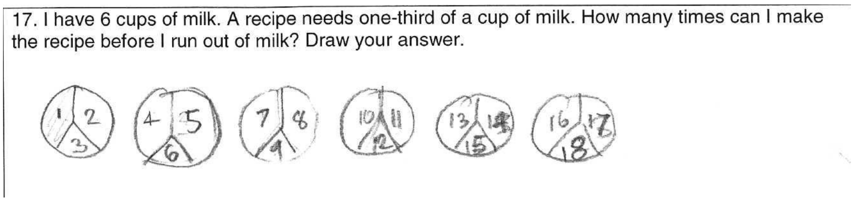



**Figure 5.47** A Year 6 response enumerating each one-third

Of course, not all representations used the difficult prototypical image of one-third of a circle. The Year 5 student's response shown in Figure 5.48 uses subdivision of rectangles to show one-third of a cup. Although the six cups, each showing three-thirds, is conveniently packaged into a 3 x 6 array structure, the individual thirds are numbered in succession. To have thought of constructing an answer to this division question in such a way, suggests a quite sophisticated appreciation of the multiplicative structure inherent in the question.

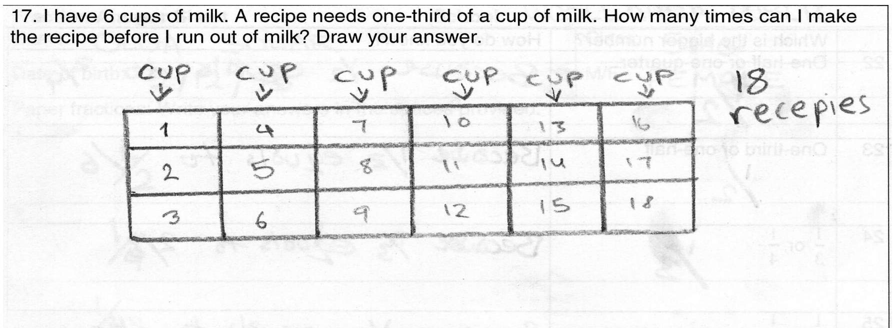



**Figure 5.48** A 3 by 6 rectangular array used with division

The response in Figure 5.49 does not explicitly make use of the multiplicative structure of an array but it does stress the equal subdivision of equal-sized units in forming six lots of three-thirds.

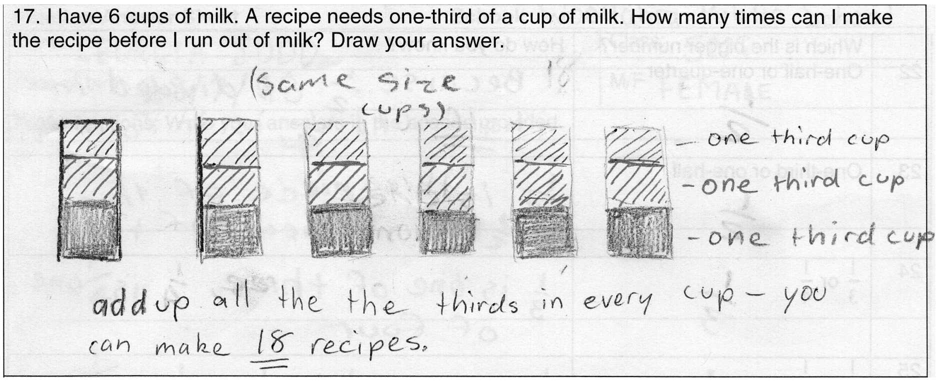



**Figure 5.49** Emphasising the equal wholes leading to equal thirds

The frequency of Category 3 (correct answer with incorrect or inadequate reasoning) was higher than in the previous questions because more students (34 altogether) provided the correct answer without any reasoning.

Students are introduced to the algorithmic approach to division by fractions in Year 7 and 8. However, having been taught the division algorithm for fractions did not appear to appreciably increase the likelihood of correctly answering Question 17. Fewer Year 8 students correctly answered the question than Year 6 students.

### *Question 18*

Question 18 also looked at quotitive or measurement division, this time using three-quarters as the unit of measure. That is, students were told that a recipe uses three-quarters of a cup of milk and were then asked to determine how many times the recipe can be made using 6 cups of milk. The percentages of students who answered the question (correctly or incorrectly) with correct or incorrect reasoning are shown in Table 18.

Table 18

*Percentage distribution of responses to Question 18, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Right answer/right reason (1) | 17 | 35 | 51 | 28 | 37 |
| Wrong answer/right reason (2) | 6 | 5 | 8 | 8 | 14 |
| Right answer/wrong reason (3) | 2 | 2 | 1 | 2 | 1 |
| Wrong answer/wrong reason (4) | 60 | 53 | 37 | 49 | 39 |
| Omitted | 15 | 5 | 4 | 14 | 9 |

In total, 33% of answers were allocated to Category 1 (correct answer and reason) and another 2.0% of students provided a correct answer with no diagram or explanation. Students who answered the question correctly frequently divided the six cups into quarters before determining the number of three-quarter units required. Indeed, 6.5% of the students drew a partitioned diagram and numbered the quarters. For example, the Year 5 student whose response is shown in Figure 5.50 initially subdivided the six cups into quarters, then formed three-quarter composites from the quarter units and coded these as recognisable composite fractions.

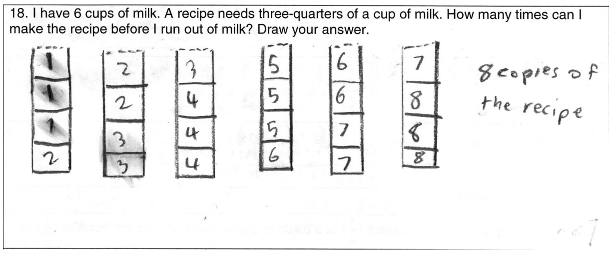



**Figure 5.50** Cups quartered and each successive three quarters numbered

Another way of working out the answer was to recognise that each whole cup contained three-quarters of a cup and one-quarter of a cup. Then, as the Year 6 student has done in Figure 5.51, the residual quarters are grouped to form two new three-quarter units.

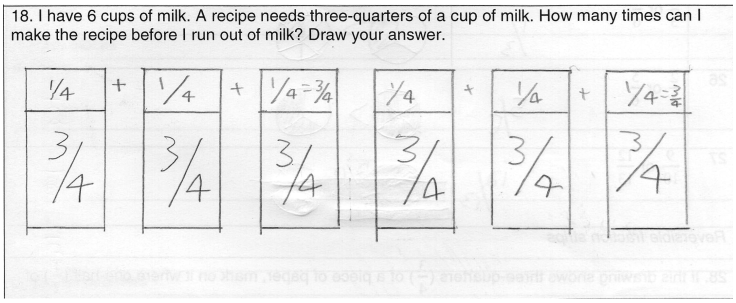



**Figure 5.51** Each cup as $\frac{3}{4}$ and $\frac{1}{4}$ with the quarters aggregated

Of the 8.1% of students who did not answer the question correctly although their reasoning about units was correct (Category 2), the majority used the correct unit and number of units but either did not accumulate the units or made an error accumulating them.

Approximately one-half of all the answers were coded as incorrect. The diversity of incorrect answers made it difficult to immediately discern any patterns in the data. For example, the method depicted in Figure 5.52 suggests an incorrect but internally consistent strategy or evoked concept image.

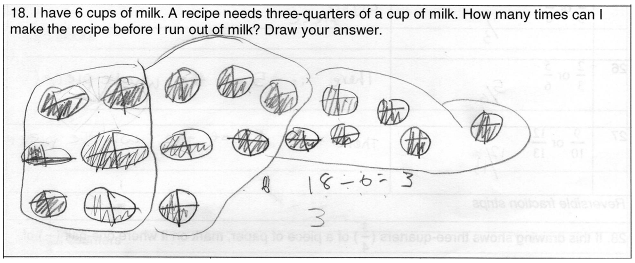



**Figure 5.52** An example of a reasoned incorrect answer

Presumably this response blended elements of multiplication and division involving the three-quarter unit. However, the interpretation of the strategy applied by this student in answering Question 18 depended upon a shared understanding of the symbolism used in the drawing. Logically, the drawing of circles with three-quarters shaded suggests representations of three-quarters of a cup. Unfortunately, examining this student's response to the previous question (Figure 5.53) shows the same three-quarters shaded circle was used to represent one-third. Multiplication is clearly used to arrive at an answer for Question 17 and this may have influenced the answer to Question 18.

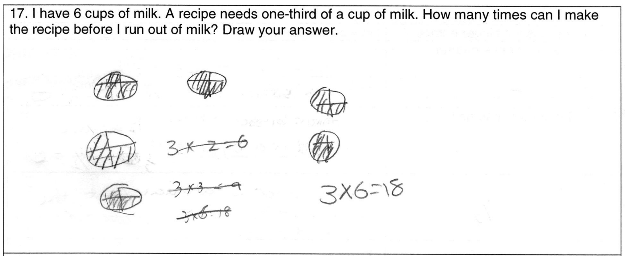



**Figure 5.53** A different meaning attributed to three-quarters of a circle shaded

When a count of incorrect answers was made for Question 18, 34 students had the incorrect answer "24". The students who simply partitioned each cup into quarters (Figure 5.54) reasonably provided this answer.

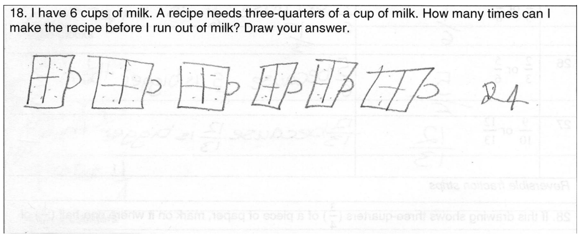



**Figure 5.54** A response showing counted quarters

Even when the individual three-quarters appear to have been constructed, as in Figure 5.55, the multiplicative relationship between quarters and the number of cups could still lead to an answer of 24. Just as in Figure 5.52 where 18 circles were drawn in response to a question involving 6 cups and *three*-quarter cups (6 x 3), the 24 vessels appear to result from the product of the 6 cups and the *quarters* (6 x 4). Indeed, close examination of the diagram suggests that the shaded portion of the cups corresponds to three scale units rather than three-quarters.

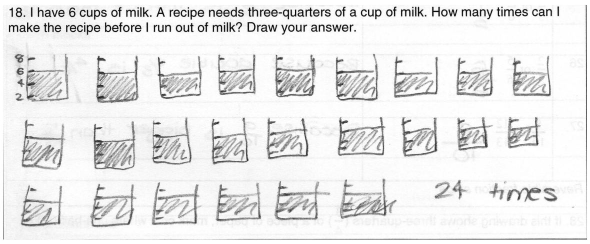



**Figure 5.55** A multiplicative response: (6 x 4) “three” units

Another 28 students had the incorrect answer “4 $\frac{1}{2}$”. They presumably interpreted the question as requiring multiplication and multiplied three-quarters by six (Figure 5.56).

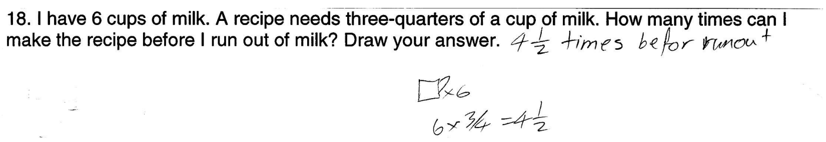



**Figure 5.56** Transforming the question to multiplication

Some students, such as the Year 7 student whose answer in shown in Figure 5.57, progressively accumulated three-quarter units to obtain an answer of 4 $\frac{1}{2}$.

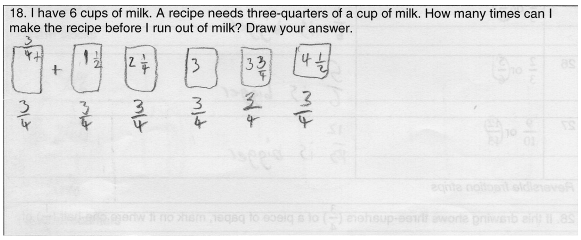



**Figure 5.57** A Year 7 response showing six lots of $\frac{3}{4}$

The greatest number of students who misinterpreted the required process was in Year 6, where 14 students answered “4 $\frac{1}{2}$”. Overall, 3.6% of students interpreted this measurement division context as requiring multiplication.

The omission rates for Years 4, 7, 8 were quite high. While it is to be expected that Year 4 would omit a question requiring division by a non-unit fraction due to lack of specific curriculum access, this should not apply to Years 7 and 8. It may well be lack of familiarity with the problem context that contributed to the higher than expected omit rates at that level. High school students are infrequently asked problems that require division by a fraction, although they may be asked to complete an algorithm requiring division by a fraction.

Comparing Tables 17 and 18, the introduction of three-quarters into the measurement (quotitive) division problem has resulted in a marked reduction in the percentage of correct answers. The use of a composite fraction with measurement division (quotition) might contribute to an increased level of challenge, as the question forms are the same. Although for continuous quantity one quarter is easier to construct than one-third, managing three-quarters in a quotitive division is clearly more difficult than managing one-third in the same context. Aggregating the three-quarter units is conceptually more difficult (see Figure 5.58) than aggregating the one-third units (see Figure 5.49). In fact, the accumulation of any composite fraction unit relies upon decomposition and continuous re-unitising.

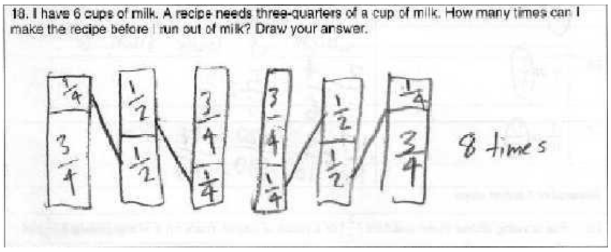



**Figure 5.58** Three-quarter units split and combined

Conceptually, the continuous representation in Figure 5.58 also appears a little different from subdividing each cup into quarters and identifying groups of three discrete units (Figure 5.50).

The difference in performance between Question 17 and Question 18 increased across the school years. In particular, the proportion of high school students arriving at the wrong answer after correctly partitioning the six cups more than doubled between Question 17 and 18. In Figure 5.59 the partitioning into quarters is visible in this Year 8 student's explanation, as well as some arithmetic. Using the shading as a guide, the correct answer is very close and interpreting the tally marks may well have been the stumbling block.

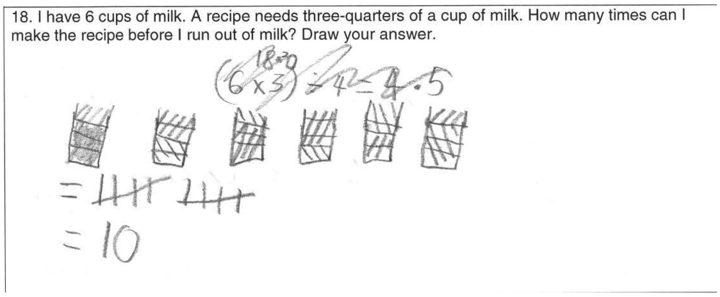



**Figure 5.59** Partitioning into quarters and losing track of the $\frac{3}{4}$ units

**Looking back**

In Questions 17 and 18, contexts requiring measurement division by a unitary fraction ($\frac{1}{3}$) and a composite fraction ($\frac{3}{4}$) challenged the students. Although 45% were able to successfully reason the number of one-third cups in six cups, the number who could similarly reason the number of three-quarter cups in six cups fell to 33%. The largest decrease in the percentage correct was in Year 8, which also saw an increase in the proportion of students arriving at the wrong answer after correctly partitioning the six cups. This problem of managing the accumulation of composite fractions may be the result of an increased focus on manipulation of integers associated with the fraction algorithms in high schools. Despite the absence of division contexts involving fractions from the Primary curriculum, at least 50% of Year 6 students involved in this study were able to confidently answer this type of question.

## 5.8 Locating positions corresponding to fractional parts on a continuous model: Q19–21

Questions 19, 20 and 21 required students to locate a position on an equilateral triangle that corresponded to a given fraction of the perimeter.

### *Question 19*

Question 19 asked the students to mark a point half way around an equilateral triangle, starting from the apex. The results for Question 19 are shown in Table 19.

Table 19

*Percentage distribution of responses to Question 19, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Correct | 83 | 92 | 94 | 91 | 94 |
| Incorrect | 12 | 8 | 6 | 9 | 4 |
| Omitted | 4 | 0 | 0 | 1 | 2 |

Over 80% of each grade correctly located a point half way around an equilateral triangle when the axis of symmetry was vertical. The selection of the vertex used as the starting point for the journey around the equilateral triangle appears to have a marked impact upon how students responded to the task. A similar question in a study by Clements and Del Campo (1987), using a different starting point (the bottom left vertex), produced a much lower percentage of correct answers than Question 19.

The category of incorrect answers included the responses of six students who marked multiple points on the triangle. The next question used the same context of an ant crawling around the outside of a triangle.

### *Question 20*

In Question 20 students were asked to determine where an ant would be if it started from the apex of an equilateral triangle and travelled clockwise one-third of the way around the perimeter. The results for are shown in Table 20.

Table 20

*Percentage distribution of responses to Question 20, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Correct | 37 | 51 | 66 | 49 | 57 |
| Incorrect | 58 | 47 | 33 | 49 | 41 |
| Omitted | 5 | 2 | 0 | 2 | 2 |

There was a marked reduction in the percentage of correct answers in each grade when students were asked to find one-third of the distance around an equilateral triangle rather than half way around. In the Clements and Del Campo (1987) study, less than 10% of all the children interviewed obtained the correct answer to locating one-third of the distance around the outside of an equilateral triangle. Indeed the vertex for two-thirds was nominated more often in their study than the vertex for one-third. Although the percentage able to correctly answer the question in this study was much higher with 52% of the students correctly answering the question, the percentage correct was much lower than the 91% who were able to locate half way around. Again, the different starting place appears to have influenced the increased percentage of correct answers. The direction of travel was marked on the triangles in Questions 19 to 21. Needing to hold a "trace" of the visual description of clockwise[13] may have added to the difficulty of identifying one-third in a clockwise direction from the bottom left vertex in the Clements and Del Campo study.

Although the percentage of each grade able to locate one-third of the distance around an equilateral triangle was comparable to the percentage able to shade one-third of a circle (Question 13), the incorrect methods students used to obtain answers were often equally surprising. In Figure 5.60, the Year 7 student used equi-distant partitioning of the equilateral triangle along a vertical axis to create thirds. This is similar to the horizontal partitioning and equi-distant partitioning discussed in relation to Questions 11 to 13.

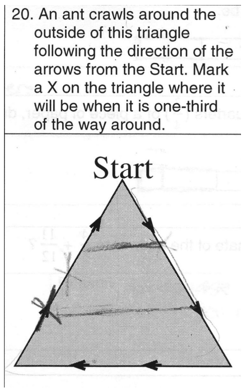



**Figure 5.60** Forming horizontal thirds

An alternative approach to horizontal partitioning was to divide the area of the triangle into three as in Figure 5.61.

[13] No arrows were used on the sides of the triangle in the Clements and Del Campo study.

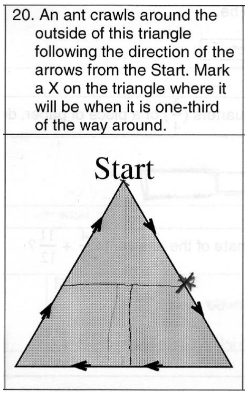



**Figure 5.61** Dividing the area into "thirds"

Both of these methods appear to align with processes associated with one-third of a regional model. The dividing lines used to create a "third" are horizontal in Figure 5.60 and, vertical and horizontal in Figure 5.61. The methods of attempting to find a distance one-third of the way around the triangle are reminiscent of the diagrams used to show one-third of a circle, a regional model of fractions, in Question 13 (cf. Figure 5.22). For example, in Figure 5.62 the Year 8 student's geometric subdivision of the triangle can be interpreted as the formation of "thirds" in a regional sense to determine the location.

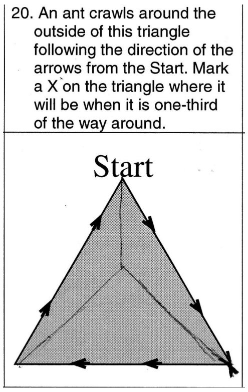



**Figure 5.62** A regional subdivision to find one-third of the perimeter

As students were not asked to show how they found where one-third of the way around was, it is possible that some incorrect methods were used to obtain correct answers.

## *Question 21*

Question 21 required students to determine the position corresponding to a distance of one-quarter of the way around an equilateral triangle, starting from the apex. The major response categories are summarised in Table 21.

Table 21

*Percentage distribution of responses to Question 21, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Correct | 19 | 31 | 35 | 25 | 30 |
| $\frac{1}{6}$ [a] | 30 | 31 | 31 | 36 | 32 |
| $\frac{1}{3}$ | 5 | 5 | 5 | 4 | 5 |
| $\frac{5}{6}$ | 6 | 4 | 3 | 4 | 2 |
| Other | 35 | 28 | 25 | 26 | 27 |
| Omitted | 5 | 2 | 1 | 4 | 4 |

[a] This response category includes 18 students who marked a location $\frac{1}{6}$ of the way around the triangle through a process of "quartering" the triangle using vertical and horizontal dividers (see Figure 5.65).

Finding one-quarter of three sub-units is a challenging question and overall only 28% of students could correctly determine the answer, even allowing generous tolerances in the range of acceptable answers. Those students who provided an indication of how they achieved the correct response, generally used one of two strategies. One method of determining the correct answer was to divide each side of the triangle in quarters, or more economically, to quarter the right-hand side of the triangle. The Year 6 students' answers in Figure 5.63 make use of three-quarters of one-third of the perimeter to determine the location of one-quarter of the perimeter.

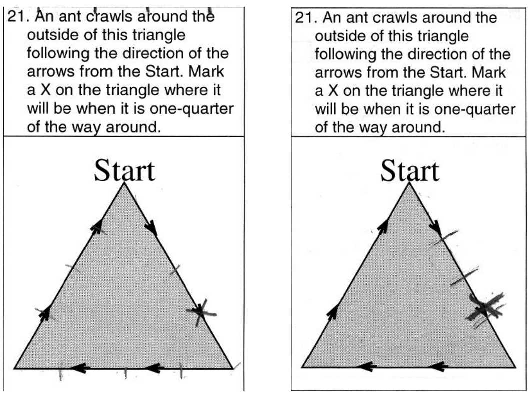



**Figure 5.63** Quartering each of the unit sides (left) and quartering only one side (right)

Another approach was to find half of one-half, recognising that half of the perimeter corresponded to one and a half sides. The response from the Year 6 student (Figure 5.64) divides 2 into 1.5 (sides) to determine an answer of .75 (sides). This is a sophisticated use of repeated halving while being aware of the unit of interest and moving flexibly between decimal and fraction representations.

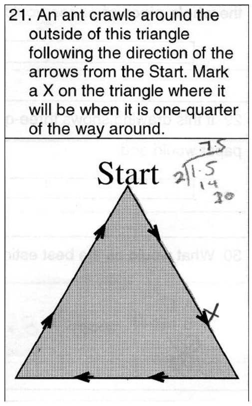



**Figure 5.64** Finding half of one and a half sides

In accepting a response as correct, allowance was made for natural perceptual variability of students' responses, whilst attempting to exclude students using a "visual quartering" of the shape. The act of visual quartering of the triangle continued to draw upon visual preferences in the vertical and horizontal directions. That some students quartered the shape rather than the perimeter can be seen from the construction lines evident in Figure 5.65. However, as many students may have used a visual quartering method and not shown the construction lines, such a response could not be distinguished from $\frac{1}{6}$, therefore this category includes quartering.

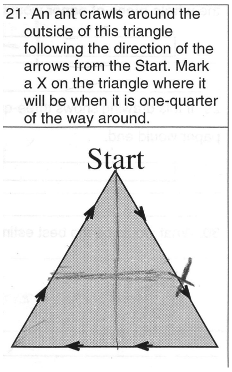



**Figure 5.65** A Year 7 response showing quartering the region

The tolerances used to determine an acceptable correct answer were set to exclude answers of one-fifth or less of the perimeter. That is, one-sixth formed by 'quartering' or otherwise marking half of the right-hand side, is excluded. The answer of $\frac{1}{6}$ was the most popular response with 32% of all responses included in this category. Although less than 2% of responses showed construction lines, the insight into the processes afforded by those who did was most helpful. Quartering the triangle contributed to the popularity of $\frac{1}{6}$ as the location marked and no doubt also influenced the number of responses coded as $\frac{5}{6}$.

With approximately 5% of each year indicating that the vertex corresponding to one-third would represent one-quarter of the distance around the equilateral triangle, it is tempting to dismiss this category of responses as the result of guessing. The vertices of the triangle are readily identified and would constitute a likely position for a guess. For those students who incorrectly answered Question 20, where the required answer was a distance of one-third of the perimeter, an answer of one-third to Question 21 is a plausible guess. Indeed, the pattern for the majority of the responses (83%) of those who answered one-third to Question 21 was to have incorrectly answered Question 20. However, 14 students correctly indicated the location corresponding to one-third in Question 20 and yet used the same location in Question 21 to show one-quarter of the way around the triangle. Although quite rare, some students respond as if they believe that one-third and one-quarter are the same.

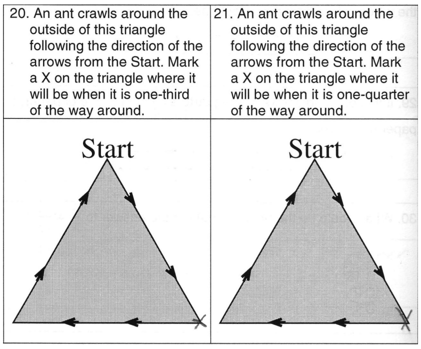



**Figure 5.66** One-third and one-quarter marked in the same position

The Year 7 student's response in Figure 5.66 records one-third of the way around the triangle as being the same position as one-quarter. Looking back at other responses from this student, the answer to Question 2 and Question 8 are both recorded as "$\frac{1}{3}$ – quater" (see Figure 5.67).

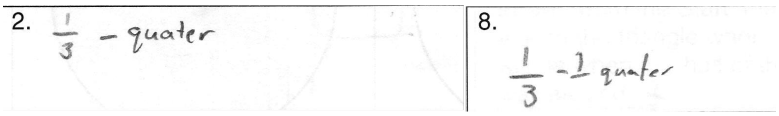



**Figure 5.67** The same response for $\frac{1}{4}$ of a strip of paper and 2 out of 8 counters

Even shading one-third of a circle (Figure 5.68) appeared as one-quarter.

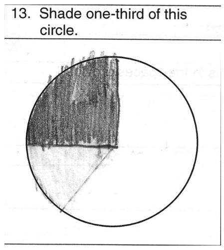



**Figure 5.68** The same Year 7 student shading a quarter for one-third

**Looking back**

In locating fractions of the perimeter of an equilateral triangle, general cutting methods appear to carry over from work with regional models. Vertical cutting to determine half way from the apex of an equilateral triangle produced a much higher percentage of success than locating half way obliquely (cf. Clements & Del Campo, 1987). The visual preference for vertical and horizontal cuts also appeared to influence the strategies some students adopted in locating one-third and one-quarter of the distance around an equilateral triangle. Quartering as a process of using vertical and horizontal cuts was not constrained only to subdividing regional models to partition area but also appeared when some students attempted to subdivide the perimeter of an equilateral triangle. There were also occasions where quite robust misconceptions had been formed, such as one-third and one-quarter being considered to be the same, which could only be detected by comparing a range of answers from the same student.

## 5.9 Justifying a quantitative comparison of fractions as mathematical objects

In Questions 22 to 27 students were asked to determine which of a pair of fractions was the bigger number and to justify their answers. The fractions were presented as mathematical objects—abstract quantity fractions—devoid of any context. The students' explanations gave rise to a number of descriptive categories, which seek to capture common strategies used in the explanations of how the relative size of the fractions was determined.

The first two Questions, 22 and 23, expressed both fractions in words. As the sequence of questions progressed, the fractions increased in abstractness in that as well as introducing the fraction notation, less familiar fractions were used. Although questions involving non-unit fractions were designed to evoke a change in solution method, a persistent application of one strategy was commonly noted. The descriptions of the major explanatory categories are included with the discussion of the questions where they were most evident.

### *5.9.1 Comparison of unit fractions (Questions 22 to 25)*

Question 22 presented the fractions to be compared in words, one-half and one-quarter, to keep the focus on the size of the fractions as quantities rather than fractions formed by pairs of numerals. Beyond identifying the bigger fraction students were asked to justify their answers. The results are presented in Table 22 organised by whether the answers were correct or incorrect, aligned with the apparent reasoning. In addition to the three major categories (correct answers with correct reasoning, correct answers with incorrect or inadequate reasoning, and incorrect answers with incorrect reasoning), four subcategories of correct answer with correct reasoning are outlined. No answers were identified for the category of incorrect answers with correct reasoning (Category 2) for this question.

Table 22

*Percentage distribution of responses to Question 22, by year*

| | | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|---|
| Right answer/right reason | (1) | | | | | |
| Calculated equivalence | | 21 | 27 | 39 | 29 | 28 |
| Regional model | | 19 | 21 | 21 | 16 | 18 |
| Bigger is smaller | | 0 | 2 | 5 | 4 | 6 |
| Number of parts/shares | | 4 | 8 | 6 | 6 | 7 |
| Right answer/wrong reason | (3) | 47 | 38 | 28 | 42 | 39 |
| Wrong answer/wrong reason | (4) | 5 | 4 | 1 | 3 | 2 |
| Omit | | 3 | 0 | 0 | 0 | 1 |

In total, 48% of the answers were allocated to the first category of correct answer with correct reasoning. Answers such as “one-quarter is half of one-half” or changing to a common denominator, converting to a per cent or decimal, or showing the relative location on a number line were all included in the first subcategory of “calculated equivalence” for right answer/right reason. When the reasoning made correct use of a regional model, the response was included in the regional model subcategory. Regional models were not always associated with correct answers. The Year 6 student’s response in

Figure 5.69 shows correct use of a regional model in Question 22 and incorrect use of a regional model with Question 23.

| | Which is the bigger number? | How do you know? |
|---|---|---|
| 22 | One-half or one-quarter<br>One-half | One half is bigger<br>One half One quarter |
| 23 | One-third or one-half<br>One-third | One-third is bigger<br>One third One half |

**Figure 5.69** Using a regional model correctly and incorrectly to justify answers

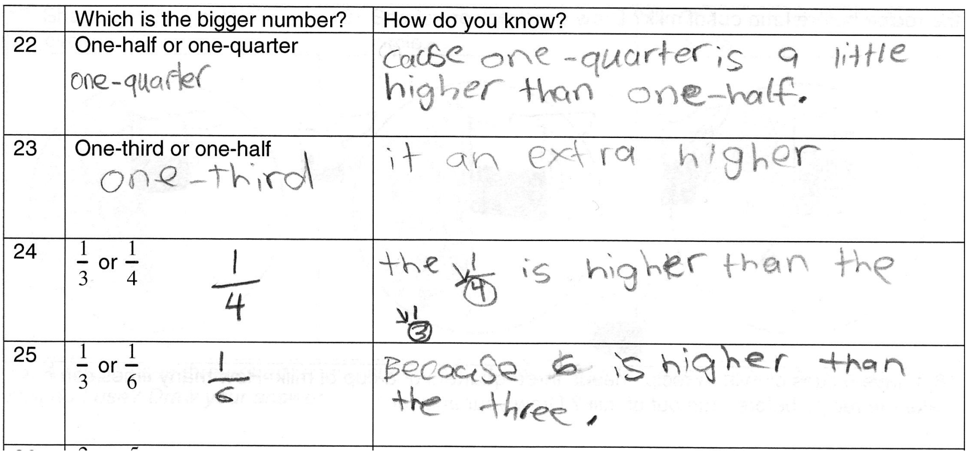
A number of students appealed to an inverse relationship essentially the same as one used in coding the reasoning in Question 11 (see also Table 4.3), and characterised as "bigger is smaller" (BIS). The category bigger is smaller was used to describe reasoning about the inverse relationship between the size of the denominator and the size of the fraction. For example, responses such as "the bigger the bottom number the lower it is" or "the smaller the bottom number the bigger" were included in this category. Bigger is smaller was coded as correct reasoning for comparisons involving unitary fractions. Responses that focused on the number of parts, such as "a half has only 2 parts and a quarter has 4" or "you are only cutting the whole thing one time instead of twice", were typical of the subcategory of "number of parts" (NOP). An example of the category of responses that referred to the number of parts or shares in response to "How do you know?" is provided in Figure 5.70. There is evidence of a transition of strategy in Figure 5.70 with the number of parts referred to in Question 22 (implying the smaller the unit of measure the more of the units needed), shifting in the following questions to an emphasis on the number of spaces remaining to be filled (a focus on the complement of the unit fraction). Students' solution methods often became clearer when considered across the range of questions.

| | Which is the bigger number? | How do you know? |
|---|---|---|
| 22 | One-half or one-quarter<br>one-half | ~~a~~ $\frac{1}{2}$ is 2 peices $\frac{1}{4}$ is 4 peices |
| 23 | One-third or one-half<br>One half | one half is 2 one ~~3~~ third is 3 |
| 24 | $\frac{1}{3}$ or $\frac{1}{4}$<br>$\frac{1}{3}$ | because $\frac{1}{3}$ only has ~~3~~ 2 more places to fill & $\frac{1}{4}$ has 3 spaces to fill. |
| 25 | $\frac{1}{3}$ or $\frac{1}{6}$<br>$\frac{1}{3}$ | $\frac{1}{3}$ has 2 spaces to fill $\frac{1}{6}$ has 5 to fill. |

**Figure 5.70** A progression in reasoning from the number of parts to empty spaces

The relatively large proportion of answers (39%) classified as being correct with wrong or inadequate reasoning was due to the large number of reasons that provided only a statement of fact (33% of all answers). These statements included “one half is bigger”, “it’s a fact”, “I just know”, “I worked it out”, “becose my teacher said so” and “Its common sense”. This type of reasoning was used by 41% of Year 4 students, with the remaining 3% making up the category of right answer/wrong reason not providing any justification at all for the choice of one-half. Many students did not appear to be familiar with the need to provide mathematical backing for their answers.

A direct relationship between the size of the denominator and the size of the fraction was described as “Whole number in the denominator” (WND) and led to incorrect answers in Question 22 and 23. A focus on the size of the whole number in the denominator was not commonly observed in Question 22 and, when it was, contributed to the category of the wrong answer with an incorrect reason. The Year 5 student’s response in Figure 5.71 is an example of equating the size of the fraction with the size of the number in the denominator. This strategy is described as whole number denominator because the focus appears to be only on the size of the whole number in the denominator, and to distinguish this strategy from others that incorporated the numerator. This intense focus on the denominator is made clear in the Year 5 student’s subsequent responses where the denominators are circled and indicated with an arrow.



| | Which is the bigger number? | How do you know? |
|---|---|---|
| 22 | One-half or one-quarter<br>one-quarter | Cause one-quarter is a little higher than one-half. |
| 23 | One-third or one-half<br>one-third | it an extra higher |
| 24 | $\frac{1}{3}$ or $\frac{1}{4}$ $\quad \frac{1}{4}$ | the $\frac{1}{4}$ is higher than the $\frac{1}{3}$ |
| 25 | $\frac{1}{3}$ or $\frac{1}{6}$ $\quad \frac{1}{6}$ | Because 6 is higher than the three. |

**Figure 5.71** Reasoning based on the size of the whole number in the denominator

### *Question 23*

Question 23 compared the size of one-third and one-half as mathematical objects, with the fractions expressed in words. The results for Question 23 are shown in Table 23, organised by category. A small number of responses were identified where the wrong answer was determined despite correct reasoning, giving rise to some examples from the category of wrong answer/right reason.

Table 23

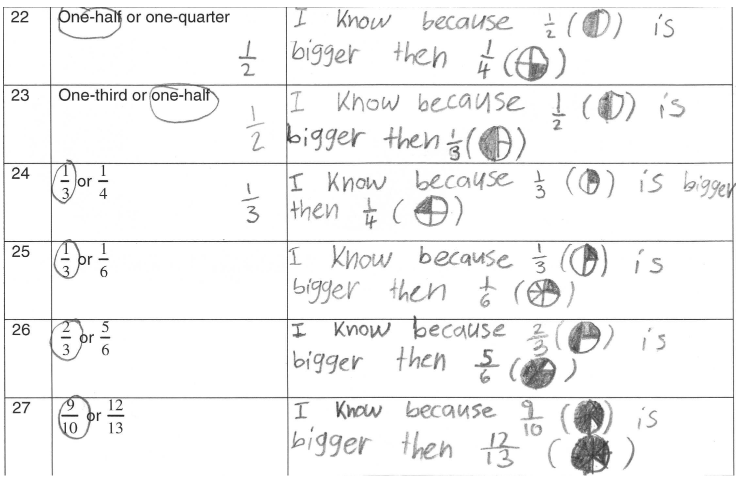
*Percentage distribution of responses to Question 23, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Right answer/right reason (1) | | | | | |
| Calculated equivalence | 5 | 13 | 21 | 4 | 7 |
| Regional model | 14 | 18 | 21 | 11 | 15 |
| Bigger is smaller | 1 | 3 | 5 | 4 | 6 |
| Number of parts/shares | 5 | 7 | 10 | 9 | 10 |
| Wrong answer/right reason (2) | 0 | 0 | 0 | 1 | 1 |
| Right answer/wrong reason (3) | 45 | 39 | 32 | 55 | 44 |
| Wrong answer/wrong reason (4) | 26 | 18 | 11 | 16 | 14 |
| Omit | 4 | 1 | 1 | 1 | 3 |

Although the total percentage of students who selected the larger of the two fractions was quite high in Question 23, so too was the percentage of answers that lacked correct or adequate reasoning to support the answer. Overall, 37% provided a correct answer with a sound supporting reason, with the most common correctly reasoned response—16% of all students—making use of a regional model. Where it may have been reasonable to expect a greater use of equivalent fractions (part of the subcategory of calculated equivalence) in reasoning by Year 7 and 8, there was actually a reduction from the previous question in the use of processes to create equivalent fraction forms. The subcategory of calculated equivalence dropped from 28% for Year 7 and 8 in Question 22 to 6% in Question 23. The strategy of converting to a common denominator to compare fractions was not the method of choice of high school students to compare one-third and one-half. Quite a number of explanations in Years 7 and 8 relied more upon sense-making than learnt procedures. For example, the Year 8 student's explanation, "because if a number is divided 3 times it will have a lesser value than if it is divided 2 times", demonstrated an appreciation of the relationship between fractions and division. Similarly, another Year 8 student stated "$\frac{1}{3}$ goes into 1 3 times, $\frac{1}{2}$ goes into 1 2 times". This answer was categorised as using the number of parts or shares to determine the larger fraction. The overall application of reasoning that the bigger the denominator was the smaller the fraction would be, corresponded to 3.8% of answers.

