ROBUST REINSURANCE CONTRACTS WITH RISK CONSTRAINT

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Abstract

This thesis investigates a class of optimal reinsurance contract problems in continuous time. We use the principal-agent framework to incorporate the bargaining between the insurer and the reinsurer. To this end, We extend the reinsurer's relative safety loading factor which is usually a pre-specified constant in the traditional expected value principle to be time-varying and to represent the reinsurance premium.

Since the insurance companies should satisfy the regulators' capital requirements and the computation of capital requirements is based on Value-at-Risk (VaR) under Solvency II regime, we introduce the dynamic version of VaR and impose a dynamic VaR constraint on the insurer. As for the reinsurer, we assume that she is ambiguity-averse and aims to maximize the expected utility of her terminal wealth under the worst-case scenario of the alternative measures. The dynamic programming technique is applied to derive the principal's Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation and the agent's Hamilton-Jacobi-Bellman (HJB) equation. Additionally, Karush-Kuhn-Tucker (KKT) conditions are utilized to settle the constrained optimization problem of the agent. Explicit expressions for the optimal retained proportional of the claims, the optimal reinsurance premium and the corresponding value functions of the insurer and the reinsurer are derived.

Finally, we analyze several numerical examples to illustrate economic intuition. Our results show that the reinsurer's ambiguity aversion and the insurer's risk constraint increase the optimal reinsurance premium, which decreases the optimal reinsurance demand of the insurer.

Keywords Model ambiguity, dynamic Value-at-Risk (VaR), proportional reinsurance, principal-agent problem, reinsurance premium

Statement of Originality

This work has not previously been submitted for a degree or diploma in any university. To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.

Signed: Ning Wang   Date: 13/10/2018

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Introduction

Insurance plays a significant role in many aspects of economic activities. An individual or a corporation who purchases an insurance policy may receive some compensation or benefit from an insurance company if certain unexpected adverse events covered by the policy occur. However, as evident in recent decades, the severity and frequency of catastrophes such as natural disasters may result in the sudden collapse of an insurance company within a few months. These events highlight the importance of adopting proper methodologies in risk diversification for insurance companies to manage their underwriting risks. Reinsurance has been considered as a useful technique to avoid an insurance company's possible bankruptcy due to extreme events. Picard and Besson (1982) defined reinsurance as follows:

"A reinsurance operation is a contractual arrangement between a reinsurer and a professional insurer (called cedant), who alone is fully responsible to the policy holder, under which, in return for remuneration, the former bears all or part of the risks assumed by the latter and agrees to reimburse according to specified conditions all or part of the sums due or paid by the latter to the insured in case of claims."

We model the underlying risk faced by the insurer as a non-negative random variable, say X, with a known distribution. If the insurer decides to purchase a reinsurance contract, it means that $f(X)$ would be the risk covered by the reinsurer, while the insurer would pay the remaining loss $X - f(X)$. The function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is called ceded loss function and it defines a reinsurance contract. We denote the reinsurance premium charged by the reinsurer for risk coverage as $\delta_f(X)$, which is a function of $f(X)$. Under this model setting, the total loss of the insurer is denoted by $T_f(X) = X - f(X) + \delta_f(X)$. According to the form of the ceded loss function f, the reinsurance contracts can be classified into two types: proportional reinsurance and non-proportional reinsurance. Two of the most commonly applied forms of non-proportional reinsurance are excess-of-loss reinsurance and stop-loss reinsurance, which deal with individual risks and aggregate risks, respectively. In this thesis, we will only focus on investigating the proportional reinsurance treaty which is the most simple structure of reinsurance, wherein the reinsurer would cover a fixed share of the liabilities arising from the original insurance contracts.

Generally speaking, the reinsurance premium is an increasing function of the ceded loss, so there is a natural trade-off between the benefit of receiving indemnity and the cost of reinsurance premium. The problem in optimal reinsurance design involves how risk should be shared between the insurer and the reinsurer, as it aims to determine the optimal form of a reinsurance treaty under specific optimization criteria. The commonly used optimization criteria in the literature include minimizing the variance of an insurer's total loss, minimizing an insurer's ruin probability or equivalently maximizing his survival probability, maximizing the expected utility function of the terminal wealth of an insurance company and minimizing the risk exposure quantified by risk measures such as Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR).

Most of the existing investigation on optimal reinsurance design are from the perspective of the insurer. However, a reinsurance treaty is an agreement between the insurer and the reinsurer. Therefore, an optimal reinsurance policy that only considers the insurer's interest may be unacceptable to the reinsurer. Borch (1960) was the first to study the optimal proportional and stop-loss reinsurance contracts from both the perspectives of the insurer and the reinsurer by maximizing the product of the expected utility functions of the two parties' wealth. This thesis applies principal-agent framework to formulate the interaction between the insurer and the reinsurer. For this study, we extend the pre-specified positive constant representing the reinsurer's safety loading factor in the expected value principle to be time-varying, which represents the reinsurance premium. In our study, the reinsurer is the principal and the insurer is the agent. Moreover, the optimal reinsurance contract derived under this framework maximizes a combination of the objective functions of the insurer and the reinsurer, thus considering mutual benefits for both the insurer and the reinsurer. The optimal reinsurance contract designed in this thesis is not only implementable for the insurer but also maximizes the reinsurer's utility.

With the recent huge prosperity of financial markets, practitioners and regulators pay more attention to risk management. Risk measures such as VaR and CVaR have become increasingly popular for quantifying risk among insurance companies. VaR is the maximum expected loss over a given horizon period at a given confidence level. From a regulatory perspective, the insurance company should satisfy solvency capital requirements, which provides a buffer for potential losses. Solvency II, a unified regime applicable to all insurance companies in the European Union insurance market, suggests an insurer to compute his capital efficiency with VaR. In view of this, we impose in this thesis a dynamic VaR constraint on the insurer to guarantee that he satisfies the capital requirements at all times. The framework of Solvency II is specified in European Commission (2009), and the determination of capital requirements is described as follows:

"The Solvency Capital Requirement should be determined as the economic capital to be held by insurance and reinsurance undertakings in order to ensure that ruin occurs no more often than once in every 200 cases or, alternatively, that those undertakings will still be in a position, with a probability of at least 99.5 %, to meet their obligations to policy holders and beneficiaries over the following 12 months. That economic capital should be calculated on the basis of the true risk profile of those undertakings, taking account of the impact of possible risk-mitigation techniques, as well as diversification effects."

In the subjective expected utility theory, the probability distribution and the utility function are chosen by a rational decision maker. However, Ellsberg (1961) violated the subjective expected utility theory with a paradox which can be explained by ambiguity aversion. In the traditional optimal reinsurance design, both the insurer and the reinsurer are assumed to have complete confidence in their models. But in reality the probability measures describing the real-world model may have some small perturbations due to misspecification in areas such as data collection or measuring errors. In this thesis, we assume that the principal is ambiguity-averse and aims to seek robust decision rules under probability noises, while the agent is assumed as not having model ambiguity. This is a reasonable assumption because it is the insurer who collects the information of the policy-holders and the reinsurer may doubt about the claim process estimated by the insurer. Regarding the reinsurer who distrusts the model, we use the equivalent probability measures of the reference measure under which the insurance market is defined to capture model uncertainty, and these equivalent measures are called alternative measures. The robust reinsurance contract is designed to maximize the reinsurer's expected utility in the worst-case scenario of the alternative measures, subject to the insurer's incentive constraint.

To the best of our knowledge, there is little literature incorporating risk constraint as well as ambiguity aversion attitudes in the optimal reinsurance contract design. In this thesis, we aim to fill this gap and extend the results in the study conducted by Hu et al. (2018b), by imposing a dynamic VaR constraint on the insurer to guarantee his financial solvency. As did Hu et al. (2018b), we employ the principal-agent framework to model transferring risk exposure through reinsurance contracts and suppose that the reinsurer is ambiguous about the claim process. We apply the Lagrange multiplier techniques to solve the agent's constrained optimization problem. By solving the agent's HJB equation and the principal's HJBI equation, we derive closed-form solutions to the robust optimal reinsurance contract and the optimal value functions of the contracting parties. The results show that our model is more general and can reduce to the unconstrained model considered by Hu et al. (2018b) when the parameters of the insurer and the reinsurer satisfy specific conditions making the dynamic VaR constraint inactive for the insurer. It is shown in our numerical examples that risk constraint and ambiguity aversion have material impacts on the optimal reinsurance contract. Specifically, the reinsurer facing model uncertainty becomes more conservative to the risk than an ambiguity-neutral reinsurer, which induces her to increase the reinsurance premium and so the insurer would decrease his demand for reinsurance. This conclusion is also obtained in the work of Hu et al. (2018b). The result that seems to be novel is that when the dynamic VaR constraint is active for the insurer, the reinsurer offers a higher reinsurance premium than that of inactive constraint case, which then makes the insurer retain more insurance risk. The reason behind this phenomenon is that both the insurer and the reinsurer should have higher levels of risk aversion to satisfy some conditions making the risk constraint active on the insurer and the effect of the reinsurer's risk aversion parameter magnifies the effect of her attitude towards ambiguity. From this conclusion we can see that the reinsurer dominates over the insurer in reinsurance agreements, which is in line with the existing literature, for example, Chen and Shen (2018).

The remaining parts of this thesis are structured as follows. The next chapter provides a brief literature review. In Chapter 2, we present the research methods used in this thesis. Chapter 3 demonstrates how we formulate the model in this thesis, and includes the definition of dynamic VaR, the problems of the principal and the agent, and the reinsurer's ambiguity aversion attitudes. Chapter 4 provides the explicit solutions to the robust optimal proportional reinsurance contracts and the optimal value functions when the VaR constraint is binding and inactive for the insurer. In Chapter 5, we present special cases of our proposed model. We thereafter analyze the reinsurer's expected utility loss in Chapter 6. Detailed numerical simulations are conducted in Chapter 7 to demonstrate the results. Some concluding remarks and discussion of further research are made in Chapter 8.

Chapter 1

Review of the Literature

There exists a large collection of literature on the problems related to optimal portfolio selection, where the main goal is to allocate the investor's assets among different financial securities for maximizing the total investment return of the portfolio. In the pioneering work of Markowitz (1952), a measure of investment risk was provided by adopting the definition of variance in probability theory. In a single-period setting, Markowitz considered the trade-off between the mean and the variance of the rate of return, which was the celebrated mean-variance approach. By applying stochastic optimal control techniques, Merton (1969) firstly gave a closed-form solution to the optimal portfolio selection problem in a continuous-time framework, which opened the study for continuous-time finance theory. The mean-variance portfolio allocation problem in dynamic settings was originally solved by Zhou and Li (2000), where the stochastic linear-quadratic control theory was employed. Some recent works on optimal portfolio selection problems include the works of Elliott et al. (2010), Siu (2011), Fu et al. (2014) and Zhang et al. (2017), amongst others. In recent years, insurance companies have actively participated in the financial markets to increase their profits. The optimal investment problems for an insurance company have been frequently studied in the literature. For example, Browne (1995) obtained optimal investment strategies by minimizing the insurer's ruin probability and maximizing the expected exponential utility of its terminal wealth. Yang and Zhang (2005) derived an optimal investment policy of an insurance company whose surplus process was described by a jump-diffusion process with the help of stochastic control theory. Wang et al. (2007) established an explicit solution to the optimal investment problem of an insurer by applying the martingale approach. Badaoui et al. (2018) considered an investment optimization problem of an insurer when the market was incomplete and the volatility process of the risky asset was captured by an external factor model.

