Three studies in higher category theory: fibrations, skew monoidal structures and excision of extremals

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Declaration

I hereby declare that, unless otherwise stated, the material presented in this thesis is the result of original research. It has not been submitted, either in whole or in part, at this university or elsewhere for the award of a degree.

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Abstract

Many fundamental constructions from ordinary category theory can be generalised to higher categories. Obvious examples include adjunctions, monads, algebras, limits, and colimits. This thesis explores three cases where a construction from category theory is extended to higher categories.

We first consider (Grothendieck) fibrations and the Grothendieck construction. We generalise fibrations to the contexts of 2-categories and bicategories. A fibration of bicategories exhibits many of the usual properties of ordinary fibrations. The main result is the Grothendieck construction which presents a correspondence between fibrations of bicategories and contravariant trihomomorphisms into the tricategory of bicategories.

We next consider skew monoidal categories. Our goal is to uncover a definition of skew monoidal bicategory (a definition which is non-trivial due to the absence of a coherence theorem for skew monoidal categories). We do this by introducing the Catalan simplicial set $\mathbb{C}$ and show that simplicial maps from $\mathbb{C}$ into an appropriate nerve of Cat are precisely skew-monoidal categories. This simplicial set has a similar classifying property for skew-monoidales internal to any monoidal bicategory. By examining simplicial maps from $\mathbb{C}$ to a suitable nerve of Bicat we obtain a definition of skew-monoidal bicategory that is consistent with existing definitions of monoidal bicategory.

Finally, we consider Street's paper *Parity Complexes*. Parity complexes are multi-dimensional graph-like objects that exhibit the minimal structure required to build free n-categories such as the orientals. Due to its detailed combinatorial nature, the material in this paper can be difficult to follow and quite hard to verify. Indeed, there are minor errors in the original text that were later corrected. We present a formalisation, in Coq, of this theory up to the excision of extremals algorithm in Section 4. We have verified that Street's work is fundamentally sound and that there are no further errors. We summarise the main content of the theory, and the basic intuition involved in its construction. We also discuss some technical aspects of the formalisation, and comment on which portions of the theory could benefit from some refinement.

Contents

Chapter 1

Introduction

Many fundamental constructions from ordinary category theory can be generalised to higher categories. Obvious examples include adjunctions, monads, algebras, limits, and colimits. In most cases, even when the intuition is relatively clear, a detailed account of the theory at higher dimensions is not easy to write down and requires a lot of careful work to make it precise. This thesis explores three cases where a construction from category theory may be extended to higher categories. The topics are fibrations, skew-monoidal categories, and presentations of free categories. In each case we find that the intuition behind a generalisation is not too hard to follow, but supplying accurate definitions and rigorous proofs requires substantial patience and care.

Each chapter of the thesis reproduces a paper that was published or submitted for publication during the time of the author's PhD candidature.

Fibrations

Suppose $P\colon E \to B$ is a functor. A map $f\colon a \to b$ in E is cartesian when, for each $g\colon c \to b$ with $Pg = Pf.h$ there exists a unique $\hat{h}\colon c \to a$ with $P\hat{h} = h$ and $g = f.\hat{h}$. A functor P is a (Grothendieck) fibration when for all $e \in E$ and $f\colon b \to Pe$ in B there is a cartesian map $h\colon a \to e$ with $Ph = f$. The main consequence of this definition is that every map $f\colon a \to b$ in B creates a 'pullback' functor $E_a \leftarrow E_b$ between the fibres in E over a and b. Our goal in Chapter 2 is to define a notion of fibration of bicategory in such a way that the usual theory for fibrations is essentially repeated at this higher dimension. We regard this as an important first important step toward understanding fibrations for higher categories in general.

We are not the first to investigate fibrations of bicategories. Claudio Hermida gave the first definition of 2-fibration (fibration of 2-categories) by examining the structure of the 2-functor Fib $\to$ Cat. In the process he gave new definitions of cartesian 1- and 2-cell, and also presented a number of results concerning the behaviour of those cells. This was partially extended by Igor Baković who generalised those definitions to bicategories and homomorphisms of a certain kind. He also sketched out the Grothendieck construction for such fibrations but publicly available proofs were incomplete at the time of our research.

Our investigation in this area revealed some holes in the original literature. First, the full Grothendieck construction was not proved and basic results concerning composition and pullback of fibrations were not given. Second, there was an axiom missing from the definition of fibration that prevented the Grothendieck construction from being a triequivalence.

Before we go any further, consider the following list of standard results concerning ordinary fibrations and cartesian arrows.

1. Fibrations are closed under composition.

2. If we take the pullback of a fibration $P\colon E \to B$ along any functor $F\colon A \to B$ then $A \times_B E \to A$ is a fibration and $A \times_B E \to E$ preserves cartesian arrows.

3. A map $f\colon y \to z$ is cartesian for $P\colon E \to B$ if and only if the following square is a pullback in Set.

$$\begin{tikzcd} E(x,y) \arrow[r, "f_*"] \arrow[d, "P_{x,y}"'] & E(x,z) \arrow[d, "P_{x,z}"] \\ B(Px,Py) \arrow[r, "Pf_*"'] & B(Px,Pz) \end{tikzcd}$$

4. There is a biequivalence between fibrations over B and contravariant pseudo-functors from B to Cat.

$$\mathrm{Fib}(B) \simeq \mathrm{PsFun}(B^{\mathrm{op}}, \mathrm{Cat})$$

   The right-to-left functor underlying this equivalence is called the *Grothendieck construction.*

5. Let $\mathrm{Fam}(B)$ be the comma category $(\mathrm{Set} \downarrow B)$. The projection $\mathrm{Fam}(B) \to \mathrm{Set}$ is a fibration.

6. If $F\colon E \to B$ is any functor then the projection $(B \downarrow F) \to B$ is a fibration. It has a particular universal property that makes it the *free fibration* on F.

We expect that a reasonable definition of fibration of bicategories should satisfy these theorems when they are properly expressed in a bicategorical context. In particular, the Grothendieck construction should exist and provide a triequivalence between a certain tricategory of fibrations of bicategories over a fixed base and the tricategory of contravariant trihomomorphisms from that base into Bicat. In order to make the other results in our list precise we must also introduce and study a number of general bicategorical constructions, such as appropriate bicategorical pullbacks and commas.

Following intuitions developed from the list of desirable properties given above, as informed by the work of our 2-categorical predecessors [Her99] and [Bak12], we obtain the following definition:

Definition 2.3.5. Suppose that $\mathscr{B}$ is a bicategory, $P\colon \mathscr{E} \to \mathscr{B}$ is a homomorphism, and cartesian 1-cells and 2-cells are appropriately defined in $\mathscr{E}$. We say that P is a *fibration of bicategories* when

i. for any $f\colon b \to Pe$ in $\mathscr{B}$ there is a cartesian 1-cell $h\colon a \to e$ in $\mathscr{E}$ with $Ph = f$;

ii. each hom-functor $P_{x,y}\colon \mathscr{E}(x,y) \to \mathscr{B}(Px,Py)$ is a fibration; and

iii. the horizontal composite of any two cartesian 2-cells is cartesian.

This definition gives rise to a natural theory of bicategorical fibrations in which the definitions of accompanying structures, such as cartesian homomorphisms between fibrations, follow directly. These then allow us to prove the following generalisations of the usual theorems:

1. **Proposition 2.4.1.** Fibrations of bicategories are closed under composition.

2. **Proposition 2.4.15.** The 'pullback' of a fibration along any homomorphism is a fibration and the homomorphism along the top of the pullback square preserves cartesian 1- and 2-cells.

3. **Proposition 2.3.2.** A 1-cell $f\colon y \to z$ is cartesian for $P\colon \mathscr{E} \to \mathscr{B}$ if and only if the following square is a bipullback in Cat.

$$\begin{tikzcd}
\mathscr{E}(x,y) \arrow[r, "f_*"] \arrow[d, "P_{x,y}"'] \arrow[dr, phantom, "\cong"] & \mathscr{E}(x,z) \arrow[d, "P_{x,z}"] \\
\mathscr{B}(Px,Py) \arrow[r, "Pf_*"'] & \mathscr{B}(Px,Pz)
\end{tikzcd}$$

4. **Theorem 2.3.32.** There is a triequivalence between fibrations over $\mathscr{B}$ and contravariant trihomomorphisms from $\mathscr{B}$ to Bicat

$$\mathrm{Fib}(\mathscr{B}) \simeq \mathrm{Trihom}(\mathscr{B}^{\mathrm{coop}}, \mathrm{Bicat})$$

   The right-to-left functor underlying this triequivalence is called the Grothendieck construction for fibrations of bicategories.

5. **Proposition 2.3.36.** Define $\mathrm{Fam}(\mathscr{B})$ to be a certain comma bicategory $(\mathrm{Cat} \downarrow \mathscr{B})$. The projection $\mathrm{Fam}(\mathscr{B}) \to \mathrm{Cat}$ is a fibration.

6. **Proposition 2.4.6.** If $F\colon \mathscr{E} \to \mathscr{B}$ is any homomorphism then the projection $(\mathscr{B} \downarrow F) \to \mathscr{B}$ from a certain comma bicategory is a fibration of bicategories. It has a particular universal property that makes it the *free fibration* on F.

This material clears up many of the details left behind by previous authors. Cartesian 1-cells are indeed defined by bipullback, the local actions of a fibration are in themselves fibrations, fibrations are closed under pullback, and most importantly the Grothendieck construction behaves precisely as well as one might hope. The fact that these fibrations exhibit so many of the properties of ordinary fibrations gives us confidence that our formulation is both sound and complete. It might now be noted that the conceptual framework we have built to inform our generalisation from dimension one to dimension two should also apply in the passage to dimensions three and above. It is our view that this structured approach may well lead to more routine generalisation of these notions to tricategories, iterated Segal spaces and so forth.

Skew-monoidal bicategories

A skew monoidal category consists of a category $\mathcal{A}$, two functors

$$\otimes\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A} \quad \text{and} \quad I\colon 1 \to \mathcal{A},$$

and three natural transformations α, λ, ρ, with components

$$\begin{aligned} \alpha_{abc}\colon (a \otimes b) \otimes c &\to a \otimes (b \otimes c), \\ \lambda_a\colon I \otimes a &\to a, \text{ and} \\ \rho_a\colon a \otimes I &\to a \end{aligned}$$

which are not necessarily invertible. These natural transformations must satisfy Mac Lane's five axioms for a monoidal category (see p.95). This definition is due to Kornel Szlachányi, who showed that bialgebroids over a ring R can be characterised in terms of certain skew-monoidal structures on the category of R-modules. It is clear that when α, λ, and ρ are invertible a skew-monoidal category is precisely a monoidal category. However, it is important to note that without invertibility of α, λ, and ρ, the usual coherence theorem does not hold and it is not the case that all diagrams commute. In particular $\rho_I \circ \lambda_I\colon I \otimes I \to I \to I \otimes I$ is not generally equal to $\mathrm{id}_{I \otimes I}$.

Just as it is possible to describe monoidales (pseudo-monoids) internal to any monoidal bicategory $\mathcal{H}$, and monoidal categories are monoidales in Cat, it is also possible to define skew monoidales internal to any monoidal bicategory, and skew-monoidal categories are skew monoidales in Cat. Skew monoidales were introduced by Stephen Lack and Ross Street who showed that if $\mathcal{V}$ is a suitably complete braided monoidal category then quantum categories are certain skew monoidales in Comod($\mathcal{V}$).

Following their lead, we would like to understand what a skew-monoidal bicategory would look like. This then might act as a prelude to a general understanding of what a skew-monoidal n-category might be.

To that end, let us first consider how to describe monoidal bicategories as a generalisation of monoidal categories. A monoidal category $\mathcal{V}$ has a tensor functor $\otimes$, a unit functor I and natural isomorphisms α, λ, ρ that satisfy three axioms. When we move to monoidal bicategories we make the following changes: natural isomorphisms α, λ, ρ are replaced with pseudo-natural equivalences, the three axioms are replaced with invertible modifications, and three further axioms are introduced to govern all of this data. As we make these changes we find that, because each new coherence map is essentially invertible, we do not need to worry about their orientation. Also, since we expect to prove a coherence theorem where 'all diagrams commute', once we introduce a minimum few axioms for the data, we are free to add any other axioms without changing the theory. The case for skew-monoidal bicategories is not so simple. When we introduce new coherence data we are forced to choose an orientation for each map. And when we introduce the coherence axioms we need to be careful that we do not introduce too few, or too many (since we do not expect that all diagrams will commute). It is not obvious how these choices should be made.

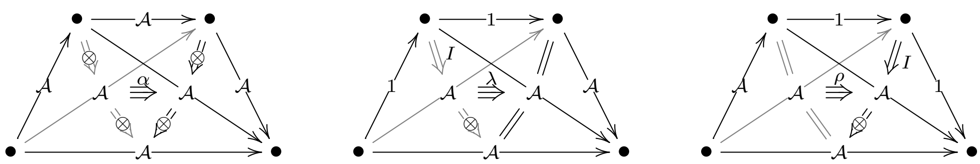
We have a solution to this problem that is quite surprising. Our observation (first noted by Mike Johnson) is that the data for a skew-monoidal category can be described using 1-, 2-, and 3-simplexes. We regard the underlying category as a 1-simplex

$$\bullet \xrightarrow{\mathcal{A}} \bullet$$

and the tensor and unit as 2-simplexes

$$\begin{tikzcd} & \bullet \arrow[dr, "\mathcal{A}"] & \\ \bullet \arrow[ur, "\mathcal{A}"] \arrow[rr, "\mathcal{A}"'] & \Downarrow\otimes & \bullet \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[dr, "1"] & \\ \bullet \arrow[ur, "1"] \arrow[rr, "\mathcal{A}"'] & \Downarrow I & \bullet \end{tikzcd}$$

then each coherence natural transformation takes the form of a 3-simplex



and each of the five axioms can be described with a 4-simplex. This presentation encodes not only the domains and codomains, but the orientation of each map.

If we regard Cat as a 1-object bicategory then these are simplexes in an appropriate nerve of Cat. In that case, a skew-monoidal category may be identified with a simplicial map $\mathbb{C} \to \mathrm{N}(\mathrm{Cat})$ whose domain $\mathbb{C}$ it remains for us to describe formally. Coherence concerns lead us to observing that $\mathbb{C}$ is precisely the 2-coskeletal simplicial set whose non-degenerate 0-, 1- and 2-simplexes are precisely those whose shape is displayed in the diagrams above. In studying this structure we are led to the remarkable observation that for all integers n the number of $(n-1)$-simplexes of $\mathbb{C}$ is simply the n^{th} Catalan number. This realisation eventually leads us to the following simple and explicit structural characterisation:

Proposition 3.2.5. *The simplicial set $\mathbb{C}$ is uniquely isomorphic to the monoidal nerve of the poset $\mathbf{2} = \bot \leqslant \top$, seen as a monoidal category under disjunction.*

Aside from its interesting combinatorial properties, this simplicial set also allows us to prove the following classification results.

Proposition 3.4.3. *To give a simplicial map $f\colon \mathbb{C} \to \mathrm{N}_p(\mathrm{Cat})$ is equally to give a small monoidal category; to give a simplicial map $f\colon \mathbb{C} \to \mathrm{N}_\ell(\mathrm{Cat})$ is equally to give a small skew-monoidal category (N_p and N_ℓ are pseudo- and lax-nerve functors).*

Theorem 4.4.3. *For any monoidal bicategory $\mathcal{H}$ there is a biequivalence*

$$\mathrm{sSet}(\mathbb{C}, \mathbf{N}\mathcal{H}) \simeq \mathrm{SkMon}(\mathcal{H})$$

where the bicategory on the right is the bicategory of skew monoidales in $\mathcal{H}$.

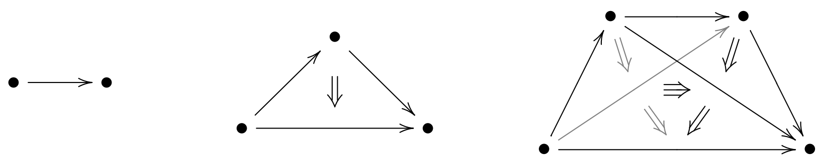
We do not understand exactly *why* this result holds though we have a few thoughts (see Remark 4.4.7). In light of these two classification results, it seems likely that simplicial maps $\mathbb{C} \to \mathrm{N}(\mathrm{Bicat})$ will be skew-monoidal bicategories.

In Chapter 3 we introduce and study this Catalan simplicial set, explore some of its combinatorial properties and prove the first classification result above. In Chapter 4 we prove the second classification result and derive a definition of skew-monoidal bicategory by examining simplicial maps $\mathbb{C} \to N(\mathrm{Bicat})$, where the nerve of Bicat is appropriately defined. When the coherence maps for a skew-monoidal bicategory are equivalences (respectively isomorphisms) we recover the usual definition of monoidal bicategory. We do not prove a classification result for skew-monoidal bicategories because the proof from earlier cases does not generalise directly.

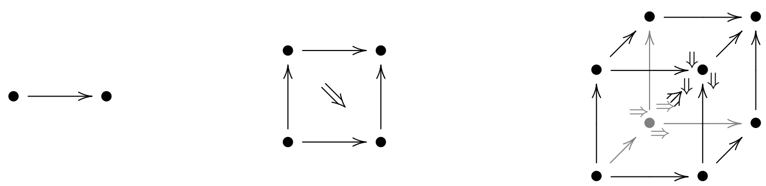
The Catalan simplicial set appears to encode the combinatorics of skew monoidal structure at multiple dimensions. At the very least it successfully classifies skew monoidal categories and skew-monoidales. The definition of skew-monoidal bicategory obtained from $\mathbb{C}$ is sensible and consistent with existing definitions of monoidal bicategory; this suggests that $\mathbb{C}$ might describe skew-monoidal structure at arbitrary dimensions. At the very least, it seems clear that $\mathbb{C}$ is a part of some underlying process concerning skew-monoidal categories. We hope that future work will uncover more about this process and produce a characterisation of skew-monoidal structure for higher categories in general.

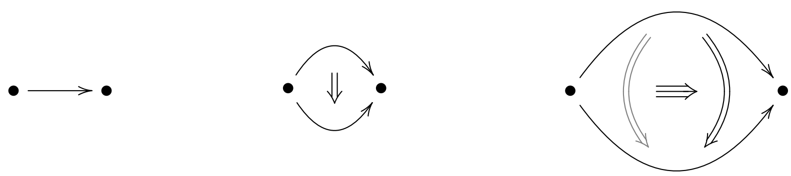
Parity Complexes

An n-simplex is a geometric figure that generalises the notion of triangle or tetrahedron to n-dimensional space. Simplexes have a number of properties that make them useful in algebraic topology, algebraic geometry and homotopy theory where they often play a foundational role. Each face of an n-simplex can be oriented in such a way that the n-simplex freely generates a strongly "loop free" n-category. We include below the cases for $n = 1$, 2 and 3.



This orientation is easily described at all dimensions, but as n increases beyond three it becomes more of a challenge to give an explicit description of the n-category generated by the n-simplex. Its cells are certain large *pasting diagrams* of oriented faces of the n-simplex, whose fine structure depends upon some detailed combinatorial and structural analyses of loop freeness as higher dimensions. It turns out that simplexes are not the only structures that are amenable to this kind of analysis; cubes and globes can be similarly oriented and given a category structure in low dimensions. We include below the cases for $n = 1$, 2 and 3.





Again, in the case for cubes, the orientation of the faces of these objects are easily described at all dimensions but as n increases beyond three it becomes very difficult to describe the corresponding free n-category structures. The free n-category structure on each globe can be described quite easily.

In the early 1980s Ross Street examined the case for simplexes and showed how to form a free n-category $\mathcal{O}_n$ from each n-simplex. These categories were dubbed the *orientals*. At the same time, the case for cubes was examined by Iain Aitcheson who showed how to form a free n-category from each n-cube. Following these successes, Ross Street (*Parity complexes* [Str91]) and Richard Steiner (*Directed complexes* [Ste93]) described some general conditions under which one could directly build free categories from these oriented combinatorial structures. Each paper describes the essential combinatorial structure, exhibited by the examples above, that enable the construction of free categories in this way. Parity complexes and directed complexes are each closed under geometrical product and join operations, so that all examples of which we are aware are covered.

To orient the reader here, we might simply mention that a parity complex is a kind of multi-dimensional directed graph. It is a graded set $C = \sum_{n=0}^{\infty} C_n$ with face-set operators $(\text{-})^+, (\text{-})^- : C_{n+1} \to \mathcal{P}C_n$ satisfying some basic axioms that make it suitably well-formed and loop-free. Each set C_n contains the elements of the structure at dimension n and when $x \in C_{n+1}$ the set $x^+ \subseteq C_n$ is the set of 'positive faces' of x (x^- plays a dual role). Every parity complex C gives rise to a free ω-category $\mathcal{O}(C)$. When appropriately oriented, simplexes, cubes, and globes all become natural examples of parity complexes. A key theorem in Street's paper is the *excision of extremals algorithm*, which shows how to present each cell of $\mathcal{O}(C)$ as a composite of atomic cells; this is, in large part, the sense in which $\mathcal{O}(C)$ is free. This algorithm can also be used to generate explicit algebraic descriptions of the cells in $\mathcal{O}(C)$.

The goal of Chapter 5 is to present a formal verification, in Coq, of Street's *Parity Complexes* [Str91]. Our motivation is two-fold. First, some of the combinatorial arguments in Street's text can be difficult to follow and can easily conceal small yet significant mistakes; in fact, some key corrections were issued shortly after the original publication [Str94]. The work presented here provides a formal verification that Street's theorems hold once the necessary amendments of that latter paper are made. Second, a computer-verified encoding provides a good resource for understanding the intricacies of these complicated structures and opens a path to further refinement of the material.

Chapter 5 begins by outlining the basic definitions (mostly set-theoretic) that are required to describe parity complexes. We then summarise the main content of sections 1, 2 and 3 of Street's paper and end with the excision of extremals algorithm (Theorem 4.2). Throughout this summary we explain how various definitions were encoded, and the essential nature of each proof. Further details can be uncovered by examining the code itself at `https://github.com/MitchellBuckley/Parity-Complexes`. We end the chapter by commenting on the difficulties associated with formalising such a detailed piece of mathematics.

Our encoding of Parity Complexes verifies that the main body of Street's work on the subject is sound. By working through every detail we are able to discover which parts of the theory follow directly from definitions in an elegant way, and which parts of the theory might benefit from some refinement and further investigation. The encoding also serves as an example of formal verification of mathematics. The code is freely available online for anyone to check.

Chapter 2

Fibred 2-categories and bicategories

Abstract

We generalise the usual notion of fibred category; first to *fibred 2-categories* and then to *fibred bicategories*. Fibred 2-categories correspond to 2-functors from a 2-category into 2Cat. Fibred bicategories correspond to trihomomorphisms from a bicategory into Bicat. We describe the Grothendieck construction for each kind of fibration and present a few examples of each. We give constructions of oplax comma bicategory and equiv-comma bicategory by analogy with ordinary comma categories and iso-comma categories. Then fibrations in our sense, between bicategories, are not only closed under composition but when $F\colon \mathscr{C} \to \mathscr{B}$ is a homomorphism and $P\colon \mathscr{E} \to \mathscr{B}$ is a fibration of bicategories then the projection from the equiv-comma $\mathscr{C} \times_{\simeq} \mathscr{E} \to \mathscr{C}$ is also a fibration of bicategories. If $F\colon \mathscr{A} \to \mathscr{B}$ is a homomorphism then the projection from the oplax comma bicategory $(1_B \downarrow F) \to \mathscr{B}$ is the free such fibration on F.

Contribution by the author

As the sole author, this paper is entirely my own work. It is a direct reproduction of the original which was published in the Journal of Pure and Applied Algebra. Any differences from that publication are limited to cosmetic changes such as citation numbering.

2.1 Introduction

Fibred categories were first developed by Grothendieck [Gro71; Gro60] to describe notions of descent in algebraic geometry. Some of this material was then extended by Gray as tool for understanding Čech cohomology[Gra66] and by Giraud for non-abelian cohomology [Gir64; Gir71]. Later, Street described fibrations internal to any bicategory [Str74; Str80] together with internal two-sided fibrations. Some more recent work on internal fibrations can be found in [Rie10]. Fibrations have found strong application in categorical logic: in describing comprehension schema for categories [Gra69; Law70] and via indexed categories [PS78]. For an overview of applications to categorical logic and type theory see [Jac99]. Fibrations were also used by Bénabou [Bén85] to describe some foundations of category theory.

Fibred 2-categories (also called 2-fibrations) were investigated by Hermida [Her99] where the projection Fib $\to$ Cat was used as a canonical example. A definition of 2-fibration is given that very nearly (but not entirely) captures the full structure required for a Grothendieck construction. This definition was extended in a preprint by Baković [Bak12] to strict homomorphisms of bicategories. In that paper he describes the action-on-objects of a Grothendieck construction sending trihomomorphisms $B^{\mathrm{coop}} \to$ Bicat to fibrations of bicategories. The paper also presents a large number of examples and from each strict homomorphism of bicategories he constructs a 'canonical fibration' associated to that homomorphism. Fibrations of bicategories are characterised by a certain right biadjoint right inverse. The action-on-objects of a pseudo-inverse to the Grothendieck construction is also partially described.

Our goal is to establish precise definitions of 2-fibration and fibration of bicategories by describing a complete Grothendieck construction in each case. By 'complete' we mean a construction that is provably a 3-equivalence or triequivalence. In the most general case fibrations of bicategories should be triequivalent to trihomomorphisms into Bicat. Among other things this means dealing with the fibres of non-strict homomorphisms and understanding the properties of cartesian 1- and 2-cells. Our second goal is to understand in what sense these fibrations are closed under 'pullback' and composition. Third, we aim to describe the 'free fibration' on a homomorphism of bicategories.

In doing this we find that the existing definitions of 2-fibration need to be adjusted: cartesian 2-cells must be preserved by both pre- and post-composition with any 1-cell. Until now definitions have only required that cartesian 2-cells be closed under pre-composition. Without adding the post-composition property it is not possible to construct a pseudo-inverse to the Grothendieck construction and fibrations of bicategories do not truly correspond to trihomomorphisms into Bicat. This important observation is made in Remark 2.2.9 and this definition of fibration is used in the rest of the paper.

In general, fibrations of bicategories can be non-strict homomorphisms $P\colon \mathscr{E} \to \mathscr{B}$ and we use the fact that fibrations are locally iso-lifting to show that any fibration is equivalent to a strict (and better behaved) fibration. We prove many of the usual results concerning cartesian 1-cells in this new context. Our main result is a proof that the Grothendieck construction described partially in [Bak12] can be extended to a full triequivalence once the necessary adjustments are made. We give the construction of the free fibration (Baković's canonical fibration) using the oplax comma construction (to be defined). We also show that fibrations are closed under composition and stable underequiv-comma (also to be defined).

In Section 1 we give an introduction and remind the reader of the basic theory of fibred categories. Subsection 1.1 includes the basic definitions of cartesian arrow, fibration, cleavage, et cetera. We then outline some standard properties of cartesian arrows and give a brief description of the original Grothendieck construction.

In Section 2 we outline the theory of fibred 2-categories. We say that a 2-category is fibred when it is the domain of a *2-fibration*. In line with the usual theory, we require that these 2-functors have *cartesian 1-cells* which are cartesian in the normal sense but also have a 2-dimensional lifting property. A 2-fibration P also has *cartesian 2-cells* which are cartesian as 1-cells for the action of P on hom-sets; we ask also that cartesian 2-cells be closed under horizontal composition. In subsection 2.1 we give definitions of cartesian 1-cell, cartesian 2-cell and 2-fibration. We prove 2-categorical versions of the standard results concerning cartesian 1-cells. Subsection 2.2 gives the Grothendieck construction for 2-categories: this is an equivalence that sends 2-fibrations $P\colon E \to B$ to 2-functors from $B^{\mathrm{coop}} \to 2\mathrm{Cat}$. Subsection 2.3 contains some examples of 2-fibrations. Many of the results found in this section correspond well with the classical theory and become somewhat routine once the right foundations are established.

In Section 3 we outline the theory of fibred bicategories. We say that a bicategory is fibred when it is the domain of a (bicategorical) *fibration*. Fibrations of bicategories have the same structure as 2-fibrations except that cartesian 1-cells have a much weaker lifting property defined by *bipullback* in Cat. This significantly weakens the usual results concerning cartesian 1-cells: multiple invertible 2-cells are introduced into every calculation and many uniqueness properties are reduced to 'unique up to isomorphism' or weaker. Despite these complications the usual results can be stated in a form that is consistent with this bicategorical setting. Subsection 3.1 covers these new definitions and results. The fact that fibrations are locally fibred means that they locally have the iso-lifting property; as a result many of the complications mentioned above can be simplified. In subsection 3.2 we show that every fibration is equivalent to one with a somewhat simpler structure. Subsection 3.3 describes the Grothendieck construction for fibrations between bicategories: a triequivalence that sends fibred bicategories to trihomomorphisms into Bicat! Some of the heavier calculations have been omitted as they are not immediately helpful to the reader. Subsection 3.4 contains a number of examples.

In Section 4 we investigate how fibrations between bicategories behave under composition, pullback and comma. Fibrations are closed under composition (4.1) and stable under (bi-)pullback (4.2). We define the oplax comma of two homomorphisms (4.3) and show that the free fibration on a homomorphism is the projection from an appropriate oplax comma bicategory.

2.1.1 Standard notation

Here we present the basic definitions and properties of fibrations and cartesian maps. For a more complete account see Chapter 1 of [Jac99] and Section 8 in [Bor94].

Throughout the entire chapter we will use single arrows $f\colon a \to b$ to denote 1-cells or functors, 2-functors, or homomorphisms of bicategories (which are 1-cells in a higher category). We will use double arrows $\alpha\colon f \Rightarrow g$ to denote 2-cells or natural transformations, 2-natural transformations, or transformations of homomorphisms (which are 2-cells in a higher category). We will use triple arrows $\Gamma\colon \alpha \Rrightarrow \gamma$ to denote modifications which are a commonly used notion of map between 2-natural transformations or transformations of homomorphisms.

Suppose $P\colon E \to B$ is a functor. A map $f\colon a \to b$ in E is *cartesian* when

$$\begin{tikzcd} E(z,a) \arrow[r, "f_*"] \arrow[d, "P_{za}"'] & E(z,b) \arrow[d, "P_{zb}"] \\ B(Pz,Pa) \arrow[r, "Pf_*"'] & B(Pz,Pb) \end{tikzcd}$$

is a pullback. This is the same as saying that for each $g\colon c \to b$ with $Pg = Pf.h$ there exists a unique $\hat{h}\colon c \to a$ with $P\hat{h} = h$ and $g = f.\hat{h}$.

A functor P is a *fibration* when for all $e \in E$ and $f\colon b \to Pe$ in B there is a cartesian map $h\colon a \to e$ with $Ph = f$. In this case we say that h is a cartesian lift of f. Informally, we say that E is a *fibred category* when it is the domain of a fibration. A *cleavage* for a fibration P is a function $\varphi(\text{-},\text{-})$ that describes for each e and $f\colon b \to Pe$ as above a choice of cartesian lift of f which is denoted $\varphi(f,e)$. That is, $\varphi(f,e)\colon f^*e \to e$ is the chosen cartesian lift of f at e. A fibration equipped with a cleavage is called a *cloven fibration.* Every fibration can be equipped with a cleavage using the axiom of choice so we implicitly regard all fibrations as cloven. If a cloven fibration has $\varphi(g.f,e) = \varphi(g,e).\varphi(f,g^*e)$ where $g\colon a \to b$, and $\varphi(1_{Pe},1) = 1_e$ then we say the cleavage is *split* and call it a *split fibration.*

Strictly speaking, we should annotate each cleavage so we know for which fibration it makes choices of cartesian lift. In practice there is generally little confusion and we will only make such annotations when omitting them would create ambiguity.

The following results are easy to verify for any fibration $P\colon E \to B$. Cartesian lifts of any 1-cell in B are unique up to unique isomorphism in the slice over their common codomain. If f and g are composable 1-cells in E and are both cartesian then $g.f$ is also cartesian. If f and g are composable 1-cells in E and $g.f$ and g are cartesian then f is also cartesian. If f is a cartesian 1-cell in E and Pf is an isomorphism, then f is an isomorphism.

Let Fib be the 2-category whose objects are fibrations P, Q; 1-cells are pairs of functors (F, G) such that $QF = GP$ and F preserves cartesian maps; and 2-cells are pairs of natural transformations (α, β) such that $Q\alpha = \beta P$.

$$\begin{tikzcd} E \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[d, "P"'] & D \arrow[d, "Q"] \\ B \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd}$$

In this case we say that a functor F underlying a 1-cell (F, G) is *cartesian*. That is, a functor $F: E \to D$ between domains of fibrations P and Q is cartesian (with respect to G) when it preserves cartesian maps and has $QF = GP$. In practice, it is the preservation of cartesian maps that is most relevant and G is either an identity or is otherwise clear from context. If such a functor also preserves choice of cartesian map ($F(\varphi(v, e)) = \varphi(v, Fe)$) then we say F is *split*. We say that a natural transformation α underlying a 2-cell (α, β) is *vertical* (with respect to β). In practice β is either an identity or is otherwise clear from context.

We use Fib(B) to denote the sub-2-category of Fib whose objects, arrows and 2-cells have second component B, 1_B, and 1_{1_B} respectively.

Let Cat be the 2-category of small categories. We use $\mathrm{Hom}(B^{\mathrm{op}}, \mathrm{Cat})$ to denote the 2-category of pseudo-functors, pseudo-natural transformations and modifications. The *Grothendieck construction* is a 2-functor $\mathrm{el}: \mathrm{Hom}(B^{\mathrm{op}}, \mathrm{Cat}) \to \mathrm{Fib}(B)$. It sends each pseudo-functor $F: B^{\mathrm{op}} \to \mathrm{Cat}$ to the obvious projection $\mathrm{el}\, F \to B$ from the *category of elements*. The category of elements has objects pairs (a, x) where $a \in B$ and $x \in Fa$; arrows are pairs $(f, u): (a, x) \to (b, y)$ where $f: a \to b$ and $u: x \to Ff(y)$. The Grothendieck construction is an equivalence.

Suppose the following square is a pullback. If P is a fibration then P' is a fibration and F' is cartesian; F' also reflects cartesian maps.

$$\begin{tikzcd} D \arrow[r, "F'"] \arrow[d, "P'"'] & E \arrow[d, "P"] \\ C \arrow[r, "F"'] & B \end{tikzcd}$$

Fibrations are also closed under composition. There is a 2-monad on Cat/B whose category of algebras is $\mathrm{Fib}(B)$. The monad acts by taking each functor $F\colon A \to B$ to the comma category $(B \downarrow F)$; the projection $(B \downarrow F) \to B$ is the *free fibration on F*.

A functor $P\colon E \to B$ is a *Street fibration* when for all $e \in E$ and $f\colon b \to Pe$ in B there is a cartesian map $h\colon a \to e$ where Ph is isomorphic to f in the slice over Pe. Morphisms of Street fibrations $P\colon E \to B$,$Q\colon C \to D$ are pairs of functors $(F\colon C \to E, F'\colon D \to B)$ together with an isomorphism $\alpha\colon PF \cong F'Q$ where F preserves cartesian maps. This is only a slight variation on the ordinary notion of fibration but is useful for considering fibrations internal to a 2-category.

Remark 2.1.1. Suppose that F and P are functors (or 2-functors or homomorphisms) between categories (or 2-categories or bicategories) with a common codomain and there is some construction that forms a square as pictured below.

$$\begin{tikzcd} C \star E \arrow[r, "F^\star"] \arrow[d, "P^\star"'] & E \arrow[d, "P"] \\ C \arrow[r, "F"'] & B \end{tikzcd}$$

The symbol $\star$ does not refer to any specific concept, it is just a place-holder here. Usually the square will commute or contain a transformation, and have a universal property. The usual example is the pullback construction. Suppose now that there is some class of functors (or 2-functors or homomorphisms) $\mathscr{W}$ and for all F and P we know that $P \in \mathscr{W}$ implies $P^\star \in \mathscr{W}$. In this case we say that $\mathscr{W}$ is *closed under* or *stable under* the construction in question. Thus ordinary fibrations are stable under pullback and we will find that 2-fibrations and fibrations of bicategories are stable under analogous constructions in their 2-categorical and bicategorical contexts.

Remark 2.1.2. It is generally the case that cartesian lifts are unique up to unique isomorphism (or equivalence in later sections). It is for this reason that we have invoked the axiom of choice above and are not generally concerned with choice of cartesian lifts. This attitude continues when we consider fibrations of 2-categories and of bicategories. There has in the past been some controversy attached to this approach (see the appendix of [Bén85]). If there is any confusion over choice, or uniqueness, or use of 'up-to-isomorphism' terminology then one should equip every fibration with a specific cleavage and observe how each construction incorporates such a choice of cartesian lifts. We make no claim that any maps are 'canonical' and acknowledge that there may be cases where such choices have non-trivial consequences.

2.2 Fibred 2-categories

We present the basic data and properties of fibrations of 2-categories. All notions in this section are completely 2-categorical unless otherwise indicated.

2.2.1 Definitions and properties of cartesian 1- and 2-cells

We wish to define fibrations of 2-categories in a way that fits with the usual definition of fibration of categories. It is thus necessary to describe cartesian arrows (and in this case cartesian 2-cells).

Definition 2.2.1. Suppose $P\colon E \to B$ is a 2-functor. We shall say a 1-cell $f\colon x \to y$ in E is *cartesian* when it has the following two properties.

1. For all $h\colon z \to y$ and $u\colon Pz \to Px$ with $Ph = Pf.u$, there is a unique $\hat{u}\colon z \to x$ with $P\hat{u} = u$ and $h = f.\hat{u}$;

$$\begin{tikzcd} z \arrow[d, dotted, "\hat{u}"'] \arrow[dr, "h"] & \\ x \arrow[r, "f"'] & y \end{tikzcd} \qquad \begin{tikzcd} Pz \arrow[d, "u"'] \arrow[dr, "Ph"] & \\ Px \arrow[r, "Pf"'] & Py \end{tikzcd} .$$

We call $\hat{u}$ the *lift* of u.

2. For all $\sigma\colon h \Rightarrow k$, $\tau\colon u \Rightarrow v$ with $P\sigma = Pf.\tau$ and lifts $\hat{u}$,$\hat{v}$ of u,v, there is a unique $\hat{\tau}\colon \hat{u} \Rightarrow \hat{v}$ with $P\hat{\tau} = \tau$ and $\sigma = f.\hat{\tau}$.

$$\begin{tikzcd} z \arrow[d, bend right=40, "\hat{v}"'] \arrow[d, bend left=40, "\hat{u}"] \arrow[d, phantom, "\overset{\Leftarrow}{\hat{\tau}}"] \arrow[dr, bend left=30, "h"] \arrow[dr, bend right=10, "k"'] \arrow[dr, phantom, "\Downarrow\sigma"] & \\ x \arrow[r, "f"'] & y \end{tikzcd} \qquad \begin{tikzcd} Pz \arrow[d, bend right=40, "v"'] \arrow[d, bend left=40, "u"] \arrow[d, phantom, "\overset{\Leftarrow}{\tau}"] \arrow[dr, bend left=30, "Ph"] \arrow[dr, bend right=10, "Pk"'] \arrow[dr, phantom, "\Downarrow P\sigma"] & \\ Px \arrow[r, "Pf"'] & Py \end{tikzcd}$$

We call $\hat{\tau}$ the *lift* of τ.

The pairs of pastings shown above have the property that the right hand side is the image of the left hand side via the functor P. This is a convenient way to illustrate various lifting properties of cartesian maps and these kinds of diagrams will be repeated throughout the chapter.

It is not hard to prove that:

Proposition 2.2.2. *Suppose $P\colon E \to B$ is a 2-functor. A 1-cell $f\colon x \to y$ in E is cartesian if and only if*

$$\begin{tikzcd} E(z,x) \arrow[r, "f_*"] \arrow[d, "P_{zx}"'] & E(z,y) \arrow[d, "P_{zy}"] \\ B(Pz,Px) \arrow[r, "Pf_*"'] & B(Pz,Py) \end{tikzcd}$$

is a pullback in Cat.

Definition 2.2.3. Suppose $P\colon E \to B$ is a 2-functor. A 2-cell $\alpha\colon f \Rightarrow g\colon x \to y$ in E is *cartesian* if it is cartesian as a 1-cell for the functor $P_{xy}\colon E(x,y) \to B(Px,Py)$.

We take the time here to establish a few basic properties of cartesian maps.

Proposition 2.2.4. *Suppose $P\colon E \to B$ is a 2-functor.*

1. *Suppose that E', B' are the 1-categories obtained from E, B by forgetting 2-cells, and that $P'\colon E' \to B'$ agrees with P on 0 and 1-cells. If $f\colon x \to y$ in E is cartesian for P, then it is cartesian for P'.*
2. *If $f\colon x \to y$ and $f'\colon z \to y$ are cartesian in E and $Pf = Pf'$ then there exists a unique isomorphism $h\colon z \to x$ with $f' = h.f$ and $Ph = 1_{Pa}$.*
3. *If $f\colon x \to y$ is cartesian in E and Pf is an isomorphism then f is an isomorphism.*
4. *Suppose $f\colon x \to y$ and $g\colon y \to z$ in E. If f and g are cartesian then $g.f$ is cartesian.*
5. *Suppose $f\colon x \to y$ and $g\colon y \to z$ in E. If g and $g.f$ are cartesian then f is cartesian.*

Proof. (1) is true because cartesian 1-cells have the ordinary lifting property for 1-cells. (2) and (3) are a consequence of (1). For (4), notice that since f and g are cartesian, the two commuting squares below are pullbacks. Hence, the outer rectangle is a pullback and $g.f$ is cartesian.

$$\begin{tikzcd}
E(w,x) \arrow[r, "f_*"] \arrow[d, "P_{wx}"'] & E(w,y) \arrow[r, "g_*"] \arrow[d, "P_{wy}"] & E(w,z) \arrow[d, "P_{wz}"] \\
B(Pw,Px) \arrow[r, "Pf_*"'] & B(Pw,Py) \arrow[r, "Pg_*"'] & B(Pw,Pz)
\end{tikzcd}$$

For (5), use the same diagram as above. Since g and $g.f$ are cartesian, the right square and outer rectangle are pullbacks. Hence, the left square is a pullback and f is cartesian. $\square$

Proposition 2.2.5. *Suppose $P\colon E \to B$ is a 2-functor and that $h\colon y \to z$, $\alpha\colon f \Rightarrow g\colon x \to y$ in E. If h and $h\alpha$ are cartesian then α is cartesian.*

Proof. Since h is cartesian, the following square is a pullback.

$$\begin{tikzcd}
E(x,y) \arrow[r, "h_*"] \arrow[d, "P_{zx}"'] & E(x,z) \arrow[d, "P_{zy}"] \\
B(Px,Py) \arrow[r, "Ph_*"'] & B(Px,Pz)
\end{tikzcd}$$

It is a property of pullbacks that h_* reflects cartesian maps. Now since $h\alpha$ is cartesian, α is cartesian. $\square$

Definition 2.2.6. A 2-functor $P\colon E \to B$ is a *2-fibration* if

1. for any $e \in E$ and $f\colon b \to Pe$, there is a cartesian 1-cell $h\colon a \to e$ with $Ph = f$;

2. for any $g \in E$ and $\alpha\colon f \Rightarrow Pg$, there is a cartesian 2-cell $\sigma\colon h \Rightarrow g$ with $P\sigma = \alpha$; and

3. the horizontal composite of any two cartesian 2-cells is cartesian.

We will often say that E is a *fibred 2-category* when it is the domain of a 2-fibration $P\colon E \to B$. This terminology is informal and should only be used when it is clear from context which 2-fibration makes E a fibred 2-category.

Remark 2.2.7. The second condition in Definition 2.2.6 could equivalently be stated as "$P_{xy}\colon E(x,y) \to B(Px,Py)$ is a fibration for all x, y in E". In this case we say that P is *locally fibred.*

Remark 2.2.8. The third condition in Definition 2.2.6 could equivalently be stated as "cartesian 2-cells are closed under pre-composition and post-composition with arbitrary 1-cells". This is a consequence of the middle-four interchange and the fact that cartesian 2-cells are closed under vertical composition.

Remark 2.2.9. The first definition of 2-fibration was given by Hermida in [Her99]. His local characterisation of 2-fibrations (Theorem 2.8) is identical to our definition except that it only requires cartesian 2-cells to be closed under pre-composition with any 1-cell. We insist that cartesian 2-cells also be closed under *post*-composition with any 1-cell. The two definitions are not equivalent. This is illustrated by the following example.

Remember that Fib is the 2-category whose objects are fibrations P, Q in Cat; 1-cells are pairs of functors (F,G) such that $QF = GP$ and F preserves cartesian maps; and 2-cells are pairs of natural transformations (α,β) such that $Q\alpha = \beta P$.

$$\begin{tikzcd}
E \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[d, "P"'] & D \arrow[d, "Q"] \\
B \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & C
\end{tikzcd}$$

There is an obvious projection cod: Fib $\to$ Cat that sends P to B, (F,G) to G and (α,β) to β.

This is the 2-functor Hermida takes as a prototype for 2-fibrations. Its cartesian 1-cells are pullback squares and its cartesian 2-cells are (α,β) where α is point-wise cartesian. Pre-composition with 1-cells obviously preserves cartesian 2-cells because it amounts to re-indexing the natural transformations. Post-composition with 1-cells also preserves cartesian 2-cells because the 1-cells in Fib are cartesian in their first component.

Suppose we modify Fib by not requiring that the first component of each 1-cell preserves cartesian maps. In this case cod is a Hermida-style 2-fibration and not a 2-fibration by our definition. Thus the definitions are not equivalent.

The post-composition property is essential for building a pseudo-inverse to the Grothendieck construction. It arises when defining the action of a trihomomorphism on 2-cells. We will see in Proposition 2.2.19 and Construction 2.3.25 below where the property is explicitly required.

Definition 2.2.10. A *cleavage* for a 2-fibration $P\colon E \to B$ is a function $\varphi(\text{-},\text{-})$ that describes a choice of cartesian lifts. If $e \in E$ and $f\colon b \to Pe$ is a 1-cell in B then $\varphi(f,e)\colon f^*e \to e$ is a cartesian 1-cell in $\mathscr{E}$ with $P\varphi(f,e) = f$ which we call the *chosen cartesian lift of f at e*. The function also acts on 2-cells: if $k\colon a \to b$ is a 1-cell in $\mathscr{E}$ and $\alpha\colon h \Rightarrow Pk$ is a 2-cell in E then $\varphi(\alpha,k)\colon \alpha^*k \Rightarrow k$ is a cartesian 2-cell in $\mathscr{E}$ with $P\varphi(\alpha,k) = \alpha$ which we call the *chosen cartesian lift of α at k*. A 2-fibration with a cleavage is called a *cloven* 2-fibration.

A *split* 2-fibration is a cloven 2-fibration where $\varphi(\text{-},\text{-})$ satisfies the following five equations. For all $e \in \mathscr{E}$, $g\colon b \to Pe$ and $f\colon a \to b$,

$$\varphi(gf,e) = \varphi(g,e).\varphi(f,g^*e)\ . \tag{2.2.1}$$

For all $k\colon c \to d$ in $\mathscr{E}$, $\beta\colon h \to Pk$ and $\alpha\colon j \to j$,

$$\varphi(\beta\alpha,k) = \varphi(\beta,k).\varphi(\alpha,\beta^*e)\ . \tag{2.2.2}$$

For all $k\colon x \to y$, $l\colon y \to z$ in $\mathscr{E}$, $\alpha\colon h \to Pk$ and $\gamma\colon m \to Pl$,

$$\varphi(\gamma * \alpha, lk) = \varphi(\gamma,l) * \varphi(\alpha,k)\ . \tag{2.2.3}$$

For all e and $k\colon c \to d$ in $\mathscr{E}$,

$$\varphi(1_{Pe},e) = 1_e\ , \tag{2.2.4}$$

$$\varphi(1_{Pk},k) = 1_k\ . \tag{2.2.5}$$

These conditions specify that chosen cartesian maps be closed under all forms of composition and that chosen cartesian lifts of identities are identities. We say that P is *locally split* when each P_{xy} is split (conditions (2.2.2) and (2.2.4)). We say that P is *horizontally split* when chosen cartesian 2-cells are closed under horizontal composition (condition (2.2.3)).

Remark 2.2.11. Every 2-fibration can be equipped with a cleavage using the axiom of choice. Since cartesian lifts are unique up to isomorphism the choice of cleavage for a

2-fibration does not significantly affect its behaviour. As a result we usually suppose that every 2-fibration is cloven and rarely distinguish between one cleavage and another.

Remark 2.2.12. There is some subtlety in equation (2.2.3) in the definition of split 2-fibration. We could equally well ask that chosen cartesian 2-cells be closed under pre- and post-composition with arbitrary 1-cells. That is, for every pair of composable 1-cells j and k, if $\alpha\colon h \Rightarrow k$ then

$$\varphi(\alpha, k)j = \varphi(\alpha Pj, kj) \tag{2.2.6}$$

and if $\gamma\colon l \Rightarrow j$ then

$$k\varphi(\gamma, j) = \varphi(Pk\gamma, kj)\ . \tag{2.2.7}$$

The equivalence of these two statements relies on equations (2.2.1), (2.2.2) and the middle-four interchange.

Definition 2.2.13. Suppose $P\colon E \to B$ and $Q\colon D \to B$ are 2-fibrations. A 2-functor $\eta\colon E \to D$ between fibred 2-categories is *cartesian* when it preserves all cartesian maps and $Q\eta = P$. A 2-natural transformation $\alpha\colon \eta \Rightarrow \tau$ is *vertical* when $Q\alpha = 1_P$. A modification $\Gamma\colon \alpha \Rrightarrow \beta$ is *vertical* when $Q\Gamma = 1_{1_P}$. Together, these constitute the 0,1,2,3-cells of a 3-category $\mathrm{Fib}(B)$

A cartesian 2-functor $\eta\colon E \to D$ is *split* when it also preserves choice of cartesian maps. That is, for all $f : b \to Pe$

$$\eta(\varphi(f, e)) = \varphi(f, \eta(e)) \tag{2.2.8}$$

and for all $\alpha\colon h \Rightarrow k$

$$\eta(\varphi(\alpha, k)) = \varphi(\alpha, \eta(k))\ . \tag{2.2.9}$$

Suppose that B is a 2-category. Let $\mathrm{Fib}_s(B)$ be the sub-3-category of $\mathrm{Fib}(B)$ containing the split 2-fibrations over B, split cartesian 2-functors, vertical 2-natural transformations and vertical modifications. Let 2Cat be the 3-category of 2-categories, 2-functors, 2-natural transformations and modifications. Let $[B^{\mathrm{coop}}, 2\mathrm{Cat}]$ be the 3-category of contravariant 2-functors from B to 2Cat, 2-natural transformations, modifications and perturbations. A *perturbation* is a morphism of modifications as defined in [GPS95; Gur06].

The following two results will prove useful.

Proposition 2.2.14. *Suppose that $P\colon E \to B$ is a 2-functor and that α and β are cartesian 2-cells in E with domains and codomains as indicated in the diagram below. If $\beta * \alpha$ is cartesian then all cartesian 2-cells over $P\alpha$ and $P\beta$ are closed under horizontal composition.*

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \arrow[r, bend left, "g"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & c \end{tikzcd}$$

Proof. Suppose that α, β and $\beta * \alpha$ are cartesian and γ and δ are cartesian lifts of $P\alpha$ and $P\beta$. There exist unique isomorphisms η and τ such that $\gamma = \alpha.\eta$ and $\delta = \beta.\tau$. Then

$\delta * \gamma = (\beta.\tau) * (\alpha.\eta) = (\beta * \alpha).(\tau * \eta)$. Since $\beta * \alpha$ is cartesian and $\tau * \eta$ is an isomorphism $\delta * \gamma$ is cartesian. □

Proposition 2.2.15. *Suppose that P and Q are 2-fibrations, F is a 2-functor and the following diagram commutes.*

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "P"'] & & D \arrow[dl, "Q"] \\ & B & \end{tikzcd}$$

If F sends chosen cartesian maps to cartesian maps then it preserves all cartesian maps.

Proof. Suppose that $\alpha \colon f \Rightarrow h$ is cartesian in D. It factors as $\alpha = \varphi(P\alpha, h).\sigma$ where σ is an isomorphism. Then $F\alpha = F\varphi(P\alpha, h).F\sigma$ and since F preserves chosen cartesian maps and isomorphisms $F\alpha$ is cartesian. The reasoning for cartesian 1-cells is exactly the same. □

2.2.2 The Grothendieck construction

We will describe the Grothendieck construction for split fibred 2-categories: for every 2-category B, an equivalence

$$\mathrm{el} \colon [B^{\mathrm{coop}}, 2\mathrm{Cat}] \to \mathrm{Fib}_s(B).$$

We are mostly concerned with its action on objects and use "Grothendieck construction" to mean both the action on objects and the whole 3-functor. This generalises the Grothendieck construction for ordinary fibrations. We found that the Grothendieck construction for general 2-fibrations could not be described so neatly as for split 2-fibrations; this is discussed in Remark 2.2.28. In Section 2.3 we will give a more general result: the Grothendieck construction for fibred bicategories.

Construction 2.2.16 (The Grothendieck construction for 2-categories)**.** Suppose that $F \colon B^{\mathrm{coop}} \to 2\mathrm{Cat}$ is a 2-functor. Let $\mathrm{el}\, F$ be the 2-category:

- 0-cells are pairs (x, x_-) where $x \in B$ and $x_- \in Fx$.
- 1-cells are pairs $(f, f_-) \colon (x, x_-) \to (y, y_-)$ where $f \colon x \to y$ and $f_- \colon x_- \to Ff(y_-)$.
- 2-cells are pairs $(\alpha, \alpha_-) \colon (f, f_-) \Rightarrow (g, g_-) \colon (x, x_-) \to (y, y_-)$ where $\alpha \colon f \Rightarrow g$ and

$$\begin{tikzcd} x_- \arrow[r, "f_-"] \arrow[dr, "g_-"'] \arrow[r, phantom, shift right=4, "\Downarrow \alpha_-"] & Ff(y_-) \\ & Fg(y_-) \arrow[u, "F\alpha_{y_-}"'] \end{tikzcd} .$$

- If (α, α_-) as above and $(\gamma, \gamma_-)\colon (g, g_-) \Rightarrow (h, h_-)\colon (x, x_-) \to (y, y_-)$ then the composite $(\gamma, \gamma_-).(\alpha, \alpha_-)$ has first component $\gamma.\alpha$ and second component

$$\begin{tikzcd}
x_- \arrow[r, "f_-"] \arrow[dr, "g_-"'] \arrow[ddr, bend right, "h_-"'] \arrow[dr, phantom, bend left, "\Downarrow\alpha_-"] \arrow[ddr, phantom, bend right=15, "\Downarrow\gamma_-"] & Ff(y_-) \\
& Fg(y_-) \arrow[u, "F\alpha_{y_-}"'] \\
& Fh(y_-) \arrow[u, "F\gamma_{y_-}"']
\end{tikzcd} .$$

- If (α, α_-) as above and $(\beta, \beta_-)\colon (j, j_-) \Rightarrow (k, k_-)\colon (y, y_-) \to (z, z_-)$ then the composite $(\beta, \beta_-) * (\alpha, \alpha_-)$ has first component $\beta * \alpha$ and second component

$$\begin{tikzcd}
x_- \arrow[r, "f_-"] \arrow[dr, "g_-"'] \arrow[dr, phantom, bend left, "\Downarrow\alpha_-"] & Ff(y_-) \arrow[r, "Ff(j_-)"] \arrow[dr, "Ff(k_-)"'] \arrow[dr, phantom, bend left, "\Downarrow Ff(\beta_-)"] & FfFj(z_-) \\
& Fg(y_-) \arrow[u, "F\alpha_{y_-}"'] \arrow[dr, "Fg(k_-)"'] & FfFk(z_-) \arrow[u, "Ff(F\beta_{z_-})"'] \\
& & FgFk(z_-) \arrow[u, "F\alpha_{Fk(z_-)}"']
\end{tikzcd} .$$

- Identity 1-cells are $(1_x, 1_{x_-})\colon (x, x_-) \to (x, x_-)$ and identity 2-cells are $(1_f, 1_{f_-})\colon (f, f_-) \Rightarrow (f, f_-)$.

By projecting onto the first component of $\mathrm{el}\, F$ we obtain a 2-functor $P_F\colon \mathrm{el}\, F \to B$.

Proposition 2.2.17. *For any 2-functor $F\colon B^{\mathrm{coop}} \to 2\mathrm{Cat}$, the projection $P_F\colon \mathrm{el}\, F \to B$ of Construction 2.2.16 is a split 2-fibration.*

Proof. It is easy to show that $\mathrm{el}\, F$ is a 2-category. Associativity and unit laws rely on the fact that F is a 2-functor and that it maps into 2Cat. Now notice that in the first component of $\mathrm{el}\, F$ composition is just composition in B. Thus P_F is a 2-functor.

We need to show that P_F is a 2-fibration. Suppose that (y, y_-) is an object of $\mathrm{el}\, F$ and $f\colon x \to y$ in B. We claim that $(f, 1_{Ff(y_-)})\colon (x, Ff(y_-)) \to (y, y_-)$ is cartesian over f. This can be verified by examining the following commuting diagrams where the diagram on the left maps down to the diagram on the right under the action of P_F

$$\begin{tikzcd}
(z, z_-) \arrow[d, dotted, bend right, "(h', g_-')"'] \arrow[d, dotted, bend left, "(h, g_-)"] \arrow[d, phantom, "\overset{\Leftarrow}{(\beta,\alpha_-)}"] \arrow[dr, "(g', g_-')"'] \arrow[dr, bend left, "(g, g_-)"] \arrow[dr, phantom, bend left=15, "\Downarrow(\alpha,\alpha_-)"] & \\
(x, Ff(y_-)) \arrow[r, "(f, 1_{Ff(y_-)})"'] & (y, y_-)
\end{tikzcd}
\qquad
\begin{tikzcd}
z \arrow[d, bend right, "h'"'] \arrow[d, bend left, "h"] \arrow[d, phantom, "\overset{\Leftarrow}{\beta}"] \arrow[dr, "g'"'] \arrow[dr, bend left, "g"] \arrow[dr, phantom, bend left=15, "\Downarrow\alpha"] & \\
x \arrow[r, "f"'] & y
\end{tikzcd}$$

and showing that the indicated lifts are unique.