A small number of students (0.5%) formed correct regional models of the fractions but then recorded the wrong fraction as the larger of the two. The inclusion of the category wrong answer/right reason is justified on the basis of logical completeness of the categories rather than relative number of responses. Any incorrect application of a regional model that resulted in a correct answer was included in the category of right answer/wrong reason. That is, an incorrect regional model, such as representing $\frac{1}{3}$ incorrectly, was considered to be part of incorrect reasoning. Responses that used incorrect regional models were initially coded in a separate subcategory before recoding on the basis of whether the incorrect representations led to correct or incorrect answers. Simple statements of fact, such as "one half is bigger than one-third", were considered to be inadequate reasoning and included in the category of right answer/wrong reason.

For this question, the regional model of fractions became less reliable than in Question 22 as a means of obtaining the correct answer. Incorrect regional models could result in either fraction being declared the larger. In Figure 5.72, the Year 4 student used circles to represent fractional parts. However, the argument she made was based on the number of parts representing the fraction and not the area. For example, in Question 23 the area of one-third does not look less than the area of one-half of a circle.



| | | |
|---|---|---|
| 22 | One-half or one-quarter $\frac{1}{2}$ | I Know because $\frac{1}{2}$ ( ) is bigger then $\frac{1}{4}$ ( ) |
| 23 | One-third or one-half $\frac{1}{2}$ | I Know because $\frac{1}{2}$ ( ) is bigger then $\frac{1}{3}$ ( ) |
| 24 | $\frac{1}{3}$ or $\frac{1}{4}$ $\frac{1}{3}$ | I Know because $\frac{1}{3}$ ( ) is bigger then $\frac{1}{4}$ ( ) |
| 25 | $\frac{1}{3}$ or $\frac{1}{6}$ | I Know because $\frac{1}{3}$ ( ) is bigger then $\frac{1}{6}$ ( ) |
| 26 | $\frac{2}{3}$ or $\frac{5}{6}$ | I Know because $\frac{2}{3}$ ( ) is bigger then $\frac{5}{6}$ ( ) |
| 27 | $\frac{9}{10}$ or $\frac{12}{13}$ | I Know because $\frac{9}{10}$ ( ) is bigger then $\frac{12}{13}$ ( ) |

**Figure 5.72** An example of regional models used with number of parts interpretations

Sometimes the incorrect regional model did influence the reasoning based on area. The Year 6 student whose answer is shown in Figure 5.73 relied upon the evoked concept image for one-third to argue that one-half is bigger than one-third. This student's evoked concept image for one-third was also consistent with the image presented in an earlier question (Q13).

| | | |
|---|---|---|
| 23 | One-third or one-half<br>1/2 | because you need 2 one-thirds to make one-half |

**Figure 5.73** A response with an incorrect regional model supporting a correct answer

Occasionally the process of halving appeared to dominate a student's fraction knowledge. The Year 8 student's response in Figure 5.74 emphatically states that $\frac{1}{3}$ is half of $\frac{1}{4}$, a conclusion possibly reached by successive halving.

| | Which is the bigger number? | How do you know? |
|---|---|---|
| 22 | One-half or one-quarter<br>1/2 | ~~on~~ 1/4 is only half of 1/2 |
| 23 | One-third or one-half<br>1/2 | 1/3 is half of 1/4 and 1/4 is half of 1/2 |

**Figure 5.74** A Year 8 response describing one-third as half of one-quarter

A new feature of students' regional models started to appear in Question 23. Some students drew regional models to represent fractions using different shapes. The drawings suggest that these students didn't appreciate the need to use common shapes to form the unit whole. In Figure 5.75, the Year 7 student compared the area of half a circle to that of one-third of an equilateral triangle to conclude that one-half is a bigger number than one-third. The lack of recognition of the need for equality of the whole was described as "no equal whole" and coded as NEW. This response was included in the category of right answer/wrong reason.

| | | |
|---|---|---|
| 23 | One-third or one-half<br>1/2 | 1/3 |

**Figure 5.75** Regional comparisons without an equal whole

As in Question 22, focusing on the size of the whole number in the denominator (WND) was not commonly observed with one per cent or less of each year group using this strategy to arrive at an incorrect answer. The low level of use of WND may have been influenced by the fractions still being presented as words. Indeed, sometimes the reference to the whole number identified by the denominator, treated the denominator as if it were a unit in another system. For example, some of the recorded reasons in response to Question 23 made reference to one-third being equal to *three*-quarters or *three*-eighths. This reference to one-third being equal to three-quarters or three-eighths could be explicit, such as the Year 6 student who stated "$\frac{1}{3}$ is $\frac{3}{4}$, $\frac{1}{2}$ is $\frac{2}{4}$", or implicit, such as the Year 5 student who states "one third is a quater bigger than a half". This equating of one-third with three fraction units is a *number of parts* interpretation of fractions where the name of the parts is quarters or eighths. The focus on the number of parts rather than the size of those parts reinforces the interpretation identified in responses to earlier question.

### *Question 24*

Question 24 introduced the standard fraction notation for the first time in asking students to determine which is the bigger number, $\frac{1}{3}$ or $\frac{1}{4}$. The results for Question 24 are shown in Table 24, organised by category.

Table 24

*Percentage distribution of responses to Question 24, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Right answer/right reason (1) | | | | | |
| Calculated equivalence | 6 | 13 | 26 | 6 | 11 |
| Regional model | 15 | 15 | 19 | 11 | 16 |
| Bigger is smaller | 2 | 8 | 7 | 7 | 8 |
| Number of parts/shares | 2 | 5 | 5 | 8 | 6 |
| Wrong answer/right reason (2) | 0 | 0 | 0 | 1 | 1 |
| Right answer/wrong reason (3) | 24 | 25 | 24 | 34 | 35 |
| Wrong answer/wrong reason (4) | | | | | |
| Whole number denominator | 12 | 8 | 2 | 6 | 2 |
| Regional model | 3 | 3 | 4 | 2 | 1 |
| Other | 31 | 20 | 12 | 24 | 18 |
| Omit | 5 | 2 | 1 | 1 | 2 |

Although expressing equivalent forms of the two fractions, such as describing $\frac{1}{3}$ and $\frac{1}{4}$ in terms of twelfths, increased in Question 24, using regional models remained the dominant method of correctly answering the question and accounted for 15% of all answers. Students reasoning that the bigger the denominator was, the smaller the fraction would be, increased to 6.6% with the introduction of the two-number fraction notation. The overall use of regional models (correct or incorrect) remained at a similar level to Question 33 with 15% of all students correctly answering this question using a regional model to justify their answer.

Again the category corresponding to an incorrect answer with correct reasoning (Category 2) was added to cover the four logical categories of answers as only 0.5% of answers fell into this category. All but three responses in the wrong answer/right reason category used regional models. Approximately one-quarter of the students in Years 4–6 obtained the correct answer for the wrong reason and this rose to over one-third of the students in Years 7–8. In total, 5.1% of the students provided a correct answer without any reasoning, a decline from the 6.7% who omitted recording a reason in Question 23. Similarly, 5.6% of the students provided the wrong answer without any reasoning, whereas 6.1% of the students arrived at the wrong answer by equating the size of the fraction to the size of the whole number in the denominator. The introduction of the formal fraction notation into the questions coincides with a marked increase in reasoning about the size of a fraction being based directly on the size of the denominator. Approximately 1% of each year group used this reasoning to determine the wrong answer to Question 23 whereas in Question 24, whole number denominator reasoning corresponds to over 6% of all answers.

### *Question 25*

In Question 25, which asked students to determine which fraction is bigger, $\frac{1}{3}$ or $\frac{1}{6}$, the denominators of the two fractions are clearly linked in that one is double the other. The results for Question 25 are shown in Table 25, organised by category.

Table 25

*Percentage distribution of responses to Question 25, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Right answer/right reason (1) | | | | | |
| Calculated equivalence | 7 | 19 | 32 | 9 | 13 |
| Regional model | 12 | 11 | 18 | 9 | 13 |
| Bigger is smaller | 2 | 7 | 6 | 10 | 8 |
| Number of parts/shares | 2 | 3 | 5 | 5 | 6 |
| Wrong answer/right reason (2) | 1 | 0 | 0 | 0 | 0 |
| Right answer/wrong reason (3) | 25 | 28 | 22 | 39 | 42 |
| Wrong answer/wrong reason (4) | | | | | |
| Whole number denominator | 11 | 10 | 3 | 7 | 3 |
| Regional model | 2 | 3 | 3 | 2 | 1 |
| Other | 31 | 18 | 10 | 18 | 11 |
| Omit | 5 | 2 | 1 | 1 | 3 |

Overall 16% of the students correctly used equivalent fractions in their reasoning. This method was concentrated in the Year 6 students' responses with almost one-third of the Year 6 students correctly using equivalent fractions to reason that one-third was larger than one-sixth. This high level of correct usage of equivalent fractions in Year 6 compared to Year 7 and Year 8 strengthens the likelihood that Year 6 was a more able sample than the Year 7 sample. Using regional models was no longer the dominant method of correctly answering the question although it still accounted for almost 13% of all answers. A correct number of parts argument described the inverse relationship between the number of parts and the size of the parts coming from a common whole, and accounted for 4.5% of student responses.

Some responses, such as the Year 6 student's response in Figure 5.76, used drawings of regional models as well as some principled reasoning.

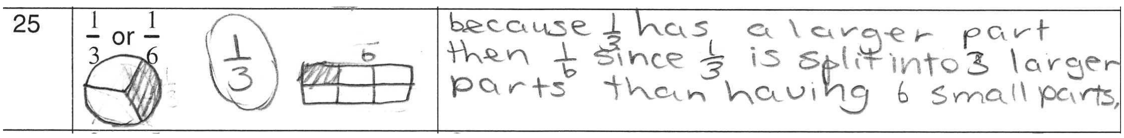



**Figure 5.76** Applying the compensatory principle without equal wholes

Responses that displayed a lack of the need for equality of the whole when using a regional model (see Figure 5.76) were included in the right answer/wrong reason category. Although the use of the rectangle and the circle with the regional models suggest that the student had no appreciation of the need for an equal whole, the explanation suggests that to her the shape used with the regional model may simply have been an irrelevant feature.

The use of the size of the whole number in the denominator as representative of the size of the fraction in Question 25 was strongest in Year 4. Students would be less likely to encounter the fractions $\frac{1}{3}$ and $\frac{1}{6}$ in their class work in Year 4 than in later years. Overall, the use of WND accounted for 6.7% of all answers to Question 25. Representations of fractions based on discrete items may also have contributed to direct comparison between the size of the denominator and the size of the fraction (WND).

| | | |
|---|---|---|
| 24 | $\frac{1}{3}$ or $\frac{1}{4}$ | $\frac{1}{4}$ because it is one out of 4 and the other is 1 out o 3 |
| 25 | $\frac{1}{3}$ or $\frac{1}{6}$ | 1/6 because it is one out of six and the other is 1 out of three |

**Figure 5.77** An explanation suggesting the size is linked to WND

The reasoning of the Year 7 student in Figure 5.77 suggests that a "one out of…" description, frequently associated with models used in teaching fractions, may aid the development of the idea that the fraction grows in size directly as the denominator increases. This link between the discrete model and whole number in the denominator appears more clearly in the Year 4 student's reasoning in Figure 5.78 and it seems reasonable to consider that representations of fractions involving discrete items may contribute to the belief that the size of a fraction is directly described by its denominator.

| | | |
|---|---|---|
| 25 | $\frac{1}{3}$ or $\frac{1}{6}$<br>$\frac{1}{6}$ | 6 is larger than 3<br>●○○○○○ ●○○ |

**Figure 5.78** A representation of a discrete model and WND

Sometimes regional diagrams were used to display a number of parts interpretation of the regional model. In Figure 5.79 the circle does not appear to link to a regional representation of fractions. The number of units marked is used as an indication of the denominator only.

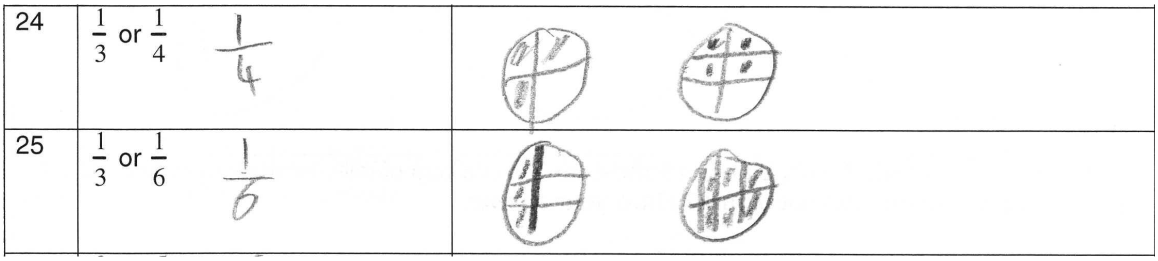



**Figure 5.79** A response matching the denominator with a number of parts in a circle

As well as using a continuous quantity to represent the relationship between a part and the whole of which it is part, fractions can be represented using predominantly identical discrete objects. The discrete model of fractions may also contribute to the development of a "difference in the count" as a way of comparing the size of fractions. For example, in comparing one-third and one-sixth, a student can draw and shade one of three compared to one of six, as in Figure 5.80.

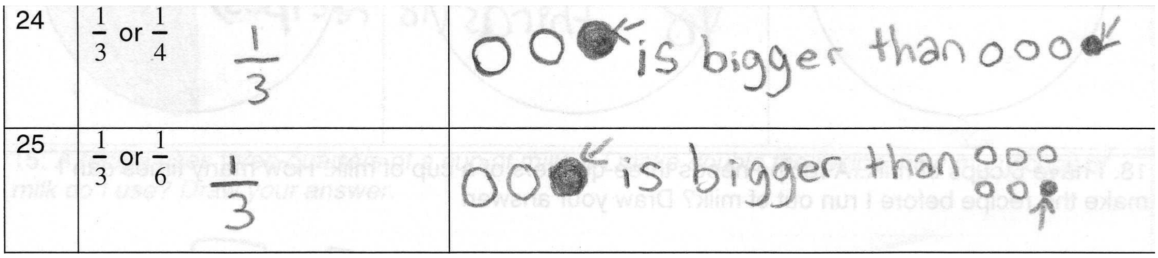
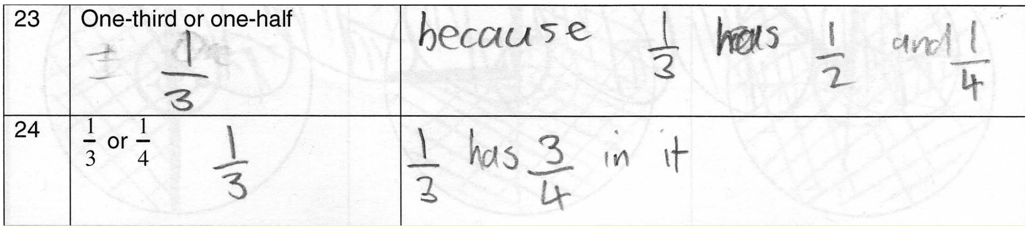



**Figure 5.80** An example of comparing fractions with a discrete model

How then, do you make a comparison when equal wholes are not apparent? Comparing unitary fractions using discrete models may emphasise the size of the complement. The one (part) remains constant while both the whole and the complement (non-shaded elements) continue to grow. The difference in the count of the discrete items can influence, if not become, the relational number for some students, although in this case the size of the individual units also varies.

**Looking back**

In examining responses across questions it is clear that some strategies can be consistently applied in different fraction contexts. Some of the strategies, such as bigger is smaller and whole number denominator, were the same methods identified in reasoning involving discrete representations of fractions in Question 11. Some students also appear to have used unusual fraction equivalents for one-third, such as three-eighths. For example, in comparing the size of one-half and one-third a Year 8 student stated "$\frac{1}{2}$ is bigger because $\frac{1}{3}$ is one and a half quarters". Some secondary students also reasoned that one-half is larger than one-third because "one-third is three-quarters of one-half". Other responses to Questions 23 and 24 confirmed that some students believe that one-third is the same as three-quarters. The Year 8 student's reasoning in Figure 5.81, indicates a belief that one-third is the same as three-quarters which is consistent with the same student's earlier response, showing three-quarters of a circle shaded for one-third.

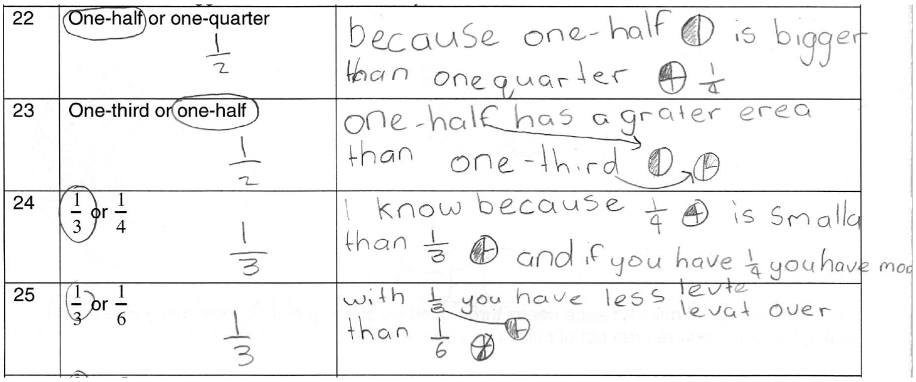


| | | |
|---|---|---|
| 23 | One-third or one-half $\frac{1}{3}$ | because $\frac{1}{3}$ has $\frac{1}{2}$ and $\frac{1}{4}$ |
| 24 | $\frac{1}{3}$ or $\frac{1}{4}$ $\frac{1}{3}$ | $\frac{1}{3}$ has $\frac{3}{4}$ in it |

**Figure 5.81** A Year 8 response treating $\frac{1}{3}$ as equivalent to $\frac{3}{4}$

Those students who focus on the number of parts can interpret one-third of a circle as three-eighths (i.e., three parts corresponds to one-third). Misconstruing three-eighths as one-third can be reinforced by the visual similarity of one-third of a circle and three-eighths of a circle. Students who focus on the number of parts could equally interpret one-third of a circle as three quarter parts.

Inaccurate regional models of one-third could still result in correct conclusions (Figure 5.82), which would contribute to inaccurate concept images being reinforced.



| | | |
|---|---|---|
| 22 | One-half or one-quarter $\frac{1}{2}$ | because one-half is bigger than one quarter $\frac{1}{4}$ |
| 23 | One-third or one-half $\frac{1}{2}$ | one-half has a grater erea than one-third |
| 24 | $\frac{1}{3}$ or $\frac{1}{4}$ $\frac{1}{3}$ | I know because $\frac{1}{4}$ is smalla than $\frac{1}{3}$ and if you have $\frac{1}{4}$ you have mo |
| 25 | $\frac{1}{3}$ or $\frac{1}{6}$ $\frac{1}{3}$ | with $\frac{1}{3}$ you have less levat over than $\frac{1}{6}$ |

**Figure 5.82** Incorrect regional models combined with spurious reasoning

All of the first four of the fraction comparison questions (Q22–Q25) used fractions that students had encountered in earlier questions and so students' evoked concept images could be cross-referenced to these earlier questions. The evoked fraction concept images were frequently used inconsistently. For example, in Question 13, a Year 8 student constructed one-third as a segment of a circle yet ten questions later the same student accessed an essentially correct regional model for one-third as a sector and used it to compare the size of fractions (Figure 5.83).

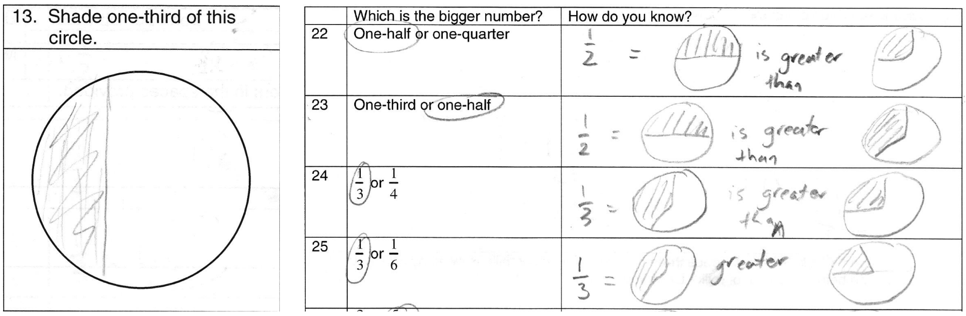



**Figure 5.83** Different evoked regional models of one-third

Inconsistent concept images such as $\frac{1}{3}$ being both half of $\frac{1}{4}$ and half of $\frac{1}{6}$ were also presented without apparent tension in the minds of students.

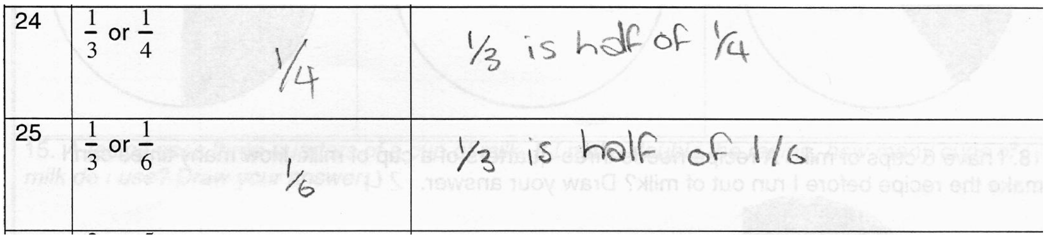



**Figure 5.84** A Year 8 response indicating $\frac{1}{3}$ was half of two different fractions

As well as attending to non-regional features of regional models, students would also evoke different concept images of the same fraction at different times in the same test. Incorrect regional models also contributed to the evoked concept images of many students. However, incorrect regional models exist beside correct regional models in students' fraction concept images, as in Figure 5.85.

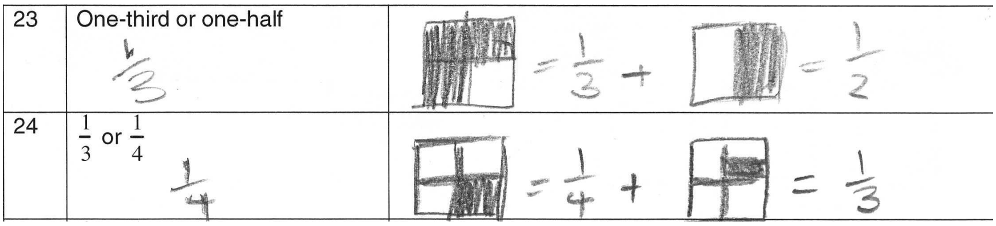



**Figure 5.85** An example of correct and incorrect regional models

Earlier questions influenced the evoked concept images used by some students in later questions. Even the difficult representation of one-third and one-half as distance around an equilateral triangle was used in a student's explanation. In Figure 5.86 a Year 6 student compared one-half of the perimeter of an equilateral triangle to one-third of the perimeter.

| | | |
|---|---|---|
| 23 | One-third or one-half  | Because 1 half has because if you had a triagle 1 half would be a line and a half but $\frac{1}{3}$ would be 1 line. |

**Figure 5.86** Comparing one-half and one-third of the perimeter of a triangle

In general, using two different shapes to compare fractions may indicate more than a lack of appreciation of the need for equal wholes to compare quantity fractions. It may also mean that for some students, the shape used with a regional model is not a significant feature of the fraction concept image. For the student who compared one-third of a circle to one-sixth of a rectangle (Figure 5.76), the regional model may have only contributed to the idea that the greater the number of parts, the smaller those parts are. Clearly this compensatory principle applies irrespective of the shape used as can be seen in the same Year 6 student's representation of one-third and one-sixth (see Figure 5.87).

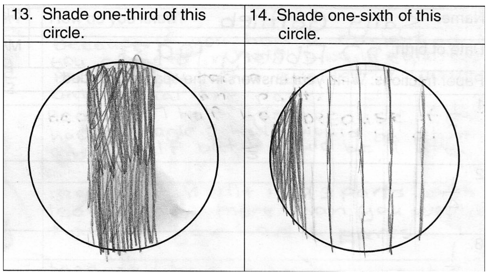



**Figure 5.87** Linear partitioning of a circle showing increasing the number of parts

The number of parts may become the dominant feature of the fraction concept image. In fact, the compensatory principle need not refer to area at all, but rather may be applied to linear subdivision. If the regional model leads to abstracting only the compensatory principle of measurement (the smaller the unit of measure the more of the units needed), the particular shape used may not matter. The compensatory principle applied without accessing an appreciation of the "universal whole" or "universal form of one" is not sufficient to enable students to appreciate fractions as mathematical objects. This limited appreciation of fractions sees them remain more as partitioned fractions rather than quantity fractions.

### *5.9.2 Comparison of non-unit fractions (Questions 26 and 27)*

Both Question 26 and 27 were designed to provide opportunities for students to reason from the size of the complement. That is, students could determine which fraction is the larger number by reasoning about how much remained to make the unit. They could reason from regional models, create equivalent fractions or use the previous answer to determine how much remained.

### *Question 26*

Question 26 asked students to compare two non-unit fractions, the complements of the fractions used in Question 25. In Question 26, students were asked to determine which of $\frac{2}{3}$ and $\frac{5}{6}$ was the bigger number. The results for Question 26 are shown in Table 26, organised by category.

Table 26

*Percentage distribution of responses to Question 26, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Right answer/right reason (1) | | | | | |
| Calculated equivalence | 4 | 15 | 28 | 5 | 9 |
| Regional model | 8 | 12 | 17 | 8 | 14 |
| Size of complement | 0 | 1 | 3 | 3 | 2 |
| Wrong answer/right reason (2) | 2 | 2 | 1 | 1 | 1 |
| Right answer/wrong reason (3) | | | | | |
| Both larger numbers | 5 | 3 | 2 | 5 | 1 |
| Whole number numerator | 4 | 2 | 2 | 2 | 1 |
| Whole number denominator | 3 | 3 | 1 | 1 | 1 |
| Other | 46 | 40 | 27 | 40 | 32 |
| Wrong answer/wrong reason (4) | | | | | |
| Bigger is smaller | 4 | 4 | 3 | 6 | 5 |
| Fraction in difference | 1 | 2 | 3 | 6 | 4 |
| Regional model | 6 | 2 | 4 | 1 | 0 |
| Other | 12 | 12 | 8 | 20 | 24 |
| Omit | 6 | 2 | 1 | 2 | 4 |

Overall, 26% of the students worked out the correct answer and provided an acceptable explanation compared to 39% in Question 25. This reduction in performance must be attributed to the introduction of the non-unit fractions, as the denominators of the two fractions were the same in both questions. Again the performance of the Year 6 sample, with the highest percentage of any year using equivalent fractions, was much stronger than the Year 7 sample on this item. Apart from Year 4, where the proportion of students who could provide a reasoned correct answer using equivalent fractions or regional models reduced from 20% for Question 25 to 11% for Question 26, the major difference in the right answer/right reason category appeared to be the loss of the bigger is smaller and number of parts methods from correct reasoning.

Having determined that one-third was greater than one-sixth in Question 25, it was possible to use this to compare two-thirds and five-sixths. The description of the Year 7 student (Figure 5.88) with reference to "space to move" appears to draw from elements of a regional model.

| | | | |
|---|---|---|---|
| 26 | $\frac{2}{3}$ or $\frac{5}{6}$ | $\frac{5}{6}$ | because $\frac{1}{6}$ is smaller than $\frac{1}{3}$ but $\frac{5}{6}$ is bigger than $\frac{2}{3}$ because $\frac{1}{6}$ has less space to move. |

**Figure 5.88** Comparing the relative size of the complements of the fractions

In the Year 6 student's explanation in Figure 5.89, the fraction notation is used to convey the arithmetical link to the size of the complement.

| | | | |
|---|---|---|---|
| 26 | $\frac{2}{3}$ or $\frac{5}{6}$ | $\frac{5}{6}$ | $\frac{5}{6}$ is $1-\frac{1}{6}$ $\frac{1}{6}$ is smaller than<br>$\frac{2}{3}$ is $1-\frac{1}{3}$ $\frac{1}{3}$ |

**Figure 5.89** A Year 6 student's explanation of the size of the complement reasoning

The percentage of students using the size of the complement (SOC) in their reasoning was relatively small, particularly compared to those who recorded the correct answer with inadequate or incorrect reasoning. Excluding the group of students who did not provide any reason, 36% of all students provided an incorrect reason when identifying why $\frac{5}{6}$ was greater than $\frac{2}{3}$. The use of the unit (1) as a benchmark to compare the size of fractions was rare with less than 2% of all responses using the size of the difference from one in their reasoning.

For the first time, Category 3 (right answer/wrong reason) provided a number of clear erroneous strategies that resulted in the correct answer. As an extension of using the size of the whole number in the denominator of a fraction as a measure of its size (WND), some students argued that $\frac{5}{6}$ was bigger than $\frac{2}{3}$ because "5 & 6 are larger numbers than 2 & 3". This form of reasoning, which can build on the focus on the size of the denominator, was described as "both larger numbers" (BLN). Overall, 3.3% of responses used the argument that five-sixths was the larger fraction because both the 5 and 6 are larger than the 2 and 3 in two-thirds. Further, 8.4% of all of the responses provided the larger of the two fractions without any reasoning.

Those students who considered the size of a fraction to be determined by the size of the numerator only were categorised as "whole number numerator" (WNN). A Year 8 student's reasoning that "2 of something is always less than 5 of something" is typical of reasoning focusing only on the size of the numerator. The whole number numerator reasoning and indeed most methods of whole number reasoning were most prevalent in Year 4 and usually declined across the years, with 2.1% of all responses classified in this subcategory. The focus on the size of the whole number in the denominator corresponded to 1.7% of all responses and also declined in the secondary school years.

The reasoning subcategory bigger is smaller accounted for 4.4% of all answers and was the main incorrect stated reason leading to the wrong answer. For example, a Year 5 student argued that $\frac{2}{3}$ was bigger than $\frac{5}{6}$ because "thirds are bigger than sixths which means there's more area in a third than in a sixth" which is essentially a bigger is smaller argument as it only addresses the size of the denominator.

The idea that the fraction is defined by the difference between the numerator and the denominator was described as "fraction in the difference" (FID). In Question 26, faced with two non-unit fractions where the numerator was one less than the denominator, 3.0% of all of the students concluded that the fractions were the same.

| | | |
|---|---|---|
| 26 | $\frac{2}{3}$ or $\frac{5}{6}$ Same Same | They both only have 1 space to fill |
| 27 | $\frac{9}{10}$ or $\frac{12}{13}$ same | they both have one place to fill. |

**Figure 5.90** Neither fraction taken to be bigger

Rather than reducing in the secondary school years, the number of responses within this subcategory (FID) actually increased in Years 7 and 8. While only 1% in Year 4 argued that the size of the fraction was articulated by the difference between the numerator and the denominator (FID), this increased to 6% in Year 7. It may be that having been taught to operate algorithmically with fractions in secondary schools has contributed to some students developing their own algorithmic approach to determining the size of fractions (i.e., finding the difference between the numerator and the denominator).

Some of the reasoning categorised as fraction in the difference (FID) recognised that the fractions were both one away from having the same numbers in the numerator and denominator. For example, it is tempting to read into the notation (Figure 5.91) an appreciation that $\frac{3}{3}$ and $\frac{6}{6}$ are both one whole and to question why the sizes of the pieces determined by the denominators have been ignored. However, nowhere does the student indicate that $\frac{3}{3}$ or $\frac{6}{6}$ are one whole.

| | | |
|---|---|---|
| 26 | $\frac{2}{3}$ or $\frac{5}{6}$ They are the Same | because you need 1 more on the top and you will get 3/3 6/6 |
| 27 | $\frac{9}{10}$ or $\frac{12}{13}$ They are the Same | because you need 1 more on the top and you will get 13/13 10/10 |

**Figure 5.91** Year 6 response indicating that the fractions are the same

Incorrect regional models combined with the wrong answer contributed to 2.7% of all responses.

### *Question 27*

The next pair of non-unit fractions to be compared was similarly constructed to differ in the numerator and denominator by one. Question 27 asked students to compare two non-unit fractions, $\frac{9}{10}$ and $\frac{12}{13}$, and determine which was the bigger number. The range of strategies observed in response to Question 26 was again observed with Question 27 (see Table 27).

Table 27

*Percentage distribution of responses to Question 27, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Right answer/right reason (1) | | | | | |
| Calculated equivalence | 1 | 8 | 18 | 1 | 4 |
| Regional model | 5 | 5 | 8 | 4 | 5 |
| Size of complement | 1 | 2 | 5 | 2 | 3 |
| Wrong answer/right reason (2) | 2 | 2 | 1 | 2 | 2 |
| Right answer/wrong reason (3) | | | | | |
| Both larger numbers | 6 | 3 | 2 | 6 | 2 |
| Whole number numerator | 2 | 2 | 2 | 2 | 1 |
| Whole number denominator | 3 | 5 | 1 | 2 | 1 |
| Other | 44 | 38 | 29 | 35 | 26 |
| Wrong answer/wrong reason (4) | | | | | |
| Fraction in difference | 2 | 4 | 5 | 10 | 6 |
| Bigger is smaller (BIS & BSN) | 1 | 4 | 2 | 4 | 4 |
| Size of denominator | 1 | 2 | 2 | 2 | 2 |
| Number of parts | 1 | 1 | 2 | 1 | 2 |
| Regional model | 5 | 5 | 5 | 1 | 2 |
| Other | 17 | 16 | 15 | 23 | 31 |
| Omit | 9 | 5 | 3 | 5 | 9 |

The overall percentage of correct answers with correct reasons fell from 26% in Question 26 to 14% in Question 27. As this question appeared more difficult than other questions in this sequence, the underlying ability differences of the Year 6 sample compared to the Year 7 sample appears amplified in the responses. Whereas three times as many Year 6 students as Year 7 students were able to communicate a correctly reasoned answer to Question 26, this rose to over five times as many for Question 27.

Those students able to argue correctly from the size of the complement (SOC) seemed to have a strong quantitative sense of fractions. The mean total score for the 37 items of all students was 19.4 with a standard deviation of 8.9. For the 40 students who used the size of the complement in Question 27, the mean score was 30.6 with a standard deviation of 4.7. This mean is significantly higher than the mean for the entire sample ($\underline{z} = 14.95$, $\underline{p} < 0.001$). That is, in Question 27 the students who used the size of the complement when reasoning which fraction was bigger had a significantly stronger quantitative sense of fractions, as determined by the set tasks.

The number of students who had the wrong answer although their reasoning was correct was higher in Question 27 than in earlier questions. Most of the students whose answers fell into this category used one of the three strategies associated with a correct answer.

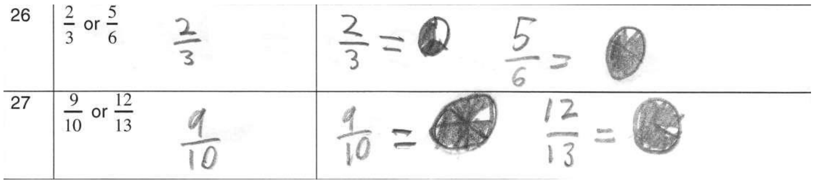



**Figure 5.92** Incorrect answer associated with correct reasoning

The Year 4 student's response depicted in Figure 5.92 has used regional models but nominated the smaller region as the answer to the question. As with Question 26, a number of students (7.9%) obtained the correct answer from the wrong reason of choosing the fraction with the larger numbers in the numerator, denominator, or both. Those who used a regional model without regard to the need for equal wholes (see Figure 5.93) also contributed to the incorrect reasoning categories.

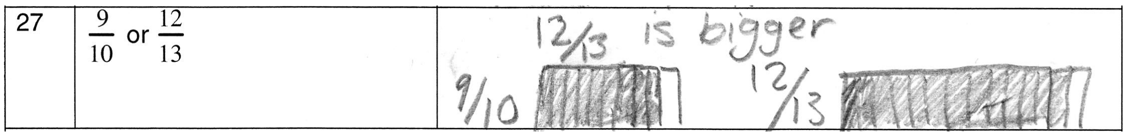



**Figure 5.93** A correct answer achieved without using equal-sized units (NEW)

The similarity of the proportions of correctly reasoned answers for Year 4 compared to Year 7, and Year 5 compared to Year 8 suggest that the high school samples were not demonstrating a more mature understanding of fractions compared to the primary school samples. Based on the state syllabus, the Year 4 students would not have been formally taught how to compare fractions of the type used in the question and yet their performance is very similar to the Year 7 students. The percentages of students' responses in each of the categories in Year 4 and Year 7 are remarkably similar except when it comes to the use of the fraction in the difference (FID) strategy. In the secondary school years (Years 7–8), 8% argued that the two fractions were the same size (FID) whereas in the primary years (Years 4–6) only 3% used the fraction in the difference strategy. As each of the questions asked which was the bigger number, arguing that the two are the same size requires a higher level of conviction than simply taking a guess. The increased use of this reasoning in the secondary school years suggests that being taught algorithmic operations with fractions may increase students' acceptance of incorrect procedures to compare fractions.

A new incorrect strategy operated in Question 27. This new method, described as "both smaller numbers" (BSN), was a generalisation of the bigger is smaller strategy. Just as the focus on the size of the denominator described by the whole number in the denominator (WND) was extended to apply to both numerator and denominator in the both larger numbers (BLN) method, bigger is smaller was also applied to both numerator and denominator. The strategy was used by only five students, and consequently it has been added to the category of students who used bigger is smaller.
Another 1.9% incorrectly argued that the size of the fraction was determined solely by the size of the denominator and 1.4% similarly based their argument on the number of parts. Incorrect regional models leading to an incorrect answer accounted for 3.7% of all responses.