On the other hand, reinsurance is an effective tool for the insurer to transfer or avoid the underwriting risk. In some cases, large claims resulting from catastrophic events such as hurricanes or earthquakes may lead to bankrupt or ruin of an insurance company. It is, therefore, imperative for the insurance company to mitigate his risk exposure by buying an insurance from a third party, say a reinsurance company. This provides a basic description for the concept of reinsurance. To transfer part of insurance risks to a reinsurer, the insurer needs to pay a reinsurance premium. Loosely speaking, the larger the insurance risk transferred to the reinsurer, the more expensive the reinsurance contract becomes. This demonstrates the trade-off between the cost of the reinsurance premium and retaining risk. The goal of optimal reinsurance design is to find a specified form of the ceded loss function to optimize such trade-offs according to some optimality criteria.

By examining the existing literature, a substantial part of the research has been devoted to the derivation of the optimal ceded loss functions under different assumptions. The actuarial methods of problems in reinsurance optimization can be classified into two categories. In the first category, one aims to find the form of the optimal ceded loss function within a family of general and broad ceded loss functions (i.e., satisfying specific properties). With respect to these problems, the seminal work is attributed to Borch (1960), in which it was shown that stop-loss reinsurance was optimal under the criterion of minimizing the variance of the insurer's retained loss. Arrow (1963) aimed to maximize the expected utility of the insurer's terminal wealth under the expected value principle and obtained the result that stop-loss reinsurance was also optimal. More recently, Chi and Tan (2011) assumed that the reinsurance premium principle satisfied three basic conditions: distribution invariance, risk loading and preservation of stop-loss order. Under these assumptions, they showed that the layer reinsurance was optimal under both VaR and CVaR criteria. Chi (2012) investigated the optimal reinsurance strategies by minimizing the risk-adjusted value of an insurer's liability where capital at risk was calculated by VaR or CVaR risk measure, and he showed that it was best for the insurer to cede two separate layers under a more general reinsurance premium principle assumption. Chi and Zhou (2016) investigated an optimal reinsurance model under a mean-variance reinsurance premium principle from the perspective of an insurance company who has a general mean-variance preference, and they demonstrated that any admissible reinsurance policy would be dominated by a change-loss reinsurance or a dual change-loss reinsurance. Some other models belonging to this category can also be found in the works of Cai et al. (2008), Cheung (2010) and Chi and Meng (2014).

The second category of problems in optimal reinsurance design is to determine the optimal parameters for a given form of the ceded loss function. The commonly used forms of the reinsurance treaty include proportional, quota-share, excess-of-loss, stop-loss, change-loss, etc. Højgaard and Taksar (1998) studied an optimal proportional reinsurance policy by maximizing a discounted return function. Yuen et al. (2015) derived an optimal proportional reinsurance strategy by maximizing the expected exponential utility when the dependence between different classes of insurance risks existed. With the widespread popularity of risk measures that quantify risk and set regulatory capitals, Cai and Tan (2007) derived optimal retentions for the stop-loss reinsurance under the optimization criteria of minimizing the VaR and the conditional tail expectation (CTE) of the total risks of an insurer. Tan et al. (2009) extended the results of the research conducted by Cai and Tan (2007) and examined the optimality of the quota-share and the stop-loss reinsurance arrangements under general reinsurance premium principles. Hu et al. (2015) determined an optimal retention in a stop-loss reinsurance treaty wherein the information of the total loss was incomplete. The objective function in their paper was chosen as minimizing an upper bound for the VaR of the insurer's total loss.

Besides, the investigation of the insurer's optimal investment and reinsurance strategies simultaneously has attracted considerable attention in the area of actuarial science. For example, Bai and Guo (2008) considered the optimal investment-reinsurance problem by maximizing the expected exponential utility of the terminal wealth under the no-shorting constraint. Shen and Zeng (2015) applied the mean-variance criterion to derive the insurer's optimal reinsurance and investment strategies. Zhang et al. (2016) aimed to seek the optimal reinsurance and investment strategies by minimizing the probability of run, where the reinsurance premium is calculated through the generalized mean-variance principle. For more literature, the reader may refer to Gu et al. (2012), Guan and Liang (2014), Zhao et al. (2016) and Sun and Guo (2018), just to name a few.

Along with the proliferation of the risk measures such as VaR and Tail Value-at-Risk (TVaR) in banking and insurance industries, some risk-measure-based constraints have been formulated and imposed on the classical reinsurance and/or investment optimization problems for practical consideration. In this aspect, Yiu et al. (2010) considered the optimal portfolio selection problem when the model parameters changed according to the states of an underlying economy by maximizing the expected discounted utility, subject to the maximum Value-at-Risk (MVaR) constraint. They also employed some numerical methods to solve the Hamilton-Jacobi-Bellman (HJB) equations. Chen et al. (2010) incorporated the dynamic VaR constraint into investigating the optimal reinsurance and investment strategies of an insurance company to minimize its ruin probability. Liu et al. (2013) imposed the maximal conditional Value-at-Risk (MCVaR) constraint on the optimal investment-reinsurance problem of an insurer whose risky investment security was governed by a Markovian regime-switching model. Liu and Yiu (2013) studied a family of stochastic differential reinsurance and investment games between two competing insurance companies subject to risk constraints. Guan and Liang (2016) studied the optimal investment problem of defined contribution (D-C) pension plans under the loss aversion and VaR constraint. Zhang et al. (2016) studied the optimal proportional and excess-of-loss reinsurance strategies under dynamic VaR constraint by maximizing the insurer's survival probability.

In the traditional settings of optimal reinsurance and/or investment problems, the decision makers believe with certainty in the models that describe the real-world probability. However, it should be noted that there exist many uncertainties in the financial markets and insurance industries. Because it is controversial to determine which model is completely true so as to represent the real-world probability and be used in the optimization problems. Thus, we should take model uncertainty or ambiguity into account. According to Knight (1921), unlike risk which referred to a variable having a known probability distribution, the uncertainty due to lack of information on the probability measure referred to ambiguity. Based on the experimental results, Ellsberg (1961) showed the inadequacy of the subjective expected utility theory and argued that decision makers are ambiguity-averse. Another reason for us to consider model uncertainty is that the parameters, especially the drift parameters, are difficult to estimate with precision. Thus, it is reasonable to assume that the decision maker is concerned about model misspecification. In the literature, one popular approach to describe model ambiguity was proposed by Anderson et al. (2003), and they studied asset pricing problems in stochastic continuous-time settings by incorporating the investor's consideration of model misspecification. Under their assumption, the investor regarded the specific probability measure as their reference measure and could then find robust strategies that work over the nearby measures known as alternative measures. The deviation between the reference measure and an alternative measure was determined by relative entropy which had a wide application for model detection in statistics and econometrics. Specifically, the relative entropy was used to construct a penalty term in the robust optimization problems. Since then, due to its analytical tractability, the formulation of the robust optimization procedures conducted by Anderson et al. (2003) has been adopted in portfolio selection, asset pricing and optimal reinsurance-investment problems. For example, Maenhout (2004) obtained the optimal portfolio decision for an investor with ambiguity aversion attitudes. Zhang and Siu (2009) considered an optimal reinsurance-investment problem in the presence of model uncertainty and formulated the problem into a zero-sum stochastic differential game between the insurer and the market. Pun and Wong (2015) discussed the robust optimal reinsurance-investment problem for a general class of utility functions when the risky asset followed a multiscale stochastic volatility (SV) model. Li et al. (2018) articulated the optimal investment and excess-of-loss reinsurance problem for an insurer with model ambiguity concerning the diffusion and jump components arising from financial and insurance markets. Wang and Li (2018) incorporated ambiguity aversion into an optimal investment problem for a DC pension plan, where stochastic interest rate and stochastic volatility were introduced to describe the financial market consisting of a risk-free asset, a rolling bond and a stock. Gu et al. (2018) investigated a robust optimal investment and proportional reinsurance problem for an ambiguity-averse insurer who could invest his surplus into one risk-free asset, one market index and a pair of mispriced stocks.

In the aforementioned literature, the optimal reinsurance problems are solely treated from the insurer's point of view and the interests of the reinsurer are neglected completely, whereas a reinsurance contract is supposedly a mutual agreement between the insurer and the reinsurer. As pointed out by Borch (1969), an optimal reinsurance treaty for an insurer might not be optimal and even be unacceptable for a reinsurer. Therefore, it would be more reasonable and interesting to analyze the reinsurance problem from the perspectives of an insurer and a reinsurer. Motivated by this concept, Cai et al. (2013) designed the optimal quota-share and stop-loss reinsurance policies by maximizing the joint survival and profitable probabilities of the insurer and the reinsurer under different premium principles. Cai et al. (2016) took the goals of the insurer and the reinsurer into account in the design of the optimal reinsurance contract by minimizing the convex combination of the VaR risk measures of both the insurer's and the reinsurer's losses. Lo (2017) proposed a unifying approach to develop the optimal reinsurance treaties in the presence of practical constraints and studied three motivating models, one of which employed minimizing the distortion risk measure of the insurer's total risk exposure and considered the reinsurer's risk constraint to make the reinsurance arrangement mutually acceptable. Zhang et al. (2018) developed the optimal quota-share reinsurance agreements by applying the optimization criteria and utility increment constraints that reflected the consideration of mutual beneficiary.

Although the consideration of the optimal reinsurance contracts that benefit two parties in the agreements is extensive in the discrete-time single-period settings, there exists rare literature that focuses on designing the reciprocally optimal reinsurance treaty in dynamic settings. The two-party agreement nature of reinsurance suggests that it is natural to model this relationship between the insurer and the reinsurer as principal-agent relationship, where the insurer is the agent and the reinsurer is the principal. The detailed surveys of the dynamic contracting problems under principal-agent framework in continuous time can be found in the work conducted by Cvitanić and Zhang (2012). According to the amount of information available to the principal and the agent, there are three categories of contract problems that have been studied in the literature, and they are Risk Sharing (RS), Hidden Action (HA) (also called moral hazard) and Hidden Type (HT) (also called adverse selection), and the contracts obtained under these three cases are called first best solution, second best solution and third best solution, respectively. Cvitanić et al. (2006) applied the stochastic maximum principle to derive the necessary and sufficient conditions for the first-best contracts when the agent and the principal had complete information. Miao and Rivera (2016) introduced the principal's robustness preference into the contracting problem under the agent's hidden action in continuous time. Sung (2005) considered a principal-agent problem in the presence of moral hazard and adverse selection with a risk-neutral principal and a risk-averse agent. Along the direction of the principal-agent framework, Hu et al. (2018a,b) designed the robust proportional reinsurance and excess-of-loss reinsurance contracts by maximizing the expected exponential utility in the worst-case scenario of the alternative measures. Moreover, game theory provides us with mathematical views to study the interactive decision situations. Some scholars apply game theory to investigate the interactive roles played by the insurer and the reinsurer in reinsurance agreements. For instance, Chen and Shen (2018) formulated the optimal proportional reinsurance problem under the stochastic Stackelberg differential game framework and applied backward stochastic differential equation (BSDE) approach to derive the reinsurer's optimal reinsurance premium and the insurer's optimal retained proportion of the insurance risk.

In the following chapters, the utilized methods, constructed model and obtained results will be presented.

