CHAPTER 1

BACKGROUND TO THE STUDY

Understanding place value is not a matter of simply 'cracking' an arbitrary written code following adult explanation or some degree of exposure to computation. It is indissolubly linked to understanding the number system itself. Grasping it implies understanding a multiplicative recursive structure ...

(Sinclair, Garin, & Tieche-Christinat, 1992, p. 193)

Research on mathematics learning and teaching has flourished over the past three decades and there now exists a recognisable body of research that takes into account the importance of developing number sense as fundamental to learning mathematics. Number sense is generally considered a major goal of mathematics learning and instruction (Australian Education Council, 1990; Curriculum Corporation, 1997; National Council of Teachers of Mathematics, 1989; National Research Council, 1989). In the Curriculum and Evaluation Standards for School Mathematics (National Council of Teachers of Mathematics, 1989), number sense is described as "an intuition about number that is drawn from all the meanings of number" (p. 39). Children who have this intuitive feel for number are recognised because of their ability to choose appropriate strategies in mental computation and their general comfort with numbers in real world contexts. Markovits and Sowder (1994) list the characteristics of number sense as "using numbers flexibly when mentally computing, estimating, judging number magnitude, and judging reasonableness of results; moving between number representations; and relating numbers, symbols, and operations, all stemming from a disposition to make sense of numerical situations" (p. 5). A broader use of this is *multidigit number sense* which refers to "intuitive feelings for numbers and their uses as well as the ability to make judgements about the reasonableness of multidigit numbers in diverse problem situations" (Jones, Thornton & Putt, 1994, p. 118).

Adequate number sense is essential for effective functioning in a technologically-orientated society. Reform in educational practice has been fuelled by evidence that children's lack of number sense influences mathematical development generally and inhibits the effective use of mathematics in daily life. Recent and ongoing research (Cooper, Heirdsfield, & Irons, 1996; English, 1996a, b; Jones, Thornton, Putt, Hill, Mogill, Rich, & van Zoest, 1996; Markovits & Sowder, 1994; Olivier, Murray, & Human, 1990; Resnick, Bill, & Lesgold, 1992; Shuard, Walsh, Goodwin, & Worcester, 1991; Thornton, 1990; Thornton & Jones, 1994) has demonstrated that children develop effective number sense and can learn to solve number problems in ways that develop a range of numerical strategies.

Developing a coherent understanding of the number system as an extendable system based on units of ten is a fundamental aspect of number sense. The development of number concepts and relationships leading to an understanding of the number system must grow through the primary school grades so that children can understand and use numbers beyond those involving 'ones, tens and hundreds' meaningfully.

1.1 BACKGROUND TO THE PROBLEM

Over the past 25 years, mathematics educators have been concerned about the problem that many children complete primary and secondary school without developing adequate numeration and place value knowledge. "The initial introduction of the decimal system and the positional notation system based on it is, by common agreement of educators, the most difficult and important instructional task in mathematics in the early school years" (Resnick, 1983a, p. 126). Without a grasp of the number system, problems with number concepts and operations are prolonged into adulthood with serious consequences. For example, results of student achievement on the Third International Mathematics and Science Survey (Lokan, Ford & Greenwood, 1997) indicated that 24 % of Australian 9 year olds and 14% of 11 year olds were unable to identify the largest of four 4-digit numbers. Similar concerns have been raised about the difficulties experienced by young children in solving place value tasks such as those reported by the New South Wales Basic Skills Testing results (1989-1992 data): 36% of Grade 3 children were unable to solve a task where children had to identify a pictorial representation for the numeral 31, 31% were unable to identify the place value of the hundreds digit in a 4-digit numeral, 44% were unable to identify the next number after 103 when counting by tens, and 30% were unable to show four 3-digit numbers in order from largest to smallest (Leeson, Lindsey & Doig, 1997). Similar concerns about national standards have been raised by educators in Western countries such as the United States, New Zealand and the United Kingdom (Lokan, Ford & Greenwood, 1997).

A fundamental problem here concerns the number system itself. Children need to develop both conceptual understanding of the number system and skills in using numeration, including both the system of spoken number words and the corresponding written numeral system. Initially, children learn to count to ten and beyond, as they learn to develop an abstract concept of number. A second layer of the number system is constructed when the two meanings of ten, as 'ten units of one' and 'one unit of ten', are integrated within a multiplicative structure to determine the names and symbols for two-digit numbers. At this level, there is an added complexity because the English spoken number word structure for the teen numbers and multiples of ten does not match the symbolic system. A third layer of the number system is constructed when the three meanings of a hundred, as 'a hundred ones', 'ten tens' and 'one unit of a hundred', are integrated within a multiplicative structure to determine the names and symbols of three-digit numbers. This structure of layers continues with each new layer involving the next power of ten (or the group of ten of the preceding multiplicative unit) as the extra multiplicative unit.

Thus, a key aspect of the research on number concepts has focussed on the introduction of multidigit numbers which involves the construction of multiunits and the concept of place value; children must progress from counting where one is the iterable unit to counting where successive powers of ten become the iterable unit (Fuson, 1990b; Gray, 1991; Jones et al., 1994; Jones et al., 1996; Steffe & Cobb, 1988). Children must also understand and master the structure of the decimal system of numeration in order to use it in calculation (Hiebert & Wearne, 1992; Ross, 1989a). However, research shows that many children fail to develop adequate numeration and place value skills with difficulties in using large numbers and decimals continuing well into adulthood (Hiebert, 1992; Irwin, 1996b; Swan, 1990).

1.2 ACCOUNTING FOR CHILDREN'S DIFFICULTIES IN UNDERSTANDING NUMERATION AND PLACE VALUE

The information processing load required when children relate concrete materials to notational representations has been shown to be a problem for children learning numeration and place value (Boulton-Lewis & Halford, 1992). Their research showed that some representations and strategies used in teaching place value to young children impose an unnecessary processing load which can interfere with conceptual learning. It is further suggested (Boulton-Lewis, Wilss & Mutch, 1996) that "in order to reduce the load it is important to ensure that the child understands the operation, the relation between quantity, numeration and place value, and any symbolic and concrete representation of the task" (p. 138). They suggested that in order to use concrete representations successfully, processing load cannot be increased, and this can only occur if children know the material being used well.

Another difficulty is highlighted by Fuson (1990a) when she argues that current textbook presentations of place value that depend primarily on a skills analysis approach contribute to the failure of children to build adequate multiunit conceptual structures. As a result of this textbook approach "multidigit addition and subtraction are learned as procedures carried out on columns of single digits, and meanings other than single-digit meanings are not constructed" (Fuson, 1990a, p. 274).

Baroody (1990) suggests a need for changes to these existing practices. "Introducing multiunit meanings concretely as soon as children begin using two-digit numbers in school and discussing them throughout the primary grades, ... may develop a more secure basis for understanding multiunit concepts" (Baroody, 1990, p. 282). He outlines a sequence of concrete and pictorial models which become increasingly abstract but refers to research of Ross (1989a, b) that warns that children can learn procedures with concrete materials that do not reflect any understanding.

Further concern has been expressed by educators that teaching practice, including the use of concrete materials and text books, has not taken account of the most recent research findings in the area of children's development of number (Bednarz & Janvier, 1982; Fuson, 1990a, b; Steffe & Cobb, 1988; Wright, 1991a). Of particular interest to this discussion are the tendencies for children to develop the operations separately and for teachers to introduce written algorithms for these operations at an earlier age level. Commenting on school text series in the U.S., Fuson (1990a) observed that "rote rules are given initially for multidigit addition and subtraction, and inadequate support is provided for multiunit conceptual structures" (p. 274). Bednarz and Janvier (1988) agree that "a lot of children's misconceptions can even be explained by analysing the perception of numeration found in school text books, curriculum ... " (p. 300).

The problem here is that children do not recognise that notational values and each place value position of numeration are part of a structured system. Thompson (1982b; 1992) explains that this knowledge is compartmentalised. Children learn elements of the system as discrete parts but they do not understand the interconnections that give rise to this structure. These basic elements of the numeration system have been categorised and described as counting, partitioning, grouping and ordering numbers (Jones et al., 1996), and acquiring these elements is critical to an understanding of the number system. However, any compartmentalisation of knowledge encourages children to behave inconsistently from numeration task to task. Connections need to be built between the elements of the system and for any concrete representations to be used effectively, when learning some notational procedure, representation and notation need to be seen as reflections of each other (Thompson, 1992).

Both Labinowicz (1985) and Thompson (1982a) have suggested that difficulties children experience with numeration and place value are embedded in teaching and learning. Providing children only with ready-made groups of ten as concrete representations of number, which are narrowly linked to the numerals, may retard the process of constructing a structure for the number system. It can be questioned whether the problem with understanding numeration and place value lies in the rigid lock-step approach to curriculum that treats the number system up to a hundred as a collection of unrelated pieces of information. It appears that children are not encouraged, or helped to make the connections between the pieces of information. Groups of ten, which children form to assist counting large numbers of objects should become the basis for the construction of a versatile counting unit within the oral counting system (and the related numeral system). This lack of understanding of structure becomes critical because children are now expected to use larger numbers much earlier in everyday use than advocated in traditional curricula.

1.3 STATEMENT OF THE PROBLEM

Understanding the multiplicative nature of the base 10 system is critical to the development of children's "number sense", i.e. the facility to mentally decompose and reconstruct a number. The rules for using place value (involving how zero acts as a place holder, the regular ten-for-one trade rules that make each larger value in the next position to the left, and the continued generation of larger numbers by trading to the left again and again) are needed in order to carry out operations with multidigit numbers. **A main problem is that children do not recognise that the numbers they use are part of a system, and thus they do not have the multiunit structures to understand how the numbers are regrouped in mental and written algorithms.** Further, understanding of the use of powers of ten is needed in order to construct the multiunit conceptual structures for multidigit numbers. Whereas the powers of ten might be the 'landmarks' in the development of number sense, the way numbers are constructed depends upon understanding additive and multiplicative structures, the generation of patterns and construction of new multiunits (Rubin & Russell, 1992).

1.3.1 Research on numeration and place value

There has been continuing strong research interest in children's development of number concepts and processes (Bideaud, Meljac & Fischer, 1992; Davis & Maher, 1993; English & Halford, 1995; Fuson, & Briars, 1990; Jones et al., 1996; Leinhardt, Putnam & Hattrup, 1992; Mulligan & Mitchelmore, 1996a; Steffe, Nesher, Cobb, Goldin & Greer, 1996; Steffe & Wood, 1990; Thompson, 1997). This research has resulted in detailed reporting of the way children develop number concepts and operations. Several studies have described the development of ten as a unit (Boulton-Lewis & Halford, 1992; Cobb & Wheatley, 1988; Steffe & Cobb, 1988; Thompson, 1982a), and the understanding of place value (Bednarz & Janvier, 1988; Kamii, M., 1982; Ross, 1986, 1989a, b, 1990; Thompson, 1992). It is suggested that the construction of new conceptual multiunit structures is an ongoing process that occurs within the classroom environment that includes many elements other than just the representational objects for number (Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter, & Fennema 1997).

The importance of mental models that reflect conceptual structure of numeration has been highlighted by Boulton-Lewis and Halford (1992). It is asserted that "concrete teaching aids are useful only if children clearly recognise the correspondence between the structure of the material and the structure of the concept" (p. 21). The teacher's role is important in helping children construct mental models of the numerical relationships presented to them with concrete material (Cobb, Yackel, & Wood, 1992). Thompson (1992) suggested that much greater attention should be paid to numerical relationships within base-ten numeration. He also asserted that "mathematical reasoning at all levels is firmly grounded in imagery" (1996, p. 267).

1.3.2 Classroom-based studies

Much of the groundwork focussing on children's difficulties with numeration and place value has given rise to classroom-based studies (Bednarz & Janvier, 1988; Carpenter, Fennema, & Romberg, 1993; Fuson, Fraivig, & Burghardt, 1992; Hiebert & Wearne, 1992) and learning frameworks (Denvir & Brown, 1986a, Fuson et al., 1997; Jones et al., 1996; Resnick, 1983a) aimed at improving the teaching and learning of numeration, place value and multidigit operations. Case studies and teaching experiments have investigated young children's development of two and three-digit numeration (Cobb & Bauersfeld, 1995; Jones et al., 1994). New approaches for teaching and learning numeration, place value and arithmetical operations have been developed from the analyses of children's strategies and use of multiunit structures. Sinclair, Garin, and Tieche-Christinat (1992) assert the importance of instruction in the structure of the written numerals and the attendant understanding of place value. "The assumption that a grasp of the number system, and its attendant understanding of place value will not take place without instruction seems unquestionable" (p. 193). Furthermore, Fuson shows that children face difficulties when moving from what is seen as unitary conceptual structures needed for basic fact knowledge to more complex multiunit structures needed for addition and subtraction of larger numbers.

> *Reading and writing two-digit numerals up to 20 or 30 and addition and subtraction of all single-digit sums and differences (i.e., all sums through 9+9 and their subtraction inverses) can be based on unitary conceptual structures. Understanding of place value is multifaceted and prolonged and accompanies and follows understanding of multidigit addition and subtraction.*
>
> (Fuson, 1990a, p. 274)

Classroom-based studies on the learning and teaching of numeration have both stressed the need for direct instruction (Sinclair et al., 1992) and alternatively, relied on understanding multidigit addition and subtraction (Fuson, 1990a).

1.3.3 New directions for research on numeration

From the developing body of research on numeration and place value we know that a child's understanding of the numeration system is complex, is not necessarily lock-step and develops over many years. There is also a fundamental change in the way a child understands number from the early notion of number as a counting unit, to the construction of composite units (Steffe, 1994) and the reinitialising of units (Confrey, 1994). The process that starts with treating a collection as a whole and then develops as a system that is built on the iteration of grouping collections, requires significant cognitive reorientations. Hiebert and Behr (1988) suggested that "competence with middle school number concepts requires a break with simpler concepts of the past, and a reconceptualisation of number itself" (p. 8).

Research on how children extend their early number understanding and skills to cater for the expanding system of generating number names and symbols that we know as the Hindu-Arabic numeration system has been far less comprehensive than the work on early number. In the early 1990's there seemed a need to understand more fully why children failed to develop a structure for the numeration system at a time when concrete materials and more conceptually-based teaching approaches had been advocated for over a decade in Australia (Booker, Irons & Jones, 1980; Dienes, 1960, 1963, 1964).

Despite much research on counting and place value in the 1980's, by the 1990's researchers could not assert any firm explanations about why children fail to grasp the structure of the number system. Sinclair, Garin, and Tieche-Christinat (1992) argued that place value knowledge "is culturally constructed and socially transmitted through education" (p. 193). However it must also be questioned whether there is something more natural (intuitive) about the number ten as a base for the numeration system. Is there an intuitive move towards using groupings of 2, 5 or 10 as a tool for calculating the quantity of a collection because of perceptual or kinaesthetic relationships i.e. ten fingers? We know that children move from a perceptual system of counting (using fingers as perceptual items) to a system that is multiunit, based on the recursive nature of tens. The foundation of this structure is based on counting, ordering, notating, and the use of ten as a composite unit.

Labinowicz (1985, p.253) says that "by providing them with ready-made groups we fail to learn how children construct groupings naturally." He poses some research questions relating to children's counting and enumeration of increasingly larger collections of objects. He questions how children realise the importance of uniform groupings and what sized groups children tend to arrange initially. When does the notion of ten occur to them and when does the value of grouping the groups occur to them? Is the idea of grouping made easier by grouping into smaller number of objects initially? Should grouping by tens be delayed until grouping in general is understood?

In view of these questions, a number of researchers have conducted studies examining various aspects of numeration and place value. From an information processing model, Boulton-Lewis and Halford (1992) pursued the question of how children were influenced by concrete analogs in their developing numeration and place value skills, particularly in relation to counting and place value. At the same time, Hiebert and Wearne (1992) investigated children's representations of numeration including their understanding of decimals. Other studies endeavoured to categorise in developmental levels, key aspects of place value knowledge (Jones et al., 1994; Resnick, 1983a, b; Ross, 1986). However, few studies have consistently examined the relative influence of these key elements of children's understanding of the structure of number system. It seemed both appropriate and timely in 1992 to formulate a study that would describe the strategies children used in acquiring numeration and place value knowledge, and to ascertain how children form structure, or lack thereof, during this process. Further research also seemed necessary in order to investigate children's construction of their internal mental representations when developing basic counting, partitioning and grouping skills.

1.4 PURPOSE OF THE STUDY

This study will attempt to ascertain which aspects of developing number knowledge contribute to the apparent failure of children to make sense of numeration as a number system. A broad descriptive approach will be used to provide evidence of qualitative differences in the way children use their strategies and relate key elements of the numeration system.

The purpose of this study is to initially compare performance across a wide age range (Kindergarten to Grade 6 children) on a range of related tasks designed to investigate critical elements of numeration as a system. This can provide opportunity to compare these elements and to track the range and consistency of strategies used. Additionally children's representations of the counting sequence 1-100 will be described to gain more insight into the relationship between counting and numeration as a multiunit system. It is unclear how traditional instruction influences the development of critical elements of numeration and whether children's representations play a more important part in this development than has previously been thought.

1.5 RESEARCH QUESTIONS

Three basic research questions are addressed in this study in relation to children's developing understanding of the numeration system:

(i) **What strategies do children use in solving numeration tasks involving key elements of counting, grouping, and structuring place value?**

(ii) **How are critical aspects of counting and grouping related to understanding the structure of the numeration system?**

(iii) **How do children's representations of the counting sequence 1-100 reflect their developing structure of the numeration system?**

These questions are pursued within the broad research framework described earlier in this chapter.

This thesis is organised in nine chapters. The first chapter has provided the background to the research and has described the research problem and purpose of the study in terms of three key research questions. Chapter 2 provides a theoretical orientation for the study. Chapter 3 reviews a range of literature pertaining to the field of research on numeration and place value. Based on a preliminary study, Chapter 4 presents a summary of a cross-sectional pilot study in which tasks were trialled across Kindergarten to Grade 4 in order to provide a basis for planning the main cross-sectional study. The research design and methodology of the main study are described in Chapter 5. Chapter 6 provides a synthesis of key findings of the cross-sectional data by analysing performance overall and describing the range of solution strategies employed. Chapter 7 provides a discussion of results, and examines some interrelationships between performance and strategies used to solve tasks. Following the discussion of the main findings, further in-depth analysis of children's representations of the counting sequence 1-100 is reported and exemplified by a range of pictorial representations drawn from the cross-sectional study and a follow up study of children from Grades 4 to 6. Finally, Chapter 9 summarises the conclusions of the study, its limitations and the implications for teaching, learning and assessment, and for further research.

The next chapter will provide an overview of theoretical perspectives related to mathematics education research on numeration.

CHAPTER 2

THEORETICAL PERSPECTIVES: RESEARCH IN NUMERATION

Chapter 1 has established that the problem to be addressed in this thesis is that children find difficulty developing the multiunit structures needed to understand numeration as a system.

In this chapter a number of theoretical perspectives about the development of numeration are compared. Mathematics education research adopting developmental (Denvir & Brown, 1986a; Jones et al., 1996; Resnick, 1992; Ross, 1989a, b) and constructivist approaches (Cobb, Yackel, & Wood, 1992; Hiebert & Wearne, 1992; Steffe & Cobb, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983; Thompson, 1992, 1996; von Glasersfeld, 1987, 1996) are contrasted. Theories of cognitive processing (Boulton-Lewis, 1993a, b; English & Halford, 1995; Resnick, 1992), and representational thinking (Goldin, 1992a; Goldin & Kaput, 1996; Janvier, 1987; Pirie & Kieren, 1994) are discussed in terms of common constructs that underpin the research in this thesis. This discussion provides a theoretical basis for the research in terms of helping to explain how researchers investigate, analyse and interpret children's development of the numeration system.

2.1 THEORETICAL PERSPECTIVES

A number of theoretical approaches related to research in children's development of mathematics concepts are advanced in this chapter. Broadly, four interrelated approaches will be examined: developmental, constructivist, cognitive processing, and representational thinking. These approaches have been categorised for the purpose of comparison even though common aspects are found across approaches.

Kieran (1994), writing in the 25th anniversary year edition of the *Journal for Research in Mathematics Education*, described views of mathematical learning, over that time, from "an essentially Piagetian, individual-cognitive, theoretical framework to one that includes a social-interactionist Vygotskian orientation" (p. 584). Mathematical learning is now seen as "learning with understanding and that understanding (no longer equated with mathematical rigor and correctness) is an ongoing activity, not an achievement" (p. 605). Vygotsky (1978) emphasised the social basis of learning; instruction is most effective when children cooperatively engage in activities within a supportive learning environment and there is social interaction with more competent peers and adults. Although there are distinctions between the traditional and Vygotskian approaches, research in number learning has in the last decade focussed to a large extent on the cognitive aspects of the learner.

Much of theory-based mathematics education research has used the models and methodologies of cognitive science, particularly the theory of how humans process information. Most psychologists of cognition, whether they come from a developmental, constructivist, or information processing perspective, share the view that it is essential to try to identify individual's thinking and reasoning competencies independently of their performances on any particular occasion. They acknowledge that both specific knowledge and general competencies are needed to account for the varied performances of individuals but assume that competency can be defined without reference to context.

From a developmental approach, frameworks for number learning are developed where the child is encouraged to construct their own system of counting, building a system of tens and hundreds. Constructivist approaches focus upon the child's construction of numeration including the child's conceptualisation of units of ten and one hundred. Importance is placed on the way children construct and make sense of their own knowledge, where learners are regarded as active constructors of knowledge from their own experiences. In the 1990's social-constructivist theories acknowledged the importance of mathematical learning as an interactive as well as a constructivist activity (Cobb, 1996; Cobb, Yackel & Wood, 1992).

Research using a cognitive processing perspective has focussed more explicitly on the construction of cognitive structures and procedures. It is considered that an explicit recognition and understanding of the relationships inherent in our number system is essential if children are to deal meaningfully with multidigit numbers. Children are expected to 'internalise' the base-ten system of numeration and associated procedural rules for the operations on whole numbers. They are expected to extend these methods to decimal, fractional, and algebraic systems of formal representation. Further, children's internal and external representations of the number system provides yet another way that researchers can analyse mathematical relations and meanings, where 'internal' representations give a framework for describing individual knowledge structures and problem solving processes.

These theoretical perspectives concerning research in numeration discussed in this Chapter have many commonalities and much of the research cited is based on several perspectives. However each of these perspectives has its origins to some extent in Piagetian theory. The following section discusses what is termed broadly 'developmental' approaches in order to focus discussion of research on frameworks and models of numeration and place value.

2.2 DEVELOPMENTAL APPROACHES

Piagetian research throughout the 1970's focussed on children's acquisition of prenumber concepts which were considered necessary for readiness to use number. Developmental psychologists investigated the complexities of conservation, seriation, and classification.

Inconsistent results and criticisms of methodology (Clements & Callahan, 1983; Donaldson, 1978; Markman, 1979; Pennington, Wallach, & Wallach, 1980) questioned the notion of a developmental barrier to a child's number understanding. As a result, a major research interest in children's development of counting as it relates to number conceptualisation occurred during the 1980's (Fuson, 1988; Fuson, Richards, & Briars, 1982; Resnick, 1983a; Steffe & Cobb, 1988) and was dominated by the work of the constructivist research group (Steffe, von Glasersfeld, Richards, & Cobb, 1983; Steffe & Cobb, 1988). This research on counting had widespread influence on research into other mathematical concepts and processes (Lesh & Landau, 1983). A fuller review is provided in Chapter 3 and some aspects are discussed in Section 2.3 following.