Suppose that $(g, g_-)\colon (x, x_-) \to (y, y_-)$ in $\mathrm{el}\, F$ and $\alpha\colon f \Rightarrow g\colon x \to y$ in B. We claim that $(\alpha, 1_{F\alpha_{y_-}g_-})\colon (f, F\alpha_{y_-}g_-) \Rightarrow (g, g_-)$ is cartesian over α. This can be verified by examining the following commuting diagrams where the diagram on the left maps down to the diagram on the right under the action of P_F

$$\begin{tikzcd}
(h, h_-) \arrow[d, Rightarrow, dotted, "{(\sigma,\beta_-)}"'] \arrow[dr, Rightarrow, "{(\beta,\beta_-)}"] & \\
(f, F\alpha_{y_-}g_-) \arrow[r, Rightarrow, "{(\alpha,1_{F\alpha_{y_-}g_-})}"'] & (g, g_-)
\end{tikzcd}
\qquad
\begin{tikzcd}
h \arrow[d, Rightarrow, "\sigma"'] \arrow[dr, Rightarrow, "\beta"] & \\
f \arrow[r, Rightarrow, "\alpha"'] & g
\end{tikzcd}$$

and showing that the indicated lift is unique.

When

$$(x, x_-) \xrightarrow{(f,f_-)} (y, y_-) \xrightarrow{(g,g_-)} (z, z_-)$$

is a diagram in $\mathrm{el}\, F$ which maps to the codomain of the following composite in B

$$\begin{tikzcd}
x \arrow[r, bend left, "f'"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, bend left, "g'"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & z
\end{tikzcd} ,$$

the chosen cartesian lifts of α and σ compose to give

$$\begin{aligned}
(\alpha, 1_{F\alpha_{y_-}f_-}) * (\sigma, 1_{F\sigma_{z_-}g_-}) &= (\alpha * \sigma, Ff'(1_{F\sigma_{z_-}g_-}) * 1_{F\alpha_{y_-}f_-}) \\
&= (\alpha * \sigma, 1_{Ff'(F\sigma_{z_-}g_-)F\alpha_{y_-}f_-}) \\
&= (\alpha * \sigma, 1_{F(\sigma*\alpha)_{z_-}Ff(g_-)f_-})
\end{aligned}$$

because Ff is a 2-functor, $F\alpha$ is 2-natural and F is a 2-functor. Thus *chosen* cartesian 2-cells are closed under horizontal composition and by Proposition 2.2.14 *all* cartesian 2-cells are closed under horizontal composition. Thus P_F is a 2-fibration.

We need to show that P_F is split. The equations above demonstrate that P_F is horizontally split. The other two conditions are a matter of routine verification. □

Construction 2.2.18 (Pseudo-inverse to the Grothendieck construction)**.** Suppose that $P\colon E \to B$ is a split 2-fibration. We define a functor $F_P\colon B^{\mathrm{coop}} \to 2\mathrm{Cat}$ as follows:

on 0-cells: For all $b \in B$, set $F_P b = E_b$ where E_b is the fibre of P over b i.e. the sub-category of E with 0-, 1-, and 2-cells being those that map to b, 1_b and 1_{1_b}.

on 1-cells: For all $f\colon b \to b'$ in B, set $F_P f = f^*\colon E_{b'} \to E_b$ to be the 2-functor described by the following diagram.

$$
\begin{tikzcd}
f^*e \arrow[r, "\varphi(f,e)"] \arrow[d, bend right, dotted, "f^*h"'] \arrow[d, bend left, dotted, "f^*k"] \arrow[d, phantom, "f^*\alpha"] & e \arrow[d, bend right, "h"'] \arrow[d, bend left, "k"] \arrow[d, phantom, "\Rightarrow\alpha"] \arrow[r, phantom, ""] & {} \arrow[d, phantom, "\overset{P}{\longmapsto}"] & b \arrow[r, "f"] \arrow[d, bend right, "1_b"'] \arrow[d, bend left, "1_b"] \arrow[d, phantom, "="] & b' \arrow[d, bend right, "1_{b'}"'] \arrow[d, bend left, "1_{b'}"] \arrow[d, phantom, "="] \\
f^*e' \arrow[r, "\varphi(f,e')"'] & e' & {} & b \arrow[r, "f"'] & b'
\end{tikzcd}
$$

It sends e to the domain of $\varphi(f,e)$. It sends h, k to the unique f^*h, f^*k over 1_b generated by the cartesian 1-cell $\varphi(f,e')$ and α to the unique $f^*\alpha$ over 1_{1_b}.

on 2-cells: For all $\sigma\colon f \Rightarrow g\colon b \to b'$ in B, set $F_P\sigma = \sigma^*\colon g^* \Rightarrow f^*\colon E_{b'} \to E_b$ is the 2-natural transformation described by the following diagram.

$$
\begin{tikzcd}
& f^*e \arrow[dr, "\varphi(f,e)"] \arrow[d, phantom, "\Downarrow \varphi(\sigma,\varphi(g,e))"] & & & \\
g^*e \arrow[ur, dotted, "\sigma_e^*"] \arrow[rr, bend right, "\varphi(g,e)"'] & {} & e \arrow[r, mapsto, "P"] & b \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & b'
\end{tikzcd}
$$

We take the cartesian lift of σ at $\varphi(g,e)$ and uniquely factorise its domain as $\varphi(f,e)\sigma_e^*$. Then $(F_P\sigma)_e = \sigma_e^*$. This unique factorisation is explained in Proposition 2.2.20.

Proposition 2.2.19. *Suppose that $P\colon E \to B$ is a split 2-fibration, then $F_P\colon B^{\mathrm{coop}} \to$ 2Cat defined in Construction 2.2.18 is a 2-functor.*

Proof. First of all, it is clear that $Fb = E_b$ is well defined as a 2-category. Second, when $f\colon b \to b'$ in B and α, β are 2-cells in E with domains and codomains as indicated in the diagram below, we get $\varphi(f,e')(f^*\alpha.f^*\beta) = (\alpha.\beta)\varphi(f,e)$. Thus by the uniqueness of liftings along $\varphi(f,e)$ we have $f^*(\alpha.\beta) = f^*\alpha.f^*\beta$. The diagrams are:

$$
\begin{tikzcd}
f^*e \arrow[r, "\varphi(f,e)"] \arrow[d, bend right=50, "f^*g"'] \arrow[d, "f^*h" description] \arrow[d, bend left=50, "f^*k"] \arrow[d, phantom, "\overset{f^*\beta}{\Leftarrow}\ \overset{f^*\alpha}{\Leftarrow}"] & e \arrow[d, bend right=50, "g"'] \arrow[d, "h" description] \arrow[d, bend left=50, "k"] \arrow[d, phantom, "\overset{\beta}{\Leftarrow}\ \overset{\alpha}{\Leftarrow}"] & & f^*e \arrow[r, "\varphi(f,e)"] \arrow[d, bend right, "f^*h"'] \arrow[d, bend left, "f^*h"] \arrow[d, phantom, "\underset{f^*1_h}{\Leftarrow}"] & e \arrow[d, bend right, "h"'] \arrow[d, bend left, "h"] \arrow[d, phantom, "\underset{1_h}{\Leftarrow}"] \\
f^*e' \arrow[r, "\varphi(f,e')"'] & e' & & f^*e' \arrow[r, "\varphi(f,e')"'] & e'
\end{tikzcd}
$$

Similarly, we have $\varphi(f,e')1_{f^*h} = 1_h\varphi(f,e)$ and thus $f^*1_h = 1_{f^*h}$. Very similar arguments tell us that if α and β are horizontally composable 2-cells in $\mathscr{E}$ then $f^*(\beta * \alpha) = f^*\beta * f^*\alpha$ and $f^*(1_e) = 1_{f^*e}$ for all $e \in \mathscr{E}$. Thus f^* is a 2-functor.

On 2-cells, if $\sigma\colon g \Rightarrow f\colon x \to y$ is a 2-cell in $\mathscr{B}$ we display the 2-natural transformation

σ^* using the following diagram.

$$
\begin{tikzcd}
 & & g^*e \arrow[drr, "\varphi(g,e)"] \arrow[dd, dotted, bend right] \arrow[dd, dotted, bend left] \arrow[dd, phantom, "\Leftarrow"] & & \\
f^*e \arrow[urr, "\sigma^*_e"] \arrow[rrrr, "\varphi(f,e)"'] \arrow[rrrr, phantom, shift left=3, "\Downarrow\varphi(\sigma,\varphi(g,e))"] \arrow[dd, bend right, "f^*k"'] \arrow[dd, bend left, "f^*h"] \arrow[dd, phantom, "\Leftarrow f^*\alpha"] & & & & e \arrow[dd, bend right, "k"'] \arrow[dd, bend left, "h"] \arrow[dd, phantom, "\Leftarrow \alpha"] \\
 & & g^*e' \arrow[drr, "\varphi(g,e')"] & & \\
f^*e' \arrow[urr, "\sigma^*_{e'}"] \arrow[rrrr, "\varphi(f,e')"'] \arrow[rrrr, phantom, shift left=3, "\Downarrow\varphi(\sigma,\varphi(g,e'))"] & & & & e'
\end{tikzcd}
$$

The unlabelled 2-cell at the back is $g^*\alpha$. We want to show that $g^*\alpha.\sigma^*_e$ equals $\sigma^*_{e'}.f^*\alpha$. For simplicity, let $\eta = \varphi(\sigma, \varphi(g, e))$ and $\tau = \varphi(\sigma, \varphi(g, e'))$. Since $g^*\alpha\sigma^*_e$ and $\sigma^*_{e'}f^*\alpha$ both map down to 1_b, and $\varphi(g, e')$ is cartesian, and lifts along cartesian 1-cells are equal, it is enough to show that $\varphi(g, e')g^*\alpha\sigma^*_e = \varphi(g, e')\sigma^*_{e'}f^*\alpha$. Similarly, since τf^*k is cartesian, it is enough to show that $\tau f^*k.\varphi(g, e')(g^*\alpha)\sigma^*_e = \tau f^*k.\varphi(g, e')(\sigma^*_{e'})f^*\alpha$. In the equations below we use various combinations of the middle-four interchange, the fact that cartesian 2-cells are closed under pre- and post-composition, that cartesian 2-cells are split, and that both front and back-right squares commute.

$$
\begin{aligned}
\tau f^*k.\varphi(g, e')(\sigma^*_{e'})f^*\alpha &= \varphi(f, e')f^*\alpha.\tau f^*h \\
&= \alpha\varphi(f, e).h\eta \\
&= \alpha\varphi(g, e)\sigma^*_e.k\eta \\
&= \varphi(g, e')(g^*\alpha)\sigma^*_e.k\eta \\
&= \tau f^*k.\varphi(g, e')(g^*\alpha)\sigma^*_e\ .
\end{aligned}
$$

Thus the back-left composites are equal and σ^* is a 2-natural transformation.

Note that the 2-naturality of σ^* relies heavily on the post-composition property of cartesian 2-cells (Remark 2.2.9). We needed to show that $\tau f^*h = h\eta$. We proved it by saying that since both η and τ are cartesian, τf^*h and $h\eta$ are cartesian, but they both sit over σ, so by the splitness property they must be equal. Without post-composition, we couldn't say that $h\eta$ is cartesian and hence that σ^* is 2-natural. Without the post-composition property F_P would not be well-defined on 2-cells.

Now we need to show that F_P preserves composition and identities. If $\alpha \colon f \Rightarrow g$ and

$\beta\colon h \Rightarrow f$ in $\mathscr{B}$ and $e \in E$ then we can form two different composites

$$\begin{tikzcd} & h^*e \arrow[dr, "\varphi(h,e)"] & \\ g^*e \arrow[ur, "(\alpha\beta)^*_e"] \arrow[rr, bend right, "\varphi(g,e)"'] \arrow[rr, phantom, "\Downarrow\varphi(\alpha\beta,\varphi(g,e))"] & & e \end{tikzcd} \qquad \begin{tikzcd} & & h^*e \arrow[ddr, "\varphi(h,e)"] & \\ & f^*e \arrow[ur, "\beta^*_e"] \arrow[drr, "\varphi(f,e)"'] \arrow[rr, phantom, "\Downarrow\varphi(\beta,\varphi(f,e))"] & & \\ g^*e \arrow[ur, "\alpha^*_e"] \arrow[rrr, bend right, "\varphi(g,e)"'] \arrow[rr, phantom, "\Downarrow\varphi(\alpha,\varphi(g,e))"] & & & e \end{tikzcd}$$

and assert that $\varphi(\alpha\beta, \varphi(g,e)) = \varphi(\alpha, \varphi(g,e)).\varphi(\beta, \varphi(f,e))\alpha^*_e$ because of the splitness conditions. Since $(\alpha\beta)^*_e$ is defined via unique factorisation it follows directly that $(\alpha\beta)^* = \beta^*\alpha^*$. Similarly, $\varphi(1_g, \varphi(g,e)) = 1_{\varphi(g,e)}$ and hence $(1_g)^* = 1_{g^*}$. Thus 2-cell composition is preserved.

For composition of 1-cells, suppose that f and g are composable 1-cells in $\mathscr{B}$ and $\alpha\colon h \Rightarrow k\colon e \to e'$ is a 2-cell in $\mathscr{E}$, then we get the following pasting diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge] g^*f^*e \arrow[r, "\varphi(g,f^*e)"] \arrow[d, bend right, "g^*f^*k"'] \arrow[d, bend left, "g^*f^*h"] \arrow[d, phantom, "\overset{g^*f^*\alpha}{\Rightarrow}"] & f^*e \arrow[r, "\varphi(f,e)"] \arrow[d, bend right, "f^*k"'] \arrow[d, bend left, "f^*h"] \arrow[d, phantom, "\overset{f^*\alpha}{\Rightarrow}"] & e \arrow[d, bend right, "h"'] \arrow[d, bend left, "k"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\ g^*f^*e' \arrow[r, "\varphi(g,f^*e')"'] & f^*e' \arrow[r, "\varphi(f,e')"'] & e' \end{tikzcd}$$

and assert that $\varphi(gf, e) = \varphi(f,e).\varphi(g, f^*e)$ because of the splitness conditions. It follows directly that $f^*g^* = (gf)^*$. Similarly, $\varphi(1_{b'}, e) = 1_e$ and hence $(1_{b'})^* = 1_{E_{b'}}$. Thus 1-cell composition is preserved.

For horizontal composition of 2-cells, if $\sigma\colon h \Rightarrow g$ and $\tau\colon k \Rightarrow f$ are horizontally composable 2-cells in $\mathscr{B}$ and $e \in \mathscr{E}$ we get diagrams

$$\begin{tikzcd} & & h^*k^*e \arrow[dr, "\varphi(h,k^*e)"] & & \\ & h^*f^*e \arrow[ur, "h^*\tau^*_e"] \arrow[dr, "\varphi(h,f^*e)"'] & & k^*e \arrow[dr, "\varphi(k,e)"] & \\ g^*f^*e \arrow[ur, "\sigma^*_{f^*e}"] \arrow[rr, bend right, "\varphi(g,f^*e)"'] \arrow[rr, phantom, "\Downarrow\varphi(\sigma,\varphi(g,f^*e))"] & & f^*e \arrow[ur, "\tau^*_e"] \arrow[rr, bend right, "\varphi(f,e)"'] \arrow[rr, phantom, "\Downarrow\varphi(\tau,\varphi(f,e))"] & & e \end{tikzcd}$$

and

$$\begin{tikzcd} & hk^*e \arrow[dr, "\varphi(hk,e)"] & \\ gf^*e \arrow[ur, "(\sigma\tau)^*_e"] \arrow[rr, bend right, "\varphi(gf,e)"'] \arrow[rr, phantom, "\Downarrow\varphi(\sigma\tau,\varphi(gf,e))"] & & e \end{tikzcd} .$$

The top-left 1-cells are the components of $(\sigma\tau)^*$ and $\tau^*\sigma^*$. By the splitness conditions on 2-cells these two diagrams are equal as 2-cells. By the splitness conditions on 1-cells the top-right 1-cells are equal. Finally, by uniqueness of factorisation, the top-left 1-cells are equal. □

To prove that the Grothendieck construction is surjective up to isomorphism, we will need the following two results.

Proposition 2.2.20. *Suppose $P\colon E \to B$ is a 2-fibration. Every $f\colon x \to z$ in E factors uniquely as*

$$\begin{tikzcd} x \arrow[d, "\hat{f}"'] \arrow[dr, "f"] & \\ y \arrow[r, "\varphi(Pf,z)"'] & z \end{tikzcd}$$

where $P\hat{f} = 1_{Px}$.

Proof. Simply note that

$$\begin{tikzcd} x \arrow[dr, "f"] & \\ y \arrow[r, "\varphi(Pf,z)"'] & z \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} Px \arrow[d, "1_{Px}"'] \arrow[dr, "Pf"] & \\ Px \arrow[r, "Pf"'] & Pz \end{tikzcd}$$

and there exists a unique $\hat{f}\colon x \to y$ with $P\hat{f} = 1_{Px}$ and $f = \varphi(Pf,z)\hat{f}$. □

Proposition 2.2.21. *Suppose $P\colon E \to B$ is a 2-fibration and $\alpha\colon f \Rightarrow g$ is a 2-cell in $\mathscr{E}$. There exist unique $\hat{f}, \hat{g}, \hat{\alpha}$ such that*

$$\begin{tikzcd} w \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & z \end{tikzcd} = \begin{tikzcd} w \arrow[rr, bend left, "\hat{f}"] \arrow[dr, "\hat{g}"'] \arrow[rr, phantom, "\Downarrow\hat{\alpha}"] & & y \arrow[dr, "\varphi(Pf,z)"] & \\ & x \arrow[ur, "\hat{h}"] \arrow[rr, bend right, "\varphi(Pg,z)"'] \arrow[rr, phantom, "\Downarrow\varphi(P\alpha,\varphi(Pg,z))"] & & z \end{tikzcd},$$

$P\hat{f} = P\hat{g} = P\hat{h} = 1_{Pw}$ and $P\hat{\alpha} = 1_{1_{Pw}}$.

Proof. The proof is similar to that above but somewhat more involved. Begin by uniquely factoring $f = \varphi(Pf,z) \cdot \hat{f}$ and $g = \varphi(Pg,z) \cdot \hat{g}$ by Proposition 2.2.20. Now take the cartesian lift of $P\alpha$ at $\varphi(Pg,z)$:

$$\begin{tikzcd} x \arrow[r, bend left, "h"] \arrow[r, bend right, "\varphi(Pg,z)"'] \arrow[r, phantom, "\Downarrow\varphi(P\alpha,\varphi(Pg,z))"] & z \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} Px \arrow[r, bend left, "Pf"] \arrow[r, bend right, "Pg"'] \arrow[r, phantom, "\Downarrow P\alpha"] & Pz \end{tikzcd}.$$

Then $\varphi(P\alpha, \varphi(Pg, z))\hat{g}$ is cartesian over $P\alpha$ and

$$\begin{tikzcd}
f \arrow[dr, Rightarrow, "\alpha"] & \\
h\hat{g} \arrow[r, Rightarrow, "\varphi(P\alpha,\ldots)\hat{g}"'] & g
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
Pf \arrow[d, Rightarrow, "1_{Pf}"'] \arrow[dr, Rightarrow, "P\alpha"] & \\
Pf \arrow[r, Rightarrow, "P\alpha"'] & Pg
\end{tikzcd}$$

so there exists a unique η with $P\eta = 1_{Pf}$ and

$$\begin{tikzcd}
w \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] & z
\end{tikzcd}
=
\begin{tikzcd}
w \arrow[rr, bend left, "\hat{f}"] \arrow[dr, "\hat{g}"'] \arrow[rr, phantom, "\Downarrow\eta"] & & y \arrow[dr, "\varphi(Pf,z)"] & \\
& x \arrow[rr, bend left, "h"{name=H}] \arrow[rr, bend right, "\varphi(Pg,z)"'{name=G}] \arrow[from=H, to=G, phantom, "\Downarrow\varphi(P\alpha,\varphi(Pg,z))"] & & z
\end{tikzcd} .$$

By Proposition 2.2.20 we factor $h = \varphi(Pf, z) \cdot \hat{h}$ uniquely where $P\hat{h} = 1_{Px}$. Finally, we observe that

$$\begin{tikzcd}
w \arrow[d, bend right, "\hat{h}\hat{g}"'] \arrow[d, bend left, "\hat{f}"] \arrow[dr, bend right] \arrow[dr, bend left, "f"] \arrow[dr, phantom, "\Leftarrow\eta"] & \\
y \arrow[r, "\varphi(Pf,z)"'] & z
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
Pw \arrow[d, bend right, "1_{Pw}"'] \arrow[d, bend left, "1_{Pw}"] \arrow[d, phantom, "\Leftarrow \hat{1}_1"] \arrow[dr, bend right, "Pf"'] \arrow[dr, bend left, "Pf"] \arrow[dr, phantom, "\Leftarrow 1_{Pf}"] & \\
Pw \arrow[r, "Pf"'] & Pz
\end{tikzcd}$$

so there exists a unique $\hat{\alpha}\colon \hat{f} \Rightarrow \hat{h}\hat{g}$ over $1_{1_{Pw}}$ with $\eta = \hat{\alpha}\varphi(Pf, z)$ and hence a unique factorisation of α as stated. $\square$

Remark 2.2.22. This last result (Proposition 2.2.21) is recognised by Hermida in proposition 2.4 of [Her99]. He doesn't explicitly mention the uniqueness of such factorisations.

Remark 2.2.23. The factorisations of Proposition 2.2.20 and 2.2.21 are unique up to choice of cartesian lift. In each of these results we have implicitly supposed that P is cloven and that the factorisation occurs through the chosen cartesian lift. In fact, there is a unique factorisation for every cleavage on the fibration.

Construction 2.2.24 (Surjective up to isomorphism)**.** In order to show that the Grothendieck construction is surjective up to isomorphism, we need to find for every 2-fibration $P\colon E \to B$ an invertible map of fibrations H with

$$\begin{tikzcd}
\operatorname{el} F_P \arrow[rr, "H"] \arrow[dr, "\pi"'] & & E \arrow[dl, "P"] \\
& B &
\end{tikzcd} .$$

Here we will use π in place of P_{F_P} which is somewhat notationally confusing. First, what is $\mathrm{el}\, F_P$? Its data consists of:

0-cells: pairs (x, x_-) where $x_- \in E$ and $Px_- = x$.

1-cells: pairs $(f, f_-)\colon (x, x_-) \to (y, y_-)$ where $f\colon x \to y$ in B and $f_- :\ x_- \to\ f^*(y_-)$ in E_x.

2-cells: pairs $(\alpha, \alpha_-)\colon (f, f_-) \Rightarrow (g, g_-)\colon (x, x_-) \to (y, y_-)$ where $\alpha\colon f \to g$ in B and $\alpha_-\colon f_- \to \alpha^*_{y_-} g_-$ in E_x.

$$\begin{tikzcd}
x_- \arrow[r, "f_-"] \arrow[dr, "g_-"'] \arrow[dr, phantom, shift left=3, "\Downarrow \alpha_-"] & f^*(y_-) \\
& g^*(y_-) \arrow[u, "\alpha^*_{y_-}"']
\end{tikzcd}$$

We define $H\colon\ \mathrm{el}\, F_P \to E$ by

$$\begin{tikzcd}
(x, x_-) \arrow[r, bend left, "{(f,f_-)}"] \arrow[r, bend right, "{(g,g_-)}"'] \arrow[r, phantom, "\Downarrow (\alpha,\alpha_-)"] & (y, y_-) & \longmapsto & x_- \arrow[r, bend left, "{\varphi(f,y_-)f_-}"] \arrow[r, bend right, "{\varphi(g,y_-)g_-}"'] \arrow[r, phantom, "\Downarrow \varphi(\alpha,y_-)g_- \cdot \varphi(f,y_-)\alpha_-"] & y_- \ .
\end{tikzcd}$$

The action on 2-cells is to send (α, α_-) to

$$\begin{tikzcd}
x_- \arrow[rr, bend left, "f_-"] \arrow[dr, "g_-"'] \arrow[rr, phantom, shift right=2, "\Downarrow \alpha_-"] & & f^* y_- \arrow[dr, "{\varphi(f,y_-)}"] & \\
& g^* y_- \arrow[ur, "\alpha^*_{y_-}"] \arrow[rr, bend right, "{\varphi(g,y_-)}"'] \arrow[rr, phantom, "\Downarrow \varphi(\alpha,\varphi(g,y_-))"] & & y_-
\end{tikzcd} .$$

The reader can verify that this is 2-functorial.

Proposition 2.2.25. *H is a split cartesian isomorphism.*

Proof. First,

$$\begin{aligned}
\pi(x, x_-) &= x = Px_- = PH(x, x_-) \\
\pi(f, f_-) &= f = Pf_- = PH(f, f_-) \\
\pi(\alpha, \alpha_-) &= \alpha = P\alpha_- = PH(\alpha, \alpha_-)
\end{aligned}$$

so $\pi = PH$. Second, the chosen cartesian maps in $\mathrm{el}\, F_P$ are those with identities in the second component. Since H acts by post-composition with chosen cartesian maps it is split cartesian. Third, for every $e \in E$ there exists a unique $(Pe, e) \in \mathrm{el}\, F_P$ with $H(Pe, e) = e$ so H is bijective on objects. Then Proposition 2.2.20 tells us that for every

$f \in E$ there exists a unique $\hat{f}$ with $H(f, \hat{f}) = \varphi(f, e)\hat{f} = f$ so H is bijective on 1-cells. Proposition 2.2.21 gives the same result on 2-cells. Thus H is an isomorphism. $\square$

Theorem 2.2.26. *For every 2-category B the Grothendieck construction is the action on objects of a 3-functor*

$$\mathrm{el}\colon [B^{\mathrm{coop}}, 2\mathrm{Cat}] \to 2\mathrm{Fib}_s(B)$$

and this is an equivalence.

Proof. We have already shown that el is surjective up to isomorphism on objects (Proposition 2.2.25). To show that el is an equivalence we need to define its action on 1,2,3-cells and show that it is locally an isomorphism.

Suppose $\eta\colon F \Rightarrow G$ is a 2-natural transformation in $[B^{\mathrm{coop}}, 2\mathrm{Cat}]$. We define $\mathrm{el}\,\eta\colon\ \mathrm{el}\,F \to \mathrm{el}\,G$ by

$$\begin{tikzcd} (x, x_-) \arrow[d, bend right=60, "{(f,f_-)}"'] \arrow[d, bend left=60, "{(g,g_-)}"] \arrow[d, phantom, "\overset{(\alpha,\alpha_-)}{\Rrightarrow}"] \\ (y, y_-) \end{tikzcd} \longmapsto \begin{tikzcd} (x, \eta_x x_-) \arrow[d, bend right=60, "{(f,\eta_x f_-)}"'] \arrow[d, bend left=60, "{(g,\eta_x g_-)}"] \arrow[d, phantom, "\overset{(\alpha,\eta_x\alpha_-)}{\Rrightarrow}"] \\ (y, \eta_y y_-) \end{tikzcd}.$$

This is a split cartesian 2-functor from P_F to P_G.Suppose $\Gamma\colon \eta \Rrightarrow \epsilon$ is a modification in $[B^{\mathrm{coop}}, 2\mathrm{Cat}]$. We define $\mathrm{el}\,\Gamma\colon\ \mathrm{el}\,\eta \Rightarrow \mathrm{el}\,\epsilon$ by

$$\mathrm{el}\,\Gamma_{(x,x_-)} = (1_x, (\Gamma_x)_{x_-})\colon (x, \eta_x x_-) \to (x, \epsilon_x x_-).$$

where $\mathrm{el}\,\Gamma_{(x,x_-)}\colon\ \mathrm{el}\,\eta(x, x_-) \to \mathrm{el}\,\epsilon(x, x_-)$. This is a vertical 2-natural transformation. Suppose $\zeta\colon \Gamma \to \Lambda$ is a perturbation in $[B^{\mathrm{coop}}, 2\mathrm{Cat}]$. We define $\mathrm{el}\,\zeta\colon\ \mathrm{el}\,\Gamma \Rrightarrow \mathrm{el}\,\Lambda$ by

$$\mathrm{el}\,\zeta_{(x,x_-)} = (1_{1_x}, (\zeta_x)_{x_-})\colon (1_x, (\Gamma_x)_{x_-}) \Rightarrow (1_x, (\Lambda_x)_{x_-})$$

where $\mathrm{el}\,\zeta_{(x,x_-)}\colon\ \mathrm{el}\,\Gamma_{(x,x_-)} \Rightarrow \mathrm{el}\,\Lambda_{(x,x_-)}$. This is a vertical modification. This defines el on 1,2,3-cells and it is 3-functorial.

Now suppose that $\eta\colon\ \mathrm{el}\,F \to \mathrm{el}\,G$ is a split cartesian 2-functor. Define $\overline{\eta}\colon F \Rightarrow G$ by $\overline{\eta}_x(a) = \pi_2(\eta(x, a))$, $\overline{\eta}_x(f) = \pi_2(\eta(1_x, f))$, $\overline{\eta}_x(\sigma) = \pi_2(\eta(1_{1_x}, \sigma))$. This is 2-natural because η is a split cartesian 2-functor. Then $\mathrm{el}(\overline{\eta}) = \eta$ and is unique with that property. Thus el is bijective on 1-cells.

Suppose that $\Gamma\colon\ \mathrm{el}\,\eta \Rightarrow \mathrm{el}\,\epsilon$ is a vertical 2-natural transformation. Define $\overline{\Gamma}\colon \eta \Rrightarrow \epsilon$ by $(\overline{\Gamma}_x)_{x_-} = \pi_2(\Gamma_{(x,x_-)})$. This is a modification because Γ is 2-natural and η, ϵ are split cartesian. Then $\mathrm{el}(\overline{\Gamma}) = \Gamma$ and is unique with that property. Thus el is bijective on 2-cells.

Suppose that $\theta\colon\ \mathrm{el}\,\Gamma \Rrightarrow \mathrm{el}\,\Lambda$ is a vertical modification. Define $\overline{\theta}\colon \Gamma \Rrightarrow \Lambda$ by $(\overline{\theta}_x)_{x_-} = \pi_2(\theta_{(x,x_-)})$. This is a perturbation because θ is a modification and η, ϵ are split cartesian. Then $\mathrm{el}(\overline{\theta}) = \theta$ and and is unique with that property. Thus el is bijective on 3-cells.

This makes el locally an isomorphism and thus an equivalence. □

Remark 2.2.27. The action of the Grothendieck construction on objects is described by Baković in [Bak12] Section 6. Section 5 of the same paper gives a partial description of the action on objects of the pseudo-inverse. With some adjustments, we have completed the second construction (Theorem 5.1) and shown that together they form an equivalence of 3-categories. It was in completing Baković's description of the pseudo-inverse to the Grothendieck construction that we discovered that cartesian 2-cells must be closed under post-composition with all 1-cells.

Remark 2.2.28 (Non-split 2-fibrations). The Grothendieck construction for non-split 2-fibrations is somewhat more complicated than demonstrated above. We chose to first build the Grothendieck construction for fibred bicategories and then to observe how the arguments simplify when restricted to 2-fibrations. We found that non-split 2-fibrations correspond to a slightly odd kind of trihomomorphism (see Remark 2.3.33). We found however that 2-fibrations that are locally and horizontally split correspond to homomorphisms of 2Cat-enriched bicategories $B^{\mathrm{coop}} \to 2\mathrm{Cat}$ (enriched in 2Cat as a monoidal bicategory). This is somewhat more pleasing.

Remark 2.2.29 (Dual constructions). All of these results could be adjusted to describe three other kinds of fibrations: *op-2-fibrations*, *co-2-fibrations* and *coop-2-fibrations*. They correspond to 'op'-contravariant, 'co'-contravariant and covariant 2-functors into 2Cat. The dual Grothendieck constructions are obtained by reversing the direction of the second components of 1- and 2-cells in $\mathrm{el}\, F$ (in 2-cells, in 1-cells or in both). We could reasonably refer to coop-2-fibrations as *2-opfibrations*. In that case cartesian 1- and 2-cells would be defined using pullbacks associated with pre-composition instead of post-composition.

2.2.3 Examples

When C is a category, $\mathrm{Fam}(C)$ is the category of families of objects of C. The objects of $\mathrm{Fam}(C)$ are pairs (I, X) where I is a set and $X\colon I \to C$ is a functor. The 1-cells are pairs $(u, \alpha)\colon (I, X) \to (J, Y)$ where $u\colon I \to J$ is a function and α is a natural transformation.

$$\begin{tikzcd} I \arrow[rr, "u"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & J \arrow[dl, "Y"] \\ & C & \end{tikzcd}$$

Composition and identities are defined in the obvious way. There is a functor $\pi\colon \mathrm{Fam}(C) \to \mathrm{Set}$ that is the projection onto the first component of $\mathrm{Fam}(C)$. This is a well-known example of a fibration.

Construction 2.2.30 (Families)**.** When B is a 2-category we define $\mathrm{Fam}(B)$ to be the 2-category of '1-cell diagrams' in B. The objects of $\mathrm{Fam}(B)$ are pairs (C, X) where

C is a small category and $X\colon C^{\mathrm{op}} \to B$ is a pseudo-functor. The 1-cells are pairs $(F,\alpha)\colon (C,X) \to (D,Y)$ where $F\colon C \to D$ is a functor and $\alpha\colon X \Rightarrow YF^{\mathrm{op}}$ is a pseudo-natural transformation. The 2-cells are pairs $(\sigma,\Sigma)\colon (F,\alpha) \Rightarrow (G,\beta)$ where $\sigma\colon F \Rightarrow G$ is a natural transformation and Σ is a modification

$$\begin{tikzcd} C^{\mathrm{op}} \arrow[rr, "F^{\mathrm{op}}"] \arrow[dr, "X"'] \arrow[rr, phantom, shift right=5, "\overset{\alpha}{\Rrightarrow}"] & & D^{\mathrm{op}} \arrow[dl, "Y"] \\ & B & \end{tikzcd} \quad \overset{\Sigma}{\Rrightarrow} \quad \begin{tikzcd} C^{\mathrm{op}} \arrow[rr, bend left, "F^{\mathrm{op}}"] \arrow[rr, bend right, "G^{\mathrm{op}}"'] \arrow[rr, phantom, "\Uparrow\sigma^{\mathrm{op}}"] \arrow[dr, "X"'] \arrow[rr, phantom, shift right=6, "\overset{\beta}{\Rrightarrow}"] & & D^{\mathrm{op}} \arrow[dl, "Y"] \\ & B & \end{tikzcd}.$$

Composition and identities are defined in the obvious way. There is a 2-functor $\pi\colon \mathrm{Fam}(B) \to \mathrm{Cat}$ defined by projection onto the first component of $\mathrm{Fam}(B)$.

Proposition 2.2.31. $\pi\colon \mathrm{Fam}(B) \to \mathrm{Cat}$ *is a 2-fibration.*

Proof. Suppose (D,Y) in $\mathrm{Fam}(B)$ and $F\colon C \to D$ in Cat. Its cartesian lift is $(F, 1_{YF^{\mathrm{op}}})\colon (C, YF^{\mathrm{op}}) \to (D,Y)$. Suppose that $(\sigma,\Sigma)\colon (G,\beta) \Rightarrow (H,\gamma)\colon (E,Z) \to (D,Y)$ and $\sigma F = \lambda$ where $\lambda\colon J \Rightarrow K$. The unique lift of λ is $(\lambda,\Sigma)\colon (J,\beta) \Rightarrow (K,\gamma)\colon (E,Z) \to (C, YF^{\mathrm{op}})$. The diagrams are

$$\begin{tikzcd} (E,Z) \arrow[d, dotted, bend right, "(K,\gamma)"'] \arrow[d, dotted, bend left, "(J,\beta)"] \arrow[d, phantom, "\overset{\Leftarrow}{(\lambda,\Sigma)}"] \arrow[drr, bend left, "(G,\beta)"] \arrow[drr, "(H,\gamma)"'] \arrow[drr, phantom, bend left=15, "\Swarrow(\sigma,\Sigma)"] & & \\ (C, YF^{\mathrm{op}}) \arrow[rr, "(F,1)"'] & & (D,Y) \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} E \arrow[d, bend right, "K"'] \arrow[d, bend left, "J"] \arrow[d, phantom, "\overset{\Leftarrow}{\lambda}"] \arrow[drr, bend left, "G"] \arrow[drr, "H"'] \arrow[drr, phantom, bend left=15, "\Swarrow\sigma"] & & \\ C \arrow[rr, "F"'] & & D \end{tikzcd}.$$

Suppose $(G,\beta)\colon (C,X) \to (D,Y)$ in $\mathrm{Fam}(B)$ and $\sigma\colon F \Rightarrow G\colon C \to D$ in Cat. Its cartesian lift is $(\sigma, 1_{Y\sigma.\beta})\colon (F, Y\sigma.\beta) \Rightarrow (G,\beta)$. Suppose that $(\lambda,\Lambda)\colon (H,\gamma) \Rightarrow (G,\beta)$ and $\sigma\omega = \lambda$. The unique lift of ω is $(\omega,\Lambda)\colon (H,\gamma) \Rightarrow (F, Y\sigma^{\mathrm{op}}.\beta)$.

$$\begin{tikzcd} (H,\gamma) \arrow[d, Rightarrow, dotted, "(\omega,\Lambda)"'] \arrow[dr, Rightarrow, "(\lambda,\Lambda)"] & \\ (F, Y\sigma^{\mathrm{op}}.\beta) \arrow[r, Rightarrow, "(\sigma,1)"'] & (G,\beta) \end{tikzcd} \qquad \begin{tikzcd} H \arrow[d, Rightarrow, "\omega"'] \arrow[dr, Rightarrow, "\lambda"] & \\ F \arrow[r, Rightarrow, "\sigma"'] & G \end{tikzcd}$$

Suppose that $\sigma\colon F \Rightarrow G$ and $\tau\colon H \Rightarrow K$ with $(G,\beta)\colon (C,X) \to (D,Y)$ and $(K,\delta)\colon (D,Y) \to (E,Z))$ and we compose the chosen cartesian lifts of σ and τ. They are $(\sigma,1)$ and $(\tau,1)$ their composite $(\tau,1) * (\sigma,1)$ is isomorphic to $(\tau * \sigma, 1)$ which is cartesian. Thus by Proposition 2.2.14 all cartesian 2-cells are closed under composition. $\square$

Remark 2.2.32. $\pi\colon \mathrm{Fam}(B) \to \mathrm{Cat}$ can be obtained by applying the Grothendieck construction to $F\colon \mathrm{Cat}^{\mathrm{coop}} \to 2\mathrm{Cat}$ defined on objects by $F(C) = \mathrm{Hom}(C^{\mathrm{op}}, B)$.

Remark 2.2.33. The above construction yields a 2-fibration that is split under composition of cartesian 1-cells but it is not split in any other sense. If we modify this construction by replacing pseudo-functors and pseudo-natural transformations with 2-functors and 2-natural transformations then the result is split in every way. This variation can be obtained by applying the Grothendieck construction to F defined by $F(C) = [C^{\mathrm{op}}, B]$.

Definition 2.2.34. We say that an arrow $p\colon e \to b$ in a 2-category B is a (split) fibration when $p_*\colon B(c,e) \to B(c,b)$ is a (split) fibration for all c and the commuting square

$$\begin{tikzcd} B(c,e) \arrow[r, "f^*"] \arrow[d, "p_*"'] & B(c',e) \arrow[d, "p_*"] \\ B(c,b) \arrow[r, "f^*"'] & B(c',b) \end{tikzcd}$$

is a (split) morphism of fibrations for all $f\colon c' \to c$.

Definition 2.2.35. A morphism between (split) fibrations $p\colon e \to b$ and $q\colon e' \to b'$ in a 2-category B is a pair $(f\colon e \to e', g\colon b \to b')$ where $q.f = g.p$ and

$$\begin{tikzcd} B(c,e) \arrow[r, "f_*"] \arrow[d, "p_*"'] & B(c,e') \arrow[d, "q_*"] \\ B(c,b) \arrow[r, "g_*"'] & B(c,b') \end{tikzcd}$$

is a (split) morphism of fibrations for all c. In this case we say that f is *cartesian*.

Construction 2.2.36 (Internal fibrations)**.** The category of fibrations internal to a 2-category B is denoted by Fib_B. The objects are fibrations $p\colon e \to b$ in B. The 1-cells are morphisms of fibrations. The 2-cells are pairs of 2-cells $(\alpha,\beta)\colon (f,g) \Rightarrow (f',g')$ with $q\alpha = \beta p$. Composition and identities are the same as in $B^{\mathbb{2}}$.

There is a 2-functor $\mathrm{cod}\colon \mathrm{Fib}_B \to B$ defined by projection onto the codomain:

$$\begin{tikzcd} e \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[d, "p"'] & e' \arrow[d, "q"] & & \\ b \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & b' & \stackrel{\mathrm{cod}}{\longmapsto} & b \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & b' . \end{tikzcd}$$

When $B = \mathrm{Cat}$ we omit the subscript and $\mathrm{Fib}_{\mathrm{Cat}}$ is just Fib. It is the category of fibrations in Cat and the codomain 2-functor is $\mathrm{cod}\colon \mathrm{Fib} \to \mathrm{Cat}$.

Proposition 2.2.37. *When B has 2-pullbacks,* $\mathrm{cod}\colon \mathrm{Fib}_B \to B$ *is a 2-fibration.*

Proof. Suppose $q\colon e' \to b'$ in $\mathrm{Fib}(B)$ and $g'\colon b \to b'$ in B, then there exists a map $(g, g')\colon (g')^*q \to q$ defined by taking the 2-pullback

$$\begin{tikzcd} e \arrow[r, "g"] \arrow[d, "g^*q"'] \arrow[dr, phantom, very near start, "\lrcorner"] & e' \arrow[d, "q"] \\ b \arrow[r, "g'"'] & b' \end{tikzcd} .$$

Since both pullbacks and fibrations in B are defined representably and fibrations in Cat are closed under pullback, fibrations in B must also be closed under pullback. The same argument ensures that g is cartesian. Thus (g, g') is well-defined as a 1-cell. To see that this is cartesian, suppose that $(h, h')\colon r \to q$ in $\mathrm{Fib}(B)$ and $h' = g'.f'$ in B. Then there exists a unique f with $p.f = f'.r$ and $h = g.f$ and hence a unique (f, f') with $(h, h') = (g, g')(f, f')$ and $\pi(f, f') = f'$. We know that f is cartesian because h is cartesian and g reflects cartesian maps (again because pullbacks in Cat reflect cartesian maps). This same argument works for 2-cells into e' so (g, g') is cartesian. The diagram is

$$\begin{tikzcd} e'' \arrow[d, "r"'] \arrow[dr, dotted, "f"] \arrow[drr, "h"] & & \\ b'' \arrow[dr, "f'"'] \arrow[drr, "h'"] & e \arrow[r, "g"] \arrow[d, "p"] \arrow[dr, phantom, very near start, "\lrcorner"] & e' \arrow[d, "q"] \\ & b \arrow[r, "g'"'] & b' \end{tikzcd} .$$

Suppose that $(g, g')\colon p \to q$ in $\mathrm{Fib}(B)$ and $\alpha'\colon f' \Rightarrow g'$ in B. Since q is cartesian we can take the cartesian lift of $\alpha' p$ at g (call it α) and get a 2-cell $(\alpha, \alpha')\colon (f, f') \Rightarrow (g, g')$. To show that this is cartesian, suppose that $(\gamma, \gamma')\colon (h, h') \Rightarrow (g, g')$ and $\gamma' = \eta'\alpha'$. Since α is cartesian for q and $q\gamma = \gamma' p = \eta' p.\alpha' p = \eta' p.q\alpha$ there exists a unique $\eta\colon h \Rightarrow f$ and hence (η, η') with $(\gamma, \gamma') = (\alpha, \alpha').(\eta, \eta')$. Thus (α, α') is cartesian.

$$\begin{tikzcd} h \arrow[d, Rightarrow, dotted, "\eta"'] \arrow[dr, Rightarrow, "\gamma"] & \\ f \arrow[r, Rightarrow, "\alpha"'] & g \end{tikzcd} \qquad \begin{tikzcd} qh \arrow[d, Rightarrow, "\eta' p"'] \arrow[dr, Rightarrow, "q\gamma=\gamma' p"] & \\ qf \arrow[r, Rightarrow, "q\alpha=\alpha' p"'] & qg \end{tikzcd}$$

Suppose that we take the cartesian lifts of $\alpha'\colon f' \Rightarrow g'$ and $\gamma'\colon h' \Rightarrow k'$ at (g, g') and

(k, k') as indicated below.

$$
\begin{tikzcd}
e \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[d, "p"'] & e' \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\gamma"] \arrow[d, "q"] & e'' \arrow[d, "r"] \\
b \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & b' \arrow[r, bend left, "h'"] \arrow[r, bend right, "k'"'] \arrow[r, phantom, "\Downarrow\gamma'"] & b''
\end{tikzcd}
$$

Since cartesian 2-cells for r are closed under pre-composition with any 1-cell γg is cartesian. Also since h preserves cartesian maps for q we know that $h\alpha$ is cartesian. Then because cartesian 2-cells are closed under vertical composition $\gamma * \alpha = \gamma g.h\alpha$ is cartesian. Thus by Proposition 2.2.14 cartesian 2-cells for cod are closed under composition. □

Remark 2.2.38. If we apply the pseudo-inverse to the Grothendieck construction to $\mathrm{cod}\colon \mathrm{Fib}_B \to B$ we get $F\colon B^{\mathrm{coop}} \to 2\mathrm{Cat}$ defined by $F(b) = \mathrm{Fib}_B/b$, the category of fibrations over b. Its action on 1-cells is to send $f\colon b \to b'$ to $f^*\colon \mathrm{Fib}_B/b' \to \mathrm{Fib}_B/b$ defined by pullback.

Remark 2.2.39. Let Fib^s_B be the sub-2-category of Fib_B containing split fibrations and split maps. Suppose also that we can choose 2-pullbacks in B in such a way that they are closed under composition in $B^{\mathbb{2}}$ (not just up to isomorphism). Then the proof above requires only slight adjustments to show that $\mathrm{cod}\colon \mathrm{Fib}^s_B \to B$ is a split 2-fibration.

Example 2.2.40 (Enriched Categories)**.** There is a 2-functor Mon $\to$ 2Cat that maps each monoidal category $\mathscr{V}$ to $\mathscr{V}$-Cat. We can use a dual to the Grothendieck construction to get a 2-opfibration Enr $\to$ Mon. The objects of the total category Enr are enriched categories: pairs $(\mathscr{V}, A)$ where $\mathscr{V}$ is a monoidal category and A is a $\mathscr{V}$-enriched category. The rest of the structure can be deduced from the dual Grothendieck construction.

Example 2.2.41 (Algebras)**.** Let $\mathrm{Mnd}(K)$ be the 2-category of 2-monads on a 2-category K. There is a 2-functor $F\colon \mathrm{Mnd}(K)^{\mathrm{coop}} \to 2\mathrm{Cat}$ that maps each 2-monad T to the 2-category $T\text{-}\mathrm{Alg}_l$ of strict T-algebras, lax algebra morphisms and algebra 2-cells. Each monad morphism $\lambda\colon S \to T$ gives a 2-functor that acts on $m\colon TA \to A$ by pre-composition with $\lambda_A\colon SA \to TA$. Each monad 2-cell $\Gamma\colon \lambda \Rightarrow \tau$ gives a 2-natural transformation whose component at $m\colon TA \to A$ is $(1, 1, m.\Gamma_A)$ as shown below.

$$
\begin{tikzcd}
SA \arrow[r, "1"] \arrow[d, "\tau_A"'] & SA \arrow[d, "\lambda_A"] \\
TA \arrow[d, "m"'] \arrow[r, phantom, "\Downarrow m.\Gamma_A"] & TA \arrow[d, "m"] \\
A \arrow[r, "1"'] & A
\end{tikzcd}
$$

We know that F is contravariant on 2-cells because we are using lax morphisms of algebras.

If F mapped each monad to $T\text{-Alg}_{oplax}$ containing the oplax morphisms then it would be covariant on 2-cells. Further details on 2-monads can be found in [Kel74].

We can use the Grothendieck construction to construct a 2-fibration $\mathrm{Alg} \to \mathrm{Mnd}$. The objects of the total category Alg are algebras of a 2-monad: pairs $(S, (A, m))$ where $m\colon SA \to A$ is an S-algebra. The 1-cells from $(S, (A, m))$ to $(T, (B, n))$ are pairs $(\lambda, (f, \theta_f))$ where λ is a monad morphism from S to T and $(f, \theta_f)\colon \lambda(A, m) \to F\lambda(B, n)$ is a lax algebra morphism

$$\begin{tikzcd} SA \arrow[rr, "m"] \arrow[d, "Sf"'] & \arrow[d, phantom, "\overset{\theta_f}{\nearrow}"] & A \arrow[d, "f"] \\ SB \arrow[r, "\lambda_B"'] & TB \arrow[r, "n"'] & B \end{tikzcd}.$$

The 2-cells of Alg are pairs $(\Gamma, \alpha)\colon (\lambda, (f, \theta_f)) \to (\tau, (g, \theta_g))$ where $\Gamma\colon \lambda \to \tau$ is a monad 2-cell and α is an algebra 2-cell

$$\begin{tikzcd} (A, m) \arrow[rr, "{(f,\theta_f)}"] \arrow[dr, "{(g,\theta_g)}"'] & \arrow[d, phantom, "\Downarrow\alpha"] & F\lambda(B, n) \\ & F\tau(B, n) \arrow[ur, "F\Gamma_{(B,n)}"'] & \end{tikzcd}.$$

The 2-fibration is projection on the first component of Alg. By construction the fibre over T is $T\text{-Alg}_l$.

2.3 Fibred bicategories

What follows is the theory of fibrations developed specifically for bicategories. The concepts are not significantly different from Section 2.2 but the details are *much* more complicated.

2.3.1 Definitions and properties of cartesian 1- and 2-cells

Definition 2.3.1. Suppose $P\colon \mathscr{E} \to \mathscr{B}$ is a homomorphism of bicategories. We say a 1-cell $f\colon x \to y$ in $\mathscr{E}$ is *cartesian* when it has the following two properties:

1. Suppose that $g\colon z \to y$ in $\mathscr{E}$ with $h\colon Pz \to Px$ and an isomorphism $\alpha\colon Pf.h \Rightarrow Pg$

$$\begin{tikzcd} z \arrow[dr, "g"] & \\ x \arrow[r, "f"'] & y \end{tikzcd} \qquad \begin{tikzcd} Pz \arrow[d, "h"'] \arrow[dr, "Pg"] \arrow[dr, phantom, shift right=3ex, "\overset{\alpha}{\nearrow}"] & \\ Px \arrow[r, "Pf"'] & Py \end{tikzcd}.$$

Then there exists an $\hat{h}\colon z \to x$ and isomorphisms $\hat{\alpha}\colon f\hat{h} \Rightarrow g$, $\hat{\beta}\colon P\hat{h} \Rightarrow h$ such that $\alpha.Pf\hat{\beta} = P\hat{\alpha}.\phi_{hf}$. We say that $(\hat{h}, \hat{\alpha}, \hat{\beta})$ is a *lift* of (h, α).

2. Suppose that $\sigma\colon g \Rightarrow g'$ in $\mathscr{E}$ and $h, h'\colon Pz \to Px$ with isomorphisms $\alpha\colon Pf.h \Rightarrow Pg$ and $\alpha'\colon Pf.h' \Rightarrow Pg'$. Suppose also that (h, α) and (h', α') have lifts $(\hat{h}, \hat{\alpha}, \hat{\beta})$ and $(\hat{h'}, \hat{\alpha'}, \hat{\beta'})$. For any $\delta\colon h \Rightarrow h'$ in $\mathscr{B}$ with $\alpha'.Pf\delta = P\sigma.\alpha$

$$\begin{tikzcd}
z \arrow[d, bend right, "\hat{h}"'] \arrow[d, dotted, bend left, "\hat{h}'"] \arrow[dr, "g"' near end] \arrow[dr, bend left=40, "g'"] \arrow[dr, phantom, bend left=20, "\overset{\sigma}{\Rrightarrow}"] \\
x \arrow[r, "f"'] \arrow[ur, phantom, "\overset{\hat{\alpha}}{\Rightarrow}" near start] & y
\end{tikzcd}
\qquad
\begin{tikzcd}
Pz \arrow[d, bend right, "h"'] \arrow[d, dotted, bend left, "h'"] \arrow[d, phantom, "\overset{\delta}{\Rrightarrow}"] \arrow[dr, "Pg"' near end] \arrow[dr, bend left=40, "Pg'"] \arrow[dr, phantom, bend left=20, "\overset{P\sigma}{\Rrightarrow}"] \\
Px \arrow[r, "Pf"'] \arrow[ur, phantom, "\overset{\alpha}{\Rightarrow}" near start] & Py
\end{tikzcd}$$

there exists a unique $\hat{\delta}\colon \hat{h} \Rightarrow \hat{h'}$ such that $\hat{\alpha'}.f\hat{\delta} = \sigma.\hat{\alpha}$ and $\delta.\hat{\beta} = \hat{\beta'}.P\hat{\delta}$.

Informally this can be stated by saying that f lifts 1-cells up to isomorphism and lifts 2-cells coherently with the lifted isomorphisms. The uniqueness of lifted 2-cells implies that lifted 1-cells are unique up to a coherent isomorphism.

Proposition 2.3.2. *Suppose $P\colon \mathscr{E} \to \mathscr{B}$ is a homomorphism of bicategories. A 1-cell $f\colon x \to y$ in $\mathscr{E}$ is cartesian if and only if*

$$\begin{tikzcd}
\mathscr{E}(z,x) \arrow[r, "f_*"] \arrow[d, "P_{zx}"'] \arrow[dr, phantom, "\cong"] & \mathscr{E}(z,y) \arrow[d, "P_{zy}"] \\
\mathscr{B}(Pz,Px) \arrow[r, "Pf_*"'] & \mathscr{B}(Pz,Py)
\end{tikzcd}$$

is a bipullback for all z. The pictured isomorphism is the coherence map for P on composition.

Remark 2.3.3. In Proposition 2.3.2 we use bipullback in the sense of Street and Joyal [JS93]: a weakly-universal iso-square over Pf_* and P_{zy}. That is, there is a pseudo-natural equivalence

$$\mathrm{Hom}(A, \mathscr{E}(z,x)) \simeq \mathrm{Hom}(A, \mathscr{B}(Pz,Px)) \times_{\cong} \mathrm{Hom}(A, \mathscr{B}(Pz,Py))$$

where the right expression is a pseudo-pullback (iso-comma-category).

Definition 2.3.4. Suppose $P\colon \mathscr{E} \to \mathscr{B}$ is a homomorphism of bicategories. A 2-cell $\alpha\colon f \Rightarrow g\colon x \to y$ in $\mathscr{E}$ is *cartesian* if it is cartesian as a 1-cell for the functor $P_{xy}\colon \mathscr{E}(x,y) \to \mathscr{B}(Px,Py)$.

As in Section 2.2 we say that P is *locally fibred* when $P_{xy}\colon \mathscr{E}(x,y) \to \mathscr{B}(Px,Py)$ is a fibration for all x, y in $\mathscr{E}$.

Definition 2.3.5. Let $P\colon \mathscr{E} \to \mathscr{B}$ be a homomorphism. We say that P is a *fibration* when

1. for any $e \in \mathscr{E}$ and $f\colon b \to Pe$, there is a cartesian 1-cell $h\colon a \to e$ with $Ph = f$;
2. P is locally fibred; and
3. the horizontal composite of any two cartesian 2-cells is cartesian.

Informally, we say that $\mathscr{E}$ is a *fibred bicategory* when it is the domain of a fibration.

Remark 2.3.6. In the first condition of Definition 2.3.5 we could insist that cartesian 1-cells only have $Ph \cong f$ and the definition above would not be any weaker. When P is a fibration it is locally fibred and thus locally has the iso-lifting property. Now cartesian 1-cells isomorphic to a cartesian 1-cell are cartesian (see Proposition 2.3.8 below). Thus if there is a cartesian lift h with $Ph \cong f$ then there is a cartesian lift h' with $Ph' = f$. The converse is trivial so the two definitions are equivalent.

Remark 2.3.7. Corollary 1 in [JS93] states that if one leg of a cospan has the iso-lifting property then the pullback of that cospan is a bipullback. When P is a fibration it is locally fibred and thus locally has the iso-lifting property. It follows that if a 1-cell is 2-categorically cartesian (Definition 2.2.1) then it is bicategorically cartesian (Definition 2.3.1). As a result fibred 2-categories are also fibred bicategories.

We take the time here to establish a few basic properties of cartesian maps.

