

CHAPTER 7

DISCUSSION OF RESULTS

This chapter will provide an in-depth discussion of the results reported in Chapter 6. The qualitative and quantitative analyses are discussed in terms of what they indicate about the key aspects of children's development of structure of the numeration system. The results are then discussed in relation to other studies on numeration and place value.

This chapter is organised in two main sections as shown below.

7.1. A discussion of key aspects of the children's development of structure of the numeration system:

- (i) Recognition of place value structure;
- (ii) Arithmetic calculations drawing on place value knowledge;
- (iii) Significance of recursive groups by tens, and
- (iii) Understanding of the meaning of multiplication.

7.2. A discussion of results in relation to other studies organised under the six task categories:

- (i) Counting;
- (ii) Number sense;
- (iii) Grouping;
- (iv) Notation and place value;
- (v) Regrouping and mental computation, and
- (vi) Structure.

7.1 CHILDREN'S DEVELOPMENT OF STRUCTURE OF THE NUMERATION SYSTEM

Following the reporting of the results for each task in Chapter 6 it was considered appropriate to cluster some particular tasks and to synthesise the results for what they say about key aspects of the children's understanding of structure of numeration. This section will compare and discuss the recognition of place value structure, the way arithmetic calculations draw on place value knowledge, the significance of recursive groups of tens and understanding the meaning of multiplication.

7.1.1 Recognition of Place Value Structure

The first aspect will review key results pertaining to place value. Figure 7.1 shows children's performance on the various tasks which tested their recognition of place value

structure. These tasks simply asked children to recognise or represent a number or to use place value structure in counting. No calculation was involved.

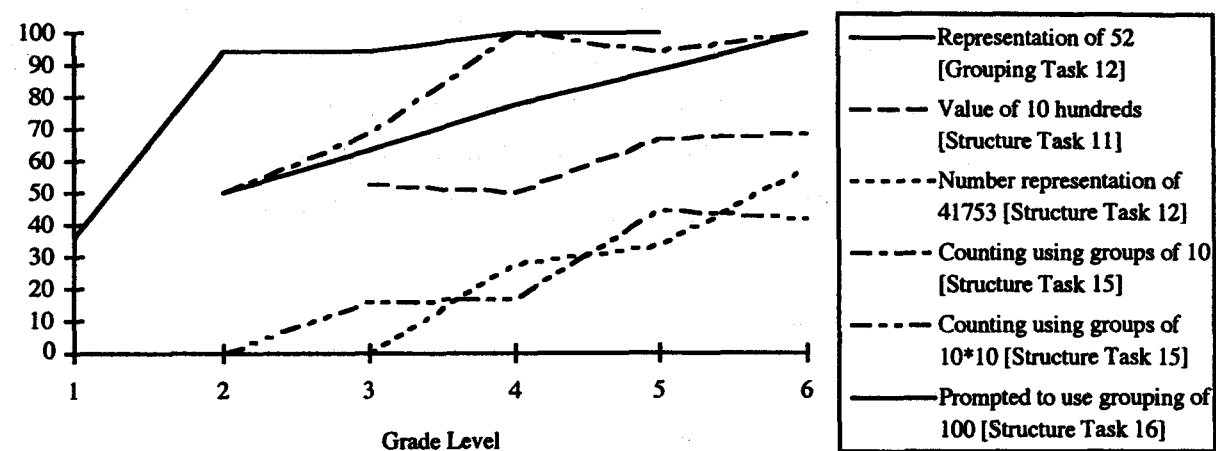


Figure 7.1: Percentage of sample correctly performing place value recognition tasks

By the end of Grade 2, most children could represent the 52 shells using the pregrouped material [Grouping Task 12]. Surprisingly a large number of children (increasing from 50% at Grade 3 to 68% at Grade 6) recognised that a box of lollies (containing 10 bags of 10 rolls of 10 lollies) held 1000 lollies [Structure Task 11]. The number of children who could recognise the number of lollies in a collection when they were also packed in cases of 10 boxes [Structure Task 12], grew steadily from none in Grade 3 to 58% in Grade 6. These findings are consistent with the results of Place Value Task 14, in which children were asked to identify the value of individual digits in given numbers. Children showed a high level of competency up to the thousands place (68%, 94%, 100% and 100% in Grades 3 to 6 respectively), but at Grade 6 there were still 42% who could not identify the ten thousands place. It would appear that most children master place value in numbers with up to 4 digits by Grade 3, but progress after that is much slower.

Performance on Structure Tasks 15 and 16 suggests a modification to the above conclusion. Given a picture of 144 marks grouped in tens (circled in black) with ten groups of ten circled in red, the number of children in Grades 2 and 3 who used the tens grouping to count the marks [Structure Task 15] was considerably fewer than the number who used the tens grouping in Grouping Task 12; Structure Task 15 did not become nearly as easy as Grouping Task 12 until Grade 4. The number of children who recognised and used the fact that the red circle enclosed 100 marks [also part of Structure Task 15] was much smaller, and consistently smaller than the number of students who recognised the structure of 1000 [Structure Task 11]. When prompted to consider the red circle [Structure Task 16] the number of children who used the enclosure of 100 marks was consistently below the number of those who used tens groupings in Grouping Task 12 or Structure Task 15.

(Unfortunately, there was no task in which students were simply asked to represent a three-digit number; but such a task would probably be easier than Structure Task 11). One explanation for these discrepancies is that students have learned to interpret certain concrete materials (bags, blocks, bundles, etc.) representing the number system but have not reached the general level of understanding needed to interpret unfamiliar groupings (circled marks) in the same way.

7.1.2 Arithmetic calculations drawing on place value knowledge

Figure 7.2 reports the performance of three calculation tasks where students could or must have used their knowledge of place value.

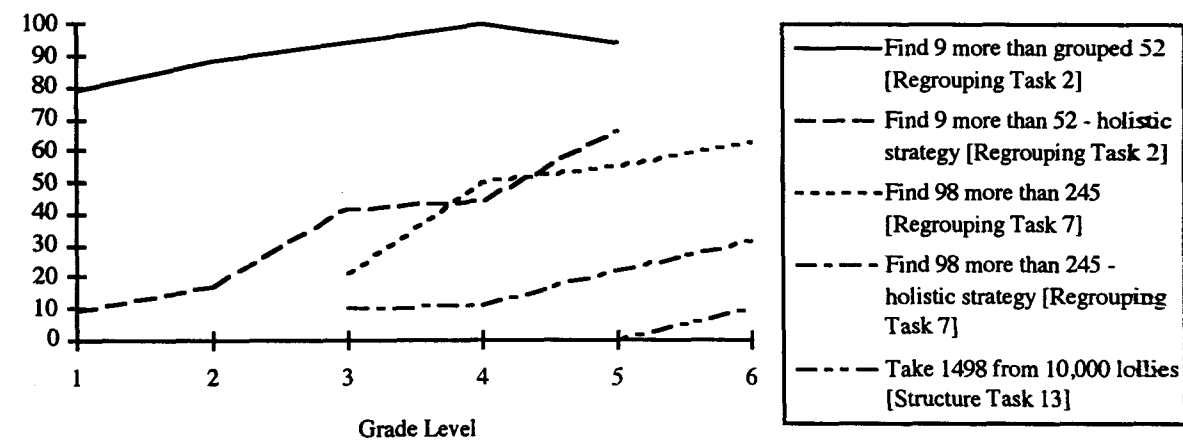


Figure 7.2: Percentage of sample correctly performing place value calculation tasks

By Grade 3 over 90% of the children were successful in adding 9 to the grouped collection of 52 shells [Regrouping Tasks 2]. However, up to Grade 4 the majority counted on 9 in ones and even in Grade 5 only 67% added 10 and subtracted 1 or otherwise made use of the base ten structure.

Many strategies were used for adding 98 to 245 in Regrouping Task 7, but most children counted on in tens and then ones (counting strategy) or separated one or both numbers into places values and added each part separately (separation and aggregation strategies). The number of children who took advantage of the base ten structure to find more efficient methods (holistic strategies) increased from 11% in Grade 3 to 32% in Grade 6. These children (especially those in Grade 3, who had no instruction in subtracting three-digit numbers) were probably using methods they had invented themselves.

Overall, very few children (none in Grades 3 to 5) were able to calculate the subtraction $10000 - 1498$ [Structure Task 13], although many more had apparently been able to recognise the value of a five-digit numeral [Structure Task 12].

These results suggest that most students learn to follow various procedures which are based on the place value system, but have not understood the system deeply enough to invent alternative methods when appropriate or to deal with larger numbers outside their common experience.

7.1.3 Significance of recursive grouping by tens

Several tasks sought to find if children would spontaneously group by tens recursively in order to “make counting easier”. These tasks (Structure Tasks 10, 11 and 14) are to be distinguished from the tasks discussed so far, where a grouping by tens was given by the interviewer. The results are shown in Figure 7.3. It will be noted that the second part of Structure Task 14 was only asked of children in Grades 4 -6.

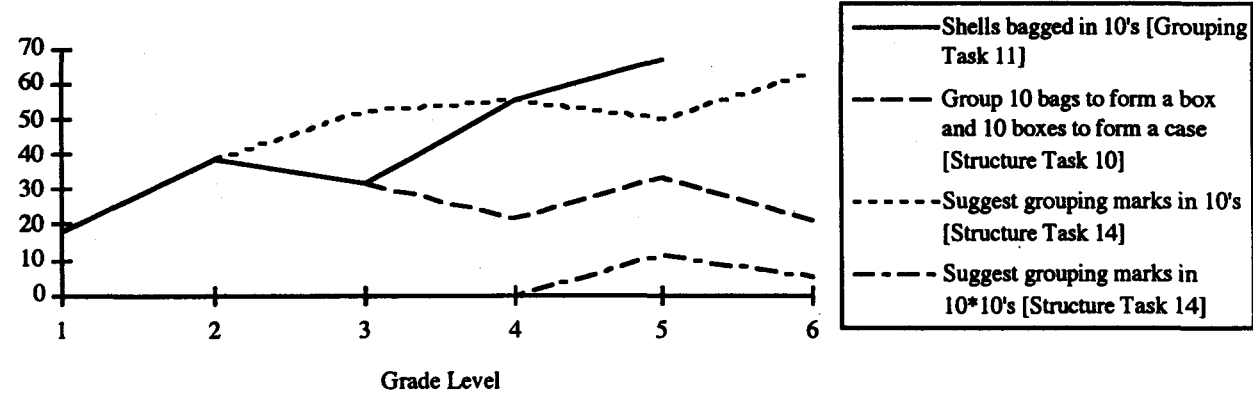


Figure 7.3: Percentage of sample suggesting grouping by tens in various tasks

The number of children suggesting grouping by tens for counting shells [Grouping Task 11], or counting marks [Structure Task 14], show a gradual increase from about 20% in Grade 1 to about 60% in Grade 6. A range of grouping numbers were considered appropriate. Most of those who suggested grouping by tens could not offer a reason for their choice. A notable exception was given by Andrew, who responded:

Andrew (Grade 1): *About ten in each ... then we only have to put 10 tens to make 100.*

Given the use in schools of a variety of concrete materials to model the place value system – all of which are, of course, based on grouping in tens – this is indeed a surprising result. It suggests that many children are not aware, even at the most basic level, of the purpose or usefulness of our place value system.

The number of children suggesting recursive grouping by tens for packing lollies [Structure Task 10] or counting marks [Structure Task 14] was even smaller. It is again surprising that so few children are aware of this fundamental characteristic of our numeration system.

Results from Structure Tasks 1 and 6 would also seem to confirm that many children are unable to use the structure of the numeration system effectively. By Grade 2, most children's drawings of the numbers 1-100 [Structure Task 1] presented either a linear or array structure, but there was little change in the mix of visualisations from then until Grade 6. Only 32% of the drawings in Grades 2 to 6 showed an array structure. In Structure Task 6, children were asked to use a hundred square to find the number to be added to 84 to make 100. The number of children who were able to find the answer without counting on in ones increased from 32% in Grade 3 to 79% in Grade 6.

If the array structure of the numbers 1-100 is poorly known, it is not surprising that students have difficulty with larger numbers where grouping by tens is recursively repeated.

7.1.4 Understanding the meaning of multiplication

The action of grouping by tens, so basic to the place value system, is closely associated with the operation of multiplication. Recursive grouping by tens is linked with repeated multiplication and/or the exponential function. Three tasks (Grouping Task 7 and Structure Tasks 11 and 22) related to students' understanding of multiplication. The results are shown in Figure 7.4. Note that Structure Tasks 11 and 22 were only asked of children in Grades 3 to 6.

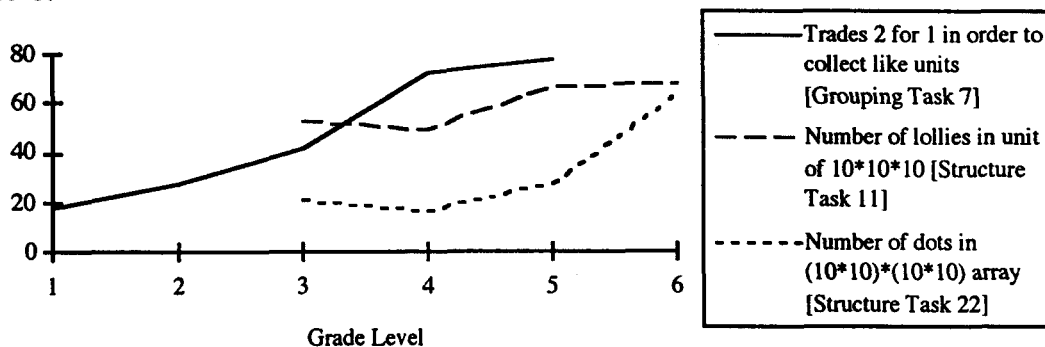


Figure 7.4: Percentage of sample correctly performing multiplication tasks

The performance of children on Grouping Task 7 (calculating the result of trading two stickers for one) increased steadily from Grades 1 to 5. This task involved a multiplicative notion of ratio (Mulligan & Mitchelmore, 1996b). This task is similar to that of relating the values of successive places in a numeral, and the shape of the curve showing the profile of success on this task (see Figure 7.4) resembles that of several of the place value tasks shown in Figures 7.1 - 7.3.

The task of calculating the number of lollies in 10 bags of 10 rolls of 10 lollies [Structure Task 11] has already been mentioned (see Figure 7.1): there is relatively little improvement between Grade 3 and Grade 6. Structure Task 22 involved a further recursion: the pattern

could be regarded as made up of 10 rows of 10 groups of 10 rows of 10 dots. Successful students invented several different strategies. For example, after they had determined that there were 100 dots in each square, some counting by 100s, 100 times; some counted by 100s to find that there were 1000 in the first row of squares and then counted in 1000s, 10 times; and some determined that there were 100 squares and multiplied 100 by 100. Most of the children in Grades 4 to 6 (89%, 89% and 100% respectively) recognised the pattern of 100s, but many (72%, 61% and 37%) could not complete the calculation, that is, they were unable to cope with the recursion. In these three grades, only about one third of the successful students used the most sophisticated strategy of multiplying 100 by 100.

Performance on Structure Task 22 vividly indicates that many children experience difficulties relating recursive grouping to multiplication. It may be conjectured that children have little experience with arrays or with multiplication. It is no wonder that they also have increasing difficulties coping with the place value system as the numbers become exponentially larger.

The key aspects of children's development of structure in the number system are the recognition and use of: place value structure, recursive grouping by tens, and multiplicative structure. Results discussed here have shown that by Grade 6 children have a good knowledge of place value up to the thousands place provided familiar representations are used. Groupings of ten are not understood as part of a recursive system and place value beyond the thousand's place is not used by many Grade 6 children. The number system is not well enough understood by children in order for them to use the structure in mental computation nor to invent their own alternative methods for calculating with larger numbers. Many Grade 6 children do not identify multiplication as a strategy for quantifying an array of objects. This lack of recognition of structure means that children have increasing difficulties coping with numbers as they become larger.

7.2 A DISCUSSION OF RESULTS IN RELATION TO OTHER STUDIES

The key aspects of children's development of structure of the numeration system are now discussed in relation to other studies. The discussion will be organised according to the categories of tasks that were used: counting; number sense; grouping; notation and place value; regrouping and mental computation, and structure.

The fundamental basis of numeration is the notion of treating a group as a unit (Cobb & Wheatley, 1988; Fuson, 1988; Steffe & Cobb, 1988). Hiebert and Wearne (1992) described children's understanding of numeration as "building connections between key ideas of place value such as quantifying sets of objects by grouping by 10 and treating the groups as units... and using the structure of the written notation to capture the information about groupings" (p. 99). Understanding of place value can be difficult and slow to develop for

children (Kamii & DeClark, 1985; Ross, 1989b). Boulton-Lewis (1993b) showed that children's levels of counting were significantly related to their knowledge of, and ability to explain, place value in the first three years of school. Other studies have focussed on structural aspects of numeration, identifying the child's ability to group and regroup composite units of ten and to relate this to a general structure of the base ten system (Cobb & Wheatley, 1988; Kamii, 1989). Jones et al. (1996) developed a framework for multidigit number sense which provides indicators that can be used to characterise children's thinking and to monitor their understanding with respect to clearly enunciated expectations.