A developmental approach to researching children's mathematics is built generally on an underlying belief that there is a sequential development of mathematical skills and understandings. Unlike Piaget's developmental theory it is recognised that learning is not rigid, there are no defined barriers to children's learning and there are multiple learning pathways possible. A common aim of researchers is to build a plausible account of conceptual development from a cognitive science point of view. Theories of conceptual understanding have been built up on the basis of detailed analyses of procedures used by children in performing designated tasks. Various models have been developed by an expanding and successively elaborated set of processes or schemata.

While some researchers built models using a hierarchy of skills (Denvir & Brown, 1986a; Gagne, 1965) others attempted to define stages of learning (DeBlois, 1996; Resnick, 1983a; Ross, 1989a, b; Schaeffer, Eggleston & Scott, 1974; Wright, 1990) or levels of understanding (Bergeron & Herscovics, 1990; Bruner, 1962; Clark & Kamii, 1996; van Hiele, 1986). In common with a constructivist point of view, each stage incorporates a previous stage, the stages form an invariant sequence, and each new stage involves a conceptual reorganisation resulting from reflection and abstraction. A level "does *not* refer to a stretch of time ... it indicates a certain elevation and ... invariably implies a specific degree or height of some measurable feature or performance" (von Glasersfeld & Kelly, 1983, p. 157).

Some researchers have designed frameworks to show developmental sequences for understanding aspects of the numeration system (Denvir & Brown, 1986a; Fuson et al., 1997; Jones et al., 1996; Labinowicz, 1985; Resnick, 1983a; Ross, 1986, 1989b). Kamii (1982) asserted that developmental patterns emerge in terms of the order in which ideas are constructed. Ross (1986, 1989b) proposed a five-stage model of the interpretations children assign to two-digit numerals. Clark and Kamii (1996) later identified levels in children's progression from additive to multiplicative thinking. Similarly, Resnick (1992) and Irwin (1996a) identified a sequence that learners follow in understanding quantitative relationships like covariation (effects on the whole of changes to a part) and compensation. DeBlois (1996) developed a three stage model for the notational system of numeration and Jones et al. (1996) proposed a framework for children developing understanding of multi-digit numbers. Closely related to this framework, Fuson et al. (1997) have produced a developmental sequence of conceptual structures for two-digit numbers. An overview of each of these frameworks will be discussed in turn in relation to the development of numeration to provide a background to a fuller review of the literature in Chapter 3.

2.2.1 Developmental framework of Resnick

Resnick (1983a) suggests a theoretical framework for the development of number representation that involves three broad periods: preschool, early primary and primary. The preschool period is when "counting and quantity comparison competencies" (p. 109) provide the basis for a child's use of 'number', through using a mental number line. The early primary period is when the development of the part-whole relationship (schema) enables the child's "invention of sophisticated mental computational procedures and the mastery of certain forms of story problems" (p. 109). The primary period is when the representation of number is modified to reflect the *structure* of the number system. Resnick goes on to detail the development of understanding the decimal, place value numeration system through the "successive elaboration of the part-whole schema for numbers" (p. 126). In view of a lack of sufficient research at the time Resnick indicated that her framework was tentative but in 1990 Resnick and Greeno (Resnick, 1992) more fully developed the notion of building mathematical competence. They identified four kinds of mathematical thinking about quantity and number:

(i) mathematics of protoquantities - reasoning is about amounts of physical material, language used is descriptive and comparative;
(ii) mathematics of quantities - reasoning is about numerically quantified amounts of material and actions on those quantities, number words are used as adjectives;
(iii) mathematics of number - numbers are conceptual entities that can be manipulated and acted upon, number words are used as nouns, and
(iv) mathematics of operators - numbers and operations on numbers are conceptual entities that can be reasoned about.

From the view of this framework, understanding the structure of the number system would appear to be available only to children who can reason in terms of the mathematics of operators and relations. Resnick (1992) suggested that in order to develop these competencies children must "participate in situations of reasoning and talking about numbers and their relations and about operators *without immediate reference to counted or measured quantities of material*" (p. 419). She also suggested that formal notations, carefully linked to children's experiences, language and invented computational procedures, are needed to support talk about numbers and their compositions.

2.2.2 A descriptive framework proposed by Denvir and Brown

From the results of individual interviews with 7-9 year old low achievers in the *Low Attainers in Mathematics* project, Denvir and Brown (1986a) made predictions about hierarchical skills that might exist between different aspects of number. A descriptive framework for number was identified from the observation of behaviour, reflection on the responses, logical analyses of the mathematics and evidence from the research literature. The part of the framework which shows the relationship between skills in the place value strand (leading to mentally carrying out 2-digit subtraction with regrouping) is given in Figure 2.1. Relationships between skills in which one skill is a prerequisite for the other are indicated by arrows, whereas when skills appeared to be strongly connected but acquisition may be in either direction the relationships are shown by dotted lines.

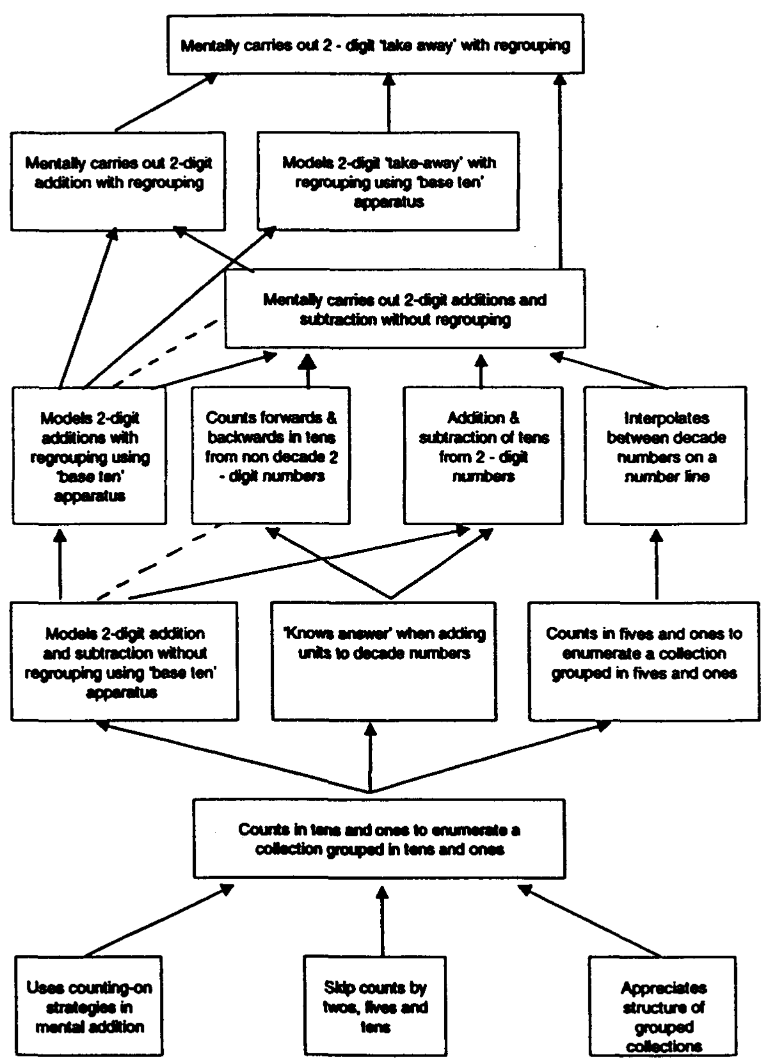


Figure 2.1: Descriptive Framework for Understanding Number - Place Value Strand (adapted from Denvir & Brown, 1986a, p. 33)

Denvir & Brown (1986a) reported that the strongest hierarchical strand in their framework appeared to be the counting strand: counting-on; counting collections grouped in tens; mentally adding units to decades, and mentally adding tens to 2-digit numbers. It is of interest to note the important role that aspects of counting play in this framework. These counting skills can be related to the conceptualisation of tens as units discussed by Fuson (1990a, b) and Cobb and Wheatley (1988) and are further discussed in Chapter 3 (Section 3.2).

In two follow-up teaching studies Denvir and Brown (1986b) found the hierarchy to be most useful for describing a child's knowledge of number which was then used as a basis for extending the child's understanding and establishing which cognitive skills the child was most likely to learn (Denvir & Brown, 1986b). The hierarchy was not found to be a precise predictor of which, or how many skills a child will learn and so it was concluded that "teaching should not be too prescriptive or rigid in its assumptions about what may be learnt" (p. 157).

2.2.3 Developmental stages of Ross

Ross (1986) built on the work of Mieko Kamii (1982) in a comprehensive study of children's understanding of place value. Ross proposed a five stage model to explain the development of children's understanding of our numeration system for two-digit numbers and this is discussed in more detail in Section 3.2.3 of Chapter 3. It was suggested that understanding of our numeration system requires the learner to build on the early notions of the part-whole concept involving partitioning, and the combining and rearranging of single-digit numbers. The standard place value partitioning of a 2-digit number is then directly linked to the meanings of the individual digits. The renaming of numbers in computational algorithms is seen as a nonstandard tens and ones partitioning. It is asserted that in many teaching situations a child's understanding of place value would only be challenged when confronted by regrouping in algorithmic work.

2.2.4 Steffe's counting stages

Investigations by Steffe and colleagues of children's early arithmetical development (Steffe, 1991a; Steffe et al., 1983; Steffe & Cobb, 1988) resulted in the formulation of a five-stage model of children's construction and elaboration of the number sequence as shown in Table 2.1. Steffe analysed the process of creating units as a complex developmental sequence of making the units, treating the composite as a unit and iterating it, and finally creating an anticipatory scheme that allowed the learner to imagine mental actions, particularly the action of unitising (Steffe, 1994). With each level in the counting stages the units become more abstract and less dependent on objects. Progress in counting results from the children's need to count-on or count-back larger quantities in addition or subtraction problem situations. At the fourth stage children simultaneously keep track of units counted-on (or back) as they are being counted. At the fifth stage the counting sequence can be used as a complex and flexible tool for calculating. The construction of the part-whole relation enables mental activity for addition and subtraction where constituent numbers are separated into parts and other numbers are formed built on units of ten. These counting abilities are related to the emergence of logical thinking capacities characteristic of Piaget's preoperational, concrete and abstract operational stages.

Table 2.1: Steffe's Counting stages

Stage 1: Perceptual counting
Children are dependent in their counting on what they can see and touch, or perceptually available objects.

Stage 2: Figural counting
Children generate (imagine) mental representations of figural patterns for hidden objects. Counting is still dependent on perceptual activity, such as pointing or nodding at expected positions of hidden objects.

Stage 3: Initial number sequence
Children use the counting words as units to be counted. Children count on, rather than count from one when solving tasks involving hidden items.

Stage 4: Tacitly-nested number sequence
Children simultaneously keep track of the units counted-on as they are being counted (double counting). The development towards greater internalisation of the counting act is also reflected in children's counting-back methods. Children can choose the most efficient of count-down-from and count-down-to strategies to solve subtraction problems.

Stage 5: Explicitly-nested number sequence
Children are simultaneously aware of two number sequences when subtracting numbers and can compare the smaller composite and containing composite units. The part-whole relation has been constructed and so children are explicitly aware of the constituent composite units and the containing composite unit in addition and subtraction situations. Children use strategies such as compensation, equal adjustments, using a known fact, adding to ten, and subtraction as the inverse of addition.

2.2.5 Developmental approach by Kamii

A developmental approach by Kamii (1986) and Kamii and DeClark (1985) emphasised the need to allow children time to construct their system of ones as a prerequisite for building their system of tens. In the tradition of Piagetian research, it is asserted that number concepts belong to logical-mathematical knowledge, the source of which lies in the child's mental action. Number is the synthesis of the relationships of order and hierarchical inclusion, first at the level of a system of ones, and then at the level of a system of tens. Figure 2.2 shows how the system of tens requires the mental cutting of the first system of ones into equal parts of tens while keeping the first system intact. There is a system of ones with each new counting number being at the same time the 'next number' and the 'number of items so far', i.e. hierarchical inclusion. There is also a system of tens constructed on the system of ones. Figure 2.2 also shows how counting in tens (10, 20, 30, ...) or counting tens (1, 2, 3, ...) involves coordinating the relations of order and hierarchical inclusion where each new counting number is at the same time the 'next ten' and the 'number of tens so far'. Kamii and Livingston (1994) asserted that the systems of ones, tens, hundreds and so on can not be acquired by empirical abstraction from external models but must be "constructed by constructive (reflecting) abstraction on previously constructed relationships" (p. 120).

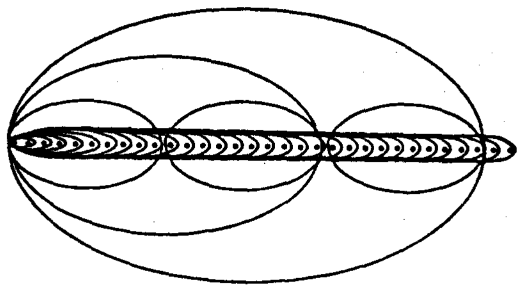


Figure 2.2: The construction of the system of tens on the system of ones (Kamii, 1986, p. 79).

Kamii and De Clark (1985) suggested that "given what we know about the developmental course of children's thinking, we ought to ask ourselves whether it would be wiser to delay place value instruction until children have solidly constructed the number series (by repetition of the +1 operation) and can partition wholes in many different ways (part-whole relationships)" (p. 63). It is explained (Kamii, 1989) that the written numerals are social knowledge, the decision to use ten as the base is a convention, and the hierarchical part-whole relationships shown in Figure 2.2 belong to logical-mathematical knowledge.

2.2.6 Developmental sequence for the notational system, DeBlois

DeBlois (1996) proposed a three stage development of the notational system of numeration from her case study research with eight to eleven year old children experiencing learning difficulties. The stages are:

1. *Preliminary concept* - intuitive, procedural and abstract composition;
2. *Emerging concept* - procedural and abstract composition, and
3. *Formal composition of the emergent concept* - formalisation relative to conventions and quantities.

The stages of development describe how children progress from a stage where notation is seen as a spatial pattern between the figures, to the second stage which relates the digits to the count of elements and grouping, and then to the third stage where the relations between the different quantitative units are known. DeBlois (1996) argued the importance of children with learning difficulties developing ways of creating relations between the different objects of their learning. It was suggested that many of these children do not get the positive reinforcement from instruction for the number relations that they do make nor the support for the construction of new connections.

2.2.7 Developmental framework of Jones, Thornton, Putt, Hill, Mogill, Rich and van Zoest

More recently, Jones et al. (1996) adopted a developmental perspective when identifying four constructs which are central to the development of multidigit number sense - counting, grouping, partitioning, and number relationships. A framework for nurturing and assessing multidigit number sense was developed by considering each of these major components at four levels of place value thinking as follows:

Level 1: pre-place value;
Level 2: initial place value;
Level 3: developing place value;
Level 4: extended place value, and
Level 5: essential place value.

The "key learning element in moving from level 1 to level 3 across all four constructs is the ability to recognise and readily apply composite units, that is, to think in terms of both single units and groups based on ten as a unit" (p. 325). The distinction between levels 3 and 4 is associated with the transition from 2-digit to 3-digit number representations. Mental calculations with 3-digit numbers can be carried out using units based on hundreds, tens and ones. Level 5 children can find or determine equivalent standard and nonstandard representations of 3-digit numbers. The constructs of the framework appeared to be highly stable within each of the five levels and across the full range of responses exhibited in twelve case studies (Grade 1 and 2 children) that were undertaken to validate the framework. The descriptors which define the thinking patterns at each level of the framework have covered the early elements of the developing structure of the numeration system and can be related to the transition from constructing numerical composite units to constructing abstract composite units for two-digit and then three-digit numbers. Further discussion of the developmental components of the framework form a central part of the discussion in Chapter 3, Section 3.2 on children's understanding of numeration.

2.2.8 A developmental sequence of conceptual structures for two-digit numbers (Fuson)

Fuson, Smith and Lo Cicero (1997) used their research on the use of conceptual supports that help children build number meanings to produce a sequence of five correct conceptions that children have of two-digit numbers. The developmental sequence of conceptual structures for two-digit numbers is known as the UDSSI (unitary, decade, sequence, separate, integrated) model as follows:

(i) Unitary conception in which children count a two-digit number quantity by ones, the entire number word or numeral refers to the whole quantity;
(ii) Decades-and-ones conception where each word and each digit takes on a meaning as a decade or as the extra ones;
(iii) Sequence-tens-and-ones conception where units of ten single units are formed and in which children count by tens and then by ones;
(iv) Separate-tens-and-ones conception in which the units of tens and the units of ones are counted separately by ones, and
(v) Integrated sequence-separate conception in which the sequence-tens and separate-tens conceptions are related to each other.

Individual learners may construct either sequence-tens or separate-tens conceptions first, and number words seem to facilitate the sequence-tens conception whereas written numerals facilitate the separate-tens conception.

The developmental frameworks constructed by Resnick (1983a), Denvir and Brown (1986a), Ross (1986), DeBlois (1996), Jones et al. (1996) and Fuson et al. (1997) have been shown to be useful in assisting teachers assess understanding of numeration and for developing suitable experiences for extending learning. Each teaching program involved elements of a constructivist approach such as group work, problem solving, discussion, reporting and reflective activities. Resnick (1992) questioned the idea of hierarchical learning being used as a basis for teaching. The alternative she suggested was a *distributed curriculum* in which multiple topics are developed with increasing levels of sophistication and demand rather than a strictly sequential curriculum. Jones et al. (1996) reported that their framework enabled teachers, implementing a teaching program, to assess and monitor children's learning and to have a sharper focus of the instructional program for multidigit number sense than normally would be the case. Fuson et al. (1997) emphasised how complex the teaching-learning process is. "The learning zones (zones of proximal development) and individual constructive paths of children in our classes varied so much that it is difficult for a teacher to meet all the needs for assistance at the same time" (p. 762).

In this section, various developmental models of numeration have been discussed. It must be acknowledged that the research undertaken in formulating these models has in each case involved elements of what can be also described to some extent as a constructivist approach to learning. The following section examines some constructivist frameworks.

2.3 CONSTRUCTIVIST APPROACHES

During the 1980's and 1990's researchers were strongly influenced by the constructivist approach and much of the work on early number contributed towards theoretical frameworks for describing children's construction of numeration.

Broadly, constructivism is a philosophical position which defines learning as active construction. All knowledge is constructed and children's mathematical understandings are constructed through their interactions with mathematical contexts and environments (von Glasersfeld, 1996). Constructivism is heavily influenced by considerations of the importance of the context of learning, both physical and social. From a constructivist perspective, the learner seeks equilibrium and problems disrupt that equilibrium. Once the learner acknowledges a problem exists he/she works towards re-establishing the equilibrium. Knowledge is not simply accumulated information, it is the construction of cognitive structures that are applicable, enabling and generative. The process of reflective abstraction (Piaget, 1971) occurs when an action or operation which is seen as repeatedly successful, is set aside by various processes of naming, objectifying and making it a tool for further action (Confrey, 1991). Problem solving is an essential part of the process of the construction of knowledge (Confrey, 1985; 1994).

While there are certainly common aspects that are shared between developmental and constructivist approaches, several researchers (Confrey, 1991, 1994; Fuson, 1992a, b; Kamii, M., 1982; Kamii & DeClark, 1985; Labinowicz, 1985; Thompson, 1982a) have explored numeration from broadly a constructivist perspective.

Thompson (1982a) characterised the concepts of numeration in terms of relational networks. The nodes of the networks are aspects of the concept (routines, special words, meanings), and relationships among the aspects are shown. Configurations for the concepts of ten and place value are shown in Figures 2.3 and 2.4.

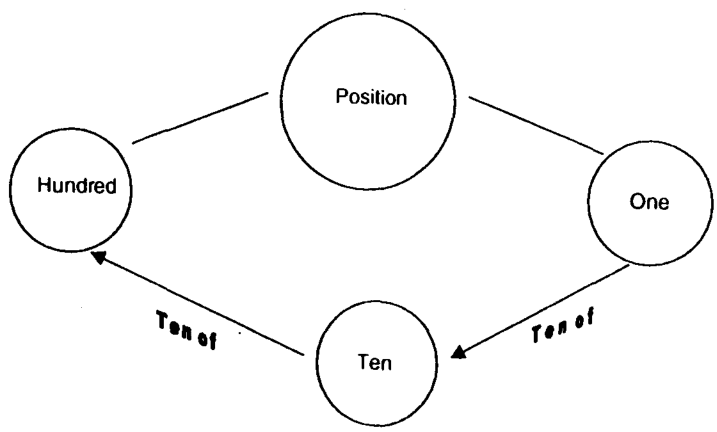


Figure 2.3: Concept of place value (Thompson, 1982a, p. 99)

Figure 2.3 shows how the concept of place value in the notational system is exemplified by the values given to positions that digits (individual symbols in notation) occupy. The relationships between the positions is the multiplicative relation of 'ten of'. The concept of place value in numeration is schematised here by showing this relationship between ones, tens and hundreds, all of which need the concept of position for notational representation. Thompson (1982a) explained that in the concept of place value, "order is a result of recursion, in that the successor to a unit in the sequence of numeration units is constructed by taking ten of that unit as a unit" (p. 100).