Proposition 2.3.8. *Suppose $P\colon \mathscr{E} \to \mathscr{B}$ is a homomorphism of bicategories. If 1-cells f and g in $\mathscr{E}$ are isomorphic then f is cartesian if and only if g is cartesian.*

Proof. Suppose that f is cartesian and $\alpha\colon f \Rightarrow g$ is an isomorphism. In the diagram below: the inner isomorphism is the coherence of P on composition with f. The isomorphisms above and below are induced by α and $P\alpha$ and thus the pasting is equal to the coherence of P on composition with g.

$$\begin{tikzcd}
\mathscr{E}(z,x) \arrow[r, bend left, "g_*"] \arrow[r, bend right, "f_*"'] \arrow[r, phantom, "\cong"] \arrow[d, "P_{zx}"'] \arrow[dr, phantom, "\cong"] & \mathscr{E}(z,y) \arrow[d, "P_{zx}"] \\
\mathscr{B}(Pz,Px) \arrow[r, bend left, "Pf_*"] \arrow[r, bend right, "Pg_*"'] \arrow[r, phantom, "\cong"] & \mathscr{B}(Pz,Py)
\end{tikzcd}$$

By definition of cartesian 1-cell the inner isomorphism is a bipullback. Bipullbacks are closed under the pasting of isomorphisms as indicated. Thus whole diagram is a bipullback and g is cartesian. $\square$

Proposition 2.3.9. *Suppose $P\colon \mathscr{E} \to \mathscr{B}$ is a homomorphism of bicategories and $f\colon w \to x$ and $g\colon x \to y$ are 1-cells in $\mathscr{E}$. If f and g are cartesian then gf is cartesian.*

Proof. Suppose that f and g are cartesian. In the diagram below: the inner two isomorphisms are coherence of P on composition with f and g. The outer isomorphisms are induced by associativity of composition. Thus the pasting is equal to the coherence of P on composition with gf.

$$\begin{tikzcd}
\mathscr{E}(z,w) \arrow[r, "f_*"] \arrow[d, "P_{zw}"'] \arrow[rr, bend left, "(gf)_*"] \arrow[dr, phantom, "\cong"] & \mathscr{E}(z,x) \arrow[r, "g_*"] \arrow[d, "P_{zx}"] \arrow[dr, phantom, "\cong"] & \mathscr{E}(z,y) \arrow[d, "P_{zy}"] \\
\mathscr{B}(Pz,Pw) \arrow[r, "Pf_*"'] \arrow[rr, bend right, "(Pgf)_*"'] & \mathscr{B}(Pz,Px) \arrow[r, "Pg_*"'] & \mathscr{B}(Pz,Py)
\end{tikzcd}$$

The inner two isomorphisms are bipullbacks by definition of cartesian 1-cell. Bipullbacks are closed under the inner pasting as indicated, as well as the pasting of isomorphisms on top and bottom. Thus the whole diagram is a bipullback and gf is cartesian. $\square$

Proposition 2.3.10. *Suppose $P\colon \mathscr{E} \to \mathscr{B}$ is a homomorphism of bicategories and $f\colon w \to x$, $g\colon x \to y$ are 1-cells in $\mathscr{E}$. If g and gf are cartesian then f is cartesian.*

Proof. This proof is essentially the same as Proposition 2.3.9. The only other thing we need to know is that bipullbacks have the same cancellation property as pullbacks. $\square$

Proposition 2.3.11. *Suppose $P\colon \mathscr{E} \to \mathscr{B}$ is a fibration of bicategories and $f\colon x \to y$ is a 1-cell in $\mathscr{E}$. If f is an equivalence then it is cartesian for P.*

Proof. Suppose that f is part of an adjoint equivalence $(f, f^\bullet, \eta, \epsilon)$ and

$$\begin{tikzcd}
z \arrow[dr, "g"] & \\
x \arrow[r, "f"'] & y
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
Pz \arrow[d, "h"'] \arrow[dr, "Pg"] \arrow[d, phantom, shift left=3, "\Nearrow\alpha"] & \\
Px \arrow[r, "Pf"'] & Py
\end{tikzcd}.$$

Let $\hat{h} = f^\bullet g$ and let $\hat{\alpha}$ be the composite $f.(f^\bullet.g) \cong (f.f^\bullet).g \cong 1.g \cong g$. Let $\hat{\beta}\colon P\hat{h} \Rightarrow h$ be the composite $P(f^\bullet.g) \cong Pf^\bullet.Pg \cong Pf^\bullet.(Pf.h) \cong (Pf^\bullet.Pf).h \cong P(f^\bullet.f).h \cong P1.h \cong 1.h \cong h$. This is a lift of (h, α).

To check the 2-cell property suppose that $\gamma\colon g \Rightarrow g'$ and

$$\begin{tikzcd}
Pz \arrow[d, bend right=40, "h"'] \arrow[d, bend left=20, "h'"] \arrow[dr, "Pg'"] \arrow[d, phantom, "\overset{\sigma}{\Rightarrow}"] & \\
Px \arrow[r, "Pf"'] \arrow[ur, phantom, "\alpha'\Nearrow"] & Py
\end{tikzcd}
\quad \text{equals} \quad
\begin{tikzcd}
Pz \arrow[d, "h"'] \arrow[dr, bend right=20, "Pg"] \arrow[dr, bend left=30, "Pg'"] & \\
Px \arrow[r, "Pf"'] \arrow[ur, phantom, "\alpha\Nearrow"] & Py \arrow[ul, phantom, "P\gamma\Nearrow"]
\end{tikzcd} .$$

Then suppose that there are lifts $(\hat{h}, \hat{\alpha}, \beta)$, $(\hat{h'}, \hat{\alpha'}, \beta')$ of (h, α) and (h', β) as above. Let $\hat{\sigma}$ be the composite

$$\begin{tikzcd}
h \arrow[r, Rightarrow, "l"] & 1.h \arrow[r, Rightarrow, "\eta.h"] & (f^{-}\bullet.f).h \arrow[r, Rightarrow, "a"] & f^{-}\bullet.(f.h) \arrow[r, Rightarrow, "f^{\bullet}.\hat{\alpha}"] & f^{\bullet}.g \arrow[d, Rightarrow, "f^{\bullet}.\gamma"] \\
h' & 1.h' \arrow[l, Rightarrow, "l"] & (f^{\bullet}.f).h' \arrow[l, Rightarrow, "\eta^{-1}.h'"] & f^{\bullet}.(f.h') \arrow[l, Rightarrow, "a"] & f^{\bullet}.g' \arrow[l, Rightarrow, "f^{\bullet}.\hat{\alpha'}"]
\end{tikzcd} .$$

It is unique with the property that $\hat{\alpha'}.f\hat{\sigma} = \gamma.\hat{\alpha}$ and $\sigma.\hat{\beta} = \hat{\beta}'.P\hat{\sigma}$. □

Proposition 2.3.12. *Suppose that $P\colon \mathscr{E} \to \mathscr{B}$ is a fibration of bicategories and $f\colon x \to y$ is cartesian in $\mathscr{E}$. If Pf is an equivalence then f is an equivalence.*

Proof. Suppose Pf is part of an adjoint equivalence $(Pf, Pf^{\bullet}, \eta, \epsilon)$. Then we can lift ϵ to obtain $\hat{\epsilon}\colon f.h \cong 1$.

$$\begin{tikzcd}
y \arrow[d, dotted, "h"'] \arrow[dr, "1"] & \\
x \arrow[r, "f"'] \arrow[ur, phantom, "\hat{\epsilon}\Nearrow"] & y
\end{tikzcd}
\qquad
\begin{tikzcd}
Py \arrow[d, "(Pf)^{\bullet}"'] \arrow[dr, bend right=20, "1"] \arrow[dr, bend left=30, "P1"] & \\
Px \arrow[r, "Pf"'] \arrow[ur, phantom, "\epsilon\Nearrow"] & Py \arrow[ul, phantom, "\cong"]
\end{tikzcd}$$

Now since 1_y is an equivalence it is cartesian. Then since $f.h \cong 1$, fh is cartesian. Then Proposition 2.3.10 tells us that h is cartesian. We can then lift $Ph.Pf \cong (Pf)^{\bullet}.Pf \cong 1 \cong P1$ as picture on the right to obtain $\hat{\eta}\colon h.k \cong 1$.

$$\begin{tikzcd}
x \arrow[d, dotted, "k"'] \arrow[dr, "1"] & \\
y \arrow[r, "h"'] \arrow[ur, phantom, "\hat{\eta}\Nearrow"] & x
\end{tikzcd}
\qquad
\begin{tikzcd}
Px \arrow[d, "Pf"'] \arrow[dr, bend right=10, "1"] \arrow[dr, bend left=30, "P1"] & \\
Py \arrow[r, bend left=10, "(Pf)^{\bullet}"] \arrow[r, bend right=10, "Ph"'] \arrow[r, phantom, "\cong"] \arrow[ur, phantom, "\eta\Nearrow"] & Px \arrow[ul, phantom, "\cong"]
\end{tikzcd}$$

Finally, $f \cong f.1 \cong f.(h.k) \cong (f.h.).k \cong 1.k \cong k$ and then $h.f \cong h.k \cong 1$. Thus f is an equivalence. □

Proposition 2.3.13. *Suppose $P\colon \mathscr{E} \to \mathscr{B}$ is a homomorphism of bicategories, $f\colon a \to b$ is a cartesian 1-cell in $\mathscr{E}$ and σ, τ are isomorphisms in $\mathscr{B}$ as pictured. In this case the unique lift of σ along f, called $\hat{\sigma}$, is an isomorphism.*

$$
\begin{tikzcd}
z \arrow[d, bend right=40, "\hat{h}'"'] \arrow[d, bend left=20, "\hat{h}"] \arrow[d, phantom, "\overset{\hat{\sigma}}{\Rightarrow}"] \arrow[dr, bend left=30, "g"] \arrow[dr, bend right=10, "g'"'] \arrow[dr, phantom, "\overset{\tau}{\Rightarrow}"] \\
x \arrow[r, "f"'] & y
\end{tikzcd}
\quad \textit{over} \quad
\begin{tikzcd}
Pz \arrow[d, bend right=40, "h'"'] \arrow[d, bend left=20, "h"] \arrow[d, phantom, "\overset{\sigma}{\Rightarrow}"] \arrow[dr, bend left=30, "Pg"] \arrow[dr, bend right=10, "Pg'"'] \arrow[dr, phantom, "\overset{P\tau}{\Rightarrow}"] \\
Px \arrow[r, "Pf"'] & Py
\end{tikzcd}
$$

There are isomorphisms on the front and back of each pasting that we have not illustrated in this diagram.

Proof. If σ and τ are both invertible then the above lifting can be done with σ^{-1} and τ^{-1}. This gives a map $\hat{\sigma}^{\bullet}\colon \hat{h} \Rightarrow \hat{h}'$. If we paste these diagrams together then $\hat{\sigma}\hat{\sigma}^{\bullet}$ is a lift of $\sigma\sigma^{-1} = 1_h$. However $1_{\hat{h}}$ is also a lift of 1_h and thus by uniqueness $\hat{\sigma}\hat{\sigma}^{\bullet} = 1_{\hat{h}}$. Pasting the diagrams together the other way gives $\hat{\sigma}^{\bullet}\hat{\sigma} = 1_{\hat{h}'}$. □

Corollary 2.3.14. *Suppose $P\colon \mathscr{E} \to \mathscr{B}$ is a homomorphism of bicategories. The lift of any 1-cell in $\mathscr{B}$ along any cartesian 1-cell in $\mathscr{E}$, as in Definition 2.3.1 (1), is unique up to a unique invertible 2-cell.*

Proof. Use the above result with $\sigma = 1_h$ and $\tau = 1_g$. □

Proposition 2.3.15. *Suppose $P\colon \mathscr{E} \to \mathscr{B}$ is a homomorphism of bicategories and $f\colon a \to b$ is cartesian over $Pf\colon Pa \to Pb$. Among all cartesian 1-cells that map to Pf, it is unique up to an equivalence 1-cell and isomorphism 2-cell. This equivalence and isomorphism are unique up to an isomorphism 2-cell.*

Proof. Suppose that $g\colon c \to b$ is cartesian over Pf. Then

$$
\begin{tikzcd}
c \arrow[d, dotted, "\hat{h}"'] \arrow[dr, "g"] \arrow[dr, phantom, "\overset{\hat{r}}{\Leftarrow}"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
Pc \arrow[d, "1"'] \arrow[dr, "Pg"] \arrow[dr, phantom, "\overset{r}{\Leftarrow}"] \\
Pa \arrow[r, "Pf"'] & Pb
\end{tikzcd}
$$

and f is cartesian so there exists a lift $(\hat{h}, \hat{r}, \hat{\beta})$. By Corollary 2.3.14 this lift is unique up to a unique isomorphism. We have yet to show that $\hat{h}$ is an equivalence.

If we draw the same diagram with g in the base then since g is cartesian there exists a lift $(\hat{h}', \hat{l}', \hat{\beta}')$ of $r\colon Pg.1 \Rightarrow Pf$. These two lifts can be pasted together to form a lift pictured on the left. The base forms a commuting shell by coherence in a bicategory. Then there exists a unique lift of $r\colon 1.1 \Rightarrow 1$. It is a 2-cell $\hat{h}\hat{h}' \Rightarrow 1$ and it is an isomorphism by

Proposition 2.3.13.

$$
\begin{tikzcd}
x \arrow[d, "\hat{h}'"] \arrow[dd, bend right, "1"'] \arrow[ddr, "f"] \arrow[ddr, bend left, "f"] \arrow[ddr, phantom, bend left=15, "="] \\
z \arrow[d, "\hat{h}"] \arrow[dr, "g"] \arrow[ur, phantom, "\hat{r}'\Nearrow"] \\
x \arrow[r, "f"'] \arrow[u, phantom, "\hat{r}\Nearrow"] & y
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
Px \arrow[d, "1"] \arrow[dd, bend right, "1"'] \arrow[ddr, "Pf"] \arrow[ddr, bend left, "Pf"] \arrow[ddr, phantom, bend left=15, "="] \\
Pz \arrow[d, "1"] \arrow[dr, "Pg"] \arrow[ur, phantom, "r\Nearrow"] \\
Px \arrow[r, "Pf"'] \arrow[u, phantom, "r\Nearrow"] & Py
\end{tikzcd}
$$

The isomorphisms omitted in this diagram are all r. If we paste the lifts together the other way and follow the same reasoning we get another isomorphism $\hat{h}'\hat{h} \Rightarrow 1$. Thus $\hat{h}$ is an equivalence. □

Proposition 2.3.16. *Suppose that P, Q, F are homomorphisms with with $P = QF$.*

1. *If P is locally fibred and chosen cartesian 2-cells are closed under horizontal composition then all cartesian 2-cells are closed under horizontal composition.*

2. *If P and Q are fibrations and F preserves chosen cartesian maps then F preserves all cartesian maps.*

Proof. The proofs are essentially the same as for Proposition 2.2.14 and 2.2.15. □

2.3.2 Fibrations with stricter properties

Every fibration is locally fibred and thus locally has the iso-lifting property. We can take advantage of this to make fibrations much easier to handle.

Proposition 2.3.17. *Suppose $P\colon \mathscr{E} \to \mathscr{B}$ is a homomorphism of bicategories. When P is locally fibred every lift $(\hat{h}, \hat{\alpha}, \hat{\beta})$ of (h, α) along a cartesian 1-cell can be chosen so that $\hat{\beta} = 1_h$. That is, lifts along cartesian 1-cells can be chosen so $P\hat{h} = h$.*

Proof. Suppose that f is cartesian and $(\hat{h}, \hat{\alpha}, \hat{\beta})$ is a lift of (h, α). That is,

$$
\begin{tikzcd}
z \arrow[d, "\hat{h}"'] \arrow[dr, "g"] \\
x \arrow[r, "f"'] \arrow[u, phantom, "\hat{\alpha}\Nearrow"] & y
\end{tikzcd}
\qquad
\begin{tikzcd}
Pz \arrow[d, "h"'] \arrow[dr, "Pg"] \\
Px \arrow[r, "Pf"'] \arrow[u, phantom, "\alpha\Nearrow"] & Py
\end{tikzcd}
$$

where $\alpha.Pf\hat{\beta} = P\hat{\alpha}.\phi_{hf}$. Let $\sigma\colon h^! \Rightarrow h$ be the cartesian lift of $\hat{\beta}$ at h and let $\alpha^! = \hat{\alpha}.f\sigma$ (σ is an isomorphism because it is a cartesian 2-cell over an isomorphism). Then $(h^!, \alpha^!, 1_h)$

is a lift of (h,α) for f. This is proved by $P\alpha^!.\phi_{h^!f} = P\hat{\alpha}.P(f\sigma).\phi_{h^!f} = P\hat{\alpha}.\phi_{hf}.PfP\sigma = \alpha$. $\square$

Proposition 2.3.18. *Every locally fibred homomorphism $P\colon \mathscr{E} \to \mathscr{B}$ is isomorphic in the slice over $\mathscr{B}$ to a locally fibred $P'\colon \mathscr{E}' \to \mathscr{B}$ that preserves identities and composition strictly. Call the isomorphism S as pictured. If P is a fibration then P' is a fibration and S is cartesian.*

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "S"] \arrow[dr, "P"'] & & \mathscr{E}' \arrow[dl, "P'"] \\ & \mathscr{B} & \end{tikzcd}$$

Proof. Suppose $P\colon \mathscr{E} \to \mathscr{B}$ is locally fibred. Let $\mathscr{E}'$ have the same 0-, 1-, and 2-cells as $\mathscr{E}$ with the same vertical composition and 2-cell identities. Then let P' have the same action as P on 0-, 1-, and 2-cells so that S is the identity on 0-, 1-, and 2-cells. We now define horizontal composition in $\mathscr{E}'$. If $f\colon e \to e'$ and $g\colon e' \to e''$ in $\mathscr{E}'$ then $g \circ f$ is the domain of the cartesian lift of $\phi_{gf}\colon PgPf \Rightarrow P(gf)$ at gf. If $\alpha\colon f \Rightarrow f'$, $\beta\colon g \Rightarrow g'$ then $\beta \circ \alpha$ is the unique map above $P\beta * P\alpha$ such that

$$\begin{tikzcd} g \circ f \arrow[r, Rightarrow, "\varphi(\phi,gf)"] \arrow[d, Rightarrow, dotted, "\beta\circ\alpha"'] & gf \arrow[d, Rightarrow, "\beta*\alpha"] \\ g' \circ f' \arrow[r, Rightarrow, "\varphi(\phi,g'f')"'] & g'f' \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} PgPf \arrow[r, Rightarrow, "\phi"] \arrow[d, Rightarrow, "P\beta*P\alpha"'] & P(gf) \arrow[d, Rightarrow, "P(\beta*\alpha)"] \\ Pg'Pf' \arrow[r, Rightarrow, "\phi"'] & P(g'f') \end{tikzcd}$$

This has the effect that $P'(g \circ f) = P'g.P'f$ and $P'(\beta \circ \alpha) = P\beta * P\alpha = P'\beta * P'\alpha$. Similarly, let the identity 1-cells $\hat{1}_e$ be the domain of the cartesian lift of $\phi_e\colon 1_{Pe} \Rightarrow P(1_e)$ at 1_e. Then $P'(\hat{1}_e) = 1_{P'e}$. The coherence isomorphisms for $\mathscr{E}'$ are obtained as the unique maps in the following diagrams.

$$\begin{tikzcd} (h \circ g) \circ f \arrow[r, Rightarrow, "\varphi"] \arrow[d, Rightarrow, dotted, "\hat{a}"'] & (h \circ g).f \arrow[r, Rightarrow, "\varphi.f"] & (h.g).f \arrow[d, Rightarrow, "a"] \\ h \circ (g \circ f) \arrow[r, Rightarrow, "\varphi"'] & h.(g \circ f) \arrow[r, Rightarrow, "h.\varphi"'] & h.(g.f) \end{tikzcd}$$

$$\begin{tikzcd} f \arrow[d, Rightarrow, dotted, "\hat{r}"'] \arrow[drr, Rightarrow, "r"] & & \\ f \circ \hat{1} \arrow[r, Rightarrow, "\varphi"'] & f.\hat{1} \arrow[r, Rightarrow, "f.\varphi"'] & f.1 \end{tikzcd} \qquad \begin{tikzcd} f \arrow[d, Rightarrow, dotted, "\hat{l}"'] \arrow[drr, Rightarrow, "l"] & & \\ \hat{1} \circ f \arrow[r, Rightarrow, "\varphi"'] & \hat{1}.f \arrow[r, Rightarrow, "\varphi.f"'] & 1.f \end{tikzcd}$$

The reader can verify that this is a bicategory and that P' is a homomorphism that preserves composition and identities. Let S be the identity on 0-, 1-, and 2-cells. The coherence morphisms for S are the isomorphisms $\varphi(\phi, gf)\colon g \circ f \cong gf$ and $\varphi(\phi_e, 1_e)\colon \hat{1}_e \cong$

1_e described above. It is easy to check that $P'S = P$ and since S is the identity on 0-, 1-, and 2-cells it is an isomorphism.

Suppose that P is a fibration. The cartesian 1-cells in $\mathscr{E}'$ are precisely those we obtain in $\mathscr{E}$. Lifts along cartesian 1-cells are $(\hat{h}, \alpha^!, \hat{\beta})$ obtained by taking lifts $(\hat{h}, \hat{\alpha}, \hat{\beta})$ in $\mathscr{E}$ and letting $\alpha^! = \hat{\alpha}\varphi(\phi, f\hat{h})$. Thus P' has cartesian lifts of 1-cells. Since the action of P' on hom-categories is the same as P, P' is locally fibred. Suppose that β, α are cartesian 2-cells. Their composite in $\mathscr{E}'$ is defined using the diagram

$$\begin{tikzcd}
g \circ f \arrow[r, Rightarrow, "\varphi"] \arrow[d, Rightarrow, "\beta\circ\alpha"'] & g.f \arrow[d, Rightarrow, "\beta*\alpha"] \\
g' \circ f' \arrow[r, Rightarrow, "\varphi"'] & g'.f'
\end{tikzcd} .$$

Since P is a fibration $\beta * \alpha$ is cartesian. Then the cancellation property of cartesian 1-cells tells us that $\beta \circ \alpha$ is also cartesian. Thus P' is a fibration. Chosen cartesian 1- and 2-cells in $\mathscr{E}'$ are the same as in $\mathscr{E}$. Since S is the identity on 0-, 1-, and 2-cells it is cartesian. $\square$

Remark 2.3.19. These last two results rely on the local iso-lifting property of P. The first result corresponds to the lifting of an isomorphism that arises from the bipullback definition of cartesian 1-cell.

$$\begin{tikzcd}
\mathbb{2} \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[dr, phantom, "\cong", near end, shift right=4ex] & & \\
& \mathscr{E}(z,x) \arrow[r, "f_*"] \arrow[d, "P_{zx}"] \arrow[dr, phantom, "\cong"] \arrow[ur, phantom, "\cong"] & \mathscr{E}(z,y) \arrow[d, "P_{zy}"] \\
& \mathscr{B}(Pz,Px) \arrow[r, "Pf_*"'] & \mathscr{B}(Pz,Py)
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\mathbb{2} \arrow[dr, bend right] \arrow[dr, bend left] \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[dr, phantom, "\cong"] & & \\
& \mathscr{E}(z,x) \arrow[r, "f_*"] \arrow[d, "P_{zx}"] \arrow[dr, phantom, "\cong"] \arrow[ur, phantom, "\cong"] & \mathscr{E}(z,y) \arrow[d, "P_{zy}"] \\
& \mathscr{B}(Pz,Px) \arrow[r, "Pf_*"'] & \mathscr{B}(Pz,Py)
\end{tikzcd}$$

The isomorphism on the left is the $\hat{\beta}$ associated with lifts along cartesian f. The second result corresponds to the lifting of the isomorphisms associated with the action of P on composition and identities.

$$\begin{tikzcd}
\mathscr{E}(y,z) \times \mathscr{E}(x,y) \arrow[r, "*"] \arrow[d, "P_{yz}\times P_{xy}"'] \arrow[dr, phantom, "\cong"] & \mathscr{E}(x,z) \arrow[d, "P_{xz}"] \\
\mathscr{B}(Py,Pz) \times \mathscr{B}(Px,Py) \arrow[r, "*"'] & \mathscr{B}(Px,Pz)
\end{tikzcd}
\longmapsto$$

$$\begin{tikzcd}
\mathscr{E}(y,z)\times\mathscr{E}(x,y) \arrow[rr, bend left, "*"] \arrow[rr, bend right, "\circ"'] \arrow[rr, phantom, "\cong"] \arrow[d, "P_{yz}\times P_{xy}"'] & & \mathscr{E}(x,z) \arrow[d, "P_{xz}"] \\
\mathscr{B}(Py,Pz)\times\mathscr{B}(Px,Py) \arrow[rr, "*"'] & & \mathscr{B}(Px,Pz)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "1_x"] \arrow[dr, bend right, "1_{Px}"'] \arrow[dr, phantom, "\cong"] & \mathscr{E}(x,x) \arrow[d, "P_{xx}"] \\
& \mathscr{B}(Px,Px)
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\mathbb{1} \arrow[r, bend left, "1_x"] \arrow[r, bend right, "\hat{1}_x"'] \arrow[r, phantom, "\cong"] \arrow[dr, bend right, "1_{Px}"'] & \mathscr{E}(x,x) \arrow[d, "P_{xx}"] \\
& \mathscr{B}(Px,Px)
\end{tikzcd}$$

We've borrowed this convenient visual representation of iso-lifting from [JS93].

From this point on we will suppose that all fibrations preserve composition and identities and have the stronger lifting property of Proposition 2.3.17. Working under this supposition will vastly simplify future calculations. We can do this without loss of generality because all fibrations are isomorphic to a fibration of this kind.

Definition 2.3.20. Suppose $P\colon \mathscr{E} \to \mathscr{B}$ and $Q\colon \mathscr{D} \to \mathscr{B}$ are fibrations of bicategories. A homomorphism $\eta\colon \mathscr{E} \to \mathscr{D}$ is called *cartesian* when it preserves cartesian maps and $Q\eta = P$.

Let Fib($\mathscr{B}$) be the tricategory whose objects are fibrations over $\mathscr{B}$, whose 1-cells are cartesian homomorphisms,2-cells are pseudo-natural transformations $\Gamma\colon \eta \Rightarrow \epsilon$ that have $Q\Gamma = 1_P$; and 3-cells are modifications $\zeta\colon \Gamma \Rrightarrow \Lambda$ that have $Q\zeta = 1_{1_P}$. Let Bicat be the tricategory of bicategories, homomorphisms, transformations and modifications. Let $[\mathscr{B}^{\mathrm{coop}}, \mathrm{Bicat}]$ be the tricategory of contravariant trihomomorphisms from $\mathscr{B}$ to Bicat, tritransformations, trimodifications and perturbations. As in Section 2.2, a *perturbation* is a morphism of modifications as defined in [GPS95; Gur06]..

2.3.3 The Grothendieck construction

Before describing the Grothendieck construction for bicategories we will unpack the tricategory structure of Bicat and see what a contravariant trihomomorphism from $\mathscr{B}$ to Bicat really means.

Remark 2.3.21. An algebraic definition of tricategory can be found in [Gur06, p. 23]. There is more than one tricategory structure on Bicat. We choose the following:

- Composition of 1-cells is the usual composition of homomorphisms: $GF(x) = G(F(x))$ and the usual coherence isomorphisms.

- Composition of 2-cells: When $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$ we let $(\beta\alpha)_x = \beta_x\alpha_x$ and

$(\beta\alpha)_f$ be the associated pasting of 2-cells. When

$$\begin{tikzcd} \mathscr{B} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{C} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathscr{D} \end{tikzcd}$$

we let $(\beta * \alpha)_x$ be

$$GF(x) \xrightarrow{\beta_{F(x)}} G'F(x) \xrightarrow{G'(\alpha_x)} G'F'(x) \cdot$$

$(\beta * \alpha)_f$ be the associated pasting of 2-cells.

- Composition of 3-cells is similar to that for 2-cells.
- Identity 1-cells are the obvious identity homomorphisms $1_{\mathscr{B}} \colon \mathscr{B} \to \mathscr{B}$. Identity 2-cells are transformations $1_F \colon F \Rightarrow F$ with 1-cell components $(1_F)_x = 1_{F(x)}$. Identity 3-cells are modifications 1_α with $(1_\alpha)_x = 1_{(\alpha_x)}$.

The rest of the data consists firstly of pseudo-natural equivalences governing associativity and the action of identities. The final data are invertible modifications that sit in place of the usual axioms. The details can be found in [GPS95] and are summarised below.

- Associativity of composition is governed by a pseudo-natural equivalence

$$\begin{tikzcd} (HG)F \arrow[r, Rightarrow, "a_{HGF}"] \arrow[d, Rightarrow, "(\gamma\beta)\alpha"'] \arrow[dr, phantom, "\cong"] & H(GF) \arrow[d, Rightarrow, "\gamma(\beta\alpha)"] \\ (H'G')F' \arrow[r, Rightarrow, "a_{H'G'F'}"'] & H'(G'F') \end{tikzcd} .$$

The 2-cells components are identity transformations $1 \colon H(GF) \Rightarrow (HG)F$. The 3-cell components are invertible modifications whose components are a composite of coherence isomorphisms in the image of $H'(G'F')$.

- The unity of identities is governed by two pseudo-natural transformations

$$\begin{tikzcd} 1.F \arrow[r, Rightarrow, "l_F"] \arrow[d, Rightarrow, "1_1.\alpha"'] \arrow[dr, phantom, "\cong"] & F \arrow[d, Rightarrow, "\alpha"] \\ 1.F' \arrow[r, Rightarrow, "l'_F"'] & F' \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, Rightarrow, "r_F"] \arrow[d, Rightarrow, "\alpha"'] \arrow[dr, phantom, "\cong"] & F.1 \arrow[d, Rightarrow, "\alpha.1_1"] \\ F' \arrow[r, Rightarrow, "r'_F"'] & F'.1 \end{tikzcd} .$$

The 2-cells components are identity transformations $1 \colon 1F \Rightarrow F$ and $1 \colon F \Rightarrow F1$. The 3-cell components are invertible modifications whose components are a composite of coherence isomorphisms in the image of F'.

- There are invertible modifications π, μ, λ, ρ that relate various composites of a, l,

r above. In this case they are built from the coherence isomorphisms of assorted bicategories and homomorphisms.

- There are three axioms that π, μ, λ, ρ are required to satisfy. The coherence theorem for bicategories guarantees that they hold.

There is a related tricategory structure given by an alternate composition rule for 2,3-cells. Choosing one or the other will not significantly effect the nature of our results.

Remark 2.3.22. An algebraic definition of trihomomorphism can be found in [Gur06, p. 29]. A trihomomorphism $F\colon \mathscr{B}^{\mathrm{coop}} \to \mathrm{Bicat}$ consists of the following data:

- An object function $F_0\colon \operatorname{ob}\mathscr{B} \to \operatorname{ob}\mathrm{Bicat}$.

- For objects a, b of $\mathscr{B}$, a pseudo-functor $\mathscr{B}(a,b) \to \mathrm{Bicat}(Fa, Fb)$. This means that 2-cell composition and identities in $\mathscr{B}$ are preserved up to natural isomorphisms satisfying standard coherence axioms. The data is just isomorphisms $\phi_{\beta\alpha}\colon F(\beta.\alpha) \Rrightarrow F\alpha.F\beta$ and $\phi_f\colon F(1_f) \Rrightarrow 1_{Ff}$. The components are

$$\begin{tikzcd} Ff(x) \arrow[rr, bend left, "F(\beta.\alpha)_x"] \arrow[dr, "F\alpha_x"'] & \arrow[d, phantom, "\Downarrow\phi_{\beta\alpha}"] & Fh(x) \\ & Fg(x) \arrow[ur, "F\beta_x"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Ff(x) \arrow[r, bend left, "F(1_f)_x"] \arrow[r, bend right, "1_{Ff(x)}"'] \arrow[r, phantom, "\Downarrow\phi_f"] & Ff(x) \end{tikzcd} .$$

- For objects a, b, c of $\mathscr{B}$, an adjoint equivalence $\chi\colon \otimes' \circ (F\times F) \Rightarrow F\circ\otimes$. This means that horizontal composition is preserved up to adjoint equivalence. The data is

$$\begin{tikzcd} Ff.Fg \arrow[r, Rightarrow, "\chi_{gf}"] \arrow[d, Rightarrow, "F\alpha.F\beta"'] \arrow[dr, phantom, "\Swarrow\chi_{\beta\alpha}"] & F(gf) \arrow[d, Rightarrow, "F(\beta\alpha)"] \\ Ff'.Fg' \arrow[r, Rightarrow, "\chi_{g'f'}"'] & F(g'f') \end{tikzcd}$$

which amounts to

$$\begin{tikzcd} FfFg(x) \arrow[r, "\chi_{gf\,x}"] \arrow[d, "FfFg(h)"'] \arrow[dr, phantom, "\Swarrow\chi_{gf\,h}"] & Fgf(x) \arrow[d, "Fgf(h)"] \\ FfFg(y) \arrow[r, "\chi_{gf\,y}"'] & Fgf(y) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FfFg(x) \arrow[r, "\chi_{gf\,x}"] \arrow[d, "F\alpha_{Fg(x)}"'] & Fgf(x) \arrow[dd, "F(\beta\alpha)_x"] \\ Ff'Fg(x) \arrow[d, "Ff'(F\beta_x)"'] \arrow[r, phantom, "\Swarrow\chi_{\beta\alpha\,x}"] & \\ Ff'Fg'(x) \arrow[r, "\chi_{g'f'\,x}"'] & Fgf(x) \end{tikzcd} .$$

The data on the left indicates that χ_{gf} is a pseudo-natural equivalence and the data on the right is the component at x of the modification $\chi_{\beta\alpha}$.

- For objects a of $\mathscr{B}$, an adjoint equivalence transformation $\iota\colon I'_{Fa} \Rightarrow F \circ I_a$. This means that identity 1-cells are preserved up to adjoint equivalence. The data is

$$\begin{tikzcd} 1_{Fa} \arrow[r, Rightarrow, "\iota_a"] \arrow[d, Rightarrow, "1_{1_{Fa}}"'] \arrow[dr, phantom, "\Swarrow\iota_{1_a}"] & F1_a \arrow[d, Rightarrow, "F(1_{1_a})"] \\ 1_{Fa} \arrow[r, Rightarrow, "\iota_a"'] & F1_a \end{tikzcd} .$$

This means

$$\begin{tikzcd} x \arrow[r, "\iota_{a\,x}"] \arrow[d, "h"'] \arrow[dr, phantom, "\Swarrow\iota_{a\,h}"] & F1(x) \arrow[d, "F1(h)"] \\ y \arrow[r, "\iota_{a\,y}"'] & F1(y) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x \arrow[r, "\iota_{a\,x}"] \arrow[d, "1_x"'] \arrow[dr, phantom, "\Swarrow\iota_{1_a\,x}"] & F1(x) \arrow[d, "F(1_1)_x"] \\ x \arrow[r, "\iota_{a\,x}"'] & F1(x) \end{tikzcd} .$$

The data on the left indicates that ι_a is a pseudo-natural equivalence and the data on the right is the component at x of the modification ι_{1_a}.

- For objects a, b, c, d of $\mathscr{B}$, an invertible modification ω whose component at fgh is itself an invertible modification whose component at x is an invertible pseudo-natural transformation

$$\begin{tikzcd} & FfF(h.g)(x) \arrow[r, "\chi_x"] \arrow[d, phantom, "\Downarrow\omega_{fgh,x}"] & F((h.g).f)(x) \arrow[dr, "Fa"] & \\ FfFgFh(x) \arrow[ur, "Ff(\chi_x)"] \arrow[dr, "1"'] & & & F(h.(g.f))(x) \\ & FfFgFh(x) \arrow[r, "\chi_{Fh(x)}"'] & F(g.f)Fh(x) \arrow[ur, "\chi_x"'] & \end{tikzcd} .$$

- For objects a, b of $\mathscr{B}$, an invertible modification γ whose component at f is itself an invertible modification whose component at x is an invertible pseudo-natural transformation

$$\begin{tikzcd} & FfF1(x) \arrow[r, "\chi_x"] \arrow[d, phantom, "\Downarrow\gamma_x"] & F(1.f)(x) \arrow[dr, "Fl_x"] & \\ Ff(x) \arrow[ur, "Ff(\iota_x)"] \arrow[rrr, "1"'] & & & Ff(x) \end{tikzcd} .$$

- For objects a, b of $\mathscr{B}$, an invertible modification δ whose component at f is itself an invertible modification whose component at x is an invertible pseudo-natural transformation

$$\begin{tikzcd}
Ff(x) \arrow[rr, "Fr_x"] \arrow[dr, "1"'] & \arrow[d, phantom, "\Downarrow\delta_x"] & F(f.1)(x) \\
& Ff(x) \xrightarrow[\iota_{Ff(x)}]{} F1Ff(x) \arrow[ur, "\chi_x"'] &
\end{tikzcd}.$$

- There are two axioms involving ω, δ, γ, ι, χ above. They can be found in [Gur06; GPS95].

We will describe the Grothendieck construction for fibred bicategories: a triequivalence

$$\mathrm{el}\colon [\mathscr{B}^{\mathrm{coop}}, \mathrm{Bicat}] \to \mathrm{Fib}(\mathscr{B})$$

that generalises the result given in Section 2.2. Again, we are mostly concerned with its action on objects and use "Grothendieck construction" to mean both the action on objects and the whole trihomomorphism.

Construction 2.3.23 (The Grothendieck construction for bicategories)**.** Suppose $F\colon \mathscr{B}^{\mathrm{coop}} \to \mathrm{Bicat}$ is a trihomomorphism. We define a fibration $P_F\colon \mathrm{el}\, F \to \mathscr{B}$ as follows. $\mathrm{el}\, F$ is the bicategory with:

- 0-cells are pairs (x, x_-) where $x \in \mathscr{B}$ and $x_- \in Fx$.

- 1-cells are pairs $(f, f_-)\colon (x, x_-) \to (y, y_-)$ where $f\colon x \to y$ and $f_-\colon x_- \to Ff(y_-)$.

- 2-cells are pairs $(\alpha, \alpha_-)\colon (f, f_-) \Rightarrow (g, g_-)\colon (x, x_-) \to (y, y_-)$ where $\alpha\colon f \Rightarrow g$ and

$$\begin{tikzcd}
x_- \arrow[rr, "f_-"] \arrow[dr, "g_-"'] & \arrow[d, phantom, "\Downarrow\alpha_-"] & Ff(y_-) \\
& Fg(y_-) \arrow[ur, "F\alpha_{y_-}"'] &
\end{tikzcd}.$$

- If $(g, g_-)\colon (y, y_-) \to (z, z_-)$ then the composite $(g, g_-).(f, f_-)$ has first component $g.f$ and second component

$$x_- \xrightarrow{f_-} Ff(y_-) \xrightarrow{Ff(g_-)} FfFg(z_-) \xrightarrow{\chi_{gf}} Fgf(z_-).$$

- If $(\gamma, \gamma_-)\colon (g, g_-) \to (h, h_-)\colon (x, x_-) \to (y, y_-)$ then the vertical composite has first component $\gamma.\alpha$ and second component

$$\begin{tikzcd}
x_- \arrow[rrr, "f_-"] \arrow[rr, "g_-"'] \arrow[dr, "h_-"'] & & \arrow[d, phantom, "\Downarrow\alpha_-"] & Ff(y_-) \\
& \arrow[r, phantom, "\Downarrow\gamma_-"] & Fg(y_-) \arrow[ur, "F\alpha_{y_-}"'] \arrow[d, phantom, "\Downarrow\phi_{\alpha\gamma y_-}"] & \\
& Fh(y_-) \arrow[ur, "F\gamma_{y_-}"] \arrow[uurr, bend right, "F\gamma\alpha_{y_-}"'] & &
\end{tikzcd}$$

- If $(\beta,\beta_-)\colon (j,j_-) \to (k,k_-)\colon (y,y_-) \to (z,z_-)$ then the horizontal composite has first component $\beta * \alpha$ and second component

$$\begin{tikzcd}
x_- \arrow[r, "f_-"] \arrow[dr, "g_-"'] \arrow[dr, phantom, "\Downarrow\alpha_-"] & Ff(y_-) \arrow[r, "Ff(j_-)"] \arrow[dr, "Ff(k_-)"'] \arrow[dr, phantom, "\Downarrow Ff(\beta_-)"] & FfFj(z_-) \arrow[r, "\chi_{jf}"] & Fjf(z_-) \\
& Fg(y_-) \arrow[u, "F\alpha_{y_-}"] \arrow[dr, "Fg(k_-)"'] \arrow[r, phantom, "\Downarrow F\alpha_{k_-}"] & FfFk(z_-) \arrow[u, "Ff(F\beta_{z_-})"] \arrow[r, phantom, "\Downarrow\chi_{\beta\alpha}"] & \\
& & FgFk(z_-) \arrow[u, "F\alpha_{Fk(z_-)}"] \arrow[uu, bend right, "(F\alpha F\beta)_{z_-}"'] \arrow[r, "\chi_{kg}"'] & Fkg(z_-) \arrow[uu, bend right, "F\beta\alpha_{z_-}"']
\end{tikzcd} .$$

The 2-cell labelled $Ff(\beta_-)$ is strictly the composite $\phi.Ff(\beta_-)$ where ϕ is an isomorphism associated with Ff. In order to simplify this diagram and those that follow, all isomorphisms associated with such homomorphisms have been omitted.

- Identity 1-cells are $1_{(x,x_-)} = (1_x, (i_x)_{x_-})$, the second component is

$$1_{Fx}(x_-) \xrightarrow{(i_x)_{x_-}} F1_x(x_-) \cdot$$

- Identity 2-cells are $1_{(f,f_-)} = (1_f, \phi_f f_-.l_{f_-})$, the second component is

$$\begin{tikzcd}
x_- \arrow[r, "f_-"] \arrow[dr, bend right, "f_-"'] \arrow[dr, phantom, "\Downarrow l_{f_-}"] & Ff(y_-) \\
& Ff(y_-) \arrow[u, bend left, "1"] \arrow[u, bend right, "F1_{f y_-}"'] \arrow[u, phantom, "\overset{\phi_f}{\Longrightarrow}"]
\end{tikzcd} .$$

- For associativity isomorphisms, 2-cells with first component a_{fgh} and second component given by the following composite.

$$\begin{tikzcd}
w_- \arrow[r, "f_-"] & Ff(x_-) \arrow[r, "Ff(g_-)"] & FfFg(y_-) \arrow[r, "\chi_{y_-}"] \arrow[dr, "FfFg(h_-)"'] & Fgf(z_-) \arrow[r, "Fgf(h_-)"] \arrow[d, phantom, "\Downarrow\chi_{h_-}"] & FgfFh(z_-) \arrow[r, "\chi"] \arrow[dr, phantom, "\Downarrow\omega^{-1}_{fgh}"] & Fh(gf)(z_-) \\
 & & & FfFgFh(z_-) \arrow[r, "Ff(\chi)"'] \arrow[ur, "\chi_{Fh(z_-)}"] & FfFhg(z_-) \arrow[r, "\chi"'] & F(hg)f(z_-) \arrow[u, "Fa_{z_-}"']
\end{tikzcd}$$

- For left unit isomorphisms, 2-cells with first component l_f and second component given by the following composite.

$$\begin{tikzcd}
x_- \arrow[r, "f_-"] & Ff(y_-) \arrow[rr, "1"] \arrow[dr, "Ff(i_y)"'] & \arrow[d, phantom, "\Downarrow\gamma_f^{-1}"] & Ff(y_-) \\
 & & FfF1_y(y_-)Ff(y_-) \arrow[r, "\chi"'] & F1_y f(y_-) \arrow[u, "Fl_{y_-}"']
\end{tikzcd}$$

- For right unit isomorphisms, 2-cells with first component r_f and second component given by the following composite.

$$\begin{tikzcd}
x_- \arrow[r, "i"] \arrow[drr, bend right, "f_-"'] & F1(x_-) \arrow[r, "F1(f)"] \arrow[d, phantom, "\Downarrow i"] & F1Ff(y_-) \arrow[r, "\chi"] \arrow[dr, phantom, "\Downarrow\delta_f^{-1}"] & Ff1(y_-) \\
 & \; & Ff(y_-) \arrow[u, "i_{Ff}"'] \arrow[ur, bend right, "Fr_{y_-}"'] &
\end{tikzcd}$$

By projecting onto the first component of $\mathrm{el}\, F$ we obtain a homomorphism $P_F \colon \mathrm{el}\, F \to \mathscr{B}$.

Proposition 2.3.24. *The homomorphism $P_F \colon \mathrm{el}\, F \to \mathscr{B}$ defined above is a fibration.*

Proof. To show that $\mathrm{el}\, F$ is a bicategory we need to use the axioms for a trihomomorphism.

- $\mathrm{el}\, F((x, x_-), (y, y_-))$ is a category: the definition of vertical composition uses the coherence isomorphisms for the homomorphism F_{xy}. The three axioms for this homomorphism give associativity and left and right identity in $\mathrm{el}\, F((x, x_-), (y, y_-))$.
- Horizontal composition is functorial: the definition of horizontal composition uses the isomorphism $\chi_{\beta\alpha}$ from the transformation χ. The two axioms for this transformation make horizontal composition in $\mathrm{el}\, F$ functorial.
- Coherence axioms: the coherence isomorphisms make use of invertible modifications ω, γ and δ. The two axioms for these modifications ensure that the axioms for a bicategory hold.

Observe that composition in $\mathrm{el}\, F$ in the first component is just composition in $\mathscr{B}$, thus P_F is a homomorphism that preserves composition and identities strictly. We need to show that $P_F \colon \mathrm{el}\, F \to \mathscr{B}$ is a fibration.

Suppose that $f \colon x \to y$ in $\mathscr{B}$ and (y, y_-) is in $\mathrm{el}\, F$. We claim that $\varphi(f, (y, y_-)) = (f, 1_{Ff(y_-)}) \colon (x, Ff(y_-)) \to (y, y_-)$ is cartesian over f.

$$Ff(y_-) \xrightarrow[1_{Ff(y_-)}]{} Ff(y_-)$$

Whenever

$$\begin{tikzcd}
(z, z_-) \arrow[d, "{(h,\ )}"'] \arrow[dr, "{(g, g_-)}"] \arrow[d, phantom, shift left=4ex, "{(\alpha,\ )\ \Nearrow}"] & \\
(x, Ff(y_-)) \arrow[r, "{(f,1)}"'] & (y, y_-)
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
z \arrow[d, "h"'] \arrow[dr, "g"] \arrow[d, phantom, shift left=3ex, "\alpha\ \Nearrow"] & \\
x \arrow[r, "f"'] & y
\end{tikzcd}$$

we can choose $h_- = \chi^{\bullet}.F\alpha_{y_-}.g_-$ and α_- equal to

$$\begin{tikzcd}
z \arrow[r, "g_-"] \arrow[drrrr, bend right, "g_-"'] & Fg(y_-) \arrow[r, "F\alpha_{y_-}"] \arrow[d, phantom, "\Downarrow l"] & Ffh(y_-) \arrow[r, "\chi^{\bullet}"] \arrow[rrrr, bend right, "1"'] & FhFf(y_-) \arrow[r, shift left, "Fh(1)"] \arrow[r, shift right, "1"'] \arrow[rrr, bend right, "\chi"'] \arrow[d, phantom, "\Downarrow \epsilon"] & FhFf(y_-) \arrow[r, "\chi"] \arrow[d, phantom, "\Downarrow r"] & Ffh(y_-) \\
& & & & Fg(y_-) \arrow[ur, "F\alpha_{y_-}"'] &
\end{tikzcd}$$

to give a lifting of (h, α) as required. Showing the 2-cell property is not very difficult but requires large diagrams that we will not include here. Thus P_F has cartesian 1-cells.

Suppose that $\alpha \colon f \Rightarrow g \colon x \to y$ in $\mathscr{B}$ and (g, g_-) is in $\mathrm{el}\, F$. We claim that $\varphi(\alpha, (g, g_-)) = (\alpha, 1_{F\alpha_{y\_}g}) \colon (f, F\alpha_{y\_}g) \Rightarrow (g, g_-) \colon (x, x_-) \to (y, y_-)$ is cartesian over α.

$$\begin{tikzcd}
x \arrow[r, "F\alpha_{y_}g"] \arrow[dr, "g"'] & Ff(y_-) \arrow[d, phantom, shift right=6ex, "\Downarrow 1"] \\
& Fg(y_-) \arrow[u, "F\alpha_{y_-}"']
\end{tikzcd}$$

Whenever

$$\begin{tikzcd}
(h, h_-) \arrow[d, Rightarrow, "{(\delta,\ )}"'] \arrow[dr, Rightarrow, "{(\gamma, \gamma_-)}"] & \\
(f, F\alpha_{y_}g \arrow[r, Rightarrow, "{(\alpha,1)}"'] & (g, g_-)
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
h \arrow[d, Rightarrow, "\delta"'] \arrow[dr, Rightarrow, "\gamma"] & \\
f \arrow[r, Rightarrow, "\alpha"'] & g
\end{tikzcd}$$

we find that $\delta_- = \gamma_-.\phi^{-1}_{\delta\alpha} g$ and that this uniquely makes $(\gamma, \gamma_-) = (\alpha, 1).(\delta, \delta_-)$. This occurs precisely because $1_{F\alpha_{y\_}g}$ is an isomorphism. Thus P_F has cartesian 2-cells.

The horizontal composite of the two lifts $(\alpha, 1_{F\alpha_{y\_}g})$, $(\beta, 1_{F\beta_{z\_}k})$ has first component $\beta * \alpha$ and second component a pasting of $\chi_{\beta\alpha}$ and $F\alpha_{g_-}$. Since the second component is an isomorphism we can use the argument above to show that this is cartesian. Thus by

Proposition 2.3.16 cartesian 2-cells are closed under horizontal composition. □

Construction 2.3.25 (Pseudo-inverse to the Grothendieck construction)**.** Suppose that $P\colon \mathscr{E} \to \mathscr{B}$ is a fibration. We define a trihomomorphism $F_P\colon \mathscr{B}^{\mathrm{coop}} \to \mathrm{Bicat}$ as follows:

on 0-cells: $Fb = \mathscr{E}_b$ for all $b \in B$. $\mathscr{E}_b$ is the fibre of P over b. Its 0-, 1- and 2-cells are those in $\mathscr{E}$ that map to b, 1_b and 1_{1_b}. Horizontal composition of 1-cells is defined by $g \hat{*} f = r^*(g.f)$: the domain of the cartesian lift of $r\colon 1_b \to 1_b.1_b$ at $g.f$. Composition of 2-cells is defined to be the unique 2-cell in the following diagram.

$$\begin{tikzcd}
g\hat{*}f \arrow[r, Rightarrow, "\varphi"] \arrow[d, Rightarrow, dotted, "\alpha\hat{*}\beta"'] & g.f \arrow[d, Rightarrow, "\alpha*\beta"] & \text{over} & 1 \arrow[r, Rightarrow, "r"] \arrow[d, Rightarrow, "1_1"'] & 1.1 \arrow[d, Rightarrow, "1_1*1_1"] \\
k\hat{*}h \arrow[r, Rightarrow, "\varphi"'] & k.h & & 1 \arrow[r, Rightarrow, "r"'] & 1.1
\end{tikzcd}$$

Identities are given by

$$\begin{tikzcd}
\hat{1} \arrow[r, Rightarrow, "\varphi"] \arrow[d, Rightarrow, dotted, "\hat{1_1}"'] & 1 \arrow[d, Rightarrow, "1_1"] & \text{over} & 1 \arrow[r, Rightarrow, "\phi"] \arrow[d, Rightarrow, "1_1"'] & P1 \arrow[d, Rightarrow, "1_1"] \\
\hat{1} \arrow[r, Rightarrow, "\varphi"'] & 1 & & 1 \arrow[r, Rightarrow, "\phi"'] & P1
\end{tikzcd}$$

and coherence isomorphisms by

$$\begin{tikzcd}
h\hat{*}(g\hat{*}f) \arrow[r, Rightarrow, "\varphi"] \arrow[d, Rightarrow, dotted, "\hat{a}"'] & h.(g\hat{*}f) \arrow[r, Rightarrow, "h\varphi"] & h.(g.f) \arrow[d, Rightarrow, "a"] & \text{over} & 1 \arrow[r, Rightarrow, "r"] \arrow[d, Rightarrow, "1_1"'] & 1.1 \arrow[r, Rightarrow, "1.r"] & 1.(1.1) \arrow[d, Rightarrow, "a"] \\
(h\hat{*}g)\hat{*}f \arrow[r, Rightarrow, "\varphi"'] & (h\hat{*}g).f \arrow[r, Rightarrow, "\varphi f"'] & (h.g).f & & 1 \arrow[r, Rightarrow, "r"'] & 1.1 \arrow[r, Rightarrow, "r.1"'] & (1.1).1
\end{tikzcd}$$

et cetera. The uniqueness of these 2-cells guarantees that the middle-four interchange holds and that these isomorphisms satisfy the axioms for a bicategory.

on 1-cells: $Ff = f^*\colon \mathscr{E}_{b'} \to \mathscr{E}_b$ is the homomorphism described using the following diagram (isomorphisms omitted).

$$\begin{tikzcd}
f^*e \arrow[r, "\varphi(f{,}e)"] \arrow[d, dotted, bend right, "f^*h"', ""{name=A}] \arrow[d, dotted, bend left, "f^*k", ""{name=B}] \arrow[from=A, to=B, phantom, "\overset{f^*\alpha}{\Rightarrow}"] & e \arrow[d, bend right, "h"', ""{name=C}] \arrow[d, bend left, "k", ""{name=D}] \arrow[from=C, to=D, phantom, "\overset{\alpha}{\Rightarrow}"] & \text{over} & b \arrow[r, "f"] \arrow[d, bend right, "1_b"', ""{name=E}] \arrow[d, bend left, "1_b", ""{name=F}] \arrow[from=E, to=F, phantom, "="] & b' \arrow[d, bend right, "1_{b'}"', ""{name=G}] \arrow[d, bend left, "1_{b'}", ""{name=H}] \arrow[from=G, to=H, phantom, "="] \\
f^*e' \arrow[r, "\varphi(f{,}e')"'] & e' & & b \arrow[r, "f"'] & b'
\end{tikzcd}$$

Ff sends e to the domain of $\varphi(f, e)$. Using the cartesian 1-cell $\varphi(f, e')$ we send h, k to

f^*h, f^*k over 1_b with an iso-square on the front and back and α is send to the unique $f^*\alpha$ over 1_{1_b}. The action of f^* on 1-cells is only defined up to a unique isomorphism. The coherence isomorphisms for f^* are precisely the unique 2-cells that arise when comparing $f^*h' \hat{*} f^*h$ to $f^*(h' * h)$ and $\hat{1}_{f^*e}$ to $f^*(1_e)$.

Again, the uniqueness of these maps ensures that f^* preserves vertical composition of 2-cells in the fibres and that the coherence isomorphisms satisfy the appropriate axioms.

on 2-cells: $F\sigma = \sigma^* \colon g^* \Rightarrow f^* \colon E_{b'} \to E_b$ is the transformation described by the following diagrams (isomorphisms omitted).

$$
\begin{tikzcd}
 & f^*e \arrow[dr, "\varphi(f,e)"] & \\
g^*e \arrow[ur, dotted, "\sigma_e^*"] \arrow[rr, bend left, "g_e"] \arrow[rr, bend right, "\varphi(g,e)"'] \arrow[rr, phantom, "\Downarrow\varphi(\sigma,\varphi(g,e))"] & & e
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
 & b \arrow[dr, "f"] & \\
b \arrow[ur, "1"] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Downarrow\sigma"] & & b'
\end{tikzcd}
$$

We take the cartesian lift of σ at $\varphi(g,e)$ and factor its domain as $\varphi(f,e).\sigma_e^*$ (see Proposition 2.3.28 below). Then $(F\sigma)_e = \sigma_e^*$. Now suppose $k \colon e \to e'$ and consider the action of f^* and g^* on k. We construct $F\sigma_k = \sigma_k^*$ as the unique isomorphism in the diagram

$$
\begin{tikzcd}
\varphi(f,e').f^*k * \sigma_e^* \arrow[r, Rightarrow, "1\varphi"] \arrow[d, Rightarrow, dotted, "1\sigma_k^*"'] & \varphi(f,e').f^*k.\sigma_e^* \arrow[r, Rightarrow, "\tau_f 1"] & k.\varphi(f,e).\sigma_e^* \arrow[r, Rightarrow, "1\varphi"] & k.g_e \arrow[d, Rightarrow, "\tau_g"] \\
\varphi(f,e').\sigma_{e'}^* * g^*k \arrow[r, Rightarrow, "1\varphi"'] & \varphi(f,e').\sigma_{e'}^*.g^*k \arrow[rr, Rightarrow, "\varphi 1"'] & & g_{e'}.g^*k
\end{tikzcd}
\tag{2.3.1}
$$

over

$$
\begin{tikzcd}
f.1 \arrow[r, Rightarrow, "f.r"] \arrow[d, Rightarrow, "f.1_1"'] & f.1.1 \arrow[r, Rightarrow, "lr.1"] & 1.f.1 \arrow[r, Rightarrow, "1.r"] & 1.f \arrow[d, Rightarrow, "lr"] \\
f.1 \arrow[r, Rightarrow, "f.r"'] & f.1.1 \arrow[rr, Rightarrow, "r.1"'] & & f.1
\end{tikzcd} .
$$

where τ_f is the isomorphism associated with the action of f^* on k. It is obtained using the cartesian property of $\varphi(f,e')$ and the indicated diagrams. Again, the uniqueness of the σ_k^* ensures that σ^* is actually a transformation.

Note that the 2-cell σ_k^* exists because the bottom composite in (2.3.1) is cartesian. It is an isomorphism because the upper composite is cartesian. This relies on the fact that each 1φ is cartesian (the post-composition property). Without the post-composition property this map doesn't exist, σ^* is not natural in any sense and F_P is not well-defined. This demonstrates again the importance of the post-composition property (see Remark 2.2.9).

F is locally a homomorphism: Suppose $\alpha\colon f \Rightarrow g\colon b \to b'$ and $\beta\colon g \Rightarrow h$. Then

$$\begin{tikzcd}
& & f^*e \arrow[ddr, "\varphi(f,e)"] & \\
& g^*e \arrow[ur, "\alpha_e^*"] \arrow[r, phantom, "\cong"] \arrow[drr, bend left, "g_e"] \arrow[drr, bend right, "\varphi(g,e)"'] \arrow[drr, phantom, "\Downarrow\varphi(\alpha,\ )"] & & \\
h^*e \arrow[ur, "\beta_e^*"] \arrow[r, phantom, "\cong"] \arrow[rrr, bend left, "h_e"] \arrow[rrr, bend right, "\varphi(h,e)"'] \arrow[rrr, phantom, "\Downarrow\varphi(\beta,\ )"] & & & e
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& f^*e \arrow[dr, "\varphi(f,e)"] \arrow[d, phantom, "\cong"] & \\
h^*e \arrow[ur, "(\alpha.\beta)_e^*"] \arrow[rr, bend left, "h_e"] \arrow[rr, bend right, "\varphi(h,e)"'] \arrow[rr, phantom, "\Downarrow\varphi(\alpha.\beta,\ )"] & & e
\end{tikzcd}$$

both sit over

$$\begin{tikzcd}
& b \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow r"] & \\
b \arrow[ur, "1"] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "h"'] \arrow[rr, phantom, "\Downarrow\alpha.\beta"] & & b'
\end{tikzcd}$$

so there exists a unique isomorphism $(\phi_{\beta\alpha})_e\colon (\alpha.\beta)^*_e \Rightarrow \alpha^*_e.\beta^*_e$. This is the component of a modification and is one of the coherence isomorphisms for $F_{bb'}\colon \mathscr{B}(b,b') \to \mathrm{Bicat}(\mathscr{E}_b, \mathscr{E}_{b'})$. The isomorphism for identities $\phi_f\colon (1_f)^* \Rightarrow (1_{f^*})$ is formed in a similar way. Their uniqueness ensures that they satisfy the appropriate axioms.