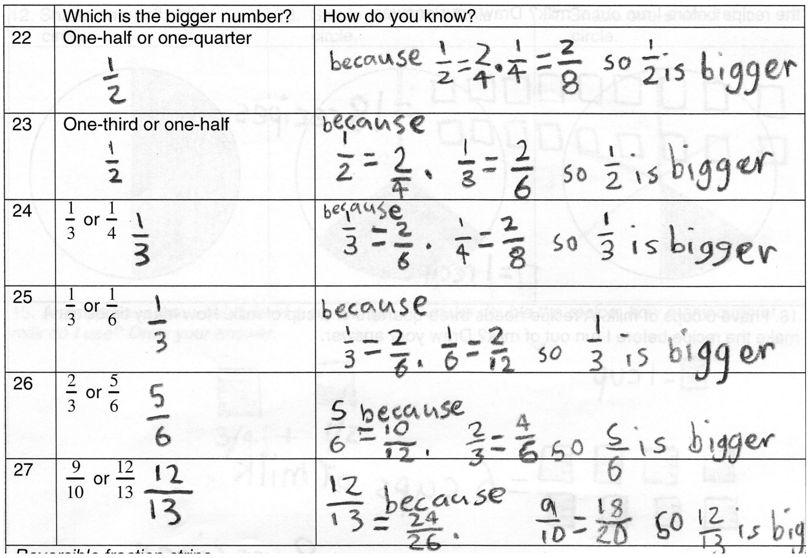
Sometimes students appeared to have created very situation-specific rules. The word "nonce" means for one occasion or purpose. I have called these situation specific rules nonce rules as they were not sufficiently general to represent either a taught or a commonly discovered strategy. For example, in response to Question 27, a Year 6 student reasoned that twelve-thirteenths was bigger than nine-tenths as follows: "If both sets have a one-number difference and one set has larger numbers, the set with larger numbers is larger." While this nonce rule is correct, it is extremely specific and difficult to see how a student could justify it. The Year 5 student whose reasoning appears in Figure 5.94 appears to be using a less explicitly stated version of this nonce rule to answer Question 27. The rule was not used in answering Question 26, as the equivalent fraction form of $\frac{4}{6}$ was available for this student.

| | | |
|---|---|---|
| 26 | $\frac{2}{3}$ or $\frac{5}{6}$    $5/6$ | because $4/6$ can simplyfly into $2/3$ but it is $5/6$ so $5/6$ is a bit bigger |
| 27 | $\frac{9}{10}$ or $\frac{12}{13}$    $12/13$ | becaus if it is a question like this which is bigger $\frac{99}{100}$ or $\frac{100}{101}$ it would be $\frac{100}{101}$ the bigger the number and the denomintnater is the bigger |
| *Reversible fraction strips* | | |

**Figure 5.94** An implicit nonce rule

Nonce rules were always very low incidence strategies. Another nonce rule with less justification in experience appeared to rely on a form of "magic doubling". In Question

27, the Year 5 student (see Figure 5.95) used a form of "magic doubling" by converting $\frac{12}{13}$ to $\frac{24}{26}$ and $\frac{9}{10}$ to $\frac{18}{20}$. This strategy may have been prompted by the previous question, where this process would work because the denominator of one fraction was indeed double the denominator of the other fraction. Magic doubling appears as the first step in all of the questions and is presumably combined with the strategies of bigger is smaller for Questions 22 to 25 and both larger numbers for Questions 26 and 27.



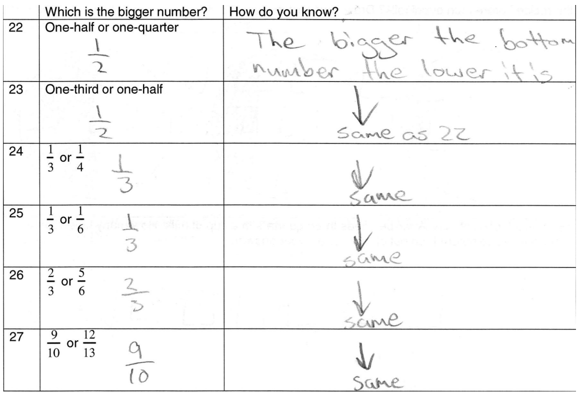
| | Which is the bigger number? | How do you know? |
|---|---|---|
| 22 | One-half or one-quarter $\frac{1}{2}$ | because $\frac{1}{2}=\frac{2}{4}$, $\frac{1}{4}=\frac{2}{8}$ so $\frac{1}{2}$ is bigger |
| 23 | One-third or one-half $\frac{1}{2}$ | because $\frac{1}{2}=\frac{2}{4}$, $\frac{1}{3}=\frac{2}{6}$ so $\frac{1}{2}$ is bigger |
| 24 | $\frac{1}{3}$ or $\frac{1}{4}$ $\frac{1}{3}$ | because $\frac{1}{3}=\frac{2}{6}$, $\frac{1}{4}=\frac{2}{8}$ so $\frac{1}{3}$ is bigger |
| 25 | $\frac{1}{3}$ or $\frac{1}{6}$ $\frac{1}{3}$ | because $\frac{1}{3}=\frac{2}{6}$, $\frac{1}{6}=\frac{2}{12}$ so $\frac{1}{3}$ is bigger |
| 26 | $\frac{2}{3}$ or $\frac{5}{6}$ $\frac{5}{6}$ | $\frac{5}{6}$ because $\frac{5}{6}=\frac{10}{12}$, $\frac{2}{3}=\frac{4}{6}$ so $\frac{5}{6}$ is bigger |
| 27 | $\frac{9}{10}$ or $\frac{12}{13}$ $\frac{12}{13}$ | $\frac{12}{13}$ because $=\frac{24}{26}$. $\frac{9}{10}=\frac{18}{20}$ so $\frac{12}{13}$ is big |

**Figure 5.95** "Magic doubling" as a nonce rule

**Looking back**

The comparison of fractions as mathematical objects (abstract quantity fractions) elicited a range of strategies. The need to justify which was the larger of two fractions highlighted a number of learnt methods that were not part of the intended curriculum. Many of these strategies, such as comparing the size of both the numerator and denominator separately (BLN) or the difference between the two (FID), have been independently developed by more than 5% of students. Rather than decreasing or disappearing when students entered high school, the proportion of students using reasoning devoid of any quantitative sense of fractions increased within this sample. It is plausible that the formal introduction of algorithmic manipulation of fraction symbols in secondary schools may have accelerated the loss of a quantitative sense of fractions.

Some strategies that led to correct answers when comparing unit fractions, such as bigger is smaller, were overgeneralised. In Figure 5.96, a Year 8 student applied the bigger is smaller strategy to all of the Questions from 22 to 27.



| | Which is the bigger number? | How do you know? |
|---|---|---|
| 22 | One-half or one-quarter $\frac{1}{2}$ | The bigger the bottom number the lower it's |
| 23 | One-third or one-half $\frac{1}{2}$ | Same as 22 |
| 24 | $\frac{1}{3}$ or $\frac{1}{4}$ $\frac{1}{3}$ | Same |
| 25 | $\frac{1}{3}$ or $\frac{1}{6}$ $\frac{1}{3}$ | same |
| 26 | $\frac{2}{3}$ or $\frac{5}{6}$ $\frac{2}{3}$ | same |
| 27 | $\frac{9}{10}$ or $\frac{12}{13}$ $\frac{9}{10}$ | same |

**Figure 5.96** Consistently applying the strategy "bigger is smaller"

This example not only demonstrates the persistence of the strategy but also shows the transition of bigger is smaller from the correct reasoning categories for unit fractions to the incorrect reasoning categories for non-unit fractions.

The features of the regional models used in teaching fractions that students have attended to as being salient, are often not those essential to the model representing fractions as mathematical objects. When students partition a region by drawing equally spaced vertical lines, the area may be an irrelevant feature of the model. Even the need for equal spacing may not be a salient feature of a regional model for students, such as the more than 10% of students who described one quarter of a rectangle as one-third, due to the number of parts (Question 3). Further, the additive components of the discrete model may contribute to whole number reasoning such as $\frac{1}{6}$ is larger than $\frac{1}{3}$ because "6 is larger than 3" (see Figure 5.78). The discrete model of fractions may also encourage the development of a "difference in the count" as a way of comparing the size of fractions. The need for equality of the whole when comparing the size of fractions was another feature of regional fraction models overlooked by some students.

The inadequate quantitative reasoning associated with the concept images of some students is highlighted in the lower percentage of students who could correctly reason with non-unit fractions in Questions 26 and 27 compared to those who achieved the right answer for the wrong reason. In Question 25, which asked students to compare $\frac{1}{3}$ and $\frac{1}{6}$, 39% had the right answer with the right reason and 31% had the right answer with the wrong reason. When students were subsequently asked to compare $\frac{2}{3}$ and $\frac{5}{6}$, 26% achieved the correct answer with appropriate supporting reasoning whereas 44% recorded the right answer with wrong or inadequate reasoning.

## 5.10 Reconstructing the unit from a continuous part: Q28–29

Questions 28 and 29 were designed to look for evidence of the use of a reversible fraction conception. A reversible fraction conception refers to "partitioning of a non-unit fraction ($\frac{n}{m}$) into $n$ parts to produce the unit fraction ($\frac{1}{m}$) from which the non-unit fraction was composed in the first place" (Tzur, 2004). These two questions actually require more than this conception. They require students to construct the unit fraction, reconstruct the whole and then construct a new fraction by iterating the unit fraction or partitioning the reconstructed whole.

Although not explicitly stated, the hope was that Question 28 and 29 would produce a record of students' constructions. However, many students did not record any intermediate working out and simply provided a location on the rectangle in response to the question. Answers without construction marks did not provide evidence of the intent of the student. Consequently, there was little to be gained by coding the responses other than dichotomously.

The decision to use dichotomous coding was also influenced by how close the application of incorrect reasoning could come to producing a seemingly correct answer. For example, in Figure 5.97 this Year 4 student has constructed three-quarters of the rectangle and left the construction marks in place. For Question 28, the rectangle was 6 cm long and answers within the range 3.6 cm – 4.4 cm were accepted as correct. This range allows for a variation (±10%) in determining two-thirds of the length of the rectangle, as most students did this visually. Had this Year 4 student simply marked three-quarters of the way along the rectangle and been just one millimetre short of the location, the answer would have been coded as correct.

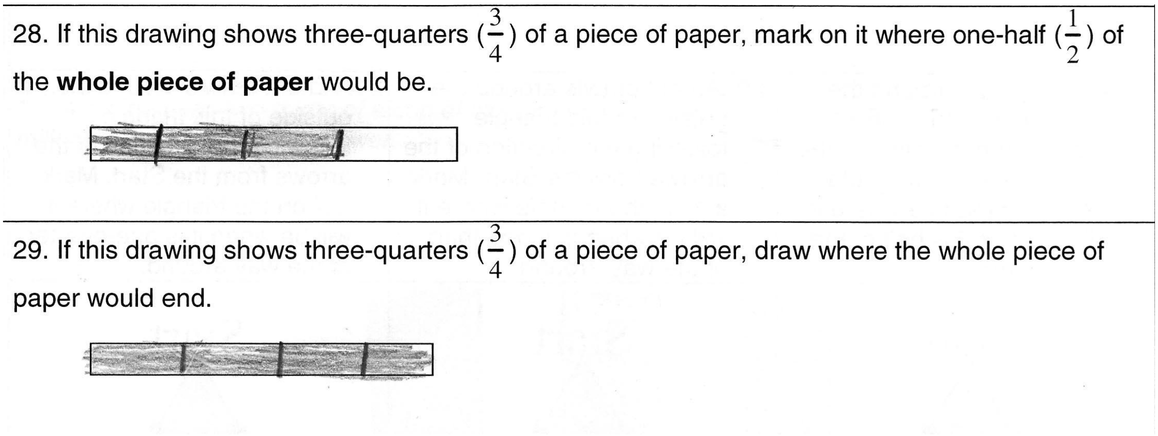



**Figure 5.97** An incorrect method showing construction marks

Further, as the question did not state which end to start from in marking one-half of the whole piece of paper, it was necessary to accept both one-third and two-thirds of the rectangle. With the additional space to the right of the rectangle almost all of the answers coded as correct marked the rectangle at a point two-thirds of the way along from the left. The Year 7 student's response in Figure 5.98 was a notable exception.

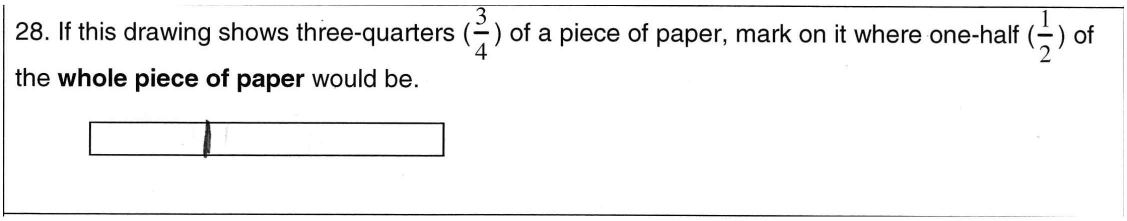



**Figure 5.98** A correct location for Question 28 working from the right

### *Question 28*

Question 28 provided a drawing of a rectangle representing three-quarters of a piece of paper and students were asked to mark on the rectangle where one-half of the whole piece of paper would be. The results for Question 28 are shown in Table 28.

Table 28

*Percentage distribution of responses to Question 28, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Correct | 28 | 42 | 59 | 43 | 49 |
| Incorrect | 64 | 56 | 40 | 54 | 46 |
| Omit | 8 | 2 | 1 | 3 | 5 |

Overall, 44% of the responses were accepted as being within the range 3.6 cm–4.4 cm. Some students, such as the Year 5 student whose answer is shown in Figure 5.99, added appropriate fraction notations to their answers. The inclusion of the notation suggested that this student had linked the idea of a partitioned fraction with the quantity fraction notation.

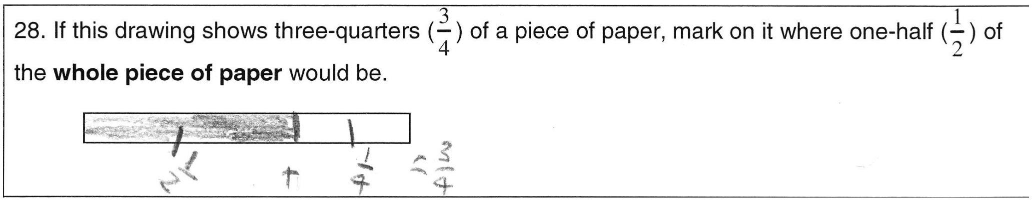



**Figure 5.99** A Year 5 response with added fraction notation

When Tzur (2004) investigated the content-specific reversible fraction conception, he found that when asked to produce the original whole from an unmarked non-unit fraction of that whole such as $\frac{5}{8}$, quite often a learner would partition that fraction into 8 parts instead of 5 ($\frac{1}{8}$) parts. The same type of reasoning appears to have contributed to many students dividing the rectangle into quarters instead of thirds. As well as marking three-quarters of the rectangle, another incorrect answer involved marking half of the rectangle provided (Figure 5.100). Thus, both fractions mentioned in the questions contributed to the range of incorrect answers.

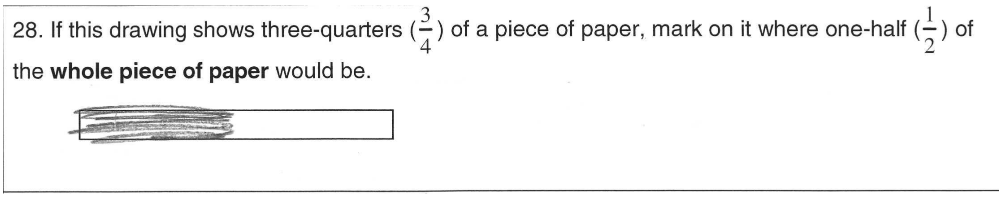



**Figure 5.100** A response with one-half of the rectangle shaded

Even when a student could correctly partition the three-quarter strip as in Figure 5.101, the requested one-half of the whole piece of paper was not always formed from two quarter-strips.

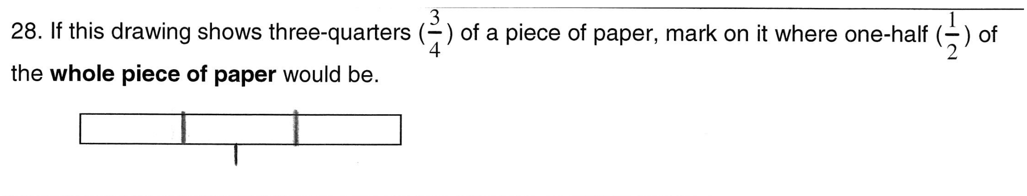



**Figure 5.101** Correctly marking the three-quarters but not using two quarters for $\frac{1}{2}$

The dip in the percentage of correct answers from Year 6 to Year 7 continued to be present for Question 28. However, the reversible fraction concept is a sophisticated quantitative fraction concept and one that, given student groups of similar ability, should improve with age. The ability estimates of the students who correctly answered Question 28, provided by the Rasch analysis, indicated that the reversible fraction concept was slightly stronger in students in high school despite the apparent dip in the percentage correct. That is, when students who have the same level of a quantitative sense of fractions are compared across the school years, a high school student was more likely on average to answer Question 28 correctly than a primary school student with the same level of ability. The probability of correctly answering the question for students of similar ability across the different stages of schooling can be seen in Figure 5.102. Each school stage corresponds to two year-groups, with Stage 4 representing Years 7 and 8, and Stage 3 representing Years 5 and 6. The mathematics curriculum content is described for each stage of schooling, making school stage a reasonable way to organise the data to compare group performance.

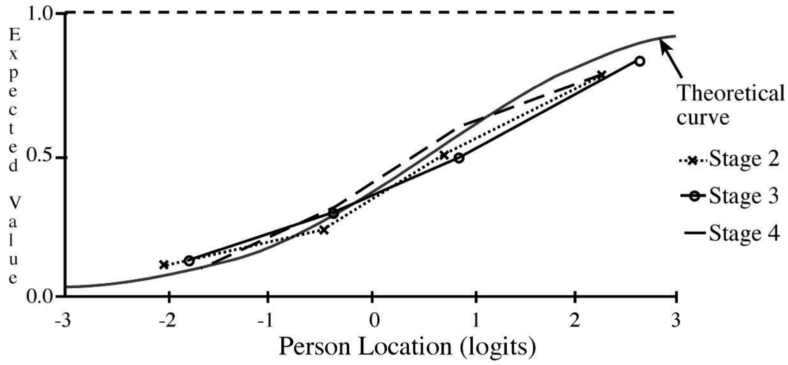



**Figure 5.102** Like-ability comparison of responses to Q28 by stage of schooling

## *Question 29*

Question 29 provided a drawing of a rectangle representing three-quarters of a piece of paper and students were asked to mark on the rectangle where the whole piece of paper would end. The length of the rectangle for Question 29 was 55 mm, a little smaller than the rectangle used in Question 28. The responses were similarly coded correct or incorrect, with answers in the range of 1.5 cm – 2.3 cm beyond the end of the rectangle, coded correct. This range was used to attempt to exclude answers obtained by adding one-quarter of the rectangle to its length. Again, a slight overestimate of one-quarter of the rectangle could result in an incorrect answer being coded as correct. Of course, if the construction marks were present the intent became clear and this reduced the need to rely only on the location of the endpoint. The results for Question 29 are summarised in Table 29.

Table 29

*Percentage distribution of responses to Question 29, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| Correct | 24 | 43 | 52 | 36 | 47 |
| Incorrect | 64 | 54 | 47 | 59 | 45 |
| Omit | 12 | 3 | 2 | 5 | 8 |

In total, 40% of the responses recorded a location for the end of the rectangle within the accepted range. When students added subdivision marks to the rectangle as in Figure 5.103, it increased the confidence that the response indicated a sound appreciation of the relationship between fractional parts and the whole.

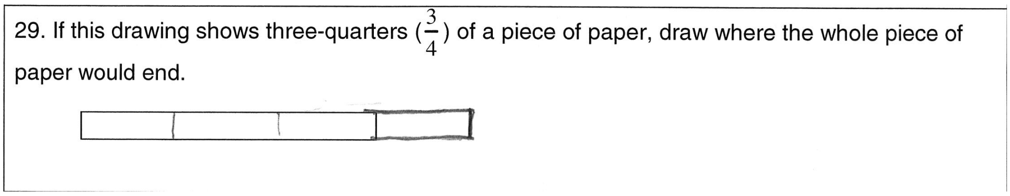



**Figure 5.103** A response showing the three quarter subdivisions

In Figure 5.104, this Year 7 student appeared to be able to subdivide the three-quarters but did not provide an indication of recognising the related whole. Going beyond the end of the rectangle provided in Question 29 acted as a stumbling block for quite a few students. For example, of the 59% of Year 7 responses that were coded as incorrect, 129 Year 7 students (38% of Year 7) created answers to Question 29 that did not extend beyond the rectangle. Of these 129 Year 7 responses, 56 (16% of Year 7) indicated the end of the rectangle as their answer. Only 73 of the incorrect responses (21% of Year 7) had an answer that extended beyond the rectangular shape representing three-quarters. Within the context of this problem, being able to access the reversible fraction conception did not always lead to iterating beyond the presented three-quarter boundary.

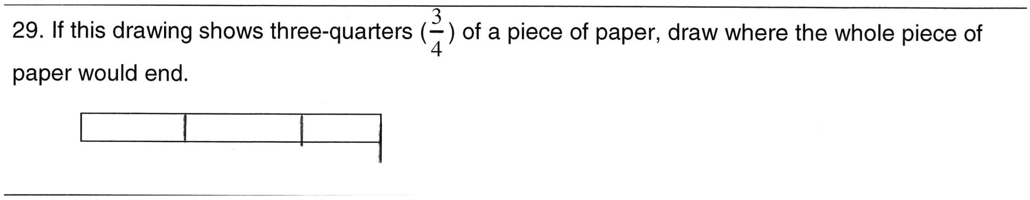



**Figure 5.104** A Year 7 response that failed to go beyond the end of the rectangle

**Looking back**

It is worth noting that although the two questions provided similar levels of challenge, a correct response to Question 28 did not guarantee a correct response to Question 29.

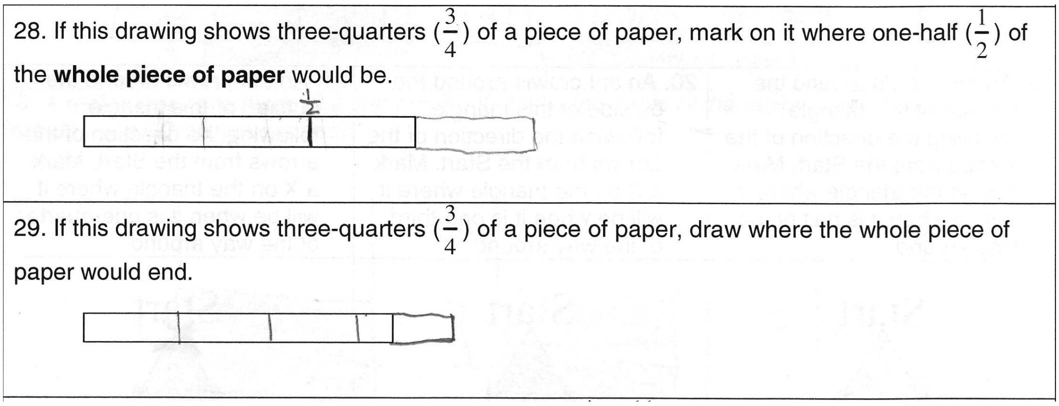



**Figure 5.105** Correct partitioning of the $\frac{3}{4}$-unit in Q28 with a different use in Q29

In Figure 5.105, this Year 7 student has correctly interpreted and answered Question 28. However, in Question 29, the attempt to show the three quarters in the rectangle appears to have been miscalculated and the necessary adjustments to divide the rectangle into three equal parts correctly, abandoned. The inaccurately formed quarter-units, three of which no longer equal the length of the rectangle, appear to be used to form a total of four quarter-units. It may be that in attempting to partition the three-quarter unit and iterate the constructed one-quarter unit, attention to the creation of equal quarters has contributed to loosing sight of the importance of the initial three-quarter rectangle.

Comparing Table 28 with Table 29, the percentage of correct responses in each year group on the two items was quite stable. Moreover, 15% of Year 4, 31% of Year5, 42% of Year 6, 29% of Year 7 and 37% of Year 8 answered both questions correctly. The mean ability estimates of those correctly answering the two questions were also similar (1.38 logits and 1.48 logits, respectively). The reversible fraction concept, as assessed in Questions 28 and 29 within this sample, appeared to be present in about one-third of upper primary and lower secondary students.

## 5.11 Estimating the size of the sum of two fractions

Although a task requiring students to add two fractions presented in the standard numerical notation is likely to encourage the use of an algorithm in response, it was felt that a task which required estimating the sum would be more likely to draw upon a quantitative sense of fractions. To further encourage estimation rather than calculation, the question was presented in multiple-choice format.

### *Question 30*

Question 30 required students to select the best estimate for the sum of $\frac{4}{5} + \frac{11}{12}$ from the options 17, 15, 2 and 1. The results for Question 30 are shown in Table 30.

Table 30

*Percentage distribution of responses to Question 30, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| 2 | 12 | 26 | 36 | 10 | 23 |
| 17 | 38 | 24 | 21 | 37 | 23 |
| 15 | 38 | 43 | 31 | 44 | 40 |
| 1 | 2 | 5 | 10 | 4 | 7 |
| Other | 2 | 0 | 1 | 1 | 1 |
| Omit | 7 | 1 | 2 | 5 | 6 |

Overall, 21% of students provided the answer "2" as the best estimate to the sum of two fractions, each near one. The responses to this question suggested that the symbol system related to fractions did not link to a quantitative sense of fractions for the majority of students. Over two-thirds (68%) of all of the students answered 17 or 15 as the best estimate of the sum of two fractions each near one[14].

Although students were not asked to provide an explanation for their answer to this question, some students did. The Year 8 student whose answer is shown in Figure 5.106, added the whole-number numerators and the whole-number denominators in the classic additive misinterpretation, and does not appear to have a quantitative sense of the answer $\frac{15}{17}$.

[14] A similar question was used as part of the Secondary Numeracy Assessment Program in 2006 (NSW Department of Education and Training, 2006). In responding to the question, "$\frac{4}{5} + \frac{11}{12}$ is about 1, 2, 15 or 17?", 57% of 81 784 Year 7 students chose 15 or 17.

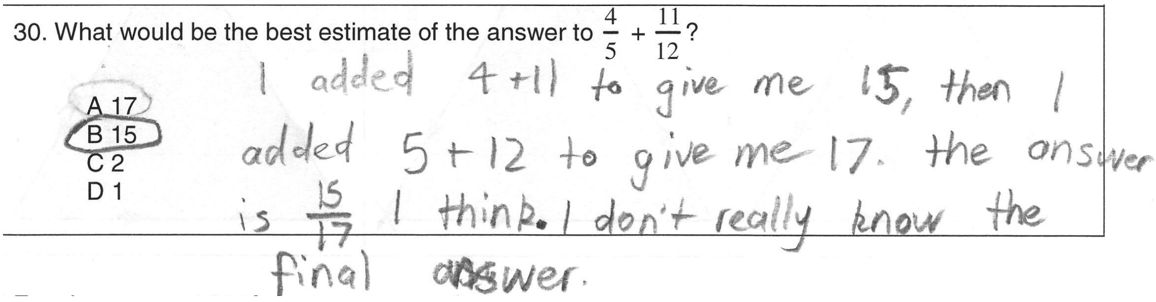



**Figure 5.106** An additive response without a quantitative sense of the answer

Even more exotic interpretations of the rules for addition of fractions, such as the Year 8 student (Figure 5.107) who added diagonally, did not result in a quantitative sense of fractions influencing the answer. The explanation in Figure 5.107, "A or B 1 away from both" suggests that $\frac{16}{16}$ is considered to be 16 rather than 1.

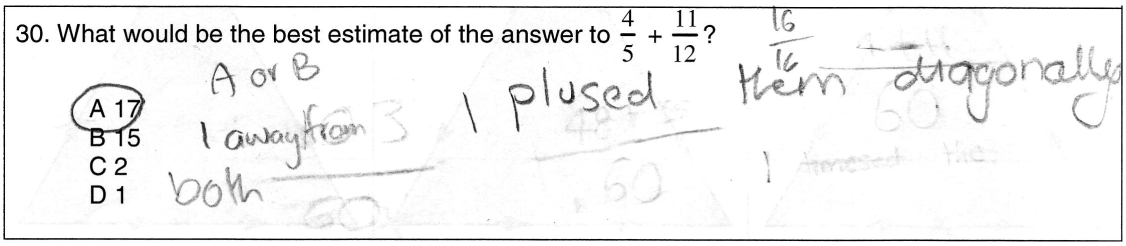



**Figure 5.107** An example of adding fractions diagonally as an erroneous algorithm

Although the question asked for an estimate, some students made use of the standard addition algorithm for fractions (see Figure 5.108). However, even the correct use of the algorithm did not necessarily result in the correct answer.

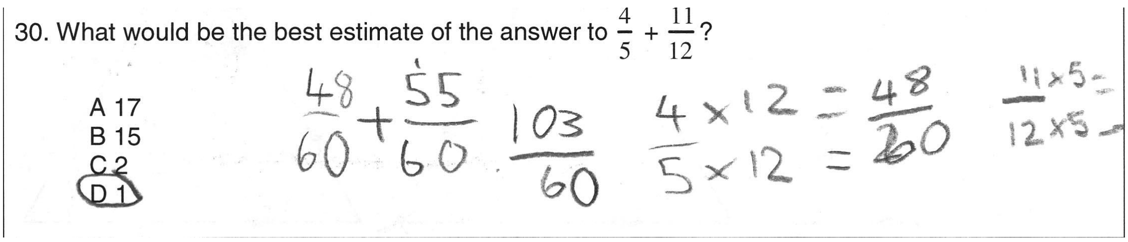



**Figure 5.108** A correct application of the algorithm resulting in the wrong answer

Knowing how to apply the algorithm for addition of fractions did not equate to displaying a quantitative sense of fractions. Interpreting $\frac{103}{60}$ required some sense of the size of fractions. The Year 8 student whose answer is shown in Figure 5.109 has been able to correctly convert both fractions to equivalent fractions with a denominator of 60 before carrying out a division. However, the remainder is treated as a decimal fraction, causing the student to round down.

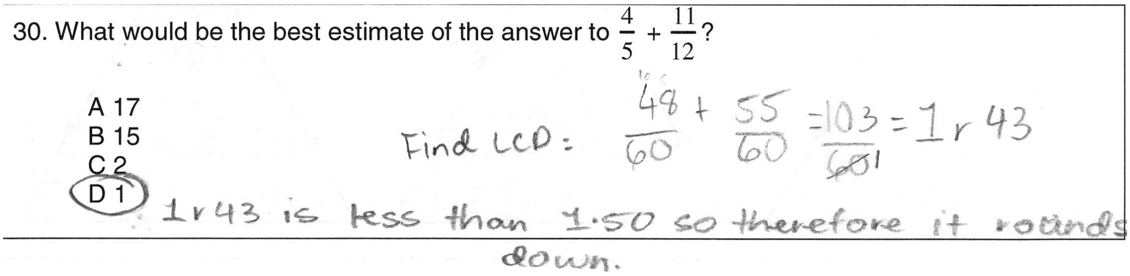



**Figure 5.109** A Year 8 response that confused the remainder with a decimal fraction

When the remainder was recorded as the correct fraction (Figure 5.110), the lack of a quantitative sense of fractions needed to compare $\frac{43}{60}$ and one-half appeared to let the Year 8 student down in the following response.

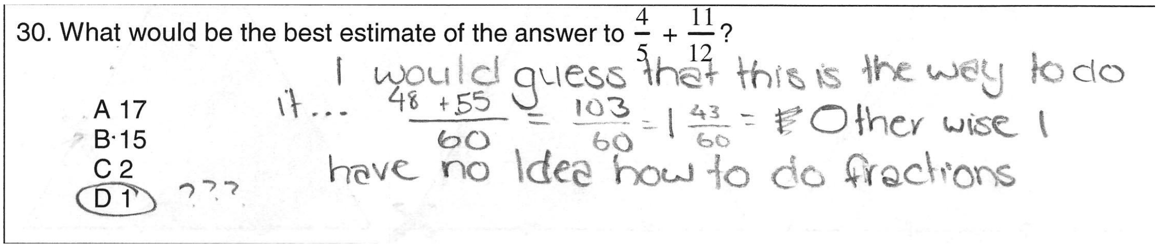



**Figure 5.110** An example of algorithmic knowledge of fractions without sense of size

If a student had a quantitative sense of fractions, using an additive misinterpretation of the symbolism to obtain $\frac{15}{17}$ should lead to "1" being a popular answer. An answer of "1" is a much better indicator of some vestige of a quantitative sense of fractions than either 17 or 15. It is at least a near miss (see Figure 5.111).

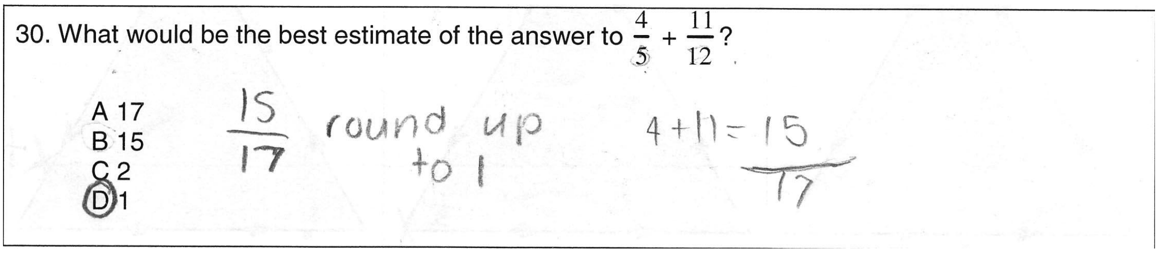
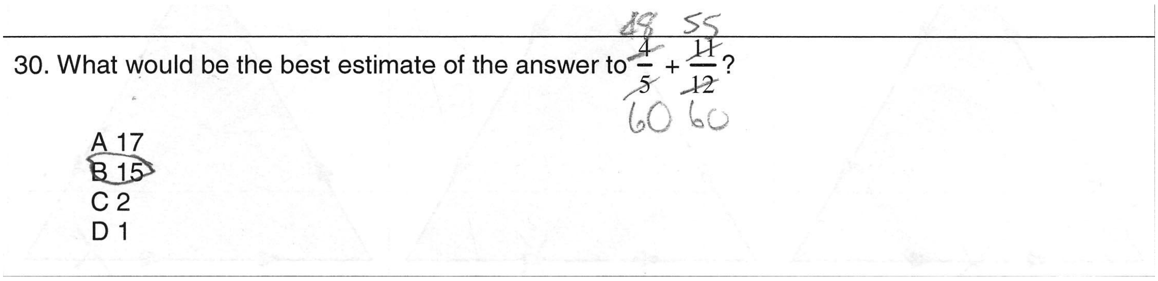



**Figure 5.111** An example of rounding up following an additive misinterpretation

The emphasis in the teaching of fractions of the instrumental over the relational, or method over meaning, clearly led to fragmented fraction concept images for many students. The low percentage of correct responses to this task highlights the problem. In Figure 5.112, the Year 8 student was able to correctly calculate equivalent fractions with the lowest common denominator but this process does not appear to have influenced the selection of the wrong answer. It is undoubtedly possible to learn methods of calculating with fractions without developing a quantitative sense of fractions.

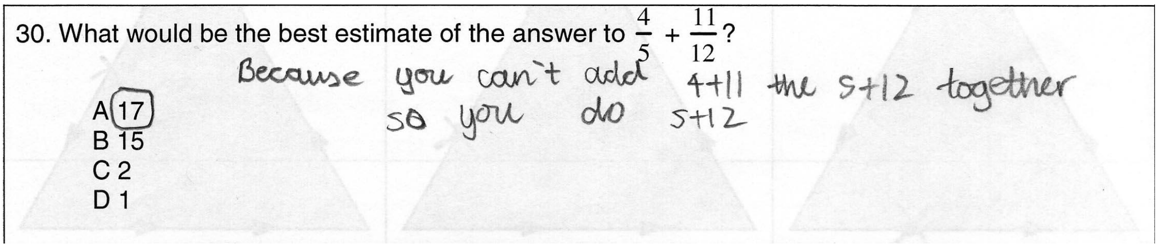

30. What would be the best estimate of the answer to $\frac{4}{5} + \frac{11}{12}$?

$\frac{48}{60}$ $\frac{55}{60}$

A 17
B 15
C 2
D 1

**Figure 5.112** Correct conversion to equivalent fractions

The poor performance of Year 7 on this task suggests that the introduction of the formal operations on fractions may have a deleterious effect on quantitative fraction sense. Sometimes students' responses suggested that they might have recalled being told what you cannot do with fractions. For example, the Year 7 student's response in Figure 5.113 appears to argue that because you cannot add the numbers on the top, you add the numbers on the bottom instead.

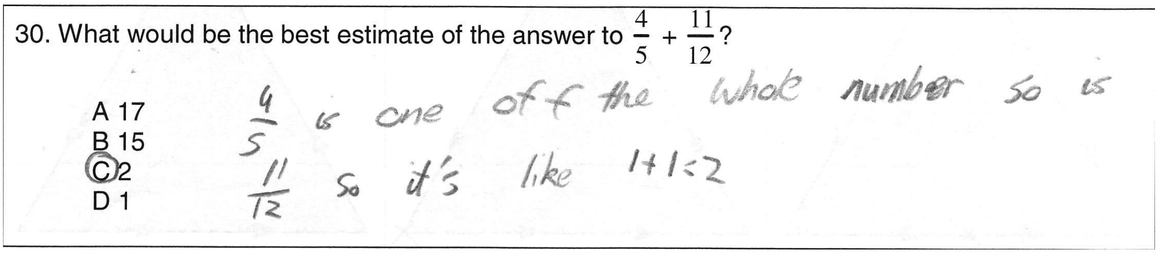

30. What would be the best estimate of the answer to $\frac{4}{5} + \frac{11}{12}$?

Because you can't add 4+11 the 5+12 together so you do 5+12

A 17
B 15
C 2
D 1

**Figure 5.113** A Year 7 response explaining that you cannot add the numerators

Unfortunately, even the correct responses do not by themselves provide an indication of a quantitative sense of fractions. The Year 7 student in Figure 5.114 argues that it is "like 1 + 1 = 2" because both fractions are one off the whole number.


30. What would be the best estimate of the answer to $\frac{4}{5} + \frac{11}{12}$?

$\frac{4}{5}$ is one off the whole number so is $\frac{11}{12}$ so it's like 1+1=2

A 17
B 15
C 2
D 1

**Figure 5.114** A Year 7 response apparently backed by reasoned estimates

Initially, this argument appears to suggest a sense of the size of the two fractions, as both fractions are taken to be near one. However, the quantitative sense of the size of fractions is seen to be rather more limited when you consider that this student saw $\frac{2}{3}$ and $\frac{5}{6}$ as being the same size in Question 26 as they are both "one off the whole number". A student who believes that the size of a fraction is related (solely) to the difference between the numerator and the denominator does not have a robust quantitative sense of fractions.