Chapter 2

Methodology

In this thesis, a continuous-time model is considered to investigate a class of reinsurance contract problems. Mathematically, this is formulated as a stochastic optimal control problem. There exist several approaches to discuss stochastic optimal control problems. Three approaches are mentioned here. The first approach is dynamic programming principle and HJB equation, and the second approach is martingale approach which is based on equivalent martingale measure and martingale representation theorem. The third approach is based on backward stochastic differential equation (BSDE). In this thesis, we apply the first approach and the main results in that direction would be presented in Section 2.1. Considering that the reinsurance arrangements are basically contracts, we can adopt the methods in economic contract theory to study the optimal reinsurance problems. This may hopefully provide a new perspective to look at the problems. The use of principal-agent framework for designing optimal reinsurance does not seem to have been well-explored in the literature, and this framework will be introduced in Section 2.2. Another main element of the problem in this thesis is the incorporation of the risk constraints. Solvency II which has come into effect on 1 January 2016 requires the insurance companies in Europe to hold adequate capitals to withstand or reserve risks, and certain risk measures have been regarded as important tools to determine the regulatory capitals for the insurers. On account of this, we impose a risk constraint based on Value-at-Risk on the insurer to reduce the chance of insolvency, which leads us to the insurer's constrained optimization problem. In Section 2.3 we will review the popular KKT conditions, and we usually employ them to solve the optimization problems with inequality constraints. Finally, we hold the view that it is the insurer who collects the information from the policyholders and the reinsurer seldom deals with the customers directly. The asymmetric information between the insurer and the reinsurer may lead the reinsurer to distrust the claim process estimated by the insurer. To articulate this situation, we incorporate the reinsurer's ambiguity aversion attitudes into the model, and this is why the reinsurance contract designed in this thesis is called robust. When formulating robust contracts, we have to answer the question on how to describe model uncertainty or ambiguity. In Section 2.4, we shall review the change of measure techniques, and in the later chapters these techniques will be applied to describe how the reinsurer does not believe the approximating model and considers the alternative models.

2.1 Dynamic programming principle and HJB equation

In this section, we will discuss the dynamic programming principle and HJB equation, which play important roles in the stochastic optimal control problems. We borrow the notations and conclusions from Pham (2008), and omit the proofs of the theorems in the following part. Other works on this topic include, for example, Yong and Zhou (1999), Fleming and Soner (2006) and Schmidli (2008).

Assume that $T \in (0, \infty)$ is a finite horizon, and we consider the following controlled diffusion system:

$$\mathrm{d}X(s) = b\left(X(s), \alpha(s)\right) \mathrm{d}s + \sigma\left(X(s), \alpha(s)\right) \mathrm{d}W(s), \tag{2.1.1}$$

where $X(s)$ is the state of the system at time s, and $W(s)$ is a d-dimensional Brownian motion. The control $\alpha(\cdot)$ is a progressively measurable process valued in a convex set A. The measurable functions $b : \mathbb{R}^n \times A \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times A \to \mathbb{R}^{n\times d}$ satisfy a uniform Lipschitz condition, i.e., for $\forall\ x, y \in \mathbb{R}^n$, $\forall\ a \in A$, we have

$$|b(x, a) - b(y, a)| + |\sigma(x, a) - \sigma(y, a)| \leq C|x - y|,$$

for some non-negative constant C. We use $\mathcal{A}$ to denote the set of control processes satisfying

$$\mathbb{E}\left[\int_0^T |b(0,\alpha(t))|^2 + |\sigma(0,\alpha(t))|^2 \mathrm{d}t\right] < \infty.$$

For $\forall\ (t,x) \in [0,T] \times \mathbb{R}$, $\mathcal{A}(t,x)$ collects the elements in $\mathcal{A}$ such that

$$\mathbb{E}\left[\int_t^T |f\left(s, X_{t,x}(s), \alpha(s)\right)| \mathrm{d}s\right] < \infty.$$

where $X_{t,x}(s)$ is the strong solution to the stochastic differential equation (2.1.1) starting from state x and time t. Let $f : [0,T] \times \mathbb{R}^n \times A \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be two functions, and we suppose that

(i) g is lower-bounded, or

(ii) g satisfies a quadratic growth condition:

$$|g(x)| \leq K\left(1 + |x|^2\right), \quad \forall\ x \in \mathbb{R}^n,$$

for some constant K independent of x.

For all $(t,x) \in [0,T] \times \mathbb{R}^n$ and $a \in \mathcal{A}(t,x)$, we can then define the gain functional as follows:

$$J(t,x;\alpha(\cdot)) := \mathbb{E}\left[\int_t^T f\left(s, X_{t,x}(s), \alpha(s)\right) \mathrm{d}s + g(X_{t,x}(T))\right]. \tag{2.1.2}$$

The standard stochastic optimal control problem is usually stated as follows:

Problem P1. For given $(t,x) \in [0,T] \times \mathbb{R}^n$, maximize (2.1.2) subject to (2.1.1) over $\mathcal{A}(t,x)$.

In order to solve **Problem P1**, we first define the associated value function

$$V(t,x) := \sup_{\alpha \in \mathcal{A}(t,x)} J(t,x;\alpha(\cdot)),$$

and we say that $\alpha^* \in \mathcal{A}(t,x)$ is an optimal control if $V(t,x) = J(t,x;\alpha^*(\cdot))$. The value function $V(t,x)$ will play an important role in obtaining the optimal controls, which will be seen in the later chapters. Based on the previous assumptions, we present the following standard theorem, which is called the dynamic programming principle.

Theorem 2.1.1 *(Dynamic programming principle) For any $(t,x) \in [0,T] \times \mathbb{R}^n$,*

$$V(t,x) = \sup_{\alpha \in \mathcal{A}(t,x)} \mathbb{E}\left[\int_t^\theta f\left(s, X_{t,x}(s), \alpha(s)\right) \mathrm{d}s + V(\theta, X_{t,x}(\theta))\right], \quad \forall\ 0 \leq t \leq \theta \leq T. \tag{2.1.3}$$

Remark 2.1.1 *The dynamic programming principle is originated in the early 1950s by Bellman, the key idea of which is to consider a family of optimal control problems having varying initial states and times. Although we have already known that the value function should satisfy the dynamic programming equation* (2.1.3), *it is uneasy to solve the value function from* (2.1.3) *directly because of the complicated operation involved. So we will give a further exploration in the next theorem.*

We define $C^{1,2}([0,T]\times\mathbb{R}^n) := \{f : [0,T]\times\mathbb{R}^n \to \mathbb{R} | f(t,\,\cdot\,)$ is once continuously differentiable on $[0,T]$ and $f(\,\cdot\,,x)$ is twice continuously differentiable on $\mathbb{R}^n\}$. The following theorem states that solving the stochastic optimal control problem **P1** can be converted to solving a certain partial differential equation (PDE) when the value function satisfies some conditions.

Theorem 2.1.2 *Assume $V(t,x)\in C^{1,2}([0,T]\times\mathbb{R}^n)$, then $V(t,x)$ is a solution to the following terminal value problem of a first-order PDE:*

$$\begin{cases} -\frac{\partial V}{\partial t}(t,x) - H(t,x,D_xV(t,x),D_x^2V(t,x)) = 0, \quad \forall\ (t,x)\in[0,T)\times\mathbb{R}^n, \\ V(T,x) = g(x), \qquad \forall\ x\in\mathbb{R}^n, \end{cases} \tag{2.1.4}$$

where for $(t,x,p,M)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times S_n$,

$$H(t,x,p,M) = \sup_{\alpha\in A}\left[b(x,\alpha)p + \frac{1}{2}tr\left(\sigma(x,\alpha)\sigma'(x,\alpha)M\right) + f(t,x,\alpha)\right],$$

with $tr(\cdot)$ denoting the trace of a matrix, S_n the set of symmetric $n\times n$ matrices, D_x the gradient vector and D_x^2 the Hessian matrix of a function.

We call the PDE in (2.1.4) the Hamilton-Jacobi-Bellman (HJB) equation, and $V(t,x)$ solves (2.1.4) is a classical solution to the HJB equation.

2.2 Principal-agent framework in reinsurance contract

In the principal-agent problem, our goal is to determine the optimal contract between two parties, namely the principal and the agent. We will henceforth also call the principal "she" and the agent "he". Whether the actions of the agent are observable/contractable or not plays decisive roles in what form of contract is optimal. If the agent and the principal share the same set of information, we will get the first best case, or risk sharing. In this case, the agent has to implement the contract that the principal offers to maximize her objective function, otherwise the agent would be penalized for his dysfunctional behavior. However, in many realistic examples, we can see that sometimes it is costly or even impossible for the principal to monitor the agent's action, she cannot order the agent to perform the actions she prefers and will only be able to attain the second best solution, or moral hazard. Finally, if the type of the agent is hidden, we will obtain the third best case or adverse selection. We only study the first type of principal-agent problem in this thesis. For further details of contract theory and its applications, the reader is referred to Bolton and Dewatripont (2005) and Cvitanić and Zhang (2012), which are excellent works and devoted to discrete-time models and continuous-time models, respectively.

In our optimal reinsurance contract problem the reinsurer is the principal and the insurer is the agent. One reinsurance contract consists of two important components, i.e., the reinsurance demand and the reinsurance premium. If we consider the models in continuous time, we might as well use $a(t)$ to denote the retention in the case of proportional reinsurance and reinsurance level in the case of non-proportional reinsurance at time t. Moreover, we suppose that the reinsurer offers a reinsurance premium $b(t)$ at time t. Note that $a(t)$ reflects the reinsurance demand and $b(t)$ reflects the reinsurance premium. Under these assumptions, we know that a reinsurance contract is determined by $(a(t), b(t))$, and we aim to find the optimal reinsurance contract $(a^*(t), b^*(t))$ under a certain optimization criterion. From the mathematical perspective, we can describe the strategic interactions between the insurer and the reinsurer as the following two steps:

(1) For a given reinsurance premium $b(t)$, we first find an $a^*(t) = \alpha^*(t, b(t))$ to solve the agent's optimization problem:

$$J_A(\alpha^*(t, b(t)), b(t)) := \sup_{a(t)} J_A(a(t), b(t)), \tag{2.2.1}$$

where $J_A(\cdot, \cdot)$ denotes the agent's objective function.

(2) Then we come to solve the principal's optimization problem: find a $b^*(t)$ such

that

$$J_P(\alpha^*(t, b^*(t)), b^*(t)) := \sup_{b(t)} J_P(\alpha^*(t, b(t)), b(t)),$$

where $J_P(\cdot, \cdot)$ denotes the principal's objective function.

The pair $(a^*(t), b^*(t)) = (\alpha^*(t, b^*(t)), b^*(t))$ is then called the optimal reinsurance contract. Following Hu et al. (2018a), we call (2.2.1) the incentive compatibility constraint, and a reinsurance contract is incentive compatible if and only if the retained risk level $a(t)$ satisfies (2.2.1). From the above procedures, it can be seen that under our framework, the principal indirectly influences the agent to select an optimal reinsurance demand by offering an appropriate reinsurance premium subject to the agent's incentive compatibility constraint instead of directly ordering the agent to undertake the actions she prefers.

2.3 KKT conditions in the constrained optimization problems

The constrained optimization problems which are usually referred to as mathematical programming models have a wide range of applications in economic decision makings. The Karush-Kuhn-Tucker (KKT) conditions are important instruments in qualitative economic analysis. It appears that Karush (1939) has been the first to discuss optimization problems with inequalities as side constraints. Since Kuhn and Tucker (1951) further used the optimality conditions to deal with the nonlinear programming problems, the KKT conditions have spurred a significant amount of interest in optimization problems in the presence of inequality constraints. The classical optimization problems usually have constraints in equation forms, which can be solved by applying the Lagrange-multiplier approach. Since equality constraints can be considered as special cases of inequality constraints, the optimization problem with inequality constraints is an extension of the classical optimization problem and KKT conditions are generations of the Lagrange theorem. In this section, we shall review the KKT necessary optimality conditions and these results can be found in Bazaraa et al. (2013) and Luptáčik (2010).