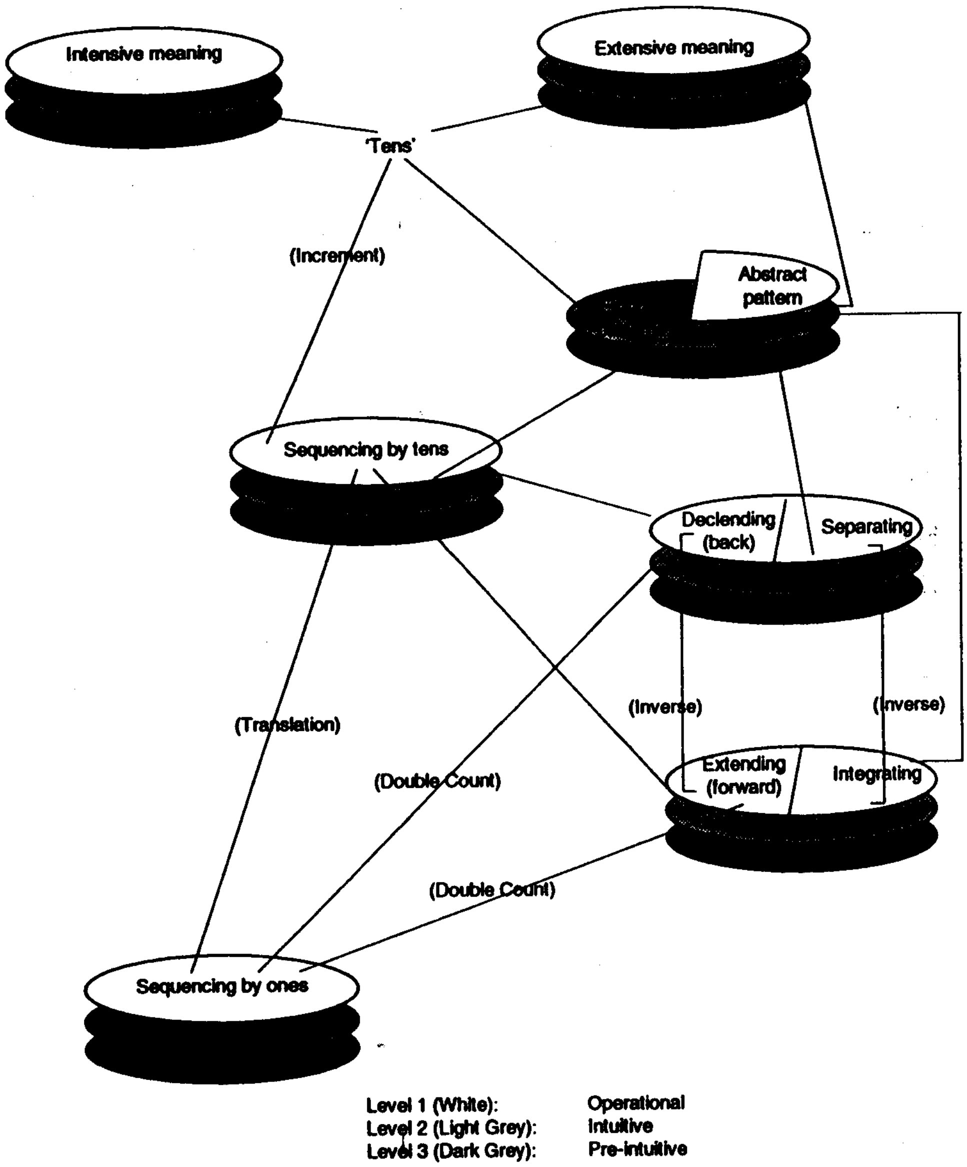


Figure 2.4: Relational network for the concept of ten (Thompson, 1982a, p. 95)

Figure 2.4 shows the relational network for 'ten' which conveys the idea that the concept develops through pre-intuitive, intuitive and operational levels for each aspect of the concept as they interact. A child can give 'ten' either an intensive or extensive meaning. Intensive meaning is related to sequencing by one whereas extensive meaning is related to subitising by way of figural or abstract patterns. The construction of the concept of ten involves a complex network of counting (forward and backward), patterning, separating and combining operations interconnected through the relationships of inverse, translation, increment and direct mappings. It was further shown that construction of the concept of a hundred parallels that of the concept of ten, but it is essentially a new construction. These conceptualisations of ten and a hundred are recorded in notation with the added conceptualisation of place value as shown in Figure 2.3.

Mieko Kamii (1982) related the conceptual development of number to both the culturally derived conventional representations of language (verbal system) and notation (graphic system), and children's personal constructions of notational recording. These personal constructions are the externalisation of individually arrived at ideas but also reflect the child's interactions with the conventional verbal and graphic systems. It is asserted that the relations shown in the Figure 2.5 (Kamii, 1982, p. 95) would be constructed by children over time, where successive understandings form developmental levels. It is suggested that numerical quantities can be represented symbolically by personally motivated means or they can be represented in conventional notation. The acquisition of the conventional system includes learning the digits, but more importantly, a reconstruction of the numerical and notational principles that organise the digits into our numeration system. Place value is regarded as the most important property of the written system.

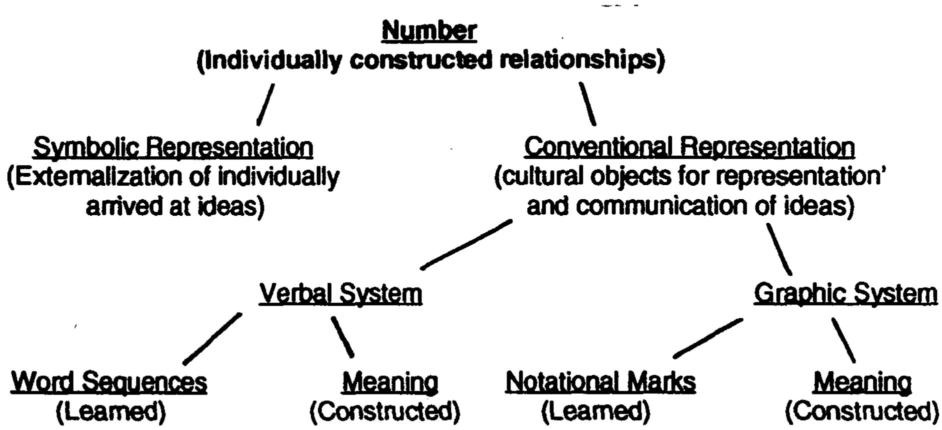


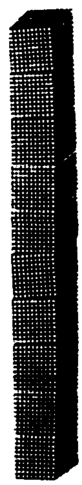
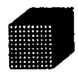
Figure 2.5: General relations between number and numerical representation (Kamii, 1982, p. 95)

From his study, Labinowicz (1985) developed a framework for numeration based on counting by ones, tens and hundreds. It was suggested (1985, p. 265) that children extend the number-name sequence by abstracting intuitive procedures and rules from sound patterns in number names. During a period of *disequilibrium* children may produce numbers which at first glance appear careless, but are really serious attempts to extend the number-name sequence (e.g., they say ninety-ten or write 1003 for one hundred and three). "Furthermore, in dealing with the multiple meanings of ten, children are faced with the dilemma of whether to count by tens or to count the tens. Once these multiple meanings are integrated by the child, focus can be placed on either view of ten without losing sight of the other view" (Labinowicz, 1985, p. 266). Children later construct a mental unit of a hundred and integrate the value of the hundred group into the counting process. Transition points such as a hundred or a thousand were shown to cause particular problems for children and this has implications for working with larger numbers.

The extensive work of Karen Fuson (Fuson, 1986, 1990a, b; Fuson & Briars, 1990; Fuson, Fraillig, & Burghardt, 1992) has explored the way children construct multiunit conceptual structures for multi-digit numbers. She examined the way relationships between number words, notation and base-ten blocks are constructed, and how these relationships are maintained while exploring multi-digit addition. It was found that most children did not spontaneously make links between the quantitative features of the blocks, notation and number words. The successful use of the blocks requires a process of internalisation of the features of and actions on these pedagogical objects. The process of internalisation depends on the conceptual structures the child already has, and the amount of sensitively adapted conceptual support in the learning environment. It is suggested that a teacher's support of linking between the blocks and notation may be needed. "Children need much of the learning time spent on experiences in which the blocks world and marks world are tightly connected in order for the marks to take on the quantitative meanings supported by the blocks" (Fuson et al., 1992, p. 105).

Fuson (1990b) analysed the conceptual structures for multiunit numbers. Table 2.2 shows the features of the notation, the features of the number words and multiunit structures. The system of number symbols requires the perception of a visual layout of horizontal positions into which the digits are written to show the number of each kind of multiunit and learning that these positions are ordered in increasing value from the right. The multiunit structures can be thought of in several different ways: as collections of single units (multiunit quantities); as generated by a ten-for-one trade from the next smaller multiunit; as values of cumulative ten-for-one trades; as cumulative multiples of ten, and as exponential word or symbol expressions of the multiples of ten. The multiunit quantities must be constructed, and these multiunit quantities have to be associated with the conceptual structures for the symbols and the number words. Table 2.2 explains the nature of these conceptual structures.

Table 2.2: Conceptual structures for multiunit numbers (Fuson, 1990b, p. 348)

Name of conceptual structure	Nature of conceptual structure				
Features of the notation					
Visual layout	___	___	___	___	___
Positions ordered in increasing value from the right	fifth	fourth	third	second	first
Features of the words					
Multiunit value word	ten thousand	thousand	hundred	ten	one
Multiunit structures					
Multiunit quantities					
Regular ten-for-one trades	ten thousands	ten hundreds	ten tens	ten ones	ones
Positions as cumulative trades	four trades	three trades	two trades	one trade	no trades
Positions as cumulative multiples of ten	four tens multiplied	three tens multiplied	two tens multiplied	one multiple of ten	no multiples of ten
Positions as exponential words for multiples of ten	ten to the fourth power	ten to the third power	ten to the second power	ten to the first power	ten to the zero power
Positions as exponential symbols for multiples of ten	10^4	10^3	10^2	10^1	10^0

The 'multiunit quantities' conceptual structure shown in Table 2.2 must be constructed, and these multiunit quantities have to be associated with the conceptual structures for the notation and words. The 'regular ten-for-one' and 'one-for ten trades' conceptual structures can be used in any situation in which one has too many or not enough of a given multiunit. These conceptual structures guide the trades that can be made without changing the quantity of the overall multiunit number. The last four conceptual structures shown in Table 2.2 require increasing reflection on the whole multiunit structure and movement from additive to multiplicative reasoning. Each of the four structures builds on the structure above it. The system can be extended by using the 'regular ten-for-one trades' structure (or one of the later conceptual structures of Table 2.2) to ascertain the multiunit value of any given position to the left. To get larger multiunits in a named-value system of words, one needs a new name for each new larger multiunit. Most systems of number words meet this challenge by creating larger multiunits within which a small list of multiunit names is reused. In American English, numbers are chunked into large multiunits of a thousand, and smaller multiunits of hundred and ten are used within these thousand-unit chunks to form a base thousand structure.

An alternative notion of the structure of numeration has been proposed by Confrey (1994). From observations of children working with number she proposed a splitting structure as an alternative to the usual counting structure for number. Confrey argued that there exist two relatively independent primitive structures that children can use: counting and splitting. When using counting (the repeated addition of 1) as a basic structure the operations of addition and subtraction are basic operations, constructed as inverses. In the construction of the splitting structure of numbers, counting numbers are used to count the number of splitting actions and to name the result of a split. For example, two 3-splits involve the repeated actions of splitting something into 3 pieces, resulting in nine pieces. In an n-splitting world, where the successor action results in multiplication by n, the unit is n. It is claimed that "this primitive and intuitive structure can be useful in creating a numeric system, and that it has a unit concept embedded in it" (Confrey, 1994, p. 312). Multiplication and division are constructed as the result of repeated splits and are the basic operations rather than addition and subtraction. Using the counting numbers to index the splitting numbers is equivalent to mapping the positive whole numbers onto geometric sequences and this is the genesis of the exponential function. Therefore, when ten is used as the first split, the basis is constructed for a multiplicative generation of the base-ten numeration system. The place values of positions of the digits in a numeral, counting from right to left, are generated by the results of the repeating 10-split. This approach puts much more emphasis on the multiplicative relation in the construction of multiunit values.

A constructivist outlook is based essentially on the assumption that any conceptual learning involves some kind of integration of new experiences into the existing cognition of the learner. The acquisition of knowledge is considered a complex process and so there are many different but interrelated constructivist models for learning the system of numeration. The constructivist research work of Confrey, Fuson, Kamii, Labinowicz, and Thompson has been discussed here and together they contribute to a more coherent understanding of how children construct their understandings of the system of numeration, albeit there is still much to learn about how the structure of this system is extended.

2.4 COGNITIVE PROCESSING APPROACHES

Cognitive processing approaches explore the construction of cognitive structures and involves analysis of the nature of knowledge.

2.4.1 Cognition and learning

Generally, from a cognitive processing perspective, learning is the construction of cognitive structures that reflect and enable the concepts, principles, and procedures of the content domain to develop. Some cognitive researchers (Boulton-Lewis & Halford, 1992; English & Halford, 1995; Gelman & Gallistel, 1978) assert that in learning a new concept children must construct mental models that reflect the structure of the concept. Understanding place value requires the conceptualisation of a collection of ten as a unit and knowledge (assimilation) of both relational mappings (e.g., the relation between number names and objects being counted) and of part-whole relationships at a system level (e.g., understanding that 14 can be represented as 1 ten and 4 ones). The development of conceptual structures for ten moves from the formation of a collection of ten objects to the construction of a numerical composite unit that allows a child to consider a perceptual collection of ten objects as one unit, while maintaining its numerosity. The unit is then developed as an abstract composite unit where tens and ones are coordinated when counting-on in mental calculations, as proposed by the constructivists (Steffe, & Cobb, 1988).

Halford's (1992, 1993) structural mapping theory of cognitive development aims to account for cognitive development in terms of structural complexity and has four levels: element mapping; relational mapping; system mapping; and multiple system mapping. Each higher level of mapping permits more complex and abstract concepts to be represented but imposes higher processing loads on the learner. Boulton-Lewis & Halford (1992) argue that the construction of the system of tens on the basis of knowledge of ones, place value and abstract counting, all need cognitive processing at the level of system mappings. The mappings that are undertaken between the spoken word, actual single objects grouped as tens and singles, and symbolic representations for understanding a 2-digit number require the conceptualisation of addition.

English and Halford (1995) apply principles for learning by analogy, to the development of children's mental models of place value. This theory uses concrete analogs and previously established mental models as the base for the formation of new target concepts or procedures. Mappings are associations between the salient relations in the base with the corresponding relations in the target. It is suggested that the number of mappings that the child must make from base to target are kept to a minimum and that they are clear and direct. Uniformity of mapping procedures and language are achieved by making the verbal explanations accompanying the use of an analog (concrete model) clearly reflect the analog. These principles provide a framework of carefully sequenced steps, with limited appropriate modelling where explicit recognition is given to relationships within the numeration system.

On the basis of Halford's (Boulton-Lewis & Halford, 1992) structural mapping theory of cognitive development, Boulton-Lewis (1993b) showed that the level of sequence counting was more closely associated with the ability to explain place value than with the ability to explain counting itself. The increasing ability of children during the first three years of school to explain the counting sequence (i.e., from no explanation, to sequence and to place value or generative explanations) followed rather than developed with increasing length of the counting sequence. In contrast, a child's level of explanation of materials used to represent 2-digit numbers as related to place value (i.e., no explanation, global or one-to-one response, and evidence of place value knowledge) was shown to be closely associated with increasing length of the counting sequence. From this research, a framework for numeration would involve challenging children to count as far as they can rather than using any artificial limitations by grade level, and include discussion of the structure of the counting sequence so children learn as soon as possible the generative rule for counting and how this relates to place value.

2.4.2 Conceptual and procedural knowledge

The analyses of children's mathematical competencies should also include the relationship between the procedures or skills and the concepts, understandings, or intuitions of formal and informal mathematics. Procedures that are learned with meaning are those that are linked to conceptual knowledge. Both conceptual and procedural knowledge are considered necessary aspects of mathematical understanding (Resnick, 1983b; Wearne & Hiebert, 1988). Conceptual knowledge is characterised as knowledge that can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Procedural knowledge, on the other hand, is made up of the formal language or symbol representation system and the step by step rules for completing tasks.

Although Skemp (1971, 1976) made the distinction between relational and instrumental understanding there was a later move towards regarding understanding in terms of a spectrum rather than simply in terms of rightness or wrongness (Kieran, 1994). The move was away from a focus on what is right or wrong to a learner's conceptual frameworks. Researchers became more interested in children's individual thinking as a reflection of their personal level of understanding.

Hiebert and Wearne (1992, 1996) investigated the features of instruction that facilitate conceptual understanding of place value. The instruction supported children's efforts to build relationships between number and actions on number that were represented physically, pictorially, verbally and symbolically. From a cognitive point of view, it was argued that building connections between external representations supports more coherent and useful internal representations. The results showed that place value understanding did not translate directly into procedures but that understanding and procedures interacted together to give increased flexibility and power. This interaction flourished more when instruction attempted to facilitate children's understanding rather than procedural proficiency.

Cognitive processing research focusses generally upon understanding of the conceptual structures of number that are constructed by children. Researchers discussed here (Boulton-Lewis, 1993a, b; Boulton-Lewis & Halford, 1992; English & Halford, 1995; Hiebert & Wearne, 1992; Skemp, 1971, 1979) have explored the conceptual structures for numeration and have concluded that such constructions for multidigit numbers are influenced by instruction that *builds connections between the critical aspects of counting, grouping and place value.*

2.5 A REPRESENTATIONAL VIEW OF NUMERATION

In this thesis, theoretical models for describing children's internal and external representational systems of number are also drawn upon. This section is limited to some theoretical views pertinent to this thesis.

In the last decade researchers have also approached the study of children's understanding of mathematical concepts as a dynamic process. This has involved investigating how children represent numerical ideas (Goldin & Herscovics, 1991a, b; Kamii & Livingston, 1994; Pirie & Keiren, 1992, Thompson, 1992). A representation can be described as some kind of configuration that corresponds to or symbolises something else. Goldin and Kaput (1996) point out that representations do not exist in isolation but "usually belong to highly structured systems, either personal and idiosyncratic or cultural and conventional" (p. 398). These have been termed 'representational systems' by Goldin (1987) and a distinction is made between internal and external systems. Internal representations refer to the mental configurations that learners construct and are not directly observable. Goldin and Kaput (1996) use the term *internal representation* to mean the construct arrived at by an observer from observed behaviour rather than the actual internal configurations of the mind that might or might not exist. In contrast, *external representations* are the "physically embodied, observable configurations such as words, graphs, pictures, equations, or computer microworlds" (p. 400).

Bednarz and Janvier (1982) define numeration "as a process which consists of moving from the number (associated with a given collection) to the representation of that number. It is that process which will enable us to speak of a collection, to gather information about the collection, and to work with the collection, that is, to compute" (p.34). They provided a diagram (Figure 2.6) to illustrate the networks of skills which interact as components of this process. A pathway in the diagram might require a number of specific skills in passing from one element to another. A numeration task might require the utilisation of several different pathways involving a whole complex of basic skills.

Bednarz and Janvier (1982) describe the basic skills used in numeration as:

- making groups;
- separating the elements of groups (decomposing);
- making groups of groups;
- regrouping;
- coding the symbol system for a collection organised in groupings;
- decoding, and
- the rule of grouping.

Figure 2.6 shows the network of relationships between various representations of number. Any pathway in the network may require a number of the basic skills in order to pass from one element to another.

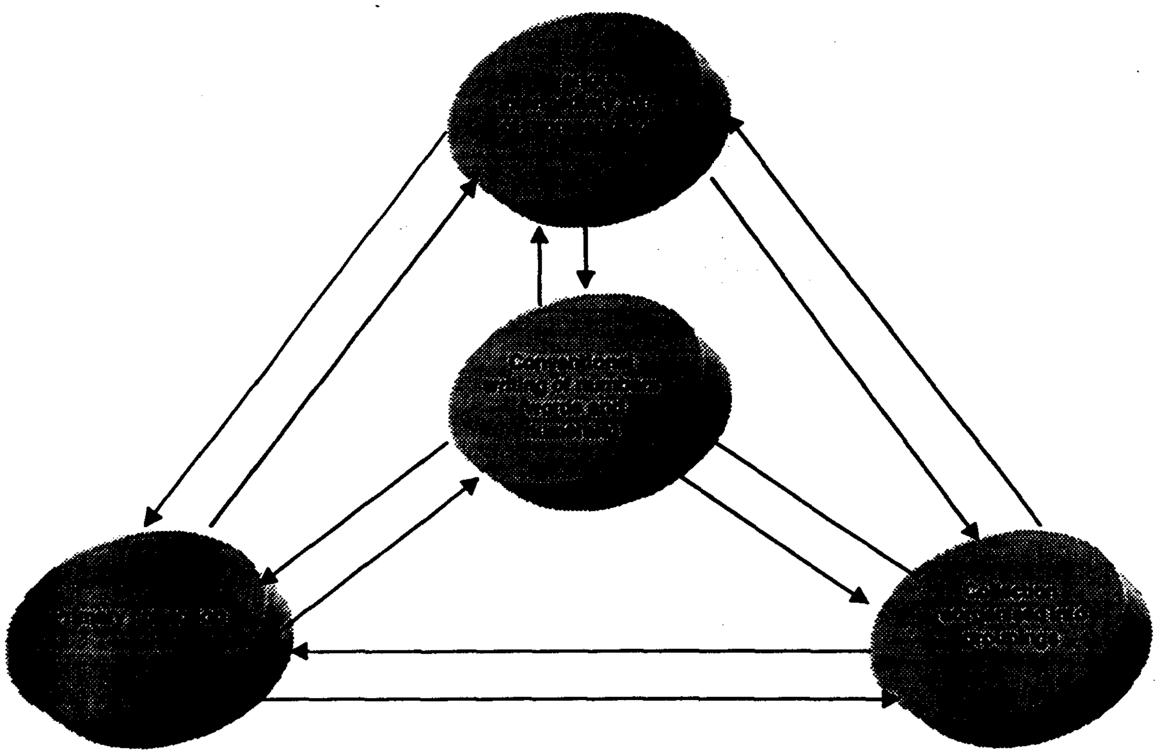


Figure 2.6: Numeration as a network of relationships between various representations of number (Bednaz & Janvier, 1982, p. 35)

Bednarz and Janvier (1982) used their theoretical framework for numeration to design a comprehensive study of children's (ages 6 to 10) understanding. They probed into children's ideas about hundreds, tens and ones with a large variety of tasks involving objects and pictures. Even in third and fourth grades, most children could not understand place value. It was found that when an item involved non-numerical representations of number, children had great difficulty in handling simultaneously two different kinds of groupings. A fundamental issue is that difficulties were often linked *with the modes of representation used in the situations and with the translations needed from one mode to another*.

Goldin (1992a) asserted that much less attention is given to the imagistic systems, to heuristic systems of planning and decision making for effective problem solving, or to the development of supportive affect. Meira (1992) has stressed that the act of construction of external representations is critical to the construction of internal representations. This means that the quality of the learning environment (including interactions, language used, concrete modelling, problems/investigations solved, visualisation, recordings, etc.) is important to the development of powerful internal representational systems.

Goldin (1983, 1987) proposed a model for mathematical problem-solving competency structures which involve five types of mature, internal cognitive representational systems. These systems have been developed further in the context of mathematical learning, conceptual development and assessment (Goldin, 1988, 1992a, b; Goldin & Herscovics, 1991a, b). These are

(a) verbal/syntactic systems,
(b) imagistic systems,
(c) formal notational systems,
(d) a system of planning, monitoring and executive control, and
(e) a system of affective representation.

Goldin further identified stages in the construction of internal systems. Three main stages in the development of representational systems were summarised as: an inventive-semiotic stage; a period of structural development, and an autonomous stage. The development of internal representational systems through such stages requires interaction with external structures including spoken language, pictorial and concrete representations, and formal notation.

Goldin's model has been applied to conceptual development of early number (Goldin, & Herscovics, 1991a; Thomas, Mulligan & Goldin, 1994, 1996). In this thesis Goldin's model will be applied partially to the analysis of the visualisation of the number sequence 1 to 100 as described in Chapter 8. Children are generally expected to internalise the base-ten system of numeration and the associated operations with most attention paid to formal symbolic systems of representation.

The work of Bednarz and Janvier (1982) and Goldin (1992a) has been contrasted in order to show how children's understanding of numeration can be described through the construct of representation. Broadly, external representations are a part of the world of number that we all operate within, and internal representational systems are useful theoretical tools for characterising the constructive processes in the learning of the numeration system. Goldin and Kaput (1996) suggested that research is needed into what makes children's systems of representation powerful.

2.6 SUMMARY

The theoretical perspectives described in this chapter (developmental, constructivist, cognitive processing and a representational view) have all contributed to the knowledge base of our understanding of how children acquire numeration and place value. There are many common elements in these perspectives and many studies have drawn from more than one perspective. Common to all perspectives, learning numeration is regarded as involving understanding of a structure or system, both conceptually and procedurally. A particular child's understanding of a concept may also be unique and can be reflected within a conceptual framework.

The research questions outlined in Chapter 1 will be investigated from a theoretical orientation which can be broadly described as developmental but which draws on aspects from each of the perspectives discussed in this chapter. Hence these theoretical perspectives frame the context for a review of the literature on numeration that is now discussed in Chapter 3.