Researchers have reported on the intuitive capacity of young children in the pre-school and early school years to construct meanings for numbers (Hughes, 1986; Kamii & DeClark, 1985; Ross, 1989a, b; Steffe, 1991a). As children's mathematics continues to develop through the primary school, they should acquire the mental images and connections that enable them to work with larger numbers and more complex mathematical operations in a meaningful way. However, research evidence has shown that these connections and understanding of procedures are often lacking in children's algorithms (Cobb & Wheatley, 1988; Fuson, 1990a, b; Kamii, 1986) and it is thought that this reflects a lack of understanding of the number system. Children appear not to transfer their initial understandings of the number system to operations with numbers and other aspects of mathematics.

There is extensive evidence (Fuson, 1986, 1992a, b; Fuson & Briars, 1990; Wearne & Hiebert, 1988) that many children do not make the connection between the partitioning of a number into tens and ones and the decomposition of numbers in written algorithms. Using multidigit numbers meaningfully requires understanding of how to compose and decompose these multidigit numbers into multiunit structures in which numbers are thought of as collections of objects or collections of collections of objects. It is suggested (Fuson, 1990b) that emphasis on two-digit number operations without trading, or delaying opportunities for interaction with numbers in the hundreds and thousands, encourages children to form misconceptions about the number system. The problem appears to be that many children base their number operations on the idea that numbers can only be added or subtracted, rather than an understanding of the number system based on multiplication.

7.2.1 Counting tasks

The results of the present study can be compared initially with those of Steffe and Wright. The children's responses from the first three counting tasks reported in Chapter 6 are analysed according to the stages of early arithmetical learning (Steffe, 1992; Steffe & Cobb, 1988; Steffe, von Glasersfeld, Richards & Cobb, 1983; Wright, 1991b, 1994a, 1996). For the purposes of this study the two stages of tacitly- and explicitly-nested number sequences

are combined as the advanced number sequence (Wright, 1996). Explanations of the stages are set out in Table 7.1.

Table 7.1: Stages of Early Arithmetic Learning

Arithmetical stages	Explanations
1. Perceptual counting	Children are limited to counting items they can perceive.
2. Figurative counting	Children count figural, motor or verbal unit items. They typically count from one in problem orientated contexts.
3. Initial number sequence	Children count abstract unit items and are said to have constructed numerical concepts. They typically count -on or down from a given number in problem orientated contexts.
4. Advanced number sequence	Children can conceptualise the numerosity of the whole and at least one part in a subtractive problem. They typically can choose the most appropriate subtractive strategy of counting-down-from or counting-down-to in problem orientated contexts. They also use holistic strategies such as compensation, relating to a known result, and bridging tens.

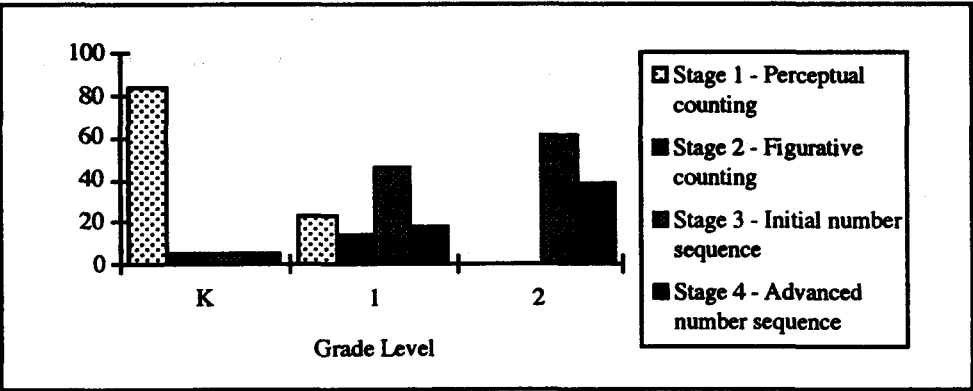


Figure 7.5: Stages of early arithmetical learning: Percentage of sample at each grade achieving the counting stages

It would be expected that the use of the advanced number sequence (stage 4) would soon become dominant in the higher grades (Figure 7.5). Wright's 1992 study (1994a) showed a strong growth in the attainment of higher stages by Kindergarten children involved in a year-long teaching experiment. The progress notably exceeded that of an earlier 1990 study where the children followed a regular instructional program. Interestingly, counting stages reached by children in the 1992 study after specially designed instruction showed 50% reaching the figurative counting stage, 35% reaching the advanced counting stage but only 6% reaching the initial number sequence stage. Wright (1996, p. 50) also reports relatively large gains were made by the children who were initially the most advanced. In the present cross-sectional study the percentage of children at the initial number sequence stage progressed from 6% at Kindergarten, 46% at Grade 1 to 61% at Grade 2. A strong reliance upon counting-on skills seems to have developed with children in the cross-sectional sample. However the children in Wright's teaching experiment who underwent accelerated learning as a result of instruction were more advanced and used more sophisticated counting strategies.

The basis for Steffe's investigations (Steffe, 1992; Steffe & Cobb, 1988; Steffe, von Glasersfeld, Richards & Cobb, 1983; Wright, 1991b, 1994a, 1996) is the counting scheme, which is not limited to counting by ones but can be elaborated to include composite units. Counting multiples of a given number also involves the construction of these composite units. Many children apparently develop the skills for counting by twos and fives in Grades 1 and 2 but this present study has shown that 94% of the Grade 2 children did not count by threes unless they used rhythmic counting (e.g. "one, two, three, ... four, five, six, ... " etc). Less than a third of Grade 2 children used their rhythmic counting skills to find the total number of objects in the collection. Double counting in conjunction with rhythmic counting probably involves a processing load that is too high for most children to cope with. In this study, a progression was shown from rhythmic counting the groups, without any means to keep track of the count of threes (at the end of Grade 1), to double counting (using fingers or at an abstract level) at the end of Grade 4. When the children completed the task at an abstract level, they either kept track silently or said they knew when to stop because "it has to be eighteen". There was shown to be very little difference in performance for rhythmic and double counting across Grades 4, 5 and 6.

7.2.2 Number sense tasks

An ability to recognise the part-whole relationships of numbers 5 and 10 enables these numbers to be benchmark numbers (Bobis, 1996) for developing flexible and creative strategies for calculating with number. The high performance levels of Grade 2 and 3 children in the novel situation of reacting to the incorrect written algorithm (Task 7), contrasted with their relatively low performance when using the part-whole relationships with ten and one hundred (Tasks 2, 3, 6, and 8).

It appears from the present study that most Grade 1 and 2 children (96% and 78% respectively) are reliant on unitary counting strategies to add small numbers. These children did not recognise how they could use the ten frames to combine the two collections of dots; they appeared not to see the relationships between the numbers as represented by the collections of dots and the 'ten spaces' of the ten frames. By contrast, in an intervention study Bobis (1996) reported that after a year of instruction all of the children in two kindergarten classes were able to represent numbers from 0 to 10 using the ten frame and to give explanations that revealed an understanding of part-whole relationships.

7.2.3 Grouping tasks

A fundamental idea in understanding numeration is the ability to think in groups. The flexibility in dealing with quantities is needed when children construct ten as a structure composed of ones. Using ten as an iterable unit (Steffe & Cobb, 1988) means that one can

solve tasks by counting by tens and ones abstractly, that composite units of ten are constructed or deconstructed as required in mental calculation. The idea of units and multiunits (or units of more than one) plays an important role in the structure of the numeration system. The way children deal with the units of one and ten influences their understanding of larger numbers (Cobb & Wheatley, 1988; Steffe & Cobb, 1988). A child who uses ten as a singleton unit might be able to recite the decade numbers (skip count in tens) but makes no sense of the increments of ten. The unit of one and the unit of ten may co-exist but they are not coordinated. Children who coordinate the units and multiunits can use these units in mental strategies for operations on larger numbers.

In a teaching experiment, Davydov (1982) showed that some first grade children were successful with a similar problem to the sticker problem (Grouping Task 7) but using containers of water that were either full or half full. Davydov's contention is that in this task children will only realise that the unit of measure is arbitrary and can be exchanged if they have a clear understanding of the abstract properties of quantity. It is suggested that children's difficulties with the operations on numbers can be explained by the assumption that all quantities are represented by units of one (a quantity of singleton units). The strategies of bridging ten, or adding ten and taking away one, rely on an ability to partition nine or ten in ways which make the combining sensible in terms of the numeration system and so children simultaneously construct these numbers as composite and singleton units. It is inferred that ten has been used as an iterable unit (Steffe & Cobb, 1988; Cobb & Wheatley, 1988) because the unit of ten, as well as the unit of one, are taken for granted without the need to count.

It appears from the present study that the children are familiar with the use of grouping materials to represent number but many rely on unitary counting and do not have the partitioning skills to be flexible in their manipulation of number. Similar to the results in this present study, Ross (1986) reported children at Grades 2, 3 and 4 having difficulty with showing a non-canonical representation using Dienes blocks. Although this sample of children was more adept with using the pregrouped material to represent a 2-digit number than those children in the Ross sample, they had greater difficulty functioning simultaneously with the system of ones and the system of tens. This was shown by the poor performance in finding non-standard representations.

For the uncovering tens and ones task (Grouping Task 13) in the present study the percentage of children who successfully counted-on by tens and ones as appropriate steadily increased through the Grades 1 to 4 (9%, 39%, 53% and 89% respectively). Labinowicz (1985) reported in a related task (where there was only a ten or group of ones uncovered at any one time) that 12 out of 29 Grade 3 children (41%) had successfully counted using the different units as needed when using Dienes' blocks. Some of these children had been given

a prompt to remind them that a 'long' was a 'ten' group. As with the Labinowicz study, children in the present study showed a range of unsuccessful strategies. Many children attempted to count all by ones or count-on from ten by ones (64% of Grade 1 children), and sometimes by counting the number of single units in a 'long' unsuccessfully. Only one child miscounted ones as tens. The majority of Grade 1 children could only count-on successfully by ones. Labinowicz (1985) stated that "they are likely to deal only with the face value of digits in different columns and to find answers by following a procedural rule for computation" (p. 269). Cobb and Wheatley (1988) reported that on the same uncovering task with a group of 14 Grade 2 children, five of the 14 children (36%) counted the tens and ones separately (ten as an abstract singleton unit) whereas only 3 children in the present study (21%) coordinated counting by tens and ones (ten as an abstract composite unit). This compares with 33% and 39% in this present study. In a similar task in the Labinowicz (1985) study, only five Grade 3 children (17%) were successful without reverting to count smaller units. A majority of children in the grades below Grade 4 could not coordinate the counting of hundreds, tens and ones and so could not deal with the multiple meanings of a hundred and ten. Labinowicz (1985) would predict that these children had a difficulty with using three-digit place value notation.

7.2.4 Notation and place value tasks

As the traditional teaching of numeration in Grades 1 to 3 focusses on the ability to read and write numbers and to point out place values of digits in any number up to one thousand, it is not unexpected that the children would show a high level of competency with recognising place values from ones to thousands. Clearly the children were much less competent when identifying place value of digits outside this range. Teaching practice in each of the schools in this study emphasised the use of Base 10 Material (Dienes blocks) which has four different sized blocks that are almost exclusively used to represent ones, tens, hundreds and thousands. Strong emphasis is put on handling the material and the transition from the material to conventional notation. When children are judged sufficiently advanced in their learning of numeration, instruction focusses on computation and operations using concrete Base 10 material, pictures of these materials and the conventional algorithms. However, despite the use of these materials, many children in this present study did not develop even a basic knowledge of the numeration system outside the ones to thousands range by the end of Grade 6. It is possible that the emphasis on Dienes blocks as classroom materials and the limited interpretation of the component pieces in the instruction process influenced adversely the development of structure of the numeration system beyond the thousands place.

In order to investigate children's understanding of place value, Ross (1986, 1989b) probed the meanings children attribute to two-digit numerals through a series of digit-correspondence tasks. It will be recalled from Chapter 3 that an earlier study of 60 second

through fifth grade children (Ross, 1986), 26 (43%) children were successful in giving the meanings of the digits in the number 25 when they were shown a collection of 25 sticks (not grouped in any way). The children who were successful described in a variety of ways that the 5 represented five of the sticks and the 2 represented the other twenty. Another twelve children thought that the individual digits had nothing to do with how many sticks were in the collection, fourteen described invented numerical meanings and eight thought that the digits were represented by the quantities of their face values. It was found that when the task was altered by representing the number 52 with a standard place value partitioning of base-ten blocks (5 longs and 2 shorts) many more of the children (44 out of the 60) were successful. When 52 was represented using 4 longs and 12 shorts in a further task, the number of successful children dropped to 20. Similar results were found in two tasks where 48 beans were partitioned in standard and non-standard partitionings.

A follow-up study (Ross, 1989a, b) was designed to examine whether some children did in fact use the face value interpretation to assign meaning to individual digits. In this task 30 Grade 3 children each grouped a collection of 26 objects into fours giving 6 groups of four and 2 left over and then were asked to give the meanings of the individual digits in terms of the material in front of them. Nearly half of the Grade 3 children related the digits to the matching prominent groupings and so incorrectly responded that the 2 in 26 stood for two of the objects and the 6 represented the six groups.

This present study (Place Value Task 1) found that only 24 out of the 95 (25%) Kindergarten to Grade 4 children correctly gave total value quantities to each of the digits when asked the same digit correspondence task (non-standard groupings in 26). Only four children (4%) used the face value response of relating digits to the numerically related observable groupings but a further 55 children (58%) used another face value response where the meaning of each digit was given by the quantity of the face value of that digit (2 shells and 6 shells) and the remaining objects did not account for any meaning.

Although there were some differences in the way children responded to digit correspondence task over the three studies, it appears that it is not until Grade 4 at the earliest, that a majority of children have a clear understanding of the meaning of individual digits in two digit numerals. This must have implications for the development of understanding of the more complex larger numbers and for the ability to undertake mental computations or understand written algorithms.

Many young children have difficulty with writing the numerals of numbers with zeros and in particular, interpreting the role of zero as a place holder. The present study resulted in 64% of Grades 1 through 4 children identifying '01' as one or first (Place Value Task 2) but only 22% could give the meaning of zero as no tens. This compared with 74% of Grades 1

through 4 children in the Sierink and Watson study (1990) who identified '01' as one. However the Sierink and Watson study did not ask children for an explanation of the meaning of zero in the given label. Grades 1 and 2 children in the present study (73% and 50% respectively) were much more likely to interpret '01' as ten or were unsure, than in the Sierink and Watson study (43% and 23% respectively).

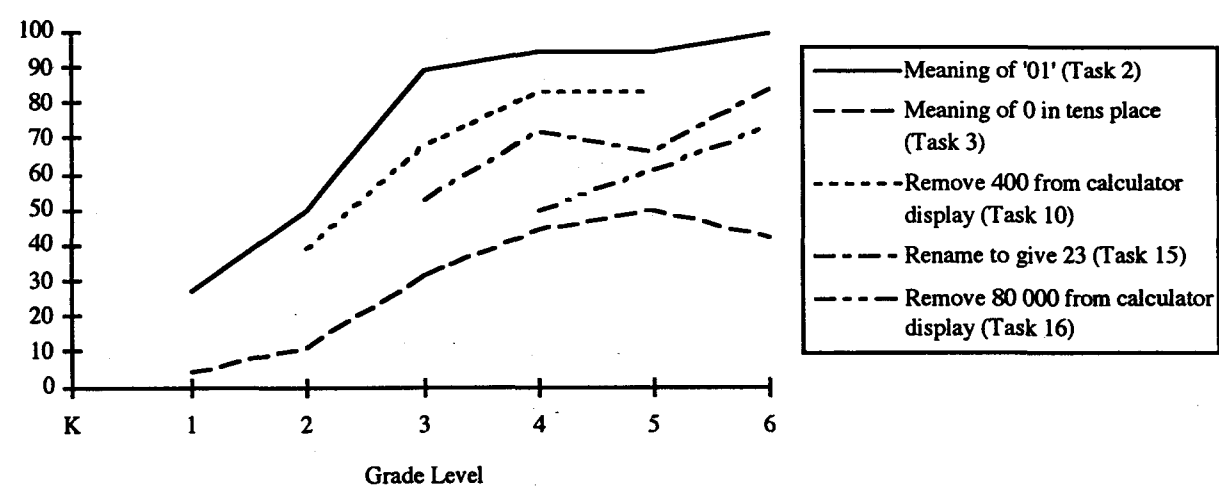


Figure 7.6: Comparison of performance: Percentage of sample correctly interpreting digits or numerals on Place Value Tasks 2, 3, 10, 15 and 16

Figure 7.6 compares the performance of children on five digit and numeral interpretation tasks (Place Value Tasks 2, 3, 10, 15 and 16). It can be seen that there was greater difficulty experienced by children with explaining a meaning for the zero as a place holder than any of the other tasks. Whereas success on Tasks 2, 10, 15 and 16 increased for children through the grades, success on Task 3 (meaning for the zero as a place holder) decreased to only 42% for Grade 6 children.

