Parity structure on associahedra and other polytopes

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Contents

Summary **v**

Declaration **vi**

Introduction **vii**

1 Parity structures **1**
- 1.1 Higher categories . . . 2
  - 1.1.1 Many-sorted definition of categories . . . 3
  - 1.1.2 Single-sorted definition of categories . . . 5
- 1.2 Parity complexes . . . 6
- 1.3 Pasting schemes . . . 14
- 1.4 Pasting schemes via parity structures . . . 16

2 Associahedra **19**
- 2.1 Partial bracketings . . . 19
  - 2.1.1 The poset of bracketings . . . 20
- 2.2 Higher left bracketing functions . . . 23
  - 2.2.1 The poset of HLBFs . . . 25
  - 2.2.2 Isomorphism with bracketings . . . 27
  - 2.2.3 HLBFs $x \leq y$ with $|S_y \backslash S_x| = 1$ . . . 30
  - 2.2.4 HLBFs $x \leq y$ with $|S_y \backslash S_x| = 2$ . . . 36
  - 2.2.5 Properties of the poset of HLBFs . . . 39

3 Polytopes **47**
- 3.1 Label structures . . . 47
  - 3.1.1 Axioms 1*, 2 and 3(a) . . . 51
  - 3.1.2 Characterisation of π and μ . . . 53
  - 3.1.3 Linearity axiom . . . 58
  - 3.1.4 Cellularity axiom . . . 67
- 3.2 Products of label structures . . . 72
- 3.3 Examples . . . 75
  - 3.3.1 Hypergraph polytopes . . . 75
  - 3.3.2 Hypercubes . . . 77
  - 3.3.3 Associahedra . . . 81
  - 3.3.4 Permutohedra . . . 84

Conclusion **91**

Appendix A 4-dimensional associahedron **93**

Appendix B LGC-complexes **103**

Bibliography **109**

Summary

A parity structure is a name given to a formalism of pasting diagrams, among these are: parity complexes in the sense of Street, pasting schemes in the sense of Johnson, and directed complexes in the sense of Steiner. The idea behind these formalisms is to take a set of faces and attach an orientation to each face so that we obtain a presentation of a (strict) free ω-category.

The above formalisms include examples such as the simplexes and hypercubes, but are not sufficiently general to allow for other reasonable examples. In this thesis, our main goal is to construct a parity structure on other polytopes. An example of interest is the polytope family known as the associahedra. Notable work on the associahedra includes that of Tamari and Stasheff.

One approach to associahedra is via left bracketing functions (lbf) due to Huang-Tamari. We will introduce a generalisation of an lbf which we call a higher left bracketing function (hlbf), and show that they correspond to the faces of the associahedron. We are able to construct a parity structure on the hlbfs. This parity structure does not satisfy the parity complex axioms due to Street. However, it does satisfy a modified set of axioms given by Campbell. It follows from the results of Campbell that we have a loop free pasting scheme in the sense of Johnson.

The construction of a parity structure on the associahedron is generalised to a more basic structure known as an abstract pre-polytope. To achieve this, we introduce the idea of a label structure on an abstract pre-polytope. From a label structure, we obtain a parity structure which is then proven to satisfy the axioms due to Campbell. Finally, we use this construction on other polytopes such as the hypercubes and permutohedra.

Declaration

This thesis is an account of research undertaken between January 2013 and July 2017 at the Department of Mathematics, Macquarie University, North Ryde, Australia.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any other university.

Christopher Nguyen
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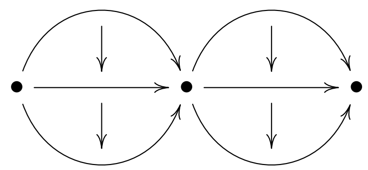
Introduction

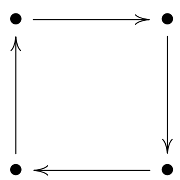
A category [15] consists of objects and morphisms between these objects, together with a way to compose these morphisms. A morphism f between objects x and y is denoted by an arrow $f : x \longrightarrow y$. We say that x is the source (domain) of f, and y is the target (codomain) of f. Pairs of morphisms $f : x \longrightarrow y$ and $g : y \longrightarrow z$ are composable whenever the target of f is the source of g. The composite is denoted by the morphism $gf : x \longrightarrow z$. Furthermore, we require that composition satisfy the associative and identity laws.

In Chapter 1, we introduce the idea of a higher category as a generalisation of a category to include higher dimensional arrows called n-cells. An n-cell is denoted by an arrow $\alpha : f \longrightarrow g$ between $(n-1)$-cells f and g. We say that f is the source of α, and g is the target of α. An ω-category consists of all n-cells for each $n \in \omega$ (the first infinite ordinal). The notion of composition is generalised to allow n-cells to be composed. Depending on the axioms we require on these compositions, we either obtain a weak ω-category [14] or a strict ω-category [23]. In this thesis, we will consider strict ω-categories in the sense of Street [23].

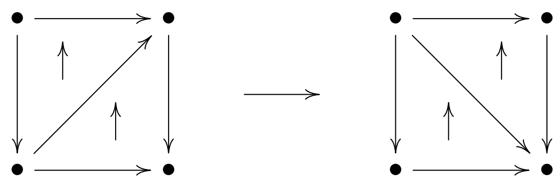
A pasting diagram is a collection of n-cells which is presented as arrows that have not yet been composed, instead they are pasted together. It is required that the source and target of these arrows fit together in such a way that for every possible order of composition, we obtain the same n-cell. For example, the interchange law for 2-categories can be expressed by the following pasting diagram.



For example, we exclude loops such as in the following.



The following is a pasting diagram of a 3-simplex.



A parity structure in the sense of Street [24] is a formal definition of a pasting diagram. The central idea behind this formalism is that a pasting diagram is built up from a collection of n-dimensional geometric shapes (called the faces), each of which represents an n-cell. We may define a parity structure as a graded set $C = \sum_{n \geq 0} C_n$ where C_n consists of the n-dimensional faces. For each face x, we partition the boundary of x (subfaces of one dimension lower) into two disjoint non-empty subsets x^- and x^+. We call x^- the negative faces of x, and x^+ the positive faces of x. The positive faces correspond to the target of the corresponding cell in the pasting diagram, and the negative faces correspond to the source.

In the literature, there are at least three different formalisms of the notion of a well behaved pasting diagram. These include: parity complexes due to Street [24], pasting schemes due to Johnson [12] and directed complexes due to Steiner [21]. In each formalism, the underlying structure is equivalent to a parity structure as described above. However, each requires slightly different axioms to hold. Each formalism gives a presentation of a free ω-category.

A computad, first defined by Street [22], is a generalisation of a quiver. A free category is generated by a quiver. Analogously, a free n-category is generated by a computad. Parity complexes, pasting schemes and directed complexes are among the computads that generate free ω-categories.

The common examples of parity complexes, pasting schemes and directed complexes include the simplexes and hypercubes. We observe that the existing formalisms of pasting diagrams are not sufficiently general to cover all examples of well behaved pasting diagrams. The following pasting diagram generates a free 3-category, however it is not a parity complex as demonstrated in Example 1.22.

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\alpha\Uparrow", pos=0.2, shift right] \arrow[dr, phantom, "\beta\Uparrow", pos=0.75, shift left] & d \arrow[d, "g"] \\
b \arrow[r, "k"'] \arrow[ur, "l"] & c
\end{tikzcd}
\xrightarrow{\;s\;}
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\gamma\Uparrow"] & d \arrow[d, "g"] \\
b \arrow[r, "k"'] & c
\end{tikzcd}
$$

A (convex) polytope is a generalisation of a polygon to arbitrary dimensions. There are numerous definitions of a polytope, some of which can be found in [28]. For the purposes of investigating parity structure, the combinatorial properties of these convex polytopes are more important than the geometric properties. The set of all faces of a polytope together with the subface relation forms a partially ordered set, we often call this the face poset. An abstract polytope in the sense of McMullen et al. [16] is a generalisation of a polytope which involves an abstraction of the face poset. An abstract polytope is a bounded, graded poset which satisfies the diamond property, and the connectedness condition. A formal statement of the diamond property is given in Definition 1.30(c), and connectedness is given in Definition 1.32(d).

In Chapter 2, we investigate a polytope known as the associahedron. Associahedra have a very rich history dating back to the early work by Tamari [26] and Stasheff [20]. In [26], Tamari defines the associahedron as a polytope with binary bracketings of n-fold products for the vertices, and partial bracketings of n-fold products for the higher dimensional faces. Throughout this thesis, all trees are planar unless stated otherwise. In [20], Stasheff defines the associahedron as a polytope with binary trees with n leaves for the vertices, and non-binary trees for the higher dimensional faces.

Another interesting polytope is the permutohedron. Ziegler claims in [28] that the permutohedron was first studied by Schoute [19]. Guilbaud and Rosenstiehl are credited for coining the name permutohedra [10]. A permutohedron is defined as a polytope with vertices corresponding to permutations of n letters.

We will discuss, in Chapter 1, an implicit partial ordering on the underlying graded set of a parity structure. This partial order is generated by the relation $x \leq y$ iff $x \in y^+ \cap y^-$. Such a partial ordering is considered by Johnson in [12] and Steiner in [21], but not by

Street in [24]. A consequence of this implicit partial ordering is that a parity structure can actually be defined on a graded poset (instead of a graded set). This will be used as our motivation to investigate parity structures defined on an abstract polytope. We then describe the idea of a Hasse diagram, which is a useful illustrative tool when studying parity structures on abstract polytopes. In such a setting, we then give a characterisation of axioms 1* and 2 (which are closely related to Street's parity complex axioms).

In [4], Campbell gives a comparison between parity complexes and pasting schemes. Campbell describes a sufficient set of conditions on a parity structure to obtain a loop free pasting scheme in the sense of Johnson. We will call such a parity structure an LGC-complex. For the purposes of our thesis we will describe and then use an equivalent list of conditions. We also incorporate our poset perspective of parity structures into Campbell's results. Our main thesis aim is to construct an LGC-complex on polytopes such as the associahedron and permutohedron.

To give a parity structure on the associahedron, we require a suitable presentation of the associahedron. The vertices of an associahedron can be represented by the left bracketing functions (lbf) due to Huang-Tamari [11]. We generalise the notion of an lbf to that of a higher left bracketing function (hlbf); these correspond to higher dimensional faces of an associahedron. To motivate the definition of an hlbf, we will consider a construction called a broom which is due to Verity [27]. A broom is constructed from a tree, and so we may exploit its geometric properties to describe a specific construction on a broom. This construction gives an lbf when applied to a broom arising from a binary tree. The same construction when applied to a broom arising from a non-binary tree will give an hlbf. We give a formal definition of an hlbf in Chapter 2.

In Chapter 1, we investigate the relationship between parity complexes in the sense of Street, and pasting schemes in the sense of Johnson. We first include our point of view of parity structures in terms of abstract polytopes. Then, we discuss Campbell's point of view that an LGC-complex gives rise to a loop free pasting scheme. This leads to Chapter 2, where we give a description of the poset of hlbfs and investigate its properties. We prove numerous properties of this poset which are then used to motivate the definition of a label structure in Chapter 3. Each hlbf x can be associated with a pair (ℓ_x, S_x) as defined in Proposition 2.15. A label structure involves labelling each face of an abstract polytope with a pair (φ, Φ) which is compatible with the partial ordering as prescribed by a set of axioms. From a label structure, we can obtain a parity structure as described in Proposition 3.3. The axioms of a label structure are then used to prove that we have an LGC-complex.

Towards the end of our thesis we will construct label structures for our main polytope examples: hypercubes, associahedra and permutohedra. These polytope families are part of a larger class of polytopes known as hypergraph polytopes due to Došen and Petrić [7]. Hypergraph polytopes are also known as nestohedra in the sense of Postnikov [18]. In terms of defining a label structure, we will show that it is convenient to work in a hypergraph polytope setting.

The contents of Chapter 1 and Chapter 2 are essentially independent of each other. Chapter 3 relies on many ideas from Chapter 1, but depends on Chapter 2 only for certain specific aspects about faces of the associahedron.

List of Symbols

Standard

$\wp$	Powerset
$[n]$	The linearly ordered set $\{0, 1, \ldots, n\}$ where n is a natural number
$\wedge$	Smash product
$\oplus$	Ordinal sum

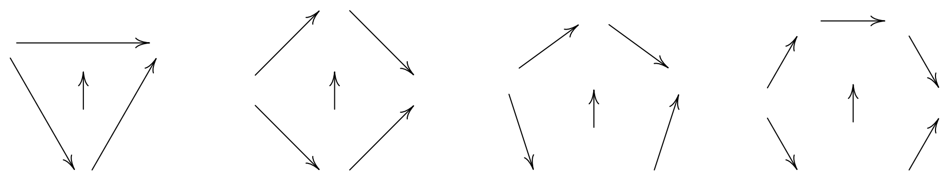
Specific to thesis

$S^{\mp}, S^{\pm}$	$S^{\mp} = S^{-} \backslash S^{+}$ and $S^{\pm} = S^{+} \backslash S^{-}$, pg. 7
$\prec$	Preorder on a parity structure defined by $u \prec v$ iff $u^{+} \cap v^{-} \neq \varnothing$, pg. 7
$\lhd$, $\blacktriangleleft$	Partial orders defined on a parity structure, pg. 7, 16
$\leq, <_1$	$<_1$ is the covering relation with respect to the partial ordering $\leq$, pg. 10
$\perp_P$	The least element (if it exists) of a poset P, pg. 11
$\top_P$	The greatest element (if it exists) of a poset P, pg. 11
$\widetilde{P}$	The poset P after removing its least element, pg. 11
$\mathsf{B}, \mathsf{E}, \mathsf{R}$	Misc. subsets of a pasting scheme, pg. 14
B, E, R	Misc. subsets of a parity structure, pg. 16
M/i	The set $\{k \in M \mid k \leq i\}$, pg. 25
$\mathcal{H}_M$, $\mathcal{B}_M$	The poset of all hlbfs, bracketings (respectively) on M, pg. 28
$\mathcal{L}$	Definition 3.12, pg. 53
$x \rightsquigarrow$	A special path described in Proposition 3.22, pg. 60
$x \rightsquigarrow y$	A special path described in Definition 3.25, pg. 61
$y \leftsquigarrow x$	Equivalent to $x \rightsquigarrow y$
V	$VX = \{\{x\} \mid x \in X\}$ for any set X, pg. 77

Chapter 1

Parity structures

In this chapter, we will discuss higher categories as a generalisation of categories to include higher dimensional arrows known as n-morphisms or n-cells. We are interested in special types of higher categories known as strict ω-categories. In the context of higher categories, a pasting diagram is a collection of n-morphisms with well behaved properties. These diagrams capture the idea of composition whereby morphisms are 'pasted' along their common boundaries. Intuitively, n-morphisms can be understood as n-dimensional geometric shapes known as polytopes. The pasting diagrams below represent the 2-dimensional simplex (triangle), hypercube (square), associahedron (pentagon), and permutohedron (hexagon) respectively.



In the literature, there are at least three different notions of parity structures - parity complexes due to Street [24], pasting schemes due to Johnson [12], directed complexes due to Steiner [21]. In [24], Street observes that for any parity complex, there is an associated chain complex built up from free abelian groups. This point of view is central to the theory of directed complexes due to Steiner [21]. A unifying theme among these formalisms is that they capture the intuitive idea that a pasting diagram consists of n-dimensional faces of a polytope that are pasted together in a certain way. For each n-dimensional face there is a boundary consisting of $(n-1)$-dimensional faces. For a given face we partition its boundary into two sets which are called the positive and negative faces; this action gives parity to such a face. It is required that a set of conditions must hold. These conditions vary among the existing formalisms. However each allows us to construct a free ω-category.

The main examples of each of the above formalisms include the simplexes and hypercubes. Our main thesis aim is to extend this list to include the polytope family known as the associahedra. We will construct a parity on the associahedron and check if it is an example of a parity complex or pasting scheme. It turns out that we can show that the pasting scheme axioms hold, however one of the parity complex axioms does not. There are three parity complex axioms, of which the third has two halves, called 3(a) and 3(b). We are able to prove a stronger version of axiom 1, which we call axiom 1*, as well as axioms 2 and 3(a), but it turns out that axiom 3(b) does not hold for the associahedra.

This chapter will serve as an introduction to the formalisms of parity complexes and pasting schemes which we will be used within this thesis. Firstly, we will provide a formalism for the process of giving parity as discussed earlier; these are called parity structures. Then we describe a parity complex [24] as a parity structure which satisfies the parity complex axioms. We will then provide a discussion on the shortcomings of axiom 3(b) which is one of the parity complex axioms.

Following our discussion on parity complexes, we introduce our point of view of parity structures in terms of polytopes. We show that the underlying graded set of parity structure due to Street [24] has an implicit partial ordering. In the examples of parity complexes, the simplexes and hypercubes, this partial ordering is in fact the face inclusion ordering. Following this, we introduce the notion of an abstract polytope due to McMullen et al. [16]. We then give our characterisation of axioms 1* and 2 for parity structures on abstract polytopes.

Next, we describe pasting schemes due to Johnson [12]. For our purposes of constructing a parity on the associahedron, we found it convenient to consider pasting schemes as defined via parity structures. Here we will be following the approach of Campbell [4] who gives a sufficient set of conditions on a parity structure in order to obtain a loop free pasting scheme; these are called the linearity, globularity and cellularity axioms denoted by **(L)**, **(G)** and **(C)** respectively. We shall call a parity structure satisfying these axioms an LGC-complex.

Lastly, we will present an equivalent set of axioms for an LGC-complex. It follows from Theorem B.3.3. of [4] that an LGC-complex satisfies axioms 1*, 2 and 3(a). We will show that the globularity axiom **(G)** follows from the cellularity axiom **(C)** together with parity complex axioms 1*, 2 and 3(a). Thus we have an equivalent set of axioms for an LGC-complex; these are axioms 1*, 2, **(L)** and **(C)**.

1.1 Higher categories

A category is a collection of objects and morphisms together with a way to compose these morphisms. A natural example is the category of sets and functions where composition is the usual composition of functions. It is customary to represent a morphism by an arrow. For example, a morphism f is a function $f : x \longrightarrow y$ between sets x and y. We represent this morphism by the following diagram

$$x \xrightarrow{f} y$$

which consists of an arrow labelled by f. In this case, we call x the domain (or source) of f and y the codomain (or target) of f.

A higher category is a generalisation of a category which includes additional structure in the form of higher dimensional arrows. The concept of a morphism, which is an arrow between objects, is generalised to an n-cell. A 0-cell is an object and a 1-cell is a morphism. For $n > 1$, an n-cell is an arrow between $(n-1)$-cells. For example, a 2-cell is an arrow between morphisms as demonstrated by the following diagram

$$\begin{tikzcd} x \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & y \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

where f and g are morphisms (or 1-cells), and α is a 2-cell represented by an arrow from f to g. A 2-category consists of 0-cells, 1-cells and 2-cells together with two types of compositions (horizontal and vertical composition).

In this thesis, we will only consider strict higher categories such as strict n-categories and strict ω-categories. Weak ω-categories have been studied in the literature via several different models, including those in the survey [14]. Strictness refers to the strict equalities appearing in the axioms. We will describe two equivalent ways of defining an ω-category; the many-sorted definition and the single-sorted definition.

1.1.1 Many-sorted definition of categories

In the many-sorted definition of an n-category, we explicitly define sets $C_0, C_1, \ldots, C_n$ consisting of the i-cells for each $i \leq n$. Let ω be the first infinite ordinal. An ω-category consists of sets C_n for each $n \in \omega$. We will first give the definition of a category.

Definition 1.1 A *category* C consists of a set of objects C_0 and a set of morphisms C_1 together with the following

* functions $s, t : C_1 \longrightarrow C_0$ called the source and target maps respectively.
* a function $id : C_0 \longrightarrow C_1$ which sends x to an identity morphism id_x for each $x \in C_0$.
* a function $\circ : \{(f, g) \in C_1 \times C_1 \mid t(f) = s(g)\} \longrightarrow C_1$ called composition which sends composable pairs f, g to $g \circ f$.

which satisfy the following conditions.

1. For composable pairs (f, g), we have $s(g \circ f) = s(f)$ and $t(g \circ f) = t(g)$.
2. (Identity) For each $x, y \in C_0$ and $f \in C_1$, we have $f \circ id_x = f = id_y \circ f$ where $x = s(f)$ and $y = t(f)$.
3. (Associativity) For composable pairs (f, g) and (h, g), we have $(f \circ g) \circ h = f \circ (g \circ h)$.

Remark For a morphism $f \in C_1$ we say that the source (or domain) of f is $x = s(f)$ and the target (or codomain) of f is $y = t(f)$ and denote this by the diagram $x \xrightarrow{f} y$.

We denote the set of composable pairs of morphisms by $C_1 \times_{C_0} C_1 = \{(f, g) \in C_1 \times C_1 \mid t(f) = s(g)\}$. In following diagram, we summarise all the data of a category.

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\circ"] & C_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & C_0 \arrow[l, "id"'] \end{tikzcd}$$

We will often use the following diagram to represent the data of a category; the arrows indicate the source and target maps.

$$\begin{tikzcd} C_1 \arrow[r, shift left] \arrow[r, shift right] & C_0 \end{tikzcd}$$

Example 1.2 (a) There are many categories consisting of finitely many objects, among these are the following examples. The empty category which consists of no objects or morphisms. **1**; there is one object together with an identity morphism on this object. **2**; there are two objects a and b together with an identity on each object, and a morphism $f : a \longrightarrow b$.

(b) A group can be viewed as a one object category. The morphisms of this category correspond to elements of the group. For such a category, the composition corresponds to multiplication of the group.

(c) A partially ordered set $(P, \leq)$ can be viewed as a category whose objects are the elements of P and there is a morphism $x \longrightarrow y$ whenever $x \leq y$. Note that for any $x \in P$ there is an identity due to reflexivity; $x \leq x$. The associativity law holds due to transitivity of $\leq$; $x \leq y \leq z$ implies $x \leq z$.

(d) **Set** objects are sets and morphisms are functions between sets,

Grp the objects are groups and the morphisms are homomorphisms of groups,

Top the objects are topological spaces and the morphisms are continuous maps,

Cat the objects are categories and the morphisms are functors,

and many other which can be found in pg. 10 of [15].

Definition 1.3 A *2-category* C consists of the sets of objects (0-cells) C_0, morphisms (1-cells) C_1 and 2-cells C_2 together with categories

$$C_1 \underset{t_0}{\overset{s_0}{\rightrightarrows}} C_0 \qquad C_2 \underset{t_0 t_1}{\overset{s_0 s_1}{\rightrightarrows}} C_0 \qquad C_2 \underset{t_1}{\overset{s_1}{\rightrightarrows}} C_1$$

These categories (from left to right) have composition denoted by $\circ$, $\circ_0$ (horizontal composition of 2-cells along 0-cells), and $\circ_1$ (vertical composition of 2-cells along 1-cells). Furthermore, the following conditions hold.

1. (Globular) $s_0 s_1 = s_0 t_1$ and $t_0 t_1 = t_0 s_1$.
2. (Interchange law) $(\alpha \circ_0 \beta) \circ_1 (\gamma \circ_0 \delta) = (\alpha \circ_1 \gamma) \circ_0 (\beta \circ_1 \delta)$ for all composable 2-cells $\alpha, \beta, \gamma, \delta$.

We will often express the information of a 2-category by a diagram of the following form

$$C_2 \rightrightarrows C_1 \rightrightarrows C_0$$

Example 1.4 (a) It was mentioned earlier that **Cat** is a category. In fact, it becomes a 2-category when we define the 2-cells to be natural transformations.

(b) Consider the following pasting diagram

$$\begin{tikzcd}
a \arrow[r, bend left=70, "f"] \arrow[r, "g"] \arrow[r, bend right=70, "h"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha"] \arrow[r, phantom, shift right=4, "\Downarrow\gamma"] & b \arrow[r, bend left=70, "k"] \arrow[r, "l"] \arrow[r, bend right=70, "m"'] \arrow[r, phantom, shift left=4, "\Downarrow\beta"] \arrow[r, phantom, shift right=4, "\Downarrow\delta"] & c
\end{tikzcd}$$

which is a presentation of a 2-category. The idea of pasting in a pasting diagram corresponds to composition in the 2-category. The cells are generated by pasting arrows together. For example, the 1-cells $a \xrightarrow{f} b$ and $b \xrightarrow{k} c$ can be pasted together to obtain another 1-cell $a \xrightarrow{kf} c$; this corresponds to the horizontal composition. The 2-cells $f \xrightarrow{\alpha} g$ and $g \xrightarrow{\gamma} h$ can be pasted together to obtain another 2-cell $f \xrightarrow{\gamma\alpha} h$; this corresponds to the vertical composition. We also include identities and insist the required axioms hold.

We will now give a definition of an ω-category which is a generalisation of a 2-category to include n-cells for all $n \in \omega$. We will first introduce the idea of a globular set which is a generalisation of condition 1 in the definition of a 2-category.

Definition 1.5 A *globular set* C is a collection of sets $\{C_n\}_{n\in\omega}$ together with source and target maps as shown in the following diagram

$$\cdots \quad C_3 \underset{t_2}{\overset{s_2}{\rightrightarrows}} C_2 \underset{t_1}{\overset{s_1}{\rightrightarrows}} C_1 \underset{t_0}{\overset{s_0}{\rightrightarrows}} C_0$$

which satisfy the equations $s_n s_{n+1} = s_n t_{n+1}$ and $t_n t_{n+1} = t_n s_{n+1}$ for all $n \in \omega$.

Definition 1.6 An *ω-category* C consists of a globular set

$$\cdots \quad C_3 \underset{t_2}{\overset{s_2}{\rightrightarrows}} C_2 \underset{t_1}{\overset{s_1}{\rightrightarrows}} C_1 \underset{t_0}{\overset{s_0}{\rightrightarrows}} C_0$$

together with categories

$$C_m \rightrightarrows C_n$$

for all $n < m$. Furthermore, we require that

$$C_l \rightrightarrows C_m \rightrightarrows C_n$$

are 2-categories for all possible $n < m < l$.

Remark The unlabelled arrows are understood as composites of the appropriate source or target maps. The n-cells are the elements of C_n. We may define an n-category as an ω-category for which the i-cells are identities for all $i > n$.

1.1.2 Single-sorted definition of categories

The single-sorted definition of a category is used by Street [24] and Johnson [12]. As our discussion is related to the works of Street and Johnson, for completeness will also give this definition of an ω-category. In the single-sorted approach, we consider a single set whose elements are called cells. As a consequence we do not explicitly state what the n-cells are, rather we may infer them from the identity cells.

Definition 1.7 A *category* $(C, s, t, *)$ consists of a set C together with

* functions $s, t : C \longrightarrow C$ called the source and target maps respectively.
* composition function $* : \{(x, y) \in C \times C \mid s(x) = t(y)\} \longrightarrow C$ which sends composable pairs x, y to $x * y$.

which satisfy the following conditions.

1. $ss = s = ts$, $tt = t = st$.
2. If $s(x) = t(y)$, then $s(x * y) = s(y)$ and $t(x * y) = t(x)$.
3. (Identity) $s(x) = t(y) = y$ implies $x * y = x$,
   $x = s(x) = t(y)$ implies $x * y = y$.
4. (Associativity) $s(x) = t(y)$ and $s(y) = t(z)$ implies $(x * y) * z = x * (y * z)$.

Remark An identity morphism is an element $x \in C$ with either $s(x) = x$ or $t(x) = x$ (and hence both); we identify these identity morphisms as the objects. We have an arrow $u \xrightarrow{x} v$ whenever $s(x) = u$ and $t(x) = v$.

Definition 1.8 A *2-category* $(C, s_0, t_0, *_0, s_1, t_1, *_1)$ consists of categories $(C, s_0, t_0, *_0)$ and $(C, s_1, t_1, *_1)$ which satisfy the following conditions.

1. $s_1 s_0 = s_0 = s_0 s_1 = s_0 t_1$, $t_1 t_0 = t_0 = t_0 t_1 = t_0 s_1$.
2. If $s_0(a) = t_0(a')$, then $s_1(a *_0 a') = s_1(a) *_0 s_1(a')$ and $t_1(a *_0 a') = t_1(a) *_0 t_1(a')$.
3. (Interchange law) If $s_1(a) = t_1(b)$, $s_1(a') = t_1(b')$ and $s_0(a) = t_0(a')$, then $(a *_1 b) *_0 (a' *_1 b') = (a *_0 a') *_1 (b *_0 b')$.

Remark We identify the the identities of $*_0$ and $*_1$ as the 0-cells and 1-cells respectively.

We may now give the definition of an ω-category due to Street [23].

Definition 1.9 An ω-category is a set C together with categories $(C, s_n, t_n, *_n)$ for each $n \in \omega$ such that $(C, s_n, t_n, *_n, s_m, t_m, *_m)$ is a 2-category for all $n < m$.

Remark For each $n \in \omega$, the identities of $*_n$ are identified as the n-cells; these are elements $x \in C$ with $s_n x = t_n x = x$. We may define an n-category as an ω-category C with no non-identity i-cells for $i > n$.

Note that the above definition does not exclude the existence infinite dimensional cells. This differs from the definition of an ω-category given in Definition 1.6 which requires that every cell is finite dimensional.

Example 1.10 (a) In [24, 12, 21], certain pasting diagrams have been shown to generate ω-categories; these include the simplexes and hypercubes.

(b) In Section 1 of [24], the ω-glob described generates an ω-category.

(c) In Chapter 3, we will present a general method of constructing ω-categories using our notion of a label structure. Our examples include the hypercubes, associahedra and permutohedra.

We will now give the definition of a free ω-category. Let C_n be the elements of C that are identities for $*_n$ but not for $*_k$ where $k < n$. Denote $C^{(n)} = \sum_{k=0}^{n} C_k$ as the n-skeleton of C.

Definition 1.11 An ω-functor $f : (C, (s_n, t_n, *_n)_{n\in\omega}) \longrightarrow (C', (s'_n, t'_n, *'_n)_{n\in\omega})$ is a function $f : C \longrightarrow C'$ which respects all sources, targets and compositions

Definition 1.12 An ω-category C is *freely generated* by a subset A when, for all ω-categories X, for all $n \in \omega$, for all ω-functors $f : C^{(n)} \longrightarrow X$, and, for all functions $g : A \cap C^{(n+1)} \longrightarrow X$ such that $s_n g = f s_n$, $t_n g = f t_n$, there exists a unique ω-functor $h : C^{(n+1)} \longrightarrow X$ whose restriction to $C^{(n)}$ is is f and whose restriction to $A \cap C^{(n+1)}$ is g as indicated in the following commutative diagram.

$$
\begin{tikzcd}
A \cap C^{(n+1)} \arrow[d] \arrow[dr, "g"] & \\
C^{(n+1)} \arrow[r, dashed, "h"] \arrow[d, shift right, "s_n"'] \arrow[d, shift left, "t_n"] & X \arrow[d, shift right, "s_n"'] \arrow[d, shift left, "t_n"] \\
C^{(n)} \arrow[r, "f"'] & X
\end{tikzcd}
$$

1.2 Parity complexes

In this section, we will formalise the notion of a parity as discussed in the introduction; we will call this a parity structure. This will allow us to give a definition of a parity complex as defined in [24, 25]. We will then discuss an implicitly defined partial ordering on a parity structure and how we plan to use this for the purposes of our thesis.

The definition below is due to Street [24].

Definition 1.13 A *parity structure* is a graded set

$$C = \sum_{n=0}^{\infty} C_n$$

together with, for each element $x \in C_n$ with $n > 0$, two disjoint non-empty finite subsets $x^-, x^+ \subseteq C_{n-1}$. The elements of x^- are called negative faces of x, and those of x^+ are called positive faces of x.

The following are various notations and conventions introduced by Street [24].

* Greek symbols such as ε, η will be used to denote signs $-$ or $+$.
* Each subset $S \subseteq C$ admits a grading by $S_n = S \cap C_n$. The *n-skeleton* of $S \subseteq C$ is denoted by

$$S^{(n)} = \sum_{m=0}^{n} S_m$$

We say that S is *n-dimensional* when it is equal to its n-skeleton.

* Let S^- denote the set of elements of C which occur as negative faces of some $x \in S$, and similarly for S^+. In symbols,

$$S^\varepsilon = \bigcup_{x \in S} x^\varepsilon$$

Whenever $S = \{x\}$ we may write S^ε as x^ε. Often we will work with sets of the following form $x^{\varepsilon\eta} = \{z \in y^\eta \mid y \in x^\varepsilon\}$.

* Let $S^\mp$ denote the set of negative faces of elements of S which are not positive faces of any element of S, and similarly for $S^\pm$. In symbols,

$$S^\mp = S^- \backslash S^+$$
$$S^\pm = S^+ \backslash S^-$$

The following definition is due to Street [24].

Definition 1.14 Let C be a parity structure. A subset $S \subseteq C$ is called *well formed* when S_0 has at most one element, and, for all $x, y \in S_n$ for $n > 0$, if $x \neq y$ then $x^\varepsilon \cap y^\varepsilon = \varnothing$ for each $\varepsilon \in \{-, +\}$.

Example 1.15 Consider the parity structure arising from the following pasting diagram of a 2-simplex.

$$\begin{tikzcd}
0 \arrow[rr, "02"] \arrow[dr, "01"'] & \arrow[d, phantom, "012 \Downarrow"] & 2 \\
& 1 \arrow[ur, "12"'] &
\end{tikzcd}$$

The subset $\{01, 12\}$ is well formed since we can observe that the arrows 01 and 12 share no heads or tails. On the other hand, the subset $\{02, 12\}$ is not well formed since $2 \in 02^+ \cap 12^+$.

Let $x \prec y$ whenever $x^+ \cap y^- \neq \varnothing$. This implies that $x \neq y$ since x^-, x^+ were assumed disjoint. For any $S \subseteq C$, let $\lhd_S$ denote the preorder obtained on S as the reflexive transitive closure of the relation $\prec$ on S. We will drop the subscript whenever $S = C$ so that $\lhd = \lhd_C$. In general, $\lhd_S$ is contained in, but not equal to, the restriction of $\lhd$ to S. Whenever order properties of a subset S of C are referred to in this work, it will be implicitly understood that the order $\lhd_S$ is intended.

We may now give the definition of a parity complex due to Street [24].

Definition 1.16 A *parity complex* is a parity structure C satisfying the following conditions.

Axiom 1 $x^{--} \cup x^{++} = x^{-+} \cup x^{+-}$

Axiom 2 x^- and x^+ are well formed

Axiom 3 (a) $x \lhd y \lhd x \implies x = y$

(b) $x \lhd y, x \in z^{\eta}, y \in z^{\varepsilon} \implies \eta = \varepsilon$

As observed by Street [24], for any parity complex C there is an associated chain complex built up out of the free abelian groups on each C_n, with differential defined by

$$d(x) = \sum_{y \in x^+} y - \sum_{z \in x^-} z.$$

This point of view was central to the theory of Steiner [21], which was further developed by Crans-Steiner [5] and Ara-Maltsiniotis [1, 2, 3].

We will now give the definition of the cells for a parity complex. The following definitions are due to Street [24].

Definition 1.17 Let C be a parity complex, S, M, P are subsets of C. Say S *moves* M to P when

$$P = (M \cup S^+) \backslash S^- \text{ and } M = (P \cup S^-) \backslash S^+.$$

Example 1.18 Consider the pasting diagram as given in Example 1.15. It is clear that $\{012\}$ moves $\{2, 02\}$ to $\{2, 01, 12\}$.

Definition 1.19 A *cell* of a parity complex C is a pair (M, P) of non-empty well formed finite subsets M, P of C such that M and P move M to P. The set of cells is denoted by $\mathcal{O}(C)$. The n-source and n-target of (M, P) are defined by

$$s_n(M, P) = (M^{(n)}, M_n \cup P^{(n-1)}), t_n(M, P) = (M^{(n-1)} \cup P_n, P^{(n)}).$$

An ordered pair of cells $(M, P), (N, Q)$ is called n-composable when

$$t_n(M, P) = s_n(N, Q),$$

in which case their n-composite is defined by

$$(M, P) *_n (N, Q) = (M \cup (N \backslash N_n), (P \backslash P_n) \cup Q).$$

Call $(M, P) \in \mathcal{O}(C)$ an n-cell when $M \cup P$ is n-dimensional. This is the same as the requirement that the cell is equal to its n-source (and/or its n-target), and so $M_n = P_n$.

Example 1.20 Consider the pasting diagram as given in Example 1.15. Here $M = \{0, 02, 012\}$, $P = \{2, 01, 12, 012\}$ are well formed subsets for which both M and P moves M to P. So we have (M, P) is a 2-dimensional cell.

Theorem 1.21 (Street) *Let C be a parity complex. Then $\mathcal{O}(C)$ is an ω-category.*

Additional axioms on a parity complex C as specified in Section 4 of [24] and corrected in [25] are then used to prove that $\mathcal{O}(C)$ is a free ω-category. We have decided to not include this part in our discussion of parity complexes as this does not reflect the scope of our thesis. To understand why, we must consider axiom 3(b) of a parity complex.

In order to investigate axiom 3(b), we will first consider the relationship between a pasting diagram and a parity structure. For a pasting diagram, the n-cells correspond to the elements of C_n, and for any morphism the negative and positive faces correspond to the domain and codomain respectively. We will show this by an example to help develop a sense of intuition behind the concept of a pasting diagram. Next, we will give an example of a 3-dimensional parity structure which does not satisfy axiom 3(b) in [24].

Example 1.22 Consider the following pasting diagram.

$$\begin{tikzcd}
a \arrow[rr, "f"] \arrow[dd, "h"'] \arrow[dr, phantom, "\alpha\Uparrow"] & & d \arrow[dd, "g"] & & & a \arrow[rr, "f"] \arrow[dd, "h"'] \arrow[ddrr, phantom, "\gamma\Uparrow"] & & d \arrow[dd, "g"] \\
& & & {} \arrow[r, "s"] & {} & & & \\
b \arrow[rr, "k"'] \arrow[uurr, "l"] & & c \arrow[ul, phantom, "\beta\Uparrow"] & & & b \arrow[rr, "k"'] & & c
\end{tikzcd}$$

We obtain a graded set $C = \sum_{n=0}^{3} C_n$ where $C_0 = \{a, b, c, d\}$, $C_1 = \{f, g, h, k, l\}$, $C_2 = \{\alpha, \beta, \gamma\}$ and $C_3 = \{s\}$. For any element $x \in C$, we let x^- and x^+ be the domain and codomain (respectively) of the arrow x. For instance $\gamma^- = \{h, k\}$ and $\gamma^+ = \{f, g\}$. It can be verified that this is a parity structure which satisfies axioms 1, 2 and 3(a).

Note that $h \lhd l \lhd g$ since $h^+ \cap l^- = \{b\}$ and $l^+ \cap g^- = \{d\}$. However $h \in \gamma^-$ and $g \in \gamma^+$ and so axiom 3(b) does not hold. Nevertheless, C generates a free ω-category, and furthermore it satisfies Campbell's LGC axioms (which will be discussed in the last section of this chapter).

In Appendix A, we will provide a counterexample of axiom 3(b) for the 4-dimensional associahedron. Despite this issue with axiom 3(b) we are nevertheless able to show that our parity structure on the associahedron satisfies axioms 1, 2 and 3(a). In fact, we are able to show that the unions of axiom 1 are disjoint,

Axiom 1* $x^{--} \cup x^{++} = x^{-+} \cup x^{+-}$
$x^{--} \cap x^{++} = x^{-+} \cap x^{+-} = \varnothing$

As a consequence of our above observations, in the remainder of this chapter we will be interested in parity structures which satisfy axioms 1*, 2 and 3(a). Note that Street proves in Proposition 1.1 of [24] that disjointness of the unions of axiom 1 follows from axiom 3(b). The result below is the second half of Proposition 1.1 of [24] which follows from the disjointness of unions of axiom 1*.

Proposition 1.23 *Let C be a parity structure satisfying axiom 1*. Then*

$$x^{-\mp} = x^{--} \cap x^{+-} = x^{+\mp}$$

$$x^{-\pm} = x^{-+} \cap x^{++} = x^{+\pm}$$

for all $x \in C_n$ with $n \geq 2$.

We will now introduce a partial order which is implicit in the definition of parity structure. This partial order has been considered by Steiner [21] and Johnson [12], however Street [24] does not consider this partial ordering. We will make observations for this partial order which will then serve as our motivation for considering abstract polytopes in the sense of McMullen et al. which can be found on pg. 22 of [16].

Proposition 1.24 *Let C be a parity structure. Let $\leq$ be the preorder on C generated by the relation $x \leq y$ iff $x \in y^- \cup y^+$. Then $(C, \leq)$ is a partially ordered set.*

Proof. All that needs to be shown is that the preorder $\leq$ is anti-symmetric. Suppose there exist $x, y \in C$ with $x \neq y$ and $x \leq y \leq x$; we aim to show a contradiction. Note that $x \leq y$ implies there exists a sequence $x = x_0, \ldots x_n = y$ where $n > 0$ such that $x_i \leq x_{i+1}$ are generators for all i. Let $x \in C_k$ for some k; then by the above calculation we have $y \in C_m$ where $m = k + n > k$. Similarly, since $y \leq x$, we have $x \in C_k$ where $k > m$, which is a contradiction. $\square$

We discovered that the above partially ordered set is a highly useful viewpoint for the purposes of our thesis: constructing parity structures on polytopes such as the associahedron. We will use this poset approach of parity structures in Chapter 3 to make a very meaningful connection between parity structures and geometric shapes known as polytopes. In the latter parts of this section, we will introduce an abstraction of the poset of faces of a (geometric) polytope; this is known as an abstract polytope [16].

A natural property that is included in the definition of an abstract polytope is that it is a graded poset. We now give a definition of a graded poset and then show that a parity structure with the partial ordering defined in Proposition 1.24 is a graded poset.

Definition 1.25 Let $(C, \leq)$ be a partially ordered set.

(a) Given $x, y \in C$, write $x <_1 y$ whenever $x \leq y$, $x \neq y$ and $x \leq z \leq y$ implies $x = z$ or $z = y$.

(b) C is *graded* when there exists a rank function $\operatorname{rank} : C \longrightarrow \mathbb{Z}$ satisfying the following condition. Let $x, y \in C$ with $x \leq y$. Then $x <_1 y$ iff $\operatorname{rank} y = \operatorname{rank} x + 1$.

Proposition 1.26 *Let C be a parity structure together with the partial ordering defined in Proposition 1.24. Then $(C, \leq)$ is a graded poset.*

Proof. Recall that a parity structure is a graded set $C = \sum_{n=0}^{\infty} C_n$. Let $\operatorname{rank} x = n$ for each $x \in C_n$; this defines a rank function on C.

All that needs to be shown is that $x <_1 y$ iff $\operatorname{rank} y = \operatorname{rank} x + 1$. Let $x, y \in C$ with $x \leq y$ be given. Note that $x \leqslant y$ implies that $x \in C_n$ and $y \in C_m$ where $n \leq m$. It suffices to show that $x <_1 y$ iff $m = n + 1$. Suppose that $x <_1 y$; then by definition of $\leq$, we must have $m = n + 1$.

We will now prove the converse. Suppose that $m = n + 1$; we aim to show that $x <_1 y$. Let $z \in C_p$ with $x \leq z \leq y$. Note that $n \leq p \leq m = n + 1$, it follows that either $n = p$ or $p = n + 1$ and so either $x = z$ or $z = y$. Thus we have shown that $x <_1 y$. □

Proposition 1.27 *Let C be a parity structure satisfying axioms 1* and 2. Consider the partially ordered set $(C, \leq)$ defined in Proposition 1.24. Let $x, y, z \in C$ with $x <_1 y <_1 z$ be given. Then there exists a unique $y' \in C$ with $y' \neq y$ and $x <_1 y' <_1 z$.*

Proof. Let $x \in y^\beta$ and $y \in z^\alpha$ for some $\alpha, \beta \in \{-, +\}$; it follows that $x \in z^{\alpha\beta}$. By axiom 1* we have $z^{--} \cup z^{++} = z^{-+} \cup z^{+-}$. Now since these unions are disjoint, it follows that either $x \in z^{\varepsilon(\alpha)\beta}$ or $x \in z^{\alpha\varepsilon(\beta)}$ where $\varepsilon(\eta)$ is the opposite sign to η.

Consider the case of $x \in z^{\varepsilon(\alpha)\beta}$. There exists $y' \in z^{\varepsilon(\alpha)}$ such that $x \in y'^\beta$ and so $x <_1 y' <_1 z$. By definition of a parity structure, z^α and $z^{\varepsilon(\alpha)}$ are disjoint subsets and so we must have $y' \neq y$.

All that remains is to show that y' is unique with the properties $y' \neq y$ and $x <_1 y' <_1 z$. Suppose there exists $u \neq y'$ with the above property; we aim to show a contradiction. Recall that $x \in z^{\varepsilon(\alpha)\beta}$ so by the disjointness part of axiom 1*, we must have $x \notin z^{\alpha\varepsilon(\beta)}$. It follows that $u \in z^{\varepsilon(\alpha)}$ and so $x \in u^\beta$. By axiom 2, $z^{\varepsilon(\alpha)}$ is well formed. However $x \in y^\beta \cap u^\beta$ so by well formedness we have $y = u$, which is a contradiction. Hence the result follows.

For the case of $x \in z^{\alpha\varepsilon(\beta)}$ we can use a similar argument as above. □

The above result is informally known as the 'diamond property' which is equivalent to axiom (P4) as appearing on pg. 25 of [16]. In order to see the motivation behind this name, we must first consider the concept of a Hasse diagram.

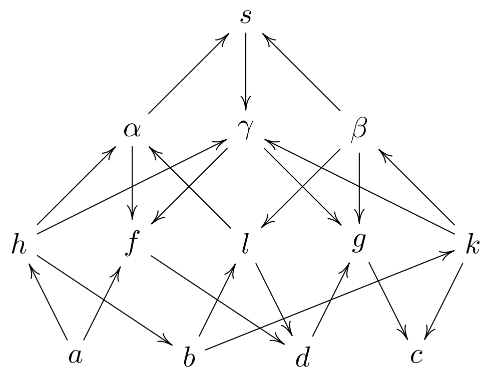
Definition 1.28 Let $(C, \leq)$ be a graded poset. A *Hasse diagram* of C is a graph with the set of vertices are elements of C, together with edges given by $\{(x, y) \mid x <_1 y \text{ or } y <_1 x\}$.

For any given parity structure on C we assign direction to the edges of its Hasse diagram in the following way.

$$\left.\begin{array}{r} y <_1 x \\ x \longrightarrow y \end{array}\right\} \iff y \in x^+, \quad \left.\begin{array}{r} y <_1 x \\ y \longrightarrow x \end{array}\right\} \iff y \in x^-$$

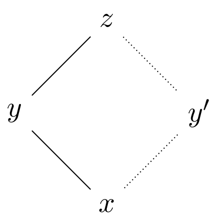
We will now express the parity structure given in Example 1.22 in terms of Hasse diagrams.

Example 1.29 Consider the parity structure $C = \sum_{n=0}^{3} C_n$ given by the pasting diagram in Example 1.22. The following is the Hasse diagram of the poset $(C, \leq)$ where we have also included the directions as to indicate parity.



Remark Strictly speaking, a Hasse diagram is a directed graph; the directed edges are given by the partial ordering. In this thesis, we shall represent this direction by relative position on the page, so that if $x <_1 y$ then x appears below y. This leaves us free to use the direction of the arrows on the edges (either up or down) to represent parity.

The above Proposition 1.27 can now be visualised by the following Hasse diagram.



The uniqueness of y' ensures that there are no more edges between x and z. We have drawn the edges $x <_1 y'$ and $y' <_1 z$ with a dotted line to indicate the existence of y'. Although the above diagram does not indicate the logic of Proposition 1.27, we can observe that it takes the shape of a diamond which motivates the name 'diamond property'.

In the following, we introduce our working definition of a polytope. We will then consider parity structures on a polytope and then characterise axioms 1* and 2.

In geometry, there exist notions of polytopes which are geometric objects that arise from a suitable generalisation of shapes known as polygons; a polytope then consists of higher dimensional faces together with a subface relation. The set of faces of a polytope together with the subface relation is a partially ordered set; this is known as the face poset. We now introduce a notion of polytopes known as abstract polytopes which is due to McMullen et al. [16]. In our thesis, a polytope is understood to be an abstract polytope. We begin by making the following preliminary definitions.

Definition 1.30 Let $(P, \leq)$ be a finite partially ordered set (whose elements are referred to as faces).

(a) P is *bounded* if there exists a least face and a greatest face; denote these by $\perp_P$ and $\top_P$ respectively. Write $\widetilde{P} = P \backslash \{\perp_P\}$.

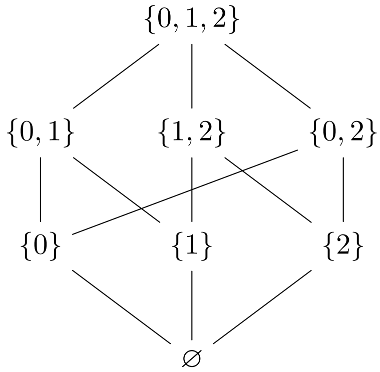
(b) An *interval* of P is the sub-poset denoted by $[f, g] = \{h \in P \mid f \leqslant h \leqslant g\}$ where f, g are faces with $f \leq g$; for a bounded poset P we have $P = [\bot_P, \top_P]$.

(c) P is a *line segment* if it is isomorphic to $\{\varnothing, \{0\}, \{1\}, \{0,1\}\}$ with the subset relation.

Remark Recall the definition of a graded poset given in Definition 1.25. For a graded poset P, a face $f \in P$ with $\operatorname{rank} f = k$ is called a *k-face*. An interval $[f, g]$ is called *k-interval* where $k = \operatorname{rank} g - \operatorname{rank} f - 1$. By convention, the least face (whenever it exists) has rank equal to -1.

Example 1.31 The poset $\wp\{0, \dots n\}$ with the subset relation is a model of the n-simplex. For any face $X \subseteq \{0, \dots n\}$, we let $\operatorname{rank} X = |X| - 1$.

In the following, we give the Hasse diagram of a 2-simplex (triangle) $\wp\{0, 1, 2\}$.



The 0-faces (vertices) are $\{0\}, \{1\}, \{2\}$, and the 1-faces (edges are $\{0, 1\}, \{1, 2\}, \{0, 2\}$. It is clear from the Hasse diagram that the least face is $\varnothing$, and the greatest face is $\{0, 1, 2\}$.

We will now introduce our working definition of an abstract polytope. For the purposes of our thesis we found it convenient to define an abstract polytope in the following way. Firstly, we need to make the following definitions.

Definition 1.32 Let $(P, \leq)$ be a graded poset.

(a) A *chain* of P is a linearly ordered subset.

(b) A *flag* of P is a maximal chain with respect to the subset ordering; the number of elements of a flag is called its *length*.

(c) Faces f and g are said to be *incident* when $f \leq g$ or $g \leq f$.

(d) A k-interval $[f, g]$ of P is *connected* when either

  (i) $k \leq 1$,

  (ii) $k > 1$; for faces $x, y \in [f, g] \backslash \{f, g\}$ there exists a sequence of faces $x = h_1, \dots, h_k = y$ in $[f, g] \backslash \{f, g\}$ such that h_i and h_{i+1} are incident for all possible i.

Definition 1.33 An *abstract pre-polytope* is a finite partially ordered set P satisfying the following conditions.

(P1) P is bounded.

(P2) P is a graded poset.

(P3) Every 1-interval is a line segment.

For any 1-interval $[x, z]$ we have $\operatorname{rank} z - \operatorname{rank} x = 2$. Note that (P3) is equivalent to saying that $[x, z] \backslash \{x, z\}$ has exactly two elements. This is equivalent to the diamond property; $x <_1 y <_1 z$ implies there exists a unique y' with $y' \neq y$ and $x <_1 y' <_1 z$.

Definition 1.34 An *abstract polytope* is an abstract pre-polytope P satisfying the condition.

(P4) Every interval is connected.

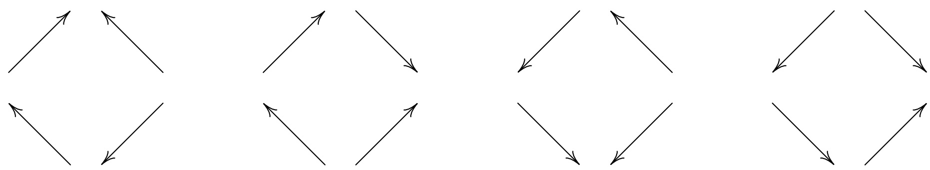
On pg. 22 of [16], condition (P2) is replaced by the requirement that all flags have the same length. But this is also shown to be equivalent to our (P2) in the presence of the other axioms.

Example 1.35 (a) The face poset of a (geometric) polytope is an abstract polytope; among these are regular polygons, platonic solids and other convex polytopes.

(b) Hypergraph polytopes as defined in [7] are abstract polytopes; among these are the simplexes, hypercubes, associahedra and permutohedra.

Observe that for a partially ordered set, a parity structure is equivalent to assigning orientations to each edge of the Hasse diagram as described in Definition 1.28. In the following result, we will use this observation to give a characterisation of axioms 1* & 2.

Theorem 1.36 *Let P be an abstract pre-polytope with a parity structure on $\widetilde{P}$. Consider the directions described in Definition 1.28. Then $\widetilde{P}$ satisfies axioms 1* & 2 iff every line segment of $\widetilde{P}$ has one of the following configurations.*



Proof. Consider the graded set,

$$\widetilde{P} = P \backslash \{\bot_P\} = \sum_{k=0}^{n} P_k$$

where $P_k = \{f \in P \mid \operatorname{rank} f = k\}$. It is clear that axiom 1* implies that every line segment has one of the above configurations. We now proceed with showing the converse. Firstly, we will show that axiom 1* holds; $x^{--} \cup x^{++} = x^{-+} \cup x^{+-}$ for each $x \in P_k$ with $k \geq 2$. Let $x \in \widetilde{P}$ with $y \in x^{--} \cup x^{++}$; then either $y \in x^{--}$ or $y \in x^{++}$. If $y \in x^{--}$, then there exists $a \in x^-$ such that $y \in a^-$. Apply (P3) to $y <_1 a <_1 x$ to obtain a face b with $b \neq a$ and $y <_1 b <_1 x$. By the hypothesis, there are two possible configurations as shown below.

$$\begin{tikzcd} & x & \\ a \arrow[ur] & & b \arrow[ul] \arrow[dl] \\ & y \arrow[ul] & \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr] & \\ a \arrow[ur] & & b \\ & y \arrow[ul] \arrow[ur] & \end{tikzcd}$$

It follows that either $y \in b^+ \subseteq x^{-+}$ or $y \in b^- \subseteq x^{+-}$, and so $y \in x^{-+} \cup x^{+-}$. Now if $y \in x^{++}$, then by similar calculations as above, $y \in x^{-+} \cup x^{+-}$. We have shown that $x^{--} \cup x^{++} \subseteq x^{-+} \cup x^{+-}$; the reverse inequality follows similarly. All that remains is to show that the unions of $x^{--} \cup x^{++} = x^{-+} \cup x^{+-}$ are disjoint. Suppose that $x^{--} \cap x^{++} \neq \varnothing$; then there exists $y \in x^{--} \cap x^{++}$ which gives the following diagram.

$$\begin{tikzcd}
& x \arrow[dr] & \\
a \arrow[ur] & & b \arrow[dl] \\
& y \arrow[ul] &
\end{tikzcd}$$

However this configuration is not permitted, thus $x^{--} \cap x^{++} = \varnothing$. By a similar calculation as above, $x^{-+} \cap x^{+-} = \varnothing$. Hence we have shown that axiom 1* holds.

We will now show that axiom 2 holds; x^- and x^+ are well formed for each $x \in P_k$ with $k \geq 1$. Let $x \in P_k$ with $k \geq 1$ and $a, b \in x^-$ with $a \neq b$ be given.

Consider the case of $k > 1$; note that $a, b \notin P_0$. Recall that x^- is well formed when $a^\eta \cap b^\eta = \varnothing$ for each $\eta \in \{-, +\}$. Suppose that $y \in a^+ \cap b^+$; then by (P3), the interval $[y, x]$ is given by the following diagram.

$$\begin{tikzcd}
& x & \\
a \arrow[ur] \arrow[dr] & & b \arrow[ul] \arrow[dl] \\
& y &
\end{tikzcd}$$

However this is not a permitted configuration, thus $a^+ \cap b^+ = \varnothing$. By a similar calculation as above, $a^- \cap b^- = \varnothing$, and so x^- is well formed.

Now consider the case of $k = 1$; note that $a, b \in P_0$. Recall that x^- is well formed when x^- is a singleton. Note that by (P3), the interval $[\perp_P, x]$ is given by the following diagram.

$$\begin{tikzcd}
& x & \\
a \arrow[ur] \arrow[dr, dash] & & b \arrow[ul] \arrow[dl, dash] \\
& \perp_P &
\end{tikzcd}$$

It follows that $x^- = \{a, b\}$ and $x^+ = \varnothing$, which contradicts the non-emptiness condition of a parity structure. Hence we have shown that x^- is well formed. Finally, well-formedness of x^+ follows by similar argument as for x^-. $\square$

1.3 Pasting schemes

In this section, we will describe the notion of a loop free pasting scheme due to Johnson [12]. This consists of two parts: firstly, we give the definition of a pasting scheme, secondly, we will give the definition of loop freeness.

Definition 1.37 A *pasting scheme* is a triple $(A, \mathsf{E}, \mathsf{B})$ consisting of a graded set $A = \sum A_i$ together with finitary relations $\mathsf{E}^i_j, \mathsf{B}^i_j$ for $j \leq i$ which satisfy the following conditions.