Horizontal composition is preserved up to pseudo-natural equivalence: Suppose $\alpha\colon f \Rightarrow g\colon b \to b'$ and $\beta\colon h \Rightarrow k\colon b' \to b''$ then since cartesian 1-cells are unique up to equivalence we get an equivalence $\chi_{hf\,e}$

$$\begin{tikzcd}
(hf)^*e \arrow[d, "\chi_{hf\,e}"'] \arrow[drr, "\varphi(hf,e)"] \arrow[dr, phantom, "\hat{r}\,\Nearrow"] & & \\
f^*h^*e \arrow[r, "\varphi(f,h^*e)"'] & h^*e \arrow[r, "\varphi(h,e)"'] & e
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
b \arrow[d, "1"'] \arrow[drr, "hf"] \arrow[dr, phantom, "r\,\Nearrow"] & & \\
b \arrow[r, "f"'] & b' \arrow[r, "h"'] & b''
\end{tikzcd}$$

that is unique up to isomorphism. It is the 1-cell component of a transformation $\chi_{hf}\colon f^*h^* \Rightarrow (hf)^*$. The 2-cell component of χ_{hf} is obtained as the unique 2-cell comparing two 1-cell lifts along a given cartesian 1-cell. Essentially every 2-cell isomorphism in this construction is obtained this way and all the the relevant axioms hold by uniqueness. For example, the invertible modification $\chi_{\beta\alpha}$ is described the same way and satisfies the axioms for a modification by uniqueness.

Identity 1-cells are preserved up to pseudo-natural equivalence: Suppose $1_b\colon b \to b$

then since cartesian 1-cells are unique up to equivalence we get an equivalence ι_{be}

$$\begin{tikzcd} e \arrow[d, "\iota_{be}"'] \arrow[drr, "1"] \arrow[dr, phantom, "\hat{r}\Nearrow"] & & \\ 1^*e \arrow[rr, "\varphi(1,e)"'] & & e \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} b \arrow[d, "1"'] \arrow[drr, "1"] \arrow[dr, phantom, "r\Nearrow"] & & \\ b \arrow[rr, "1"'] & & b'' \end{tikzcd}$$

that is unique up to isomorphism. It is the 1-cell component of a transformation $\iota_b \colon (1_b)^* \Rightarrow 1_{\mathscr{E}_b}$. The 2-cell component of ι_b and the modification ι_{1_b} are constructed in a similar way to that above.

Invertible modifications ω, γ, δ: As above, these are obtained using the 2-cell property for cartesian 1-cells. For ω_{fgh_e} we use

$$f^*(g^*(h^*e)) \xrightarrow{\varphi(f,g^*(h^*e))} g^*(h^*e) \xrightarrow{\varphi(g,h^*e)} h^*e \xrightarrow{\varphi(h,e)} e$$

and the appropriate liftings from the definitions of χ_{gf} et cetera. The other two are done in a similar way. We then use the uniqueness property to show that they satisfy the relevant axioms.

This gives us the following result.

Proposition 2.3.26. *The* $F_P \colon \mathscr{B}^{\mathrm{coop}} \to \mathrm{Bicat}$ *defined above is a trihomomorphism.*

Remark 2.3.27 (Fibres). We defined the fibre over an object b by insisting that each 1-cell and 2-cell sit exactly above 1_b and 1_{1_b}. Then the composition is such that this actually forms a bicategory. We could give a different kind of fibre by asking that 0-cells only have $Px \simeq b$ and 1-cells only have $Pf \cong 1_b$ and so on. The construction would work either way. However, when these simpler fibres are transported back and forth across the Grothendieck construction, the result is a bicategory that is not just biequivalent to the original, but almost exactly the same bicategory. This makes some of the proofs easier.

To prove that the Grothendieck construction is surjective up to equivalence we will need the following two results.

Proposition 2.3.28. *Suppose* $P \colon \mathscr{E} \to \mathscr{B}$ *is a fibration. Every* $f \colon x \to z$ *in* $\mathscr{E}$ *can be factored as*

$$\begin{tikzcd} x \arrow[d, "\hat{f}"'] \arrow[dr, "f"] \arrow[d, phantom, shift left=4, "\hat{l}\Nearrow"] & \\ y \arrow[r, "\varphi(Pf,z)"'] & z \end{tikzcd} \quad \textit{over} \quad \begin{tikzcd} Px \arrow[d, "1_{Px}"'] \arrow[dr, "Pf"] \arrow[d, phantom, shift left=4, "l\Nearrow"] & \\ Px \arrow[r, "Pf"'] & Pz \end{tikzcd}$$

where $\hat{f}$ *is unique up to unique isomorphism.*

Proof. The cartesian property of $\varphi(Pf,z)$ gives a factorisation as shown above. Suppose that there is another factorisation $(\hat{f}',\hat{l}')$ and consider the commuting shell formed with

1_{Pf} and $1_{1_{Px}}$ in the base. There is then a unique isomorphism $\tau\colon \hat f \cong \hat f'$ with $\hat l = \hat l'.\varphi(Pf,z)\tau$ and $P\tau = 1_{1_{Px}}$. ☐

Proposition 2.3.29. *Suppose $P\colon \mathscr{E} \to \mathscr{B}$ is a fibration. Every $\alpha\colon f \Rightarrow g\colon w \to z$ in $\mathscr{E}$ can be factored as*

$$\begin{tikzcd}
w \arrow[rr, bend left, "\hat f"] \arrow[rr, bend right, "\hat h * \hat g"'] \arrow[rr, phantom, "\Downarrow\hat\alpha"] \arrow[dr, "\hat g"'] \arrow[drrr, bend left=60, "f"] \arrow[drrr, bend right=60, "g"'] & & x \arrow[dr, "\varphi(Pf,z)"] & \\
& y \arrow[ur, "\hat h"] \arrow[rr, bend left, "h"] \arrow[rr, bend right, "\varphi(Pg,z)"'] \arrow[rr, phantom, "\Downarrow\varphi(P\alpha,\varphi(g,z))"] & & z
\end{tikzcd}
\quad \textit{over} \quad
\begin{tikzcd}
Pw \arrow[rr, bend left, "1"] \arrow[rr, bend right, "1"'] \arrow[dr, "1"'] \arrow[drrr, bend left=60, "Pf"] \arrow[drrr, bend right=60, "Pg"'] & & Pw \arrow[dr, "Pf"] & \\
& Pw \arrow[ur, "1"] \arrow[rr, bend left, "Pf"] \arrow[rr, bend right, "Pg"'] \arrow[rr, phantom, "\Downarrow P\alpha"] & & Pz
\end{tikzcd}$$

where $\hat\alpha$ is unique up to choice of $\hat f$, $\hat g$ and $\hat h$. (Invertible 2-cells have been omitted in each diagram).

Proof. The structure of this proof is the same as Proposition 2.2.21. Begin by factoring $f \cong \varphi(Pf,z)\cdot \hat f$ and $g \cong \varphi(Pg,z)\cdot\hat g$ using Proposition 2.3.28. Now take the cartesian lift of $P\alpha$ at $\varphi(Pg,z)$. Then $\varphi(P\alpha,\varphi(Pg,z))\hat g$ is cartesian over $P\alpha.1$ and

$$\begin{tikzcd}
\varphi(Pf,z)\hat f \arrow[r, Rightarrow, "\tau_f"] \arrow[d, Rightarrow, dotted, "\eta"'] & f \arrow[r, Rightarrow, "\alpha"] & g \arrow[d, Rightarrow, "\tau_g"] \\
h\hat g \arrow[rr, Rightarrow, "\varphi(P\alpha,\varphi(Pg,z))\hat g"'] & & \varphi(Pg,z)\hat g
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
Pf.1 \arrow[r, Rightarrow, "r"] \arrow[d, Rightarrow, "1"'] & Pf \arrow[r, Rightarrow, "P\alpha"] & Pg \arrow[d, Rightarrow, "r"] \\
Pf.1 \arrow[rr, Rightarrow, "P\alpha.1"'] & & Pg.1
\end{tikzcd}$$

so there exists a unique $\eta\colon \varphi(Pf,z)\hat f \Rightarrow h\hat g$ with $P\eta = 1_{Pf.1}$ and

$$\begin{tikzcd}
w \arrow[r, "\hat f"] \arrow[d, "\hat g"'] \arrow[dr, bend left, "f"] \arrow[dr, bend right, "g"'] \arrow[dr, phantom, "\Downarrow\alpha"] & y \arrow[d, "\varphi(Pf,z)"] \\
x \arrow[r, "\varphi(Pg,z)"'] & z
\end{tikzcd}
=
\begin{tikzcd}
w \arrow[r, "\hat f"] \arrow[d, "\hat g"'] \arrow[dr, phantom, "\Downarrow\eta"] & y \arrow[d, "\varphi(Pf,z)"] \\
x \arrow[r, bend left, "h"] \arrow[r, bend right, "\varphi(Pg,z)"'] \arrow[r, phantom, "\Downarrow\varphi(P\alpha,\ldots)"] & z
\end{tikzcd}.$$

By Proposition 2.3.28 we factor $h \cong \varphi(Pf,z)\cdot\hat h$ where $P\hat h = 1_{Px}$. We then form the fibre-composite of $\hat{h}$ and $\hat{g}$ by lifting the isomorphism $1 \cong 1.1$. Finally, we observe that

$$\begin{tikzcd}
\varphi(Pf,z)\hat{f} \arrow[r, Rightarrow, "\eta"] \arrow[dd, Rightarrow, "1"'] & h\hat{g} \arrow[d, Rightarrow, "\tau_h\hat{g}"] \\
& \varphi(Ph,z)\hat{h}\hat{g} \arrow[d, Rightarrow, "\varphi(Ph,z)\varphi(r,-)"] \\
\varphi(Pf,z)\hat{f} \arrow[r, Rightarrow, dotted, "\varphi(Ph,z)\hat{\alpha}"'] & \varphi(Ph,z)(\hat{h}\hat{*}\hat{g})
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
Pf.1 \arrow[r, Rightarrow, "1"] \arrow[dd, Rightarrow, "1"'] & Ph.1 \arrow[d, Rightarrow, "r1"] \\
& Pf.1.1 \arrow[d, Rightarrow, "Pfr"] \\
Pf.1 \arrow[r, Rightarrow, "Pf.1_1"'] & Pf.1
\end{tikzcd}$$

so there exists a unique $\hat{\alpha}\colon \hat{f} \Rightarrow \hat{h}\hat{*}\hat{g}$ over $1_{1_{Pw}}$ with $\varphi(Ph,z)\varphi(r,1).\tau_h\hat{g}.\eta = \varphi(Pf,z)\hat{\alpha}$ and hence a decomposition of α as stated.

Suppose that there was another decomposition of α that obtained $\hat{\alpha}'$ using $\hat{f}'$, $\hat{g}'$ and $\hat{h}'$. Then there exist unique τ_f, τ_g and τ_h as in Proposition 2.3.28. Using the fact that $\hat{\alpha}'$ is unique, we find that $\hat{\alpha}' = (\tau_h\hat{*}\tau_g).\hat{\alpha}.\tau_f$. □

Construction 2.3.30 (Surjective up to biequivalence)**.** In order to show that the Grothendieck construction is surjective up to biequivalence we need to find for every P a biequivalence of fibrations H.

$$\begin{tikzcd}
\mathrm{el}\,F_P \arrow[rr, "H"] \arrow[dr, "\pi"'] & & E \arrow[dl, "P"] \\
& B &
\end{tikzcd}$$

Here π is used in place of P_{F_P}.

First, what is $\mathrm{el}\,F_P$? Its data consists of:

0-cells: pairs (x, x_-) where $x_- \in \mathscr{E}$ and $Px_- = x$.

1-cells: pairs $(f, f_-)\colon (x, x_-) \to (y, y_-)$ where $f\colon x \to y$ in $\mathscr{B}$ and $f_- : \ x_- \to \ f^*(y_-)$ in $\mathscr{E}_x$.

2-cells: pairs $(\alpha, \alpha_-)\colon (f, f_-) \Rightarrow (g, g_-)\colon (x, x_-) \to (y, y_-)$ where $\alpha\colon f \Rightarrow g$ in $\mathscr{B}$ and $\alpha_-\colon f_- \Rightarrow \alpha^*_{y_-}\hat{*}g_-$ in $\mathscr{E}_x$.

$$\begin{tikzcd}
x_- \arrow[rr, bend left, "f_-"] \arrow[rr, bend right, "\alpha^*_{y_-}*g_-"'] \arrow[rr, phantom, "\Downarrow\alpha_-"] \arrow[dr, "g_-"'] & & f^*(y_-) \\
& g^*(y_-) \arrow[ur, "\alpha^*_{y_-}"'] &
\end{tikzcd}$$

Then $H\colon \mathrm{el}\,F_P \to \mathscr{E}$ is defined on 0-cells by $H(x, x_-) = x_-$. On 1-cells by $H(f, f_-) = r^*(\varphi(f, y_-).f_-)$, the domain of the cartesian lift of $r\colon f \Rightarrow f.1$ at

$$x_- \xrightarrow{f_-} f^*y_- \xrightarrow{\varphi(f,y_-)} y_- \,.$$

On 2-cells $H(\alpha, \alpha_-)$ is the composite

$$
\begin{tikzcd}
x_- \arrow[rr, bend left, "f_-"] \arrow[rr, bend right, "\alpha^*_{y_-} * \hat{g}"'] \arrow[rr, phantom, "\Downarrow\alpha_-"] \arrow[dr, "g_-"'] \arrow[drrr, bend left=60, "H(f,f_-)"] \arrow[drrr, bend right=60, "H(g,g_-)"'] & & f^* y_- \arrow[dr, "\varphi(f,y_-)"] & \\
& y \arrow[ur, "\alpha^*_{y_-}"] \arrow[rr, bend left, "h"] \arrow[rr, bend right, "\varphi(g,y_-)"'] \arrow[rr, phantom, "\Downarrow\varphi(\alpha,\varphi(g,y_-))"] & & y_-
\end{tikzcd}
.
$$

The coherence isomorphisms are the unique fillers in the following diagrams.

$$
\begin{tikzcd}
H(g,g_-).H(f,f_-) \arrow[r, Rightarrow, "\hat{r}*\hat{r}"] \arrow[dd, dotted] & \varphi(Pg,\ ).g_-.\varphi(Pf,\ ).f_- \arrow[d, Rightarrow, "\varphi(Pg,\ ).\cong.f_-"] \\
& \varphi(Pg,\ ).\varphi(Pf,\ ).f^*(g_-).f_- \arrow[d, Rightarrow, "\cong.f^*(g_-).f_-"] \\
H((g,g_-).(f,f_-)) \arrow[r, Rightarrow, "\hat{r}"'] & \varphi(Pgf,\ ).\chi.f^*(g_-).f_-
\end{tikzcd}
$$

$$
\begin{tikzcd}
H((1,(i_x)_{x_-}) \arrow[r, Rightarrow, "\hat{r}"] \arrow[d, dotted] & \varphi(P1,\ )(i_x)_{x_-} \arrow[d, Rightarrow, "\cong"] \\
1_{H(x,x_-)} \arrow[r, Rightarrow, "1"'] & 1_{x_-}
\end{tikzcd}
$$

Note that the coherence maps χ and ι are those given by the inverse to the Grothendieck construction.

Proposition 2.3.31. *The functor H is a cartesian biequivalence.*

Proof. First,

$$
\begin{aligned}
\pi(x, x_-) &= x = PH(x, x_-) \\
\pi(f, f_-) &= f = PH(f, f_-) \\
\pi(\alpha, \alpha_-) &= \alpha = PH(\alpha, \alpha_-)
\end{aligned}
$$

so $PH = \pi$.

Second, the chosen cartesian maps in $\operatorname{el} F_P$ are those with identities in the second component. Since H acts by post-composition with cartesian maps it is cartesian on chosen cartesian maps, thus H is cartesian.

Third, for every $e \in \mathscr{E}$ there exists $(Pe, e) \in \operatorname{el} F_P$ with $H(Pe, e) = e$ so H is surjective on objects. Then each 1-cell in the image of H is a composite of a factorisation according to Proposition 2.3.28. Since such factorisations are unique up to isomorphism, H is surjective up to isomorphism on 1-cells. Finally, all 2-cells in the image of H are composites of factorisations according to Proposition 2.3.29. Since such factorisations are unique (up to the given factorisation of the 1-cells), H is appropriately bijective on 2-cells. Thus H is a biequivalence. $\square$

Theorem 2.3.32. *The Grothendieck construction is the action on objects of a triequivalence*

$$\mathrm{el}\colon [\mathscr{B}^{\mathrm{coop}}, \mathrm{Bicat}] \to \mathrm{Fib}(\mathscr{B}).$$

Proof. We have already shown that on objects el is surjective up to biequivalence (Proposition 2.3.31). Showing that it is locally a biequivalence requires many pages of verification. We will present most of the required data but omit many of the details.

Suppose $\eta\colon F \Rightarrow G$ is a tritransformation in $[\mathscr{B}^{\mathrm{coop}}, \mathrm{Bicat}]$. We define $\mathrm{el}\,\eta\colon \mathrm{el}\,F \to \mathrm{el}\,G$ by $\mathrm{el}\,\eta(x, x_-) = (x, \eta_x(x_-))$ on objects and $\mathrm{el}\,\eta(f, f_-) = (f, \eta_{f\,y_-}.\eta_x(f_-))$ on 1-cells. The first component of $\mathrm{el}\,\eta(\alpha, \alpha_-)$ is α and the second is the composite

$$\begin{tikzcd}
\eta_x(x_-) \arrow[r, "\eta_x(f_-)"] \arrow[dr, "\eta_x(g_-)"'] & \eta_x Ff(y_-) \arrow[r, "\eta_{f\,y_-}"] \arrow[d, phantom, "\Downarrow\eta_x(\alpha_-)"] & Gf\eta_y(y_-) \arrow[dl, phantom, "\Downarrow\eta_{\alpha\,y_-}"] \\
 & \eta_x Fg(y_-) \arrow[u, "\eta_x F\alpha_{y_-}"'] \arrow[r, "\eta_{g\,y_-}"'] & Gg\eta_y(y_-) \arrow[u, "G\alpha\eta_{y\,y_-}"']
\end{tikzcd} .$$

The coherence isomorphisms ϕ_{fg} have first component 1_{fg} and second component

$$\begin{tikzcd}
\eta_x(x_-) \arrow[d, "\eta_x(f_-)"] & & & & \\
\eta_x Ff(y_-) \arrow[r, "\eta_{f\,y_-}"] \arrow[dr, "\eta_x Fg(f_-)"'] & Gf\eta_y(y_-) \arrow[r, "Gf(g_-)"] \arrow[d, phantom, "\Downarrow\eta_f"] & Gf\eta_y Fg(z_-) \arrow[r, "Gf\eta_{g\,y}"] \arrow[d, phantom, "\Downarrow\Pi"] & GfGg\eta_z(z_-) \arrow[r, "\chi^1"] & Ggf\eta_z(z_-) \arrow[dl, phantom, "\cong"] \\
 & \eta_x FgFf \arrow[ur, "\eta_f Fg"'] \arrow[rr, "\eta_x\chi"'] & & \eta_x Fgf \arrow[ur, "\eta_{gf}"] \arrow[r, "\eta_{gf}"'] & Ggf\eta_z \arrow[u, "G1_{\eta_z\,z_-}"']
\end{tikzcd}$$

where Π is part of the data of a tritransformation. The coherence isomorphisms ϕ_x have first component 1_{1_x} and second component

$$\begin{tikzcd}
\eta_x(x_-) \arrow[r, "\eta_x(i)"] \arrow[dr, "\iota_{\eta_x(x_-)}"'] & \eta_x F1(x_-) \arrow[r, "\eta_1"] \arrow[d, phantom, "\Downarrow M"] & G1\eta_x(x_-) \\
& G1\eta_x(x_-) \arrow[ur, "G1\eta_{x x_-}"'] &
\end{tikzcd} .$$

M is part of the data of a tritransformation.

This is a cartesian homomorphism from P_F to P_G and defines the action of el on 1-cells.

Suppose $\Gamma\colon \eta \Rrightarrow \epsilon$ is a trimodification in $[\mathscr{B}^{\mathrm{coop}}, \mathrm{Bicat}]$. We define a transformation $\mathrm{el}\,\Gamma\colon\ \mathrm{el}\,\eta \to \mathrm{el}\,\epsilon$. The first component of $\mathrm{el}\,\Gamma_{(x,x_-)}$ is 1_x and second component is

$$\begin{tikzcd}
\eta_x(x_-) \arrow[r, "\Gamma_{x x_-}"] & \tau_x(x_-) \arrow[r, "i"] & G1\tau_x(x_-)
\end{tikzcd} .$$

The first component of $\mathrm{el}\,\Gamma_{(f,f_-)}$ is lr and second component is

$$\begin{tikzcd}
\eta_x(x_-) \arrow[r, "\Gamma_x"] \arrow[dr, "\eta_x(f_-)"'] & \tau_x(x_-) \arrow[r, "\iota\tau"] \arrow[dr, "\tau_x(f_-)"] \arrow[d, phantom, "\cong"] & G1\tau_x(x_-) \arrow[r, "G1\tau_x(f_-)"] \arrow[d, phantom, "\cong"] & G1\tau_x Ff(y_-) \arrow[r, "G1\tau_f"] \arrow[d, phantom, "\cong"] & G1Gf\tau_x(y_-) \arrow[r, "\chi 1"] \arrow[dr, phantom, "\Downarrow\delta"] & Gf1\tau_x(y_-) \\
& \eta_x Ff(y_-) \arrow[r, "\Gamma_x Ff"] \arrow[dr, "\eta_f"'] & \tau_x Ff(y_-) \arrow[ur, "\iota\tau Ff"'] \arrow[dr, "\tau_f"] \arrow[d, phantom, "\Downarrow m"] & & & Gf\tau_y(y_-) \arrow[u, "Gr\tau_{y_-}"'] \\
& & Gf\eta_y(y_-) \arrow[r, "Gf\Gamma_y"'] & Gf\tau_y(y_-) \arrow[uur, "\iota Gf\tau"] \arrow[urr, "1"] \arrow[r, "Gf\iota\tau_y"'] & GfG1\tau_y(y_-) \arrow[r, "\chi 1"'] \arrow[ur, phantom, "\Downarrow\gamma"] & G1f\tau_y(y_-) \arrow[u, "Gl\tau_{y_-}"'] \arrow[uu, bend right=60, "Glr\tau_{y y_-}"']
\end{tikzcd}$$

where m is part of the data of a trimodification. The unlabelled isomorphisms are associated with ι. This is a vertical transformation from $\mathrm{el}\,\eta$ to $\mathrm{el}\,\tau$ and defines el on 2-cells.

Suppose $\zeta\colon \Gamma \to \Lambda$ is a perturbation in $[\mathscr{B}^{\mathrm{coop}}, \mathrm{Bicat}]$. We define a modification $\mathrm{el}\,\zeta\colon\ \mathrm{el}\,\Gamma \to \mathrm{el}\,\Lambda$. The first component of $\mathrm{el}\,\zeta_{(x,x_-)}$ is 1_{1_x} and the second component is

$$\begin{tikzcd}
\eta_x(x_-) \arrow[r, bend left, "\Gamma_x"{name=A}] \arrow[r, bend right, "\Lambda_x"'{name=B}] \arrow[from=A, to=B, phantom, "\Downarrow\zeta_x"] & \tau_x(x_-) \arrow[r, "\iota"] \arrow[dr, "\iota"'] \arrow[r, phantom, shift right=6, "\cong"] & G1\tau_x(x_-) \\
& & G1\tau_x(x_-) \arrow[u, bend left, "1"] \arrow[u, bend right, "G1_1\tau_{x x_-}"'] \arrow[u, phantom, "\cong"]
\end{tikzcd} .$$

This is a vertical modification from $\mathrm{el}\,\Gamma$ to $\mathrm{el}\,\Lambda$ and defines el on 3-cells.

It can be verified that el is a trihomomorphism.

Suppose $\alpha\colon\ \mathrm{el}\,F \to \mathrm{el}\,G$ is a cartesian homomorphism. We will define a tritransformation $\hat{\alpha}\colon F \Rightarrow G$ with $\mathrm{el}\,\hat{\alpha} \simeq \alpha$. This means homomorphisms $\hat{\alpha}_x\colon Fx \to Gx$ and pseudo-natural equivalences $\hat{\alpha}_x.F(\text{-}) \Rightarrow G(\text{-}).\hat{\alpha}_y$

$$\begin{tikzcd}
\hat{\alpha}_x.Fg \arrow[r, Rightarrow, "\hat{\alpha}_g"] \arrow[d, Rightarrow, "\hat{\alpha}_x F\gamma"'] \arrow[dr, phantom, "\overset{\hat{\alpha}_\gamma}{\Rrightarrow}"] & Gg.\hat{\alpha}_y \arrow[d, Rightarrow, "G\gamma\hat{\alpha}_x"] \\
\hat{\alpha}_x.Ff \arrow[r, Rightarrow, "\hat{\alpha}_f"'] & Gf.\hat{\alpha}_y
\end{tikzcd}$$

together with two invertible modifications Π and M.

Suppose $\sigma\colon h \Rightarrow k\colon a \to b$ in Fx. Define $\hat{\alpha}_x\colon Fx \to Gx$ to be the composite $Fx \to \mathrm{el}\, F_x \to \mathrm{el}\, G_x \to Gx$ where $\mathrm{el}\, F_x$ is the fibre over x. The first map sends

$$\begin{tikzcd}
a \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\sigma"] & b
\end{tikzcd} \mapsto \begin{tikzcd}
(x,a) \arrow[r, bend left, "{(1_x,\iota.h)}"] \arrow[r, bend right, "{(1_x,\iota.k)}"'] \arrow[r, phantom, "\Downarrow(1_{1_x},\delta)"] & (x,b)
\end{tikzcd} \quad \text{where } \delta \text{ is} \quad \begin{tikzcd}
a \arrow[r, "h"] \arrow[dr, "k"'] \arrow[r, phantom, shift right=4, "\Downarrow\sigma"] & b \arrow[r, "\iota_b"] \arrow[dr, phantom, "\Downarrow\iota_{1_b}"] & F1(b) \\
& b \arrow[u, "1"] \arrow[r, "\iota_b"'] & F1(b) \arrow[u, "F(1_1)_b"']
\end{tikzcd}.$$

The second map is α restricted to fibres and the final map is π_2, the projection onto the second component. We choose not to include descriptions of $\hat{\alpha}_f$, $\hat{\alpha}_\sigma$, Π or M.

We find that $\alpha(x,x_{\text{-}})$ equals $\mathrm{el}\,\hat{\alpha}(x,x_{\text{-}})$ and that $1_{\alpha(x,x_{\text{-}})}$ are the 1-cell components of a pseudo-natural equivalence. Thus el is locally surjective up to equivalence.

Suppose $\Gamma\colon\ \mathrm{el}\,\alpha \to \mathrm{el}\,\beta$ is a vertical transformation. We will define a trimodification $\hat{\Gamma}\colon \alpha \Rightarrow \beta$ with $\mathrm{el}\,\hat{\Gamma} \cong \Gamma$. This means transformations $\hat{\Gamma}_x\alpha_x \Rightarrow \beta_x$ together with invertible modifications m.

Remember that the action of $\mathrm{el}\,\alpha$ on objects is $\mathrm{el}\,\alpha(x,x_{\text{-}}) = (x, \alpha_x(x_{\text{-}}))$. Thus the second component of $\Gamma_{(x,x_{\text{-}})}\colon (x,\alpha_x(x_{\text{-}})) \to (x,\beta_x(x_{\text{-}}))$ is a 1-cell $\alpha_x(x_{\text{-}}) \to G1\beta_x(x_{\text{-}})$. Then let $(\hat{\Gamma}_x)_{x_{\text{-}}}$ be this 1-cell composed with $\iota^{\bullet}\colon G1\beta_x(x_{\text{-}}) \to \beta_x(x_{\text{-}})$. We choose not to include descriptions of $(\hat{\Gamma}_x)_f$ or m.

We find that $(\mathrm{el}\,\hat{\Gamma})_{(x,x_{\text{-}})} = (1_x, \iota.(\hat{\Gamma}_x)_{x_{\text{-}}}) = (1_x, \iota.\iota^{\bullet}.\pi_2\Gamma_{(x,x_{\text{-}})}) \cong (1_x, \pi_2\Gamma_{(x,x_{\text{-}})}) = \Gamma_{(x,x_{\text{-}})}$ and that this invertible 2-cell is the component of a modification. Thus el is surjective up to isomorphism on 2-cells.

Suppose $\zeta\colon\ \mathrm{el}\,\Gamma \to \mathrm{el}\,\Lambda$ is a vertical modification. We will define a perturbation $\hat{\zeta}\colon \Gamma \Rightarrow \Lambda$ with $\mathrm{el}\,\hat{\zeta} = \zeta$. This means modifications $\hat{\zeta}_x\colon \Gamma_x \Rrightarrow \Lambda_x$ satisfying the appropriate axioms.

The 2-cell components of $\zeta_{(x,x_{\text{-}})}$ have first component 1_{1_x} and second component a 2-cell pictured on the left. We define $\hat{\zeta}$ by letting $(\hat{\zeta}_x)_{x_{\text{-}}}$ equal the pasting given on the right.

$$
\begin{tikzcd}
\alpha_x(x_-) \arrow[r, "\iota\Gamma_x"] \arrow[dr, "\iota\Lambda_x"'] \arrow[r, phantom, shift right=4ex, "\Downarrow"] & G1\beta_x(x_-) \\
& G1\beta_x(x_-) \arrow[u, "G1_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha_x(x_-) \arrow[r, "\iota\Gamma_x"] \arrow[dr, "\iota\Lambda_x"'] \arrow[rr, bend left, "\Gamma_x"] \arrow[drr, bend right=60, "\Lambda_x"'] & G1\beta_x(x_-) \arrow[r, "\iota^{\cdot}"] \arrow[dr, phantom, "\Downarrow\iota"] & \beta_x(x_-) \\
& G1\beta_x(x_-) \arrow[u, "G1_1"'] \arrow[r, "\iota^{\cdot}"'] & \beta_x(x_-) \arrow[u, "1"']
\end{tikzcd}
$$

This makes $\hat{\zeta}$ a perturbation and it is unique with the property that $\mathrm{el}\,\hat{\zeta} = \zeta$. Thus el is bijective on 3-cells. $\square$

Remark 2.3.33 (Variations on the Grothendieck construction). The Grothendieck constructions given in sections 2.2 and 2.3 are closely related but are purpose-built for their 2-categorical and bicategorical settings. What happens if we apply the bicategorical construction to a 2-categorical fibration? Suppose that $P\colon E \to B$ is a 2-fibration of 2-categories (2.2.6) and apply the inverse Grothendieck construction (2.3.25) with the ordinary notion of fibre. When we inspect the reasoning we find:

- $Fb = E_b$ is a 2-category.
- $Ff = f^*$ is a 2-functor.
- $F\alpha = \alpha^*$ is a pseudo-natural transformation (2-natural when P is horizontally split).
- F is still locally a homomorphism (locally a 2-functor when P is locally split and horizontally split).
- χ is a pseudo-natural isomorphism and χ_{gf} is 2-natural (χ is 2-natural when P is horizontally split and χ_{gf} is an identity when P is split on 1-cells).
- ι is a pseudo-natural isomorphism and ι_b is 2-natural (ι is 2-natural when P is locally split and ι_b is an identity when P is split on 1-cells).
- ω, δ and γ are identities.

This amounts to a trihomomorphism $F\colon B^{\mathrm{coop}} \to \mathrm{Gray}$ where χ, ι are invertible, ω, δ and γ are identities and χ_{gf}, ι_a are 2-natural. Trihomomorphisms of this kind certainly do give 2-fibrations under the Grothendieck construction.

When P is locally and horizontally split these trihomomorphisms map into 2Cat and are just homomorphisms of “2Cat-enriched bicategories”. Functors of this kind certainly do give (appropriately split) 2-fibrations under the Grothendieck construction.

Remark 2.3.34. As in Section 2: the action of the Grothendieck construction on objects is described by Baković in [Bak12] Section 6. Section 5 of the same paper gives a partial description of the action on objects of the pseudo-inverse. With some adjustments, we have completed the second construction (Theorem 5.1) and shown that together they form an equivalence of 3-categories. In Proposition 2.3.25 we indicate where the post-composition property (mentioned in Remark 2.2.9) is explicitly required.

2.3.4 Examples

Construction 2.3.35 (Families)**.** When $\mathscr{B}$ is a bicategory we define $\mathrm{Fam}(\mathscr{B})$ as the bicategory of '1-cell diagrams' in $\mathscr{B}$. An object of $\mathrm{Fam}(\mathscr{B})$ is a pair $(\mathscr{C}, X)$ of where $\mathscr{C}$ is a category and $X\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{B}$ is a pseudo-functor. A 1-cell is a pair $(F,\alpha)\colon (\mathscr{C},X) \to (\mathscr{D}, Y)$ where $F\colon \mathscr{C} \to \mathscr{D}$ is a functor and $\alpha\colon X \Rightarrow YF^{\mathrm{op}}$ is a pseudo-natural transformation. A 2-cell is a pair $(\sigma,\Sigma)\colon (F,\alpha) \Rightarrow (G,\beta)$ where $\sigma\colon F \Rightarrow G$ is a natural transformation and $\Sigma\colon \alpha \Rrightarrow Y\sigma^{\mathrm{op}}.\beta$ is a modification as pictured here.

$$
\begin{tikzcd}
\mathscr{C}^{\mathrm{op}} \arrow[rr, "F^{\mathrm{op}}"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & \mathscr{D}^{\mathrm{op}} \arrow[dl, "Y"] \\
& \mathscr{B} &
\end{tikzcd}
\overset{\Sigma}{\Rrightarrow}
\begin{tikzcd}
\mathscr{C}^{\mathrm{op}} \arrow[rr, bend left, "F^{\mathrm{op}}"] \arrow[rr, bend right, "G^{\mathrm{op}}"'] \arrow[rr, phantom, "\Uparrow\sigma^{\mathrm{op}}"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] & \mathscr{D}^{\mathrm{op}} \arrow[dl, "Y"] \\
& \mathscr{B} &
\end{tikzcd}
$$

Composition and identities are not hard to describe. The coherence isomorphisms a, l and r are modifications obtained from the corresponding coherence isomorphisms in $\mathscr{B}$. There is an obvious functor $\pi\colon \mathrm{Fam}(\mathscr{B}) \to \mathrm{Cat}$ defined by projection from the first component.

Proposition 2.3.36. $\pi\colon \mathrm{Fam}(\mathscr{B}) \to \mathrm{Cat}$ *is a fibration.*

Proof. First, suppose that $(\mathscr{D}, Y)$ is an object in $\mathrm{Fam}(\mathscr{B})$ and $F\colon \mathscr{C} \to \mathscr{D}$ a 1-cell in Cat. Let the cartesian lift of F at $(\mathscr{D}, Y)$ be $(F, 1_{YF^{\mathrm{op}}})\colon (\mathscr{C}, YF^{\mathrm{op}}) \to (\mathscr{D}, Y)$. Now suppose that $(G,\beta)\colon (\mathscr{C},X) \to (\mathscr{D},Y)$ is a 1-cell in $\mathrm{Fam}(\mathscr{B})$ and $\sigma\colon F \Rightarrow G$ is a 2-cell in Cat. Let the cartesian lift of σ at (G,β) be $(\sigma, 1_{Y\sigma.\beta})\colon (F, Y\sigma.\beta) \Rightarrow (G,\beta)$. The details here are somewhat more complicated but the basic behaviour is the same as Proposition 2.2.31. $\square$

Definition 2.3.37. Suppose that $\mathscr{B}$ is a bicategory. An arrow $p\colon a \to b$ in $\mathscr{B}$ is called a *Street fibration* when $p_*\colon \mathscr{B}(c,e) \to \mathscr{B}(c,b)$ is a Street fibration for all c and the square

$$
\begin{tikzcd}
\mathscr{B}(c,e) \arrow[r, "f^*"] \arrow[d, "p_*"'] \arrow[dr, phantom, "\cong"] & \mathscr{B}(c',e) \arrow[d, "p_*"] \\
\mathscr{B}(c,b) \arrow[r, "f^*"'] & \mathscr{B}(c',b)
\end{tikzcd}
$$

is a morphism of Street fibrations for all $f\colon c' \to c$.

This means that for each 1-cell $g\colon e \to a$ and 2-cell $\alpha\colon h \to pg$ in $\mathscr{B}$ there exists a 'cartesian' 2-cell $\varphi(\alpha,g)\colon \alpha^*g \Rightarrow g$ and isomorphism $\eta\colon h \Rightarrow p\alpha^*g$ where $\varphi(\alpha,g)$ is cartesian for p_* and

$$\begin{tikzcd} e \arrow[r, "g"] \arrow[dr, bend right, "h"'] \arrow[r, phantom, shift right=3ex, "\alpha \Nearrow"] & a \arrow[d, "p"] \\ & b \end{tikzcd} \quad \text{equals} \quad \begin{tikzcd} e \arrow[r, bend left, "g"] \arrow[r, bend right, "\alpha^* g"'] \arrow[r, phantom, "\Uparrow \varphi(\alpha,g)"] \arrow[dr, bend right, "h"'] \arrow[dr, phantom, shift left=1ex, "\eta \Nearrow"] & a \arrow[d, "p"] \\ & b \end{tikzcd} .$$

It also means that the 'cartesian' 2-cells are closed under pre-composition with arbitrary 1-cells.

Definition 2.3.38. Suppose that $\mathscr{B}$ is a bicategory. A morphism between Street fibrations $p\colon e \to b$ and $q\colon e' \to b'$ in $\mathscr{B}$ is a pair of 1-cells $(f\colon e \to e', g\colon b \to b')$ in $\mathscr{B}$ where $q.f \cong g.p$ and the induced natural transformation

$$\begin{tikzcd} \mathscr{B}(c,e) \arrow[r, "f_*"] \arrow[d, "p_*"'] \arrow[dr, phantom, "\cong"] & \mathscr{B}(c,e') \arrow[d, "q_*"] \\ \mathscr{B}(c,b) \arrow[r, "g_*"'] & \mathscr{B}(c,b') \end{tikzcd}$$

is a morphism of Street fibrations for all c.

Construction 2.3.39 (Internal fibrations)**.** Suppose that $\mathscr{B}$ is a bicategory. We define $\mathrm{Fib}_{\mathscr{B}}$ to be the bicategory whose:

- Objects are Street fibrations $g\colon a \to b$ in $\mathscr{B}$. These are sometimes written as a triple (a, g, b).

- 1-cells are triples $(h, \sigma, h')\colon (a, g, b) \to (c, g', d)$ where $h\colon a \to c$, $h'\colon b \to d$,

$$\begin{tikzcd} a \arrow[r, "h"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\sigma"] & c \arrow[d, "g'"] \\ b \arrow[r, "h'"'] & d \end{tikzcd}$$

  is an isomorphism and h is a cartesian 1-cell.

- 2-cells are pairs of 2-cells $(\alpha, \alpha')\colon (h, \sigma, h') \Rightarrow (k, \tau, k')$ where $\alpha\colon h \Rightarrow k$, $\alpha'\colon h' \Rightarrow k'$, and

$$\begin{tikzcd} a \arrow[r, "h"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\sigma"] & c \arrow[d, "g'"] \\ b \arrow[r, bend left, "h'"] \arrow[r, bend right, "k'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & d \end{tikzcd} \quad \text{equals} \quad \begin{tikzcd} a \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\tau"] & c \arrow[d, "g'"] \\ b \arrow[r, "k'"'] & d \end{tikzcd}$$

in $\mathscr{B}$. We sometimes write $(\alpha, 1, \alpha')$ where 1 is representative of the commuting condition.

Composition and identities are easy to describe.

There is a homomorphism $\mathrm{cod}\colon \mathrm{Fib}_{\mathscr{B}} \to \mathscr{B}$ defined by projection onto the third component. It is called the codomain functor because it projects onto the codomain of the objects of $\mathrm{Fib}_{\mathscr{B}}$. It is modelled on $\mathrm{cod}\colon \mathrm{Fib} \to \mathrm{Cat}$ which was used by Hermida to guide his definition of 2-fibrations.

Proposition 2.3.40. *When a bicategory $\mathscr{B}$ has bipullbacks* $\mathrm{cod}\colon \mathrm{Fib}_{\mathscr{B}} \to \mathscr{B}$ *is a fibration.*

Proof. Suppose that (c, q, d) is an object of $\mathrm{Fib}_{\mathscr{B}}$ and $h\colon b \to d$ is a 1-cell in $\mathscr{B}$. The cartesian lift of h is obtained by taking the bipullback of q and h which is pictured below. The 2-cell σ is an isomorphism.

$$\begin{tikzcd} a \arrow[r, "\hat{h}"] \arrow[d, "\hat{q}"'] \arrow[dr, phantom, "\Downarrow\sigma"] & c \arrow[d, "q"] \\ b \arrow[r, "h"'] & d \end{tikzcd} .$$

See [Str74] for a proof that internal Street fibrations are closed under bipullback.

Now suppose that (h, σ, h') is a 1-cell in $\mathrm{Fib}_{\mathscr{B}}$ and $\alpha'\colon k' \Rightarrow g'$ is a 2-cell in $\mathscr{B}$. Then $\sigma.\alpha' g\colon k'g \Rightarrow g'h$ and since g' is a Street fibration, we get a cartesian lift $\alpha\colon k \Rightarrow h$ and an isomorphism $\eta\colon g'k \Rightarrow k'g$ satisfying

$$\begin{tikzcd} a \arrow[d, "g"'] \arrow[dr, bend left, "h"] \arrow[dr, phantom, "\sigma\Nearrow"] & \\ b \arrow[r, "h'"] \arrow[dr, bend right, "k'"'] \arrow[dr, phantom, "\alpha'\Nearrow"] & c \arrow[d, "g'"] \\ & d \end{tikzcd} = \begin{tikzcd} a \arrow[d, "g"'] \arrow[dr, bend left, "h"] \arrow[dr, "k"'] \arrow[dr, phantom, "\alpha\Nearrow"] & \\ b \arrow[dr, bend right, "k'"'] \arrow[dr, phantom, "\eta\Nearrow"] & c \arrow[d, "g'"] \\ & d \end{tikzcd} .$$

Then the cartesian lift of α' is $(\alpha, 1, \alpha')\colon (k, \eta, k') \to (h, \sigma, h')$. $\square$

Example 2.3.41 (Algebras)**.** Let $\mathrm{Mnd}(\mathscr{K})$ be the bicategory of pseudo-monads on a bicategory $\mathscr{K}$ (called *doctrines* in [Str80]). There is a trihomomorphism $\mathrm{Mnd}(\mathscr{K})^{\mathrm{coop}} \to \mathrm{Bicat}$ that maps each pseudo-monad T to the bicategory T-Alg of T-algebras, lax algebra morphisms and algebra 2-cells. We can use the Grothendieck construction to construct a fibration $\mathrm{Alg} \to \mathrm{Mnd}(\mathscr{K})$. The objects of the total category Alg are algebras for a pseudo-monad: pairs $(S, (A, m))$ where $m\colon SA \to A$ is an S-algebra for a pseudo-monad S. The 1-cells from algebras $(S, (A, m))$ to $(T, (B, n))$ are pairs $(\lambda, (f, \theta_f))$ where λ is a monad morphism from S to T and $(f, \theta_f): (A, m) \to (B, n\lambda_B)$ is a lax algebra morphism.

$$
\begin{tikzcd}
SA \arrow[rr, "m"] \arrow[d, "Sf"'] \arrow[drr, phantom, "\overset{\theta_f}{\Nearrow}"] & & A \arrow[d, "f"] \\
SB \arrow[r, "\lambda_B"'] & TB \arrow[r, "n"'] & B
\end{tikzcd} .
$$

The 2-cells of Alg are pairs $(\Gamma, \alpha): (\lambda, (f, \theta_f)) \Rightarrow (\tau, (g, \theta_g))$ where $\Gamma: \lambda \Rightarrow \tau$ is a monad 2-cell and α is an algebra 2-cell

$$
\begin{tikzcd}
(A, m) \arrow[rr, "{(f,\theta_f)}"] \arrow[dr, "{(g,\theta_g)}"'] & \arrow[d, phantom, "\Downarrow\alpha"] & (B, n\lambda_B) \\
& (B, n\tau_B) \arrow[ur, "{(1_B, n\Gamma_B)}"'] &
\end{tikzcd} .
$$

The fibration is projection on the first component of Alg. By construction the fibre over T is equivalent to T-Alg.

Example 2.3.42 (Equivalence lifting)**.** A homomorphism $P: \mathscr{E} \to \mathscr{B}$ is said the have the equivalence lifting property when:

1. for each object $e \in \mathscr{E}$ and equivalence $f: b \to Pe$ in $\mathscr{B}$ there is an equivalence $\hat{f}: a \to e$ in $\mathscr{E}$ with $P\hat{f} = f$; and

2. for each arrow $h: e \to e'$ in $\mathscr{E}$ and isomorphism $\alpha: g \to Ph$ in $\mathscr{B}$ there is an isomorphism $\hat{\alpha}: k \to h$ with $P\hat{\alpha} = \alpha$.

When P is strict, these are precisely the fibrations in Lack's Quillen model structure on Bicat_s [Lac04].

Every fibration has the equivalence lifting property. Further more, when $\mathscr{E}$ and $\mathscr{B}$ are bigroupoids (bicategories in which all 1-cells are equivalences and all 2-cells are isomorphisms) every homomorphism with the equivalence lifting property is a fibration.

2.4 Composition, commas and pullbacks

In this section we show that fibrations of bicategories are closed under composition and closed under equiv-comma, and that projections from oplax comma bicategories are fibrations.

2.4.1 Composition

Proposition 2.4.1. *If $P: \mathscr{D} \to \mathscr{B}$ and $Q: \mathscr{E} \to \mathscr{D}$ are fibrations of bicategories then $PQ: \mathscr{E} \to \mathscr{B}$ is a fibration of bicategories.*

Proof. We first need to show that it has cartesian lifts of 1-cells. Suppose that $e \in E$ and $f\colon a \to PQe$ in B, then let $f'' = \varphi(\varphi(f, Qb), b)$. This is the cartesian double-lift:

$$\begin{tikzcd}
f'^{*}b \arrow[r, "f''"] & b \\
f^{*}Qb \arrow[r, "F'"] & Qb \\
a \arrow[r, "f"] & PQb
\end{tikzcd}$$

where f' is cartesian for P over f and f'' is cartesian for Q over f'. We need to show that f'' is cartesian for PQ over f. This is trivial when we consider the definition of cartesian 1-cell by bipullback.

$$\begin{tikzcd}
E(z,x) \arrow[r, "f''_*"] \arrow[d, "Q_{zx}"'] \arrow[dr, phantom, "\cong"] & E(z,y) \arrow[d, "Q_{zy}"] \\
D(Qz,Qx) \arrow[r, "f'_*"] \arrow[d, "P_{QzQx}"'] \arrow[dr, phantom, "\cong"] & D(Qz,Qy) \arrow[d, "P_{QzQy}"] \\
B(PQz,PQx) \arrow[r, "f_*"'] & B(PQz,PQy)
\end{tikzcd}$$

Since both squares below are bipullbacks, the composite is a bipullback and f'' is a cartesian 1-cell over f. This could also be proved by explicitly using the properties of cartesian 1-cells.

Now we need to show that PQ locally fibred. Since PQ_{xy} is defined by the composite

$$E(x,y) \xrightarrow{Q_{xy}} D(Qx,Qy) \xrightarrow{P_{QxQy}} B(PQx,PQy)$$

and P and Q are locally fibred, PQ_{xy} is a fibration. Thus PQ is locally fibred. The cartesian lifts of 2-cells in B are the double-lifts similar to those described above.

Finally, we need to show that cartesian 2-cells closed under horizontal composition. Suppose that $\alpha\colon h \Rightarrow k$ and $\beta\colon f \Rightarrow g$ are the chosen cartesian lifts of $PQ\alpha$ and $PQ\beta$; then $\alpha * \beta$ is cartesian over $Q(\alpha * \beta)$ because Q is a fibration. Notice also that $Q\alpha * Q\beta$ is cartesian over $P(Q\alpha * Q\beta)$ because P is a fibration. But

$$Qhf \overset{Q(\alpha*\beta)}{\Longrightarrow} Qkg \;=\; Qhf \overset{\phi}{\Longrightarrow} QhQf \overset{Q\alpha * Q\beta}{\Longrightarrow} QkQg \overset{\phi^{-1}}{\Longrightarrow} Qkg$$

where each ϕ is a composition coherence isomorphism for Q. Now since isomorphisms are cartesian, $Q(\alpha * \beta)$ is cartesian over $PQ(\alpha * \beta)$. Thus $\alpha * \beta$ is a cartesian lift of a cartesian lift and so it is cartesian for PQ. We have proven this for the chosen cartesian lift only but by Proposition 2.2.14 this is enough. $\square$

Proposition 2.4.2. *If $P\colon D \to B$ and $Q\colon E \to D$ are 2-fibrations then $PQ\colon E \to B$ is a 2-fibration.*

Proof. This proof is essentially the same as that for fibrations of bicategories. There are no major differences. □

2.4.2 Oplax comma bicategories

Construction 2.4.3 (Oplax comma bicategory)**.** Suppose that $F\colon \mathscr{C} \to \mathscr{D}$, $G\colon \mathscr{B} \to \mathscr{D}$ are homomorphisms. Let $(F \downarrow G)$ be the bicategory whose objects are (x, x', τ_x) where $x \in \mathscr{C}$, $x' \in \mathscr{B}$ and $\tau_x\colon Fx \to Gx'$. The arrows are triples $(f, f', \tau_f)\colon (x, x', \tau_x) \to (y, y', \tau_y)$ where $f\colon x \to y$, $f'\colon x' \to y'$ and $\tau_f\colon \tau_y Ff \Rightarrow Gf'\tau_x$. The 2-cells are triples $(\alpha, \alpha', \tau_\alpha)\colon (f, f', \tau_f) \Rightarrow (g, g', \tau_g)$ where $\alpha\colon f \Rightarrow g$, $\alpha'\colon f' \Rightarrow g'$ and τ_α is an equality:

$$\begin{tikzcd} Fx \arrow[r, "\tau_x"] \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Fg"] \arrow[d, phantom, "\overset{F\alpha}{\Rightarrow}"] & Gx' \arrow[d, "Gg'"] \arrow[dl, phantom, "\tau_g \Nearrow"] \\ Fy \arrow[r, "\tau_y"'] & Gy' \end{tikzcd} = \begin{tikzcd} Fx \arrow[r, "\tau_x"] \arrow[d, "Ff"'] & Gx' \arrow[d, bend right, "Gf'"'] \arrow[d, bend left, "Gg'"] \arrow[d, phantom, "\overset{G\alpha'}{\Rightarrow}"] \arrow[dl, phantom, "\tau_f \Nearrow"] \\ Fy \arrow[r, "\tau_y"'] & Gy' \end{tikzcd}.$$

Composition and identities are given in the obvious way. By projection onto the first and second components we obtain pseudo-functors d_0 and d_1 as displayed below. Both of these preserve composition and identities on the nose.

This gives rise to an oplax natural transformation τ whose components are given by projecting $(F \downarrow G)$ onto its third component.

$$\begin{tikzcd} (F \downarrow G) \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\tau \Nearrow"] & \mathscr{B} \arrow[d, "G"] \\ \mathscr{C} \arrow[r, "F"'] & \mathscr{D} \end{tikzcd}$$

Remark 2.4.4 (Lax comma bicategories)*.* There is an obvious dual to this construction in which τ above is a lax natural transformation. This variation is called "2-comma-category" in [Gra69] and "lax comma category" in [Kel74].

Remark 2.4.5 (Weighted limits)*.* The oplax comma bicategory construction could also be defined as some kind of weighted limit. The same is true of other constructions later in this section. We leave the details to the interested reader.

Proposition 2.4.6. *For any pair of homomorphisms $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{B} \to \mathscr{D}$, $d_0\colon (F \downarrow G) \to \mathscr{C}$ is a fibration.*

Proof. We need to show that d_0 has cartesian lifts of 1-cells. Suppose that (y, y', σ_y) in $(F \downarrow G)$ and $f\colon x \to y$ in $\mathscr{C}$. Then there exists $(f, 1, \sigma_{f1})\colon (x, x', \sigma_x) \to (y, y', \sigma_y)$ where

$x' := y'$, $\sigma_x := \sigma_y.Ff$ and σ_{f1} is

$$\begin{tikzcd} Fx \arrow[r, "\sigma_x"] \arrow[d, "Ff"'] \arrow[dr, "\sigma_x"'] & Gy' \arrow[d, bend right, "1"'] \arrow[d, bend left, "G1"] \arrow[d, phantom, "\cong"] \\ Fy \arrow[r, "\sigma_y"'] & Gy' \end{tikzcd}$$

where the unlabelled isomorphisms are coherence data for G and $\mathscr{D}$. This is a cartesian lift of f.

We need to show that d_0 has cartesian lifts of 2-cells. Suppose that $(g, g', \sigma_g) : (x, x', \sigma_x) \to (y, y', \sigma_y)$ in $(F \downarrow G)$ and $\alpha : f \Rightarrow g$ in $\mathscr{C}$. Then there exists $(\alpha, 1_{g'}) : (f, f', \sigma_f) \Rightarrow (g, g', \sigma_g)$ where $f' := g'$, $\sigma_f := \sigma_g.\sigma_y F\alpha$ and thus

$$\begin{tikzcd} Fx \arrow[r, "\sigma_x"] \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Fg"] \arrow[d, phantom, "\overset{F\alpha}{\Rightarrow}"] & Gx' \arrow[d, "Gg'"] \\ Fy \arrow[r, "\sigma_y"'] \arrow[ur, phantom, "\sigma_g \Nearrow"] & Gy' \end{tikzcd} = \begin{tikzcd} Fx \arrow[r, "\sigma_x"] \arrow[d, "Ff"'] & Gx' \arrow[d, bend right, "Gg'"'] \arrow[d, bend left, "Gg'"] \arrow[d, phantom, "\overset{G1}{\Rightarrow}"] \\ Fy \arrow[r, "\sigma_y"'] \arrow[ur, phantom, "\sigma_f \Nearrow"] & Gy' \end{tikzcd} .$$

This is a cartesian lift of α.

We need to check that cartesian 2-cells are closed under horizontal composition. Examining the chosen cartesian 2-cells we find that $(\alpha, 1_{g'}) * (\beta, 1_{k'}) = (\alpha * \beta, 1_{g'k'})$ and thus they're closed under composition. We've proven this for the chosen cartesian lifts only, but by Proposition 2.2.14 this is enough. □

Remark 2.4.7. Suppose that $G : \mathscr{B} \to \mathscr{D}$ is a homomorphism. We use the notation $\mathscr{D}/G \to \mathscr{D}$ instead of $(1_D \downarrow G) \to \mathscr{D}$. This is referred to by Baković [Bak12] as the "canonical fibration associated to F". We call it the free fibration on F (see the following remark).

Remark 2.4.8 (Free fibrations). If $H : \mathscr{A} \to \mathscr{B}$ is a homomorphism then define $FH := d_0 : \mathscr{B}/H \to \mathscr{B}$ and call it the free fibration on H. For each fibration of bicategories P there is a biequivalence

$$(\mathrm{Fib}/\mathscr{B})(FH, P) \simeq (\mathrm{Bicat}/\mathscr{B})(H, UP)$$

where UP is P regarded simply as a homomorphism. The details are very similar to the standard result for ordinary fibrations; the only extra result we need is that arbitrary 2-cells can be lifted along cartesian 1-cells (The defining property of cartesian 1-cells given in Definition 2.3.1 point 1 holds even when the α given there is not invertible). This relies on the fact that fibrations are locally fibred.

The naturality of this biequivalence is very weak: it is the 1-cell component of a tritransformation

$$\begin{tikzcd}(\mathrm{Bicat}/\mathscr{B})^{\mathrm{op}} \times \mathrm{Fib}/\mathscr{B} \arrow[r, bend left, "\mathrm{Fib}/\mathscr{B}(F\text{-},\text{-})"] \arrow[r, bend right, "\mathrm{Bicat}/\mathscr{B}(\text{-},U\text{-})"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{Bicat}\end{tikzcd} .$$

Remark 2.4.9 (Two-sided fibrations). Suppose that $F \colon \mathscr{C} \to \mathscr{D}$, $G \colon \mathscr{B} \to \mathscr{D}$ are homomorphisms. The oplax comma construction also makes d_1 a 'coop-fibration' and the span (d_0, d_1) over $\mathscr{C}$ and $\mathscr{B}$ has some of the characteristics of a two-sided discrete fibration. By 'coop-fibration' we mean the notion dual to fibration: a homomorphism that is locally an opfibration, has opcartesian lifts of 1-cells and opcartesian 2-cells are closed under horizontal composition.