**Looking back**

It is possible that the true level of correct responses to Question 30 might be lower, as some of the correct responses were accompanied by confessions that the student had guessed. An unanticipated method of manipulation of the whole numbers involved also resulted in the correct answer for at least one Year 7 student. This Year 7 student's explanation of how the correct answer was achieved (Figure 5.115) strengthens the concern that even some of the answers treated as correct, may have been achieved through totally erroneous methods.

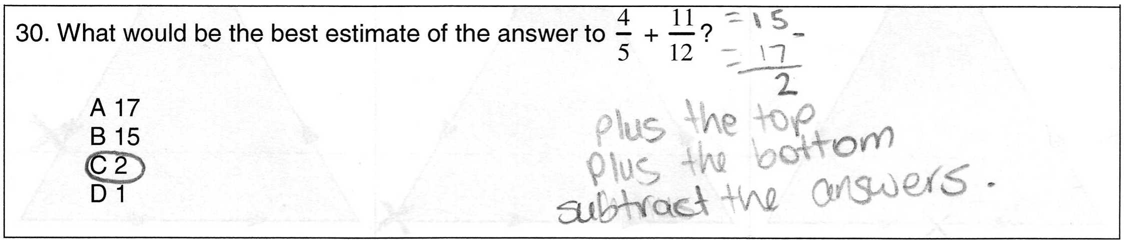



**Figure 5.115** An erroneous method leading to the correct answer

## 5.12 Measurement comparison of two quantities (fraction remainder): Q31–35

For questions 31 to 35, students were provided with different sized rectangular strips of coloured paper with a common width and asked to identify the fraction relationships between the two lengths.

### *Question 31*

In Question 31, students were asked to determine exactly how many of the short green strip of paper would be needed to be equal to the length of the blue strip of paper. The results for Question 31 are shown in Table 31.

Table 31

*Percentage distribution of responses to Question 31, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $2\frac{1}{2}$ | 78 | 82 | 85 | 89 | 93 |
| 2 | 5 | 4 | 2 | 3 | 1 |
| 3 | 5 | 5 | 4 | 1 | 0 |
| $2\frac{1}{4}$ | 3 | 2 | 2 | 2 | 1 |
| Other | 9 | 8 | 5 | 5 | 5 |
| Omitted | 0 | 0 | 1 | 0 | 0 |

The percentage of correct answers in each year group shows a consistent growth across the school years. This pattern of growth in the correct answer category (about a 4% increase each year) was not evident in any other question. Although the explanations did not always unambiguously illuminate the method students used, explanations suggested at least two different approaches to this question. Most students used one of the strips of paper as the unit of measure, as intended by the question (see Figure 5.116).

| Question | **How do you know?** |
|---|---|
| P1. Exactly how many of the short green strip of paper equals the length of the blue strip of paper? | I Know that, two and a half is the Answer because I mesured it. With the green strip. |

**Figure 5.116** A Year 4 explanation of measuring with the green strip of paper

A smaller group of students couched their explanations in terms of the lengths of each strip of paper, usually expressed in centimetres. The division carried out through the intermediary unit of centimetres is evident in the Year 4 student's explanation in Figure 5.117.

| Question | **How do you know?** |
|---|---|
| P1. Exactly how many of the short green strip of paper equals the length of the blue strip of paper? $2\frac{1}{2}$ times | The blue strip is 20 cm an the green one is 8 cm. 8 goes into 20 twice and 4 cm is left which is half of the green one. |

**Figure 5.117** An explanation of measuring in centimetres and then dividing

It was not always possible to tell the method that had been used as students often wrote only that they had measured. The answers “2” and “3” could be attributed to a failure to deal with the fraction unit remaining from measurement. A response from a Year 5 student has “2” crossed out and replaced by “3”, with the explanation that “two isn’t enough”. The same type of reasoning for Question 31 (P1) is evident in the response from the Year 7 student in Figure 5.118.

| Question | How do you know? |
|---|---|
| P1. Exactly how many of the short green strip of paper equals the length of the blue strip of paper?<br>3 | beacause if you only had 2 it would not be enough |

**Figure 5.118** Rounding up to the next whole number because 2 is not enough

The small percentage of students who answered $2\frac{1}{4}$ might have been influenced by the ubiquitous nature of “one-quarter” when some students refer to a fractional part.

### *5.12.1 Halving the unit of measure*

As well as dealing with the idea of a fractional remainder resulting from measuring, the simple multiplicative relationship between the size of the unit being used to measure and the number of units needed was also investigated.

#### *Question 32*

In Question 32, students were asked to determine exactly how many strips of paper half as long as the green strip of paper would be needed to equal the blue strip of paper. The green strip was 8 cm long and the blue strip was 20 cm long. The students were asked to imagine a red strip of paper half as long as the green strip of paper and to determine how many red strips would be equal to the length of the blue strip of paper. The results for Question 32 are shown in Table 32.

Table 32

*Percentage distribution of responses to Question 32, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| 5 or “doubled” | 69 | 79 | 84 | 80 | 84 |
| $4\frac{1}{2}$ | 4 | 2 | 3 | 3 | 6 |
| 2 | 3 | 3 | 2 | 3 | 2 |
| 6 | 4 | 2 | 1 | 2 | 2 |
| 4 | 3 | 2 | 1 | 2 | 1 |
| Other | 17 | 11 | 8 | 10 | 5 |
| Omitted | 0 | 1 | 1 | 0 | 1 |

Overall, 79% of the students could halve the unit used to measure to determine the number of half-units required. Some 18% of students stated that you simply double the previous answer while another 23% indicated that they folded the green strip in half and measured (see Figure 5.119).

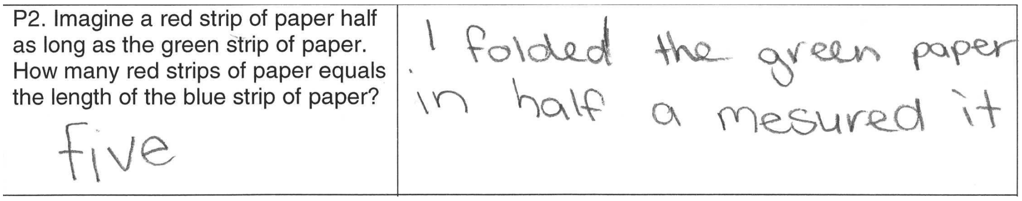



**Figure 5.119** An example of halving the unit and measuring

Many others (16%) indicated that they measured to find the answer without providing further detail and 4% (67) provided no reason. In total, 6% or 102 students used drawings, as shown in Figure 5.120, to indicate their reasoning.

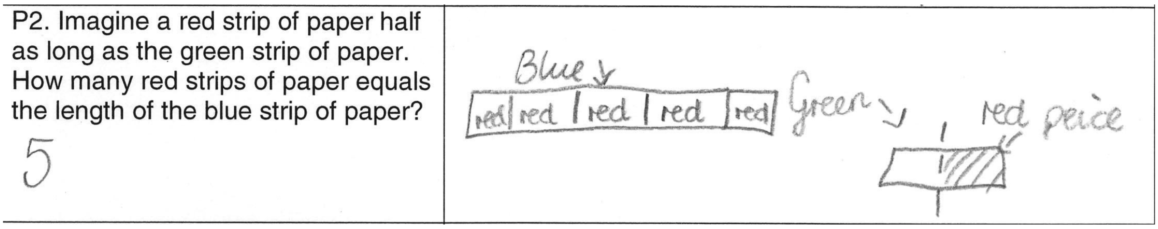
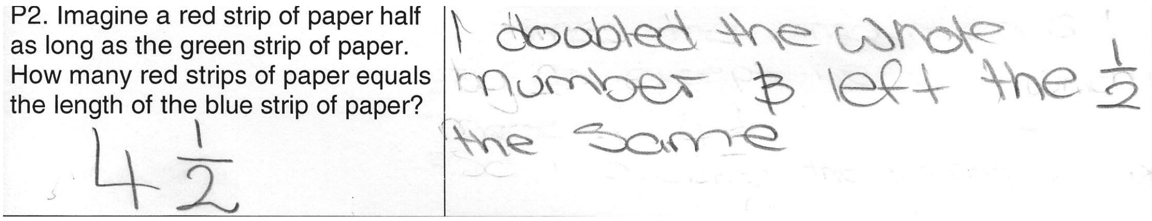



**Figure 5.120** An explanation provided through drawings

If students provided the answer of “3” to Question 31 (P1) but then explained that they doubled this answer for Question 32, they were included in the “doubled” category.

The answer $4\frac{1}{2}$ following on from a correct answer to Question 31 initially appears unexpected. However, this answer often arose from the way "one-half" from Question 31 was handled. The Year 8 student's explanation in Figure 5.121 treats the two wholes separately from the half.

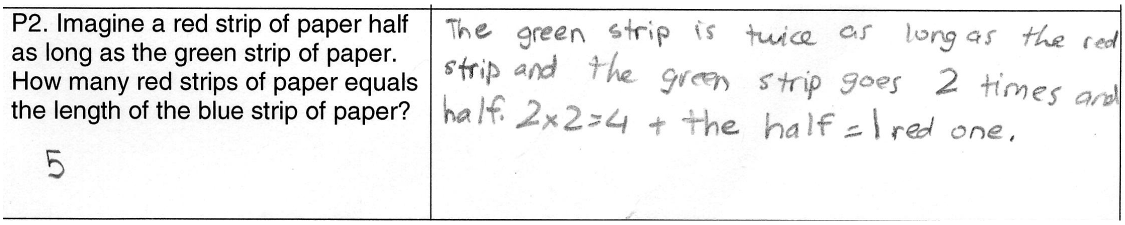

P2. Imagine a red strip of paper half as long as the green strip of paper. How many red strips of paper equals the length of the blue strip of paper?

$4\frac{1}{2}$

I doobled the whole number & left the $\frac{1}{2}$ the same

**Figure 5.121** An explanation of how $4\frac{1}{2}$ resulted from doubling the whole part only

Dealing with the whole number and the fractional part from Question 31 separately did not need to result in the wrong answer. Although on the surface the Year 4 student's explanation in Figure 5.122 appears similar, the "half" is exchanged for "1 red one" resulting in a correct answer.

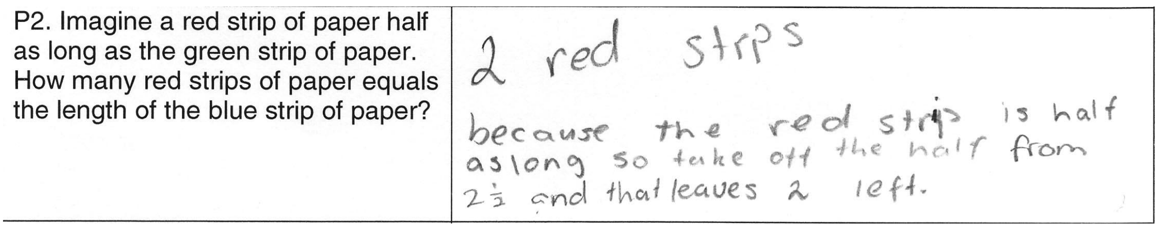

P2. Imagine a red strip of paper half as long as the green strip of paper. How many red strips of paper equals the length of the blue strip of paper?

5

The green strip is twice as long as the red strip and the green strip goes 2 times and half. 2x2=4 + the half = 1 red one.

**Figure 5.122** An example of multiplication followed by an exchange of units

Some of the answers of "2" were formed by an additive change of one-half a unit rather than a multiplicative change. For example, rather than replacing each whole with a unit half as long, the Year 5 student (Figure 5.123) has simply subtracted half a unit.


P2. Imagine a red strip of paper half as long as the green strip of paper. How many red strips of paper equals the length of the blue strip of paper?

2 red strps

because the red strip is half as long so take off the half from $2\frac{1}{2}$ and that leaves 2 left.

**Figure 5.123** An example of taking off half to obtain an answer of 2

Most students who answered "2" appeared to lose track of the exchange process, sometimes at the first step of recognising that each green strip corresponded to two red strips.

| | |
|---|---|
| P2. Imagine a red strip of paper half as long as the green strip of paper. How many red strips of paper equals the length of the blue strip of paper? 2 | ½ of something means that that thing is cut twice into halves. Two halves make a whole, so two red strips equals the length of the blue strip |

**Figure 5.124** An example of losing track of the exchange

As well as losing track of the exchange where each whole is replaced by two halves, a number of less common errors contributed to the response category "2". In Figure 5.125, the student's linguistic problems appear to link a phonemic error to a semantic error. Taking "half of it" appears to mean taking "half off it". Similar errors occurred in some responses to Question 11.

| | |
|---|---|
| P2. Imagine a red strip of paper half as long as the green strip of paper. How many red strips of paper equals the length of the blue strip of paper? 2 | Because Question Ones answer was 2½ so I took half of it. |

**Figure 5.125** An explanation confusing taking half *of* it with half *off* it

**Looking back**

What was particularly interesting from a cross-item analysis, was that of the 51 students who answered "2" to Question 31, the first of the practical tasks, 34 answered "5" to the next question. Similarly, of the 52 students who answered "3" to this question, 13 answered "5" to the next question and 27 answered "6". The response to Question 32 (P2) depended on what method was used. Halving the strip and carrying out the measurement could produce the correct answer to Question 32, whereas doubling the value from Question 31 would only produce the correct answer if the initial number of units had been correctly determined.

## 5.13 Reversing the measurement comparison

In Questions 33 to 35, one strip of paper was used as a measurement unit before determining the fractional relationships between the strip measured and the measurement unit.

### *Question 33*

In Question 33, students were asked to determine exactly how many of the yellow paper strips would equal the length of the blue paper strip. The yellow strip was 15 cm long and the blue strip was 20 cm long. The results for Question 33 are shown in Table 33.

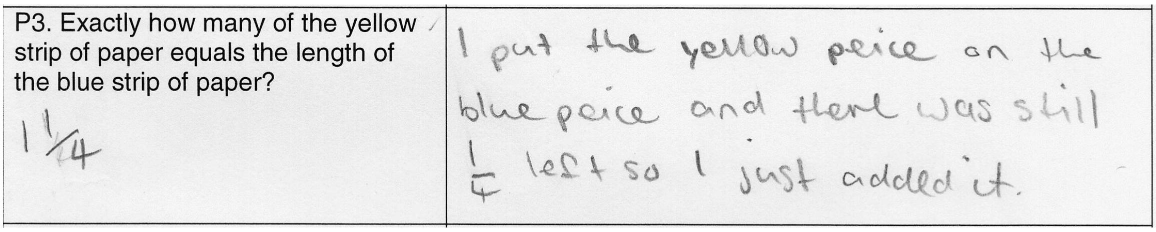
Table 33

*Percentage distribution of responses to Question 33, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $1\frac{1}{3}$ | 23 | 38 | 51 | 37 | 48 |
| $1\frac{1}{4}$ | 41 | 37 | 33 | 43 | 34 |
| $1\frac{1}{2}$ | 9 | 4 | 4 | 4 | 3 |
| 1 | 5 | 3 | 2 | 4 | 1 |
| $1\frac{3}{4}$ | 2 | 2 | 0 | 2 | 2 |
| $\frac{1}{4}$ | 3 | 2 | 1 | 0 | 2 |
| Other | 17 | 12 | 8 | 9 | 9 |
| Omitted | 1 | 1 | 1 | 1 | 1 |

In Question 33, 39% of the students correctly answered the question with the percentage of each year correct peaking in Year 6 and declined in Year 7. As many students had difficulty in correctly identifying one-third of a continuous unit (see Questions 2, 3, 6 and 13), it is not surprising that less than half of the students could correctly identify $1\frac{1}{3}$ of the yellow strips as the required measurement.

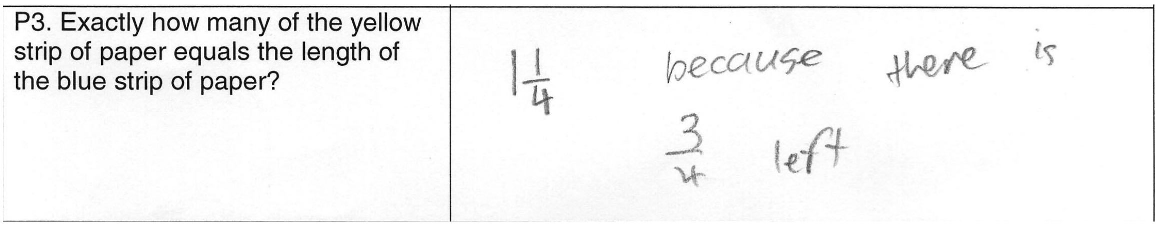
The answer $1\frac{1}{4}$ was almost as popular as the correct answer, with 38% of the students responding with this answer. The major reason for answering $1\frac{1}{4}$ appeared to be losing track of the referent unit of measurement. That is, having identified one unit, the remainder often appeared to reference the strip being measured rather than the strip used to measure. The "$\frac{1}{4}$ left" visible when the yellow strip is placed on the blue strip is one-quarter of the blue strip, not one-quarter of the yellow strip. Consequently, the referent unit of measurement is readily confused with the object being measured, as in Figure 5.126.



| | |
|---|---|
| P3. Exactly how many of the yellow strip of paper equals the length of the blue strip of paper? 1 1/4 | I put the yellow peice on the blue peice and there was still 1/4 left so I just added it. |

**Figure 5.126** An explanation of obtaining $1\frac{1}{4}$ by adding the "$\frac{1}{4}$ left"

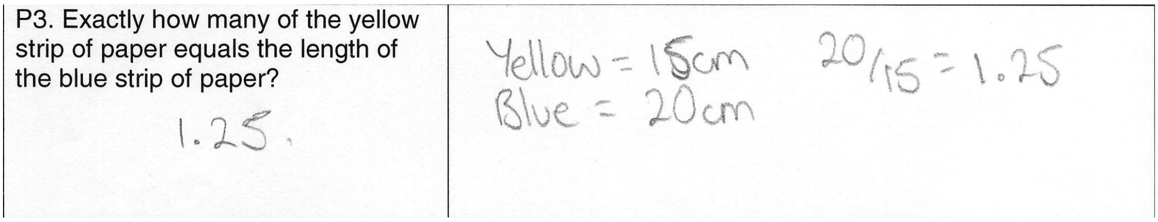
The confusion between the unit being measured and the unit used to measure contributed to the wrong answers in a number of ways. In Figure 5.127, the "$\frac{3}{4}$ left" referred to is the three-quarters of the blue strip corresponding to the length of the yellow strip. The "$\frac{1}{4}$" is calculated by subtracting the three-quarters from the length being measured.



| | |
|---|---|
| P3. Exactly how many of the yellow strip of paper equals the length of the blue strip of paper? | $1\frac{1}{4}$ because there is $\frac{3}{4}$ left |

**Figure 5.127** An example of taking the yellow strip as $\frac{3}{4}$ of the blue with remainder $\frac{1}{4}$

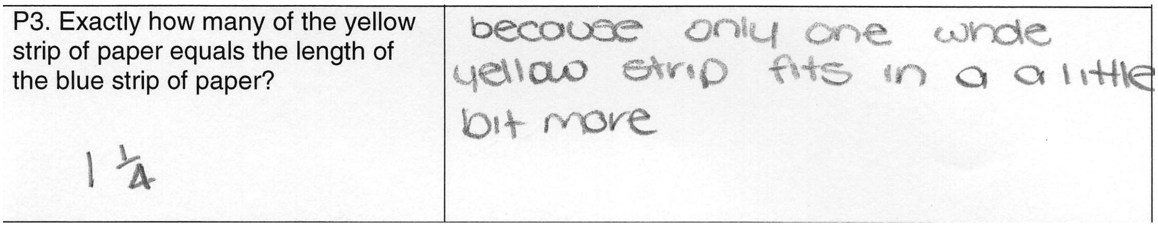
Even when indirect measurement was used coupled with a correct statement of division, the divisor could change in dealing with the remainder as in Figure 5.128.



| | |
|---|---|
| P3. Exactly how many of the yellow strip of paper equals the length of the blue strip of paper? 1.25. | Yellow = 15cm Blue = 20cm 20/15 = 1.25 |

**Figure 5.128** An example of interchanging the divisor and the dividend

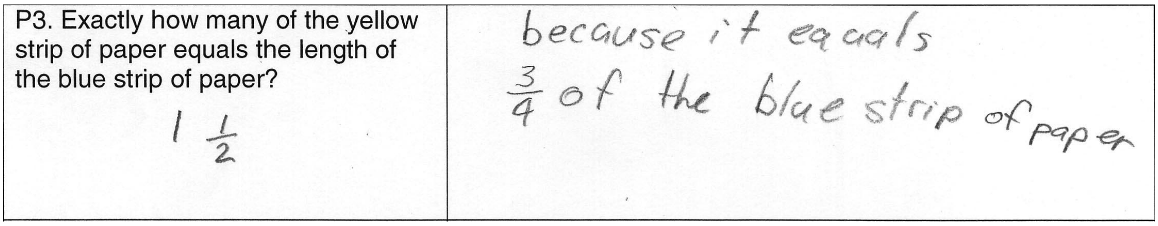
The ubiquitous nature of one-quarter when dealing with a "little bit" may have also contributed to the high percentage of students who answered $1\frac{1}{4}$. The Year 8 student whose answer is shown in Figure 5.129 described the remainder as "a little bit more" and quantified this as one-quarter.



| | |
|---|---|
| P3. Exactly how many of the yellow strip of paper equals the length of the blue strip of paper? 1 1/4 | becouse only one whole yellow strip fits in a a little bit more |

**Figure 5.129** A Year 8 response treating $\frac{1}{4}$ as a "little bit more"

Losing track of the unit of measurement being referenced is also likely to have contributed to answers of $\frac{1}{4}$ and $1\frac{3}{4}$. The answer of $1\frac{1}{2}$ was caused by partitioning the blue strip into quarters and the yellow strip into parts corresponding to three of these quarters. The Year 8 student (Figure 5.130) identified the yellow strip as corresponding to three-quarters of the blue strip. All of the sub-units were viewed as quarters and a reorganisation of one-quarter from the blue and "one-quarter" from the yellow contributed to a constructed half. Most of the errors appeared to relate to inconsistent application of the unit of measure in switching to the fractional component of the remainder **on the object being measured**.



| | |
|---|---|
| P3. Exactly how many of the yellow strip of paper equals the length of the blue strip of paper? $1\frac{1}{2}$ | because it eqauls $\frac{3}{4}$ of the blue strip of paper |

**Figure 5.130** An example of losing the unit of measurement and exchanging units

## *Question 34*

In Question 34, students were asked to determine what fraction of the blue strip of paper the yellow strip of paper represented. The yellow strip was 15 cm long and the blue strip was 20 cm long. These were the same strips of paper used in Question 33. The results for Question 34 are shown in Table 34.

Table 34

*Percentage distribution of responses to Question 34, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{3}{4}$ | 34 | 51 | 68 | 52 | 61 |
| $\frac{1}{4}$ | 10 | 6 | 3 | 9 | 6 |
| $\frac{2}{3}$ | 4 | 5 | 5 | 5 | 6 |
| $1\frac{1}{4}$ | 7 | 9 | 4 | 3 | 4 |
| $\frac{1}{3}$ | 4 | 6 | 1 | 6 | 3 |
| $1\frac{1}{3}$ | 3 | 4 | 6 | 3 | 5 |
| Other | 21 | 14 | 10 | 14 | 8 |
| Omitted | 17 | 6 | 2 | 8 | 7 |

In total, 53% of the students correctly determined that the yellow strip of paper was three-quarters of the blue strip. Recognising the fractional relationship between the two strips of paper (Question 34) was easier than measuring one strip by using the other strip of paper (Question 33). Further, an incorrect answer of $1\frac{1}{4}$ to Question 33 was not an obstacle to providing a correct response to Question 34.

The correct answer $\frac{3}{4}$ was arrived at in a number of different ways. For example, the Year 4 student whose reasoning is displayed in Figure 5.131 focused first on the remaining length.

| | |
|---|---|
| P4. What fraction of the length of the blue strip of paper is the yellow strip of paper? 3/4 | 20 - 15 = 5 20 ÷ 5 = 4 so 5 is a quarter 5 × 3 = 15 so 15 equals 3 quarters |

**Figure 5.131** A Year 4 response using a unitary method involving fractional units

Reasoning through an exchange of units involving subdividing the blue strip, could be achieved through repeated halving. One Year 6 student (Figure 5.132) divided the blue strip into eight parts rather than four to arrive at the answer.

| | |
|---|---|
| P4. What fraction of the length of the blue strip of paper is the yellow strip of paper? 3/4 | Fold the blue strip into 8 parts, place yellow strip on top. The yellow strip is EXACTLY 6/8 of blue strip. 6/8 = 3/4 |

**Figure 5.132** An explanation of subdividing the blue strip into eighths to answer Q34

As well as dividing the blue strip to act as a ruler in measuring the yellow strip as in Figure 5.132, other students (Figure 5.133) constructed a ratio comparison by dividing both strips into different numbers of parts.

| | |
|---|---|
| P4. What fraction of the length of the blue strip of paper is the yellow strip of paper? 3/4 | Folded the blue strip into 1/4's and the yellow strip into 1/3's and they were equale. |

**Figure 5.133** An explanation of creating a ratio comparison

This type of response was rare as it relies in part on being able to fold the yellow strip into thirds. The low percentage of students able to do this from Question 33 explains why this was not a high frequency response. Repeated halving is a much easier way to divide a unit than creating thirds. Consequently, most correct answers divided the blue strip rather than the yellow strip.

A few students (Figure 5.134) recognised the reciprocal relationship between the questions, although this was not a common response.

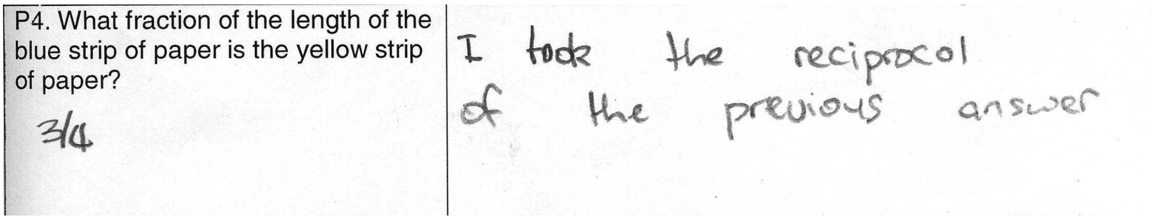



**Figure 5.134** A Year 5 response acknowledging the reciprocal relationship

For those who answered $\frac{1}{4}$ the focus appeared to be on what was left. For example, in Figure 5.135 the Year 8 student simply stated that the answer is $\frac{1}{4}$ because that is the amount left.

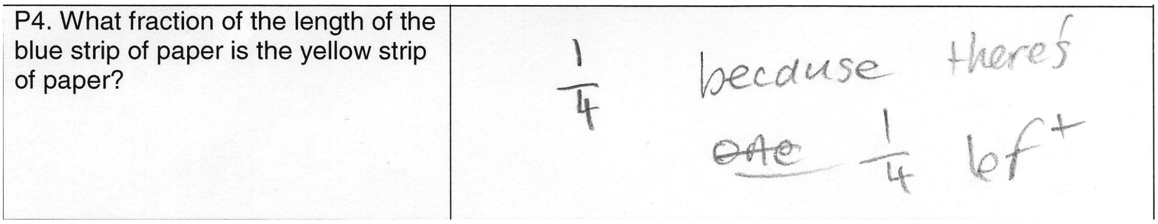



**Figure 5.135** A response describing the fraction remaining

Again, many of the incorrect answers appeared to be due to shifting focus from the referent whole. Students who answered $\frac{2}{3}$ also appeared to be referencing the remainder but comparing the remainder of the blue strip to the yellow. For example, the response in Figure 5.136 compares the visible remainder of the blue strip to the length of the yellow strip.

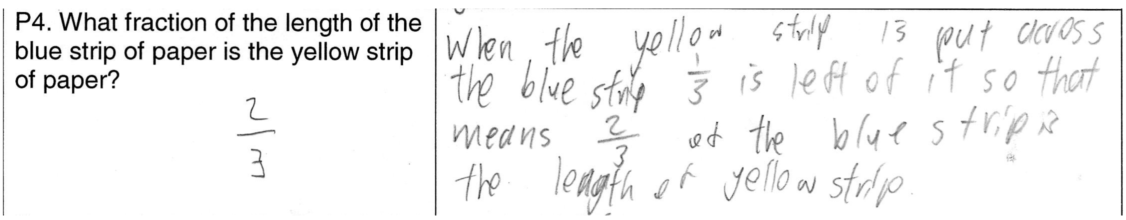



**Figure 5.136** An example of confounding the part-whole comparison resulting in $\frac{2}{3}$

The incorrect answers of $1\frac{1}{4}$, $\frac{1}{3}$ and $1\frac{1}{3}$ can also be attributed to confounding the part-whole comparisons and losing focus on the referent whole.

*Question 35*

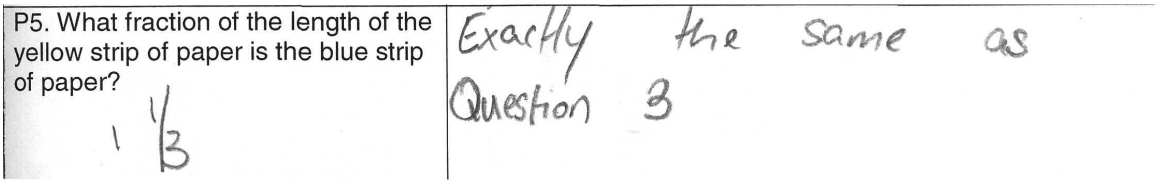
In Question 35, students were asked to reverse the comparison and determine what fraction of the yellow strip of paper the blue strip of paper represented. The same yellow strip (15 cm long) and blue strip (20 cm long) were used as in the previous question. The results for Question 35 are shown in Table 35.

Table 35

*Percentage distribution of responses to Question 35, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{4}{3}$ | 7 | 18 | 32 | 10 | 18 |
| $1\frac{1}{4}$ | 13 | 22 | 27 | 17 | 17 |
| $\frac{3}{4}$ | 20 | 17 | 11 | 20 | 22 |
| $\frac{1}{4}$ | 4 | 5 | 5 | 9 | 6 |
| $\frac{1}{3}$ | 2 | 5 | 1 | 4 | 4 |
| Other | 27 | 24 | 18 | 25 | 18 |
| Omitted | 27 | 10 | 6 | 15 | 15 |

Although some students recognised that Question 35 was a restatement of Question 33, the majority of students found this question very difficult. Overall, 17% of students were able to correctly answer Question 35 (this was the second most difficult item of the 37 questions).

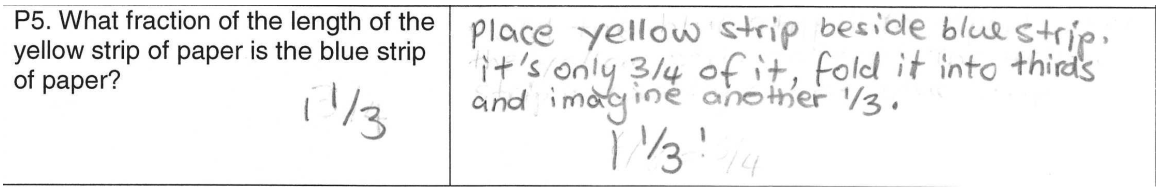

P5. What fraction of the length of the yellow strip of paper is the blue strip of paper?

1 1/3

Exactly the same as Question 3

**Figure 5.137** A response recognising the fraction link to measuring in Q33

Some students clearly recognised the link between measuring with a given unit and fractions (Figure 5.137). Still others had a reversible sense of comparisons of units (Figure 5.138) that underpins the reciprocal operator.


P5. What fraction of the length of the yellow strip of paper is the blue strip of paper?

1 1/3

place yellow strip beside blue strip, it's only 3/4 of it, fold it into thirds and imagine another 1/3.

1 1/3!

**Figure 5.138** An example of a reversible comparison of units

Many of the problems identified in the incorrect responses were similar to those in Question 33. When the fraction involved exceeded one whole, many students had difficulty in maintaining their focus on the correct unit being referenced. Instead of considering the remainder as a fraction of the yellow strip, those who answered $1\frac{1}{4}$ considered the remainder as a fraction of the thing being measured, rather than the unit of measurement.

| | |
|---|---|
| P5. What fraction of the length of the yellow strip of paper is the blue strip of paper? $1\frac{1}{4}$ | seeing as the yellow paper is smaller by a $\frac{1}{4}$ to the blue paper, it is $1\frac{1}{4}$ because its also 1 length longer. |

**Figure 5.139** A Year 8 response that confounds the referent whole

Again, answers such as $\frac{3}{4}$ came about from reversing the order of comparison, $\frac{1}{4}$ from referencing the remainder with a reversed comparison and $\frac{1}{3}$ from only referencing the remainder. As they were dealing with the remainder, those students who answered $\frac{1}{4}$ or $\frac{1}{3}$ tended to discount the idea that a fraction could be something greater than one.

Given the practical similarity between Questions 33 and 35, it is interesting to note that the two most frequent categories were the same. After these two categories, the pure measurement aspect of Question 33 appears to have influenced the next three most frequent categories. The answers $1\frac{1}{2}$, 1 and $1\frac{3}{4}$ did not appear in the top five categories for Question 35. The measurement aspect of Question 33 may have resulted in greater acceptance of "rounded" answers. Of the 25 students who answered $1\frac{1}{2}$ to Question 35, 9 also answered $1\frac{1}{2}$ to Question 33.

Some students may have read Questions 34 and 35 as being the same question. In total, 9% of all students gave the same answer to Questions 34 and 35. It is not possible to tell whether this was due to a reading processing problem or a lack of understanding that the order of comparison is important in determining fractions. That is, as the words were identical in the two questions, did these students fail to notice the reversed location of the words "yellow" and "blue", or was it simply considered to result in the same fraction?

**Looking back**

In making a measurement comparison between two lengths, many students lost track of the unit being used for the measurement and treated the remainder with reference to the object being measured. That is, the referent unit of measurement was readily confused with the object being measured. When the remainder was one-third of the unit of measure, students were equally likely to confuse the unit of measurement with the remainder. Fractional remainders arising when measuring with a specified unit were sometimes ignored. Even when the remaining fraction is half of the unit of measure, about 5% of students' responded with whole numbers. Reciprocal relationships between lengths appeared to be very difficult, especially comparison that resulted in an answer greater than one. When the fraction exceeded one whole, the unit of reference appeared to shift for many students.

## 5.14 Composition of partitioning: Q36–37

Questions 36 and 37 were designed to look for evidence of composition of partitioning, Pothier and Sawada's (1983) proposed fifth level of partitioning. The questions used paper shapes (a half circle and a square) and required students to make fraction parts that could result from appreciating repeated partitioning.

### *Question 36*

In Question 36, students were provided with half of a circular piece of paper and asked to fold the paper and to shade the part of the folded paper corresponding to one-sixth of the whole circle. The half circular disk had a diameter of 12 cm. To complete the process of composition of partitioning students needed to recognise that one-sixth of a circle could be constructed by finding one-third of one-half of a circle. The first partition was considered completed, as students were provided with a half-circle. The results for Question 36 are shown in Table 36.

Table 36

*Percentage distribution of responses to Question 36, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{1}{6}$ | 12 | 26 | 41 | 19 | 33 |
| $\frac{1}{8}$ | 20 | 21 | 18 | 22 | 23 |
| $\frac{1}{16}$ | 7 | 8 | 5 | 10 | 6 |
| $\frac{1}{4}$ | 10 | 7 | 4 | 7 | 5 |
| $\frac{3}{16}$ | 2 | 4 | 2 | 6 | 6 |
| Parallel partition | 18 | 16 | 13 | 9 | 4 |
| Other | 28 | 18 | 16 | 28 | 23 |
| Omitted | 3 | 1 | 1 | 0 | 1 |

Overall, 26% of the students could complete a process similar to composition of partitioning to create one-sixth. Most of the students who managed to create one-sixth folded the half-circle into thirds, sometimes with obvious adjustments to make three equal parts. An ingenious variation to this method was to construct two-thirds of one-quarter, as shown in Figure 5.140. The two-thirds was formed by folding a sector equivalent to half the resulting shaded sector.

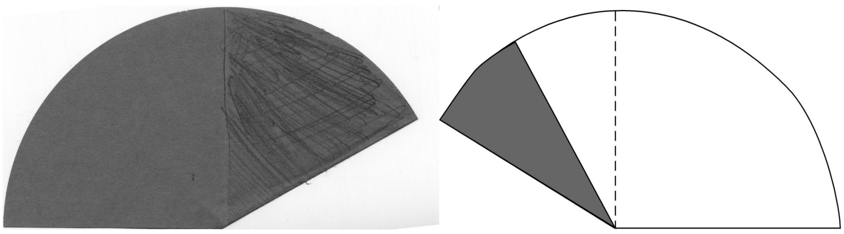


**Figure 5.140** Example of $\frac{1}{6}$ as $\frac{2}{3}$ of $\frac{1}{4}$ with a diagram of the reverse side folds

The method of creating $\frac{2}{3}$ of $\frac{1}{4}$ uses a visual halving to create one-third of the quarter of a circle. An overlap (the grey sector in the diagram on the right of Figure 5.140) is progressively moved until the resulting image creates two identical pieces, equal in area to the third uncovered piece.

Folding the half of a circle into halves, and then halving one of these quarters of the whole circle (Figure 5.141) created one-eighth of the circle. This method resulted in breaking the half circle into three pieces.

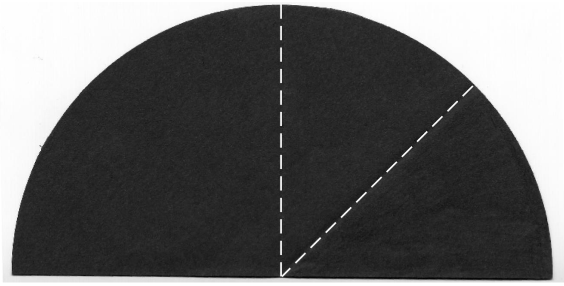


**Figure 5.141** An example of forming half of a quarter as three pieces

A symmetric fold into three parts could also result in a quarter being bounded by two one-eighths, as in Figure 5.142.