7.2.5 Regrouping and mental computation tasks

Various research studies (Baroody, 1985; Beishuizen, 1993; Cooper, Heirdsfield & Irons, 1996; Heirdsfield, 1995; McIntosh, 1996; Thornton, 1978, 1990) have shown that children use a variety of strategies in mental computation. Cooper, Heirdsfield and Irons (1996) reported that children used a variety of strategies and that these strategies changed across the interviews in the two year longitudinal study of 104 children starting at the beginning of Grade 2 (6 interviews). When solving 2-digit computation with regrouping, right to left separation overtook left to right separation or aggregation as the most successful strategies used by the children; aggregation strategies were seldom used.

For Regrouping Tasks 4, 5, 7, 10 and 11 in the present study, separation strategies were generally preferred over aggregation strategies. However, a substantial number of children

in Grades 4 and 5 used aggregation strategies, including both right to left and left to right progressions. Unlike earlier studies which used verbal or written questions (Beishuizen, 1993; Cobb & Merkel, 1989; Cobb & Wheatley, 1988; Lindquist, 1989; Reys, Reys, Nohda, & Emori, 1995), or word problems (Carraher, Carraher & Schliemann, 1985; Cooper, Heirdsfield & Irons, 1996; Heirdsfield & Cooper, 1995), this present study required children to use a physical model for at least one of the numbers in the computation which could have encouraged some children to use aggregation. This meant that children focussed on one number and split the other in order to combine (or separate) the numbers.

7.2.6 Structure tasks

Children's progression to using two-digit numbers involves the new dimension of place value and is the start of the development of an understanding of number as part of a system. Whereas 'one' had been the basis of an unitary counting sequence now 'ten' is also an iterable unit (Fuson, 1990a, b; Jones, Thornton & Putt, 1994; Steffe & Cobb, 1988). As the number of digits increase the 'powers of ten' also become iterable units in the system. Although many studies (as outlined earlier in this chapter), have investigated grouping, regrouping and place value as related to two- and three-digit numbers there has not been a sufficient focus on how this system is consistent and infinitely extendable. The linguistic complexity of the numeration system for English-speaking students has been well documented (Fuson, 1990b; Labinowicz, 1985; Kamii & Livingston, 1994) and comparisons made with that of other language groups (Bell, 1990; Fuson & Kwon, 1992; Miura, Okamoto, Kim, Steere & Fayol, 1993). The development of understanding the numeration system is a complex process involving the construction of multiunit conceptual structures (Fuson, 1990a, b; Labinowicz, 1985; Kamii & Livingston, 1994; Steffe & Cobb, 1988) and the formation of connections between these structures and the corresponding symbols.

It was found that at Grades 2 and 3 there were a substantial number of children (50% and 32% respectively) who were not successful in recognising and using groupings of ten to quantify a collection (Structure Tasks 14, 15 and 16). Bednarz and Janvier (1988) reported a higher percentages of Grade 3 children (39% and 41% in two classes) than in this present study, not seeing the use for the marked groupings of ten from their control groups in a comparative teaching experiment. However, only 4% of their teaching experiment group (twenty three Grade 3 children) were unable to solve successfully the pertinence of grouping task after 3 years of participating in the class with a constructivist teaching approach. They also reported that 30% of the experimental group were unable to recognise and used two forms of grouping compared with none of the control groups. This compares with 16% of the Grade 3 sample in this present study who were unable to recognise and used two forms of grouping. Almost half of the Grade 3 children could recognise the pattern of hundreds in

an array of dots (Figure A.6, Appendix A) and 21% could use the multiplicative structure as represented in the array to quantify the dots. For example, these children recognised the pattern of hundreds, counted these as multiunits in tens or recognised the relationship to a pattern of thousands, and then counted to give ten thousand. This shows how some Grade 3 children have developed an intuitive understanding of the way ten thousand is structured as ten to the power of four.

7.3 SUMMARY

It appears from this present study that many children in Grades 1 to 6 are familiar with concrete materials used to represent grouping of numbers, but still rely on unitary counting. They may show good performance on 2-digit calculations, but generally use poor methods and cannot extend their success to numbers with larger numbers of digits. There is in general a weak awareness of structure and, in particular, of the multiplicative nature of this structure. Nevertheless, some children acquire a good understanding of place value and develop their own efficient strategies spontaneously, confirming many similar findings among younger children (e.g., Cooper, Heirdsfield & Irons, 1995).

The results emphasise the importance of units and multiunits (units of more than one) in understanding the structure of the numeration system. The way that children deal with the units of one and ten influences their understanding of larger numbers (Cobb & Wheatley, 1988; Steffe & Cobb, 1988). A child who uses ten as a singleton unit might be able to recite the decade numbers (i.e., skip count in tens) but makes no sense of the increments of ten; the units of one and ten co-exist but are not coordinated. Only children who can coordinate the units and various multiunits can use these units in mental strategies for operations on larger numbers.

In the present study, there were a substantial number of children in Grades 2 and 3 who were not successful in recognising and using groupings of ten to quantify a collection of objects. Bednarz and Janvier (1988) report a similar high percentage in regular Grade 3 classes. Children will also not realise that multiunits are related and can be exchanged if they do not understand the abstract properties of quantity (Davidov, 1982), one of the conceptual underpinnings of multiplication. In a recent study, Clark and Kamii (1996) reported that although some children develop multiplicative thinking as early as Grade 2, most children still cannot demonstrate consistent thinking in Grade 5. The present study confirms these results. A substantial minority (about 20%) of the Grade 3 students had developed such an intuitive understanding of powers of ten that they could use the recursive multiplicative structure of the array of 10,000 dots to count the number of dots successfully. But by Grade 6, there was still a significant number who could not even count 10 groups of 10 groups of 10.

Children do not appear to make the connections between the key elements of the structure of the number system. The ideas of partitioning and grouping, the additive and multiplicative relations, the conceptualisation of place value and the conventions of notation are all basic to the structure of the number system. Most Grade 6 children have not made the connections between these elements which enables the generation of multiunits by the recursive relation of 'grouping by tens' and the corresponding generation of the notation system.

The following chapter will provide an in-depth descriptive analysis of children's representations of the number sequence 1-100 (Structure Task 1).

CHAPTER 8

CHILDREN'S REPRESENTATIONS AND STRUCTURAL DEVELOPMENT OF THE COUNTING SEQUENCE 1-100

It will be recalled from Chapter 1, that this thesis is primarily concerned with children's understanding of the structure of the numeration system and how critical aspects of counting and grouping relate to the base ten structure. Children's proficiency with using the counting sequence depends on their awareness of the patterns within this sequence. As discussed in Chapter 3, Rubin and Russell (1992) considered using multiples of ten and one hundred as key landmarks in the number system. The overall structure is a recursive structure based on the pattern of tens within the sequence 1 to 100. That is, at the next level of structure, we have the pattern of hundreds within the sequence 1 to 1000, and so the system extends by building on the powers of ten.

This chapter presents an in-depth, descriptive analysis of children's representations of the number sequence 1-100. It will be recalled that this was one of the tasks administered in the pilot study, and because the subjects produced a range of very rich and unconventional imagistic responses, it was considered important to include this task in the main study. The task was also administered to a sample of Grade 4 to 6 high ability children in an exploratory study which investigated links between their understanding of the numeration system and their representations of this number sequence (Thomas & Mulligan, 1995). Analysis of data from both samples was carried out in order to explore how the children structured their representations of 1-100 in developing understanding of numeration. Before discussing this analysis, a report is made of pilot classroom-based work carried out by the researcher which explored how a group of Grade 2 children conceptualised the pattern of tens in the counting sequence 1 to 100.

8.1 BACKGROUND

It will be recalled from Chapter 3 that using ten as an abstract composite unit is critical to children's understanding of the number system (Cobb & Wheatley, 1988; Steffe & Cobb, 1988). The discussion of research identified that many children have difficulty using more than one grouping of number when counting (Denvir & Brown, 1986a, b), and that they often do not understand the structure of tens and hundreds (Thompson, 1982a, b). It is apparent that the numeration units of ten and one hundred are each constructed separately. Also children's ability to correctly read and write numerals and to name the number of tens or hundreds in a numeral have little relationship to their knowledge of these numeration

units. Thompson states that a shortcoming of his study was that there was no consideration of how children might construct a general understanding of whole number numeration. He went on to reflect that at some point children might establish an operational routine for creating numeration units "so that the need for an elaborate construction of each succeeding numeration unit is obviated" (p. 324).

The importance of imagery in the construction of mathematical meaning has been investigated by various researchers (Brown & Presmeg, 1993; Brown & Wheatley, 1990; Presmeg, 1992; Reynolds & Wheatley, 1992; Thompson, 1996; Wheatley & Brown, 1994). Imagery is much more than the creation of mental pictures. Piaget and Inhelder (1967) distinguished three types of images: images associated with the creation of objects; images recording the process and outcomes of actions, and images that support reasoning. Pirie and Kieren (1992, 1994) suggest that the act of imagining itself influences our images. Thompson (1996) asserts that "mathematical reasoning at all levels is firmly grounded in imagery" (p. 267). Thompson further suggests that the two aspects of imagery that have a significant influence on the development of mathematical reasoning are a child's immediate understandings of an action or situation, and his or her development of mental operations. It is suggested that "internal, imagistic representation is essential to virtually all mathematical insight and understanding... interactions with external, imagistic representations are important to facilitating the construction of powerful internal imagistic systems in students" (Goldin & Kaput, 1996, p. 415). The role of dynamic imagery in mathematical reasoning has also received much attention (Presmeg, 1992; Brown & Wheatley, 1990; Thomas & Mulligan, 1995).

Based on the idea that the relationship between different forms of representation can be seen through the presentation and solution of arithmetic facts (Dehaene & Cohen, 1995), Gray, Pitta and Tall have studied the role of imagery in basic number processing. They conclude that qualitatively different outcomes arise from numerical processing because children concentrate on different objects of thought or different aspects of the objects. Children choose what imagery to create from concrete and mental activity and it is suggested that this not only has consequences for the quality of the actions that follow but also affects the quality of the object which dominates the child's imagery (Gray & Pitta, 1996; Gray & Tall, 1991, 1994; Pitta & Gray, 1996). While this notion is important in terms of the data reported here, these studies were not available at the time, and so did not influence the analysis that was carried out.

8.2 EXPLORATORY STUDY

An exploration of children's visualisation of the number sequence 1-100 began with a classroom study on investigating how the pattern of tens within the counting sequence was conceptualised. This exploratory study was part of a program of professional development for practising teachers from one Catholic Primary School in country New South Wales. Children's work samples and anecdotal records were collected in order to evaluate mathematical understanding. These classroom recordings were used as evidence of understanding in numeration and focussed on identifying representations that children used to show the pattern of tens in the counting numbers. A class of Grade 2 children were asked to think about the pattern of tens in the numbers 1 to 100. They were then asked to draw a picture of what they saw. All children in the class participated in the study. The representations that children used influenced the researcher's thinking about the significance of children's representations in their conceptualisation of the structure of the number system. The following three drawings illustrate how the researcher was able to infer aspects of the children's understanding of number from their external representations.

Emilie (Figure 8.1) showed the multiples of ten, each illustrated with tally marks. The pattern of tens is interpreted as counting by tens and each multiple of ten is illustrated with the appropriate number of tally marks which were meticulously counted and recorded. Although Emilie was familiar with the multiples of ten she used unitary counting in her recording. There is no notion of 'units of ten' within the recording.

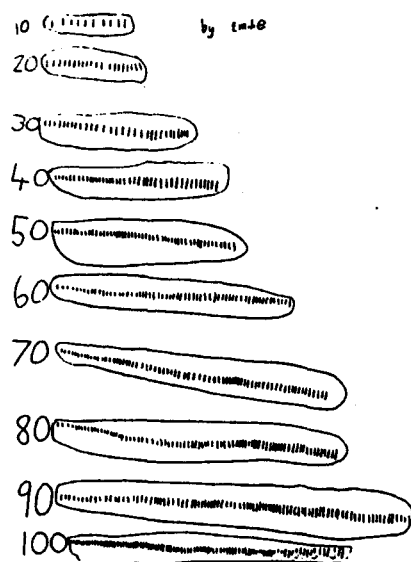


Figure 8.1 Emilie (Grade 2)

Evan (Figure 8.2) showed the 'add ten' relationship in a sequence of number sentences which elaborated each of the vertically recorded multiples of ten. The pattern of tens was shown with this quite sophisticated sequence of number sentences, each one showing a

connection between consecutive multiples of ten. In attempting to illustrate an 'adding ten' relationship for 100, the last number in the sequence, Evan incorrectly made an increment of another hundred to give two hundred. In contrast Victoria (Figure 8.3) related her imagery to classroom experiences of drawing arrays, although she actually showed 7 rows of 9 blocks in a grid. Victoria illustrated the pattern of tens as a nine by seven array which showed a sense of 9 groupings (columns) of groupings of 7 (rows). This grid pattern relates to classroom experiences with the hundred square but without accuracy in drawing units of ten.

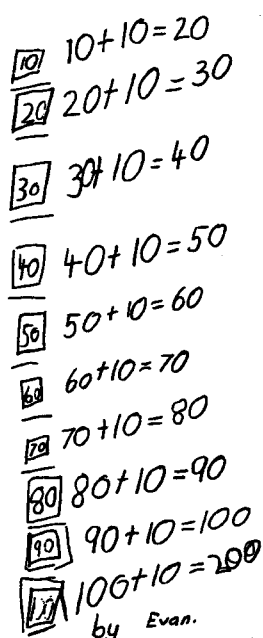


Figure 8.2 Evan (Grade 2)

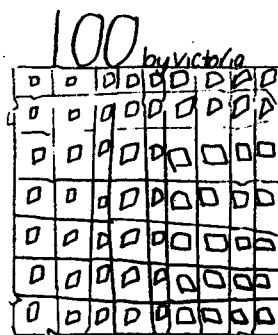


Figure 8.3 Victoria (Grade 2)

These representations of the pattern of tens show diversity in children's thinking about the way ten becomes an iterable unit within the counting sequence. The external representations which are drawn and explained reflect a variety of ways that children 'see' grouping in equal lots of ten as part of the number sequence. It should be noted though, that the majority of the class did not illustrate the pattern of tens in their recordings, with many children not giving any recordings. It is inferred that these children did not have any imagery to describe pattern within the number sequence and either said there was no pattern or recorded part of the unitary number sequence.

Because of the richness of the drawings produced by some of these young children, it was deemed useful to include this visualisation task in a larger Pilot Study (Chapter 4). Further, as described in Chapter 5, the cross-sectional study also included visualisation tasks for the number sequences 1-100 and 1-1000. A range of visualisation tasks was also included in another related study incorporating a teaching program for groups of high ability children attending enrichment classes.

From the initial analysis of the Pilot Study data, it was considered that the analyses of children's representations could be based on the model for children's problem-solving competency structures proposed by Goldin (1983, 1987, 1988, 1992a, b). After preliminary coding and discussion about the representations of numbers 1-100 in terms of pictorial, ikonic, notational and structural aspects, the researcher discussed this analysis with Professor Gerald Goldin of Rutgers University in July 1993. Following his advice, the analysis of representations remained essentially in these categories, however it was considered important to distinguish between internal and external representations. In particular, the imagery given in the children's responses was able to be explained in terms of external signs from which inferences could be made about internal imagistic systems of representation. Goldin's model for children's problem-solving competency structures is further described in the next section.

8.3 ANALYSES OF CHILDREN'S REPRESENTATIONS OF THE NUMBER SEQUENCE 1-100

The analysis is based essentially on a model for children's problem-solving competency structures proposed by Goldin (1987, 1988, 1992a, b). However the methods of analysing imagery in numerical representations integrates aspects used by other researchers (Brown & Presmeg, 1993; Brown & Wheatley, 1989, 1990; Mason, 1992; Pirie & Kieren, 1994; Presmeg, 1986a, b). Although these researchers analyse data from different perspectives, they agree that imagery, in a variety of forms, plays a critical role in developing mathematical understanding. Research associated with children's representations has been discussed in Chapter 3 but will be summarised here to preview the discussion.

Goldin's model distinguishes cognitive representational systems internal to problem solvers (a theoretical construct to describe the child's inner cognitive processing) from (external) task variables and task structures (Goldin & McClintock, 1984). This model is intended to help organise and explain observations of student actions, drawings and descriptions, or explanations, rather than to be a model of the mind (Goldin & Kaput, 1996). The construct of representation enables the description of what children can and cannot do; it helps the analysis of structural properties in mathematics and the effects due to the media in which external configurations are embodied. "One of the major challenges that theories of mathematics learning based on representation should address is that of modelling (the) constructive process, understanding the characteristics of external and internal representations that affect it, and facilitating its effective occurrence in students" (Goldin & Kaput, 1996, p. 409).