CHAPTER 3

REVIEW OF THE LITERATURE

Chapter 1 has provided a broad overview of the direction of the thesis from a mathematics education perspective. Children's difficulties with understanding and using the numeration system were raised as central problems addressed in this thesis. The research problem focusses on relationships between fundamental elements of counting, grouping, place value and the multiplicative structure of whole number numeration. Chapter 2 provided theoretical perspectives on research into children's understanding of numeration. Grounded essentially on a developmental approach, this study aims to contribute to a framework for numeration and place value based on children's numerical thinking.

This chapter provides an overview of the research literature pertaining to children's development of numeration by examining four related focus areas as follows:

1. the Hindu-Arabic numeration system;
2. children's conceptual development of numeration;
3. children's representations of the numeration system, and
4. classroom-based studies on teaching and learning numeration.

Within the first focus area (Section 3.1), the Hindu-Arabic number system is described as a specific content domain and three aspects are discussed. First, numeration is defined as a multiunit system. This is a critical issue and is discussed in some detail in Section 3.1.1. Second, numeration as a spoken language is outlined in terms of a distinguishable system in the English language (Section 3.1.2). Third, linguistic differences in numeration of several languages are compared in order to explore effects of different patterns of language structure (Section 3.1.3). Overall, these three aspects of the discussion on the Hindu-Arabic number system provide a basis for this study and provide a background for reviewing the research literature on children's conceptual development, and representations of numeration in Sections 3.2 and 3.3 respectively.

The second focus area (Section 3.2) reviews research on children's acquisition of numeration and place value concepts. Three issues are discussed: the role of counting, the construction of multiunit structures and the acquisition of place value concepts. This area of review focuses on children's numerical processes rather than research on improving learning and instruction, which is the focus of the fourth area of review (Section 3.4). The review of research of children's understanding of numeration also provides a basis for discussing the role of children's representations of the number system as the third focus area (Section 3.3).

3.1 THE HINDU-ARABIC NUMERATION SYSTEM

Our numeration system is a consistent and infinitely extendable base ten system that facilitates mental and written notational forms of number for both whole number and decimal fractions. This notational number system, known as the Hindu-Arabic system, evolved through centuries of human thought and use. It is a base ten place value system; the base of a number system is the number of units in a given digit's place which has to be taken to denote '1' in the next higher place. In base ten, the maximum number of units in the units place is nine, the next number being recorded by a '1' in the next higher place (tens). Place value, which is implied in this definition of base, is the property of the notational system which allows us to use a limited number of digits to record any number. The value of any digit in a numeral is determined by the product of the face value of the digit and the position it holds relative to the decimal point. In a base ten system, the values of positions to the left of the decimal point are given by increasing powers of ten starting with ten to the zero power (1). The values of positions of the digits to the right of the decimal point are given by negative powers of ten. Zero has the function of both recording nothing and a place holder as it can show the absence of a particular power of ten in a number; it holds open a position in order to define the place value of other positions in the numeral.

Whole number numeration is the basis from which the number system expands to incorporate common fractions, decimal fractions and percentages. This system of rational numbers can later be expanded by including irrational numbers to form the real number system. All calculations with whole numbers and decimal fractions (mental, written and calculator assisted) make use of the properties of the numeration system. These properties form the structure of the system which children need to develop and extend throughout their schooling, beginning with the early counting numbers and generalising into large numbers and decimal fractions.

Rational numbers can be expressed as the ratio of two integers, where the denominator can not be zero. They can be expressed in common fraction, decimal fraction or percentage notational forms. Decimal fraction notation provides a way of extending the rules for whole number notation to record rational numbers which are less than one. The decimal point (dot or comma) is used to separate the integral part of the number from the fractional part in the decimal number system. The first position to the right of the point (representing tenths) is the first decimal place; the second position (representing hundredths) is the second decimal place and so on for further positions. Whereas the earliest recorded occurrence of the numeration system that became known as the Hindu Arabic system was in India in AD 595, it was not until the 16th century in Europe that positional notation was first used to represent fractions (Daintith & Nelson, 1989). The first systematic treatment of decimal fractions as instruments of calculation is believed to be that given by Simon Stevin in his book *De*

Thiende ('The Tenth') in 1582, although the notation involved identifying the value of digits representing tenths by a circled '1', hundredths by a circled '2', and so on (Flegg, 1983). The use of a comma or a dot to distinguish between the whole and fractional parts of the number appeared during the last decade of the 16th century and it was not until the 19th century that the comma (in Continental Europe) and the dot (in Great Britain) became common practice.

3.1.1 Numeration as a multiunit system

Mathematics education researchers (Bednarz & Janvier, 1982, 1988; Carpenter & Fennema, 1990; Cobb & Wheatley, 1988; Denvir & Brown, 1986a; Fuson, 1990a, b, 1992a, b; Kamii, 1986; Kamii & DeClark, 1985; Olivier, Murray, & Human, 1990; Ross, 1986) have reported difficulties young children have learning place value concepts and developing flexibility in using multidigit numbers. These multiunit numbers are whole numbers composed of one or more kinds of multiunits and possibly some single units, such as the number 'three hundred and forty five' being composed of the multiunits hundreds and tens as well as single units. Multiunit numbers are represented by both number words (e.g. three hundred and forty five) and written numerals (e.g. 345). Many children have difficulties with numeration as a spoken language and how the words relate to the developing recorded symbolic system for the multiunit numbers.

In the structural development of the number system, the system of representing units (ones) serves to drive the representation of collections partitioned into groupings of ten. The 'ten', while still remaining ten ones, becomes an iterable 'unit of ten'. Similarly a system of 'hundreds' is later constructed on the system of tens, and so forth, recursively. Children's conceptual structures for number words are now "multiunit conceptual structures in which the meanings or referents of the number words are collections of entities ... or a collection of collections of objects" (Fuson, 1990a, p.273). The recursive system of groupings, and hence place values, is thus built up based on the powers of ten. The rules for interpreting the associated numerals are the 'ten ones for one ten trade' rule that gives the value of the next position to the left and the use of zero as a place holder to keep each symbol in its correct relative position. However, it is not clear how children generalise the multiplicative structure of the numeration system (Sinclair, Garin, & Tieche-Christinat, 1992, p. 193). This problem will be discussed further in Section 3.2.2.

3.1.2 Numeration as a spoken language

In the English language, a distinguishable system is used for oral counting and written numbers. The oral number labels and accompanying written words are reorganised when each of the powers of ten is reached. There are different words for each of the numbers up to ten. From the number ten on, words or word combinations are formed from the count of tens and units separately. This is not so apparent in the number words from eleven to nineteen, e.g. twelve being 'two and ten' and sixteen being 'six and ten' (which incorporates an additive structure), or for the decades twenty to ninety, e.g. thirty being 'three tens' (which incorporates a multiplicative structure). The additive structure in the word combinations for all other numbers under one hundred is much more transparent, e.g. sixty four being 'sixty and four'.

For the numbers eleven to nineteen, the number words are derivatives of word combinations where the ones are named first and tens second, whereas later numbers use word combinations in the reverse order. The way that the numbers eleven to nineteen are read from the associated numerals is also the reverse to the way young children are taught to read. For these reasons the 'teen' numbers can cause some difficulties for children. It has been shown though, in a comparative teaching experiment, that the alternative 'teens last teaching sequence' with Grade 1 children did not lead to superior performance (Scales, 1992). Scales (1992) further showed in a statewide survey of children's early numeration skills, in Victoria, Australia, that digit order reversal was not a problem for children by the stage of entering Grade 3.

From the number one hundred (ten tens) until the next power of ten (one thousand), word combinations name the hundreds first, then the tens and then the ones in a multiplicative fashion. This pattern of word combinations continues with the addition of number words (placed in front of existing word combination), for each successive power of ten. The powers of ten can then be considered as counting units themselves which are formed by successive groupings of ten. Within the naming scheme for very large numbers, there is a pattern of groups of 'three powers of ten'; ones, tens and hundreds. In the British tradition, the periods in this pattern are ones, thousands, millions, thousand millions, billions, thousand billions, million billions, and so on. It should be noted that the American English naming scheme varies from this at the fourth level where the word 'billions' is used to denote thousand millions and this meaning is increasingly becoming the common practice, including in Australia where it is used somewhat ambiguously.

HTU	**HTU**	**HTU**	**HTU**	**HTU**	**HTU**
thousand billions	**billions**	**thousand millions**	**millions**	**thousands**	**ones**

Number periods in British English

HTU	**HTU**	**HTU**	**HTU**	**HTU**	**HTU**
quillions	**trillions**	**billions**	**millions**	**thousands**	**ones**

Number periods in American English

Figure 3.1: Periods of 'ones, tens and hundreds' in number naming scheme

Figure 3.1 contrasts the periods of 'hundreds, tens and units' (HTU) for British and American number naming schemes. The number word combination orders the period names from the largest to the smallest (left to right) and the 'hundreds, tens, units' (HTU) pattern is used with each period name. The oral number systems used in Britain and the United States of America (US) are thus governed by conventions for using number words based on grouping units in tens. The US system has become the most commonly used naming scheme in Australia rather than the English system which influenced curricula over 20 years ago. However some confusion exists generally in the community regarding the way large numbers are named.

3.1.3 Linguistic differences in numeration and other languages

Languages other than English use different words and structures to generate their number names. Several researchers (Bell, 1990; Fuson, & Kwon, 1992; Miura, Kim, Chang, & Okamoto, 1988) have explored the differences in linguistic patterns used in other languages. Bell (1990) claimed that transparent standard number word sequences (SNWS) such as those of Japanese, Chinese and Vietnamese languages which follow the numerical representation pattern by naming tens and units in order, support sophisticated mental algorithms. "In contrast to Asian SNWS's, the English SNWS is poorly understood and poorly applied by many young children" (Bell, 1990, p. 12) and it is asserted that activities need to be developed to overcome the inherent linguistic difficulties of the English SNWS.

Miura and her colleagues (Miura et al., 1988) compared the cognitive representations of number for American, Chinese, Japanese first graders and Korean Kindergarten and first graders, to determine if there might be variations resulting from numerical language characteristics that differentiate Asian and non-Asian languages. It was found that Chinese, Japanese and Korean children preferred to use a construction of tens and ones to show numbers. In contrast, English speaking children preferred to use a collection of ones. As well as apparently showing more awareness of place values in their initial constructions, the Asian children, compared with US children, were more likely to construct each number in different ways, which suggested greater flexibility of mental number manipulation (Miura et al., 1988, p. 1445). It was concluded that the young Asian language speakers appeared to represent numbers differently from English-speaking children. In a later study, Miura, Okamoto, Kim, Steare, and Fayol (1993) indicated that place value understanding of children in countries like Japan and Korea, where the numerals correspond exactly to the spoken number words, was better than in countries like the United States, France and Sweden, where the number names do not correspond in the same way. "Learning to count in an Asian language appears to promote an understanding of number that is congruent with the traditional Base 10 numeration system, and this in turn, may provide the basis for making arithmetic fundamentals relatively simple for Asian-language speakers to learn" (Miura et al., 1993, p. 27).

Fuson and Kwon (1992) found that prior to instruction of ten-structured methods, a group of Korean Grade 1 children "showed remarkable competence at solving the single-digit addition and subtraction combinations with sums between 10 and 18" (p. 157). Although the Korean children used renaming methods structured around ten or known facts it was asserted that, in contrast, American children built their understandings and skills based on unitary conceptions of numbers in which ten played no special role. This use of unitary methods can be traced through many studies focussing on children's difficulties with place value and grouping. English and Halford (1995) attribute the use of unitary methods rather than those based on understanding place value to the linguistic complexity of the place-value system for English-speaking children. It is suggested that the numbers between ten and twenty pose particular problems because of irregularities with the English number names. The arbitrary names of 'eleven' and 'twelve' (derived from the old English 'en lefan' meaning one left and 'twe lefan' meaning two left) give no indication of the composition of the corresponding numbers, 'thirteen' and 'fifteen' do not conform to the 'digit-teen' pattern, and in contrast to other two-digit numbers, the 'teen' numbers are read in the opposite way to which their numerals are written. Further there is not a direct mapping from the number names, 'two', 'three' and 'five', to the names 'twenty', 'thirty' and 'fifty'.

Some aspects relating to linguistic differences in numeration have been discussed in this section. This will provide some background to further discussion about differences in children's construction of multiunit structures for numeration in Section 3.2.2. Difficulties in performance of English speaking children seem to point to reliance on unitary methods rather than using multiunits such as tens.

Summary

The Hindu-Arabic numeration system has been described in terms of a base ten structure. The multiplicative recursive structure inherent in the whole number system needs to be constructed by children in order that they might develop the sense of number that allows them to work flexibly with number in mental, written and calculator assisted calculations and to extend the system to include decimal numbers. Some initial difficulties might arise because of linguistic patterns in the spoken language but much more critical is the long-term problem of not generalising the structure of the numeration system itself. It is recognised that the initial introduction of the decimal system, and the positional notation system based on it, is a most difficult and important aspect of mathematics in the early school years (Resnick, 1982). Children need to build on their early acquisition of the linguistic system, their understanding of meanings represented by number names and their ability to use the place value notation system in order to understand and generalise the structure of the whole number numeration system.

3.2 CHILDREN'S CONCEPTUAL DEVELOPMENT OF NUMERATION

A number of research groups have investigated related aspects of children's development of numeration and place value. They have identified a complex range of numerical properties that children need to acquire in order to use and understand the numeration system effectively (Fuson, 1990a, 1990b; Hiebert, 1992; Hiebert & Wearne, 1992; Ross, 1989a, b; Steffe & Cobb, 1988) These can be classified into seven main areas as follows :

(i) Counting properties - the development of unitary and skip counting skills;
(ii) Grouping properties - the formation of multiunit conceptual structures;
(iii) Place value properties - the value of an individual digit in a numeral is determined by the position it holds in the numeral;
(iv) Base-ten property - values of the places in a numeral increase in powers of ten from right to left and ten digits are required (0, 1, 2, 3, 4, 5, 6, 7, 8, 9);
(v) Zero as a place holder property - zero is used to identify an empty space between groupings;
(vi) Multiplicative property - the value of an individual digit is obtained by multiplying the face value of the digit by the value assigned to its position, and
(vii) Additive property - the value of the whole numeral is the sum of the values represented by the individual digits.

Acquiring these properties and relating them together to build the structure of the number system is critical to a complete understanding of the numeration system. Over the past 25 years research attention has focussed on three main areas:
(i) the role of counting;
(ii) the developing multiunit structures, and
(iii) place value properties.

Counting is fundamental for acquiring an understanding of multiunit numbers. The use of grouping properties and the base-ten property to construct multiunit structures is essential for developing flexible use of multiunit numbers. Children must develop representations of number which reflect structure and so learning requires the understanding of place value. Multiunit structures and place value concepts develop slowly over time and form the basis for multidigit number sense. As discussed in Chapter 2 Piaget (1952, 1970) and Resnick (1983a, 1992) have emphasised the importance of perceptual schemas for early learning. Part-part-whole and comparison schemas, and place value together with additive and multiplicative compositions form a major foundation for multidigit number development

(Jones et al., 1996). These aspects which form an integrated part of many studies on numeration, will be examined in Sections 3.2.1, 3.2.2 and 3.2.3. Section 3.2.4 will discuss evidence of lack of understanding of numeration from studies of student achievement.

3.2.1 Research on the role of counting in developing numeration

In this section the development of counting processes is described, taking into consideration research showing the increasing complexity of counting in relation to grouping, unit construction and representations involving place value. Researchers (Bell, 1990; Fuson, 1990a, b; Kamii & DeClark, 1985; Steffe & Cobb, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983; Wright, 1991a, b) have emphasised the importance of counting as a basis for acquiring an understanding of unit and multiunit numbers.

In the constructivist tradition, Steffe et al. (1983) proposed a model of the development of children's counting schemes which identified a progression of types of unit items that children construct when counting. These are described in order of sophistication as: perceptual, figural, motor, verbal and abstract unit items. Counting is seen to involve the acquisition of the number word sequence, the isolation of discrete countable items and coordination of number words with these experiential items. Counting needs to progress from using one as the counting unit, to the formation of ten as the iterable unit and the use of tens and ones.

Steffe & Cobb (1988) carried out a teaching experiment with six children from when they entered first grade for the next two years which documented the cognitive changes in their counting schemes. A five stage model was proposed to explain the development of the number sequence: the perceptual counting scheme; the figurative counting scheme; the initial number sequence; the tacitly nested number sequence, and the explicitly nested number sequence. The initial number sequence is not a nested sequence, there is no notion of cardinality, and the number words symbolise individual numbers in sequence but not a unit containing that sequence. Transition to the explicitly nested number sequence involves the construction of an inclusion relation - any segment of a number sequence can be disembedded from its inclusion in the sequence and treated as a unit. Therefore the part-whole relation is a characteristic of the explicitly-nested number sequence stage in this model of a child's numerical development. It is only when children have reached the explicitly-nested number sequence stage in their counting that they are able to use ten as the iterable unit in mental operations.

Steffe defines *integration* as a uniting operation that constructs numbers as composite units from distinct unitary items (Steffe et al., 1983, p. 67, Steffe et al., 1988, p. 11). So it is possible to consider the integration and part-part-whole operations as complementary combining and decomposing actions on numbers. Integration is essential for the formation of the iterable unit of ten and multiunit structures.

The system of counting can be extended when counting procedures are freed from dependence on perceptual support of grouped objects. A feel for the magnitude of numbers that have not been experienced is achieved through the abstraction of "one more", "ten more", ... ,"one less", "ten less", ... relationships and the development of counting procedures into a system. Also succeeding levels of groups are constructed, as collections of ten of the preceding group, to give the ones, tens, hundreds, ... sequence (recursive relationship).

Resnick (1982) reported examples where children who are competent at counting by ones but cannot coordinate several counting strings within a single counting task. In a study where all third grade children could count objects when only grouped in ones, tens or hundreds, over half of them became confused when two or more groupings were included in the same count. When a child coordinated counting by ones, tens and hundreds, the typical method used was to count the hundreds first and then count on using each successive counting string as required. The most sophisticated approach was to count each grouping by ones, multiply by the appropriate value and then add to give the total. Resnick's framework will be further discussed in Section 3.2.2.

In a study exploring the understanding of the relationship between parts and wholes, Irwin (1996a) found that children could demonstrate understanding of the effect on the whole of changes to uncounted parts before they could predict this relationship for counted quantities. "The results showed that 4-year old children could usually predict the effect of changes to one or more parts of an uncounted whole, but were less competent in predicting changes to counted quantities" (p. 25). The results supported the order of development that Resnick (1992) suggested learners follow in understanding quantitative relationships: from those done without exact quantification to quantitative to numerical. Further it was asserted that Piaget's (1952) claim, that seriation must be understood before compensatory relationships, may be true for counted quantity but is not true for uncounted quantity.

Spatial recognition

Gelman and Gallistel (1978) reported that children count perceptual patterns at a very young age to form semantic links between patterns and number words. Young children can also develop numerosity from spatial patterns. Starkey and Cooper (1980) reported that infants around the age of 6 months are able to discriminate between two and three dots. There appears to be only limited research into how the connection is made between spatial patterns and number words and the importance of this association to the child's early number development. Steffe et al. (1988) stated that " ... this connection will provide the first and most immediate opportunity for the revelation that the number word refers both to a unitary thing and to a collection of units" (p. 15). This raises the question of the potential importance of spatial patterns to the development of an understanding of ten as part of the place value numeration system.

A study of children entering the first year of school by Wright (1991b) showed that there was a great variation in the achievement levels attained on a spatial pattern recognition task involving numbers 3 to 6. Although children in this study were all developing the number word sequence and the associated counting strategies, Wright suggests that it is "theoretically possible ... to construct an abstract concept of number without developing number word sequences. Thus in cultures where enumeration was not used and number word sequences were not developed, abstract concepts of number are likely to have developed along a different path - one in which spatial patterns were much more prominent" (Wright, 1991b, p. 10).

Number name sequence

Labinowicz (1985) explored how twenty nine beginning third grade children acquired the conventional number name sequence through their experimentation with the sound patterns involved and the development of intuitive procedural rules for generating words. It was asserted that a child's success with counting skills results from an active search for sound patterns in the spoken language of numbers. Fuson (1990a) identified the complexities created by the irregularities in the English system for naming multiunit numbers and discussed the associated difficulties for counting. The errors that children make show the struggle involved in developing procedural rules for generating the words. Labinowicz (1985) has highlighted how children construct groupings naturally in their strategies for counting larger numbers of objects. It is asserted that teachers often fail to observe these strategies when children are given ready made groups.

Brown (1981) reported from the English *Concepts in Secondary Mathematics and Science* (CSMS) study that there were significant uncertainties among 11 to 15 year old children when confronted with numbers over a thousand. "One particular area where children seemed weaker than expected was with whole numbers over a thousand. It may be that secondary teachers tend to assume children have mastered these ideas before the age of 11, but this seems not to be the case" (Brown, 1981, p. 64). Also the "weakest group of children, when interviewed, often showed a superficial knowledge of place-names which could have easily led one to conclude that they understood the ideas of place value" (Brown, 1981, p.49). Results showed that 12% of 15 year olds and 32% of 12 year olds did not have the facility to give the answer to "What is one more than 6399?". The responses of children from seven to fourteen years to this same question resulted in the Cockcroft Report coining the phrase "the seven year difference" to highlight the age disparity of children in mathematical attainment (Cockcroft, 1986, p. 100).

Bell, Costello and Kuchemann (1983) when discussing the APU (Assessment of Performance Unit) surveys and the CSMS research state:

> *These figures give some clue about the development of children's understanding of place value, suggesting that this understanding is achieved in general terms by a slight majority of 11-year-olds, and identifying the size of the minority who still do not understand the notation at age 15. The grasp of place value is, however, often very tenuous. Questions containing numbers over a thousand, especially if these involve a number of zero place-holders, are more difficult and expose children's uncertainties.*
>
> (Bell et al., 1983, p. 107)

Results of the studies by Resnick (1983a) and Denvir and Brown (1986a) show that children find it easier to count when only one grouping has to be counted than when two different groupings are present (ones and tens). There is some difficulty for a child to stop the tens number sequence and begin the ones number sequence or vice versa. Cobb and Wheatley (1988) also showed that there was a further difficulty in making more than one switch in counting ones and counting tens.

It is difficult to observe children's natural recordings of number because of their constant exposure to conventional notation. Common errors made by young children in writing numerals for numbers such as 'four hundred and eighty nine' show a tendency to use rules consistent with more primitive additive notation systems. Bell and Burns (1981) report that 14 of 35 beginning second graders wrote 400809, 40089 or 4089 instead of 489. Labinowicz (1985) reported 3 of 29 beginning third graders wrote 4089. All 29 third graders were able to write the correct numerals for 'seventeen' and 'seventy one', showing that they had accepted the irregularities in oral number names of the teen numbers. Other questions given to the second grade children included reading numbers from three digit numerals, interpreting place value of designated digits (12 children could verbalise a reason), comparing three-digit numerals, and writing numerals that are one more than, or ten more than, a given three-digit numeral. Fifteen of the second graders could not give the number one more than 342 whilst nineteen could not give the number 10 more than 243.