1. E^i_j is a relation between A_i and A_j.
2. E^i_i is the identity relation on A_i.
3. For $k > 0$ and any $x \in A_k$ there exists $y \in A_{k-1}$ with $x\mathsf{E}^k_{k-1}y$.
4. For $k < n$, $w\mathsf{E}^n_k x$ iff there exists u, v such that $w\mathsf{E}^n_{n-1}u\mathsf{E}^{n-1}_k x$ and $w\mathsf{E}^n_{n-1}v\mathsf{B}^{n-1}_k x$.
5. If $w\mathsf{E}^n_{n-1}z\mathsf{E}^{n-1}_k x$ then either $w\mathsf{E}^n_k x$ or there exists a v with $w\mathsf{B}^n_{n-1}v\mathsf{E}^{n-1}_k x$.

Remark The following notation is due to Johnson [12]. Firstly, we will define a relation R^i_j between A_i and A_j. Let $x\mathsf{R}^i_j y$ whenever there exists a sequence $x = x_1, x_2, \ldots, x_j = y$ of elements of A satisfying $x_k D^p_q x_{k+1}$ where D^p_q is either E^p_q or B^p_q.

For any n-dimensional subset $X \subseteq A$, we let $\mathsf{E}(X) = \sum_{k\leq n} \mathsf{E}(X)_k$, $\mathsf{B}(X) = \sum_{k\leq n} \mathsf{B}(X)_k$ and $\mathsf{R}(X) = \sum_{k\leq n} \mathsf{R}(X)_k$ where

$$\begin{aligned}
\mathsf{E}(X)_k &= \{x \in A_k \mid w\mathsf{E}^n_k x \text{ for some } w \in X_n\},\\
\mathsf{B}(X)_k &= \{x \in A_k \mid w\mathsf{B}^n_k x \text{ for some } w \in X_n\},\\
\mathsf{R}(X)_k &= \{x \in A_k \mid w\mathsf{R}^n_k x \text{ for some } w \in X_n\}.
\end{aligned}$$

We will write $\mathsf{R}(x)$ for $\mathsf{R}(\{x\})$, and similarly for $\mathsf{E}(x)$ and $\mathsf{B}(x)$.

We now proceed to define a loop free pasting scheme. The proposition below observes that pasting schemes have an implicit parity structure. This result is due to Campbell [4].

Proposition 1.38 *Let $(A, \mathsf{E}, \mathsf{B})$ be a pasting scheme. Then the graded set A together with $x^- = \{y \in A_{k-1} \mid x\mathsf{B}^k_{k-1}y\}$ and $x^+ = \{y \in A_{k-1} \mid x\mathsf{E}^k_{k-1}y\}$ defines a parity structure.*

The above proposition will be used to describe a loop free pasting scheme [12] in terms of parity structures. Since A is a parity structure, we have the preorder $\lhd$ on A as defined in the previous section. This allows us to make the following definitions.

Definition 1.39 A pasting scheme A has *no direct loops* whenever for any k and $x, y \in A_k$, $\mathsf{B}(x) \cap \mathsf{E}(x) = \{x\}$ and $x \lhd y$ implies $\mathsf{B}(x) \cap \mathsf{E}(y) = \varnothing$.

Definition 1.40 Let B be a pasting scheme and $X \subseteq B$ be a subgraded set of B. We say that X is a *subpasting scheme* of B whenever $y \in \mathsf{R}(X)$ implies $y \in X$.

Remark By viewing a pasting scheme as a parity structure as described in Proposition 1.38, a subpasting scheme is equivalently a down-closed subset with respect to the partial ordering defined in Proposition 1.24.

Definition 1.41 Let B be a pasting scheme and $A \subseteq B$ be an n-dimensional subgraded set of B. Define maps called 'domain' and 'codomain' respectively;

$$\begin{aligned}
\mathrm{dom}A &= A\backslash\mathsf{E}(A_n),\\
\mathrm{cod}A &= A\backslash\mathsf{B}(A_n).
\end{aligned}$$

Define source and target maps respectively;

$$\begin{gathered}
s_kA = A = t_kA \text{ for } k \geq n,\\
s_kA = \mathrm{dom}^{n-k}A \text{ and } t_kA = \mathrm{cod}^{n-k}A \text{ for } k < n.
\end{gathered}$$

We will now give an equivalent to the definition of a well formed pasting scheme given in Section 3 of [12]. This equivalent definition, which is given below, is due to Campbell [4].

Definition 1.42 Let B be a pasting scheme and $A \subseteq B$ is an n-dimensional subgraded set of B. We call A *compatible* when A is a subpasting scheme of B and A_n is a well formed subset of B. Furthermore, A is called a $n-(J)$-cell when both s_kA and t_kA are compatible for all k.

We now give the definition of a loop free pasting scheme which is due to Johnson [12].

Definition 1.43 A pasting scheme B is loop free if it satisfies the following conditions.

1. B has no direct loops.
2. for any $x \in B$, $\mathsf{R}(x)$ is a (J)-cell.
3. for any n-(J)-cell A and $x \in B$ such that $s_n\mathsf{R}(x) \subseteq A$, then $(s_n\mathsf{R}(x))_n$ is a $\lhd$-interval of A_n.

In [12], Johnson uses the single-sorted definition of an ω-category. The $n-(J)$-cells as defined earlier are taken to be the cells of an ω-category. We will now summarise the main result of Johnson which can be found in Theorem 2.12 and Theorem 2.13 of [12].

Theorem 1.44 (Johnson) *Let A be a loop free pasting scheme. The n-(J)-cells together with source and target maps s_n and t_n and composition defined by union is an ω-category. Furthermore, this is an ω-category freely generated the $n-(J)$-cells $\mathsf{R}(x)$, for $x \in A$.*

1.4 Pasting schemes via parity structures

In this section, we consider an approach of pasting schemes via parity structures; this approach is due to Campbell [4]. The purpose of our discussion is to summarise the results in [4] which involves giving a sufficient set of conditions on parity structures so that we obtain a loop free pasting scheme. For the purposes of our thesis, we will consider a slightly different set of conditions as considered by Campbell, however our approach is equivalent. We will now give a definition which is due to Campbell [4].

Definition 1.45 Let C be a parity structure. A subset $A \subseteq C$ is a *subcomplex* of C whenever $x^- \cup x^+ \subseteq A$ for all $x \in A$.

For each $x \in C_n$, define $R(x)$ for the smallest subcomplex of C containing x. For arbitrary subsets X we define $R(X)$ as the union of $R(x)$ over all $x \in X$.

For each $x \in C_n$, define $E(x) = R(x)\backslash R(x^-)$ and $B(x) = R(x)\backslash R(x^+)$. For arbitrary subsets X we define $E(X)$ and $B(X)$ similarly to the above.

Recall the partial ordering $\leq$ on C defined in Proposition 1.24. We can understand the set $R(x) = \{y \in C \mid y \leq x\}$; the down-closure of x in the poset $(C, \leq)$. Consequently, a subcomplex is a down-closed subset. We will make use of this in the next section and in Chapter 3.

Definition 1.46 Let C be a parity structure and $x \in C$. Define the subsets $\pi(x) = R(x)\backslash R(x)^-$, $\mu(x) = R(x)\backslash R(x)^+$.

Definition 1.47 Let C be a parity structure and $x \in C$. Define the relation $\blacktriangleleft$ as the preorder on C generated by the relation $x \blacktriangleleft y$ iff either $x \in y^-$ or $y \in x^+$.

Definition 1.48 Let C be a parity structure, $A \subseteq C$ be a subcomplex of C. Then

$$s_nA = A^{(n+1)}\backslash E(A_{n+1}),$$
$$t_nA = A^{(n+1)}\backslash B(A_{n+1}).$$

Remark The above maps will turn out to be the source and target maps for a pasting scheme which will be defined in Theorem B.6. Furthermore, they are equivalent to the boundary maps due to Steiner and can be found in Definition 2.3 of [21].

It is shown in Theorem B.4.6. of [4] that a parity structure C which satisfies the axioms below gives a loop free pasting scheme. In [4], axiom **(L)** is called the linearity axiom, axiom **(G)** is called the globularity axiom, and axiom **(C)** is called the cellularity axiom. We will call a parity structure satisfying axioms **(L)**, **(G)** and **(C)** an *LGC-complex*.

Axiom *(L)* The preorder $\blacktriangleleft$ on C is a linear order.

Axiom *(G)* For each $x \in C_n$ where $n \geq 2$,

$$s_{n-2}s_{n-1}R(x) = s_{n-2}t_{n-1}R(x),$$
$$t_{n-2}s_{n-1}R(x) = t_{n-2}t_{n-1}R(x).$$

Axiom *(C)* For each $x \in C$,

(a) $\mu(x)$ and $\pi(x)$ are well formed,

(b) $s_nR(x)$ and $t_nR(x)$ are subcomplexes for all n.

In [4], Campbell proves that the above axioms imply axioms 1*, 2, 3(a). In our approach, we will use this results of [4] to show that parity structures satisfying axioms 1*, 2, 3(a), **(L)** and **(C)** also satisfies **(G)** and so gives a loop free pasting scheme. Here we are motivated to use these axioms since they were the most convenient axioms to prove for the associahedron. Note that axiom 3(a) follows from axiom **(L)**.

Before we proceed, we will introduce the notion of duality of a parity structure as described by Street in Section 1 of [24]. For a subset $K \subseteq \mathbb{N}\backslash\{0\}$, the K-dual of a parity structure C amounts to interchanging the sets x^- and x^+ for all $x \in C_n$ where $n \in K$. In this section, the dual of a parity structure C is understood to be the $(\mathbb{N}\backslash\{0\})$-dual of C. Note that axioms 1*, 2, 3(a), **(L)** and **(C)** are self dual (that is, these axioms are true for C iff they are true for the dual of C). Thus for every proposition that we prove, the dual proposition will also hold.

We will now prove our assertion that axiom **(G)** follows from axioms 1*, 2, 3(a) and **(C)**. Recall the earlier remark of Definition 1.45 that $R(x)$ is the down-closure of x in the poset $(C, \leq)$. In the proof of the following results, we will make use of the poset $(C, \leq)$ perspective, in particular we will use Hasse diagrams as an illustrative tool.

Proposition 1.49 *Let C be a parity structure satisfying axioms 1*, 2 and 3(a). Let $z \in C_{n+1}$ be given. Then $s_{n-1}s_nR(z) \subseteq R(z^{-\mp})$ and $s_{n-1}t_nR(z) \subseteq R(z^{+\mp})$.*

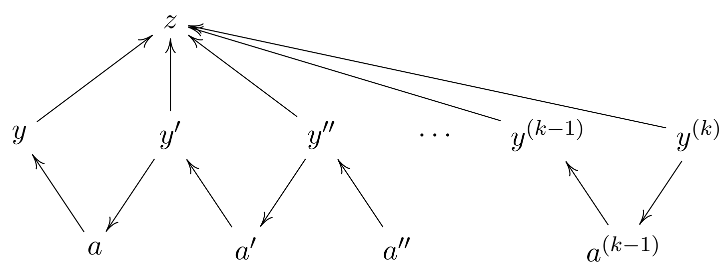
Proof. We will first show that $s_{n-1}s_nR(z) \subseteq R(z^{-\mp})$. By the definition of source,

$$\begin{aligned} s_{n-1}s_nR(z) &= s_{n-1}R(z^-) \\ &= \{x \in R(z^-)^{(n-1)} \mid x \in R(y) \implies x \in R(y^-) \text{ for all } y \in z^-\}. \end{aligned}$$

Let $x \in s_{n-1}s_nR(z)$ be given. Note that there exists $y \in z^-$ such that $x \in R(y^-)$ and so there exists $a \in y^-$ such that $x \leq a$. Suppose that $x \notin R(z^{-\mp})$ or equivalently $x \not\leq b$ for all $b \in x^{-\mp}$; we aim to show a contradiction.

Note that $a <_1 y <_1 z$ so by Proposition 1.27, there exists y' with $y' \neq y$ and $a <_1 y' <_1 z$. Suppose that $y' \in z^+$ and $a \in y'^-$; then $a \in z^{--} \cap z^{+-} = z^{-\mp}$ and so $x \in R(z^{-\mp})$, which is a contradiction. Thus $y' \in z^-$ and $a \in y'^+$.

Now $x \in s_{n-1}s_nR(z)$ and $x \in R(y')$ so by definition of source we have $x \in R(y'^-)$. It follows that there exists $a' \in y'^-$ such that $x \leq a'$. Repeating the above argument, we obtain a sequence $y, y', \ldots, y^{(k)}$ satisfying $y^{(k)} \lhd \ldots \lhd y' \lhd y$. We summarise this in the following diagram.



By axiom 3(a), $y, y', \dots, y^{(k)}$ are distinct and so by finiteness the above process must terminate. This occurs for some $k \geq 1$ where $y^{(k)} \in z^+$ and $a^{(k-1)} \in y^{(k)-}$ and so $x \leq a^{(k-1)} \in z^{-\mp}$, which is a contradiction. Hence $z \in R(z^{-\mp})$ and so $s_{n-1}s_nR(z) \subseteq R(z^{-\mp})$ as required.

Finally, $s_{n-1}t_nR(z) \subseteq R(z^{+\mp})$ follows by a similar argument. □

Proposition 1.50 *Let C be a parity structure satisfying axioms 1*, 2 and 3(a). Let $z \in C_{n+1}$ be given. Then $s_{n-1}R(z) \subseteq s_{n-1}s_nR(z)$ and $s_{n-1}R(z) \subseteq s_{n-1}t_nR(z)$.*

Proof. We will first show that $s_{n-1}R(z) \subseteq s_{n-1}s_nR(z)$. By definition of source,

$$\begin{aligned} s_{n-1}R(z) &= \{x \in R(z)^{(n-1)} \mid x \in R(y) \implies x \in R(y^-) \text{ for all } y <_1 z\} \\ s_{n-1}s_nR(z) &= s_{n-1}R(z^-) \\ &= \{x \in R(z^-)^{(n-1)} \mid x \in R(y) \implies x \in R(y^-) \text{ for all } y \in z^-\}. \end{aligned}$$

Note that $R(z)^{(n-1)} = R(x^-)^{(n-1)} \cup R(x^+)^{(n-1)}$. Let $x \in s_{n-1}R(z)$ be given. We will now show that $x \in s_{n-1}s_nR(z)$. By the above formula for $s_{n-1}s_nR(z)$, it suffices to show that $x \in R(z^-)^{(n-1)}$. Suppose that $x \notin R(z^-)^{(n-1)}$; then $x \in R(z^+)^{(n-1)}$. It follows that $x \in s_{n-1}t_nR(z)$ so by Proposition 1.49, $x \in R(z^{+\mp}) = R(z^{-\mp}) \subseteq R(z^-)^{(n-1)}$, which is a contradiction. Thus $x \in R(z^-)^{(n-1)}$ and so $x \in s_{n-1}s_nR(z)$ as required.

Finally, $s_{n-1}R(z) \subseteq s_{n-1}t_nR(z)$ follows by a similar argument. □

Theorem 1.51 *Let C be a parity structure satisfying axioms 1*, 2, 3(a) and the cellularity axiom **(C)**. Then*

$$\begin{aligned} s_{n-1}s_nR(z) = s_{n-1}R(z) = s_{n-1}t_nR(z) \\ t_{n-1}s_nR(z) = t_{n-1}R(z) = t_{n-1}t_nR(z) \end{aligned}$$

*for any $z \in C_{n+1}$, and so **(G)** holds.*

Proof. Note that by duality, it suffices to prove $s_{n-1}s_nR(z) = s_{n-1}R(z) = s_{n-1}t_nR(z)$. Firstly, by Proposition 1.23, $z^{--} \cap z^{+-} = z^{-\mp} \subseteq s_{n-1}R(z)$. By axiom **(C)**, $s_{n-1}R(z)$ is down-closed so it follows that $R(z^{--} \cap z^{+-}) \subseteq s_{n-1}R(z)$.

Since $z^{-\mp} = z^{--} \cap z^{+-} = z^{+\mp}$, then by Proposition 1.49,

$$\begin{aligned} s_{n-1}s_nR(z) \subseteq R(z^{-\mp}) = R(z^{--} \cap z^{+-}) \subseteq s_{n-1}R(z), \\ s_{n-1}t_nR(z) \subseteq R(z^{+\mp}) = R(z^{--} \cap z^{+-}) \subseteq s_{n-1}R(z). \end{aligned}$$

The result follows immediately from Proposition 1.50. □

Hence we have shown that axiom **(G)** follows from our set of axioms. It follows by the results of [4] that an LGC-complex gives a loop free pasting scheme. We summarise this as the following result.

Corollary 1.52 *A parity structure satisfying axioms 1*, 2, **(C)** and **(G)** is an LGC-complex.*

At the time of writing this thesis, the reference by Campbell [4] is not readily available. We have included all the applicable results in Appendix B.

Chapter 2

Associahedra

Associahedra were first defined by Tamari [26] and (independently) Stasheff [20]. In [26], Tamari defines the n-dimensional associahedron as the poset of bracketings of $n+2$ letters. The (complete) bracketings correspond to the vertices, and the partial bracketings correspond to higher dimensional faces. In [20], Stasheff defines the n-dimensional associahedron as the poset of rooted trees with $n+2$ leaves. The binary trees correspond to the vertices, and the non-binary trees correspond to higher dimensional faces.

There are numerous other equivalent definitions of the associahedron [17, 13]. One of importance to this thesis is the left bracketing functions (lbfs) due to Huang-Tamari [11]. Unlike bracketings or trees, it is not immediately clear how the notion of left bracketing functions could be generalised. We will describe a process of constructing left bracketing functions from binary trees. When the same process is performed on non-binary trees what we obtain can be thought of as a generalisation of the left bracketing functions. This motivates our formal definition of *higher left bracketing functions (hlbfs)*.

In this chapter, we will define the poset of hlbfs and then prove that it is a model for the associahedron. The contents of this chapter are motivated by our goal of constructing a parity structure on the associahedron. An important result that will be proven is that the poset of hlbfs is an abstract pre-polytope which was defined in Chapter 1. This has already been proved for various other models of the associahedra. Various other properties are then used to motivate a notion called a label structure. In Chapter 3, we will define a label structure on an abstract pre-polytope. From a label structure, we can obtain a parity structure which we then show is an LGC-complex in the sense of Campbell [4].

2.1 Partial bracketings

A *bracketing* is a way of parenthesising a string of letters such that within every parenthesis there contains exactly two bracketings. We may understand a bracketing as providing a specific order in which to perform binary multiplication. For example $((01)(23))((45)6)$ is a bracketing on the letters 0123456.

A *partial bracketing* is a way of parenthesising a string of letters. Unlike with bracketings, we no longer require the condition that every parenthesis contains exactly two bracketings. As a result, a partial bracketing can be understood as providing a specific order in which to perform multiplication of higher arities. For example $(01)(23)(45)6$ is a partial bracketing.

There is a natural partial ordering on the set of all bracketings; $a \leq b$ whenever a is obtain by inserting parentheses in b. For example $((01)(23))((45)6) \leq (01)(23)(45)6$.

2.1.1 The poset of bracketings

In this section, we will consider bracketings on a finite linearly ordered set M. We will first formalise the notion of bracketings on M and then describe a partial ordering on the set of all bracketings on M.

Definition 2.1 Let M be a finite linearly ordered set. We define recursively a notion of *bracketing* on M.

(1) If $|M| = 1$, then there is a unique bracketing.

(2) If $|M| > 1$, then a bracketing on M is a partition $M = M_1 \cup \ldots M_r$ of M into at least two non-empty intervals of M, together with a bracketing on each M_i.

Remark A partition of M is equivalent to giving the subset $m = \{\min M_1, \ldots, \min M_r\} \subseteq M$, where the subset contains the least element of M and at least one other element. We can then associate to this the right adjoint $b : M \longrightarrow m$ to the inclusion, which sends each $x \in M$ to $\min M_i$, where M_i is the part which contains x.

Let $\psi : m \longrightarrow M$ defined by $\psi(i) = i$ denote the inclusion. A right adjoint $b : M \longrightarrow m$ preserves the greatest element i.e. $b(\max M) = \max m$. By the adjunction $\psi \dashv b$, $\psi(i) \leq j$ iff $i \leq b(j)$ for all $i \in m$ and $j \in M$. It follows that $\min b^{-1}(i) = \min\{j \mid i = b(j)\} = \min\{j \mid \psi(i) \leq j\} = \psi(i)$. Now since b is order-preserving we must have the interval $b^{-1}(i) = [\psi(i), \psi(i+1))$. Thus the fibres $b^{-1}(i)$ specify a partition of M which corresponds to the outer most parentheses of an actual bracketing. Note that we require the condition $|m| > 1$ to ensure that we have a non trivial partition of M.

Example 2.2 The bracketing 0(123)(45)6 is the surjective order-preserving map $b : M \longrightarrow m$ where $M = \{0, 1, 2, 3, 4, 5, 6\}$, $m = \{0, 1, 4, 6\}$. The fibres of b are $b^{-1}(0) = \{0\}$, $b^{-1}(1) = \{1, 2, 3\}$, $b^{-1}(4) = \{4, 5\}$, $b^{-1}(6) = \{6\}$.

Let $\mathcal{B}_M$ denote the set of all bracketings on a finite linearly ordered set M. In the following we will formalise the partial ordering of bracketings given by insertion of bracketings that we described above.

First, we introduce the following notation. Let $f : X \longrightarrow Y$ be a bracketing and $J \subseteq Y$ a subset with $|J| > 1$. Denote with $f_J : f^{-1}(J) \longrightarrow J$ for the function defined by $k \mapsto f(k)$.

Proposition 2.3 *Let $f : X \longrightarrow Y$ be a bracketing and $g : Y \longrightarrow Z$ be a surjective map and $i \in Z$ with $|g^{-1}(i)| > 1$. Then there is a bracketing given by $f_{g^{-1}(i)} : f^{-1}(g^{-1}(i)) \longrightarrow g^{-1}(i)$ together with bracketings $\left(f_{g^{-1}(i)}\right)_j = f_j$ for each $j \in g^{-1}(i)$.*

Proof. Note that by definition $f_{g^{-1}(i)}$ is a surjective order-preserving map. We will first show that $g^{-1}(i) \subseteq f^{-1}(g^{-1}(i))$. Note that for any $j \in g^{-1}(i) \subseteq Y$ by the above remark we have $j = \psi(j) = \min f^{-1}(j)$. It follows that $j \in f^{-1}(j)$ and so $g^{-1}(i) \in f^{-1}(g^{-1}(i))$.

Now note that the adjointness condition follows immediately from the fact that f is right adjoint to the inclusion $Y \subseteq X$. Finally, note that by definition, $f^{-1}_{g^{-1}(i)}(j) = f^{-1}(j)$ and so the bracketings $\left(f_{g^{-1}(i)}\right)_j = f_j$ are defined on $f^{-1}_{g^{-1}(i)}(j)$ as required. □

Example 2.4 Consider the bracketing described in Example 2.2, and the subset $J = \{1, 4\} \subseteq m$. The bracketing b_J is the surjective order-preserving map $b_J : \{1, 2, 3, 4, 5\} \longrightarrow \{1, 4\}$ whose fibres are $b_J^{-1}(1) = \{1, 2, 3\}$, $b_J^{-1}(4) = \{4, 5\}$. In terms of parentheses, this bracketing is (123)(45).

Definition 2.5 Let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be bracketings. The *composite* of the bracketings f and g is the map $gf : X \longrightarrow Z$ given by $(gf)(k) = g(f(k))$ together with bracketings $(gf)_i$ for each $i \in Z$ which are defined recursively as follows.

1. If $|g^{-1}(i)| > 1$, then $(gf)_i = g_i f_{g^{-1}(i)}$.
2. If $|g^{-1}(i)| = 1$, then $(gf)_i = f_j$ where $g(j) = i$.

Remark We will show that composition is well-defined i.e. $gf : X \longrightarrow Z$ is a bracketing.

Note that the composite of surjective order-preserving maps $gf : X \longrightarrow Z$ is also a surjective order-preserving map. Since g is a bracketing we have $|Z| > 1$. Note that since f, g are bracketings, $Z \subseteq Y \subseteq X$.

Let $\psi_{Z,Y} : Z \longrightarrow Y$ and $\psi_{Y,X} : Y \longrightarrow X$ denote the inclusions $Z \subseteq Y$ and $Y \subseteq X$ respectively. Consider the composite $\psi_{Z,X} = \psi_{Y,X}\psi_{Z,Y}$. Since f, g are bracketings we have $f \dashv \psi_{Y,X}$ and $g \dashv \psi_{Z,Y}$. It follows that gf is the right adjoint to the inclusion $\psi_{Z,X}$

All that remains is to show that $(gf)_i$ are bracketings on $(gf)^{-1}(i)$ for each $i \in Z$. Let $i \in Z$ and consider the following cases.

Case 1: $|g^{-1}(i)| = 1$.

Consider the bracketing $f_j : f^{-1}(j) \longrightarrow m_{f_j}$ where $j = g(i)$. Note that $(gf)_i = f_j$ is a bracketing on $f^{-1}(j) = f^{-1}(g^{-1}(i))$ as required.

Case 2: $|g^{-1}(i)| > 1$.

Consider the bracketings $g_i : g^{-1}(i) \longrightarrow m_{g_i}$ and $f_{g^{-1}(i)} : f^{-1}(g^{-1}(i)) \longrightarrow g^{-1}(i)$. By recursion, the composite $(gf)_i = g_i f_{g^{-1}(i)}$ is a bracketing on $f^{-1}(g^{-1}(i))$ as required.

Proposition 2.6 *Let $f : W \longrightarrow X$, $g : X \longrightarrow Y$, $h : Y \longrightarrow Z$ be bracketings and $i \in Z$. If $|h^{-1}(i)| > 1$, then $(gf)_{h^{-1}(i)} = g_{h^{-1}(i)} f_{(hg)^{-1}(i)}$.*

Proof. Note that the bracketings $g_{h^{-1}(i)}$ and $f_{(hg)^{-1}(i)}$ are composable. All that needs to be shown is $\left(g_{h^{-1}(i)} f_{(hg)^{-1}(i)}\right)_j = \left((gf)_{h^{-1}(i)}\right)_j$ for all $j \in h^{-1}(i)$. Let $j \in h^{-1}(i)$ be given and consider the following cases.

Case 1: $|g^{-1}(j)| > 1$.

For any $k \in g^{-1}(j)$ we have $k \in g^{-1}(j) \subseteq g^{-1}(h^{-1}(i)) = (hg)^{-1}(i)$. It follows that $\left(f_{g^{-1}(j)}\right)_k = f_k = \left(f_{(hg)^{-1}(i)}\right)_k = \left(\left(f_{(hg)^{-1}(i)}\right)_{g^{-1}(j)}\right)_k$. Thus $f_{g^{-1}(j)} = \left(f_{(hg)^{-1}(i)}\right)_{g^{-1}(j)}$ and so $\left(g_{h^{-1}(i)} f_{(hg)^{-1}(i)}\right)_j = \left(g_{h^{-1}(i)}\right)_j \left(f_{(hg)^{-1}(i)}\right)_{g^{-1}(j)} = g_j f_{g^{-1}(j)} = (gf)_j = \left((gf)_{h^{-1}(i)}\right)_j$.

Case 2: $|g^{-1}(j)| = 1$.

Let $k \in g^{-1}(j)$ be the unique element and so $g(k) = j$. Note that $(hg)(k) = h(g(k)) = h(j) = i$. It follows that $\left(g_{h^{-1}(i)} f_{(hg)^{-1}(i)}\right)_j = \left(f_{(hg)^{-1}(i)}\right)_k = f_k = (gf)_j = \left((gf)_{h^{-1}(i)}\right)_j$.

Hence in either case we have shown that $\left(g_{h^{-1}(i)} f_{(hg)^{-1}(i)}\right)_j = \left((gf)_{h^{-1}(i)}\right)_j$ as required. $\square$

Proposition 2.7 *Composition of bracketings is associative.*

Proof. Let $f : W \longrightarrow X$, $g : X \longrightarrow Y$, $h : Y \longrightarrow Z$ be bracketings. Note that composition of functions is associative so all that needs to be shown is that $(h(gf))_i = ((hg)f)_i$ for all $i \in Z$. Let $i \in Z$ be given and consider the following cases.

Case 1: $|h^{-1}(i)| > 1$.

Note that $|(hg)^{-1}(i)| = |g^{-1}(h^{-1}(i))| > 1$ and so by Proposition 2.6, $(h(gf))_i = h_i\, (gf)_{h^{-1}(i)} = h_i g_{h^{-1}(i)} f_{(hg)^{-1}(i)} = (hg)_i\, f_{(hg)^{-1}(i)} = ((hg)f)_i$.

Case 2: $|h^{-1}(i)| = 1$ and $|(hg)^{-1}(i)| > 1$.

Let $j \in h^{-1}(i)$ be the unique element and so $h(j) = i$. Note that $|g^{-1}(h^{-1}(i))| = |g^{-1}(j)| > 1$ and so by Proposition 2.6, $(h(gf))_i = (gf)_j = g_j f_{g^{-1}(j)} = (hg)_i\, f_{(hg)^{-1}(i)} = ((hg)f)_i$.

Case 3: $|h^{-1}(i)| = 1$ and $|(hg)^{-1}(i)| = 1$.

Let $j \in h^{-1}(i)$ be the unique element and so $h(j) = i$. It follows that $|g^{-1}(j)| = 1$. Let $k \in g^{-1}(j)$ be the unique element and so $g(k) = j$. Thus $(hg)(k) = h(g(k)) = h(j) = i$, $(h(gf))_i = (gf)_j = f_k = ((hg)f)_i$.

Hence in each case we have shown that $(h(gf))_i = ((hg)f)_i$ as required. □

Proposition 2.8 *Composition of bracketings is unital.*

Proof. Let $f : X \longrightarrow Y$ be a bracketing. Let 1_X and 1_Y be the trivial bracketings on X and Y respectively. To show that composition is unital, it suffices to show that $(1_Y f)_i = f_i$ and $(f 1_X)_i = f_i$ for all $i \in Y$. This follows immediately from the definition of composition. □

Definition 2.9 Let $f : M \longrightarrow m_f$, $g : M \longrightarrow m_g$ be bracketings. Denote $f \leqslant g$ whenever there exists a bracketing $\pi : m_g \longrightarrow m_f$ with $\pi_i = 1_{\pi^{-1}(i)}$, $f(k) = (\pi g)(k)$ for all $k \in M$ and $f_i \leqslant (\pi g)_i$ for all $i \in m_f$.

Proposition 2.10 $\mathcal{B}_M$ *with the relation given in Definition 2.9 is a partially ordered set.*

Proof. We will show that $\mathcal{B}_M$ is a partially ordered set using induction on $|M|$.

We will first prove that the relation is reflexive. Let $b : M \longrightarrow m$ be a bracketing and 1_m be the trivial bracketing on m. It needs to be shown that $b \leq b$. Now since composition is unital, $(1_m b)_i = b_i$. By induction on $|M|$, $b_i \leqslant b_i = (1_m b)_i$ for all $i \in m$ and so $b \leqslant b$.

We will now prove that the relation is anti-symmetric. Let a, b be bracketings on M with $a \leq b \leq a$. It needs to be shown that $a = b$. By definition, there exists bracketings $\pi : m_b \longrightarrow m_a$ and $\mu : m_a \longrightarrow m_b$. Also note that $m_a \subseteq m_b \subseteq m_a$ and so $m_a = m_b$. It follows that $\pi = \mu = 1$ and so $a(k) = b(k)$ for all $k \in M$. Now since $a_i \leqslant b_i$ and $a_i \leqslant b_i$ for all $i \in m_a = m_b$. By induction on $|M|$, we have $a_i = b_i$ for all $i \in m_a = m_b$ and so $a = b$.

Finally we will prove that the relation is transitive. Let a, b, c be bracketings on M. It needs to be shown that $a \leq c$. By definition, there exists bracketings $\pi : m_b \longrightarrow m_a$ and $\mu : m_c \longrightarrow m_b$ such that $\pi b(k) = a(k)$ and $\mu c(k) = b(k)$ for all $k \in M$. Consider the composite map $\pi\mu : m_c \longrightarrow m_a$. Note that $((\pi\mu)c)(k) = (\pi(\mu c))(k) = \pi((\mu c)(k)) = \pi(b(k)) = a(k)$ for all $k \in m_a$. It suffices to show that $a_i \leq ((\pi\mu)c)_i$ for all $i \in m_a$. Let $i \in m_a$ be given and consider the following cases.

Case 1: $|\pi^{-1}(i)| > 1$.

Note that $|(\pi\mu)^{-1}(i)| = |\mu^{-1}(\pi^{-1}(i))| > 1$. Let $j \in \pi^{-1}(i)$ be given and consider the following subcases.

If $|\mu^{-1}(j)| > 1$, then $\left(\mu c_{(\pi\mu)^{-1}(i)}\right)_j = \mu_j \left(c_{(\pi\mu)^{-1}(i)}\right)_{\mu^{-1}(j)} = \left(c_{\mu^{-1}(\pi^{-1}(i))}\right)_{\mu^{-1}(j)} = c_{\mu^{-1}(j)} = \mu_j c_{\mu^{-1}(j)} = (\mu c)_j \geqslant b_j = \left(b_{\pi^{-1}(i)}\right)_j$.

If $|\mu^{-1}(j)| = 1$, then let $k \in \mu^{-1}(j)$ be the unique element and so $\mu(k) = j$. Note that $k \in (\pi\mu)^{-1}(i)$. Now $\left(b_{\pi^{-1}(i)}\right)_j = b_j \leqslant (\mu c)_j = c_k = \mu_k \left(c_{(\pi\mu)^{-1}(i)}\right)_k = \left(\mu c_{(\pi\mu)^{-1}(i)}\right)_j$. It follows that $b_{\pi^{-1}(i)} \leqslant c_{(\pi\mu)^{-1}(i)}$. Now since $a_i \leqslant (\pi b)_i = b_{\pi^{-1}(i)} \leqslant c_{(\pi\mu)^{-1}(i)}$ then by induction on $|M|$, $a_i \leqslant c_{(\pi\mu)^{-1}(i)} = (\pi\mu)_i \, c_{(\pi\mu)^{-1}(i)} = ((\pi\mu)c)_i$.

Case 2: $|\pi^{-1}(i)| = 1$.

Let $j \in \pi^{-1}(i)$ be the unique element and so $\pi(j) = i$. Note that $(\pi\mu)^{-1}(i) = \mu^{-1}(\pi^{-1}(i)) = \mu^{-1}(j)$. Consider the following subcases.

If $|\mu^{-1}(j)| > 1$, then $a_i \leqslant b_j \leqslant (\mu c)_j = \mu_j c_{\mu^{-1}(j)} = c_{(\pi\mu)^{-1}(i)}$. By induction on $|M|$, $a_i \leqslant c_{(\pi\mu)^{-1}(i)} = (\pi\mu)_i \, c_{(\pi\mu)^{-1}(i)} = ((\pi\mu)c)_i$.

If $|\mu^{-1}(j)| = 1$, then let $k \in \mu^{-1}(j)$ be the unique element and so $\mu(k) = j$. Note that $(\pi\mu)(k) = \pi(\mu(k)) = \pi(j) = i$. Now since $a_i \leqslant b_j \leqslant c_k = ((\pi\mu)c)_i$ then by induction on $|M|$, $a_i \leqslant ((\pi\mu)c)_i$. □

Example 2.11 We will show that $(0(12))3 \leq (012)3$ in terms of our formal definitions.

Let the bracketing $(012)3$ be represented by the surjective order-preserving map $b : M \longrightarrow m_b$ where $M = \{0,1,2,3,4\}$, $m_b = \{0,3\}$. The fibres of b are $b^{-1}(0) = \{0,1,3\}$ and $b^{-1}(3) = \{3\}$.

Let the bracketing $(0(12))3$ be represented by the surjective order-preserving map $a : M \longrightarrow m_a$ where $m_a = \{0,3\}$. The fibres of a are $a^{-1}(0) = \{0,1,2\}$ and $a^{-1}(3) = \{3\}$.

The map $\pi : m_b \longrightarrow m_a$ is the identity. Here $(\pi b)_0$ corresponds to the bracketing (012), and a_0 corresponds to $(0(12)$ so we have $a_0 \leq (\pi b)_0$.

2.2 Higher left bracketing functions

In this section, we will describe a way of generalising the notion of a *left bracketing function* and then investigate its properties. For a natural number n, let $[n]$ be the linearly ordered set $\{0, 1, \ldots, n\}$.

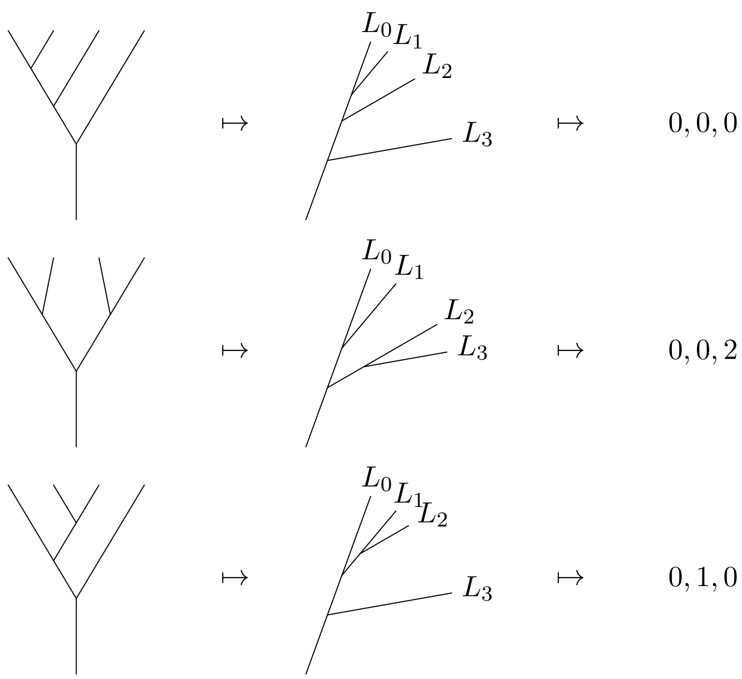
The following definition is due to Huang-Tamari [11].

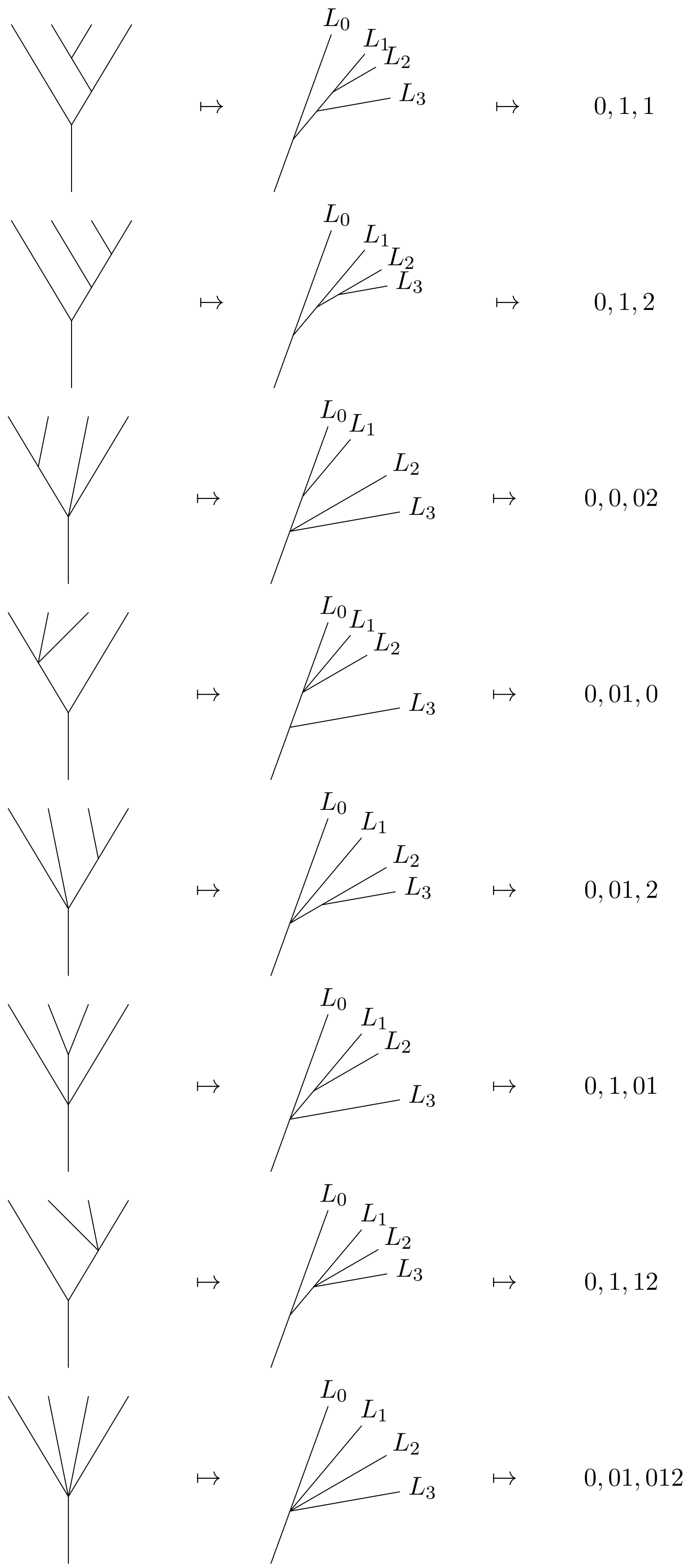
Definition 2.12 A *left bracketing function* (*lbf*) on $[n]$ is a function $\ell : [n] \longrightarrow [n]$ satisfying the following conditions.

(L1) $\ell(i) \leqslant i$ for all $i \in [n]$.

(L2) $\ell(j) \leqslant i \leqslant j \implies \ell(j) \leqslant \ell(i)$ for all $i, j \in [n]$.

We begin by introducing a construction called a *broom* which is due to Verity [27]. A broom is obtained from a rooted tree with $n+2$ leaves in the following way. Given a rooted tree, for each node we straighten the leftmost line when viewed from the root. By convention we draw brooms such that the gradient of each line is strictly decreasing from left to right. We illustrate this construction for all rooted trees with $n+2 = 4$ leaves in the following diagram. Note that the additional labels and the second calculation will be explained in the next paragraph.



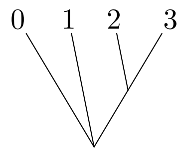


We now describe how to encode Verity's construction combinatorially. For each broom we label the lines from left to right with $L_0, L_1, \ldots, L_n$. In the following definition, we shall view a broom as being embedded in $\mathbb{R}^2$ and make use of its geometric properties. For a given broom we may define a function $x : [n] \longrightarrow \wp[n]$ which is given by $i \mapsto x(i) = \{j \in [n] \mid j \leqslant i, L_j \text{ intersects } L_{i+1}\}$ for each $i \in [n]$. Note that it is always true that $x(0) = \{0\}$. In the diagram above we illustrate these calculations, here we are using a shorthand for representing the function defined above. For example $0, 01, 0$ represents the function given by $0 \mapsto \{0\}$, $1 \mapsto \{0, 1\}$ and $2 \mapsto \{0\}$.

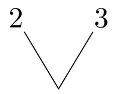
Observe that for the first 5 brooms listed above, our construction give left bracketing functions. In the next section, we will use the above calculations to motivate our generalisation of a left bracketing function.

Alternatively, we may define hlbfs without relying on any geometric properties of trees. Firstly, we begin with a rooted tree T with its $n + 2$ leaves labelled with $0, 1, \ldots, n + 1$ from left to right. For such a tree, we define a function $x : [n] \longrightarrow \wp[n]$ given by $i \mapsto x(i)$ where $x(i)$ is defined as follows. Write T_i for the smallest subtree which contains leaves i and $i + 1$. Write $T_i \backslash R$ for the forest (collection of root trees) obtained when removing the root of T_i. We may write $T_i \backslash R = \sum t_k$ where t_k is a tree of the forest $T_i \backslash R$. Let $\min t_k$ be the smallest leaf of the tree t_k. Let $x(i)$ is the set consisting of all $\min t_k$ with value no greater than i.

For example, consider following tree T,



Here $T_0 = T_1 = T$ and T_2 is the following subtree,



Removing the root of T we obtain the tree T_2 and points $\{0\}$ and $\{1\}$. From this we read off $x(0) = \{0\}$ and $x(1) = \{0, 1\}$. Removing the root of T_2 we obtain points $\{2\}$ and $\{3\}$. From this we read off $x(2) = \{2\}$.

2.2.1 The poset of HLBFs

In this section, we will formalise the notion of higher left bracketing functions as discussed above. We will then describe a partial ordering on the set of all hlbfs on $[n]$.

Definition 2.13 A *higher left bracketing function* (*hlbf*) on $[n]$ is a function $x : [n] \longrightarrow \wp[n]$ satisfying the following conditions.

(H1) $x(i)$ is non-empty for each $i \in [n]$, and so we may define $\ell_x(i) = \min x(i)$.

(H2) The assignation $i \mapsto \ell_x(i)$ defines an lbf $\ell_x : [n] \longrightarrow [n]$.

(H3) If $|x(i)| > 1$, then there exists $j < i$ such that $\ell_x(j) = \ell_x(i)$. Furthermore, if h is the greatest such j, then $x(i) = x(h) \cup \{h + 1\}$.

For each tree displayed in the previous section, a corresponding hlbf was given. Before proceeding we will introduce the following notation. Given a linearly ordered set M and an element $i \in M$, let $M/i = \{k \in M \mid k \leqslant i\}$. For an hlbf x, let $m_x(i) = \max x(i)$ for each $i \in [n]$, and $S_x = \{i \in M \mid |x(i)| > 1\}$.

Proposition 2.14 *If x is an hlbf on $[n]$, then $x(i) \subseteq [n]/i$ for all $i \in [n]$.*

Proof. We will prove this result using induction on i. Note that for $i = 0$ we have $x(0) = \{0\} = [n]/0$. Let $i > 0$ then it follows from (H3) that $x(i) = x(h) \cup \{h + 1\}$ where $h < i$. By the inductive hypothesis, we have $x(h) \subseteq [n]/h \subseteq [n]/i$ and so the result follows. $\square$

We will now define a relation on the set of all hlbfs on $[n]$. Let $x \leqslant y$ whenever $x(i) \subseteq y(i)$ for all $i \in [n]$. This is a partial ordering since the subset relation is a partial ordering on sets. In the remaining parts of this section, we study this partial ordering in terms of the pairs (ℓ, S) as described in Proposition 2.15 below.

We will describe a way of constructing an hlbf on $[n]$ by first constructing an lbf on $[n]$ and then use (H3) to extend the lbf to an hlbf. Before calculating examples, we will formalise this observation. We are able to characterise hlbfs in the following way.

Proposition 2.15 *The assignment* $x \mapsto (\ell_x, S_x)$ *defines a bijection between hlbfs on* $[n]$ *and pairs* (ℓ, S) *consisting of an lbf* $\ell : [n] \longrightarrow [n]$ *and a subset* $S \subseteq [n]$ *with the property that if* $i \in S$*, then there exists an* $h < i$ *such that* $\ell(h) = \ell(i)$*.*

Proof. We have seen that the map $x \mapsto (\ell_x, S_x)$ is well-defined. Condition (H3) shows how, recursively, x may be reconstructed from ℓ_x and S_x. Thus the map is clearly injective. On the other hand, given (ℓ, S), observe that by (H3) $0 \notin S$ so $x(0) = \{0\}$, and now use (H3) to define inductively

$$x(i) = \begin{cases} x(h) \cup \{h+1\} & \text{if } i \in S \text{ and } h = \max\{j < i \mid \ell(j) = \ell(i)\}, \\ \{\ell(i)\} & \text{otherwise.} \end{cases}$$

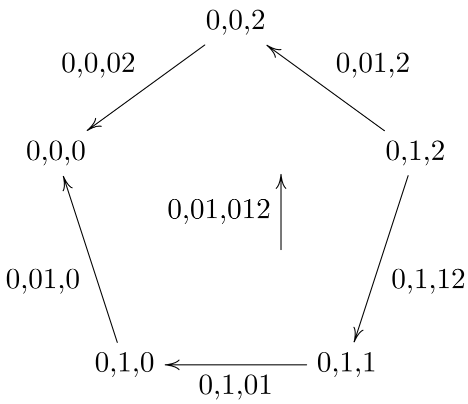
It follows inductively that $\ell_x = \ell$, which is therefore an lbf. Clearly $S_x = S$, and x is an hlbf mapped to (ℓ, S) as required. □

This gives a systematic method of calculating all hlbfs on $[n]$. We first calculate all pairs (ℓ, S) as described in the above result. Then by the formula given in the above proof, we obtain the hlbfs. For example, if we have $\ell = 0, 0, 1, 2, 0$ and $S = \{1, 4\}$ then we obtain an hlbf $0, 01, 1, 2, 012$.

We claim that the set of all hlbfs on $[n]$ with the pointwise inclusion is a model for the n-dimensional associahedron. Furthermore, an hlbf x is a face with dimension equal to $|S_x|$. This will be proven in the next section.

Example 2.16 We now proceed to calculate all hlbfs on [2] and [3], which will then be used to label the faces of the 2-dimensional and 3-dimensional associahedra respectively. At this stage, the directions of the arrows have not been explained; they come from the parity structure defined in Chapter 3.

We start with the 2-dimensional associahedron (pentagon). Note that the hlbfs are exactly those that we calculated earlier with brooms.

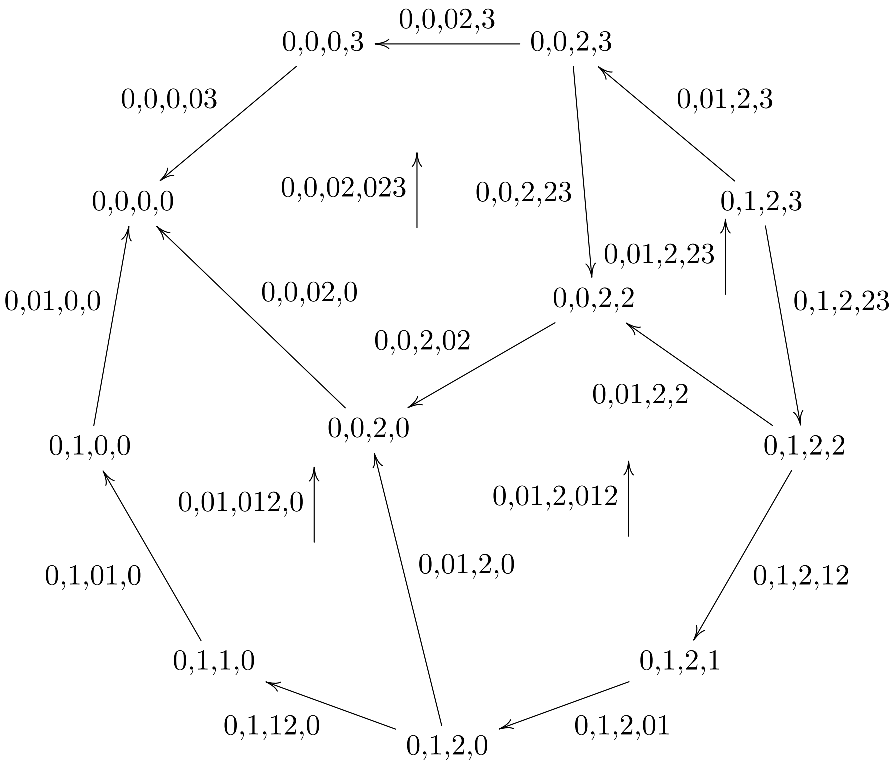


Next we turn to the 3-dimensional associahedron.

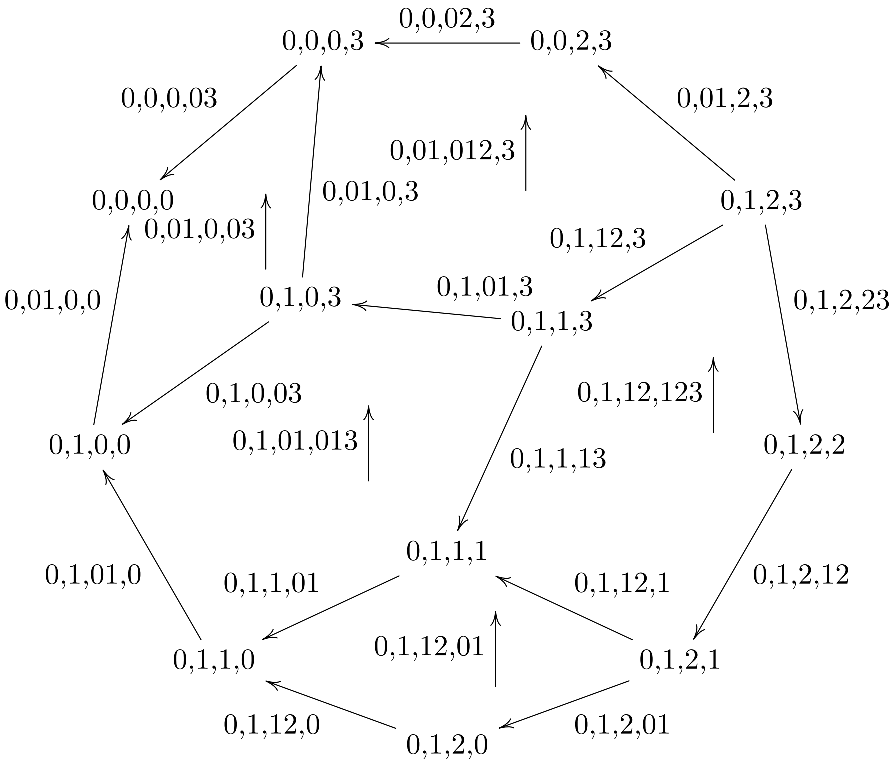
$$A_1 \xleftarrow{0,01,012,0123} A_2$$

where A_1 and A_2 are the diagrams below.

A_1



A_2



2.2.2 Isomorphism with bracketings

Let M be a finite linearly ordered set. Write $M + 1$ for the set obtained from M by adjoining a new maximum element.

Recall that in the previous section we defined an hlbf on $[n]$. Note that M is isomorphic to $[n]$ whenever $|M| = n + 1$. It is often convenient to consider hlbfs defined on M.

In this section, we will show that the poset of hlbfs is a model for the associahedron.

To achieve this, we will prove that the poset of on M hlbfs is isomorphic to the poset of bracketings on $M+1$.

Proposition 2.17 *Let x be an hlbf on M, $m = x(h) \cup \{h+1\}$ where $h = \max\{i \in M \mid \ell_x(i) = \perp_M\}$. Then there exists a right adjoint $b_x : M+1 \longrightarrow m$ to the inclusion $\psi : m \longrightarrow M+1$. Furthermore, b_x is a surjective order-preserving map.*

Proof. Note that $\perp_M \in m \subseteq M+1$. Consider the inclusion $\psi : m \longrightarrow M+1$ which is a injective order-preserving map. Now since $\perp_M \in m$ we have $\psi(\perp_M) = \perp_M = \perp_{M+1}$. Hence there exists a right adjoint $b_x : M+1 \longrightarrow m$. It follows that b_x is necessarily a surjective order-preserving map. □

In the next result, we define sets X_i which form a partition of the set M. The purpose of such a decomposition will be apparent when we give a map from $\mathcal{H}_M$ to $\mathcal{B}_{M+1}$ in Definition 2.19.

Proposition 2.18 *Let x be an hlbf on M. Consider the right adjoint $b_x : M+1 \longrightarrow m$ as in Proposition 2.17. Let $X_i = b_x^{-1}(i)\backslash\{\max(b_x^{-1}(i))\}$ for each $i \in m$. If $X_i \neq \varnothing$, then the function x_i defined by $x_i(k) = x(k)$ for all $k \in X_i$ is an hlbf on X_i.*

Proof. Let $i \in m$ and $k \in X_i$ be given. Note that x_i satisfies axioms (H1), (H2) and (H3) since x is an hlbf. Thus it suffices to show that x_i is a function from X_i to $\wp(X_i)$ or equivalently $x_i(k) \subseteq X_i$. By assumption, $X_i \neq \varnothing$ so we have $\min X_i = \min b_x^{-1}(i) = \psi(i)$. We need to show that $\min x_i(k) \geq \min X_i$ and $\max x_i(k) \leq \max X_i$.

Suppose that $\ell_x(k) = \min x_i(k) < \min X_i = \psi(i)$; we seek to prove a contradiction. Note that since x is an hlbf we must have $\psi(i) > \ell_x(k) \geq \perp_M$. Now $\ell_x(k) \leqslant \psi(i) - 1 < k$ and so $\ell_x(k) \leqslant \ell_x(\psi(i)-1) = \perp_M$ since $\psi(i) = i$ is a non-minimal element of m. It follows that $\ell_x(k) = \perp_M$.

Note that $k+1 \in b_x^{-1}(i)$ so $k+1 \neq \min b_x^{-1}(j)$ for any $j \neq i$. Also either $k+1 = \max X_i \neq \min X_i$ or $k+1 \in X_i$ so $k+1 \neq \min X_i$. Thus $k+1 \notin m$.

By definition of h, since $\ell_x(k) = \perp_M$ it follows that $k \leqslant h$. If $k = h$, then $k+1 = h+1 \in m$, which is a contradiction. If $k < h$, then consider $x(h) = \{\perp\} \cup \{i'+1 \leqslant j+1 < h+1 \mid \ell_x(j) = \perp\}$ where $i'+1 \in x(h)$ is the least non-minimal element. Note that $i'+1 \leqslant i = \min X_i < k+1 < h+1$ and so $k+1 \in x(h) \subseteq m$, which is a contradiction.

Finally, we have $\max x_i(k) = m_x(k) \leqslant k \leqslant \max X_i$. Hence x_i is an hlbf on X_i. □

Where M is a finite linearly ordered set, we will use the notation $\mathcal{H}_M$ for the poset of all hlbfs on M, and $\mathcal{B}_M$ for the poset of all bracketings on M.