Goldin further proposed that learning goals should be formulated in terms of the kind of internal systems of representation that support powerful problem solving. For example,

children are expected to 'internalise' our base-ten system of numeration and the associated processes for operations. They are expected to generalise and extend these understandings and processes to decimal fraction and algebraic systems. In order to do this, Goldin asserted that children need to develop powerful imagistic, heuristic and affective systems. The purpose of mathematical education is "to develop broad, powerful cognitive systems that can enable the student to grapple with new situations as they arise, to represent them internally in a variety of ways, and to think mathematically by making use of the representations" (Goldin, 1992b, p. 83). Goldin proposed that these internal systems develop through the three main stages of: inventive-semiotic; structural development, and autonomous stages. Children's cognitive structural development of numeration can be described by this model.

In the structural development of the number system, the system of representing units (1's) must serve to drive the representation of assemblages partitioned into groupings of ten. The "ten", while still remaining ten ones, becomes an iterable "unit of ten". Similarly a system of "hundreds" is later constructed on the system on tens, and so forth (recursively). Children's conceptual structures for number words are now "multiunit conceptual structures in which the meanings or referents of the number words are collections of entities ... or a collection of collections of objects" (Fuson, 1990a, p. 273). This process is not just a verbal or notational one; the role of imagery in it is essential.

Several other researchers (Mason, 1992; Pirie & Kieren, 1994; Presmeg, 1986) have also emphasised the importance of imagery and the development of imagistic systems. When analysing the role of imagery in the development of children's conceptual understandings, Presmeg (1986) identified five main types of visual imagery used by students:

- (i) concrete, pictorial imagery (pictures in the mind);
- (ii) pattern imagery depicting pure relationships;
- (iii) memory images of formulae;
- (iv) kinaesthetic imagery involving muscular activity, and
- (v) dynamic (moving) imagery.

In describing how children use spatial stimulus, Mason (1992) distinguished between images that are eidetic (fully formed from something presented), and those that are constructed (i.e. built up from other images). The meaning-constructing process continues as the 'mental picture' is described, drawn, compared and discussed. He suggests that for students to access images they must actively process them, "looking through" rather than "looking at" the "mental screen", regardless of the mode of external representation. Imagery is also an important aspect of Pirie and Kieren's (1994) model for the growth of mathematical understanding. They describe 'image making' and 'image having' as early levels of understanding which can be observed. 'Image making', is the development of

mental representations from initial learning experiences. At the next level of 'image having' a learner is able to manipulate and use the image in mathematical thinking.

Personal visuo-spatial representations of number (number-forms) were described long ago by Galton (1880). Seron, Pesenti, Noel, Deloche & Cornet (1993) suggest that the number-form is a more accomplished development of a general disposition of people to encode numbers in a visual way. They conclude that number-forms are used to code the number sequence, and that the function (if it exists) of this phenomenon should be examined in number and calculation processing. Dehaene (1993) further proposed a functional model of number processing. This triple-code model assumes that there are three categories of mental representations in which number can be manipulated in the human brain. The first is a visual arabic number form, in which numbers are represented as strings of digits on an internal visuo-spatial scratchpad. Second there is a verbal word frame, in which numbers are represented as syntactically organised sequence of words. The meaning of numbers is represented only in the third category of the model, the analogical magnitude representation. Here the quantity or magnitude associated with a given number is retrieved and can be put in relation with other numerical quantities.

The studies of students' representations cited above, indicate consistently that students use imagery in the construction of mathematical meaning. With a theoretical base provided by these researchers, we next describe the study and some of its outcomes. The observations of child actions are interpreted with respect to the developing theoretical model for mathematical learning and problem solving based on characteristics of representations. The children's structural development of numeration is examined through their spontaneous representations of the counting sequence.

8.4 METHOD

8.4.1 Sample

Data presented in this chapter are selected from children's responses in the pilot study and main cross-sectional study. As well, some examples are drawn from a high ability sample consisting of 92 children from Grades 3-6, assessed by teachers for participation in a Program for Gifted and Talented students from 84 country and city schools in NSW. The children were participants in a mathematics enrichment program conducted by the researcher over five two day workshops over a three year period. The workshops were conducted on weekends in May 1992, February, July and October, 1993, and May 1994. All the students were selected by classroom teachers on the basis that they were high achievers in

mathematics within their school. Students were interested in participating in the workshops and displayed positive attitudes to mathematics learning.

8.4.2 The Visualisation Task

It will be recalled that in one of the visualisation tasks, children were asked to close their eyes and to imagine the numbers from one to one hundred. Then they were asked to draw the pictures that they saw in their minds. They were also asked to explain the image and their drawing. The visualisation task was asked first, prior to other numeration tasks, so that responses could not be influenced by representations used by the researcher in other tasks. The analyses of the visualisation task in the Pilot and Cross-sectional studies are discussed in Chapters 4 and 5 respectively.

8.4.3 Procedure

Children in each of the three samples were interviewed individually on selected numeration and visualisation tasks. Procedures for the Pilot and Cross-sectional studies have been described in detail in Chapters 4 and 5 respectively.

After the high ability sample children had been interviewed individually on the visualisation task, they were questioned verbally about their attitudes to learning mathematics. Each child was encouraged to draw or explain in writing his or her mental image. If the child did not show any array type structural features in their recording, the researcher asked the child to think of the numbers from 1-100 in rows and columns, and draw or describe that image. Every child made an attempt to explain their thinking and no interviews were terminated. A selected set of eight numeration tasks was administered to each group of children at each workshop. Each group comprised approximately 15-16 children who were seated individually in a random pattern. The children were provided with paper and pencils, and told that they could record their thinking and their responses if they chose to do so. From the external representations produced by the high ability children, dynamic aspects of their internal representation were inferred, and from this in turn some description of the structural development of the system that has taken place.

8.5 ANALYSIS OF CHILDREN'S REPRESENTATIONS

In order to describe individual representations of number in as much detail as possible, the 245 interview transcripts from the three samples, together with the external pictorial and notational representations produced by the children, were obtained and analysed. The external representations were considered with respect to three dimensions as described in

Chapter 5. These dimensions are:

- (a) the type of representation (pictorial, ikonik and notational), from which characteristics of each child's internal imagistic representation were inferred;
- (b) the level of structural development of the number system evidenced in the representation, and
- (c) evidence of the static or dynamic nature of the image.

As stated earlier in this chapter, the discussions with Professor Gerald Goldin in 1993 affirmed the method of analysis of the data based on the above three dimensions. Intercoder reliability was later established by an independent coder. A 3% discrepancy in coding representations was found and apparent errors corrected. Professor Goldin also suggested that his theory governing the construction of internal systems, through the three main stages of inventive-semiotic, structural and autonomous development, might be applied to the conceptual development in numeration. Although the data is not longitudinal, it was still deemed useful to investigate the data in relation to developing stages.

In the following sections, children's external representations of the number sequence will be analysed according to the type of representation (pictorial, ikonik or notational), structural development and the static or dynamic nature of the representations. Further analysis will focus on how dynamic imagery in children's representations shows the linear or array structure of the numeration system. Finally examples of children's representations are used to illustrate cognitive structural development of the numeration system.

8.6 DESCRIPTION OF EXTERNAL REPRESENTATIONS OF THE NUMBER SEQUENCE

The external representations of the number sequence were categorised according to three dimensions of type of representation, structural development and nature as static or dynamic. Examples discussed here are drawn from the pilot study, the cross-sectional sample and from the high ability group (see below).

8.6.1 Types of external representations

The type of representation was analysed according to whether it was pictorial, ikonik or notational. Pictorial recordings were defined as pictures drawn, or oral descriptions of objects given by a child, e.g. a drawing of a truck, a dinosaur labelled with the numeral 100, a description (with some drawings) of one hundred people each labelled with the numerals 1 through 100, and a description of one hundred objects lying on the floor. Ikonik recordings were defined to include drawings of tally marks, squares, circles or dots that represented the counting sequence. Notational recordings were distinguished by the predominant use of

numerals drawn in various formations such as a number line, array, 100 cm ruler or vertical column.

Figures 8.4 to 8.7 show the drawings of Anthony (Grade 1), Andrew (Grade 1), Candice (Grade 3), and Timothy (Grade 4) respectively.

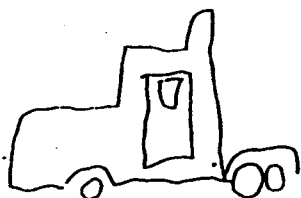


Figure 8.4 Anthony (Grade 1)

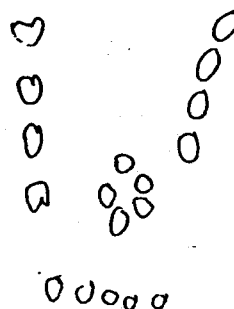


Figure 8.5 Andrew (Grade 1)

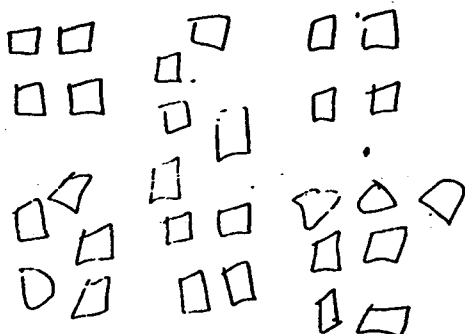


Figure 8.6 Candice (Grade 3)

9

Figure 8.7 Timothy (Grade 4)

The truck drawn by Anthony (Figure 8.4), and reported in Chapter 4, reflects the association of an image of his dad's truck with the number 100. Anthony's reason for the image referred to both "how heavy it is... " and "cause my Dad's truck does a hundred". This suggests we can infer an inventive semiotic internal representation relating the truck-image to mass and speed. This is highly idiosyncratic, but quite meaningful. Andrew saw a picture of 100 shells (Figure 8.5) and explained as he drew, "some were in rows, some were in diagonals and some across like that". This was interpreted as an ikonic internal representation of number with some evidence of structural development of the number system. Analysis of Andrew's protocol showed some partial development of grouping, with capability of dealing with three or ten as a unit, but without the recursive capability of keeping track of how many units.

Candice also drew an ikonic representation using squares, with some evidence of developing structure. Her internal representation was evoked by her prior experience of using square counters, and she attempted to draw these in groups representing the numbers four, five or

six. Candice explained that her drawing was "square counters to count with" and further evidence showed that she was unable to treat numbers as iterable units. Candice used a highly imagistic representation to explain how she saw numbers only "as squares" rather than notational symbols. Timothy's representation (Figure 8.7) similarly displayed little structure but in this case was notational, focusing on just the one numeral 9. This example alone gave insufficient evidence to interpret Timothy's level of structural development. However, in other portions of this protocol, Timothy revealed his counting capabilities explaining that "his numbers stopped at 9 and this was the biggest number". It is inferred from this idiosyncratic example a relatively undeveloped understanding of numeration.

8.6.2 Structural development of the number system

The structural development of the number system was inferred from structural elements (i.e. grouping, regrouping, partitioning and patterning) found in the recordings of the numbers 1 to 100. Evidence of emerging structural development of number was found across a range of recordings. There were a number of cases where children showed no evidence of structure and these were typified by drawings showing a single object, a random pattern of dots or a single numeral. Emerging structure was typified by numerals organised in a counting sequence, recorded continuously in a horizontal, vertical, curved or spiral formation. Children showing evidence of a more developed multiplicative system recorded a multiple counting sequence, and marks or pictures in a partial or complete ten-by-ten array structure.

Figures 8.8 to 8.11 show the drawings of Warren (Grade 2), Joshua (Grade 2), Cassie (Grade 4) and Kimberley (Grade 2) respectively.



Figure 8.8 Warren (Grade 2)

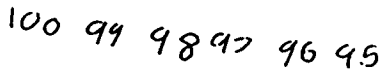


Figure 8.9 Joshua (Grade 2)

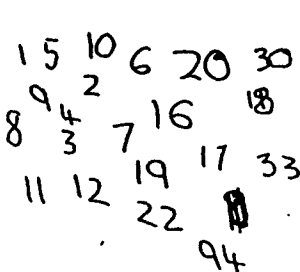


Figure 8.10 Cassie (Grade 4)

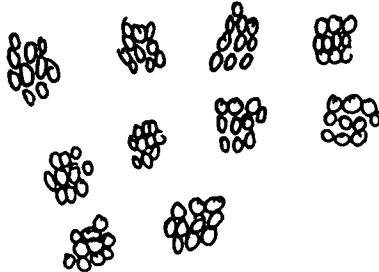


Figure 8.11 Kimberley (Grade 2)

Warren, and Joshua (Figures 8.8, 8.9) produced horizontal, linear structured representations of numbers. Warren's picture of a line of marks was ikonic, whereas Joshua used conventional notation writing the counting sequence of numerals counting-back from 100. Warren's ikonic representation was related to his concentration on counting "by ones" with the marks representing his internal process of counting on by ones. Further evidence from Warren's protocol revealed his ability to count in threes, but his mental image of this remained ikonic rather than seeing numerals. In contrast, Joshua was able to elaborate on his drawing by counting forwards and backwards, grouping in tens and using multiple counting efficiently. A high level of structural development is inferred from these examples. Cassie (Figure 8.10) wrote the numerals in counting sequence in a spiral configuration initially and then became random in sequence and spatial setting. From this is inferred an internal representation with a non-conventional structure of the number sequence. The structure in Kimberley's recording (Figure 8.11) was more explicit as she saw numbers in groups of ten, but could not identify the general structure explaining her drawing as "just circles".

Figures 8.12 and 8.13 show drawings made by Mellissa and Robert, both from Grade 2.

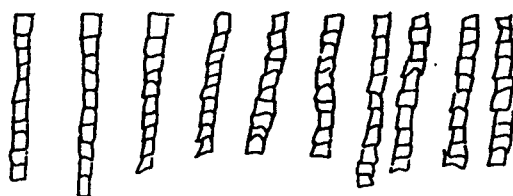


Figure 8.12 Mellissa (Grade 2)

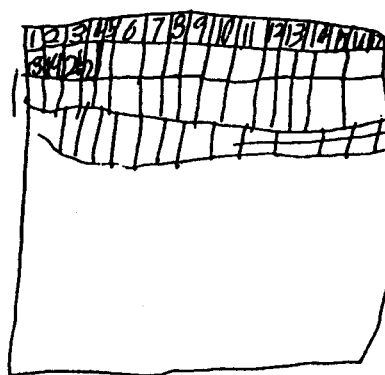


Figure 8.13 Robert (Grade 2)

Figure 8.12 shows how Mellissa (imagery previously reported in Chapter 4) drew ten ten-rods to produce another ikonic representation of grouping in tens. This gives evidence of a highly structural imagistic internal representation for the developing numeration system. Robert (Figure 8.13) drew a square and subdivided rows of separate squares, each square not being aligned to adjacent squares, and then recorded numerals for the numbers in squares, 1 to 17 being in the first row. This partial array displays an emerging notational structure, but Robert showed further evidence of difficulty with using ten as an iterable unit, saying: "you just put the numbers in the boxes as far as you can go ... and you count in ones".

0 2 3 4 5 6 7 8 9 10 Etc. Etc. Etc. Etc

Figure 8.14 Adrian (Grade 3)

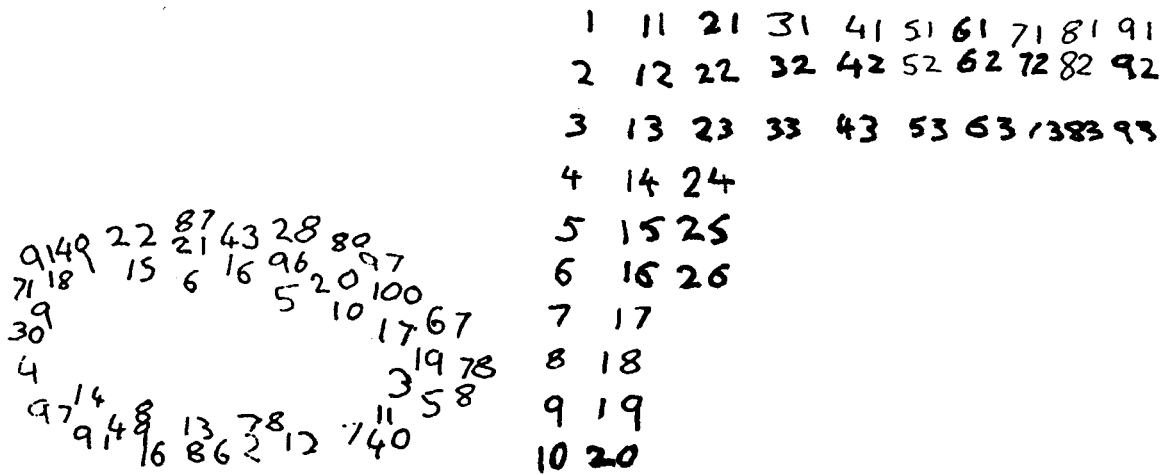


Figure 8.15 Caedyn (Grade 4)

Figure 8.16 Caedyn (prompted response)

Adrian (Figures 8.14) produced the sequence of numerals where each numeral moved past very quickly and Caedyn (Figure 8.16) produced the numerals in a swirl moving anti-clockwise. When both children were asked to again close their eyes and think of the numbers in rows and columns, Adrian said he "couldn't see them in rows because they were moving so fast" whereas Caedyn produced an array with the sequence going down in columns of ten. Adrian's dynamic visualisation was so dominant that he could not organise the numerals as directed whereas it was inferred that Caedyn's internal representation of the numeration system involved a structure of ten tens. Caedyn's imagery was more flexible and a higher level of structure in the number system available to her can be inferred.