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, for $i = 1, 2, \ldots, m$, be some functions. We consider the following nonlinear programming problem:

Problem P2.

$$\begin{cases} \min \ f(x), \\ \text{subject to} \quad g_i(x) \le 0, \ \text{for } i = 1, 2, \ldots, m. \end{cases}$$

Definition 2.3.1 *Let X be a nonempty open set in $\mathbb{R}^n$. A vector $x \in X$ satisfying all the constraints is called a feasible solution to* **Problem P2**.

Assume that point x^* is a feasible solution of **Problem P2**, and denote by $I(x^*) = \{i : g_i(x^*) = 0\}$ the set of binding (or active, or tight) constraints.

Definition 2.3.2 *We say that feasible solution x^* satisfies the linear independence constraint qualification (LICQ) if the set of gradients of the active constraints at x^*, denoted by $\nabla g_i(x^*)$, for $i \in I(x^*)$, are linearly independent.*

Theorem 2.3.1 *(KKT conditions) Assume that f and g_i, for $i \in I(x^*)$, are differentiable at x^* and that LICQ holds at feasible solution x^*. Then there exist scalars λ_i, for $i \in I(x^*)$, such that KKT conditions hold at x^* if f attains minimum at point x^* :*

$$\begin{cases} \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) = 0, & (2.3.1a) \\ \lambda_i g_i(x^*) = 0, \qquad \text{for } i = 1, 2, \ldots, m, & (2.3.1b) \\ g_i(x^*) \le 0, \qquad \text{for } i = 1, 2, \ldots, m, & (2.3.1c) \\ \lambda_i \ge 0, \qquad \text{for } i = 1, 2, \ldots, m, & (2.3.1d) \end{cases}$$

where ∇f and ∇g_i denote the gradient vectors (the vectors of first-order partial derivatives) of f and g_i, respectively.

The scalars in Theorem 2.3.1 are called Lagrange multipliers, since the optimal point x^* minimizes f over X, and hence the subdifferential of f at x^* must equal 0, and (2.3.1a) is called the stationarity condition (SC); the requirement in (2.3.1b) is referred to as the complementary slackness (CS) condition; the restriction in (2.3.1c) implying that x^* is feasible to **Problem P2** is known as the primal feasibility (PF) condition; finally (2.3.1d) is called the dual feasibility (DF) condition. Any point x^* for which there exists a Lagrange multiplier λ^* such that (x^*, λ^*) satisfies the KKT conditions is called a KKT point. Theorem 2.3.1 tells us that the KKT conditions summarized in (2.3.1a)-(2.3.1d) are necessary conditions for an optimal solution to **Problem P2**, and we remark here that the KKT conditions are also sufficient when **Problem P2** is a convex optimization problem where the objective function and the constraint functions are convex functions. The related details are referred to Boyd and Vandenberghe (2004).

2.4 Change of probability measures

In this section we shall see how to change the original probability measure to an equivalent one. The main instrument in change of measure for stochastic process is Girsanov's theorem. The materials presented here are standard in stochastic analysis. We begin with a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, where T is a fixed terminal time. Let $\{W(t)\}_{t\in[0,T]}$ be a standard Brownian motion, and β be a measurable function on $[0, T] \times \Omega$.

Definition 2.4.1 *We say two probability measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, denoted as $\mathbb{P} \sim \widetilde{\mathbb{P}}$, if they have the same null sets, i.e., $\mathbb{P}(A) = 0$ if and only if $\widetilde{\mathbb{P}}(A) = 0$.*

Definition 2.4.2 *$\widetilde{\mathbb{P}}$ is called absolutely continuous with respect to $\mathbb{P}$, denoted as $\widetilde{\mathbb{P}} \ll \mathbb{P}$, if $\widetilde{\mathbb{P}}(A) = 0$ whenever $\mathbb{P}(A) = 0$. $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are called equivalent if $\widetilde{\mathbb{P}} \ll \mathbb{P}$ and $\mathbb{P} \ll \widetilde{\mathbb{P}}$.*

Theorem 2.4.1 *(Radon-Nikodym) If $\widetilde{\mathbb{P}} \ll \mathbb{P}$, then there exists a random variable Λ, such that $\Lambda \geq 0$, the expectation of Λ under probability measure $\mathbb{P}$, written as $\mathbb{E}_{\mathbb{P}}\Lambda$, equals one, and the following formula holds for any measurable set A :*

$$\widetilde{\mathbb{P}}(A) = \int_A \Lambda \mathrm{d}\mathbb{P}. \tag{2.4.1}$$

Conversely, if there exists a random variable Λ having the above properties and $\widetilde{\mathbb{P}}$ is defined by (2.4.1)*, then $\widetilde{\mathbb{P}}$ is a probability measure such that $\widetilde{\mathbb{P}} \ll \mathbb{P}$.*

The random variable Λ in Theorem 2.4.1 is called the Radon-Nikodym derivative of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and is denoted as $\dfrac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}$. The proof of Theorem 2.4.1 can be found in Klebaner (2005).

Lemma 2.4.1 *We define*

$$Z(t) = \exp\left[\int_0^t \beta(u)\mathrm{d}W(u) - \frac{1}{2}\int_0^t \beta^2(u)\mathrm{d}u\right], \tag{2.4.2}$$

and so that $Z(t)$ is a martingale under $(\mathcal{F}_t, \mathbb{P})$.

Remark 2.4.1 *A sufficient condition which is known as Novikov's condition for $Z(t)$ to be a martingale is as follows:*

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \beta^2(t)\mathrm{d}s\right)\right] < \infty,$$

where $\mathbb{E}_{\mathbb{P}}[\,\cdot\,]$ denotes the expectation under probability measure $\mathbb{P}$.

Theorem 2.4.2 *(Girsanov's theorem) Define a new probability measure on $\mathcal{F}_T$ by putting*

$$\widetilde{\mathbb{P}}(A) = \int_A Z(T, w)\mathrm{d}\mathbb{P}, \quad \textit{for all } A \in \mathcal{F}_T,$$

then $\widetilde{\mathbb{P}}$ is a probability measure on $(\Omega, \mathcal{F}_T)$. Furthermore,

$$\widetilde{W}(t) = W(t) - \int_0^t \beta(u)\mathrm{d}u, \quad 0 \le t \le T$$

is a Brownian motion in the probability space $(\Omega, \mathcal{F}_T, \widetilde{\mathbb{P}})$ with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$.

From Girsanov's theorem we know that $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$, and the form of Radon-Nikodym derivative of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$ is given by (2.4.2). For the proofs of Lemma 2.4.1 and Theorem 2.4.2, please see Shreve (2004) and Elliott and Kopp (2005).

Chapter 3

Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space which is equipped with a filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$, where $T > 0$ is a positive constant denoting the time horizon for investment and reinsurance. We start from the classic Cramér-Lundberg model, in which the risk process of the insurer is described as:

$$U(t) = u_0 + pt - \sum_{i=1}^{N(t)} Z_i,$$

where $u_0 \geq 0$ is the initial surplus, p is the constant insurance premium rate, the claim arrival process $\{N(t)\}_{t\in[0,T]}$ is a Poisson process with constant intensity $\lambda > 0$, and the claim sizes $Z_i, i = 1, 2, \ldots,$ are i.i.d. random variables independent of $N(t)$. Suppose that the claim size has finite first and second moments defined as μ_1 and μ_2, respectively. The insurance premium rate p is for simplicity determined by the expected value principle, i.e., $p = (1 + \theta)\lambda\mu_1$, where $\theta > 0$ is the relative safety loading factor of the insurer.

To manage the risk exposures, an insurance company could purchase reinsurance protection. Specifically, we let $f(Z_i)$ denote the portion of the claim retained by the insurer for an incoming claim Z_i. That is, the insurer cedes $Z_i - f(Z_i)$ to the reinsurer. In return, the insurer needs to allocate the corresponding fraction of his insurance premium rate to the reinsurer, which is called reinsurance premium rate and denoted as p^f. Then, after considering reinsurance the insurer's surplus at time t changes to

$$U^f(t) = u_0 + \int_0^t (p - p^f(s))\mathrm{d}s - \sum_{i=1}^{N(t)} f(Z_i). \tag{3.0.1}$$

The reinsurance premium is also calculated via the expected value principle, in contrast to the existing studies where the relative safety loading factor of the reinsurer is a pre-specified and positive constant at all times, we assume that the reinsurer's safety loading could be adjusted according to the reinsurance demand, i.e.,

$$p^f(t) = (1+\eta(t))\mathbb{E}_{\mathbb{P}}(\,\cdot\,), \tag{3.0.2}$$

where $\mathbb{E}_{\mathbb{P}}$ denotes the expectation under probability measure $\mathbb{P}$. Unlike charging the same premium per unit exposure per unit time in the traditional expected value principle, the assumption (3.0.2) has the advantage to show the bargaining process between the insurer and the reinsurer. From (3.0.2) we can see that the reinsurance premium $p^f(t)$ depends on the safety loading factor $\eta(t)$, and $\eta = \{\eta(t) > 0 : 0 \leq t \leq T\}$ is the choice variable of the reinsurer. In the setting considered here, once the safety loading factor is determined, the reinsurance premium is determined. For convenience, we may use these two terms interchangeably unless otherwise stated. Thus, the reinsurance premium payable to the reinsurer at time t is

$$\begin{aligned} p^f(t) &= (1+\eta(t))\mathbb{E}_{\mathbb{P}}\left(\sum_{i=1}^{N(t)}(Z_i - f(Z_i))\right) \\ &= (1+\eta(t))\lambda\left(\mu_1 - \mathbb{E}_{\mathbb{P}}(f(Z_i))\right), \end{aligned}$$

and the net premium rate for the insurer becomes

$$\begin{aligned} p - p^f(t) &= (1+\theta)\lambda\mu_1 - (1+\eta(t))\lambda(\mu_1 - \mathbb{E}_{\mathbb{P}}(f(Z_i))) \\ &= (\theta - \eta(t))\lambda\mu_1 + (1+\eta(t))\lambda\mathbb{E}_{\mathbb{P}}(f(Z_i)). \end{aligned}$$

According to Grandell (1990), (3.0.1) can be approximated by the following diffusion process:

$$U^f(t) = u_0 + \int_0^t \left[(\theta - \eta(s))\lambda\mu_1 + \lambda\eta(s)\mathbb{E}_{\mathbb{P}}(f(Z_i))\right] \mathrm{d}s + \int_0^t \sqrt{\lambda\mathbb{E}_{\mathbb{P}}(f(Z_i))^2}\mathrm{d}W(s),$$

where $\{W(t)\}_{t\in[0,T]}$ is a standard Brownian motion.