Labinowicz (1985) suggested that the initial notion of "comes after" which is related to the sound pattern of number words has to be restructured by children in terms of the "one more than" relation. Some of the children who gave the correct response did so by counting on by ones. Most of the errors were sensible in relation to some particular perspective of the situation and so serve to remind us of the complexities of the numeration system. It was also observed that children often gave inconsistent responses across different tasks showing some instability in understanding. These children appear to "have structured ideas for different perspectives of number and numerals" but have yet to "integrate these ideas into a coherent organization" (Labinowicz, 1985, p. 297).

Bentley (1987) argued that the error patterns of school children in the use and interpretation of place value notation reflects the conflict between the counting principles learnt in preschool and later place value principles. "The order-irrelevance principle forbids assigning differing values to digits based on their relative position. The one-one principle with its unique tagging requirements forbids recycling of digits (there must be a unique visual tag, i.e. symbol, for each unique count word)." (Bentley, 1987, p. 11). It is asserted that school children spontaneously generate incorrect how-to-represent rules for number because of the how-to-count principles which have been learnt in the preschool years.

Labinowicz (1985) suggests that when children have constructed and integrated the two meanings of ten (ten ones and one ten) as demonstrated by the efficient counting on and back of tens and ones then two-digit notation becomes a natural representation of what is already known. "Children who only count-on by ones do not have the conceptual basis for understanding place value notation" (Labinowicz, 1985, p. 269).

Wright (1991a, 1992a, 1994a, 1996) has used interview-based assessments to document the early number knowledge of children in Kindergarten and Grade 1. Levels of arithmetical knowledge are determined for five aspects of number: facility with forward and backward number word sequences; counting and other strategies to solve addition and subtraction problems (based on Steffe's (1988) counting types; subitising, and identification of 1-, 2-, and 3-digit numerals). Wright (1994a) asserted that current curricula underestimate kindergarten children's levels of arithmetic knowledge. Wright (1992b) further criticised teaching approaches which cultivated low-level strategies, and overuse of activities involving structured materials and exercises based on pictorial representations.

Summary

Children's counting skills are an important component of their understanding of numeration which reflect the developing multiunit structure of number and enable the use of counting strategies in mental arithmetic and problem solving. The researchers discussed have highlighted the need for children to move from counting by ones, counting-on and counting-back from given numbers, to using powers of ten as iterable units and to coordinating these units in the counting process.

3.2.2 Research on children's construction of multiunit conceptual structures

This section discusses children's construction of multiunit conceptual structures comparing research by Bednarz and Janvier (1982), Cobb and Wheatley (1988), Fuson (1990a, b), Kamii and DeClark (1985), Resnick (1983a, b), Steffe and Cobb (1988), Steffe et al. (1983), and Thompson (1982a, b, 1992). Several aspects are discussed as follows: constructivist research on numeration; the use of ten as an iterable unit, and conceptual structures for multiunit numbers.

As discussed in Section 3.1.1 children conceptualise multidigit numbers through the development of multiunits of ones, tens, hundreds, thousands, and so on. These multiunits and the relationships form the multiunit structures and are built out of the increasing abstractness of counting. Along with counting, grouping and partitioning are important constructs in this process. The way children deal with units of one and ten, influences their understanding of larger numbers and the numeration system generally (Cobb & Wheatley, 1988; Kamii, 1990; Kamii & DeClark, 1985; Resnick, 1983a, b; Steffe et al., 1983; Steffe & Cobb, 1988).

The conceptualisation of ten is central to the development of multiunit structures. Tens as singleton units (Cobb & Wheatley, 1988) exist with units of one within a child's knowledge of counting numbers but they may not be coordinated. Decade numbers may be recited but there may be no sense of increments by ten. The one-as-many and many-as-one schemes are needed as complementary aspects in children's construction of abstract composite units (Steffe & Cobb, 1988). The idea of an abstract composite unit of ten means that the unit is a unit of ten and ten units of one at the same time, but there may still be no sense of increments of ten when counting by tens. A two-digit number may be constructed as composed of a number of ten-units and ones or as a single entity composed of ones but not both simultaneously. The many-as-one coordination is needed to make sense of multiplication as repeated addition and the one-as-many coordination to make sense of division as partition. Researchers have also reported that children's abilities to construct units have played important roles in their understanding of fractions (Confrey & Smith, 1995; Watanabe, 1995), ratios and proportions (Lamon, 1993; Lo & Watanabe, 1993), exponential functions (Confrey & Smith, 1995), and geometry (Wheatley & Reynolds, 1996).

Constructivist research on numeration

In the early 1980's, as part of the radical constructivist research group at the University of Georgia, Thompson contributed importantly to work on counting and numeration. Thompson (1982a) reported his observation of tremendous inconsistencies in the kinds of whole number numeration-related problems that young children could and could not solve. In an in-depth study, eight first and second grade children were each given three interviews over a period of two weeks, and their responses were applied to Thompson's theoretical framework in order to explain their understanding of whole number numeration. Many children had merely learned pencil and paper algorithms with little understanding. As a result children often appeared to know more about numeration than is actually the case, and hence the disparities observed in children's performances with tasks. Children could tell how many tens were in fifty seven, but at the same time needed to count by ones to determine the number represented by a collection of Dienes' longs and unit cubes. This raised the question of how we account for the fact that many children can reliably read two and three digit numerals but are not able to use these numbers in any but the most trivial problems.

As outlined earlier in Chapter 2, Thompson (1982a) developed a framework for analysing a child's understanding of numeration. Young children's understanding of whole number numeration was analysed in terms of four domains of knowledge: reading and writing numerals; the language of number names; subitising, and numerical operations. The framework involved showing the concepts of ten, one hundred, and place value in terms of relational networks. The nodes of the networks were routines and meanings associated with the concepts. The networks provided a means of explaining the ways that the concepts could be expressed in children's behaviours. It was anticipated that the framework would provide a "task environment" from which to construct explanations of children's responses to a variety of numeration tasks. As a result of the analysis of case studies it was asserted that the units of 'ten' and 'one hundred' each appear to be constructed independently. With ever increasing units to be constructed, there is a need for children to establish an operational routine for creating these numeration units.

A detailed analysis was conducted by Thompson (1982b) in order to explain the way that children develop the system of numerals and corresponding number names. A schema was proposed for the construction of the system of number names for the numerals. This schema did not involve any meanings for these numbers but rather just provided a method of generating words for the symbols. It was shown how a set of rules can be developed in the child's mind which enables the reading of numerals. Similarly a set of rules can be generated for writing numerals. "The essential points ... are that a child's ability to read and write numerals need not imply anything about his or her understanding of numeration, and that children can 'succeed' in much of elementary school mathematics by knowing only a few basic addition and subtraction facts and by having well-formed schemas for reading and writing numerals. " (p. 8).

Thompson (1982b) further questioned how a child increments by ten from a non-nil and non-decade starting point and suggested a schema for producing number names in sequence by ten which he called the 'skip-ten' schema. He concluded that counting-on ten is partly linguistic but also involves comparing the starting and ending number names after counting-on ten in place of counting-on by ones, ten times.

Thompson (1982a) raised many important questions concerning children's construction of multiunits as a numeration system. What are the conceptual requirements for having well formed concepts of ten and one hundred as numeration units? How is it possible for children to behave so inconsistently from problem to problem? The variation in performance on apparently related tasks was explained by Thompson's suggestion that the children's knowledge of numeration is compartmentalised. It is further suggested by Thompson that these inconsistencies might vanish when those compartments become meaningfully related. This pointed to the need for further research to focus more explicitly on how aspects of numeration can "become meaningfully related". He asserts that numeration should be developed through the interaction of the knowledge and understanding implicit in the schema domains of the numeration framework.

Using ten as an iterable unit

Following on the work of Steffe et al., (1983) and Thompson (1982a), Steffe, and Cobb (1988) identified three concepts of ten that children construct after they reach the stage of the construction of abstract unit items when counting by ones:

(i) Ten as a numerical composite - there are either ten ones or a single entity called "ten" but not both together;

(ii) Ten as an abstract composite unit - the numerical composite is a single entity at the same time as maintaining its tenness, each counting act by ten is a short cut to ten counting acts by ones, there exist visual images of the abstract composite units of ten which are presented (hidden or displayed), but a numerical whole and the units of ten and ones can not be constructed at the same time, and

(iii) Ten as an iterable unit - can count by ten and by one in the absence of any representation of composite and single units, each counting act by ten (forward or backward) is the same as the addition (or subtraction) of ten ones. Iterable units of ten are formed when the composite units of ten are constructed during the counting process and each 'one' increments the total by ten.

They suggested that the use of units of different rank to measure or label quantities and hence understand the positional system of numeration required the construction of ten as an iterable unit. "When the unit of ten is iterable, rather than coordinate the use of two separate schemes, the children use one scheme that contains two different units" (Steffe & Cobb, 1988, p. 231). Also the verbal number sequence used by children should be developed as the basis for the written symbol system. Children should be counting "by ten and by one to solve problems", thereby "modifying their schemes counting-on, counting-up-to, counting-off-from, and counting-down-to to include the unit of ten. These modified schemes are included in the *child generated algorithms* that should be encouraged" at the explicitly nested number sequence stage (Steffe & Cobb, 1988, p. 320). Steffe and Cobb assert that "children should also construct the decimal system of numeration and concepts of place value and total value of a digit in a multiple-digit number" (p. 320).

The study further explored children's attempts to organise 'groups of ten groups' after having understood the construction of 'groups of ten' as a powerful strategy for counting. The children could organise the acts of counting by ten into modules of ten. But they had difficulty finding how many were in a group of ten groups, how many groups of ten were left over after organising a collection in this way and how many objects there were in such an organised collection. "Units of units of units were truly a novelty and a whole new sequence of constructions seemed to be necessary" (Steffe & Cobb, 1988, p. 224). This raised the question of how difficult it is for a child to equate a counting by ten act with an addition of ten more ones and hence to understand that each decade is a number sequence of numerosity ten.

Cobb and Wheatley (1988) further studied children's constructions of the concept of ten. In a study of 14 second graders, children attempted to make sense of typical textbook-based school instruction. The tasks included:

(i) Use of counting and thinking strategies in addition and subtraction tasks;
(ii) Horizontal addition number sentences;
(iii) Tens tasks using individual cardboard squares and strips of ten squares - uncovering board with sequence of strips and squares, addition where one addend is hidden and missing addend tasks, and
(iv) A worksheet with eleven two-digit vertical addition tasks including a sequence of six related additions, one addition involving no regrouping and the earlier horizontal number sentences presented in vertical form.

The children were placed at three different levels with respect to addition and subtraction on the basis of their performance on the first set of tasks. These levels are closely related to the Steffe and Cobb (1988) concepts of ten, i.e. (i) ten as a numerical composite, (ii) ten as an abstract composite unit and, (iii) ten as an iterable unit. Eight children were placed at level 1, three children at level 2 and three children at level 3. On the basis of the thinking strategies used in responses to the horizontal number sentences and the tens tasks, the analysis of the concept of ten was expanded to include further conceptualisations. Children's mental addition strategies for two 2-digit numbers and their use of 'ten' was categorised in one of five ways:

(i) *Numerical composite* - add by counting on by ones, construct ten as a numerical composite only in the presence of appropriate materials;

(ii) *Abstract composite unit* - coordinate counting by ten and by one starting in the middle of a decade in the presence of visible or hidden materials, organise counting by ones into units of ten with support of suitable materials;

(iii) *Iterable unit* - coordinate counting by ten and by one in the absence of suitable materials, reorganisation of counting by ones;

(iv) *Abstract collectable unit* - construct units of ten and of one when meaning is given to each numeral and then adding units of the same rank, abstract composite units are manipulated in the absence of materials, and

(v) *Abstract singleton unit* - units of ten are constructed as singletons, can coordinate counting by tens and ones but there is no understanding of the cardinal value of ten, distinction is made on the basis that they are different 'objects' to be counted.

The strategies children used on the tens tasks were in contrast to those strategies listed by Steffe and Cobb (1988), as there were no situations where the use of ten was identified as a numerical composite or an iterable unit. All the children at level 1, one at level 2 and two at level 3 constructed ten as an abstract singleton in the uncovering tasks. "Their construction of ten did not involve a sense of tenness, of ten as composed of individual units" (Cobb & Wheatley, 1988, p. 17). All three children at level 3 were identified as constructing abstract collectable units when they solved the tasks involving screened collections. "This issue has considerable instructional relevance given that ten as an abstract singleton and as an abstract collectable unit seem to reflect a prior instructional emphasis on the value of each digit of a two-digit number" (Cobb & Wheatley, 1988, p. 18).

It was also noted that the children appeared to see the horizontal number sentences and tens tasks in different contexts and so most used contrasting methods of solution - based on counting-on and counting abstract singleton units respectively. "Ten as an abstract singleton does not seem to be derived from counting-on by ones but instead appears to be a consequence of school instruction" (Cobb & Wheatley, 1988, p. 18).

It is further suggested (Cobb & Wheatley, 1988) that instruction which emphasised assigning values to digits based on their position as preparation for a didactic approach to the standard addition and subtraction algorithms caused the misconceptions of the meaning of ten. In the worksheet tasks (vertical algorithm format) all the children, except those at level 3, saw each task as two single digit tasks or as separate tens and ones tasks. For example, the meaning given to

$$\begin{array}{r} 39 \\ +53 \\ \hline \\ \hline \end{array}$$

was either as two single-digit addition tasks, 3+5 and 9+3, or as two separate tens and ones tasks corresponding to 30+50 and 9+3. These children could not assess the appropriateness of their wrong answers (generally 82 or 812) because they did not conceptualise the questions in terms of adding two numbers, each as a single entity. It was reported that the 14 second grade children who were interviewed had been introduced to the notion of grouping in tens and ones and had been taught to add and subtract 2-digit numbers without regrouping prior to the study. Typical textbook instruction had taken place where rules were taught for assigning value to digits based on their position. Cobb and Wheatley (1988) contend that this approach to algorithms "is seriously flawed. Even when drawings of bundles of sticks and manipulatives are used, this approach encourages children to construct ten as an abstract singleton rather than as a unit that is itself composed of ten ones" (p. 23).

Cobb and Wheatley conclude by making a strong call for a teaching approach "that builds on children's self-generated methods" (p. 24). In numeration this means the construction of ten as an iterable unit and an abstract collectable unit which lead to the construction of self-generated algorithms. An instructional approach based on the construction of increasingly powerful units of ten is suggested. This involves the use of thinking strategies for finding sums and differences, activities to help children construct composite units, then the construction of more sophisticated units of ten leading to the realisation that 'counting tens' is the same as the addition of ten ones (similarly for subtraction).

The work of Cobb, Steffe, Thompson and Wheatley during the 1980's has explored the construction by children of ten as an iterable unit. Classroom studies based on this research will be discussed in Section 3.4.

Resnick's (1983a) framework for the development of multiunit structures and number representation

As discussed in Section 2.2 Resnick developed a framework for the development of number representation. The framework was described in the following three stages.

Stage 1: Multidigit numbers are uniquely partitioned into ones and tens. Each number has only one base-ten representation, with no more than nine of each object in a grouping (canonical). The earliest stage of decimal knowledge is mentally represented as a ten-by-ten matrix or 100 square. This 100 square (0 to 99) can be used to illustrate the way numbers are simultaneously related by ones and tens. The "one more" relationship exists between numbers horizontally across the matrix from left to right, "one less" being the reverse direction, "ten more" is vertically down and "ten less" is vertically up the matrix (the "Next structure"). Adding and subtracting other numbers also form patterns within this matrix.

Stage 2: Multiple partitionings of multidigit numbers are initially constructed through counting and then through the application of an exchange which maintains equivalence of the whole. Non-canonical representations are now allowed.

Stage 3: Exchange principles are applied to written numbers to enable the construction of computational algorithms.

The part-whole schema is central to this model for understanding number. It is suggested that the Piagetian class inclusion problems "are part-whole problems without a requirement of specific numerical quantification" (Resnick, 1983a, p.125) and so it is likely that the part-whole schema develops from very early situations in a child's life. The first elaboration of the schema is in the use of the procedures of counting on and counting down to find sums and differences. Numbers are simultaneously both positions on the mental number line and compositions of other numbers. In constructing base-ten representations of two-digit numbers, application of the part-whole schema has the restriction that one of the parts must always be a multiple of ten. The schema is then developed to include the operations of trading, exchanging and regrouping (multiple partitioning) material, and renaming numbers.

It is also suggested that the early notion of cardinality, as specified by the final word reached when a set of objects is counted, is developed to a higher stage of understanding by the absorption of the part-whole schema. The application of the part-whole operation allows cardinality to reside "in the total quantity, no matter how it is displayed or partitioned" (Resnick, 1983a, p. 148). Transformations under the operations of trading, exchanging or regrouping, involve a change in the actual number of objects present as the notion of value is incorporated and so a higher level of abstraction of cardinality is required.

Whereas Resnick (1983a) focussed on the importance of partitioning representations of multidigit number (both concrete and notational), other researchers have explored grouping as the basis of numeration.

Grouping model of Bednarz and Janvier (1982)

Bednarz and Janvier (1982) carried out a research project with primary level children to clarify the notion of numeration and what is meant by understanding numeration. A three-dimensional reference framework was designed to guide the development of diagnostic activities and learning situations. The criteria and corresponding classifications used in the framework were: degree of complexity (easy, average or difficult); context (familiar or not familiar), and representation related to a rule of grouping (apparent, disguised or conventional). The reference framework was used to construct two sets of structured interview items, one for first year children (6-7 year olds) and the other for third and fourth year children (8-10 year olds). Forty Year 1 children were each given 14 items individually and for the third year (75) and fourth year (45) children 12 items were given individually.

Some of these children (42 and 36 respectively) were also given 6 items in small-group structured interviews which were designed to probe understanding of numeration. Individual interview items for the third and fourth year children involved determining:

- the importance given to the position of a digit in a written number by constructing and comparing numbers using digit cards;
- the rôle attributed to position when writing numerals;
- what meaning is attached by the child to the words hundreds, tens and ones;
- the meaning given to regrouping in subtraction, and
- the perception of groupings, and groupings of groupings, in concrete and pictorial representations.

It was shown that most of the children could compare two 3-digit numbers but they found it much more difficult to construct a larger number from random consecutive digits. Although many children appeared to have the skills to compare the numbers, they reverted to a digit-by-digit strategy in the construction situation and so were shown not to see these numbers as entities. Their understanding of 3-digit numeration was shown to be superficial with many children giving a meaning to ones, tens and hundreds in terms of order rather than groupings. Most of the children had mastered decomposition procedures for subtraction but generally they are rules using the language of 'borrowing' and are rarely linked to exchanging.

When the question involved an illustration of materials, one of the important difficulties was the impossibility of handling simultaneously two different kinds of groupings. The main difficulties encountered by children were problems with zero, working with groupings of groupings, working simultaneously with two groupings, unmaking groupings and handling the process of exchanging. As discussed by Thompson (1982a) these results suggest that children's knowledge of numeration is compartmentalised.

Bednarz and Janvier (1982) suggested that there is a close relationship between associating a grouped collection with a number in numeration and associating the magnitude of a physical object with a number in measurement. "The action of making groupings corresponds to that of covering with a connected chain of basic units. We easily see that counterparts of groupings, and groupings of groupings are working with basic units and working with units built from basic units" (Bednarz & Janvier, 1982, p. 55). As a result of this study, Bednarz and Janvier cast doubt on the effectiveness of current textbook approaches to the teaching of numeration. The theoretical and reference frameworks were used to develop a two and a half year longitudinal study of a teaching experiment starting with two classes in first year (39 children) through one class in second grade (22 children) to third grade (23 children) (Bednarz & Janvier, 1988) which is discussed later in Section 3.4.

Baroody (1987) supported the view of Resnick (1983a) that knowledge of the base-ten numeration system develops gradually and builds upon previous counting knowledge. With experience, children can think about multidigit numbers with more flexibility. Children initially represent numbers as the composition of tens and ones in standard form. Later, multiple partitioning of multidigit numbers occurs when children use nonstandard partitions. Baroody (1990) asserted that "multidigit arithmetic instruction is too piecemeal" (p. 281). He suggested that children should be introduced to two-digit numbers through using meaningful models, and linking these models to written symbols before written multiunit procedures. Baroody further advocated that problems with and without trading, and using multidigit numbers should then be introduced concurrently.

Kamii and DeClark (1985) reported on a study of 100 children, Grades 1 through 5, from a school in Switzerland. The children were asked to estimate the number of plastic chips which were shown briefly (73-78 for Grade 1 and 98-120 for older children), spontaneously count a large number of chips and then to count a different amount of chips by tens. The Grade 1 children were also challenged to consider the attitude (of a fictitious first grader) that it is hard to count the chips when they are in heaps. Results showed that all first graders and most of the others counted the large quantity of chips by ones. Counting by tens appeared first in the fourth grade and counting by twos was a frequent technique. Four levels of response were identified for the counting by tens task:

Level 1: responses showed no idea of how to count by tens;
Level 2: made heaps of ten without conservation of the whole;
Level 3: counted by tens without separating the whole into parts, and
Level 4: counted by making heaps of ten with conservation of the whole.

Kamii found that the percentage of children who performed at levels 1, 2, and 3 were 33, 29, and 38 respectively, but that no children were at level 4. "Most first graders can think about the whole and the parts *successively* but not *simultaneously*. When they made heaps of ten, for them, the whole disappeared, and the only things left to count were seven heaps or ten chips in each heap" (Kamii, 1986, p. 82). The quantitative part-whole relationship is necessary for these children to construct the system of tens and so be ready to understand place value. Also the first graders appear to be confused by the separation of the whole into parts as was shown by the result that 66% of them agreed that it was better to mix up the heaps before counting. The construction of a system of tens, and hence place value instruction, could only be considered for those Grade 2 children who counted the heaps to determine the total number of chips (level 4). This is supported by Piaget (1952, 1972) who found that children are not ready to understand the part-whole relationship until around seven or eight years of age.

Constance Kamii adopted a developmental approach, exploring the way children need time to construct their system of ones as a prerequisite for building a system of tens. Resnick also adopted a developmental approach, but explored the way children partition multidigit numbers and then use partitioning in computation. This can be contrasted with the way that Thompson analysed the conceptualisation of tens and hundreds that children construct.

Fuson's conceptual structures for multiunit numbers

Fuson (1990a, b) specified ten conceptual structures that are contained within the numeration system. Some structures only contain knowledge of the surface features of the written marks and of the spoken words, whereas others contain knowledge of how the multiunits are put together to give large numbers and some arise from reflecting on earlier structures. An important conceptual structure identified is the multiunit quantities structure which contains knowledge about the size of each multiunit in terms of single units. It is asserted (Fuson, 1992b) that children construct two different kinds of conceptual structure for use in multidigit addition and subtraction: collected multiunits and sequence multiunits. "The *collected multiunits* are collections of single units: A ten-unit item is a collection of ten single unit items, a hundred unit item is a collection of one hundred single unit items, a thousand unit item is a collection of one thousand unit items, and so forth" (Fuson, 1992b, p. 142). These multiunits are then combined or separated in mental addition and subtraction. Alternately, using *sequence multiunits* in addition and subtraction is an extension of unitary sequence solution procedures in which sequence pattern skills of counting up or down by the multiunits (tens, hundreds, etc.) are utilised. The quantitative sequence counting on and back is connected to cardinal values and so can be used to solve addition and subtraction situations; e.g., 27 plus 35 could be solved by counting on the tens and then the ones (20, 30, 40, 50, 57, 58, 59, 60, 61, 62).