8.6.3 Static or dynamic nature of the image

The static or dynamic nature of the image was defined according to whether the recordings and the children's explanations of their representations described fixed or moving (or changing) entities. In the cross-sectional sample 3% of the children displayed dynamic images of the number sequence and in the sample of high achieving children 10% had dynamic images. Examples of dynamic images included numerals flashing one at a time, groups of numerals moving around, and numerals rolling down. Figures 8.17, 8.18 and 8.19 show drawings produced by Nik (Grade 4), Jane (Grade 1) and David (Grade 4).

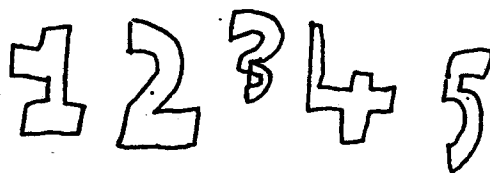


Figure 8.17 Nik (Grade 4)

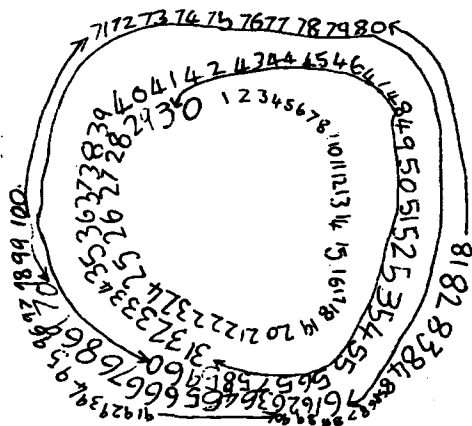


Figure 8.18 Jane (Grade 1)

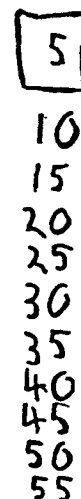


Figure 8.19 David (Grade 4)

Nik (Grade 4), from the high ability sample, explained that "big thick numbers were flashing and flying past, each taking 2 seconds" and indicated that all the numbers would have come if he had closed his eyes for long enough. Jane (Figure 8.18) recorded the number sequence using conventional notation but explained that she saw the numbers moving in a spiral formation "going on forever". Some evidence of a developing system of grouping by tens was revealed in her segments of number strings in tens (e.g. 71-80, 81-90). David's picture (Figure 8.19) showed numerals flashing one at a time, using multiple counting in fives, up to 100. This dynamic notational model gave evidence of an emerging structure for the system of numeration. The formal notations of Nik, Jane and David were organised imagistically in a non-conventional manner. Analysis of their protocols showed that these images were highly creative and unrelated to these children's conventional experiences in the classroom.

8.6.4 Summary

Examples of children's representations of the number sequence 1-100 were discussed in relation to various descriptive categories. From these external representations evidence was shown of internal imagistic representations, structural development of the number system, and dynamic imagery. Children who were deemed to show evidence of structural development described imagery that reflected a multiples of ten counting sequence or pictures/marks/numerals in a ten-by-ten array structure. Sometimes these children (e.g.

Caedyn - Figures 8.15 and 8.16) had multiple images available to them and responded according to the prompt that might be given. Dynamic imagery often reflected highly creative thinking which was unrelated to the children's conventional classroom experiences.

8.7 DYNAMIC IMAGERY IN CHILDREN'S REPRESENTATIONS OF NUMBER

Because of the unusual nature of dynamic imagery shown in children's representations, it was decided to analyse this data more explicitly. Questions were raised as to whether dynamic images are conventional or uniform in nature, and whether children have a range of images available. The examples of dynamic imagery are analysed according to whether they illustrate linear structure, emerging array structure, array structure or no structure. The analysis of the children's dynamic imagery raises questions regarding the nature of the imagery (conventional, idiosyncratic or creative) and whether there is a range of imagery available.

8.7.1 Linear structure

Figure 8.20 shows Clint's initial response to the visualisation task where "the numbers were moving around like people" whereas Figure 8.21 shows his response when asked to think of the numbers in rows and columns. Evidence of linear structure was demonstrated in Clint's external imagery. His initial response showed the linear sequence in a highly creative context and his prompted response demonstrated his conventional experiences with arrays but in a creative way showing a separation of odd and even numbers. David (Grade 6) described a pictorial representation of groupings of objects,

... as I thought, everything suddenly became more numerous ... like first there was one of everything, then there was two of everything and then there were three trees, suns, cats, dogs, people and it kept on going.

Figure 8.22 shows the 6 by 6 grid with numerals following a Fibonacci pattern that David drew when prompted to think of numbers in rows and columns. David's initial imagery focused on the cardinal aspect of the numbers as they appeared in sequence and the structure appearing in the array was creative and individualistic. There was no evidence of a numeration structure based on groupings of tens.

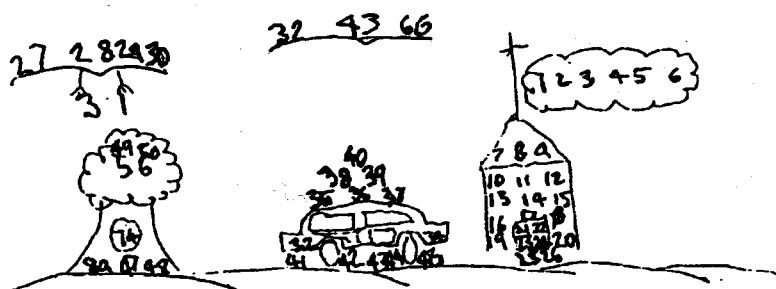


Figure 8.20 Clint (Grade 5)

1	5	7	3	9
11	13	15	17	19
21	23	25	27	29
31	33	35	37	39
4	6	8	10	
12	14	16	18	20
22	24	26	28	30
32	34	36	38	40

Figure 8.21 Clint (Grade 5)

0	1	1	2	3	5
8	13	21	34	55	

Figure 8.22 David (Grade 6)

Rosalie (Grade 6) described "a screen moving one number at a time" (Figure 8.23). When asked to think of the numbers in rows and columns Rosalie drew a picture of a 10 by 10 grid with the numerals 1 to 100 in rows of ten. The initial visualisation was dynamic but this did not suggest advanced structural development. When later prompted to think in rows and columns, Rosalie produced a conventional static array image for the number sequence. From this it is inferred that she is developing an autonomous structure of numeration.

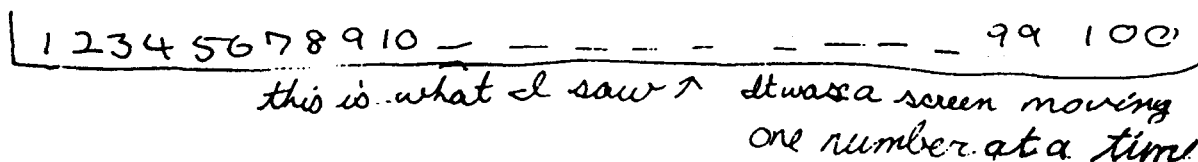


Figure 8.23 Rosalie (Grade 6)

Colin (Grade 5) described the numbers moving along a wave like line (Figure 8.24) and Christopher (Grade 5) explained:

I saw all the numbers from 1 to 100 beaming at me and lighting up like neon signs ... the numbers did this in order then disappeared when I opened my eyes ... they were moving around ... then they would flash ... once they had done that they disappeared.

Figures 8.25 and 8.26 show the prompted responses from Colin and Christopher. Colin started at the bottom left corner of the grid and then filled in diagonals of increasing size moving up to the top right corner and Christopher displayed the numbers backwards from

100 to 1 in rows of ten. Both boys gave dynamic linear imagery for the initial representations and filled in the ten-by-ten grid in non-conventional ways.

I saw the numbers to 1-100 but they #were like 1 2 3 4 5 6 7 8 9 10 11 12 and so on

Figure 8.24 Colin (Grade 5)

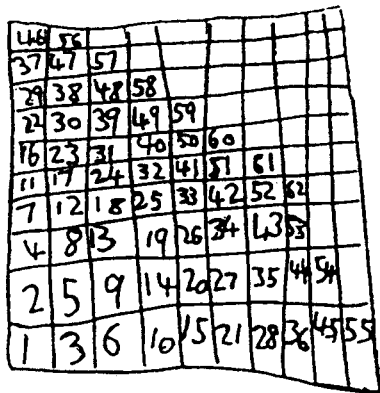


Figure 8.25 Colin (Grade 5)

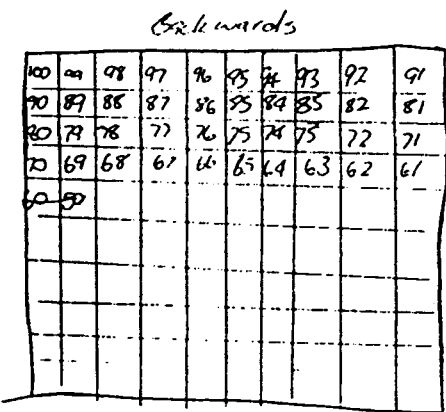


Figure 8.26 Christopher (Grade 5)

Tessa (Grade 5) described the numbers: "walking round in circles ... circles getting bigger and bigger" and when prompted to think of the numbers in rows and columns, described them "like soldiers marching into rows with ten as the commander" (Figure 8.27). When prompted the linear structure was readily changed to dynamic imagery with a tens structure.

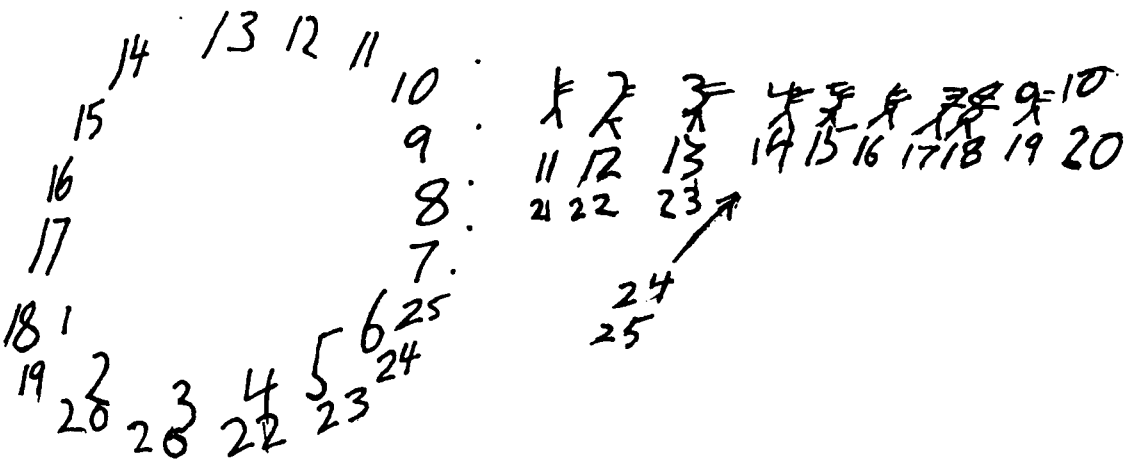


Figure 8.27 Tessa (Grade 5)

The drawing produced by Tim (Figure 8.28) did not reveal anything about his structure of the number system. He explained his image as "3-D numbers flying across in front of my eyes in order ... of 1 first ... and 100 last" and this was classified as a linear notational

image. When Tim was asked to close his eyes and think of the numbers in rows and columns he drew a conventional grid with the numbers 1 to 100 in rows of ten.



Figure 8.28 Tim (Grade 6)

Both Tessa and Tim show a developing structure of numeration which is not applied consistently in the use of large numbers. Although they have the flexibility to think of the numbers in a ten tens structure, can operate successfully with numbers up to one thousand, and can rename numbers based on noncanonical representations, they have not made the connections which would enable them to consistently work with large numbers.

8.7.2 Emerging structure

Michelle (Figure 8.29) gave an array structure for her initial visualisation of the numbers 1 to 100. She described the columns of six as "moving up and down" such that she "was getting dizzy". Michelle's emerging structure was the sequence of numbers separated into columns of six.

1	7	13	19
2	8	14	20
3	9	15	21
4	10	16	22
5	11	17	23
6	12	18	24

Figure 8.29 Michelle (Grade 6)

Leah saw the sequence of the multiples of ten presented in two columns. This changed to a sequence counting by tens from five. Other sequences were then obtained by starting from all other one-digit numbers. Leah drew a picture of a tree and explained that "without all the features of a tree, there would be no leaves, trunks, branches etc ... without numbers no hundreds." The developing autonomous nature of Leah's numeration structure was reflected by her understanding of the importance of tens and hundreds in the numeration system.

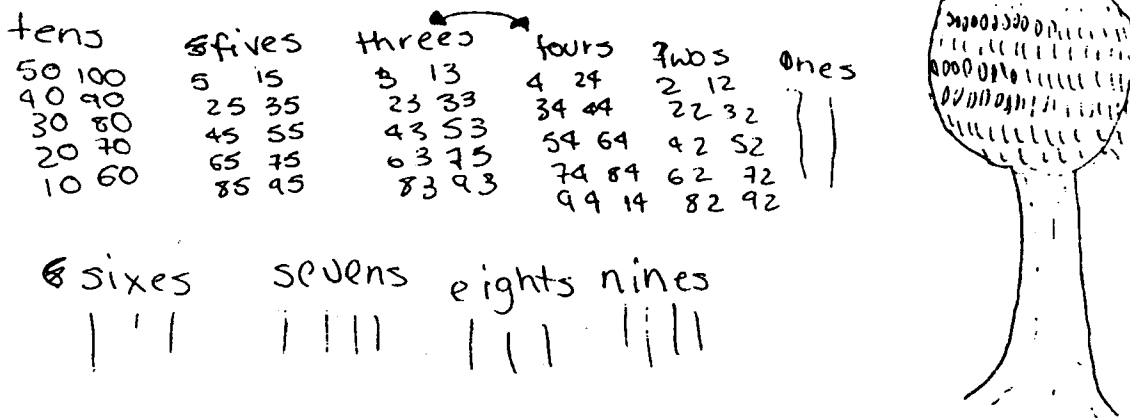


Figure 8.30 Leah (Grade 6)

8.7.3 Array Structure

Figures 8.31 and 8.32 show drawings produced by Renee and Ben. Renee (Grade 5) initially recorded a standard array which was as a "board ... moving to the right". Ben explained that he "saw ten rows of numbers each containing the set of ten numbers that come next in the sequence ... the rows move along to be replaced by the next few." Renee and Ben gave dynamic images of the number sequence which were otherwise conventional, reflecting their classroom experiences.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42			45	46	47	48	49	50
51	52				56	57	58	59	60
61	62				66	67	68	69	70
71	72				76	77	78	79	80
81	82				86	87	88	89	90
91	92				96	97	98	99	100

Figure 8.31 Renee (Grade 5)

12 3 4 5 6 7 8 9 10
 11 12 13 14 15 16 17 18 19 20
 etc. etc. etc.

Figure 8.32 Ben (Grade 5)

In contrast Edward (Figure 8.33) who described the numbers as "floating down in rows" showed a tens pattern in this initial visualisation.

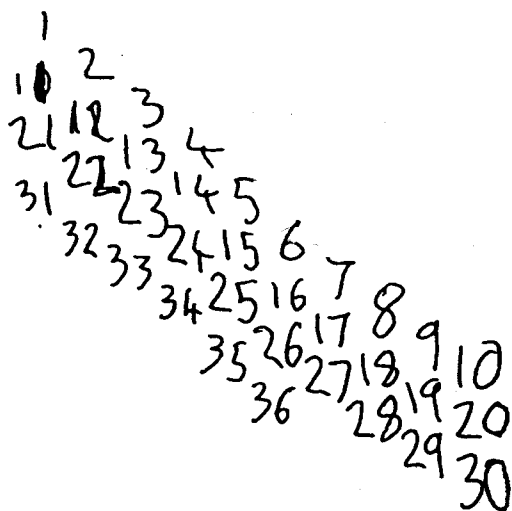


Figure 8.33 Edward (Grade 6)

Dario (Figure 8.34) gave an array structure for his initial visualisation of the numbers 1 to 100. Dario described how he "first saw all the whole numbers ... then ... all the fractions all in a line doubling, tripling". Dario's imagery changed from a standard tens array to a further tens array based on tenths.