In this thesis, we focus on proportional reinsurance policy, and leave the study on non-proportional reinsurance treaty such as excess-of-loss reinsurance to our future research. Under the proportional reinsurance, we have $f(Z_i) = q(t)Z_i$, where $\{q(t)\}_{t\in[0,T]}$ is $\{\mathscr{F}_t\}_{t\in[0,T]}$-adapted. Besides the reinsurance exchange, the insurer and reinsurer are allowed to invest all their surpluses in one risk-free asset continuously compounded with risk-free interest rate $r > 0$. The price process $S(t)$ of the risk-free asset is given by the following ordinary differential equation (ODE):

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t.$$

Therefore, under the proportional reinsurance strategy $q = \{q(t) \in [0,1] : 0 \leq t \leq T\}$ and the investment opportunity, the insurer's controlled surplus process is governed by

$$\begin{aligned} \mathrm{d}X^q(t) &= rX^q(t)\mathrm{d}t + \left[\lambda\mu_1(\theta - \eta(t)) + \lambda\mu_1\eta(t)q(t)\right]\mathrm{d}t + \sqrt{\lambda\mu_2}q(t)\mathrm{d}W(t) \\ &= rX^q(t)\mathrm{d}t + \lambda\mu_1(\theta - \eta(t) + \eta(t)q(t))\mathrm{d}t + \sqrt{\lambda\mu_2}q(t)\mathrm{d}W(t). \end{aligned} \tag{3.0.3}$$

Similarly, the surplus process of the reinsurer is expressed as follows:

$$\begin{aligned} \mathrm{d}Y^\eta(t) &= rY^\eta(t)\mathrm{d}t + (1+\eta(t))\lambda(\mu_1 - \mu_1 q(t))\mathrm{d}t - \lambda\mu_1(1-q(t))\mathrm{d}t \\ &\quad + \sqrt{\lambda\mu_2}(1-q(t))\mathrm{d}W(t) \\ &= rY^\eta(t)\mathrm{d}t + \lambda\mu_1\eta(t)(1-q(t))\mathrm{d}t + \sqrt{\lambda\mu_2}(1-q(t))\mathrm{d}W(t). \end{aligned} \tag{3.0.4}$$

3.1 Dynamic VaR constraint

Solvency II, which is referred to as "Basel for insurers", is a major regulatory framework for capital adequacy for the insurance industry in the European Union (EU) region. As pointed out by Asimit et al. (2016) and Weber (2018), one of the main aspects of Solvency II is that the insurance company should satisfy the Solvency Capital Requirement (SCR) and the Minimum Capital Requirement (MCR). These requirements are helpful to promote the solvency of the insurance company and the stability of the insurance market. Considering that VaR is one of the most commonly used risk measures in risk management and that it is used to compute the capital requirements under Solvency II, we apply VaR to determine the necessary capitals that an insurance company must hold to withstand risks in our model setting.

Applying the Itô's formula, SDE (3.0.3) admits a solution

$$\begin{aligned} X^q(s) &= e^{r(s-t)}X^q(t) + \int_t^s e^{r(s-l)}\lambda\mu_1(\theta - \eta(l) + \eta(l)q(l))\mathrm{d}l \\ &\quad + \int_t^s e^{r(s-l)}q(l)\sqrt{\lambda\mu_2}\mathrm{d}W(l). \end{aligned} \tag{3.1.1}$$

For a small enough $h > 0$, taking $s = t + h$, and for any l in the interval $[t, t+h)$, inspired by Yiu et al. (2010), we approximate $q(l)$ by $q(t)$, i.e., $q(l) \doteqdot q(t)$, for $\forall\, l \in [t, t+h)$. This is a reasonable approximation because the reinsurance policy can only be adjusted at discrete time and the decision is made based on the surplus at time t. Hence, (3.1.1) becomes

$$\begin{aligned} X^q(t+h) \doteqdot e^{rh}X^q(t) + \frac{e^{rh}-1}{r}[\lambda\mu_1(\theta - \eta(t) + \eta(t)q(t))] \\ + \sqrt{\lambda\mu_2}q(t)\int_t^{t+h} e^{r(t+h-l)}\mathrm{d}W(l). \end{aligned}$$

Following Chen et al. (2010), we define the loss in interval $[t, t+h)$ by

$$\Delta X^q(t) := e^{rh}X^q(t) - X^q(t+h).$$

For a given probability level $\alpha \in (0,1)$ and a given time horizon $h > 0$, the VaR at time t, denoted by $\mathrm{VaR}_t^{\alpha,h}$, is defined by

$$\begin{aligned} \mathrm{VaR}_t^{\alpha,h} &: = \inf\{L \geq 0 : \mathbb{P}(\Delta X^q(t) \geq L|\mathscr{F}_t) < \alpha\} \\ &= (Q_t^{\alpha,h})^-, \end{aligned}$$

where

$$Q_t^{\alpha,h} := \sup\{L \in \mathbb{R} : \mathbb{P}(-\Delta X^q(t) \leq L|\mathscr{F}_t) < \alpha\}.$$

We denote $x^- = \max\{0, -x\}$, $\Phi(\cdot)$ the cumulative distribution of a standard normal random variable, and $\Phi^{-1}(\cdot)$ the inverse function of $\Phi(\cdot)$.

Proposition 3.1.1 *Given time interval h and probability level $\alpha \in (0,1)$, we have*

$$\mathrm{VaR}_t^{\alpha,h} = \left[-\frac{e^{rh}-1}{r}\lambda\mu_1(\theta - \eta(t) + \eta(t)q(t)) - \Phi^{-1}(\alpha)q(t)\sqrt{\frac{e^{2rh}-1}{2r}\lambda\mu_2}\right]^+.$$

Proof. The proof is standard, please see Appendix A. ■

Then the constraint that the level of dynamic VaR is subject to a constant $R > 0$ at all times is given by

$$\left[-\frac{e^{rh}-1}{r}\lambda\mu_1(\theta - \eta(t) + \eta(t)q(t)) - \Phi^{-1}(\alpha)q(t)\sqrt{\frac{e^{2rh}-1}{2r}\lambda\mu_2}\right]^+ \leq R. \quad (3.1.2)$$

3.2 Agent's problem

In most of the literature, the relative safety loading of the reinsurer is often assumed to be fixed whatever the amount of reinsurance purchased by the insurer is, which does not seem to reflect well the strategic interaction between the reinsurer and the insurer. Under our assumption, the risk loading factor in the reinsurance premium principle is time-varying, as described in Section 2.2, which aims to provide flexibility to analyze the optimal risk sharing and reinsurance pricing in the principal-agent modelling framework. In this section we describe the agent's problem, and we would move on to the principal's problem in the next section.

In order to derive explicit results, we suppose that the insurer has an exponential utility, i.e.,

$$U_I(x) = -\frac{1}{n}e^{-nx},$$

where $n > 0$ is the insurer's constant absolute risk aversion (CARA) coefficient. The insurer's objective is to maximize the expected utility of his terminal surplus by choosing the optimal risk retention level $q(t)$ for a given reinsurance premium $\eta(t)$, that is, his value function is defined as

$$V^I(t,x) := \sup_q \mathbb{E}_{\mathbb{P}}\left[-\frac{1}{n}\exp\left(-nX^q(T)\right)\right],$$

where $q(t)$ is supposed to satisfy the solvency requirement (3.1.2).

Assumption 3.2.1 *In the current thesis, we assume that the insurer's value function, denoted by $V^I(t,x)$, satisfies the following two conditions:*

$$\frac{\partial V^I(t,x)}{\partial x} > 0, \qquad \frac{\partial^2 V^I(t,x)}{\partial x^2} < 0.$$

We define an operator

$$\mathcal{L}^q V^I(t,x) := \left[rx + \lambda\mu_1(\theta - \eta + \eta q)\right] V_x^I(t,x) + \frac{1}{2}\lambda\mu_2 q^2 V_{xx}^I(t,x),$$

where $V_x^I(t,x)$ and $V_{xx}^I(t,x)$ represent the first-order and second-order partial derivatives of $V^I(t,x)$ with respect to x, respectively. Applying the stochastic dynamic programming method presented in Section 2.1, we can derive the following HJB equation that $V^I(t,x)$ should satisfy

$$-V_t^I(t,x) - \sup_q \mathcal{L}^q V^I(t,x) = 0, \tag{3.2.1}$$

subject to the terminal condition $V^I(T,x) = -\frac{1}{n}e^{-nx}$, where $V_t^I(t,x)$ is the first-order partial derivative of $V^I(t,x)$ with respect to t. Here, we require that the value function $V^I(t,x)$ and its partial derivatives $V_t^I(t,x), V_x^I(t,x), V_{xx}^I(t,x)$ are continuous on $[0,T]\times R$. Consequently, a classical solution to the HJB equation for $V^I(t,x)$ in (3.2.1) is considered. Therefore, the agent's optimization problem can be formulated as follows:

$$\begin{cases} \sup\limits_q \mathcal{L}^q V^I(t,x), \\ \text{subject to} \begin{cases} -\frac{e^{rh}-1}{r}[\lambda\mu_1(\theta-\eta+\eta q)] - \Phi^{-1}(\alpha) q\sqrt{\frac{e^{2rh}-1}{2r}\lambda\mu_2} \le R, \\ (X^q(t), q) \text{ satisfy (3.0.3)}. \end{cases} \end{cases} \tag{3.2.2}$$

We say a reinsurance contract is incentive compatible if and only if the insurer's risk retention level q solves the constrained optimization problem (3.2.2). In order to solve (3.2.2), we first rewrite

$$\sup_q \left\{\mathcal{L}^q V^I(t,x)\right\} = \inf_q \left\{-\mathcal{L}^q V^I(t,x)\right\},$$

and then introduce the Lagrange function

$$\begin{aligned} L(q,\tilde{\lambda}) = &- \left[rx + \lambda\mu_1(\theta-\eta+\eta q)\right] V_x^I(t,x) - \frac{1}{2}\lambda\mu_2 q^2 V_{xx}^I(t,x) \\ &+ \tilde{\lambda}\left\{-\frac{e^{rh}-1}{r}[\lambda\mu_1(\theta-\eta+\eta q)] - \Phi^{-1}(\alpha) q\sqrt{\frac{e^{2rh}-1}{2r}\lambda\mu_2} - R\right\}, \end{aligned}$$

where $\tilde{\lambda}$ is the Lagrange multiplier. By the Karush-Kuhn-Tucker (KKT) conditions introduced in Section 2.3, we have

$$\begin{cases} \dfrac{\partial L(q,\tilde{\lambda})}{\partial q} = -\lambda\mu_1\eta V_x^I(t,x) - \lambda\mu_2 q V_{xx}^I(t,x) \\ \qquad\qquad + \tilde{\lambda}\left[-\dfrac{e^{rh}-1}{r}\lambda\mu_1\eta - \Phi^{-1}(\alpha)\sqrt{\dfrac{e^{2rh}-1}{2r}\lambda\mu_2}\right] = 0, & (3.2.3a) \\ \tilde{\lambda}\left\{-\dfrac{e^{rh}-1}{r}[\lambda\mu_1(\theta-\eta+\eta q)] - \Phi^{-1}(\alpha) q\sqrt{\dfrac{e^{2rh}-1}{2r}\lambda\mu_2} - R\right\} = 0, & (3.2.3b) \\ -\dfrac{e^{rh}-1}{r}[\lambda\mu_1(\theta-\eta+\eta q)] - \Phi^{-1}(\alpha) q\sqrt{\dfrac{e^{2rh}-1}{2r}\lambda\mu_2} - R \le 0, & (3.2.3c) \\ \tilde{\lambda} \ge 0. & (3.2.3d) \end{cases}$$

We discuss the following two possible scenarios to find the candidate optimal points.