Locally, the span (d_0, d_1) is a discrete two-sided fibration. Suppose that $(\alpha, \alpha') \colon (f, f', \sigma_f) \Rightarrow (g, g', \sigma_g)$ is a 2-cell in $(F \downarrow G)$ and opcartesian lifts for d_1 are described using the same notation as cartesian lifts for d_0. Then

$$\begin{aligned} d_1(\varphi(\alpha, (g, g', \sigma_g))) &= 1_{g'} \\ d_0(\varphi(\alpha', (f, f', \sigma_f))) &= 1_f \\ \varphi(\alpha', (f, f', \sigma_f)).\varphi(\alpha, (g, g', \sigma_g))) &= (\alpha, \alpha') \ . \end{aligned}$$

This means that the cartesian lifts for d_0 are identities under the action of d_1; and vice versa; and that the cartesian lift of $d_0(\alpha, \alpha')$ and the opcartesian lift of $d_1(\alpha, \alpha')$ compose to give (α, α').

On 1-cells the behaviour is somewhat weaker. Suppose $(f, f', \sigma_f) \colon (x, x', \sigma_x) \to (y, y', \sigma_y)$ is a 1-cell in $(F \downarrow G)$. Then

$$\begin{aligned} d_1(\varphi(f, (y, y', \sigma_y))) &= 1_{y'} \\ d_0(\varphi(f', (x, x', \sigma_x))) &= 1_x \end{aligned}$$

and

$$\begin{tikzcd}(x, x', \sigma_x) \arrow[rrr, "{(f,f',\sigma_f)}"] \arrow[dr, "{(1,f',r.\phi)}"'] & & & (y, y', \sigma_y) \\ & (x, y', Gf'\sigma_x) \arrow[r, "{(1,\phi.l.\sigma_f.r.\phi)}"'] \arrow[r, phantom, shift left=6, "\Downarrow(r.r,l.l)"] & (x, y', \sigma_y Ff) \arrow[ur, "{(f,1,\phi.l)}"'] & \end{tikzcd}$$

where the ϕ are coherence isomorphisms for F and G.

Proposition 2.4.10. *For any 2-functors $F \colon C \to D$ and $G \colon B \to D$, $d_0 \colon (F \downarrow G) \to C$ is a 2-fibration.*

Proof. This proof is essentially the same as that for fibrations of bicategories. There are no major differences. □

2.4.3 Pullbacks

Construction 2.4.11 (equiv-comma)**.** Suppose $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{B} \to \mathscr{D}$ are homomorphisms. The equiv-comma $\mathscr{C} \times_{\simeq} \mathscr{B}$ is the subcategory of $(F \downarrow G)$ containing: all objects (x, x', τ_x) where $\tau_x\colon Fx \to Gx'$ is an *equivalence*; all 1-cells (f, f', τ_f) where τ_f is an *isomorphism*; and all 2-cells. The functors G' and F' are the projections onto the first and second components of $\mathscr{C} \times_{\simeq} \mathscr{B}$. This gives a pseudo-natural equivalence

$$\begin{tikzcd}
\mathscr{C} \times_{\simeq} \mathscr{B} \arrow[r, "F'"] \arrow[d, "G'"'] \arrow[dr, phantom, "\overset{\tau}{\nearrow}"] & \mathscr{B} \arrow[d, "G"] \\
\mathscr{C} \arrow[r, "F"'] & \mathscr{D}
\end{tikzcd} .$$

Proposition 2.4.12. *Let $\mathscr{A} \times_{\simeq} \mathscr{E}$ be the equiv-comma of homomorphisms $P\colon \mathscr{E} \to \mathscr{B}$ and $F\colon \mathscr{A} \to \mathscr{B}$. If P is a fibration then P' is a fibration and F' is cartesian.*

Proof. We want to show that P' is a fibration. We first need to show that it has cartesian lifts of 1-cells. Suppose (y, y', τ_y) is a 0-cell in $\mathscr{A} \times_{\simeq} \mathscr{E}$ and $f\colon x \to y$ is a 1-cell in $\mathscr{B}$ and let $\hat{\tau}\colon \tau^* y' \to y'$ be the cartesian lift of $\tau_y.Ff\colon Fx \to Fy \to Py'$ in $\mathscr{B}$. Then $(f, \hat{\tau}, \eta)\colon (x, \tau^* y', 1_{Fx}) \to (y, y', \tau_y)$ is a 1-cell in $\mathscr{A} \times_{\simeq} \mathscr{E}$ where η is

$$\begin{tikzcd}
Fx \arrow[r, "1"] \arrow[dr, "Ff"'] \arrow[dd, "Ff"'] & Fx \arrow[d, "Ff"] \arrow[d, phantom, shift right=3ex, "\Downarrow r"] \\
 & Fy \arrow[d, "\tau_y"] \\
Fy \arrow[r, "\tau_y"'] & Py'
\end{tikzcd} .$$

This is a cartesian lift of f for P'.

We need to show that there are cartesian lifts of 2-cells. Suppose $(g, g', \tau_g)\colon (x, x', \tau_x) \to (y, y', \tau_y)$ is a 1-cell in $\mathscr{A} \times_{\simeq} \mathscr{E}$ and $\alpha\colon f \Rightarrow g\colon x \to y$ is a 2-cell in $\mathscr{B}$. Suppose also that τ_x is an adjoint equivalence. Now let $\hat{\alpha}\colon f' \to g'$ be the cartesian lift of the following composite at g'.

$$\begin{tikzcd}
 & Fx \arrow[r, bend left, "Ff"] \arrow[r, "Fg"'] \arrow[r, phantom, shift left=2ex, "\Downarrow F\alpha"] \arrow[dr, "\tau_x"'] \arrow[d, phantom, "\Downarrow \epsilon"] & Fy \arrow[dr, "\tau_y"] \arrow[d, phantom, "\Downarrow \tau_g"] & \\
Px' \arrow[ur, "\tau_x^\bullet"] \arrow[rr, "1"] \arrow[rrr, bend right, "Pg'"'] & & Px' \arrow[r, "Pg'"] & Py'
\end{tikzcd}$$

Then $(\alpha, \hat{\alpha})\colon (f, f', \tau_f) \Rightarrow (g, g', \tau_g)$ is a 2-cell where τ_f is

$$\begin{tikzcd}
Fx \arrow[r, "\tau_x"] \arrow[dr, "1"'] \arrow[ddd, "Ff"'] & Px' \arrow[d, "\tau_x^{\bullet}"'] \arrow[ddd, bend left, "Pf'"] \arrow[dl, phantom, "\eta \Nearrow"] \\
& Fx' \arrow[d, "Ff"'] \\
\arrow[r, phantom, "r \Nearrow"] & Fy' \arrow[d, "\tau_y"'] \\
Fy \arrow[r, "\tau_y"'] & Py'
\end{tikzcd} .$$

This is a cartesian lift of α for P'.

We need to show that cartesian 2-cells are closed under horizontal composition. The chosen cartesian lifts are cartesian precisely because their second component is cartesian for P. Since P is a fibration we know that the second component of $(\alpha, \hat{\alpha}) * (\beta, \hat{\beta}) = (\alpha * \beta, \hat{\alpha} * \hat{\beta})$ is also cartesian. Thus the chosen cartesian lifts are closed under horizontal composition and by Proposition 2.2.14 this is enough. This makes P' a fibration.

Notice that cartesian lifts for P' have cartesian maps in their second component so F' is cartesian. □

Remark 2.4.13. We say that a functor F reflects cartesian maps when Ff cartesian implies f cartesian. It is worth noting that the F' above reflects cartesian maps.

Remark 2.4.14. Suppose $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{B} \to \mathscr{D}$ are homomorphisms, the usual notion of pullback of F and G is not well-defined as a bicategory. If (f, f') and (g, g') are 1-cells in $\mathscr{C} \times \mathscr{B}$ with $Ff = Gf'$ and $Fg = Gg'$ and f, g are composable and f', g' are composable then $(gf, g'f')$ only has $F(gf) \cong G(g'f')$ and not equality; thus the usual notion of composition of 1-cells is not well-defined. The following construction is in some sense the closest we can get to pullback of homomorphisms. The "pullback" of F and G is the subcategory of $(F \downarrow G)$ containing: all objects (x, x', τ_x) where $\tau_x\colon Fx \to Gx'$ is an *identity*; all 1-cells (f, f', τ_f) where τ_f is an *isomorphism*; and all 2-cells. The functors G'' and F'' are the projections onto the first and second components of $\mathscr{C} \times_{\mathscr{D}} \mathscr{B}$. This gives a pseudo-natural equivalence τ as displayed.

$$\begin{tikzcd}
\mathscr{C} \times_{\mathscr{D}} \mathscr{B} \arrow[r, "F''"] \arrow[d, "G''"'] \arrow[dr, phantom, "\tau \Nearrow"] & \mathscr{B} \arrow[d, "G"] \\
\mathscr{C} \arrow[r, "F"'] & \mathscr{D}
\end{tikzcd}$$

This is actually an iso-comma object in Bicat_2: the 2-category of bicategories, homomorphisms and icons in the sense of [Lac10].

Suppose now that G is a fibration. We can prove that $G^{''}$ is a fibration using essentially the same argument as Proposition 2.4.12. Alternatively, we can prove an analogue of a result by Joyal and Street: when G is a fibration the pullback has the same universal property as the equiv-comma (in the sense that the induced comparison is a biequivalence). Thus G' and G'' are biequivalent in the slice over $\mathscr{C}$ and G'' is a fibration.

We've proved the following theorem.

Proposition 2.4.15. *Let $\mathscr{A} \times_{\mathscr{B}} \mathscr{E}$ be the "pullback" of $P\colon \mathscr{E} \to \mathscr{B}$ and $F\colon \mathscr{A} \to \mathscr{B}$. If P is a fibration then P' is a fibration and F' is cartesian.*

A quick investigation reveals that 2-fibrations are not closed under equiv-comma. They are however closed under iso-comma as defined below.

Construction 2.4.16 (Iso-comma)**.** Suppose $F\colon C \to D$ and $G\colon B \to D$ are 2-functors. The iso-comma $C \times_{\cong} B$ is the subcategory of $(F \downarrow G)$ containing: all objects (x, x', τ_x) where $\tau_x\colon Fx \to Gx'$ is an *isomorphism*; all 1-cells $(f, f', \tau_f)\colon (x, x', \tau_x) \to (y, y', \tau_y)$ where τ_f is an *identity*; and all 2-cells. The functors G' and F' are the projections onto the first and second components of $\mathscr{C} \times_{\cong} \mathscr{B}$. This gives a 2-natural isomorphism τ as follows.

$$\begin{tikzcd} C \times_{\cong} B \arrow[r, "F'"] \arrow[d, "G'"'] \arrow[dr, phantom, "\overset{\tau}{\nearrow}"] & B \arrow[d, "G"] \\ C \arrow[r, "F"'] & D \end{tikzcd}$$

Proposition 2.4.17. *Let $A \times_{\cong} E$ be the iso-comma of $P\colon E \to B$ and $F\colon A \to B$. If P is a 2-fibration then P' is a 2-fibration and F' is cartesian.*

Proof. This proof is essentially the same as that for equiv-commas. There are no major differences. ☐

Construction 2.4.18 (Pullback)**.** Suppose $F\colon C \to D$ and $G\colon B \to D$ are 2-functors. The pullback $C \times_D B$ is the subcategory of $(F \downarrow G)$ containing: all objects (x, x', τ_x) where $\tau_x\colon Fx \to Gx'$ is an *identity*; all 1-cells $(f, f', \tau_f)\colon (x, x', \tau_x) \to (y, y', \tau_y)$ where τ_f is an *identity*; and all 2-cells. The functors G' and F' are the projections onto the first and second components of $C \times B$. This gives commuting square.

$$\begin{tikzcd} C \times_D B \arrow[r, "F'"] \arrow[d, "G'"'] & B \arrow[d, "G"] \\ C \arrow[r, "F"'] & D \end{tikzcd}$$

Proposition 2.4.19. *Let $A \times_B E$ be the pullback of $P\colon E \to B$ and $F\colon A \to B$. If P is a 2-fibration then P' is a 2-fibration and F' is cartesian.*

Proof. The proof is essentially the same as Proposition 2.4.17 but much easier because the isomorphisms are equalities. □

Chapter 3

The Catalan simplicial set

Abstract

The Catalan numbers are well known to be the answer to many different counting problems, and so there are many different families of sets whose cardinalities are the Catalan numbers. We show how such a family can be given the structure of a simplicial set. We show how the low-dimensional parts of this simplicial set classify, in a precise sense, the structures of monoid and of monoidal category. This involves aspects of combinatorics, algebraic topology, quantum groups, logic, and category theory.

Contribution by the author

This paper was co-authored with Richard Garner, Steve Lack, and Ross Street. My initial interest was prompted by a talk of Professor Michael Johnson at the Australian Categories Seminar. It began as a discussion with Professor Street as we considered how simplicial sets could describe skew monoidal categories. It quickly evolved into a four-person effort as Dr Lack and Dr Garner identified the presence of the Catalan numbers and number of key results. As one author in four my contribution should be considered 25%.

What follows is is a direct reproduction of the original which was published in the Mathematical Proceedings of the Cambridge Philosophical Society. Any differences from that publication are limited to cosmetic changes such as citation numbering.

3.1 Introduction

The nth Catalan number C_n, given explicitly by $\frac{1}{n+1}\binom{2n}{n}$, is well-known to be the answer to many different counting problems; for example, it is the number of bracketings of an

$(n+1)$-fold product. Thus there are many $\mathbb{N}$-indexed families of sets whose cardinalities are the Catalan numbers; Stanley [Sta99; Sta13] describes at least 205 such.

A Catalan family of sets may bear extra structure that is invisible in the mere sequence of Catalan numbers. For example, one presentation of the nth Catalan set is as the set of functions $f\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$ which preserve order and satisfy $f(k) \leqslant k$ for each k. The set of such functions is a monoid under composition, and in this way we obtain the *Catalan monoids* [Sol96] which are of importance to combinatorial semigroup theory. For another example, a result due to Tamari [Tam62] makes each Catalan set into a lattice, whose ordering is most clearly understood in terms of bracketings of words, as the order generated by the basic inequality $(xy)z \leqslant x(yz)$ under substitution.

The first main objective of this paper is to describe another kind of structure borne by Catalan families of sets. We shall show how to define functions between them in such a way as to produce a simplicial set $\mathbb{C}$, which is the "Catalan simplicial set" of the title. The simplicial structure can be defined in various ways, but the most elegant makes use of what seems to be a new presentation of the Catalan sets that relies heavily on the Boolean algebra $\mathbf{2}$.

Simplicial sets are abstract, combinatorial entities, most often used as models of spaces in homotopy theory, but flexible enough to also serve as models of higher categories [Lur09; Ver08]. Therefore, we might hope that the Catalan simplicial set had some natural role to play in homotopy theory or higher category theory. Our second objective in this paper is to affirm this hope, by showing that the Catalan simplicial set has a *classifying property* with respect to certain kinds of categorical structure. More precisely, we will consider simplicial maps from $\mathbb{C}$ into the nerves of various kinds of higher category (the *nerve* of such a structure is a simplicial set which encodes its cellular data). We will see that:

(a) Maps from $\mathbb{C}$ to the nerve of a monoidal category $\mathscr{V}$ are the same thing as monoids in $\mathscr{V}$;

(b) Maps from $\mathbb{C}$ to the nerve of a bicategory $\mathscr{B}$ are the same thing as monads in $\mathscr{B}$;

(c) Maps from $\mathbb{C}$ to the *pseudo* nerve of the monoidal bicategory Cat of categories and functors are the same thing as monoidal categories;

(d) Maps from $\mathbb{C}$ to the *lax* nerve of the monoidal bicategory Cat are the same thing as *skew-monoidal categories*.

Skew-monoidal categories generalise Mac Lane's notion of monoidal category [Mac63] by dropping the requirement of invertibility of the associativity and unit constraints; they were introduced recently by Szlachányi [Szl12] in his study of bialgebroids, which are themselves an extension of the notion of quantum group. The result in (d) can be seen as a coherence result for the notion of skew-monoidal category, providing an abstract justification for the axioms. Thus the work presented here lies at the interface of several mathematical disciplines:

- combinatorics, in the form of the Catalan numbers;
- algebraic topology, via simplicial sets and nerves;
- quantum groups, through recent work on bialgebroids;
- logic, through the distinguished role of the Boolean algebra **2**; and
- category theory.

Nor is this the end of the story. Monoidal categories and skew-monoidal categories can be generalised to notions of *monoidale* (pseudo-monoid) and *skew monoidale* in a monoidal bicategory; this has further relevance for quantum algebra, since Lack and Street showed in [LS12] that quantum categories in the sense of [DS04] can be described using skew monoidales. In a sequel to this paper, we will generalise (c) and (d) to prove that:

(e) Maps from $\mathbb{C}$ to the pseudo nerve of a monoidal bicategory $\mathscr{W}$ are the same thing as monoidales in $\mathscr{W}$; and

(f) Maps from $\mathbb{C}$ to the lax nerve of a monoidal bicategory $\mathscr{W}$ are the same thing as skew monoidales in $\mathscr{W}$.

The results (a)–(f) use only the lower dimensions of the Catalan simplicial set, and we expect that its higher dimensions in fact encode *all* of the coherence that a higher-dimensional monoidal object should satisfy. We therefore hope also to show that:

(g) Maps from $\mathbb{C}$ to the pseudo nerve of the monoidal tricategory Bicat of bicategories are the same thing as monoidal bicategories;

(h) Maps from $\mathbb{C}$ to the homotopy-coherent nerve of the monoidal simplicial category ∞-Cat of ∞-categories are the same thing as monoidal ∞-categories in the sense of [Lur14];

together with appropriate skew analogues of these results.

Finally, a note on the genesis of this work. We have chosen to present the Catalan simplicial set as basic, and its classifying properties as derived. This belies the method of its discovery, which was to look for a simplicial set with the classifying property (d); the link with the Catalan numbers only later came to light. The notion that a classifying object as in (d) might exist is based on an old idea of Michael Johnson's on how to capture not only associativity but also unitality constraints simplicially. He reminded us of this in a recent talk [Joh] to the Australian Category Seminar.

3.2 The Catalan simplicial set

In this section we define and investigate the Catalan simplicial set. We begin by recalling some basic definitions. We write Δ for the *simplicial category*, whose objects are non-empty finite ordinals $[n] = \{0, \dots, n\}$ and whose morphisms are order-preserving functions, and write SSet for the category of presheaves on Δ. Objects X of SSet are called *simplicial sets*; we think of them as glueings-together of discs, with the n-dimensional discs in that glueing labelled by the set $X_n := X([n])$ of *n-simplices* of X. We write $\delta_i \colon [n-1] \to [n]$ and $\sigma_i \colon [n+1] \to [n]$ for the maps of Δ defined by

$$\delta_i(x) = \begin{cases} x & \text{if } x < i \\ x+1 & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma_i(x) = \begin{cases} x & \text{if } x \leqslant i \\ x-1 & \text{otherwise.} \end{cases}$$

The action of these morphisms on a simplicial set X yields functions $d_i \colon X_n \to X_{n-1}$ and $s_i \colon X_n \to X_{n+1}$, which we call *face* and *degeneracy* maps. An $(n+1)$-simplex x is called *degenerate* when it is in the image of some s_i, and *non-degenerate* otherwise. The face and degeneracy maps of a simplicial set satisfy the following *simplicial identities*:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{for } i < j \\ s_i s_j &= s_{j+1} s_i && \text{for } i \leqslant j \end{aligned} \qquad d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j \\ \mathrm{id} & \text{for } i = j, j+1 \\ s_j d_{i-1} & \text{for } i > j+1; \end{cases}$$

and in fact, a simplicial set may be completely specified by giving its sets of n-simplices, together with face and degeneracy maps satisfying the simplicial identities.

Definition 3.2.1. The *Catalan simplicial set* $\mathbb{C}$ has its n-simplices given by *Dyck words* of length $2n+2$; these are strings comprised of $(n+1)$ U's and $(n+1)$ D's such that the ith U precedes the ith D for each $1 \leqslant i \leqslant n+1$. The face maps $d_i \colon \mathbb{C}_n \to \mathbb{C}_{n-1}$ act on a word W by deleting the ith U and ith D; the degeneracy maps $s_i \colon \mathbb{C}_{n-1} \to \mathbb{C}_n$ act on a word W by repeating the ith U and ith D.

Each Dyck word corresponds to a sequence of moves up and down a ladder, starting from ground-level. In each Dyck word W, each U denotes a step up on the ladder and D indicates a step down on the ladder. The condition that the ith U precedes the ith D ensures that one cannot take a step below ground-level. The fact that there are equal numbers of Us and Ds ensures that the sequence starts and ends at ground-level.

The sets of Dyck words of length $2n$ are a Catalan family of sets—corresponding to (i) or (r) in Stanley's enumeration [Sta99]—and so we have that $|\mathbb{C}_n| = C_{n+1}$, the $(n+1)$st Catalan number.

Remark 3.2.2. The sets of n-simplices of $\mathbb{C}$ are not quite a Catalan family, due to the dimension shift causing us to omit the 0th Catalan number. We may rectify this by viewing $\mathbb{C}$ as an *augmented* simplicial set. An augmented simplicial set is a presheaf on Δ_+, the category of all finite ordinals and order-preserving maps; it is equally given by a simplicial set X together with a set X_{-1} of (-1)-simplices and an "augmentation" map $d_0 \colon X_0 \to X_{-1}$ satisfying $d_0 d_0 = d_0 d_1 \colon X_1 \to X_{-1}$. By allowing n to range over $\{-1\} \cup \mathbb{N}$ in the definition of the Catalan simplicial set $\mathbb{C}$, it becomes an augmented simplicial set with the property that its sets of $(n-1)$-simplices (for $n \in \mathbb{N}$) are a Catalan family.

In order to understand the Catalan simplicial set as a simplicial set, we must understand the face and degeneracy relations between its simplices. In low dimensions, we see directly that $\mathbb{C}$ has:

- A unique 0-simplex UD, which we write as $\star$;
- Two 1-simplices $UUDD$ and $UDUD$, the first of which is $s_0(\star)$ and the second of which is non-degenerate; we write these as $e = s_0(\star) \colon \star \to \star$ and $c \colon \star \to \star$;
- Five 2-simplices: three degenerate ones $UUUDDD$, $UUDDUD$ and $UDUUDD$, and two non-degenerate ones $UUDUDD$ and $UDUDUD$. We depict these, and their faces, by:

$$
\begin{gathered}
\begin{tikzcd}[column sep=small]
& \star \arrow[dr, "e"] \arrow[d, phantom, "\substack{s_0(e)\\=s_1(e)}"] & & & \star \arrow[dr, "c"] \arrow[d, phantom, "s_0(c)"] & & & \star \arrow[dr, "e"] \arrow[d, phantom, "s_1(c)"] & \\
\star \arrow[ur, "e"] \arrow[rr, "e"'] & {} & \star & \star \arrow[ur, "e"] \arrow[rr, "c"'] & {} & \star & \star \arrow[ur, "c"] \arrow[rr, "c"'] & {} & \star
\end{tikzcd} \\
\begin{tikzcd}[column sep=small]
& \star \arrow[dr, "c"] \arrow[d, phantom, "t"] & & & \star \arrow[dr, "e"] \arrow[d, phantom, "i"] & \\
\star \arrow[ur, "c"] \arrow[rr, "c"'] & {} & \star & \star \arrow[ur, "e"] \arrow[rr, "c"'] & {} & \star
\end{tikzcd} .
\end{gathered}
\tag{3.2.1}
$$

In higher dimensions, the simplices of $\mathbb{C}$ will be determined by *coskeletality.* A simplicial set is called *r-coskeletal* when every n-boundary with $n > r$ has a unique filler; here, an *n-boundary* in a simplicial set is a collection of $(n-1)$-simplices $(x_0, \dots, x_n)$ satisfying $d_j(x_i) = d_i(x_{j+1})$ for all $0 \leqslant i \leqslant j < n$; a *filler* for such a boundary is an n-simplex x with $d_i(x) = x_i$ for $i = 0, \dots, n$.

Proposition 3.2.3. *The Catalan simplicial set is* 2*-coskeletal.*

Proof. For each natural number n, let $\mathbb{K}_n$ be the set of binary relations $R \subset \{0, \dots, n\}^2$ such that

(i) $i \mathrel{R} j$ implies $i < j$;

(ii) $i < j < k$ and $i \mathrel{R} k$ implies $i \mathrel{R} j$ and $j \mathrel{R} k$.

For each $n \geqslant 0$, there is a bijection $\mathbb{C}_n \to \mathbb{K}_n$ which sends a Dyck word W to the set of those pairs $i < j$ such that the $(j+1)$st D precedes the $(i+1)$st U in W; these bijections induce a simplicial structure on the $\mathbb{K}_n$'s, and it suffices to prove that this induced structure is 2-coskeletal.

We may identify the faces of an n-simplex $R \in \mathbb{K}_n$ with the restrictions of R to the $(n+1)$ distinct n-element subsets of $\{0, \ldots, n\}$. An arbitrary collection $(R_0, \ldots, R_n)$ of such relations, seen as elements of $\mathbb{K}_{n-1}$, comprises an n-boundary just when each R_i and R_j agree on the intersections of their domains. In this situation, there is a a unique relation $R \subset \{0, \ldots, n\}^2$ restricting back to the given R_i's, and satisfying (i) since each R_i does. If $n > 2$, then each triple $0 \leqslant i < j < k \leqslant n$ will lie entirely inside the domain of some R_ℓ, and so the relation R will satisfy (ii) since each R_ℓ does, and thus constitute an element of $\mathbb{K}_n$. Thus for $n > 2$, each n-boundary of $\mathbb{K} \cong \mathbb{C}$ has a unique filler. $\square$

We now give one further description of the Catalan simplicial set, perhaps the most appealing: we will exhibit it as the monoidal nerve of a particularly simple monoidal category, namely the poset $\mathbf{2} = \bot \leqslant \top$, seen as a monoidal category with tensor product given by disjunction.

We first explain what we mean by this. Recall that if $\mathscr{A}$ is a category, then its *nerve* $\mathrm{N}(\mathscr{A})$ is the simplicial set whose 0-simplices are objects of $\mathscr{A}$, and whose n-simplices for $n > 0$ are strings of n composable morphisms. Since the face and degeneracy maps are obtained from identities and composition in $\mathscr{A}$, the nerve in fact encodes the entire category structure of $\mathscr{A}$.

Suppose now that $\mathscr{A}$ is a *monoidal category* in the sense of [Mac63]—thus, equipped with a tensor product functor $\otimes\colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$, a unit object $I \in \mathscr{A}$, and families of natural isomorphisms $\alpha_{ABC}\colon (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, $\lambda_A\colon I \otimes A \cong A$ and $\rho_A\colon A \cong A \otimes I$, satisfying certain coherence axioms which we recall in detail in Section 3.4 below. In this situation, the nerve of $\mathscr{A}$ as a category fails to encode any information concerning the monoidal structure. However, by viewing $\mathscr{A}$ as a one-object bicategory (=weak 2-category), we may form a different nerve which *does* encode this extra information.

Definition 3.2.4. Let $\mathscr{A}$ be a monoidal category. The *monoidal nerve* of $\mathscr{A}$ is the simplicial set $\mathrm{N}_\otimes(\mathscr{A})$ defined as follows:

- There is a unique 0-simplex, denoted $\star$.
- A 1-simplex is an object $A \in \mathscr{A}$; its two faces are necessarily $\star$.
- A 2-simplex is a morphism $A_{12} \otimes A_{01} \to A_{02}$ in $\mathscr{A}$; its three faces are A_{12}, A_{02} and A_{01}.

- A 3-simplex is a commuting diagram

$$\begin{tikzcd}
(A_{23} \otimes A_{12}) \otimes A_{01} \arrow[rr, "\alpha"] \arrow[d, "A_{123}\otimes 1"'] & & A_{23} \otimes (A_{12} \otimes A_{01}) \arrow[d, "1\otimes A_{012}"] \\
A_{13} \otimes A_{01} \arrow[r, "A_{013}"'] & A_{03} & A_{23} \otimes A_{02} \arrow[l, "A_{023}"]
\end{tikzcd} \tag{3.2.2}$$

  in $\mathscr{A}$; its four faces are $A_{123}, A_{023}, A_{013}$ and A_{012}.

- Higher-dimensional simplices are determined by 3-coskeletality.

The degeneracy of the unique 0-simplex is the unit object $I \in \mathscr{A}$; the two degeneracies $s_0(A), s_1(A)$ of a 1-simplex are the respective coherence constraints $\rho_A^{-1} : A \otimes I \to A$ and $\lambda_A : I \otimes A \to A$; the three degeneracies of a 2-simplex are simply the assertions that certain diagrams commute, which is so by the axioms for a monoidal category. Higher degeneracies are determined by coskeletality.

Note that, because the monoidal nerve arises from viewing a monoidal category as a one-object bicategory, we have a dimension shift: objects and morphisms of $\mathscr{A}$ become 1- and 2-simplices of the nerve, rather than 0- and 1-simplices.

Proposition 3.2.5. *The simplicial set $\mathbb{C}$ is uniquely isomorphic to the monoidal nerve of the poset $\mathbf{2} = \bot \leqslant \top$, seen as a monoidal category under disjunction.*

Proof. In any monoidal nerve $\mathrm{N}_\otimes(\mathscr{A})$, each 3-dimensional boundary has at most one filler, existing just when the diagram (3.2.2) associated to the boundary commutes. Since every diagram in a poset commutes, the nerve $\mathrm{N}_\otimes(\mathbf{2})$, like $\mathbb{C}$, is 2-coskeletal. It remains to show that $\mathbb{C} \cong \mathrm{N}_\otimes(\mathbf{2})$ in dimensions $0, 1, 2$. In dimension 0 this is trivial. In dimension 1, any isomorphism must send $s_0(\star) = e \in \mathbb{C}_1$ to $s_0(\star) = \bot \in \mathrm{N}_\otimes(\mathbf{2})_1$ and hence must send c to $\top$. In dimension 2, the 2-simplices of $\mathrm{N}_\otimes(\mathbf{2})$ are of the form

$$\begin{tikzcd}
& \star \arrow[dr, "x_{12}"] & \\
\star \arrow[ur, "x_{01}"] \arrow[rr, "x_{02}"'] & & \star
\end{tikzcd}$$

where $x_{12} \vee x_{01} \leqslant x_{02}$ in $\mathrm{N}_\otimes(\mathbf{2})$. Thus in $\mathrm{N}_\otimes(\mathbf{2})$, as in $\mathbb{C}$, there is at most one 2-simplex with a given boundary, and by examination of (3.2.1), we see that the same possibilities arise on both sides; thus there is a unique isomorphism $\mathbb{C}_2 \cong \mathrm{N}_\otimes(\mathbf{2})_2$ compatible with the face maps, as required. $\square$

We conclude this section by investigating the non-degenerate simplices of the Catalan simplicial set; these will be of importance in the following sections, where they will play the role of basic coherence data in higher-dimensional monoidal structures. We will see that these non-degenerate simplices form a sequence of Motzkin sets. The *Motzkin numbers* [DS77] $1, 1, 2, 4, 9, \ldots$ are defined by the recurrence relations

$$M_0 = 1 \qquad \text{and} \qquad M_{n+1} = M_n + \textstyle\sum_{k=0}^{n-1} M_k M_{n-1-k} \ .$$

An $\mathbb{N}$-indexed family of sets is a *sequence of Motzkin sets* if there are a Motzkin number of elements in each dimension.

Example 3.2.6. A *Motzkin word* is a string in the alphabet $\{U, C, D\}$ which, on striking out every C, becomes a Dyck word. The sets $\mathbb{M}_n$ of Motzkin words of length n are a sequence of Motzkin sets.

Proposition 3.2.7. *The family* $(\operatorname{nd}\mathbb{C}_n : n \in \mathbb{N})$ *of non-degenerate simplices of* $\mathbb{C}$ *is a sequence of Motzkin sets.*

Proof. It suffices to construct a bijection $\operatorname{nd}\mathbb{C}_n \cong \mathbb{M}_n$ for each n. In one direction, we have a map $\operatorname{nd}\mathbb{C}_n \to \mathbb{M}_n$ sending a non-degenerate Dyck word W to the Motzkin word $M_1 \ldots M_n$ defined as follows: if the ith and $(i+1)$st U's are adjacent in W, then $M_i = U$; if the ith and $(i+1)$st D's are adjacent in W, then $M_i = D$; otherwise $M_i = C$. (Note that the first two cases are disjoint; a Dyck word W satisfying both would have to be in the image of the ith degeneracy map).

In the other direction, suppose given a Motzkin word $M = M_1 \ldots M_n$. Let $a_1 < \cdots < a_k$ enumerate all i for which M_i is D or C, and let $b_1 < \cdots < b_k$ enumerate all i for which M_i is U or C. The inverse mapping $\mathbb{M}_n \to \operatorname{nd}\mathbb{C}_n$ now sends M to the Dyck word

$$U^{a_1} D^{b_1} U^{a_2 - a_1} D^{b_2 - b_1} \ldots U^{a_k - a_{k-1}} D^{b_k - b_{k-1}} U^{n+1-a_k} D^{n+1-b_k} \ . \qquad \square$$

Using this result, we may re-derive a well-known combinatorial identity relating the Catalan and Motzkin numbers.

Corollary 3.2.8. *For each* $n \geqslant 0$*, we have* $C_{n+1} = \sum_k \binom{n}{k} M_k$.

Proof. Recall that the *Eilenberg–Zilber lemma* [GZ67, §II.3] states that every simplex $x \in X_n$ of a simplicial set X is the image under a unique surjection $\phi \colon [n] \twoheadrightarrow [k]$ in Δ of a unique non-degenerate simplex $y \in X_k$. Since there are $\binom{n}{k}$ order-preserving surjections $[n] \twoheadrightarrow [k]$,

$$C_{n+1} = |\mathbb{C}_n| = \textstyle\sum_{\phi \colon [n] \twoheadrightarrow [k]} |\operatorname{nd}\mathbb{C}_k| = \sum_k \binom{n}{k} |\operatorname{nd}\mathbb{C}_k| = \sum_k \binom{n}{k} M_k$$

as required. $\square$

3.3 First classifying properties

We now begin to investigate the *classifying properties* of the Catalan simplicial set, by looking at the structure picked out by maps from $\mathbb{C}$ into the nerves of certain kinds of categorical structure.

For our first classifying property, recall that a *monoid* in a monoidal category $\mathscr{A}$ is given by an object $A \in \mathscr{A}$ and morphisms $\mu\colon A \otimes A \to A$ and $\eta\colon I \to A$ rendering commutative the three diagrams

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "\alpha"] \arrow[d, "\mu\otimes 1"'] & & A \otimes (A \otimes A) \arrow[d, "1\otimes\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A & A \otimes A \arrow[l, "\mu"]
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "\rho_A"] \arrow[d, equal] & A \otimes I \arrow[d, "1\otimes\eta"] \\
A & A \otimes A \arrow[l, "\mu"]
\end{tikzcd}
\quad
\begin{tikzcd}
I \otimes A \arrow[r, "\lambda_A"] \arrow[d, "\eta\otimes 1"'] & A \\
A \otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Proposition 3.3.1. *If $\mathscr{A}$ is a monoidal category, then to give a simplicial map $f\colon \mathbb{C} \to \mathrm{N}_\otimes(\mathscr{A})$ is equally to give a monoid in $\mathscr{A}$.*

Proof. Since $\mathrm{N}_\otimes(\mathscr{A})$ is 3-coskeletal, a simplicial map $f\colon \mathbb{C} \to \mathrm{N}_\otimes(\mathscr{A})$ is uniquely determined by where it sends non-degenerate simplices of dimension $\leqslant 3$. We have already described the non-degenerate simplices in dimensions $\leqslant 2$, while in dimension 3, there are four such, given by

$$a = (t, t, t, t) \qquad \ell = (i, s_1(c), t, s_1(c))$$
$$r = (s_0(c), t, s_0(c), i) \qquad k = (i, s_1(c), s_0(c), i)\ .$$

Here, we take advantage of 2-coskeletality of $\mathbb{C}$ to identify a 3-simplex x with its tuple $(d_0(x), d_1(x), d_2(x), d_3(x))$ of 2-dimensional faces. We thus see that to give $f\colon \mathbb{C} \to \mathrm{N}_\otimes(\mathscr{A})$ is to give:

- In dimension 0, no data: f must send $\star$ to $\star$;
- In dimension 1, an object $A \in \mathscr{A}$, the image of the non-degenerate simplex $c \in \mathbb{C}_1$;
- In dimension 2, morphisms $\mu\colon A \otimes A \to A$ and $\eta'\colon I \otimes I \to A$, the images of the non-degenerate simplices $t, i \in \mathbb{C}_2$;
- In dimension 3, commutative diagrams

$$f(a) = \begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "\alpha"] \arrow[d, "\mu\otimes 1"'] & & A \otimes (A \otimes A) \arrow[d, "1\otimes\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A & A \otimes A \arrow[l, "\mu"]
\end{tikzcd}$$

$$f(\ell) = \begin{tikzcd}
(I \otimes I) \otimes A \arrow[rr, "\alpha"] \arrow[d, "\eta' \otimes 1"'] & & I \otimes (I \otimes A) \arrow[d, "1 \otimes \lambda_A"] \\
A \otimes A \arrow[r, "\mu"'] & A & I \otimes A \arrow[l, "\lambda_A"]
\end{tikzcd}$$

$$f(r) = \begin{tikzcd}
(A \otimes I) \otimes I \arrow[rr, "\alpha"] \arrow[d, "\rho_A^{-1} \otimes 1"'] & & A \otimes (I \otimes I) \arrow[d, "1 \otimes \eta'"] \\
A \otimes I \arrow[r, "\rho_A^{-1}"'] & A & A \otimes A \arrow[l, "\mu"]
\end{tikzcd}$$

$$f(k) = \begin{tikzcd}
(I \otimes I) \otimes I \arrow[rr, "\alpha"] \arrow[d, "\eta' \otimes 1"'] & & I \otimes (I \otimes I) \arrow[d, "1 \otimes \eta'"] \\
A \otimes I \arrow[r, "\rho_A^{-1}"'] & A & I \otimes A \arrow[l, "\lambda_A"]
\end{tikzcd}$$

the images as displayed of the non-degenerate 3-simplices of $\mathbb{C}$.

On defining $\eta = \eta' \circ \rho_A \colon I \to I \otimes I \to A$, we obtain a bijective correspondence between the data (A, μ, η') for a simplicial map $\mathbb{C} \to \mathrm{N}_\otimes(\mathscr{A})$ and the data (A, μ, η) for a monoid in $\mathscr{A}$. Under this correspondence, the axiom $f(a)$ for (A, μ, η') is clearly the same as the first monoid axiom for (A, μ, η); a short calculation with the axioms for a monoidal category shows that $f(\ell)$ and $f(r)$ correspond likewise with the second and third monoid axioms. This leaves only $f(k)$; but it is easy to show that this is automatically satisfied in any monoidal category. Thus monoids in $\mathscr{A}$ correspond bijectively with simplicial maps $\mathbb{C} \to \mathrm{N}_\otimes(\mathscr{A})$ as claimed. □

Remark 3.3.2. A generalisation of this classifying property concerns maps from $\mathbb{C}$ into the nerve of a *bicategory* $\mathscr{B}$ in the sense of [Bén67]. Bicategories are "many object" versions of monoidal categories, and the nerve of a bicategory is a "many object" version of the monoidal nerve of Definition 3.2.4. An easy modification of the preceding argument shows that simplicial maps $\mathbb{C} \to \mathrm{N}(\mathscr{B})$ classify monads in the bicategory $\mathscr{B}$.

3.4 Higher classifying properties

The category Cat of small categories and functors bears a monoidal structure given by cartesian product, and monoids with respect to this are precisely small *strict* monoidal categories—those for which the associativity and unit constraints α, λ and ρ are all identities. It follows by Proposition 3.3.1 that simplicial maps $\mathbb{C} \to \mathrm{N}_\otimes(\mathrm{Cat})$ classify small strict monoidal categories. The purpose of this section is to show that, in fact, we may also classify both

(i) Not-necessarily-strict monoidal categories; and

(ii) *Skew-monoidal* categories in the sense of [Szl12]

by simplicial maps from $\mathbb{C}$ into suitably modified nerves of Cat, where the modifications at issue involve changing the simplices from dimension 3 upwards. The 3-simplices will no longer be commutative diagrams as in (3.2.2), but rather diagrams commuting up to a natural transformation, invertible in the case of (i) but not necessarily so for (ii). The 4-simplices will be, in both cases, suitably commuting diagrams of natural transformations, while higher simplices will be determined by coskeletality as before. Note that, to obtain these new classification results, we do not need to change $\mathbb{C}$ itself, only what we map it into. The change is from something 3-coskeletal to something 4-coskeletal, which means that the non-degenerate 4-simplices of $\mathbb{C}$ come into play. As we will see, these encode precisely the coherence axioms for monoidal or skew-monoidal structure.

Before continuing, let us make precise the definition of skew-monoidal category. As explained in the introduction, the notion was introduced by Szlachányi in [Szl12] to describe structures arising in quantum algebra, and generalises Mac Lane's notion of monoidal category by dropping the requirement that the coherence constraints be invertible.

Definition 3.4.1. A *skew-monoidal category* is a category $\mathscr{A}$ equipped with a unit element $I \in \mathscr{A}$, a tensor product $\otimes \colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$, and natural families of (not necessarily invertible) constraint maps

$$\begin{gathered} \alpha_{ABC} \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C) \\ \lambda_A \colon I \otimes A \to A \quad \text{and} \quad \rho_A \colon A \to A \otimes I \end{gathered} \tag{3.4.1}$$

subject to the commutativity of the following diagrams—wherein tensor is denoted by juxtaposition—for all $A, B, C, D \in \mathscr{A}$:

$$\begin{tikzcd}
& (AB)(CD) \arrow[dr, "\alpha"] & \\
((AB)C)D \arrow[ur, "\alpha"] \arrow[d, "\alpha 1"'] & (5.1) & A(B(CD)) \\
(A(BC))D \arrow[rr, "\alpha"'] & & A((BC)D) \arrow[u, "1\alpha"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(AI)B \arrow[r, "\alpha"] \arrow[dr, phantom, "(5.2)"] & A(IB) \arrow[d, "1\lambda"] \\
AB \arrow[u, "\rho 1"] \arrow[r, "\mathrm{id}"'] & AB
\end{tikzcd}$$

$$\begin{tikzcd}
& I(AB) \arrow[dr, "\lambda"] \arrow[d, phantom, "(5.3)"] & \\
(IA)B \arrow[ur, "\alpha"] \arrow[rr, "\lambda 1"'] & {} & AB
\end{tikzcd}
\qquad
\begin{tikzcd}
& (AB)I \arrow[dr, "\alpha"] \arrow[d, phantom, "(5.4)"] & \\
AB \arrow[ur, "\rho"] \arrow[rr, "1\rho"'] & {} & A(BI)
\end{tikzcd}
\qquad
\begin{tikzcd}
& II \arrow[dr, "\lambda"] \arrow[d, phantom, "(5.5)"] & \\
I \arrow[ur, "\rho"] \arrow[rr, "\mathrm{id}"'] & {} & I\,.
\end{tikzcd}$$

A skew-monoidal category in which α, λ and ρ are invertible is exactly a monoidal category; the axioms (5.1)–(5.5) are then Mac Lane's original five axioms [Mac63], justified by the fact that they imply the commutativity of *all* diagrams of constraint maps. In the skew case, this justification no longer applies, as the axioms no longer force every diagram of constraint maps to commute; for example, we need not have $1_{I\otimes I} = \rho_I \circ \lambda_I \colon I\otimes I \to I\otimes I$. The classification of skew-monoidal structure by maps out of the Catalan simplicial set can thus be seen as an alternative justification of the axioms in the absence of such a result.

Before giving our classification result, we describe the modified nerves of Cat which will be involved. The possibility of taking natural transformations as 2-cells makes Cat not just a monoidal category, but a *monoidal bicategory* in the sense of [GPS95]. Just as one can form a nerve of a monoidal category by viewing it as a one-object bicategory, so one can form a nerve of a monoidal bicategory by viewing it as a one-object tricategory (=weak 3-category), and in fact, various nerve constructions are possible—see [CH12]. The following definitions are specialisations of some of these nerves to the case of Cat.

Definition 3.4.2. The *lax nerve* $\mathrm{N}_\ell(\mathrm{Cat})$ of the monoidal bicategory Cat is the simplicial set defined as follows:

- There is a unique 0-simplex, denoted $\star$.
- A 1-simplex is a (small) category $\mathscr{A}_{01}$; its two faces are both $\star$.
- A 2-simplex is a functor $A_{012} \colon \mathscr{A}_{12} \times \mathscr{A}_{01} \to \mathscr{A}_{02}$.
- A 3-simplex is a natural transformation

$$\begin{tikzcd}
(\mathscr{A}_{23} \times \mathscr{A}_{12}) \times \mathscr{A}_{01} \arrow[rr, "\cong"] \arrow[d, "A_{123}\times 1"'] \arrow[drr, phantom, "\overset{A_{0123}}{\Rightarrow}"] & & \mathscr{A}_{23} \times (\mathscr{A}_{12} \times \mathscr{A}_{01}) \arrow[d, "1\times A_{012}"] \\
\mathscr{A}_{13} \times \mathscr{A}_{01} \arrow[r, "A_{013}"'] & \mathscr{A}_{03} & \mathscr{A}_{23} \times \mathscr{A}_{02} \arrow[l, "A_{023}"]
\end{tikzcd}$$

  with 1-cell components

$$(A_{0123})_{a_{23},a_{12},a_{01}} \colon A_{013}(A_{123}(a_{23},a_{12}),a_{01}) \to A_{023}(a_{23},A_{012}(a_{12},a_{01}))\ .$$

- A 4-simplex is a quintuple of appropriately-formed natural transformations

$(A_{1234}, A_{0234}, A_{0134}, A_{0124}, A_{0123})$ making the pentagon

$$\begin{tikzcd}[column sep=small]
& A_{024}(A_{234}(a_{34}, a_{23}), A_{012}(a_{12}, a_{01})) \arrow[ddr, "A_{0234}"] & \\
A_{014}(A_{124}(A_{234}(a_{34}, a_{23}), a_{12}), a_{01}) \arrow[ur, "A_{0124}"] \arrow[d, "A_{014}(A_{1234},1)"'] & & \\
& & A_{034}(a_{34}, A_{023}(a_{23}, A_{012}(a_{12}, a_{01}))) \\
A_{014}(A_{134}(a_{34}, A_{123}(a_{23}, a_{12})), a_{01}) \arrow[dr, "A_{0134}"'] & & \\
& A_{034}(a_{34}, A_{013}(A_{123}(a_{23}, a_{12}), a_{01})) \arrow[uur, "A_{034}(1,A_{0123})"'] &
\end{tikzcd}$$

commute in $\mathscr{A}_{04}$ for all $(a_{01}, a_{12}, a_{23}, a_{34}) \in \mathscr{A}_{01} \times \mathscr{A}_{12} \times \mathscr{A}_{23} \times \mathscr{A}_{34}$.

- Higher-dimensional simplices are determined by 4-coskeletality, and face and degeneracy maps are defined as before.

The *pseudo nerve* $\mathrm{N}_p(\mathrm{Cat})$ is defined identically except that the natural transformations occurring in dimensions 3 and 4 are required to be invertible.

We are now ready to give our higher classifying property of the Catalan simplicial set.

Proposition 3.4.3. *To give a simplicial map $f\colon \mathbb{C} \to \mathrm{N}_p(\mathrm{Cat})$ is equally to give a small monoidal category; to give a simplicial map $f\colon \mathbb{C} \to \mathrm{N}_\ell(\mathrm{Cat})$ is equally to give a small skew-monoidal category.*

Proof. First we prove the second statement. Since $\mathrm{N}_\ell(\mathrm{Cat})$ is 4-coskeletal, a simplicial map into it is uniquely determined by where it sends non-degenerate simplices of dimension $\leqslant 4$. In dimensions $\leqslant 3$, to give $f\colon \mathbb{C} \to \mathrm{N}_\ell(\mathrm{Cat})$ is to give:

- In dimension 0, no data: f must send $\star$ to $\star$;
- In dimension 1, a small category $\mathscr{A} = f(c)$;
- In dimension 2, a functor $\otimes = f(t)\colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ and an object $I \in \mathscr{A}$ picked out by the functor $f(i)\colon 1 \times 1 \to \mathscr{A}$;
- In dimension 3, natural transformations

$$f(a) = \begin{tikzcd}
(\mathscr{A} \times \mathscr{A}) \times \mathscr{A} \arrow[rr, "\cong"] \arrow[d, "\otimes \times 1"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & \mathscr{A} \times (\mathscr{A} \times \mathscr{A}) \arrow[d, "1 \times \otimes"] \\
\mathscr{A} \times \mathscr{A} \arrow[r, "\otimes"'] & \mathscr{A} & \mathscr{A} \times \mathscr{A} \arrow[l, "\otimes"]
\end{tikzcd}$$

$$
f(\ell) = \begin{tikzcd}
(1\times 1)\times\mathscr{A} \arrow[rr, "\cong"] \arrow[d, "f(i)\times 1"'] & \arrow[d, phantom, "\overset{\lambda}{\Rightarrow}"] & 1\times(1\times\mathscr{A}) \arrow[d, "1\times\cong"] \\
\mathscr{A}\times\mathscr{A} \arrow[r, "\otimes"'] & \mathscr{A} & 1\times\mathscr{A} \arrow[l, "\cong"]
\end{tikzcd}
$$

$$
f(r) = \begin{tikzcd}
(\mathscr{A}\times 1)\times 1 \arrow[rr, "\cong"] \arrow[d, "\cong\times 1"'] & \arrow[d, phantom, "\overset{\rho}{\Rightarrow}"] & \mathscr{A}\times(1\times 1) \arrow[d, "1\times f(i)"] \\
\mathscr{A}\times 1 \arrow[r, "\cong"'] & \mathscr{A} & \mathscr{A}\times\mathscr{A} \arrow[l, "\otimes"]
\end{tikzcd}
$$

$$
f(k) = \begin{tikzcd}
(1\times 1)\times 1 \arrow[rr, "\cong"] \arrow[d, "f(i)\times 1"'] & \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] & 1\times(1\times 1) \arrow[d, "1\times f(i)"] \\
\mathscr{A}\times 1 \arrow[r, "\cong"'] & \mathscr{A} & 1\times\mathscr{A} \arrow[l, "\cong"]
\end{tikzcd}
$$

which are equally well natural families α, λ and ρ as in (3.4.1) together with a map $\kappa_\star\colon I \to I$.

So the data in dimensions $\leqslant 3$ for a simplicial map $\mathbb{C} \to \mathrm{N}_\ell(\mathrm{Cat})$ is the data $(\mathscr{A}, \otimes, I, \alpha, \lambda, \rho)$ for a small skew-monoidal category augmented with a map $\kappa_\star\colon I \to I$ in $\mathscr{A}$. It remains to consider the action on non-degenerate 4-simplices of $\mathbb{C}$. There are nine such, given by:

$$
\begin{aligned}
A1 &= (a, a, a, a, a) & A6 &= (s_0(i), \ell, k, r, s_2(i)) \\
A2 &= (r, s_1(t), a, s_1(t), \ell) & A7 &= (k, \ell, s_0s_1(c), r, k) \\
A3 &= (\ell, \ell, s_2(t), a, s_2(t)) & A8 &= (r, s_1(t), s_0(t), r, k) \\
A4 &= (s_0(t), a, s_0(t), r, r) & A9 &= (k, \ell, s_2(t), s_1(t), \ell) \\
A5 &= (s_1(i), s_2(i), k, s_0(i), s_1(i)) & &
\end{aligned}
$$

where as before, we take advantage of coskeletality of $\mathbb{C}$ to identify a 4-simplex with its tuple of 3-dimensional faces. The images of these simplices each assert the commutativity of a pentagon of natural transformations involving α, ρ, λ or κ; explicitly, they assert that for any $A, B, C, D \in \mathscr{A}$, the following pentagons commute in $\mathscr{A}$:

$$
\begin{tikzcd}[column sep=small]
& (AB)(CD) \arrow[dr, "\alpha"] & \\
((AB)C)D \arrow[ur, "\alpha"] \arrow[d, "\alpha 1"'] & \text{(A1)} & A(B(CD)) \\
(A(BC))D \arrow[rr, "\alpha"'] & & A((BC)D) \arrow[u, "1\alpha"']
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& AB \arrow[dr, "1"] & \\
AB \arrow[ur, "1"] \arrow[d, "\rho 1"'] & \text{(A2)} & AB \\
(AI)B \arrow[rr, "\alpha"'] & & A(IB) \arrow[u, "1\lambda"']
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& I(AB) \arrow[dr, "\lambda"] & \\
(IA)B \arrow[ur, "\alpha"] \arrow[d, "\lambda 1"'] & \text{(A3)} & AB \\
AB \arrow[rr, "1"'] & & AB \arrow[u, "1"']
\end{tikzcd}
$$

$$\begin{tikzcd}
 & & (AB)I \arrow[drr, "\alpha"] & & \\
AB \arrow[urr, "\rho"] \arrow[dr, "1"'] & & \text{(A4)} & & A(BI) \\
 & AB \arrow[rr, "1"'] & & AB \arrow[ur, "1\rho"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & I \arrow[drr, "1"] & & \\
I \arrow[urr, "1"] \arrow[dr, "1"'] & & \text{(A5)} & & I \\
 & I \arrow[rr, "\kappa_\star"'] & & I \arrow[ur, "1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & II \arrow[drr, "\lambda_I"] & & \\
I \arrow[urr, "\rho_I"] \arrow[dr, "1"'] & & \text{(A6)} & & I \\
 & I \arrow[rr, "\kappa_\star"'] & & I \arrow[ur, "1"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & II \arrow[drr, "\lambda_I"] & & \\
I \arrow[urr, "\rho_I"] \arrow[dr, "\kappa_\star"'] & & \text{(A7)} & & I \\
 & I \arrow[rr, "1"'] & & I \arrow[ur, "\kappa_\star"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & AI \arrow[drr, "1"] & & \\
A \arrow[urr, "\rho"] \arrow[dr, "\rho"'] & & \text{(A8)} & & AI \\
 & AI \arrow[rr, "1"'] & & AI \arrow[ur, "1\kappa"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & IA \arrow[drr, "\lambda"] & & \\
IA \arrow[urr, "1"] \arrow[dr, "\kappa 1"'] & & \text{(A9)} & & A \\
 & IA \arrow[rr, "1"'] & & IA \arrow[ur, "\lambda"'] &
\end{tikzcd}\ .$$

Note first that (A5) forces $\kappa_\star = 1_I \colon I \to I$. Now (A1)–(A4) express the axioms (5.1)–(5.4), both (A6) and (A7) express axiom (5.5), whilst (A8) and (A9) are trivially satisfied. Thus the 4-simplex data of a simplicial map $\mathbb{C} \to \mathrm{N}_\ell(\mathrm{Cat})$ exactly express the skew-monoidal axioms and the fact that the additional datum $\kappa_\star \colon I \to I$ is trivial; whence a simplicial map $\mathbb{C} \to \mathrm{N}_\ell(\mathrm{Cat})$ is precisely a small skew-monoidal category.

The same proof now shows that a simplicial map $\mathbb{C} \to \mathrm{N}_p(\mathrm{Cat})$ is precisely a small monoidal category, under the identification of monoidal categories with skew-monoidal categories whose constraint maps are invertible. $\square$

Chapter 4

The Catalan simplicial set II

Abstract

The Catalan simplicial set $\mathbb{C}$ is known to classify skew-monoidal categories in the sense that a map from $\mathbb{C}$ to a suitably defined nerve of Cat is precisely a skew-monoidal category [Buc+15]. We extend this result to the case of skew monoidales internal to any monoidal bicategory $\mathcal{B}$. We then show that monoidal bicategories themselves are classified by maps from $\mathbb{C}$ to a suitably defined nerve of Bicat and extend this result to obtain a definition of skew-monoidal bicategory that aligns with existing theory.

Contribution by the author

As the sole author, this paper is entirely my own work. It is a direct reproduction of the original which was submitted for publication. Any differences from that submission are limited to cosmetic changes such as citation numbering.

4.1 Introduction

Skew-monoidal categories generalise Mac Lane's notion of monoidal category [Mac63] by dropping the requirement of invertibility of the associativity and unit constraints. They were introduced recently by Szlachányi [Szl12] in his study of bialgebroids, which are themselves an extension of the notion of quantum group. Monoidal categories and skew-monoidal categories can be further generalised to notions of *monoidale* and *skew monoidale* in a monoidal bicategory; this has further relevance for quantum algebra, since Lack and Street showed in [LS12] that quantum categories in the sense of [DS04] can be described using skew monoidales.

Skew-monoidal categories are only one of many possible generalisations of monoidal category: the orientation of the coherence maps and the number and shape of the axioms could reasonably be chosen otherwise. The connection with bialgebroids and quantum categories motivates the particular generalisation in current usage, but until recently there was no abstract justification for such a choice.

The Catalan simplicial set $\mathbb{C}$ was introduced in [Buc+15] where it was shown that, apart from a number of interesting combinatorial properties, it classifies skew-monoidal categories in the sense that simplicial maps from $\mathbb{C}$ into a suitably-defined nerve of Cat are the same thing as skew-monoidal categories. This provides some abstract justification for the choices made in describing coherence data for skew-monoidal categories.

The first main goal of this paper is to demonstrate that $\mathbb{C}$ has a further classifying property: for any monoidal bicategory $\mathcal{B}$, simplicial maps from $\mathbb{C}$ into a suitably defined nerve of $\mathcal{B}$ are the same as skew monoidales in $\mathcal{B}$. More precisely, we construct a biequivalence between the $\mathrm{sSet}(\mathbb{C}, \mathbf{N}\mathcal{B})$, the bicategory whose objects are simplicial maps from $\mathbb{C}$ to the nerve of $\mathcal{B}$, and $\mathrm{SkMon}(\mathcal{B})$, the bicategory of skew-monoidales, lax monoidal morphisms and monoidal transformations.