Definition 2.19 We define recursively maps $\mathcal{H}_M \xrightarrow{\phi} \mathcal{B}_{M+1}$ for all finite linearly ordered sets M.

(1) If $|M| = 1$, then $\mathcal{H}_M$ and $\mathcal{B}_{M+1}$ are singletons; so there is a unique map $\mathcal{H}_M \xrightarrow{\phi} \mathcal{B}_{M+1}$ where $\phi(x) = 1_M$ where x is the unique hlbf in $\mathcal{H}_M$ and 1_{M+1} is the unique trivial bracketing in $\mathcal{B}_{M+1}$.

(2) If $|M| > 1$, then for any hlbf $x \in \mathcal{H}_M$, let $\phi(x) = b_x$ together with bracketings $(b_x)_i$ for each $i \in m$ defined as follows. If $|X_i| = \varnothing$, then let $(b_x)_i$ be the trivial bracketing. Otherwise $|X_i| \neq \varnothing$, then by recursion let $(b_x)_i = \phi(x_i)$.

Theorem 2.20 $\mathcal{H}_M \xrightarrow{\phi} \mathcal{B}_{M+1}$ *is a bijection.*

Proof. In this proof, we will use induction on $|M|$ to prove that ϕ is a bijection. Note that for the case of $|M| = 1$, ϕ is a bijection follows immediately from its definition. Let $|M| > 1$; we will now prove the inductive step.

We will first prove injectivity: if $\phi(x) = \phi(y)$ then $x = y$.

Given $x, y \in \mathcal{H}_M$ then we have $\phi(x) = b_x : M+1 \longrightarrow m_x$, $\phi(y) = b_y : M+1 \longrightarrow m_y$. If $\phi(x) = \phi(y)$, then $m_x = m_y$ and $b_x = b_y = b$.

We will now consider the sets X_i and Y_i as given in Proposition 2.18 for the bracketings b_x and b_y respectively. Note that $X_i = b_x^{-1}(i)\backslash\{\max b_x^{-1}(i)\} = b_y^{-1}(i)\backslash\{\max b_y^{-1}(i)\} = Y_i$. Since $(b_x)_i = (b_y)_i$ we have $\phi(x_i) = \phi(y_i)$. By induction we have $x_i = y_i$ and so $x(k) = y(k)$ for all $k \in X_i = Y_i$.

Now since $m_x = m_y$ we have $h_x + 1 = h_y + 1$ where h_x and h_y are the h's as calculated in ϕ. It follows that $h_x = h_y = h$ which implies that $x(h) = y(h)$. Note that b preserves the maximum element so $\min b^{-1}(h+1) = \max(M+1) \notin M$.

All that remains is to show that $x(k) = y(k)$ for each $k = \max b^{-1}(i)$ with $i \in x(h)$. We will now prove this by induction on the elements of $x(h)$. Note that $\ell_x(\max b^{-1}(\perp_M)) = \ell_x(\psi(\perp_M + 1) - 1) = \perp_M$, and similarly $\ell_y(\max b^{-1}(\perp_M)) = \perp_M$. Now since it has been shown that $x(k) = y(k)$ for all $k < \max b^{-1}(\perp_M)$, it follows from the definition of an hlbf that $x(\max b^{-1}(\perp_M)) = y(\max b^{-1}(\perp_M))$.

Now for the inductive step consider the following. Note that $k = \max b^{-1}(i) = \min b^{-1}(i+1) - 1 = \psi(i+1) - 1$. Let $i \neq \perp_M$ be given. Note that we have $\ell_x(\psi(i) - 1) = \perp_M$. Suppose that $\psi(i) - 1 < j < \psi(i+1) - 1$; then $j \in X_i$ and so $\ell_x(j) \geq \min X_i > \perp_M$. It follows that $\max\{j < \psi(i+1) - 1 \mid \ell_x(j) = \ell_x(\psi(i+1) - 1)\} = \psi(i) - 1$ and so $x(\psi(i+1) - 1) = x(\psi(i) - 1) \cup \{\psi(i)\}$. Similarly we have $y(\psi(i+1) - 1) = y(\psi(i) - 1) \cup \{\psi(i)\}$. Now if $x(\psi(i) - 1) = y(\psi(i) - 1)$, then $x(\psi(i+1) - 1) = y(\psi(i+1) - 1)$ for $i \neq \perp_M$ as required. Hence we have shown that $x = y$.

We will now prove surjectivity: for each bracketing $b \in \mathcal{B}_{M+1}$ there exists an hlbf $x \in \mathcal{H}_M$ such that $\phi(x) = b$.

Let $b \in B_{M+1}$ be given; this consists of $M + 1 \xrightarrow{b} m$ and $b_i \in B_{b^{-1}(i)}$ for each $i \in m$. Consider $X_i = b^{-1}(i)\backslash\{\max b^{-1}(i)\}$, then by induction there exists $x_i \in H_{X_i}$ such that $\phi(x_i) = b_i$. Let $x : M \longrightarrow \wp M$ be given by

$$x(k) = \begin{cases} x_i(k) & k \in X_i \\ \psi\{j \in m \mid j \leqslant i\} & k = \max b^{-1}(i). \end{cases}$$

We will now show that x is an hlbf as defined by conditions (H1), (H2), (H3). Firstly we will show that condition (H1) holds. If $k \in X_i$, then $x(k) = x_i(k)$ which is non-empty since x_i is an hlbf. If $k = \max b^{-1}(i)$, then $x(k) = \psi\{j \in m \mid j \leqslant i\}$ is non-empty since $i \in \psi\{j \in m \mid j \leqslant i\}$. Hence (H1) holds and so we can compute $\ell_x(k) = \min x(k)$ as follows,

$$\ell_x(k) = \begin{cases} \ell_{x_i}(k) & k \in X_i \\ \perp & k = \max b^{-1}(i). \end{cases}$$

We now show that condition (H2) holds. If $k \in X_i$ and $\ell_x(k) \leqslant j \leqslant k$, then $\ell_{x_i}(k) \leqslant j \leqslant k$ and so $\ell_{x_i}(k) \leqslant \ell_{x_i}(j)$ since ℓ_{x_i} is an lbf. Now since $j \in X_i$ we have $\ell_x(j) = \ell_{x_i}(j)$ and so it follows that $\ell_x(k) \leqslant \ell_x(j)$. If on the other hand $k = \max b^{-1}(i)$, then $\ell_x(k) = \perp \leqslant \ell_x(j)$. Hence we have shown that $\ell_x(k) \leqslant j \leqslant k \implies \ell_x(k) \leqslant \ell_x(j)$ as required.

Finally, we now show that condition (H3) holds. If $k \in X_i$ then $x(k) = x_i(k) = x(m_{x_i}(k) - 1) \cup \{m_{x_i}(k)\}$ since x_i is an hlbf. Note that $m_x(k) = m_{x_i}(k)$ and $m_{x_i}(k) - 1 \in X_i$, and so $x(k) = x(m_x(k) - 1) \cup \{m_x(k)\}$. If $k = \max b^{-1}(i)$, then $x(k) = \psi\{j \in m \mid j \leqslant i\} = \psi\{j \in m \mid j \leqslant i - 1\} \cup \{\psi(i)\}$. Now since $m_x(k) - 1 = \psi(i) - 1 = \min b^{-1}(i) - 1 = \max b^{-1}(i-1)$, we have $x(m_x(k) - 1) = \psi\{j \in m \mid j \leqslant i - 1\}$ It follows that $x(k) = x(m_x(k) - 1) \cup \{m_x(k)\}$ as required. Hence $M \xrightarrow{x} \wp M$ is an hlbf.

All that remains is to show that $\phi(x) = b$. Note that $h = \max\{i \in M \mid \ell_x(i) = \perp_M\} = \max b^{-1}(\max(m\backslash\{\max m\}))$. It follows from the definition of x that $x(h) \cup \{h+1\} = m$ and so $\phi(x) = b$ as required. □

Theorem 2.21 $\mathcal{H}_M \xrightarrow{\phi} \mathcal{B}_{M+1}$ *is an isomorphism of posets.*

Proof. We have shown in Theorem 2.20 that ϕ is a bijection of the sets $\mathcal{H}_M$ and $\mathcal{B}_{M+1}$. To show that ϕ is an isomorphism of the posets $\mathcal{H}_M$ and $\mathcal{B}_{M+1}$ all that remains to be proven is $x \leqslant y$ iff $\phi(x) \leqslant \phi(y)$. We will prove this using induction on $|M|$. Note for the case of $|M| = 1$, $\mathcal{H}_M$ and $\mathcal{B}_{M+1}$ are singletons so the condition to check is empty. Let $|M| > 1$; we will now prove the inductive step.

We will first prove that $x \leqslant y$ implies $\phi(x) \leqslant \phi(y)$.

Let $x, y \in H_M$ be given and consider $\phi(x) = b_x : M+1 \longrightarrow m_x$, $\phi(y) = b_y : M+1 \longrightarrow m_y$. Let $h_x = \max\{i \in M \mid \ell_x(i) = \perp_M\}$ and $h_y = \max\{i \in M \mid \ell_y(i) = \perp_M\}$. We will firstly show that $m_x \subseteq m_y$. Note that $\ell_y(i) = \perp_M$ implies that $\ell_x(i) = \perp_M$ and so we have $h_x \leqslant h_y$. If $h_x = h_y$, then $m_x = x(h_x) \cup \{h_x + 1\} \subseteq y(h_y) \cup \{h_y + 1\} = m_y$ as required. Otherwise $h_x < h_y$, then suppose that $h_y \notin S_y$. Now $\ell_y(h_y) = \ell_x(h_y) \neq \perp$, which is a contradiction. Consider $y(h_y) = \{\perp\} \cup \{j+1 \leqslant k+1 < h_y + 1 \mid \ell_y(k) = \perp\}$ where $j+1 \in y(h_y)$ least non-minimal element. Since $j \notin S_y$, $\ell_x(j) = \ell_y(j) = \perp$ so by definition $j \leqslant h_x$. It follows that $h_x + 1 \in y(h_y)$ non-minimal. Hence $m_x = x(h_x) \cup \{h_x + 1\} \subseteq y(h_y) \subseteq m_y$ as required.

By adjointness, there is a map $m_y \xrightarrow{\pi} m_x$ such that $\pi b_y(k) = b_x(k)$. Consider the function y' defined as follows.

$$y'(k) = \begin{cases} y_j(k) & k \in Y_j \\ \psi_y\{h \in \pi^{-1}(i) \mid h \leqslant j\} & k = \max b_y^{-1}(j) \end{cases}$$

Note that $\phi(y') = (\pi b_y)_i$. Let $k \in X_i$ be given. If $k \in Y_j$, then $x_i(k) = x(k) \subseteq y(k) = y_j(k) = y'(k)$. If $k = \max b_y^{-1}(j)$, then note that $x_i(k) = x(k) \subseteq y(k) = \psi_y\{h \in m_y \mid h \leqslant j\}$. However $y'(k) = \psi_y\{h \in \pi^{-1}(i) \mid h \leqslant j\} = \psi_y\{h \in m_y \mid i \leqslant h \leqslant j\}$ and so $\ell_{x_i}(k) \geq \min X_i = \psi_x(i) = i = \psi_y(i) = \ell_{y'}(k)$. It follows that $x_i(k) \subseteq y'(k)$. By induction, we have $x_i \leqslant y'$ so $\phi(x_i) = (b_x)_i \leqslant (\pi b_y)_i = \phi(y')$ as required.

We will now prove that $\phi(x) \leqslant \phi(y)$ implies $x \leqslant y$. Let $x, y \in H_M$ be given and consider $\phi(x) = b_x : M+1 \longrightarrow m_x$, $\phi(y) = b_y : M+1 \longrightarrow m_y$ with $b_x \leqslant b_y$. Now there exists $m_y \xrightarrow{\pi} m_x$ such that $(b_x)_i \leqslant (\pi b_y)_i$. Note that $\phi(x_i) = (b_x)_i \leqslant (\pi b_y)_i$ and so by induction, $x_i \leqslant y'$ where

$$y'(k) = \begin{cases} y_j(k) & k \in Y_j \\ \psi_y\{h \in \pi^{-1}(i) \mid h \leqslant j\} & k = \max b_y^{-1}(j) \end{cases}$$

Let $k \in X_i$ be given and consider the subcases. (i) $k \in Y_j$ then $x(k) = x_i(k) \subseteq y'(k) = y_j(k) = y(k)$. (ii) $k = \max b_y^{-1}(j)$ then $x(k) = x_i(k) \subseteq y'(k) = \psi_y\{h \in \pi^{-1}(i) \mid h \leqslant j\} \subseteq \psi_y\{h \in m_y \mid h \leqslant j\} = y(k)$.

Let $k = \max b_x^{-1}(i)$; then $x(k) = \psi_x\{h \in m_x \mid h \leqslant i\}$. Note that $\max b_x^{-1}(i) = \max b_y^{-1}(\max \pi^{-1}(i))$ so $y(k) = \psi_y\{h \in m_y \mid h \leqslant \max \pi^{-1}(i)\}$. For $h \in x(k)$ we have $h \in m_x \subseteq m_y$ and also $h \leqslant i = \min \pi^{-1}(i) \leqslant \max \pi^{-1}(i)$. Hence $x(k) \subseteq y(k)$. □

2.2.3 HLBFs $x \leq y$ with $|S_y \backslash S_x| = 1$

In the previous section, we have shown that the poset of hlbfs on $[n]$ is a model for the n-dimensional associahedron. We will now be working towards constructing a parity structure on the poset of hlbfs. The remaining parts of this chapter involve studying the properties of the partial ordering on hlbfs as defined in Section 2.2.1.

The formulation of hlbfs given in Definition 2.13 allows us to make very precise statements. However these results are highly technical, and furthermore such technicalities may be avoided by using other models for the associahedron such as bracketings. Nevertheless, for our goal of constructing parity, these technicalities ultimately allow us, in Chapter 3, to construct a parity structure on the poset hlbfs.

In this section, we will analyse hlbfs $x \leq y$ with $|S_y \backslash S_x| = 1$. We are able to make very precise statements in Theorem 2.27 and Theorem 2.30. In subsequent sections, with the help of further results, we will be able to prove that the condition $|S_y \backslash S_x| = 1$ characterises the covering relation of $\leq$.

It is convenient to study the pointwise subset partial ordering of hlbfs in conjunction with the characterisation given in Proposition 2.15. The following results provide a useful starting point.

Proposition 2.22 *If x, y are hlbfs with $x \leqslant y$, then $\ell_y \leqslant \ell_x$ and $S_x \subseteq S_y$.*

Proof. Since $x(i) \subseteq y(i)$ then we have $\ell_y(i) = \min y(i) \leqslant \min x(i) = \ell_x(i)$ for all $i \in [n]$. Now given $i \in S_x$ then it follows that $|y(i)| \geqslant |x(i)| > 1$ so $i \in S_y$. Hence we have shown that $S_x \subseteq S_y$ and $\ell_y \leq \ell_x$ as required. □

Lemma 2.23 *Let x, y be hlbfs with $x \leqslant y$. If k is minimal with the property that $\ell_y(k) < \ell_x(k)$, then $k \in S_y \backslash S_x$.*

Proof. By Proposition 2.22, $\ell_y(i) \leqslant \ell_x(i)$ for all i. Suppose $k \in S_x$; then there exists a non-minimal $i+1 \in x(k)$, which is also non-minimal in $y(k)$ since $\ell_y(k) \leqslant \ell_x(k) < i+1$. Now by Proposition 2.14, $i < i+1 \leqslant k$ so by the minimality of k, $\ell_y(k) = \ell_y(i) = \ell_x(i) = \ell_x(k)$, which is a contradiction. Thus $k \notin S_x$.

Now suppose $k \notin S_y$; then, by Proposition 2.22, $k \notin S_x$ and so $x(k)$, $y(k)$ are both singletons. Since $x(k) \subseteq y(k)$ it follows that $x(k) = y(k)$ and so $\ell_x(k) = \ell_y(k)$, which is a contradiction. Thus $k \in S_y$ so we have shown that $k \in S_y \backslash S_x$ as required. □

Corollary 2.24 *Let x, y be hlbfs with $x \leq y$. If $S_x = S_y$, then $x = y$.*

Proof. Suppose there exists a minimal $i \in [n]$ such that $x(i) \neq y(i)$. Note that by minimality of i, $x(k) = y(k)$ for all $k < i$. By Proposition 2.22, $\ell_y(i) \leq \ell_x(i)$. If $\ell_y(i) = \ell_x(i)$, then by (H3), it follows that $x(i) = y(i)$, which is a contradiction. If $\ell_y(i) < \ell_x(i)$, then by Lemma 2.23, $i \in S_y \backslash S_x = \varnothing$, which is a contradiction. Hence we have shown that $x = y$. □

Recall the notation $<_1$ described in Definition 1.25. We write $x <_1 y$ whenever $x \leqslant y$, $x \neq y$, and if there exists z such that $x \leqslant z \leqslant y$ it follows that $z = x$ or $z = y$. One of the results that we are interested in proving is the following. Let x, y be hlbfs with $x \leq y$, then $x <_1 y$ iff $|S_y \backslash S_x| = 1$. We provide a proof for this in Theorem 2.39.

This section will be devoted to the investigation of hlbfs x, y with $x <_1 y$. We discovered that the most convenient approach is to begin with an hlbf x and characterise hlbfs y such that $x <_1 y$. There are two cases, $\ell_y = \ell_x$ and $\ell_y < \ell_x$, which appear in Theorem 2.27 and Theorem 2.30 respectively.

The following result gives a relationship between ℓ_x and m_x. This leads to Lemma 2.26 which is a highly useful result that will appear in many calculations within this chapter.

Proposition 2.25 *Let x be an hlbf. Then we have the following formula*

$$m_x(k) = \begin{cases} 1 + \max\{j < k \mid \ell_x(j) = \ell_x(k)\} & \text{if } k \in S_x, \\ \ell_x(k) & \text{otherwise.} \end{cases}$$

Proof. The formula is clearly true for the case of $k \notin S_x$. Let $k \in S_x$; then by (H3) we have $x(k) = x(h) \cup \{h+1\}$ where $h = \max\{j < k \mid \ell_x(j) = \ell_x(k)\}$. Thus $m_x(k) = h+1 = 1 + \max\{j < k \mid \ell_x(j) = \ell_x(k)\}$ as required. □

Lemma 2.26 *Let x be an hlbf and $i+1 \in x(k)$ be a non-minimal element. Then*

(a) $x(i) = x(k)/i$,

(b) $x(k) = x(i) \cup \{h+1 \mid i+1 \leqslant h+1 < k+1,\, \ell_x(h) = \ell_x(i)\}$,

(c) $\ell_x(i) = \ell_x(k)$.

Proof. Firstly, we will prove (a). Let k be minimal with the property that $x(k)$ has a non-minimal element $i+1 \in x(k)$ and $x(i) \neq x(k)/i$. Since $x(k)$ has a non-minimal element it follows that $|x(k)| > 1$, and so $x(k) = x(m_x(k)-1) \cup \{m_x(k)\}$. If $i+1 = m_x(k)$ then $x(i) = x(m_x(k)-1) = x(k)/(m_x(k)-1) = x(k)/i$, where the second last equality holds by (H3). Otherwise, $i+1 \in x(m_x(k)-1)$ which is also non-minimal since $\ell_x(k) = \ell_x(m_x(k)-1)$. By minimality of k we have $x(i) = x(m_x(k)-1)/i = x(k)/i$. This proves (a), and (c) is an immediate consequence.

We will now prove (b) holds. Let $E_{i,k} = \{h+1 \mid i+1 \leqslant h+1 < k+1, \ell_x(h) = \ell_x(i)\}$. If $h+1 \in x(k)$ and $h+1 \notin x(i)$ then certainly $h+1$ is non-minimal in $x(k)$. Note that $\ell_x(h) = \ell_x(k) = \ell_x(i)$ and so $h+1 \in E_{i,k}$. Thus $x(k) \subseteq x(i) \cup E_{i,k}$.

Note that $x(i) = x(k)/i \subseteq x(k)$ so all that remains is to show that $E_{i,k} \subseteq x(k)$. We will prove this using induction on the cardinality of $x(k)\backslash x(i)$. This number is at least one, since the set contains $i+1$. Also $i+1 \leqslant i+1 < k+1$ and $\ell_x(i) = \ell_x(k)$, so $i+1 \in E_{i,k}$. For the inductive step, suppose that $x(k)\backslash x(i)$ contains more than one element. If $i+1 = m_x(k)$ then the result follows since $E_{m_x(k),k} = \{m_x(k)\} \subseteq x(k)$. Otherwise let $h+1$ be the least such element strictly greater than $i+1$. Since $\ell_x(h) = \ell_x(k)$, we have $E_{i,k} = E_{i,h} \cup E_{h,k}$. By (a), $x(i) \subseteq x(h) \subseteq x(k)$, and in fact each of these subsets is strict: the first because of the element $i+1$, and the second because of $h+1$. Thus by inductive hypothesis, we know that $E_{i,h} \subseteq x(h)$ and so $E_{h,k} \subseteq x(k)$ as required. □

Theorem 2.27 *If x is an hlbf and $j \notin S_x$, then there is at most one hlbf y with $x \leqslant y$, $S_y = S_x \cup \{j\}$ and $\ell_x(j) = \ell_y(j)$. There is such a y iff there exists an $h < j$ with $\ell_x(h) = \ell_x(j)$. Furthermore, in this case we have $\ell_y = \ell_x$ and*

$$m_y(k) = \begin{cases} 1 + \max\{h < j \mid \ell_x(h)) = \ell_x(j)\} & \textit{if } k = j, \\ m_x(k) & \textit{otherwise.} \end{cases}$$

$$y(k) = \begin{cases} x(m_y(j)-1) \cup \{m_y(j)\} & \textit{if } k = j, \\ x(k) \cup y(j) & \textit{if } \{\ell_x(j), j+1\} \subseteq x(k), \\ x(k) & \textit{otherwise.} \end{cases}$$

Proof. We will first prove the uniqueness part of our result. Note that if $\ell_x \neq \ell_y$ then $\ell_y(k) < \ell_x(k)$ for some k, and the minimal such could only be j. Thus if $\ell_x(j) = \ell_y(j)$ then $\ell_x = \ell_y$. The uniqueness of y is now immediate from Proposition 2.15.

Now we will prove the existence part of our result. Suppose there exists an hlbf y with $x \leqslant y$, $S_y = S_x \cup \{j\}$, and $\ell_x(j) = \ell_y(j)$. Since $j \in S_y$ there exists $h < j$ such that $\ell_x(h) = \ell_y(h) = \ell_y(j) = \ell_x(j)$ so the condition on j holds. Thus all that remains to be shown is the converse.

Suppose the condition on j holds, and define m_y and y as in the above formulas. It follows that $\ell_x(m_y(j)-1) = \ell_x(j)$ by definition of $m_y(j)$, and so $x(j) = \{\ell_x(j)\} \subseteq y(j)$ and $\ell_y(j) = \ell_x(j)$. It is clear from the above formula that $\ell_x(k) = \ell_y(k)$ and $x(k) \subseteq y(k)$

for all $k \neq j$, and that $S_y = S_x \cup \{j\}$. Thus all that remains is to show is that y defines an hlbf.

Note that $x(k) \subseteq y(k)$ so $y(k)$ is non-empty, and $\ell_y = \ell_x$ is an lbf. It suffices to check condition (H3) in the definition of an hlbf. Let $k \in S_y = S_x \cup \{j\}$ and consider the following cases.

Case 1: $k = j$.
The result follows from the definition of $y(j)$.

Case 2: $k \neq j$ and $\{\ell_x(j), j+1\} \nsubseteq x(k)$.
Suppose that $m_x(k) = j+1$; then by Lemma 2.26(c), $\ell_x(j) = \ell_x(k)$ and so $\{\ell_x(j), j+1\} \subseteq x(k)$, which is a contradiction. Thus $m_x(k) - 1 \neq j$ and so $\{\ell_x(j), j+1\} \nsubseteq x(m_x(k)-1)$. Now note that $k \neq j$ so we have $k \in S_x$ and so it follows that

$$\begin{aligned} y(k) = x(k) &= x(m_x(k)-1) \cup \{m_x(k)\} \\ &= y(m_x(k)-1) \cup \{m_x(k)\} = y(m_y(k)-1) \cup \{m_y(k)\}. \end{aligned}$$

Case 3: $k \neq j$ and $\{\ell_x(j), j+1\} \subseteq x(k)$.
Note that by Lemma 2.26(c), $\ell_x(k) = \ell_x(j)$ and so $k \in S_x$. It follows from the fact that $j+1 \in x(k)$ that $m_x(k) \geqslant j+1$. If $m_x(k) > j+1$, then since $x(k) = x(m_x(k)-1) \cup \{m_x(k)\}$ we also have $\{\ell_x(j), j+1\} \subseteq x(m_x(k)-1)$, and now

$$\begin{aligned} y(k) = x(k) \cup y(j) &= x(m_x(k)-1) \cup \{m_x(k)\} \cup y(j) \\ &= y(m_x(k)-1) \cup \{m_x(k)\} = y(m_y(k)-1) \cup \{m_y(k)\}. \end{aligned}$$

Otherwise $m_x(k) = j+1$, then $x(k) = x(m_x(k)-1) \cup \{m_x(k)\} = \{\ell_x(j), j+1\}$ since $j \notin S_x$ and so $x(j) = \{\ell_x(j)\}$. It follows that

$$\begin{aligned} y(k) = x(k) \cup y(j) &= \{\ell_x(j), j+1\} \cup y(j) = y(j) \cup \{j+1\} \\ &= y(m_x(k)-1) \cup \{m_x(k)\} = y(m_y(k)-1) \cup \{m_y(k)\}. \end{aligned}$$ □

Proposition 2.28 *Let x, y be hlbfs with $x \leqslant y$, $S_y = S_x \cup \{j\}$ and $\ell_y(j) < \ell_x(j)$. The following formula holds*

$$\ell_y(k) = \begin{cases} \ell_x(\ell_x(j)-1) & \text{if } k = j \text{ or } \{\ell_x(j), j+1\} \subseteq x(k), \\ \ell_x(k) & \text{otherwise.} \end{cases}$$

Furthermore, if $k > j$ and $\ell_x(k) = \ell_x(j)$, then $j+1 \in x(k)$.

Proof. The formula for $\ell_y(k)$ is true for $k < j$ by Lemma 2.23, and is obviously true if $k \notin S_x \cup \{j\}$. For $k = j$ we have $\ell_x(j) \in y(j)$ is non-minimal, and so $\ell_y(j) = \ell_y(\ell_x(j)-1)$ but this is $\ell_x(\ell_x(j)-1)$ since $\ell_x(j)-1 < j$. If $\{\ell_x(j), j+1\} \subseteq x(k)$ we have $j+1 \in x(k)$ is non-minimal and so also non-minimal in $y(k)$, and so $\ell_y(k) = \ell_y(j) = \ell_x(\ell_x(j)-1)$. For all remaining cases we have $k \in S_x$, $k > j$, and $\{\ell_x(j), j+1\} \nsubseteq x(k)$. If $\ell_x(k) \neq \ell_y(k)$ for some such k, then consider the minimal such k for which $\ell_x(k) \neq \ell_y(k)$. Since $k \in S_x$ there exists a non-minimal $i+1 \in x(k)$; then $x(i) \subseteq x(k)$ by Lemma 2.26, and $\{\ell_x(j), j+1\} \nsubseteq x(i)$. If $i \neq j$ then $\ell_x(k) = \ell_x(i) = \ell_y(i) = \ell_y(k)$. But if $i = j$ then $j+1$ is non-minimal in $x(k)$ and so also $\ell_x(j) \in x(k)$, giving a contradiction. This proves the formula for ℓ_y.

Suppose that $k > j$ and $\ell_x(k) = \ell_x(j)$. Then $\ell_y(k) \leqslant \ell_x(k) = \ell_x(j) \leqslant j < k$ and so $\ell_y(k) \leqslant \ell_y(j) = \ell_x(\ell_x(j)-1)$. We now have $\ell_y(k) \leqslant \ell_x(\ell_x(j)-1) < \ell_x(j) = \ell_x(k)$ and $k \neq j$; this is possible only if $\{\ell_x(j), j+1\} \subseteq x(k)$; in particular, $j+1 \in x(k)$. □

Proposition 2.29 *Let x, y be hlbfs with $j \notin S_x$. Suppose that $j+1 \in x(k)$ whenever $k > j$ and $\ell_x(k) = \ell_x(j)$. Then the formula in Proposition 2.28 defines an lbf ℓ_y.*

Proof. We will first show that $\ell_y(k) \leqslant \ell_x(k)$ for all k. Note that by definition of ℓ_y it suffices to prove this for k such that $k = j$ or $\{\ell_x(j), j+1\} \subseteq x(k)$. For the case of $k = j$, $\ell_y(j) = \ell_x(\ell_x(j) - 1) \leq \ell_x(j) - 1 \leq \ell_x(j)$. For the case of $\{\ell_x(j), j+1\} \subseteq x(k)$, by Lemma 2.26(c), we have $\ell_y(k) = \ell_x(\ell_x(j) - 1) \leq \ell_x(j) - 1 \leq \ell_x(j) = \ell_x(k)$. Hence we have shown that $\ell_y(k) \leqslant \ell_x(k) \leq k$ for all k.

All that remains is to prove that if $\ell_y(k) \leqslant i < k$, then $\ell_y(k) \leqslant \ell_y(i)$. Suppose that $\ell_y(k) \leqslant i < k$ and consider the following cases.

Case 1: $k = j$ or $\{\ell_x(j), j+1\} \subseteq x(k)$.

It will suffice to prove that $\ell_x(\ell_x(j) - 1) \leqslant \ell_x(i)$. Consider the following subcases. If $i < \ell_x(j)$, then $\ell_x(\ell_x(j) - 1) \leqslant i \leqslant \ell_x(j) - 1$ and so $\ell_x(\ell_x(j) - 1) \leqslant \ell_x(i)$. If $\ell_x(j) \leqslant i \leqslant j$, then $\ell_x(j) \leqslant \ell_x(i)$ and now $\ell_x(\ell_x(j) - 1) \leqslant \ell_x(j) - 1 \leqslant \ell_x(i)$. Finally if $i > j$, then clearly $k = j$ is impossible so we must have $\{\ell_x(j), j+1\} \subseteq x(k)$. By Lemma 2.26(c), $\ell_x(k) = \ell_x(j) \leqslant j < i < k$ and so $\ell_y(k) \leqslant \ell_x(k) \leqslant \ell_x(i)$ as required.

Case 2: $k \neq j$ and $\{\ell_x(j), j+1\} \nsubseteq x(k)$.

Note that $\ell_x(k) = \ell_y(k) \leqslant i < k$ and so $\ell_y(k) = \ell_x(k) \leqslant \ell_x(i)$. Consider the case of $i \neq j$ and $\{\ell_x(j), j+1\} \nsubseteq x(i)$. Then $\ell_y(k) = \ell_x(k) \leq \ell_(i) = \ell_y(i)$ as required.

Now consider the case of $i = j$ or $\{\ell_x(j), j+1\} \subseteq x(i)$. If $j+1 \in x(i)$, then $j+1 \leq i$ and so $j \leq i$; by Lemma 2.26(c), $\ell_x(j) = \ell_x(i)$. Thus in either case, $j > i$ and $\ell_x(j) = \ell_x(i)$.

Suppose that $\ell_x(k) = \ell_x(i)$; we aim to show a contradiction. Note that $j \leq i < k$ and $\ell_x(k) = \ell_x(i) = \ell_x(j)$ so by the condition stated in the proposition, $j+1 \in x(k)$. By Lemma 2.26(c), $\ell_x(j) = \ell_x(k)$ and so it follows that $\{\ell_x(j), j+1\} \subseteq x(k)$, which is a contradiction. Thus we have shown that $\ell_x(k) \neq \ell_x(i)$.

Finally, we will now show that $\ell_y(k) \leq \ell_y(i)$. Since $\ell_x(k) \leqslant \ell_x(i) - 1 = \ell_x(j) - 1 < j \leq i < k$ it follows that $\ell_x(k) \leqslant \ell_x(\ell_x(j) - 1) = \ell_y(i)$. Hence $\ell_y(k) \leqslant \ell_x(k) \leqslant \ell_y(i)$ as required. $\square$

Theorem 2.30 *If x is an hlbf and $j \notin S_x$, then there is at most one hlbf y with $x \leqslant y$, $S_y = S_x \cup \{j\}$ and $\ell_y(j) < \ell_x(j)$. There is such a y iff, whenever $\ell_x(k) = \ell_x(j)$ and $k > j$, we have $j+1 \in x(k)$. Furthermore, in this case we have*

$$y(k) = \begin{cases} x(j) \cup x(\ell_x(j) - 1) & k = j, \\ x(k) \cup y(j) & k \in T, \\ x(k) \cup \{j+1\} \cup \{h+1 \mid h < k,\ h \in T\} & \ell_x(j) \textit{ is non-minimal in } x(k), \\ x(k) & \textit{otherwise} \end{cases}$$

where $T = \{h \mid \{\ell_x(j), j+1\} \subseteq x(h)\}$.

Proof. We will first prove the uniqueness part of our result. Let y be an hlbf with $x \leq y$ and $S_y = S_x \cup \{j\}$. By Proposition 2.28, we have a formula for ℓ_y. It follows that y must be unique by the bijection described in Proposition 2.15.

Now we will prove the existence part of our result. Suppose there exists an hlbf y with $x \leqslant y$, $S_y = S_x \cup \{j\}$ and $\ell_y(j) < \ell_x(j)$. By Proposition 2.28, we have the condition on j as required. All that remains to be shown is the converse.

Suppose the condition on j holds and let y be given by the above formula. We will now prove that $\ell_y(k) = \min y(k)$ is given by the formula in Proposition 2.28. This is clear in all but the case when $\ell_x(j)$ is non-minimal in $x(k)$. Note that then $\ell_x(k) < \ell_x(j) \leqslant j < j+1$ so it follows that $\min y(k) = \min x(k) = \ell_x(k) = \ell_y(k)$ since neither $k = j$ nor $\{\ell_x(j), j+1\} \subseteq x(k)$. We have verified that ℓ_y is given as in Proposition 2.28.

It is clear from the formula that $x(k) \subseteq y(k)$ for all k, and also that $S_y = S_x \cup \{j\}$. We also verify from the formula in Proposition 2.28 that $\ell_y(j) = \ell_x(\ell_x(j) - 1) < \ell_x(j)$ so all that remains is to show is that y defines an hlbf.

Note that $x(k) \subseteq y(k)$ so $y(k)$ is non-empty, while ℓ_y is an lbf by Proposition 2.29, thus it suffices to check that y satisfies condition (H3). Let $m_y(k) = \max y(k)$ for each $k \in S_y$. To verify condition (H3), it suffices to show that $y(k) = y(m_y(k)-1) \cup \{m_y(k)\}$ for each $k \in S_y$.

Let $k \in S_y = S_x \cup \{j\}$ be given. In the following, we consider the four cases as given in the above formula of y.

Case 1: $k = j$.

Note that $m_y(j) = m_x(j) = \ell_x(j)$ and so

$$\begin{aligned} y(j) = x(\ell_x(j)-1) \cup x(j) &= y(\ell_x(j)-1) \cup \{\ell_x(j)\} \\ &= y(m_y(j)-1) \cup \{m_y(j)\}. \end{aligned}$$

Case 2: $k \neq j$ (still with $k \in S_x$).

Then $m_y(k) = m_x(k)$; call this $m+1$. Since $m+1$ is non-minimal in $x(k)$ we have $x(m) \subseteq x(k)$. If $m = j$, then $j+1 = m+1$ is non-minimal in $x(k)$ and so $\ell_x(k) = \ell_x(j)$ which would imply that $k \in T$, taking us outside of this case.

Now $k \notin T$ so $m \notin T$; also $\ell_x(j)$ is not a non-minimal element of $x(k)$ so is not a non-minimal element of $x(m)$ either. Thus $y(m) = x(m)$, and so

$$y(k) = x(k) = x(m) \cup \{m+1\} = y(m) \cup \{m+1\}.$$

Case 3: $k \in T$ (in other words $\{\ell_x(j), j+1\} \subseteq x(k)$).

Note that $m_y(k) = m_x(k)$. Now since $j+1 \in x(k)$ we have $m_x(k) \geqslant j+1$. Consider the case of $m_x(k) = j+1$. Note that $j \notin S_x$ so we have $x(k) = x(j) \cup \{j+1\} = \{\ell_x(j), j+1\}$. It follows that

$$\begin{aligned} y(k) = y(j) \cup x(k) &= y(j) \cup \{\ell_x(j), j+1\} \\ &= y(j) \cup \{j+1\} = y(m_y(k)-1) \cup \{m_y(k)\}. \end{aligned}$$

Now consider the case of $m_x(k) > j+1$. Recall that $x(k) = x(m_x(k)-1) \cup \{m_x(k)\}$ and $\{\ell_x(j), j+1\} \subseteq x(k)$. Note that $j+1 \neq m_x(k)$ so we must have $\{\ell_x(j), j+1\} \subseteq x(m_x(k)-1)$. It follows that

$$\begin{aligned} y(k) = x(k) \cup y(j) &= x(m_x(k)-1) \cup \{m_x(k)\} \cup y(j) \\ &= y(m_x(k)-1) \cup \{m_x(k)\} = y(m_y(k)-1) \cup \{m_y(k)\}. \end{aligned}$$

Case 4: $\ell_x(j)$ non-minimal in $x(k)$ ($k \neq j$ and $\{\ell_x(j), j+1\} \not\subseteq x(k)$).

Let $m = m_y(k)-1$; then $m+1 \geqslant j+1 > \ell_x(j)$. Note that $m+1 \in y(k) = x(k) \cup \{j+1\} \cup \{h+1 \mid h < k, h \in T\}$. Consider the following subcases.

(a) $m+1 \in x(k)$.

Then $m_y(k) = m+1 = m_x(k)$ and so $x(k) = x(m) \cup \{m+1\}$. Now since $\ell_x(j) > \ell_x(k) = \ell_x(m)$ we have $\ell_x(j)$ non-minimal in $x(m)$, and so

$$\begin{aligned} y(k) &= x(k) \cup \{j+1\} \cup \{h+1 \mid h < k, h \in T\} \\ &= x(k) \cup \{j+1\} \cup \{h+1 \mid h < m, h \in T\} \\ &= x(m) \cup \{j+1\} \cup \{h+1 \mid h < m, h \in T\} \cup \{m+1\} \\ &= y(m) \cup \{m+1\} \end{aligned}$$

with $\{h+1 \mid h < k, h \in T\} = \{h+1 \mid h < m, h \in T\}$ since $m_y(k) = m_x(k)$.

(b) $m+1=j+1$.

$$\begin{aligned} y(m)\cup\{m+1\} &= y(j)\cup\{j+1\} \\ &= x(j)\cup x(\ell_x(j)-1)\cup\{j+1\} \\ &= \{\ell_x(j), j+1\}\cup x(\ell_x(j)-1). \end{aligned}$$

Now $\ell_x(j)$ is non-minimal in $x(k)$ so $x(\ell_x(j)-1)\subseteq x(k)\subseteq y(k)$. We also have that $\ell_x(j)\in x(k)\subseteq y(k)$, and $j+1\in y(k)$, and so $y(m)\cup\{m+1\}\subseteq y(k)$.

Now consider the reverse inclusion. If $h<k$ and $h\in T$, then $h+1>j+1$, contradicting the assumption that $m+1$ is maximal. Also $j+1=m+1\in y(m)\cup\{m+1\}$, so it will suffice to show that $x(k)\subseteq y(m)\cup\{m+1\}$. Since $\ell_x(j)$ is non-minimal in $x(k)$, we have $x(k)/(\ell_x(j)-1)=x(\ell_x(j)-1)\subseteq y(j)=y(m)$; also $\ell_x(j)=\ell_x(m)\in x(m)\subseteq y(m)$. Thus it will suffice to consider $h\in x(k)$ with $h>\ell_x(j)$. Since $j+1=m+1>m_x(k)$ we have $\ell_x(j)<h\leqslant j$ and so $\ell_x(j)\leqslant \ell_x(h)=\ell_x(k)=\ell_x(\ell_x(j)-1)<\ell_x(j)$, which is a contradiction.

(c) $m\in T$.

Note that $\{\ell_x(j), j+1\}\subseteq x(m)$ and so $\ell_x(m)=\ell_x(j)>\ell_x(k)$. It follows that $\ell_x(m)$ is non-minimal in $x(k)$, and so

$$\begin{aligned} y(m)\cup\{m+1\} &= x(m)\cup y(j)\cup\{m+1\} \\ &= x(m)\cup\{\ell_x(j)\}\cup x(\ell_x(j)-1)\cup\{m+1\} \\ &= x(m)\cup x(\ell_x(j)-1)\cup\{m+1\} \\ &= x(j)\cup\{h+1\mid j+1\leqslant h+1<m+1, \ell_x(h)=\ell_x(j)\} \\ &\quad \cup x(\ell_x(j)-1)\cup\{m+1\} \end{aligned}$$

Now $\ell_x(j)$ is non-minimal in $x(k)$ so $x(\ell_x(j)-1)\subseteq x(k)\subseteq y(k)$, and certainly $x(j)=\{\ell_x(j)\}\subseteq x(k)\subseteq y(k)$ and $j+1, m+1\in y(k)$. Suppose that $j+1<h+1<m+1$ and $\ell_x(h)=\ell_x(j)$; then by the condition on j we have $j+1\in x(h)$ and so $h\in T$ and $h+1\in y(k)$. Hence we have proven that $y(m)\cup\{m+1\}\subseteq y(k)$.

It remains to prove the reverse inclusion $y(k)\subseteq y(m)\cup\{m+1\}$. If $h\in T$ and $h+1<m+1$ then $j+1<h+1<m+1$ and $\ell_x(h)=\ell_x(j)$, and so $h+1\in x(m)\subseteq y(m)$. Also $j+1\in x(m)\subseteq y(m)$ so it will suffice to show that $x(k)\subseteq y(m)\cup\{m+1\}$. Now $\ell_x(j)$ is non-minimal in $x(k)$, so $x(k)/(\ell_x(j)-1)=x(\ell_x(j)-1)\subseteq y(j)\subseteq y(m)$. Also $\ell_x(j)\in x(m)\subseteq y(m)$.

Finally if $h+1\in x(k)$ and $h+1>\ell_x(j)$ we must show that $h+1\in y(m)\cup\{m+1\}$. We may as well suppose that $h+1<m+1$. If $h\leqslant j$ then $\ell_x(j)\leqslant h\leqslant j$ and so $\ell_x(j)\leqslant \ell_x(h)=\ell_x(k)<\ell_x(j)$, which is a contradiction. Thus $h>j$ and so $j+1<h+1<m+1$ and $\ell_x(m)=\ell_x(j)$. It follows that $\ell_x(m)=\ell_x(j)\leqslant h<m$ and $\ell_x(j)=\ell_x(m)\leqslant\ell_x(h)=\ell_x(k)<\ell_x(j)$, giving a contradiction once again. □

2.2.4 HLBFs $x\leq y$ with $|S_y\backslash S_x|=2$

As discussed in the previous section, we are investigating the pointwise subset partial order on hlbfs in terms of the characterisation of hlbfs by pairs (ℓ, S) given in Proposition 2.15. In the previous section, we were able to give an explicit relationship between hlbfs x, y with $x\leqslant y$ and $|S_y\backslash S_x|=1$.

In this section, we will investigate hlbfs x, y with $x\leqslant y$ and $|S_y\backslash S_x|=2$. As alluded to in the previous section, such results arising from this investigation will allow us to characterise the covering relation of $\leq$. This is motivated by our interest in proving that the poset of hlbfs is a graded poset as defined in Definition 1.25.

Although we are not able to determine formulas as in the previous section, we are able to determine a formula for ℓ_y in terms of ℓ_x. We begin by proving the following lemma.

Lemma 2.31 *Let x, y be hlbfs with $x \leqslant y$. If $\ell_y(k) < \ell_x(k)$ and $k \notin S_y \backslash S_x$, then there exists $j \in S_y \backslash S_x$ with $\ell_y(j) < \ell_x(j)$ and $\{\ell_x(j), j+1\} \subseteq x(k)$.*

Proof. Let k with $\ell_y(k) < \ell_x(k)$ and $k \notin S_y \backslash S_x$ be given. Suppose that $k \notin S_y$; then $k \notin S_x$ and so it follows that $\ell_y(k) = \ell_x(k)$, which is a contradiction. Thus $k \in S_y$ and so since $k \notin S_y \backslash S_x$ we must have $k \in S_x$.

Let $j+1 \in x(k)$ be a non-minimal element; then it is also non-minimal in $y(k)$, and so $\ell_y(j) = \ell_y(k) < \ell_x(k) = \ell_x(j)$. Note that $\ell_x(j) = \ell_x(k) \in x(k)$ so we have $\{\ell_x(j), j+1\} \subseteq x(k)$ (since $j+1 \in x(k)$). Now since $\ell_y(j) < \ell_x(j)$ we must have $j \in S_y$. If $j \notin S_x$ we are done; otherwise repeat the above argument. □

Proposition 2.32 *Let x, y be hlbfs with $x \leqslant y$, $S_y = S_x \cup \{j_1, j_2\}$, $\ell_y(j_1) < \ell_x(j_1)$ and $\ell_y(j_2) < \ell_x(j_2)$ where $j_1 < j_2$.*

If $\ell_x(j_2) - 1 \neq j_1$ and $\{\ell_x(j_1), j_1+1\} \nsubseteq x(\ell_x(j_2)-1)$, then we have the following formula

$$\ell_y(k) = \begin{cases} \ell_x(\ell_x(j_1) - 1) & k = j_1 \text{ or } \{\ell_x(j_1), j_1+1\} \subseteq x(k), \\ \ell_x(\ell_x(j_2) - 1) & k = j_2 \text{ or } \{\ell_x(j_2), j_2+1\} \subseteq x(k), \\ \ell_x(k) & \text{otherwise.} \end{cases}$$

If $\ell_x(j_2) - 1 = j_1$ or $\{\ell_x(j_1), j_1+1\} \subseteq x(\ell_x(j_2) - 1)$, then we have the following formula

$$\ell_y(k) = \begin{cases} \ell_x(\ell_x(j_1) - 1) & k = j_1 \text{ or } \{\ell_x(j_1), j_1+1\} \subseteq x(k) \\ & k = j_2 \text{ or } \{\ell_x(j_2), j_2+1\} \subseteq x(k), \\ \ell_x(k) & \text{otherwise.} \end{cases}$$

Furthermore, if $k > j_2$ and $\ell_x(k) = \ell_x(j_2)$, then $j_2+1 \in x(k)$. Suppose that $\ell_x(j_1) \neq \ell_x(j_2)$. If $k > j_1$ and $\ell_x(k) = \ell_x(j_1)$, then $j_1+1 \in x(k)$.

Proof. By Lemma 2.23, $\ell_y(k) = \ell_x(k)$ for all $k < j_1$.

We will first prove the above formulas hold. For the case of $k = j_1$, note that $\ell_y(j_1) < \ell_x(j_1)$ and $\ell_x(j_1) \in x(j_1) \subseteq y(j_1)$. It follows that $\ell_x(j_1)$ is a non-minimal element of $y(j_1)$ and so $\ell_y(j_1) = \ell_y(\ell_x(j_1) - 1) = \ell_x(\ell_x(j_1) - 1)$ since $\ell_x(j_1) - 1 < j_1$.

For the case of $\{\ell_x(j_1), j_1+1\} \subseteq x(k)$, note that $j_1 + 1 \in x(k) \subseteq y(k)$. It follows that $j_1 + 1$ is a non-minimal element of $y(x)$ and so $\ell_y(k) = \ell_y(j_1) = \ell_x(\ell_x(j_1) - 1)$.

For the case of $k = j_2$, note that $\ell_y(j_2) < \ell_x(j_2)$ and $\ell_x(j_2) \in x(j_2) \subseteq y(j_2)$. It follows that $\ell_x(j_2)$ is a non-minimal element of $y(j_2)$ and so $\ell_y(j_2) = \ell_y(\ell_x(j_2) - 1)$.

For the case of $\{\ell_x(j_2), j_2+1\} \subseteq x(k)$, note that $j_2 + 1 \in x(k) \subseteq y(k)$. It follows that $j_2 + 1$ is a non-minimal element of $y(k)$ and so $\ell_y(k) = \ell_y(j_2)$.

For the otherwise case, we have $k \notin S_y \backslash S_x = \{j_1, j_2\}$, $\{\ell_x(j_1), j_1+1\} \nsubseteq x(k)$, $\{\ell_x(j_2), j_2+1\} \nsubseteq x(k)$. By Lemma 2.31, we have $\ell_y(k) = \ell_x(k)$.

All that remains is to consider the cases $k = j_2$ and $\{\ell_x(j_2), j_2+1\} \subseteq x(k)$. Note that there are two formulas each will be considered in the following cases.

Case 1: $\ell_x(j_2) - 1 \neq j_1$ and $\{\ell_x(j_1), j_1+1\} \nsubseteq x(\ell_x(j_2) - 1)$.

For the case of $k = j_2$, note that $\ell_x(j_2) - 1 \neq j_1$ and $\ell_x(j_2) - 1 < j_2$. It follows that $\ell_x(j_2) - 1 \notin S_y \backslash S_x = \{j_1, j_2\}$. By Lemma 2.31, we have $\ell_y(j_2) = \ell_y(\ell_x(j_2) - 1) = \ell_x(\ell_x(j_2) - 1)$.

For the case of $\{\ell_x(j_2), j_2+1\} \subseteq x(k)$ we have determined above that $\ell_y(k) = \ell_y(j_2) = \ell_x(\ell_x(j_2) - 1)$.

Case 2: $\ell_x(j_2) - 1 = j_1$ or $\{\ell_x(j_1), j_1+1\} \subseteq x(\ell_x(j_2) - 1)$.

We will first show that $\ell_y(j_1) = \ell_y(\ell_x(j_2) - 1)$. Note that when $\ell_x(j_2) - 1 = j_1$ this follows immediately. Consider when $\{\ell_x(j_1), j_1+1\} \subseteq x(\ell_x(j_2) - 1)$. Note that $j_1 + 1$ is a non-minimal element of $x(\ell_x(j_2)-1)$ so we have $\ell_x(\ell_x(j_2)-1)=\ell_x(j_1)\leq j_1$. Now note that $\ell_y(\ell_x(j_2)-1)\leq \ell_x(\ell_x(j_2)-1)<j_1+1$ and $j_1+1\in x(\ell_x(j_2)-1)\subseteq y(\ell_x(j_2)-1)$. It follows that j_1+1 is a non-minimal element of $y(\ell_x(j_2)-1)$ and so $\ell_y(j_1)=\ell_y(\ell_x(j_2)-1)$.

For the case of $k=j_2$, it follows that $\ell_y(j_2)=\ell_y(\ell_x(j_2)-1)=\ell_y(j_1)=\ell_x(\ell_x(j_1)-1)$.

For the case of $\{\ell_x(j_2),j_2+1\}\subseteq x(k)$ we have determined above that $\ell_y(k)=\ell_y(j_2)=\ell_x(\ell_x(j_1)-1)$.

Hence we have shown that the above formulas hold.

If $k>j_2$ and $\ell_x(k)=\ell_x(j_2)$, then since $\ell_y(k)\leqslant \ell_x(k)=\ell_x(j_2)\leqslant j_2<k$ it follows that $\ell_y(k)\leqslant \ell_y(j_2)<\ell_x(j_2)=\ell_x(k)$. By the above formula, either $k\in S_y\backslash S_x$, $\{\ell_x(j_1),j_1+1\}\subseteq x(k)$ or $\{\ell_x(j_2),j_2+1\}\subseteq x(k)$. Note that $k\notin S_y\backslash S_x$ since $k>j_2$. If $\{\ell_x(j_2),j_2+1\}\subseteq x(k)$, then $j_2+1\in x(k)$. Otherwise $\{\ell_x(j_1),j_1+1\}\subseteq x(k)$, then $j_1+1\in x(k)$ is non-minimal, and so $x(k)=x(j_1)\cup\{h+1\mid j_1+1\leqslant h+1<k+1,\ell_x(h)=\ell_x(j_1)\}$. Now $\ell_x(j_1)=\ell_x(k)=\ell_x(j_2)$ which implies $j_2+1\in x(k)$.

Suppose that $\ell_x(j_1)\neq\ell_x(j_2)$. If $k>j_1$ and $\ell_x(k)=\ell_x(j_1)$, then since $\ell_y(k)\leqslant\ell_x(k)=\ell_x(j_1)\leqslant j_1<k$ it follows that $\ell_y(k)\leqslant\ell_y(j_1)<\ell_x(j_1)=\ell_x(k)$. By the above formula, either $k\in S_y\backslash S_x$, $\{\ell_x(j_1),j_1+1\}\subseteq x(k)$ or $\{\ell_x(j_2),j_2+1\}\subseteq x(k)$. If $k\in S_y\backslash S_x$, then it must be that $k=j_1$, and so $\ell_x(j_1)=\ell_x(j_2)$, which is a contradiction. If $\{\ell_x(j_2),j_2+1\}\subseteq x(k)$, then $j_2+1\in x(k)$ is non-minimal, and so $\ell_x(k)=\ell_x(j_2)\neq\ell_x(j_1)$, which is a contradiction. All that remains is $\{\ell_x(j_1),j_1+1\}\subseteq x(k)$ which implies $j_1+1\in x(k)$. $\square$

Proposition 2.33 *Let x,y be hlbfs with $x\leqslant y$, $S_y=S_x\cup\{j_1,j_2\}$, $\ell_y(j_1)=\ell_x(j_1)$ and $\ell_y(j_2)<\ell_x(j_2)$ where $j_1<j_2$. The we have the following formula.*

$$\ell_y(k)=\begin{cases}\ell_x(\ell_x(j_2)-1) & k=j_2 \text{ or } \{\ell_x(j_2),j_2+1\}\subseteq x(k),\\ \ell_x(k) & \text{otherwise.}\end{cases}$$

Furthermore, if $k>j_2$ and $\ell_x(k)=\ell_x(j_2)$, then $j_2+1\in x(k)$. Also there exists $h<j_1$ with $\ell_x(h)=\ell_x(j_1)$.

Proof. By Lemma 2.23, $\ell_y(k)=\ell_x(k)$ for all $k<j_2$. Note that $\ell_y(j_2)<\ell_x(j_2)$, and so $\ell_x(j_2)\in x(j_2)\subseteq y(j_2)$ is non-minimal. It follows that $\ell_y(j_2)=\ell_y(\ell_x(j_2)-1)$ which equals $\ell_x(\ell_x(j_2)-1)$ since $\ell_x(j_2)-1<j_2$. If $\{\ell_x(j_2),j_2+1\}\subseteq x(k)$, then $j_2+1\in x(k)\subseteq y(k)$ is non-minimal, and so $\ell_y(k)=\ell_y(j_2)=\ell_x(\ell_x(j_2)-1)$.

If $k\neq j_2$ and $\{\ell_x(j_2),j_2+1\}\nsubseteq x(k)$, then by Lemma 2.31, $\ell_y(k)=\ell_x(k)$. We have now obtained the above formula.

If $k>j_2$ and $\ell_x(k)=\ell_x(j_2)$, then since $\ell_y(k)\leqslant\ell_x(k)=\ell_x(j_2)\leqslant j_2<k$ it follows that $\ell_y(k)\leqslant\ell_y(j_2)<\ell_x(j_2)=\ell_x(k)$. By the above formula, either $k=j_2$ or $\{\ell_x(j_2),j_2+1\}\subseteq x(k)$. Note that $k\neq j_2$ since $k>j_2$. All that remains is $\{\ell_x(j_2),j_2+1\}\subseteq x(k)$ which implies $j_2+1\in x(k)$.

Note that $j_1\in S_y$ so there exists $h<j_1$ with $\ell_y(h)=\ell_y(j_1)$. By the above formula, $\ell_y(h)=\ell_x(h)$. Hence there exists $h<j_1$ with $\ell_x(h)=\ell_x(j_1)$. $\square$

Proposition 2.34 *Let x,y be hlbfs with $x\leqslant y$, $S_y=S_x\cup\{j_1,j_2\}$, $\ell_y(j_1)<\ell_x(j_1)$ and $\ell_y(j_2)=\ell_x(j_2)$ where $j_1<j_2$. The we have the following formula.*

$$\ell_y(k)=\begin{cases}\ell_x(\ell_x(j_1)-1) & k=j_1 \text{ or } \{\ell_x(j_1),j_1+1\}\subseteq x(k),\\ \ell_x(k) & \text{otherwise.}\end{cases}$$

Furthermore, if $k>j_1$ and $\ell_x(k)=\ell_x(j_1)$, then $j_1+1\in x(k)$. Also there exists $h<j_2$ with $\ell_x(h)=\ell_x(j_2)$.

Proof. The above result follows using the same method as with Proposition 2.33. $\square$

2.2.5 Properties of the poset of HLBFs

An abstract pre-polytope is a poset which satisfies some axioms which are specified in Definition 1.33 of Chapter 1. In this section, we will first prove that the poset of hlbfs is an abstract pre-polytope. Recall that our aim is to construct a parity structure on the poset of hlbfs. The final results of this section are used to motivate a notion called a label structure which will be defined in Chapter 3. In Chapter 3, we will demonstrate how to construct a parity structure from a label structure.

We will first work towards proving Theorem 2.39 which implies axiom (P2) for an abstract pre-polytope. To prove this, we will make use of the decomposition result given in Theorem 2.38 which requires the following results.

Lemma 2.35 *Let x, y, z be hlbfs with $x \leqslant y$, $S_y = S_x \cup \{j\}$ and $\ell_y = \ell_x$. Then $y \leqslant z$ iff the following conditions hold: $x \leqslant z$, $j \in S_z$ and $m_y(j) \in z(j)$.*

Proof. ($\Longrightarrow$) If $y \leqslant z$, then by transitivity $x \leqslant y \leqslant z$ implies $x \leqslant z$. Since $S_y \subseteq S_z$, $j \in S_y$ implies $j \in S_z$. Note that $y(j) \subseteq z(j)$, and so $m_y(j) \in y(j) \subseteq z(j)$.