(I) If the VaR constraint is binding, i.e., $\tilde{\lambda} > 0$, the insurer's optimal response to a given η can be obtained from (3.2.3b) :

$$q^* = \frac{C_2 + \lambda\mu_1(\theta - \eta)}{C_1 - \lambda\mu_1\eta}, \tag{3.2.4}$$

where

$$\begin{cases} C_1 = -\Phi^{-1}(\alpha)\sqrt{\frac{r\lambda\mu_2(e^{2rh}-1)}{2(e^{rh}-1)^2}}, \\ C_2 = \frac{Rr}{e^{rh}-1}. \end{cases}$$

Accordingly, we can calculate the derivative of q^* with respect to η as follows:

$$\frac{\partial q^*}{\partial \eta} = \frac{\lambda^2\mu_1^2\theta + \lambda\mu_1(C_2 - C_1)}{(C_1 - \lambda\mu_1\eta)^2}. \tag{3.2.5}$$

(II) If the VaR constraint is inactive, i.e., $\tilde{\lambda} = 0$, (3.2.3a) reduces to

$$-\lambda\mu_1\eta V_x^I(t,x) - \lambda\mu_2 q V_{xx}^I(t,x) = 0,$$

which implies that the insurer's optimal risk retention level q responding to reinsurance premium η is given by

$$q^* = -\frac{\mu_1\eta V_x^I(t,x)}{\mu_2 V_{xx}^I(t,x)}. \tag{3.2.6}$$

Recalling Assumption 3.2.1, we know that the insurer's optimal retained proportion q^* of the risk in (3.2.6) is linearly increasing with respect to the given reinsurance premium η. This conclusion is in line with the economic intuition that the insurer determines the optimal reinsurance demand when he is given the information about the reinsurance premium, and the optimal reinsurance demand $1 - q^*$ should be decreasing with respect to the reinsurance premium, which is equivalent to that the optimal risk retention proportion q^* is an increasing function of the reinsurance premium η. This appears to be consistent with the law of demand in economic theory. To guarantee that the expression of q^* in (3.2.4) also has this property, we require that the expression in (3.2.5) is positive, i.e.,

$$\frac{\lambda^2\mu_1^2\theta + \lambda\mu_1(C_2 - C_1)}{(C_1 - \lambda\mu_1\eta)^2} > 0,$$

which is equivalent to

$$\lambda\mu_1\theta + C_2 - C_1 > 0. \tag{3.2.7}$$

3.3 Principal's problem

We also assume that the reinsurer has a CARA utility function denoted as

$$U_R(y) = -\frac{1}{m}e^{-my},$$

where $m > 0$ represents the risk preference of the reinsurer. In the classical frameworks of the optimal reinsurance contract problems, the reinsurer is also assumed to be ambiguity-neutral in her decision, which implies that she completely believes that the model provided by statistical estimation in probability measure $\mathbb{P}$ is the true model; in this thesis we call this model as the reference model. Under this setup, the ambiguity-neutral reinsurer (ANR) aims to seek the optimal reinsurance premium η^* to maximize the expected utility of her terminal wealth $Y^\eta(T)$, which is expressed as the following optimization problem:

$$\sup_\eta \mathbb{E}_\mathbb{P}\left[-\frac{1}{m}\exp\left(-mY^\eta(T)\right)\right]. \tag{3.3.1}$$

However, in a realistic environment, there may exist some misspecification errors in the model parameters, which induces that the reinsurer doesn't have full confidence in the reference model. For this reason, it is interesting to take model uncertainty into account and study the optimal reinsurance contract problem when the reinsurer is ambiguity-averse.

In this thesis, we adopt the diffusion model to approximate the true claims model, where the latter is often described by a Poisson jump model. So the reinsurer actually has ambiguity aversion attitudes towards the approximating claim process under the probability measure $\mathbb{P}$, and she would like to evaluate her expected utility under the alternative probability measures. We define a class of probability measures which are equivalent to $\mathbb{P}$:

$$\mathcal{Q} := \{\mathbb{Q}|\mathbb{Q} \sim \mathbb{P}\}.$$

According to Girsanov's Theorem stated in Section 2.4, for $\forall\ \mathbb{Q} \in \mathcal{Q}$, there exists a measurable real-valued process $\nu(t)$ such that

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \nu(t),$$

where

$$\nu(t) = \exp\left\{\int_0^t l(s)\mathrm{d}W(s) - \frac{1}{2}\int_0^t l^2(s)\mathrm{d}s\right\} \tag{3.3.2}$$

is a $\left(\{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$-martingale. Following Miao and Rivera (2016), we call $l(t)$ the density generator which satisfies the Novikov's condition

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T l^2(t)\mathrm{d}t\right)\right] < \infty.$$

Under a new probability measure $\mathbb{Q}$, the stochastic process $\{W_{\mathbb{Q}}(t)\}_{t\in[0,T]}$ satisfying

$$\mathrm{d}W_{\mathbb{Q}}(t) = \mathrm{d}W(t) - l(t)\mathrm{d}t$$

is a standard Brownian motion. Accordingly, the reinsurer's surplus process under the measure $\mathbb{Q}$ is governed by the following stochastic differential equation (SDE):

$$\begin{aligned}\mathrm{d}Y^{\eta}(t) = {} & rY^{\eta}(t)\mathrm{d}t + \left[\lambda\mu_1\eta(t)(1-q(t)) + \sqrt{\lambda\mu_2}(1-q(t))l(t)\right]\mathrm{d}t \\ & + \sqrt{\lambda\mu_2}(1-q(t))\mathrm{d}W_{\mathbb{Q}}(t).\end{aligned}$$

We use the concept of ambiguity aversion proposed by Maenhout (2004) in such a way that the reinsurer aims to find the robust reinsurance contract by looking for the optimal reinsurance premium η under the worst-case scenario of the alternative measures. Thus, the reinsurer's value function in the presence of robustness preference is given by

$$V^R(t,y) := \sup_{\eta}\inf_{\mathbb{Q}}\mathbb{E}_{\mathbb{Q}}\left[-\frac{1}{m}\exp(-mY^{\eta}(T)) + P(\mathbb{P}\|\mathbb{Q})\right],$$

where $\mathbb{E}_{\mathbb{Q}}$ denotes the expectation under the alternative probability measure $\mathbb{Q}$, and $P(\mathbb{P}\|\mathbb{Q}) \geq 0$ is a penalty function measuring the divergence of $\mathbb{Q}$ from $\mathbb{P}$. In the perspective of an ambiguity-averse reinsurer (AAR), the model under probability measure $\mathbb{P}$ is an estimation of the true model in the real world, and she is sceptical about this reference model because of the misspecification error and aims to consider the alternative models under probability measure $\mathbb{Q}$. In the case of $P(\mathbb{P}\|\mathbb{Q}) \to \infty$, the reinsurer is convinced that the reference model is the true model and any alternative models deviating from this reference model will be heavily penalized. On the other hand, if $P(\mathbb{P}\|\mathbb{Q}) \to 0$, i.e., the penalty function term vanishes, the reinsurer will not penalize any deviation from the reference model, which implies that the decision maker is extremely ambiguous. With a view to deriving a closed-form solution of the optimal reinsurance contract, we apply the assumption in Maenhout (2004) and use a relative entropy to measure the deviation of alternative measure $\mathbb{Q}$ from reference measure $\mathbb{P}$. In Appendix B, it can be seen that the increase in relative entropy form t to $t + \mathrm{d}t$ equals $\frac{1}{2}l^2(t)$, under this circumstance, the state-dependent penalty function is defined as

$$P(\mathbb{P}\|\mathbb{Q}) := \int_t^T \Psi\left(s, l(s)\right) \mathrm{d}s,$$

with

$$\Psi\left(s, l(s)\right) = -\frac{ml^2(s)V^R(s,y)}{2\beta},$$

where $\beta \geq 0$ is the ambiguity aversion coefficient of the reinsurer, which describes the degree of her ambiguity attitude with respect to the diffusion risk. When $\beta = 0$, $P(\mathbb{P}\|\mathbb{Q}) \to \infty$ and the reinsurer is ambiguity-neutral towards the diffusion risk.

According to the dynamic programming principle, the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation satisfied by the value function $V^R(t,y)$ of the reinsurer can be derived as

$$\sup_{\eta} \inf_{l} \left\{ \mathcal{L}^{q^*,\eta,l} - \frac{ml^2V^R(t,y)}{2\beta} \right\} = 0, \tag{3.3.3}$$

with boundary condition $V^R(T,y) = -\frac{1}{m}e^{-my}$, and the operator $\mathcal{L}^{q,\eta,l}$ is defined as

$$\begin{aligned}\mathcal{L}^{q,\eta,l}V^R(t,y) := V_t^R(t,y) + \left[ry + \lambda\mu_1\eta(1-q) + \sqrt{\lambda\mu_2}(1-q)l\right] V_y^R(t,y) \\ + \frac{1}{2}\lambda\mu_2(1-q)^2 V_{yy}^R(t,y).\end{aligned}$$

The reinsurer designs the reinsurance contract to meet her own objectives, but also wants the contract to be attractive for the insurer. Hence, she recommends the risk retention level q^* that satisfies the incentive compatibility constraint (3.2.2) and the reinsurance premium η^* that solves the optimization problem (3.3.3).

Chapter 4

Robust proportional reinsurance contract

In this chapter, we shall derive the robust optimal reinsurance contract by solving the insurer's HJB equation and the reinsurer's HJBI equation. Firstly, the insurer's value function $V^I(t,x)$ is assumed to have the following form

$$V^I(t,x) = -\frac{1}{n}\exp\left\{-nxe^{r(T-t)} + f(t)\right\},$$

where $f(t)$ is a deterministic function with $f(T) = 0$. This is a trial solution to the value function of the insurer. By some straightforward calculations, we have

$$\begin{cases} V_t^I(t,x) = \left(rnxe^{r(T-t)} + f'(t)\right)V^I(t,x), \\ V_x^I(t,x) = -ne^{r(T-t)}V^I(t,x), \\ V_{xx}^I(t,x) = n^2e^{2r(T-t)}V^I(t,x). \end{cases} \tag{4.0.1}$$

Similarly, we conjecture that the reinsurer's value function $V^R(t,y)$ has the following form

$$V^R(t,y) = -\frac{1}{m}\exp\left\{-mye^{r(T-t)} + g(t)\right\},$$

where $g(t)$ is a function to be determined with $g(T) = 0$. This is, again, a trial solution to the value function of the reinsurer. Note that the conjectured forms of the value functions for the insurer and the reinsurer are imposed based on the form of the exponential utility function. A direct calculation also yields

$$\begin{cases} V_t^R(t,y) = \left(rmye^{r(T-t)} + g'(t)\right)V^R(t,y), \\ V_y^R(t,y) = -me^{r(T-t)}V^R(t,y), \\ V_{yy}^R(t,y) = m^2e^{2r(T-t)}V^R(t,y). \end{cases} \tag{4.0.2}$$

If we insert the derivatives in (4.0.1) into (3.2.1), we can get the HJB equation of the insurer as follows

$$\sup_q \left\{ f'(t) - ne^{r(T-t)}\lambda\mu_1 \left[\theta + \eta(q-1)\right] + \frac{1}{2}\lambda\mu_2 n^2 q^2 e^{2r(T-t)} \right\} = 0. \tag{4.0.3}$$

4.1 Case I: VaR constraint is binding for the insurer

Substituting the expression of q^* in (3.2.4) and the corresponding derivatives in (4.0.2) to the HJBI equation (3.3.3), we can solve the robust optimal reinsurance premium when the dynamic VaR constraint is active for the insurer. The HJBI equation turns into

$$\begin{aligned} \sup_\eta \inf_l \Bigg\{ & g'(t) - \frac{me^{r(T-t)}\left(C_1 - C_2 - \lambda\mu_1\theta\right)}{C_1 - \lambda\mu_1\eta} \left(\lambda\mu_1\eta + \sqrt{\lambda\mu_2}l\right) \\ & + \frac{1}{2}\lambda\mu_2 m^2 e^{2r(T-t)} \left(\frac{C_1 - C_2 - \lambda\mu_1\theta}{C_1 - \lambda\mu_1\eta}\right)^2 - \frac{ml^2}{2\beta} \Bigg\} = 0. \end{aligned} \tag{4.1.1}$$