Our second main goal is to investigate whether $\mathbb{C}$ has this classifying property for higher-dimensional categories; in particular, whether simplicial maps from $\mathbb{C}$ to a suitably defined nerve of Bicat are the same as monoidal or skew-monoidal bicategories. We first describe a nerve for Bicat by informally regarding it as a monoidal tricategory. We then find that simplicial maps from $\mathbb{C}$ into this nerve contain some unexpected data which go beyond what is required for a monoidal bicategory. In the case for monoidal bicategories, when the coherence data are invertible, the unexpected data are essentially trivial and the classification result holds. In the case for skew-monoidal bicategories, the data are not invertible, and the unexpected data appear to be a problem. We address this by identifying certain simplices in $\mathbb{C}$ and insist that they be mapped to trivial coherence data. By considering only simplicial maps satisfying this condition, it is easy to compute the data and axioms of a skew-monoidal bicategory.

In Section 4.2 we define skew-monoidal categories and re-introduce the Catalan simplicial set. In Section 4.3 we provide an introduction to monoidal bicategories, skew monoidales, and nerves of monoidal bicategories. In Section 4.4 we describe a biequivalence between maps from $\mathbb{C}$ to the nerve of a monoidal bicategory $\mathcal{B}$ and skew monoidales in $\mathcal{B}$. In Section 4.5 we describe a nerve for Bicat, and define skew-monoidal bicategories by examining certain maps from $\mathbb{C}$ to the nerve of Bicat.

4.2 Preliminaries

In this section we recall the definition of skew-monoidal category, outline our notation for simplicial sets, and re-introduce the Catalan simplicial set defined in [Buc+15].

4.2.1 Skew-monoidal categories

A *skew-monoidal category* is a category $\mathcal{A}$ equipped with a a unit element $I \in \mathcal{A}$ and a tensor $\otimes \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ with natural families of maps:

$$\begin{aligned} &\lambda_A \colon I \otimes A \to A \qquad \text{and} \qquad \rho_A \colon A \to A \otimes I \quad (\text{for } A \in \mathcal{A}) \\ &\alpha_{ABC} \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C) \quad (\text{for } A, B, C, \in \mathcal{A}) \end{aligned} \tag{4.2.1}$$

satisfying five axioms:

$$\begin{tikzcd} & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha"] & \\ ((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha"] \arrow[d, "\alpha \otimes D"'] & & A \otimes (B \otimes (C \otimes D)) \\ (A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "A \otimes \alpha"'] \end{tikzcd} \tag{4.2.2}$$

$$\begin{tikzcd} & (A \otimes I) \otimes B \arrow[r, "\alpha"] & A \otimes (I \otimes B) \arrow[dr, "A \otimes \lambda"] & \\ A \otimes B \arrow[ur, "\rho \otimes B"] \arrow[rrr, "\mathrm{id}"'] & & & A \otimes B \end{tikzcd} \tag{4.2.3}$$

$$\begin{tikzcd} & I \otimes (A \otimes B) \arrow[dr, "\lambda"] & \\ (I \otimes A) \otimes B \arrow[ur, "\alpha"] \arrow[rr, "\lambda \otimes B"'] & & A \otimes B \end{tikzcd} \tag{4.2.4}$$

$$\begin{tikzcd} & (A \otimes B) \otimes I \arrow[dr, "\alpha"] & \\ A \otimes B \arrow[ur, "\rho"] \arrow[rr, "A \otimes \rho"'] & & A \otimes (B \otimes I) \end{tikzcd} \tag{4.2.5}$$

$$\begin{tikzcd} & I \otimes I \arrow[dr, "\lambda"] & \\ I \arrow[ur, "\rho"] \arrow[rr, "\mathrm{id}"'] & & I \, . \end{tikzcd} \tag{4.2.6}$$

These five axioms are the same as those given in Mac Lane's original formulation of monoidal categories [Mac63]. Thus, when α, λ and ρ are invertible, $\mathcal{A}$ is precisely a monoidal category. In that case, Kelly [Kel64] showed that the final three axioms can be derived from the first two, in light of which, some definitions of monoidal category choose to include only those first two axioms. The same result does not hold for skew-monoidal categories and so we must list all five.

On a similar note, when $\mathcal{A}$ is a monoidal category the commutativity of these particular diagrams in fact implies the commutativity of *all* such diagrams; this is one form of the coherence theorem for monoidal categories [Mac71]. Skew-monoidal categories, by contrast, do not have the property that all coherence diagrams commute. For example, the composite $\rho_I \lambda_I \colon I \otimes I \to I \otimes I$ does not generally equal the identity on $I \otimes I$.

4.2.2 Simplicial sets

We write Δ for the simplicial category; the objects are $[n] = \{0, \ldots, n\}$ for $n \geq 0$ and the morphisms are order-preserving functions. Objects X of $\mathrm{SSet} = [\Delta^{\mathrm{op}}, \mathrm{Set}]$ are called *simplicial sets*; we write X_n for $X([n])$ and call its elements n*-simplices* of X. We use the notation $d_i \colon X_n \to X_{n-1}$ and $s_i \colon X_n \to X_{n+1}$ for the face and degeneracy maps, induced by acting on X by the maps $\delta_i \colon [n-1] \to [n]$ and $\sigma_i \colon [n+1] \to [n]$ of Δ, the respective injections and surjections for which $\delta_i^{-1}(i) = \emptyset$ and $\sigma_i^{-1}(i) = \{i, i+1\}$. An $(n+1)$-simplex x is called *degenerate* when it is in the image of some s_i, and *non-degenerate* otherwise.

A simplicial set is called r*-coskeletal* when it lies in the image of the right Kan extension functor $[(\Delta^{(r)})^{\mathrm{op}}, \mathrm{Set}] \to [\Delta^{\mathrm{op}}, \mathrm{Set}]$, where $\Delta^{(r)} \subset \Delta$ is the full subcategory on those $[n]$ with $n \leq r$. In elementary terms, a simplicial set is r-coskeletal when every n-boundary with $n > r$ has a unique filler; here, an n*-boundary* in a simplicial set is a collection of $(n-1)$-simplices $(x_0, \ldots, x_n)$ satisfying $d_j(x_i) = d_i(x_{j+1})$ for all $0 \leqslant i \leqslant j < n$; a *filler* for such a boundary is an n-simplex x with $d_i(x) = x_i$ for $i = 0, \ldots, n$.

4.2.3 The Catalan simplicial set

The Catalan simplicial set $\mathbb{C}$ was introduced and studied in [Buc+15]; its name derives from the fact that it has a Catalan number of simplices in each dimension. There are many ways to characterise $\mathbb{C}$ up to isomorphism; perhaps the most concise and elegant is as the nerve of the monoidal poset $(2, \vee, \perp)$. Here, we will take the following description as basic, since it is most helpful for seeing the connection with skew-monoidal categories.

Definition 4.2.1. The *Catalan simplicial set* $\mathbb{C}$ is the simplicial set with:

- A unique 0-simplex $\star$;

- Two 1-simplices $s_0(\star) \colon \star \to \star$ and $c \colon \star \to \star$;

- Five 2-simplices as displayed in:

$$
\begin{tikzcd}
& \star \arrow[dr, "s_0(\star)"] \arrow[d, phantom, "{\substack{s_0(s_0(\star))\\ =s_1(s_0(\star))}}"] & \\
\star \arrow[ur, "s_0(\star)"] \arrow[rr, "s_0(\star)"'] & {} & \star
\end{tikzcd}
\quad
\begin{tikzcd}
& \star \arrow[dr, "c"] \arrow[d, phantom, "s_0(c)"] & \\
\star \arrow[ur, "s_0(\star)"] \arrow[rr, "c"'] & {} & \star
\end{tikzcd}
\quad
\begin{tikzcd}
& \star \arrow[dr, "s_0(\star)"] \arrow[d, phantom, "s_1(c)"] & \\
\star \arrow[ur, "c"] \arrow[rr, "c"'] & {} & \star
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \star \arrow[dr, "c"] \arrow[d, phantom, "t"] & \\
\star \arrow[ur, "c"] \arrow[rr, "c"'] & {} & \star
\end{tikzcd}
\quad
\begin{tikzcd}
& \star \arrow[dr, "s_0(\star)"] \arrow[d, phantom, "i"] & \\
\star \arrow[ur, "s_0(\star)"] \arrow[rr, "c"'] & {} & \star
\end{tikzcd}
\;;
$$

- Higher-dimensional simplices determined by 2-coskeletality.

Since $\mathbb{C}$ is 2-coskeletal, all simplices above dimension one are uniquely determined by their faces and as such, every n-simplex a for $n \geq 2$ can be identified with the $(n+1)$-tuple of faces $(d_0(a), d_1(a), \ldots, d_n(a))$. By direct computation we find that there are four non-degenerate 3-simplices

$$
\begin{aligned}
a &= (t, t, t, t) \\
\ell &= (i, s_1(c), t, s_1(c)) \\
r &= (s_0(c), t, s_0(c), i) \\
k &= (i, s_1(c), s_0(c), i) \;;
\end{aligned}
$$

and nine non-degenerate 4-simplices

$$
\begin{array}{ll}
A1 = (a, a, a, a, a) & A6 = (s_0(i), r, k, \ell, s_2(i)) \\
A2 = (r, s_1(t), a, s_1(t), \ell) & A7 = (k, r, s_0 s_1(c), \ell, k) \\
A3 = (r, r, s_2(t), a, s_2(t)) & A8 = (\ell, s_1(t), s_0(t), \ell, k) \\
A4 = (s_0(t), a, s_0(t), \ell, \ell) & A9 = (k, r, s_2(t), s_1(t), r) \\
A5 = (s_1(i), s_2(i), k, s_0(i), s_1(i)) &
\end{array}
\;.
$$

The simplices above dimension four will play more of a role in Section 4.5.

Now consider a simplicial map $F \colon \mathbb{C} \to \mathrm{NCat}$, it is completely determined by its behaviour on non-degenerate simplices. At dimension 0, $F\star$ is the unique 0-simplex in the nerve of Cat. At dimension 1, we get a category Fc. At dimension 2, we get two functors $Ft \colon Fc \times Fc \to Fc$ and $Fi \colon I \times I \to Fc$. At dimension 3, we get four natural transformations.

$$
\begin{tikzcd}
(Fc \times Fc) \times Fc \arrow[rr, "\cong"] \arrow[d, "Ft \times 1"'] & & Fc \times (Fc \times Fc) \arrow[d, "1 \times Ft"] \\
Fc \times Fc \arrow[r, "Ft"'] & Fc \arrow[u, phantom, "\overset{Fa}{\Rightarrow}"] & Fc \times Fc \arrow[l, "Ft"]
\end{tikzcd}
$$

$$\begin{tikzcd}
(I \times I) \times Fc \arrow[rr, "\cong"] \arrow[d, "Fi \times 1"'] & \arrow[d, phantom, "\overset{F\ell}{\Rightarrow}"] & I \times (I \times Fc) \arrow[d, "1 \times \cong"] \\
Fc \times Fc \arrow[r, "Ft"'] & Fc & I \times Fc \arrow[l, "\cong"]
\end{tikzcd}$$

$$\begin{tikzcd}
(Fc \times I) \times I \arrow[rr, "\cong"] \arrow[d, "\cong \times 1"'] & \arrow[d, phantom, "\overset{Fr}{\Rightarrow}"] & Fc \times (I \times I) \arrow[d, "1 \times Fi"] \\
Fc \times I \arrow[r, "\cong"'] & Fc & Fc \times Fc \arrow[l, "Ft"]
\end{tikzcd}$$

$$\begin{tikzcd}
(I \times I) \times I \arrow[rr, "\cong"] \arrow[d, "Fi \times 1"'] & \arrow[d, phantom, "\overset{Fk}{\Rightarrow}"] & I \times (I \times I) \arrow[d, "1 \times Fi"] \\
Fc \times I \arrow[r, "\cong"'] & I & I \times Fc \arrow[l, "\cong"]
\end{tikzcd}$$

The unnamed isomorphisms are canonical maps arising from the monoidal category structure on Cat. Already we can see the strong resemblance with skew-monoidal categories. At dimension 5 we get nine axioms concerning transformations Fa, $F\ell$, Fr, Fk. Among those nine are the Mac Lane pentagon and the four other axioms for a skew-monoidal category.

There is some work to do in sorting out the details, but there is a perfect bijection between skew-monoidal categories and simplicial maps $F\colon \mathbb{C} \to \mathrm{NCat}$. This is the final classification result presented in [Buc+15] and the result which we seek to generalise.

4.3 Monoidal bicategories and skew monoidales

One way to generalise monoidal categories is to consider monoidales in a monoidal bicategory $\mathcal{B}$. In this case a monoidal category is precisely a monoidale in Cat. In the same way, it is possible to generalise skew-monoidal categories by describing skew monoidales in a monoidal bicategory $\mathcal{B}$, in which case, a skew-monoidal category is precisely a skew monoidale in Cat. This generalisation was put to use by Lack and Street in [LS12], where it was shown that quantum categories in the sense of [DS04] are skew monoidales in a monoidal bicategory of comodules.

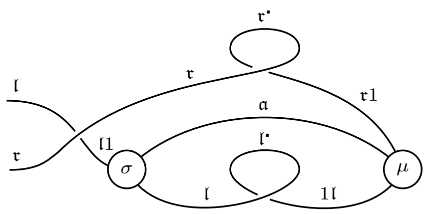
In the following section, we will show that skew-monoidales in a monoidal bicategory $\mathcal{B}$ correspond with simplicial maps from $\mathbb{C}$ into a suitably defined nerve of $\mathcal{B}$. Our result will take the form of a biequivalence

$$\mathrm{SkMon}(\mathcal{B}) \simeq \mathrm{sSet}(\mathbb{C}, \mathbf{N}\mathcal{B}) \ . \tag{4.3.1}$$

The purpose of the present section is to define the bicategories appearing on each side of (4.3.1). We begin by fixing definitions and notation for monoidal bicategories. We then define skew monoidales and describe the bicategory $\mathrm{SkMon}(\mathcal{B})$ of skew monoidales in $\mathcal{B}$ appearing to the left of (4.3.1). Finally we describe a nerve construction for monoidal bicategories assigning to each monoidal bicategory $\mathcal{B}$ a simplicial set $\mathrm{N}\mathcal{B}$, and explain how this simplicial set underlies a simplicial bicategory $\mathbf{N}\mathcal{B}$; now homming into this simplicial bicategory from $\mathbb{C}$ yields the bicategory $\mathrm{sSet}(\mathbb{C}, \mathbf{N}\mathcal{B})$ on the right-hand side of (4.3.1).

4.3.1 Monoidal bicategories

A *monoidal bicategory* is a one-object tricategory in the sense of [Gur06]; it thus comprises a bicategory $\mathcal{B}$ equipped with a unit object I and tensor product homomorphism $\otimes\colon \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ which is associative and unital only up to pseudo-natural equivalences $\mathfrak{a}$, $\mathfrak{l}$ and $\mathfrak{r}$. The coherence of these equivalences is witnessed by invertible modifications π, μ, σ and τ, whose components are 2-cells with boundaries those of the axioms (4.2.2)–(4.2.5) above, and an invertible 2-cell θ whose boundary is that of (4.2.6). The modifications π, μ, σ and τ are as in [Gur06], though we write σ and τ for what there are called λ and ρ; whilst $\theta\colon \mathfrak{r}_I \circ \mathfrak{l}_I \Rightarrow 1_{I\otimes I}\colon I \otimes I \to I \otimes I$ can be defined from the remaining coherence data as the composite



.

The axioms for a tricategory also imply that each of σ and τ are also completely determined by π and μ.

Here, and elsewhere in this paper, we use string notation to display composite 2-cells in a bicategory, with objects represented by regions, 1-cells by strings, and generating 2-cells by vertices. We orient our string diagrams with 1-cells proceeding down the page and 2-cells proceeding from left to right. If a 1-cell ψ belongs to a specified adjoint equivalence, then we will denote its specified adjoint pseudo-inverse by $\psi^\bullet$, and as usual with adjunctions, will draw the unit and counit of the adjoint equivalence in string diagrams as simple caps and cups. In representing the monoidal structure of a bicategory, we notate the tensor product $\otimes$ by juxtaposition, notate the structural 1-cells $\mathfrak{a}, \mathfrak{l}, \mathfrak{r}$ and 2-cells $\pi, \mu, \sigma, \tau, \theta$ explicitly, and use string crossings to notate pseudo-naturality constraint 2-cells, and also instances of the pseudo-functoriality of $\otimes$ of the form $(f \otimes 1) \circ (1 \otimes g) \cong (1 \otimes g) \circ (f \otimes 1)$ (the interchange isomorphisms). String splittings and joinings are used to notate pseudo-functoriality of $\otimes$ of the form $f \otimes g \cong (f \otimes 1) \circ (1 \otimes g)$ and $(1 \otimes g) \circ (f \otimes 1) \cong f \otimes g$ respectively.

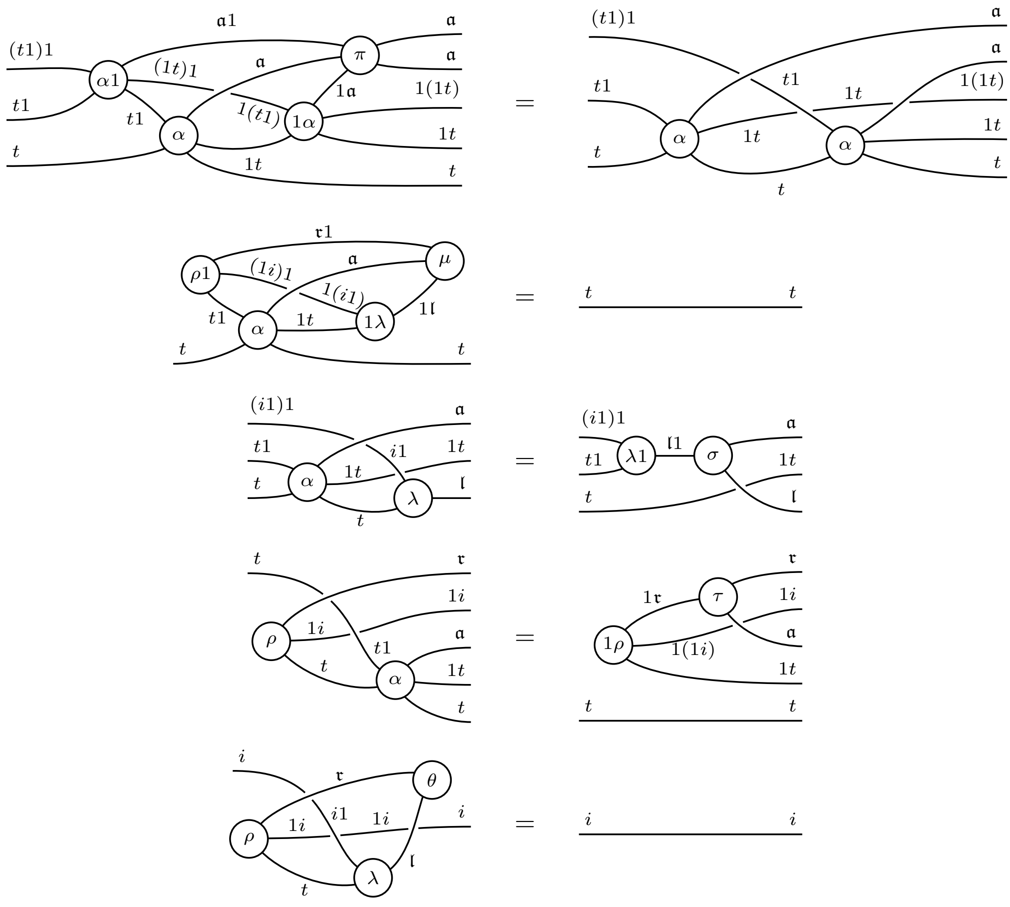
4.3.2 Skew monoidales

Let $\mathcal{B}$ be a monoidal bicategory.

Definition 4.3.1. A *skew monoidale* in $\mathcal{B}$ is an object $A \in \mathcal{B}$ together with morphisms $i\colon I \to A$ and $t\colon A \otimes A \to A$, and (non-invertible) coherence 2-cells

$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "\mathfrak{a}"] \arrow[d, "t\otimes A"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & A\otimes(A\otimes A) \arrow[d, "A\otimes t"] \\
A\otimes A \arrow[r, "t"'] & A & A\otimes A \arrow[l, "t"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
I\otimes A \arrow[r, "\mathfrak{l}"] \arrow[d, "i\otimes A"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & A \arrow[r, "\mathfrak{r}"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\rho}{\Rightarrow}"] & A\otimes I \arrow[d, "A\otimes i"] \\
A\otimes A \arrow[r, "t"'] & A & A\otimes A \arrow[l, "t"]
\end{tikzcd}$$

subject to the following five axioms, the appropriate analogues of (4.2.2)–(4.2.6).

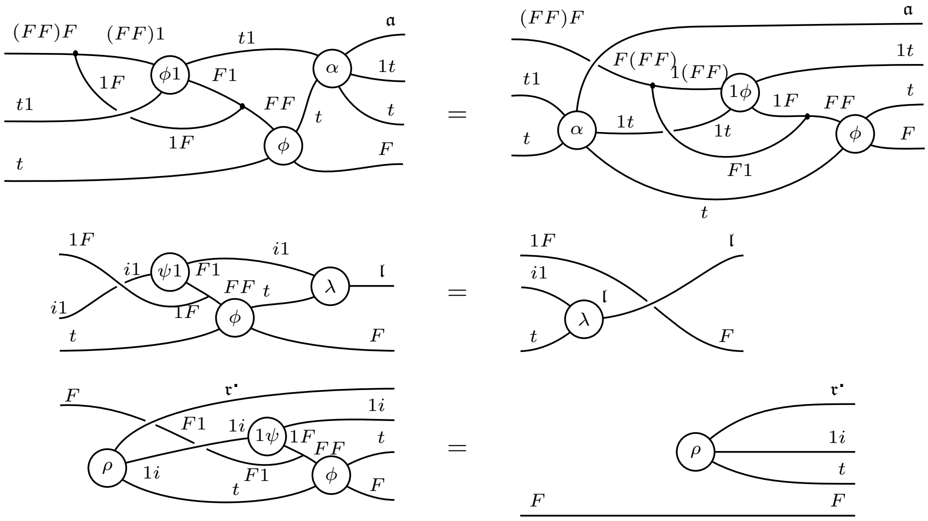


Definition 4.3.2. Let A and B be skew monoidales in $\mathcal{B}$. A *lax monoidal morphism from*

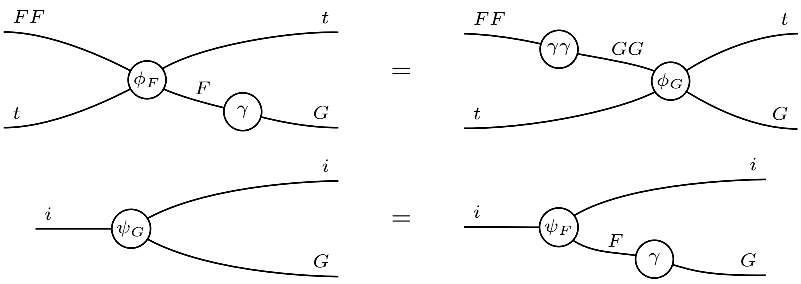
A to B consists of a morphism $F\colon A \to B$ together with 2-cells

$$\begin{tikzcd}
A \otimes A \arrow[r, "t"] \arrow[d, "F\otimes F"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & A \arrow[d, "F"] \\
B \otimes B \arrow[r, "t"'] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
I \arrow[r, "i"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\psi}{\Rightarrow}"] & A \arrow[d, "F"] \\
I \arrow[r, "i"'] & B
\end{tikzcd}$$

subject to the following three axioms.



Definition 4.3.3. Let F and G be lax monoidal morphisms from A to B. A *monoidal transformation from F to G* consists of a 2-cell $\gamma\colon F \Rightarrow G$ satisfying the following two axioms.



Together, skew monoidales, lax monoidal morphisms, and monoidal transformations in $\mathcal{B}$ form a bicategory $\mathrm{SkMon}(\mathcal{B})$. Suppose $(F, \phi_F, \psi_F)\colon\ A\ \to\ B$ and $(G, \phi_G, \psi_G)\colon B \to C$

are lax monoidal morphisms; their composite is GF together with 2-cells

$$
\begin{tikzcd}
A \otimes A \arrow[rr, "t"] \arrow[dr, "F \otimes F"] \arrow[dd, bend right=60, "GF \otimes GF"'] \arrow[dd, phantom, shift left=2ex, "\cong"] & & A \arrow[dr, "F"] \arrow[d, phantom, "\overset{\phi_F}{\Rightarrow}"] & \\
& B \otimes B \arrow[rr, "t"] \arrow[dl, "G \otimes G"'] & {} \arrow[d, phantom, "\overset{\phi_G}{\Rightarrow}"] & B \arrow[dl, "G"] \\
C \otimes C \arrow[rr, "t"'] & & C &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
I \arrow[r, "i"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\psi_F}{\Rightarrow}"] & A \arrow[d, "F"] \\
I \arrow[r, "i"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\psi_G}{\Rightarrow}"] & A \arrow[d, "G"] \\
I \arrow[r, "i"'] & B
\end{tikzcd}
.
$$

The unnamed isomorphism arises from the pseudo-functoriality of $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$. The identity morphism on a skew monoidale A is $1_A \colon A \to A$ together with 2-cells

$$
\begin{tikzcd}
A \otimes A \arrow[r, "t"] \arrow[d, bend right=40, "1 \otimes 1"'] \arrow[d, bend left=40, "1"] \arrow[d, phantom, "\cong"] \arrow[dr, "t"] \arrow[dr, phantom, shift right=2ex, "\cong"] \arrow[dr, phantom, shift left=2ex, "\cong"] & A \arrow[d, "1"] \\
A \otimes A \arrow[r, "t"'] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
I \arrow[r, "i"] \arrow[d, equal] \arrow[dr, "i"] \arrow[dr, phantom, shift left=2ex, "\cong"] & A \arrow[d, "1"] \\
I \arrow[r, "i"'] & A
\end{tikzcd}
.
$$

The unnamed isomorphisms arise from the pseudo-functoriality of $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and coherence cells in the bicategory. If α and β are composable transformations, their composite is $\beta\alpha$. The identity transformation on a lax monoidal morphism F is 1_F. Coherence 2-cells for $\mathrm{SkMon}(\mathcal{B})$ are inherited from $\mathcal{B}$. That is, if F, G, H are composable lax monoidal morphisms then the coherence isomorphisms $(HG)F \cong H(GF)$, $1F \cong F$ and $F \cong F1$ in $\mathcal{B}$ are already monoidal transformations.

4.3.3 Nerves of monoidal bicategories

As noted above, a monoidal bicategory is a one-object tricategory in the sense of [GPS95]. There are several known constructions of nerves for tricategories; the one of interest to us is essentially Street's ω-categorical nerve [Str87], restricted from dimension ω to dimension 3, and generalised from strict to weak 3-categories. An explicit description of this nerve is given in [CH12]; we now reproduce the details for the case of a monoidal bicategory $\mathcal{B}$.

Definition 4.3.4. Suppose that $\mathcal{B}$ is a monoidal bicategory. The *nerve of* $\mathcal{B}$, denoted $\mathrm{N}\mathcal{B}$, is the simplicial set with:

- A unique 0-simplex $\star$.
- A 1-simplex is an object A_{01} of $\mathcal{B}$; its two faces are necessarily $\star$.
- A 2-simplex is given by objects A_{12}, A_{02}, A_{01} of $\mathcal{B}$ together with a 1-cell

$$A_{012} \colon A_{12} \otimes A_{01} \to A_{02} \ ;$$

its three faces are A_{12}, A_{02}, and A_{01}.

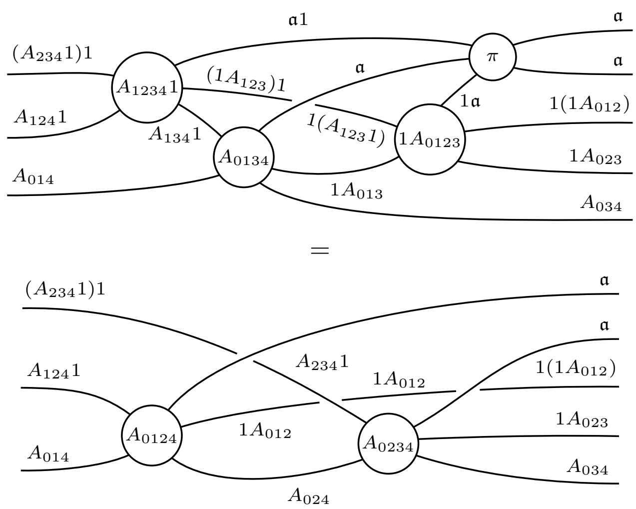
- A 3-simplex is given by:
  - Objects A_{ij} for each $0 \leqslant i < j \leqslant 3$;
  - 1-cells $A_{ijk} \colon A_{jk} \otimes A_{ij} \to A_{ik}$ for each $0 \leqslant i < j < k \leqslant 3$;
  - A 2-cell

$$
\begin{tikzcd}
(A_{23} \otimes A_{12}) \otimes A_{01} \arrow[rr, "\mathfrak{a}"] \arrow[d, "A_{123}\otimes 1"'] & \arrow[d, phantom, "\overset{A_{0123}}{\Longrightarrow}"] & A_{23} \otimes (A_{12} \otimes A_{01}) \arrow[d, "1\otimes A_{012}"] \\
A_{13} \otimes A_{01} \arrow[r, "A_{013}"'] & A_{03} & A_{23} \otimes A_{02} \arrow[l, "A_{023}"]
\end{tikzcd}
\;;
$$

  its four faces are A_{123}, A_{023}, A_{013} and A_{012}.

- A 4-simplex is given by:
  - Objects A_{ij} for each $0 \leqslant i < j \leqslant 4$;
  - 1-cells $A_{ijk} \colon A_{jk} \otimes A_{ij} \to A_{ik}$ for each $0 \leqslant i < j < k \leqslant 4$;
  - 2-cells $A_{ijk\ell} \colon A_{ij\ell} \circ (A_{jk\ell} \otimes 1) \Rightarrow A_{ik\ell} \circ (1 \otimes A_{ijk}) \circ \mathfrak{a}$ for each $0 \leqslant i < j < k < \ell \leqslant 4$

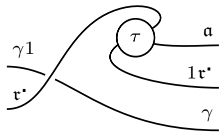
  such that the 2-cell equality

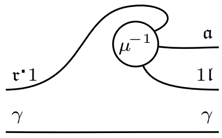


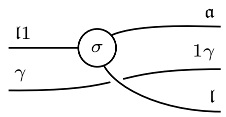
  holds. The five faces of this simplex are A_{1234}, A_{0234}, A_{0134}, A_{0124} and A_{0123}.

- Higher-dimensional simplices are determined by the requirement that $\mathrm{N}\mathcal{B}$ be 4-coskeletal.

It remains to describe the degeneracy operators. The degeneracy of the unique 0-simplex is the unit object $I \in \mathcal{B}$; the two degeneracies $s_0(A), s_1(A)$ of a 1-simplex $A \in \mathcal{B}$ are the unit constraints $\mathfrak{r}^{\bullet}\colon A\otimes I \to A$ and $\mathfrak{l}\colon I\otimes A \to A$; the three degeneracies $s_0(\gamma), s_1(\gamma)$ and $s_2(\gamma)$ of a 2-simplex $\gamma\colon B\otimes C\to A$ are the respective 2-cells





and


.

The four degeneracies of a 3-simplex are simply the assertions of certain 2-cell equalities; that these hold is a consequence of the axioms for a monoidal bicategory. Higher degeneracies are determined by coskeletality.

All simplicial identities except $s_0(I) = s_1(I)$ (i.e. $r^{\bullet}_I = \ell_I$) hold automatically. There is however a canonical isomorphism $r^{\bullet}_I \cong \ell_I$, see [Gur06] A.3.1. Thus ℓ_I is a pseudo-inverse for r_I and we can suppose that $r^{\bullet}_I = \ell_I$ without any loss of generality.

Definition 4.3.5. The *pseudo nerve of* $\mathcal{B}$, called $\mathrm{N_p}\mathcal{B}$, is the same as $\mathrm{N}\mathcal{B}$ with the extra requirement that 3-simplex components A_{0123} be invertible.

Remark 4.3.6. The assignation $\mathcal{B} \mapsto \mathrm{N}(\mathcal{B})$ sending a monoidal bicategory to its nerve can be extended to a functor $\mathrm{N}\colon \mathrm{MonBicat}_s \to \mathrm{SSet}$, where $\mathrm{MonBicat}_s$ is the category of monoidal bicategories and morphisms which strictly preserve all the structure. When seen in this way, the nerve is a right adjoint. This holds equally well for $\mathrm{N_p}$.

Now that the nerve is well defined, we can properly examine simplicial maps $F\colon \mathbb{C} \to \mathrm{N}\mathcal{B}$. In the lowest few dimensions the data for such an F consist of the following.

- A single object Fc in $\mathcal{B}$.
- Two 1-cells in $\mathcal{B}$

  $$Fc\otimes Fc \xrightarrow{Ft} Fc \quad \text{and} \quad I\otimes I \xrightarrow{Fi} Fc$$

  since $F(s_0(\star)) = I$.

Before we can examine the higher data we already notice a problem: while Ft has the right form to provide a multiplication $Fc\otimes Fc \to Fc$, the map $Fi\colon I\otimes I \to Fc$ has the wrong domain to be a unit map for Fc. While this problem is easily resolved using the canonical equivalence of $I\otimes I$ with I, the fact that $I\otimes I$ and I are only *equivalent* and not *isomorphic* means that the correspondence we're investigating cannot be a literal bijection between $\mathrm{sSet}(\mathbb{C},\mathrm{N}\mathcal{B})$ and the set of skew monoidales in $\mathcal{B}$. It will, however, be surjective up to equivalence when $\mathrm{sSet}(\mathbb{C},\mathrm{N}\mathcal{B})$ is regarded as a *bicategory*.

4.3.4 The bicategory $\mathrm{sSet}(\mathbb{C},\mathbf{N}\mathcal{B})$

In order to construct the bicategory $\mathrm{sSet}(\mathbb{C},\mathbf{N}\mathcal{B})$, we will first show that $\mathrm{N}\mathcal{B}$ underlies a simplicial bicategory (a bicategory object internal to simplicial sets). Then since the representable $\mathrm{sSet}(\mathbb{C},\text{-})\colon \mathrm{sSet} \to \mathrm{Set}$ preserves limits, $\mathrm{sSet}(\mathbb{C},\mathrm{N}\mathcal{B})$ becomes the set of objects of a bicategory.

Observation 4.3.7. The nerve of a monoidal bicategory $\mathrm{N}\mathcal{B}$ is the object of objects of a bicategory internal to sSet

$$\mathrm{N}(\mathcal{B} \Downarrow \mathcal{B}) \rightrightarrows \mathrm{N}(\mathcal{B} \downarrow \mathcal{B}) \rightrightarrows \mathrm{N}(\mathcal{B}) \tag{4.3.2}$$

where $(\mathcal{B} \downarrow \mathcal{B})$ and $(\mathcal{B} \Downarrow \mathcal{B})$ are monoidal bicategories defined below. We call this internal bicategory $\mathbf{N}\mathcal{B}$; see Table 4.1 for an explicit description of 0, 1 and 2-cells in the lowest few dimensions. We construct it by first building a bicategory

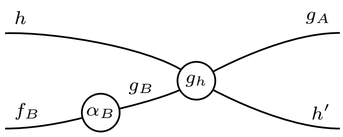
$$(\mathcal{B} \Downarrow \mathcal{B}) \rightrightarrows (\mathcal{B} \downarrow \mathcal{B}) \rightrightarrows (\mathcal{B})$$

internal to $\mathrm{MonBicat}_s$ and then use the fact that N preserves limits (because it is a right adjoint, Remark 4.3.6).

The *oplax-comma monoidal bicategory* $(\mathcal{B} \downarrow \mathcal{B})$ is defined as follows. Its objects are arrows $h\colon A \to B$. A morphism from h to $h'\colon A' \to B'$ is a triple (f_A, f_B, f_h), where $f_A\colon A \to A'$, $f_B\colon B \to B'$, and where

$$\begin{tikzcd} A \arrow[r, "f_A"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{f_h}{\Rightarrow}"] & A' \arrow[d, "h'"] \\ B \arrow[r, "f_B"'] & B' \end{tikzcd} \tag{4.3.3}$$

A 2-cell from (f_A, f_B, f_h) to (g_A, g_B, g_h) is a pair (α_A, α_B), where $\alpha_A\colon f_A \Rightarrow g_A$ and $\alpha_B\colon f_B \Rightarrow g_B$ satisfy



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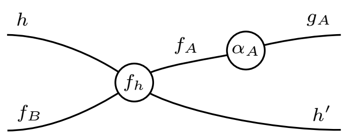
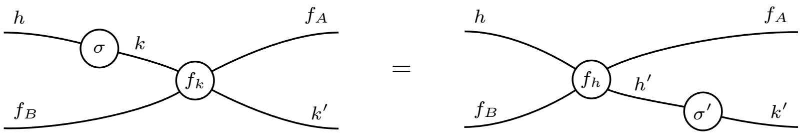


. (4.3.4)

Composition and identities are defined in the obvious way. The tensor for the monoidal structure is defined on 0 and 2-cells by tensoring the underlying data in $\mathcal{B}$. On 1-cells, we need $(f_A, f_B, f_h) \otimes (p_A, p_B, p_h) = (f_A \otimes p_A, f_B \otimes p_B, \varphi_0 \circ (f_h \otimes p_h) \circ \varphi_1)$ where φ_0 and φ_1 are appropriate coherence maps associated to $\otimes\colon \mathcal{B} \times \mathcal{B} \to \mathcal{B}$.

The monoidal bicategory $(\mathcal{B} \Downarrow \mathcal{B})$ is defined as follows. Its objects are 2-cells $\sigma\colon h \Rightarrow k\colon A \to B$ in $\mathcal{B}$. A morphism from σ to $\sigma'\colon h' \Rightarrow k'\colon A' \to B'$ is a 4-tuple (f_A, f_B, f_h, f_k)

where (f_A, f_B, f_h) and (f_A, f_B, f_k) take the same form as (4.3.3) and satisfy



.

A 2-cell $(f_A, f_B, f_h, f_k) \Rightarrow (g_A, g_B, g_h, g_k)$ is a pair (α_A, α_B), $\alpha_A \colon f_A \Rightarrow g_A$, $\alpha_B \colon f_B \Rightarrow g_B$ satisfying (4.3.4) for both h and k. Composition and identities are defined in the obvious way. Again, the tensor for the monoidal structure is defined on 0 and 2-cells by tensoring the underlying data in $\mathcal{B}$. On 1-cells, we need $(f_A, f_B, f_h, f_k) \otimes (p_A, p_B, p_h, p_k) = (f_A \otimes p_A, f_B \otimes p_B, \varphi_0 \circ (f_h \otimes p_h) \circ \varphi_1, \varphi_2 \circ (f_h \otimes p_h) \circ \varphi_3)$ where each φ_i is an appropriate coherence map associated to $\otimes \colon \mathcal{B} \times \mathcal{B} \to \mathcal{B}$.

The internal bicategory structure

$$(\mathcal{B} \Downarrow \mathcal{B}) \Rrightarrow (\mathcal{B} \downarrow \mathcal{B}) \Rrightarrow (\mathcal{B})$$

is given by first defining composition of 1-cells (4.3.3) to be ‘down-the-page’. Domain maps, codomain maps, identities and 2-cell composition follow easily from there.

Observation 4.3.8. For any simplicial set X, $\mathrm{sSet}(X, \mathbf{N}\mathcal{B})$ is a bicategory. To see this, note that the representable $\mathrm{sSet}(X, \text{-}) \colon \mathrm{sSet} \to \mathrm{Set}$ preserves limits. Now if $\mathbf{Y}$ is a bicategory

$$Y_2 \Rrightarrow Y_1 \Rrightarrow Y_0$$

internal to sSet, the 2-globular set

$$\mathrm{sSet}(X, Y_2) \Rrightarrow \mathrm{sSet}(X, Y_1) \Rrightarrow \mathrm{sSet}(X, Y_0)$$

is a bicategory which we call $\mathrm{sSet}(X, \mathbf{Y})$.

The following observation is useful for understanding the nature of 0,1 and 2-cells in $\mathrm{sSet}(X, \mathbf{N}\mathcal{B})$.

Observation 4.3.9. Since $\mathrm{N}(\mathcal{B})$ is 4-coskeletal, its simplices at dimension 5 and above are uniquely determined by their boundary.

This is also true for $\mathrm{N}(\mathcal{B} \downarrow \mathcal{B})$, but it has a stronger property: a 4-simplex in $\mathrm{N}(\mathcal{B} \downarrow \mathcal{B})$ is uniquely determined by its boundary 3-simplices and its source and target 4-simplices in $\mathrm{N}(\mathcal{B})$.

The simplicial set $\mathrm{N}(\mathcal{B} \Downarrow \mathcal{B})$ has both of these properties and an even stronger one: each 3-simplex is uniquely determined by its boundary 2-simplices and its source and target 3-simplices in $\mathrm{N}(\mathcal{B} \downarrow \mathcal{B})$. This means that the essential data of these simplicial sets are contained in their lowest $4, 3, 2$ dimensions respectively. In particular this means that a

dim.	$\mathrm{N}(\mathcal{B})$	$\mathrm{N}(\mathcal{B}\downarrow\mathcal{B})$	$\mathrm{N}(\mathcal{B}\Downarrow\mathcal{B})$
0	$\star$	$\star$	$\star$
1	A	$\begin{tikzcd} A \arrow[d, "p"'] \\ A' \end{tikzcd}$	$\begin{tikzcd} A \arrow[d, bend right, "p"'] \arrow[d, bend left, "\hat{p}"] \arrow[d, phantom, "\overset{\sigma}{\Rightarrow}"] \\ A' \end{tikzcd}$
2	$A \otimes B \xrightarrow{f} C$	$\begin{tikzcd} A \otimes B \arrow[r, "f"] \arrow[d, "p\otimes q"'] \arrow[dr, phantom, "\overset{\zeta}{\Rightarrow}"] & C \arrow[d, "r"] \\ A' \otimes B' \arrow[r, "f'"'] & C' \end{tikzcd}$	$\begin{tikzcd} AB \arrow[rr, "f"] \arrow[d, bend right, "pq"'] \arrow[d, bend left, "\hat{p}\hat{q}"] \arrow[d, phantom, "\overset{\sigma\tau}{\Rightarrow}"] & \overset{\zeta}{\Rightarrow} \quad \overset{\hat{\zeta}}{\Rightarrow} & C \arrow[d, bend right, "r"'] \arrow[d, bend left, "\hat{r}"] \arrow[d, phantom, "\overset{\mu}{\Rightarrow}"] \\ A'B' \arrow[rr, "f'"'] & & C' \end{tikzcd}$
3	$\begin{tikzcd} (AB)D \arrow[rr, "\mathfrak{a}"] \arrow[d, "fD"'] & \overset{\gamma}{\Rightarrow} & A(BD) \arrow[d, "Ah"] \\ CD \arrow[r, "g"'] & E & AF \arrow[l, "k"] \end{tikzcd}$	$\begin{tikzcd} & A(BD) \arrow[rr, "Ah"] \arrow[dd, "p(qs)"'] & & AF \arrow[dr, "k"] \arrow[dd, "pt"] & \\ (AB)D \arrow[ur, "\mathfrak{a}"] \arrow[dd, "(pq)s"'] \arrow[drr, gray] & \cong & \overset{p\xi}{\Rightarrow} & & E \arrow[dd, "u"] \\ & A'(B'D') \arrow[rr, "B'h'"'] & \color{gray}{CD} \arrow[urr, gray] \arrow[dd, gray] & A'F' \arrow[dr, "k'"] & \overset{\phi}{\Rightarrow} \\ (A'B')D' \arrow[ur, "\mathfrak{a}"] \arrow[drr, "f'D'"'] & & \Uparrow\gamma' & & E' \\ & & C'D' \arrow[urr, "g'"'] & & \end{tikzcd}$	
4	See (4.3.4) on p.103		
≥ 5			

Table 4.1: 0, 1 and 2-cells in the lowest few dimensions of the simplicial bicategory $\mathbf{N}\mathcal{B}$. Grey entries are uniquely determined by coskeletality conditions, see Observation 4.3.9. The 2-cells on the front face of the pentagonal prism have been omitted; they can easily be filled in by the reader. Unlabelled isomorphisms are composites of basic coherence data in $\mathcal{B}$. The 2-cell $p\xi$ is schematic for the obvious pasting of ξ with p and coherence isomorphisms.

0-cell in $\mathrm{sSet}(X, \mathbf{N}\mathcal{B})$, a map $X \to \mathrm{N}(\mathcal{B})$, is completely determined by its behaviour up to dimension 4. A 1-cell in $\mathrm{sSet}(X, \mathbf{N}\mathcal{B})$, a map $X \to \mathrm{N}(\mathcal{B} \downarrow \mathcal{B})$, is completely determined by its behaviour up to dimension 3 and its source and target. And a 2-cell in $\mathrm{sSet}(X, \mathbf{N}\mathcal{B})$, a map $X \to \mathrm{N}(\mathcal{B} \Downarrow \mathcal{B})$, is completely determined by its behaviour up to dimension 2 and its source and target.

In order to be more rigorous we make the following definition. Let $F\colon X \to Y$ be a map of simplicial sets. We say that F *is m-coskeletal* or that X *is m-coskeletal over* Y when F has the unique right-lifting property with respect to boundary inclusions $\delta\Delta_n \to \Delta_n$ for all $n > m$. That is, for all u, v as below there exists a unique k making both triangles commute.

$$\begin{tikzcd} \partial\Delta_n \arrow[r, "u"] \arrow[d, hook] & X \arrow[d, "F"] \\ \Delta_n \arrow[r, "v"'] \arrow[ur, dashed, "k"] & Y \end{tikzcd} .$$

A simplicial set X is m-coskeletal precisely when it is m-coskeletal over $\mathbb{1}$. We can now restate the observation as:

- $\mathrm{N}(\mathcal{B})$ is 4-coskeletal;
- $\mathrm{N}(\mathcal{B} \downarrow \mathcal{B})$ is 3-coskeletal over $\mathrm{N}\mathcal{B} \times \mathrm{N}\mathcal{B}$ via $(\mathrm{N}s, \mathrm{N}t)$; and
- $\mathrm{N}(\mathcal{B} \Downarrow \mathcal{B})$ is 2-coskeletal over $\mathrm{N}(\mathcal{B} \downarrow \mathcal{B}) \times_{(\mathrm{N}s,\mathrm{N}t)} \mathrm{N}(\mathcal{B} \downarrow \mathcal{B})$ via $(\mathrm{N}s, \mathrm{N}t)$.

To justify our observation, note the following. The nerve functor $\mathrm{N}\colon \mathrm{MonBicat}_s \to \mathrm{sSet}$ has the property that it sends every locally faithful functor to a 3-coskeletal map and every locally fully faithful functor to a 2-coskeletal map. The maps $(s, t)\colon (\mathcal{B} \downarrow \mathcal{B}) \to \mathcal{B} \times \mathcal{B}$ and $(s, t)\colon (\mathcal{B} \Downarrow \mathcal{B}) \to (\mathcal{B} \downarrow \mathcal{B}) \times_{(s,t)} (\mathcal{B} \downarrow \mathcal{B})$ are locally faithful and locally fully faithful respectively.

4.4 Classifying skew monoidales

In this section we show that simplicial maps from $\mathbb{C}$ to $\mathrm{N}\mathcal{B}$ are skew monoidales in $\mathcal{B}$ in the sense that there is a biequivalence

$$\mathrm{sSet}(\mathbb{C}, \mathbf{N}\mathcal{B}) \simeq \mathrm{SkMon}(\mathcal{B}) .$$

Before we formally construct this biequivalence let us examine the data of a simplicial map $F\colon \mathbb{C} \to \mathrm{N}\mathcal{B}$ and highlight the difficulties that arise. The data for such an F consist of the following:

- A single object A in $\mathcal{B}$

- Two 1-cells in $\mathcal{B}$

$$A \otimes A \xrightarrow{t} A \quad \text{and} \quad I \otimes I \xrightarrow{i} A$$

- Four 2-cells

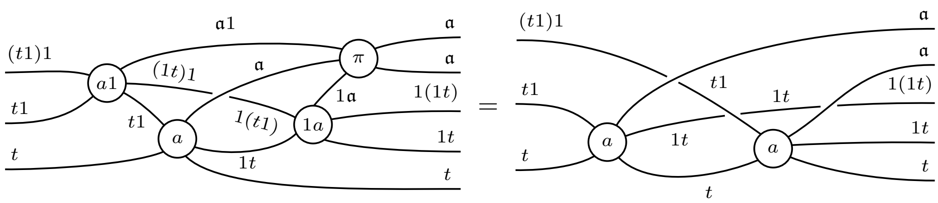
$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "\mathfrak{a}"] \arrow[d, "t\otimes A"'] & {} \arrow[d, phantom, "\overset{a}{\Rightarrow}"] & A\otimes(A\otimes A) \arrow[d, "A\otimes t"] \\
A\otimes A \arrow[r, "t"'] & A & A\otimes A \arrow[l, "t"]
\end{tikzcd}$$

$$\begin{tikzcd}
(A\otimes I)\otimes I \arrow[rr, "\mathfrak{a}"] \arrow[d, "\mathfrak{r}^\bullet\otimes 1"'] & {} \arrow[d, phantom, "\overset{r}{\Rightarrow}"] & A\otimes(I\otimes I) \arrow[d, "A\otimes i"] \\
A\otimes I \arrow[r, "\mathfrak{r}^\bullet"'] & A & A\otimes A \arrow[l, "t"]
\end{tikzcd}$$

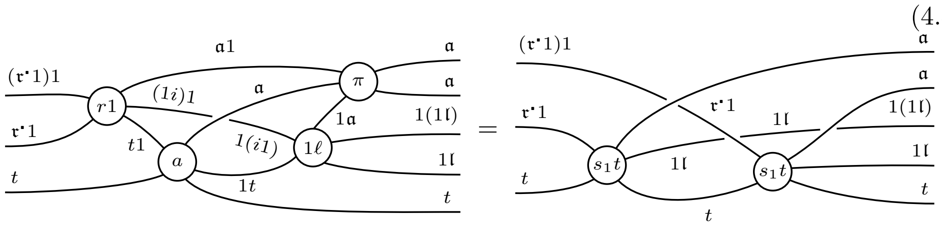
$$\begin{tikzcd}
(I\otimes I)\otimes A \arrow[rr, "\mathfrak{a}"] \arrow[d, "i\otimes 1"'] & {} \arrow[d, phantom, "\overset{\ell}{\Rightarrow}"] & I\otimes(I\otimes A) \arrow[d, "1\otimes\mathfrak{l}"] \\
A\otimes A \arrow[r, "t"'] & A & I\otimes A \arrow[l, "\mathfrak{l}"]
\end{tikzcd}$$

$$\begin{tikzcd}
(I\otimes I)\otimes I \arrow[rr, "\mathfrak{a}"] \arrow[d, "i\otimes 1"'] & {} \arrow[d, phantom, "\overset{k}{\Rightarrow}"] & I\otimes(I\otimes I) \arrow[d, "1\otimes i"] \\
A\otimes I \arrow[r, "\mathfrak{r}^\bullet"'] & A & I\otimes A \arrow[l, "\mathfrak{l}"]
\end{tikzcd}$$

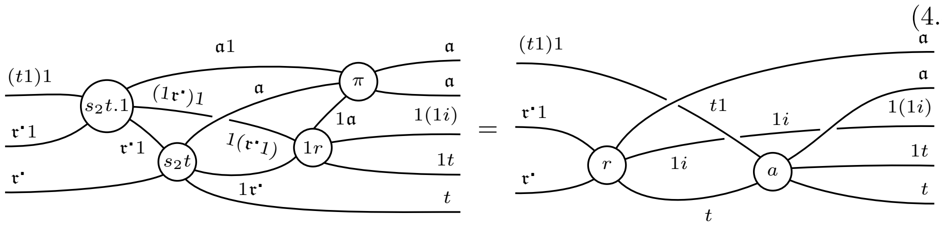
- And nine equalities



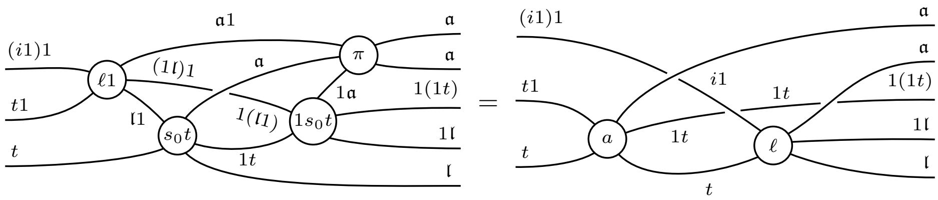
(4.4.1)



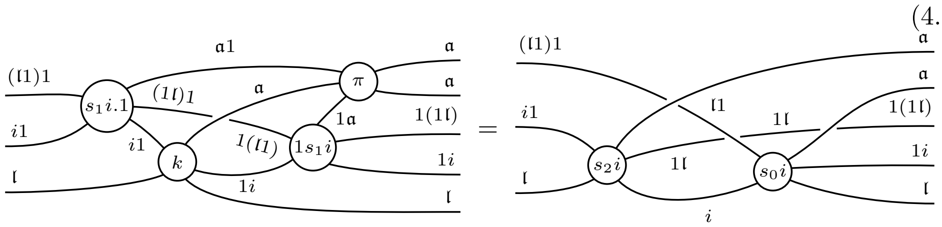
(4.4.2)



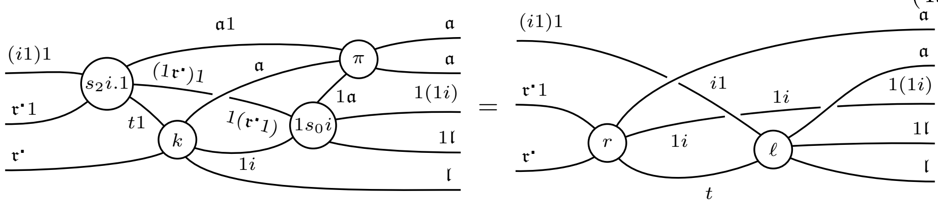
(4.4.3)



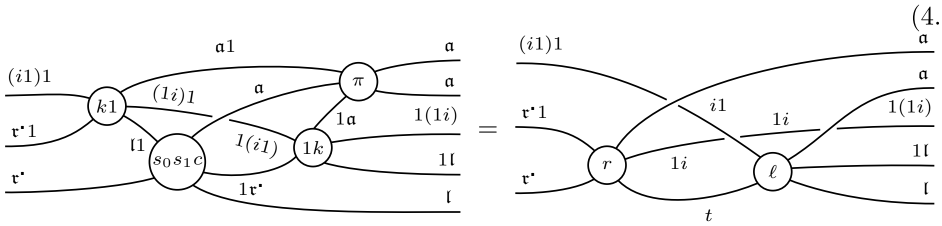
(4.4.4)



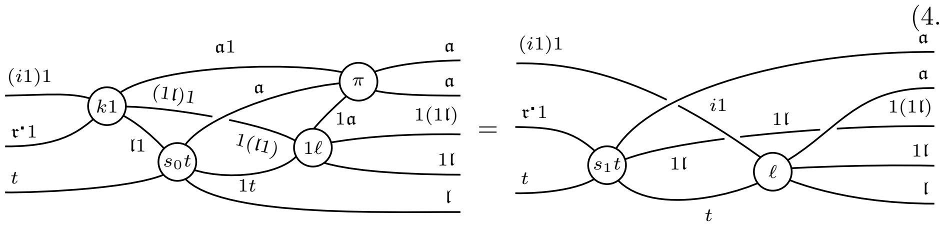
(4.4.5)



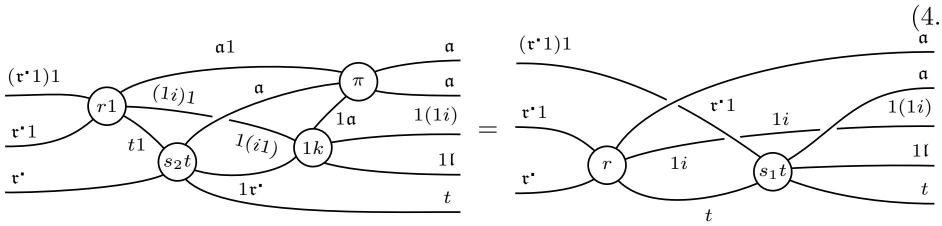
(4.4.6)



(4.4.7)



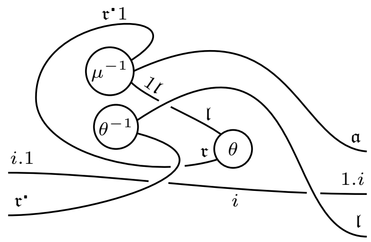
(4.4.8)



(4.4.9)

The similarity with skew monoidales in $\mathcal{B}$ is strong but there are some problems.

As previously mentioned, the unit map for a skew monoidale is of the form $I \to A$ but i is a map $I \otimes I \to A$. Similarly, the left and right unit constraints for a skew monoidale have different domains and codomains than the r and ℓ shown here. These differences amount to the fact that $I \otimes I$ does not equal I; we will deal with this momentarily. The second problem is that there is an extra coherence 2-cell k. Fortunately, the equality in (4.4.5) together with the monoidal bicategory axioms force k to be equal to the pasting



(4.4.10)

and thus completely specified by the coherence data of $\mathcal{B}$. The third problem is that there are too many axioms! Fortunately, (4.4.8) and (4.4.9) hold trivially in any monoidal bicategory. The remaining six equalities are precisely the five axioms we require (axioms (4.4.6) and (4.4.7) are equivalent).

We have yet to resolve the first problem: i, ℓ, r have the wrong shape. This is resolved by constructing a new monoidal bicategory $\mathcal{B}^*$ which is like $\mathcal{B}$, but with unit object $I \otimes I$ and appropriately modified coherence data. Then simplicial maps from $\mathbb{C}$ to $\mathrm{N}\mathcal{B}$ are exactly skew monoidales in $\mathcal{B}^*$: there is an isomorphism of bicategories

$$\mathrm{sSet}(\mathbb{C}, \mathbf{N}\mathcal{B}) \cong \mathrm{SkMon}(\mathcal{B}^*) \ .$$

Then since $\mathcal{B}^* \simeq \mathcal{B}$ we know that $\mathrm{SkMon}(\mathcal{B}^*) \simeq \mathrm{SkMon}(\mathcal{B})$ and we have the required correspondence.