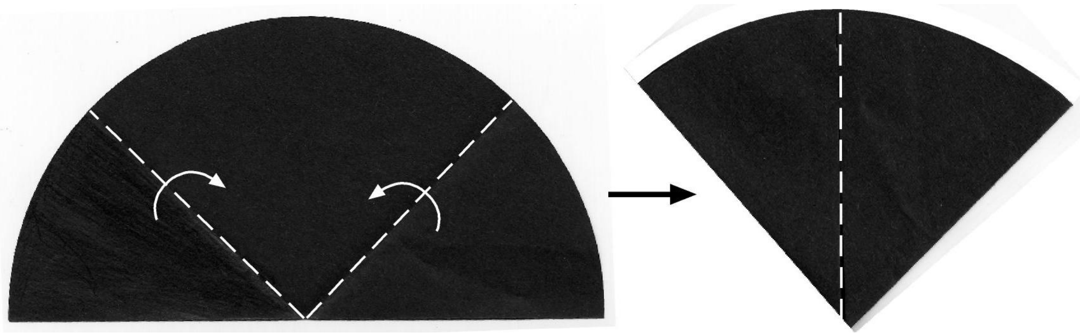
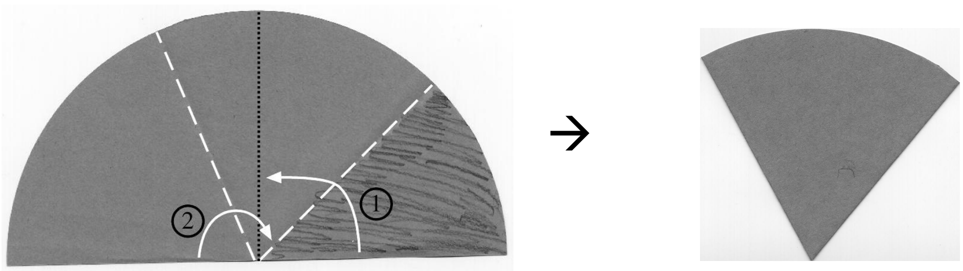


**Figure 5.142** A symmetric fold producing $\frac{1}{4}$ and two $\frac{1}{8}$ sections

Depending on which part was shaded this could result in an answer of one-eighth or one-quarter. Other folds subdividing the half circle into three parts resulted in one-eighth and three-sixteenths. By folding one edge to the vertical (fold 1 on the left of Figure 5.143), one-eighth could be created. Next, folding the remaining straight edge to align with the first fold line (fold 2) halved the three-eighths resulting in three parts: one-eighth and two three-sixteenths.




The two folds to create $\frac{1}{8}$ and $\frac{3}{16}$ | The folded result showing $\frac{3}{16}$

**Figure 5.143** Creating sectors representing $\frac{1}{8}$ and $\frac{3}{16}$

Depending on which piece was shaded, this could result in an answer of one-eighth or three-sixteenths.

As well as repeated halving to create one-sixteenth, there were those students who shaded one-sixteenth as one of six parts of the half-circle (Figure 5.144).

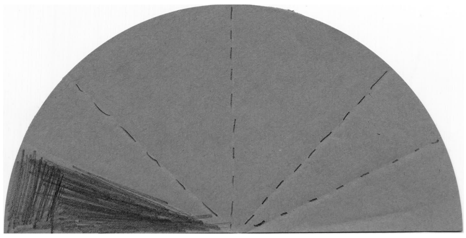


**Figure 5.144** An example of one-sixteenth as one of six parts

Consequently, one-sixteenth could have resulted from repeated halving or have been one way of depicting “one-sixth” of the half-circle. Folding the half circle usually resulted in attempts to create three parts or six parts, with the equality of the parts often being a secondary constraint. Parallel partitioning was evident in similar levels to responses to Questions 13 and 14. Folding one-third of one-half also appeared to be more difficult than shading one-third of a circle in Question 13, presumably because the task removes the advantage of having a prototypic image of one-third of a circle.

## *Question 37*

Question 37 continued the challenge of composition of partitioning or recursive partitioning. Students were provided with a coloured square of paper, edge length 10 cm, and asked to fold it to show one-ninth of the square. Partitions requiring an initial cut other than a median cut require thinking about how the number of partitions can fit into the unit and were expected to be more difficult than those achieved by repeated halving. Consequently, it was expected that folding a square to show one-ninth would be more challenging than if students had been asked to show one-sixteenth of a square. The results for Question 37 are shown in Table 37.

Table 37

*Percentage distribution of responses to Question 37, by year*

| | Year 4 (N=331) | Year 5 (N=330) | Year 6 (N=331) | Year 7 (N=342) | Year 8 (N=342) |
|---|---|---|---|---|---|
| $\frac{1}{9}$ | 14 | 30 | 45 | 21 | 37 |
| $\frac{1}{8}$ | 12 | 12 | 11 | 15 | 14 |
| $\frac{1}{16}$ | 4 | 8 | 5 | 14 | 6 |
| 1 of 9 unequal parts | 5 | 9 | 9 | 4 | 2 |
| $\frac{9}{16}$ | 3 | 4 | 1 | 4 | 3 |
| Incomplete | 21 | 13 | 5 | 18 | 13 |
| Other | 35 | 23 | 23 | 25 | 24 |
| Omit | 5 | 1 | 1 | 1 | 1 |

Overall, 30% of the responses included a shaded part of the square corresponding to one-ninth. As the focus of the question was on composition of partitioning, the way that students created one-ninth of the square was important.

**Methods of partitioning the square**

Folding can be used in a number of different ways to partition a square. Folding in one direction, usually vertically or horizontally, is described here as uni-directional or 1-directional folding. Bi-directional or 2-directional folding was typically achieved by folding in two perpendicular directions aligned to the sides, as in Figure 5.150. Quilting was a term initially used to describe 4-directional folding. In Figure 5.145, 4-directional folding (quilting) has been used before a part corresponding in area to one-sixteenth of the square has been shaded.

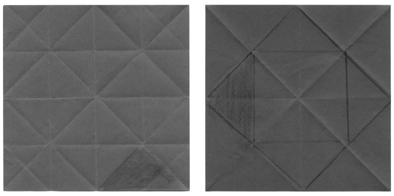


**Figure 5.145** Examples of 4-directional folding (quilting) showing $\frac{1}{16}$

Uni-directional folding included repeated halving, fan-folds and roll-folds, as described in Section 4.9 *Coding folding*. The uni-directional roll-fold invariably resulted in the individual sections increasing in size, as each new section was formed by being rolled around the outside of an accumulating width of paper. As a partitioning strategy, the roll-fold was likely to result in the correct answer more by good fortune rather than good practice.

The request to construct one-ninth of a square sought to determine how frequently students used composition of partitioning, particularly in two directions. Sometimes students managed to correctly partition the square in to three equal parts in one direction but failed to do this is the perpendicular direction.

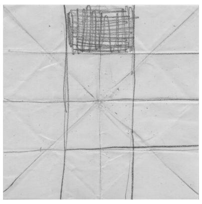


**Figure 5.146** An example of correct partitioning in one of two directions

In Figure 5.146, this Year 8 student has constructed one-twelfth through the composition of a 3-partition with a 4-partition. Although this was a low frequency response, a similar composition of partitioning error of a 4-partition followed by a 3-partition using uni-directional folding was recorded. The relative ease of creating a 4-partition meant that it was evident even in the presence of a correct 3-partition.

Folding methods frequently did not produce the same answers from the same type of folding. Indeed, 1-directional, 2-directional and 4-directional folding contributed to all of the major categories of responses recorded in Table 37. In total, 20% of the students created one-ninth by folding in thirds in one direction and then folding in thirds in a perpendicular direction (2-directional folding). Less than 3% successfully carried out 1-directional folding to record one-ninth and 6.3% achieved it by other methods, often combining inaccurate folding methods with final "visual adjustments" to the shaded part of the square. The composition of partitions resulting in one-ninth of the square could be achieved through 1-directional folding. In Figure 5.147, the student has used 1-directional folding to construct thirds before folding to form one-third of a third.

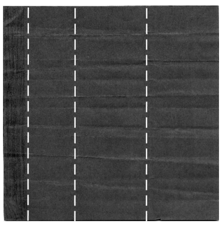


**Figure 5.147** An example of recursive 1-directional folding to create $\frac{1}{3}$ of $\frac{1}{3}$

Overall, 13% of students constructed one-eighth in response to Question 37 and this was the second highest coherent category. The diverse range of methods that could result in one-eighth may have influenced the size of this category, as well as the primitive process of algorithmic halving. The ease of repeated halving undoubtedly influenced the creation of partitions leading to eighths and sixteenths and led to 7.6% of all answers indicating $\frac{1}{16}$ of the square. In addition to being much easier to construct, eighths are also very close in size and number to the required ninths.

Initially, $\frac{2}{16}$ of the square was coded as a separate response from $\frac{1}{8}$ of the square. Bi-directional folding resulting in $\frac{2}{16}$ being shaded occurred in two distinct ways. One method was to create a 4 x 4 partition and shade in two parts. The second type of folding, a 2 x 8 partition, resulted in what was described as “long sixteenths”. Overall, 2.0% of all responses were initially coded as two-sixteenths before being combined in the one-eighth category.

In addition to the correct answer, shading one part out of nine parts also led to the creation of the category of shading one of nine unequal parts. When students were not able to create nine equal parts by partitioning the square they sometimes nevertheless knew that they had to show one part of nine. The Year 8 student’s method in Figure 5.148 uses repeated halving to form eighths before halving one of the eighths to form nine parts.

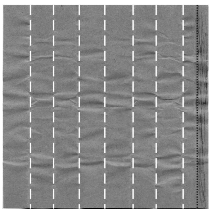


**Figure 5.148** An example of representing $\frac{1}{9}$ as half of $\frac{1}{8}$

When students folded the square into eighths and then halved one of the eighths to make nine parts, often the recorded shaded part was one-eighth rather than one-sixteenth. As with “one-quarter” in Questions 13 and 14, the intent appeared to be to shade one part out of nine parts rather than to shade either one-eighth or one-sixteenth. Answers of $\frac{1}{8}$ or $\frac{1}{16}$ were recoded according to how they were formed.

The intent to shade one part out of nine parts is quite clear in Figure 5.149, despite an area of one-eighth being shaded. The shaded area clearly does not capture the student’s intent.

The first eight pieces are equal but cover the whole square. Halving one of the corners forms the ninth piece, which effectively has the same area as the initial pieces.

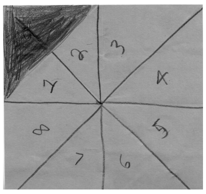


**Figure 5.149** An example of numbering the parts to show one part of nine parts

When eighths were formed prior to halving one of the eighths, this was coded as one part of nine (halving). These responses contributed to the new category of “1 of 9 unequal parts”. Overall, 5.8% of students represented $\frac{1}{9}$ as one of nine unequal parts.

**A number of parts interpretation**

Partitioning a square to create a regional representation of a fraction usually resulted in the formation of smaller squares or rectangles. When each new rectangle or piece was formed it became a countable item and could also be used to represent a unit fraction as a number of parts corresponding to the denominator. The answer of $\frac{9}{16}$ reflected this number of parts interpretation of the fraction $\frac{1}{9}$. In Figure 5.150, the Year 8 student has formed sixteenths using 2-directional folding before counting and shading nine of the squares to represent one-ninth.

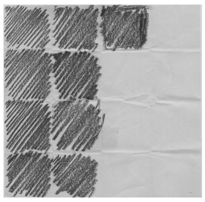


**Figure 5.150** An example of shading $\frac{9}{16}$ to represent $\frac{1}{9}$

The “incomplete” category corresponded to a large number of students who tried different folding patterns that did not result in an interpretable response. Incomplete responses involved folding patterns of all different types without any section shaded. One Year 8 student indicated that she couldn’t “get 9” through folding. The size of the ‘other’ category suggests that she may not have been alone. The “other” category included a diverse range of responses including those who measured 9mm to create a $\frac{1}{9}$-unit, producing $\frac{1}{11}$ instead.

**Looking back**

The utility of algorithmic halving influenced students' use of composition of partitioning. When provided with half of a unit and asked to create one-sixth of the unit, the dominance of algorithmic halving contributed to the majority of answers being formed through halving (eighths, sixteenths, quarters or parallel partitions). Even when representing one-sixth as one of six pieces, halving was sometimes used to form the pieces. Similarly, when asked to create one-ninth of a square, algorithmic halving resulted in incorrect responses being dominated by answers involving eighths and sixteenths.

In general, students tried to represent $\frac{1}{9}$ of the square by shading one part out of nine parts (equal or unequal parts), shading nine parts, or using repeated halving before shading one part (producing $\frac{1}{8}$ and $\frac{1}{16}$). The category of answers attempting creating nine parts was heavily infused with algorithmic halving. In addition to the prevalence of halving as a partitioning strategy, students' responses indicated that the fraction interpretations of "parts out of total parts" or number of parts (e.g., $\frac{9}{16}$) persisted.

# Chapter 6 RASCH ANALYSIS

*The Rasch analysis of the fraction tasks treats the responses as dichotomous in terms of whether they have communicated a sense of the size of fractions. Tasks meeting the conditions of the Rasch model can be ranked from easiest to hardest in a way that does not depend upon the ability of the students who attempted the tasks. The Rasch model also provides a method of determining how well the items describe a unidimensional construct and identify specific tasks for further investigation.*

**Preview**

This chapter describes the measures used with the Rasch model to indicate how well items fit the model, in particular "infit" and "outfit", and how they are used to investigate the construct of a quantitative sense of fractions. The relative difficulty of the fraction questions is summarised in an item map that uses the logarithm of the odds of answering the question correctly as a common scale. An analysis and discussion of the relative difficulty of the various elements of the fraction concept evoked by the tasks follows.

## 6.1 Introduction

The second phase of analysis employed the simple logistic model to carry out a Rasch analysis. Rasch analysis, a specific form of Item Response Modelling, can be used to understand the data rather than to simply model the data. The Rasch model is based on the property that the chance that a student will answer a question correctly depends on his or her ability, in this case ability to deal with fractions quantitatively, and the difficulty of the question. In particular, Rasch analysis is used to answer two questions:

1. Is there evidence of a quantitative sense of fractions as a construct?
2. What are the relative difficulties of the items used to assess a quantitative sense of fractions?

## 6.2 An overview of the Rasch model

The Rasch item response model, or simple logistic model, is a probabilistic model of measurement. For the simple logistic model, the questions are coded as right or wrong (dichotomous). The probability that a student will answer an item correctly is a function of an ability parameter (pertaining to that student) and a difficulty parameter (pertaining to the item). The item difficulty parameter is sometimes referred to as a threshold. The probability of answering a question correctly can be described as a function of the difference between the ability of a person and the difficulty of the item. The Rasch model assumes that the probability that a student will correctly answer a question is a logistic function of the difference between the student's ability [θ] and the difficulty of the question [δ] (i.e. the ability required to answer the question correctly) (Wright & Stone, 1979).

$$p\left(X_{ni} = 1 \middle| \theta_n, \delta_i\right) = \frac{e^{(\theta_n - \delta_i)}}{1 + e^{(\theta_n - \delta_i)}}$$

The Rasch one-parameter logistic model enables precise measurement of the difficulty of each task or item and a way to determine the association of each item with the construct being measured. The ability and difficulty scale arising from the Rasch one parameter logistic model have a common log-odds measurement unit called the *logit*. This common unit creates the opportunity to plot the items on a scale according to their difficulty (as determined by the students' responses) and to locate the students according to their measured sense of fractions on the same scale. The logit is the natural logarithm of the odds ratio of success. If a person has a 0.5 probability of answering a question correctly the odds ratio is 1:1 (or "even money") and the natural logarithm of this ratio is zero. In Figure 6.1 a person whose ability was equal to the difficulty of the item would be located at the zero mark on the horizontal scale and those with greater ability would have a greater probability of correctly answering the question.

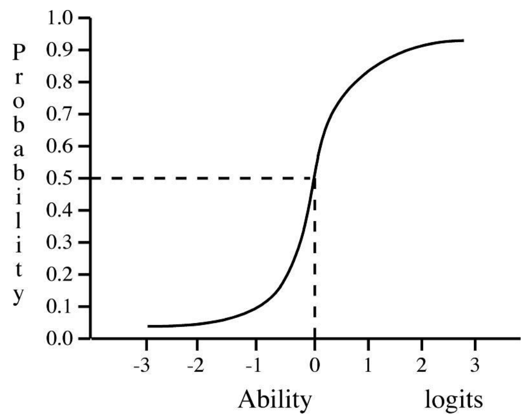



**Figure 6.1** A logistic response curve

The logits form the basis of an interval scale and can be averaged on a like with like basis to determine the average score of a sense of fractions of students making a particular response.

## 6.3 Measures of data fit to the Rasch model

Not all items perform in a way that is similar to the idealised logistic response curve. To determine how well the items used in this study fit the model, the students' responses were analysed using the Quest program, a Rasch modelling tool developed by Adams and Khoo (1999). The Quest program produces an estimate of the relative difficulty of each item and a series of measures of how well the data fit the model. The agreement between the model and the data can be measured by the chi-square goodness-of-fit index. Each time a student answers a question there is the expected value of his or her response, according to the Rasch model, and the observed value of the response. A residual is described as the difference between the observed value and the expected value. The squares of these residuals (squared to take account of the magnitude of the values when summed) are combined in average measures of fit. The two mean-square residual summary statistics commonly used with the Rasch model are referred to as measures of "infit" and "outfit" (Wright & Linacre, 1994).

Infit was devised as a statistic that down-weights outliers and focuses more on the response string close to the item difficulty (or person ability). The infit statistic is based on the chi-square statistic with each observation weighted by its statistical information (model variance) and may be described as the *average of the squared standardised deviations of observations from their expectations.* Infit is sensitive to irregular patterns of responses for items that are close to an individuals ability level. For ease of interpretation, infit is reported in mean-square form by dividing the weighted chi-square by the sum of the weights.

Outfit is an outlier-sensitive fit statistic, based on the conventional chi-square statistic, which picks up rare events that have occurred in an unexpected way. Outfit is more sensitive than infit to unexpected responses by persons on items that are relatively very easy or very hard for them. More formally, outfit is the *average of the squared standardised deviations of the observed performance from the expected performance*. For ease of interpretation, this chi-square is divided by its degrees of freedom to have a mean-square form and reported as outfit.

Values of infit and outfit close to 1.0 indicate that the variability in the responses is close to the variance expected by the model. Values greater than 1.0 (underfit) indicate that the variability in the responses is greater than the variability expectations of the model, and values below 1.0 (overfit) suggest that the variability in responses is influenced by a covariance term. Values less than 1.0 indicate that the model predicts the data too well, causing summary statistics, such as reliability statistics, to report inflated statistics. A mean-square residual statistic such as 1.2 indicates that there is 20% more noise in the data than modelled.

The items were specifically designed to address students' quantitative sense of fractions, that is, the items were designed to have face validity, and the fit statistics give an indication of the fit and the width of the variable under consideration. In particular the infit statistics quantify coherence with the core variable —a quantitative sense of fractions— underlying the interactions between respondents and items.

Compliance with the Rasch model's characteristics can be demonstrated by a high, but not too high, probability of predicting success in a task by success on other tasks. As the Rasch model is probabilistic and not deterministic, some failure of the model to predict the observed values is expected. When the discrepancy between modelled and observed values is optimal, the mean value of the weighted mean square residual has an ideal mean value of 1.0 (Wright & Masters, 1982).

INFIT MNSQ values (used in Quest to report infit mean square) that range from 0.8 to 1.2 are deemed to be acceptable infit statistics for high stakes multiple choice questions with a wider range (0.7 to 1.3) acceptable for other forms of assessment (Wright & Linacre, 1994). Values of the INFIT MNSQ greater than one indicate that the observations are too unpredictable and contain unmodelled noise. High values of INFIT MNSQ are a much greater threat to validity than low mean-squares values. Low values of infit suggest that the responses are too predictable and overfit idealised Guttman patterns. A Guttman pattern is an idealised separation of items by order of difficulty with all of the easiest items being answered correctly and all of the hardest items being answered incorrectly. A very low value of the infit measure indicates that even the random variability expected by the model is missing.

## 6.4 Performance of the item set

Only five perfect scores were recorded from the sample and there were no scores of zero. The 37 items in the current study had a mean overall infit mean square value of 0.99 with a standard deviation of 0.13 and a mean overall outfit mean square value of 1.14 with a standard deviation of 0.48. Both of these measures of the fit of the test are close to the ideal value of one. That is, globally, the items appear to relate to a construct described as a quantitative sense of fractions.

The reliability of the case estimates, sometimes referred to as the person separation index, was 0.93 on a possible range from zero to one, suggesting that in addition to the items fitting the model well, the scale is reliable in the sense of being internally consistent. (The person separation index is used in modern test theory instead of reliability indices as an estimate of internal consistency reliability.)

## 6.5 Performance of individual items

For individual items, the mean square infit statistics provided a measure of how well the item fits the model. The mean square infit statistics (reported as INFIT MNSQ) range from 0.76 for item 26 to 1.35 for item 34 (see Appendix B). The infit statistic is considered to be more robust than the outfit statistic as a test of item-model fit because the outfit statistic is particularly sensitive to a few individual responses misfitting.

As high values of INFIT MNSQ are a much greater threat to validity than low mean-squares values, and as mean-squares averaging to 1.0 result in high values influencing low values, the items with highest values of INFIT MNSQ will be examined first. Indeed, none of the items had an INFIT MNSQ value markedly below 0.8. Question 34 had the highest value of INFIT MNSQ (1.35) and warrants examining, with the next highest being 1.21 for Question 19.

Using the Rasch Unidimensional Measurement Models (RUMM) 2020 software (Andrich, Sheridan, & Luo, 2002) the behaviour of this item is compared to the theoretical item characteristic curve (ICC) in Figure 6.2.

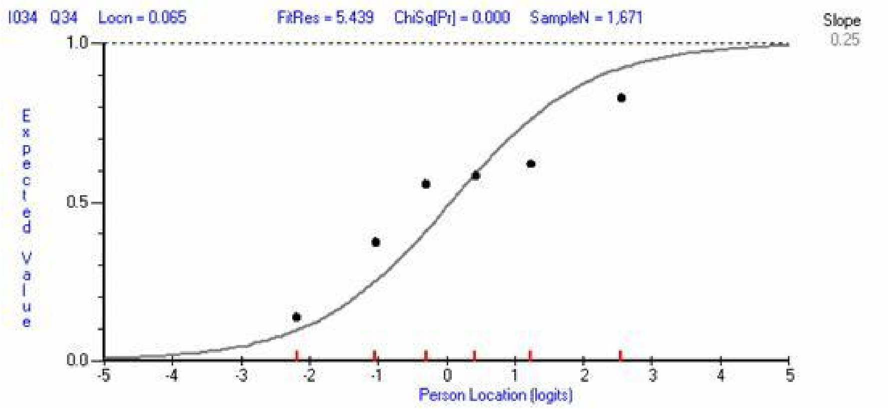



**Figure 6.2** Comparing responses on Q34 to the ICC

The actual response curve indicated by the data points tends to flatten out in the middle of the person ability range compared to the theoretical ICC. That is, lower ability students performed better than the model predicted and more able students did not perform as well as expected. To see if this effect was influenced by the performance of specific groups (school stage), a further analysis was carried out. The school mathematics curriculum is organised in 2-year stages. Stage 2 refers to Years 3 and 4, Stage 3 to Years 5 and 6, and Stage 4 to Years 7 and 8. In Figure 6.3, the flat middle section is still evident across the school stages, although it is slightly less pronounced for Stage 2 (corresponding to Year 4 in this study).

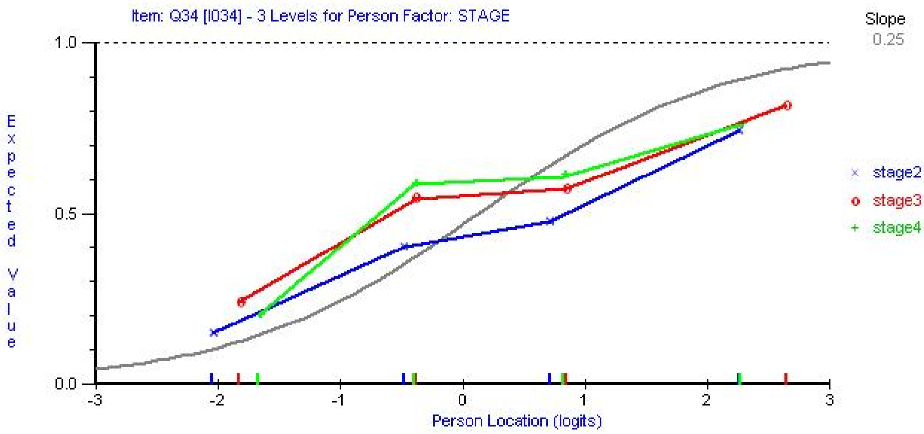



**Figure 6.3** Comparing the ICC by school stage

There appears to be a lack of discrimination for average ability students or some kind of threshold effect occurring with Question 34. As students' quantitative sense of fractions moves from negative to positive (about –0.5 logits to +0.5 logits) the probability of obtaining the correct answer is almost constant. This plateau across an interval of one logit indicates that a substantial increase in a quantitative sense of fractions is required before the likelihood of a correct response increases. Even then, more able students find the item harder than the model predicts.

Question 34 asked students what fraction of the length of the blue strip of paper (20 cm long) is the yellow strip of paper (15 cm long). The levelling out across the middle of the ability range may be due to the application of a basic method, such as repeated halving to subdivide the blue strip, being used with similar outcomes for a broad range of student ability. That is, the widespread application of the very basic but effective strategy of repeated halving of the blue strip might have lowered the discrimination of the item in the middle of the ability range. The popularity of the strategy of repeated halving of the blue strip is evident even in the incorrect answer of $\frac{1}{4}$. For those who answered $\frac{1}{4}$ (6.7%), the focus appeared to be on what was left. For example, in Figure 6.4 the Year 8 student simply states that the answer is $\frac{1}{4}$ because that is the amount left.

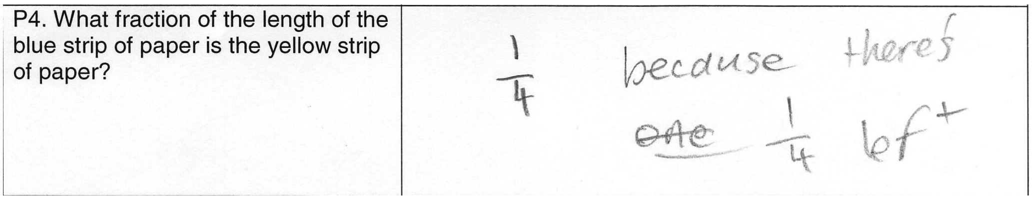



**Figure 6.4** An example of the possible influence of repeated halving

The widespread applicability of a basic solution method appears to have reduced the discrimination of this item. However, mean-square fit statistics between 0.5 and 1.5 are considered productive for the purpose of measurement (Bond & Fox, 2001) and this lack of discrimination does not warrant removing the item.

The next largest INFIT MNSQ measure is associated with Question 19 (INFIT MNSQ = 1.21). Question 19 asked students to indicate the point half way around an equilateral triangle, starting from the apex. Looking at the item characteristic curve (ICC) for Question 19 by school stage (Figure 6.5) the flatness in the middle of the curve is most pronounced for Stage 3. This may well have been influenced by the application of a simple strategy of vertical halving of the triangle. The lack of discrimination near the centre of the curve is to be expected if the solution is formed by a primitive halving action in the visually preferred vertical orientation. The use of a basic halving action may also explain why the performance of lower ability students was higher than the model predicted.

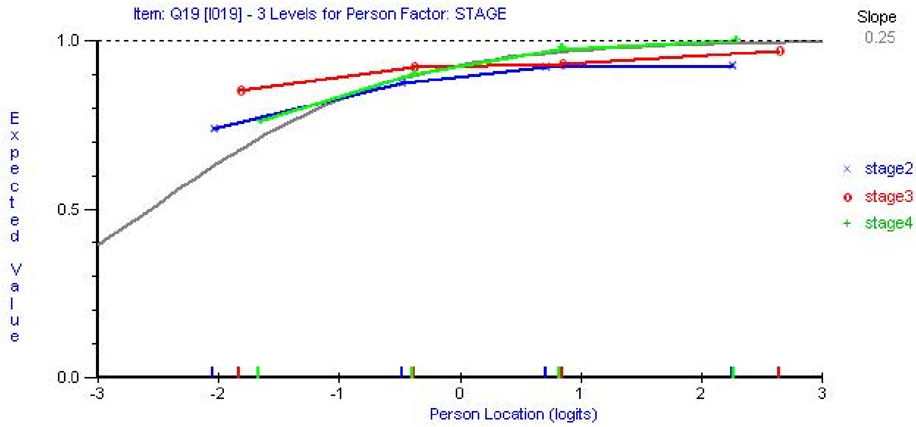



**Figure 6.5** Comparing the ICC for Q19 by school stage

Moreover, the item is relatively easy (–2.9 logits) and the overall ICC is quite flat, suggesting that high levels of discrimination in the middle of the curve would not be expected. If the purpose of the Rasch analysis had been to construct a test, these two items might have been removed and the item statistics recalculated. However, in this study the Rasch model is used, not for the purpose of test construction, but rather as an investigative tool. To that end it has highlighted questions where the responses appeared to fail to discriminate near the item location, possibly influenced by the dominant use of a basic halving strategy.

There were two items with INFIT MNSQ values marginally below 0.8 — Question 26 with an INFIT MNSQ value of 0.76 and Question 5 with an INFIT MNSQ value of 0.78. Question 26 required students to indicate whether $\frac{2}{3}$ or $\frac{5}{6}$ was the bigger number and to justify the answer. There was a marked drop in the percentage of Year 7 students successfully providing a correctly reasoned answer to Question 26 compared to Year 6. In Figure 6.6 the more able students performed better than predicted by the model and less able students performed worse than predicted. The performance of Stage 4 students on this question is also lower than predicted.

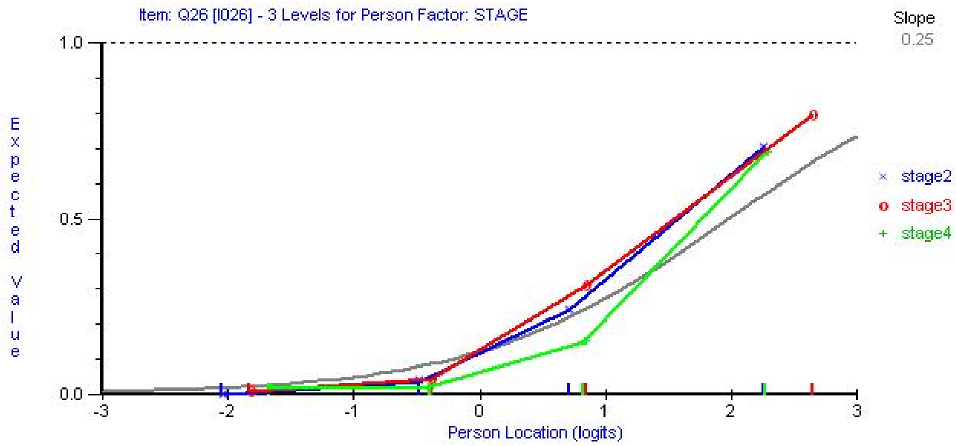



**Figure 6.6** Response curves to Q26 by school stage

In total, 49% of Year 7 suggested the correct answer to Question 26 with incorrect or inadequate reasoning. The introduction of non-unit fractions in Question 26 removed the option of successfully using a strategy (bigger denominator means smaller fraction) that might have helped to achieve correct answers to other fraction questions. Across all of the items the strategy “bigger is smaller” was usually most prevalent in Year 8, except in Question 26 were it was more prevalent in Year 7 in providing the wrong reason. Over 5% of Stage 4 students (Years 7 and 8) incorrectly used “bigger is smaller” reasoning to answer Question 26 and another 5% incorrectly used “fraction in the difference” reasoning. Having been taught how to find equivalent fractions in Stage 4 may have actually disadvantaged this group in comparing non-unit fractions. In contrast, having a range of methods to determine the correct answer may have aided the performance of the more able students on this question and contributed to the slight overfit of the item.

Question 5 required students to determine the fraction of a regular hexagon represented by an equilateral triangle. The item characteristic curve by stage for Question 5 has been discussed (see Section 5.2) in seeking to understand the marked drop in the percentage of Year 7 students correctly answering the question compared to Year 6 students. The correct answer of $\frac{1}{6}$ was less frequently recorded in Year 7 than $\frac{1}{3}$ (possibly one part out of three) or $\frac{1}{4}$ (possibly incorrect visual image for one-quarter or an over-generalisation of the term one-quarter). The low value of INFIT MNSQ suggests that the results are a little too predictable. As this question was a free-response item the predictability of the responses suggest that the question may have identified a prevalent misconception.

## 6.6 The relative difficulty of items

An overall sense of fit to the model can be obtained from an item map, showing estimates of the students' performance and the difficulty of the items mapped on a common scale. Figure 6.7 shows a map of the question difficulty and student ability in determining the relative size of fractions, generated by the Quest program. The right hand side shows questions in order of difficulty, with a brief description of the item appended to the item number, and the left hand side shows the distribution of the students' abilities based on their total score.

```
Logits     Students' performance  |  Items' difficulty
-------------------------------------------------------------------------------

5.0      High attaining students  |  Difficult items
                                  |
                            XXXX  |
                                  |
                                  |
4.0                          XXX  |
                                  |
                                  |
                          XXXXXX  |
3.0                       XXXXXX  |  Q27 Bigger9/10,12/13
                                  |
                          XXXXXX  |  Q35 Measinv4/3
                         XXXXXXX  |
                         XXXXXXX  |  Q30 Estimate4/5+11/12
2.0                       XXXXXX  |  Q26 Bigger 2/3,5/6 Q36 Semicirc1/6
                         XXXXXXX  |  Q21 Ant1/4
                  XXXXXXXXXXXXXX  |  Q37 Square1/9
                       XXXXXXXXX  |  Q18 Rec3/4in6
1.0                      XXXXXXX  |  Q5 Hexarea1/6 Q11 Cmp1/4&1/6(12) Q16 Rec3lots2/3 Q23 Bigger third, half
                  XXXXXXXXXXXXXX  |  Q24 Bigger1/3,1/4 Q25 Bigger1/3,1/6 Q29 3/4:whole Q33 Meas1 1/3
                          XXXXXX  |  Q17 Rec1/3in6 Q28 3/4 to 1/2 whole
                        XXXXXXXX  |  Q13 Areacircle1/3 Q14 Areacircle1/6
               XXXXXXXXXXXXXXXXX  |  Q6 Hexarea1/3 Q8 Counters equiv fract 1/4 Q20 Ant1/3   Q34 Meas3/4
 .0                    XXXXXXXXX  |
                      XXXXXXXXXX  |  Q15 Rec double 3/4   Q22 Bigger half, quarter
                         XXXXXXX  |
             XXXXXXXXXXXXXXXXXXX  |
-1.0                 XXXXXXXXXXX  |
                       XXXXXXXXX  |  Q10 Counters 3/4 of 12
                       XXXXXXXXX  |  Q3 Paperstrip1/4 (mid)
                       XXXXXXXXX  |  Q2 Paperstrip1/4   Q9 Counters 1/4 of 12 Q32 Meashalveunit
                                  |
-2.0                   XXXXXXXXX  |
                           XXXXX  |  Q7 Counters2/8    Q31 Meas2 1/2
                            XXXX  |
                                  |
                            XXXX  |  Q4 Hexarea1/2    Q19 Ant1/2
-3.0                              |  Q1 Paperstrip1/2
                             XXX  |
                                  |
                              XX  |
-4.0                              |
                                  |
                               X  |
                                  |
                                  |
-5.0                              |
                                  |
                                  |
                                  |
-6.0                              |
                                  |
                                  |  Q12 Area circle 1/2
                                  |
          Low attaining students  |  Easy items
```

Each X represents 7 students.

**Figure 6.7** Map of estimates of a sense of the size of fractions and item difficulty

One of the key features of the Rasch model is that the locations of the items on the item map are comparisons, and can be considered independent of the person locations. The underpinning basis of item response modelling is that individuals have various amounts of a trait that influences the probability that when they interact with an item, they will answer it correctly. Generally, a student would have a better than 50% chance of correctly answering questions below their location on the map. The unit of measurement used for the scale in Figure 6.7 is logits. The logit is a unit of measure derived from the natural logarithm of the odds of an event, where the odds of that event is defined as the ratio of the probability that the event will occur to the probability that the event will not occur. If the difficulty of a task (A) is determined to be 1.0 logit greater than the difficulty of task (B), then the odds of a student correctly answering task (B) are 2.7 times the odds of the same student responding correctly to task (A), regardless of whether this student has high or low ability. Similarly, if the ability of Student A is 1.0 logit greater than the ability of Student B, then the odds of Student A responding correctly to a task are 2.7 times the odds of Student B responding correctly to the same task, regardless of task difficulty. The relationship between student ability and task difficulty can be understood by remembering that a student with a logit score of 2.0 has a 0.5 probability of correctly answering an item with a difficulty of 2.0 logits, as the measure of difficulty of an item is located so that someone with the same location (number of logits) has a 50% chance of getting the question correct.

Although the numeric order of questions reflected the specific representation (e.g. circular regional model questions were presented together), the item difficulties clustered around the underlying fractions. For example, questions involving one-half were easiest and even the question requiring carrying out a measurement division with a half unit remainder (Question 31) was easier than questions involving one-quarter (Questions 2 and 9). Overall the items using the various representations for one-half and one-quarter are comparatively much easier than those involving other fractions, and are lower on the item map. Typically, items involving one-half and one-quarter had a negative item difficulty estimate.

**Questions involving one-half (partitioned)**

From the item map, it is clear that Question 12, shading one-half of the area of a circle, was far too easy for this group of students. This item appears to bear out Hart's statement of students viewing one-half as an honorary whole number. However, other forms of representing one-half were not nearly as easy. Interpreting one-half as one of three pieces of a folded strip of paper (Q1) decreases the odds of correctly identifying one-half of a circle (Q12) by a factor of twenty[15]. Students found identifying one half of a folded strip of paper that had been folded into one-half and two quarters (Q1: –3.0 ± 0.1 logits) much more difficult than shading in half a circle (Q12: –6.4 ± 0.4 logits). A student whose ability is the same as an item's difficulty (e.g. –3 logits for Q1) has a 50% chance of answering the question correctly, but if his or her ability is 1 logit higher this chance is raised to 73% (and if 1 logit lower it is decreased to 27%). Thus a difference of 2 logits is considered large. A 3 logit separation between the items is so great that shading half of a circle (Q12) is at a statistically very distinct level of difficulty[16] from identifying one half of a folded strip of paper (Q1).