($\Longleftarrow$) Suppose that $x \leqslant z$, $j \in S_z$ and $m_y(j) \in z(j)$. Note that $\ell_z(j) \leqslant \ell_x(j) = \ell_y(j) < m_y(j)$. Hence $m_y(j)$ is a non-minimal element of $z(j)$, and so $z(m_y(j) - 1) \subseteq z(j)$. It follows that $y(j) = x(m_y(j) - 1) \cup \{m_y(j)\} \subseteq z(m_y(j) - 1) \cup \{m_y(j)\} \subseteq z(j)$.

Now all that remains is to show that $y(k) \subseteq z(k)$ for all $k \neq j$. If $\{\ell_x(j), j + 1\} \subseteq x(k)$, then $j + 1 \in x(k) \subseteq z(k)$ is non-minimal, and so $z(j) \subseteq z(k)$. It follows that $y(k) = x(k) \cup y(j) \subseteq z(k) \cup z(j) = z(k)$.

Finally, for the remaining case, we have $y(k) = x(k) \subseteq z(k)$. Hence $y \leqslant z$. □

Lemma 2.36 *Let x, y, z be hlbfs with $x \leqslant y$, $S_y = S_x \cup \{j\}$ and $\ell_y(j) < \ell_x(j)$. Then $y \leqslant z$ iff the following conditions hold: $x \leqslant z$, $j \in S_z$ and $\ell_z(j) < \ell_x(j)$.*

Proof. ($\Longrightarrow$) If $y \leqslant z$, then by transitivity $x \leqslant y \leqslant z$ implies $x \leqslant z$. Since $S_y \subseteq S_z$, $j \in S_y$ implies $j \in S_z$. Note that $\ell_z(j) \leqslant \ell_y(j) < \ell_x(j)$.

($\Longleftarrow$) Suppose that $x \leqslant z$, $j \in S_z$ and $\ell_z(j) < \ell_x(j)$. Note that $\ell_z(j) < \ell_x(j)$ and $\ell_x(j) \in x(j) \subseteq z(j)$ implies that $\ell_x(j)$ is non-minimal in $z(j)$, and so $z(\ell_x(j) - 1) \subseteq z(j)$. It follows that $y(j) = x(j) \cup x(\ell_x(j) - 1) \subseteq z(j) \cup z(\ell_x(j) - 1) = z(j)$.

Now all that remains is to show that $y(k) \subseteq z(k)$ for all $k \neq j$. If $\{\ell_x(j), j + 1\} \subseteq x(k)$, then $j + 1 \in x(k) \subseteq z(k)$ is non-minimal, and so $z(j) \subseteq z(k)$. It follows that $y(k) = x(k) \cup y(j) \subseteq z(k) \cup z(j) = z(k)$.

If $\ell_x(j) \in x(k) \subseteq z(k)$ is non-minimal, then

$$z(k) = z(\ell_x(j) - 1) \cup \{h + 1 \mid \ell_x(j) \leqslant h + 1 < k + 1,\, \ell_z(h) = \ell_z(\ell_x(j) - 1)\}.$$

Suppose that $j \geq k$; then since $\ell_x(j) \leqslant k \leqslant j$, it follows that $\ell_x(j) \leqslant \ell_x(k)$, which is a contradiction. Thus $\ell_x(j) \leqslant j < k$ and $\ell_z(j) = \ell_z(\ell_x(j) - 1)$, which implies $j + 1 \in z(k)$ is non-minimal. Now we have $z(k) = z(j) \cup \{h + 1 \mid j + 1 \leqslant h + 1 < k + 1,\, \ell_z(h) = \ell_z(j)\}$. If $\{\ell_x(j), j + 1\} \subseteq x(h)$, then $j + 1 \leqslant h$. Since $j + 1 \in x(h)$ is non-minimal, we have $\ell_x(j) = \ell_x(h)$, and so $h + 1 \in z(k)$. It follows that $y(k) = x(k) \cup \{j + 1\} \cup \{h + 1 \mid h < k,\, h \in T\} \subseteq z(k) \cup \{j + 1\} \cup \{h + 1 \mid h < k,\, h \in T\} = z(k)$.

Finally, for the remaining case, we have $y(k) = x(k) \subseteq z(k)$. Hence $y \leqslant z$. □

Proposition 2.37 *Let x, z be hlbfs with $x \leq z$, $j \in S_z \backslash S_x$ such that $\ell_z(j) < \ell_x(j)$. Then there exists $k \in S_z \backslash S_x$ with $k \leqslant j$ such that $\ell_z(j) = \ell_x(\ell_x(k) - 1)$.*

Proof. This proof will be by induction on $j \in S_z \backslash S_x$. Note that $\ell_x(j) \in x(j) \subseteq z(j)$ is non-minimal, and so $\ell_z(j) = \ell_z(\ell_x(j) - 1)$. If $j = \min(S_z \backslash S_x)$, then by Lemma 2.23, $\ell_z(j) = \ell_z(\ell_x(j) - 1) = \ell_x(\ell_x(j) - 1)$. Let $k = j$ then the result follows. For the inductive step, consider the following cases.

Case 1: $\ell_x(j) - 1 \notin S_z \backslash S_x$

If $\{\ell_x(i), i+1\} \nsubseteq x(\ell_x(j)-1)$ for all $i \in S_z\backslash S_x$ with $\ell_z(i) < \ell_x(i)$, then by Lemma 2.31, $\ell_z(\ell_x(j)-1) = \ell_x(\ell_x(j)-1)$. It follows that $\ell_z(j) = \ell_x(\ell_x(j)-1)$. Let $k = j$ then the result follows immediately.

Otherwise, there exists $i \in S_z\backslash S_x$ such that $\ell_z(i) < \ell_x(i)$ and $\{\ell_x(i), i+1\} \subseteq x(\ell_x(j)-1)$. Note that $i+1 \in z(\ell_x(j)-1)$ is non-minimal, and so $\ell_z(i) = \ell_z(\ell_x(j)-1)$. It follows that $\ell_z(j) = \ell_z(\ell_x(j)-1) = \ell_z(i)$. Now since $i < \ell_(j)-1 < j$ then by induction on j, there exists $k \in S_z\backslash S_x$ such that $k \leq i$ and $\ell_z(i) = \ell_x(\ell_x(k)-1)$. Hence $\ell_z(j) = \ell_z(i) = \ell_x(\ell_x(k)-1)$ where $k \in S_z\backslash S_x$ and $k \leq i < j$ as required.

Case 2: $\ell_x(j) - 1 \in S_z\backslash S_x$.

If $\ell_z(\ell_x(j)-1) = \ell_x(\ell_x(j)-1)$, then let $k = j$ and the result follows immediately. Otherwise $\ell_z(\ell_x(j)-1) < \ell_x(\ell_x(j)-1)$, then let $i = \ell_x(j)-1 < j$. By induction on j, there exists $k \in S_z\backslash S_x$ such that $k \leq i$ and $\ell_z(i) = \ell_x(\ell_x(i)-1)$. Hence $\ell_z(j) = \ell_z(\ell_x(j)-1) = \ell_z(i) = \ell_x(\ell_x(k)-1)$ where $k \in S_z\backslash S_x$ and $k \leq i < j$ as required. □

Theorem 2.38 *Let x, z be hlbfs with $x \leqslant z$. If $|S_z\backslash S_x| > 1$, then there exists an hlbf y such that $x \leqslant y \leqslant z$ and $y \neq x, z$.*

Proof. Let $j = \max(S_z\backslash S_x)$. We will consider two cases in this proof, $\ell_z(j) < \ell_x(j)$ and $\ell_z(j) = \ell_x(j)$. Firstly, consider the case of $\ell_z(j) < \ell_x(j)$. Let $k > j$ such that $\ell_x(k) = \ell_x(j)$ be given; we seek to prove that $j+1 \in x(k)$. Since $\ell_z(k) \leqslant \ell_x(k) = \ell_x(j) \leqslant j < k$ it follows that $\ell_z(k) \leqslant \ell_z(j) < \ell_x(j) = \ell_x(k)$. If $k \in S_z\backslash S_x$, then $k \leqslant j$, which is a contradiction. Thus $k \notin S_z\backslash S_x$ and so by Lemma 2.31, there exists $i \in S_z\backslash S_x$ such that $\{\ell_x(i), i+1\} \subseteq x(k)$. Note that $i \in S_z\backslash S_x$ so we have $i \leqslant j$. If $i = j$, then the result follows immediately. Otherwise $i < j$, then since $i+1 \in x(k)$ is non-minimal we have

$$x(k) = x(i) \cup \{h+1 \mid i+1 \leqslant h+1 < k+1,\ \ell_x(h) = \ell_x(i)\}.$$

Note that $\ell_x(j) = \ell_x(k) = \ell_x(i)$ and $i < j < k$, and so $j+1 \in x(k)$ as required.

By Theorem 2.30, there exists an hlbf y with $x \leqslant y$, $S_y = S_x \cup \{j\}$ and $\ell_y(j) < \ell_x(j)$. Hence by Lemma 2.36, $y \leqslant z$.

We will now consider the case of $\ell_z(j) = \ell_x(j)$. We seek to prove that there exists $h < j$ such that $\ell_x(h) = \ell_x(j)$. Note that $j \in S_z$ so there exists $h' < j$ such that $\ell_z(h') = \ell_z(j) = \ell_x(j)$. Consider the following cases.

Case 1: $h' \notin S_z\backslash S_x$.

If $\{\ell_x(i), i+1\} \nsubseteq x(h')$ for all $i \in S_z\backslash S_x$ with $\ell_z(i) < \ell_x(i)$, then by Lemma 2.31, $\ell_z(h') = \ell_x(h')$. Thus $\ell_x(h') = \ell_z(h') = \ell_z(j) = \ell_x(j)$ and we may take $h = h'$.

Otherwise there exists $i \in S_z\backslash S_x$ such that $\ell_z(i) < \ell_x(i)$ and $\{\ell_x(i), i+1\} \subseteq x(h')$ and so $i+1 \in z(h')$ is non-minimal. It follows that $\ell_z(h') = \ell_z(i)$. By Proposition 2.37, there exists $i' \in S_z\backslash S_x$ with $i' \leqslant i$ such that $\ell_z(i) = \ell_x(\ell_x(i')-1)$. Let $h = \ell_x(i')-1 < i' \leqslant i < h' < j$, then $\ell_x(h) = \ell_z(i) = \ell_z(h') = \ell_z(j) = \ell_x(j)$.

Case 2: $h' \in S_z\backslash S_x$.

If $\ell_z(h') = \ell_x(h')$, then let $h = h'$ and the result follows immediately. Otherwise $\ell_z(h') < \ell_x(h')$, then by Proposition 2.37, there exists $h'' \in S_z\backslash S_x$ such that $h'' \leqslant h'$ and $\ell_z(h') = \ell_x(\ell_x(h'')-1)$. Let $h = \ell_x(h'')-1 < h'' \leqslant h' < j$, then $\ell_x(h) = \ell_z(h') = \ell_z(j) = \ell_x(j)$.

Hence in either case there exists $h < j$ such that $\ell_x(h) = \ell_x(j)$. By Theorem 2.27, there exists an hlbf y with $x \leqslant y$, $S_y = S_x \cup \{j\}$ and $\ell_y = \ell_x$.

We will now show that $m_y(j) \in z(j)$ which is required in order to use Lemma 2.35 to prove that $y \leq z$. Let $i+1 \in z(j)$ be the least non-minimal element then

$$z(j) = z(i) \cup \{h+1 \mid i+1 \leqslant h+1 < j+1, \, \ell_z(h) = \ell_z(i)\}.$$

If $i \in S_z$, then there exists a non-minimal $i' \in z(i)$, and so $i' < i+1$ in $z(j)$, which is a contradiction. Thus $i \notin S_z$ and furthermore $i \notin S_x$.

Let $m = \max\{h < j \mid \ell_x(h) = \ell_x(j)\}$ then $m_y(j) = m+1$. Note that $\ell_x(i) = \ell_z(i) = \ell_z(j) = \ell_x(j)$ so we have $i \leqslant m$. By the above formula for $z(j)$, to show that $m_y(j) \in z(j)$ it suffices to show that $\ell_z(m) = \ell_z(j)$. Consider the following cases.

Case 1: $m \notin S_z \backslash S_x$.

If $\{\ell_x(i), i+1\} \nsubseteq x(m)$ for all $i \in S_z \backslash S_x$ with $\ell_z(i) < \ell_x(i)$, then by Lemma 2.31, $\ell_z(m) = \ell_x(m) = \ell_x(j) = \ell_z(j)$.

Otherwise there exists $i \in S_z \backslash S_x$ such that $\ell_z(i) < \ell_x(i)$ and $\{\ell_x(i), i+1\} \subseteq x(m)$, then $i+1 \in x(m)$ is non-minimal. It follows that $\ell_x(i) = \ell_x(m) = \ell_x(j)$. Since $\ell_z(j) = \ell_x(j) = \ell_x(i) \leqslant i < j$ it follows that $\ell_z(j) \leqslant \ell_z(i) < \ell_x(i) = \ell_x(j)$, which is a contradiction.

Case 2: $m \in S_z \backslash S_x$.

If $\ell_z(m) = \ell_x(m)$, then $\ell_z(m) = \ell_x(m) = \ell_x(j) = \ell_z(j)$. Otherwise $\ell_z(m) < \ell_x(m)$, then since $\ell_z(j) < \ell_x(j) = \ell_x(m) \leqslant m < j$ it follows that $\ell_z(j) \leqslant \ell_z(m) < \ell_x(m) = \ell_x(j)$, which is a contradiction.

We have now shown that $m_y(j) = m+1 \in z(j)$. Hence by Lemma 2.35, $y \leqslant z$. □

Theorem 2.39 *Let x, y be hlbfs with $x \leq y$. Then $x <_1 y$ iff $|S_y \backslash S_x| = 1$.*

Proof. We will first prove that $x <_1 y$ implies $|S_y \backslash S_x| = 1$. Suppose that $|S_y \backslash S_x| > 1$; we will seek to prove a contradiction. By Theorem 2.38, there exists z such that $x \leqslant z \leqslant y$ and $z \neq x, y$, which is a contradiction. Thus $|S_y \backslash S_x| \leqslant 1$. Note that by definition, $x \neq y$ follows from $x <_1 y$ and so by Corollary 2.24, we can't have $|S_y \backslash S_x| = 0$. It follows that $|S_y \backslash S_x| = 1$ as required.

We will now prove the converse, $x \leqslant y$ with $|S_y \backslash S_x| = 1$ implies $x <_1 y$. Note that $x \neq y$ since $S_x \neq S_y$. Suppose there exists an hlbf z such that $x \leqslant z \leqslant y$. It follows from $|S_y \backslash S_x| = 1$ that we must have $S_x = S_z$ or $S_z = S_y$. In either of these cases, the contrapositive of Lemma 2.23 implies that $\ell_z = \ell_x$ or $\ell_y = \ell_z$ respectively. By the bijection in Proposition 2.15, we have $z = x$ or $z = y$. Hence $x <_1 y$ as required. □

The remainder of this section contains various results that we found to be important for calculating parity. The following result is informally known as the diamond property in the context of abstract polytopes which is condition (P3) given in Definition 1.33 in Chapter 1. Although the result below is easier to prove using bracketings, for completeness we have chosen to give a proof in terms of hlbfs.

Theorem 2.40 *Let x, z be hlbfs with $x \leqslant z$. If $S_z = S_x \cup \{j_1, j_2\}$, then $(x, z) = \{y \mid x < y < z\}$ has exactly two elements.*

Proof. Assume without loss of generality that $j_1 < j_2$.

Case 1: $\ell_z = \ell_x$.

Let $j \in S_z \backslash S_x$, then since $j \in S_z$ there exists $h < j$ such that $\ell_z(h) = \ell_z(j)$, and so $\ell_x(h) = \ell_x(j)$. By Theorem 2.27, there exists an hlbf y with $x \leqslant y$, $S_y = S_x \cup \{j\}$ and $\ell_y = \ell_x$. Note that $\max\{h < j \mid \ell_x(h) = \ell_x(j)\} = \max\{h < j \mid \ell_z(h) = \ell_z(j)\}$ so we have $m_y(j) = m_z(j)$. Moreover $m_y(j) \in z(j)$, hence by Lemma 2.35, $y \leqslant z$.

Note that for any hlbf y with $x \leqslant y \leqslant z$, since $\ell_z = \ell_x$ it must be that $\ell_y = \ell_x$. Thus by the uniqueness result of Theorem 2.27, there exists exactly two such y as calculated above, one for each $j \in S_z \backslash S_x$.

Case 2: $\ell_z(j_1) = \ell_x(j_1)$ and $\ell_z(j_2) < \ell_x(j_2)$.

We will first calculate any hlbfs y with $x \leqslant y \leqslant z$ and $S_y = S_x \cup \{j_1\}$. By Proposition 2.33, there exists $h < j_1$ with $\ell_x(h) = \ell_x(j_1)$. By Theorem 2.27, there exists an hlbf y with $x \leqslant y$, $S_y = S_x \cup \{j_1\}$ and $\ell_y = \ell_x$. Note that $m_y(j_1) = m_z(j_1)$ since $\ell_z(k) = \ell_x(k)$ for all $k \leqslant j_1$. Moreover $m_y(j_1) \in z(j_1)$, hence by Lemma 2.35, $y \leqslant z$.

Note that for any hlbf y with $x \leqslant y \leqslant z$ and $S_y = S_x \cup \{j_1\}$, $\ell_y(j_1) \geqslant \ell_z(j_1) = \ell_x(j_1)$ so we must have $\ell_y(j_1) = \ell_x(j_1)$. Now by the uniqueness result of Theorem 2.27, there exists exactly one hlbf y with $x \leqslant y \leqslant z$ and $S_y = S_x \cup \{j_1\}$, which is given by the above calculation.

We will now calculate any hlbfs y with $x \leqslant y \leqslant z$ and $S_y = S_x \cup \{j_2\}$. By Proposition 2.33, if $k > j_2$ and $\ell_x(k) = \ell_x(j_2)$ then $j_2 + 1 \in x(k)$. By Theorem 2.30, there exists an hlbf y with $x \leqslant y$, $S_y = S_x \cup \{j_2\}$ and $\ell_y(j_2) < \ell_x(j_2)$. Hence by Lemma 2.36, $y \leqslant z$.

Note that for any hlbf y with $x \leqslant y \leqslant z$ and $S_y = S_x \cup \{j_2\}$, since $j_1 < j_2$ we have $\ell_z(j_1) = \ell_x(j_1) = \ell_y(j_1)$, and so $\ell_z = \ell_y$. It follows that $\ell_y(j_2) = \ell_z(j_2) < \ell_x(j_2)$. Now by the uniqueness part Theorem 2.30, there exists exactly one hlbf y with $x \leqslant y \leqslant z$ and $S_y = S_x \cup \{j_2\}$, which is given by the above calculation.

Case 3: $\ell_z(j_1) < \ell_x(j_1)$ and $\ell_z(j_2) = \ell_x(j_2)$.

We will first calculate any hlbfs y with $x \leqslant y \leqslant z$ and $S_y = S_x \cup \{j_2\}$. By Proposition 2.34, there exists $h < j_2$ with $\ell_x(h) = \ell_x(j_2)$. By Theorem 2.27, there exists an hlbf y with $x \leqslant y$, $S_y = S_x \cup \{j_2\}$ and $\ell_y(j_2) = \ell_x(j_2)$.

We will now show that $m_y(j_2) \in z(j_2)$ which is required in Lemma 2.35. Let $i+1 \in z(j_2)$ be the least non-minimal element, then

$$z(j_2) = z(i) \cup \{h + 1 \mid i + 1 \leqslant h + 1 < j_2 + 1,\, \ell_z(h) = \ell_z(i)\}.$$

Suppose that $i \in S_z$; then there exists a non-minimal $i' \in z(i)$, and so $i' < i + 1$ in $z(j_2)$, which is a contradiction. Thus $i \notin S_z$ and furthermore $i \notin S_x$.

Let $m = \max\{h < j_2 \mid \ell_x(h) = \ell_x(j_2)\}$ then $m_y(j_2) = m + 1$. Note that $\ell_x(i) = \ell_z(i) = \ell_z(j_2) = \ell_x(j_2)$ so we have $i \leqslant m$. By the above formula for $z(j_2)$, it suffices to show that $\ell_z(m) = \ell_z(j_2)$. If $j_1 = m$, then $\ell_x(j_1) = \ell_x(j_2)$. Now by Proposition 2.34, $j_1 + 1 \in x(j_2)$, and so $j_1 + 1 = \ell_x(j_2)$, which is a contradiction. If $\{\ell_x(j_1), j_1 + 1\} \subseteq x(m)$, then $j_1 + 1 \in x(m)$ is non-minimal, and so $\ell_x(j_1) = \ell_x(m) = \ell_x(j_2)$, which again gives a contradiction. By the formula in Proposition 2.34, $\ell_z(m) = \ell_x(m) = \ell_x(j_2) = \ell_z(i)$, thus $m_y(j_2) = m + 1 \in z(j_2)$. Hence by Lemma 2.35, $y \leqslant z$.

Note that for any hlbf y with $x \leqslant y \leqslant z$ and $S_y = S_x \cup \{j_2\}$, we have $\ell_y(j_2) \geqslant \ell_z(j_2) = \ell_x(j_2)$ and so $\ell_y(j_2) = \ell_x(j_2)$. Now by the uniqueness result of Theorem 2.27, there exists exactly one hlbf y with $x \leqslant y \leqslant z$ and $S_y = S_x \cup \{j_2\}$, which is given by the above calculation.

We will now calculate any hlbfs y with $x \leqslant y \leqslant z$ and $S_y = S_x \cup \{j_1\}$. By Proposition 2.34, if $k > j_1$ and $\ell_x(k) = \ell_x(j_1)$ then $j_1 + 1 \in x(k)$. By Theorem 2.30, there exists an hlbf y with $x \leqslant y$, $S_y = S_x \cup \{j_1\}$ and $\ell_y(j_1) < \ell_x(j_1)$. Hence by Lemma 2.36, $y \leqslant z$.

Note that for any hlbf y with $x \leqslant y \leqslant z$ and $S_y = S_x \cup \{j_1\}$, since $j_1 < j_2$ we have $\ell_y(j_1) = \ell_z(j_1) < \ell_x(j_1)$. Now by the uniqueness part Theorem 2.30, there exists exactly one hlbf y with $x \leqslant y \leqslant z$ and $S_y = S_x \cup \{j_1\}$, which is given by the above calculation.

Case 4: $\ell_z(j_1) < \ell_x(j_1)$ and $\ell_z(j_2) < \ell_x(j_2)$.

By Proposition 2.32, if $k > j_2$ and $\ell_x(k) = \ell_x(j_2)$ then $j_2 + 1 \in x(k)$. By Theorem 2.30, there exists an hlbf y with $x \leqslant y$, $S_y = S_x \cup \{j_2\}$ and $\ell_y(j_2) < \ell_x(j_2)$. Hence by Lemma 2.36, $y \leqslant z$.

Note that in this case we have two subcases, $\ell_x(j_1) \neq \ell_x(j_2)$ and $\ell_x(j_1) = \ell_x(j_2)$. If $\ell_x(j_1) \neq \ell_x(j_2)$, then by Proposition 2.32, if $k > j_1$ and $\ell_x(k) = \ell_x(j_1)$ then $j_1 + 1 \in x(k)$.

By Theorem 2.30, there exists an hlbf y with $x \leqslant y$, $S_y = S_x \cup \{j_1\}$ and $\ell_y(j_1) < \ell_x(j_1)$. Hence by Lemma 2.36, $y \leqslant z$.

If $\ell_x(j_1) = \ell_x(j_2)$ then by Theorem 2.27, there exists an hlbf y with $x \leqslant y$, $S_y = S_x \cup \{j_2\}$ and $\ell_y(j_2) = \ell_x(j_2)$.

We will now show that $m_y(j_2) \in z(j_2)$. Let $i+1 \in z(j_2)$ be the least non-minimal element, then

$$z(j_2) = z(i) \cup \{h+1 \mid i+1 \leqslant h+1 < j_2+1,\, \ell_z(h) = \ell_z(i)\}.$$

where $i \notin S_z$ and furthermore $i \notin S_x$. Let $m = \max\{h < j_2 \mid \ell_x(h) = \ell_x(j_2)\}$ then $m_y(j_2) = m+1$. Note that $\ell_x(j_2) \in x(j_2) \subseteq z(j_2)$ is non-minimal, and so by the least non-minimal element, $i+1 \leqslant \ell_x(j_2)$. Note that $\ell_x(j_1) = \ell_x(j_2)$ so we have $j_1 \leqslant m$. Hence $i \leqslant \ell_x(j_2) - 1 = \ell_x(j_1) - 1 < j_1 \leqslant m$.

By the above formula for $z(j_2)$, it suffices to show that $\ell_z(m) = \ell_z(i)$. Note that $m < j_2$ so we have $\{\ell_x(j_2), j_2+1\} \nsubseteq x(m)$. If $m \neq j_1$ and $\{\ell_x(j_1), j_1+1\} \nsubseteq x(m)$, then by Lemma 2.31, $\ell_z(m) = \ell_x(m)$. Since $\ell_z(m) = \ell_x(m) = \ell_x(j_2) = \ell_x(j_1) \leqslant j_1 \leqslant m$ it follows that $\ell_x(j_1) = \ell_z(m) \leqslant \ell_z(j_1) < \ell_x(j_1)$, which is a contradiction. Now note that $\ell_x(j_2) - 1 < j_1$ so we have $\{\ell_x(j_1), j_1+1\} \nsubseteq x(\ell_x(j_2)-1)$. If $m = j_1$, then by the formula in Proposition 2.32 we have $\ell_z(j_1) = \ell_x(\ell_x(j_1)-1) = \ell_x(\ell_x(j_2)-1) = \ell_z(j_2) = \ell_z(i)$ as required. We have shown that $m_y(j_2) = m+1 \in z(j_2)$. Hence by Lemma 2.35, $y \leqslant z$.

We now need to show that there are exactly two hlbfs y with $x \leqslant y \leqslant z$ and $|S_y \backslash S_x| = 1$. Note that for the case of $S_y = S_x \cup \{j_1\}$ we must have $\ell_y(j_1) = \ell_z(j_1) < \ell_x(j_1)$. Thus we have excluded the possibility of $\ell_y(j_1) = \ell_x(j_1)$. Hence in either of the two subcases above we must have exactly two hlbfs y with $x \leqslant y \leqslant z$ and $|S_y \backslash S_x| = 1$. $\square$

We are now able to fulfil the promise to prove that the poset of hlbfs is an abstract pre-polytope.

Corollary 2.41 *Let P be the poset of all hlbfs on $[n]$ adjoined with a least face. Then P is an abstract pre-polytope.*

Proof. Recall from Definition 1.33 that for P to be an abstract pre-polytope it must satisfy axioms (P1), (P2) and (P3). Firstly, we claim that the greatest face is given by the hlbf with $\ell(i) = 0$ for all $i \in [n]$ and $S = [n] \backslash \{0\}$. Denote such an hlbf by $\top$. It follows from the definition of an hlbf that $\top(i) = \{0, \ldots, i\}$ for all $i \in [n]$. For any hlbf x we have $x(i) \subseteq \{0, \ldots, i\} = \top(i)$ and so $x \leq \top$. By definition, P has a least face, hence P satisfies (P1).

Now note that (P2) follows from Theorem 2.39, and (P3) follows from Theorem 2.40. Hence P is an abstract pre-polytope. $\square$

Recall our definition of a Hasse diagram in Definition 1.28 of Chapter 1. In the context of the poset of hlbfs, we draw edges between hlbfs x and y whenever $x \leq y$ and $S_y = S_x \cup \{j\}$. We will also label such an edge by j. The following result is related to axioms 1* and 2 as given in Chapter 1. We will describe this relationship more explicitly in Chapter 3.

Theorem 2.42 *Let x, y, z be hlbfs with $x \leqslant y \leqslant z$, $S_z = S_y \cup \{j_2\}$ and $S_y = S_x \cup \{j_1\}$. Then by Theorem 2.40, there exists a unique hlbf y' with $y' \neq y$ and $x <_1 y' <_1 z$. The following are the possible configurations.*

$$(b)\quad \begin{tikzcd} & z \arrow[dl, dash, "j_2"'] \arrow[dr, dash, "j_2"] & \\ y \arrow[dr, dash, "j_1"'] & & y' \arrow[dl, dash, "j_1"] \\ & x & \end{tikzcd} \qquad (a)\quad \begin{tikzcd} & z \arrow[dl, dash, "j_2"'] \arrow[dr, dash, "j_1"] & \\ y \arrow[dr, dash, "j_1"'] & & y' \arrow[dl, dash, "j_2"] \\ & c & \end{tikzcd}$$

(a) *If* $S_z = S_{y'} \cup \{j_2\}$, *then* $j_1 > j_2$, $\ell_y(j_1) = \ell_x(j_1) \iff \ell_{y'}(j_1) < \ell_x(j_1)$, *and* $\ell_z(j_2) = \ell_y(j_2) \iff \ell_z(j_2) = \ell_{y'}(j_2)$.

(b) *If* $S_z = S_{y'} \cup \{j_1\}$, *then* $\ell_y(j_1) = \ell_x(j_1) \iff \ell_z(j_1) = \ell_{y'}(j_1)$, *and* $\ell_z(j_2) = \ell_y(j_2) \iff \ell_{y'}(j_2) = \ell_x(j_2)$.

Proof. (a) Suppose that $j_1 < j_2$; we seek to prove that $j_1 > j_2$, $\ell_y(j_1) = \ell_x(j_1) \iff \ell_{y'}(j_1) = \ell_x(j_1)$, and $\ell_z(j_2) = \ell_y(j_2) \iff \ell_z(j_2) = \ell_{y'}(j_2)$. If $\ell_y(j_1) = \ell_x(j_1)$, then $\ell_{y'}(j_1) = \ell_z(j_1) = \ell_y(j_1) = \ell_x(j_1)$. If $\ell_{y'}(j_1) = \ell_x(j_1)$, then $\ell_y(j_1) = \ell_z(j_1) = \ell_{y'}(j_1) = \ell_x(j_1)$. However, by the uniqueness result of Theorem 2.27 and Theorem 2.30, $y = y'$, which is a contradiction. Thus we have $j_1 > j_2$ as required.

Now since $j_2 < j_1$ it is clear that $\ell_z(j_2) = \ell_y(j_2) \iff \ell_z(j_2) = \ell_{y'}(j_2)$. Finally, we will show that $\ell_y(j_1) = \ell_x(j_1) \iff \ell_{y'}(j_1) < \ell_x(j_1)$. If $\ell_y(j_1) = \ell_x(j_1)$, then by the uniqueness result of Theorem 2.27, $\ell_{y'}(j_1) < \ell_x(j_1)$. If $\ell_{y'}(j_1) < \ell_x(j_1)$, then by the uniqueness result of Theorem 2.30, $\ell_y(j_1) = \ell_x(j_1)$.

(b) Assume without loss of generality that $j_1 < j_2$. Firstly we'll show that $\ell_y(j_1) = \ell_x(j_1) \iff \ell_z(j_1) = \ell_{y'}(j_1)$. If $\ell_y(j_1) = \ell_x(j_1)$, then $\ell_z(j_1) = \ell_y(j_1) = \ell_x(j_1) = \ell_{y'}(j_1)$. If $\ell_z(j_1) = \ell_{y'}(j_1)$, then $\ell_y(j_1) = \ell_z(j_1) = \ell_{y'}(j_1) = \ell_x(j_1)$.

We will now show $\ell_z(j_2) < \ell_y(j_2) \iff \ell_{y'}(j_2) < \ell_x(j_2)$. If $\ell_{y'}(j_2) < \ell_x(j_2)$, then $\ell_z(j_2) \leqslant \ell_{y'}(j_2) < \ell_x(j_2)$. Now since $j_2 \notin S_y$ it follows that $\ell_z(j_2) < \ell_x(j_2) = \ell_y(j_2)$. If $\ell_z(j_2) < \ell_y(j_2)$, then suppose that $\ell_{y'}(j_2) = \ell_x(j_2)$. Note that $\ell_{y'}(j_2) = \ell_x(j_2) = \ell_y(j_2) = m_z(j_2)$, however $y'(j_2) \subseteq z(j_2)$ so $m_{y'}(j_2) = \ell_{y'}(j_2)$ and so $j_2 \notin S_{y'}$, which is a contradiction. Thus $\ell_{y'}(j_2) < \ell_x(j_2)$ as required. $\square$

Finally, we have the following results which we will use in Chapter 3 to motivate the definition of a label structure. The result below is easier to prove using bracketings, but for completeness we have chosen to give a proof in terms of hlbfs.

Theorem 2.43 *Let x, y be distinct hlbfs. If there exists an hlbf a with $a <_1 x$ and $a <_1 y$, then there exists an hlbf z such that $x <_1 z$ and $y <_1 z$.*

Proof. By the hypothesis, there exists $h \in S_x$ and $j \in S_y$ such that $S_x \backslash \{h\} = S_a = S_y \backslash \{j\}$. Consider the case of $h = j$, here $S_a = S_x \backslash \{h\} = S_y \backslash \{h\}$. It follows from the uniqueness result of Theorem 2.27 and Theorem 2.30 that $\ell_x = \ell_a \implies \ell_y \neq \ell_a$ and $\ell_x \neq \ell_a \implies \ell_y = \ell_a$. Assume without loss of generality that $\ell_x = \ell_a$ and $\ell_y \neq \ell_a$.

Let j be maximal with the property $j \notin S_x, j < h, \ell_x(j) = \ell_x(h)$. We seek to prove $k > j, \ell_x(k) = \ell_x(j) \implies j + 1 \in x(k)$. Suppose this is false and let k be the minimal counterexample i.e. $k > j, \ell_x(k) = \ell_x(j)$ and $j + 1 \notin x(k)$.

If $k \in S_x$, then let $i + 1 \in x(k)$ be the least non-minimal element, then

$$x(k) = \{\ell_x(k)\} \cup \{i + 1 \leqslant m + 1 < k + 1 \mid \ell_x(m) = \ell_x(k)\}.$$

Suppose that $j < i$ $(< k)$; then since $\ell_x(i) = \ell_x(k) = \ell_x(j)$, by the minimal counterexample, $j + 1 \in x(i)$. However $i \notin S_x$ so $j + 1 = \ell_x(i) = \ell_x(j)$, which is a contradiction. Thus $i \leqslant j$ and so by the above formula of $x(k)$, we have $j + 1 \in x(k)$, which is a contradiction.

If $k \notin S_x$, then either $k < h$ or $k > h$. If $k < h$, then $\ell_x(k) = \ell_x(j) = \ell_x(h)$ so by definition of j, $k \leqslant j$, which is a contradiction. If $k > h$, then $\ell_a(k) = \ell_x(k) = \ell_x(j) = \ell_x(h) = \ell_a(h)$ so $h + 1 \in a(k)$, however $k \notin S_a$ so $h + 1 = \ell_a(k) = \ell_a(h) \leqslant h$, which is a contradiction.

Hence by Theorem 2.30, there exists an hlbf z such that $x \leqslant z$, $S_z = S_x \cup \{j\}$ and $\ell_z(j) < \ell_x(j)$. By Theorem 2.40, there exists a unique hlbf w such that $a <_1 w <_1 z$ and $w \neq x$. Suppose that $S_z = S_w \cup \{h\}$; then by Proposition 2.42(b), $\ell_w(j) < \ell_a(j)$ since $\ell_z(j) < \ell_x(j)$. Note that $\ell_a(h) = \ell_x(h) = \ell_x(j) = \ell_a(j)$, and so $h + 1 \in a(j)$. However $j \notin S_a$ so we have $h + 1 = \ell_a(j) = \ell_a(h) \leqslant h$, which is a contradiction.

It remains that $S_z = S_w \cup \{j\}$, and so by Proposition 2.42(a), $\ell_w(h) < \ell_a(h)$ since $\ell_z(h) = \ell_x(h)$. By the uniqueness result of Theorem 2.30, $w = y$, and so the result follows.

Now we consider the case of $h \neq j$. Recall that $S_x \backslash \{h\} = S_a = S_y \backslash \{j\}$. Assume without loss of generality that $h < j$. Consider the following cases.

Case 1: $\ell_x = \ell_a = \ell_y$.

Note that $j \notin S_x$ so by Theorem 2.27, there exists an hlbf z with $x \leqslant z$, $S_z = S_x \cup \{j\}$ and $\ell_z(j) = \ell_x(j)$. By Theorem 2.40, there exists a unique hlbf w such that $a <_1 w <_1 z$ and $w \neq x$. Note that $h < j$ so it must be that $S_z = S_w \cup \{h\}$, and so by Proposition 2.31(a), $\ell_w(j) = \ell_a(j)$ since $\ell_z(j) = \ell_x(j)$. It follows by the uniqueness result of Theorem 2.27, $w = y$, and so the result follows.

Case 2: $\ell_x = \ell_a \neq \ell_y$.

Let $k > j$ and $\ell_x(k) = \ell_x(j)$, since $\ell_a(k) = \ell_x(k) = \ell_x(j) = \ell_a(j)$ it follows that $j + 1 \in a(k) \subseteq x(k)$. Note that $j \notin S_x$ so by Theorem 2.30, there exists an hlbf z with $x \leqslant z$, $S_z = S_x \cup \{j\}$ and $\ell_z(j) < \ell_x(j)$. By Theorem 2.40, there exists a unique hlbf w such that $a <_1 w <_1 z$ and $w \neq x$. Since $h < j$ it must be that $S_z = S_w \cup \{h\}$, and so by Proposition 2.31(a), $\ell_w(j) < \ell_a(j)$ since $\ell_z(j) < \ell_x(j)$. By the uniqueness result of Theorem 2.30, $w = y$, and so the result follows.

Case 3: $\ell_x \neq \ell_a = \ell_y$.

Note that by Theorem 2.27, there exists $i < j$ such that $\ell_a(i) = \ell_a(j)$. Suppose that $\ell_x(i) \neq \ell_a(i)$; then by the formula in Proposition 2.28, either $i = h$ or $\{\ell_a(h), h+1\} \subseteq a(i)$. In either case we have $i \geqslant h$ and $\ell_a(i) = \ell_a(h)$. It follows that $j > i \geqslant h$ and $\ell_a(j) = \ell_a(i) = \ell_a(h)$, and so by Theorem 2.30, $h+1 \in a(j)$. However $j \notin S_a$ so $h+1 = \ell_a(j) = \ell_a(h) \leqslant h$, which is a contradiction. Thus we have $\ell_x(i) = \ell_a(i) = \ell_a(j) = \ell_x(j)$.

Note that $j \notin S_x$ so by Theorem 2.27 there exists an hlbf z with $x \leqslant z$, $S_z = S_x \cup \{j\}$ and $\ell_z(j) = \ell_x(j)$. By Theorem 2.40, there exists a unique hlbf w such that $a <_1 w <_1 z$ and $w \neq x$. Since $h < j$ it must be that $S_z = S_w \cup \{h\}$, and so by Proposition 2.31(a), $\ell_w(j) = \ell_a(j)$ since $\ell_z(j) = \ell_x(j)$. It follows by the uniqueness result of Theorem 2.27, $w = y$ and so the result follows.

Case 4: $\ell_x \neq \ell_a \neq \ell_y$

In this case, we will seek to prove that $k > j, \ell_x(k) = \ell_x(j) \implies j + 1 \in x(k)$. Suppose this is false and let k be the minimal counterexample i.e. $k > j, \ell_x(k) = \ell_x(j)$ and $j + 1 \notin x(k)$.

If $k \in S_x$, then let $i + 1$ be the least non-minimal element, then

$$x(k) = \{\ell_x(k)\} \cup \{i + 1 \leqslant m + 1 < k + 1 \mid \ell_x(m) = \ell_x(k)\}.$$

Suppose that $j < i$ $(< k)$; then since $\ell_x(i) = \ell_x(k) = \ell_x(j)$, by the minimal counterexample, we have $j+1 \in x(i)$. However $i \notin S_x$ so $j+1 = \ell_x(i) = \ell_x(j)$, which is a contradiction. Thus $i \leqslant j$ and so $j + 1 \in x(k)$, which is a contradiction.

If $k \notin S_x$, then $\ell_x(k) = \ell_a(k)$. Suppose that $\ell_x(j) \neq \ell_a(j)$; then either $j = h$ or $\{\ell_a(h), h+1\} \subseteq a(j)$, which is a contradiction since $h \neq j$ and $j \notin S_a$. Thus $\ell_x(j) = \ell_a(j)$, it follows that $\ell_a(k) = \ell_x(k) = \ell_x(j) = \ell_a(j)$, and so $j + 1 \in a(k) \subseteq x(k)$, which is a contradiction. Hence we have shown that $k > j, \ell_x(k) = \ell_x(j) \implies j + 1 \in x(k)$.

Note that $j \notin S_x$ so by Theorem 2.30, there exists an hlbf z with $x \leqslant z$, $S_z = S_x \cup \{j\}$ and $\ell_z(j) < \ell_x(j)$. By Theorem 2.40, there exists a unique hlbf w such that $a <_1 w <_1 z$ and $w \neq x$. Since $h < j$ it must be that $S_z = S_w \cup \{h\}$, and so by Proposition 2.31(a), $\ell_w(j) < \ell_a(j)$ since $\ell_z(j) < \ell_x(j)$. It follows by the uniqueness result of Theorem 2.27, $w = y$, and so the result follows. $\square$

Proposition 2.44 *Let x be an hlbf. If $m_x(j) \leqslant i < j$, then $m_x(j) \leqslant \ell_x(i)$.*

Proof. If $j \notin S_x$, then since $\ell_x(j) = m_x(j) \leqslant i < j$ it follows that $m_x(j) = \ell_x(j) \leqslant \ell_x(i)$ as required. Otherwise $j \in S_x$, then let $m = \max\{h < j \mid \ell_x(h) = \ell_x(j)\}$ so that $m_x(j) = m + 1$. Now since $m + 1 \leqslant i < j$ it follows from the maximality of m that $\ell_x(i) \neq \ell_x(j)$. Suppose that $\ell_x(i) \leqslant m$; then since $\ell_x(i) \leqslant m < i$ it follows that $\ell_x(i) \leqslant \ell_x(m) = \ell_x(j)$. However we have $\ell_x(j) < m_x(j) \leqslant i < j$ so it follows that $\ell_x(j) \leqslant \ell_x(i)$, and so $\ell_x(i) = \ell_x(j)$, which is a contradiction. Hence $m < \ell_x(i)$, and so $m_x(j) \leqslant \ell_x(i)$ as required. $\square$

Proposition 2.45 *Let y be an hlbf and $j = \max S_y$. Then there exists a unique hlbf x satisfying each of the following conditions:*

(a) $x \leqslant y$, $S_y = S_x \cup \{j\}$ and $\ell_y(j) = \ell_x(j)$.

(b) $x \leqslant y$, $S_y = S_x \cup \{j\}$ and $\ell_y(j) < \ell_x(j)$.

Proof. We will first prove part (a). Consider the pair (ℓ_x, S_x) where $\ell_x = \ell_y$ and $S_x = S_y \backslash \{j\}$. Note that for any $i \in S_x \subseteq S_y$ there exists $h < i$ with $\ell_y(h) = \ell_y(j)$ and so $\ell_x(h) = \ell_y(h) = \ell_y(i) = \ell_x(i)$. By the bijection in Proposition 2.15, we deduce the existence of a unique hlbf x with ℓ_x and S_x as defined above.

All that needs to be shown is that $x \leq y$. Note that $j \in S_y$ so there exists $h < j$ with $\ell_y(h) = \ell_x(j)$. Now we have $j \notin S_x$ with the property that there exists a $h < j$ with $\ell_x(h) = \ell_y(h) = \ell_y(j) = \ell_x(j)$. By Theorem 2.27, there exists a unique hlbf y' with $x \leq y'$, $S_{y'} = S_x \cup \{j\}$ and $\ell_{y'} = \ell_x$. Note that $\ell_{y'} = \ell_x = \ell_y$ and $S_{y'} = S_x \cup \{j\} = (S_y \backslash \{j\}) \cup \{j\} = S_y$ so by the bijection of Proposition 2.15, $y' = y$. Hence $x \leq y' = y$ as required.

We will now prove part (b). Suppose there exists an x with the required properties. By the formula in Proposition 2.28, we have $\ell_x(j) = m_y(j)$ and $\ell_x(k) = \ell_y(k)$ for any $k \neq j$. Since ℓ_x is determined, it must be that such an x is unique. All that needs to be shown is the existence part of (b).

Consider the pair (ℓ_x, S_x) where $S_x = S_y \backslash \{j\}$ and ℓ_x is defined by.

$$\ell_x(k) = \begin{cases} m_y(j) & \text{if } k = j, \\ \ell_y(k) & \text{otherwise.} \end{cases}$$

It follows from Proposition 2.44 that this is an lbf. For any $i \in S_x \subseteq S_y$ there exists $h < i$ such that $\ell_y(h) = \ell_y(i) = \ell_x(i)$. If $h \neq j$, then we have $\ell_x(h) = \ell_x(i)$. Otherwise $h = j$, then $\ell_x(\ell_x(j) - 1) = \ell_y(j) = \ell_x(i)$. Hence there exists $h' < i$ such that $\ell_x(h') = \ell_x(i)$. By the bijection in Proposition 2.15, we deduce the existence of an hlbf x with ℓ_x and S_x as defined above.

It needs to be shown that $x \leqslant y$. We will first show that the condition of Theorem 2.30 holds. Suppose that $k > j$ and $\ell_x(k) = \ell_x(j)$; then we have $\ell_y(k) = \ell_x(k) = \ell_x(j) \leqslant j < k$ so it follows that $\ell_x(k) = \ell_y(k) \leqslant \ell_y(j) < \ell_x(j)$, which is a contradiction. Thus the condition of Theorem 2.30 holds trivially, and so there exists an hlbf y' with $x \leqslant y'$, $S_{y'} = S_x \cup \{j\}$ and $\ell_{y'}(j) < \ell_x(j)$.

We will now show that $\ell_{y'} = \ell_y$. By the formula in Proposition 2.28, we have $\ell_{y'}(j) = \ell_x(\ell_x(j) - 1) = \ell_x(m_y(j) - 1) = \ell_y(m_y(j) - 1) = \ell_y(j)$. If $\{\ell_x(j), j + 1\} \subseteq x(k)$, then we have $j + 1 \leqslant k$. However since $j = \max S_y$ we have $k < j$, which is a contradiction. Now finally by the otherwise case we have $\ell'_y(k) = \ell_x(k) = \ell_y(k)$ since $k \neq j$. Hence $\ell_{y'} = \ell_y$ as required.

Now since $S_{y'} = S_x \cup \{j\} = S_y$ and $\ell_{y'} = \ell_y$ so by the bijection of Proposition 2.15, $y' = y$. Hence $x \leqslant y' = y$ as required. $\square$

Chapter 3

Polytopes

A polytope is a higher dimensional generalisation of the notion of polygon and polyhedron defined in euclidean space. Often the term polytope is synonymous with convex polytopes as studied in [9] and [28]. For the purposes of constructing parity structures on a polytope, we shall make use of a combinatorial notion of polytopes called abstract polytopes [16]. In Section 1.2 of Chapter 1, we demonstrated a close relationship between parity structures and abstract polytopes.

In this chapter, we will provide a general framework for constructing parity structures on polytopes. We will define the notion of a *label structure* on an abstract polytope which gives rise to a parity structure. Much of our work within this chapter involves proving that this parity structure satisfies axioms **(L)**, **(G)** and **(C)** of Section 1.4 of Chapter 1.

In the existing formalisms of pasting diagrams, the examples include abstract polytopes such as the simplexes and hypercubes. Our theory in Chapter 2 on higher left bracketing functions is used to describe associahedra as an abstract polytope. This will serve as our motivating example for our definition of a label structure. We will then show that we can take products of label structures.

The final section will be devoted to a discussion of our main examples; hypercubes, associahedra and permutohedra. We demonstrate that these families of polytopes are a type of hypergraph polytope [7]. It will be shown that understanding these as hypergraph polytopes provides a convenient setting for the construction of label structures and subsequent verification of the axioms.

3.1 Label structures

In Section 1.2 of Chapter 1, we gave the definition of abstract polytope in the sense of McMullen [16]. An abstract pre-polytope is a bounded, graded poset which satisfies the diamond property. An abstract polytope is an abstract pre-polytope which satisfies the connectedness property.

Recall the notation $<_1$ in Definition 1.25. We describe a Hasse diagram of an abstract polytope as a graph whose vertices are the faces, together with edges between faces with $x <_1 y$. We also show how to describe a parity structure on an abstract polytope by giving direction to each edge of the Hasse diagram.

In this section, we continue our investigation into parity structures and polytopes found in Section 1.2 of Chapter 1. Recall that in the aforementioned section, we introduced our perspective on parity structures; we consider a parity structure being a graded poset as in Proposition 1.26. Recall the definitions pertaining to abstract pre-polytopes and abstract polytopes. A characterisation of axioms 1* and 2 for parity structures on abstract pre-polytopes can be found in Theorem 1.36.

We will provide a framework that allows us to construct parity structures on polytopes; to accomplish this we introduce a concept which we call label structures. The definition of a label structure is motivated by the properties of hlbfs as discussed in Chapter 2. For each hlbf x we can associate a pair (ℓ_x, S_x) as defined in Proposition 2.15. A label structure involves labelling the faces of an abstract pre-polytope with a pair (φ, Φ) which is compatible with the partial ordering. This compatibility is made precise in the axioms of the definition below. Note that we will work with abstract pre-polytopes since it is more convenient for the purposes of developing our theory.

We now give the formal definition of a label structure. Recall that the elements of P are called faces and that $\widetilde{P} = P \backslash \{\bot_P\}$; where $\bot_P$ is the least face of P.

Definition 3.1 Let P be an abstract pre-polytope in the sense of Definition 1.33. A *label structure* on P is a finite linearly ordered set M together with the assignation (φ_f, Φ_f) to each face $f \in \widetilde{P}$ which are defined as follows.

* φ_f is a function $\varphi_f : M \longrightarrow M$ satisfying $\varphi_f(i) \leq i$ for all $i \in M$.
* Φ_f is a subset of M; we require that $\Phi_{\top_P} = M \backslash \{\min M\}$.

Denote this by the triple (M, φ, Φ). Furthermore, the following conditions hold.

(C1) Let f, g be faces with $f \leqslant g$.

(a) $f <_1 g$ iff $\Phi_g = \Phi_f \cup \{j\}$ for some $j \notin \Phi_f$.

(b) $\varphi_g(i) \leqslant \varphi_f(i)$ for all $i \in M$.

(c) If $\Phi_g = \Phi_f \cup \{j\}$, then $\varphi_g(i) = \varphi_f(i)$ for all $i < j$.

Note that giving assignations $f \mapsto \Phi_f$ satisfying (C1)(a) is equivalent to labelling the edges of the Hasse diagram of $\widetilde{P}$ with elements of M. For each $f <_1 g$, label the corresponding edge on the Hasse diagram with the unique $j \in \Phi_g \backslash \Phi_f$.

(C2) The following is a Hasse diagram of a line segment (in the sense of Definition 1.30) in $\widetilde{P}$. Consider the labelled edges as shown below; this indicates that $\Phi_v = \Phi_u \cup \{j_1, j_2\}$.

$$\begin{tikzcd} & v \arrow[dl, dash, "j_2"'] \arrow[dr, dash, "j_1"] & \\ f \arrow[dr, dash, "j_1"'] & & g \arrow[dl, dash, "j_2"] \\ & u & \end{tikzcd}$$

A line segment with the above configuration satisfies

$$\begin{aligned} \varphi_f(j_1) = \varphi_u(j_1) &\iff \varphi_v(j_1) = \varphi_g(j_1), \\ \varphi_v(j_2) = \varphi_f(j_2) &\iff \varphi_g(j_2) = \varphi_u(j_2). \end{aligned}$$

(C3) Let f be a face, then for each $j \notin \Phi_f$ there is at most one face g satisfying the conditions in (a), and at most one satisfying the conditions in (b).

(a) $f \leqslant g$, $\Phi_g = \Phi_f \cup \{j\}$ and $\varphi_g(j) = \varphi_f(j)$;

(b) $f \leqslant g$, $\Phi_g = \Phi_f \cup \{j\}$ and $\varphi_g(j) \neq \varphi_f(j)$.

(C4) Let f be a face and $j = \max \Phi_f$. There exists a unique face u satisfying the conditions in (a), and a unique face satisfying the conditions in (b).

(a) $u \leqslant f$, $\Phi_f = \Phi_u \cup \{j\}$ and $\varphi_f(j) = \varphi_u(j)$;

(b) $u \leqslant f$, $\Phi_f = \Phi_u \cup \{j\}$ and $\varphi_f(j) \neq \varphi_u(j)$.

Note that given a label structure as defined above, the rank function can be taken as follows.

$$\operatorname{rank} f = \begin{cases} |\Phi_f| & \text{if } f \in \widetilde{P} \\ -1 & \text{if } f = \bot_P \end{cases}$$

Example 3.2 As mentioned earlier the poset of hlbfs is the motivating example of a label structure. Let $M = [n]$ and $\mathcal{H}_n$ be the poset of all hlbfs on $[n]$. Let P be obtained from $\mathcal{H}_n$ by adjoining a least face. By Corollary 2.41, P is an abstract pre-polytope.

Let the label structure on P be given by (M, ℓ, S) where ℓ_x is defined as in Definition 2.13, and S_x immediately below it. We will show that (M, ℓ, S) is a label structure. Now the verification of the remaining conditions are summarised as follows.

- (C1)(a) follows from Theorem 2.39
- (C1)(b),(c) are consequences of Theorem 2.27 and Theorem 2.30
- (C2) follows from Theorem 2.42
- (C3)(a) follows from Theorem 2.27
- (C3)(b) follows from Theorem 2.30
- (C4) follows from Proposition 2.45

For an abstract pre-polytope P with a label structure (M, φ, Φ) we define for each $f \in \widetilde{P}$

$$\#(j; f) = |\{i \in \Phi_f \mid i < j\}|.$$

This allows us to give our construction of parity for an abstract pre-polytope.

Proposition 3.3 *Let P be an abstract pre-polytope with a label structure (M, φ, Φ). Let f, g be faces with $f \leqslant g$ and $\Phi_f = \Phi_g \cup \{j\}$. Then*

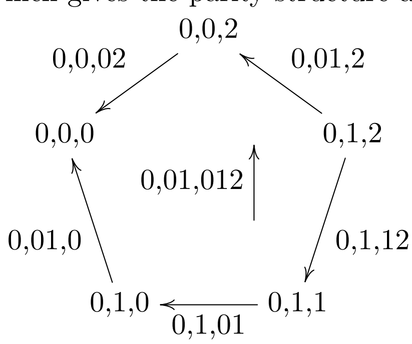
$$g \in f^\alpha \iff \begin{cases} \varphi_f(j) = \varphi_g(j) & \text{if } (-1)^{\#(j;f)} = \alpha \\ \varphi_f(j) \neq \varphi_g(j) & \text{if } (-1)^{\#(j;f)} = \varepsilon(\alpha) \end{cases}$$

where $\alpha \in \{-, +\}$ and $\varepsilon(\alpha) = -\alpha$ defines a parity structure on $\widetilde{P}$.

Proof. Note that disjointness follows immediately from the above definition. All that needs to be shown is that f^+ and f^- are non-empty. Let f be a face with $j = \max \Phi_f$ be given. If $|\Phi_f|$ is even, then $\#(j; f) = |\Phi_f \backslash \{j\}|$ is odd. By (C4)(a), there exists a face u such that $u \leqslant f$, $\Phi_f = \Phi_u \cup \{j\}$ and $\varphi_f(j) = \varphi_u(j)$. It follows that $u \in f^-$ and so f^- is non-empty. Similarly, it follows from (C4)(b) that f^+ is also non-empty.

Now if $|\Phi_f|$ is odd, then a similar argument as above can be used to show that f^- and f^+ are non-empty. $\square$

Example 3.4 Consider the 2-dimensional associahedron (pentagon) as modelled by the hlbfs on $\{0, 1, 2\}$. Recall the label structure on hlbfs discussed in Example 3.2. The following is the pasting diagram which gives the parity structure as defined in Proposition 3.3.



We will now demonstrate some calculations of parity. Let $x = 0, 1, 01$ and $y = 0, 01, 012$; then $x <_1 y$ since $S_y = \{1, 2\} = S_x \cup \{1\}$. We have $\#(1; y) = 0$ and $\ell_y(1) = 0 \neq 1 = \ell_x(1)$, thus $0, 1, 01 \in 0, 01, 012^-$.

Let $z = 0, 01, 2$; then $z <_1 y$ since $S_y = \{1, 2\} = S_z \cup \{2\}$. We have $\#(2; y) = 1$ and $\ell_y(2) = 0 \neq 2 = \ell_z(2)$, thus $0, 01, 2 \in 0, 01, 012^+$.

Recall our discussion in Section 1.4 of Chapter 1. Therein we summarised a list of axioms on a parity structure; 1*, 2, (L) and (C) which is then shown to give a loop free pasting scheme. In order to prove that the above parity structure satisfies axioms 1*, 2, (L) and (C) we will require an additional condition on the abstract pre-polytope.

Definition 3.5 Let P be an abstract pre-polytope.

(a) A *vertex figure* of P is an interval of the form $[x, \top_P]$ where $x \in P$ with $\operatorname{rank} x = 0$.

(b) P is *simple* when every vertex figure is isomorphic to a simplex.

Example 3.6 (a) The n-simplex is simple; the vertex figures are $(n-1)$-simplexes.

(b) More generally, the hypergraph polytopes of [7] and Section 3.3 below are simple.