Child-invented mental algorithms for solving two-digit addition and subtraction problems are more likely to involve a sequential multiunit conceptual structure (Cobb & Wheatley, 1988; Fuson, 1990a, b; Labinowicz, 1985). Thompson (1982a) showed that when children were provided with base-ten blocks (longs), the children used collected multiunits before sequence multiunits. There is some evidence that when adding and subtracting three- and four-digit numbers, children use collected multiunits for the hundreds and sequence multiunits for the tens (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fuson & Briars, 1990). Fuson (1990b) suggested that children from China, Japan and Korea, where the languages provide further supports for understanding number, construct multiunit structures for two-digit numbers at an earlier age than US children. The reported advantages that exist for children using these Asian languages were discussed in Section 3.1.3.

Luria (1969) suggested that children only understand the base-ten structure after they have been taught to write numerals. In contrast, an investigation in Brazil carried out by

Terezinha Nunes (Carraher, 1985) showed that for samples of 5-7 year old children and adults, none of whom had attended school, some individuals could understand the concept of units of different size in the context of money. Composing totals with units of different value, in the context of money, could be mastered from the use of the oral numeration system. Further research (Nunes & Bryant, 1996) suggested that progress in understanding addition, particularly from counting-all to counting-on strategies, rather than proficiency in one-to-one correspondence counting was important for understanding the properties of the numeration system. This work has shown that in order to build a basis for numeration it is important to involve young children in solving simple problems with visual representations as support.

Multiplicative structures of multiunit numbers

Inhelder and Piaget (1964) showed that multiplication is a more complex operation than if it is considered as just a construction of repeating addition. Multiplication is different from addition because of the level of abstraction and the number of inclusion relationships that have to be made simultaneously (Piaget & Inhelder, 1964). Repeated addition involves the inclusion relationship with only one level of successively combining groups that can be regarded as made up of ones. If the same numerical exercise of repeated addition is thought of as multiplication then the many-to-one correspondence is needed to construct abstract composite units and the inclusion relationships operate on both units of one and abstract composite units. Clark and Kamii (1996) identified four developmental levels in children's progression from additive to multiplicative thinking: (i) no serial correspondence or only qualitative notion of quantity; (ii) additive thinking with a numerical sequence of +1 or +2; (iii) additive thinking involving larger numbers, and (iv) multiplicative thinking. They found that while multiplicative thinking appeared early (among 45% of second graders) it developed very slowly.

In numeration the notion of multiplicative units is necessary in order to understand the conceptual structure of multiunit quantities (Behr, Harel, Post, & Lesh, 1994). Construction of a unit occurs through internalising repeatable actions. The unitising action of assigning a number word to each object in a collection is the basis of counting. Multiplicative structures are important because the units in the numeration system are multiplicative. An infinite sequence of multiplicative units are created by grouping units with a particular multiplying number, treating the composite as a unit and iterating them to form further units. Units can also be created by splitting existing units in a similar way (Confrey & Smith, 1995).

Behr, Harel, Post, and Lesh (1994) asserted that children should be given situations of whole number arithmetic that involve a variety of multiplicative units in order to better understand whole number arithmetic and to act as a bridge to understanding rational number concepts and operations. In discussing units in relationship to understanding fractions, Watanabe (1995) suggested that children must be provided with tasks that encourage them to coordinate units and that using the many-as-many scheme "appears to be a very fruitful approach to developing a sophisticated multiplicative structure" (p. 173). A many-as-many scheme is where the one-as-many and many-as-one schemes are coordinated to give equivalence to two different groupings. In whole number numeration the equivalence of 6 hundreds and 60 tens is a many-as-many scheme. Watanabe (1991) proposed a theoretical developmental model which contains three levels associated with the coordination of units: coordination of the unitary item being counted and the appropriate counting word; the one-to-many and many-to-one schemes that allow the construction of abstract composite units (Steffe & Cobb, 1988), and the many-to-many scheme which allows flexibility in moving between various representations of a unit. It appears that the coordination of different units is an essential aspect of understanding the numeration system, both for whole numbers and decimal fractions.

Summary

The development of multiunits is an essential element in children's construction of a system of numeration. The concept of a 'unit' and the construction of multiunits through the multiplicative relation has been the focus of the research discussed in this section. The way children deal with units of one and ten influences their understanding of larger numbers (Cobb & Wheatley, 1988; Steffe et al., 1983; Steffe & Cobb, 1988). Grouping is the basis for recognising and constructing multiunit numbers (Bednarz & Janvier, 1988) whereas partitioning numbers in different ways facilitates the renaming of multidigit numbers and the flexible use of number in mental calculations (Resnick, 1983a, b; Fuson, 1990a).

3.2.3 Research on children's construction of place value

This section will discuss research that explores why many children have difficulty with learning place value concepts. Understanding place value develops slowly over time and forms the basis for the conceptualisation and recording of multidigit numerals as a place value system. In Section 3.1.1 it was described how multiunit quantities in multiunit numbers are powers of ten, the symbols for the numbers one through nine are used to give the number of each kind of multiunit, and the multiunit value is indicated by the place of the digit. Some researchers (Kamii, 1982; Ross, 1986, 1989a, b; Sinclair, Garin, & Tieche-Christinat, 1992; Sinclair & Scheuer, 1993) have focussed explicitly on the values of digits in multiunit numbers. Related Australian research studies (Boulton-Lewis, 1993b; Scales, 1992; Sierink & Watson, 1990) have further explored children's understanding of place value.

Kamii (1982) used a digit-correspondence task to study the development of children's ability to make groups of objects and represent them with pictures and numerals. Five levels of thinking from early representation through to the place value interpretation of the digits were identified for the sample of four to nine year old children. Even at age nine, 58% could not indicate correctly the representation of each digit. Kamii found that a child's understanding of place value is built in phases over a long period of time and suggests that the best approach to describing the learning process "may lie in the notion of theory-building about the notational system as a whole, about number concepts, operations, and their interrelationships" (p. 179).

Table 3.1 shows the digit correspondence tasks used by Ross (1986, 1989a). The proportion of children who performed successfully on each of the two canonical tasks (2nd and 3rd tasks in Table 3.1) was significantly higher than the proportions successful on the other digit-correspondence tasks. Ross suggested that the reason for this difference was that responses to these two tasks did not distinguish between those children who just identified face value of the digits with a number of objects in the concrete representation (stages 3 and 4) and those who understood place value representation (stage 5). Performance on these tasks was used to identify the stage of understanding in numeration for each child. It was suggested that 'stage 3' interpretation of each digit is a developmental plateau for many children since it is sufficient for success on many classroom tasks.

Table 3.1: Digit-correspondence tasks (Ross, 1989a)

1. An ungrouped collection of sticks (25) were counted and then the child was asked to make correspondences between the digits of a 2-digit number and the objects.
2. A collection of beans (48) grouped as 4 cups of ten beans and 8 loose beans were presented to the child and the digit-correspondence questions were asked.
3. As above using a collection of 5 longs and 2 shorts (Dienes base ten blocks) to represent the number 52.
4. As above using a regrouped collection of beans to represent 48 (3 cups of ten and 18 loose beans).
5. As above using a regrouped collection of Dienes blocks to represent 52 (4 longs and 12 shorts).
6. As above using a collection of 16 wheels grouped into 4 groups of four.

As discussed in Chapter 2 Ross (1986) proposed a five stage model to explain the development of children's understanding of our numeration system for two-digit numbers. The model was developed as a result of a study of sixty children in Grades 2 to 5, randomly selected from five schools and a synthesis of the research literature available at the time. A significant finding of the study was that many children were still constructing meanings for the individual digits as late as fifth grade (8 out of 15 children). The model is described in five stages in Table 3.2.

Table 3.2: Ross Model for the Development of Understanding of Two-digit Numbers

Stage 1:	The child is able to read and write two-digit numbers and associate the whole numeral with the number it represents. The child assigns no meaning to the individual digits which comprise the two-digit numeral.
Stage 2:	The child knows that in a two-digit numeral the digit on the right is in the "ones place" and the digit on the left is in the "tens place". The child's knowledge is limited to the position of the digits and does not encompass the quantities to which each corresponds.
Stage 3:	The digits are interpreted by their face values. The set of objects represented by the tens digit are different objects than the objects represented by the ones digit. A tens digit is not recognised as the same as ten of the objects that correspond to unit digits.
Stage 4:	The child knows that the left digit in a two-digit numeral represents sets of ten objects and that the right digit represents the remaining single objects but this knowledge is tentative and characterised by unreliable task performances.
Stage 5:	Individual digits in a two-digit numeral represent a partitioning of the whole quantity into a tens part and a ones part. The quantity of objects corresponding to each digit can be determined even for collections that have been partitioned in nonstandard ways.

(Ross, 1986a, pp. 33-42)

In a follow-up study Ross (1989b) interviewed thirty Grade 3 children to examine whether they used a face value interpretation to assign meaning to the individual digits. Each child was asked to count a collection of 26 objects and to write down how many there were. The children were then asked to sort the collection of objects into groups of four and correspondences between the concrete representation and each of the digits in the numeral for twenty six were sought. The visually prominent groupings in this task encouraged an incorrect response where the meanings of the digits are reversed. It was reported that nearly half of the grade 3 children responded that the 2 in 26 stood for the two separate objects and the 6 represented the six sets of four. Here the responders do not recognise that the digit on the left (even if it is identified as in the tens place) represents a multiple of ten. A conclusion drawn from these studies is that more instructional support focussing on the nonstandard base ten grouping representation of two-digit numeration is needed in the middle grades. This is also supported by the results of others who have used digit-correspondence tasks to assess understanding of numeration (Kamii, C. 1986; Kamii, M. 1980; 1982).

On the basis of Ross's work, a study by Sierink and Watson (1990) using the non-standard groupings task (Table 3.1) with 60 children in grades Prep to 4 found that only seven children responded that the 2 in 26 stood for the two left over and the 6 stood for six groups. However, of those seven children, five were in grades 3 and 4. Contrasted with this, 17 out of 27 Prep and Grade 1 children responded with an apparent place value understanding to this task whereas 7 responded that they didn't know, wrote the number 26 incorrectly or gave an insufficient explanation. A possible explanation is that younger children respond in relation to the logic of the language, i.e. twenty with the digit 2 and six with the digit 6. The older children may have rote learned, through their grouping experiences with materials, that one of the digits stands for the number of groups of objects and the other digit for the number of objects left over.

Sinclair, Garin and Tieche-Christinat (1992) also investigated how children build up an understanding of the characteristics of place value inherent in the notational system of number. The study involved using digit correspondence tasks in clinically interviewing a cross-section of French-speaking Swiss children, thirty children from each of the 5, 6, 7 and 8 age groups and ten children aged 9. Each child's knowledge of how the values of the digits in a numeral related to the cardinality of the number was probed. The responses to the 2- and 3-digit correspondence tasks were categorised as follows:

0 no response;

1 responses did not account for the whole collection of chips;

2 gave face value responses but then took account for the whole in some way when prompted, and reverted to face value responses;

3 gave face value responses but became convinced that the whole collection must be accounted for and used various creative incorrect correspondences;

4 some collections were partitioned correctly using total values, and

5 all collections were partitioned correctly.

Results from this research showed that "the idea that the whole must equal the sum of parts is constructed quite early, well before children have any intuitions at all concerning groups of ten" (p. 205). As a result of further research with 6 year old children Sinclair and Scheuer (1993) asserted that because young children consider the system of numeration as purely additive they are likely to interpret the numerical meanings of digits by their face values.

Scales (1992) investigated the use of an alternative instructional sequence for the introduction of 2-digit numeration in response to the perceived problem in learning the 'teen' numbers. Performance outcomes from alternative instructional sequences for introducing the numbers 11-99 to Grade 1 children were compared. Empirical evidence from this research indicated that the innovative 'teens last' sequence did not lead to superior performance on either the post-tests at Grade 1, or a retention test administered after a period of eighteen months.

The research project *Concepts in Secondary Mathematics and Science* (CSMS) based at Chelsea College, University of London, in the years 1974-79 attempted to study the understanding and application of the base ten place value notation by 11 to 15 year old children (Hart, 1981). The major part of this area of the CSMS work covered the application of the numeration system to the algorithms and the area of decimals. Topics investigated included the exchanging aspects of the addition and subtraction algorithms, reading decimal scales, multiplication and division of decimals by powers of ten, multiplication and division by numbers less than one, the infinite nature of the set of real numbers, and knowledge of real situations where decimals are used. Of more interest in this review of research are the interview or written questions on larger whole numbers, place value notation and the concept of zero as a place holder, all of which showed significant lack of understanding among these older children. Some of the questions and the facility ratings are given below (Brown, 1981). Table 3.3 shows questions 1b, 3, 7, 10b and 12b which probed understanding of the numerals for large numbers.

Table 3.3: CSMS numeration / place value questions

1b. 5214		The 2 stands for 2 HUNDREDS		
521 400		The 2 stands for 2		
Age	12	13	14	15
Facility	22	32	31	43 percent
3. Write in figures:				
Four hundred thousand and seventy three				
Age	12	13	14	15
Facility	42	51	57	57 percent
7. This meter counts the people going to a football stand.				
06399				
After *one* more person has gone in, it will read				
Age	12	13	14	15
Facility	68	77	86	88 percent
10b(i). Ring the BIGGER number: 20 100 or 20 095				
Age	12	13	14	15
Facility	86	89	91	94 percent
12b. Write down any number between 4100 and 4200				
Age	12	13	14	15
Facility	69	80	85	88 percent

It appears that many of the children interviewed had not generalised the structure of the numeration system and so were relying on their knowledge of the way numerals were structured for two and three digit numbers. Brown (1981) also indicated that some children obtained correct answers for computation questions through rote learned rules (even giving correct, though stereotyped explanations) although they lacked any real understanding of place value. It was also asserted that determining whether a child's response indicated 'relational understanding' of properties of number or just 'instrumental' facility to carry out a procedure was difficult. Six levels were identified from the results of the written test results for place value and decimals; all levels except level 1 were concerned with understanding decimals, reflecting the major focus of the study. A child was judged to have attained a level if he or she answered two-thirds of the questions at that level correctly. Approximately 17% of 12 year olds, 10% of 13 year olds, 6% of 14 year olds and 8% of 15 year olds had not reached the basic first level of place value in whole numbers up to thousands. It was also concluded that children were weaker than expected in dealing with numbers over a thousand. "It may be that secondary school teachers tend to assume children have mastered these ideas before the age of 11, but this seems not to be generally the case" (Brown, 1981, p. 64).

The *Low Attainers in Mathematics* project, outlined in Chapter 2 was carried out in England to investigate the learning of number concepts by children aged seven to nine who were considered to be low attainers in mathematics (reported by Dickson, Brown, and Gibson (1984) in an overview of research). This study aimed to find a framework for describing low attainers' acquisition of number concepts, the development of a diagnostic assessment instrument and the design and implementation of a remedial teaching program. Denvir and

Brown (1986a) conducted individual interviews with 7-9 year olds, in which the tasks were presented orally with frequent use of practical materials. Throughout the pilot (5 children) and main (7 children) assessment studies predictions were made about what hierarchical dependencies there might be between acquisitions of different number concepts. The assessment interview explored each child's strategies for dealing with the aspects of the application of number as shown in Table 3.4.

Table 3.4: Aspects of numeration assessed by Denvir and Brown (1986a)

i) Addition and subtraction of small numbers given as oral number sentences and word problems. Responses were categorised as counting all with direct physical modelling, counting from one, counting on/back/up, derived fact, and recalled fact.
ii) Commutativity of addition was determined by timing responses to horizontal written sums of the form n+1 and 1+n, n<10.
iii) Enumeration of grouped collections. Responses were categorised as guessing, counting each package as one (2, 5, 10), counting in ones with guesses for the number of objects in each package, counting in ones using the correct number for each package (numerical composite), counting by tens and by ones without coordination, and counting by tens and by ones (abstract composite unit).
iv) Addition of larger numbers were used to determine an appreciation of place value. Oral addition questions involving two-digit (includes decade numbers) and single digit numbers were used. Responses were categorised as counting all with direct physical modelling, counting from one, counting on, and use of place value strategy.
v) Conservation of number and class inclusion.

From the results of the counting grouped collections tasks the following sequence of stages was proposed: counting each package as one; counting each item as one; attempt at counting by tens and by ones but lacking coordination, and successfully counting the grouped collection by tens and by ones.

Based on the framework, a diagnostic assessment interview was constructed consisting of 47 ordered skills grouped into seven levels. This instrument was administered to 41 children and each child's performance was described by detailing performance on each of the skills and by ascribing one of the levels of performance. It is noted that the skill of using counting back/up/down to do subtraction, though low in facility and therefore given a level of 6, was nevertheless achieved by children with overall low scores. From these results a descriptive framework showing the hierarchy of skills was developed. The criteria used to determine a prerequisite relationship between skills was based on the idea that one skill depended on the other and that in nearly all cases children who succeeded on the harder skill must have succeeded on the easier skill. However, while the acquisition of some skills formed a hierarchy, some were found to be quite independent.

Denvir and Brown (1986a) also reported the use of the diagnostic assessment interview to examine changes in performance of seven children interviewed approximately six monthly over a period of two years. At each interview, almost all the children had made some progress as recorded by the assessment score, although this was always very small. Whilst no two children acquired skills in the same order, when the sequence of acquisition of skills for each child was compared with the multi-pathed framework the match was in most cases according to the order of the hierarchy. This supports the case for a developmental theory to explain children's understanding of numeration. Follow-up teaching studies by Denvir and Brown (1986b) will be discussed in Section 3.4.

Summary

Place value is an essential concept that has enabled the concise form of notational recording that is known as the Hindu Arabic System. Based on an understanding of grouping and developing along with the conceptualisation of multiunits, children gradually master the sophistication of place value notation. Kamii and DeClark (1985) reminded us that at the time when young children are realising that the order in which they count the objects in a collection and spatial arrangement of these objects do not affect the quantity, they are being expected to make sense of a place value notational system in which spatial arrangement and position are critical to the values assigned these positions. This section has reviewed the research on children's understanding and use of place value.

3.2.4 Evidence from studies of student achievement

Studies of student achievement in both the USA and England have shown that there are many students in secondary school who still do not understand our numeration system. One of the biggest problems appears to be that these young adults have not mastered the structure of the numeration system and so can not effectively use number.

In the U.S.A. there have been four national assessments of achievement levels of 9, 13, and 17 year olds in mathematics (1972/73, 1977/78, 1981/82, 1987/88). For the 1972/73 assessment Carpenter, Coburn, Reys, and Wilson (1978) reported that the 9 year olds performed well on the numeration items but indicated that the items could be considered quite simple. These items assessed such things as the recognition of the digit in the tens place of a four digit number, selection of the verbal translation of a three digit number, counting by tens and selection of the translation of a three digit number from words to symbols. It should be noted that there were substantial numbers of children who could not perform on superficial tasks, 26% of 9 year olds and 9% of 13 year olds could not select the correct numeral for the sum of three hundreds, eight tens, and four ones. In the report of the 1977/78 assessment Carpenter, Corbitt, Kepner, Lindquist, and Reys (1981) indicate that although the performance of 9 year olds on counting, writing numerals from spoken words and identifying ordinals was high, this was not so when place value and grouping notions were essential. In discussing the results of the Fourth NAEP Assessment of Mathematics, Kouba, Brown, Carpenter, Lindquist, Silver, and Swafford (1988) say that although there is success shown in learning some mathematical skills, there are serious gaps in knowledge and learning is often superficial. This lack of understanding is shown by the 63% of third graders and 36% of seventh graders who were unable to give the number 100 more than 498.

Kouba et al. (1988) also reporting from the fourth mathematics assessment say that approximately 65% of the third-grade children could do items involving grouping by tens, identify the tens digit in numerals and find the number ten more than a given number. The performance of these same children fell below 50% when place value beyond tens was involved. They go on to conclude that "it appears that younger students could profit from more emphasis on understanding place value and grouping, especially with situations involving numbers greater than 100. Students could also profit from experience with situations involving place value, order, and number sense that go beyond the usual textbook examples." (Kouba et al., 1988, p. 15).

3.3 CHILDREN'S REPRESENTATIONS OF THE NUMERATION SYSTEM

This section will discuss children's representations of the numeration system. Research groups focussing on children's counting strategies (Kamii, 1986; Steffe & Cobb, 1988; Steffe, 1991a; Wright, 1991a) and conceptual development of numeration (Bednarz & Janvier, 1988; Cobb & Wheatley, 1988; Denvir & Brown, 1986a, b; Fuson, 1990a, b; Hiebert & Wearne, 1992; Kamii, 1989; Ross, 1990) have highlighted children's construction of representations of the number system. Children's representations also reveal much about the idiosyncratic and creative ways in which they structure mathematical relationships (Maher, Davis & Alston, 1991; Thomas, Mulligan & Goldin, 1994).

Rubin and Russell (1992) assert that children's counting, grouping, estimating and notating skills are essential elements in developing representations of the number system. They describe these elements in terms of "landmarks in the number system". These landmarks appear to be related to additive structure, multiplicative structure, the generation and analysis of mathematical patterns and mathematical definitions. Rubin and Russell suggest that people who are adept with number operations, e.g., computing, comparing, and estimating, "have a non-uniform view of the whole number system" (p. 136).

Children's representations of number as some form of physical, pictorial, or notational recording have been exemplified in many studies analysing children's structural development of number and understanding of the numeration system (Davis, Maher, & Noddings, 1990; Goldin & Herscovics, 1991a; Hiebert & Wearne, 1992; Hughes, 1986; Rubin & Russell, 1992).

Young children's recordings of number

Studies by Sastre and Moreno (1976) [cited in Kamii (1982)], Allardice (1977), Kamii, M. (1982) and Hughes (1986) have explored children's spontaneous and conventional methods for representing small numbers. Sastre and Moreno (1976) categorised children's representations into four types and expressed surprise that many children did not use the numerals from their school experiences. Allardice (1977) found six distinct methods of representing set numerosity and relative quantity. Both these studies demonstrated the use of pictorial and iconic representations, and the use of numerals in a one-to-one match with the objects, before using a single numeral to represent total quantity.

Kamii (1982) proposed four developmental levels in conventional representation which reflected the following sequence of acquisition: each object is represented on paper with a discrete mark; an awareness of shapes used to represent the numbers, but not always successful; children write conventional numerals for single digit numbers, and children write conventional numerals for two- and three-digit numbers. Hughes (1986) showed that pre-school children can represent small quantities, although they are based on one-to-one correspondence, and many can also represent zero. On the other hand, there is a reluctance by many 9 year old children to use conventional symbols to represent the operations of addition and subtraction.

3.3.1 Representational Systems

Goldin (1992a) proposed a unified model for the psychology of mathematical learning which he felt could accommodate the most helpful and applicable constructs from a variety of frameworks. He distinguished cognitive representational systems internal to problem solvers (a theoretical construct to describe the child's inner cognitive processing) from external task variables and task structures. The distinction is made between external representations (a structured environment with which the child is interacting that may include, for example, actual physical objects to manipulate and actions in response to that environment), and internal imagistic representations (a theoretical construct to describe the child's inner cognitive processing). In the unified framework there are five types of internal, cognitive representational systems to model competence in mathematical problem solving:

(a) verbal/syntactic systems (using mathematical vocabulary, developing precision of language, self-reflective descriptions);
(b) imagistic systems (non-verbal, non-notational representations, e.g., visual or kinaesthetic);
(c) formal notational systems (using notation, relating notation to conceptual understanding, creating new notations);
(d) systems for heuristic planning and executive control, and
(e) affective systems which monitor and evaluate the problem solving process.