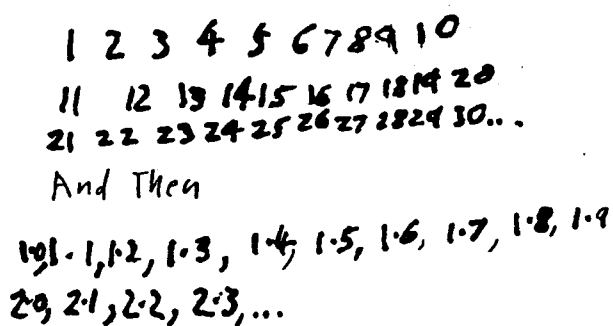


Figure 8.34 Dario (Grade 6)

8.7.4 No Structure

Michael (Figure 8.35) and Joel (Figure 8.36) both produced an apparently random cluster of numerals moving all over the place, although Joel showed his in the context of a road. When both children were asked to again close their eyes and think of the numbers in rows and columns, Michael (Figure 8.37) appeared to focus only on numerals, whereas Joel (Figure 8.38) produced columns for each decade as troops marching behind the general i.e. one hundred. Joel demonstrated a higher level of understanding of the numeration system than Michael through his awareness of the ten tens structure in one hundred.

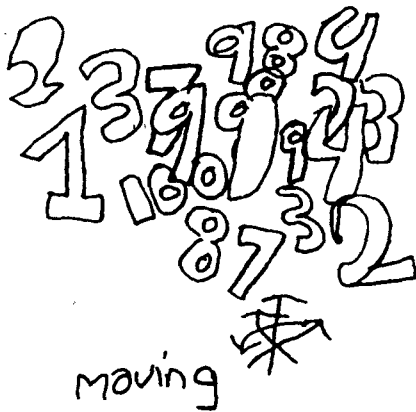


Figure 8.35 Michael (Grade 5)

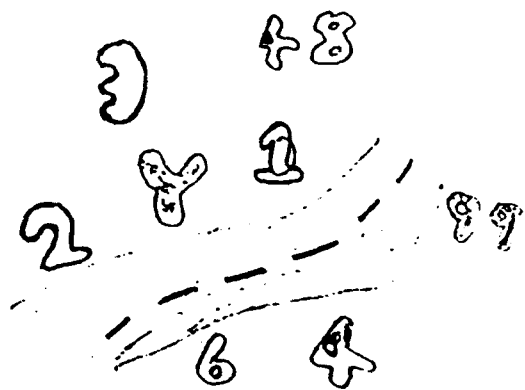


Figure 8.36 Joel (Grade 5)

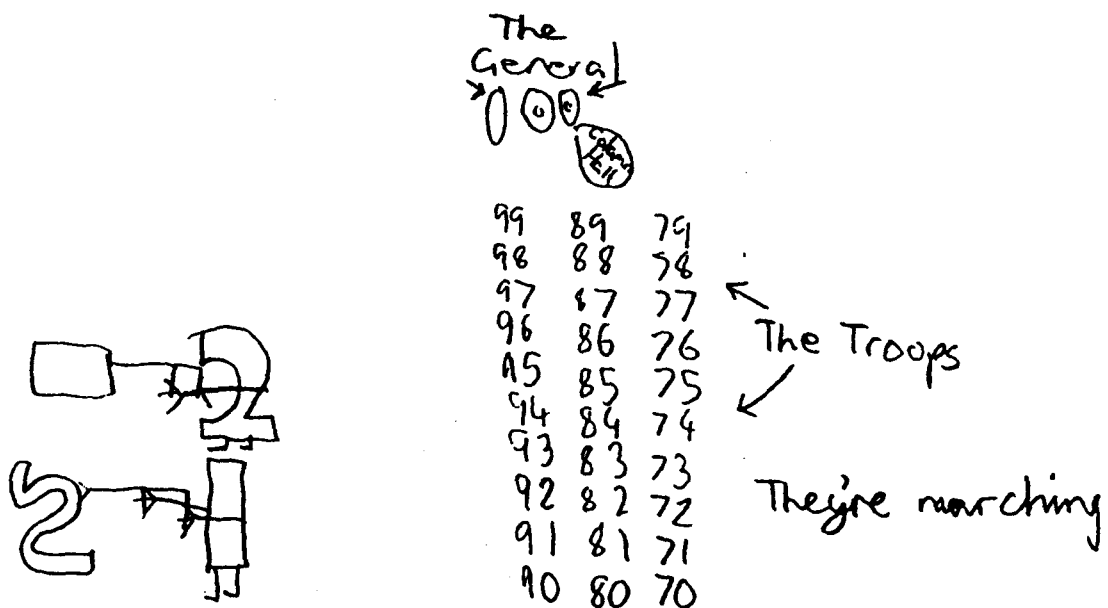


Figure 8.37 Michael (Grade 5)

Figure 8.38 Joel (Grade 5)

Figures 8.39 and 8.40 show that Keryn (Grade 5) used no structure in either the initial or prompted visualisations of the numbers 1 to 100. For the initial visualisation she explained that "they keep changing from plus to times". When Keryn was given the rows and column prompt, she placed the numerals in a grid like formation and then ruled some grid lines but did not really have the grid formation available. Michael and Keryn (Figures 8.37 and 8.40 respectively) demonstrated through their lack of structure (when prompted to think of rows and columns) that they had not developed a relational understanding of the numeration system.

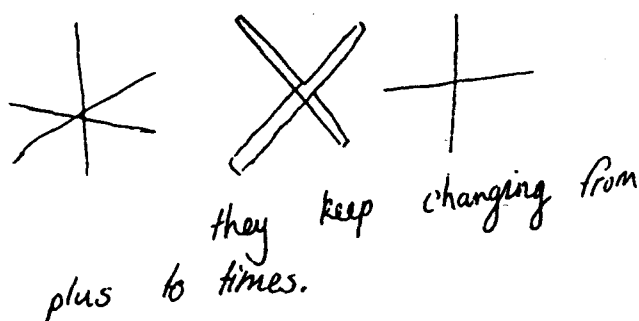


Figure 8.39 Keryn (Grade 5)

100	6	8	9	10
7	11	45	99	
90	10	20	32	

Figure 8.40 Keryn, prompted response

In an analysis of the 77 Grades 5 and 6 children who were in the high ability sample (Thomas & Mulligan, 1995) it was found that there was a wider diversity of representations of the counting sequence 1-100 displayed than might have been expected, and there was a higher percentage of dynamic imagery (30%) than for the average/lower ability children.

8.7.5 Discussion

All the examples of representations showing dynamic imagery were made by children identified by their teachers as high-achievers in mathematics and with attitudes reflecting self-confidence in their abilities. Although some of the spontaneous dynamic imagery did not reveal the pre-categorised structural characteristics, when the children were prompted, varying aspects of structure appeared, often in highly creative ways. For example, Joel initially showed no structure in his representations. When prompted to think of the numbers in rows and columns, he described the numbers "like soldiers marching in columns" (Figure 8.38), a highly creative dynamic way of 'seeing' the ten by ten grid. Here a 0 to 99 hundred square was formed where the numbers were filled down in columns from 99 in the top left cell, one hundred having a dominating position outside the grid as the general. Tessa who had an initial linear image of the number sequence, also described a creative dynamic image of the array structure (Figure 8.27). Tim and Christopher, who also had initial linear images, described more conventional static images of the array structure when prompted. These children showed evidence of having access to a variety of internal images and of developing relational understanding of the numeration system. This suggests that children would benefit from a learning environment which exposes them to a range of representations and structures and where they are prompted to think in different ways.

How does the external imagery produced by children in this visualisation task relate to their internal representations of structural features of the numeration system and to the construction of relational understanding? It is conjectured that it is possible to infer aspects of the child's internal imagistic representations of this structure, from the external

representations of the counting sequence 1-100. The resulting internal representations are fluid and changing, as evidenced by the transitory nature of imagery as learning occurs. It is further suggested that a child will benefit from having available a variety of images for use in representing mathematics, so that salient features of particular imagistic representations can be drawn on in a variety of situations, and flexibility of thought developed.

8.8 ILLUSTRATIVE EVIDENCE FOR COGNITIVE STRUCTURAL DEVELOPMENT

In Sections 8.6 and 8.7 evidence was given of how images can be classified and then dynamic images were investigated for structural aspects of the number system. In the following section, 'snapshots' of imagery used by children, will be examined for developmental features. However, the quality and features of imagery used at any particular stage is of more importance in the discussion than the identification of the stage of development. The cross-sectional data used is purely illustrative of possible points of growth which lead to meaningful understanding of structure within the number sequence.

As discussed in Chapter 3, many researchers have used the constructs of external and internal representation to describe the growth of mathematical learning and problem solving (Goldin, 1992a; Goldin & Kaput, 1996; Vergnaud, 1996). The premise is that a child's cognitive structural development is determined by the construction of internal systems of representation. Powerful internal representations of mathematical ideas or concepts develop through a complex transformation process which has been summarised by Goldin as having three main stages: inventive-semiotic; structural development; and autonomous. This model has been applied to conceptual development of early number (Goldin & Herscovics, 1991a), the exponential function (Goldin & Herscovics, 1991b), and multiplication and division (Mulligan & Mitchelmore, 1996).

On the basis of the analysis of the data described in this chapter so far, it appears that the further the representational system has developed structurally, the more coherent and well-organised is the external representation of the numeration system, and the more competent the child is numerically. This section will examine further, aspects of children's representation that develop and change at various points. Figures 8.41 to 8.47 provide examples of how the imagery the children produced is interpreted as evidence for cognitive structural development associated with various stages of development of their internal representations of numeration.

8.8.1 Evidence of inventive-semiotic stage

In the semiotic stage objects and ideas are labelled or named. The new characters, which are created, are used to symbolise aspects of prior representational systems.



Figure 8.41 Andre (Grade 3)

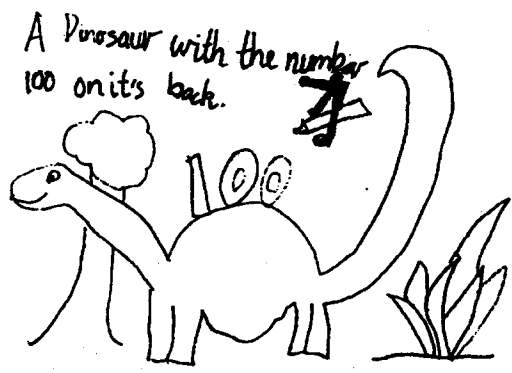


Figure 8.42 Magnus (Grade 3):

Evidence for the Inventive Semiotic Stage of Construction

Magnus drew a dinosaur with the number 100 on its back. This pictorial representation appears to reflect the association of the number 'one hundred' with something large, an early semiotic act. There is no indication of a counting sequence, but rather a focus on the part of the question most significant to the child; at least one aspect of the child's semantic content of 'one hundred' (size) is represented visually here. Andre drew idiosyncratic figures for each of the numbers 1 to 10, saying that "one faded, then two came - the people and animals moving around". In this pictorial representation, evidence was found not only of inventive 'meaning' assigned to numerical symbols, but of an emerging awareness of sequence. (The drawing is restricted to the part of number sequence with which Andre is familiar, as evidenced by his performance on other tasks). Other examples that might show evidence for inventive-semiotic constructions are provided earlier in this chapter by Anthony (Figure 8.4), Timothy (Figure 8.7), Michael (Figures 8.35 and 8.37), and Keryn (Figures 8.39 and 8.40).

8.8.2 Evidence of Structural Development Stage

During the stage of structural development, children use the symbolisation of the previous stage and prior representational systems as a template for the development of the new system. "Over time the new characters and configurations become no longer discrete, unrelated entities, but part of a larger structure (which is the new representational system)"

(Goldin & Kaput, 1996, p. 424). Patterns in the numeration system are obtained by reasoning from earlier additive, multiplicative and notational systems of representation.

Fig. 8.43 shows Naomi's drawing of ten columns of ten circles. It is inferred from this ikonic representation that Naomi is developing some structure to her internal representation of the number sequence. It is a reasonable conjecture that her external representation has been driven by concrete experiences of grouping objects into tens, leading to the internal, imagistic capability of representing such groupings.

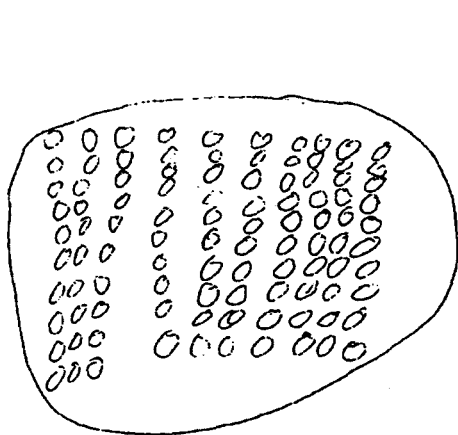


Figure 8.43 Naomi (Grade 3)

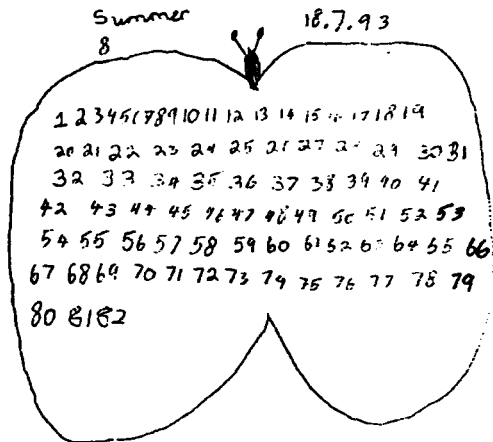


Figure 8.44 Summer (Grade 3)

Evidence for the Structural Development Stage of Construction

Summer's notational representation (Figure 8.44) also seems to display an attempt to fit the known linear sequence into an array structure. Groupings in the rows appear to relate to Summer's notion of the prominent numbers up to twenty, and we detect some semblance of decades.

Both Naomi and Summer thus show evidence of structural development in their internal representational systems for the number sequence. Naomi represents objects as units, while Summer represents numerals, and they have used different scaffolds (groups of tens, and linear sequence) in their respective visualisations. Further examples of children showing evidence of structural development are given earlier in this chapter. Although there appears little evidence of structure in the drawing produced by Andrew (Figure 8.5) and Candice (Figure 8.6), their explanations indicate elements of structural development. The imagery used by Cassie (Figure 8.10), David (Figure 8.19), and Michelle (Figure 8.29) show non-conventional structure of the number sequence. Linear structure is displayed in the imagery of Warren (Figure 8.8) and Joshua (Figure 8.9). The linear structures of Clint (Figure 8.20), Rosalie (Figure 8.23), and Colin (Figure 8.24) all displayed elements of an array

structure when prompted to think of rows and columns. Kimberley (Figure 8.11) and Mellissa (Figure 8.12) produced ikonic grouping structure, although Kimberley was unable to verbalise the groupings of ten that she used. Tessa (Figure 8.27) and Tim (Figure 8.28) initially described linear structures which, when prompted, changed to a conventional hundred square, showing a non-consistent application of the tens structure.

8.8.3 Evidence of autonomous stage

The construction of autonomous representations and the quality of the relationships between formal configurations or symbols and other internal representations lead to meaningful understanding in mathematics. Relationships exist within the representational system created, and between systems, both internal and external. The new representational system now stands separate from the systems from which it was built, and can itself be used as a template for the development of other representational systems.

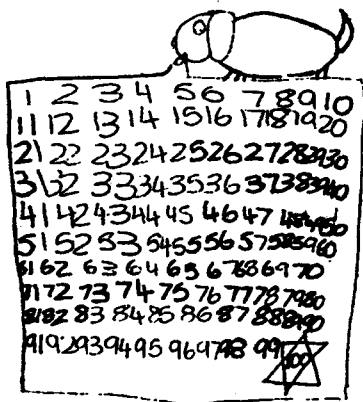


Figure 8.45 Cassie (Grade 4)

Evidence for Advanced Stage of Structural Development

Cassie (Figure 8.45) drew an array with the number sequence in rows of ten. She could describe the notational representation as, "a hundred is ten rows of ten". It is inferred that Cassie's internal representation involves both the notion of sequence and the idea of groupings by ten, including iteration of that idea relating to the notational system. Edward (Figure 8.46) also showed an array structure in his spontaneous imagery for the numbers 1 to 100. When Edward was further asked to show the patterns of ten in the numbers, he too described one hundred as ten tens (Figure 8.47).

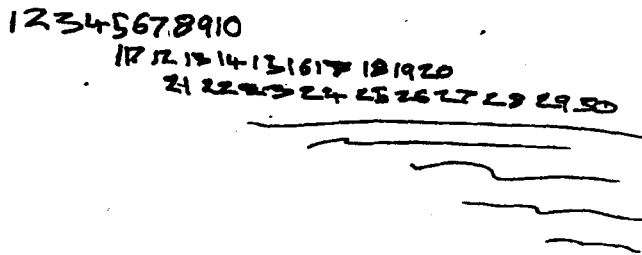


Figure 8.46 Edward (Grade 6)

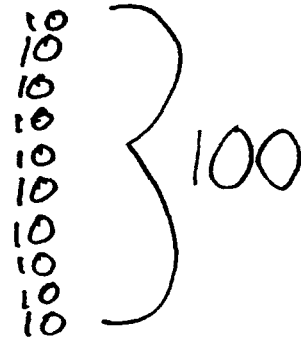


Figure 8.47 Edward (Grade 6)

Evidence for Advanced Stage of Structural Development

From the children's performance on other tasks in the study, there is evidence that Cassie and Edward are able to interpret numerical representations in a variety of contexts, so that as structured systems of internal, cognitive representation, they can reasonably be considered to have reached an autonomous stage of development. Other examples of evidence for developing autonomous constructions are provided earlier in this chapter by Leah (Figure 8.30), Renee (Figure 8.31), Ben (Figure 8.32), and Dario (Figure 8.34). Although Leah did not use an array structure, she used a variety of skip counting patterns and displayed a notion of a hundred as a unit. It could possibly be inferred that Leah is developing an autonomous numeration structure. On the other hand, the autonomous nature of the structure described by Dario is much clearer, as it is applied to a further tens array based on tenths.