We will show in Section 3.1.3 that the above condition of simplicity implies that the abstract pre-polytope with a label structure is in fact an abstract polytope. The following are properties of a simple abstract pre-polytope which we will make use of in the upcoming sections.

Proposition 3.7 *Let P be a simple abstract pre-polytope. Let $f, g \in \widetilde{P}$ with $f \leq g$ and $k = \operatorname{rank} g - \operatorname{rank} f - 1$.*

(a) The k-interval $[f, g]$ is isomorphic to the k-simplex.

(b) The k-interval $[f, g]$ has 2^{k+1} faces, among these are $k + 1$ faces with rank equal to $\operatorname{rank} g - 1$.

(c) The k-interval $[f, g]$ has $(k + 1)!$ flags.

Proof. We will first consider when $k \leq 1$. For the case of $k = -1$ and $k = 0$, $[f, g]$ has exactly 1 and 2 elements respectively. The result follows trivially. For the case of $k = 1$, the result follows immediately from (P3).

Now consider when $k > 1$. We will first show that (a) holds. Note the property that every interval of a simplex is isomorphic to a simplex. Observe that the k-interval $[f, g]$ is an interval of the vertex figure of f which by definition is isomorphic to a simplex. Furthermore, by the isomorphism we must preserve difference in rank and so $[f, g]$ is isomorphic to a k-simplex.

We will now show that (a) holds. Note that $\wp\{0, \ldots, k\}$ consists of exactly $k + 1$ faces with rank one less than that of $\{0, \ldots, k\}$; these are $\{0, \ldots, k\} \backslash \{i\}$ for each $0 \leq i \leq k$. It follows by the isomorphism that $[f, g]$ has $k+1$ faces with rank equal to $\operatorname{rank} g - 1$. Again by the isomorphism, $[f, g]$ has $|\wp\{0, \ldots, k\}| = 2^{k+1}$ faces. Hence we have shown that (a) holds.

Finally, we will now show that (b) holds. Note that to give a flag of a k-simplex is equivalent to giving a permutation ξ on $\{0, \ldots, k\}$. The faces of a flag are determined the recursive formula $x_{i+1} = x_i \cup \{\xi(i)\}$ for each $0 \leq i \leq k$, where $x_0 = \varnothing$. It follows that the k-simplex has $(k + 1)!$ flags as determined by the number of permutations on $\{0, \ldots, k\}$. Hence by the isomorphism, $[f, g]$ has has $(k + 1)!$ flags as required. $\square$

Proposition 3.8 *Let P be a simple abstract pre-polytope. Let f, g, h be distinct faces with $f \leq h$ and $g \leq h$. If there exists a face $u \in \widetilde{P}$ such that $u <_1 f$ and $u <_1 g$, then there exists a face $v \in [u, h]$ such that $f <_1 v$ and $g <_1 v$.*

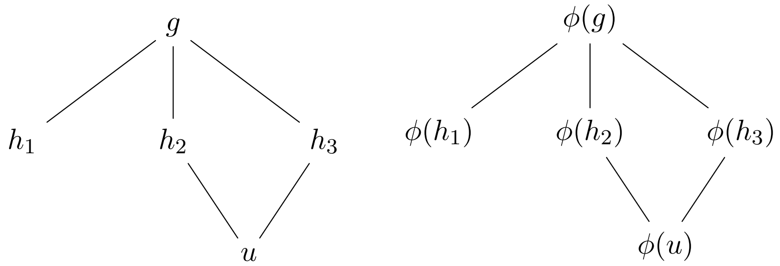
Proof. Consider the k-interval $[u,h]$ where $k = \operatorname{rank} h - \operatorname{rank} u - 1$. By Proposition 3.7(a), $[u,h]$ is isomorphic to the k-simplex. Let $\phi : [f,g] \longrightarrow \wp\{0,\ldots,k\}$ denote the isomorphism.

Note that $f,g \in [u,h]$ since $u <_1 f \leq h$ and $u <_1 g \leq h$. Applying the isomorphism ϕ, we have $\phi(h) = \{0,\ldots,k\}$ since the isomorphism must preserve the greatest face. Also $\phi(u) <_1 \phi(f)$ and $\phi(u) <_1 \phi(g)$ which implies that $\phi(f)\backslash\{j\} = \phi(u) = \phi(g)\backslash\{h\}$ for some $j \in \phi(f)$ and $h \in \phi(g)$. Consider the subset $\phi(f) \cup \{h\} = \phi(g) \cup \{j\} \subseteq \{0,\ldots,k\}$. By the isomorphism there exists a face $v = \phi^{-1}(\phi(f) \cup \{h\})$. Note that since the isomorphism preserves order, we also have $f <_1 v$, $g <_1 v$ and $v \leq h$ as required. □

Proposition 3.9 *Let P be a simple abstract pre-polytope. Let $f,g \in \widetilde{P}$ with $f \leq g$, and $h_1,h_2,h_3 \in [f,g]$ with rank equal to* $\operatorname{rank} g - 1$. *If there exists $u \in [f,g]$ with $u <_1 h_2$ and $u <_1 h_3$, then there exists a face $v \in [f,g]$ with $v <_1 u$ and $v \leq h_1$.*

Proof. Consider the k-interval $[u,h]$ where $k = \operatorname{rank} h - \operatorname{rank} u - 1$. By Proposition 3.7(a), $[u,h]$ is isomorphic to the k-simplex. Let $\phi : [f,g] \longrightarrow \wp\{0,\ldots,k\}$ denote the isomorphism.

We have the following diagram in P, and by applying the isomorphism we also have a diagram in $\wp\{0,\ldots,k\}$.



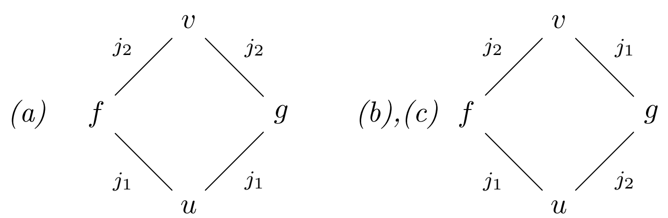
Note that we have $\phi(h) = \{0,\ldots,k\}$ since the isomorphism must preserve the greatest face. Also since $\operatorname{rank} h_i = \operatorname{rank} g - 1$, there exists distinct elements $j_1,j_2,j_3 \in \{0,\ldots,k\}$ such that $\phi(h_i) = \{0,\ldots,k\}\backslash\{j_i\}$ for each $i = 1,2,3$. It follows that $\phi(u) = \phi(h_2)\backslash\{j_3\} = \phi(h_3)\backslash\{j_2\}$.

Consider the subset $\phi(u)\backslash\{j_1\} = \phi(h_1)\backslash\{j_2,j_3\} \subseteq \{0,\ldots,k\}$. By the isomorphism, there exists a face $v = \phi^{-1}(\phi(u)\backslash\{j_1\})$. Note that since the isomorphism preserves order, we also have $v <_1 u$, $v \leq h_1$ and $v \leq g$ as required. □

3.1.1 Axioms 1*, 2 and 3(a)

For any abstract pre-polytope with a given label structure, unless stated otherwise we will consider it as a parity structure defined in Proposition 3.3. In this section, we will prove axioms 1*,2 and 3(a). We will first show that axioms 1* and 2 (as given in Section 1.2 of Chapter 1) hold.

Lemma 3.10 *Let P be an abstract pre-polytope with a label structure (M,φ,Φ). Consider a line segment $[u,v]$ in $\widetilde{P}$ with $\Phi_v = \Phi_u \cup \{j_1,j_2\}$. The following are the possible edge labellings.*



(a) If $\Phi_v = \Phi_g \cup \{j_2\}$, then $j_1 > j_2$ and

$$\begin{aligned} u \in f^\alpha &\iff u \in g^{\varepsilon(\alpha)}, \\ f \in v^\alpha &\iff g \in v^\alpha. \end{aligned}$$

(b) If $\Phi_v = \Phi_g \cup \{j_1\}$ and $j_1 < j_2$, then

$$\begin{aligned} u \in f^\alpha &\iff g \in v^\alpha, \\ f \in v^\alpha &\iff u \in g^{\varepsilon(\alpha)}. \end{aligned}$$

(c) If $\Phi_v = \Phi_g \cup \{j_1\}$ $j_1 > j_2$, then

$$\begin{aligned} u \in f^\alpha &\iff g \in v^{\varepsilon(\alpha)}, \\ f \in v^\alpha &\iff u \in g^\alpha. \end{aligned}$$

where $\alpha \in \{-, +\}$ and $\varepsilon(\alpha) = -\alpha$. Furthermore, axioms 1 and 2 hold.*

Proof. (a) If $\Phi_v = \Phi_g \cup \{j_2\}$, then suppose $j_1 < j_2$. By (C1)(c), $\varphi_f(j_1) = \varphi_v(j_1) = \varphi_g(j_1)$ so it follows that $\varphi_f(j_1) = \varphi_u(j_1) \iff \varphi_g(j_1) = \varphi_u(j_1)$. Note that $\#(j_1; f) = \#(j_1; g)$ so by (C3), $f = g$, which is a contradiction. Thus $j_1 > j_2$ and so by (C3), $\varphi_f(j_1) = \varphi_u(j_1) \iff \varphi_g(j_1) < \varphi_u(j_1)$. Now by (C1)(c), $\varphi_f(j_2) = \varphi_u(j_2) = \varphi_g(j_2)$. It follows that $\varphi_v(j_2) = \varphi_f(j_2) \iff \varphi_v(j_2) = \varphi_g(j_2)$. Note that $\#(j_1; f) = \#(j_1; g)$ since $\Phi_f = \Phi_g$. Thus the required result follows by the definition of parity.

(b), (c) If $\Phi_v = \Phi_g \cup \{j_1\}$, then consider the following subcases; $j_1 > j_2$ and $j_1 < j_2$. If $j_1 > j_2$, then $\#(j_1; v) = \#(j_1; f)$ and $\#(j_2; v) \backslash \{j_1\} = \#(j_2; g)$. If $j_1 < j_2$, then $\#(j_1; v) \backslash \{j_2\} = \#(j_1; f)$ and $\#(j_2; v) = \#(j_2; g)$. Thus the required result follows by (C2) and the definition of parity.

In each of the above cases, we have shown that the Hasse diagram of any line segment has a pair of edges with the same direction and a pair of edges with the opposite direction. Hence all line segments satisfy the condition of Proposition 1.36 and so axioms 1* and 2 hold. □

Lemma 3.11 *Let P be an abstract pre-polytope with a label structure (M, φ, Φ). Let x, z be faces with $\Phi_x = \Phi_z$. If there exists a face y such that $x \prec y \lhd z$, then there exists $j \in M$ such that $\varphi_x(j) \neq \varphi_z(j)$ and $\varphi_x(k) = \varphi_z(k)$ for all $k < j$. Furthermore, axiom 3(a) holds.*

Proof. By the hypothesis there exists a sequence of faces $y_0, y_1, \ldots, y_n$ as shown in the following diagram.

$$\begin{tikzcd}
x = y_0 \arrow[dr, "h_0"'] & & y_1 \arrow[dr, "h_1"'] & & y_2 & \cdots & y_{n-1} \arrow[dr, "h_{n-1}"'] & & y_n = z \\
& a_1 \arrow[ur, "k_1"'] & & a_2 \arrow[ur, "k_2"'] & & & & a_n \arrow[ur, "k_n"'] &
\end{tikzcd}$$

Let $j = \min\{h_0, h_1, \ldots, h_{n-1}, k_1, k_2, \ldots, k_n\}$. Note that there exists an i such that $j \in \Phi_{y_i} \subseteq M$. For $k < j$ note that $k < j \leqslant h_i$ and $k < j \leqslant k_i$. By (C1)(a), we have $\varphi_{y_i}(k) = \varphi_{a_{i+1}}(k)$ and $\varphi_{y_i}(k) = \varphi_{a_i}(k)$. It follows that $\varphi_x(k) = \varphi_{y_0}(k) = \varphi_{y_n}(k) = \varphi_z(k)$ for all $k < j$.

Suppose that $h_i \neq j$ for all i. By the definition of j, there exists k_i such that $k_i = j$. If $j \in \Phi_x$, then consider the least i such that $k_i = j$. It follows that $j \notin \Phi_{a_i}$ and so $j \notin \Phi_x$, which is a contradiction. If $j \notin \Phi_x$, then consider the greatest i such that $k_i = j$. It follows that $j \in \Phi_{y_i}$ and so $j \in \Phi_z = \Phi_x$, which is a contradiction. Thus there exists an i such that $h_i = j$.

Suppose that $k_i \neq j$ for all i. By the definition of j, there exists h_i such that $h_i = j$. If $j \in \Phi_x$, then consider the greatest i such that $h_i = j$. It follows that $j \notin \Phi_{a_{i+1}}$ and so $j \notin \Phi_z = \Phi_x$, which is a contradiction. If $j \notin \Phi_x$, then consider the least i such that $h_i = j$. It follows that $j \in \Phi_{y_{i-1}}$ and so $j \in \Phi_x$, which is a contradiction. Thus there exists an i such that $k_i = j$.

Note that since $\Phi_{y_{i+1}} = (\Phi_{y_i} \backslash \{h_i\}) \cup \{k_{i+1}\}$ and $j \leqslant h_i$, $j \leqslant k_{i+1}$ it follows that $\#(j; y_i) = \#(j; y_{i+1})$. Consider the following cases.

Case 1: $\#(j; y_0)$ is even.

If $k_i = j$, then by the definition of parity, $\varphi_{y_i}(j) < \varphi_{a_i}(j)$. Otherwise $j < k_i$ and so by (C1)(c), $\varphi_{y_i}(j) = \varphi_{a_i}(j)$. By a similar argument it can be shown that $\varphi_{x_i}(j) = \varphi_{a_{i+1}}(j)$ for all i. Since there exists an i such that $k_i = j$ it follows that $\varphi_x(j) = \varphi_{y_0}(j) > \varphi_{y_n}(j) = \varphi_z(j)$.

Case 2: $\#(j; y_0)$ is odd.

If $h_i = j$, then by definition of parity, $\varphi_{y_i}(j) < \varphi_{a_{i+1}}(j)$. Otherwise $j < h_i$ and so by (C1)(c), $\varphi_{y_i}(j) = \varphi_{a_{i+1}}(j)$. By a similar argument it can be shown that $\varphi_{x_i}(j) = \varphi_{a_i}(j)$ for all i. Since there exists an i such that $h_i = j$ it follows $\varphi_x(j) = \varphi_{y_0}(j) < \varphi_{y_n}(j) = \varphi_z(j)$.

Hence in either of the above cases we have shown that $\varphi_x(j) \neq \varphi_z(j)$. Let x, y be faces with $x \lhd y \lhd x$. Suppose that $x \neq y$; we will show that this results in a contradiction which then implies that axiom 3(a) holds. Note that this satisfies the hypothesis of the Lemma so it follows that $\varphi_x(j) \neq \varphi_x(j)$ for some j, which is contradiction. Hence $x = y$ which proves axiom 3(a) as required. $\square$

3.1.2 Characterisation of π and μ

We will now focus our attention to calculating maps π and μ introduced in Definition 1.46. Recall that for any face $z \in \widetilde{P}$, $\pi(z) = R(z) \backslash R(z)^-$ and $\mu(z) = R(z) \backslash R(z)^+$. The definition below will turn out to be a characterisation of π and μ for label structures.

Definition 3.12 Let P be an abstract pre-polytope with a label structure (M, φ, Φ). Let z be a face with $|\Phi_z| = p$. Let $\mathcal{L}_j^\alpha z = \{x \in z^\alpha \mid \Phi_z = \Phi_x \cup \{j\}\}$. Let $\alpha_i = (-1)^{i-1}$, $\beta_i = (-1)^i$ for $i \geq 1$. Let $\pi(z)$ and $\mu(z)$ be subsets given by $\pi(z)_p = \mu(z)_p = \{z\}$ and for each $1 \leqslant k \leqslant p$,

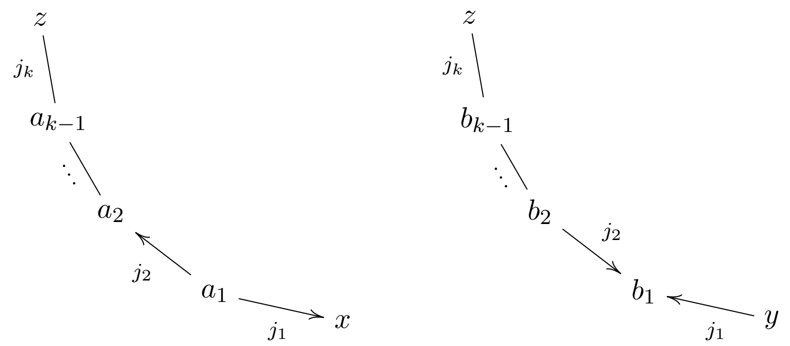
$$\pi(z)_{p-k} = \sum \mathcal{L}_{j_1}^{\alpha_1} \ldots \mathcal{L}_{j_k}^{\alpha_k} z$$

$$\mu(z)_{p-k} = \sum \mathcal{L}_{j_1}^{\beta_1} \ldots \mathcal{L}_{j_k}^{\beta_k} z$$

where the above are disjoint unions over all possible sequences $j_1 < \ldots < j_k$.

Remark The following is an important observation which we will rely on for the results within this section. If there exists an element $x \in \sum \mathcal{L}_{j_1}^{\varepsilon_1} \ldots \mathcal{L}_{j_k}^{\varepsilon_k} z$, then there exists a sequence $u_1, \ldots, u_{k-1}$ such that $u_i \in u_{i+1}^{\varepsilon_i}$ for all possible i. By the uniqueness axiom of a label structure (C3), it follows that the above sequence is uniquely determined. Note that $\{x, u_1, \ldots, u_{k-1}, z\}$ is a flag of the interval $[x, z]$.

It follows from the above remark that we can represent the flags associated with $x \in \pi(z)_{p-k}$ and $y \in \mu(z)_{p-k}$ by the following Hasse diagrams.



A dual parity structure involves interchanging the positive and negative faces. To be explicit, the dual of the parity structure given in Proposition 3.3 is given by the following.

$$g \in f^{\alpha} \iff \begin{cases} \varphi_f(j) = \varphi_g(j) & \text{if } (-1)^{\#(j;f)} = \varepsilon(\alpha) \\ \varphi_f(j) \neq \varphi_g(j) & \text{if } (-1)^{\#(j;f)} = \alpha \end{cases}$$

Note that μ is identified with π in the dual parity structure. Thus it is sufficient to prove that $\sum \mathcal{L}_{j_1}^{\alpha_1} \dots \mathcal{L}_{j_k}^{\alpha_k} z$ is a characterisation of $R(z) \backslash R(z)^-$ for label structures.

As we are working with a label structure on an abstract pre-polytope, we found it useful to work in terms of Hasse diagrams of the polytope. Each edge in the Hasse diagram can be labelled as per the comments in axiom (C1) of Definition 3.1. Furthermore, the edges have an orientation as introduced in Definition 1.28. Thus an element $x \in \sum \mathcal{L}_{j_1}^{\varepsilon_1} \dots \mathcal{L}_{j_k}^{\varepsilon_k} z$ can be represented as a flag of the interval $[x, z]$ where its Hasse diagram has edges labelled precisely by the $j_i's$ and the orientation is given by $\varepsilon_i's$.

We may apply the diamond property to two adjacent edges to obtain another flag. The Hasse diagram of such a flag may or may not have the the labels or orientation changed. The following definition serves as a shorthand for a flag which allows us to, in Lemma 3.14, describe precisely what happens when we apply the diamond property to a flag.

Definition 3.13 A *signed permutation* on $(k) = \{1, \dots, k\}$ is a permutation $\tau : (k) \longrightarrow (k)$ together with a map $\alpha : (k) \longrightarrow \{-, +\}$. A signed permutation (τ, α) is said to be *positive* whenever $\alpha(i) = (-1)^{|\{j > i | \tau(j) < \tau(i)\}|}$.

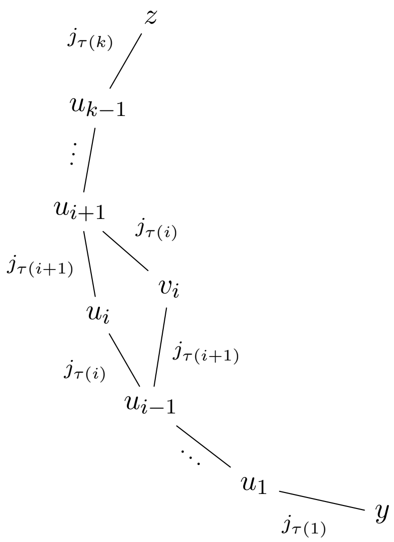
Recall our earlier remark that an element $x \in \sum \mathcal{L}_{j_1}^{\varepsilon_1} \dots \mathcal{L}_{j_k}^{\varepsilon_k} z$ can be represented as a flag of the interval $[x, z]$. By applying the diamond property iteratively, we obtain other flags in this interval; we shall keep track of these using signed permutations. We demonstrate this by the following stability result.

Lemma 3.14 *Let P be an abstract pre-polytope with a label structure (M, φ, Φ). Let $y \leq z$ with $\Phi_z \backslash \Phi_y = \{j_1, \dots, j_k\}$ where $j_1 < \dots < j_k$. Let (τ, α) be a positive signed permutation. If $y \in \mathcal{L}_{j_{\tau(1)}}^{\alpha(1)} \dots \mathcal{L}_{j_{\tau(k)}}^{\alpha(k)} z$ and there exists $i \in (k)$ such that $\tau(i) < \tau(i+1)$, then $y \in \mathcal{L}_{j_{\tau\sigma_i(1)}}^{\beta(1)} \dots \mathcal{L}_{j_{\tau\sigma_i(k)}}^{\beta(k)} z$ where $\sigma_i = (i, i+1)$ and*

$$\beta(j) = \begin{cases} -\alpha(i+1) & \textit{if } j = i, \\ \alpha(i) & \textit{if } j = i+1, \\ \alpha(j) & \textit{otherwise.} \end{cases}$$

Furthermore $(\tau\sigma_i, \beta)$ is a positive signed permutation.

Proof. Note that by (C3), $y \in \mathcal{L}_{j_{\tau(1)}}^{\alpha(1)} \dots \mathcal{L}_{j_{\tau(k)}}^{\alpha(k)} z$ determines a unique flag as shown in the following diagram.



Apply the diamond property to $u_{i-1} <_1 u_i <_1 u_{i+1}$ to obtain a face v_i with $u_{i-1} <_1 v_i <_1 u_{i+1}$. Note that $j_{\tau(i)} < j_{\tau(i+1)}$ so by Lemma 3.10, $\Phi_{u_{i+1}} = \Phi_{v_i} \cup \{j_{\tau(i)}\}$ and so $\Phi_{v_i} = \Phi_{u_{i-1}} \cup \{j_{\tau(i+1)}\}$. It follows immediately that $y \in \mathcal{L}^{\beta(1)}_{j_{\tau\sigma_i(1)}} \dots \mathcal{L}^{\beta(k)}_{j_{\tau\sigma_i(k)}} z$.

All that remains is to show that $(\tau\sigma_i, \beta)$ is a positive signed permutation. This amounts to showing that $\beta(m) = (-1)^{|\{j>m|\tau\sigma_i(j)<\tau\sigma_i(m)\}|}$ for all $m \in (k)$. Note that for $m \neq i, i+1$ we have $\tau\sigma_i(m) = \tau(m)$ and $\beta(m) = \alpha(m)$ so it suffices to prove the above equation for $m = i, i+1$.

Note that $\tau(i+1) < \tau(i)$ is equivalent to $\tau\sigma_i(i) < \tau\sigma_i(i+1)$ and so we have the following.

$$\begin{aligned}\{j > i \mid \tau\sigma_i(j) < \tau\sigma_i(i)\} &= \{j > i+1 \mid \tau\sigma_i(j) < \tau\sigma_i(i)\} \cup \{i+1\} \\ &= \{j > i+1 \mid \tau(j) < \tau(i+1)\} \cup \{i+1\}\end{aligned}$$

$$\begin{aligned}\{j > i+1 \mid \tau\sigma_i(j) < \tau\sigma_i(i)\} &= \{j > i+1 \mid \tau(j) < \tau(i)\} \\ &= \{j > i \mid \tau(j) < \tau(i+1)\}\end{aligned}$$

The case of $m = i$ follows from the first equation, and the case of $m = i+1$ follows from the second equation. □

Proposition 3.15 *Let P be an abstract pre-polytope with a label structure (M, φ, Φ). Let $y \leq z$ satisfying $y \in \mathcal{L}^{\alpha(1)}_{j_1} \dots \mathcal{L}^{\alpha(k)}_{j_k} z$ where $j_1 < \dots < j_k$ and $\alpha(i) = (-1)^{i-1}$. For each permutation τ we have $y \in \mathcal{L}^{\beta(1)}_{\tau(1)} \dots \mathcal{L}^{\beta(k)}_{\tau(k)} z$ where (τ, β) is a positive signed permutation.*

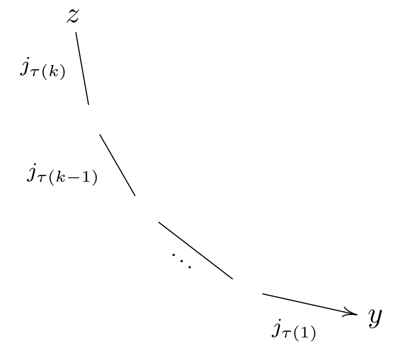
Proof. Let τ be a permutation on $\{0, \dots, k\}$. Consider the bubble sort algorithm: let i be the least such that $\tau(i) > \tau(i+1)$, then there is a permutation τ' such that $\tau = \tau'\sigma_i$ and $\tau'(i) < \tau'(i+1)$. By iterating this process, we terminate at an identity permutation. It follows that we have a sequence of permutations $\tau_s = \sigma_{i_s} \dots \sigma_{i_1}$ for each $1 \leq s \leq t$. Furthermore, we have $\tau_s(i_{s+1}) < \tau(i_{s+1} + 1)$ and $\tau_t = \tau$.

Note that $(1, \alpha)$ is a positive signed permutation. By applying Lemma 3.14 to the flag $y \in \mathcal{L}^{\alpha(1)}_{j_1} \dots \mathcal{L}^{\alpha(k)}_{j_k} z$ we obtain a flag $y \in \mathcal{L}^{\beta_1(1)}_{j_{\sigma_{i_1}(1)}} \dots \mathcal{L}^{\beta_1(k)}_{j_{\sigma_{i_1}(k)}} z$ where (σ_{i_1}, β_1) is a positive signed permutation. Using the above decomposition of τ, the result follows by iteratively applying Lemma 3.14. □

We are ready to prove that Definition 3.12 does give a characterisation of $R(z) \backslash R(z)^-$, and so deserves the name $\pi(z)$ according to Definition 1.46 found in Chapter 1.

Theorem 3.16 *Let P be a simple abstract pre-polytope with a label structure (M, φ, Φ). Let z be a face with* $\operatorname{rank} z = p$. *Then* $(R(z)\backslash R(z)^-)_{p-k} = \sum \mathcal{L}_{j_1}^{\alpha_1} \ldots \mathcal{L}_{j_k}^{\alpha_k} z$ *where* $\alpha_i = (-1)^{i-1}$.

Proof. We will first show that $\sum \mathcal{L}_{j_1}^{\alpha_1} \ldots \mathcal{L}_{j_k}^{\alpha_k} z \subseteq (R(z)\backslash R(z)^-)_{p-k}$. Consider the flag $y \in \mathcal{L}_{j_1}^{\alpha_1} \ldots \mathcal{L}_{j_k}^{\alpha_k} z$ in the interval $[y, z]$. Note that $\operatorname{rank} z - \operatorname{rank} y = k$ so by Proposition 3.7, $[y, z]$ has $k!$ flags. Now by Proposition 3.15, for each permutation τ on (k) we have $y \in \mathcal{L}_{\tau(1)}^{\beta(1)} \ldots \mathcal{L}_{\tau(k)}^{\beta(k)} z$ where (τ, β) is a positive signed permutation. Hence every permutation τ determines a unique flag of $[y, z]$. As there are $k!$ permutations, this implies that there is a bijection between the flags of $[y, z]$ and permutations on (k). Moreover, every flag has parity as indicated in the following diagram

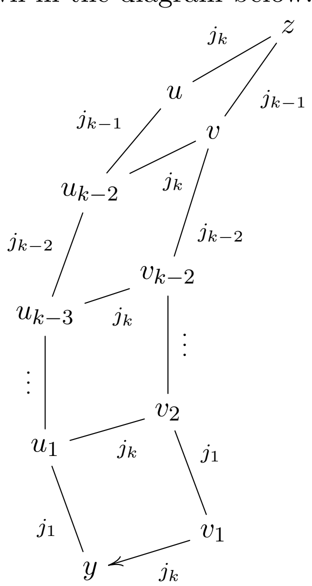


since (τ, β) is a positive signed permutation and so $\beta(1) = +1$. Hence $y \notin R(z)^-$ and so $y \in R(z)\backslash R(z)^-$.

We will now use induction on $\operatorname{rank} z$ to show that $R(z)\backslash R(z)^- \subseteq \sum \mathcal{L}_{j_1}^{\alpha_1} \ldots \mathcal{L}_{j_k}^{\alpha_k} z$. Let $y \in R(z)\backslash R(z)^-$, $\Phi_z \backslash \Phi_y = \{j_1, \ldots, j_k\}$ where $j_1 < \ldots < j_k$. Now since P is a graded poset, there exists a face u such that $y \leq u <_1 z$ and $\Phi_z = \Phi_u \cup \{j_i\}$ for some $1 \leq i \leq k$. Firstly, we will show that $y \in R(u)\backslash R(u)^-$. Note that $R(u) \subseteq R(z)$ so $R(u)^- \subseteq R(z)^-$ and so $y \notin R(u)^-$. Thus $y \in R(u)\backslash R(u)^-$ as required. Consider the following two cases.

Case 1: $i = k$.

By induction, we have $y \in \mathcal{L}_{j_1}^{\alpha_1} \ldots \mathcal{L}_{j_{k-1}}^{\alpha_{k-1}} u$. It follows that $y \in \mathcal{L}_{j_1}^{\alpha_1} \ldots \mathcal{L}_{j_{k-1}}^{\alpha_{k-1}} \mathcal{L}_{j_k}^{\eta} z$ which determines a unique flag as shown in the diagram below.

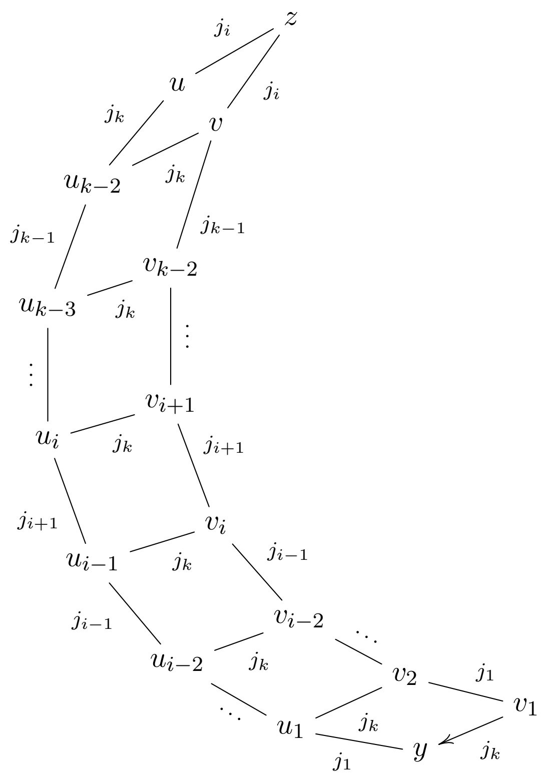


Apply the diamond property to $u_{k-2} <_1 u <_1 z$ to obtain a face v with $v \neq u$ and $u_{k-2} <_1 v <_1 z$. By Lemma 3.10, $\Phi_z = \Phi_v \cup \{j_{k-1}\}$ and so $\Phi_v = \Phi_{u_{k-2}} \cup \{j_k\}$. Repeat this type of argument to complete the above diagram.

Note that since $y \notin R(z)^-$ we have $y \in v_1^+$. By the parity relations of Lemma 3.10, it follows that $u \in z^\eta$ where $\eta = (-1)^{k-1} = \alpha_k$ as required.

Case 2: $i < k$.

By induction, we have $y \in \mathcal{L}_{j_1}^{\alpha_1} \ldots \mathcal{L}_{j_{i-1}}^{\alpha_{i-1}} \mathcal{L}_{j_{i+1}}^{\alpha_i} \ldots \mathcal{L}_{j_k}^{\alpha_{k-1}} u$ which determines a unique flag as shown in the diagram below.



Apply the diamond property to $u_{k-2} <_1 u <_1 z$ to obtain a face v with $v \neq u$ and $u_{k-2} <_1 v <_1 z$. Suppose that $\Phi_z = \Phi_v \cup \{j_i\}$. By a similar argument as in Case 1, we complete the above diagram. Now since $y \notin R(z)^-$ we have $y \in v_1^+$ and so by the parity relations of Lemma 3.10, $u_{k-2} \in v^\eta$ where $\eta = (-1)^{k-2}$. However we also have $u_{k-2} \in u^{-\eta}$, which is a contradiction since $-\eta = (-1)^{k-1} \neq (-1)^{k-2} = \alpha_{k-1}$. Thus $\Phi_z = \Phi_v \cup \{j_k\}$ and so the result follows by Case 1. □

For any abstract pre-polytope with a given label structure, we now have an explicit formula for the set π and (by duality) μ which is given by Definition 3.12. We now prove an important result regarding the rank 0 faces of the sets π and μ. By duality, we only need to prove this for π.

Proposition 3.17 *Let P be an abstract pre-polytope with a label structure (M, φ, Φ). Let z be a face with $|\Phi_z| = p$. Then $\pi(z)_0$ is a singleton.*

Proof. Note that for the case of $p = 0$, $\pi(z)_0 = \{z\}$ is a singleton. Let $p > 0$ and $\Phi_z = \{j_1, \ldots, j_p\}$ where $j_1 < \ldots < j_p$. By definition, $\pi(z)_0 = \mathcal{L}_{j_1}^{\alpha_1} \ldots \mathcal{L}_{j_p}^{\alpha_p} z$ where $\alpha_i = (-1)^{i-1}$. Note that by (C4), for any face x we have $\mathcal{L}_j^\eta x$ is a singleton for $j = \max \Phi_x$ and $\eta \in \{-, +\}$. Firstly, $\mathcal{L}_{j_p}^{\alpha_p} z$ is a singleton since $j_p = \max \Phi_z$. Suppose that $\mathcal{L}_{j_i}^{\alpha_i} \ldots \mathcal{L}_{j_p}^{\alpha_p} z$ is a singleton for some $i < p$. Let $y \in \mathcal{L}_{j_i}^{\alpha_i} \ldots \mathcal{L}_{j_p}^{\alpha_p} z$ be the unique element. Now $\Phi_y = \Phi_z \backslash \{j_i, \ldots, j_p\} = \{j_1, \ldots, j_{i-1}\}$ and so $\mathcal{L}_{j_{i-1}}^{\alpha_{i-1}} \mathcal{L}_{j_i}^{\alpha_i} \ldots \mathcal{L}_{j_p}^{\alpha_p} z$ is a singleton since $j_{i-1} = \max \Phi_y$. Hence by recursion, $\pi(z)_0 = \mathcal{L}_{j_1}^{\alpha_1} \ldots \mathcal{L}_{j_p}^{\alpha_p} z$ is a singleton. □

3.1.3 Linearity axiom

In this section, we will consider the preorder $\blacktriangleleft$ as given in Definition 1.47 of Chapter 1. To prove that the linearity axiom holds we need to show that $\blacktriangleleft$ is anti-symmetric (which makes it a partial order) and moreover it is a linear ordering. We will first show that the preorder $\blacktriangleleft$ is anti-symmetric and then use this to show that we have a linear ordering.

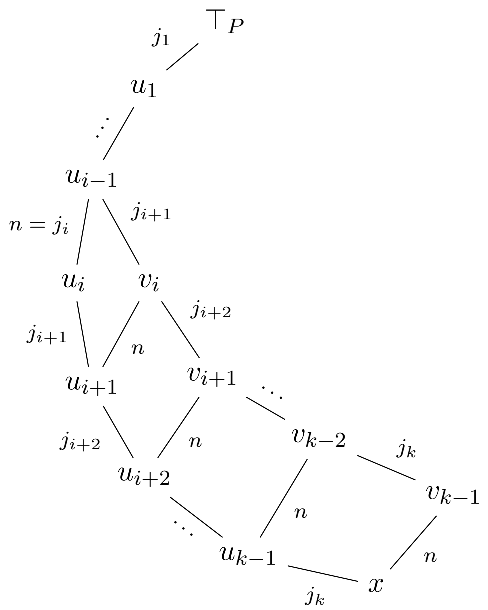
It will be shown in a later section that our examples of label structures consist of a family of polytopes; the hypercubes, associahedra and permutohedra are among the examples that we will consider. We now formalise this observation to show that simple abstract pre-polytopes are part of a nested family of polytopes indexed by their dimension. This result will then be used to prove anti-symmetry of $\blacktriangleleft$. We begin by making the following definition; this will determine a nested family of polytopes.

Definition 3.18 Let P be an abstract pre-polytope with a label structure (M, φ, Φ), and $n = \max M$. Write ∂P for the sub-poset of P containing the least face $\perp_P \in P$ and faces $x \in \widetilde{P}$ with $n \notin \Phi_x$ such that there exists a unique face y with $x \leq y$, $\Phi_y = \Phi_x \cup \{n\}$. Additionally, we require the unique face y to satisfy $\varphi_y(n) = \varphi_x(n)$.

It is immediate from the definition that the dimension of ∂P is one less than that of P. We seek to show that by applying ∂ successively we obtain a nested family of polytopes. We will show that by beginning with a simple abstract pre-polytope, we do obtain a nested family of simple abstract pre-polytopes. Furthermore, we will show that there is an induced label structure on ∂P. In the following, we prove various properties which then leads to our desired result of Proposition 3.21.

Proposition 3.19 *Let P be an abstract pre-polytope with a label structure (M, φ, Φ), and $n = \max M$. Let $x \in \widetilde{P}$ be given. If $n \notin \Phi_x$, then there exists a face y with $x \leq y$, and $\Phi_y = \Phi_x \cup \{n\}$.*

Proof. Note that $x \leq \top_P$ and so since P is a graded poset we have a flag as shown the following diagram. As indicated we have $\Phi_{\top_P} \backslash \Phi_x = \{j_1, \ldots, j_k\}$. Now since $n \in \Phi_{\top_P} = M \backslash \{\min M\}$ and that $n \notin \Phi_x$, there exists an i such that $j_i = n$.



Apply the diamond property to $u_{i+1} <_1 u_i <_1 u_{i-1}$ to obtain a face v_i with $v_i \neq u_i$ and $u_{i+1} <_1 v_i <_1 u_{i-1}$. By Lemma 3.10, $\Phi_{u_{i-1}} = \Phi_{v_i} \cup \{j_{i+1}\}$ and so $\Phi_{v_i} = \Phi_{u_{i+1}} \cup \{n\}$. Repeat this type of argument to complete the above diagram. Hence we have a face v_{k-1} with $x \leq v_{k-1}$ and $\Phi_{v_{k-1}} = \Phi_x \cup \{n\}$. $\square$

Lemma 3.20 *Let P be a simple abstract pre-polytope with a label structure (M, φ, Φ), and $n = \max M$. Let $x, y \in \widetilde{P}$ with $x \leq y$ and $n \notin \Phi_x \cap \Phi_y$. Then $x \in \partial P$ iff $y \in \partial P$.*

Proof. We will first prove that $x \in \partial P$ implies $y \in \partial P$. Note that it suffices to show this for the case of $x \leq y$ with $\Phi_y = \Phi_x \cup \{j\}$ for some $j < n$. Suppose that $y \notin \partial P$; since $n \notin \Phi_y$ then by Proposition 3.19, there exists a face z with $y \leq z$, $\Phi_z = \Phi_y \cup \{n\}$ and $\varphi_z(n) \neq \varphi_y(n)$. Apply the diamond property to $x <_1 y <_1 z$ to obtain a face y' with $y' \neq y$ and $x <_1 y' <_1 z$ as shown in the following diagram.

$$\begin{tikzcd} & z & \\ y \arrow[ur, dash, "n"] & & y' \arrow[ul, dash, "j"'] \\ & x \arrow[ul, dash, "j"] \arrow[ur, dash, "n"'] & \end{tikzcd}$$

By Lemma 3.10, we have $\Phi_z = \Phi_{y'} \cup \{j\}$ and so $\Phi_{y'} = \Phi_x \cup \{n\}$. By (C2), $\varphi_{y'}(n) \neq \varphi_x(n)$ and so it follows that $x \notin \partial P$, which is a contradiction. Hence $y \in \partial P$.

Now we will prove that $y \in \partial P$ implies $x \in \partial P$. Note that it suffices to show this for the case of $x \leq y$ with $\Phi_y = \Phi_x \cup \{j\}$ for some $j < n$. Suppose that $x \notin \partial P$; since $n \notin \Phi_x$ then by Proposition 3.19, there exists a face y' with $x \leq y'$, $\Phi_{y'} = \Phi_x \cup \{n\}$ and $\varphi_{y'}(n) \neq \varphi_x(n)$. By Proposition 3.8 where $y', y \leq \top_P$, there exists a face z with $y' <_1 z$ and $y <_1 z$ as shown in the following diagram.

$$\begin{tikzcd} & z & \\ y' \arrow[ur, dash, "j"] & & y \arrow[ul, dash, "n"'] \\ & x \arrow[ul, dash, "n"] \arrow[ur, dash, "j"'] & \end{tikzcd}$$

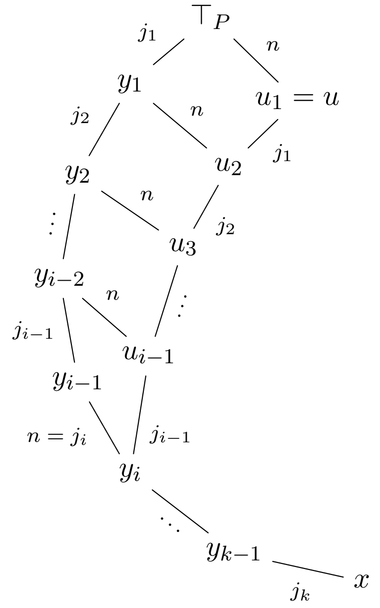
Note that $\Phi_{y'} \cup \{j\} = \Phi_z = \Phi_y \cup \{n\}$ and so by (C2), $\varphi_z(n) \neq \varphi_y(n)$. It follows that $y \notin \partial P$, which is a contradiction. Hence $u \in \partial P$. $\square$

Proposition 3.21 *Let P be a simple abstract pre-polytope with a label structure (M, φ, Φ), and $n = \max M$. Then ∂P is also a simple abstract pre-polytope. Furthermore, the assignation $\psi_f(i) = \varphi_f(i)$ and $\Psi_f = \Phi_f$ for each $f \in \widetilde{\partial P} \subseteq \widetilde{P}$ defines a label structure $(M \backslash \{n\}, \psi, \Psi)$ on ∂P.*

Proof. We will first show that ∂P is an abstract pre-polytope. By definition, ∂P is a graded poset since it is a sub-poset of P. Note that the the rank function of ∂P is the rank function of P restricted to ∂P. All that remains is to show that ∂P is bounded and satisfies the diamond property.

Firstly, we will show that ∂P is bounded. Note that by the definition of a label structure, $n = \max \Phi_{\top_P}$. By (C4)(a), there exists a unique face u with $u \leq \top_P$, $\Phi_{\top_P} = \Phi_u \cup \{n\}$ and $\varphi_{\top_P}(n) = \varphi_u(n)$. We claim that $\partial P = \{y \in P \mid y \leq u\}$ from which we can deduce that ∂P is bounded. By Lemma 3.20, $\{y \in P \mid y \leq u\} \subseteq \partial P$ since $u \in \partial P$.

Let $x \in \partial P$ be given; we will now show that $x \in \{y \in P \mid y \leq u\}$. Note that P is a graded poset so we have a flag as shown the following diagram. As indicated we have $\Phi_{\top_P} \backslash \Phi_x = \{j_1, \ldots, j_k\}$. Note that $n \notin \Phi_x$, there exists an i such that $j_i = n$.



Apply the diamond property to $y_i <_1 y_{i-1} <_1 y_{i-2}$ to obtain a face u_{i-1} with $u_{i-1} \neq y_{i-1}$ and $y_i <_1 u_{i-1} <_1 y_{i-2}$. By Lemma 3.20, $y_i \in \partial P$ since $n \notin \Phi_{y_i}$. Now by Lemma 3.10, we must have $\Phi_{y_{i-2}} = \Phi_{u_{i-1}} \cup \{n\}$ and so $\Phi_{u_{i-1}} = \Phi_{y_i} \cup \{j_{i-1}\}$. Repeat this type of argument to complete the above diagram. Hence we have $x \leq u_1 = u$ and so $x \in R(u)$ as required.

We will now show that ∂P satisfy the diamond property. Let x, y, z be in ∂P such that $x <_1 y <_1 z$. Now since ∂P is a sub-poset of P, apply the diamond property in P to obtain a unique face y' with $y' \neq y$ and $x <_1 y' <_1 z$. All that remains is to show that $y' \in \partial P$. Note that $n \notin \Phi_{y'}$ so by Lemma 3.20, $y' \in \partial P$ as required.

Hence we have shown that ∂P satisfies the axioms (P1), (P2), (P3) and so we have shown that ∂P is an abstract pre-polytope.

We will now show that ∂P is simple. Let $[x, u]_{\partial P}$ be a vertex figure of ∂P; it needs to be shown that this is isomorphic to a simplex. Denote with $[x, u]_P$ for this interval when calculated P. We claim that $[x, u]_{\partial P} = [x, u]_P$ from which we can deduce using Proposition 3.7(a) that $[x, u]_{\partial P}$ is isomorphic to a simplex. Note that $[x, u]_{\partial P} \subseteq [x, u]_P$ since ∂P is a sub-poset of P. It follows from Lemma 3.20 that $[x, u]_P \subseteq [x, u]_{\partial P}$. Hence $[x, u]_{\partial P} = [x, u]_P$ as required.

Finally, we will show that $(M \backslash \{n\}, \psi, \Psi)$ is a label structure on ∂P. Note that all of the axioms follow immediately with the exception of (C4). However, this follows from Lemma 3.20 by a similar argument used to show the diamond property. □

In the following, we define a map from P to ∂P which has been motivated by the examples considered by Street in [24]. The main idea behind such a map is that it should be well behaved with respect to the parity structure. We will prove various results which then allow us to show that this map preserves the preorder ◀; this will be our Proposition 3.29.

Proposition 3.22 *Let P be an abstract pre-polytope with a label structure (M, φ, Φ), and $n = \max M$. For any $x \in \widetilde{\partial P}$ there exists a unique maximal length path with the following configuration*

$$\begin{tikzcd}
 & z_1 \arrow[dr, dash, "n"] & & z_2 & & z_{k-1} \arrow[dr, dash, "n"] & & z_k \arrow[dr, dash, "n"] & \\
x = x_0 \arrow[ur, dash, "n"] & & x_1 \arrow[ur, dash, "n"] & & \cdots & & x_{k-1} \arrow[ur, dash, "n"] & & x_k
\end{tikzcd}$$

where $\varphi_{z_i}(n) \neq \varphi_{x_i}(n)$ and $\varphi_{x_i}(n) = \varphi_{z_{i+1}}(n)$. Denote this path by $x \rightsquigarrow$. Furthermore, for any face $y \in \widetilde{P}$, there exists a unique face $x \in \partial P$ such that y is contained in the path $x \rightsquigarrow$.

Proof. Let $x_0 = x \in \partial P \subseteq P$, then by definition there exists a unique face $z_1 \in P$ with $x_0 \leq z_1$, $\Phi_{z_1} = \Phi_{x_0} \cup \{n\}$ and $\varphi_{z_1}(n) = \varphi_{x_0}(n)$. By (C4)(b), there exists a unique face $x_1 \in P$ such that $x_1 \leq z_1$, $\Phi_{z_1} = \Phi_{x_1} \cup \{n\}$, $\varphi_{z_1}(n) \neq \varphi_{x_1}(n)$. If $x_1 \in \partial P$, then we are done. Otherwise, repeat the above calculations.

By the uniqueness part of (C3) and (C4), there exists a uniquely determined diagram in the above form for some $k \geq 1$. All that needs to be verified is that k is finite i.e. the above process terminates. Note that $\varphi_{x_i}(n) = \varphi_{z_1}(n) < \varphi_{x_{i+1}}(n)$ so it follows that $x_0, x_1, \ldots, x_k$ are distinct faces. Now by finiteness of P the above process must terminate. Hence we have shown that there exists a maximal length path $x \rightsquigarrow$.

We will now show the furthermore part of the proposition. By applying a similar argument involving (C3) and (C4), there must exist an $x \in \partial P$ such that y is contained in $x \rightsquigarrow$. Uniqueness of such an x follows from the uniqueness parts of (C3) and (C4). $\square$

Definition 3.23 Let P be an abstract pre-polytope with a label structure (M, φ, Φ), and $n = \max M$. Define a map $P \xrightarrow{\partial} \partial P$ given by $\partial \bot_P = \bot_P$ and for $x \in \widetilde{P}$, $\partial x = x_0$ where $x_0 \in \partial P$ such that x is contained in the path $x_0 \rightsquigarrow$.

Proposition 3.24 *Let P be an abstract pre-polytope with a label structure (M, φ, Φ), and $n = \max M$. Let x,y be faces with $x <_1 y$. Then $\partial x = \partial y$ iff $\Phi_y = \Phi_x \cup \{n\}$.*

Proof. Firstly, we will prove that $\partial x = \partial y$ implies $\Phi_y = \Phi_x \cup \{n\}$. By definition of the map ∂, we have the following path $\partial x \rightsquigarrow$.

$$\begin{tikzcd}[column sep=small] & z_1 \arrow[dl, dash, "n"'] \arrow[dr, dash, "n"] & & z_2 \arrow[dl, dash, "n"'] & \cdots & z_{k-1} \arrow[dr, dash, "n"] & & z_k \arrow[dl, dash, "n"'] \arrow[dr, dash, "n"] & \\ \partial x = x_0 & & x_1 & & & & x_{k-1} & & x_k \end{tikzcd}$$

Suppose that $\partial x = \partial y$; we will now show that $\Phi_y = \Phi_x \cup \{n\}$. Note that $\operatorname{rank} x + 1 = \operatorname{rank} y$ so we must have $x = x_i$ and $y = z_{i'}$ for some i, i'. It follows immediately that $\Phi_y = \Phi_x \cup \{n\}$.

We will now prove that $\Phi_y = \Phi_x \cup \{n\}$ implies $\partial x = \partial y$. Suppose that $\Phi_y = \Phi_x \cup \{n\}$. By definition of ∂, x appears in the path $\partial x \rightsquigarrow$ as shown above. It follows that $x = x_i$ for some $0 \leq i \leq k$. Recall that $x <_1 y$ and $\Phi_y = \Phi_x \cup \{n\}$ so either $y = z_{i-1}$ or $y = z_{i+1}$, and so y is contained in the path $\partial x \rightsquigarrow$. Hence by definition of ∂, $\partial y = \partial x$ as required. $\square$

Definition 3.25 Let P be an abstract pre-polytope with a label structure (M, φ, Φ), and $n = \max M$. Write $x \rightsquigarrow y$ when there exists a path of the following configuration

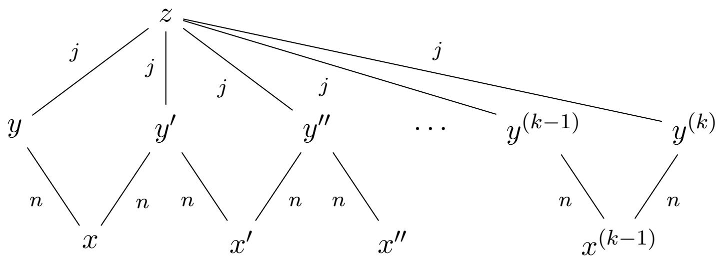
$$\begin{tikzcd}[column sep=small] & z_1 \arrow[dl, dash, "n"'] \arrow[dr, dash, "n"] & & z_2 \arrow[dl, dash, "n"'] & \cdots & z_{k-1} \arrow[dr, dash, "n"] & & z_k \arrow[dl, dash, "n"'] \arrow[dr, dash, "n"] & \\ x = x_0 & & x_1 & & & & x_{k-1} & & x_k = y \end{tikzcd}$$

where $\varphi_{z_i}(n) \neq \varphi_{x_i}(n)$ and $\varphi_{x_i}(n) = \varphi_{z_{i+1}}(n)$ and where $k \geq 0$.

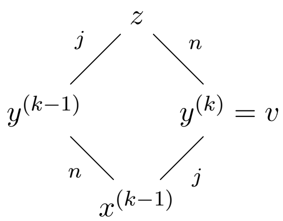
Proposition 3.26 *Let P be an abstract pre-polytope with a label structure (M, φ, Φ), and $n = \max M$. Let x, y, z be faces with $x \leq y \leq z$, $\Phi_y = \Phi_x \cup \{n\}$, $\varphi_y(n) = \varphi_x(n)$, and $\Phi_z = \Phi_y \cup \{j\}$ for some $j < n$. Let v be a face with $v \leq z$, $\Phi_z = \Phi_v \cup \{n\}$, $\varphi_z(n) = \varphi_v(n)$. Then there exists a face u such that $u \leq v$ with $\Phi_v = \Phi_u \cup \{j\}$ and $u \rightsquigarrow x$.*

Proof. Suppose that $u \not\leq v$ for all u with $u \rightsquigarrow x$. Apply the diamond property to $x <_1 y <_1 z$ to obtain a face y' with $y' \neq y$ and $x <_1 y' <_1 z$. Suppose that $\Phi_z = \Phi_{y'} \cup \{n\}$ and so $\Phi_{y'} = \Phi_x \cup \{j\}$. By (C2), we have $\varphi_z(n) = \varphi_{y'}(n)$ and so by the uniqueness part of (C4)(a), $y' = v$, which is a contradiction. Thus $\Phi_z = \Phi_x \cup \{j\}$ and so $\Phi_{y'} = \Phi_x \cup \{n\}$. Let x' be the unique face with $x' \leq y'$, $\Phi_{y'} = \Phi_{x'} \cup \{n\}$ and $\varphi_{y'}(n) = \varphi_{x'}(n)$.

Repeat the above process to obtain a sequence $y', y'', \dots, y^{(k)}$ as shown in the diagram below. Note that $\varphi_{y^{(i)}}(n) = \varphi_{x^{(i)}}(n) > \varphi_{y^{(i+1)}}(n)$ so this a sequence of distinct faces.



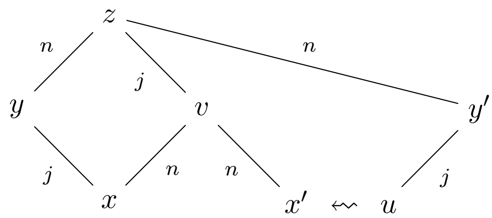
By finiteness of P, this process must terminate for some $k \geq 1$. Here termination must occur whenever $\Phi_z = \Phi_{y^{(k)}} \cup \{n\}$ as shown in the following diagram.



Note that $x^{(k-1)} \rightsquigarrow x$ and $x^{(k-1)} \leq y^{(k)} = v$, which is a contradiction. Hence there exists a face u with $u \leq v$, $\Phi_v = \Phi_u \cup \{j\}$ and $u \rightsquigarrow x$. □

Corollary 3.27 *Let P be an abstract pre-polytope with a label structure (M, φ, Φ), and $n = \max M$. Let x, y, z be faces with $x \leq y \leq z$, $\Phi_z = \Phi_y \cup \{n\}$, $\varphi_z(n) \neq \varphi_y(n)$, and $\Phi_y = \Phi_x \cup \{j\}$ for some $j < n$. Let y' be a face with $y' \leq z$, $\Phi_z = \Phi_{y'} \cup \{n\}$, $\varphi_y(n) = \varphi_x(n)$. Then there exists a face u such that $u \leq y'$ with $\Phi_{y'} = \Phi_u \cup \{j\}$ and $u \rightsquigarrow x$.*

Proof. Apply the diamond property to $x <_1 y <_1 z$ to obtain a face v with $v \neq y$ and $x <_1 v <_1 z$ By Lemma 3.10, $\Phi_z = \Phi_v \cup \{j\}$ and so $\Phi_v = \Phi_x \cup \{n\}$. By (C2), $\varphi_v(n) \neq \varphi_x(n)$. By (C4)(a), there exists a face x' with $x' \leq v$, $\Phi_v = \Phi_{x'} \cup \{n\}$ and $\varphi_v(n) = \varphi_{x'}(n)$. Now by Proposition 3.26, there exists a face u with $u \leq y'$, $\Phi_{y'} = \Phi_u \cup \{j\}$ and $u \rightsquigarrow x'$. We summarise the above in the following diagram.



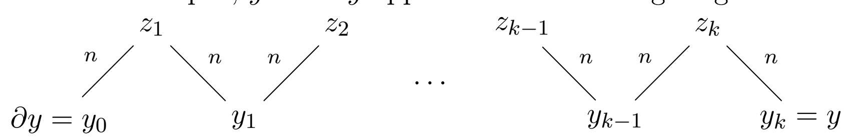
Finally, note that $u \rightsquigarrow x$ and so the result follows. □

Proposition 3.28 *Let P be a simple abstract pre-polytope with a label structure (M, φ, Φ), and $n = \max M$. Let x,y be faces with $x <_1 y$ and $\Phi_y = \Phi_x \cup \{j\}$ for some $j < n$. Then $\partial x <_1 \partial y$ with $\Psi_{\partial y} = \Psi_{\partial x} \cup \{j\}$. Furthermore, $\psi_{\partial y}(j) = \psi_{\partial x}(j) \iff \varphi_y(j) = \varphi_x(j)$.*

Proof. Consider the following cases.