We first concentrate on the minimization part in (4.1.1). Fixing η and letting the first-order derivative of the left-hand side of (4.1.1) with respect to l equal zero, we obtain

$$-me^{r(T-t)}\sqrt{\lambda\mu_2}\frac{C_1 - C_2 - \lambda\mu_1\theta}{C_1 - \lambda\mu_1\eta} - \frac{ml}{\beta} = 0,$$

which implies that the worst-case density generator is given by

$$l^* = -e^{r(T-t)}\beta\sqrt{\lambda\mu_2}\frac{C_1 - C_2 - \lambda\mu_1\theta}{C_1 - \lambda\mu_1\eta}. \tag{4.1.2}$$

Inserting (4.1.2) into the HJBI equation (4.1.1), we get

$$\begin{aligned} \sup_\eta \Bigg\{ & g'(t) - \lambda\mu_1\eta me^{r(T-t)}\frac{C_1 - C_2 - \lambda\mu_1\theta}{C_1 - \lambda\mu_1\eta} \\ & + \frac{1}{2}m\lambda\mu_2 e^{2r(T-t)}(\beta + m)\left(\frac{C_1 - C_2 - \lambda\mu_1\theta}{C_1 - \lambda\mu_1\eta}\right)^2 \Bigg\} = 0. \end{aligned} \tag{4.1.3}$$

Then the first-order optimality condition for η gives that the maximum of the left hand side of HJBI equation is attained at

$$\eta^*(t) = \frac{C_1^2 - \lambda\mu_2 e^{r(T-t)}(\beta + m)(C_1 - C_2 - \lambda\mu_1\theta)}{C_1\lambda\mu_1}. \tag{4.1.4}$$

Plugging (4.1.4) back into (3.2.4), we can obtain the following expression of $q^*(t)$

$$q^*(t) = 1 - \frac{C_1}{\lambda\mu_2 e^{r(T-t)}(\beta + m)}. \tag{4.1.5}$$

Substituting (4.1.4) into (4.1.2) yields the worst-case density generator as follows

$$l^*(t) = -\frac{C_1\beta}{\sqrt{\lambda\mu_2}(\beta + m)}.$$

In the following, we will first derive the reinsurer's value function. Putting the expression of $\eta^*(t)$ in (4.1.4) back into (4.1.3), we get the ODE that $g(t)$ should satisfy as follows:

$$g'(t) + m(C_1 - C_2 - \lambda\mu_1\theta)e^{r(T-t)} - \frac{mC_1^2}{2\lambda\mu_2(\beta + m)} = 0. \tag{4.1.6}$$

Recalling the terminal condition $g(T) = 0$, we have that the solution to (4.1.6) is

$$\begin{aligned} g(t) &= \int_t^T m(C_1 - C_2 - \lambda\mu_1\theta)e^{r(T-s)}\mathrm{d}s - \int_t^T \frac{mC_1^2}{2\lambda\mu_2(\beta + m)}\mathrm{d}s \\ &= \frac{m(C_1 - C_2 - \lambda\mu_1\theta)}{r}\left(e^{r(T-t)} - 1\right) - \frac{mC_1^2(T-t)}{2\lambda\mu_2(\beta + m)}. \end{aligned}$$

Similar to solving the reinsurer's value function, we need to solve the function $f(t)$ in the insurer's value function. (4.1.4) and (4.1.5) readily imply that the insurer's HJB equation in (4.0.3) becomes

$$\begin{aligned} f'(t) + \frac{1}{2}\lambda\mu_2 n^2 e^{2r(T-t)} - ne^{r(T-t)}\left(C_1 - C_2 + \frac{C_1 n}{\beta + m}\right) \\ + \frac{nC_1^2}{\lambda\mu_2(\beta + m)} + \frac{C_1^2 n^2}{2\lambda\mu_2(\beta + m)^2} = 0, \end{aligned} \tag{4.1.7}$$

with the terminal condition $f(T) = 0$. Solving the ordinary differential equation (ODE) in (4.1.7), we have

$$\begin{aligned} f(t) &= \int_t^T \frac{1}{2}\lambda\mu_2 n^2 e^{2r(T-s)}\mathrm{d}s - \int_t^T ne^{r(T-s)}\left(C_1 - C_2 + \frac{C_1 n}{\beta + m}\right)\mathrm{d}s \\ &\quad + \int_t^T \left(\frac{nC_1^2}{\lambda\mu_2(\beta + m)} + \frac{C_1^2 n^2}{2\lambda\mu_2(\beta + m)^2}\right)\mathrm{d}s \\ &= \frac{\lambda\mu_2 n^2\left(e^{2r(T-t)} - 1\right)}{4r} - n\left(C_1 - C_2 + \frac{C_1 n}{\beta + m}\right)\frac{e^{r(T-t)} - 1}{r} \\ &\quad + \frac{2nC_1^2(\beta + m) + C_1^2 n^2}{2\lambda\mu_2(\beta + m)^2}. \end{aligned}$$

After solving the insurer's value function $V^I(t,x)$, we proceed to determine the Lagrange multiplier $\tilde{\lambda}$. From (3.2.3a), we have

$$\tilde{\lambda} = \frac{\lambda\mu_1\eta^*(t)V_x^I(t,x) + \lambda\mu_2 q^*(t)V_{xx}^I(t,x)}{-\frac{e^{rh}-1}{r}\lambda\mu_1\eta^*(t) - \Phi^{-1}(\alpha)\sqrt{\frac{e^{2rh}-1}{2r}\lambda\mu_2}}. \tag{4.1.8}$$

Substituting $\eta^*(t)$, $q^*(t)$ and the related derivatives of $V^I(t,x)$ into (4.1.8), we derive $\tilde{\lambda}$ in the following form

$$\tilde{\lambda} = \frac{-A_1(t)ne^{r(T-t)}V^I(t,x) + A_2(t)n^2e^{2r(T-t)}V^I(t,x)}{C_3 - \frac{e^{rh}-1}{r}A_1(t)}, \tag{4.1.9}$$

where

$$\begin{cases} A_1(t) = C_1^2 - \lambda\mu_2 e^{r(T-t)}(\beta+m)(C_1 - C_2 - \lambda\mu_1\theta), \\ A_2(t) = \lambda\mu_2 C_1 - \frac{C_1^2}{e^{r(T-t)}(\beta+m)}, \\ C_3 = \frac{\left(\Phi^{-1}(\alpha)\right)^2\lambda\mu_2(e^{2rh}-1)}{2(e^{rh}-1)}. \end{cases}$$

We need to impose some additionally restricted conditions to guarantee that the Lagrange multiplier $\tilde{\lambda}$ is positive, and these conditions are presented in the following theorem.

Theorem 4.1.1 *The parameters in* (4.1.9) *should satisfy anyone of the following conditions:*

(1)

$$ne^{r(T-t)}A_2(t) < A_1(t) < \frac{rC_3}{e^{rh}-1},$$

and

$$\lambda\mu_1\theta + C_2 - C_1 > 0.$$

(2)

$$\frac{rC_3}{e^{rh}-1} < A_1(t) < ne^{r(T-t)}A_2(t),$$

and

$$\lambda\mu_1\theta + C_2 - C_1 > 0.$$

Proof. Normally, we take $0 < \alpha < \frac{1}{2}$, and hence C_1 is always positive. Additionally, to ensure $\eta^*(t)$ in (4.1.4) is positive, we must have $A_1(t) > 0$. Since $q^*(t)$ should be within $[0,1]$, we have

$$0 \le \frac{\lambda\mu_2 e^{r(T-t)}(\beta+m) - C_1}{\lambda\mu_2 e^{r(T-t)}(\beta+m)} \le 1,$$

which is equivalent to

$$0 \leq \lambda\mu_2 C_1 - \frac{C_1^2}{e^{r(T-t)}(\beta+m)} \leq \lambda\mu_2 C_1,$$

and this implies $A_2(t) \geq 0$. It is obvious that C_3 is positive and accordingly, it is sufficient to require the numerator and the denominator in (4.1.9) to be positive and negative at the same time to make a positive $\tilde{\lambda}$.

Therefore, we have two possible cases:

(1)
$$\begin{cases} -A_1(t)ne^{r(T-t)}V^I(t,x) > -A_2(t)n^2e^{2r(T-t)}V^I(t,x), \\ C_3 > \frac{e^{rh}-1}{r}A_1(t), \end{cases}$$

which is equivalent to

$$ne^{r(T-t)}A_2(t) < A_1(t) < \frac{rC_3}{e^{rh}-1},$$

and

(2)
$$\begin{cases} -A_1(t)ne^{r(T-t)}V^I(t,x) < -A_2(t)n^2e^{2r(T-t)}V^I(t,x), \\ C_3 < \frac{e^{rh}-1}{r}A_1(t), \end{cases}$$

which is equivalent to

$$\frac{rC_3}{e^{rh}-1} < A_1(t) < ne^{r(T-t)}A_2(t).$$

Finally, combining with the condition in (3.2.7), we complete the proof. ∎

We now summarize the main results of this case in the following theorem.

Theorem 4.1.2 *When the VaR constraint is binding for the insurer, the robust optimal reinsurance contract is given by*

$$\begin{cases} q^*(t) = 1 - \frac{C_1}{\lambda\mu_2 e^{r(T-t)}(\beta+m)}, \\ \eta^*(t) = \frac{C_1^2 - \lambda\mu_2 e^{r(T-t)}(\beta+m)(C_1-C_2-\lambda\mu_1\theta)}{C_1\lambda\mu_1}, \end{cases} \tag{4.1.10}$$

and the worst-case density generator is

$$l^*(t) = -\frac{C_1\beta}{\sqrt{\lambda\mu_2}(\beta+m)}. \tag{4.1.11}$$

Under this robust optimal reinsurance contract, the reinsurer's optimal value function is given by

$$V^R(t,y) = -\frac{1}{m}\exp\left\{-mye^{r(T-t)} + g_1(t)\right\}, \tag{4.1.12}$$

where

$$g_1(t) = \frac{m(C_1 - C_2 - \lambda\mu_1\theta)}{r}\left(e^{r(T-t)} - 1\right) - \frac{mC_1^2(T-t)}{2\lambda\mu_2(\beta+m)},$$

and the insurer's optimal value function is

$$V^I(t,x) = -\frac{1}{n}\exp\left\{-nxe^{r(T-t)} + f_1(t)\right\},$$

where

$$\begin{aligned} f_1(t) = & \frac{\lambda\mu_2 n^2\left(e^{2r(T-t)}-1\right)}{4r} - n\left(C_1 - C_2 + \frac{C_1 n}{\beta+m}\right)\frac{e^{r(T-t)}-1}{r} \\ & + \frac{2nC_1^2(\beta+m) + C_1^2 n^2}{2\lambda\mu_2(\beta+m)^2}. \end{aligned}$$

The Lagrange multiplier is given by (4.1.9)*, and the parameters in it should satisfy one of the conditions in Theorem* 4.1.1*.*