These systems develop over time through three stages of construction:

(i) inventive/semiotic, in which characters in a new system are first given meaning in relation to previously-constructed representations;

(ii) structural development, where the new system is driven in its development by a previously existing system, and

(iii) autonomous, where the new system of representation can function independently of its precursor.

Thomas (1992) investigated links between children's understandings of the structure of numeration and their representations of the counting sequence 1-100. Attention was drawn to some particularly creative and idiosyncratic representations for the number sequence used by young children. Thomas and Mulligan (1994, 1995) drew on Goldin's theoretical model of problem solving competencies (Goldin, 1992a, b; Goldin & Herscovics, 1991a, 1991b) when reporting a follow-up study of a cohort of high ability children. The relationship between the children's representations of the number sequence and their structural development of the numeration system was explored. It was shown that there was a wider diversity of representations than might have been expected and a higher percentage of dynamic imagery existed than for the average/lower ability children observed in earlier studies. It was also shown that those children with dynamic visualisations had higher achievement on a selection of numeration tasks than those with static visualisations.

Thomas, Mulligan, and Goldin (1994) provided further analysis of the representational characteristics of children's counting. It was suggested that external representations can be used to gain insight into some aspects of children's internal representational systems. This then provides some evidence of children's internal structure of the number system, albeit only partial description of children's internal representational capabilities. Of particular importance was the evidence that children's representations of number are highly imagistic, and that these configurations embody structural development of the number system and vary widely across children often in unconventional ways. Evidence was shown that children's internal representational systems for number develop over time through a period of structural development and become eventually powerful, autonomous systems (Thomas, Mulligan, & Goldin, 1996).

The role of imagery in the representation of number

The role of imagery in the representation of mathematical ideas has been described by a number of researchers (Bishop, 1989; Clements, 1982) although Gray and Pitta (1996) warn that "because of the disguised nature of mental images it is only possible to make conjectures about them" (p. 36).

Personal visuo-spatial representations of number (number forms) were described long ago by Galton (1880). Seron, Pesenti, Noel, Deloche, and Cornet (1993) suggest that the number form is a more accomplished development of a general disposition of people to encode numbers in a visual way. They conclude that number forms are used to code the number sequence, and that the function (if it exists) of this phenomenon should be examined in number and calculation processing. Dehaene and Cohen (1995) proposed a functional-anatomical model for number processing in mental arithmetic which involved three categories of mental representations in which numbers can be manipulated: visual Arabic number form (symbolic); verbal word frame, and analogical magnitude representations. It is suggested that transcoding between internal representations is necessary for any processing to take place. The studies of students' representations undertaken by these researchers indicate consistently that students use imagery in the construction of mathematical meaning.

Recent work (Brown & Presmeg, 1993; Brown & Wheatley, 1990; Presmeg, 1986a, b, 1992) in which individual students' thinking was probed in clinical interviews, indicated that students use imagery in the construction of mathematical meaning. Brown and Presmeg (1993) assert that learning frequently involves the use of imagery although sometimes it might be very abstract and vague forms of imagery. Presmeg (1986a) identified five types of visual imagery used by students:

(i) concrete, pictorial imagery (pictures in the mind);
(ii) pattern imagery (pure relationships depicted in a visual-spatial scheme);
(iii) memory images to recall information;
(iv) kinaesthetic imagery (involving muscular activity, e.g., fingers 'walking', and
(v) dynamic (moving) imagery involving the transformation of concrete visual images.

Mason (1992) distinguished between images that are eidetic (fully formed from something presented), and those that are constructed (i.e. built up from other images). The meaning-constructing process continues as the 'mental picture' is described, drawn, compared and discussed. He suggests that for students to access images they must actively process them, 'looking through' rather than 'looking at' the 'mental screen', regardless of the mode of external representation. Imagery is also an important aspect of Pirie and Kieren's (1994) model for the growth of mathematical understanding. They describe 'image making' and 'image having' as early levels of understanding which can be observed. 'Image making' is the development of mental representations from initial learning experiences. At the next level, 'image having', a learner is able to manipulate and use the image in mathematical thinking.

Recent findings (Brown & Presmeg, 1993) revealed wide differences in the types and facility of imagery used by students in problem solving. All students in the study of seven fifth grade and six eleventh grade students used some type of imagery to solve mathematical tasks. Students with a greater relational understanding of mathematics tended to use more abstract forms of imagery such as dynamic and pattern imagery while students with less relational understanding tended to rely on concrete, kinaesthetic and memory images.

Reynolds and Wheatley (1994) reported that fourth/fifth graders used recording to help symbolise mathematical constructions, and children's reflections on these symbolisations elaborated their mental schemes. They found that images were often not well developed but were constructed during reflection on an activity. They suggest that children benefit from these ways of externalising their own meaningful constructions, rather than having the symbolisations of others imposed on them.

Hershkowitz and Markovits (1992) emphasised the importance of visualisation of mathematical concepts and the development of advanced visual thinking. Markovits and Hershkowitz (1997) and Bobis (1993) investigated the role of visualisation in estimation of number. Markovits and Hershkowitz showed that visual imagery played a vital role for 9 year olds in doing numerical tasks, especially in problem solving. Bobis (1993) found that with practice, Kindergarten children were able to use visualising strategies to mentally combine and separate patterns. The children developed subitising skills and started to relate number patterns mentally as they enhanced part-part and part-whole relations. Ten frame imagery was found to be a useful referent for children in their visualisation of number.

The notion of procedural encapsulation (Dubinsky, 1991; Gray & Tall, 1994), which is the cognitive process of forming a conceptual entity from a mathematical process, provides a useful theoretical basis for describing how thinking changes from concrete to abstract. Although, on the surface, processes and objects are incompatible, they can be simultaneously conceived of as a mathematical entity after the process of 'encapsulation' - the idea of 'duality' (Sfard, 1989; Gray & Tall, 1994). Gray and Pitta (1996) concluded from their research that in the abstraction of numerical concepts from numerical processes (encapsulation), qualitatively different outcomes may arise between different children. "Children described as 'high achievers' provide evidence of an implicit appreciation of the information compressed into mathematical symbolism. In contrast, 'low achievers' create images strongly associated with visual stimuli suggesting that these children, far from encapsulating arithmetical processes, are mentally imitating them." (p. 35). These differences are described in terms of a procedural divide between children who rely on procedures and those who process information in a flexible way, between imagery which is associated with recollection of personal happenings and visual stimulus and that which is associated with organised knowledge and relationships. Low achievers appear to concentrate on analogies of physical actions, which places strain on the child's working memory. Sfard (1991) discussed the notion of 'encapsulation' in terms of a three phase process involving interiorisation, condensation and reification . Reification involves an ontological shift from operational to structural thinking, instrumental to relational thinking (Skemp, 1979), or procedural to conceptual knowledge (Hiebert & Lefevre, 1986). The new entity created becomes independent from the process, and information is compressed into numerical symbolism which in turn has properties through its relationships with other concepts. This state is similar to Goldin's (1992a) autonomous stage in the development of understanding.

These theoretical explanations of the role of imagery in the representation of mathematical ideas provide some basis for understanding the way children construct the numeration system. The process of developing the multiunit conceptual structure of the numeration system as discussed in Section 3.2.2 is not just a verbal or notational one; it appears that the role of imagery is essential. Thompson (1996) asserted that "without students having developed ... images that entail both figurative and operative thought, students cannot constitute the situations that their visible mathematics is supposed to be about with sufficient richness to support their reasoning" (p. 280).

3.3.2 The use of concrete materials in representing the numeration system

Although the use of concrete materials in the development of understanding and use of the numeration system is widely advocated, research evidence is not conclusive as to its benefit (Dienes, 1960; Fennema, 1972; Labinowicz, 1985; Resnick, 1982; Resnick & Omanson, 1987; Ross, 1990; Sowell, 1989; Thompson, 1992; Wearne & Hiebert, 1988). Some studies found little effect (Labinowicz, 1985; Resnick & Omanson, 1987) ; others found some positive effect for using concrete materials (Fuson & Briars, 1990; Wearne & Hiebert, 1988). The nature of children's engagement with concrete materials and their use of the materials in relation to numerical value and symbolism is a possible reason for this variability in research findings (Fuson & Briars, 1990). Children's manipulation of concrete materials in a prescribed activity may have little effect if they see themselves as following a prescription (Resnick & Omanson, 1987). Before children can make productive use of concrete material they must first be committed to making sense of their activities and be committed to expressing their sense in meaningful ways. For concrete embodiments of a mathematical concept to be used effectively in relation to learning some symbolic procedure, children must come to see each as a reflection of the other (Thompson, 1992). Concrete analogs by themselves cannot impart meaning (English & Halford, 1995) and the processing load entailed in mapping a concept onto an analog may cause difficulties (Boulton-Lewis & Halford, 1992). Importantly, sources of meaning can not be assumed to be inherent in the external representations of concrete materials but are located in "students' purposeful, socially and culturally situated mathematical activity" (Cobb, Yackel, & Wood, 1992, p. 6).

Researchers (Cobb, Perlwitz, & Underwood, 1996; Dienes, 1960; Fuson & Briars, 1990; Labinowicz, 1985) have advocated a range of concrete materials in teaching numeration and place value. Concrete representations have also been advocated when developing the number operations (Beishuizen, 1993; Thornton, Jones, & Neal, 1995). Dienes (1960, 1964) advocated the use of his *multiple embodiment principle* for developing understanding of numeration. This principle suggests that concept learning is maximised when a concept is presented in a variety of physical contexts. Number is represented by a variety of materials which differ in perceptual features, the common element to be abstracted is the targeted concept or process. During the 1960's, the use of different number bases was advocated in many teaching schemes as a result of the work of Dienes. Dienes (1960) suggested that wooden blocks should be available in a number of different bases (Multi-Base Arithmetic Blocks or MAB) in order to aid the abstraction of the concept of place value. More abstract representations were also advocated. These included the use of coloured counters in which, for instance, 5 yellow counters would be worth 1 blue, 5 blue counters would be worth 1 red counter; and an abacus model using washers arranged in columns, where each washer is given a value according to position working from the right on the abacus frame (e.g., values of 1, 5 and 25 in base five).

Later researchers have focussed on the importance of a variety of base ten models (Baroody, 1990; Bednarz & Janvier, 1985; Cobb, Yackel, & Wood, 1992). Numeration blocks (Dienes base ten blocks) are considered as part of a sequence of concrete embodiments for the multidigit numbers and to help construct the abstract 10-for-1 scheme (Baroody, 1990). Concrete representations in numeration can involve simple grouping, proportional and non-proportional grouping, linear, and proportional and non-proportional place value models which reflect different degrees of abstraction and so require different quantities of processing load. Bednarz and Janvier (1985) asserted that learning numeration is a process which involves representing numbers in a variety of meaningful ways leading to the conventional notation. Concrete materials should be used in models which provide a context that has some inherent meaning for the children. Bednarz and Janvier (1985, 1988) reported the use of the 'flower model' which relates two-dimensional flowers, three-dimensional flowers (10 superimposed two-dimensional flowers), bunches (10 two-dimensional flowers tied together) and garlands (made of flowers from 10 bunches). The 'flower model' is set within a context of the workshop of flower makers, packaging, and the store. Prior to using the 'flower model' with children, the 'cereal boxes model' was used where the relations between presented groupings was really explicit from the material, but not in a base system (e.g., cereal boxes, cases made from 6 cereal boxes, and baskets made from 3 cases). Cobb Yackel, and Wood (1992) reported the use of the 'lolly factory model' which related lollies, rolls, packets and cases (e.g., 10 lollies make a roll, 10 rolls make a packet, and 10 packets make a case). The context of the models gives meaning to the groupings, to the operations on collections, to the comparisons between collections, and to the need to use written representations. These situations facilitate children operating with, comparing, and communicating with various representations of quantity.

Other research, as described in Section 3.2.3, has shown that representations and strategies that are used can impose an unnecessary processing load which can interfere with conceptual learning (Boulton-Lewis & Halford, 1992). It is suggested that the difficulty that many children experience in conceptualising the numeration system is due to the process of mapping the tasks onto a mental model and the load that this imposes. Understanding of number is required when devising a way to perform a task but is not always necessary in performing the number task (Greeno, 1991; Boulton-Lewis & Halford, 1992). It is asserted that once procedures have been developed they may be used without invoking mental models (Greeno, Riley & Gelman, 1984; Boulton-Lewis & Halford, 1992).

Boulton-Lewis and Halford (1992) contend that, although children can physically manipulate materials appropriately, they sometimes do not recognise the structural correspondence between the concrete representation and the mathematical concept it is intended to illustrate. It is further asserted that this lack of connection is because the processing load involved in operating with the material is too high. Children need assistance in building relationships between physical, pictorial, verbal and symbolic representations (Bednarz & Janvier, 1982; Fuson & Briars, 1990; Hiebert & Wearne, 1992; Janvier, 1987; Lesh, Post, & Behr, 1987; Thompson, 1992). As a result of experience with the *Children's Mathematical Frameworks* research, Hart (1989) argued that there is often little apparent connection for children between a practical or material-based approach to mathematics and formal or symbolic mathematical language. Hart concluded that referring children to materials because they are having difficulties with symbols is of little help unless one is sure that they have a workable method of using objects to mirror symbols, and that they can make an effective mapping from the concrete representation to the symbols. It is important for teachers to assess the suitability of a particular model before it is used as a representation of number (Baroody, 1990; English & Halford, 1995; Kaput, 1987) as there is a danger that children will rote learn procedures for concrete material (Baroody, 1990; Ross, 1989b).

Numeration blocks (otherwise known as Dienes blocks or base-ten blocks) are the most commonly used concrete representations of number used in New South Wales schools (Howard, Perry, & Conroy, 1995). Dienes (1960, 1963, 1964; Dienes & Golding, 1971) introduced the currently used base-ten blocks to Australian schools in the 1960's, as part of a broader range of materials that used a number of different bases; hence the name, multibase arithmetic blocks (MAB). English and Halford (1995) explain that although the blocks are an appropriate analog for the whole numbers (albeit numbers up to 9999) "their effectiveness will be limited if children do not form the correct mappings between the analog representations and the target concepts and between their manipulations with the analog and the target procedures" (p. 106).

Whereas the base-ten blocks might provide an effective analog for whole numbers up to four-digits, they take on an added complexity when representing larger numbers. This can either be done by changing the values of the blocks or by imagining the construction of ever larger pieces of wood to represent the larger positional values. Both when the blocks are given new values (e.g., if the 'little' is assigned the value one thousand, then the 'long' is equal to ten thousand, the 'flat' is a hundred thousand and the 'block' is a million), and when the system of blocks is extended, children must use the multiplicative property to generate the representations of positional values.

Summary

Numeration involves the development of an increasingly sophisticated counting scheme and a system of notation for recording the numbers that are generated. Initially the counting scheme is a system of ones. In order to quantify a collection which is being counted, the relationship of hierarchical inclusion must be included within this scheme. The counting scheme then involves constructing a system of tens on the system of ones. The first system of numbers is constructed through the operation of +1 and the second system through the operation of +10. The system of ones must be conserved as the system is partitioned into groupings of ten. The ten ones, while still remaining ones, become an iterable unit of ten. A system of hundreds is later constructed on the system on tens. Children's conceptual structures for number words are now "multiunit conceptual structures in which the meanings or referents of the number words are collections of entities ... or a collection of collections of objects". These new structures replace the earlier "unitary conceptual structures in which the meanings or referents of the number words are as single objects ... or a collection of single objects" (Fuson, 1990a, p. 273).

Initially numbers are recorded by their verbal number sequence and later by the written symbolic system. This system of notation is not simply the representation of the numbers in the counting scheme because it involves the conceptual abstraction of place value. Place value involves more than knowing the values of digits in a numeral, it involves understanding and applying the system that generates the layers in the counting scheme that we called the system of ones, the system of tens, the system of hundreds, etc. The meaning of a digit in a numeral is given by its place value multiplied by its face value. The notion of place value and the use of zero as a place holder are needed in relation to the symbolic recording of the numbers.

Children's representations of number reflect their developing construction of the complexity of the numeration system. The constructs of internal and external representational systems are useful to help understand how learning of numeration occurs. Imagery is important in the construction of meaning. External support in the form of concrete materials needs to fit into the way a child structures number at that time. Connections need to be made between concrete, pictorial and notational representations that might be used.

3.4 CLASSROOM-BASED STUDIES ON TEACHING AND LEARNING NUMERATION

In this section classroom-based research on teaching and learning numeration will be discussed in the following subsections: background to the search for classroom practice that would provide meaningful learning; teaching approaches explored through the teaching experiments; teaching practices leading to invented systems for algorithms, and projects on the construction of multiunit structures.

3.4.1 Background

The teaching and learning of numeration and place value have been characterised by the work of educators such as Brownell, Montessori, Stern, Dienes and Piaget. Their views reflected the search for ways to develop children's meaningful understanding of the numeration system. This had been a continuing focus for teacher educators since Brownell's (1935) classic article calling for the use of the meaning theory of instruction in arithmetic. Stern (1949) was one of the first mathematics educators to advocate the importance of modelling the groupings of tens and ones as the basis for developing understanding of numeration. Multibase approaches to numeration were advocated (Dienes & Golding, 1971; Bereiter, 1968; Underhill, 1972) during the time of the 'new mathematics' movement to help children develop a general scheme for building a notation system.

In a review of literature on place value and the decimal number system, van Engen (1947) concluded that until the late 1930's the treatment of this content from the point of view of meaning was negligible. He went on to outline how children should be taught the basic principles of the decimal system and how they should be given concrete experiences from which to extract and internalise meaning. This view encouraged the use of concrete materials in the learning of number, and the verbalisation of the connections between the materials, actions and symbols. The learning theory was that actions, sufficiently practiced, would be accompanied and eventually replaced by visualisation of the actions and conceptualisation using symbols alone.

The introduction of ideas based on the theory of sets, the use of Dienes' *multiple embodiments principle* and the introduction of Cuisenaire rods dominated much of the discussion of children's arithmetic during the 1960's and early 1970's. Payne and Rathmell

(1975) defined numeration as "those concepts, skills, and understandings necessary for naming and processing numbers ten or greater" (p. 137). There was an emphasis in curricula on grouping objects into equivalent sets and naming the number of groups as preparation for numeration activities.

At about the same time, curricula and diagnostic tests based on task analyses, derived from Gagne's learning hierarchies, were being formulated (e.g., Resnick, Wang, & Kaplan, 1973; Smith, 1973). These task analyses started with a concept or skill which was then broken down into constituent elements or prerequisite capabilities. The focus was therefore on the content as a complex hierarchy to be mastered, or problem to be solved, rather than on the process of learning from the point of view of the individual child.

Treffers (1991) asserted that the structuralist approach following Bruner's enactive, ikonic, symbolic model of learning causes difficulties with numeration. This approach is reflected in the teaching of algorithms by offering concrete numeration blocks first, which are then replaced with pictures and then finally by mental images whilst using symbols. Treffers contends that the structuralist approach includes a "one-sided emphasis on the positional structure of the number sequence, that is, the notational suggestion that 75 is nothing but 7 tens and 5 units, on the one hand, and the disregard for counting and measuring aspects of number on the other hand" (p. 344). Also, this carefully structured approach to managing the learning sequence does not encourage the use of intuitive (natural) mental strategies. The focus is upon the structure of the mathematics as the teaching content. The alternative is the realistic approach in which children's informal strategies provide the starting point for "a gradual process of schematising, shortening and generalising, which eventually leads to the very arithmetical procedures" (p. 345). The focus is upon the children and the context of learning. Instruction is guidance through the model situations and the models which are offered (guided reinvention). Construction of meaning is globally determined by the learners themselves rather than directed externally.

Rathmell (1972) categorised the introduction of numeration instruction into three approaches:

(i) The teaching of numeration is based on counting with an introduction to 2-digit symbols prior to consideration of groupings of tens and ones;

(ii) More emphasis is placed on the base ideas during initial instruction, symbols are developed by relating them to groupings of tens and ones, and

(iii) Grouping is introduced early and is not limited to base ten. The concepts of base and place value are abstracted from a variety of experiences with decimal and non-decimal systems.

Rathmell (1972) concluded from his research that there was little difference between a multi-base or base ten approach to instruction in numeration. It was also reported that postponing highly symbolic representations until meaning is well established at the concrete level was not beneficial.

The use of other bases as a way of generalising structure (discussed in Section 3.3.2) fell out of favour during the 80's as the focus of curriculum reform turned to problem solving. Baroody (1987) suggested that instruction using other bases had failed because it was introduced in a highly formal manner and that it could be used in a more informal and meaningful way. In 1990, Davis raised the issue of whether aspects of the earlier attention to other bases might be worth reconsidering in the context of children's constructions. He referred to the following problem which had been given to children in the Madison Project in the 1970's.

> *Two small groups of children are asked to communicate messages back and forth, but they must pretend that nobody can count above three. One group of children - at the front of the room, say - is then given a pile of tongue depressors (let's say that you and I know that there are twenty-two tongue depressors in the pile). The children at the front must send messages to the other group of children (at the back of the room) so that the second group can assemble the same number of tongue depressors.*
>
> (Davis, 1990, p. 97)

It is suggested that this is an example of focussing instruction on the basic task, and leaving it up to the children to invent a way to solve the problem. The construction of a recording system in this problem would parallel the invention of place value numerals as an elegant solution to the problem of how to record the very large numbers. Maybe it is time to consider the value of other bases when learning the base-ten numeration system, not as a formal content area but as a focus of exploration of alternative (invented and historical) notational systems.

The Mathshop Project in Australia published a series of books designed to assist teachers in planning mathematics programs (Booker, Irons, & Jones, 1980). *Fostering Arithmetic in the Primary School - Numeration* presented a sequence of development beginning with introductory whole number concepts and continuing through decimal fractions. The structured approach was based on the outcomes of work conducted at the Learning Assistance Centre at the then Kelvin Grove College of Advanced Education. The approach used concrete manipulations and language forms to support symbolic representations.

The emphasis in the 80's was on problem solving approaches, teaching for problem solving, teaching about problem solving, and teaching through problem solving (NSW Department of Education, 1989). Using problem solving was a teaching strategy that was being advocated as a way of children inventing their own procedures and recordings before the conventional ways were introduced (Bentley, 1987; Cobb & Merkel, 1989; Kamii & Joseph, 1988; Ross, 1989b).

A constructivist view of learning, in which children reorganise their mental activity to resolve problematic situations, is grounded in the work of Piaget (1970) and extended by a number of researchers (Cobb, Wood, & Yackel, 1990; Confrey, 1985, 1991; Steffe & Cobb, 1988). The emphasis on interactive and reflective aspects of learning is also supported by Vygotsky (1978). In contrast to the constructivist assumption that children actively create their own ways of knowing, the activity theory approach of Soviet researchers (e.g. Davydov, Leont'ev, Vygotsky) claim that children need to take on the accumulated experience of human kind, and to learn to perform the social actions and use the cultural tools of mathematics appropriately (Cobb, Perlwitz, & Underwood, 1992).

Researchers have questioned what teaching methods and materials are appropriate for the meaningful learning of number. Consequently teaching experiments have been conducted (Bednarz & Janvier, 1988; Cobb & Wheatley, 1988; Denvir & Brown, 1986b; Hiebert & Wearne, 1992; Jones et al., 1996; Kamii & DeClark, 1985; Ross, 1990; Swan, 1990) to explore teaching approaches that might facilitate children constructing mathematical understanding of aspects of the numeration system.