Case 1: $n \notin \Phi_y$.

By definition of the map ∂, y and ∂y appear in the following diagram



where $\varphi_{z_i}(n) \neq \varphi_{y_i}(n)$ and $\varphi_{y_i}(n) = \varphi_{z_{i+1}}(n)$.

By Corollary 3.27, there exists a face x' with $x' \leq y_{k-1}$, $\Phi_{y_{k-1}} = \Phi_{x'} \cup \{j\}$ and $x' \rightsquigarrow x$. Repeat the above argument to obtain the following diagram.

$$\begin{tikzcd}
 & z_1 \arrow[dl, dash, "n"'] \arrow[dr, dash, "n"] & & & & z_{k-1} \arrow[dl, dash, "n"'] \arrow[dr, dash, "n"] & & z_k \arrow[dl, dash, "n"'] \arrow[dr, dash, "n"] & \\
\partial y = y_0 \arrow[d, dash, "j"'] & & y_1 \arrow[d, dash, "j"] & \cdots & y_{k-2} \arrow[d, dash, "j"'] & & y_{k-1} \arrow[d, dash, "j"] & & y_k = y \arrow[d, dash, "j"] \\
x^{(k)} \arrow[rr, squiggly] & & x^{(k-1)} & & x'' \arrow[rr, squiggly] & & x' \arrow[rr, squiggly] & & x
\end{tikzcd}$$

By Lemma 3.20, $x^{(k)} \in \partial P$ since $y_0 \in \partial P$. It follows that $\partial x = x^{(k)}$ and so $\partial x <_1 \partial y$ with $\Psi_{\partial y} = \Phi_y = \Phi_x \cup \{j\} = \Psi_{\partial x} \cup \{j\}$. Note that by (C1)(c), $\varphi_{\partial y}(j) = \varphi_y(n)$ and $\varphi_{\partial x}(n) = \varphi_x(n)$. It follows that $\varphi_y(j) = \varphi_x(j) \iff \varphi_{\partial y} = \varphi_{\partial x}(j) \iff \psi_{\partial y} = \psi_{\partial x}(j)$.

Case 2: $n \in \Phi_x$.

By definition of the map ∂, y and ∂y appear in the following diagram

$$\begin{tikzcd}
 & y_1 \arrow[dl, dash, "n"'] \arrow[dr, dash, "n"] & & y_2 \arrow[dl, dash, "n"'] & & y_{k-1} \arrow[dr, dash, "n"] & & y_k = y \arrow[dl, dash, "n"'] \\
\partial y = v_0 & & v_1 & & \cdots & & v_{k-1} &
\end{tikzcd}$$

where $\varphi_{y_i}(n) \neq \varphi_{v_i}(n)$ and $\varphi_{y_{i+1}}(n) = \varphi_{v_i}(n)$.

By (C4)(a), there exists a face u with $u \leq x$, $\Phi_x = \Phi_u \cup \{n\}$ and $\varphi_x(n) = \varphi_u(n)$. By Proposition 3.26, there exists a face u' with $u' \leq v_{k-1}$, $\Phi_{v_{k-1}} = \Phi_{u'} \cup \{j\}$ and $u' \rightsquigarrow u$. By Corollary 3.27, there exists a face u'' with $u'' \leq v_{k-2}$, $\Phi_{v_{k-2}} = \Phi_{u''} \cup \{j\}$, and $u'' \rightsquigarrow u'$. Repeat the above argument to obtain the following diagram.

$$\begin{tikzcd}
 & y_1 \arrow[dl, dash, "n"'] \arrow[dr, dash, "n"] & & & & y_{k-1} \arrow[dl, dash, "n"'] \arrow[dr, dash, "n"] & & y_k = y \arrow[dl, dash, "n"'] \arrow[dr, dash, "j"] & \\
\partial y = v_0 \arrow[d, dash, "j"'] & & v_1 \arrow[d, dash, "j"] & \cdots & v_{k-2} \arrow[d, dash, "j"'] & & v_{k-1} \arrow[d, dash, "j"] & & x \arrow[d, dash, "n"] \\
u^{(k)} \arrow[rr, squiggly] & & u^{(k-1)} & & u'' \arrow[rr, squiggly] & & u' \arrow[rr, squiggly] & & u
\end{tikzcd}$$

By Lemma 3.20, $u^{(k)} \in \partial P$ since $v_0 \in \partial P$. It follows that $\partial x = u^{(k)}$ and so $\partial x <_1 \partial y$ with $\Psi_{\partial y} = \Phi_y = \Phi_x \cup \{j\} = \Psi_{\partial x} \cup \{j\}$. Finally, by a similar argument as in Case 1 it can be shown that $\varphi_y(j) = \varphi_x(j) \iff \varphi_{\partial y} = \varphi_{\partial x}(j) \iff \psi_{\partial y} = \psi_{\partial x}(j)$. $\square$

Proposition 3.29 *Let P be a simple abstract pre-polytope with a label structure (M, φ, Φ). Consider the parity structure on $\widetilde{P}$ and $\widetilde{\partial P}$ defined in Proposition 3.3. Let $x, y \in \widetilde{P}$. Then $x \blacktriangleleft y$ implies $\partial x \blacktriangleleft \partial y$.*

Proof. In Definition 1.47, the partial ordering $\blacktriangleleft$ is defined to be generated by the relation $x \blacktriangleleft_1 y$ iff either $y \in x^+$ or $x \in y^-$. It suffices to show that $x \blacktriangleleft_1 y$ implies either $\partial x = \partial y$ or $\partial x \blacktriangleleft_1 \partial y$. Consider the case of $x \in y^-$. Note that $x \leq y$ with $\Phi_y = \Phi_x \cup \{j\}$ for some $j \in M$. If $j = \max M$, then by Proposition 3.24, $\partial x = \partial y$. Otherwise $j < \max M$, then by Proposition 3.28, $\partial x \leq \partial y$ with $\Psi_{\partial y} = \Psi_{\partial x} \cup \{j\}$ and $\psi_{\partial y}(j) = \psi_{\partial x}(j) \iff \varphi_y(j) = \varphi_x(j)$. Note that $\#(j; \partial y) = \#(j; y)$ since $\Psi_{\partial y} = \Phi_y$. It follows that $\partial x \in (\partial y)^-$ and so $\partial x \blacktriangleleft_1 \partial y$.

By a similar argument as above, it can be shown that $\partial x = \partial y$ or $\partial x \in (\partial y)^+$ for the case of $x \in y^-$. Hence we have shown that either $\partial x = \partial y$ or $\partial x \blacktriangleleft_1 \partial y$. $\square$

Theorem 3.30 *Let P be a simple abstract pre-polytope with a label structure (M, φ, Φ). Consider the parity structure on $\widetilde{P}$ defined in Proposition 3.3. The preorder $\blacktriangleleft$ on $\widetilde{P}$ is anti-symmetric.*

Proof. This proof will be by induction on the rank of P which is equal to rank $\top_P$. Consider the smallest non-trivial case of an abstract pre-polytope with rank $\top_P = 1$. Note that by the diamond property we may take P to be a line segment as shown in the following diagram. Also note that the line segment is simple since its vertex figures are a 0-simplex.

$$\begin{tikzcd}
 & \{0,1\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{0\} \arrow[dr, dash] & & \{1\} \arrow[dl, dash] \\
 & \varnothing &
\end{tikzcd}$$

We will show that there exists a label structure (M, φ, Φ) on P and show that is uniquely up to some trivial relabelling. We shall take $M = \{0, 1\}$ with the usual ordering of natural numbers. Note that since $\{0, 1\}$ is the greatest face so we require that $\Phi_{\{0,1\}} = M\backslash\{\min M\} = \{1\}$. Now we must have $\Phi_{\{0\}} = \Phi_{\{1\}} = \varnothing$ since these are the vertices. Note that for any $x \in \widetilde{P}$ we must have $\varphi_x(i) \leq i$ for all $i \in M$. It follows that $\varphi_x(0) = 0$ for all $x \in \widetilde{P}$. Note that in order to satisfy (C4) we can only have $\varphi_{\{0,1\}}(1) = 0$, and so we can assume without loss of generality that $\varphi_{\{0\}}(1) = 0$ and $\varphi_{\{1\}}(1) = 1$. We have now have a label structure (M, ϕ, Φ) on the line segment P, and it is clear that this is the only possible label structure (up to some relabelling as indicated earlier).

We now calculate the parity as defined in Proposition 3.3. It can be verified that $\{0\} \in \{0, 1\}^+$ and $\{1\} \in \{0, 1\}^-$ and so $\{1\} \blacktriangleleft_1 \{0, 1\} \blacktriangleleft_1 \{0\}$. Hence $\blacktriangleleft$ is anti-symmetric.

All that remains is to show the inductive step. Let P be a simple abstract pre-polytope with $\operatorname{rank} P > 1$, and $n = \max M$. Note that we have $|M| > 2$. Consider the following result. If $x \blacktriangleleft y \blacktriangleleft x$, then by Proposition 3.29, $\partial x \blacktriangleleft \partial y \blacktriangleleft \partial x$. It follows by induction on $\operatorname{rank} P$ that $\partial x = \partial y$.

Suppose that there exists a sequence $x = x_0 \blacktriangleleft_1 x_1 \blacktriangleleft_1 x_2 \blacktriangleleft_1 \ldots \blacktriangleleft_1 x_k = x$; we aim to show a contradiction. By the above result, we have $\partial x_i = \partial x_{i+1}$ for all i. By definition of $\blacktriangleleft$, $x_i \blacktriangleleft_1 x_{i+1}$ iff either $x_{i+1} \in x_i^+$ or $x_i \in x_{i+1}^-$. By Proposition 3.24, either $\Phi_{x_i} = \Phi_{x_{i+1}} \cup \{n\}$ or $\Phi_{x_{i+1}} = \Phi_{x_i} \cup \{n\}$. Thus (subject to relabelling of the x's) we have the following diagram.

$$\begin{tikzcd}
x = x_0 \arrow[dr, "n"'] & & x_2 \arrow[dr, "n"'] & & \cdots & & x_{k-2} \arrow[dr, "n"'] & & x_k = x \\
 & x_1 \arrow[ur, "n"'] & & x_3 & & x_{k-3} \arrow[ur, "n"'] & & x_{k-1} \arrow[ur, "n"'] &
\end{tikzcd}$$

However, by Lemma 3.11 we obtain a contradiction. Hence $\blacktriangleleft$ is anti-symmetric. □

All that remains is to prove linearity of the partial order $\blacktriangleleft$; for all faces x, y we either have $x \blacktriangleleft y$ or $y \blacktriangleleft x$. In the following, we define a map and then explain why it could be understood as a successor map.

Definition 3.31 Let P be a simple abstract pre-polytope with a label structure (M, φ, Φ). Define a map $\nu : \widetilde{P}\backslash\pi(\top_P)_0 \longrightarrow \widetilde{P}\backslash\mu(\top_P)_0$ given by the following. Let $x \in \widetilde{P}\backslash\pi(\top_P)_0$ be given. If there exists a face y with $x \in y^-$, $\Phi_y = \Phi_x \cup \{j\}$ and $j > \max \Phi_x$, then let $\nu(x)$ be such a y for the greatest possible j. Otherwise, by (C4), let $\nu(x) = y$ for the face with $y \in x^+$ and $\Phi_y = \Phi_x\backslash\{\max \Phi_x\}$.

Remark Note that by Proposition 3.17 and its dual, $\pi(\top_P)_0$ and $\mu(\top_P)_0$ are singletons. We will show that the unique element of these sets is the greatest and least element (respectively) with respect to $\blacktriangleleft$.

We may call the above map a successor map since it satisfies the property that $x \blacktriangleleft_1 \nu(x)$ for each $x \in \widetilde{P}\backslash\pi(\top_P)_0$.

In the following, we will define another map ξ which is dual to ν. We will show in Proposition 3.33 that these maps are inverses. Following this we prove in Theorem 3.34 that $\blacktriangleleft$ is a linear order.

Definition 3.32 Let P be a simple abstract pre-polytope with a label structure (M, φ, Φ). Define a map $\xi : \widetilde{P}\backslash\mu(\top_P)_0 \longrightarrow \widetilde{P}\backslash\pi(\top_P)_0$ given by the following. Let $x \in \widetilde{P}\backslash\mu(\top_P)_0$ be given. If there exists a face y with $x \in y^+$, $\Phi_y = \Phi_x \cup \{j\}$ and $j > \max \Phi_x$, then let $\xi(x)$ be such a y for the greatest possible j. Otherwise, by (C4), let $\xi(x) = y$ for the face with $y \in x^-$ and $\Phi_y = \Phi_x \backslash \{\max \Phi_x\}$.

Remark We may call the above map a predecessor map since it satisfy the property that $\xi(x) \blacktriangleleft_1 x$ for each $x \in \widetilde{P}\backslash\mu(\top_P)_0$.

Proposition 3.33 *Let P be a simple abstract pre-polytope with a label structure (M, φ, Φ). The map $\nu : \widetilde{P}\backslash\pi(\top_P)_0 \longrightarrow \widetilde{P}\backslash\mu(\top_P)_0$ is a bijection with inverse ξ.*

Proof. To prove that ν is a bijection we will show that it is both injective and surjective. Firstly, we will show that ν is injective; $\nu(x) = \nu(y)$ implies $x = y$. Let $x, y \in \widetilde{P}\backslash\pi(\top_P)_0$ with $\nu(x) = \nu(y)$ be given. Consider the following cases.

Case 1: $\operatorname{rank} x = \operatorname{rank} y$.

Note that there are two subcases. Consider the case of $\nu(x) \in x^+$ and $\nu(y) \in y^+$ which is shown in the following diagram.

$$\begin{tikzcd}
 & z \arrow[dr, dash, "\max \Phi_x"] & \\
x \arrow[ur, "\max \Phi_y"] \arrow[dr, "\max \Phi_x"'] & & y \arrow[dl, "\max \Phi_y"] \\
 & \nu(x) = \nu(y) &
\end{tikzcd}$$

Suppose that $\max \Phi_x \neq \max \Phi_y$; we assume without loss of generality that $\max \Phi_x < \max \Phi_y$. By Proposition 3.8 where $x, y \leq \top_P$, there exists a face z with $x <_1 z$ and $y <_1 z$. Note that $\Phi_z = \Phi_x \cup \{\max \Phi_y\}$ and $\Phi_z = \Phi_y \cup \{\max \Phi_x\}$. By Lemma 3.10, $x \in z^-$. However this implies that $x \in \nu(x)^-$, which is a contradiction. Thus $\max \Phi_x = \max \Phi_y$ and so by (C3), $x = y$ as required.

Consider the case of $x \in \nu(x)^-$ and $y \in \nu(y)^-$ shown in the following diagram.

$$\begin{tikzcd}
 & \nu(x) = \nu(y) & \\
x \arrow[ur, "j"] & & y \arrow[ul, "h"']
\end{tikzcd}$$

Note that $j > \max \Phi_x$ and $h > \max \Phi_y$. It follows that $j = \max \Phi_{\nu(x)} = \max \Phi_{\nu(y)} = h$. By (C4), $x = y$ as required.

Case 2: $\operatorname{rank} x \neq \operatorname{rank} y$.

Note that $|\operatorname{rank} x - \operatorname{rank} y| = 2$ so we can assume without of loss of generality that $x \in \nu(x)^-$ and $\nu(y) \in y^+$. Apply the diamond property to $x <_1 \nu(x) <_1 y$ to obtain a face u with $u \neq \nu(x)$ and $x <_1 u <_1 y$ as shown in the following diagram.

$$\begin{tikzcd}
 & y \arrow[dl, "\max \Phi_y"'] & \\
\nu(x) = \nu(y) & & u \arrow[ul, "j"'] \\
 & x \arrow[ul, "j"] \arrow[ur, "\max \Phi_y"'] &
\end{tikzcd}$$

Note that $j > \max \Phi_x$. It follows that $j = \max \Phi_{\nu(x)} = \max \Phi_{\nu(y)} < \max \Phi_y$. By Lemma 3.10, $\Phi_y = \Phi_u \cup \{j\}$ and so $\Phi_u = \Phi_x \cup \{\max \Phi_y\}$. Note that $x \in u^-$ so by maximality of j, $\max \Phi_y \leq j$, which is a contradiction.

We will now show that ν is surjective; for each $x \in \widetilde{P}\backslash\mu(\top_P)_0$ there exists $w \in \widetilde{P}\backslash\pi(\top_P)_0$ with $\nu(w) = x$.

For each $x \in \widetilde{P}\backslash\mu(\top_P)_0$ we may let $w = \xi(x) \in \widetilde{P}\backslash\pi(\top_P)_0$. We will now show that $\nu(w) = x$. Consider the following cases.

Case 1: $x \in \xi(x)^+$.

Suppose that $w \in \nu(w)^-$; then consider the following diagram.

$$\begin{tikzcd}
& \nu(w) \arrow[dr, "j"] & \\
w = \xi(x) \arrow[ur, "h"] \arrow[dr, "j"'] & & u \arrow[dl, "h"] \\
& x &
\end{tikzcd}$$

Note that $j > \max\Phi_x$ and $h > \max\Phi_w$. It follows that $j = \max\Phi_{\xi(x)} = \max\Phi_w < h$. Apply the diamond property to $x <_1 w <_1 \nu(w)$ to obtain a face u with $u \neq w$ and $x <_1 u <_1 \nu(w)$. By Lemma 3.10, $\Phi_{\nu(w)} = \Phi_u \cup \{j\}$ and so $\Phi_u = \Phi_x \cup \{h\}$. Note that $x \in u^+$ so by the maximality of j, $h \leq j$, which is a contradiction. Thus $\nu(w) \in w^+$ and so we consider the following diagram.

$$\begin{tikzcd}
& w = \xi(x) \arrow[dl, "j"'] \arrow[dr, "\max\Phi_w"] & \\
x & & \nu(w)
\end{tikzcd}$$

It follows that $j = \max\Phi_{\xi(x)} = \max\Phi_w$ so by the uniqueness part of (C4), $x = \nu(w)$ as required.

Case 2: $\xi(x) \in x^-$.

Note that $\max\Phi_x > \max\Phi_w$ so we have $w \in \nu(w)^-$ where $\Phi_{\nu(w)} = \Phi_w \cup \{j\}$ as shown in the following diagram. By the maximality of j, we have $j \geq \max\Phi_x$.

$$\begin{tikzcd}
& z \arrow[dr, "j"] & \\
\nu(w) \arrow[ur, "\max\Phi_x"] & & x \\
& w = \xi(x) \arrow[ul, "j"] \arrow[ur, "\max\Phi_x"'] &
\end{tikzcd}$$

Suppose that $j < \max\Phi_x$; then by Proposition 3.8 where $\nu(w) \leq \top_P$ and $x \leq \top_P$, there exists a face z with $\nu(w) <_1 z$ and $x <_1 z$. Note that $\Phi_z = \Phi_{\nu(w)} \cup \{\max\Phi_x\}$ and $\Phi_z = \Phi_x \cup \{j\}$. By Lemma 3.10, $x \in z^+$. However this implies that $x \in \xi(x)^+$, which is a contradiction. Thus $j = \max\Phi_x$ and so by the uniqueness part of (C3), $\nu(w) = x$ as required. $\square$

Theorem 3.34 *Let P be a simple abstract pre-polytope with a label structure (M, φ, Φ). Then $\blacktriangleleft$ is a linear order.*

Proof. Note that by Proposition 3.17, $\pi(\top_P)_0$ is a singleton so let x_1 be the unique element. Dually, let x_0 be the unique element of $\mu(\top_P)_0$.

By definition, for any $x \in \widetilde{P}\backslash\{x_1\}$ either $\nu(x) \in x^+$ or $x \in \nu(x)^-$ and so we have $x \blacktriangleleft_1 \nu(x)$. Consider the sequence $x_0, \nu(x_0), \nu^2(x_0), \ldots, \nu^k(x_0), \ldots$ which satisfies $\nu^i(x_0) \blacktriangleleft_1 \nu^{i+1}(x_0)$. By Theorem 3.30, $\blacktriangleleft$ is anti-symmetric so this sequence contains distinct faces. Hence by finiteness of P this sequence terminates with $\nu^k(x_0) = x_1$ for some k.

To prove that $\blacktriangleleft$ is a linear order, it suffices to show that for each $x \in \widetilde{P}$ there exists an n such that $x = \nu^n(x_0)$. Let $x \in \widetilde{P}\backslash\{x_0, x_1\}$ be given. For the case of $x = x_0$ or

$x = x_1$ this follows immediately. By Proposition 3.33, ν is a bijection and moreover its inverse is ξ. Recall that $\xi(x) \blacktriangleleft_1 x$ for any $x \in \widetilde{P}\backslash\{x_0\}$. Consider the sequence $x, \xi(x), \xi^2(x), \ldots, \xi^n(x), \ldots$ which satisfies $\xi^i(x) \blacktriangleleft_1 \xi^{i-1}(x)$. By a similar argument as above, the above sequence terminates with $\xi^n(x) = x_0$ for some n. Hence $x = \nu^n(\xi^n(x)) = \nu^n(x_0)$ as required. □

We have now shown that for a simple abstract pre-polytope with a given label structure, the parity structure as defined in Proposition 3.3 satisfies the linearity axiom **(L)**. We have now proven sufficient results to deduce, as promised, that our abstract pre-polytope is an abstract polytope.

Proposition 3.35 *Let P be an abstract pre-polytope with a label structure (M, φ, Φ). If P is simple, then P is an abstract polytope.*

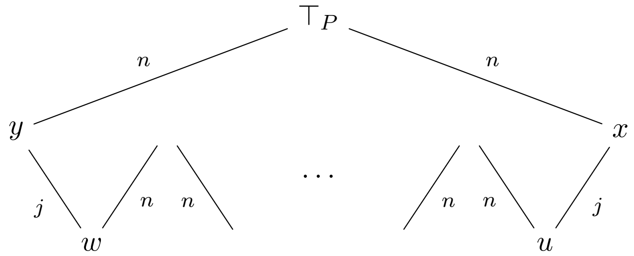
Proof. All that needs to be shown is that (P4) holds; every interval of P is connected. Note that since P is simple, every interval of P except for P itself is isomorphic to a simplex which is connected. It suffices to show that P is connected; we need to show that for any $x, y \in P\backslash\{\bot_P, \top_P\}$ there exists a sequence $x = x_0, \ldots, x_k = y$ in $P\backslash\{\bot_P, \top_P\}$ such that x_i and x_{i+1} are incident for all i. Consider the following cases.

Case 1: either $x, y \blacktriangleleft \top_P$ or $\top_P \blacktriangleleft x, y$.

By Theorem 3.34, we assume without loss of generality that $x \blacktriangleleft y$. It follows that there exists a sequence $x = x_0, \ldots, x_k = y$ with $x_i \blacktriangleleft_1 x_{i+1}$ for all i. Note that in this case we must have $x_i \neq \top_P$ for all i. By definition of $\blacktriangleleft_1$, we have either $x_i \in x_{i+1}^-$ or $x_{i+1} \in x_i^+$. Hence $x_i \leq x_{i+1}$ or $x_{i+1} \leq x_i$ and so x_i and x_{i+1} are incident as required.

Case 2: either $x \blacktriangleleft \top_P \blacktriangleleft y$ or $y \blacktriangleleft \top_P \blacktriangleleft x$.

Note that by Case 1 we may assume without loss of generality that $x = \xi(\top_P) \blacktriangleleft_1 \top_P \blacktriangleleft_1 \nu(\top_P) = y$. Let $n = \max M$; we have $\Phi_x = \Phi_y = \Phi_{\top_P}\backslash\{n\}$. Consider the case of $\varphi_x(n) = \varphi_{\top_P}(n) \neq \varphi_y(n)$. Note that (C4) ensures that there exists a face $w \leq y$ with $\Phi_y = \Phi_w \cup \{j\}$ where $j = \max \Phi_y < n$. By Corollary 3.27, there exists $u \leq x$ such that there is a path $u \rightsquigarrow w$ as shown in the following diagram.



Hence in either case we have shown that there exists a sequence $x = x_0, \ldots, x_k = y$ in $P\backslash\{\bot_P, \top_P\}$ such that x_i and x_{i+1} are incident for all i. □

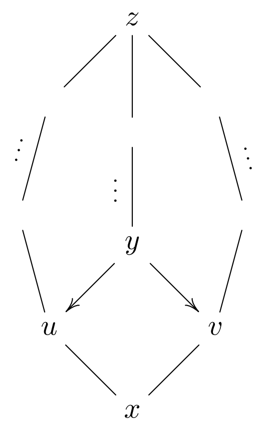
3.1.4 Cellularity axiom

We have so far shown that for a simple abstract pre-polytope with a given label structure, the parity structure defined in Proposition 3.3 satisfies axioms 1*, 2 and **(L)**. It remains to show that axiom **(C)** holds. We will now show the first part of axiom **(C)** part (a); $\pi(z)$ and $\mu(z)$ are well formed subsets. Note that by duality it suffices to show this for $\pi(z)$.

Theorem 3.36 *Let P be a simple abstract pre-polytope with a label structure (M, φ, Φ). Let z be a face with $|\Phi_z| = p$. Then $\pi(z) = R(z)\backslash R(z)^-$ is well formed.*

Proof. It suffices to show that $\pi(z)_k$ is well formed for each $0 \leqslant k \leqslant p$. By definition, $\pi(z)_p = \{z\}$ is a singleton. By Proposition 3.17, $\pi(z)_0$ is a singleton. By definition, singletons are well formed so all that remains is to prove that $\pi(z)_{p-k}$ is well formed for $0 < k < p$.

Let $u, v \in \pi(z)_{p-k}$ for some $0 < k < p$ be given. If $u^\eta \cap v^\eta \neq \varnothing$ for $\eta \in \{-,+\}$, then let $x \in u^\eta \cap v^\eta$; we seek to prove that $u = v$. Note that $x <_1 u$ and $x <_1 v$ and so Proposition 3.8 where $u \leq z$ and $v \leq z$, there exists a face $y \leq z$ with $u <_1 y$ and $v <_1 y$. We summarise the above in the following diagram.



We must have $u, v \in y^+$ since $u, v \notin R(z)^-$. Note that by axiom 2, we have y^+ is well formed. Recall that $u^\eta \cap v^\eta \neq \varnothing$ so by well formedness of y^+ we have $u = v$ as required. Hence $\pi(z)_{p-k}$ is well formed. □

We will now prove axiom **(C)** part (b); $s_n R(z)$ and $t_n R(z)$ are down-closed for all n. Again by duality, it suffices to show that $s_n R(z)$ is down-closed for all n. We will first prove the following special case which will then allow us to prove the full result in Theorem 3.39

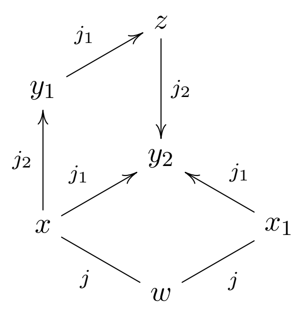
Proposition 3.37 *Let P be a simple abstract pre-polytope with a label structure (M, φ, Φ). Let z be a face, $k = \operatorname{rank} z - 2$, $x \in (s_k R(z))_k$. Then $w \in s_k R(z)$ for all $w <_1 x$.*

Proof. Note that $x \in (s_k R(z))_k = \mu(z)_k = \sum \mathcal{L}^-_{j_1} \mathcal{L}^+_{j_2} z$ as indicated in the following diagram.

$$\begin{tikzcd}
 & z \arrow[dr, "j_2"] & \\
y_1 \arrow[ur, "j_1"] & & y_2 \\
 & x \arrow[ul, "j_2"] \arrow[ur, "j_1"'] &
\end{tikzcd}$$

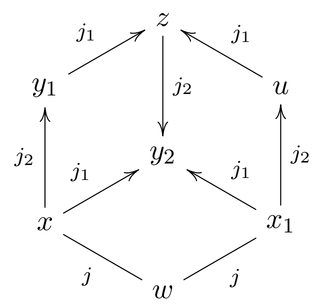
Let $w <_1 x$ with $\Phi_x = \Phi_w \cup \{j\}$ be given. We will now show that $w \in s_k R(z)$. By the definition of source, this amounts to showing that $w \in R(y^-)$ for each y with $w \leq y \leq z$ and $\operatorname{rank} y = k + 1 = \operatorname{rank} z - 1$. Consider the interval $[w, z]$. By Proposition 3.7, since $\deg z - \deg w = 3$ there exists exactly 3 faces y with $w \leq y \leq z$ and $\operatorname{rank} y = \operatorname{rank} z - 1$. Denote these faces by y_1, y_2, y_3. Let y_1 and y_2 coincide with the above diagram.

All that remains is to show that $w \in R(y_3^-)$. Apply the diamond property to $w <_1 x <_1 y_2$ to obtain a face x_1 with $x_1 \neq x$ and $w <_1 x_1 <_1 y_2$. By Lemma 3.10, we have $x_1 \in y_2^-$ as indicated in the diagram. Consider the following cases.



Case 1: $\Phi_{y_2} = \Phi_{x_1} \cup \{j_1\}$.

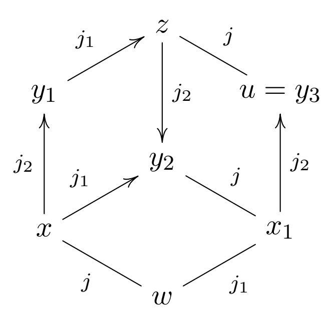
Apply the diamond property to $x_1 <_1 y_2 <_1 z$ to obtain a face u with $u \neq y_2$ and $x_1 <_1 u <_1 z$. By Lemma 3.10, $\Phi_z = \Phi_u \cup \{j_1\}$ and so we have parities as shown in the following diagram.



Note that by the uniqueness part of Proposition 3.8, $y_1 \neq u$ and so $u = y_3$. Hence $w <_1 x_1 \in y_3^-$ and so $w \in R(y_3^-)$ as required.

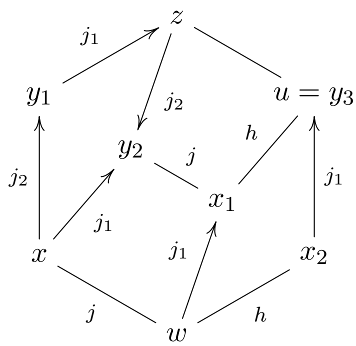
Case 2: $\Phi_{y_2} = \Phi_{x_1} \cup \{j\}$.

Apply the diamond property to $x_1 <_1 y_2 <_1 z$ to obtain a face u with $u \neq y_2$ and $x_1 <_1 u <_1 z$. Consider the case of $j < j_2$. By Lemma 3.10, $\Phi_z = \Phi_u \cup \{j\}$ and so we have parities as shown in the following diagram.



Note that $\Phi_u \neq \Phi_{y_1}$ and so $y_3 = u \neq y_1$. Hence $w <_1 x_1 \in y_3^-$ and so $w \in R(y_3^-)$ as required.

Now consider the case of $j > j_2 > j_1$. By Lemma 3.10, $w \in x_1^-$. Apply the diamond property to $x_1 <_1 y_2 <_1 z$ to obtain a face u with $u \neq y_2$ and $x_1 <_1 u <_1 z$. Note that $\Phi_u = \Phi_{x_1} \cup \{h\}$ where $h \in \{j_2, j\}$ and so $j_1 < h$. Apply the diamond property to $w <_1 x_1 <_1 u$ to obtain a face x_2 with $x_2 \neq x_1$ and $w <_1 x_2 <_1 u$. By Lemma 3.10, since $j_1 < h$ we must have $\Phi_u = \Phi_{x_2} \cup \{j_1\}$ and so $x_2 \in u^-$. We summarise the above in the following diagram.



Note that $\Phi_u \neq \Phi_{y_1}$ and so $y_3 = u \neq y_1$. Hence $w <_1 x_2 \in y_3^-$ and so $w \in R(y_3^-)$ as required. □

Lemma 3.38 *Let x, z be faces with $x \leq z$, and $k \leq \operatorname{rank} z - 1$. Then $x \in s_k R(z)$ iff $x \in s_k R(y)$ for all $y \in R(z)_{k+1}$ with $x \leq y$.*

Proof. Suppose that $x \in s_k R(y)$ for all $y \in R(z)_{k+1}$ with $x \leq y$; we aim to show that $x \in s_k R(z)$. By the definition of source,

$$s_k R(z) = \{x \in R(z)^{(k)} \mid \text{if } x \in R(y) \text{ for some } y \in R(z)_{k+1}, \text{ then } x \in R(y^-)\}.$$

Note that since $x \leq z$, there exists a $y \in R(z)_{k+1}$ such that $x \leq y$. Also note that for any $y \in R(z)_{k+1}$ we have $R(y)^{(k)} \subseteq R(z)^{(k)}$ since $y \leq z$. By our supposition we have $x \in s_k R(y)$ and so it follows that $x \in R(z)^{(k)}$. If $x \in R(u)$ for some $u \in R(z)_{k+1}$, then by our supposition we have $x \in s_k R(u)$. By the definition of source,

$$s_k R(u) = \{x \in R(u)^{(k)} \mid \text{if } x \in R(u), \text{ then } x \in R(u^-)\}$$

since $R(u)_{k+1} = \{u\}$. It follows immediately that $x \in R(u^-)$ and so $x \in s_k R(z)$ as required.

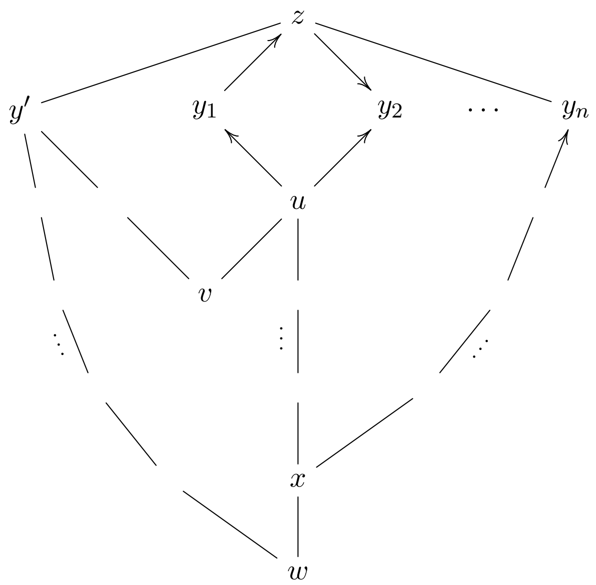
We will now show the converse. Suppose that $x \in s_k R(z)$; we aim to show that $x \in s_k R(y)$ for all $y \in R(z)_{k+1}$ with $x \leq y$. Let $y \in R(z)_{k+1}$ with $x \leq y$ be given. It needs to be shown that $x \in s_k R(y)$; this amounts to showing that if $x \in R(y)$, then $x \in R(u^-)$. This follows immediately from the supposition that $x \in s_k R(z)$. Hence we have shown that $x \in s_k R(y)$ for all $y \in R(z)_{k+1}$ with $x \leq y$. □

Theorem 3.39 *Let P be a simple abstract pre-polytope with a label structure (M, φ, Φ). Let z be a face. Then $s_k R(z)$ is down-closed for all k.*

Proof. We will prove that $s_k R(z)$ is down-closed for all k using induction on $\operatorname{rank} z$. Note that the statement holds trivially for the case of $\operatorname{rank} z = 0$ so all that needs to be shown is the inductive step.

Note that for $k \geq \operatorname{rank} z$ we have $s_k R(z) = R(z)$ which is down-closed. For $k = \operatorname{rank} z - 1$ we have $s_k R(z) = R(z^-)$ which is also down-closed. Thus it suffices to prove the statement for all $k \leq \operatorname{rank} z - 2$.

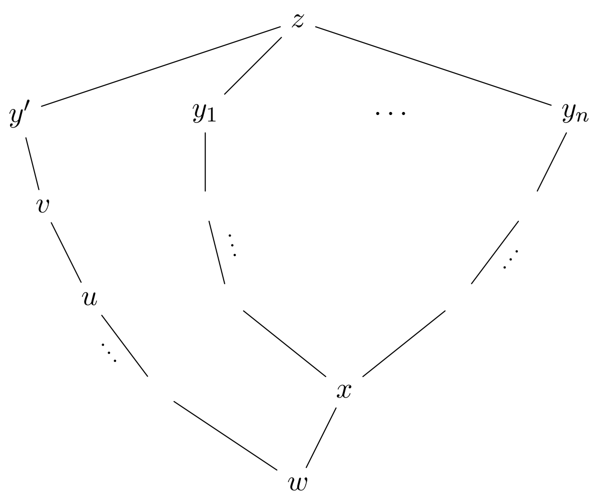
Firstly, we will show that $s_k R(z)$ is down-closed for $k = \operatorname{rank} z - 2$. This amounts to showing that for each $x \in s_k R(z)$, $w \in s_k R(z)$ for all $w <_1 x$. We will prove this using induction on $n = \operatorname{rank} z - \operatorname{rank} x$. The case of $n = 2$ follows from Proposition 3.37 so all that remains is to prove the inductive step. Let $x \in s_k R(z)$ and $w <_1 x$ with $\operatorname{rank} z - \operatorname{rank} x = n > 2$ be given. Note that by Proposition 1.49 and Proposition 1.50, $s_k R(z) \subseteq R(z^{-\mp})$ so there exists $u \in z^{-\mp}$ such that $x \leq u$ where $x \neq u$ since $\operatorname{rank} z - \operatorname{rank} x > 2$. Now by Proposition 3.7, the interval $[x, z]$ has n distinct faces $y_1, \ldots, y_n$ with rank equal to $\operatorname{rank} z - 1$. This is summarised in the following diagram.



Let $w <_1 x$ be given. By Proposition 3.7, $[w, z]$ has $n + 1$ distinct faces $y_1, \ldots, y_n, y'$ with rank equal to $\operatorname{rank} z - 1$. By Proposition 3.9, there exists a face $v \leq z$ with $v \leq y'$ and $v <_1 u$. Note that $u \in (s_k R(z))_k$ so by Proposition 3.37, $v \in s_k R(z)$ since $v <_1 u$. Now since $v \leq y' \in R(z)_{k+1}$ we have $v \in R(y'^-)$ and so $w \in R(y'^-)$. Hence $w \in s_k R(z)$ as required.

We will now show that $s_k R(z)$ is down-closed for all $k < \operatorname{rank} z - 2$. This amounts to showing that for each $x \in s_k R(z)$, we have $w \in s_k R(z)$ for all $w <_1 x$. Let $x \in s_k R(z)$ with $\operatorname{rank} z - \operatorname{rank} x = n$. Note that $\operatorname{rank} x \leq k < \operatorname{rank} z - 2$ so it follows that $n > 2$. By Proposition 3.7, there are n distinct faces $y_1, \ldots, y_n$ in $[x, z]$ with rank equal to $\operatorname{rank} z - 1$. By Lemma 3.38, $x \in s_k R(y_i)$ for all $i = 1, \ldots, n$.

Let $w <_1 x$ be given. All that remains is to show that $w \in s_k R(z)$. By Proposition 3.7, there are $n + 1$ distinct faces $y_1, \ldots, y_n, y'$ in $[w, z]$ with rank equal to $\operatorname{rank} z - 1$. The above is summarised in the following diagram.



Now since $w \leq x \in s_k R(y_i$ for all $i = 1, \ldots, n$ then by induction on $\operatorname{rank} z$, we have $w \in s_k R(y_i)$ for all $i = 1, \ldots, n$. By Lemma 3.38, it suffices to show that $w \in s_k R(y')$. Let u be a face with $w \leq u \leq y'$ and $\operatorname{rank} u = k + 1$. We will show that $w \in R(u^-)$ which implies that $w \in s_k R(y')$. Note that $u \neq y'$ (otherwise $u = y'$ so $\operatorname{rank} z - 1 = \operatorname{rank} y' = \operatorname{rank} u = k + 1$ and so $k = \operatorname{rank} z - 2$ which contradicts $k < \operatorname{rank} z - 2$). Since P is a graded poset, there exists $v <_1 y'$ with $u \leq v$. Applying the diamond property to $v <_1 y' <_1 z$ we must obtain y_i (for some i) with $v <_1 y_i <_1 z$. Now $w \leq u \leq v <_1 y_i$ and so since

$w \in s_k R(y_i)$, it follows that $w \in R(u^-)$ as required. □

We have now shown that a label structure on a simple abstract pre-polytope satisfies axioms 1*, 2, (L) and (C). It follows by the results of Section 1.4 of Chapter 1 that we have a loop free pasting scheme. This is summarised as our main result of this chapter.

Theorem 3.40 *Let P be a simple abstract pre-polytope with a label structure (M, φ, Φ). The parity structure on $\widetilde{P}$ defined in Proposition 3.3 satisfy axioms 1*, 2, 3(a), (L) and (C). Furthermore, the pasting scheme defined in Theorem B.6 is a loop free pasting scheme.*

3.2 Products of label structures

In this section, we will introduce a notion of product for abstract pre-polytopes that preserves the label structure. It will be shown that such a notion of product has a striking resemblance to Street's product of parity complexes found in Section 5 of [24]. Consequently, such a product allows us to obtain further examples of label structures simply by taking products. In the next section, we use iterated products to provide an elegant way of handling label structures for hypercubes.

We shall use the symbol $\times$ to denote the product of posets; this involves a Cartesian product of the underlying set, together with the point-wise ordering. We will now give a definition of product for abstract pre-polytopes.

Definition 3.41 Let P and Q be abstract pre-polytopes. The *smash product* of P and Q is the poset denoted by $P \wedge Q$ which is $P \times Q$ with the identification $(f, \perp_Q) \sim (\perp_P, g)$ for all $f \in P$ and $g \in Q$.

Remark The above definition corresponds to the smash product of pointed spaces, where the basepoints of P and Q are $\perp_P$ and $\perp_Q$ respectively. Note that the identified point in the smash product is the least element, and so it follows that $\widetilde{P \wedge Q} = \widetilde{P} \times \widetilde{Q}$.

Recall that our main result Theorem 3.40 involves simple abstract pre-polytopes with a given label structure. We will first prove that the smash product of simple abstract pre-polytopes is also a simple abstract pre-polytope.

Proposition 3.42 *Let P and Q be partially ordered sets. If both P and Q satisfy (P2) and (P3), then so does $P \times Q$.*

Proof. Firstly, we will prove that $P \times Q$ is a graded poset. Let $\operatorname{rank}(f,g) = \operatorname{rank} f + \operatorname{rank} g$ for all $(f,g) \in P \times Q$. If $(f,g) <_1 (h,k)$, then either $f <_1 h$ and $g = k$, or $g <_1 k$ and $f = h$. In either case it follows that $\operatorname{rank}(h,k) = \operatorname{rank}(f,g) + 1$. Suppose there exists faces (f,g) and (h,k) such that $(f,g) \leqslant (h,k)$ and $\operatorname{rank}(h,k) = \operatorname{rank}(f,g) + 1$. It follows that either $\operatorname{rank} h = \operatorname{rank} f + 1$ and $\operatorname{rank} g = \operatorname{rank} k$, or $\operatorname{rank} k = \operatorname{rank} g + 1$ and $\operatorname{rank} f = \operatorname{rank} k$. Thus either $f <_1 h$ and $g = k$, or $g <_1 k$ and $f = k$. Hence in either case $(f,g) <_1 (h,k)$ as required.

We will now prove that $P \times Q$ satisfy (P3). Let $(f,g) \leqslant (h,k)$ where $\operatorname{rank}(h,k) - \operatorname{rank}(f,g) = 2$. It needs to be shown that $[(f,g),(h,k)]$ is a line segment. Consider the following cases.

Case 1: $\operatorname{rank} f = \operatorname{rank} h$ and $\operatorname{rank} k - \operatorname{rank} g = 2$.

Note that $f = h$ and so $[(f,g),(h,k)] \cong \{f\} \times [g,k]$. Now by (P3), $[g,k]$ is a line segment. Hence $[(f,g),(h,k)]$ is a also line segment.

Case 2: $\operatorname{rank} g = \operatorname{rank} k$ and $\operatorname{rank} h - \operatorname{rank} f = 2$.

By a similar argument as in Case 1, it can be shown that $[(f,g),(h,k)] \cong [f,h] \times \{g\}$ which is a line segment.

Case 3: $\operatorname{rank} h = \operatorname{rank} f + 1$ and $\operatorname{rank} k = \operatorname{rank} g + 1$.

Note that since P and Q are graded posets we have $f <_1 h$ and $g <_1 k$. Observe that $[(f,g),(h,k)] = \{(f,g),(f,k),(h,g),(h,k)\}$ and moreover we have the following diagram.

$$\begin{tikzcd}
 & (h,k) \arrow[dl, dash] \arrow[dr, dash] & \\
(f,k) \arrow[dr, dash] & & (h,g) \arrow[dl, dash] \\
 & (f,g) &
\end{tikzcd}$$

which is a line segment. $\square$

Proposition 3.43 *Let P and Q be simple abstract pre-polytopes. Then $P \wedge Q$ is a simple abstract pre-polytope.*

Proof. We will first show that $P \wedge Q$ is an abstract pre-polytope. Let $\bot$ denote the least face of $P \wedge Q$. Let the greatest faces of P and Q be $\top_P$ and $\top_Q$ respectively. It follows that $(f,g) \leqslant (\top_P, \top_Q)$ for all $(f,g) \in \widetilde{P} \times \widetilde{Q}$. Note that by definition, $\bot \leqslant (\top_P, \top_Q)$. Hence $P \wedge Q$ is a bounded poset.

We now seek to prove that $P \wedge Q$ is a graded poset. Consider the rank function on $P \wedge Q$ defined by

$$\operatorname{rank}(f,g) = \begin{cases} \operatorname{rank} f + \operatorname{rank} g & \text{if } (f,g) \in \widetilde{P} \times \widetilde{Q} \\ -1 & \text{if } (f,g) = \bot. \end{cases}$$

Given the proof of Proposition 3.42, all that needs to be shown is $\bot <_1 (f,g)$ iff $\operatorname{rank}(f,g) = \operatorname{rank} \bot + 1$. Note that $\bot \sim (f, \bot_Q) \sim (\bot_P, g)$ so it follows that $\bot <_1 (f,g)$ iff $\bot_P <_1 f$ and $\bot_Q <_1 g$. Hence the result follows.

We will now prove that $P \wedge Q$ satisfies the diamond property. By Proposition 3.42, $\widetilde{P} \times \widetilde{Q}$ satisfies (P3). All that remains to be shown is $[\bot, (f,g)]$ is a line segment for all $(f,g) \in \widetilde{P} \times \widetilde{Q}$ with $\operatorname{rank}(f,g) - \operatorname{rank} \bot = 2$. Equivalently, we have $\operatorname{rank}(f,g) = 1$ so either $\operatorname{rank} f = 0$ and $\operatorname{rank} g = 1$, or $\operatorname{rank} f = 1$ and $\operatorname{rank} g = 0$. Consider the case of $\operatorname{rank} g = 1$. Let $[\bot_Q, g] = \{\bot_Q, a, b, g\}$ denote a line segment. It follows that $[\bot, (f,g)] = \{\bot, (f,a), (f,b), (f,g)\}$ is also a line segment. The same conclusion can be made for the case of $\operatorname{rank} f = 1$.

Now that we have shown that $P \wedge Q$ is an abstract pre-polytope, all that remains is to show that it is simple. Let $(f,g) \in P \wedge Q = \widetilde{P} \times \widetilde{Q}$ with $\operatorname{rank}(f,g) = 0$ and so $\operatorname{rank} f = \operatorname{rank} g = 0$. Note that $[(f,g),(\top_P,\top_Q)] \cong [f, \top_P] \times [g, \top_Q]$ and so is isomorphic to a simplex. Hence every vertex figure of $P \wedge Q$ is isomorphic to a simplex and so $P \wedge Q$ is simple. $\square$

We will now define a label structure on the smash product and prove that the required conditions hold.

Theorem 3.44 *Let P and Q be simple abstract pre-polytopes with label structures (M, φ, Φ) and (N, ψ, Ψ) respectively. Let $T = M \oplus (N \setminus \{\min N\})$ denote the ordinal sum of M and $N \setminus \{\min N\}$. Assign to each face $(f,g) \in \widehat{P \wedge Q}$ the pair $(\theta_{(f,g)}, \Theta_{(f,g)})$ defined by*

$$\Theta_{(f,g)} = \Phi_f \cup \Psi_g,$$

$$\theta_{(f,g)}(k) = \begin{cases} \varphi_f(k) & \textit{if } k \in M, \\ \psi_g(k) & \textit{if } k \in N \backslash \{\min N\}. \end{cases}$$

Then (T, θ, Θ) *is a label structure on* $P \wedge Q$*. Furthermore, the parity structure on* $\widetilde{P \wedge Q}$ *defined in Proposition 3.3 is given as follows. Given* $f \in P_n$*,* $g \in Q_m$ *where* $n, m > 0$*, then*

$$(f,g)^\alpha = \begin{cases} (f^\alpha \times \{g\}) \cup (\{f\} \times g^{\varepsilon(\alpha)}) & \textit{if } n \textit{ is odd}, \\ (f^\alpha \times \{g\}) \cup (\{f\} \times g^\alpha) & \textit{if } n \textit{ is even.} \end{cases}$$

where $\alpha \in \{-, +\}$ *and* $\varepsilon(\alpha) = -\alpha$*.*

Proof. We will first prove that (C1) holds. Note that (C1)(b) follows immediately from the definition of θ. Let $(f, g), (h, k) \in \widetilde{P} \times \widetilde{Q}$ with $(f, g) \leqslant (h, k)$. We now seek to prove (C1)(a). If $\Theta_{(h,k)} = \Theta_{(f,g)} \cup \{j\}$ for some $j \in \Theta_{(h,k)}$ then either $\Phi_h = \Phi_f \cup \{j\}$ for some $j \in \Phi_h$, and $\Psi_g = \Psi_k$; or $\Psi_k = \Psi_g \cup \{j\}$ for some $j \in \Psi_k$, and $\Phi_f = \Phi_h$. It follows that either $f <_1 h$ and $g = k$, or $g <_1 k$ and $f = h$. Thus $(f, g) <_1 (h, k)$. The above implications are 'if and only if' and so (C1)(a) holds.

We will now prove (C1)(c) holds. Let $(f, g), (h, k) \in \widetilde{P} \times \widetilde{Q}$ with $(f, g) <_1 (h, k)$. Note that either $f <_1 h$ and $g = k$; or $g <_1 k$ and $f = h$. By (C1)(a), $\Theta_{(h,k)} = \Theta_{(f,g)} \cup \{j\}$ for some $j \in \Theta_{(h,k)}$. It follows that $j \in \Phi_h$ or $j \in \Phi_k$. If $j \in \Phi_h$, then for all $i < j$ we have $\varphi_h(i) = \varphi_f(i)$ and so $\theta_{(h,k)}(i) = \theta_{(f,g)}(i)$. If $j \in \Psi_k$ then for $i < j$ where $i \in N$ we have $\psi_k(i) = \varphi_g(i)$, for all $i \in M$ note that $\varphi_h(i) = \varphi_f(i)$ since $f = h$. Thus $\theta_{(h,g)}(i) = \theta_{(f,g)}(i)$ for all $i < j$. Hence (C1)(c) holds.

We will now prove that (C2) holds. Note that line segments in either $\widetilde{P} \times \{g\}$ or $\{f\} \times \widetilde{Q}$ for some $f \in \widetilde{P}$ and $g \in \widetilde{Q}$ are line segments in P or Q respectively. Hence all that needs to be checked are line segments of the following configuration.

$$\begin{tikzcd}[arrows=dash] & (f,g) \arrow[dl, "j_2"'] \arrow[dr, "j_1"] & \\ (f,b) \arrow[dr, "j_1"'] & & (a,g) \arrow[dl, "j_2"] \\ & (a,b) & \end{tikzcd}$$

Note that $a <_1 f$ and $b <_1 g$ in the above diagram so $\Phi_f = \Phi_a \cup \{j_1\}$ and $\Psi_g = \Psi_b \cup \{j_2\}$. The result follows immediately.

We will now prove that (C3)(a) holds. Let $(f, g) \in \widetilde{P} \times \widetilde{Q}$ and $j \notin \Theta_{(f,g)}$ be given. Consider the case of $j \in M$; since $j \notin \Phi_f$ there exists $h \in \widetilde{P}$ such that $f <_1 h$, $\Phi_h = \Phi_f \cup \{j\}$ and $\varphi_h(j) = \varphi_f(j)$. It follows that $(f, g) <_1 (h, g)$, $\Theta_{(h,g)} = \Theta_{(f,g)} \cup \{j\}$ and $\theta_{(h,g)}(j) = \theta_{(f,g)}(j)$. We have established the existence part of (C3)(a), all that remains is to prove uniqueness. Suppose that there exists (f', g') such that $(f, g) <_1 (f', g')$, $\Theta_{(f',g')} = \Theta_{(f,g)} \cup \{j\}$ with $j \in M$ and $\theta_{(f',g')}(j) = \theta_{(f,g)}(j)$. It follows that $\Phi_{f'} = \Phi_f \cup \{j\}$ and $g = g'$. Note that h as given above is unique so $f' = h$ as required. The case of $j \in N$ follows by similar arguments as above. Finally, note that (C3)(b) follows by a similar argument.

We will now prove that (C4)(a) holds. Let $(f, g) \in \widetilde{P} \times \widetilde{Q}$ and $j = \max \Theta_{(f,g)}$ be given. If $\Psi_g \neq \varnothing$, then $j = \max \Psi_g \subseteq N$. Otherwise $\Psi_g = \varnothing$, then $j = \max \Phi_f \subseteq M$. The result follows by similar arguments as in the above proof for (C3). Finally, note that (C4)(b) follows by a similar argument. $\square$

3.3 Examples

In this section, we will give our main examples for label structures. Our examples include the hypercubes, associahedra and permutohedra. These polytopes are part of a larger class of polytopes known as the hypergraph polytopes [7]. Hypergraph polytopes are also known as nestohedra in the sense of Postnikov [18]. In this section, we will be following the hypergraph polytopes paper by Došen and Petrić [7]. We will show in the following sections that the hypergraph polytope is a convenient setting for the purposes of constructing a label structure.

3.3.1 Hypergraph polytopes

The aim of this section is to introduce a class of abstract polytopes called the hypergraph polytopes defined by Došen and Petrić [7]; the faces are given by constructs of a given hypergraph. We will now give a brief introduction to hypergraph polytopes. All definitions and results within this section, unless stated otherwise, are due to Došen and Petrić.

A hypergraph is understood as a generalisation of a graph, whereby the edges (sometimes called hyperedges) can join any number of vertices. In [7], the definition of hypergraph used is a special type of building set as defined in [8].

Definition 3.45 A *hypergraph* on a finite set H is a subset $\mathbb{H} \subseteq \wp H \backslash \{\varnothing\}$ such that $H = \bigcup \mathbb{H}$.

Remark Note that the above definition of a hypergraph is not completely general. Sometimes the definition of hypergraph does not include the condition $H = \bigcup \mathbb{H}$, but in this thesis we shall always assume it. For instance, the above definition does not include the empty hypergraph on H which can be understood as a graph with no edges, and vertices consisting of the elements of H.

Example 3.46

(a) If H is empty, then $\varnothing$ is the unique hypergraph on H.

(b) If $H = \{*\}$, then $\{\{*\}\}$ is the unique hypergraph on H.

(c) If $H = \{0, 1\}$, then the following is a complete list of the hypergraphs on H:

- $\{\{0, 1\}\}$
- $\{\{0\}, \{1\}\}$
- $\{\{0\}, \{0, 1\}\}$
- $\{\{1\}, \{0, 1\}\}$
- $\{\{0\}, \{1\}, \{0, 1\}\}$

Definition 3.47 Let $\mathbb{H}$ be a hypergraph.

(a) The *intersection graph* of $\mathbb{H}$ is the graph, denoted by $\Omega(\mathbb{H})$, whose vertices are the elements of $\mathbb{H}$ and there is an edge between $X, Y \in \mathbb{H}$ whenever the intersection $X \cap Y$ is non-empty.

(b) $\mathbb{H}$ is *connected* if $\Omega(\mathbb{H})$ is a connected graph.

(c) $\mathbb{H}$ is *atomic* when for every $x \in \bigcup \mathbb{H}$ we have $\{x\} \in \mathbb{H}$.

Example 3.48 The intersection graph of $\mathbb{H} = \{\{0\}, \{1\}, \{0, 1\}\}$ is as follows.

$$\begin{tikzcd} \{0\} \arrow[rr, dash] \arrow[dr, dash] & & \{1\} \arrow[dl, dash] \\ & \{0,1\} & \end{tikzcd}$$

This is a connected graph, so $\mathbb{H}$ is a connected hypergraph. It is clearly atomic since it contains all of the singletons.

In the following, we will work towards defining a construct of an atomic connected hypergraph. Firstly, we will introduce the following notation. Let $Y \subseteq H$; then denote $\mathbb{H}_Y = \{X \in \mathbb{H} \mid X \subseteq Y\}$. Proposition 3.49 and Corollary 3.50 are required to give the definition of a construct.

Proposition 3.49 *Let $\mathbb{H}$ be a hypergraph on $H = \bigcup \mathbb{H}$. Then $\mathbb{H}$ is atomic iff $\mathbb{H}_Y$ is a hypergraph on Y for all $Y \subseteq H$.*

Proof. Suppose that $\mathbb{H}$ is atomic. Let $Y \subseteq H$ be given; we seek to prove that $\mathbb{H}_Y$ is a hypergraph on Y. If Y is empty, then $\mathbb{H}_Y$ is empty so the result follows trivially. Let Y be non-empty and note that $\bigcup \mathbb{H}_Y \subseteq Y$. For any $x \in Y \subseteq H = \bigcup \mathbb{H}$ we have $\{x\} \in \mathbb{H}$, it follows that $\{x\} \in \mathbb{H}_Y$ since $\{x\} \subseteq Y$. Thus $Y \subseteq \bigcup \mathbb{H}_Y$ and so $Y = \bigcup \mathbb{H}_Y$. Note that $\varnothing \notin \mathbb{H}_Y$ since $\varnothing \notin \mathbb{H}$ so $\mathbb{H}_Y$ is a hypergraph on Y as required.