Definition 4.4.1. Suppose $\mathcal{B}$ is a monoidal bicategory $\mathcal{B} = (\mathcal{B}, \otimes, I, \mathfrak{a}, \mathfrak{l}, \mathfrak{r}, \pi, \mu, \sigma, \tau)$. Let $\mathcal{B}^*$ be the monoidal bicategory $(\mathcal{B}, \otimes, I \otimes I, \mathfrak{a}, \mathfrak{l}^*, \mathfrak{r}^*, \pi, \mu^*, \sigma^*, \tau^*)$ where the new data are defined as follows. The new pseudo-natural transformations $\mathfrak{l}^*$ and $\mathfrak{r}^*$ have 1-cell components

$$\mathfrak{l}^*_A = \ (I \otimes I) \otimes A \xrightarrow{\mathfrak{l} \otimes A} I \otimes A \xrightarrow{\quad \mathfrak{l} \quad} A$$

and

$$\mathfrak{r}^*_A = \ A \xrightarrow{\quad \mathfrak{r} \quad} A \otimes I \xrightarrow{A \otimes \mathfrak{r}} A \otimes (I \otimes I) \ \ .$$

Their 2-cell components can easily be deduced. The modifications μ^*, σ^* and τ^* have 2-cell components

$$\mu^*_{AB} = \begin{tikzcd}
 & & (A(II))B \arrow[r, "\mathfrak{a}"] \arrow[d, phantom, "\cong"] & A(II)B \arrow[dr, "A(\mathfrak{l}B)"] & & \\
 & A(IB) \arrow[ur, "(A\mathfrak{r})B"] \arrow[rrr, "\mathfrak{a"] & {} & & (AI)B \arrow[dr, "A\mathfrak{l}"] & \\
AB \arrow[ur, "\mathfrak{r}B"] \arrow[rrrrr, "id"'] & & {} \arrow[u, phantom, "\Downarrow\mu"] & & & AB
\end{tikzcd}$$

$$\sigma^*_{BC} = \begin{tikzcd} & (II)(BC) \arrow[rr, "\mathfrak{l}(BC)"] \arrow[d, phantom, "\cong"] & & I(BC) \arrow[dr, "\mathfrak{l}"] \arrow[d, phantom, "\Downarrow\sigma"] & \\ ((II)B)C \arrow[ur, "\mathfrak{a}"] \arrow[rr, "(\mathfrak{l}B)C"'] & & (IB)C \arrow[ur, "\mathfrak{a}"] \arrow[rr, "\mathfrak{l}C"'] & & BC \end{tikzcd}$$

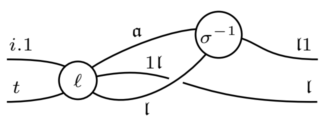
$$\tau^*_{AB} = \begin{tikzcd} & (AB)I \arrow[rr, "(AB)\mathfrak{r}"] \arrow[dr, "\mathfrak{a}"] \arrow[d, phantom, "\Downarrow\tau"] & & (AB)(II) \arrow[dr, "\mathfrak{a}"] \arrow[d, phantom, "\cong"] & \\ AB \arrow[ur, "\mathfrak{r}"] \arrow[rr, "A\mathfrak{r}"'] & & A(BI) \arrow[rr, "A(B\mathfrak{r})"'] & & A(B(II)) \end{tikzcd}$$

Unlabelled isomorphisms come from pseudo-naturality of $\mathfrak{a}$. It is not hard to check that this data satisfies the required axioms and that $\mathcal{B} \simeq \mathcal{B}^*$ as monoidal bicategories.

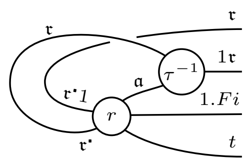
Theorem 4.4.2. *For all monoidal bicategories $\mathcal{B}$ there is an isomorphism of bicategories*

$$\mathrm{sSet}(\mathbb{C}, \mathbf{N}\mathcal{B}) \cong \mathrm{SkMon}(\mathcal{B}^*) \ .$$

Proof. Suppose that $F\colon \mathbb{C} \to \mathrm{N}\mathcal{B}$. Let us compare the image of F with the data for a skew monoidale in $\mathcal{B}^*$ and demonstrate a correspondence between the two. At dimensions one and two these data are exactly equal: a single object A, a tensor map t and a unit map $i\colon I \otimes I \to A$. At dimension two, the 2-cell a has the same form as the associativity constraint α for a skew-monoidale; whilst, as observed above, k is necessarily of the form (4.4.10). On the other hand, the data ℓ and r give rise to left and right unit constraints λ and ρ upon forming the composites



and



.

The assignments $\ell \mapsto \lambda$ and $r \mapsto \rho$ are in fact bijective, the former since it is given by composing with an invertible 2-cell, and the latter since it is given by composition with an invertible 2-cell followed by transposition under adjunction. Thus the 2-dimensional data of F and of a skew monoidale in $\mathcal{B}^*$ are in bijective correspondence.

Finally, after some calculation we find that, with respect to the α, λ and ρ defined above, equations (4.4.1), (4.4.2), (4.4.3), (4.4.4), (4.4.6) and (4.4.7) express precisely the five axioms for a skew monoidale in $\mathcal{B}$; equation (4.4.5) specifies Fk and nothing more; whilst equations (4.4.8) and (4.4.9) are both equalities which follow using only the axioms for a monoidal bicategory. Thus, every simplicial map $F\colon \mathbb{C} \to \mathrm{N}\mathcal{B}$ determines a skew monoidale in $\mathcal{B}^*$ and this assignment is bijective.

Suppose next that $\gamma\colon F \to G$ is a 1-cell in $\mathrm{sSet}(\mathbb{C}, \mathrm{N}\mathcal{B})$. By Remark 4.3.9, γ is determined by F and G and the data up to dimension 3 of a simplicial map γ satisfying

$$\begin{tikzcd} & \mathrm{N}(\mathcal{B}\downarrow\mathcal{B}) \arrow[d, "\mathrm{N}(s,t)"] \\ \mathbb{C} \arrow[ur, "\gamma"] \arrow[r, "{(F,G)}"'] & \mathrm{N}(\mathcal{B})\times\mathrm{N}(\mathcal{B}) \end{tikzcd}.$$

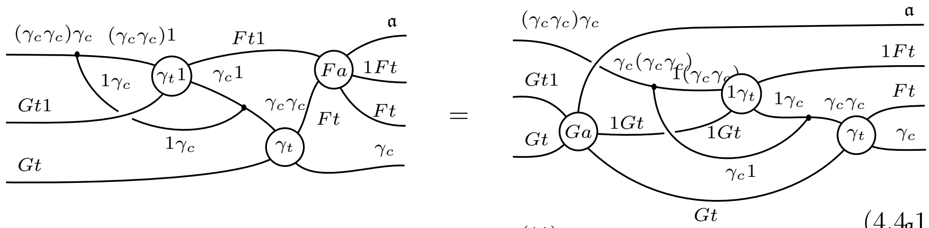
This consists of:

- A single arrow $\gamma_c \colon Fc \to Gc$.
- Two 2-cells

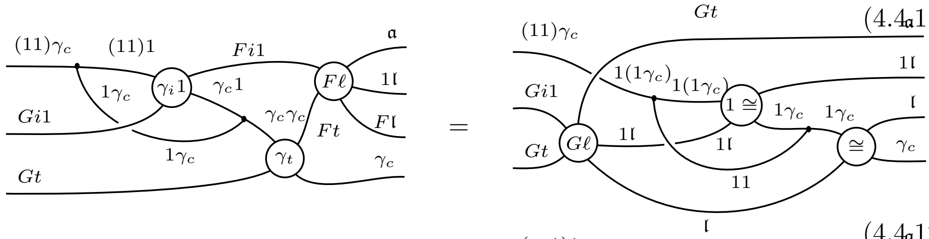
$$\begin{tikzcd} A\otimes A \arrow[r, "Ft"] \arrow[d, "\gamma_c\otimes\gamma_c"'] \arrow[dr, phantom, "\overset{\gamma_t}{\Rightarrow}"] & A \arrow[d, "\gamma_c"] \\ B\otimes B \arrow[r, "Gt"'] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} I\otimes I \arrow[r, "Fi"] \arrow[d, "1\otimes 1"'] \arrow[dr, phantom, "\overset{\gamma_i}{\Rightarrow}"] & A \arrow[d, "\gamma_c"] \\ I\otimes I \arrow[r, "Gi"'] & B \end{tikzcd}$$

  where $A = Fc$ and $B = Gc$.

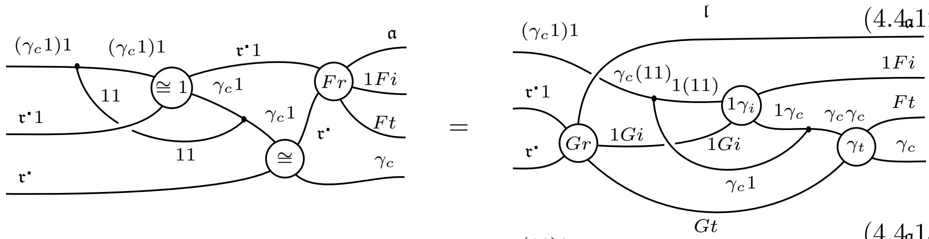
- Four equations



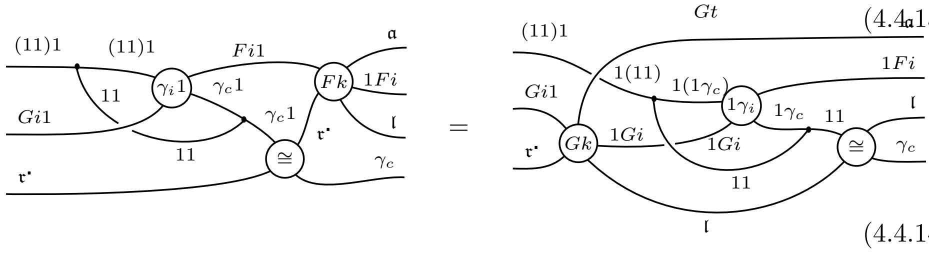
(4.4.11)



(4.4.12)



(4.4.13)



(4.4.14)

where unlabelled isomorphisms come from pseudo-naturality of $\mathfrak{l}$ and $\mathfrak{r}^{\bullet}$. Displaying these isomorphisms explicitly (rather than using string-crossings) highlights the uniformity of the axioms.

We now compare the data for γ with the data for a lax monoidal morphism in $\mathcal{B}^*$. At dimension one these data are exactly equal: a single arrow $\gamma_c\colon A \to B$. At dimension two, the 2-cell γ_t has the same form as the tensor constraint ϕ for a lax monoidal morphism; on the other hand, γ_i gives rise to unit constraint ψ upon forming the composite

$$\begin{tikzcd} I \otimes I \arrow[r, "Fi"] \arrow[d, bend right, "1"'] \arrow[d, bend left, "1\otimes 1"] \arrow[d, phantom, "\cong"] & A \arrow[d, "\gamma_c"] \arrow[dl, phantom, "\overset{\gamma_i}{\Rightarrow}"] \\ I \otimes I \arrow[r, "Gi"'] & B \end{tikzcd} .$$

The assignment $\gamma_i \mapsto \psi$ is bijective since it is just pre-composition with an invertible 2-cell.

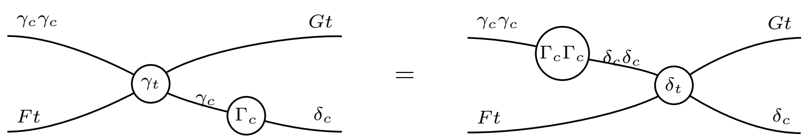
Finally, after some calculation we find that, with respect to the ϕ and ψ defined above, equations (4.4.11), (4.4.12) and (4.4.13) express precisely the three axioms for a lax monoidal morphism in $\mathcal{B}$; equation (4.4.14) is an equality which follows using only the axioms for a monoidal bicategory. Thus, every 1-cell $\gamma\colon F \to G$ in $\mathrm{sSet}(\mathbb{C}, \mathbf{N}\mathcal{B})$ determines a lax monoidal morphism in $\mathcal{B}^*$ and this assignment is bijective.

Suppose finally that $\Gamma\colon \gamma \Rightarrow \delta$ is a 2-cell in $\mathrm{sSet}(\mathbb{C}, \mathrm{N}\mathcal{B})$. By Remark 4.3.9, Γ is determined by γ and δ and the data up to dimension 2 of a simplicial map Γ satisfying

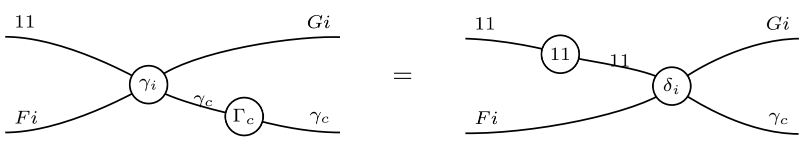
$$\begin{tikzcd} & \mathrm{N}(\mathcal{B} \Downarrow \mathcal{B}) \arrow[d, "\mathrm{N}(s,t)"] \\ \mathbb{C} \arrow[ur, "\Gamma"] \arrow[r, "(\gamma,\delta)"'] & \mathrm{N}(\mathcal{B} \downarrow \mathcal{B}) \times_{\mathrm{N}(s,t)} \mathrm{N}(\mathcal{B} \downarrow \mathcal{B}) \end{tikzcd} .$$

This consists of:

- A single 2-cell $\Gamma_c\colon \gamma_c \Rightarrow \delta_c$.
- Two equations



(4.4.15)



. (4.4.16)

This is exactly the data of a monoidal transformation in $\mathcal{B}^*$: a single 2-cell satisfying exactly the required axioms. Thus, every 2-cell $\Gamma\colon \gamma \Rightarrow \delta$ in $\mathrm{sSet}(\mathbb{C}, \mathbf{N}\mathcal{B})$ determines a monoidal transformation in $\mathcal{B}^*$ and the assignment is bijective.

□

The following result follows directly.

Theorem 4.4.3. *For all monoidal bicategories $\mathcal{B}$ there is a biequivalence*

$$\mathrm{sSet}(\mathbb{C}, \mathbf{N}\mathcal{B}) \simeq \mathrm{SkMon}(\mathcal{B}) \ .$$

Proof. From the biequivalence $\mathcal{B}^* \simeq \mathcal{B}$ we can show that $\mathrm{SkMon}(\mathcal{B}^*) \simeq \mathrm{SkMon}(\mathcal{B})$. This, together with Theorem 4.4.2, gives the desired result. □

Remark 4.4.4 (Results for dual notions)*.* The biequivalence in Theorem 4.4.3 applies to the bicategory of skew monoidales, lax monoidal morphisms and monoidal transformations. The result is also true if we replace skew monoidales with opskew monoidales or ordinary monoidales. We only need to change our definition of 3-simplices in $\mathrm{N}\mathcal{B}$ by reversing the direction of the 2-cells or making them invertible. Similarly, the result holds if we replace lax monoidal morphisms with oplax monoidal morphisms or monoidal morphisms. We only need to change our definition of 1-cells in $(\mathcal{B} \downarrow \mathcal{B})$ by reversing the direction of the 2-cells or making them invertible.

We conclude this section with some remarks on the connection between monoids and lax monoidal functors. It well known that, for a monoidal category $\mathcal{V}$, there is an equivalence

$$\mathrm{MonCat}_{\mathrm{lax}}(\mathbb{1}, \mathcal{V}) \simeq \mathrm{Mon}(\mathcal{V})$$

between lax monoidal functors $\mathbb{1} \to \mathcal{V}$ and monoids internal to $\mathcal{V}$. A similar result holds for skew monoidales internal to a monoidal bicategory.

Definition 4.4.5. Suppose that $\mathcal{B}$ and $\mathcal{E}$ are monoidal bicategories. A *lax monoidal homomorphism from $\mathcal{B}$ to $\mathcal{E}$* is a homomorphism $F\colon \mathcal{B} \to \mathcal{E}$ on the underlying bicategories together with pseudo-natural families of maps

$$\phi_{AB}\colon FA \otimes FB \to F(A \otimes B) \quad \text{and} \quad \phi_I\colon I \to FI$$

and modifications ω, γ, δ with (non-invertible) components

$$\begin{tikzcd}
(FA \otimes FB) \otimes FC \arrow[r, "a"] \arrow[d, "\phi \otimes FC"'] & FA \otimes (FB \otimes FC) \arrow[d, "FA \otimes \phi"] \\
F(A \otimes B) \otimes FC \arrow[d, "\phi"'] \arrow[r, phantom, "\overset{\omega_{ABC}}{\Rightarrow}"] & FA \otimes F(B \otimes C) \arrow[d, "\phi"] \\
F((A \otimes B) \otimes C) \arrow[r, "Fa"'] & F(A \otimes (B \otimes C))
\end{tikzcd} \tag{4.4.17}$$

$$\begin{tikzcd}
I \otimes FA \arrow[r, "\phi \otimes FA"] \arrow[ddr, bend right, "l"'] & FI \otimes FA \arrow[d, "\phi"] \\
\arrow[r, phantom, "\overset{\gamma_A}{\Rightarrow}"] & F(IA) \arrow[d, "Fl"] \\
& FA
\end{tikzcd} \tag{4.4.18}$$

$$\begin{tikzcd}
FA \arrow[r, "r"] \arrow[ddr, bend right, "Fr"'] & FA \otimes I \arrow[d, "FA \otimes \phi"] \\
\arrow[r, phantom, "\overset{\delta_A}{\Rightarrow}"] & FA \otimes FI \arrow[d, "\phi"] \\
& F(A \otimes I)
\end{tikzcd} \tag{4.4.19}$$

satisfying five axioms corresponding directly to those for skew-monoidal categories.

By giving appropriate definitions of monoidal transformation and monoidal modification one can form a bicategory $\mathrm{MonBicat}_{\mathrm{lax}}(\mathcal{B}, \mathcal{E})$ whose objects are lax monoidal homomorphisms. We state the following without proof.

Proposition 4.4.6. *For any monoidal bicategory $\mathcal{B}$, there is a biequivalence*

$$\mathrm{MonBicat}_{\mathrm{lax}}(\mathbb{1}, \mathcal{B}) \simeq \mathrm{SkMon}(\mathcal{B}) \ .$$

This result is not unexpected and easy to verify, but we do need to take care that we have defined lax monoidal homomorphisms properly. It is also relevant in light of the following remark.

Remark 4.4.7. Suppose that $\mathcal{V}$ is a monoidal category. There is an equivalence

$$\mathrm{MonCat}_{\mathrm{lax}}(\mathbb{1}, \mathcal{V}) \simeq \mathrm{Mon}(\mathcal{V})$$

mentioned above, between lax monoidal functors from $\mathbb{1}$ to $\mathcal{V}$ and monoids internal to $\mathcal{V}$.

The data for a lax monoidal functor consists of a functor $F\colon \mathbb{1} \to \mathcal{V}$ together with two natural families of maps $\phi_1\colon 1 \to F1$ and $\phi_{11}\colon F1 \otimes F1 \to F1$ satisfying certain axioms. This is precisely an object $F1$ in $\mathcal{V}$ with a monoid structure and the correspondence is easily extended to an equivalence of categories.

There is a second equivalence

$$\mathrm{MonCat}_{\mathrm{nlax}}(\mathbb{2}, \mathcal{V}) \simeq \mathrm{MonCat}_{\mathrm{lax}}(\mathbb{1}, \mathcal{V})$$

between normal lax functors from $\mathbb{2}$ to $\mathcal{V}$ and lax functors from $\mathbb{1}$ to $\mathcal{V}$. It exists as part of an adjunction where $\mathbb{2}$ is the result of taking $\mathbb{1}$ and freely adding a new unit object together with a map to the old unit object.

There is a third equivalence

$$\mathrm{sSet}(\mathrm{N}\mathbb{2}, \mathrm{N}\mathcal{V}) \simeq \mathrm{MonCat}_{\mathrm{nlax}}(\mathbb{2}, \mathcal{V})$$

obtained by observing that the nerve functor for monoidal categories is fully-faithful on normal lax functors.

Together, these form a sequence

$$\mathrm{sSet}(\mathrm{N}\mathbb{2}, \mathrm{N}\mathcal{V}) \simeq \mathrm{MonCat}_{\mathrm{nlax}}(\mathbb{2}, \mathcal{V}) \simeq \mathrm{MonCat}_{\mathrm{lax}}(\mathbb{1}, \mathcal{V}) \simeq \mathrm{Mon}(\mathcal{V}) \tag{4.4.20}$$

Since $\mathbb{C}$ is isomorphic to $\mathrm{N}\mathbb{2}$, this sequence demonstrates a correspondence between simplicial maps $\mathbb{C} \to \mathrm{N}\mathcal{V}$ and monoids internal to $\mathcal{V}$.

We fully expect that this sequence of equivalences can be generalised to the domain of monoidal bicategories and skew monoidales. Such a generalisation would require a suitable notion of normal lax functor for monoidal bicategories together with a proof that the nerve construction is essentially fully faithful on such functors. That work would lead us too far from our current goal and so we leave it for another time.

4.5 Towards skew-monoidal bicategories

In this section we give a definition of skew-monoidal bicategory by looking at simplicial maps from $\mathbb{C}$ into a suitably defined nerve of Bicat. First, we describe a nerve of Bicat by informally regarding it as a monoidal tricategory, we then examine simplicial maps from $\mathbb{C}$ into this nerve. We find that a classification result for monoidal bicategories holds almost immediately, but the corresponding result for skew-monoidal bicategories requires an extra condition on the simplicial maps in question. We then obtain a definition of skew-monoidal bicategory and find that skew-monoidal bicategories with invertible coherence data are precisely monoidal bicategories in the usual sense. The data for a skew-monoidal bicategory consists of a single bicategory, tensor and unit maps, three coherence transformations, five coherence modifications and eight axioms. This means one new coherence modification and five new axioms.

4.5.1 A nerve of Bicat

Let Bicat be the tricategory of bicategories, homomorphisms, pseudo-natural transformations and modifications. Informally regarding it as a monoidal tricategory, we take the nerve of Bicat to be the simplicial set N(Bicat) defined as follows:

- There is a unique 0-simplex $\star$.
- A 1-simplex is a bicategory
$$\mathcal{B}_{01}\ ;$$
its two faces are necessarily $\star$.
- A 2-simplex is given by bicategories $\mathcal{B}_{12}, \mathcal{B}_{02}, \mathcal{B}_{01}$ together with a pseudo-functor
$$F_{012}\colon \mathcal{B}_{12} \times \mathcal{B}_{01} \to \mathcal{B}_{02}\ ;$$
its three faces are $\mathcal{B}_{12}$, $\mathcal{B}_{02}$, and $\mathcal{B}_{01}$.
- A 3-simplex is given by:
  - Objects $\mathcal{B}_{ij}$ for each $0 \leqslant i < j \leqslant 3$;
  - functors $F_{ijk}\colon \mathcal{B}_{jk} \times \mathcal{B}_{ij} \to \mathcal{B}_{ik}$ for each $0 \leqslant i < j < k \leqslant 3$;
  - a pseudo-natural transformation γ_{0123} whose component at a, b, c is
$$F_{013}(F_{123}(a,b),c) \xrightarrow{\gamma_{0123}} F_{023}(a, F_{012}(b,c))\ ;$$

  its four faces are F_{123}, F_{023}, F_{013} and F_{012}.
- A 4-simplex is given by:
  - Objects $\mathcal{B}_{ij}$ for each $0 \leqslant i < j \leqslant 4$;
  - functors $F_{ijk}\colon \mathcal{B}_{jk} \times \mathcal{B}_{ij} \to \mathcal{B}_{ik}$ for each $0 \leqslant i < j < k \leqslant 4$;
  - transformations $\gamma_{ijk\ell}\colon F_{ij\ell} \circ (F_{jk\ell} \times 1) \Rightarrow F_{ik\ell} \circ (1 \times F_{ijk})$ for each $0 \leqslant i < j < k < \ell \leqslant 4$

– a modification Γ_{01234} whose component at a, b, c, d is

$$\begin{tikzcd}
& F_{024}(F_{012}(a,b), F_{234}(c,d)) \arrow[dr, "\gamma_{0124}"] & \\
F_{024}(F_{023}(F_{012}(a,b),c),d) \arrow[ur, "\gamma_{0234}"] \arrow[d, "F_{024}(\gamma_{0123},d)"'] \arrow[rr, phantom, "\Downarrow\Gamma_{01234}"] & & F_{014}(a, F_{124}(b, F_{234}(c,d))) \\
F_{024}(F_{013}(a, F_{123}(b,c)),d) \arrow[rr, "\gamma_{0134}"'] & & F_{014}(a, F_{134}(F_{123}(b,c),d)) \arrow[u, "F_{014}(a,\gamma_{1234})"']
\end{tikzcd} ;$$

its five faces are γ_{1234}, γ_{0234}, γ_{0134}, γ_{0124}, and γ_{0123}.

- A 5-simplex is given by six modifications $\Gamma_{ijk\ell m}$ for $0 \leqslant i < j < k < \ell < m \leqslant 5$ as above satisfying the following equality for each a, b, c, d, e.

$$
\begin{tikzcd}
 & (((ab)c)d)e \arrow[rr, "\gamma_{0345}"] \arrow[ddl, "(\gamma_{0123}d)e"'] \arrow[dr, phantom, "\cong"] & & ((ab)c)(de) \arrow[dl, "\gamma_{0123}(de)"'] \arrow[dr, "\gamma_{0235}"] & \\
 & & (a(bc))(de) \arrow[ddr, "\gamma_{0135}"] \arrow[r, phantom, "\Downarrow \Gamma_{01235}"] & {} & (ab)(c(de)) \arrow[dd, "\gamma_{0125}"] \\
((a(bc))d)e \arrow[urr, "\gamma_{0345}"] \arrow[dd, "\gamma_{0134}e"'] \arrow[drr, phantom, "\Downarrow \Gamma_{01345}"] & & & & \\
 & & & a((bc)(de)) \arrow[r, "a\gamma_{1235}"] & a(b(c(de))) \\
(a((bc)d))e \arrow[drr, "\gamma_{0145}"] \arrow[ddr, "(a\gamma_{1234})e"'] & & & & \\
 & & a(((bc)d)e) \arrow[uur, "a\gamma_{1345}"] \arrow[dr, "a(\gamma_{1234}e)"] \arrow[r, phantom, "\Downarrow a\Gamma_{12345}"] & {} & a(b((cd)e)) \arrow[uu, "a(b\gamma_{2345})"'] \\
 & (a(b(cd)))e \arrow[rr, "\gamma_{0145}"'] \arrow[ur, phantom, "\cong"] & & a((b(cd))e) \arrow[ur, "a\gamma_{1245}"'] &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
 & (((ab)c)d)e \arrow[rr, "\gamma_{0345}"] \arrow[ddl, "(\gamma_{0123}d)e"'] \arrow[ddd, "\gamma_{0234}e"] \arrow[ddrr, phantom, "\Downarrow \Gamma_{02345}"] & & ((ab)c)(de) \arrow[dr, "\gamma_{0235}"] & \\
 & & & & (ab)(c(de)) \arrow[dd, "\gamma_{0125}"] \\
((a(bc))d)e \arrow[dd, "\gamma_{0134}e"'] \arrow[dr, phantom, "\Downarrow \Gamma_{01234}"] & & & & \\
 & ((ab)(cd))e \arrow[rr, "\gamma_{0245}"] \arrow[ddd, "\gamma_{0124}e"] \arrow[ddrr, phantom, "\Downarrow \Gamma_{01245}"] & & (ab)((cd)e) \arrow[uur, "(ab)\gamma_{2345}"] \arrow[ddr, "\gamma_{0125}"] \arrow[r, phantom, "\cong"] & a(b(c(de))) \\
(a((bc)d))e \arrow[ddr, "(a\gamma_{1234})e"'] & & & & \\
 & & & & a(b((cd)e)) \arrow[uu, "a(b\gamma_{2345})"'] \\
 & (a(b(cd)))e \arrow[rr, "\gamma_{0145}"'] & & a((b(cd))e) \arrow[ur, "a\gamma_{1245}"'] &
\end{tikzcd}
$$

This one of the coherence axioms for a tricategory. It is the associahedron of dimension three, sometimes called the Stasheff polytope K_5 or the non-abelian 4-cocycle condition [GPS95].

The three unnamed isomorphisms come from the pseudo-naturality of 3-simplices $\gamma_{ijk\ell}$. We have abbreviated each $F_{ijk}(F_{\dots}(F_{\dots}(ab)c)d)e$ to $(((ab)c)d)e$. We have also chosen not to display coherence isomorphisms associated to each F_{ijk}. The six faces of this simplex are Γ_{12345}, Γ_{02345}, Γ_{01345}, Γ_{01245}, Γ_{01235}, and Γ_{01234}.

- Higher-dimensional simplices are determined by the requirement that N(Bicat) be 5-coskeletal.

We still need to describe the degenerate simplices.

- At dimension zero, $s_0(\star) = 1$, the terminal bicategory.

- At dimension one, $s_0(\mathcal{B}_{01})\colon 1 \times \mathcal{B}_{01} \to \mathcal{B}_{01}$ and $s_1(\mathcal{B}_{01})\colon \mathcal{B}_{01} \times 1 \to \mathcal{B}_{01}$ are the obvious projections.

- Each $s_j(F_{012}\colon \mathcal{B}_{12} \times \mathcal{B}_{01} \to \mathcal{B}_{02})$ for $j = 0, 1, 2$ is a pseudo-natural transformation whose 1-cell components are identities and 2-cell components are coherence data.

- At dimension four, $s_0(\gamma_{0123}\colon F_{013}(F_{123}(a,b),c) \to F_{023}(a, F_{012}(b,c)))$ is the unique composite of coherence 2-cells filling

$$\begin{tikzcd}
& F_{023}(a, F_{012}(b,c)) \arrow[dr, equal, "\mathrm{id}"] & \\
F_{013}(F_{123}(a,b),c) \arrow[ur, "\gamma_{0123}"] \arrow[d, equal, "\mathrm{id}"'] & & F_{023}(a, F_{012}(b,c)) \\
F_{013}(F_{123}(a,b),c) \arrow[rr, equal, "\mathrm{id}"'] & & F_{013}(F_{123}(a,b),c) \arrow[u, "\gamma_{0123}"']
\end{tikzcd} ;$$

  and the other three are similarly defined.

- We won't display degenerate 5-simplices; they can be computed using the simplicial identities. The equalities of pastings they describe are guaranteed to hold by coherence for bicategories.

In all of the above we have chosen to use pseudo-functors and pseudo-natural transformations rather than their lax and oplax cousins. Those other variations might work just as well, but we haven't investigated them in any detail.

Definition 4.5.1. The *pseudo nerve of* Bicat, called $\mathrm{N_p}$Bicat, is the same as NBicat with the extra requirement that each γ_{0123} be an equivalence and each Γ_{01234} an isomorphism.

4.5.2 Skew-monoidal bicategories

In Section 4.4 we showed that a simplicial map $F\colon \mathbb{C} \to \mathrm{N}\mathcal{B}$ was precisely a skew monoidale in a monoidal bicategory $\mathcal{B}$. In that case, F actually determined an extra datum k and three extra axioms (4.4.5) (4.4.8) (4.4.9). Fortunately, (4.4.5) forced k to be equal to a pasting of coherence maps already found in $\mathcal{B}$, and (4.4.8) and (4.4.9) were already true in any monoidal bicategory. When we look at simplicial maps $\mathbb{C}$ into N(Bicat) or $\mathrm{N_p}$(Bicat) we once again find more data and axioms than we might expect. Our approach to this data will depend on whether we want to describe monoidal bicategories, or skew-monoidal bicategories.

If our goal is to classify monoidal bicategories, we should consider simplicial maps from $\mathbb{C}$ into N_pBicat. In this case, because the maps in question are invertible, most of this data is over-specified and the essential extra data consists of a single equivalence $Fk\colon I \to I$ and a single isomorphism $F\delta\colon \mathrm{id}_I \Rightarrow Fk$ satisfying $F\delta Fk = FkF\delta$. Without presenting every detail: if we consider the set of all monoidal bicategories with this extra data, and also describe a suitable notion of equivalence for them, every such structure is equivalent to one where Fk and $F\delta$ are trivial. Thus we have, up to equivalence, monoidal bicategories.

If our goal is to classify skew-monoidal bicategories, we should consider simplicial maps from $\mathbb{C}$ into NBicat. Unfortunately, we cannot use the same trick as before to eliminate this extra information because the unexpected axioms do not force the unexpected data to be trivial, even up to equivalence. We don't yet understand what role these 'extra' coherence maps might play and for the moment ask that each $F\colon \mathbb{C} \to$ NBicat send the offending simplices in $\mathbb{C}$ to pastings of coherence data in Bicat. Specifically, k is mapped to the identity pseudo-natural transformation on the unit Fi and $A9$ is mapped to the unique composite of coherence data with the corresponding boundary.

With this added condition in place, we define skew-monoidal bicategories by examining the data of simplicial maps $\mathbb{C} \to$ N(Bicat). For convenience we have used the same notation as [GPS95].

Definition 4.5.2. A *skew-monoidal bicategory* consists of:

- A bicategory $\mathcal{M}$.

- Two homomorphisms

$$\otimes\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M} \quad \text{and} \quad I\colon 1 \to \mathcal{M}.$$

- Three pseudo-natural transformations

$$\begin{tikzcd} \mathcal{M}\times\mathcal{M}\times\mathcal{M} \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] \arrow[dr, phantom, "\Downarrow\mathfrak{a}"] & \mathcal{M}\times\mathcal{M} \arrow[d, "\otimes"] \\ \mathcal{M}\times\mathcal{M} \arrow[r, "\otimes"'] & \mathcal{M} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{M} \arrow[r, "I\times 1"] \arrow[dr, bend right, "\mathrm{id}"'] \arrow[dr, phantom, "\Downarrow\mathfrak{l}"] & \mathcal{M}\times\mathcal{M} \arrow[d, "\otimes"] \\ & \mathcal{M} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M} \arrow[d, "1\times I"'] \arrow[dr, bend left, "\mathrm{id}"] \arrow[dr, phantom, "\Downarrow\mathfrak{r}"] & \\ \mathcal{M}\times\mathcal{M} \arrow[r, "\otimes"'] & \mathcal{M} \end{tikzcd}$$

- Five modifications with components:

$$\begin{tikzcd} & (A\otimes B)\otimes(C\otimes D) \arrow[dr, "\mathfrak{a}"] & \\ ((A\otimes B)\otimes C)\otimes D \arrow[ur, "\mathfrak{a}"] \arrow[d, "\mathfrak{a}\otimes D"'] \arrow[r, phantom, "\Downarrow\pi"] & {} & A\otimes(B\otimes(C\otimes D)) \\ (A\otimes(B\otimes C))\otimes D \arrow[rr, "\mathfrak{a}"'] & & A\otimes((B\otimes C)\otimes D) \arrow[u, "A\otimes\mathfrak{a}"'] \end{tikzcd}$$

$$\begin{tikzcd} A\otimes B \arrow[rrr, "\mathrm{id}"] \arrow[dr, "\mathfrak{r}\otimes B"'] & \arrow[d, phantom, "\Downarrow\mu"] & & A\otimes B \\ & (A\otimes I)\otimes B \arrow[r, "\mathfrak{a}"'] & A\otimes(I\otimes B) \arrow[ur, "A\otimes\mathfrak{l}"'] & \end{tikzcd}$$

$$\begin{tikzcd} & I\otimes(A\otimes B) \arrow[dr, "\mathfrak{l}"] \arrow[d, phantom, "\Downarrow\lambda"] & \\ (I\otimes A)\otimes B \arrow[ur, "\mathfrak{a}"] \arrow[rr, "\mathfrak{l}\otimes B"'] & {} & A\otimes B \end{tikzcd}$$

$$\begin{tikzcd} & (A\otimes B)\otimes I \arrow[dr, "\mathfrak{a}"] \arrow[d, phantom, "\Downarrow\rho"] & \\ A\otimes B \arrow[ur, "\mathfrak{r}"] \arrow[rr, "A\otimes\mathfrak{r}"'] & {} & A\otimes(B\otimes I) \end{tikzcd}$$

$$\begin{tikzcd} & I\otimes I \arrow[dr, "\mathfrak{l}"] \arrow[d, phantom, "\Downarrow\sigma"] & \\ I \arrow[ur, "\mathfrak{r}"] \arrow[rr, "\mathrm{id}"'] & {} & I \end{tikzcd} .$$

- All subject to 8 axioms. Unnamed isomorphisms are either pseudo-naturality data or composites of coherence data in $\mathcal{M}$. Empty cells are actual equalities.

$$
\begin{tikzcd}
 & (((AB)C)D)E \arrow[rr, "\mathfrak{a}"] \arrow[ddl, "(\mathfrak{a}D)E"'] \arrow[rr, phantom, bend right=40, "\cong"] & & ((AB)C)(DE) \arrow[dl, "\mathfrak{a}(DE)"'] \arrow[dr, "\mathfrak{a}"] \arrow[dd, phantom, "\Downarrow \pi_{DE}"] & \\
 & & (A(BC))(DE) \arrow[ddr, "\mathfrak{a}"] & & (AB)(C(DE)) \arrow[dd, "\mathfrak{a}"] \\
((A(BC))D)E \arrow[urr, "\mathfrak{a}_{A(BC)}"] \arrow[dd, "\mathfrak{a}E"'] \arrow[rr, phantom, shift right=12, "\Downarrow \pi_{BC}"] & & & & \\
 & & & A((BC)(DE)) \arrow[r, dashed, "A\mathfrak{a}"] & A(B(C(DE))) \\
(A((BC)D))E \arrow[drr, "\mathfrak{a}"] \arrow[ddr, "(A\mathfrak{a})E"'] & & & & \\
 & & A(((BC)D)E) \arrow[uur, "A\mathfrak{a}"] \arrow[dr, "A(\mathfrak{a}E)"] \arrow[rr, phantom, "\Downarrow A\pi"] & & A(B((CD)E)) \arrow[uu, "A(B\mathfrak{a})"'] \\
 & (A(B(CD)))E \arrow[rr, "\mathfrak{a}"'] \arrow[rr, phantom, bend left=40, "\cong"] & & A((B(CD))E) \arrow[ur, "A\mathfrak{a}"'] &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
 & (((AB)C)D)E \arrow[rr, "\mathfrak{a}"] \arrow[ddl, "(\mathfrak{a}D)E"'] \arrow[dddr, "\mathfrak{a}E"] & & ((AB)C)(DE) \arrow[dr, "\mathfrak{a}"] \arrow[dd, phantom, "\Downarrow \pi"] & \\
 & & & & (AB)(C(DE)) \arrow[dd, "\mathfrak{a}"] \\
((A(BC))D)E \arrow[dd, "\mathfrak{a}E"'] & & & & \\
 & \Downarrow \pi E & ((AB)(CD))E \arrow[r, "\mathfrak{a}"] \arrow[dddl, "\mathfrak{a}E"] & (AB)((CD)E) \arrow[uur, "(AB)\mathfrak{a}"] \arrow[ddr, "\mathfrak{a}"] \arrow[r, phantom, "\cong"] \arrow[ddd, phantom, "\Downarrow \pi"] & A(B(C(DE))) \\
(A((BC)D))E \arrow[ddr, "(A\mathfrak{a})E"'] & & & & \\
 & & & & A(B((CD)E)) \arrow[uu, "A(B\mathfrak{a})"'] \\
 & (A(B(CD)))E \arrow[rr, "\mathfrak{a}"'] & & A((B(CD))E) \arrow[ur, "A\mathfrak{a}"'] &
\end{tikzcd}
\tag{4.5.1}
$$

$$
\begin{gathered}
\begin{tikzcd}
A(DE) \arrow[rrrr, bend left, "\mathrm{id}"] \arrow[dr, "\mathfrak{r}(DE)"] & & & A(I(DE)) \arrow[r, "A\mathfrak{l}"'] \arrow[dr, phantom, "\Downarrow A\lambda"] & A(DE) \\
(AD)E \arrow[u, "\mathfrak{a}"] \arrow[r, phantom, "\cong"] \arrow[dr, "(\mathfrak{r}D)E"'] & (AI)(DE) \arrow[u, phantom, "\Downarrow\mu"] \arrow[urr, "\mathfrak{a}"] \arrow[rr, phantom, "\Downarrow\pi"] & & A(I(DE)) \arrow[u, "A\mathfrak{a}"] \arrow[ur, "A(\mathfrak{l}E)"] \arrow[r, phantom, "\cong"] & (AD)E \arrow[u, "\mathfrak{a}"'] \\
& ((AI)D)E \arrow[u, "\mathfrak{a}"'] \arrow[rr, "\mathfrak{a}E"'] & & (A(ID))E \arrow[u, "\mathfrak{a}"'] \arrow[ur, "(A\mathfrak{l})E"'] &
\end{tikzcd}
\\
\|
\\
\begin{tikzcd}
A(DE) \arrow[rrrr, bend left, "\mathrm{id}"] & & & & A(DE) \\
(AD)E \arrow[u, "\mathfrak{a}"] \arrow[rrrr, "id" description] \arrow[rrrru, phantom, "\cong"] \arrow[dr, "(\mathfrak{r}D)E"'] & & \Downarrow\mu E & & (AD)E \arrow[u, "\mathfrak{a}"'] \\
& ((AI)D)E \arrow[rr, "\mathfrak{a}E"'] & & (A(ID))E \arrow[ur, "(A\mathfrak{l})E"'] &
\end{tikzcd}
\end{gathered}
\tag{4.5.2}
$$

$$
\begin{tikzcd}
 & & (AB)E \arrow[drr, "\mathfrak{a}"] \arrow[d, phantom, "\cong"] & & \\
(AB)E \arrow[urr, "\mathrm{id}"] \arrow[r, "\mathfrak{a}"] \arrow[dr, "(A\mathfrak{r})E"'] & A(BE) \arrow[rrr, "\mathrm{id}"] \arrow[dr, "A(\mathfrak{r}E)"] \arrow[d, phantom, "\cong"] & & \arrow[d, phantom, "\Downarrow A\mu"] & A(BE) \\
 & (A(BI))E \arrow[r, "\mathfrak{a}"'] & A((BI)E) \arrow[r, "A\mathfrak{a}"'] & A(B(IE)) \arrow[ur, "A(B\mathfrak{l})"'] &
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
 & & (AB)E \arrow[drr, "\mathfrak{a}"] & & \\
(AB)E \arrow[urr, "\mathrm{id}"] \arrow[r, "\mathfrak{r}E"] \arrow[dr, "(A\mathfrak{r})E"'] \arrow[r, phantom, shift right=4, "\Downarrow\rho E"] & (A(BI))E \arrow[r, "\mathfrak{a}"] \arrow[d, "\mathfrak{a}E"] \arrow[u, phantom, "\Downarrow\mu"] \arrow[dr, phantom, "\Downarrow\pi"] & A((BI)E) \arrow[u, "(AB)\mathfrak{l}"'] \arrow[dr, "\mathfrak{a}"] & \cong & A(BE) \\
 & (A(BI))E \arrow[r, "\mathfrak{a}"'] & A((BI)E) \arrow[r, "A\mathfrak{a}"'] & A(B(IE)) \arrow[ur, "A(B\mathfrak{l})"'] &
\end{tikzcd}
\tag{4.5.3}
$$

$$
\begin{tikzcd}
II \arrow[rrr, "\mathrm{id}"] \arrow[dr, "\mathfrak{r}I"] & & \arrow[d, phantom, "\Downarrow\mu"] & II \\
I \arrow[u, "\mathfrak{r}"] \arrow[dr, "\mathfrak{r}"'] \arrow[r, phantom, "\cong"] & (II)I \arrow[r, "\mathfrak{a}"] \arrow[dr, phantom, "\Downarrow\rho"] & I(II) \arrow[ur, "I\mathfrak{l}"] \arrow[r, phantom, "\Downarrow I\sigma"] & \\
 & II \arrow[u, "\mathfrak{r}"] \arrow[ur, "I\mathfrak{r}"'] \arrow[uurr, bend right, "\mathrm{id}"'] & &
\end{tikzcd}
$$

$$\Vert \tag{4.5.4}$$

$$
\begin{tikzcd}
II \arrow[rr, "\mathrm{id}"] & & II \\
I \arrow[u, "\mathfrak{r}"] \arrow[dr, "\mathfrak{r}"'] & & \\
 & II \arrow[uur, bend right, "\mathrm{id}"'] &
\end{tikzcd}
$$

$$\begin{tikzcd}
& ((AB)C)I \arrow[rr, "\mathfrak{a}"] \arrow[d, phantom, "\Downarrow\rho"] & & (AB)(CI) \arrow[dr, "\mathfrak{a}"] & \\
(AB)C \arrow[ur, "\mathfrak{r}"] \arrow[urrr, bend right, "(AB)\mathfrak{r}"'] \arrow[dr, "\mathfrak{a}"'] & & & \cong & A(B(CI)) \\
& A(BC) \arrow[urrr, bend right, "A(B\mathfrak{r})"'] & & &
\end{tikzcd}$$

$$\Vert$$

$$\begin{tikzcd}
& ((AB)C)I \arrow[rr, "\mathfrak{a}"] \arrow[dr, "\mathfrak{a}I"] & & (AB)(CI) \arrow[dr, "\mathfrak{a}"] \arrow[d, phantom, "\Downarrow\pi"] & \\
(AB)C \arrow[ur, "\mathfrak{r}"] \arrow[dr, "\mathfrak{a}"'] & \cong & (A(BC))I \arrow[r, "\mathfrak{a}"] \arrow[d, phantom, "\Downarrow\rho"] & A((BC)I) \arrow[r, "A\mathfrak{a}"] \arrow[d, phantom, "\Downarrow A\rho"] & A(B(CI)) \\
& A(BC) \arrow[ur, "\mathfrak{r}"'] \arrow[urr, bend right, "A\mathfrak{r}"'] \arrow[urrr, bend right, "A(B\mathfrak{r})"'] & & &
\end{tikzcd}$$

(4.5.5)

$$\begin{tikzcd}
& (II)I \arrow[r, "\mathfrak{a}"] \arrow[drr, bend right, "\mathfrak{l}I"'] & I(II) \arrow[dr, "\mathfrak{l}"] \arrow[d, phantom, "\Downarrow\lambda"] & \\
II \arrow[ur, "\mathfrak{r}"] \arrow[dr, "\mathfrak{l}"'] & \cong & & II \\
& I \arrow[urr, "\mathfrak{r}"'] & &
\end{tikzcd}$$

$$\Vert$$

(4.5.6)

$$\begin{tikzcd}
& (II)I \arrow[r, "\mathfrak{a}"] \arrow[d, phantom, "\Downarrow\rho"] & I(II) \arrow[dr, "\mathfrak{l}"] & \\
II \arrow[ur, "\mathfrak{r}"] \arrow[urr, bend right, "I\mathfrak{r}"'] \arrow[dr, "\mathfrak{l}"'] & & \cong & II \\
& I \arrow[urr, "\mathfrak{r}"'] & &
\end{tikzcd}$$

$$
\begin{tikzcd}
 & & (IC)(DE) \arrow[drr, "\mathfrak{a}"] \arrow[ddrr, bend right, "\mathfrak{l}(DE)"'] & & \\
((IC)D)E \arrow[urr, "\mathfrak{a}"] \arrow[ddrr, bend right, "(\mathfrak{l}D)E"'] & & & & I(C(DE)) \arrow[d, "\mathfrak{l}"] \\
 & & \cong & \Downarrow\lambda & C(DE) \\
 & & (CD)E \arrow[urr, "\mathfrak{a}"'] & &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
 & & (IC)(DE) \arrow[drrr, "\mathfrak{a}"] & & & \\
((IC)D)E \arrow[urr, "\mathfrak{a}"] \arrow[dr, "\mathfrak{a}E"] \arrow[dddrr, bend right, "(\mathfrak{l}D)E"'] & & \Downarrow\pi & & & I(C(DE)) \arrow[d, "\mathfrak{l}"] \\
 & (I(CD))E \arrow[rr, "\mathfrak{a}"] \arrow[ddr, "\mathfrak{l}E"] & & I((CD)E) \arrow[urr, "I\mathfrak{a}"] \arrow[ddl, "\mathfrak{l}"'] & \cong & C(DE) \\
 & \Downarrow\lambda E & \Downarrow\lambda & & & \\
 & & (CD)E \arrow[uurrr, "\mathfrak{a}"'] & & &
\end{tikzcd}
\tag{4.5.7}
$$

$$
\begin{tikzcd}
II \arrow[rrr, "\mathrm{id}"] \arrow[dr, "\mathfrak{r}I"] \arrow[ddrr, bend right, "\mathrm{id}"'] & & \arrow[d, phantom, "\Downarrow\mu"] & II \arrow[d, "\mathfrak{l}"] \\
& (II)I \arrow[r, "\mathfrak{a}"] \arrow[dr, "\mathfrak{l}I"'] \arrow[d, phantom, "\Downarrow\sigma I"] & I(II) \arrow[ur, "I\mathfrak{l}"] \arrow[d, "\mathfrak{l}"] \arrow[r, phantom, "\cong"] \arrow[d, phantom, shift right=4, "\Downarrow\lambda"] & I \\
& {} & II \arrow[ur, "\mathfrak{l}"'] &
\end{tikzcd}
$$

$$
\| \tag{4.5.8}
$$

$$
\begin{tikzcd}
II \arrow[rr, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & II \arrow[d, "\mathfrak{l}"] \\
& & I \\
& II \arrow[ur, "\mathfrak{l}"'] &
\end{tikzcd}
$$

Remark 4.5.3. When $\mathfrak{a}, \mathfrak{l}, \mathfrak{r}$ are equivalences and $\pi, \mu, \rho, \lambda, \sigma$ are isomorphisms this definition becomes equivalent to the usual definition of monoidal bicategory.

Beginning with a skew-monoidal bicategory with invertible coherence maps, just forget σ and axioms (4.5.4)–(4.5.8) and we have exactly a monoidal bicategory. Conversely, given a monoidal bicategory we can construct σ according to axiom (4.5.4), and axioms (4.5.5)–(4.5.8) are implied by (4.5.1)–(4.5.3) and coherence for tricategories (see Chapter 10 and Appendix C in [Gur06]).

Chapter 5

A formal verification of the theory of parity complexes

Abstract

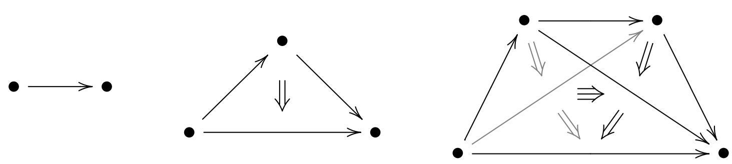
We formalise, in Coq, the opening sections of *Parity Complexes* [Str91] up to and including the all important excision of extremals algorithm. Parity complexes describe the essential combinatorial structure exhibited by simplexes, cubes and globes, that enable the construction of free ω-categories on such objects. The excision of extremals is a recursive algorithm that presents every cell in such a category as a unique composite of atomic cells, this is the sense in which the ω-category is free. Due to the complicated multi-dimensional nature of this work, the detail of definitions and proofs can be hard to follow and verify. Indeed, some corrections were required some years following the original publication [Str94]. Our formalisation verifies that all cases of each result operate as stated. In particular, we indicate which portions of the theory can be proved directly from definitions, and which require more subtle and complex arguments. By identifying results that require the most complicated proofs, we are able to investigate where this theory might benefit from further study and which results need to be considered most carefully in future work.

Contribution by the author

As the sole author, this paper is entirely my own work. It is a direct reproduction of the original which was submitted for publication. Any differences from that submission are limited to cosmetic changes such as citation numbering.

5.1 Introduction

An n-simplex Δ_n is a geometric figure that generalises the notion of triangle or tetrahedron to n-dimensional space. Simplexes have a number of properties that make them useful in algebraic topology, algebraic geometry and homotopy theory where they often play a foundational role. Each n-simplex can be oriented in such a way that it forms an n-category. We include below the cases for $n = 1$, 2 and 3.



(5.1.1)

At low dimensions, it is not hard make each of these into an n-category. At higher dimensions, say $n > 3$, it is quite hard to describe the n-category structure because the source and target of each cell are large pasting diagrams in high dimensions.

Beginning in the late 1970's Ross Street, together with John Roberts and Jack Duskin, began investigating how this process could be rigorously extended to any n. This was achieved in [Str87] where the process was described for the simplexes and the corresponding categories were dubbed the *orientals* (referring to the fact that they are oriented). The main motivation at this time stemmed from non-abelian cohomology where various constructions rely on the orientals.

At the same time, Iain Aitcheson was developing a similar series of results for n-cubes: that each cube could be given an orientation in such a way that it forms an n-category or even an ω-category [Ait86]. A third example of this phenomenon is found in n-globes where the corresponding n-categories have a very simple description. For more on the usefulness of simplexes and cubes, see Street's survey [Str95].

Following these successes, the goal was then to describe the general structure of all oriented multi-dimensional structures for which it is possible to extract free ω-categories in the style of these three examples. The early 1990's yielded a number of related solutions. Ross Street defined a structure called a *parity complex* and gave an explicit description of the ω-category associated to each [Str91]. Some minor corrections were added in [Str94]. Richard Steiner contributed *directed complexes* as a generalisation of directed graph. He showed that loop-free directed complexes generated free ω-categories in the appropriate way [Ste93]. Both of these authors also showed that their respective structures were closed under product and join and covered the three main examples of simplexes, cubes, and globes. Around the same period Mike Johnson was working on a formal description of *pasting scheme* for ω-categories [Joh88; Joh89], and was able to describe the free ω-category on such structures. He included the simplexes as his primary example, and there is a strong sense in which this addressed the same problem. Further related work can be found in [AS93; Ste04] and also in [Ver08] where a conjecture of Street–Roberts is proved in the closing chapter.

Our interest centres on *Parity Complexes* which takes a particularly 'hands-on' approach and describes the combinatorics of this construction in full detail. Our goal is to encode and verify the opening sections of this text up to the excision of extremals algorithm. The theory shows how to build, for any parity complex C an ω-category $\mathcal{O}(C)$. The excision of extremals algorithm shows that each cell can be presented as a unique composite of atomic cells; this is the sense in which $\mathcal{O}(C)$ is free. The algorithm can also be used to generate explicit algebraic descriptions of the cells in $\mathcal{O}(C)$.

Our motivation is two-fold. First, some of the combinatorial arguments in Street's text can be difficult to follow and can easily conceal errors; this is illustrated by the fact that corrections were later required. We will provide some confirmation that the corrections have addressed all issues. Second, a computer-verified encoding provides a good resource for understanding the intricacies of these complicated structures and opens a path to further refinements of the material. We have not attempted to formalised the entirety of the theory. The essential combinatorics are contained in sections 1 to 4 and culminate in the excision of extremals algorithm which is the final result that we encode.

From this point on we often refer to [Str91] as the 'original text', and to [Str94] as 'the corrigenda'.

We programmed everything in Coq [Coq14] and the code is freely available for inspection at the following location.

`https://github.com/MitchellBuckley/Parity-Complexes`

In Section 2 we outline the foundational mathematics that needs to be introduced for an encoding of parity complexes. We also outline how we chose to implement this foundation. In Section 3 we outline the content of [Str91] section-by-section. At each stage we comment on the intuition underlying each result and discuss our implementation of the definitions and results. We pay particular attention to those sections of the material that were difficult to translate into Coq. In Section 4 we discuss how formalisation has shed light on the material and make suggestions for how future work might proceed. In Section 5 we outline the few lessons we have learned in computer-verified encoding of mathematics. Section 6 contains concluding remarks.

5.2 Required Foundations

Parity complexes are described using basic set theory and partially ordered sets. In particular, we need to implement:

- sets;
- set union, set intersection, set difference, etc.;

- finite sets;
- cardinality of finite sets;
- partial orders; and
- segments of partial orders.

Many of these structures are already encoded in the Coq standard library.

5.2.1 Sets

We implement sets using the `Ensembles` standard library. This involves a universe type `U : Type` on which all our sets will be based. Then a set is an *ensemble*: an indexed proposition `U → Prop`. An element of the universe `x : U` is a member of a set `A : U → Prop` when the corresponding proposition `A x` is true. Inclusion of sets relies on logical implication.

```
Definition Ensemble := U → Prop.
Definition In (A:Ensemble) (x:U) : Prop := A x.
Definition Included (B C:Ensemble) : Prop :=
  forall x:U, In B x → In C x.
```

Set operations union, intersection, and set difference are all implemented using point-wise logical operations:

```
Union A B        := fun x => (A x ∨ B x)
Intersection A B := fun x => (A x ∧ B x)
Setminus A B     := fun x => (A x ∧ ¬(B x))
```

For the purposes of this section we suppose that we always work with a fixed universe `U`.

The Coq language has a convenient feature that allows us to introduce notation for these operations.

```
Notation "x ∈ B"      := (In A x) (at level 71).
Notation "A ⊆ B"      := (Included A B) (at level 71).
Notation "A ∪ B"      := (Union A B) (at level 61).
Notation "A ∩ B"      := (Intersection A B) (at level 61).
Notation "A '\' B"    := (Setminus A B) (at level 61).
```

Each special symbol is introduced as a utf-8 character which Coq has no problem recognising. This feature makes the code much more readable.

5.2.2 Finiteness and cardinality

Finiteness is implemented using the `Finite_sets` standard library. This contains an inductively defined proposition `Finite` stating that a set S is finite when $S = \emptyset$, or $S = \{x\} \cup S'$ where S' is finite. Cardinality is implemented in a similar way, using the same library. There is an inductively defined proposition `cardinal` stating that a set S has cardinality 0 when it is empty and has cardinality $n + 1$ when $S = \{x\} \cup S'$ and S' has cardinality n.

```
Inductive Finite : Ensemble U → Prop :=
  | Empty_is_finite : Finite (Empty_set U)
  | Union_is_finite :
      forall A:Ensemble U,
        Finite A → forall x:U, ¬ In U A x →
          Finite (Add U A x).

Inductive cardinal : Ensemble U → nat → Prop :=
  | card_empty : cardinal (Empty_set U) 0
  | card_add :
      forall (A:Ensemble U) (n:nat),
        cardinal A n → forall x:U, ¬ In U A x →
          cardinal (Add U A x) (S n).
```

When our universe has decidable equality we can show that finiteness interacts well with set operations, for example `forall A B, Finite A ∧ Finite B → Finite (A ∪ B)`. Cardinality and finiteness are related by the result `forall S, (Finite S <-> exists n, cardinal S n)`.