3.4.2 Teaching approaches

Teaching experiments have been carried out at the Shell Centre in England to test the effectiveness of diagnostic teaching methods which involve deliberate exposure and discussion of common errors and misconceptions in the teaching of decimal fractions (Bell, Swan, Onslow, Pratt, & Purdy, 1986; Swan, 1983, 1990). In one such experiment two teaching styles were investigated - 'conflict' and 'positive only' teaching styles.

> *The first of these, the conflict approach, was intended to involve the students in discussion and reflection of their own misconceptions and errors, thus creating an awareness that new or modified concepts and methods were needed. There was therefore, a destructive phase, in which old ideas were shown to be inaccurate or insufficient, before new concepts and methods were introduced. The second teaching style, the 'positive only' approach, made no attempt to expose and discuss errors, but tried to help students to develop a correct conceptual understanding from the start. Correct methods were then demonstrated and practised intensively.*
>
> (Swan, 1990, p. 61)

Lessons in 'the conflict' teaching style involved four phases: the intuitive phase; the conflict phase; the resolution phase, and the consolidation phase. Lessons in 'the positive only' teaching style involved only two phases: the teaching the concept phase and the consolidation phase. Although both groups of children (12 and 13 year olds) made substantial gains and 'the positive only' group covered the work more quickly, and so were given more demanding supplementary activities, 'the conflict' group made significantly greater gains. "Research evidence seems to show that the resultant learning [by the conflict group] is much more meaningful and permanent, and thus more likely to be of use to the students" (Swan, 1990, p. 70).

Bednarz and Janvier (1985, 1988) carried out a constructivist teaching experiment over two and a half years beginning with two classes in first year (39 children), through one second grade (22 children) to third grade (23 children). The program of instruction was planned on the basis of knowledge of the children's initial conceptual structures, continuous analysis of procedures and representations used by the children, and an analytical model of the conceptualisations of numeration (shown in Figure 2.6). The learning of numeration was regarded as a process of representation of numbers which leads to conventional notation. The materials used were collections of objects, and there were rules governing the making of groupings and the way in which operations were defined. Children used transformations of groupings to make groupings, to unmake them, to make groupings of groupings, and for exchanging. The materials used in the models were such that groupings were clearly identifiable, the relation between presented groupings was explicit, and the actions applied to groupings was explicit. The materials were used in contexts in which the operations and relations had meaning (e.g., the flower and cereal box models). The context was used to invent diverse scenarios and related tasks in order to expose the grouping structure of the situation. As children operated on real collections, they made concrete actions on groupings and observed the effect. Children were encouraged to make use of different means to communicate information about collections: oral and written language, drawings, mimes, diagrams and symbols. To make recordings move from descriptive to symbolic forms, the teaching strategy was "to work with situations where the need to communicate and operate becomes more and more hard to fill. Children are then forced to question their representations and adjust them so that they become more formal and efficient" (Bednarz & Janvier, 1985a, p. 286).

The interviews conducted at the end of each year of the study were used to examine how understanding evolved over the time of the constructivist intervention. Final evaluations were also used to determine how the children were situated in regard to common difficulties and misconceptions, and to examine to what extent there was a transfer of acquired skills to other situations which used less accessible representations. Bednarz and Janvier (1988) reported on children's development of groupings, and their ability to operate with these in more and more complex tasks. It was further shown that almost all children could operate with two groupings of different order at the same time. It was found that over 50% of children were able to transfer the acquired skills for subtraction and division within known contexts to a non-familiar situation where the representation was less accessible (an abacus). Although the children were not taught the conventional algorithms, 96% of them successfully completed the subtraction of three-digit numbers and could attribute a meaning to what they did. For division of a three-digit by two-digit number, 66% of children were successful. Bednarz and Janvier (1988) also reported the results from a comparison group, who were taught in a traditional way, which showed that the majority of participants still displayed the initially identified difficulties associated with interpretation of digit position in terms of groupings, the usefulness of groupings, and the inability to work with two different groupings or operate on groupings at the end of Grade 3.

Denvir and Brown (1986b) conducted two teaching studies with 7-9 year old low attainers, based on the descriptive framework and the children's performances in the diagnostic assessment interview. In the pilot study (over three months with seven children) the children were taught individually and specific teaching points were selected for each child. Results of the teaching program showed that "the hierarchical framework can describe children's present knowledge and suggest which further skills they are most likely to acquire and thereby inform the design of teaching activities. However it cannot predict which skill or how many skills each child will acquire, so the teaching should not be too prescriptive or rigid in its assumptions about what may be learned" (Denvir & Brown, 1986b, pp. 156-157).

In contrast a second study involved twelve children being taught as part of a group, the members of which were at a similar level of understanding but had acquired slightly different skills. The children were taught as two groups in two different schools, twice weekly for six weeks. The focus was on linking aspects of concepts which had already been grasped as well as concepts which might be acquired rather than on teaching specific outcomes. In order for the learners to make connections between ideas, the teaching program was designed so that "children should, through doing, observing, discussing and thinking:

(1) consider their methods, not just the answers;
(2) have confidence in their own methods;
(3) perceive that different methods can and do work and are permissible;
(4) try out different methods and consider whether they are equally valid, and
(5) establish links between different aspects of number" (Denvir & Brown, 1986b, p. 158).

It was found useful during the study to encourage the representation of problems or solutions in different ways. Substantial progress was made by the children when taught in groups using the framework as a basis for designing activities.

In comparing the results of the individual and group teaching studies it was shown that the overall progress of children taught in the group situation was greater than those taught individually. As teaching outcomes did not always match precisely objectives set and material taught in the individual study, it was asserted that children benefited from the group situation in the second study where there was a broader agenda and more scope for unplanned interactions. "The children who made most progress were those who appeared to involve themselves most in the practical tasks and the discussion" (Denvir & Brown, 1986b, p. 164).

Ross (1990) carried out a study to investigate whether an instructional program based on the frequent use of concrete material (including base-ten blocks) across all grades would account for any significant difference in student performance on numeration tasks. A sample of 40 children from a specially selected elementary school known for a strong concrete-based mathematics program was compared with the original Ross sample (Ross, 1989a). The first sample performed better on three tasks and the second sample performed better on six tasks. The only significant difference between samples was on the class inclusion task where 63% of sample 1 children were successful but only 38% of sample 2 children. Overall the stage model profiles of the two samples, as related to the *Model for the Development of Understanding Two-digit Numbers* (Table 3.2), were not markedly different. Noticeable features of the profiles were that second graders in sample 2 were assigned to higher levels of the stage model. Also children from sample 2 tended more often to use a stage 3 interpretation of digits than those of stage 1. Ross carried out a post hoc analysis of her data to determine whether differences in cognitive development (as measured by the class inclusion task) could account for any difference in children's understanding of place value. The limited evidence suggested that although, children from sample 2 displayed lower cognitive development, they performed equally well compared to children from sample 1. Ross concluded that neither instruction, nor cognitive development alone, accounted for the variation observed in children's understanding of place value.

Kamii and DeClark (1985) argued that premature instruction in place value was detrimental to children making sense of numeration. "Given that we know about the developmental course of children's thinking, we ought to ask ourselves whether it would not be wiser to delay place value instruction until children have solidly constructed the number series (by repetition of the +1 operation) and can partition wholes in many different ways (part-whole relationships)" (p. 63). First grade children generate the numerals by recognising the pattern of using digits in their construction. They learn through social transmission that the counting words for the teen numbers are not written as they sound. Kamii explains that six and seven year old children are still in the process of constructing the number system with the relation 'one more'. The base ten place value system requires the mental construction of one group of ten out of ten ones and then the representation by a digit in the tens place. This involves the construction of a second level to the number system. This second level also involves the idea of multiplication as the groups of ten become the new units. It is not possible for a child to construct this second level while the first level is still being built. In a sample of 29 first grade children, none were able to demonstrate place value understanding. When the same digit-correspondence task, using 16 ungrouped chips, was given to a Grade 1 class of children who had been specially taught place value, again none were able to demonstrate understanding. The test was then given to children in each of Grades 4 (51%), 6 (60%), and 8 (78%) (percentages of successful responses shown in brackets). These results showed that there were still children in high school who had never understood place value. As a result of this research, Kamii advocated a teaching approach based on developmental and constructivist theories which emphasised mental processes and the need to allow children time to construct their system of ones as a prerequisite for building their system of tens.

Bentley (1987) argued for a child's-play method of teaching the place value notation to young children. It was suggested that errors can be overcome by the use of instructional materials and procedures which eliminate the gap between school children's informal knowledge and formal written procedures which require children to apply or interpret place value notation. Experimental instructional studies have shown that children in Grades 1 and 2 can invent their own efficient algorithms without concrete material (Cobb & Merkel, 1989; Kamii & Joseph, 1988). Ross (1989b) asserted that opportunities need to be provided for children to develop a strong number sense rather than teaching place value concepts separately as a prerequisite to double-digit addition and subtraction. Mental algorithms develop as the children construct increasingly elaborate concepts of numerical part-whole relations and place value. Conventional algorithms come only after they have experienced a fertile period of inventing their own efficient procedures through problem solving.

3.4.3 Invented systems for algorithms

As discussed earlier in Section 3.2.3 Ross (1989b) suggested that teaching place value concepts separately as a prerequisite to algorithmic work was ineffective if understanding was the goal. At the same time it is considered that careful instruction with concrete material can facilitate the acquisition of the procedural knowledge to carry out computational algorithms. How then, do we get more children to demonstrate flexibility in partitioning numbers so that they might develop relational understanding of the conventional algorithms for addition and subtraction of whole numbers? Ross advocates a child-centred period of inventing procedures for solving two-digit number addition problems requiring renaming. This requires the development of a sense of number which includes the notion of the part-whole relationship and the idea that a number may have many names. Some experimental instructional studies (Cobb & Merkel, 1989; Kamii & Joseph, 1988) have shown that children as young as first and second grade level can invent their own efficient algorithms. It must be questioned whether an overly structured approach to teaching numeration and the conventional algorithms leads to the development of child misconceptions and hence, the need for unlearning.

Madell (1985) describes the natural processes that children use to solve addition and subtraction problems involving two-digit numbers. Observations of children from Grades K to 3, who had not been taught any computational algorithms, showed that the spontaneous responses to working with two-digit numbers in this way included the following: a reluctance to join tens to tens in addition; a tendency to operate with the teen numbers as a number of ones; a preference to add and subtract from left to right when in the column format, and a tendency to use strategies which focus on ten when adding or subtracting. He argued that children should create their own computational algorithms and that the teacher's role is to understand each child's method so that appropriate guidance can be given, gradually leading towards more efficient procedures.

Fuson and Briars (1990) reported a teaching/learning setting where first and second grade children used base-ten blocks and digit cards to construct multiunit conceptual structures. Children learned advanced unitary methods for adding and subtracting sums to 18. After experiences with linking the number words, block words, and digits of multidigit numerals together, the children were introduced to four-digit addition (and later subtraction). Traditional algorithms were developed through instruction using base-ten blocks on large place value charts, verbalisation of actions (block words and number words) and recording with numerals after each action with the blocks. The teaching experiment showed that with instruction, most Grade 2 children could become competent with multiunit addition and subtraction. A major focus of Fuson's analysis (Fuson, 1990b) was the learning difficulties caused by irregularities in English words for two-digit numbers. As a result it was suggested that instruction should move rapidly through single-digit numbers to four-digit numbers. It was asserted that the regularities of the hundreds and thousands enable children to construct general multiunit concepts that help generalise the system, rather than being distracted by the irregularities of the two-digit numbers.

As discussed in Chapter 2, Jones et al. (1996) have developed a framework for instruction and assessment of children's thinking in multidigit number situations. The theoretical position taken for the framework is that it enables multidigit learning to be traced and predicted across five levels of thinking. The following descriptors summarise the thinking patterns at each level of the framework:

1. *Pre-place value* requires the use of single units;
2. *Initial place value* is characterised by the movement from using only single units to using ten as a composite unit;

3. *Developing place value* extends the use of two-digit numbers to mental addition, abstract composite units of ten are used when counting, representing and comparing two-digit numbers;
4. *Extended place value* broadens the thrust to three-digit numbers, abstract composite units of one hundred and ten are used when counting, representing and comparing three-digit numbers, and
5. *Essential place value* requires that number sense be demonstrated through flexible approaches to numeric problems, including mental addition and subtraction with numbers to 1000, able to determine equivalent standard and non-standard representations of three-digit numbers.

It was shown from the case studies of twelve Grade 1 and 2 children, that the key constructs of the framework (counting, partitioning, grouping, and number relationships) were highly stable within each of the framework levels. The framework "provides the basis for generating instructional programs that build on children's prior knowledge, nurture their thinking through problem-focused experiences, and constantly monitor their understanding with respect to clearly enunciated expectations" (p. 334).

The framework was used to guide the planning, implementation and evaluation of mathematics instruction in multidigit number sense. The performances of children in two different implementations of the program were compared. The pedagogical basis for the instructional program was grounded in a socioconstructivist environment (Cobb, Wood, & Yackel, 1990; Cobb, Yackel, & Wood, 1992; Confrey, 1991; Vygotsky, 1978), involving challenging problems, children's own solutions, and collaborative problem solving. Forty children in two Grade 1 classes participated in the instructional program progressing through Grade 1 and the first semester of Grade 2. Although all four teachers involved were provided with training and support, the teachers on one class had more experience with the philosophy, intent and problem solving approach of the program. Assessment interviews were conducted with each child five times during the program. The items for the pretest at the beginning of the program were largely based on Level 1 of the framework. Subsequent assessment protocols comprised some of the initial items with additional items which increased with difficulty over time, reflecting Levels 2, 3 and 4 of the framework. Although the performances at all assessment times were in favour of the group with the more experienced teachers, these differences were not significant and were not uniform over the four assessment points. "There is evidence in this study that regular support, training, and reinforcement, which encourage teachers to be reflective practitioners, will have a powerful effect on instruction even when teachers have limited prior experience with an innovative program" (Jones et al., 1996, p. 334).

3.4.4 Projects on multiunit structures

There has been collaboration between a number of research projects which are using a problem solving approach to teaching and learning multidigit number concepts and operations. These classroom-based studies are trying new approaches that are hypothesised to support children's construction of accurate and robust conceptual structures for multidigit numbers and facilitate the use of these conceptual structures in calculations. In summary the projects are:

1. The Cognitive Guided Instruction (CGI) program, directed by Thomas Carpenter, Elizabeth Fennema, and Megan Franke, based at the University of Wisconsin;
2. A Conceptually Based Instruction (CBI) project, directed by James Hiebert and Diane Wearne, based at the University of Delaware;
3. A Problem Centered Mathematics Project (PCMP), directed by Piet Human, Hanlie Murray, and Alwyn Oliver, based at the University of Stellenbosch in South Africa, and
4. A Supporting Ten-Structured Thinking projects (STST), directed by Karen Fuson, based at Northwestern University.

All four projects take a problem-solving approach to teaching multidigit number concepts and operations. In all projects the teachers play an active role in the classroom by posing problems, coordinating the discussion of strategies used, and encouraging questions about the strategies. Fuson et al. (1997) reported on the progress made on understanding children's conceptions of multidigit numbers and their uses in addition and subtraction situations.

The Cognitive Guided Instruction Program (Carpenter, Fennema, & Romberg, 1993; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996) provides teachers with information about the different methods children themselves come up with to solve various kinds of problems. Children learn place value concepts as they explore the use of base-ten blocks and other base-ten materials to solve word problems and listen to other children explain their solutions.

Hiebert and Wearne (1992) reported on a numeration teaching program which they developed for Year 1 children. This conceptually based instruction program aims to help children construct connections between representations of number (physical, pictorial, verbal and symbolic) and to use all of the representations for recording actions on number. Different forms of representation of quantity highlight different aspects of the structure of number (e.g., grouping with physical and pictorial models, place value with symbolic models). Understanding numeration involves

... building connections between the key ideas of place value, such as quantifying sets of objects by grouping by 10, treating the groups as units ... and using the structure of the written notation to capture the information about groupings.

(Hiebert & Wearne, 1992, p. 99)

In Conceptually Based Instruction classes, the more traditional teaching sequence was followed; activities involving the development of place value ideas using base-ten blocks, then activities involving combining multidigit numbers. Children developed methods for multidigit addition and subtraction using base-ten blocks and their understanding of the meaning of written numbers.

In the Problem Centered Mathematics project, classroom activities primarily support children's construction of counting strategies (counting on, skip counting, double counting) to solve number problems. Base-ten blocks are not used because teachers in the past have used them to teach standard algorithms, but a range of other materials are used - discrete materials, bead frames, hundred square, posting cards, etc. Counting activities are extended as quickly as possible to three-digit numbers and a wide range of word problems are given from the beginning. It was found that some children who experienced difficulties with two-digit numbers performed with more understanding after exposure to three-digit numbers.

There were several Supporting Ten-Structured Thinking projects which moved more to supporting children's invention of multiple strategies. As well as standard English or Spanish words for two-digit numbers, expanded tens and ones phrases were used in later projects (to explicitly name the tens and ones and parallel the numerals). The early Target Algorithm Studies showed that the linked conceptually supported instruction resulted in considerably higher levels of correct multidigit addition and subtraction and higher levels of explanations of computational procedures using multiunit quantities than much of traditional school instruction (Fuson, 1986; Fuson & Briars, 1990). The Children's Invented Procedures Study showed that teachers need to support the linking of the multiunit objects and any written mathematical methods (Fuson et al., 1997; Fuson, Fraivillig, & Burghardt, 1992; Fuson, Smith, & Lo Cicero, 1997). Children's Math Worlds focussed on children's construction of arithmetical understandings using a wide variety of conceptual supports. All classes used tens and ones language as well as standard English or Spanish number words and all children added and subtracted two-digit numbers by making ten-sticks and unit drawings. These drawings were used in real-world problem contexts like a doughnut store, with doughnuts, boxes of ten doughnuts, and baking trays of a hundred doughnuts.

Drawing on the four projects, a common framework of conceptual structures that children construct was described. The collaboration produced the UDSSI model (Fuson et al., 1997) which describes a developmental sequence of conceptual structures of two-digit numbers that children use. The conceptions are unitary, decades and ones, sequence-tens and ones, separate-tens and ones, and integrated sequence-separate conceptions. Each of the conceptions involves a triad of two-way relationships between number words, two-digit numerals, and quantities. In all projects, most children understood the multidigit addition and subtraction methods that they used.

Discussion

These studies raise several issues for researchers and teachers. The first question is whether children need instruction with various representations of the numeration system before number is used in computation. Studies by Bednarz and Janvier (1985), Hiebert and Wearne (1992), and Jones et al. (1996) have used programs that build up understanding in a gradual way. Fuson and Briars (1990), Kamii (1989), and Carpenter et al. (1993) have shown that understanding of the numeration system can be developed through multidigit addition and subtraction. There is some evidence that sustained experience solving problems requiring multiunit addition before subtraction could be unhelpful (Olivier, Murray, & Human, 1990). Such a separation of addition and subtraction can lead to an incorrect generalisation of an addition solution method to subtraction. The second issue is whether an instructional approach should be used to support prechosen multidigit addition and subtraction procedures (Fuson & Briars, 1990) or whether a problem solving approach to support procedures invented by children should be used (Kamii, 1989; Labinowicz, 1985; Treffers, 1991).

It appears, from the success children have had in the studies discussed here, that the nature of the conceptual supports available in the classes have assisted the construction of conceptual structures for number. Although there were many differences in the teaching/learning approaches, all studies used word problem situations and emphasised discourse about the problem solutions. Fuson et al. (1997) suggested that they construct triads of connected conceptual structures that relate quantities based on ten, hundreds, etc. to number words and to numerals.

Recent research has focussed on teaching strategies and approaches appropriate to meaningful learning. The problem solving approach to teaching - where a major goal of instruction is the encouragement of intellectual autonomy and the development of relational thinking and problem solving - has been suggested as providing an appropriate environment for constructivist learning (Baroody, 1990; Cobb, Boufi, McCain, & Witenack, 1995; Cobb & Wheatley, 1988; Cobb, Wood & Yackel, 1990; Kamii, 1990; Kamii & DeClark, 1985; Pengelly, 1990; Ross, 1990; Sowder, 1992). Children's existing and relatively powerful informal knowledge is the basis from which learning takes place (Bentley, 1987; Carraher, Carraher, & Schliemann, 1987; Ginsburg, 1977; Hughes, 1986) and instruction should address points of cognitive conflict and facilitate resolution (Bentley, 1987; Swan, 1990).

Krauthausen (1996) asserted that rather than segmenting and administering bits of knowledge to children, the complexity of topics should be presented through an holistic, active-investigative approach and through productive practice.

3.5 SUMMARY

The research described in this chapter covers a wide range of studies concerned with children's understanding of number. Thompson (1982a, b) established the complexity of the construction of the early stages of the numeration system including detailed models for the conceptualisation of tens and hundreds. Jones et al. (1996) developed a framework for describing children's understanding of one to three-digit numbers. The framework comprised the constructs of counting, partitioning, grouping and number relationships, each operating at one of five levels of thinking in mutidigit number sense. Fuson (1990b) identified conceptual structures that children construct for four-digit numbers. These conceptual structures involve two aspects of the written number marks (the visual layout and the increase in value according to relative positional value from the right), and two aspects of spoken number words (the number names and the decreasing value as the names are spoken). The conceptual structures also involve six increasingly general and abstract quantity multiunit structures that give meanings to the written marks and spoken words: as collections of single units; as generated by a ten-for-one trade from the next smaller multiunit; as values of cumulative ten-for-one trade; as cumulative multiples of ten, and as exponential word expressions for the multiples of ten. These theoretical perspectives form the basis for understanding multidigit number sense. Counting, grouping, partitioning and number relationships are the constructs that are important for the development of understanding structure within the numeration system.

Although some research has been carried out on the teaching of structure and its consequences for children learning algorithms (Fuson, 1990a, b, 1992a, b; Resnick, 1983a), there has been little research on the construction of the structure of the decimal system. There have been assertions made about our lack of knowledge regarding how children develop understanding of the numeration system as a whole (Sinclair, Garin, & Tieche-Christinat, 1992; Thompson, 1982a). Jones et al. (1996) have indicated the need for further research to amplify and extend the framework that they developed to characterise children's thinking in multidigit problem situations.

This thesis addresses the need for direction in teaching/learning programs to nurture and assess children's growth in understanding and using multidigit numbers during the primary grades (Bednarz & Janvier, 1982, 1988; Carpenter & Fennema, 1990; Jones & Thornton, 1993; Jones et al., 1996; Fuson, 1990a; Kamii, 1990; Thompson, 1982a, b). There is a need for research to amplify and extend our understanding of how children construct and use the Hindu-Arabic system of numeration. This research will not only provide descriptions of children's thinking in multidigit situations, it will also provide the basis for generating instructional programs that build on children's prior knowledge, use appropriate constructing experiences, and provide indicators of growth in understanding.

While the research reported in this thesis is based on a developmental approach to numeration, the investigation makes no assumption that Piagetian stages will be shown. The research aims to blend theoretical and methodological perspectives from constructivist, developmental, cognitive processing, and representational thinking approaches to gain a more coherent picture of how children acquire numeration and place value knowledge. The aim is to broaden the research base by investigating skills and relationships necessary for understanding the structure of the numeration system.