8.8.4 Discussion

In order to conceptualise the structure of the number sequence 1-100, it is necessary during the semiotic and structural development stages to build imagistic configurations of the sequence, initially identifying or naming elements of the sequence with an image. The images of Andre (Figure 8.41) and Magnus (Figure 8.42) are symptomatic of the numbers 1 to 100, showing what is important to each of these children. During the stage of structural development, children are using external representations of the sequence which they come in contact with and other internal representational systems available to them, such as those associated with concrete materials, the number line, and the hundred square. The images they draw and describe are built up from the instructional situations involving concrete, semantic, and notational models. As these internal structures of the number sequence are developing, children's external representations may show invented imagery and unconventional detail. The examples of imagery of Naomi (Figure 8.43) and Summer

(Figure 8.44) have been created by the children in response to their concrete and notational experiences but, it is conjectured that, as yet, connections between iterable units of ten and skip counting in tens have not been formed. Evidence for an autonomous stage of development is shown by the notational imagery of Cassie (Figure 8.45) and Edward (Figure 8.46 and 8.47) where a hundred is described as ten groupings of ten and shown in a formation that illustrates the pattern within the number sequence. The internal representation of the number sequence 1-100 which is inferred for these children, provides a powerful system available for processing number operations. The method of mental calculation for a 4-digit subtraction, explained by Edward (Figure 8.48), illustrates how he 'powerfully' uses the structure of number to solve the problem.

2003-25= Edward
I took the 3 of the 2000 and took
away 25 to get 1975 and then
I just added 3 to get 1978.

Figure 8.48 Edward's explanation of mental subtraction

The formal symbolic system of representation for the number sequence 1-100, is the first 'landmark' in internalising the base-ten system of numeration and the associated procedural rules for the operations on whole numbers. This can then be extended and generalised to larger numbers, decimal, fractional, and algebraic systems of formal representation. The internal representation of the number sequence is 'powerful' when it has applicability in a wide range of context, when it has 'meaning' in relation to many different other representations. Goldin (1987, 1992a) also stresses the importance of the associated heuristic systems of planning and decision making for effective problem solving and the development of powerful affect.

Data taken in just one or two interviews do not permit the tracing of the process of structural development, i.e. the construction of internal representational systems, in individual children. But the variations observed across different children strongly suggests that such systems are not fully developed at any one time, but are built up over time. Previously developed representations may serve to provide students with a framework (scaffolding, or

template) on which new, meaningful representational configurations can be fit (new knowledge), and new cognitive structures built. *During the many steps that occur in the structural development stage of the numeration system, it is believed that the variety and meaningfulness of the images facilitates passage to an autonomous representational system of number.* While the representations may be constructed in response to specific tasks, conceptual understanding of numeration must involve many experiences with the representation of numerical ideas, across many different tasks, with meaningful semantic relationships among them. It is postulated that it is the building up of flexible internal representational systems that is important, and that the learner has to make the connections inherent in her/his experiences in order for these constructions to take place.

8.9 IMAGERY AND THE LEARNING PROCESS

From the analysis of data described in this chapter, it appears that the active processing of images plays an important part in the development of the child's understanding of numeration. It is inferred that the examples shown here, revealed each child's unique internal constructions of the number sequence at a particular time. Since images are built up from words, notations, and other images, the representations do not become autonomous until the idea makes sense to the child. That is, numeration can be used itself as a tool in mental thinking, flexibly and independently of any particular image. To facilitate this, children's mental images should be described, drawn, compared and discussed. As their internal structures are developing, the children's external representations, both static and dynamic, may not correspond to conventional mathematics, or be uniform in nature from one child to the next. They should be expected to reflect each child's unique internal constructions at that time. Such a range of available images is important; the images being constructed so that an internal representation system that 'works' can be built up. Gray and Pitta (1996) suggest that when children do not develop internal representations, it is because they only create and utilise mental images which support their procedural thinking. It is further suggested that "mental manipulation with these objects places such strain on the limits of the child's working memory that it impinges against the continuing compression required for 'constructive abstraction' and the development of perceptual thinking" (Gray & Pitta, 1996, p. 3-36). Thus the teaching/learning situation needs to provide opportunities for children to develop and represent structurally meaningful mathematics.

8.10 SUMMARY

The purpose of the analysis presented in this chapter was to describe in detail children's internal representations of the number sequence 1-100, and to investigate examples of

imagery to support this. From the external representations produced by the children, aspects of their internal representation have been inferred, and from this in turn, the structural development of the system that has taken place has been described. In general, a wider diversity of representations of the counting sequence 1-100 than might have been expected were found. This diversity occurred at each grade level, and across all samples. Evidence was found that the children's internal representations of numbers are highly imagistic, and that their imagistic configurations embody structural development of the number system to widely varying extents, and often in unconventional ways. Instances in which the formal notational symbols are organised imagistically were seen, as in Jane's spiral (Figure 8.18) and David's flashing numerals (Figure 8.19).

Those children with dynamic visualisations were shown to have higher achievement on the numeration tasks than those with static visualisations (Thomas & Mulligan, 1995). What might be important here is the unconventional, creative nature of the visualisations rather than just that the image is dynamic or static. It should be noted that other responses might have occurred if the children had been prompted in other ways, for example, to imagine moving numbers, or to group the numbers in tens. Also, other representations may have been available to the children, with just one of several possible internal image configurations having been selected for recording or elaboration. Thus, only a partial description of each child's internal representational capabilities can be inferred.

Many questions are raised but unanswered. Why do some children have the capability of spontaneously visualising the counting sequence in a dynamic way? Can static external representations represent ('carry the meaning') of dynamic internal representations? Further research is needed to shed light on how children construct their personal numeration systems, and how they structure them over time. On the basis of this present data, it is hypothesised that the further-developed the structure of a child's internal representational system for the counting numbers (e.g., kinaesthetic, auditory, or visual/spatial representation of the counting sequence that embodies grouping-by-tens) is, the more coherent and well-organised will be the child's externally-produced representations, and the wider will be its range of numerical understandings (Thomas & Mulligan, 1995).

The analysis of children's visualisation of the number sequence 1-100 gives some insight into these children's cognitive structural development of many aspects of mathematics. Understanding how children construct an independent internal representation of one aspect of the numeration system which can be used efficiently and flexibly, and interrelates effectively with other systems, is critical. The use of culturally provided conventional systems like the Hindu-Arabic Numeration system depends upon children constructing these powerful autonomous internal representations. Examples of imagery provided in this chapter have given 'snapshots' of the developmental stages that it is suggested children may

progress through. It is through the kind of close observation of children's external representations and their use of imagery, used in this thesis, that further insights into how an understanding of the structure of the numeration system can be achieved.

CONCLUSIONS AND IMPLICATIONS

The research problem investigated in this thesis focussed on children's understanding of the structure of the numeration system. A purpose of the study was to analyse key aspects of the numeration system and to determine which of these aspects were critical in assessing children's acquisition of structure. Several research questions were addressed concerning how children acquire and relate key aspects of the numeration system. What strategies do children use in solving numeration tasks involving the elements of counting, grouping, and structuring place value? How are critical aspects of counting and grouping related to understanding the base ten structure of the numeration system? These questions are critical because children need structural flexibility in counting and grouping in order to operate meaningfully with the number system. The role of visualisation of the counting sequence was also examined in view of children's representations of the numeration system.

Four interrelated perspectives on research on children's understanding of mathematical concepts and processes were considered; a constructivist, developmental, cognitive, and a representational approach. The constructivist perspective employed throughout the study highlighted children's use of mathematical strategies and these were explored through task-based interviews. Another important focus was the role of children's representations of counting and grouping in developing structural understanding of numeration.

Researchers such as Bednarz and Janvier (1982; 1988), Boulton-Lewis (1993), Jones et al. (1996), Sinclair, Garin, and Tieche-Christinat (1992), and Thompson (1982a) have at different times called for further research into children's understanding of numeration. This present study was formulated in 1992 and aimed to amplify and extend the work of these researchers, particularly investigating the way children generalise the properties of multiunit numbers within the numeration system.

The study was designed as a broad exploratory investigation: the main study employing task-based interviews of a cross-sectional sample of 126 children across Grades K to 6. Additionally, a study of children's visualisation of the counting sequence 1-100 was included in the pilot and main study. A follow-up study of 92 high ability children was conducted to provide further evidence of children's representations of the counting sequence. This analysis was undertaken within the context of Goldin's (1987, 1988, 1992) model for children's problem-solving competency structures. The overall design of the study allowed the researcher to gain some new evidence about how children relate key aspects of the numeration system.

This concluding chapter discusses the main findings of the study and the implications for further research, teaching and learning. Limitations of the research are outlined in the early section of the chapter but are discussed more thoroughly in terms of the recommendations for research, and teaching and learning.

9.1 CONCLUSIONS

In view of the research questions described in Chapter 1 and the analysis of results, the conclusions will be described under two main headings:

- (i) children's understanding of the structure of the numeration system, and
- (ii) children's representations of numeration.

The first part of the discussion will draw some tentative conclusions based on the results in Chapter 6 and on the analysis presented in Chapter 7 of the strategies children use in solving numeration tasks involving key elements of counting, grouping, and structuring place value. The critical aspects of counting and grouping are discussed in relation to understanding the base ten structure of the numeration system.

The second aspect will discuss the analysis of children's representations of their visualisation of the number sequence 1-100 presented in Chapter 8. Children's external representations of the number sequence were analysed according to the type of representation (pictorial, iconic or notational), the type of structural development, and the static or dynamic nature of the representations. Examples of children's representations were used to illustrate some aspects of the children's structural development of the numeration system. The conclusions drawn address the research questions concerning the key aspects of children's developing understanding of the numeration system and the role of imagery in facilitating structural flexibility when operating meaningfully with number.

9.1.1 Children's Understanding of the System of Numeration

The study showed that the majority of children across Grades 1-6 recognised and used concrete materials to represent grouping of numbers, could identify place values of digits in numerals, and could successfully carry out algorithmic procedures. However, many relied on unitary counting in mental calculations and they did not necessarily use structured materials meaningfully. Children may have shown good performance on 2-digit mental calculations, but generally the use of unitary counting methods prevailed and so many children could not extend their successful use of small numbers to larger numbers. There was, in general, a weak awareness of structure and, in particular, of the multiplicative nature of this structure. It appears from the data that additive relationships within the number

system are better understood and used than multiplicative relationships. The lack of conceptual understanding of the tens and hundreds structure of number means that the knowledge of ones, tens and hundreds that exists is not connected and so ability to work with larger numbers is restricted. On the other hand, some young children acquired elements of understanding of place value, represented number in ways that reflected elements of structure and developed their own efficient mental strategies.

In the present study, children's counting abilities were shown to be of fundamental importance to developing understanding of the number system. By the end of Grade 2, most children still had a strong reliance on rhythmic counting but other children had developed double counting skills. For Grade 5 and 6 children there was little discernible progress with rhythmic or double counting. At Grade 2 the majority of children (61%) were undertaking mental calculations using unitary counting methods for tasks involving 1-digit numbers. At Grade 4 there were still 11% of children using counting-on by ones in Regrouping Task 1 and 33% of Grade 5 children using unitary counting in Regrouping Task 2, both tasks involving addition of a 1-digit number. The results of the number sense tasks also showed that children in Grades 2 and 3 exhibited low performance on using the part-whole relationships with ten and one hundred.

The importance of units and multiunits (units of more than one) in understanding the structure of the numeration system was emphasised by the results of this present study. The way that children deal with the units of one and ten influences their understanding of larger numbers (Cobb & Wheatley, 1988; Steffe & Cobb, 1988). A child who uses ten as a singleton unit might be able to recite the decade numbers (i.e., skip count in tens) but makes no sense of the increments of ten — the units of one and ten co-exist but are not coordinated. The results of this study show that 22% of Grade 5 children could not deal with two different units simultaneously in Grouping Task 7. Approximately a third of Grade 6 children could not successfully add two 2-digit numbers mentally, where the first number was represented with pregrouped material (regrouping Task 7). A further third of the Grade 6 children used counting or separation strategies. The existence of a significant number of children using counting and separation strategies could be explained by their strategies reflecting classroom instruction that commonly emphasises unitary counting in the early years and written procedures for algorithms in the later grades.

In the present study, there was a substantial number of students in Grades 2 and 3 (50% and 32% respectively) who were not successful in recognising and using groupings of ten to quantify a collection of objects (Structure Task 15). Bednarz and Janvier (1988) reported a similar high percentage in regular Grade 3 classes. This result is important because only children who can coordinate units and various multiunits and have a sense of numbers in

relation to the nearest decade or century are able to use mental strategies for operations on larger numbers.

It appears that many children can only recognise numbers in terms of additive properties rather than a combination of additive and multiplicative properties. Many children in this study did not realise that multiunits are related through multiplication or that they can be exchanged. In a recent study, Clark and Kamii (1996) reported that although some children develop multiplicative thinking as early as Grade 2, most children still could not demonstrate consistent multiplicative thinking in Grade 5. The present study confirms these results. A substantial minority (about 20%) of the Grade 3 students had developed such an intuitive understanding of powers of ten that they could use the recursive multiplicative structure of the array of 10,000 dots to count the number of dots successfully. By Grade 6, however, there was still a significant number who could not even count 10 groups of 10 groups of 10 ($10 \times 10 \times 10$).

Many children were shown not to have developed a basic knowledge of the numeration system outside the ones to thousands range by the end of Grade 6. As Dienes blocks are used widely in New South Wales classrooms, it seems that there might be a connection between children's limited number knowledge and their experiences with the standard representations and interpretations of the blocks. Boulton-Lewis (1992) argued that concrete materials are only useful if "children clearly recognize the correspondence between the structure of the material and the structure of the concept" (p. 21). Although it was not a specific focus of this study, it appears that children who are familiar with Dienes blocks may have a limited understanding of the structure inherent in the blocks. Hence any use of the blocks to represent larger numbers may lead to greatly increased processing loads for the children.

Children appear to have had little experience recognising or using arrays. There was over a third of Grade 6 children who could not use their recognition of the pattern of hundreds in an array of dots to quantify the whole collection (Structure Task 22). Many children did not know how to use recognised groups of one hundred objects to quantify a collection of 10 000 objects. They did not make the connection that there was a need for equal grouping or multiplication in order to quantify the collection.

The idea that the numeration system is additive in the simplest way (multi-digits represent the total of the face value of the digits) is very strong among Grade 1 children (95.5%) and persists with some children until Grade 4 (22%) and probably beyond (Place Value Task 1). A surprisingly large number of children (61% in Grade 2 through to 31% in Grade 6) was not able to suggest grouping by tens as a means of quantifying a collection (Structure Task

14). Many children seem slow to grasp these basic elements of grouping tens, formation of multiunits and the way the position of digits plays a role in terms of quantity.

The item response analysis reported in Chapter 6 demonstrated the lack of progress generally made by children on many tasks over the Grades 4 to 6. The difficulties that children experience with understanding the structure of the number system was further highlighted by the decline in performance shown by Grade 6 children when counting using groups of 10×10 (Task 15), repeated use of groupings of 10 (Task 10), suggesting the use of 10 groups of ten (Task 14), and interpreting zero as a place holder (Place Value Task 3).

The results presented in this thesis show that understanding of numeration develops slowly over the Kindergarten to Grade 6 period and that very few children are able to generalise the multiplicative structure of the system. There is evidence of the use of abstract counting strategies by some Kindergarten children, but the performance on counting tasks in the upper grades is still poor for many children. Although the performance on the estimation task was consistently good across the grades, there was lower than expected performance on many number sense tasks because of a high reliance on counting strategies rather than pattern and holistic strategies. Although there was good performance on using grouping in quantifying and building grouped material, there were indications that children did not understand the significance of ten in the number system. This understanding is critical to their further development of understanding and use of the numeration system.

Multiunits are constructed by the recursive grouping by tens and this process is linked to repeated multiplication and the growth of powers of ten. The results of the present study show that many children are unable to use the structure of the numeration system effectively. The study highlights the difficulties that primary school children have in understanding the complex nature of the number system. Children do not understand the multiplicative relationships within the system that are the basis of place value structure and the patterns in the counting sequence. Children can count and group in tens but do not relate these processes to a base ten structure.

9.1.2 Children's Representations of Number

From the analysis of children's representations as reported in Chapter 8, it appears that the active processing of images plays an important part in the development of the child's understanding of numeration. Children's images of the structure of the numeration system involve recognition of the patterns of powers of ten, the additive and multiplicative relations between components of a number when conceptualised in terms of the powers of ten, and the related place value notational system.