We will now prove the converse. Suppose that $\mathbb{H}_Y$ is a hypergraph on Y for every subset $Y \subseteq H$. We seek to prove that $\mathbb{H}$ is atomic. Let $x \in H$ and consider $Y = \{x\}$. Now since $\mathbb{H}_{\{x\}}$ is a hypergraph on $\{x\}$ we must have $\mathbb{H}_{\{x\}} = \{\{x\}\}$. It follows that $\{x\} \in \mathbb{H}_{\{x\}} \subseteq \mathbb{H}$ as required. $\square$

Corollary 3.50 *If $\mathbb{H}$ is an atomic hypergraph and $Y \subseteq \bigcup \mathbb{H}$, then $\mathbb{H}_Y$ is an atomic hypergraph on Y.*

Write $\mathbb{H} \backslash Z = \mathbb{H}_{H \backslash Z}$. The next definition is due to Curien et al. [6], which is a characterisation of constructs as defined by Došen and Petrić [7].

Definition 3.51 Let $\mathbb{H}$ be an atomic connected hypergraph. A *construct* of $\mathbb{H}$ will be a certain non-planar rooted tree T with nodes decorated by subsets of H. We define these trees by recursion on $|H|$.

1. If $|H| = 0$, then there is a unique construct given by the empty tree.
2. If $|H| \geqslant 1$ and $X \subseteq H$ is non-empty, then let $\mathbb{H} \backslash X = \sum \mathbb{H}_i$ be the disjoint union of connected components. By Corollary 3.50, the $\mathbb{H}_i$'s are atomic connected hypergraphs. Then by recursion, T consists of a root which is decorated by X and has edges to the root of each construct T_i of the hypergraph $\mathbb{H}_i$ where $|\bigcup \mathbb{H}_i| < |H|$.

Remark The definition of a construction is given in Section 3 of [7], and the definition of construct is given in Section 5 of [7].

A construction is a special type of construct. If we require that every node is decorated by a singleton, then we obtain a construction. This definition is closely related to the definition of an f-construction given in [7].

Note that the above definition is for atomic connected hypergraphs, whereas in [7], the definition is for an atomic hypergraph. In the more general setting (atomic hypergraphs), we would have forests instead of trees.

Example 3.52 Recall the hypergraph $\mathbb{H} = \{\{0\}, \{1\}, \{0, 1\}\}$ given in Example 3.48. It was verified that $\mathbb{H}$ is an atomic connected hypergraph. We will now determine the constructs of $\mathbb{H}$. There are 3 possible non-empty subsets $X \subseteq \{0, 1\}$: $\{0\}$, $\{1\}$ and $\{0, 1\}$.

For each subset we compute $\mathbb{H}\backslash X$ as: $\{\{1\}\}$, $\{\{0\}\}$ and $\varnothing$ respectively. We obtain the constructs as shown below. Note that this corresponds to a line segment; the construct consisting a single node $\{0, 1\}$ is an edge, and the remaining constructions are the vertices.

$$\begin{array}{ccccc} \{1\} & \qquad & \{0\} & \qquad & \\ | & & | & & \\ \{0\} & & \{1\} & & \{0,1\} \end{array}$$

Let $\mathbb{H}$ be an atomic connected hypergraph. We will now describe a partial ordering on the set of all constructs of $\mathbb{H}$. Let $T <_1 U$ when U is obtained by collapsing an edge of T, for all constructs T and U. When collapsing an edge, we take the union of the decorating set of each adjacent node, this union is used to decorate the identified nodes. Note that this is well-defined since the action of collapsing an edge is equivalent to choosing a larger subset X described in Definition 3.51, and subsequently having one fewer connected component. Let $\leq$ be the preorder generated by $<_1$, and denote with $\mathcal{A}(\mathbb{H})$ for the poset of all constructs of $\mathbb{H}$ adjoined with a new least element. We now give an example of this partial ordering.

Remark We have intentionally used the symbol $<_1$ for the above generating relation. This is due to the fact that $<_1$ as defined above satisfies the conditions of Definition 1.25(a) of Chapter 1.

Example 3.53 Recall the constructs given in Example 3.52.

$$\begin{array}{ccccc} \{1\} & \qquad & \{0\} & \qquad & \\ | & & | & & \\ \{0\} & & \{1\} & & \{0,1\} \end{array}$$

If we collapse the only possible edge of first two constructs (from the left), then we obtain the rightmost construct.

The following result is Theorem 8.3 which can be found in Section 8 of [7]. Note that our statement here differs from that of Theorem 8.3 since we consider atomic connected hypergraphs (whereas the more general atomic hypergraph is considered in Theorem 8.3).

Theorem 3.54 (Došen-Petrić) *Let $\mathbb{H}$ be an atomic connected hypergraph. Then $\mathcal{A}(\mathbb{H})$ is an abstract polytope of rank $|\bigcup H| - 1$.*

Remark In Section 9 of [7], it is shown that $\mathcal{A}(\mathbb{H})$ has a geometric realisation as a convex polytope with $\mathcal{A}(\mathbb{H})$ as the face poset. Furthermore, it is shown that the geometric realisation is a simple polytope as defined in [28]. This implies that $\mathcal{A}(\mathbb{H})$ is a simple abstract polytope as defined in this thesis.

In the following sections, we exhibit the n-dimensional hypercube, associahedron and permutohedron as hypergraph polytopes on $H = [n] = \{0, 1, \ldots, n\}$. By the above results, the corresponding hypergraph polytope is a simple abstract polytope. In order to obtain a parity structure for each of the above listed polytopes, all that remains is to give a label structure and show that it satisfies the required axioms.

In the following sections, we will make use of the notation $VX = \{\{x\} \mid x \in X\}$ for a set X. In particular, $V\varnothing = \varnothing$.

3.3.2 Hypercubes

Consider the hypergraph $\mathbb{H}_n = V[n] \cup \{[i] \mid 1 \leqslant i \leqslant n\}$. It is clear that this is an atomic hypergraph; to see that it is connected we note that $[i] \cap [j] \neq \varnothing$ for $i < j$, and also $\{i\} \subseteq [i]$. We will show that $\mathcal{A}(\mathbb{H}_n)$ is a model of the n-dimensional hypercube.

The constructs of $\mathbb{H}_n$ are defined inductively on n. For $n = 0$, there is a unique construct of $\mathbb{H}_0$ given by the point $\{0\}$. For $n = 1$, the constructs of $\mathbb{H}_1 = \{\{0\}, \{1\}, \{0, 1\}\}$ are given in Example 3.52.

For $n > 1$, let $X \subseteq [n]$ be a non-empty subset. If $X = [n]$, then we obtain a construct consisting of a single node $[n]$; constructs of this type will be denoted by $[n]$. Otherwise $X \neq [n]$, then note that

$$\mathbb{H}_n \backslash X = \begin{cases} \mathbb{H}_m \cup V\{k \in [n]\backslash X \mid k > m\} & \text{if } \min X \geqslant 1 \\ V([n]\backslash X) & \text{if } \min X = 0 \end{cases}$$

where $m = \min X - 1$. Consider the following cases.

Case 1: $\min X \geqslant 1$.

Note that the connected components of $\mathbb{H}_n \backslash X$ are $\mathbb{H}_m$ and singleton hypergraphs as given above. We obtain a construct consisting of a root X which has edges connecting to nodes as shown in the following diagram,

$$\begin{tikzcd}[arrows=dash] D \arrow[drr] & \{y_1\} \arrow[dr] & \cdots & \{y_i\} \arrow[dl] & \cdots \\ & & X & & \end{tikzcd}$$

where D is a construct of $\mathbb{H}_m$ and $\{y_i\} \in V\{i \in [n]\backslash X \mid i > m\}$. The edge from X to D in the above diagram represents an edge from X the root of D.

Case 2: $\min X = 0$.

Note that the connected components of $\mathbb{H}_n \backslash X$ are singleton hypergraphs as given above. We obtain a construct consisting of a root X which has edges connecting to nodes as shown in the following diagram,

$$\begin{tikzcd}[arrows=dash] \{y_1\} \arrow[dr] & \cdots & \{y_i\} \arrow[dl] & \cdots \\ & X & & \end{tikzcd}$$

where $\{y_i\} \in V([n]\backslash X)$.

Example 3.55 Consider the hypergraph $\mathbb{H}_3$ and the process of calculating constructs described above. For Case 1, we shall consider $X = \{2\} \subseteq [3]$. Note that $\mathbb{H}_3 \backslash X = \mathbb{H}_1 \cup \{\{3\}\}$. The following are the constructs computed for such a subset.

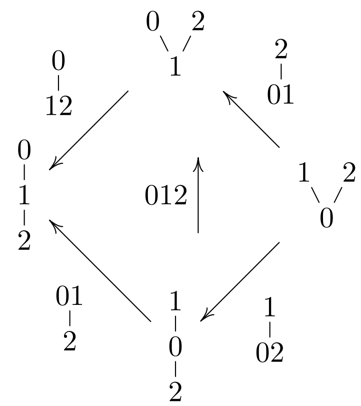
$$\begin{tikzcd}[arrows=dash] \{1\} \arrow[d] & \\ \{0\} \arrow[dr] & \{3\} \arrow[d] \\ & \{2\} \end{tikzcd} \qquad \begin{tikzcd}[arrows=dash] \{0\} \arrow[d] & \\ \{1\} \arrow[dr] & \{3\} \arrow[d] \\ & \{2\} \end{tikzcd} \qquad \begin{tikzcd}[arrows=dash] \{0,1\} \arrow[dr] & \{3\} \arrow[d] \\ & \{2\} \end{tikzcd}$$

For Case 2, we shall consider $X = \{0, 2\} \subseteq [3]$. Note that $\mathbb{H}_3 \backslash X = \{\{1\}\} \cup \{\{3\}\}$. The following is the construct computed for such a subset.

$$\begin{tikzcd}[arrows=dash] \{1\} \arrow[dr] & & \{3\} \arrow[dl] \\ & \{0,2\} & \end{tikzcd}$$

Example 3.56 We will represent $\mathcal{A}(\mathbb{H}_2)$ and $\mathcal{A}(\mathbb{H}_3)$ by a pasting diagram for the square and cube respectively. Note that these pasting diagrams consist of arrows that will be given by a label structure which will be defined in the latter half of this section. Thus at this stage the directions of the arrows are unexplained. We will drop the braces used for denoting sets, for instance we write 012 to denote the set $\{0, 1, 2\}$.

$\mathcal{A}(\mathbb{H}_2)$ is represented by the pasting diagram,

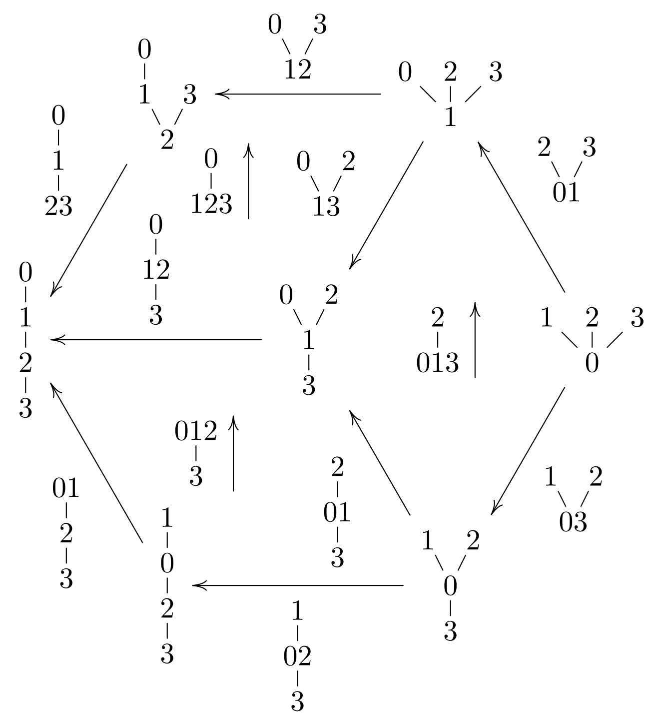


$\mathcal{A}(\mathbb{H}_3)$ is represented by the pasting diagram,

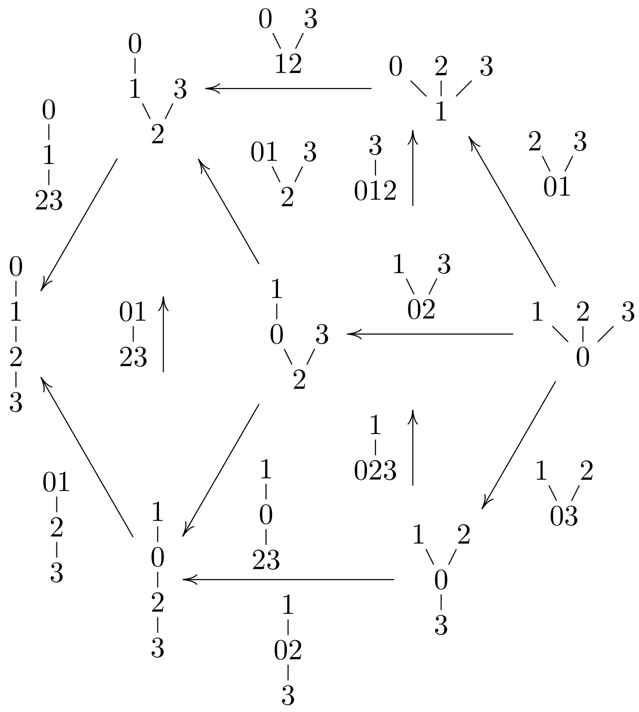
$$B_1 \xleftarrow{\;0123\;} B_2$$

where B_1 and B_2 are the diagrams given below.

B_1



B_2



Recall that in [24] a model for an n-dimensional hypercube is given by the poset of n-letter words in $\{-,+,0\}$; denote this by Q_n. The empty word is the least face of Q_n. We will define a map from $\mathcal{A}(\mathbb{H}_n)$ to Q_n, which can be shown by an inductive argument to be an isomorphism of posets. It will be convenient to express the faces of Q_n by functions from $\{1,\dots,n\}$ to $\{-,+,0\}$.

We will now define a map from $\mathcal{A}(\mathbb{H})$ to Q_n by induction on n. Let T be a construct of $\mathbb{H}_n$. For $n=1$, recall the constructs computed earlier, then let

$$\begin{array}{c}\{1\}\\ |\\ \{0\}\end{array} \mapsto + \qquad \begin{array}{c}\{0\}\\ |\\ \{1\}\end{array} \mapsto - \qquad \{0,1\} \mapsto 0$$

For $n>1$, recall our calculation in the above cases. We define a function which is given by the following. Let X be the root of T, then for all $i \in X\backslash\{\min X\}$ we have $i \mapsto 0$, and $\min X \mapsto -$. In either case found in the above description of a construct of $\mathbb{H}_n$, let $y_i \mapsto +$ for each y_i. We will now determine a function from $\{1,\dots,n\}$ to $\{-,+,0\}$. If we are in Case 2, then there is nothing else to do. If we are in Case 1, then by recursion, from the construct D we obtain a function from $\{1,\dots,m\}$ to $\{-,+,0\}$. Note that if D is a singleton $\{y\}$, then we let $y \mapsto +$. Hence we have determined a function from $\{1,\dots,n\}$ to $\{-,+,0\}$.

Example 3.57 In Example 3.56, pasting diagrams corresponding the $\mathcal{A}(\mathbb{H}_2)$ and $\mathcal{A}(\mathbb{H}_3)$ were given. Applying the above map to the pasting diagram corresponding to $\mathcal{A}(\mathbb{H}_2)$ we obtain the following.

$$\begin{tikzcd}
 & -+ \arrow[dl, "-0"'] & \\
-- \arrow[rr, phantom, "00\Uparrow"] & & ++ \arrow[ul, "0+"'] \arrow[dl, "+0"] \\
 & +- \arrow[ul, "0-"] &
\end{tikzcd}$$

Next, we will define the assignations (λ_T, Λ_T) for each face $T \in \widetilde{\mathcal{A}(\mathbb{H}_n)}$.

Definition 3.58 Let T be a construct of $\mathbb{H}_n$. Each $i \in [n]$ is contained in a unique node Y, and this Y is the root of a subtree T_Y. Let $\lambda_T : [n] \longrightarrow [n]$ be given by $\lambda_T(i) = \min T_Y$; here $\min T_Y$ means the minimum element among all the decorations of T_Y. Let $\Lambda_T = \sum Y \backslash \{\min Y\}$ which is a disjoint union over all the nodes of T.

Example 3.59 The following construct

$$\begin{array}{ccc} \{1\} & & \{3\} \\ & \diagdown \ \diagup & \\ & \{0,2\} & \end{array}$$

has λ_T given by $\lambda_T(0) = 0$, $\lambda_T(1) = 1$, $\lambda_T(2) = 0$, $\lambda_T(3) = 3$, and $\Lambda_T = \{2\}$.

We could show that $([n], \lambda, \Lambda)$ as defined above is a label structure on $\mathcal{A}(\mathbb{H}_n)$ by showing that the necessary axioms hold. However, it is more efficient to use Theorem 3.44 and the following recursive argument.

Recall the isomorphism described earlier $\mathcal{A}(\mathbb{H}_n) \cong Q_n$ For the case of $n = 1$, $\mathcal{A}(\mathbb{H}_1)$ is an line segment as shown by Example 3.52. Consider the label structure defined above for $\mathcal{A}(\mathbb{H}_1) \cong Q_1$. Observe that Q_n is the smash product of Q_{n-1} and Q_1 so by Theorem 3.44 and recursion, we have a label structure on $\mathcal{A}(\mathbb{H}_n) \cong Q_n$.

3.3.3 Associahedra

Consider the hypergraph $\mathbb{A}_n = V[n] \cup \{\{i, i+1\} \mid 0 \leq i < n\}$ described in Appendix B of [7]. It is clear that this is an atomic hypergraph; to see that it is connected we note that $\{i, i+1\} \cap \{i+1, i+2\} \neq \varnothing$ for each $0 \leqslant i < n$, and $\{i\} \subseteq [i]$. The fact that $\mathcal{A}(\mathbb{A}_n)$ is a model of the n-dimensional associahedron is due to Postnikov [18].

The constructs of $\mathbb{A}_n$ are defined inductively on n as follows. Note that for $n = 0, 1$ we have $\mathbb{A}_n = \mathbb{H}_n$ so the constructs have already been given in the previous section. For $n > 1$, let $X \subseteq [n]$ be a non-empty subset. If $X \neq [n]$, then since the hypergraph $\mathbb{A}_n$ is equivalent to the following graph,

$$0 \longrule 1 \longrule \cdots \longrule n$$

it follows that $\mathbb{A}_n \backslash X$ consists of connected components that are equivalent to $\mathbb{A}_m$ (by an appropriate relabelling) for some $m < n$. Thus a construct of $\mathbb{A}_n$ is either a single node decorated by $[n]$, or a non-planar tree consisting of a root decorated by X and has edges connecting to nodes as shown in the following diagram,

$$\begin{array}{ccccc} V_1 & V_2 & \cdots & V_i & \cdots \\ & \diagdown & \mid & \diagup & \\ & & X & & \end{array}$$

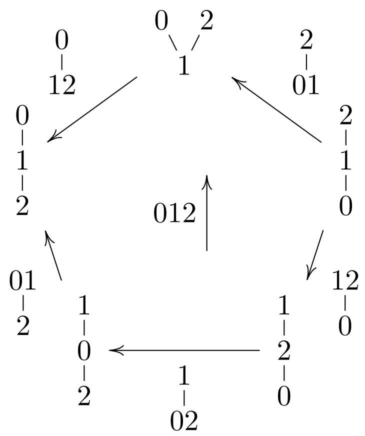
where V_i are constructs of $\mathbb{A}_{m_i}$ with relabelled vertices whenever appropriate.

Example 3.60 Consider the hypergraph $\mathbb{A}_3 = \{\{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{1,2\}, \{2,3\}\}$ and the process of calculating constructs described above. Let $X = \{1\}$; then we have $\mathbb{A}_3 \backslash X = \{\{0\}\} \cup \{\{2\}, \{3\}, \{2,3\}\}$. Note that $\{\{2\}, \{3\}, \{2,3\}\}$ has the form of $\mathbb{A}_1$. Recall that $\mathbb{A}_1 = \mathbb{H}_1$ and so the following are the constructs computed for such a subset.

$$\begin{array}{ccc} & & \{3\} \\ & & \mid \\ \{0\} & & \{2\} \\ & \diagdown \ \diagup & \\ & \{1\} & \end{array} \qquad \begin{array}{ccc} & & \{2\} \\ & & \mid \\ \{0\} & & \{3\} \\ & \diagdown \ \diagup & \\ & \{1\} & \end{array} \qquad \begin{array}{ccc} \{0\} & & \{2,3\} \\ & \diagdown \ \diagup & \\ & \{1\} & \end{array}$$

Example 3.61 We will represent $\mathcal{A}(\mathbb{A}_2)$ and $\mathcal{A}(\mathbb{A}_3)$ by a pasting diagram for the pentagon and 3-dimensional associahedron respectively. As in the previous section, these pasting diagrams consist of arrows that will be given by a label structure which will be defined in the latter half of this section. Thus at this stage the directions of the arrows are unexplained.

$\mathcal{A}(\mathbb{A}_2)$ is represented by the pasting diagram,

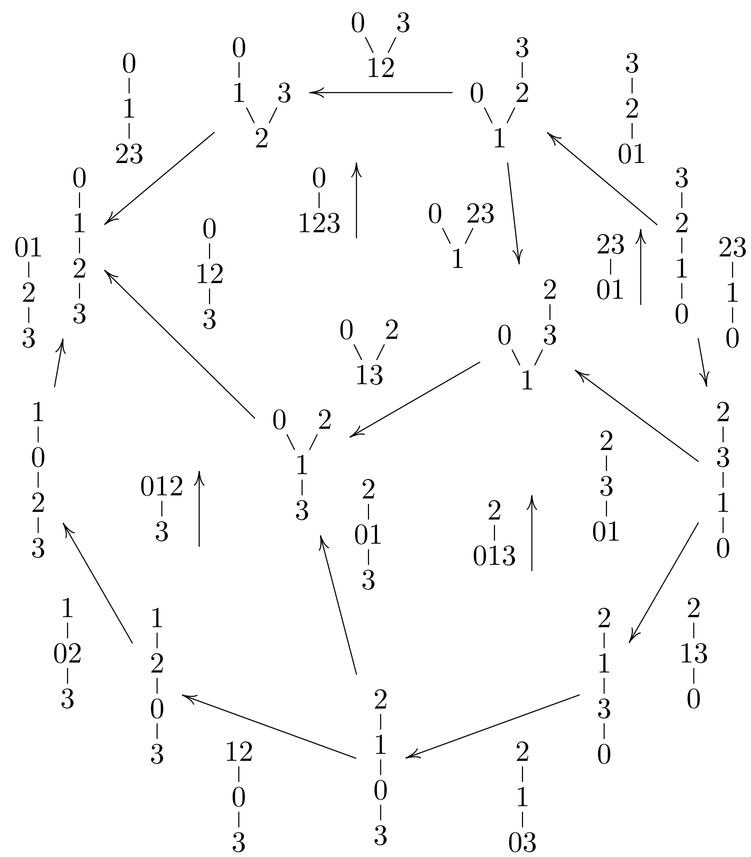


$\mathcal{A}(\mathbb{A}_3)$ is represented by the pasting diagram,

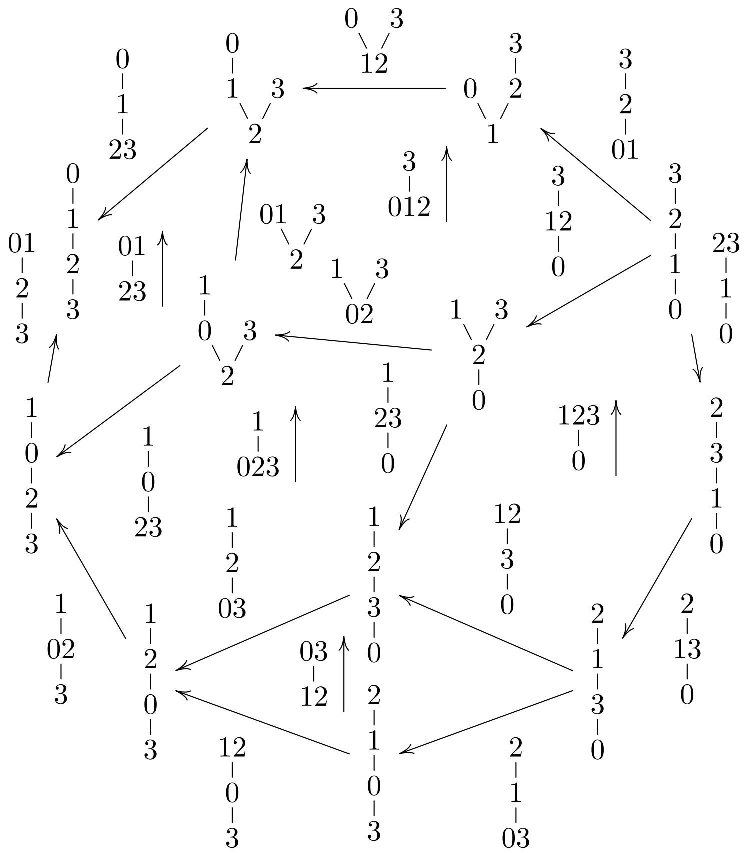
$$C_1 \xleftarrow{0123} C_2$$

where C_1 and C_2 are the diagrams given below.

C_1



C_2



Next, we define the assignations (λ_T, Λ_T) for each face $T \in \widetilde{\mathcal{A}(\mathbb{A}_n)}$. The definition of λ_T and Λ_T given below coincide with that of Definition 3.58 for hypercubes.

Definition 3.62 Let T be a construct of $\mathbb{A}_n$. Each $i \in [n]$ is contained in a unique node Y, and this Y is the root of a subtree T_Y. Let $\lambda_T : [n] \longrightarrow [n]$ be given by $\lambda_T(i) = \min T_Y$; here $\min T_Y$ means the minimum element among all the decorations of T_Y. Let $\Lambda_T \subseteq [n]$ be given by $\Lambda_T = \sum Y \backslash \{\min Y\}$ which is a disjoint union over all the nodes of T.

Example 3.63 The following construct

$$\begin{array}{ccc} 01 & & 3 \\ & \diagdown \quad \diagup & \\ & 2 & \end{array}$$

has λ_T given by $\lambda_T(0) = 0$, $\lambda_T(1) = 0$, $\lambda_T(2) = 0$, $\lambda_T(3) = 3$, and $\Lambda_T = \{1\}$.

It can be shown that for any construct T of $\mathbb{A}_n$, λ_T is an lbf and furthermore the pair (λ_T, Λ_T) satisfies the property in Proposition 2.15 thus we may use the bijection to obtain an hlbf. For the construct given above we obtain the hlbf $0, 01, 3$. If we perform this calculation for the above pasting diagrams of $\mathcal{A}(\mathbb{A}_2)$ and $\mathcal{A}(\mathbb{A}_3)$ we will obtain the same diagrams in Example 2.16.

We will now show that $([n], \lambda, \Lambda)$ is a label structure on $\mathcal{A}(\mathbb{A}_n)$ by verifying the required axioms. This will be shown indirectly, using Example 3.2 and the fact that that the poset of hlbfs is a model of the associahedron. It will be convenient to consider hlbfs as defined on a finite linearly ordered set M. It is also convenient to consider the hypergraph $\mathbb{A}_n$ as being defined on a finite linearly ordered set M with $|M| = |[n]| = n+1$. We will denote this hypergraph by $\mathbb{A}_M = VM \cup \{\{i, i+1\} \mid i \in M \backslash \{\max M\}\}$.

In the result below, we give a characterisation of hlbfs on $[n]$.

Proposition 3.64 *An hlbf x on $[n]$ is equivalently a collection of hlbfs x_i on N_i for $1 \leq i \leq k$ such that $[n] = N_1 \oplus \{n_1\} \oplus \ldots \oplus N_{k-1} \oplus \{n_{k-1}\} \oplus N_k$. Furthermore,*

$$\ell_x(j) = \begin{cases} \ell_{x_i}(j) & \textit{if } j \in N_i \\ 0 & \textit{otherwise.} \end{cases}$$

$$S_x = \left(\bigcup S_{x_i}\right) \cup \{n_2, \ldots, n_{k-1}\}$$

Proof. Let $m = x(h) \cup \{h+1\}$ where $h = \max\{i \in [n] \mid \ell_x(i) = 0\}$. By Proposition 2.17, there exists a right adjoint $b_x : [n] \oplus 1 \longrightarrow m$ to an inclusion $\psi : m \longrightarrow [n] \oplus 1$.

Let $m = \{j_1, \ldots, j_k\}$, $n_i = \max b_x^{-1}(j_i)$ and $N_i = b_x^{-1}(j_i) \backslash \{n_i\}$ for each $1 \leq i \leq k$. Note that b_x is a right adjoint so it must preserve the greatest element. It follows that n_k is the greatest element of $[n] \oplus 1$ and so we have $[n] = N_1 \oplus \{n_1\} \oplus \ldots \oplus N_{k-1} \oplus \{n_{k-1}\} \oplus N_k$. By Proposition 2.18, there are hlbfs x_i on N_i given by $x_i(j) = x(j)$ for each $1 \leq i \leq k$.

Finally, note that by definition of h and Lemma 2.26, we have $\ell_x(n_{j_i}) = 0$ for all $1 \leq i \leq k$. It follows that the above formula for ℓ_x hold. Note that $n_1 = \max b_x^{-1}(j_1) = \min b_x^{-1}(j_2) - 1 = j_2 - 1$ so $n_1 + 1 = j_2$ which is the least non-minimal element of $x(h)$. By a familiar argument involving Lemma 2.26, it follows that $n_1 \notin S_x$. Thus the above formula for S_x hold. $\square$

We will now describe a bijection between the poset of hlbfs on $[n]$ and the poset of constructs of $\mathbb{A}_n$. Recall our above discussion on the constructs of $\mathbb{A}_n$. By Proposition 3.64, given an hlbf x on $[n]$ we may let $X = \{n_1, \ldots, n_{k-1}\}$. The corresponding construct of $\mathbb{A}_n$ consist of constructs T_i (corresponding to the hlbfs x_i) on $\mathbb{A}_{N_i}$ for each $1 \leq i \leq k$ as shown in the following tree.

$$\begin{tikzcd}[column sep=small] T_1 \arrow[dr, dash] & \cdots & T_k \arrow[dl, dash] \\ & \{n_1, \ldots, n_{k-1}\} & \end{tikzcd}$$

We now verify that

$$\lambda_x(j) = \begin{cases} \lambda_{T_i}(j) & \text{if } j \in N_i, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Lambda_x = \left(\bigcup \Lambda_{T_i}\right) \cup \{n_2, \ldots, n_{k-1}\}.$$

Hence by the above observation and Example 3.2, it follows that $([n], \lambda, \Lambda)$ is a label structure on $\mathcal{A}(\mathbb{A}_n)$.

3.3.4 Permutohedra

Consider the hypergraph $\mathbb{P}_n = V[n] \cup \{X \subseteq [n] \mid |X| = 2\}$ described in Appendix B of [7]. It is clear that that this is an atomic hypergraph; to see that it is connected we note that $\{i, j\} \cap \{j, k\} \neq \varnothing$ and $\{i\}, \{j\} \subseteq \{i, j\}$.

The n-dimensional permutohedron is a polytope with faces corresponding to surjective maps with domain $[n]$, the vertices are the bijections from $[n]$ to $[n]$. In this subsection, we will present a model of the permutohedron using hypergraph polytopes. The fact that $\mathcal{A}(\mathbb{P}_n)$ is a model of the n-dimensional permutohedron is due to Postnikov [18].

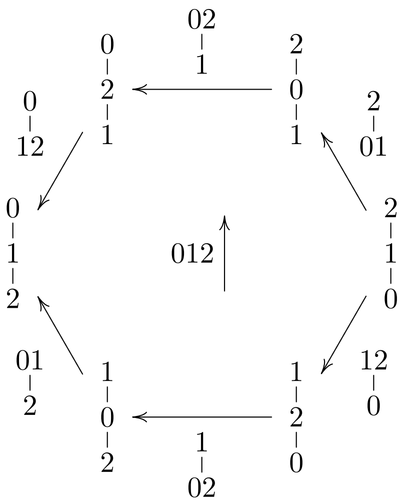
Note that for any $X \subseteq [n]$, the hypergraph $\mathbb{P}_n \backslash X$ has the form $\mathbb{P}_{n-|X|}$, which is connected. It follows that a construct of $\mathbb{P}_n$ is a tree of the form

$$\begin{array}{c} X_m \\ \vdots \\ | \\ X_1 \\ | \\ X_0 \end{array}$$

where $X_0, X_1, \ldots, X_m$ are disjoint subsets of $[n]$ satisfying $\sum_{i=0}^{m} X_i = [n]$. We shall use the notation $(X_0, X_1, \ldots, X_m)$ to denote the above tree. For such a construct, we may define a map from $[n]$ to $[m]$ by $j \in X_i \mapsto i$. Note that such a map is surjective and so the faces of $\mathcal{A}(\mathbb{P}_n)$ are in bijection with surjective maps with domain $[n]$.

Example 3.65 We will represent $\mathcal{A}(\mathbb{P}_2)$ and $\mathcal{A}(\mathbb{P}_3)$ by a pasting diagram for the hexagon and 3-dimensional permutohedron respectively. Note that these pasting diagrams consist of arrows that will be given by a label structure which shall be defined in the latter half of this subsection. Thus at this stage the directions of the arrows are unexplained.

$\mathcal{A}(\mathbb{P}_2)$ is represented by the pasting diagram,

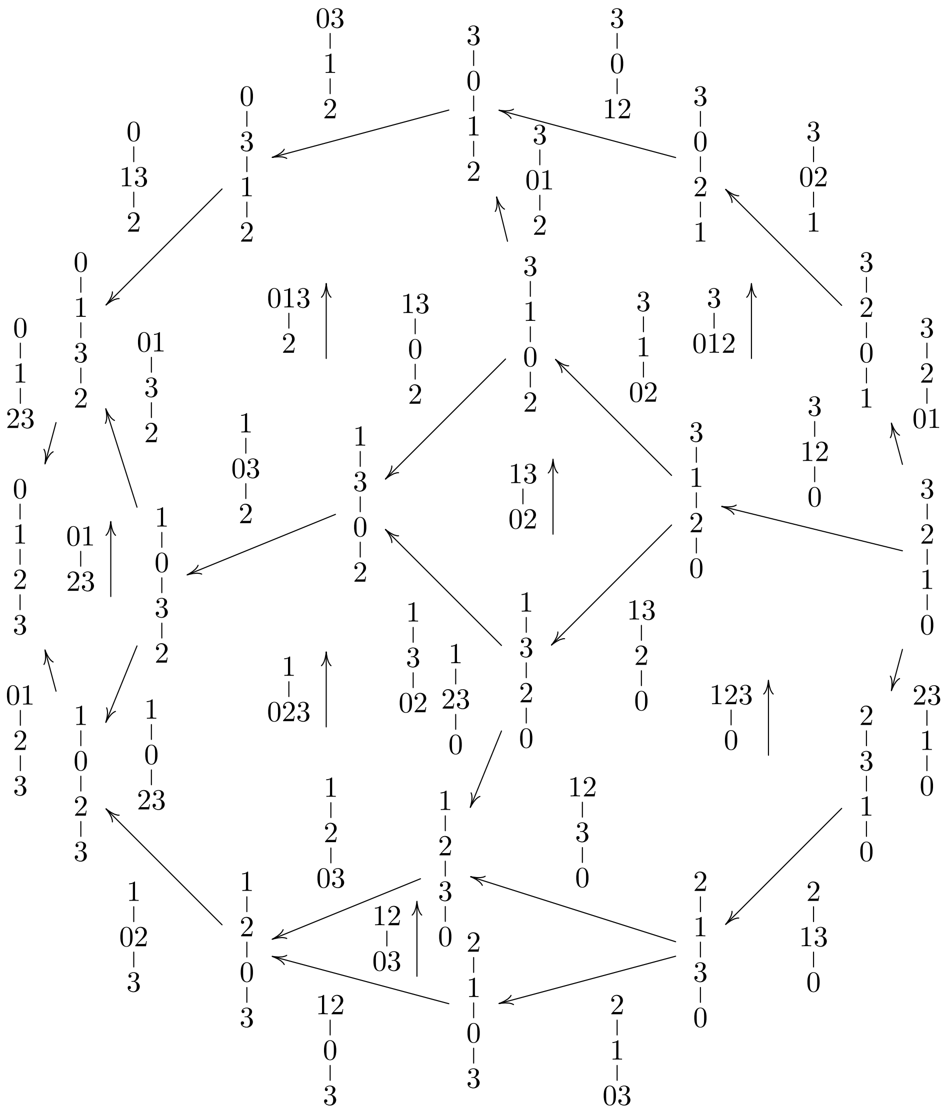


$\mathcal{A}(\mathbb{P}_3)$ is represented by the pasting diagram,

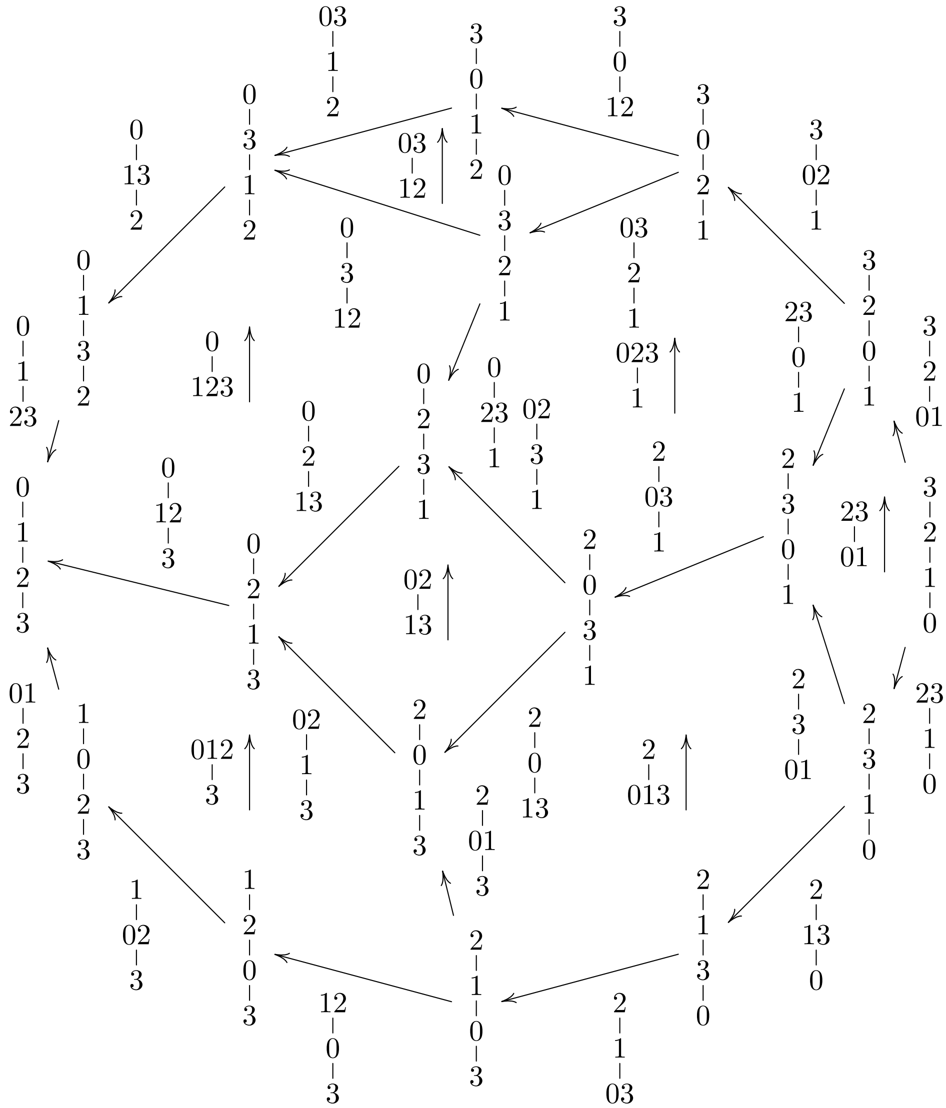
$$D_1 \xleftarrow{0123} D_2$$

where D_1 and D_2 are the diagrams below.

D_1


03-1-2
3-0-1-2
3-0-12
0-3-1-2
3-0-2-1
0-13-2
3-01-2
3-02-1
0-1-3-2
3-1-0-2
3-2-0-1
013-2
13-0-2
3-1-02
3-012
0-1-23
01-3-2
3-2-01
1-03-2
1-3-0-2
3-1-2-0
3-12-0
13-02
0-1-2-3
01-23
1-0-3-2
3-2-1-0
1-3-02
1-3-2-0
13-2-0
1-23-0
1-023
123-0
23-1-0
2-3-1-0
01-2-3
1-0-2-3
1-0-23
1-2-03
1-2-3-0
12-3-0
1-02-3
1-2-0-3
12-03
2-1-0-3
2-1-3-0
2-13-0
12-0-3
2-1-03

D_2



Next, we will define the assignations (ρ_T, Λ_T) for each face $T \in \widetilde{\mathcal{A}(\mathbb{P}_n)}$. The definition of Λ_T given below coincides with that of Definition 3.58 and Definition 3.62 for hypercubes and associahedra respectively.

Definition 3.66 Let $T = (X_0, X_1, \ldots, X_m)$ be a construct of $\mathbb{P}_n$. Each $i \in [n]$ is contained in a unique node X_r for some $r \in [k]$. Let $\rho_T : [n] \longrightarrow [n]$ be given by $\rho_T(i) = |\{0 \leq k < r \mid i > \min X_k\}|$. Let $\Lambda_T = \sum_{i=0}^{m} X_i \backslash \{\min X_i\}$.

Remark So far we have shown that $([n], \lambda, \Lambda)$ (given in Definition 3.58 and Definition 3.62) is a label structure for both the associahedron and hypercube. However for the permutohedron, we must substitute ρ for λ to obtain a label structure (this will be proven in the latter part of this section).

We will now show by a counterexample that $([n], \lambda, \Lambda)$ is not a label structure for the permutohedron. Consider the pasting diagram of $\mathcal{A}(\mathbb{P}_2)$ shown earlier in this section. Denote with T_1, T_2 and T_3 respectively for the constructs shown below.

$$\begin{array}{ccc} 0 & 0 & \\ | & | & \\ 1 & 2 & 0 \\ | & | & | \\ 2 & 1 & 12 \end{array}$$

The above constructs correspond to an edge T_3 and its adjacent vertices T_1 and T_2. Here we have $\Lambda_{T_3} = \Lambda_{T_1} \cup \{2\} = \Lambda_{T_2} \cup \{2\}$. However $\lambda_{T_1} = \lambda_{T_2} = \lambda_{T_3}$ thus axiom (C4)(b) of a label structure does not hold.

We will now show that $([n], \rho, \Lambda)$ is a label structure on $\mathbb{P}_n$ by verifying the required axioms. Firstly, we need to show that (C1) holds. Let T, U be constructs of $\mathbb{P}_n$ with $T <_1 U$. By definition of the relation $<_1$, U is obtained from T by collapsing the edge connecting nodes X_s and X_{s-1} for some $s > 0$. It follows that $T = (X_0, X_1, \dots, X_m)$ and $U = (Y_0, Y_1, \dots, Y_{m-1})$ where

$$Y_i = \begin{cases} X_i & \text{if } i < s-1, \\ X_s \cup X_{s-1} & \text{if } i = s-1, \\ X_{i+1} & \text{otherwise.} \end{cases}$$

We will now determine (ρ_U, Λ_U) in terms of (ρ_T, Λ_T). Consider the following cases.

Case 1: $\min X_s > \min X_{s-1}$.

Let $j = \min X_s$; then $j > \min X_{s-1} = \min(X_s \cup X_{s-1}) = \min Y_{s-1}$ and so $\Lambda_U = \Lambda_T \cup \{j\}$. We now determine ρ_U in terms of ρ_T. Let $i \in [n]$ be given; there exists $r \in [m]$ such that $i \in X_r$.

If $i = j$, then $r = s$ and so $j \in X_s \subseteq Y_{s-1}$. It follows that

$$\begin{aligned} \rho_U(j) + 1 &= |\{0 \le k < s-1 \mid j > \min Y_k\} \cup \{s-1\}| \\ &= |\{0 \le k < s-1 \mid j > \min X_k\} \cup \{s-1\}| \\ &= |\{0 \le k < s \mid j > \min X_k\}|, \text{ since } j > \min X_{s-1} \\ &= \rho_T(j). \end{aligned}$$

If $i < j$, then $r \neq s$ since $i < j = \min X_s$; we either have $r \le s-1$ or $r > s$. We seek to prove that $\rho_U(i) = \rho_T(i)$. Consider the case of $r \le s-1$. Note that $i \in Y_r$ and so

$$\begin{aligned} \rho_U(i) &= |\{0 \le k < r \mid i > \min Y_k\}| = |\{0 \le k < r \mid i > \min X_k\}| \\ &= \rho_T(i). \end{aligned}$$

We now consider the case of $r > s$. Note that $i \in Y_{r-1}$ and so

$$\begin{aligned} \rho_U(i) &= |\{0 \le k \le s-1 \mid i > \min Y_k\}| + |\{s-1 < k < r-1 \mid i > \min Y_k\}| \\ &= |\{0 \le k \le s-1 \mid i > \min X_k\}| + |\{s-1 < k < r-1 \mid i > \min X_{k+1}\}| \\ &= |\{0 \le k \le s-1 \mid i > \min X_k\}| + |\{s < k < r \mid i > \min X_k\}| \\ &= |\{0 \le k < r \mid i > \min X_k\}|, \text{ since } i < j = \min X_s \\ &= \rho_T(i). \end{aligned}$$

Let $i > j$; note that $i > \min X_s$ and $i > \min X_{s-1}$. By similar calculations as above, we can deduce the following. For $r \ge s$ we have $\rho_U(i) = \rho_T(i) - 1$, and for $r \le s-1$ we have $\rho_U(i) = \rho_T(i)$.

Case 2: $\min X_s < \min X_{s-1}$.

Let $j = \min X_{s-1}$; then $j > \min X_s = \min(X_s \cup X_{s-1}) = \min Y_{s-1}$ and so $\Lambda_U = \Lambda_T \cup \{j\}$. We now determine ρ_U in terms of ρ_T. Let $i \in [n]$ be given; there exists $r \in [m]$ such that $i \in X_r$.

If $i = j$, then $r = s$ and so $j \in X_s \subseteq Y_{s-1}$. It follows that

$$\begin{aligned} \rho_U(j) &= |\{0 \le k < s-1 \mid j > \min Y_k\}| = |\{0 \le k < s-1 \mid j > \min X_k\}| \\ &= |\{0 \le k < s \mid j > \min X_k\}|, \text{ since } j = \min X_{s-1} \\ &= \rho_T(j). \end{aligned}$$

By similar calculations to those used in Case 1, we can make the same conclusions for the cases where $i \neq j$.

The above cases imply that (C1)(b) and (c) hold. Also it follows from the above that $T <_1 U$ implies $|\Lambda_U \backslash \Lambda_T| = 1$. Now the converse follows from the fact that $\Lambda_U \subseteq \Lambda_T$ whenever $T \leq U$. Hence we have shown that (C1)(a) holds.

We will now show that (C2) holds. Let $T = (X_0, X_1, \ldots, X_m)$ be a construct of $\mathbb{P}_n$. By definition of the partial order, we can deduce that a line segment in $\mathcal{A}(\mathbb{P}_n)$ with T as the least element is obtained by collapsing two distinct edges of T. The two 0-faces are obtained by collapsing either edge. Given a line segment with the following configuration, we assume without loss of generality that $j_1 < j_2$.

$$\begin{tikzcd} & W \arrow[dl, dash, "j_2"'] \arrow[dr, dash, "j_1"] & \\ U \arrow[dr, dash, "j_1"'] & & V \arrow[dl, dash, "j_2"] \\ & T & \end{tikzcd}$$

Note that all we need to check is the case where W is obtained by collapsing a pair of edges of T that share a common node. Let U be obtained by collapsing the edge connecting nodes X_s and X_{s-1}, and V be obtained by collapsing the edge connecting X_{s+1} and X_s for some $s > 0$. It follows that $U = (Y_0, Y_1, \ldots, Y_{m-1})$ and $V = (Z_0, Z_1, \ldots, Z_{m-1})$ where

$$Y_i = \begin{cases} X_i & \text{if } i < s-1, \\ X_s \cup X_{s-1} & \text{if } i = s-1, \\ X_{i+1} & \text{otherwise.} \end{cases}$$

$$Z_i = \begin{cases} X_i & \text{if } i < s, \\ X_{s+1} \cup X_s & \text{if } i = s, \\ X_{i+1} & \text{otherwise.} \end{cases}$$

Suppose that $\rho_V(j_2) = \rho_T(j_2)$; we aim to show that $\rho_V(j_2) = \rho_T(j_2)$. Recall Case 1 and Case 2 considered in the above proof of (C1). Note that $\rho_V(j_2) = \rho_T(j_2)$ implies that we must be in Case 2 so it follows that $\min X_{s+1} < \min X_s = j_2$. It follows that $\min X_s < \min X_{s-1} = j_1$ since $j_1 \neq j_2$. Thus $\min Y_{s-1} = \min(X_s \cup X_{s-1}) = \min X_s = j_2 > \min X_{s+1} = \min Y_s$. Note that W can be obtained from U by collapsing the edge connecting $Y_s = X_{s+1}$ and $Y_{s-1} = X_s \cup X_{s-1}$. Hence by Case 2, $\rho_W(j_2) = \rho_U(j_2)$ as required. We can use a similar argument to deduce the converse.

Finally, note that $\rho_T(j_1) = \rho_U(j_1)$ iff $\rho_W(j_1) = \rho_V(j_1)$ follows from (C1)(c). Hence we have shown that (C2) hold.

Note that (C3) follows from Cases 1 and 2 as considered in the proof of (C1) We will now show that (C4) hold. Let $T = (X_0, X_1, \ldots, X_m)$ be a construct of $\mathbb{P}_n$ and $j = \max \Lambda_T$. Note that $j \in X_r$ where for some $r \in [m]$. By definition of Λ, $j \neq \min X_r$ and so $j > \min X_r$. Observe that there are exactly two constructs W satisfying $\Lambda_W = \Lambda_T \backslash \{j\}$ and $W <_1 T$. These constructs are given below; write W_1 for the construct on the left and W_2 for the one on the right.

$$
\begin{array}{ccc}
\vdots & \quad & \vdots \\
| & & | \\
X_{r+1} & & X_{r+1} \\
| & & | \\
\{j\} & & X_r\backslash\{j\} \\
| & & | \\
X_r\backslash\{j\} & & \{j\} \\
| & & | \\
X_{r-1} & & X_{r-1} \\
| & & | \\
\vdots & & \vdots
\end{array}
$$

Note that $W_1 <_1 T$ and $W_2 <_1 T$. Furthermore, we have $\rho_T(j) \neq \rho_{W_1}(j)$ and $\rho_T(j) = \rho_{W_2}(j)$. Hence we have shown the existence part of (C4)(a) and (b). As there are no more constructs with the desired property, we also have uniqueness. Thus we have shown that (C4) hold.

Conclusion

In the introduction to this thesis, we outlined our primary goal of constructing a parity structure on the associahedron. The most convenient formalism of parity is Johnson's loop free pastings schemes. This allows us to make use of Campbell's LGC-complexes which induces a loop free pasting scheme.

Firstly, we showed that a parity structure satisfying axioms 1*, 2, L and C is equivalent to an LGC-complex. Secondly, we defined label structures on polytopes and show that there is a induced parity structure satisfying axioms 1*, 2, L and C. Thus we have a general theory for constructing parity on polytopes. We are then able to given examples of label structures to the associahedra, hypercubes and permuothedra.

There are still various avenues of investigation remaining. These include but are not limited to the following.

- Find a common label structure for our main examples (associahedra, hypercubes and permuothedra).
- Exhibit a label structure for the simplexes.
- Define a label structure for more general objects than abstract pre-polytopes.

Appendix A

4-dimensional associahedron

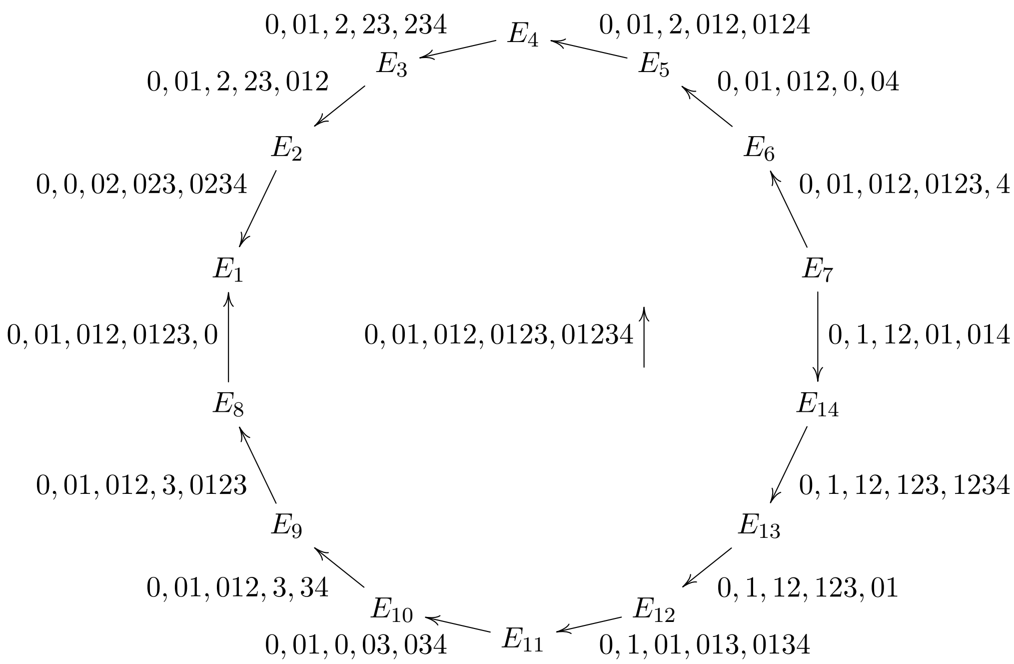
By the results of Chapter 2, the poset of hlbfs on $\{0, 1, 2, 3, 4\}$, $\mathcal{H}_4$, is a model of the 4-dimensional associahedron. By the Example 3.2 and Proposition 3.3 from Chapter 3, there is a parity structure on $\mathcal{H}_4$.

We will first give a counterexample to Street's parity complex axiom 3(b) [24]. Consider the hlbfs $x = 0, 1, 2, 12, 01$, $y = 0, 0, 2, 02, 024$ and $z = 0, 01, 2, 012, 0124$. We have $x \in z^-$, $y \in z^+$ and $x \lhd y$ as witnessed by the sequence $x = x_0 \lhd x_1 \lhd \ldots \lhd x_6 = y$ given below.

$$
\begin{aligned}
x_0 &= 0, 1, 2, 12, 01 & a_0 &= 0, 1, 2, 12, 1 \\
x_1 &= 0, 1, 12, 123, 1 & a_1 &= 0, 1, 12, 3, 1 \\
x_2 &= 0, 1, 12, 3, 123 & a_2 &= 0, 1, 12, 3, 3 \\
x_3 &= 0, 01, 012, 3, 3 & a_3 &= 0, 01, 2, 3, 3 \\
x_4 &= 0, 01, 2, 3, 34 & a_4 &= 0, 0, 2, 3, 34 \\
x_5 &= 0, 0, 2, 23, 234 & a_5 &= 0, 0, 2, 2, 24 \\
x_6 &= 0, 0, 2, 02, 024 & &
\end{aligned}
$$

we have $a_i \in x_i^+ \cap x_{i+1}^-$ and so $x_i \lhd x_{i+1}$ for each $0 \leq i \leq 5$.

The following is a pasting diagram of the 4-dimensional associahedron. The directions of the arrows are derived from the parity structure defined in Proposition 3.3.