5.2.3 Partial orders

Some material on partial orders is available in the `Relations` standard library. Our particular requirements for orders were slightly more complicated than that library could help us with. We found it simpler to explicitly prove basic results as they were needed.

5.2.4 Equality of sets

We say that two sets S and T are *equal* when they are equal as terms of the type `Ensemble U`; in that case `forall x, S x = T x`. We write $S = T$ to indicate that S and T are equal. This is the standard notion that is built into Coq and allows us to replace S with T in any expression.

There is another notion of equality: we say that S and T are *the same* when they contain the same elements. This is the usual notion of set equality used in mathematics. Equivalently, two sets are the same when they are equivalent as indexed propositions (`forall x, S x ↔ T x`), or when `S ⊆ T ∧ T ⊆ S`. We write `Same_set S T` or `S == T` to indicate that S and T are the same.

If two sets are equal then they are certainly the same but two sets can be the same without being equal. For example, the sets `fun x => x = 0` and `fun x => 1 + x = x` in `Ensemble nat` are the same but not equal. The standard library `Ensembles` contains an extensionality axiom stating that `forall A B, A == B → A = B`. In order to keep our formalisation as constructive as possible we are careful never to use the axiom in our formalisation.

The standard facilities of Coq will allow for rewriting S with T whenever S is equal to T. However, when S is the same as T we do not have any guarantee that such rewrites are legitimate. In this situation, our type of sets `Ensemble U` becomes a *setoid*: a set equipped with an equivalence relation. Then `S == T` implies that we may rewrite `S` for `T` in any expression which is built up of operations that preserve the equivalence relation. This rewrite facility is provided by the standard library `Setoid` and requires us to prove that `Same_set` is an equivalence relation and that the appropriate set operations preserve the equivalence.

Without the extensionality axiom it is not possible to prove that `Finite S` and `S == T` implies `Finite T`. Something similar happens with the definition of finite cardinality. The problem occurs when we try to show that `T == Empty_set` implies that `Finite T`. In that case, we find that neither `T = Empty_set` nor `T = {x} ∪ T'` and so neither constructor will show that `T` is finite. This problem can be solved in more than one way. We chose to solve this by adding a third constructor for `Finite` that explicitly introduces the property that `Finite S ∧ S == T → Finite T`. This modification allows us to recover this basic property of finite sets without the extensionality axiom. This illustrates how careful one must be with even the most basic of definitions.

5.2.5 More on finiteness

In many cases we augmented the standard library with extra results about finite sets that were not already present. We found that setting up this basic theory was often tedious, but occasionally an enjoyable exercise in constructive mathematics. For instance, it became clear at some point that certain basic results about sets could not be proved without supposing that equality in `U` is decidable, i.e. `forall (a b : U), (a=b) ∨ ¬(a=b)`. Since none of the examples used here or in the literature need a universe `U` without decidable equality, we have made this a further assumption in our implementation.

If one wanted to reason about, say sets of integer sequences, then the obvious universes to use would be $nat->Int$ or $StreamInt$ each of which lacks decidable equality. In that case one would find that various simple results concerning finite sets would not hold.

We have now covered the essential mathematical foundations required for a formalisation of parity complexes. More details can be found by examining the code itself.

5.3 Definitions and the simplex example

In this section and those following we summarise sections 1 to 4 of [Str91] together with modifications given in the corrigenda [Str94]. This content is sufficient to express the excision of extremals algorithm (Theorem 22). As we progress through the material we will usually reproduce definitions and terminology verbatim from [Str91; Str94]. In each case we will explain the underlying intuition of the material, comment on our implementation, and indicate where our formalisation shed light on the underlying arguments. The reproduced content has been numbered consecutively while our personal comments have been numbered within sections.

We begin by summarising the content of Section 1 of [Str91].

Definition 1. *A* parity complex *is a graded set*

$$C = \sum_{n=0}^{\infty} C_n \tag{5.3.1}$$

together with, for each $x \in C_{n+1}$ *two disjoint, non-empty, finite sets* $x^+, x^- \subseteq C_n$ *subject to Axioms 1, 2, 3A and 3B which appear below.*

From this point onward we will work exclusively within a single parity complex C as described above. When we say $S \subseteq C$ we mean that S is a subset of the underlying graded set of the parity complex. When we say $x \in C$ we mean that x is an element of the underlying graded set of the parity complex.

Before we list the axioms we will introduce some terminology. If $x \in C$ then elements of x^- are called *negative faces* of x, and those of x^+ are called *positive faces* of x. We will sometimes refer to x^- and x^+ as *face-sets* of x. Given $S \subseteq C$, let S^- denote the set of elements of C which occur as negative faces of some $x \in S$, and similarly for S^+.

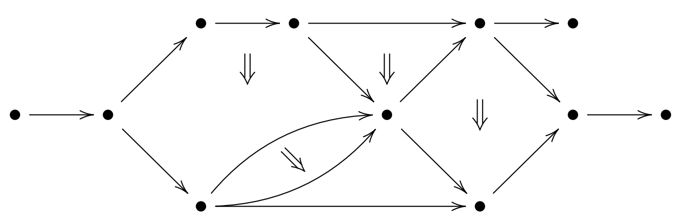
$$S^- = \bigcup_{w \in S} w^- \quad \text{and} \quad S^+ = \bigcup_{w \in S} w^+ \tag{5.3.2}$$

Each subset $S \subseteq C$ is graded via $S_n = S \cap C_n$. The *n-skeleton* of $S \subseteq C$ is defined by

$$S^n := \sum_{k=0}^{n} S_k \ . \tag{5.3.3}$$

Call S *n-dimensional* when it is equal to its n-skeleton.

The broad intuition is to see this structure as a generalisation of directed graph. Elements of C_0 are vertices, elements of C_1 are directed edges, elements of C_2 are directed 'faces', elements of C_3 are directed 'volumes', and so on. The usual notion of source and target are replaced by face-sets x^- and x^+. The following is a basic example of this structure of dimension two.



(5.3.4)

Notice that elements above dimension 1 can have more than one source-face or target-face.

Without the axioms below, this structure is very general indeed and many unusual examples can be provided. When the axioms are applied, possible examples become much better behaved. Examples of arbitrary dimension can be constructed from simplexes, cubes, and other kinds of polytopes as seen below. Of course, the simplexes provide the main motivation for understanding these kinds of structures.

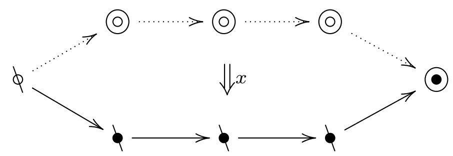
So far we have described the data of a parity complex: a graded set with a pair of face-set maps $(-)^-, (-)^+ \colon C_{n+1} \to \mathcal{P}(C_n)$. We now describe the required axioms.

Axiom 1. For all $x \in C$,

$$x^{++} \cup x^{--} = x^{-+} \cup x^{+-}$$

where $x^{++} = (x^+)^+$ etc.

This is a kind of globularity condition that ensures various face-sets are appropriately related. The following diagram is an example where $x \in C_2$ and both x^- and x^+ have four elements.

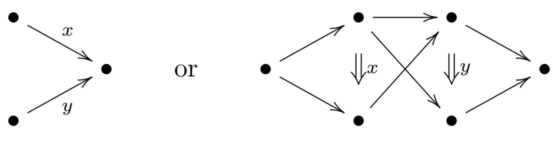


(5.3.5)

Edges marked with a dotted line belong to x^-, the other edges belong to x^+. Vertices marked with a $\bullet$ belong to x^{++}, those marked with a $\circ$ belong to x^{--}, those marked with a $\bigcirc$ belong to x^{-+}, and those marked with a $\backslash$ belong to x^{+-}. In particular, this axiom implies that $x^{++} \subseteq x^{-+} \cup x^{+-}$, that is, positive faces of positive faces must be the negative face of a positive face, or the positive face of a negative face.

Notice that in both (5.3.4) and (5.3.5) the set of source (target) faces have all elements aligned in a common direction and they do not branch apart. This behaviour is guaranteed by introducing Axiom 2 below.

Suppose that S and T are subsets of C. We write $S \perp T$ when $S^- \cap T^- = S^+ \cap T^+ = \emptyset$. This extends to elements by $x \perp y$ when $x^- \cap y^- = x^+ \cap y^+ = \emptyset$ *. A subset $S \subseteq C$ is called *well-formed* when S_0 has at most one element, and, for all $x, y \in S_n$ $(n > 0)$, if $x \neq y$ then $x \perp y$. Broadly speaking, a set is well-formed when it doesn't contain any branchings like



, (5.3.6)

and it contains at most one element of dimension 0. In each of the diagrams above we can observe that $\{x, y\}$ is not well-formed, while $\{x\}$, $\{y\}$, x^+, x^-, y^+, and y^- are all

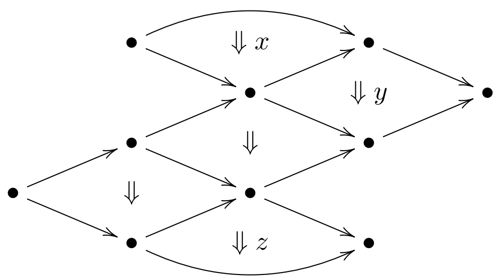
*This could equivalently be defined for elements first, and then extended to sets afterwards.

well-formed. The diagram depicts branchings in dimensions 1 and 2, but well-formedness prevents branching in all dimensions. The condition on dimension zero does not force parity complexes to have a single element of dimension zero, but that (using the axiom below) elements of dimension 1 have a single source vertex and a single target vertex.

Axiom 2. For all $x \in C$, x^- and x^+ are well-formed.

If we think of the union $x^- \cup x^+$ as forming a boundary of x, as in (5.3.5) above, then this axiom ensures that the boundary looks something like the boundary of a polytope. For those familiar with higher categories, this condition ensures that the face-sets look like valid pasting diagrams.

Suppose that $x, y \in C$. We write $x < y$ whenever $x^+ \cap y^-$ is non-empty. That is, when x and y abut by having a common element in their respective sets of positive and negative faces. This implies $x \neq y$ since x^- and x^+ are always disjoint. We then let $\lhd$ be the reflexive transitive closure of $<$. An example is



(5.3.7)

where $x < y$ and $y \lhd z$. In this case we often say that there is a path from x to z. For all $S \subseteq C$ we let $\lhd_S$ denote the reflexive transitive closure of $<$ restricted to S. When $x \lhd_S z$ we often say there is a path from x to z in S.

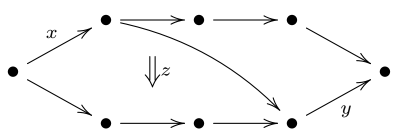
While Axioms 1 and 2 can be seen as imposing some of the basic structural behaviour of graphs, the following axiom restricts us to certain 'loop-free' graphs.

Axiom 3. For all $x, y \in C$,

A. $x \lhd y \lhd x$ implies $x = y$.

B. if $x \lhd y$ then $\forall z \in C$, $\neg(x \in z^+ \wedge y \in z^-)$ and $\neg(y \in z^+ \wedge x \in z^-)$.

Axiom 3.A says that $\lhd$ is anti-symmetric, or, that there are no paths that loop within a fixed dimension. Axiom 3.B says that there are no paths that cross between the face-sets of any element z. That is, we avoid circumstances where a path can cross from one face-set to the other face-set of an element z as in the diagram below.

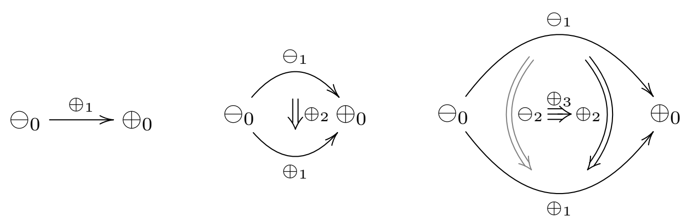


(5.3.8)

These are all the axioms for a parity complex. The following examples come from p.318–319 of [Str91].

Example 1. A 1-dimensional parity complex is precisely a directed graph with no circuits.

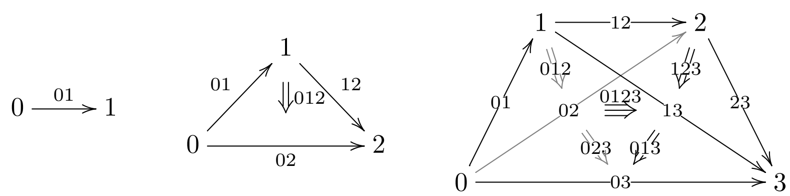
Example 2. The *ω-glob* is the parity complex $\mathcal{G}$ defined by $\mathcal{G}_n = \{(\epsilon, n) \colon \epsilon = \ominus \text{ or } \oplus\}$, and $(\epsilon, n+1)^- = \{(\ominus, n)\}$ and $(\epsilon, n+1)^+ = \{(\oplus, n)\}$. Elements of dimension 0, 1, and 2 are 'n-discs'. There are precisely two elements at each dimension, each of which has exactly one source face and exactly one target face.



(5.3.9)

We use $\ominus_n$ and $\oplus_n$ as short-hand for $(\ominus, n)$ and $(\oplus, n)$.

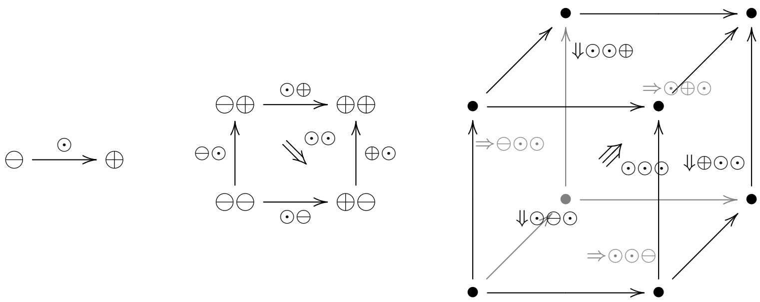
Example 3. The *ω-simplex* is the parity complex Δ described as follows. Let Δ_n denote the set of $(n+1)$-element subsets of the set of natural numbers $N = \{0, 1, 2, \dots\}$. Each $x \in \Delta_n$ is written as $(x_0, x_1, \dots, x_n)$ where $x_0 < x_1 < \dots < x_n$. Let $x\delta_i$ denote the set obtained from x by deleting x_i. Take x^- to be $\{x\delta_i : i \text{ odd}\}$ and x^+ to be $\{x\delta_i : i \text{ even}\}$. Elements of dimension 0, 1, and 2 are 'n-simplexes'.



(5.3.10)

We use $abcd$ as short-hand for (a, b, c, d) and similarly at other dimensions.

Example 4. The *ω-cube* is the parity complex $\mathcal{Q}$ described as follows. The elements are infinite sequences of the three symbols $\ominus, \odot, \oplus$ containing a finite number of $\odot$'s. The dimension of an element is the number of $\odot$'s appearing in it. Let $x\delta_i^-$ denote the sequence obtained from x by replacing the i-th $\odot$ by $\ominus$ when i is odd and by $\oplus$ when i is even. Similarly, $x\delta_i^-$ is defined by interchanging $\ominus$ and $\oplus$ in the previous sentence. For $x \in \mathcal{Q}_n$, define $x^\epsilon = \{x\delta_i^\epsilon : 1 < i < n\}$. The *$n$-cube* is the parity complex built the same way but using only lists of length n. The n-cubes of dimension 1, 2, and 3 are displayed below.



(5.3.11)

Some labels have been omitted from the last diagram in order to keep it readable.

Before continuing our exposition of Section 1 of [Str91], we will comment briefly on our implementation.

Implementation 5.3.1. The basic data for a parity complex without the axioms is sometimes called a *pre-parity complex*. We chose to implement this concept first, as there are many trivial results about preparity complexes that we will later use. A preparity complex is implemented as the following data:

```
C : Type
dim : C → nat
plus : C → Ensemble C
minus : C → Ensemble C
```

This data is technically different from our description above, but the essential structure is identical. There is a collection of objects C, each member of which has a dimension and two face-sets†. A few axioms are introduced to ensure that face-sets are finite, non-empty, and disjoint, and that they interact with dimension correctly.

```
forall (x y : C), x ∈ (plus y)  → dim y = dim x + 1
forall (x y : C), x ∈ (minus y) → dim y = dim x + 1
forall (x : C),   Finite (plus x)
forall (x : C),   Finite (minus x)
forall (x : C),   dim x > 0 → Inhabited (plus x)
forall (x : C),   dim x > 0 → Inhabited (minus x)
forall (x : C),   dim x = 0 →  plus x == Empty_set
forall (x : C),   dim x = 0 → minus x == Empty_set
forall (x : C),   Disjoint (plus x) (minus x)
```

These are given meaningful names such as `plus_Finite`, `plus_dim`, and `plus_Inhabited`. Fundamental definitions for sets such as S_n and S^n are also given and some trivial statements are also proved here. For example,

†In the actual code the type `C` is called `carrier`.

```
Definition sub (R : Ensemble C) (n : nat) : Ensemble C
 := fun (x : C) => (x ∈ R ∧ (dim x) = n).

Lemma sub_Union :
 forall T R n,
  sub (T ∪ R) n == (sub T n) ∪ (sub R n).
```

More complicated definitions like well-formedness are also given and more powerful (though almost trivial) results are also proved here. For example,

```
Definition well_formed (X : Ensemble C) : Prop :=
 (forall (x y : C), x ∈ X ∧ y ∈ X
   → dim x = 0 → dim y = 0
   → x = y)
 ∧
 (forall (x y : C), x ∈ X ∧ y ∈ X
    → (forall (n : nat), dim x = S n → dim y = S n
    → ¬ (perp x y) → x = y)).

Lemma well_formed_by_dimension :
 forall X,
  well_formed X <-> forall n, well_formed (sub X n).
```

All other basic definitions and trivial results are encoded in a similar fashion.

We now look at some basic properties of parity complexes.

Given $S \subseteq C$, let $S^{\mp}$ denote the set of negative faces of elements of S which are not positive faces of any element of S, and similarly for $S^{\pm}$. So

$$S^{\mp} = S^{-} \setminus S^{+} \quad \text{and} \quad S^{\pm} = S^{+} \setminus S^{-} \quad .$$

This extends to individual elements by $x^{\pm} := \{x\}^{\pm}$ and $x^{\mp} := \{x\}^{\mp}$. These sets capture the notion of *purely positive* and *purely negative* faces of an element x or set S.

The following propositions follow from Axioms 1, 2 and 3.

Proposition 2 (Proposition 1.1)**.** *For all $x \in C$,*

$$x^{++} \cap x^{--} = x^{-+} \cap x^{+-} = \emptyset \tag{5.3.12}$$

$$x^{-\mp} = x^{+\mp} = x^{--} \cap x^{+-} \tag{5.3.13}$$

$$x^{-\pm} = x^{+\pm} = x^{-+} \cap x^{++} \ . \tag{5.3.14}$$

Theorem 2 contains identities that one would expect from a polytope-like structure and are much like Axiom 1. The meaning is reasonably clear when the various face-sets are highlighted in an example like (5.3.5) above.

Proposition 3 (Proposition 1.2)**.** *For all $u, v, x \in C$, $u \lhd v$ and $v \in x^+$ imply*

$$u^- \cap x^{-+} = \emptyset \ . \tag{5.3.15}$$

Theorem 3 indicates that if u branches out from the source of x then a path from u to v can not end in the target of x. This is a consequence of Axiom 3.B. This has three duals obtained by reversing the roles of u and v and reversing the roles of x^- and x^+. Theorem 3 and its duals are together equivalent to Axiom 3.B.

The following observation describes a convenient technical property of well-formed sets.

Observation 4 (page 322 in [Str91])**.** *For all $T, Z \subseteq C$, if $T \cup Z$ is well-formed and $T \cap Z = \emptyset$, then $T \perp Z$.*

We say a set $R \subseteq C$ is *tight* when, for all $u, v \in C$, $u \lhd v$ and $v \in R$ implies $u^- \cap R^{\pm}$ is empty. This condition prevents a path from starting in $R^{\pm}$ and ending in R. The following two results are required for somewhat technical reasons.

Definition 5. *Suppose that $R, T \subseteq C$. We say that R* is a segment of *T when for all $x, y, z \in T$, $x \lhd y \lhd z$ and $x, z \in R$ implies $y \in R$.*

Proposition 6 (Proposition 1.4)**.** *For all $R, S \subseteq C$, if R is tight, S is well-formed, and $R \subseteq S$, then R is a segment of S.*

Observation 7 (page 359 in [Str94])**.** *For all $x \in C$, x^+ and x^- are tight.*

This concludes our exposition of Section 1.

Remark 5.3.2. The notion of tightness was introduced in the Corrigenda [Str94]. It appears to be entirely necessary, but we do not understand the full significance of the concept (see our discussion on page 156).

Implementation 5.3.3. Each axiom and proposition is readily encoded, for example

```
Axiom axiom1 :
 forall (x : C),
  (Plus ( plus x)) ∪ (Minus (minus x)) ==
  (Plus (minus x)) ∪ (Minus ( plus x)).

Lemma Prop_1_2 :
 forall u v x,
  triangle u v →
   v ∈ (plus x) →
    (minus u) ∩ (Plus (minus x)) == Empty_set.
```

We were able to prove each result from basic definitions and axioms. This is exactly as described in the original work. The proof of Theorem 6 makes use of Propositions 2 and 3.

When we look ahead we find that Axioms 1 and 2 are used frequently throughout the material. Axiom 3.A is only used to prove that $\lhd_S$ is decidable and that finite non-empty sets X have minimal and maximal elements under $\lhd_X$. Axiom 3.B is used only to prove Propositions 2 and 3.

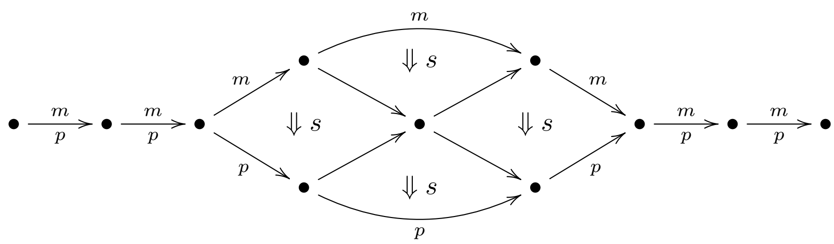
5.4 Movement

In Section 2 of [Str91] the concept of *movement* is introduced. It is a concept that is fundamental to describing cells in n-categories generated from parity complexes.

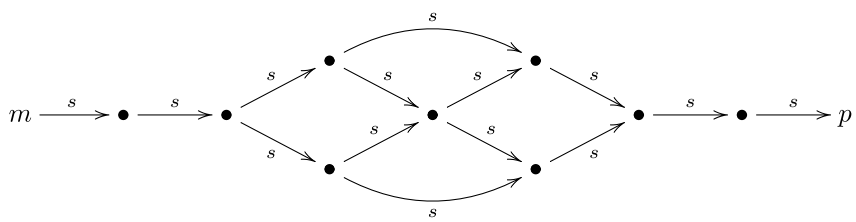
For three sets $S, M, P \subseteq C$, we say that S *moves* M *to* P, or $M \xrightarrow{S} P$, when

$$M = (P \cup S^-) \setminus S^+ \quad \text{and} \quad P = (M \cup S^+) \setminus S^- \ . \tag{5.4.1}$$

Here are some examples of movement at dimensions 2 and 1:



(5.4.2)



(5.4.3)

where lowercase labels m, p, s indicate which set each component belongs to (unlabelled elements do not belong to M, P, or S). This condition guarantees that the face-sets of S, M and P are related in the basic way we would expect of pasting diagrams in n-categories. The movement condition is intended to describe the basic combinatorial shape of cells in our yet-to-be-defined ω-category. When those cells are defined we will need to add basic finiteness and well-formedness conditions to ensure that various pathological examples of movement are excluded.

It is helpful to recognise that movement is a condition that applies dimension-by-dimension, that is, $M \xrightarrow{S} P$ if and only if $M_n \xrightarrow{S_{n+1}} P_n$ for all n. This not only aids in various proofs, but it indicates there is nothing complicated happening across dimensions.

Proposition 8 (Proposition 2.1)**.** *For all $S, M \subseteq C$, there exists $P \subseteq C$ with $M \xrightarrow{S} P$ if and only if*

$$S^{\mp} \subseteq M \quad \textit{and} \quad M \cap S^+ = \emptyset \ . \tag{5.4.4}$$

Theorem 8 illuminates a fundamental meaning of movement: that M contains the purely negative faces of S and none of the positive faces. This is illustrated below where elements of $S_2^{\mp}$ are indicated by squiggly arrows and those of S_2^{+} are indicated by dashed arrows.

$$\begin{tikzcd}[column sep=small]
 & & & \bullet \arrow[rr, squiggly, bend left, "m"] \arrow[rr, phantom, "\Downarrow s"] \arrow[dr, dashed] & & \bullet \arrow[dr, squiggly, "m"] & & & \\
\bullet \arrow[r, "m", "p"'] & \bullet \arrow[r, "m", "p"'] & \bullet \arrow[ur, squiggly, "m"] \arrow[dr, dashed, "p"'] \arrow[rr, phantom, "\Downarrow s"] & & \bullet \arrow[ur, dashed] \arrow[dr, dashed] \arrow[rr, phantom, "\Downarrow s"] & & \bullet \arrow[r, "m", "p"'] & \bullet \arrow[r, "m", "p"'] & \bullet \\
 & & & \bullet \arrow[ur, dashed] \arrow[rr, dashed, bend right, "p"'] \arrow[rr, phantom, "\Downarrow s"] & & \bullet \arrow[ur, dashed, "p"'] & & &
\end{tikzcd} \tag{5.4.5}$$

Observe that $S_2^{\mp} \subseteq M_1$ and $M_1 \cap S_2^{+} = \emptyset$ as indicated by the proposition. Theorem 8 has a dual where M and P play opposite roles.

Proposition 9 (Proposition 2.2)**.** *Suppose $S, M, P, X, Y \subseteq C$, $M \xrightarrow{S} P$ and $X \subseteq M$ has $S^{\mp} \cap X = \emptyset$. If $Y \cap S^{+} = \emptyset$, and $Y \cap S^{-} = \emptyset$, then $(M \cup Y) \cap \neg X \xrightarrow{S} (P \cup Y) \cap \neg X$.*

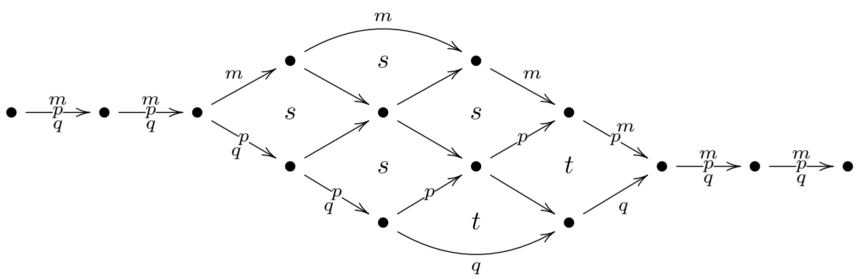
Theorem 9 indicates that some elements of M and P can be added or removed without disturbing the movement condition. The conditions on X and Y indicate that they are disjoint from the faces of S in a suitable way. Sets X and Y should be thought of as sets that are added to or removed from the movement as below.

$$\begin{tikzcd}[column sep=small]
 & & & \bullet \arrow[rr, bend left, "m"] \arrow[rr, phantom, "s"] \arrow[dr] & & \bullet \arrow[dr, "m"] & & & \\
\bullet \arrow[r, dotted, "x"] & \bullet \arrow[r, dotted, "x"] & \bullet \arrow[ur, "m"] \arrow[dr, "p"'] \arrow[rr, phantom, "s"] & & \bullet \arrow[ur] \arrow[dr] \arrow[rr, phantom, "s"] & & \bullet \arrow[r, squiggly, "y"] & \bullet \arrow[r, squiggly, "y"] & \bullet \\
 & & & \bullet \arrow[ur] \arrow[rr, bend right, "p"'] \arrow[rr, phantom, "s"] & & \bullet \arrow[ur, "p"'] & & &
\end{tikzcd} \tag{5.4.6}$$

Proposition 10 (Proposition 2.3)**.** *Suppose $M, P, Q, S, T \subseteq C$ where $M \xrightarrow{S} P$ and $P \xrightarrow{T} Q$. If $S^{-} \cap T^{+} = \emptyset$ then $M \xrightarrow{S \cup T} Q$.*

Theorem 10 describes the condition under which movements can be 'composed' or 'pasted' together. The following diagram depicts an example. Elements of sets M, S, P, T, Q

are labelled with the corresponding lower-case letters.



(5.4.7)

Proposition 11 (Proposition 2.4)**.** *Suppose* $M \xrightarrow{T \cup Z} P$ *with* $Z^{\pm} \subseteq P$*. If* $T \perp Z$ *then there exists* N *such that* $M \xrightarrow{T} N \xrightarrow{Z} P$*.*

Theorem 11 describes a condition under which movement can be decomposed. In particular, if $T \cup Z$ is well-formed then $T \perp Z$ as required in the proposition.

Implementation 5.4.1. The definition of movement and the propositions above are readily encoded. For example:

```
Definition moves_def (S M P : Ensemble C) : Prop :=
  P == ((M ∪ ( Plus S)) ∩ (Complement (Minus S)))
  ∧
  M == ((P ∪ (Minus S)) ∩ (Complement ( Plus S))).

Notation "S 'moves' M 'to' P" := (moves_def S M P) (at level 89).

Lemma Prop_2_3 : forall (S M P T Q : Ensemble C),
  S moves M to P →
  T moves P to Q →
  (Disjoint (Minus S) (Plus T)) →
    (S ∪ T) moves M to Q.
```

The `Notation` command in Coq allows us to use the statement `S moves M to P` in place of the somewhat awkward `moves_def S M P`.

It did not take long to verify that the proofs in this section proceed precisely as indicated in the original text.

Theorem 8 is proved by appealing to definitions and basic manipulation of sets. Theorems 9 to 11 are proved using Theorem 8 and basic manipulation of sets. Theorems 8 and 11 have duals that are not displayed here but are required later; they are implemented separately in our code. It is worth noting that none of these results require Axioms 1, 2 or 3. In our implementation, we prove these results before the axioms are even introduced.

This concludes our exposition of Section 2 of [Str91].

5.5 The ω-category of a parity complex

Having described the basic properties of parity complexes and the more advanced notion of movement, in Section 3 we describe the cells of an ω-category $\mathcal{O}(C)$ associated with any parity complex C.

Definition 12. *A* cell *of a parity complex C is a pair (M, P) of non-empty, well-formed, finite, subsets of C with the property that M and P both move M to P.*

If this is interpreted dimension by dimension, we get the following picture at dimension 2,

$$\begin{tikzcd}
 & & & \bullet \arrow[rr, bend left, "m"] \arrow[dr] \arrow[rr, phantom, "m \Downarrow p"] & & \bullet \arrow[dr, "m"] & & & \\
m \arrow[r, "m", "p"'] & \bullet \arrow[r, "m", "p"'] & \bullet \arrow[ur, "m"] \arrow[dr, "p"'] \arrow[rr, phantom, "m \Downarrow p"] & & \bullet \arrow[ur] \arrow[dr] \arrow[rr, phantom, "m \Downarrow p"] & & \bullet \arrow[r, "m", "p"'] & \bullet \arrow[r, "m", "p"'] & p \\
 & & & \bullet \arrow[ur] \arrow[rr, bend right, "p"'] \arrow[rr, phantom, "m \Downarrow p"] & & \bullet \arrow[ur, "p"'] & & &
\end{tikzcd} \tag{5.5.1}$$

where lowercase labels m, p, s indicate which set the elements belong to. Notice that M_1 and P_1 are neither equal nor disjoint, but each move M_0 to P_0. Notice also that $M_2 = P_2$. This kind of behaviour is uniform through all dimensions. Notice also that, aside from the movement condition, we only require that M and P be non-empty, well-formed and finite. Call (M, P) an *n-cell* when $M \cup P$ is n-dimensional. In this case we have $M_n = P_n$ as above.

Definition 13. *The n-*source *and n-*target *of a pair of sets (M, P) are defined by*

$$s_n(M, P) = (M^{n-1} \cup M_n, P^{n-1} \cup M_n) \tag{5.5.2}$$

and

$$t_n(M, P) = (M^{n-1} \cup P_n, P^{n-1} \cup P_n) \ . \tag{5.5.3}$$

If (M, P) is a cell we can show that $s_n(M, P)$ and $t_n(M, P)$ are also cells, and that they are n-dimensional. Notice that (M, P), $s_n(M, P)$ and $t_n(M, P)$ contain exactly the same elements in dimension $n - 1$ and below. We encourage the reader to consider the 1-source and 1-target of the cell depicted in (5.5.1).

Definition 14. *A pair of cells (M, P), (N, Q) are n-*composable *when*

$$t_n(M, P) = s_n(N, Q) \ , \tag{5.5.4}$$

*in which case their n-*composite *is*

$$(N, Q) *_n (M, P) := (M \cup (N \cap \neg N_n), \ (P \cap \neg P_n) \cup Q) \ . \tag{5.5.5}$$

Notice that (5.5.4) implies that the two cells agree from dimensions 0 to $n-1$ and that $P_n = N_n$ at dimension n. The resulting composite is almost exactly the pair-wise union of (M,P) and (N,Q); the set-difference ensures correct behaviour at dimension n. It is not surprising that some form of set-difference is required since most forms of composition will forget the point of contact: $A \to B \to C$ composes to $A \to C$.

For any parity complex C, let $\mathcal{O}(C)$ be the set of cells of C. We will see later (Theorem 19) that $\mathcal{O}(C)$ is an ω-category. Before this can be achieved, we need to establish some basic properties of cells.

Definition 15. *A set $S \subseteq C$ is* receptive *when for all $x \in C$,*

$$\textit{if} \quad x^{-+} \cap x^{++} \subseteq S \quad \textit{and} \quad S \cap x^{--} = \emptyset \quad \textit{then} \quad S \cap x^{+-} = \emptyset$$

and

$$\textit{if} \quad x^{+-} \cap x^{--} \subseteq S \quad \textit{and} \quad S \cap x^{++} = \emptyset \quad \textit{then} \quad S \cap x^{-+} = \emptyset \ .$$

A cell is receptive when it is receptive at every dimension.

Remark 5.5.1. The notion of receptivity is somehow important, we find later that all cells are receptive and it is a necessary condition for some central results. It appears to be entirely necessary, but we do not have an intuitive understanding of its meaning (see our discussion on page 156).

Lemma 16 (Lemma 3.1)**.** *For all $M, P \subseteq C$, $x \in C$, if $M \xrightarrow{x^+} P$ and M is receptive then $M \xrightarrow{x^-} P$.*

Theorem 16 is proved using definitions, basic manipulation of sets and Theorems 2 and 8. It has a dual which we implement in our code. We will find later that since all cells are receptive, it is not hard to find receptive subsets M of C. In fact, it is a bit difficult to illustrate why receptivity is even required because the most obvious examples of x, M, P satisfying the movement condition above are also part of a cell structure.

Lemma 17 (Lemma 3.2)**.** *Suppose $m, n \in \mathbb{N}$, all cells are receptive and (M,P) is an n-cell. Suppose also that $X \subseteq C_{n+1}$, $|X| = m$ and X is well-formed with $X^{\pm} \subseteq M_n$. Put $Y = (M_n \cup X^-) \cap \neg X^+$, then:*

B. $(M^n \cup Y, P^n \cup Y)$ *is a cell and and* $X^- \cap M_n = \emptyset$*.*

C. $(M^n \cup Y \cup X, P \cup X)$ *is a cell.*

Theorem 17 originally contained a part A which was removed in [Str94]. Theorem 17.C indicates that, if X is a well-formed set of dimension $n+1$, (M,P) is an n-cell, and X abuts (M,P) in the sense that $X^{\pm} \subseteq M_n$, then we can form an $(n+1)$-cell whose top-dimension elements are those of X and whose target is (M, P). The source of this cell has Y at its top dimension. The following diagram is labelled to illustrate this scenario.

$$\begin{tikzcd}
& & & \bullet \arrow[rr, bend left, "y"] \arrow[rr, phantom, "\Downarrow x"] \arrow[dr] & & \bullet \arrow[dr, "y"] & & & \\
\bullet \arrow[r, "my", "p"'] & \bullet \arrow[r, "my", "p"'] & \bullet \arrow[ur, "y"] \arrow[dr, "m", "p"'] \arrow[rr, phantom, "\Downarrow x"] & & \bullet \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow x"] & & \bullet \arrow[r, "my", "p"'] & \bullet \arrow[r, "my", "p"'] & \bullet \\
& & & \bullet \arrow[ur] \arrow[rr, bend right, "m", "p"'] \arrow[rr, phantom, "\Downarrow x"] & & \bullet \arrow[ur, "m", "p"'] & & &
\end{tikzcd} \tag{5.5.6}$$

There is a dual lemma obtained by reversing the direction of X in the diagram above.

This kind of result does not seem unusual, but it is surprisingly hard to prove (see Remark 5.5.3 below). The proof itself is done in three steps. To quickly summarise:

1. Theorem 17.B implies Theorem 17.C. The proof is somewhat direct and proceeds as indicated in the original paper.

2. Theorem 17.B with $m = 1$ implies Theorem 17.B in general. This is done by induction on m and follows from basic definitions and axioms.

3. Theorem 17.B holds for $m = 1$. This is done by induction on n and the argument relies on Theorem 18. The construction works as indicated, though it is not a short argument. There are particular disjointness conditions that must be established (p327 of [Str91]) and require their own special argument.

Proposition 18 (Proposition 3.3)**.** *For all $n \in \mathbb{N}$, all n-cells in C are receptive.*

This is a somewhat technical result, it is not immediately clear to us how the notion of receptivity fits naturally into the combinatorics. The proof of this result relies on Theorem 17.B.

Theorem 19 (Theorem 3.6)**.** *If C is any parity complex then $\mathcal{O}(C)$ is an ω-category. Furthermore, if (M, P), (N, Q) are n-composable cells*[‡] *then $(M_k \cup P_k)^- \cap (N_k \cup Q_k)^+ = \emptyset$ for all $k > n$.*

Theorem 19 is a central result in [Str91] since it achieves one of the main goals of the paper. In order to implement Theorem 19 we would need to implement a notion of ω-category which is not trivial. Since there is little question that this result holds, and it is not required to prove Theorem 22, we have chosen not to implement it. We similarly omit Propositions 3.4 and 3.5 which are preliminary results leading up to Theorem 19.

Remark 5.5.2. Perceptive readers will have noticed that Theorem 17.B and Theorem 18 seem to logically rely on one another. At first glance this appears to be a circular argument

[‡] $t_n(M, P) = s_n(N, Q)$

and therefore unsound. However, if we look closely we can see that each result proceeds by induction and that the two proofs can be woven together to produce a proof of both results simultaneously. Theorem 18 is restated as: *for all n, every n-dimensional cell (M, P) is receptive.* The two results are proved by mutual induction on n, the dimension of (M, P). Included in that argument is an induction on $m = |X|$. The following statements hold and are enough to show that both results hold for all n and m.

i. Theorem 17.B holds when $m = 1$ and $n = 0$.

ii. For a fixed n, if Theorem 17.B holds when $m = 1$ then it holds when $m > 1$ (by induction on m).

iii. Theorem 18 holds when $n = 0$.

iv. If Theorem 17.B and Theorem 18 hold for $n = k$, then Theorem 17.B holds for $n = k + 1$ and $m = 1$.

v. If Theorem 17.B holds for $n = k + 1$ and Theorem 18 holds for $n = k$, then Theorem 18 holds for $n = k + 1$.

This understanding is not explicit in [Str91].

Implementation 5.5.3. As in earlier sections, the definitions and statement of results are readily encoded. The main difficulty arises in encoding the proofs.

The proofs of Theorem 17 and Theorem 18 are by far the most difficult part of the entire project and consumed most of our programming effort. Consider the components of the proof given above. Each of the components follow the argument provided by Street in his paper. However the disjointness condition in iv has a dual, and i, ii, iv each have duals. Finally, we needed to uncover the logical dependence that allows us to weave these things together to produce a non-cyclic argument.

It is worth noting that the original proof of Theorem 18 uses an argument about skeletons of parity complexes (treating separate parity complexes as objects of the argument). We have translated the argument so that it is internal to any given parity complex. The combinatorial logic of our argument is exactly the same as Street's, we have only adjusted the setting slightly.

Having built the ω-category $\mathcal{O}(C)$ from a parity complex C, we now prove that it is generated from atoms.

In any parity complex C we expect that any individual element x of dimension p is the top element of some cell whose lower-dimensional structure can be computed by examining the face-sets of x and recursively taking face-sets of face-sets. This is made explicit in the following definition.

Definition 20. *For each $x \in C_p$, two subsets $\mu(x), \pi(x) \subseteq C^p$ are defined inductively as follows*

$$\mu(x)_p = \{x\} \quad \text{and} \quad \mu(x)_{k-1} = \mu(x)_k^{\mp}, \quad 1 \le k \le p$$

$$\pi(x)_p = \{x\} \quad \text{and} \quad \pi(x)_{k-1} = \pi(x)_k^{\pm}, \quad 1 \le k \le p$$

The pair $(\mu(x), \pi(x))$ is denoted by $\langle x \rangle$.

Take the following diagram for example. If $x \in C$ of dimension 2 and has boundary as illustrated here then $\langle x \rangle = (\{x, p, q, r, a\}, \{x, s, t, e\})$

$$\begin{tikzcd}
& c \arrow[r, "q"] \arrow[d, phantom, "\Downarrow x"] & b \arrow[dr, "r"] & \\
a \arrow[ur, "p"] \arrow[r, "s"'] & d \arrow[rr, "t"'] & & e
\end{tikzcd} \tag{5.5.7}$$

A priori, we have no guarantee that such a pair is actually a cell.

Definition 21. *An element $x \in C_p$ is called* relevant *when $\langle x \rangle$ is a cell. This amounts to saying that $\mu(x)_n$ and $\pi(x)_n$ are well-formed for $0 \le n < p-1$, and*

$$\mu(x)_{n-1} = \pi(x)_n^{\mp}, \qquad \pi(x)_{n-1} = \mu(x)_n^{\pm}$$

for $0 < n < p-1$. Call a cell (M, P) an atom *when it is equal to $\langle x \rangle$ for some $x \in C$. In that case we say that (M, P) is* atomic.

In all of our main examples, every $\langle x \rangle$ is a cell (all elements are relevant).

Theorem 22 (Theorem 4.1 : excision of extremals)**.** *Suppose that $\mu(x)$ is tight for all $x \in C$. Suppose (M, P) is an n-cell and $u \in M_n$ $(= P_n)$ is such that $(M, P) \neq \langle u \rangle$* [§]*. Then (M, P) can be decomposed as*

$$(M, P) = (N, Q) *_m (L, R) \tag{5.5.8}$$

where $m < n$, and (N, Q) and (L, R) are n-cells of dimension greater than m.

This is another central result of the paper. If this algorithm is applied recursively then it shows how to present an arbitrary n-cell as a composite of atoms. Thus $\mathcal{O}(C)$ is not only an ω-category, but it is generated from its atoms

The algorithm takes an n-cell (M, P) and runs as follows.

1. Find the largest $m < n$ with $(M_{m+1}, P_{m+1}) \neq (\mu(u)_{m+1}, \pi(u)_{m+1})$. This amounts to discovering the highest dimension at which the criterion for being atomic does not hold[¶]. In this case, there exists $w \in M_{m+1} \cap P_{m+1}$.

[§] Alternatively, let (M, P) be a non-atomic n-cell.

[¶] Alternatively, find the largest $m < n$ with $M_{m+1} \cap P_{m+1} \neq \emptyset$.

2. We want to decompose our cell by pulling off a cell of dimension $m+1$. Let x be a minimal element of M_{m+1} less than w, and let y be a maximal element of M_{m+1} greater than w.

3. At least one of x or y must belong to $M_{m+1} \cap P_{m+1}$. This relies on the fact that $\mu(u)_{m+1}$ is a segment of M_{m+1}, which itself relies on $\mu(u)_{m+1}$ being tight.

4. If $x \in M_{m+1} \cap P_{m+1}$ then we get a decomposition of (M, P) as

$$N = M^m \cup \{x\} \qquad Q = P^{m-1} \cup ((M_m \cup x^+) \cap \neg x^-) \cup \{x\} \tag{5.5.9}$$

$$L = ((M \cap \neg\{x\}) \cup x^+) \cap \neg x^- \qquad R = P \cap \neg\{x\} \tag{5.5.10}$$

Notice that (N, Q) is an $(m+1)$-cell whose single element at top dimension is x, and (L, R) is the n-cell obtained by cutting x out of (M, P).

$$\begin{tikzcd}
 & & & \bullet \arrow[rr, bend left, "m"] \arrow[rr, phantom, "m \Downarrow p"] \arrow[dr] & & \bullet \arrow[dr, "m"] & & & \\
\bullet \arrow[r, dashed, "x", "mp"'] & \bullet \arrow[r, "m", "p"'] & \bullet \arrow[ur, "m"] \arrow[dr, "p"'] \arrow[rr, phantom, "m \Downarrow p"] & & \bullet \arrow[ur] \arrow[dr] \arrow[rr, phantom, "m \Downarrow p"] & & \bullet \arrow[r, "m", "p"'] & \bullet \arrow[r, "m", "p"'] & \bullet \\
 & & & \bullet \arrow[ur] \arrow[rr, bend right, "p"'] \arrow[rr, phantom, "m \Downarrow p"] & & \bullet \arrow[ur, "p"'] & & &
\end{tikzcd} \tag{5.5.11}$$

5. If $y \in M_{m+1} \cap P_{m+1}$ then we get a decomposition of (M, P) as

$$N = M \cap \neg\{y\} \qquad Q = ((P \cap \neg\{y\}) \cup y^-) \cap \neg y^+ \tag{5.5.12}$$

$$L = M^{m-1} \cup ((P_m \cup y^-) \cap \neg y^+) \cup \{y\} \qquad R = P^m \cup \{y\} \tag{5.5.13}$$

This is dual to the case for x. Notice that (L, R) is an $(m+1)$-cell whose single element at top dimension is y, and (N, Q) is the cell obtained by cutting y out of (M, P).

The two hardest parts of this algorithm are parts (3) and (4). In part (3) we must show that either x or y belong to $M_{m+1} \cap P_{m+1}$. This relies on the fact that $\mu(x)_{m+1}$ is a segment of M_{m+1}, but this follows from Theorem 11 and the assumption that each $\mu(x)$ is tight. In part (4) we need to show that (N, Q) and (L, R) are well-defined cells. The various conditions of finiteness and well-formedness follow quite directly. The difficulty comes in showing that the movement conditions hold. We investigate the cells dimension by dimension and find that the movement conditions can be proved using Theorems 11 and 17.

How do we know that this algorithm terminates? The original text defines the rank of an n-cell (M, P) to be the cardinality of $M \cup P$. The algorithm produces two cells of smaller rank, so therefore must terminate. It is also possible to define the rank by

$$\operatorname{rank}(M, P) = \sum_{k=0}^{n} |M_k \cap P_k| \tag{5.5.14}$$

In this case every n-cell has a rank of at least 1 since $M_n \cap P_n$ is non-empty. A cell of rank 1 must be atomic. A cell of rank $k > 1$ can be decomposed using excision of extremals into two cells whose individual ranks are less than or equal to $k - 1$. Again, this is sufficient to guarantee termination.

Implementation 5.5.4. As already indicated, Theorem 22 is readily proved using the argument given above.

Remark 5.5.5. In order to show that $\mathcal{O}(C)$ is *freely* generated from its atoms we must show that there are no equalities among composites of cells that are not a consequence of the ω-category axioms. This is achieved in Street's Theorem 4.2 but has not been reproduced here and we have not included it in our formalisation.

Remark 5.5.6. Many of these theorems and lemmas come with a condition concerning tightness and receptivity of various sets. We can show that these conditions are satisfied by appealing to various other results. At the end of the day there may be some confusion about which conditions are ultimately required. To summarise, if a parity complex C has the property that $\mu(x)$ is tight for every $x \in C$, then all of the theorems up to this point will hold.

At this stage, we have not shown that every $\langle x \rangle$ is a cell. In fact, we have no guarantee that any cells exist at all. This is something of a loose end, it is accounted for in the following section.

Section 5 begins by describing, for any two parity complexes C and D, their *product* $C \times D$ and their *join* $C \bullet D$. That section also describes two kinds of duals for parity complexes obtained by reversing the roles of $(-)^+$ and $(-)^-$ in all dimensions or in odd dimensions only. This is of particular interest since the diagrams involved in descent are products of globes with simplexes; this is explored in Section 6.

Section 5 also addresses some issues that are as yet unresolved. First, we don't know that any elements are relevant (consequently we don't know if any cells exist at all). Second, Theorem 17 relies on the fact that all $\mu(x)$ are tight, and this was never established.

Consider the following stronger forms of Axioms 1 and 2.

For all x,

$(R1)$ $\quad \mu(x)^- \cup \pi(x)^+ = \mu(x)^+ \cup \pi(x)^-$ and

$\quad \mu(x)^- \cap \pi(x)^+ = \mu(x)^+ \cap \pi(x)^- = \emptyset$

$(R2)$ $\quad \mu(x)$ and $\pi(x)$ are well formed.

These axioms hold for Δ, $\mathcal{G}$, and $\mathcal{Q}$.

Remark 5.5.7. If a parity complex C satisfies these axioms then every $\langle x \rangle$ is a cell (every x is relevant). Thus all elements of Δ, $\mathcal{G}$, and $\mathcal{Q}$ are relevant.

In a parity complex C, write $x \prec y$ when either $y \in x^+$ or $x \in y^-$. Let $\blacktriangleleft$ denote the reflexive transitive closure of the relation $\prec$. Notice that $x < y$ means there exists $z \in x^+ \cap y^-$, so this implies $x \prec y$. Hence, $x \lhd y$ implies $x \blacktriangleleft y$. The relation $\lhd$ compares elements of the same dimension, whereas $\blacktriangleleft$ compares elements of all dimensions. We introduce this as an optional axiom.

$$(AS) \qquad \blacktriangleleft \text{ is anti-symmetric.} \tag{5.5.15}$$

This axiom holds in Δ, $\mathcal{G}$, and $\mathcal{Q}$ where $\blacktriangleleft$ is also total.

Proposition 23 (Proposition 5.2)**.** *If each x is relevant and (AS) holds then each $\mu(x)$ is tight. Thus, every $\mu(x)$ in Δ, $\mathcal{G}$, and $\mathcal{Q}$ are tight.*

Section 5 of [Str91] contains two examples of parity complexes where $\blacktriangleleft$ is not anti-symmetric. These are small pasting diagrams that are explicitly illustrated in the article and are quite elementary. This is why (AS) was not insisted upon in general.

Remark 5.5.8. It might remain unclear which conditions are required for which results (see Remark 5.5.6). To summarise, if a parity complex C satisfies (R1) and (R2) and (AS) then every theorem and proposition covered in this paper holds. In particular, every theorem and proposition holds for the parity complexes Δ, $\mathcal{G}$, and $\mathcal{Q}$.

Remark 5.5.9. There seems to be a fundamental relationship between parity complexes and 'directed graphs of multiple dimension'. Note that this notion of higher-dimensional graph would not be the same as an n-graph since each component of an n-graph has a single source and single target rather than a source set and a target set. Some of the axioms for parity complexes are just those of this 'graph' structure and others restrict us to graphs of a certain kind. Axioms (R1), (R2) and (AS) place further restrictions. Since we are mainly interested in examples that satisfy all of these conditions, we do not need to worry too much about this narrowing of our focus. More generally though, it would be good to know which of these conditions are associated with the graph structure of parity complexes, and which of the conditions allow for the (free) ω-category construction. This could be the focus of some future research.

5.6 Implications for further work

5.6.1 Confirmed material

The process of formalisation reveals that Sections 1 and 2 and Theorem 22 can be implemented with very little deviation from the original text. This is a testament to Street's insight and suggests that the definitions and results from those sections are well-expressed and useful tools for understanding these complicated combinatorial structures.

5.6.2 Adjusting the axioms

In private conversation Christopher Nguyen pointed out that Axiom 3.B is only used to prove Theorem 2, and Theorem 3 and its duals. We have commented already that Theorem 3 and its duals are equivalent to Axiom 3.B. A quick examination of our code then reveals that Theorem 3 is only used to prove that x^+ is tight and the disjointness condition described on p327. We haven't investigated this in any detail, but it might be possible to replace Axiom 3.B with something slightly weaker (or stronger) but which has the same implications in the relevant proofs. This is of particular use in light of the fact that Axiom 3.B is not always preserved under products and joins (see remark on page 334).

Note that the stronger axioms (R1) and (R2) subsume Axioms 1 and 2, and the examples of primary interest also have antisymmetry for $\blacktriangleleft$. So it is worth considering the implications of adding these conditions from the very beginning. There are however good examples of parity complexes that do not have these stronger properties. It is not yet clear if these examples are for some reason unimportant, or if parity complexes should not always be closed under product and join, or if there is even a third explanation.

5.6.3 Finding relevant elements

The excision of extremals shows that every cell can be presented as a composite of atomic cells. Unfortunately, not all cells are relevant, so we must explicitly describe some cells before we can use excision of extremals. And we're not even sure yet that any cells exist. In the presence of (R1) and (R2), the problem is solved since every element is relevant and every $\langle x \rangle$ is a cell. It is not clear whether these stronger conditions are in fact completely natural and should replace axioms 1 and 2, or whether they restrict our examples too much, or in fact, whether they are not strictly any stronger.

5.6.4 Why do we need $\mu(x)$ tight?

Theorem 22 relies on the fact that each $\mu(x)$ is tight and therefore a segment in the required place. This is readily proved when the ordering on $\blacktriangleleft$ is antisymmetric and when (R1) and (R2) hold. So, if we wish to use excision of extremals, we need to have these stronger conditions holding. So we ask, is the tightness condition strictly necessary? Is there another way to ensure that $\mu(x)$ is a segment in that proof? Or, are we happy to exclude examples of parity complexes where $\blacktriangleleft$ is not antisymmetric?

5.6.5 Understanding receptivity and tightness

Section 3 of [Str91] is particularly hard to understand and the proofs there are not always straight-forward. The notions of tightness and receptivity are both a bit opaque and Theorem 17 is very hard to prove. This provides some motivation to closely examine Theorem 17 and see whether alternative arguments might be made to prove it. In particular, it is not clear whether the stronger properties in Δ, $\mathcal{G}$, and $\mathcal{Q}$ will allow for a simpler argument. Or whether the various locations where tightness and receptivity are used, a different, more elegant argument might be possible. Or whether, on closer inspection tightness and receptivity can be seen as perfectly natural properties.

5.7 Some lessons in coded mathematics

5.7.1 Duals

We were often forced to prove dual results where x^+ and x^- were interchanged, or where the direction of a movement $M \xrightarrow{S} P$ was reversed. In these cases we were forced to explicitly restate and reprove the result, even though the underlying logic had not changed whatsoever. It would have been better if, from the beginning we had encoded plus and minus as duals to each other, then the theorems would dualise automatically. One way to do this is to define `faceset : bool → C → Ensemble C` and then set `minus := faceset false` and `plus := faceset true`. From this base-point, it should be easy to combine dual results into one.

5.7.2 Notation

Coq has a `Notation` facility which allows the user to introduce custom notation for specific expressions. We used this to make set operations easier to read and write. For example, an expression such as `Union A B` is displayed as `A ∪ B`, and similarly for intersection, inclusion, etc. This made our code much easier to read.

5.7.3 Tactics

Coq has a tactic language which allows for partial automation of proofs. The language allows the user to describe simple proof strategies that can be automatically applied when little innovative thinking is required. A particular built-in tactic called `intuition` will automatically deal with simple proofs that require only knowledge of first-order logic. We used the tactic language to describe a proof tactic called `basic` that automatically applied further logical steps such as $(x \in A \cap B) \to (x \in A \wedge x \in B)$. In many cases this vastly simplified proofs by applying `repeat (basic; intuition)` to automatically prove some trivial facts.

5.7.4 Setoid rewrite

Whenever two terms are definitionally equal (`a = b`), we can use the rewrite command to replace `a` with `b` in any expression. Whenever we use a weaker notion of equality such as `Same_set`, we do not necessarily have definitional equality and we can't replace `a` with `b` in every expression. This problem was solved using setoid rewrites as indicated in Section 5.2.4. Given our decision to eliminate the axiom of extensionality, this facility worked very well.

5.7.5 Axiom of extensionality for sets

We chose to remove the axiom of extensionality because we wanted to deal with sets in a completely constructive fashion. This was a choice of style. In many ways, retaining the axiom would not have weakened our encoding and we would not have needed to implement setoid rewrite for ensembles.

5.7.6 Compiling the excision of extremals algorithm

Our choice to implement sets using ensembles has made it impossible to directly compile an executable version of the excision of extremals. This is unfortunate: we have proved that such an algorithm can run but we can't actually compile or run it without further coding. The mathematical significance of our work is not undermined, however some more careful planning could have yielded this pleasant side-effect.

5.8 Conclusion

We have formalised Ross Street's 'Parity Complexes' up to the excision of extremals algorithm in Section 4. In particular, Sections 1 and 2 together with Theorem 22 are proved as indicated in the original text. Section 3 is also formalised with the same essential arguments as [Str91], but with many additional dual theorems, and a technical but meaningful change to the logical flow of Theorem 17 and Theorem 18.

We have indicated where the material is most effective at capturing the difficult combinatorics, and where future work might make improvements. We have explicitly outlined the logical dependence of the central results. We have also outlined some lessons learned in the encoding of this mathematics.

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