From the drawings and notations of counting 1-100 produced by the children, aspects of their internal representations have been inferred. A wider diversity of representations of the counting sequence 1-100 than might have been expected were found. This diversity occurred at each grade level, and across both samples. Evidence was found that the children's internal representations of numbers are highly imagistic, and that their imagistic configurations embody structural development of the number system to widely varying extents, and often in unconventional ways.

Children's representations of the counting sequence were found to be an indication of their developing structure of the numeration system. Although children from Grades 2 to 6 showed no increase in the use of structure through the grades, the stages of structural development as proposed by Goldin (1992) were found to be useful in describing the differences between individual children. Initially children described imagery that focussed on a particular number (e.g. 100) or the numbers 1 to 9. In this semiotic stage objects and ideas are labeled or named. Following this, some notion of structural development is given by attempts to form the known linear sequence into some form of groupings (such as an array) with either pictures or numerals. Evidence of an autonomous stage is given when children describe imagery showing the tens structure and are able to discuss and interpret numerical representations in a variety of contexts.

The study shows that the conceptualisation of the number sequence 1 to 100 is the first 'landmark' in internalising the base-ten system of numeration. It is asserted that during the semiotic and structural development stages it is necessary to build imagistic configurations of the sequence. Since images are built up from words, notations, and other images, the representations of number may not become autonomous until key aspects of numeration makes sense. That is, numeration can be used itself as a tool in mental thinking, flexibly and independently of any particular image. To facilitate this, children's mental images should be described, drawn, compared and discussed. As their internal structures are developing, the children's external representations, both static and dynamic, may not correspond to conventional mathematics or be uniform in nature from one child to the next. They should

be expected to reflect each child's unique internal constructions at that time. Such a range of available images is healthy; the images are constructed so that an internal representational system can be built up for that child that 'works'. Thus the teaching/learning situation needs to provide opportunities for children to develop and represent structurally meaningful mathematics.

While children's representations may be constructed in response to specific tasks, conceptual understanding of numeration must involve many experiences with the representation of numerical ideas, and across many different tasks, with meaningful semantic relationships among them. It is postulated that it is the building up of flexible internal representational systems that is important, and that the learner has to make the connections inherent in her/his experiences in order for these constructions to take place. It appears from the analysis of data in this study that the further the representational system has developed structurally, the more coherent and well-organised is the external representation of the numeration system, and the more competent the child is numerically.

On the basis of data presented in Chapter 8, several research papers have been developed in collaboration with Professor Gerald Goldin (Thomas & Mulligan, 1995; Thomas, Mulligan, & Goldin, 1994; Thomas, Mulligan, & Goldin, 1996). From this work there is evidence that the further-developed the structure of a child's internal representational system for the counting numbers (e.g., kinaesthetic, auditory, or visual/spatial representation of the counting sequence that embodies grouping-by-tens), the more coherent and well-organised will be the child's externally-produced representations, and the wider will be their range of numerical understandings. Further research is needed to shed light on how children construct their personal numeration systems, and how they structure them over time.

9.2 LIMITATIONS OF THE CROSS-SECTIONAL STUDY

Limitations of the cross-sectional study are discussed in terms of methodology and relate to data collection, the sample, classification schemes for strategy use and interview procedures. At the time of formulating the study, cross-sectional data collection for Grades K - 6 was considered appropriate to gain a broad picture of children's understanding of the number system. However, there is still a need for longitudinal data to show children's development of numeration over a number of years because patterns of growth in key processes need to be monitored. Also, the study did not take into account the effects of the particular instruction that children received across the seven classes in each of the eight schools. The types of experiences that children may have had with various concrete materials could influence the ways that they responded to the tasks. Also, the eight schools may not be typical of schools generally in Australia. There were also limitations in a study of this type

on the length of time taken for each interview and the number of interviews that could be undertaken.

Chapter 8 provided many examples of children's representations of number as part of this exploratory and descriptive study. Although the study involved large numbers of children, it was not designed or intended as a controlled experiment permitting immediate generalisation. The methods of inferring aspects of children's internal representations from their externally produced representations are still exploratory, and future studies will need to be subject to further tests of validity and inter-researcher reliability. Further, data taken in just one or two interviews does not allow tracing the process of construction of internal representational systems in individual children. Longitudinal studies are needed to trace the development of numerical representations more systematically.

In spite of these limitations, there is growing evidence in this study that structural aspects of numeration associated with number sequences, groupings by tens, recursive grouping and equal grouping structures are critical for understanding the structure of the numeration system. Also the analysis of representations shows that understanding of the number system is reflected by children's visualisations of the number sequence.

9.3 IMPLICATIONS FOR FUTURE RESEARCH

While this study has extended our understanding of children's lack of structure in terms of the numeration system, several considerations for further research are advanced. These are discussed in terms of: (i) the research methodology; (ii) children's variations and inconsistencies in performance across grades; (iii) further exploration of the use children make of imagery in the construction of numerical relationships, and (iv) investigation of children's representational systems including the affective domain.

The cross-sectional study employed task-based interviewing as an effective method of determining children's solution strategies and why the strategies were used. Although this method was time consuming, a more thorough analysis of children's developing understanding of the numeration system was ascertained than would have been possible with paper and pencil tests, or procedures using whole class testing. It is therefore suggested that further studies should employ task-based interviewing given the wealth and clarity of the information that was obtained in this study. This could be conducted using smaller samples as case studies over a period of time, now that a knowledge base about how children develop understanding of the numeration system is becoming more coherent.

As discussed in the limitations of the study, the range of task objectives and structures was suitable and adequate for children in the age range used. However, there is scope for far

wider use of tasks probing visualisation of number properties, for example, asking children what they visualise between zero and one, or the values of the positions in a numeral (place value). There is more research needed to ascertain how children's visualisations reflect their internal representations of numeration (Gray & Pitta, 1996).

Since data analysis was completed for the present study two related studies have been conducted and longitudinal data are presently being analysed. A study at Rutgers University involves the mathematical development of 22 children in Grades 3-6 from New Jersey. Task-based interview data were gathered during 1992-94, and are presently being analysed (Goldin & Passantino, 1996). At Macquarie University a study involving 120 Australian children in Grades 2-3 is analysing children's construction of numerical relationships (Mulligan, Mitchelmore, Outhred & Bobis, 1996; Mulligan, Mitchelmore, Outhred & Russell, 1997). It is anticipated that these studies will shed further light on key aspects of the numeration system, clarifying how children's internal systems of representation develop.

9.4 IMPLICATIONS FOR TEACHING AND CURRICULUM

9.4.1 Implications for teaching and curricula

In the present study, it was found that children have difficulties with using numeration as a number system and this could be a result of limited instructional experiences. Common teaching practice focusses on the numbers 1 to 1000 (the limits of the usual representations with Dienes blocks) and algorithm-related techniques using place values separately. There is an emphasis in instruction on multiplication procedures, but children are not sufficiently exposed to the range of meanings of multiplication and division in various contexts. Multiplication and division need to be more closely linked, and more experiences bringing out the recursive nature of repeated groupings need to be provided.

Simply asking for the expanded form of numbers, as so often happens in class exercises, is not sufficient to develop the necessary understanding of structure. The grouping and regrouping tasks showed that many children have not constructed the system of tens out of the system of ones and then the system of hundreds out of the system of tens, and so on. Many children could not extend the structure or produce it in the structure tasks, and were unable to construct the structure of $10 \times 10 \times 10 \times 10$ as 100×100 or 1000×10 . There was also some evidence that they could not partition numbers meaningfully into components based on place values. More investigation is needed of how children naturally group and partition numbers. Ten is a natural grouping number for children but one hundred is not. It appears that children have a natural affinity for grouping in fives and tens (fingers), but when the system needs to be extended they do not readily use the recursive process of

grouping of groupings. Furthermore it seems that children cannot visualise the extension of the numeration system, that is, they are not able to generate the next multiunit through multiplying or dividing by ten.

At the time that children are being taught place value of hundreds and thousands, they should also be acquiring and relating multiplicative skills to multiunit values in the numeration system - but the focus of instruction is on addition and subtraction algorithms. Place values as multiunits need to be constructed through using multiplication by 10 as a recursive relation. Children may be learning multiplication and division tables but are not necessarily learning how to extend the grouping structure of the numeration system. If we have evidence that children in Grade 3 do not have an equal grouping structure for multiplication and division, then we can explain their lack of understanding of the base ten structure which requires groupings of groupings of groupings of ten. This recursive structure is far more difficult to imagine or represent than simple equivalent groups, which can be readily represented using repeated addition (Mulligan & Mitchelmore, 1997).

If place value is just learned as assigned values for the positions and not as a multiplicative relationship (Rubin & Russell, 1992), then the learning of place value is compartmentalised. Multiplication is related to partitioning - the need to partition numbers into equal groups. In dealing with the numeration system it is necessary to partition one hundred into ten groups of ten as well as a thousand into ten groups of a hundred. Therefore it is asserted that the basis of developing the structure of numeration should be on multiplicative grouping structures and not only on counting strategies that relate to computation.

Could it be that there is too much emphasis in the early years on addition and not sufficient on multiplication as it relates to partitioning and grouping? The focus of the work with Dienes blocks is usually on addition and subtraction and not on equal groupings. The problem is also that Dienes blocks are structured to easily reflect the values of ones, tens, hundreds and thousands; in order to use them to extend the system, one needs to rename the pieces or to build extra, imaginary pieces. This is not commonly done in classrooms. Students must see concrete embodiments (like Dienes blocks) and the notational system as a reflection of each other (Thompson, 1992). This raises the question, of whether children should do addition and subtraction with Dienes blocks, as well as learning written algorithms, before multiplicative groupings. Perhaps young children need to experience more activities that use multiplicative structures, for example, groupings of groupings leading to the notion of powers.

On the basis of the analysis of representations, children might be encouraged to construct images of the numeration system and to represent these images (Wheatley & Brown, 1994). This might be encouraged at the same time as they are counting, grouping, adding,

multiplying, regrouping, and using place value, zero as a place holder and powers of ten. Through this process, children build up a sense of numbers being part of a consistent and infinitely extendable system that can be represented using their own internal system.

Children's understanding of the elegance and power of the structure of the Hindu-Arabic numeration system must contribute greatly to their efficient use of number. An appreciation of the power of the base ten system develops over an extended time as children construct an understanding and the system eventually becomes transparent. The base ten structured numeration system is multiplicative, using powers of ten as multiunits of the system. This successive repetition of the process of forming ever larger place values for each multiunit is then reflected in both the verbal and notational structures of number. A child's competence with counting, using equivalent groups and regrouping is the basis of multiplicative structures, and for working with units and multiunits. Research reported in this chapter shows that many young children need more active help in developing an adequate understanding of the structure of the numeration system.

Many researchers have pointed out that teaching practices do not take account of children's ways of understanding their experiences in the classroom. "Shortcomings of traditional place value instruction are that it focuses only on the cardinal aspects of one ten, two tens, etc. and completely overlooks the relationship between the system of tens and that of ones ... " (Kamii, 1986, p. 84). This present study shows that instruction also overlooks the relationship between the system of hundreds and that of tens; that the multiplicative structure of the number system needs to be addressed in curricula and instruction. Fuson (1990a) argues that certain characteristics of textbooks "contribute to the failure of U.S. children to build adequate multiunit conceptual structures" (p. 274). Bednarz and Janvier (1986, p. 23) assert that traditional teaching, by ignoring the sources of errors, frequently consolidate or accentuate cognitive difficulties. Teaching practices need to take account of what children know.

As discussed in Chapters 2 and 3, a multidigit number-sense framework was developed by Jones, Thornton, and Putt (1994) and extended by Jones et al. (1996). The framework enables multidigit number learning to be traced across five levels of thinking: pre-place value; initial place value; developing place value; extended place value, and essential place value. Level 5 requires that number sense be demonstrated through flexible approaches to mental calculations involving 3-digit numbers. On the basis of this present study it is suggested that the framework should be extended to include a level which is characterised by the movement from flexible use of numbers up to 1000 to an awareness that this same flexibility can be applied to larger numbers. This can be attributed to the recursive way that further multiunit values and the associated place values are incorporated into the number system. It is argued that this Level 6 (system place value) would incorporate the constructs of counting,

partitioning, grouping and number relations *as a system*. Counting on or counting back by any power of ten and partitioning larger numbers would be possible because of awareness of the generalised number system. The multiplicative recursive process of generating the multiunit values, and hence the values of position in numerals as movement is made to the left, would be the basis of regrouping and renaming representations for larger numbers. Ordering and estimation involving larger numbers would then be possible because of the awareness of the structure of number.

Further research is needed investigating children's structural development of the numeration system. With the extension of the framework for nurturing and assessing multiunit number sense (Jones et al., 1996) beyond 4-digit numbers, an instructional program could be planned and implemented in class settings to investigate how children might construct numeration as a generalised system.

9.4.2 Implications for assessment

Analysis of the task-based interview data in this present study has highlighted the complexity of the process of understanding children's thinking. Pencil and paper tests give only an indication of possible difficulties that children might be experiencing. Clinical assessment is a process of finding out how children are constructing mathematical concepts and processes. It is not enough for teachers to simply observe children's responses; it is essential to know what to look for and then how to interpret this information in the context of the child's developing range of ideas.

The questions on early numeration described in this present study attempt to capture the processes that children go through in developing an understanding of the number system. Furthermore, the assessment of children's spontaneous and idiosyncratic mathematical ideas can reveal the limits of their cognitive capabilities that may otherwise never be detected. There are implications for the role of task-based interviews in classroom assessment practices and how assessment might be integrated into the teaching/learning program. Assessment must provide accurate descriptions of individual children's representations, concepts and processes, and problem-solving methods, whether they are partially or fully developed. It should also be "reflective, allowing the student not only to grapple with mathematical discovery and conceptual constructions but to reflect on these processes" (Goldin, 1993, p. 82). Children's self-descriptions of their mathematics and how they use their mathematics in the context of life experiences is a critical part of this process.

There is a growing body of research that shows how children construct and represent mathematical ideas in their own way. As well, it is known that children interact as they contribute mathematics to the learning situation. The real challenge for professionals is how

to interpret and make sense of children's own mathematical ideas in order to help them further develop their understandings.

The results of the present study highlight the importance of children developing arithmetical strategies (Wright, 1991a, b) in the first years of schooling. The *Count Me in Too Numeracy Project* (NSW Department of School Education, 1997) being conducted in 160 New South Wales schools provides a learning framework for developing children's mental arithmetical strategies which are based on counting skills. The results of the present study supports the increased emphasis on counting and decade related strategies that are being used in this Project.

Some aspects of the approach used in this present study have been implemented in an assessment program for children bridging the transition from primary to secondary schooling. The *Year 7 Mathematics Recovery Program* (Thomas & Donaldson, 1995) was developed to cater for the needs of Grade 7 children with difficulties using place value and multidigit numbers. This program for children entering secondary school is designed to:

- (i) identify students who are having difficulties with numeration;
- (ii) carry out 1:1 diagnostic testing with the identified students, and
- (iii) structure recovery teaching packages which suggest appropriate activities and direction for instruction in understanding the number system using on-site teachers.

The initial project involved CSU - Mitchell teacher education staff, Year 4 BEd students, and staff and Year 7 students at three local high schools. It was designed and implemented in a fully collaborative manner with all non-student members having input in the design and implementation of the assessment phase. However, there was only limited success in implementing the follow-up teaching phase. The *Year 7 Mathematics Recovery Program* is continuing with further focus on the teaching phase. This program has demonstrated how individual task-based assessment can be used effectively to program appropriate work for children entering secondary school who have problems in understanding numeration.

The results of this study have highlighted the importance of task-based interviews in classroom assessment practices. Although national and state-based initiatives in assessment are becoming more focussed on standards and achievement of outcomes as shown by the National Mathematics Profiles (Australian Education Council, 1993) and the National Numeracy Benchmarks (Curriculum Corporation, 1997) there is a recognition of the importance of clinical assessment and teacher judgement in gaining explicit and systematic assessment information (Mulligan & Thomas, 1995). While paper and pencil testing in numeracy may be inevitable given current government priorities, the need for appropriate alternative forms of assessment has been recognised by education authorities.

9.5 CONCLUSION

In conclusion, this study has focussed attention on the key aspects of children's understanding of the numeration system. Conclusions drawn from this research provide more insight into the complexities of children's developing understanding of the multiplicative recursive structure of the numeration system. It appears that children do not develop sufficient understanding of numeration as a system; what is needed is for children to make the connections between the range of representations for multiunit numbers being used. Key components of counting, partitioning, grouping and number relations, and the visualisation of structure need to be considered more critically. Strategies that children use to solve numeration tasks are too often based on unitary counting and very few Grade 6 children are able to use the holistic strategies that derive from the structure of the number system. Children's representations of counting frequently reflects a lack of structure, grouping is not sufficiently linked to the formation of multiunits, and additive rather than multiplicative relations dominate the interpretation of multidigit numbers. Further research might focus on a more holistic approach to investigating children's mathematical processes.