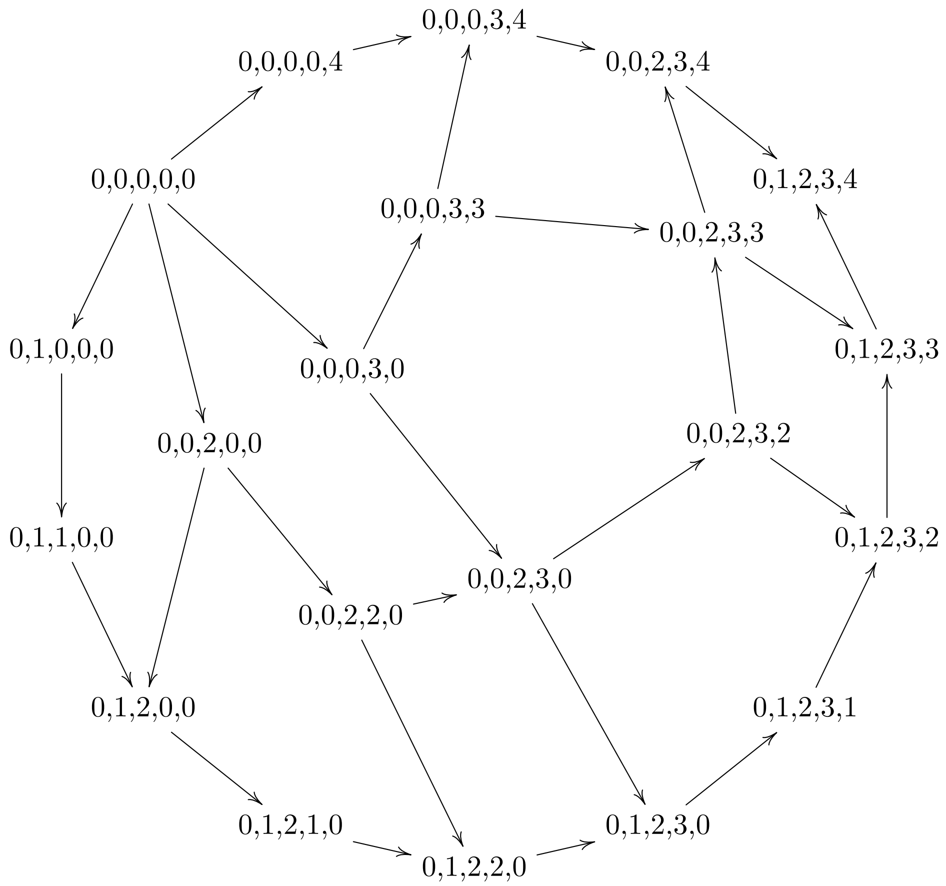


where E_i are the diagrams below.

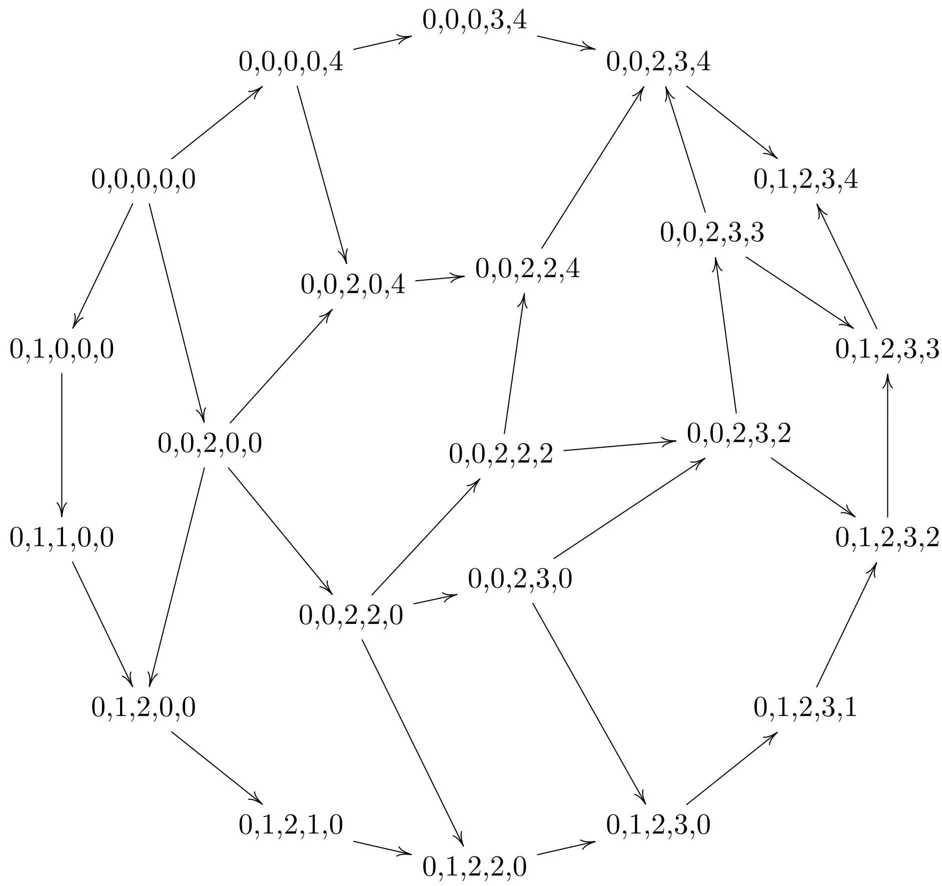
In the diagrams below, we have left the labels of edges and faces unlabelled. For these unlabelled faces, we can obtain the corresponding hlbf by taking pointwise unions of all subfaces one dimension lower. For example, in diagram E_1 we have an edge $0,0,0,0,04$ between the vertices $0,0,0,0,0$ and $0,0,0,0,4$.

We have also omitted the arrows to indicate direction for each 2-dimensional face. These arrows follow a simple pattern as we will now describe. Observe that each diagram E_i has the same outermost path between the hlbfs $0,0,0,0,0$ and $0,1,2,3,4$. In fact, there are exactly two paths from $0,0,0,0,0$ and $0,1,2,3,4$, one shorter than the other. The missing arrows will point away from the longer path and towards the shorter path.

E_1



E_2


0,0,0,3,4
0,0,0,0,4
0,0,2,3,4
0,0,0,0,0
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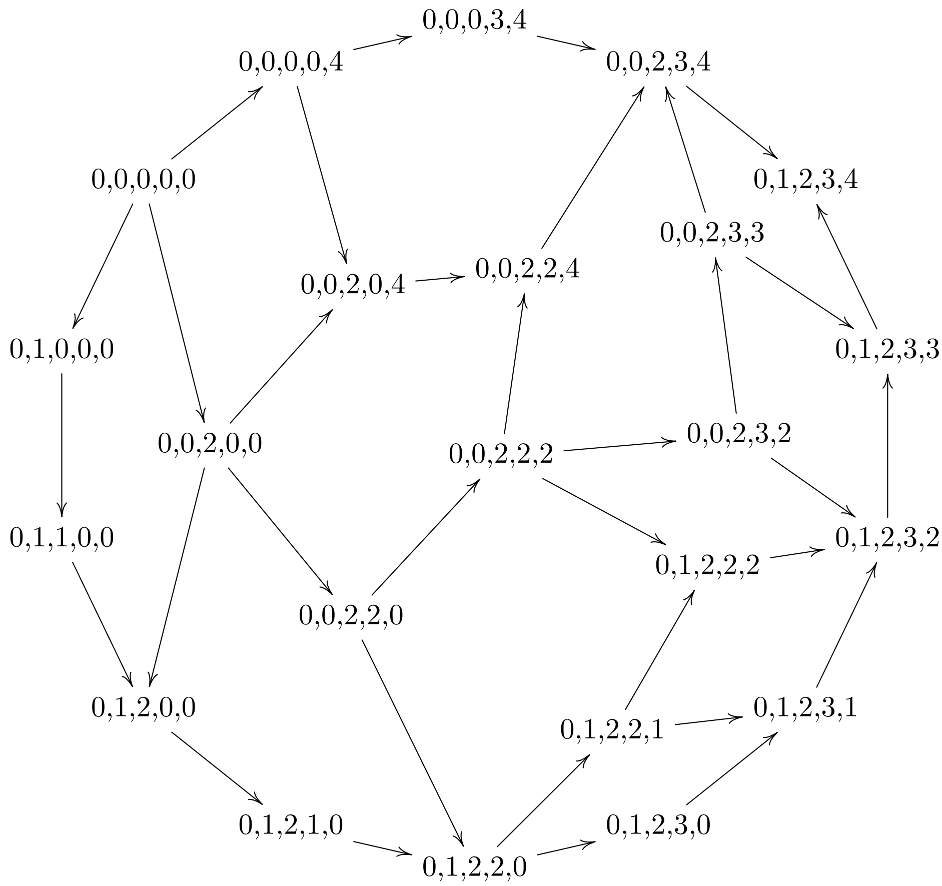
E_3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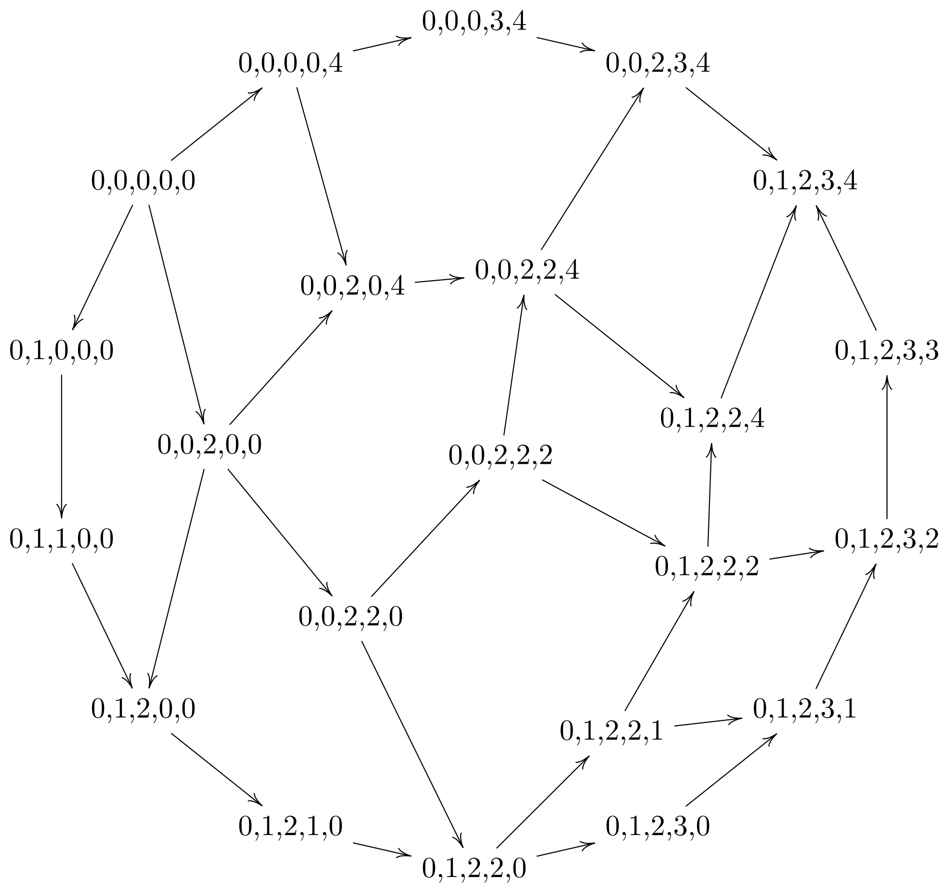
E_4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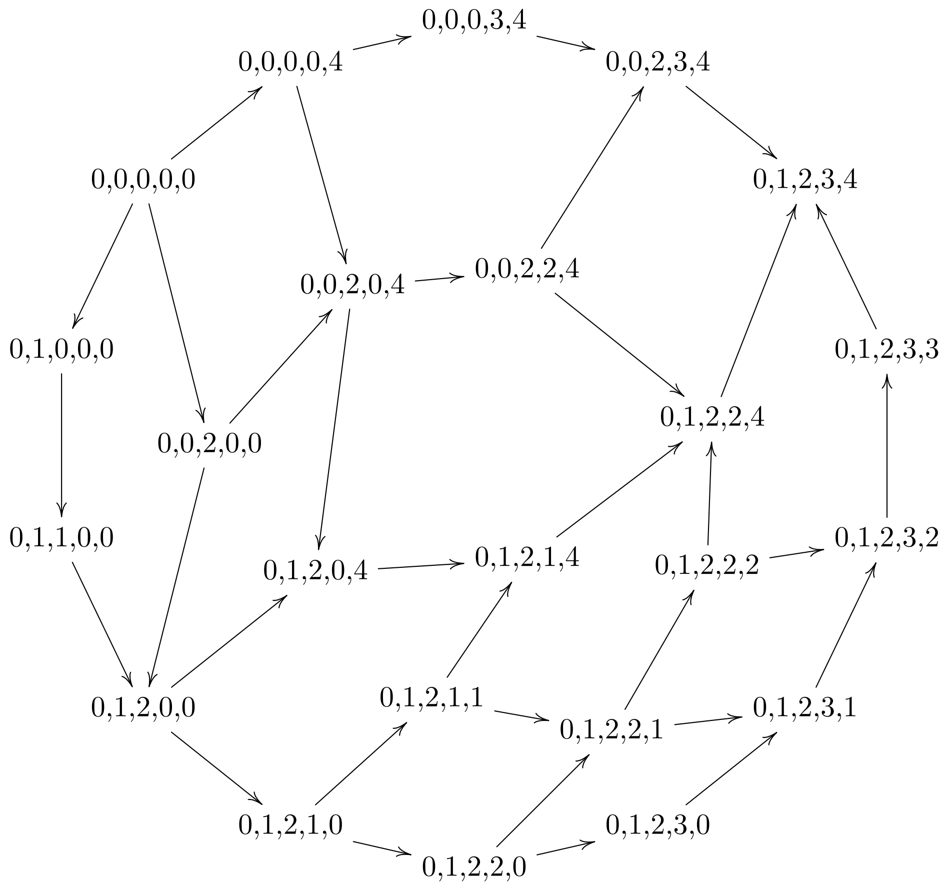
E_5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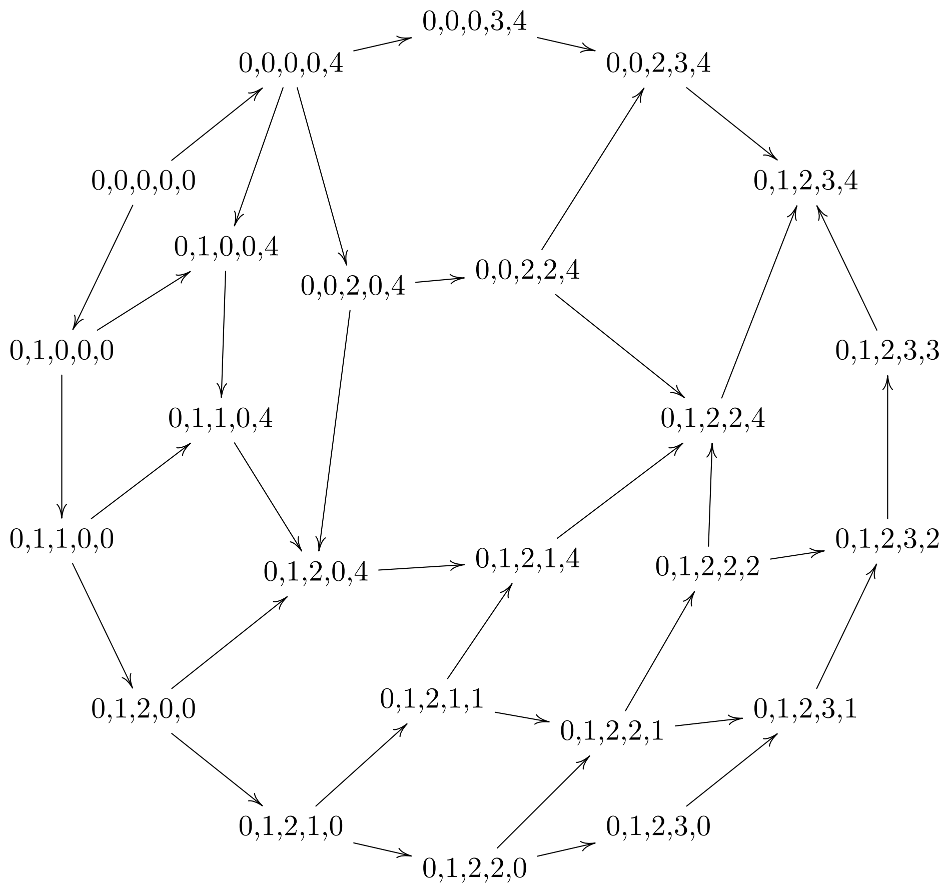
E_6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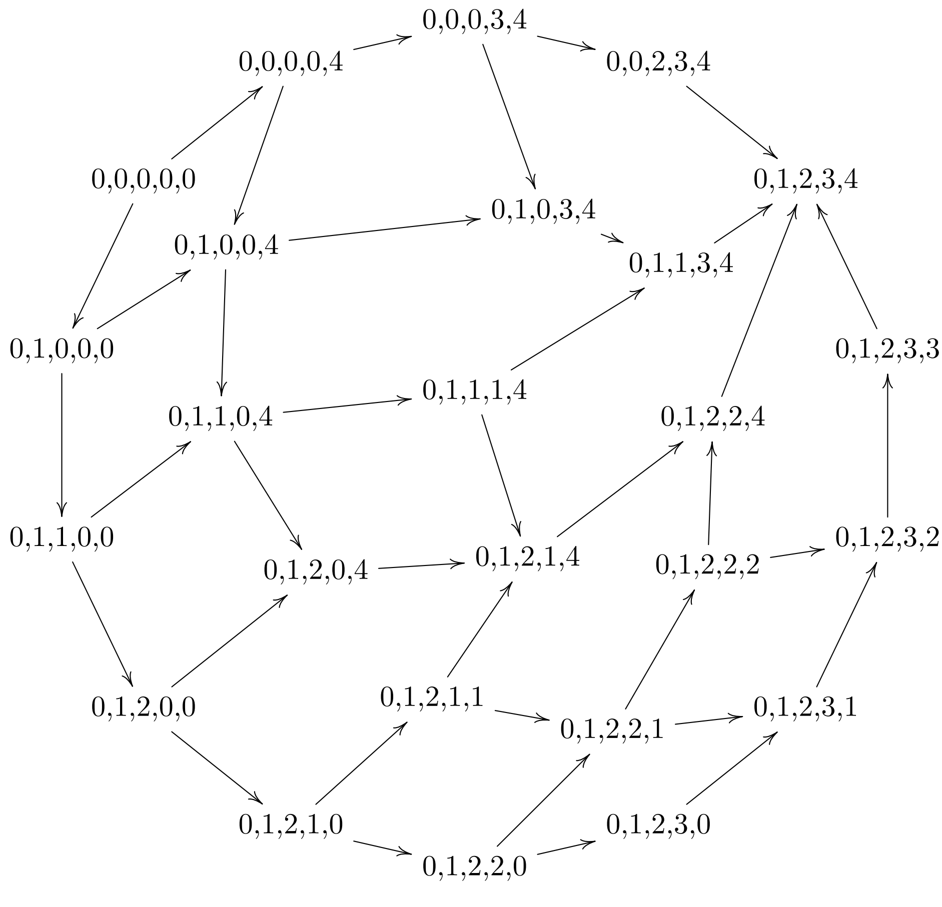
E_7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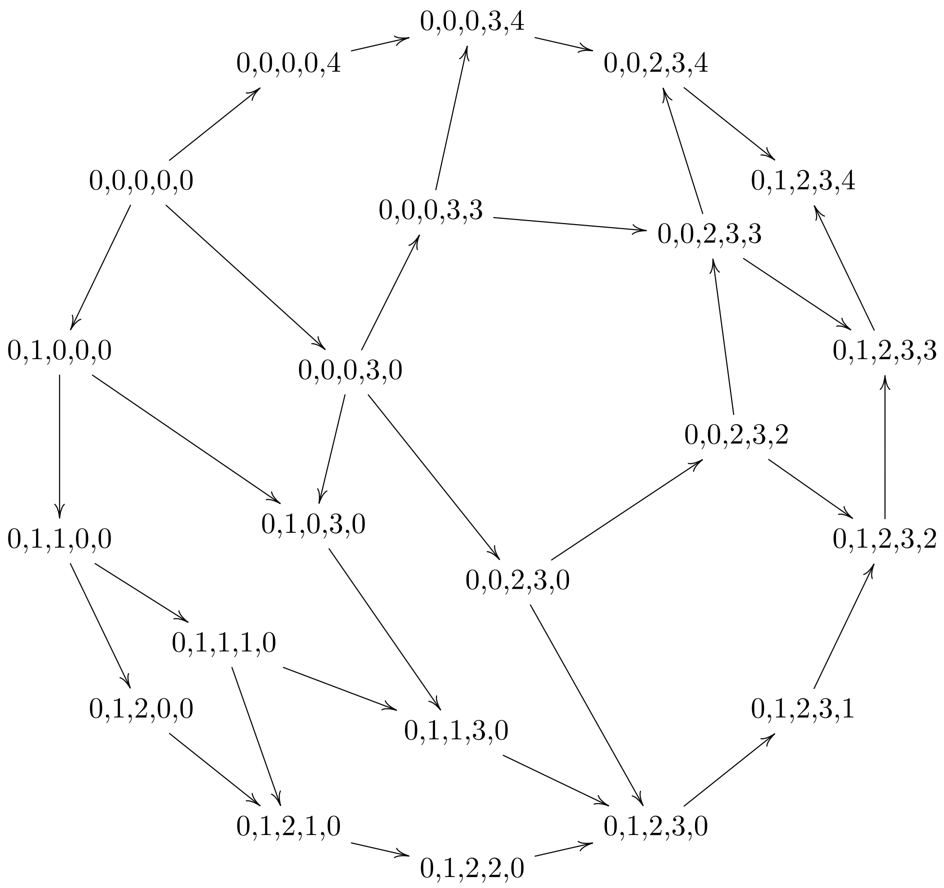
E_8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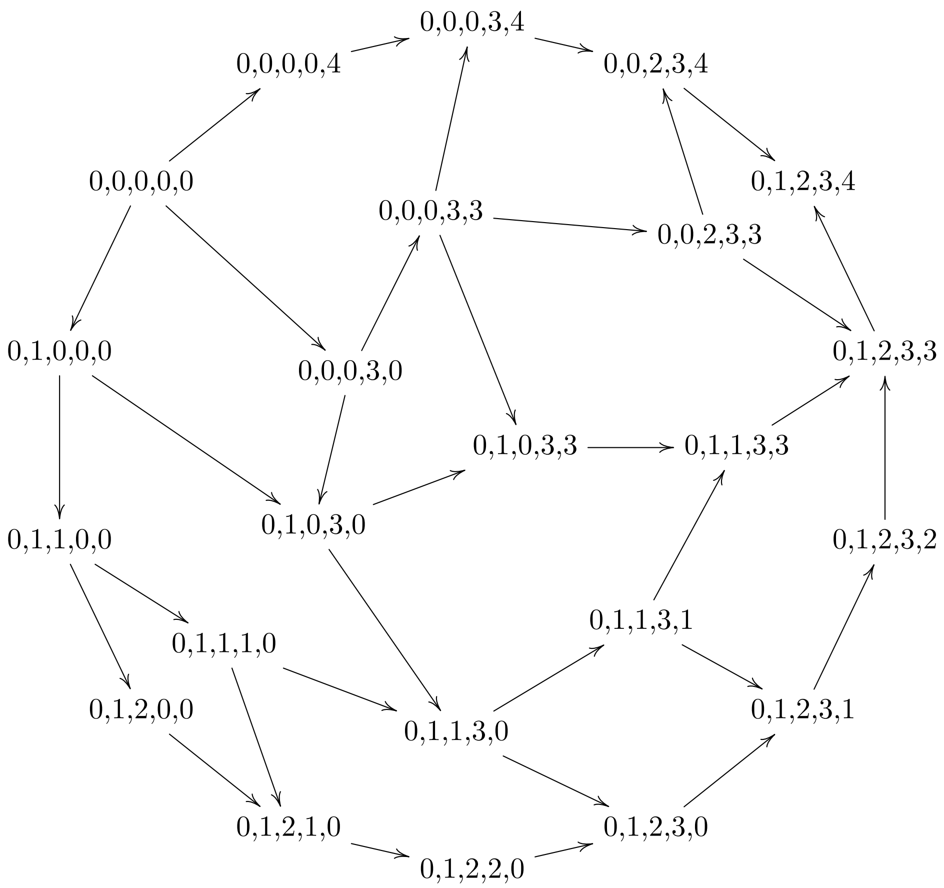
E_9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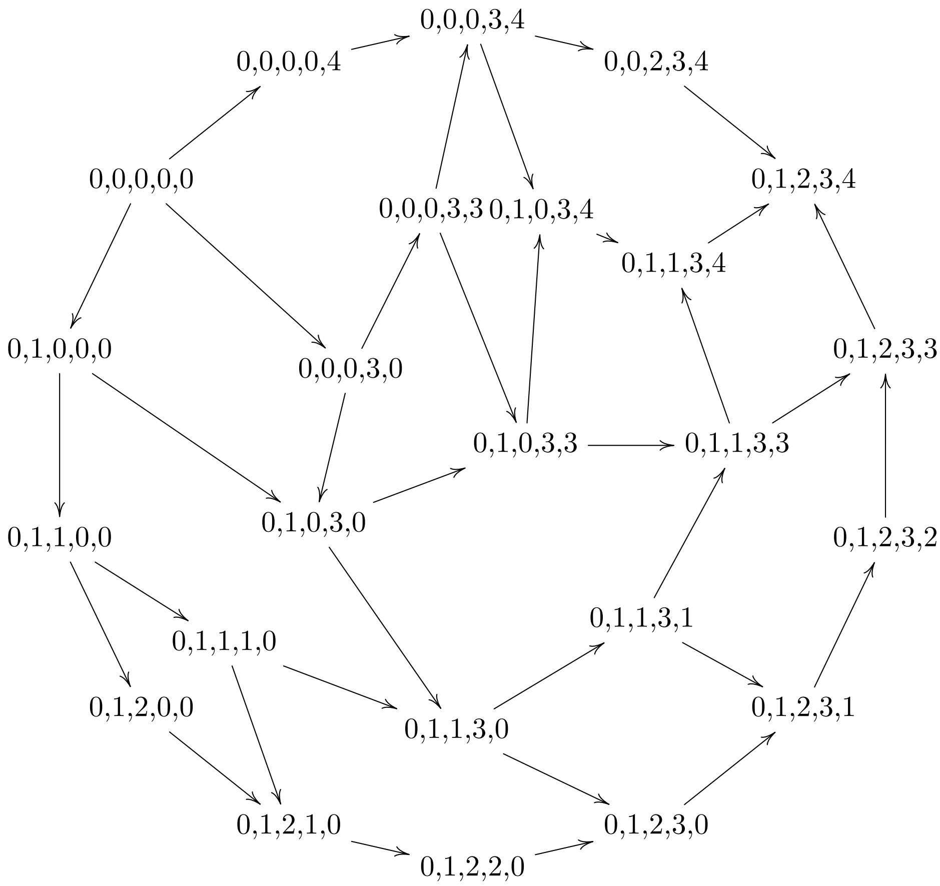
E_{10}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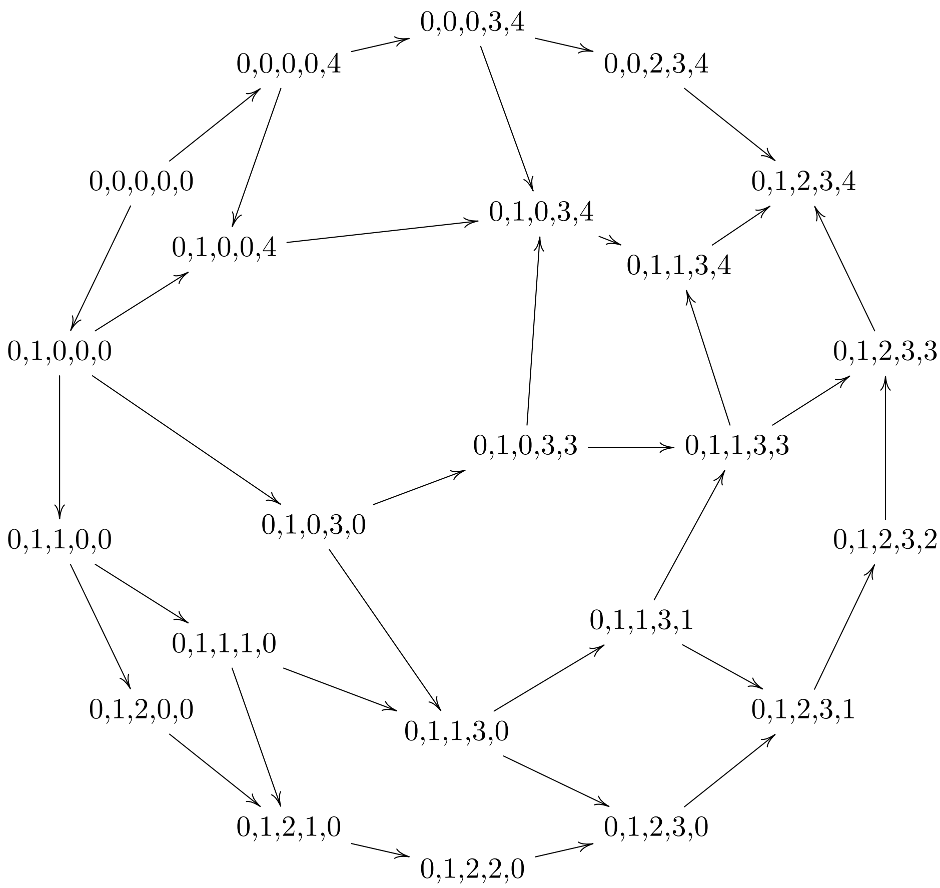
E_{11}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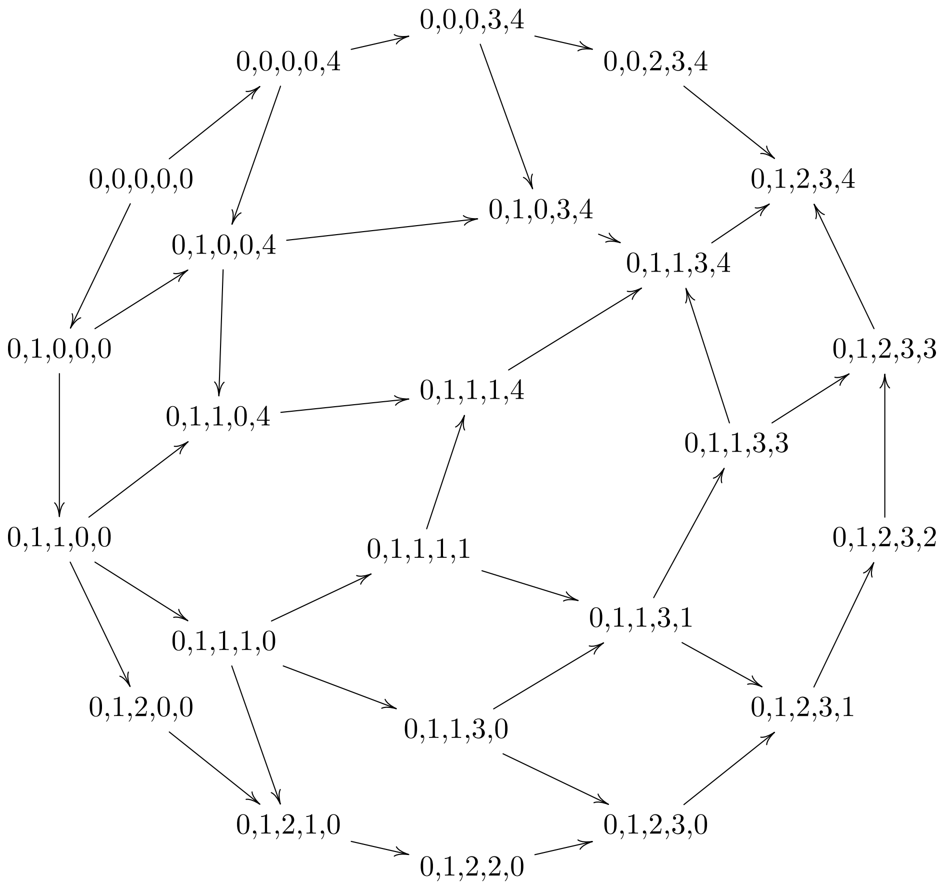
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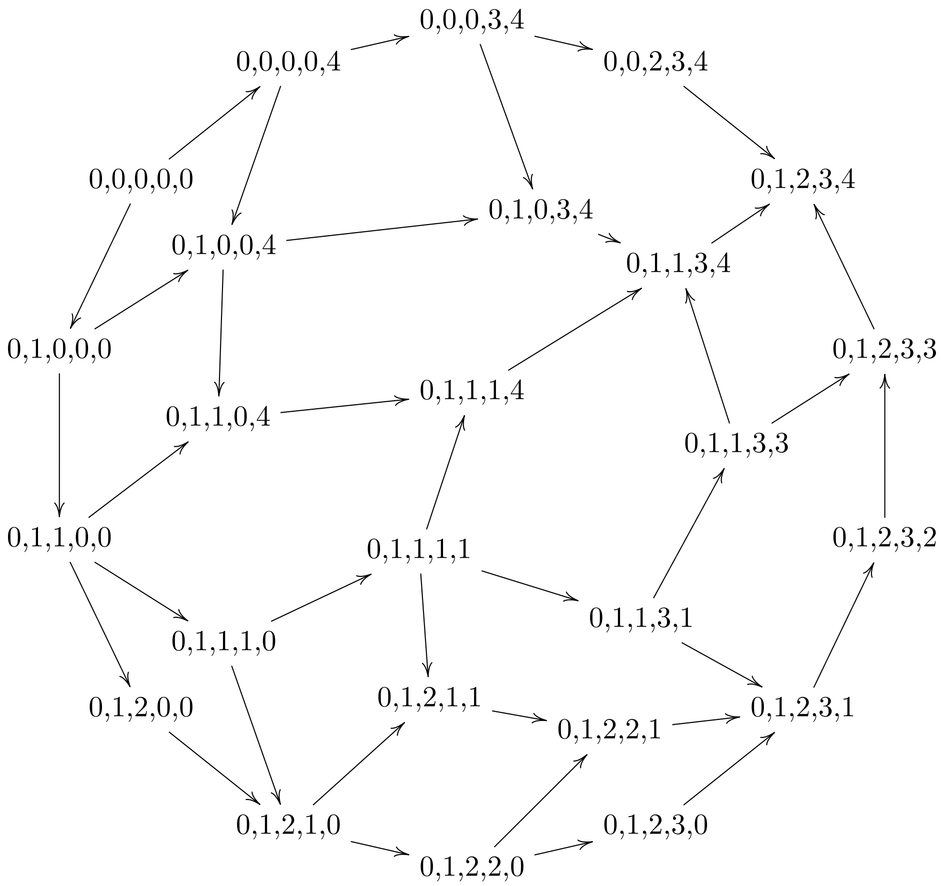
E_{13}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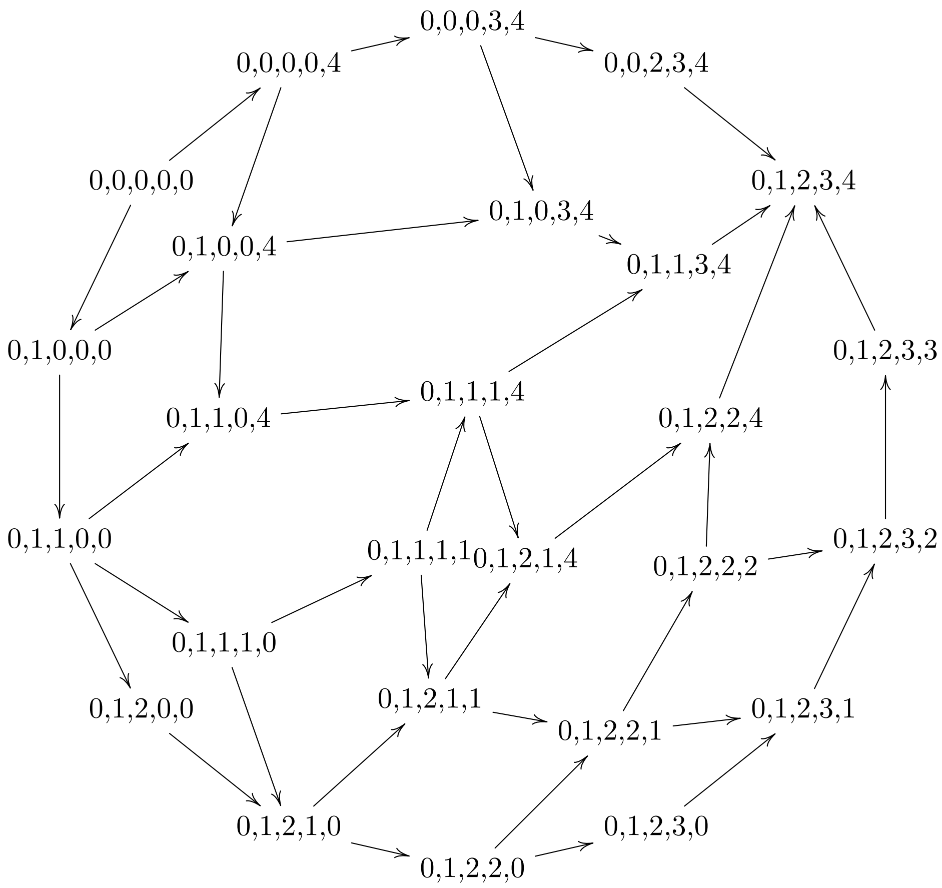
E_{14}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Appendix B

LGC-complexes

We continue our discussion of LGC-complexes in Section 1.4 of Chapter 1. At the time of writing, our reference [4] is not readily available. This appendix serves to include the necessary results and proofs which are due to Campbell [4].

The following results will use axioms **(L)** and **(G)**. These results will then be used in Theorem B.6 to deduce a pasting scheme in the sense of Johnson [12]. Firstly, we will give a characterisation of the globularity axiom **(G)** which is due to Campbell [4].

Proposition B.1 *A parity structure C satisfies axiom **(G)** iff the following equations hold*

$$E(x) + E(x^-) = B(x) + E(x^+)$$
$$E(x) + B(x^-) = B(x) + B(x^+)$$

for all $x \in C$.

Proof. We will first prove that axiom **(G)** is equivalent to

$$E(x) \cup E(x^-) = B(x) \cup E(x^+)$$
$$E(x) \cup B(x^-) = B(x) \cup B(x^+)$$

for all $x \in C$. Let $x \in C_n$ be given. Note that the above equations for the case of $n = 0, 1$ are trivial, so we let $n \geq 2$. Then

$$s_{n-1}R(x) = R(x)\backslash E(x) = R(x^-)$$

and so

$$s_{n-2}s_{n-1}R(x) = s_{n-2}R(x^-) = R(x^-)\backslash E(x^-) = R(x)\backslash(E(x) \cup E(x^-)).$$

Similarly, we also have

$$s_{n-2}t_{n-1}R(x) = R(x)\backslash(E(x) \cup E(x^+)).$$

Hence the equation $s_{n-2}s_{n-1}R(x) = s_{n-2}t_{n-1}R(x)$ is equivalent to the equation $E(x) \cup E(x^-) = B(x) \cup E(x^+)$. By a dual argument, it follows that the equation $t_{n-2}s_{n-1}R(x) = t_{n-2}t_{n-1}R(x)$ is equivalent to the equation $E(x) \cup B(x^-) = B(x) \cup B(x^+)$.

It remains to show that the above equations consist of disjoint unions. Note that $E(X) \subseteq R(X)$ and $B(X) \subseteq R(X)$ for any subset $X \subseteq C$. It follows that

$$E(x) \cap E(x^-) \subseteq (R(x)\backslash R(x^-)) \cap R(x^-) = \varnothing,$$

and similarly, $E(x) \cap B(x^-) = \varnothing$, $B(x) \cap E(x^+) = \varnothing$ and $B(x) \cap B(x^+) = \varnothing$. $\square$

The following propositions are due to Campbell [4]. In the order of appearance, they correspond to Proposition B.2.1, Proposition B.2.5, Proposition B.2.3 and Proposition B.2.6.

Proposition B.2 *Let C be a parity structure. Then $B(x) \cap E(x) = \{x\}$ for all $x \in C$.*

Proof. Note that $R(x) = \{x\} \cup R(x^+) \cup R(x^-)$ so $B(x) \cap E(x) = R(x) \backslash (R(x^-) \cup R(x^+)) = \{x\}$. □

Proposition B.3 *Let C be a parity structure satisfying axioms **(L)** and **(G)**. Let $x, y \in C$ be given. If $x \blacktriangleleft y$ and $x \neq y$, then $B(x) \cap E(y) = \varnothing$.*

Proof. Let $a \in B(x)$ and $b \in E(x)$ be given. To prove that $B(x) \cap E(y) = \varnothing$, it suffices to show that $a \neq b$. Firstly, we will show that $a \blacktriangleleft x$. This is certainly true for the case of $a = x$. Consider the case of $a \neq x$. By Proposition B.1 and Proposition B.2, $a \in B(x) \backslash \{x\} \subseteq B(x^-)$. Applying these propositions iteratively, we deduce that a has dimension less than x and is a negative face of a negative face of $\ldots$ x, and so $a \blacktriangleleft x$ as required.

By a dual argument, we have that $y \blacktriangleleft b$. It follows that $a \blacktriangleleft x \blacktriangleleft y \blacktriangleleft b$. Suppose that $a = b$; we aim to prove a contradiction. By anti-symmetry of $\blacktriangleleft$, we have $x = a = y$, which is a contradiction. Hence we have shown that $a \neq b$. □

Proposition B.4 *Let C be a parity structure satisfying axioms **(L)** and **(G)**. Then*

$$E(x^+) + B(x^-) = E(x^-) + B(x^+)$$

for all $x \in C$.

Proof. Let $x \in C$ be given. Firstly, we will show that $E(x^+) \cup B(x^-) = E(x^-) \cup B(x^+)$. Note that by Proposition B.2, $E(x) \backslash \{x\} = E(x) \backslash B(x)$. By Proposition B.1,

$$E(x^+) \backslash E(x) = (E(x) + E(x^-)) \backslash (B(x) + E(x)) \subseteq E(x^-)$$
$$E(x^+) \cap E(x) \subseteq E(x) \backslash \{x\} = E(x) \backslash B(x) \subseteq B(x^+)$$

Thus $E(x^+) \subseteq E(x^-) \cup B(x^+)$.

Note that by Proposition B.2, $B(x) \backslash \{x\} = B(x) \backslash E(x)$. By Proposition B.1,

$$B(x^-) \backslash B(x) = (B(x) + B(x^+) \backslash (E(x) + B(x)) \subseteq B(x^+)$$
$$B(x^-) \cap B(x) \subseteq B(x) \backslash \{x\} = B(x) \backslash E(x) \subseteq E(x^-)$$

Thus $E(x^+) \subseteq E(x^-) \cup B(x^+)$. Hence we have shown that $E(x^+) \cup B(x^-) \subseteq E(x^-) \cup B(x^+)$. By a dual argument, we can deduce the converse.

It remains to show that the unions are disjoint. We will first prove that $B(x^-) \cap E(x^+) = \varnothing$. Let $a \in x^-$ and $b \in x^+$ be given. Note that by disjointness of x^- and x^+, $a \neq b$. Also note that $a \blacktriangleleft x \blacktriangleleft b$. By Proposition B.3, $B(a) \cap E(b) = \varnothing$. It follows immediately that $B(x^-) \cap E(x^+) = \varnothing$.

We will now prove that $E(x^-) \cap B(x^+) = \varnothing$. By Proposition B.1 and Proposition B.2,

$$\begin{aligned} E(x^-) \cap B(x^+) &= ((B(x) + E(x^+)) \backslash E(x)) \cap ((E(x) + B(x^-)) \backslash B(x)) \\ &= (E(x^+) \backslash E(x)) \cap (B(x^-) \backslash B(x)) \\ &\subseteq E(x^+) \cap B(x^-) = \varnothing. \end{aligned}$$ □

Proposition B.5 *Let C be a parity structure satisfying axioms **(L)** and **(G)**. Then*

$$E(x) \backslash \{x\} = E(x^+) \cap B(x^+)$$
$$B(x) \backslash \{x\} = E(x^-) \cap B(x^-)$$

for all $x \in C$.

Proof. We will first prove that $E(x)\backslash\{x\} = E(x^+)\cap B(x^+)$. Note that by Proposition B.2, $E(x)\backslash B(x) = E(x)\backslash\{x\}$. By Proposition B.1,

$$\begin{aligned}
E(x^+)\cap B(x^+) &= ((E(x)+E(x^-))\backslash B(x))\cap((E(x)+B(x^-))\backslash B(x))\\
&= (E(x)\backslash\{x\}+E(x^-)\backslash B(x))\cap(E(x)\backslash\{x\}+B(x^-)\backslash B(x))\\
&= E(x)\backslash\{x\}+(E(x^-)\cap B(x^-))\backslash B(x)\\
&= E(x)\backslash\{x\}+x^-\backslash B(x), \text{ by Proposition B.2}\\
&= E(x)\backslash\{x\}, \text{ since } x^-\subseteq B(x).
\end{aligned}$$

By a dual argument, we can deduce the other equation $B(x)\backslash\{x\} = E(x^-)\cap B(x^-)$. □

Theorem B.6 *Let C be a parity structure satisfying axioms **(L)** and **(G)**. The relations E^n_k and B^n_k where $k \le n$*

$$\begin{aligned}
x\mathsf{E}^n_k y &\iff y\in E(x)_k\\
x\mathsf{B}^n_k y &\iff y\in B(x)_k
\end{aligned}$$

define a pasting scheme $(C,\mathsf{E},\mathsf{B})$. Furthermore, $\mathsf{E}(x) = E(x)$, $\mathsf{B}(x) = B(x)$ and $\mathsf{R}(x) = R(x)$ for all $x\in C$.

Proof. All that needs to be shown are conditions 4 and 5 of Definition 1.37. Note that condition 4 follows from Proposition B.5, and condition 5 follows from Proposition B.4. □

We will now show that loop freeness of the pasting scheme given in Theorem B.6 follows from the additional axiom **(C)**. The theorem below deduces a more general form globularity axiom **(G)**. This result is due to Campbell which can be found in Theorem B.2.8 of [4].

Theorem B.7 *Let C be a parity structure satisfying axiom **(G)**. Then*

$$\begin{aligned}
s_{n-2}s_{n-1}A = s_{n-2}A = s_{n-2}t_{n-1}A\\
t_{n-2}s_{n-1}A = t_{n-2}A = t_{n-2}t_{n-1}A
\end{aligned}$$

or equivalently,

$$\begin{aligned}
E(A_n)\cup E(A_{n-1}\backslash A_n^+) = A_n\cup E(A_{n-1}) = B(A_n)\cup E(A_{n-1}\backslash A_n^-)\\
E(A_n)\cup B(A_{n-1}\backslash A_n^-) = A_n\cup B(A_{n-1}) = B(A_n)\cup B(A_{n-1}\backslash A_n^+)
\end{aligned}$$

for any subcomplex $A\subseteq C$ and $n\ge 2$.

Proof. We will first prove that

$$s_{n-2}s_{n-1}A = s_{n-2}A = s_{n-2}t_{n-1}A \tag{*}$$

is equivalent to

$$E(A_n)\cup E(A_{n-1}\backslash A_n^+) = A_n\cup E(A_{n-1}) = B(A_n)\cup E(A_{n-1}\backslash A_n^-) \tag{**}$$

for any subcomplex $A\subseteq C$ and $n\ge 2$.

By definition,

$$\begin{aligned}
s_{n-2}A = A^{(n-1)}\backslash(E(A_{n-1}),\\
s_{n-1}A = A^{(n)}\backslash E(A_n),\\
(s_{n-1}A)_{n-1} = A_{n-1}\backslash A_n^+.
\end{aligned}$$

It follows that

$$\begin{aligned} s_{n-2}s_{n-1}A &= s_{n-2}(A^{(n)}\backslash E(A_n)) \\ &= A^{(n)}\backslash E(A_n)\backslash E(A_{n-1}\backslash A_n^-) \\ &= A^{(n)}\backslash (E(A_n) \cup E(A_{n-1}\backslash A_n^-)). \end{aligned}$$

Similarly, we also have

$$s_{n-2}t_{n-1}A = A^{(n)}\backslash (B(A_n) \cup E(A_{n-1}\backslash A_n^+)).$$

Thus equations (*) are equivalent to equations (**). Denote equations (**) by (1) = (2) = (3).

Note that by duality, it suffices to prove that (**) hold. We will first prove that (1) = (2). To prove that (1) $\subseteq$ (2), it suffices to show $E(A_n) \subseteq A_n \cup E(A_{n-1})$. Let $x \in E(A_n)$ be given; then there exists $y \in A_n$ such that $x \in E(y)$. By Proposition B.1, $x \in B(y)+E(y^+)$. If $x \in B(y)$, then by Proposition B.2, $x \in B(y) \cap E(y) = \{y\} \subseteq A_n$ as required. Otherwise $x \in E(y^-)$, then since A is a subcomplex $y^- \subseteq A_n^- \subseteq A_{n-1}$ and so $x \in E(y^-) \subseteq E(A_{n-1})$ as required.

To prove that (2) $\subseteq$ (1), it suffices to show $E(A_{n-1}) \subseteq E(A_n) \cup E(A_{n-1}\backslash A_n^+)$. Let $x \in E(A_{n-1})$ be given. Let $u \in A_{n-1}$ be minimal (with respect to $\blacktriangleleft$) with the property $x \in E(u)$. For the case of $u \notin A_n^+$, we have $u \in A_{n-1}\backslash A_n^+$ and so $x \in E(u) \subseteq E(A_{n-1}\backslash A_n^+)$ as required.

Consider the case of $u \in A_n^+$. Note that there exists $y \in A_n$ such that $u \in y^+$ and so $x \in E(u) \subseteq E(y^+)$. By Proposition B.1, $x \in E(y) + E(y^-)$. Suppose that $x \in E(y^-)$; we aim to prove a contradiction. Note that there exists $v \in y^-$ such that $x \in E(v)$. However $v \blacktriangleleft y \blacktriangleleft u$ and since A is a subcomplex $v \in y^- \subseteq A_n^- \subseteq A_{n-1}$, which contradicts minimality of u. Thus $x \in E(y)$ and so $x \in E(y) \subseteq E(A_n)$ as required.

Hence we have shown that (1) = (2). Finally, (2) = (3) follows by a dual argument. $\square$

The following results are due to Campbell [4]. The result below is Proposition B.2.10 of [4].

Proposition B.8 *Let C be a parity structure satisfying axiom **(G)**. Then*

$$\begin{aligned} \mu(x)_n^{\mp} &= \mu(x)_{n-1} = \pi(x)_n^{\mp} \\ \mu(x)_n^{\pm} &= \pi(x)_{n-1} = \pi(x)_n^{\pm} \end{aligned}$$

for all $x \in C$.

Proof. Note that by duality, it suffices to prove that $\mu(x)_n^{\mp} = \mu(x)_{n-1} = \pi(x)_n^{\mp}$. By definition, $(s_{n-1}A)_{n-1} = A_{n-1}\backslash A_n^+$. For any subcomplex $A \subseteq C$ we have

$$(s_{n-1}A)_{n-1} = A_{n-1}\backslash A_n^+ = (A_n^- \cup A_n^+)\backslash A_n^+ = A_n^-\backslash A_n^+ = A_n^{\mp}. \qquad (\dagger)$$

By Theorem B.7, $s_{n-1}s_nR(x) = s_{n-1}R(x) = s_{n-1}t_nR(x)$. Consider the subcomplex $A = s_nR(x)$. By definition, $(s_nR(x))_n = R(x)_n\backslash R(x)_{n+1}^+ = \mu(x)_n$. It follows that $A_n = (s_nR(x))_n = \mu(x)_n$ and $(s_{n-1}s_nR(x))_{n-1} = (s_{n-1}R(x))_{n-1} = \mu(x)_{n-1}$. By substituting the above into ($\dagger$), we obtain $\mu(x)_{n-1} = \mu(x)_n^{\mp}$.

Consider the subcomplex $A = t_nR(x)$. By definition, $(t_nR(x))_n = R(x)_n\backslash R(x)_{n+1}^- = \pi(x)_n$. Recall that $(s_nR(x))_n = \mu(x)_n$. It follows that $A_n = (t_nR(x))_n = \pi(x)_n$ and $(s_{n-1}t_nR(x))_{n-1} = (s_{n-1}R(x))_{n-1} = \mu(x)_{n-1}$. By substituting the above into ($\dagger$), we obtain $\mu(x)_{n-1} = \pi(x)_n^{\mp}$. $\square$

The result below is a special case of Corollary B.2.11 of [4]. We found that this is sufficient to prove Proposition B.10.

Corollary B.9 *Let C be a parity structure satisfying axioms **(L)** and **(G)**. If $u \lhd v$, $v \in \mu(x)$, then $u^- \cap \pi(x)^+ = \varnothing$.*

Proof. Firstly, note that $u \blacktriangleleft v$ follows from $u \lhd v$. Suppose that $y \in u^- \cap \pi(x)^+$; we aim to prove a contradiction. Note that there exists $w \in \pi(x)$ such that $y \in w^+$, and so we have $w \blacktriangleleft y \blacktriangleleft u$. By Proposition B.8, $\pi(x)_{n-1} = \pi(x)_n^{\pm}$ so w is a positive face of a positive face $\ldots$ of x, and so $x \blacktriangleleft w$. Similarly, we have $v \blacktriangleleft x$. It follows that $x \blacktriangleleft w \blacktriangleleft y \blacktriangleleft u \blacktriangleleft v \blacktriangleleft x$ which contradicts anti-symmetry of $\blacktriangleleft$. Thus $u^- \cap \pi(x)^+ = \varnothing$ as required. □

The result below is Proposition B.3.2 of [4].

Proposition B.10 *Let C be a parity structure satisfying axiom **(G)**. Let $x \in C$ and $S \subseteq C$ be a well formed subset such that $\mu(x) \subseteq S$. Then $\mu(x)$ is an $\lhd$-interval of S.*

Proof. We will show that $\mu(x)_n \subseteq S_n$ is an $\lhd$-interval of S_n. Let $w, v \in \mu(x)_n$ be given. It suffices to show that $u \in \mu(x)_n$ for any $u \in S_n$ with $w \prec u \lhd v$. By Corollary B.9, $u^- \cap \pi(x)_n^+ = \varnothing$. Let $y \in w^+ \cap u^-$; it follows that $y \notin \pi(x)_n^+$. Note that $y \in w^+ \subseteq \mu(x)_n^+$. Suppose that $y \notin \mu(x)_n^-$; we aim to prove a contradiction. By Proposition B.8, $y \in \mu(x)_n^{\pm} = \pi(x)_{n-1} = \pi(x)_n^{\pm} \subseteq \pi(x)_n^+$, which is a contradiction. Thus $y \in \mu(x)_n^-$ and so there exists $z \in \mu(x)_n$ such that $y \in z^-$. Note that $y \in u^- \cap z^-$ so by well formedness of S_n, $u = z \in \mu(x)_n$ as required. □

The result below is Proposition B.4.4 of [4].

Proposition B.11 *Let C be a parity structure satisfying axioms **(L)** and **(G)**. Consider the pasting scheme defined in Theorem B.6. A subset $A \subseteq C$ is a (J)-cell (in the sense of Definition 1.42) iff A satisfy the following conditions*

(a) $s_k A$ and $t_k A$ are subcomplexes for all k,

(b) $A \backslash A^+$ and $A \backslash A^-$ are well formed.

Proof. We will first verify that Definition 1.48 gives a characterisation of the source and target maps as defined in Definition 1.41. Recall that by Theorem B.6, we have $\mathsf{E}(x) = E(x)$ and $\mathsf{B}(x) = B(x)$. The source and target maps given in Definition 1.41 are as follows. For any n-dimensional subset $A \subseteq C$, the source and target given by

$$\widetilde{s}_k = \mathrm{dom}^{n-k} A \text{ and } \widetilde{t}_k = \mathrm{cod}^{n-k} A$$

where $\mathrm{dom}A = A \backslash E(A_n)$ and $\mathrm{cod}A = A \backslash B(A_n)$. It needs to be shown that $\widetilde{s}_k A = s_k A$ and $\widetilde{t}_k A = t_k A$.

Note that by duality, it suffices to show that $\widetilde{s}_k A = s_k A$ for all $k \leq n$. We will prove this using induction on k. For the case of $k = n$ the result follows by definition of dom. We will now prove the inductive step. Let $k < n$ be given; it needs to be shown that $\widetilde{s}_{k-1} A = s_{k-1} A$. Then

$$\begin{aligned}
\widetilde{s}_{k-1} A &= \mathrm{dom}^{n-(k-1)} A = \mathrm{dom}(\mathrm{dom}^{n-k} A) = \mathrm{dom}(A^{(k+1)} \backslash E(A_{k+1})) \\
&= (A^{(k+1)} \backslash E(A_{k+1})) \backslash E(A_k \backslash A_{k+1}^+) = A^{(k+1)} \backslash (E(A_{k+1}) \cup E(A_k \backslash A_{k+1}^+)) \\
&= A^{(k+1)} \backslash (A_{k+1} \cup E(A_k)), \text{ by Theorem B.7} \\
&= A^{(k)} \backslash E(A_k) = s_{k-1} A
\end{aligned}$$

as required.

Note that $\mathsf{R}(x) = R(x)$ so the notion of a subcomplex is equivalent to the notion of a subpasting scheme. Also note that $A \backslash A^{+} = \sum (s_k A)_k$ and $A \backslash A^{-} = \sum (t_k A)_k$. In order to prove the proposition, it suffices to check that $s_k A$ is k-dimensional for all $k \leq n$. By the definition, it is at most k-dimensional, and so it suffices to show that $(s_k A)_k = A_k \backslash A^{+}_{k+1}$ is non-empty. Let $x \in A_k$ be minimal with respect to $\blacktriangleleft$. Suppose that $x \in A^{+}_{k+1}$; we aim to prove a contradiction. Note that there exists $y \in A_{k+1}$ such that $x \in y^{+}$. Let $u \in y^{-}$ be given; since A is a subcomplex we have $u \in y^{-} \subseteq A^{-}_{k+1} \subseteq A_k$. However $u \blacktriangleleft y \blacktriangleleft x$, which contradicts the minimality of x. Thus $x \notin A^{+}_{k+1}$ and so $x \in A_k \backslash A^{+}_{k+1}$ as required. □

Theorem B.12 (Campbell) *Let C be an LGC-complex. The pasting scheme in Theorem B.6 is loop free.*

Proof. We need to show the three conditions of loop freeness given in Definition 1.43. Condition 1 follows from Proposition B.3. Condition 2 follows from axiom **(C)** and Proposition B.11. Condition 3 follows from Proposition B.10. □

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