This finding is supported by comparing the relative difficulty of identifying a trapezoidal pattern block as one-half of the area of a hexagonal pattern (Q4: –2.8 ± 0.1 logits) with shading in half a circle. Further, the continuous length representation of one-half in Question 19, where students were required to locate a position half way around an equilateral triangle starting from the apex[17] also has a similar level of difficulty (–2.9 ± 0.1 logits). Questions 1, 4 and 19 all involved one-half and had an identified difficulty of –3 logits, despite being located in different contexts in different parts of the assessment.

The features within a task requiring interpretation when dealing with one-half have an impact on how robust students' understanding of one-half appear. Applying a basic process of halving to a circle or accessing a prototypic image of half a circle is far simpler than identifying one-half when non-examples of one-half are presented (i.e. one-half as one of three pieces as in Q1 and Q4). When locating a position half way around an equilateral triangle, the three equal sides of the triangle play a similar role in creating non-examples of one-half (i.e. one-half as one and a half of three equal sides).

---

[15] The difference in difficulty between Q1 and Q12 is about 3 logits. Logits are natural logarithms of the odds ratios and $e$ is the numeric base of natural logarithms. A difference of 3 logits means that the odds of a student responding correctly to Q1 are 20 times the odds, say $\delta$, of the same student responding correctly to Q12 ( $\log_e 20\delta = \log_e 20 + \log_e \delta = \log_e \delta + 3$ as $e^3 \approx 20$).

[16] Statistically distinct levels of item difficulty are defined as being located at least 3 standard errors apart (Wright & Masters, 1982). The average (standard) measurement error is the root-mean-square of the standard errors of the items (i.e. the statistical average). The average measurement error of the items was 0.09 logits.

[17] As described earlier, the vertical bilateral axis of symmetry associated with finding half of the way around an equilateral triangle when you start from the top makes halving easier than if you start from one of the base vertices.

### Questions involving partitioned quarters

In general, questions involving one-quarter were located around –1.5 logits on the common scale. Both Question 2 and Question 3 asked students to identify one-quarter of a strip of paper and should have very similar difficulty levels as they assess the same fraction. The only difference between the items was the location of the quarters on the strip of paper. The estimated difficulty locations for Question 2 and Question 3 were –1.6 ± 0.1 logits and –1.3 ± 0.1 logits (respectively) and their correlation is 0.7 (significant at the 0.01 level). However, locating a quarter on the end of a strip of paper appears to be a little easier than locating a quarter towards the middle of the strip of paper. That is, the position of fractional parts of a regional model may exert some influence on their correct identification.

Questions 9 and 10—one-quarter and three-quarters of 12—also have similar locations on the item map (with Question 10 at –1.2 ± 0.1 logits slightly more difficult than Question 9 at –1.5 ± 0.1 logits) because finding three-quarters of twelve counters usually relies on being able to find one-quarter of twelve counters. However, comparing one-half and one-quarter (Question 22) has a higher difficulty threshold (–0.15 ± 0.06 logits) than other items relating to one-half or one-quarter because this question required a reasoned response and referred to quantity fractions. Abstract quantity fractions (those referencing an abstract unit whole) appear to be more difficult than similar questions involving partitioned (embodied) fractions.

### One-third and one-sixth (partitioned)

The item difficulty estimates from the Rasch analysis indicated that questions relating to one-half and one-quarter were less difficult than those involving one-third. The estimates of the item difficulties for questions involving one-third are all positive. Indicating that a rhombus was one-third of a regular hexagon made from three different shapes, a trapezium, a rhombus and an equilateral triangle (Q6), and marking a point one-third of the distance around an equilateral triangle (Q20) were a similar level of difficulty and were only slightly less difficult than drawing one-third of a circle (Q13). Representing one-third and one-sixth of a circle (Questions 13 and 14) appeared to be equally difficult (0.5 ± 0.1 logits). Although students found a quarter an easier fraction to work with than one-third, the hexagonal regional model in Question 5 resulted in a high proportion of students who incorrectly described one-sixth of a hexagon as one-quarter, which suggests that students' knowledge of one-quarter was not sufficiently robust to discriminate $\frac{1}{4}$ and $\frac{1}{6}$ in unfamiliar contexts. The results of determining how many one-third cups in 6 cups (Q17: 0.6 $\pm$ 0.1 logits) and shading one-third of a circle (Q13: 0.5 $\pm$ 0.1 logits) are conceptually quite similar as both questions make use of the evoked concept image for one-third.

The emphasis placed on correct reasoning in coding responses and the inclusion of only one multiple-choice question should have reduced the risk of students gaining the correct answer by guessing. However, the design of the questions did not totally remove the possibility that a student could obtain the correct answer through incorrect reasoning, especially for questions, such as Question 6, which did not require an explanation of students' reasoning. The difficulty estimate (0.15 $\pm$ 0.06 logits) for Question 6 may well have been influenced by the inclusion of students who saw the rhombus as one part out of three parts. That is, some students may have reasoned that the rhombus was one-third of the area of the hexagon by referring only to the number of parts and ignoring the area. However, it is unlikely that the application of an incorrect method has dramatically lowered the difficulty of this item, as its difficulty is quite close to the difficulty of representing one-third of a circle. The correct answer to representing one-third of a circle could not be obtained by a limited "one part of three" interpretation.

Questions involving one-sixth presented in a regional model, a quantity of discrete counters and as an abstract quantity fraction all registered comparable levels of difficulty. Students found recognising an equilateral triangle as forming one-sixth of a regular hexagon (Q5: 1.06 $\pm$ 0.06 logits), reasoning whether one-quarter or one-sixth of 12 counters is the greater amount (Q11: 1.00 $\pm$ 0.06 logits), and comparing the size of a third and a half (Q23: 1.08 $\pm$ 0.06 logits) equally difficult. To reason which is the larger, one-third or one-half, one solution method compares the two through the common medium of sixths.

**Measurement division**

The Rasch analysis confirmed that Question 18, requiring a measurement division using a non-unit fraction (determining the number of three-quarter cups in six cups), was the most difficult of the four questions relating to the number of cups of milk used in a recipe. The measurement division involving three-quarters of a cup (Q18: 1.35 ± 0.06 logits) was more difficult than a structurally identical measurement division question involving one-third of a cup (Q17: 0.59 ± 0.06 logits). A student with a 50% chance of correctly answering the measurement division question involving one-third of a cup had this probability reduce to 32% when the question involved three-quarters of a cup. As questions involving partitioned quarters are easier than questions involving partitioned thirds, this result provides some support for the premise that students may find measurement division questions involving non-unit fractions significantly more difficult than those involving unit fractions. This is further supported by the finding that the measurement division involving three-quarters of a cup was more difficult than determining multiples of non-unit fractions, for example, doubling three-quarters of a cup (Q15: –0.24 ± 0.06 logits) or finding three lots of two-thirds of a cup (Q16: 1.03 ± 0.06 logits).

**Measurement comparisons**

Although Question 33, 34 and 35 all related to comparisons between the same two strips of paper, their difficulty estimates varied markedly. Question 34 (0.10 ± 0.06 logits), which asked students to determine that one strip was three-quarters of the length of the other strip, turned out to be the easiest of the three. With an item difficulty estimate that identified that Question 34 was comparable to writing an equivalent fraction for $\frac{2}{8}$ (Q8), the ready identification of the fraction $\frac{3}{4}$ through halving no doubt influenced the item's location, compared to Question 33 and 35. Question 28 (0.65 ± 0.06 logits) and Question 29 (0.89 ± 0.06 logits) also involved $\frac{3}{4}$, but each question started with $\frac{3}{4}$ and then expected students to recreate the whole or a different fractional part.

Question 33 (0.95 ± 0.06 logits), which asked students to measure one paper strip with the other to determine that the longer strip was $1\frac{1}{3}$ times the length of the shorter strip, was at a similar level of difficulty to Question 29. Question 29 provided students with a drawing that represented $\frac{3}{4}$ of a piece of paper and asked them to draw where the whole piece of paper would end. Conceptually, Question 29 is very similar to Question 33, which provided students with two pieces of paper, one of which was three-quarters of the other, and asked students to determine exactly how many of the shorter pieces of paper would be needed to be equal to the length of the longer strip. Both questions require subdividing the based unit into three equal parts and then being able to determine that the (other) whole is another one of those sub-units. Question 29 and Question 33 (P3) were in separate parts of the assessment, one on the paper and pencil component and the other on the practical component using the provided materials. The relative location of the two items on the common logit scale in this study lends support to the argument that two questions that are conceptually homomorphic require similar amounts of the underlying trait of a sense of the size of fractions for their solution.

Question 35 (2.7 ± 0.1 logits), reversed the comparison used in Question 34 between the two strips of paper. That is, Question 35 asked students what fraction of the length of the shorter strip of paper the longer strip represented. The idea that a fraction can be greater than one is clearly a difficult idea for students, as the odds that a student could correctly double $\frac{3}{4}$ in Question 15 (–0.24 ± 0.06 logits) were 18 times the odds that the same student could find the multiplicative inverse of $\frac{3}{4}$ when comparing the lengths of two strips of paper.

**Abstract quantity fractions**

Comparing the size of fractions expressed as abstract quantities (i.e., numbers) was investigated using two questions that presented the fractions as words (Q22–23), and four questions that presented the fractions in $\frac{a}{b}$ notation (Q24–27). Since the abstract quantity fraction one-third is present in both Questions 23 and 24—determining the bigger number, one-third or one-half (Q23) and $\frac{1}{3}$ or $\frac{1}{4}$ (Q24)—albeit in different symbolic forms, having a sense of the size of one-third should be the main influence on obtaining a correct answer. Indeed, the item difficulties are quite similar for Questions 23 and 24 (1.1 ± 0.1 logits and 1.0 ± 0.1 logits respectively). Having an understanding of the size of one-third appears to be more important in determining the difficulty of an item than whether one-third was written in words or fraction notation.

Two questions, Questions 24 and 25, compared fractions presented in the standard fraction notation and differed only in one of the fractions used. Determining the bigger number, $\frac{1}{3}$ or $\frac{1}{4}$, was at the same level of difficulty (Q24: 0.95 ± 0.06 logits) as reasoning which is the bigger number, $\frac{1}{3}$ or $\frac{1}{6}$, Question 25. These two items also remained remarkably similar in terms of difficulty within each school year.

The two questions that asked students to determine the larger of two non-unitary fractions treated as numbers (Q26: 1.9 ± 0.1 logits and Q27: 3.0 ± 0.1 logits), were more difficult than similar questions comparing unitary fractions (Q24 and Q25). The odds of a student correctly comparing $\frac{2}{3}$ and $\frac{5}{6}$ (Q26) to determine the larger fraction and provide a reason were 2.7 times the odds of the same student similarly determining the larger fraction from $\frac{1}{3}$ and $\frac{1}{6}$.

The most difficult item was Question 27 (3.0 ± 0.1 logits), which required students to determine which was the larger number between $\frac{9}{10}$ and $\frac{12}{13}$, and to justify their answer. The odds of correctly determining one-half of a paper strip folded into one-half and two quarters (Question 1) were 400 times the odds of correctly answering Question 27. Question 30, which was a multiple choice question asking students to estimate $\frac{4}{5} + \frac{11}{12}$, was also quite difficult (2.3 ± 0.1 logits). Although it is tempting to postulate that the difficulty of questions 27 and 30 is largely determined by the types of fractions involved in each question, this would overlook the differences between the questions. Question 27 required a reason whereas Question 30, as the only multiple choice question, did not.

**Comparing discrete and continuous representations**

In total, the set of items contained four questions that looked at fractions as part-whole comparisons of collections of discrete items. Students were asked what fraction of a line of 2 yellow counters and 6 red counters was represented by the yellow counters (Q7); how many counters would correspond to $\frac{1}{4}$ and $\frac{3}{4}$ of 12 identical counters presented in two equal columns (Q9 & Q10); and whether you would have more counters if you had one-quarter or one-sixth of the 12 counters (Q11) and why.

Collectively, there appears to be little difference in the performance of these items for discrete representations of halves and quarters compared to questions involving continuous representations of these fractions. For example, identifying one-quarter of a strip of paper divided into one-half and two quarters was as difficult as finding one-quarter of twelve counters[18]. However, the structured arrangement of the objects in the discrete representations (rows and columns) needs to be considered when interpreting this result. Repeated halving of 12 counters presented in two equal columns is likely to be easier than finding one-quarter of 12 counters presented in a random pattern, as the question presents the initial halving. The presentation of the discrete units, in the questions used in this study to explore a quantitative sense of fractions, may have supported students in identifying the requisite units.

**Composition of partitioning**

The difficulty of composition of partitioning appears to be influenced by the individual partitions required, as well as the materials used. Finding one-third of one-half of a circle (Q36: 1.9 ± 0.1 logits) is more difficult than making one-ninth of a square (Q37: 1.6 ± 0.1 logits). However, making one-ninth of a square using partitioning in only one direction did not always involve composition of partitioning. Requiring students to determine one-quarter of the distance around an equilateral triangle (Q21: 1.7 ± 0.1 logits) can also be thought of as composition of partitioning. The total perimeter of the triangle is composed of three equal units and finding one-quarter of this can be achieved by finding one-quarter of one side iterated three times, that is, three-quarters of one side. The estimated difficulty of Question 21 (one-quarter of the distance around an equilateral triangle) lies between the other two questions involving composition of partitioning. Composition of partitioning appears to be more difficult than reconstructing the whole, given three-quarters of the length of a rectangle (Q29: 0.89 ± 0.06 logits), and measurement division involving a non-unit fraction (Q18: 1.35 ± 0.06 logits).

## 6.7 Summary

The questions used in this study to investigate students' sense of the size of fractions collectively fit the requirements of Rasch measurement. An analysis of specific items at the extremes of the fit statistics suggests that the application of basic solution strategies, such as primitive halving, might impact on the discrimination of a question. That is, more capable students appear indistinguishable to less able students if they are both using the

[18] Other research (Behr, Wachsmuth, & Post, 1988; Clements & Del Campo, 1987; Novillis, 1976) suggests that the representation of fractions through either discrete or continuous materials influences the difficulty of the questions, with the discrete set model more difficult than the continuous area model.

same primitive, albeit effective, solution methods. The comparative analysis of the performance of primary students and secondary students on the same questions suggests that having been taught how to find equivalent fractions in high school may have actually disadvantaged this group in comparing non-unit fractions.

The specific underpinning fraction concept appeared to determine the relative difficulty of the questions far more than the mode of representation of the fraction within the questions. For example, the mode of representing one-sixth—shading part of a circle, a part of 12 discrete counters or in fraction notation—did not affect the level of difficulty of the questions. However, two different fractions, $\frac{1}{3}$ and $\frac{1}{6}$, behaved far more alike than $\frac{1}{2}$ and $\frac{1}{4}$. Although questions involving abstract quantity representations of fractions were more difficult than partitioned fractions of circles involving the same fractions, one-third and one-sixth create a comparable level of challenge within a given representation.

The overarching purpose of locating the questions on a logit scale was to produce a relative ranking. The questions are ordered with respect to difficulty irrespective of the students' abilities. Generally, questions involving one-half were easier than questions involving one-quarter and questions involving operating with three-quarters, particularly division, were more difficult than those that asked students to identify one-quarter. Questions requiring identifying one-third or one-sixth were at a similar level of difficulty to each other and more difficult than those involving one-quarter. The relative difficulty of one-half and one-quarter compared to one-third is consistent with other research (e.g. Wing & Beal, 2004). Measurement division involving non-unit fractions was more difficult than measurement division involving unit fractions. In general, beyond being able to create correct representations of fractions, the difficulty of questions requiring operating with fractions as conceptual units depends upon the complexity of the operations with the sub-units more than the specific fraction involved. For example, managing measurement division by a three-quarter unit involved reorganising the parts and the whole units in a similar way to changing the unit of reference in composition of partitioning.

In Chapter 7 the results of this study will be discussed in relation to the existing research on fractions that was synthesised in Chapter 2.

# Chapter 7 DISCUSSION

*Students' evoked fraction concept images provide insight into their quantitative sense of fractions. The "parts of a whole" model commonly used in teaching fractions combined with students' greater visual acuity in the vertical and horizontal directions can result in students' concept images containing only certain features of a parts-of-a-whole model. For example, attending to equidistant partitioning as the salient feature of regional models supports a "number of parts" sense of fractions. The fraction notation also engenders a range of number-based strategies for determining the relative size of fractions. Further, the difficulty of interpreting discrete representations of fractions may rely upon the structure of the arrangement of the objects. Rather than treating apparent misconceptions as resulting from whole number bias, it may be more productive to consider that many students construct fragmented parts-of-a-whole concept images of fractions by attending to only some aspects of the part-whole relationship.*

**Preview**

This chapter begins by outlining the apparent influence of the use of a parts-of-a-whole model in the teaching of fractions on the development of a quantitative sense of fractions. The part-whole fraction representation is described in detail together with the implications of students equating specific features of the part-whole model with the total fraction concept. The multiplicative reasoning intended by the part-whole model is readily replaced by additive reasoning when only features of the model are taken to be the part-whole relationship. It is then argued that rather than representing an abstract quantitative fraction concept, for some students the fraction notation engenders a range of strategies involving the numbers contained in the notation, again often using the additive properties of the numbers in the strategies. Next, the implications of overgeneralising specific fraction concepts are described before an empirical developmental map of a quantitative sense of fractions is described.

## 7.1 Introduction

> More attention needs to be given to the limitations of the 'part of a whole' model of a fraction. In particular, distinction needs to be drawn between the embodiment and the idea. For many children, it appeared that the idea of a fraction was inextricably linked with a picture of a partly shaded shape; this is a very limiting view of fractions. Shapes are not fractions; they merely illustrate them. Moreover, it seems likely that the use of the 'part of a whole' model can inhibit the development of the more general idea of a

> fraction. Two instances of this have already been indicated. The difficulty experienced by children in thinking of the fraction a/b as a ÷ b can be attributed to their unwillingness to abandon the 'part of a whole' model. Similarly, the use of shapes partitioned in different ways to illustrate equivalence does not appear to help in the understanding of the usual arithmetic way of constructing equivalent fractions. …
> Most importantly, it seems that the dependence on diagrams inhibits the appreciation of the idea that fractions are numbers. (Kerslake, 1986, pp. 96-97)

As outlined in Chapter 2, research on children's fraction thinking has documented some persistent misconceptions (Behr, Wachsmuth, & Post, 1984; Kerslake, 1986; Streefland, 1991). These misconceptions include confounding the number of pieces in a partition with the size of each piece (e.g., $\frac{1}{6}$ is bigger than $\frac{1}{3}$ because 6 is bigger than 3), counting non-congruent parts to name a fraction (e.g., calling one-quarter one-third in a circle partitioned into a half and two quarters) and adding across numerators and denominators to add fractions. These misconceptions have often been attributed to an overgeneralisation or a misapplication of whole-number knowledge (Ni & Zhou, 2005). It may even be that these misconceptions are exacerbated, if not caused by instruction processes limited to the part of a whole model of a fraction (Kerslake, 1986; Kieren, 1988).

Although a quantitative sense of fractions exists as an identifiable construct within students in Years 4 to 8 classes, splintered negative images of the construct are widespread. Students have encountered fractions through common "part of a whole" representations and evoke fraction concept images that are rich in the parts, but not strong in the relationship between the parts and the whole. Despite the recorded concerns over the part of a whole model (Behr et al., 1992; Freudenthal, 1983; Kieren, 1988) it has continued to be the main model used to introduce fractions in schools.

A quantitative sense of fractions stands in stark contrast to the limitations of the part of a whole model. In essence, a quantitative sense of fractions is an appreciation that fractions are relational numbers that reference a constant unit or one. Students have commonly been introduced to fractions through a part of a whole model. Rather than characterising students' evoked concept images as over-generalisations of additive whole number thinking, they may be better described as arising from a fragmentation of the part of a whole model through each student's interpretation of the salient features of the model. Students evoked concept images may be used as indicators of the features of the model activated by questions designed to elicit their quantitative sense of fractions. The dependence on partitioned fractions (parts of things) can in effect create for students fraction concepts markedly different from those intended by the teacher's use of the part of a whole model.

## 7.2 Students' reconstruction of part-whole

Students' reconstruction of multiplicative part-whole relationships develops from their interaction with the models, tools and notations offered in fraction learning opportunities. However, there are two distinct common uses of part-whole relations: part-whole (additive) as it applies to whole number, and part-whole (multiplicative) as it may be applied to fractional subdivisions through equal partitioning. Part-whole (additive) knowledge relates to the principle of additive composition by which whole numbers are combined to form other whole numbers (Meron & Peled, 2004; Resnick, 1989; Riley & Greeno, 1988; Sophian & McCorgray, 1994) where the whole is the sum of the parts (see Section 2.2.2). For fractions, the multiplicative part-whole relation is such that the whole is a product of the one part (Empson et al., 2005; Sáenz-Ludlow, 1994; Vergnaud, 1988). Yet studies have found that students do not always attend to the size of the wholes from whence parts come (Armstrong & Novillis Larson, 1995; Charles & Nason, 2000; H. Yoshida & Sawano, 2002). Students attend to various aspects of the part-whole relationship implicit in mathematical analogs used to teach fraction concepts (Charles, Nason, & Cooper, 1999). Students can also apply the additive part-whole knowledge associated with whole number instead of the multiplicative part-whole knowledge required by fractions (Lamon, 1999; Ni & Zhou, 2005; Post et al., 1993).

### *7.2.1 A focus on one dimension*

Although the intent of presenting regional fraction models is to have students see the area as the determining characteristic of the model, many students do not. Even the earliest experiences of halving, such as watching an apple being cut in half, emphasises the vertical cut to form two parts. For the child to know that the apple is in two *equal* halves requires a way to compare two volumes, a quite complex task. In a classroom, the symbolic representation of halving the apple is to draw a line to cut a regional model of an apple in half. That is, we have moved from a three-dimensional comparison of volume to a two-dimensional representation of area. For some students, this cutting line becomes itself the representation of one-half (Brizuela, 2005) and the half-way location stands in place of the equal area representation.

Studies that have looked at how young students make judgments about area quantity when asked to compare two areas have found that one-dimensional aspects of the rectangular areas are initially used, with both dimensions not being equally weighted until sixth or seventh-grade level (Armstrong & Novillis Larson, 1995; Verge & Bogartz, 1978). That students partition regional models without specifically attending to the area is consequently not surprising. Parallel partitioning, also described as segmenting, appears as a form of equi-distant cutting with attention usually given to the distance between the cuts rather than the resulting area of the pieces. Therefore, one-dimensional aspects of the area may be used in partitioning, rather than the two-dimensional aspect expected of a regional model. This form of partitioning strategy, described by Pothier and Sawada in a study involving students from Kindergarten to Grade 3 (1983) as *algorithmic halving*, persists well past Grade 3 with 10% of this sample of Year 4–8 students using this type of partitioning to represent one-third of a circle.

### *7.2.2 Vertical and horizontal visual preferences*

The evoked concept images of many students include a visual image as a prototype of the fractional quantity, usually influenced by visual preferences in the vertical and horizontal directions. In addition to a tendency to focus on length (one dimension) when partitioning regional models of fractions, partitioning in specific directions is also influenced by visual preferences. As discussed in Section 4.4, we see better in the vertical and horizontal directions (Howard & Templeton, 1966) which influences students' approaches to halving and quartering regional models of fractions.

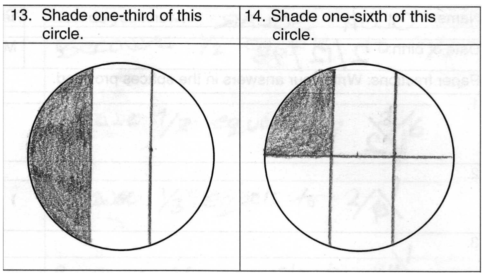



**Figure 7.1** Year 5 student from a class for talented students

The equidistant vertical division of the circle against a virtual horizontal diameter is evident in the Year 5 student's creation of thirds in Figure 7.1. The approach the student uses to halve the thirds to produce sixths also makes use of the vertical and horizontal preferences. The application of vertical parallel lines that can partition a rectangular shape being inappropriately applied to circular regions to produce thirds, fourths or fifths has been previously noted (Pothier & Sawada, 1983). However, responses such as Figure 7.1 are readily understood when a student focuses on length, rather than area, to construct parts and emphasises the vertical and horizontal directions. Students' greater visual acuity in the horizontal and vertical orientations may even influence their ability to attend to angle sizes as the basis of interpreting sectors with circle models of fractions.

The impact of horizontal and vertical visual preferences is not restricted to regional models of fractions. In Section 5.8, the comparison of the results of Questions 19–21 with a study by Clements and Del Campo (1987) that used similar questions, indicated that locating fractional parts of a perimeter of a triangle, in particular half-way, depended upon the orientation of the line of symmetry.

In addition to the improved correct response rate to finding half way around an equilateral triangle using a vertical axis of symmetry compared to an oblique axis, a number of students in this study appeared to use "vertical and horizontal quartering" to locate a distance one-quarter of the way around the equilateral triangle.

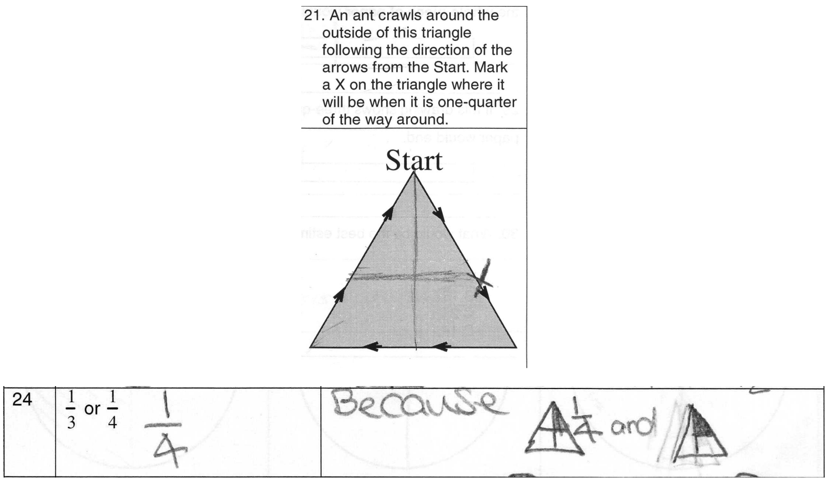



**Figure 7.2** Examples of vertical and horizontal quartering

**The impact of visual preferences on discrete item tasks**

Research is divided on which task type, continuous or discrete, is more difficult. Some argue that discrete tasks are easier (Nik Pa, 1989; Spinillo & Bryant, 1991, 1999). Others have found that discrete tasks are more difficult (Behr et al., 1988; Sophian & Wood, 1997). For example, Clements and Del Campo found "…that 'discrete' $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{3}$

questions, whether in pen-and-paper or interview form, are much more difficult than 'continuous' questions" (1987, p. 109). The Rasch analysis of the discrete item type questions in this study (see Section 6.6) indicated little difference in the performance of items using discrete representations of halves and quarters compared to questions involving continuous representations of these fractions. However, the relative difficulty of discrete and continuous tasks need not be related to the inherent classification of the task. It might be that solving continuous and discrete model problems sometimes requires different schemes, and at other times not. Some discrete fraction tasks may be solved through dealing, a primitive form of partitioning (Davis & Pitkethly, 1990), whereas students' greater visual acuity in the vertical and horizontal directions may have assisted iterated halving schemes (Pitkethly & Hunting, 1996) for quartering discrete objects arranged in two equal columns. The iterated halving schemes may be essentially the same between continuous and discrete contexts when the discrete objects are arranged in a pattern that supports visual quartering.

### *7.2.3 A discrete interpretation of the part-whole model*

The process of subdividing into parts can also change a continuous representation into a discrete representation where the number of parts appears more important than the size of the parts. Fractions as parts of a whole are readily subject to an additive interpretation. The context in which basic fraction concepts are assessed has a substantial bearing on the responses students make. In response to a figure fully partitioned into quarters, a focus only on the number of parts would result in an answer that would be indistinguishable from a part-whole relational interpretation.

How common was a focus on the number of parts in the first three questions in this study? Approximately 3% of all students described one-half as one-third in the first question. As one-half is the first fraction that students learn and is encountered in many different contexts, confusing one-half with one-third is suggestive of a very strong "number of parts" interpretation of fractions operating for these students. Perhaps more telling was that 12% of the students answered $\frac{1}{3}$ to Question 2 and 11% provided this response to Question 3, even though the correct answers of $\frac{1}{4}$ is one of the earliest fractions students encounter.

If the most basic of fractions, one-half and one-quarter, can be misinterpreted, it is perhaps not surprising that the parts-of-a-whole model unravels for some students. Despite the number of studies describing children's partitioning strategies (Charles & Nason, 2000; Lamon, 1996; Pothier & Sawada, 1983), little research has examined how children's informal partitioning supports the construction of fractions as multiplicative structures. One exception is a recent study by Empson et al. (2005) which found two major categories of strategies: parts quantities strategies and ratio quantities strategies. Parts strategies involved partitions of continuous units and ratio strategies involved the creation of associated sets of discrete quantities. The number combinations used with the problems, describing the number of objects to be shared and the number of sharers, elicited a wider range of strategies when the combinations had several common factors. In the current study, students appeared to apply elements of discrete number strategies to continuous contexts. Rather than separating students' strategies into discrete and continuous categories, it may be more productive to consider discrete interpretations of the part-whole context, including contexts intended to represent part-whole continuous models.

The process of interpreting a part-whole diagram has been described as involving: (i) counting the number of pieces shaded, (ii) counting the total number of pieces, and (iii) then writing one whole number on top of the other (Amato, 2005; Hart et al., 1981). This is a very basic analysis of using a part-whole diagram. A more detailed analysis of the part-whole relationship, involving both discrete and continuous elements, can be considered to have several key components:

1. Number of parts
2. There are equal parts
3. All parts must be equal to each other
4. The sum of the parts makes the whole
5. Wholes need to be the same size to compare part-whole fraction representations.

Each of these key components is discussed in turn in the following sections as a way of characterising students' responses to various part-whole contexts. The five components must be conjointly applied to form an effective interpretation of a parts-of-a-whole model.

**Number of parts**

The responses to the fraction tasks indicate that many students' sense of the size of fractions is initially independent of the whole. In response to a task designed to elicit a student's use of an area model for fractions, a student may evoke a concept image of a fraction as countable units. A student can construct one-third of a circle as one-quarter (area) if the other parts are not marked. One-third is simply one of three things, made discrete by the student. Similarly, one-sixth for some students is one of six things. This interpretation is consistent with findings from studies involving younger students (Empson, 1999) and the number of parts interpretation persists to at least Year 8.

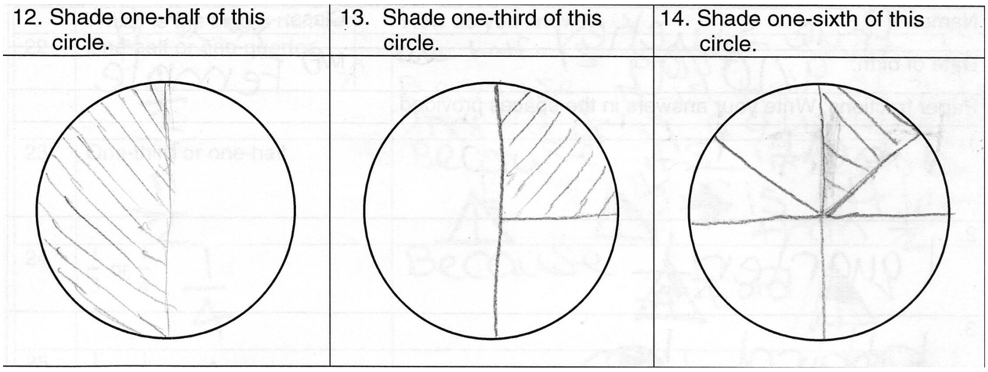



**Figure 7.3** A regional model interpreted as countable units

Rather than colouring in a pre-partitioned figure, the student in Figure 7.3 expressed what she saw as important. For this Year 6 student, the number of parts was the important feature and the whole only a notional backdrop.

With 10% or more of each grade indicating that the number of parts was to them the more salient feature of a rectangular strip representation (Q2) than a comparison of the area to the whole, the traditional regional model of a fraction has not resulted in a connected parts-of-a-whole interpretation of fractions. Similarly, a number of parts interpretation influenced almost a quarter of all students, who recorded one-sixth of a regular hexagon as one-third, one part out of three (Q5). The three parts were clearly not equal parts as they were different shapes, and distinguished by different colours. The number of parts may become the defining component of a regional model when equality of the parts is not an available feature of the representation.

Segmenting through parallel partitioning of circles (see Section 5.5) can also be interpreted as giving precedence within the part-whole relationship to the number of parts. The analysis of students attempts to create regional models for one-third and one-sixth indicated that about 10% of students in this sample used "parallel partitioning" where the area of the parts was not the salient feature, and focused on the number of parts more than the equality of the area of the parts. The same focus on the number of parts is seen in responses to creating one-ninth of a square by folding were students created and identified nine unequal parts. Even in responses to folding one-half of a circular piece of paper to show one-sixth of the original circle, a focus on the number of parts was evident in over 15% of students' responses.

**There are equal parts**

Recognising that there are equal parts as a component of parts-of-a-whole does not of itself imply that all of the parts need to be equal to each other. For example, in Figure 7.3, the representation of one-sixth has one part shaded out of six parts, with some (but not all) parts equal in size. That is, there are equal parts as the diagram has four parts equal in size and the other two parts different from the four, yet equal in size to each other.

The earliest viable strategies for partitioning geometric regions are based on halving and repeated halving (Confrey, 1994; Piaget et al., 1960; Pothier & Sawada, 1983). The link between the most basic form of partitioning, halving, and the creation of equal parts means that some equal parts are likely to be present in most simple partitions. Consequently, equal parts are a readily identifiable feature of fraction prototypes. Yet algorithmic halving as described by Pothier and Sawada (see Section 2.2.3) can result in the formation of equal parts as in Figure 7.4. There are clearly equal parts, and one part is shaded out of the six parts, yet something is missing from the parts-of-a-whole model.

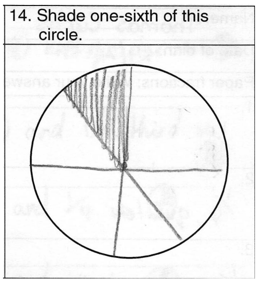



**Figure 7.4** There are equal parts formed by algorithmic halving

**All parts must be equal to each other**

As with the earlier components of the part-whole relationship, this component is most evident when it is not present in a student's response. The creation of a representation of one-ninth of a square by forming eighths through repeated halving and then halving one-eighth to form nine pieces provides an example of a response where the student does not appear to believe that all parts must be equal to each other.

The focus on the idea that all parts must be equal to each other to the exclusion of some other components of the part-whole relationship is reflected in the number of students in this study who constructed $\frac{1}{16}$ or $\frac{1}{8}$ in response to the request to create $\frac{1}{9}$ of a square. Although the number of parts and the equality of those parts are frequently related, an answer of $\frac{1}{16}$, produced by more than 5% of each grade in Years 5 to 8, appears to give precedence in those students' constructions to the idea that all of the parts must be equal, over creating the correct number of parts.

**The sum of the parts makes the whole**

A focus on the number of equal parts may occur when students overlook the relationship between the equal parts and the whole. That is, the whole needs to be the sum of the equal parts. For example, shading in a number of equal parts corresponding to the denominator as in Figure 7.5, ignores the relationship between the parts and the whole.

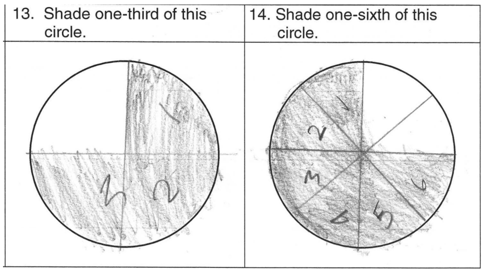



**Figure 7.5** The sum of the parts does not equal the whole

Here the number of parts indicated corresponds to the number identified by the denominator and all of the parts are equal to each other, but the number of parts does not exhaust the whole. Similarly, shading $\frac{9}{16}$ for one-ninth of a square matches the number of parts identified with the denominator and again all of the parts are equal to each other, but the sum of the parts does not equal the whole.

Students might associate cutting something up into equal pieces with forming a fraction. However, each piece then becomes a countable item and it is possible to count equal pieces before identifying a number of pieces with the denominator. Whole number bias is an alternative explanation for responses that form and shade a number of equal parts corresponding to the denominator. However, when a parts-of-a-whole model is used with fractions, whole number bias provides little explanation why the whole is ignored. If a student divides a circle into four equal parts and then shades one of those four parts to represent one-third (one part shaded and three unshaded) it could also be argued that this is whole number bias. The student has identified equal parts and represented numerator and denominator, but the above four aspects of the parts-of-a-whole model are not conjointly applicable.

**Equal wholes**

The idea of the equal whole underpins understanding fractions as mathematical objects, as it enables the ordering of fractions as relational numbers. The need to reference equal wholes is a necessary precursor to working with equivalent fractions and operations on fractions. If $\frac{1}{3}$ and $\frac{1}{4}$ do not refer to equal wholes, we cannot use procedures for determining equivalent fractions, a precursor to comparing or combining fractions, nor operate with these fractions as mathematical objects. Students have to learn to consider an equal whole when comparing fractions as numbers and to recognise that when comparing fractions of different quantities, the size of the quantities (implicit wholes) influences the size of the fractional parts.

The equal-whole component of parts-of-a-whole relies on recognising that magnitudes of one as a whole should be the same in all fractions (see Section 2.4.1). In an earlier study, when asked to order fractions, many Grade 4 students drew representations (rectangles) in which the size of the whole that each fraction represented was in direct proportion to the size of the denominator, that is, each representation of one was a different size (H. Yoshida & Kuriyama, 1995). In another study of students' strategies used to compare partitioned rectangles, it was also found that some of the students ignored the size of the whole when they made comparisons (Armstrong & Novillis Larson, 1995). Further, an earlier study found that when children were asked if one-quarter of an amount of pocket money could be greater than one-half of an amount of pocket money, a large number responded in the negative reasoning that $\frac{1}{2}$ is (always) bigger than $\frac{1}{4}$. This application of the equal-whole scheme to a situation where it is not warranted was described as an over-generalisation of the equal-whole scheme (Hart et al., 1981).

The Year 6 student's evoked concept image for one-third and one-sixth shown in Figure 7.6 has the correct number of equal parts, with all parts within a model equal to each other and making up the whole.

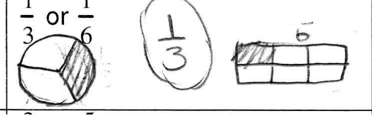
| | | |
|---|---|---|
| 24 | $\frac{1}{3}$ or $\frac{1}{4}$  | Because $\frac{1}{3}$ is split into 3 parts which leaves a bit more than you would have if you had 4 parts. |
| 25 | $\frac{1}{3}$ or $\frac{1}{6}$  | because $\frac{1}{3}$ has a larger part then $\frac{1}{6}$ since $\frac{1}{3}$ is split into 3 larger parts than having 6 small parts, |

**Figure 7.6** Regional models accessing different wholes

However, the diagrams used with the explanations in Figure 7.6 make it clear that this student is not referring to equal wholes in comparing fractional parts. The need to use equal wholes when comparing fractions is not explicitly addressed within the primary Mathematics curriculum in New South Wales, and students may not have been taught that wholes must be identical if comparisons are to be made. The syllabus unit addressing the idea fractions can be ordered, only uses decimal fractions (NSW Department of Education, 1989, p. 274).

A concept image related to a parts-of-a-whole interpretation of fractions may not have the five features outlined above fully integrated, which could lead to students sometimes appearing to have the concept and other times not (Flavel, Miller, & Miller, 1993). Distinguishing between additive and multiplicative reasoning is an important theme in understanding students' responses to fraction tasks but it is not the only way to characterise students' interpretations of the parts-of-a-whole model used with teaching fractions.

## 7.3 Parts-of-a-whole and multiplicative reasoning

Additive counting may be interwoven with part-whole relationships in students' evoked fraction concept images. In other words, additive and multiplicative structures do not constitute a simple dichotomy. In coming to develop the complex relationship of fractions as relational numbers, additive whole number knowledge plays a role in the forming and counting of parts. A comparison between the parts when constructing and counting equal parts is an advance on additive thinking, which can be described as forming and counting like parts. The relationship to the whole exists in a weak form so, although the whole is perceived to be made up of the equal parts, the inverse relationship between the size of the parts and the number of parts may develop independently.

Children can partition independently of their understanding of fractions and in many contexts the notions of equi-partitioning and a quantitative sense of fractions may appear unrelated in students' evoked concept images. Partitioning of continuous quantities has several related components that are not always present together in students' reasoning. Subdividing a shape may result in a number of pieces being formed, but often the number of pieces appears to hold more significance than the area relationship between those pieces. Indeed, the area relationship between the pieces can be quite complex, as in the case of parallel partitioning of a circle. The comparison between the size of the pieces and the relationship between the total number of pieces and the fraction quantity are often not evident. If incomplete parts-of-a-whole models represent fractions, the multiplicative relationship associated with successive partitioning can also be incomplete.

Any parts-of-a-whole model used in teaching fractions needs to enable students to see

- the whole as contributing the total number of parts,
- the whole is subdividable into any number of parts all equal to each other,
- the equality of the parts is a function of the relationship between the parts and the whole,
- there is an inverse relationship between the number of equal parts and the size of those parts,
- the subdivided equal parts can group to form new composite parts,
- the equal parts can themselves be further subdivided,
- the whole can be reconstructed from a part, and
- the transferability of the whole (i.e. the equal whole) as the common basis for comparison of abstract quantity fractions and subsequently operating with fractions.

Counting discrete elements, whether in discrete models or sub-divisions of regional models, does not establish the relationship between the parts and whole. The second point, that the whole is subdividable into any number of parts all equal to each other, is clearly a feature of a measurement model, and a limitation of discrete models of fractions. Further, if regional models are used, knowing that all of the parts are equal to each other based on area is conceptually very challenging. Even the most basic of fractions, the half, is difficult to interpret in regional models.

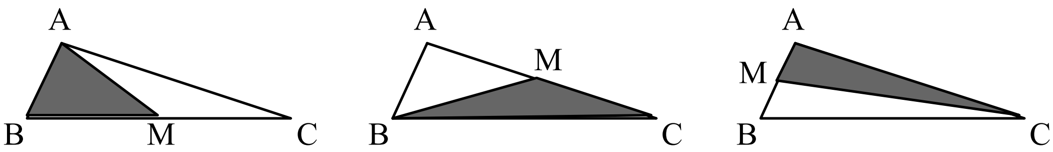



**Figure 7.7** Three different representations of one-half on the one regional model

In Figure 7.7 each copy of the triangle ABC has one half shaded. To be able to compare the area of the regions requires a very sophisticated understanding of area. A focus on area as the basis for decisions related to regional models requires a sound appreciation of area. Otherwise, number of parts is the only accessible component of the model.

Despite the limitations of regional models there are significant advantages to introducing students to fractions from the viewpoint of measuring quantities (Davydov & Tsvetkovich, 1991). Perhaps the most obvious advantage is the need to attend to the representation of the whole. As fractions are indicated divisions (Usiskin, 1979), division appears the most direct path to developing fractions. Measurement division, where it is specifically taught, is introduced as one of the two types of whole number division, partitive and measurement (also known as quotitive). Rather than using whole numbers, a better introduction to measurement division and one that is more closely aligned to measurement would be to use continuous length units.

Using measurement division with continuous length units, initially without remainder, provides a natural context for the introduction of fractional remainders as well as the relational link between the fractional remainder and the unit of measure. A single linear model applied to division, fractions and measurement could lower the risk of double counts, which are likely to arise from the use of discrete fraction representations or the interpretation of regional models as representing discrete elements. Another advantage of using a linear measurement division model to introduce fractions is that it can overcome the limits of the regional parts-of-a-whole model, namely that whilst 3 parts out of 5 has meaning, 5 out of 3 does not. The limitations of the treatment of fractions as "*n* out of *m* parts" can be reduced by an emphasis on measurement.

Davydov and Tsvetkovich (1991) suggested introducing the idea of fractions by starting from comparison of units in measurement. They argue that fractions appear quite naturally from measuring when a quantity is not an exact multiple of the unit of measure. With this approach, the size of the unit is always in the foreground, making the part-whole multiplicative relationship more obvious. This process provides a measurement-based comparison of two quantities and a natural context for fractional remainders. That is, a focus on one dimension rather than two dimensions may be preferable in introducing fractions (linear measurement model).

As a focus on the whole number in the denominator appears more frequently in the earlier grades (in this study 12% in Year 4 reducing to 2% in Year 8 on Question 24) it is important to use linear measurement as a context for introducing and discussing fractions as soon as practicable. Viewing fractional parts as arising from acts of measurement enables students to consider a composite and measurement unit as a referent whole and to partition that unit by splitting quantities in ways that reflect the multiplicative nature of the problem (Morris, 2000). Fractions developed from acts of linear measurement also emphasise the inverse relationship between the number of equal parts and the size of those parts. Further, non-unit fractions develop as collections of unit fractions and it is possible to collect unit fractions that surpass the whole. When a collection of unit fractions such as five quarters surpasses the whole, the reorganisation of the unit parts to form the whole relies upon a clear sense of the whole. Having a unit of measure can make the whole easier to describe.

Multiplicative reasoning with fractions is central to the process of composition of partitioning. Pothier and Sawada (1983) proposed composition as a fifth level of partitioning where a child would, for example, realise that to obtain ninths a region can be partitioned into thirds and then each part divided into thirds. Composition of partitioning has been elaborated in descriptions of recursive partitioning (Steffe, 2003) and unitising (Empson, 1999; Lamon, 1996). Both composition and recursive partitioning are unit fraction multiplying schemes.

Paper folding activities can provide models for composition of partitioning (Lamon, 1999). Although this composition of partitions clearly employs a multiplicative principle, the multiplicative principle is not dependent upon equality of the parts in achieving the correct number of parts. Thus a student may attempt to create one-ninth of a square by folding into three non-equal parts in one direction followed by folding into three non-equal parts in a perpendicular direction. Despite fractions expressed as partitions of continuous units being measurable, their treatment as parts-of-a-whole can evoke countable concept images that carry over to composition of partitioning.

The item map (Figure 6.7) indicates that composition of partitioning, such as creating one-sixth of a whole by making one-third of a half, is quite difficult. When supporting the transition from additive to multiplicative reasoning, often described as a significant hurdle in children's thinking, attention needs to be given to the role of the units in the multiplicative reasoning, as multiplicative reasoning for some students can be associated with unequal parts. Students need to see the equality of the parts as a function of the relationship between the parts and the whole before dealing with further subdivision of the parts.

**Looking back**

Although a parts-of-a-whole model has been used to introduce the idea of a fraction, for a substantial proportion of students in Years 4 to 8 only some features have survived in the evoked concept images of students. Some students have attended to the number of parts, the size (and the equality) of the parts and the transferability of the whole (i.e. the equal whole). Yet rather than establishing the requisite relationship across these components, one or two components have come to stand in place of the whole model. Operations involving partitioned fractions, such as multiplication modelled on composition of partitioning, are also prone to the application of these incomplete concept images.

## 7.4 Strategies evoked by the fraction notation

In this study, Questions 24 to 30 introduced the standard fraction notation. Analysis of students' responses provided indications of a number of erroneous strategies, some of which had been previously observed in small numbers of students involved in teaching experiments (Behr, Wachsmuth, Post et al., 1984; Mack, 1990). For example, when an ordering of fractions consistent with whole number arithmetic was identified, it was described in the Behr et al. (1984) study as an example of "whole number dominance". Similarly, when an explanation focused only on the denominator, such as "the bigger the number is, the smaller the pieces get", it was recorded in their study as *denominator only*. In this cross-sectional study, equating the size of the fraction with the size of the whole number in the denominator was described as "whole number denominator" to emphasise that the fraction is treated as if it *is* the whole number denominator. Further, an argument that "the bigger the number is, the smaller the pieces get" was described in this study as "bigger is smaller" to again emphasise the nature of the argument.

When asked to determine which was the larger fraction $\frac{1}{3}$ or $\frac{1}{6}$, 6.7% of students in this study chose $\frac{1}{6}$ and justified their choice by an argument based on the size of the whole number denominator (WND). Another 5.2% answered $\frac{1}{6}$ without providing a reason. Confusion between phonemically similar fractional and integer values, described here as phonemic miscuing, may contribute to students' interpretation of fractions as being equivalent to the *whole number denominator*. It was clear from students' explanations such as "one sixth is 6" (see Section 5.4) that phonemic miscuing may underpin or reinforce the equating of the size of the fraction with the whole number in the denominator.

In the same question (comparing $\frac{1}{3}$ and $\frac{1}{6}$), 6.7% of students provided the correct answer using a bigger is smaller (BIS) argument as their justification. In the very next question, which asked students to determine the larger fraction out of $\frac{2}{3}$ or $\frac{5}{6}$, 4.4% chose the wrong fraction and justified their answer with a BIS argument. Part of the reason for the reduction in the percentage of students using BIS between the two questions was the increase in new strategies. The use of the fraction notation appeared to dramatically increase the number of specific heuristics or interpretive strategies evident in students' responses. Students now started to reason that $\frac{5}{6}$ was larger than $\frac{2}{3}$ because both the numerator and the denominator were larger numbers (BLN), the number on the top was bigger (WNN), the number on the bottom was bigger (WND) or the fractions were actually the same size as the difference between the numerator and the denominator (FID) was the same. The last argument, that the size of the fraction was determined by the difference between the numerator and the denominator increased from 3.0% when comparing $\frac{2}{3}$ and $\frac{5}{6}$, to 5.3% when comparing $\frac{9}{10}$ and $\frac{12}{13}$ in the following question. The three modes of reasoning BLN, WNN and WND accounted for 7.1% of responses when comparing $\frac{2}{3}$ and $\frac{5}{6}$, and 7.9% of response when comparing $\frac{9}{10}$ and $\frac{12}{13}$.

The fraction notation appears to have engendered a range of strategies involving the numbers in the notation, often variably reinforced, as strategies that yield the correct answer with some questions do not work with others. As well as developing strategies associated with the size of the whole number in the denominator, some students used only the numerator (e.g. “2 of something is always less than 5 of something” when comparing $\frac{2}{3}$ and $\frac{5}{6}$) or both the numerator and denominator. The five major strategies influenced by the use of the fraction notation in this study are bigger is smaller (BIS), whole number denominator (WND), whole number numerator (WNN), both larger numbers (BLN) and fraction in the difference (FID). Several of these strategies appear to be under-represented in the research literature because large-scale studies rarely focus on elicited solution strategies.

### *7.4.1 What do children attend to in expressing a fraction?*

The images used by students in responding to questions designed to elicit their quantitative sense of the size of fractions suggest that for some students, initially a fraction as a region may be independent of the features we would normally attribute to the whole. Although the main reason for using a regional model of fractions is ease of demonstrating equivalent fractions by comparison of area, area is not always the feature attended to in regional models of fractions. Indeed, students’ answers to questions in this study (e.g., Section 5.5) only made sense when the areas used in their representations of fractions were ignored. The distinction between a student attending to the area in a regional model or the number of parts is readily confused. The effect of interpreting the parts-of-a-whole regional model as the number of parts can be indistinguishable from using a whole number denominator rule, as it is the number not the area that is attended to.

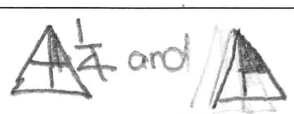
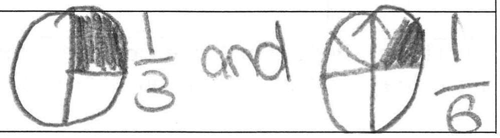
| | | | |
|---|---|---|---|
| 24 | $\frac{1}{3}$ or $\frac{1}{4}$ | $\frac{1}{4}$ | Because   |
| 25 | $\frac{1}{3}$ or $\frac{1}{6}$ | $\frac{1}{6}$ |   |

**Figure 7.8** Number of parts equivalent to WND

The area is not the defining characteristic of the representation of a traditional regional model shown in Figure 7.8. The number of parts appears to determine the magnitude, as the same area is portrayed for both $\frac{1}{4}$ and $\frac{1}{3}$ in Question 24 and yet $\frac{1}{4}$ is judged to be larger. With a limited range of examples provided in common textbooks it is quite feasible that a student attending to the number of parts rather than the area of the parts would obtain correct answers to many fraction questions.

The use of equidistant partitioning of *length* as a common partitioning strategy for regional models also indicates that area is frequently not the attribute attended to in regional models of fractions. The partitioning of circles with vertical lines, corresponding to approximately 10% of students' responses in this study, is an important indicator that area is not the feature of the regional model the student has accommodated into his or her fraction concept image.

The number of parts (sometimes equal parts) in a representation of a fraction defines the fraction *without reference to the relationship between the part and the whole*. That is, fractions can exist as countable units unfettered by being explicitly related to the whole. This lack of a relationship between the parts and the whole contributes to a focus on the number of parts and the related heuristic of interpreting fractions as being determined by the size of the whole number denominator appearing as an early expression of part-whole relationships.

### *7.4.2 Over-generalising concepts*

Determining when a student understands a particular concept is not a simple process. The variation in a student's responses to a number of questions on the same concept often reflects the dynamic and organic nature of learning. "In studying children's thinking we usually find all sorts of in-between patterns of performance: children who succeed on some versions of the task but not on others, and who thus seem to *have* the concept and at other times to *not have* it." (Flavel et al., 1993, p. 321) Providing the correct answer of one-quarter in response to representations of one-quarter is necessary, but not sufficient evidence of "knowing one-quarter". A student who knows what one-quarter is should also know what one-quarter is not.

To incorrectly assign the description of one-quarter to non-examples is sometimes described as over-generalising the concept. A quantitative sense of fractions requires having a sense of the size of fractions without over-generalising the fractions. Although over-generalisation of part-whole is referred to in some studies (Hart et al., 1981; Kerslake, 1986), it is usually treated only as a possible source of errors. Within a conceptual analysis of fractions, overgeneralising the fraction is not simply a source of possible errors; it impacts upon the boundary conditions of the fraction concept image and thus has an impact upon what it means to understand that fraction.

According to the classical view of categories (Lakoff, 1987), categories should be clearly defined, mutually exclusive and collectively exhaustive. Fractions as mathematical objects are idealised categories. That is, an abstract quantity fraction such as one-quarter can be described as a set of equivalence classes. The equivalence classes allow $\frac{2}{8}$ and $\frac{25}{100}$ to also be one-quarter. However, $\frac{2}{5}$ is clearly not one-quarter! Examples and counterexamples help to identify the boundaries of a concept. Even within an Aristotelian view of categories, the properties of the category establish the conditions that are both necessary and sufficient to capture meaning. Taking individual fraction concept images to be distinct categories, there is a need to explore the boundaries of the concept images. To have a concept of a fraction requires more than recognition of various embodiments of the fraction, it requires that the concept not be over generalised.

Does it change our perception of basic fractions such as one-half—the only fraction that is so well known it has been considered "…to be an honorary whole number" (Hart, 1989, p. 216)—or one-quarter, if we apply this *necessary and sufficient* description of a quantitative sense of fractions?

**Having a quantitative sense of one-half or one-quarter**

Considering a student's choice of the correct answer as a positive instance of the concept that the item is trying to assess is standard practice. The item distracters can be selected to characterise common misconceptions or negative instances of the concept. One limitation of this type of multiple-choice item analysis is that it can at best capture only the use of the concept in this instance as being more probable than a common misconception.

The diversity of responses provided by students to the free-response items in this study exceeds any reasonable attempt to capture the range within a multiple-choice structure. Even the single multiple-choice item used in this study evoked unexpected solution methods, such as subtracting the sum of the numerators from the sum of the denominators to select the correct answer (Figure 5.115). A correct answer may be misleading because the student has guessed, meant something different, or believed something different (Clements & Ellerton, 1995; Gay & Thomas, 1993).When an incorrect method is used to obtain the correct answer, no group of distracters in a multiple-choice question can be effective. Consequently using only positive instances or correct answers as evidence of a quantitative sense of fractions is inadequate.

In Questions 4 to 6, students were asked to identify in turn what fraction of the whole shape corresponds to each of the individual shapes. The individual shapes were identified by pointing to them and also distinguished by colour. The answer to Question 4 was one-half and the answer to Question 6, the rhombus, was one-third (Figure 7.9).

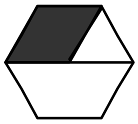


**Figure 7.9** What fraction of the whole shape is this part?

As identified by the item map (Figure 6.7), answering one-half to Question 4 was not particularly difficult. However, any student who had a well-formed concept image for one-half would not describe the rhombus as also being one-half. This stronger form of a quantitative sense of a fraction is a useful extension of a conceptual analysis of responses to fraction items. Using a necessary and sufficient condition approach to students' quantitative sense of fractions provides a way of identifying a robust form of the quantitative sense of fractions. It may also contribute to addressing a common problem in analysing students' responses. That is, given the variety of responses a single student might give to items that are ostensibly addressing the same concept, how do you know that a student has the concept of a specific fraction?

**A quantitative sense of one-quarter**

Applying the necessary and sufficient condition to the robust fraction one-half resulted in a reduction of those appearing to have the concept by approximately 5%. The strong form of a quantitative sense of fractions could also be applied to the fraction one-quarter, arguably the most basic fraction after one-half.

In Question 2, students were asked to identify one-quarter from a strip of paper folded into one-half and two quarters. Using the strong form of quantitative sense of fractions, any student who had a well-formed concept image for one-quarter would not describe the rhombus in Question 6 as also being one-quarter. Some students may identify an equilateral triangle as one-quarter of a regular hexagon because of the proximity of the shape (and orientation) to a visual prototype of one-quarter. However, if the orientation-dependent visual prototype of a quarter were the only available component of a concept image for one-quarter, such a student would have an inadequate concept of one-quarter. Alternatively, as these two questions draw on different regional models (area of a rectangular strip of paper compared with pattern block shapes) the differences in the models may have influenced the evoked concept images of one-quarter. However, a generalised regional part-whole comparison of one-quarter should apply to both.

The total reduction of those appearing to have the concept of a quarter was 18% in this study. Working with a strong form of a quantitative sense of fractions overcomes some of the problems of students sometimes appearing to have a concept and other times not. It also refines what it means to understand fractions as mathematical objects.

## 7.5 A partition-based developmental model

Acknowledging that opportunities to learn (that is, access to the curriculum) influences the order in which students are able to demonstrate their proficiency with various fraction forms, the analysis of the items across Years 4 to 8 suggests a broad developmental hierarchy. Partitioned fractions precede abstract quantity fractions on the item map and in students' learning. Within the partitioned fractions, the application of the process of halving results in halves and quarters being more readily created and used than fractions such as one-third. This is consistent with research into developmental sequences for partitioning geometric regions, which starts with halving and repeated halving (Piaget et al., 1960; Pothier & Sawada, 1983). The primitive forms of halving are conflated with visual preferences in the vertical and horizontal directions for spatial structuring, which can also apply to partitioning collections of discrete objects. Halving and quartering discrete objects simply arranged in rows and columns can draw upon the same visual partitioning techniques as used with continuous objects. Quartering 12 dots arranged in two columns of 6 dots as in Questions 9–11, can be achieved using the same quartering partitions as applied to quartering a circle (Figure 7.10).

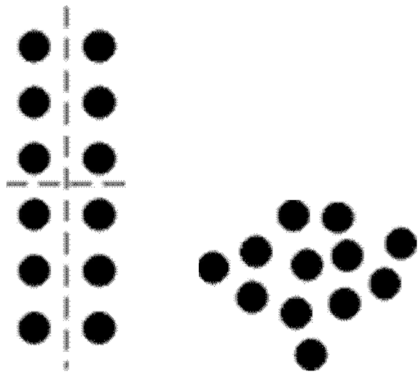


**Figure 7.10** Linking visual preference with spatial structuring of discrete items

Without the spatial structure provided by the rows and columns the 12 dots on the right of Figure 7.10 would need to be quartered using a different strategy. Distributive dealing, the one-by-one exhaustive equal sharing of items amongst places (Davis & Pitkethly, 1990) is a markedly different type of partitioning scheme that is not derived from repeated halving. In Figure 7.11, the allocation of 12 counters to 4 places has little in common with quartering a circle or rectangle.

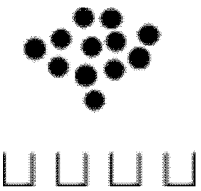


**Figure 7.11** Distributive dealing requiring identified locations

Distributive counting and dealing can be used in structured contexts without an awareness of partitioning to create equal units (Davis & Pitkethly, 1990). Discrete items are included in the proposed model of the development of fractions only when essentially the same partitioning strategy is applied to both continuous and discrete models of fractions.

The students' responses to the quantitative fraction sense items suggest that "even partitioning" through repeated halving is simpler than "odd partitioning", such as in creating thirds involving continuous representations, and that the comparative difficulty of halves, quarters and thirds persists in dealing with multiples of these as non-unit fractions. Iterating a unit when measuring was also much easier when the remainder was one-half compared to measuring when the remainder was one-third. Dealing with non-unit fractions in measurement division appeared to over-ride this hierarchical order.

A developmental progression from partitioned fractions to abstract quantity fractions emphasising partitioning and the development of the equal whole, is outlined in Figure

7.12. Fractions represented by the set model or discrete items are not included as a separate component but rather are referred to when the partition strategy is essentially the same as for continuous model representations. In an earlier study (Clements & Del Campo, 1987) it was clear that many children simply could not think of a discrete set of objects as a unit that could be partitioned. Reforming the unit whole begins with the first partitioned fractions students form and continues to develop with odd partitioned fractions. The hierarchy in Figure 7.12 is intended as a summary of the development of the key strategies or conceptual levels evident in the development of a quantitative sense of fractions, and linked to the Rasch item map in Figure 7.13.

## Partitioned fractions

1. **Halving**
Using halving to create the 2-partition and the 4-partition with vertical or horizontal preference and, interpreting halves and quarters formed. The same visual partitioning strategies (vertical and horizontal) can be applied to collections of discrete objects.

2. **Odd partitions**
Creating measurable unit fractions without repeated halving. In particular, creating thirds by subdividing a given whole.

3. **Re-unitising**
Iterating a fraction part beyond the whole and re-forming the unit whole (halves and quarters first), and reconstructing the unit whole given a (unitary) fractional part.

## Quantity fractions

4. **Quantity fractions using the equal whole**
Demonstrating a sense of the universal equal whole and using the multiplicative structure to create equivalences.

5. **Coordinated composition of partitioning**
Coordinating composition of partitioning and carrying out measurement division involving non-unit fractions.

6. **Reversible part-whole**
Reversing the part-whole comparison. That is, describing the improper fraction resulting from changing the unit-whole being referenced (e.g. if $a$ is $\frac{3}{4}$ of $b$, then $b$ is $\frac{4}{3}$ times $a$).

7. **Notational quantity fractions**
Recognising the relationship between the fraction notation and the abstract unit-whole (e.g. $\frac{12}{13}$ is closer to 1 than $\frac{9}{10}$ because $\frac{1}{13}$ is smaller than $\frac{1}{10}$).

**Figure 7.12** Partitioning and the unit whole

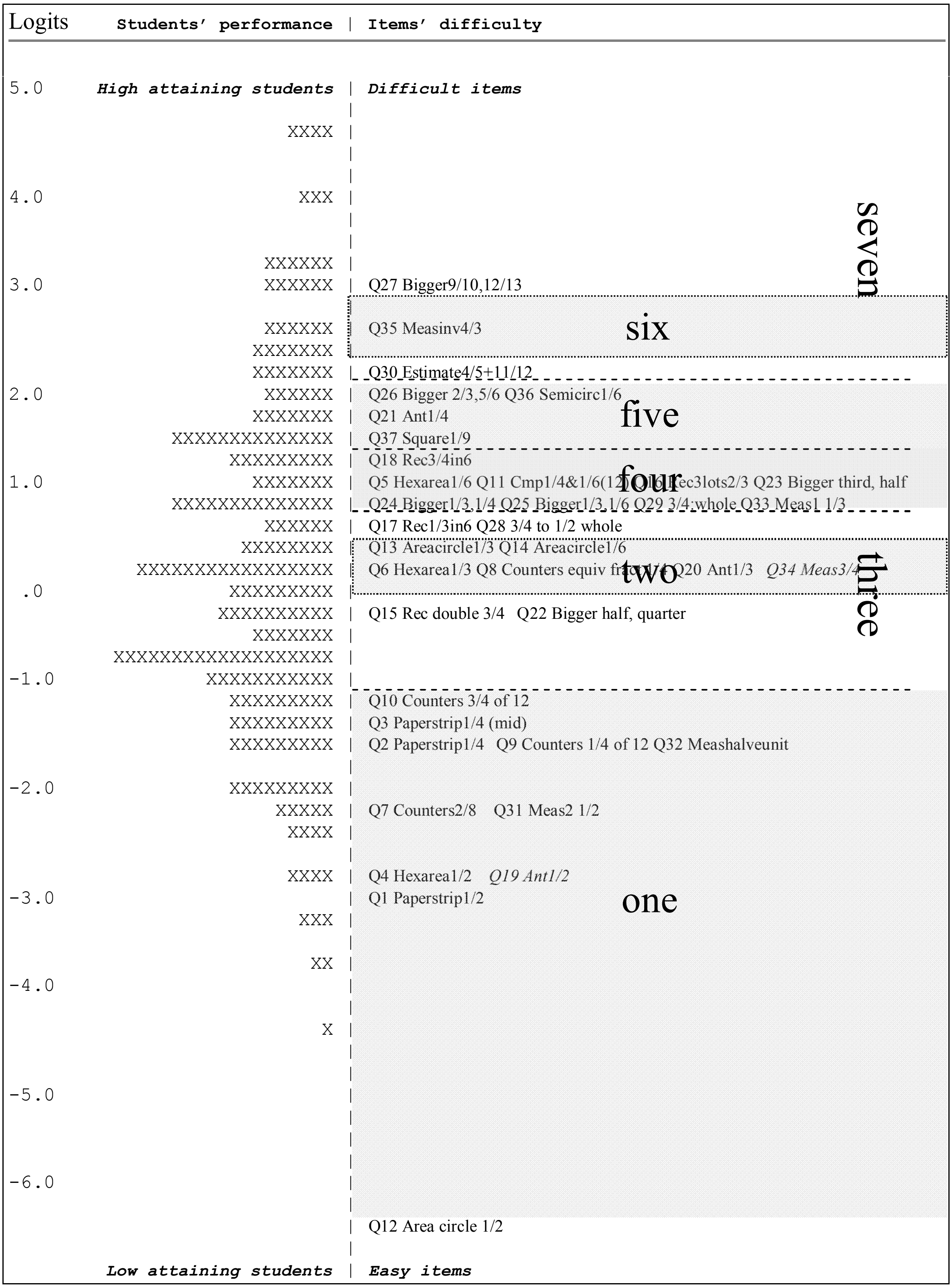



**Figure 7.13** The item map showing links to the model of development

The focus of the descriptions of the levels is the relationship between partitioning and the formation of units which reference a progressively more abstract unit whole. Strategies evoked by the fraction notation that reflect misconceptions have not been included as they represent departures from the pathway to a quantitative sense of fractions. Having created a 4-partition as a measurable fraction, a student can coordinate the count of those quarter units before being able to create other measurable unit fractions without repeated halving. That is, non-unit fractions of quarters (such as Q10: determining three-quarters of 12 counters arranged in two columns) can be more readily appreciated by students than odd partitions. The basic nature of halving and doubling is such that what is manageable with halves and quarters, precedes the same processes with odd-partition fractions. Composition of partitioning and the reversible part-whole are necessary conceptual foundations for quantity fractions and the operations involving them. Coordinating composition of partitioning, also described as recursive partitioning (Steffe, 2003), appears to be even more challenging than developing a working knowledge of the equal whole, required for quantity fractions. Quantity fractions are ultimately mathematical objects (numbers) and part of a powerful notational system. The description of notational quantity fractions recognises the link between the notation system and the abstract unit whole that characterises a quantitative sense of fractions. The meaning of the notational system develops alongside multiplicative relationships in quantity fractions.

The model describes a progression from partitioned fractions (fractions of physical entities) with initial success associated with halving and quartering before odd partitions such as thirds. The challenge and adjustments associated with creating thirds are of a similar level of difficulty to creating sixths, as the process of halving is both a precursor and of a lower order of difficulty to creating thirds. The process of re-unitising develops in parallel to creating partitioned fractions such as thirds and sixths. Next the transition to quantity fractions associated with the process of measurement takes place on the way to developing a quantitative sense of fractions. The standard unit of comparison used in measurement helps to characterise the equal whole needed to compare fractional quantities. Multiplicative composition of fractions beyond repeated halving then contributes to the reversible part-whole conception and the meaning of the notation system can develop cojointly. All developmental models, however, are subject to the order in which material is presented providing students with opportunities to learn.

### *7.5.1 Comparisons with other descriptions of fractions*

Three broad unifying elements have been suggested as providing some theoretical coherence to rational numbers: identification of the unit, partitioning, and the notion of quantity (T. P. Carpenter, Fennema, & Romberg, 1993; Pitkethly & Hunting, 1996). Partitioning that is not achieved by repeated halving is a significant reorganisation of thinking, as it requires anticipating the number of cuts required to produce the desired number of parts and adjustment of those divisions to create equality of the parts. Iterating the unit may play a role in the transition through partitioning to the notion of quantity referencing the unit whole (Tzur, 2000). Steffe and Olive (cited in Steffe, 2002) hypothesised that through abstracting the activity of partitioning combined with the activity of iterating a unit, a child would base his or her first conception of fractions, the *equi-partitioning scheme*, on the equality of all parts and on the number of times that the unit was iterated to produce the partitioned whole. By applying the equi-partitioning scheme the learner may come to anticipate that partitioning a given whole brings about a unique-size quantity relative to the size of the whole, and the inverse relationship between the number and size of parts. However, 'more parts resulting in smaller parts' is only necessarily true if the whole remains constant. Consequently, partitioning a whole and iterating the partitioned unit to re-form the whole are likely to strengthen the multiplicative part-whole relationship. Re-forming units (re-unitising) is then the next important step in developing a robust understanding of fractions.

Steffe (2002) also suggested that children can reorganise their fraction knowledge through iterating unit fractions to produce non-unit fractions, those with a numerator greater than one such as $\frac{3}{5}$ and $\frac{7}{5}$. This more advanced conception has been described as the *iterative fraction scheme.* The re-forming of the unit whole necessary in iterating a unit fraction beyond one can be an obstacle in developing the iterative fraction scheme (Tzur, 1999). That is, $\frac{1}{8}$ iterated nine times can be incorrectly thought of as $\frac{9}{9}$ by students and each part considered $\frac{1}{9}$ rather than re-forming the unit whole ($\frac{8}{8}$) to arrive at $1\frac{1}{8}$. The re-unitising associated with equivalent fractions has also been described as "chunking" (Empson, 1999; 2001). Re-forming units is an important step in the development of fractions, in particular in moving from partitioned fractions towards quantity fractions, as it emphasises the role of the whole. The whole can be subdivided in various ways without changing the nature of the whole. Being able to reverse the process of partitioning, to recreate the whole given an identified fractional part (such as determining the whole object given three-quarters of it) is more challenging than creating fractions such as one-third. The process of determining the whole from a fractional part has been described as the *reversible fraction conception*—a transformation in children's iteration-based fraction concepts (Tzur, 2004).

The general reversible fraction conception is also more elaborate than being able to recreate the unit-whole from a unitary fraction, such as determining the whole given one-quarter.

Based on teaching experiments (Steffe, 2002; Tzur, 2000, 2004), a number of schemes have been proposed in the development of children's fraction knowledge:

1. equi-partitioning scheme
2. partitive fraction scheme
3. iterative fraction scheme
4. reversible fraction conception.

These schemes emphasise the role of partitioning a whole and iterating a part to develop a multiplicative part-whole relationship for fractions. The reversible fraction conception which encompasses both the whole-to-part and part-to-whole operations (Sáenz-Ludlow, 1994) comes closest to recognising the need for a fraction as a mathematical object to reference an abstract stable unit whole and is reflected in the description of reversible part-whole in the model (Figure 7.12). The abstract equal whole, a "one" that is always the same size, is a necessary conceptualisation to enable comparisons of fractions as numbers and is a precursor to operating on quantity fractions.

*Partitioning* and *equivalence* can be thought of as fundamental processes that arise from the idea of equal sharing. These processes contribute to the equi-partitioning scheme and the equal-whole scheme. Within the above characterisation of the schemes involved in the fraction concept, anticipating iterating and iterating a unit are necessary to produce the partitioned whole. The operations of iterating and partitioning may well be parts of the same psychological structure (Steffe, 2002) and reversibility as a feature of equi-partitioning also links iterating and equivalences with equi-partitioning. Indeed, as the part-whole interpretation of fractions depends directly on being able to partition into equal-sized subparts, it can be thought of as an integration of the processes of partitioning and equivalence. The whole and part-whole relationships come into contention in forming the equal partition.

Creating equivalence is also important in addressing the distinct problem of fractions having multiple representations of the one quantity ($\frac{1}{3}=\frac{2}{6}=\frac{3}{9}$) within the same representational system. As Lamon (1999, p. 22) has suggested, the hardest part for some students is understanding that "what looks like the same amount might actually be represented by different numbers." The notational equivalence of fractions is implicitly dependent on the existence of a universal one. Existing models of fraction subconstructs or fraction schemes do not make clear the transition from partitioned fractions (parts of objects) to abstract quantity fractions (fractions as numbers, parts of the abstract unit).

The epistemological subconstructs of the fraction concept (measure, quotient, ratio number and operator) are related to the various schemes through the action base of the relevant scheme. For example, the quotient subconstruct is the result of division and the process of dividing is dependent upon the action schemes of splitting, equi-partitioning and sharing. The quotient subconstruct might be considered to result from an integration of whole-number knowledge of division with fraction schemes. Similarly, a fraction can be the result of a measurement, which in turn is the result of the successful partitioning of a unit. Describing fractions as part of the conceptual field of multiplicative structures (Vergnaud, 1994) highlights the way that fields are comprised of two basic components: elements and operators. Consequently, it is possible to distinguish the operators associated with fractions (e.g. partitioning, unitising and operators such as stretcher-shrinker) separately from the elements within the field (e.g. objects such as quotient, the result of a division).

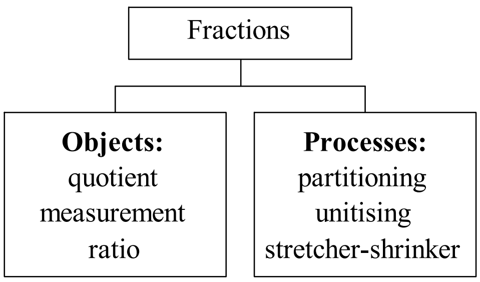



**Figure 7.14** Fractions as objects and processes

Fractions, as mathematical objects, are the individual numbers formed as the result of an indicated division or comparison. The division may be partitive or quotitive (measurement division) and the comparison for fractions as ratio numbers is part to whole or whole to part. The principal value in examining fractions as part of the conceptual field of multiplicative structures and the subconstructs of the fraction concept is in the comprehensive design and categorisation of fraction tasks made possible by this description of fractions.

Designing teaching from the stance of mathematical structures (fields and operators) that have taken thousands of years to develop is unlikely to align with the ways that children learn. Students struggle to construct meaning for representations of fractions in the absence of instruction which builds on their own informal knowledge (Mack, 1990, 1993, 1995). Similarly, defining fractions in terms of subconstructs has little value for teaching operations with fractions, when fractions are described as the amalgam of those subconstructs. Fractions are only abstract mathematical objects when they are independent of their embodiment. Indeed, rational number subconstructs (such as measure, quotient, ratio number and operator) start from a unique conception of fractions as object: our current modern formulation of fractions. The transition from parts of wholes to fractions as mathematical objects needs a clearer map.

The hierarchy outlined in Figure 7.12 considers fractions as either partitioned fractions or quantity fractions. Part-whole relationships are characterized by partition fractions. For both discrete and continuous quantities, the unit-whole is often implied, especially in everyday situations, making it difficult for students to identify the unit-whole. For example, when Grade 2, 3, 4 and 5 students in Australia were asked to find a fraction of a discrete set of objects, many students had difficulty identifying the unit-whole (Clements & Del Campo, 1987). Therefore, when students subsequently incidentally encounter quantity fractions, they may have difficulties. The essential idea of abstract quantity fractions is that fractions themselves can express abstract quantities that can be compared and ordered.

The view in which fractions correspond to two integers, the result of a double count, is typical of the partitioned fraction (Behr, Wachsmuth, Post et al., 1984). The various subconstructs (part-whole, quotient, ratio number, operator, and measure) that contribute to the domain of rational numbers can be also be considered from the action basis of scheme development, and in particular, unitising. The various subconstructs can be understood as compositions and recompositions of units. Fractions expressed as partitions of continuous units are effectively measurable and those expressed as partitions of discrete units are countable (Pitkethly & Hunting, 1996). When the partitions of a continuous unit are double counted, replacing the process of measuring, the resulting static double count meaning of a part-whole relationship can inhibit the development of the quantity concept of fractions.

A binary classification of fraction embodiments (such as measurable and countable) is a very useful way to think about the development of a quantitative sense of fractions. When fractions are first studied in Japan (Takahashi, Watanabe, & Yoshida, 2004) two interpretations are given for a fraction such as 2/3:

1. Representing two of three equally divided parts.
2. Representing the quantity resulting from a measurement, such as 2/3 metre.

The first of these interpretations does not specify the whole as an immutable entity. Two out of three equally divided parts cannot be compared to three out of six equally divided parts without the introduction of a universal equal whole. Thus the transition from partitioned fractions to quantity fractions is a necessary progression.

Progressing from partitioned fractions to quantity fractions must rely upon the introduction of the equal whole. The unique number two-thirds is more than two-thirds of a group of marbles or two-thirds of a circle; it is itself a number that can be involved in division, multiplication, addition and subtraction. Fractions as mathematical objects are dependent upon the introduction of the universal equal whole into the fraction concept image to enable comparisons of abstract quantities and equivalent forms of these quantities.

The development of the universal whole is essential in forming fractions as relational numbers. Using a continuous regional model, the equal whole defines the size and type of regions to be partitioned. Employing a discrete set model to compare the size of two fractions brings into question what is being compared. Consider comparing one-third with one-quarter using a discrete set model as in Figure 7.15.

**Figure 7.15** Comparing one-third and one-quarter

Students may conclude one quarter is larger because there are more elements in the set than for one-third. The universal whole needs to be constructed to allow comparison of fractions with a discrete model. That is, equivalences must be invoked to compare discrete models addressing the universal whole.

**Summary**

The instructional intent of partitioning shapes to focus on equal area comparisons, commonly used with regional models of fractions, appears to have been successful for some students in developing a quantitative sense of fractions. For other students, area has not been the salient feature of the regional model contributing to their fraction concept images. Our greater visual acuity in the vertical and horizontal directions than other directions has meant that when shapes are partitioned, vertical partitioning draws predominantly on the horizontal dimension to determine the distance between the partitions. Partitioning of shapes, used with the instructional intent of representing partitioned area, can in practice result in equidistant partitioning of the horizontal dimension, the attribute of length. This in turn may lead some students to attend to the number of parts formed by parallel partitioning of shapes.

A simple interpretation of a parts-of-a-whole model used in teaching fractions can lead to separate components of the model forming features of students' fraction concept images. In particular, fraction concept images may contain

- parts formed from parallel partitioning;
- a focus on the number of parts, not necessarily equal to each other;
- the number of equal parts corresponding to the denominator (but not forming the whole); and
- heuristics related to the size of the numbers used to record the fraction.

Though fractions expressed as partitions of continuous units could be thought of as measurable, their treatment as parts-of-a-whole often evokes countable concept images. When the partitions of a continuous unit are double counted, the double count meaning of a part-whole relationship can work against the development of the quantity concept of fractions. The fraction notation of one whole number over another, frequently aligned with the parts-of-a-whole model of fractions, can also give rise to count-related heuristics. In determining the size of a fraction expressed as a number, it is not unusual for students to focus on the size of the whole number in the denominator, the numerator or both. Frequently the perceived relationship between the size of the denominator and the fraction is a direct relationship rather than an inverse one. Even the correct inverse relationship between the size of the denominator and the size of a unitary fraction can be overgeneralised to non-unitary fractions.

While it is possible to describe fraction tasks according to the materials used to represent the fraction (discrete, continuous or composite) a simple description of the evoked concept image of fractions as measurable or countable helps to outline the development of a quantitative sense of fractions. The transition from partitioned fractions (i.e., fractions as parts of things) to quantity fractions (i.e., fractions as numbers related to the universal whole) is a necessary progression in developing an appreciation of fractions as mathematical objects. Rather than treating fraction models as imperfect embodiments of fractions as mathematical objects (numbers), the progression from partitioned fractions to quantity fractions must be appreciated as a necessary development. Fractions as mathematical objects are dependent upon the development of the universal equal whole as a component of the quantitative sense of fractions.

The implications of the model of partitioning and the equal whole for teaching and students' discrete interpretations of parts-of-a-whole models are outlined in Chapter 8. In particular, understanding the limits of regional models where measurable features may be interpreted as countable features suggests that introductory fraction tasks need to emphasise the measurement component more than the counting component, to avoid fractions as double counts and simplistic interpretations of the fraction notation.

# Chapter 8 IMPLICATIONS FOR TEACHING

*The essential attribute of the regional model, namely area, cannot provide an adequate basis for interpreting a parts-of-a-whole representation of fractions when students do not have a robust understanding of area. Students' greater visual acuity in the vertical and horizontal directions, combined with a lack of attention to the essential components of a parts-of-a-whole representation has contributed to the delevopment of a range of pseudo fraction concepts rather than a quatitative sense of fractions. Those for whom equidistant partitioning is the salient feature of regional models develop a "number of parts" sense of fractions, and the fraction notation also engenders a range of number-based strategies for determining the relative size of fractions. Many students construct fragmented parts-of-a-whole concept images of fractions by attending to only some aspects of the part-whole relationship. Teachers must introduce the fraction concept using a model that relies on an attribute with which students are familiar to ensure that the students can make the part-whole comparison required in developing fractions as relational numbers. Moreover, the equal whole used to make the transition from partitioned fractions to quantity fractions should be explicitly taught.*

**Preview**

This chapter outlines the implications for teaching arising from mapping students' quantitative sense of fractions. The need to address the identified limitations of current models of fractions used in teaching on developing a quantitative sense of fractions in students is outlined in Section 8.2. It is argued that non-examples should play a role in limiting the over-generalisation of fraction concepts, as well as helping to make explicit the relevant features of a model. Linear and area models of fractions are compared in Section 8.3, where it is argued that if distance is the attribute students use to reason from a regional model (e.g. the distance between regional boundaries), then teaching may benefit from focusing on the attribute of length in measurement division to provide a more cohesive conceptual model for fraction learning. This is followed by outlining ways to develop the idea of an equal whole, as this idea underpins the transition from models of fractions as made up of parts (partitioned fractions) to the appreciation of fractions as quantitative measures (quantity fractions).

## 8.1 Introduction

The documented concerns over the limitations of the parts-of-a-whole model of fractions have not been sufficient to reduce the use of this model in the teaching of fractions, or to shift the focus from counting. In part, this failure to address the limitations of the parts-of-a-whole model may be due to the lack of a clear replacement. Teaching fractions as the amalgamation of all of the different constructs and personalities of fractions is not a viable alternative. Detailed semantic analysis of the variety of interpretations of fractions does not readily translate to manageable teaching approaches to introducing fractions that allow the operations with fractions to develop from the various interpretations. The diversity of approaches to fractions within the research literature has not been helpful to improving the learning of fractions. Indeed, it is unsatisfactory in regard to designing instruction for an integrative understanding of fractions (Thompson & Saldanha, 2003). The diversity of approaches and the volume of fraction research have contributed to a stasis in the practice of fraction teaching in many countries.

Common fractions have multiple embodiments and representations, with a symbol system that is very powerful but is not transparently linked to the fraction concept. Consequently, the teaching of fractions as part concept and part algorithmic procedures has led many students to attend to different specific features of the models used in the teaching of fractions. Teachers might use parts-of-a-whole area models to introduce fractions, in line with syllabus recommendations, but some students focus on the countable features of the area models, and not the measurable features. Instead of the relationship between the area of the part and the area of the whole being interpreted as the essential feature of the model of a fraction, other components of the model contribute to students' fraction concept images.

The intended fraction embodiment is often not the interpreted fraction embodiment. The standard fraction notation itself encourages a count interpretation of the regional parts-of-a-whole model. Use of discrete entities to model a fraction does not reduce a focus on countable features. The use of a continuous representation of fractions needs to focus more clearly on the measurement property as distinct from discrete counts of regions. In particular, any parts-of-a-whole model used in teaching fractions should enable students to see

- the whole contributes to the total number of parts,
- the whole is subdividable into any number of parts all equal to each other,
- the equality of the parts is a function of the relationship between the parts and the whole,

- an inverse relationship exists between the number of equal parts and the size of those parts,
- the subdivided equal parts can group to form new composite parts,
- the equal parts can themselves be further subdivided,
- the whole can be reconstructed from a part, and
- the universal whole acts as the common basis for comparison of abstract quantity fractions and subsequent operations with fractions.

The final feature of parts-of-a-whole models described above is frequently overlooked and yet characterises the transition from partitioned objects or collections acting as models of fractions to the notation of fractions signifying abstract quantities or fractions as mathematical objects.

## 8.2 Surmounting the limits of models of partitioned fractions

The attempt to follow a traditional developmental pathway from the concrete to the abstract has contributed to the early introduction of various models of fractions in classrooms. That is, fractions are introduced as partitions of things, typically objects with subdividable regions or collections of identical objects that can be distributed. The act of partitioning takes different forms depending upon the type of quantity involved. A unit represented by a continuous quantity is partitioned by being cut into separate pieces whereas a discrete quantity is partitioned by being sorted into separate collections (Ohlsson, 1988). Fractional units expressed by partitioning continuous quantities are effectively measurable and the discrete expressions are countable (Pitkethly & Hunting, 1996). The role played by the whole when partitioning discrete items to form fractional quantities is also markedly different from the role of the continuous whole. Neither distributive dealing nor the process of fairly sharing discrete objects requires attention to the total constituting the whole—both are exhaustive processes.

The current parts-of-a-whole regional models and discrete models used to introduce the naming of fractions do not articulate the more explicit distinction between models of fractions and fractions as numbers. There are two different conceptions of fractions — one with an explicit but variable whole (partitioned) and one that is a mathematical object (quantity). To develop fractions as relational numbers it is necessary to move from partitioned fractions to abstract quantity fractions. The analysis of students' responses to questions designed to evoke their quantitative sense of the size of fractions indicates that for some students the regional model of fractions is being interpreted as a number of parts model. Although the main reason for using a regional model of fractions is ease of demonstrating equivalent fractions by comparison of area, area is not always the feature attended to in regional models of fractions. Indeed, many students have difficulties in appreciating the attribute of area (Hirstein, Lamb, & Osborne, 1978; Outhred & Mitchelmore, 2000). To reduce the risk of misinterpreting the essential features of regional models, namely the relationship between the area of a part and the area of the whole, simpler models of partitioned fractions that focus on one-dimensional attributes rather than (two-dimensional) area should be more effective. Two-dimensional area models of partitioned fractions can be introduced after students have developed a sound appreciation of area as units of units. It is sensible to introduce a new concept with a familiar model.

Few teachers would introduce the idea of area before they have introduced length. Yet the same is not true when it comes to teaching fractions. For example, the fractions one-half and one-quarter are introduced early in the primary school curriculum. Even before the formal notation is introduced, simple uses of regional models can result in unexpected concepts developing for one-half and one-quarter. Some students can construe an example presented to show partitioning area, as partitioning length. That is, when forming one half or one quarter, the subdivision of area may be considered a by-product of subdividing one side by parallel partitioning. This limited concept image may not be apparent as it can result in responses that appear indistinguishable from partitioning the region. Parallel partitioning can then exist undetected until it appears later as a misconception in response to creating fractional parts of a circle. Consequently, it is important that teachers have manageable ways of monitoring students' changing conceptions of fractions.

The use of partitioned geometric shapes is common in mathematics textbook exercises involving fractions. Typically a geometric shape will be shown with a number of equal parts shaded, and students directed to determine what fraction of the shape has been shaded. The answer is often readily determined by counting the number of parts shaded and the total number of parts, before writing one whole number over the other. Using pre-partitioned shapes in teaching and assessing can mask an incomplete or incorrect appreciation of fractions as relational numbers. Many teachers appear to be unaware that students are adopting only part of a regional parts-of-a-whole model of fractions. That is, some students focus on the number of pieces named by a fraction and others the number of equal pieces named, without addressing the relationship between the area of the parts compared to the area of the whole region. Students' responses to partitioning regional models indicate that partitioning is frequently carried out without specific reference to the attribute of area. Equidistant partitioning of length is quite a common partitioning strategy for regional models and the focus on distance provides a better way of introducing fractions through partitioning length, leading to measurement division.

The evoked concept images of many students include a visual image as a prototype of the fractional quantity, usually influenced by our greater visual acuity in the vertical and horizontal directions. The tendency of students to see better in the vertical and horizontal orientations needs to be acknowledged in the design of learning tasks that rely on partitioning using halving or recognising axes of symmetry. Partitioning length by cuts that are perpendicular to the length should make use of the greater visual acuity without confounding the attribute being partitioned with the process of partitioning. Currently, the process of quartering a square or a circle is too readily transferred by the student to quartering a triangle when the process of quartering is independent of the attribute being partitioned. It is also far easier to compare the size of the parts formed when the attribute being partitioned is length, not area. Early comparison of area frequently relies on dissection and rearrangement which is far more difficult than using a common baseline for comparison of length.

### *8.2.1 Non-examples*

In establishing what a fraction such as one-quarter is, it is important to support students in becoming aware of what one-quarter is not. That is, when forming the idea of a fraction such as one-quarter it is necessary to clearly form the boundaries of the concept. If one-quarter is always portrayed as one part of four equal parts a student may reasonably take only some of the essential components of the concept from the representation. A student may reasonably focus on the number of parts or the equality of the parts or even the number of division marks. Involving students in 'creating the parts' can strengthen the relationship between the parts and the whole. In creating equal parts of a whole (a partitioned fraction), the process of partitioning is clearer than working with pre-partitioned shapes.

However, if the focus of the regional model is area, then students need opportunities to match the areas of the parts and non-examples should also look at area. As the purpose of examples and non-examples is to help to clarify the concept boundary, both examples and non-examples need to be generated keeping in mind their proximity to that boundary. For example, a rectangle divided into four equal parts with one part shaded is a common example of a regional model of a fraction. A useful non-example might hold the number features constant—the total number of parts and the number of parts—while varying the feature of area.

**Figure 8.1** Example and non-example of a quarter

Of course, the feature of area need not be specifically used in interpreting the distinguishing differences between the example and the non-example in Figure 8.1. As the heights of the two rectangles are the same, it is the distance between the section markers that varies between the example and non-example. The attribute used in the part-to-whole judgement needs to be considered very carefully and examples such as the rectangles used in Figure 8.2 strengthen the argument for introducing fractions from comparison of length. Students' explanations are critical in determining what feature of a model of fractions they actually attend to. If an apple is used to represent the whole, the feature or attribute of the quarter of an apple being compared is volume. The attribute of the model being used also needs to be clear in the choice of non-examples. Although examples such as half an apple or a quarter of a sandwich are commonly used to introduce the idea of a partitioned fraction, the underpinning unit of comparison is volume—even though developing a method to compare volume is not encountered until much later in the school curriculum. For the practical purpose of teaching, fraction models need to be evaluated in terms of what attribute is used to convey the comparison and whether students are confident with the use of that attribute (e.g. length, area, volume, angle or number).

Vertical and horizontal partitioning can be used to link quartering of regional models and discrete objects arranged in rows and columns, as long as non-examples are also considered. In Figure 8.2, the discrete items and the rectangular regional model are both quartered using vertical and horizontal partitioning.

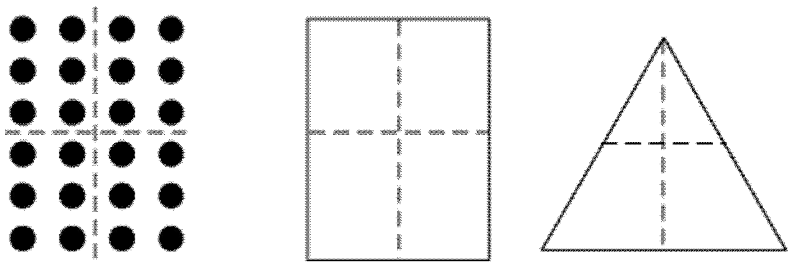


**Figure 8.2** Quartering discrete and regional models and a non-example

The final example of the equilateral triangle is used of a non-example of creating quarters by partitioning in vertical and horizontal directions. The results of this study (see Section 5.4.4) indicate that students need to work with non-examples to reduce overgeneralising the process of quartering.

### *8.2.2 Measurable or countable regional models?*

Teaching experiences need to provide opportunities for students to come to know fractions as more than a simple count. The definition of fractions as two simple counts should be avoided in introducing fractions. The limitations of a parts-of-a-whole or an "*a* out of *b*" introduction to fractions have been documented (Brown, 1993; Kerslake, 1986; Mack, 1995). This approach leaves students challenged to give meaning to fractions greater than one, such as $\frac{7}{3}$. To introduce a partitioned fraction, such as one-quarter of a length of string, iterating the one-quarter unit to form the whole moves from the unitary fraction to accumulation of the unit fraction (two-quarters, three-quarters) to the re-formation of the whole (four-quarters) and beyond. This progression of partitioning a whole and iterating a part to re-form the whole and then moving beyond the whole will help to develop a multiplicative part-whole relationship for fractions. More importantly, establishing the multiplicative relationship between the parts and the whole does not reduce the naming of a fraction to an act of counting. This is particularly important in avoiding the limitations of part-whole models (Behr et al., 1992; Freudenthal, 1983; Kerslake, 1986; Kieren, 1988). Forming quarters through repeated halving (half of one-half) is also a useful process of composition of partitioning that can leave verification of the size of the pieces formed and their relationship to the whole. Exploring the impact of repeated halving and the relationship of the parts formed to the whole can contribute to the idea that as two halves make one whole and one-quarter is half of a half, four quarters make a whole.

In dealing with linear, area (regional), volume and discrete models or embodiments of fractions it is important to realise that a student cannot make use of the model for fractions if they have not developed a sense of units in the underlying feature of the model. To make use of a regional model a student needs to be able to deal with units of area and to utilise a discrete model a student needs to have a sense of number as abstract composite units. Consequently, the linear features of fraction comparisons (length) need to be introduced before area. In particular, to elicit the feature that the student attends to when working with a fraction model, key questions are "What is equal?" and "How do you know they are equal?"

### *8.2.3 Phonemic distinctions*

The English language naming of fractions brings with it some specific challenges. The statements students made in this study to the effect that "one-sixth is six" emphasise the need to be explicit in teaching the phonemic differences between sixth and six and other fraction/integer pairs. This phonemic distinction needs to support the differentiation between fractional quantities and ordinals. Practices of counting involved with parts-of-a-whole interpretations of fractions can strengthen the notion that fractions such as one-sixth are interchangeable with a similar sounding integer, six. If you count six things and then say "one-sixth" it is relatively easy to hear it as "one six", or at least to believe that they mean the same thing.

Reading makes use of three cueing systems – graphophonic (visual), syntactic (structure) and semantic (meaning). The English naming of fractions using ordinal descriptions creates additional challenges of auditory discrimination. The English fraction naming system appears to influence the construction of some students' fraction concepts. For example, one Year 5 student argued that one-third is bigger than one-half because "a half sounds smaller".

Everyday use of fraction terms, such as in the phrase "take the bigger half", can lead to the creation of informal and incorrect units of fraction measure. This non-exact use of fraction terms goes beyond half to include quarters. Students' reasoned explanations in comparing the size of fractions suggest that the term "quarter" may sometimes be used as a substitute for "pieces". Imprecise use of fraction terms can contribute to some students developing the idea that fractions are made up of a type of common unit, called quarters. Even the process of quartering may take on a more general meaning than dividing into four equal pieces. Reducing the risk of developing misconceptions, such as using "quarter" as another term for a fractional piece, can also be achieved through the use of non-examples.

## 8.3 From partitioned fractions to quantity fractions

The identification of the whole is essential to interpreting both continuous and discrete embodiments of fractions. In Figure 8.3, the quarter of a rectangle and the quarter of the counters are both partitioned fractions in that they are a quarter of something. To compare the quarter of the rectangle to the quarter of the counters is meaningless as they are quarters of different things, that is, different wholes. Continuous and discrete models of (partitioned) fractions have different mathematical features (e.g., measurable versus countable) as well as more or less readily identified wholes.

**Figure 8.3** One-quarter in continuous and discrete forms

Some students construe the traditional part-whole teaching of fractions as part-whole number rather than part-whole area. The counting of parts may for them have been the common feature of discrete and continuous models of fractions. The use of regional part-whole models can result in what was designed to be measurable becoming countable for many students. The number of parts can readily determine the magnitude of a fraction within a regional part-whole model.

Beyond the embodiment used to introduce fractions is the interpretation students are able to construe from the offered embodiment. Even within regional models, not all shapes are equal. Ball (1993) found that partitioning of rectangular region models was easier for children in her study than the partitioning of circular region models. Students should find it easier to produce thirds with a rectangle than with a circle because the attribute used to create one-third is length rather than angle size, area or even arc length.

When the partitioning of regional models is considered carefully, it becomes clear that the area or region is often not the feature being considered when partitioning. In practice, the distance between cuts, the angle at the centre of a circle, or the length of the arc formed is the feature used, not the area. Dealing with a regional model of fractions should not be the focus of teaching until students have developed a sense of area that allows them to coordinate units in two perpendicular directions.

As with the teaching of decimals (Stacey, Helme, Archer, & Condon, 2001), area may be a less effective model of fractions than length. Students can have difficulty in working with the attribute of area in regional models. At some point, either the end of primary school or the start of secondary school, the leap from partitioned fractions (i.e. fractions with an explicit concrete whole) to quantity fractions (i.e. fractions whose size refers to an abstract constant unit) is made. In some curriculum documents this leap appears to occur without conscious decision. The algorithmic operations with fractions are usually taught as operations that deal only with whole number components of fractions. For example, the fraction sum $\frac{2}{3}+\frac{1}{8}$ is solved by operating with 1, 2, 3 and 8. Simplifying the difficulties associated with operating with fractions to basic manipulations of whole numbers is the strength of the fraction algorithms. Yet partially remembered operations with whole numbers associated with fraction notation is a substantial threat to developing a quantitative sense of fractions. The partitioned fractions may be used to justify the equivalence of quantity fractions, but the development of quantity fractions as abstract mathematical objects (often referred to as rational numbers) is the intended goal. This transition from partitioned fractions to quantity fractions is a time of high instructional risk and many students do not make this transition successfully[19].

For students to learn to operate with fractions as mathematical objects they need opportunities to recognise the role played by the unit or equal whole in our number system. When is it possible to add one-quarter and one-half? The universal whole plays the same role as the standard unit in measurement in allowing us to make comparisons. The link between the teaching of measurement and fractions (Davydov & Tsvetkovich, 1991) can be strengthened by introducing fractions in measurement contexts, as is the practice in the Japanese curriculum (Ministry of Education (Monbusho), 1999).

To gain an understanding of fractions as mathematical objects through the traditional curriculum pathway using a parts-of-a-whole model, there are some fundamental

[19] For example, as described in Section 5.11, a large proportion of high school students in NSW appeared to have little understanding of quantity fractions when estimating the sum of two fractions.

understandings that need to be developed. Within a part-whole relationship model of fractions, the role of the whole is frequently overlooked.

### *8.3.1 The lack of attention to the equal whole*

If students are to develop an understanding of fractions as mathematical objects, the idea that quantity fractions reference a constant whole needs to be developed. This idea is typically absent from current curriculum documents, or at best, is only implied. Students need opportunities to recognise that partitioned fractions are always dependent upon the whole of which they are part. It is essential to establish the idea of the equal whole as part of the concept image of students before any meaning can be given to operations on fractions. The symbolism $\frac{1}{2}+\frac{1}{4}$ has no meaning without reference to a universal whole. This universal whole is similar to the role of the standard unit of measurement within a metric space.

The distinction between partitioned fractions and quantity fractions highlights the role played by the universal whole in developing fractions as mathematical objects. As students can often achieve procedurally correct answers to fraction questions without an appreciation of the equal whole (see Figure 8.4), both teachers and students will need to be made aware of its role in the development of quantity fractions.

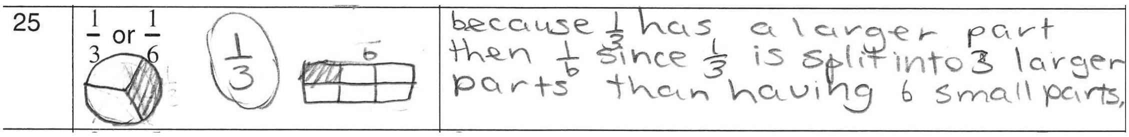



**Figure 8.4** An example of lack of awareness of the need for equal wholes

If the regional model has led to abstracting only the compensatory principle of measurement (the smaller the unit of measure, the more of the units needed), the particular shape used may not matter. The compensatory principle applies irrespective of the shape used. The compensatory principle without access to the universal whole is not sufficient to enable students to appreciate fractions as mathematical objects. In Figure 8.4, the fractions appear more as partitioned fractions rather than quantity fractions. The referenced unit whole needs to be introduced from the earliest work with fractions. The purposeful selection of the unit of area that enables a comparison of partitioned fractions is a non-trivial issue. In the introduction to fractions, arguably to provide perceptual variability for the concept, all regional models are afforded equal merit. Colouring in three-eighths of a regular octagon is afforded equal merit to shading in three-eighths of a circle. There is no apparent need for students to attend to the representation of the whole. The choice of the whole used to make comparisons of fractions needs to be brought to the fore through the use of questions such as, "Can one-quarter ever be bigger than one-half?" Comparison of the wholes is more manageable as lengths using a common baseline than comparison of areas of different shapes.

The equal whole underpins the successful transition from partitioned fractions to quantity fractions. Without the notion of the equal whole, equivalent fractions could not exist and operating with fractions as mathematical objects would not be possible. The conceptual basis of equivalent fractions needs to be used to develop a relational understanding of equivalent fractions.

## 8.4 A linear path to fractions

Students' explanations of their reasoning about the size of fractional parts can be used as a source of information regarding their fraction concept images. Any representation of a fraction can carry unintended features that students can add to their concept images. All regional models can carry the unintended feature that a fraction is represented only by a number of parts out of a total number of parts. Areas of parts of shapes are often difficult to determine and compare, with the result that number of parts is a more readily observed feature. Moreover, the expected measurable aspect of the models used to introduce fractions (area, volume or angle) are usually not introduced into teaching sequences until after students have been expected to use them to give meaning to fraction models.

The notation used for fractions also carries the unintended feature that fractions are two separate whole numbers. Discrete models not only carry this unintended feature but also suggest that the whole grows with unit fractions of decreasing magnitude (Figure 8.5).

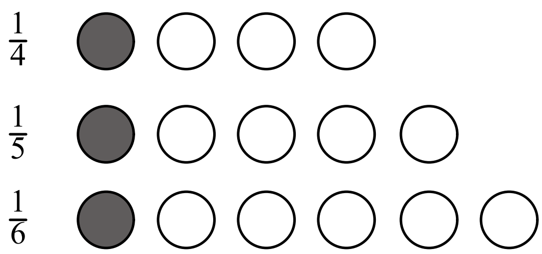



**Figure 8.5** Representing unit fractions with discrete items

Equivalent fractions in discrete form are difficult to represent in a way that makes their equivalence easily recognised. The two representations of one-quarter in Figure 8.6 are not readily seen as equal as the eye is encouraged to compare lengths.

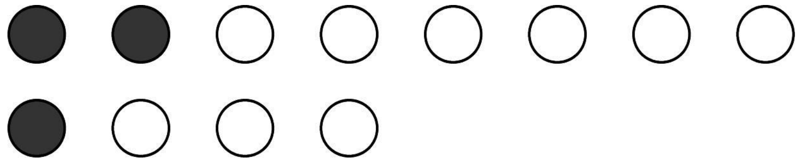


**Figure 8.6** Representing equivalent fractions with discrete items

The unwanted secondary features of instructional models used with fractions need to be addressed through the careful use of non-examples and by using a model, such as linear measurement division, within the conceptual zone of the learner. The comparison of lengths and the related linear subdivision of a unit form a better introduction to fractions than regional or discrete models. Using strips of paper (essentially equal width rectangles), instruction can focus on the linear feature when forming halves, quarters or other fractional parts. This can be thought of as a *tape measure* model of fractions as the width of the tape measure is irrelevant to its primary function. This approach has support in current research (Morris, 2000) and with the development of computer tools to support the learning of fractions (Kong & Kwok, 2003; Olive, 2000).

Although some students have been able to develop a quantitative sense of fractions through current curriculum sequences, many have not. A more cohesive conceptual approach to the teaching of fractions using models of fractions where the underpinning attribute is part of students' conceptual repertoire should reduce the risk of extraneous or erroneous features becoming entrenched in students' concept images and improve their quantitative sense of fractions.

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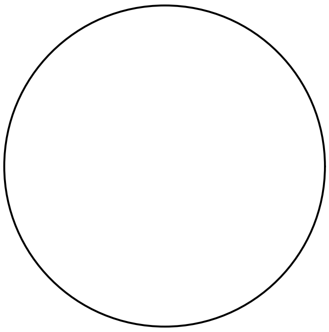
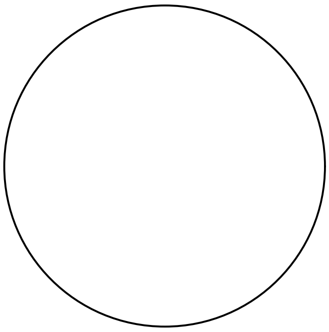
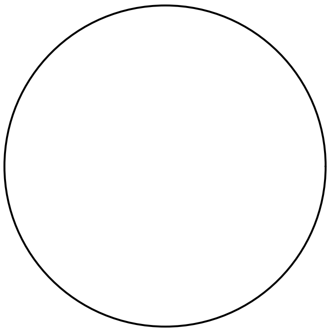
## APPENDIX A. The recording sheets

**Fraction Answer Sheet** **School:**

| | |
|---|---|
| Name: | Class: |
| Date of birth: | M/F |
| Paper fractions: Write your answers in the spaces provided. | |
| 1. | |
| 2. | |
| 3. | |
| Pattern blocks | |
| 4. | |
| 5. | |
| 6. | |
| Coloured counters | |
| 7. | |
| 8. | |
| 9. | |
| 10. | |
| 11. | Reason: |

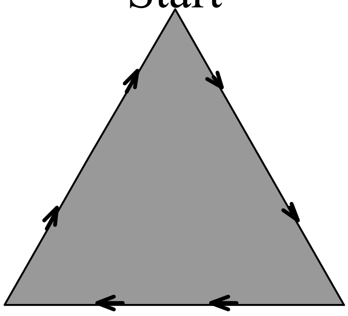
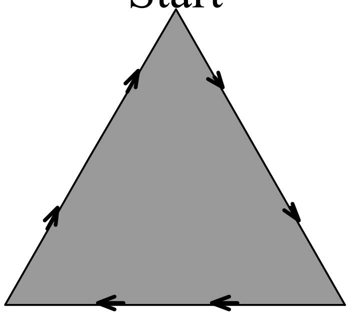
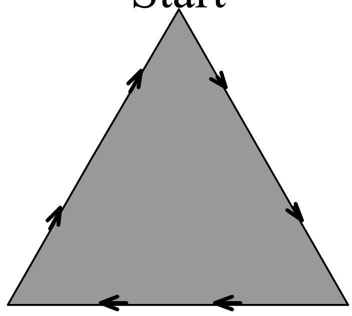
| 12. Shade one-half of this circle. | 13. Shade one-third of this circle. | 14. Shade one-sixth of this circle. |
|---|---|---|
|  |  |  |

15. A recipe uses three-quarters of a cup of milk. If I make double the recipe, how many cups of milk do I use? Draw your answer.

16. A recipe uses two-thirds of a cup of milk. If I make three lots of the recipe, how many cups of milk do I use? Draw your answer.

17. I have 6 cups of milk. A recipe needs one-third of a cup of milk. How many times can I make the recipe before I run out of milk? Draw your answer.

18. I have 6 cups of milk. A recipe needs three-quarters of a cup of milk. How many times can I make the recipe before I run out of milk? Draw your answer.

| 19. An ant crawls around the outside of this triangle following the direction of the arrows from the Start. Mark a X on the triangle where it will be when it is half of the way around. | 20. An ant crawls around the outside of this triangle following the direction of the arrows from the Start. Mark a X on the triangle where it will be when it is one-third of the way around. | 21. An ant crawls around the outside of this triangle following the direction of the arrows from the Start. Mark a X on the triangle where it will be when it is one-quarter of the way around. |
|---|---|---|
| Start  | Start  | Start  |

| | Which is the bigger number? | How do you know? |
|---|---|---|
| 22 | One-half or one-quarter | |
| 23 | One-third or one-half | |
| 24 | $\frac{1}{3}$ or $\frac{1}{4}$ | |
| 25 | $\frac{1}{3}$ or $\frac{1}{6}$ | |
| 26 | $\frac{2}{3}$ or $\frac{5}{6}$ | |
| 27 | $\frac{9}{10}$ or $\frac{12}{13}$ | |

*Reversible fraction strips*

28. If this drawing shows three-quarters ($\frac{3}{4}$) of a piece of paper, mark on it where one-half ($\frac{1}{2}$) of the **whole piece of paper** would be.

29. If this drawing shows three-quarters ($\frac{3}{4}$) of a piece of paper, draw where the whole piece of paper would end.

30. What would be the best estimate of the answer to $\frac{4}{5} + \frac{11}{12}$?

A 17

B 15

C 2

D 1

**Test 2**

**Answer sheet** **School:**

<table>
<tr><td colspan="2">Name:</td><td>Class:</td></tr>
<tr><td colspan="2">Date of birth:</td><td>M/F</td></tr>
<tr><td colspan="3">Materials: Three strips of paper, one blue, one yellow and one green.</td></tr>
<tr><td>Question</td><td colspan="2">How do you know?</td></tr>
<tr><td>P1. Exactly how many of the short green strip of paper equals the length of the blue strip of paper?</td><td colspan="2"></td></tr>
<tr><td>P2. Imagine a red strip of paper half as long as the green strip of paper. How many red strips of paper equals the length of the blue strip of paper?</td><td colspan="2"></td></tr>
<tr><td>P3. Exactly how many of the yellow strip of paper equals the length of the blue strip of paper?</td><td colspan="2"></td></tr>
<tr><td>P4. What fraction of the length of the blue strip of paper is the yellow strip of paper?</td><td colspan="2"></td></tr>
<tr><td>P5. What fraction of the length of the yellow strip of paper is the blue strip of paper?</td><td colspan="2"></td></tr>
<tr><td colspan="3">Material: half of a paper circle.</td></tr>
<tr><td colspan="3">P6. Fold the half of a paper circle to show one sixth of the full circle. Shade your answer on the paper.</td></tr>
<tr><td colspan="3">Material: a square piece of paper</td></tr>
<tr><td colspan="3">P7. Fold the square piece of paper to show one-ninth of the square. Shade your answer on the paper so that it is clear where the ninth of the square is.</td></tr>
</table>

## APPENDIX B: Item difficulty estimates

*Estimates of item difficulty and fit statistics (QUEST)*

| Item number | Estimate of item difficulty logits (Error) | Weighted Fit (Infit) MNSQ | Infit t-values | Unweighted Fit (outfit) MNSQ | Outfit t-values |
|---|---|---|---|---|---|
| 1 | -3.04 (0.10) | 0.90 | -1.5 | 1.11 | 0.5 |
| 2 | -1.63 (0.07) | 1.05 | 1.5 | 1.28 | 1.9 |
| 3 | -1.34 (0.07) | 1.04 | 1.1 | 1.22 | 1.7 |
| 4 | -2.78 (0.09) | 0.99 | -0.1 | 2.76 | 4.8 |
| 5 | 1.06 (0.06) | 0.78 | -7.3 | 0.68 | -4.7 |
| 6 | 0.15 (0.06) | 0.99 | -0.5 | 0.91 | -1.3 |
| 7 | -2.12 (0.08) | 0.93 | -1.6 | 1.09 | 0.6 |
| 8 | 0.14 (0.06) | 0.88 | -4.3 | 0.78 | -3.4 |
| 9 | -1.51 (0.07) | 0.85 | -4.4 | 1.00 | 0.1 |
| 10 | -1.17 (0.07) | 0.90 | -3.1 | 1.15 | 1.3 |
| 11 | 1.00 (0.06) | 0.89 | -3.8 | 0.82 | -2.5 |
| 12 | -6.38 (0.38) | 0.99 | 0.1 | 0.36 | 0.0 |
| 13 | 0.52 (0.06) | 1.01 | 0.2 | 1.11 | 1.5 |
| 14 | 0.49 (0.06) | 0.97 | -1.0 | 1.16 | 2.2 |
| 15 | -0.24 (0.06) | 1.07 | 2.5 | 1.16 | 2.0 |
| 16 | 1.03 (0.06) | 0.98 | -0.7 | 1.09 | 1.2 |
| 17 | 0.59 (0.06) | 0.85 | -5.4 | 0.80 | -3.1 |
| 18 | 1.35 (0.06) | 0.98 | -0.6 | 1.06 | 0.7 |
| 19 | -2.89 (0.10) | 1.21 | 3.1 | 2.04 | 3.0 |
| 20 | 0.17 (0.06) | 0.93 | -2.4 | 0.91 | -1.3 |
| 21 | 1.74 (0.07) | 1.19 | 5.1 | 2.09 | 8.0 |
| 22 | -0.15 (0.06) | 1.02 | 0.8 | 0.94 | -0.8 |
| 23 | 1.08 (0.06) | 0.87 | -4.3 | 0.81 | -2.6 |
| 24 | 0.95 (0.06) | 0.84 | -5.4 | 0.71 | -4.2 |
| 25 | 0.95 (0.06) | 0.81 | -6.7 | 0.66 | -5.1 |
| 26 | 1.92 (0.07) | 0.76 | -7.0 | 0.56 | -4.6 |
| 27 | 2.98 (0.08) | 0.87 | -2.7 | 0.52 | -2.9 |
| 28 | 0.65 (0.06) | 1.10 | 3.2 | 1.34 | 4.4 |
| 29 | 0.89 (0.06) | 1.10 | 3.0 | 1.11 | 1.5 |
| 30 | 2.27 (0.07) | 1.05 | 1.2 | 1.28 | 1.9 |
| 31 | -2.24 (0.08) | 1.14 | 2.9 | 2.11 | 4.4 |
| 32 | -1.67 (0.07) | 1.18 | 4.5 | 1.54 | 3.3 |
| 33 | 0.95 (0.06) | 1.11 | 3.3 | 1.09 | 1.2 |
| 34 | 0.10 (0.06) | 1.35 | 10.8 | 1.53 | 6.5 |
| 35 | 2.68 (0.08) | 1.06 | 1.2 | 1.36 | 2.0 |
| 36 | 1.88 (0.07) | 1.00 | 0.0 | 0.86 | -1.3 |
| 37 | 1.63 (0.07) | 1.06 | 1.7 | 1.10 | 1.0 |