4.2 Case II: VaR constraint is inactive for the insurer

Inserting the expressions of $V^I_{xx}(t,x)$ and $V^I_x(t,x)$ in (4.0.1) into (3.2.6), we have that the optimal risk retention level for the insurer when the VaR constraint is inactive is given by

$$q^* = \frac{\mu_1\eta}{\mu_2 n e^{r(T-t)}}. \tag{4.2.1}$$

If we put (4.2.1) and the associated derivatives in (4.0.2) back into (3.3.3), the HJBI equation for the reinsurer becomes

$$\begin{aligned} \sup_\eta \inf_l \Bigg\{ & g'(t) - \frac{m\left(\mu_2 n e^{r(T-t)} - \mu_1\eta\right)}{\mu_2 n}\left(\lambda\mu_1\eta + \sqrt{\lambda\mu_2}\,l\right) \\ & + \frac{\lambda m^2\left(\mu_2 n e^{r(T-t)} - \mu_1\eta\right)^2}{2n^2\mu_2} - \frac{ml^2}{2\beta}\Bigg\} = 0. \end{aligned} \tag{4.2.2}$$

Fixing η and according to the first-order conditions, we can easily obtain the minimum point l^* given by

$$l^* = -\frac{\beta\sqrt{\lambda}\left(\mu_2 n e^{r(T-t)} - \mu_1\eta\right)}{n\sqrt{\mu_2}}. \tag{4.2.3}$$

Substituting (4.2.3) into HJBI equation (4.2.2), we get the following maximization problem with respect to $\eta(t)$,

$$\begin{aligned}\sup_{\eta}\Bigg\{ g'(t) - \frac{m(\mu_2 n e^{r(T-t)} - \mu_1\eta)}{\mu_2 n}\left(\lambda\mu_1\eta - \frac{\beta\lambda\left(\mu_2 n e^{r(T-t)} - \mu_1\eta\right)}{n}\right) \\ + \frac{\lambda m(m-\beta)\left(\mu_2 n e^{r(T-t)} - \mu_1\eta\right)^2}{2n^2\mu_2}\Bigg\} = 0.\end{aligned} \tag{4.2.4}$$

Applying the first-order conditions over η yields the optimal robust reinsurance premium as follows

$$\eta^*(t) = \frac{\mu_2 n(m+\beta+n)}{\mu_1(m+\beta+2n)} e^{r(T-t)}, \tag{4.2.5}$$

correspondingly, inserting (4.2.5) into (4.2.1) and (4.2.3) yield the optimal reinsurance strategy

$$q^*(t) = \frac{m+\beta+n}{m+\beta+2n} \tag{4.2.6}$$

and the worst-case density generator

$$l^*(t) = -\beta\sqrt{\lambda\mu_2}e^{r(T-t)}\frac{n}{m+\beta+2n}.$$

Now we can solve the optimal value functions of the insurer and the reinsurer. In order to solve the optimal value function of the reinsurer, we need to solve the function $g(t)$ first. Plugging the expression of $\eta^*(t)$ in (4.2.5) into (4.2.4), we obtain the ODE that $g(t)$ should satisfy

$$g'(t) - \frac{\lambda m n^2 \mu_2 e^{2r(T-t)}}{2(m+\beta+2n)} = 0.$$

Combining with the terminal condition $g(T) = 0$, we get

$$g(t) = -\int_t^T \frac{\lambda m n^2 \mu_2 e^{2r(T-s)}}{2(m+\beta+2n)} \mathrm{d}s = \frac{\lambda m n^2 \mu_2}{4r(m+\beta+2n)}\left(1 - e^{2r(T-t)}\right).$$

Putting (4.2.5) and (4.2.6) back into the insurer's HJB equation (4.0.3), we obtain the following ODE that $f(t)$ should satisfy

$$f'(t) - \lambda\mu_1\theta n e^{r(T-t)} + \frac{\lambda\mu_2 n^3(m+\beta+n)e^{2r(T-t)}}{(m+\beta+2n)^2} + \frac{\lambda\mu_2 n^2 e^{2r(T-t)}}{2}\left(\frac{m+\beta+n}{m+\beta+2n}\right)^2 = 0.$$

Combining with the boundary condition $f(T)=0$, we get

$$\begin{aligned}
f(t) &= -\int_t^T \lambda\mu_1\theta n e^{r(T-s)}\mathrm{d}s + \int_t^T \frac{\lambda\mu_2 n^3(m+\beta+n)e^{2r(T-s)}}{(m+\beta+2n)^2}\mathrm{d}s \\
&\quad + \int_t^T \frac{\lambda\mu_2 n^2 e^{2r(T-s)}}{2}\left(\frac{m+\beta+n}{m+\beta+2n}\right)^2 \mathrm{d}s \\
&= \frac{\lambda\mu_1\theta n}{r}\left(1-e^{r(T-t)}\right) + \frac{\lambda\mu_2 n^2(m+\beta+n)(m+\beta+3n)\left(e^{2r(T-t)}-1\right)}{4r(m+\beta+2n)^2}.
\end{aligned}$$

Additionally, the robust optimal reinsurance contract $(q^*(t),\eta^*(t))$ should satisfy the VaR constraint (3.2.3c), and so we have

$$\begin{aligned}
-\frac{e^{rh}-1}{r}&\left[\lambda\mu_1\left(\theta - \frac{\mu_2 n(m+\beta+n)}{\mu_1(m+\beta+2n)}e^{r(T-t)} + \frac{\mu_2 n(m+\beta+n)^2}{\mu_1(m+\beta+2n)^2}e^{r(T-t)}\right)\right] \\
&\qquad - \Phi^{-1}(\alpha)\frac{m+\beta+n}{m+\beta+2n}\sqrt{\frac{e^{2rh}-1}{2r}\lambda\mu_2} - R \le 0,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\lambda\Big(\theta\mu_1(m+\beta+2n)^2 - \mu_2 n(m+\beta+n)(m+\beta+2n)e^{r(T-t)} \\
&+ \mu_2 n(m+\beta+n)^2 e^{r(T-t)}\Big) - C_1(m+\beta+n)(m+\beta+2n) + C_2(m+\beta+2n)^2 \ge 0.
\end{aligned} \tag{4.2.7}$$

Finally, summing up the previous calculations, we arrive at the following theorem of this case.

Theorem 4.2.1 *When the VaR constraint is inactive for the insurer, the robust optimal reinsurance contract is given by*

$$\begin{cases} q^*(t) = \frac{m+\beta+n}{m+\beta+2n}, \\ \eta^*(t) = \frac{\mu_2 n(m+\beta+n)}{\mu_1(m+\beta+2n)}e^{r(T-t)}, \end{cases}$$

and the worst-case density generator is

$$l^*(t) = -\beta\sqrt{\lambda\mu_2}e^{r(T-t)}\frac{n}{m+\beta+2n}. \tag{4.2.8}$$

Under this robust optimal reinsurance contract, the reinsurer's optimal value function is given by

$$V^R(t,y) = -\frac{1}{m}\exp\left\{-mye^{r(T-t)} + g_2(t)\right\},$$

where

$$g_2(t) = \frac{\lambda m n^2\mu_2}{4r(m+\beta+2n)}\left(1-e^{2r(T-t)}\right),$$

and the insurer's optimal value function is

$$V^I(t,x) = -\frac{1}{n}\exp\left\{-nxe^{r(T-t)} + f_2(t)\right\},$$

where

$$f_2(t) = \frac{\lambda\mu_1\theta n}{r}\left(1 - e^{r(T-t)}\right) + \frac{\lambda\mu_2 n^2(m+\beta+n)(m+\beta+3n)\left(e^{2r(T-t)}-1\right)}{4r(m+\beta+2n)^2}.$$

Furthermore, the condition in (4.2.7) *should be satisfied due to the solvency capital requirement.*

Remark 4.2.1 *It should be worth noting that if the dynamic VaR constraint on the insurer to set up capital requirement is not imposed, and under this case the condition in* (4.2.7) *would be discarded, then the results in Theorem* 4.2.1 *coincide with the Theorem 1 in Hu et al. (2018b), i.e., our model extends the robust optimal reinsurance contract in Hu et al. (2018b) to the situation where the risk constraint is present.*

Remark 4.2.2 *From Theorem* 4.1.2*, when the VaR constraint is active for the insurer, the premium rate of reinsurance is given by*

$$(1+\eta(t))\lambda\mu_1 = \lambda\mu_1 + \frac{C_1^2 - \lambda\mu_2(\beta+m)(C_1 - C_2 - \lambda\mu_1\theta)e^{r(T-t)}}{C_1}.$$

On the other hand, according to Theorem 4.2.1*, the premium rate of reinsurance when the VaR constraint is inactive for the insurer is*

$$(1+\eta(t))\lambda\mu_1 = \lambda\mu_1 + \frac{\lambda\mu_2 n(m+\beta+n)e^{r(T-t)}}{m+\beta+2n}.$$

From these two expressions, we can see that the extended expected value premium principle not only encompass the advantages of the net premium principle, the variance premium principle, and the zero utility premium principle, but also incorporate the effects of the time value of money because they contain the discounted term $e^{r(T-t)}$*. In particular, when the risk constraint is binding, the premium rate of reinsurance is also influenced by the parameters arising from the risk constraint such as the confidence level and the capital allowance level.*

Theorem 4.2.2 *The optimal reinsurance premium* $\eta^*(t)$ *given in Theorem 4.1.2 and Theorem 4.2.1 solve the maximization problem in HJBI equation* (3.3.3) *when VaR constraint is active and inactive for the insurer, respectively.*

Proof. We gather the term of η in the left-hand side of HJBI equation (3.3.3) and define

$$h(\eta) := \left[\lambda\mu_1\eta(1-q)+\sqrt{\lambda\mu_2}(l-ql)\right]V_y^R(t,y)+\frac{1}{2}\lambda\mu_2(1-q)^2V_{yy}^R(t,y)-\frac{ml^2V^R(t,y)}{2\beta}.$$

Furthermore, we have

$$\begin{aligned} h'(\eta) = &\left[\lambda\mu_1(1-q) - \lambda\mu_1\eta\frac{\partial q}{\partial\eta} + \sqrt{\lambda\mu_2}(1-q)\frac{\partial l}{\partial\eta} - \sqrt{\lambda\mu_2}l\frac{\partial q}{\partial\eta}\right]V_y^R(t,y) \\ &- \lambda\mu_2(1-q)\frac{\partial q}{\partial\eta}V_{yy}^R(t,y) - \frac{mlV^R(t,y)}{\beta}\frac{\partial l}{\partial\eta}, \end{aligned}$$

and

$$\begin{aligned} h''(\eta) = \Bigg[&- 2\lambda\mu_1\frac{\partial q}{\partial\eta} - \lambda\mu_1\eta\frac{\partial^2 q}{\partial\eta^2} + (1-q)\sqrt{\lambda\mu_2}\frac{\partial^2 l}{\partial\eta^2} - 2\sqrt{\lambda\mu_2}\frac{\partial l}{\partial\eta}\frac{\partial q}{\partial\eta} \\ &- \sqrt{\lambda\mu_2}l\frac{\partial^2 q}{\partial\eta^2}\Bigg]V_y^R(t,y) + \lambda\mu_2\left(\frac{\partial q}{\partial\eta}\right)^2V_{yy}^R(t,y) - \lambda\mu_2(1-q)\frac{\partial^2 q}{\partial\eta^2}V_{yy}^R(t,y) \\ &- \frac{mlV^R(t,y)}{\beta}\frac{\partial^2 l}{\partial\eta^2} - \frac{mV^R(t,y)}{\beta}\left(\frac{\partial l}{\partial\eta}\right)^2. \end{aligned} \tag{4.2.9}$$

1. When VaR constraint is inactive for the insurer, inserting q^* in (4.2.1) and l^* in (4.2.3) into (4.2.9), we have

$$\begin{aligned} h''(\eta) &= \left[-\frac{2\lambda\mu_1^2}{\mu_2ne^{r(T-t)}} - \frac{2\lambda\beta\mu_1^2}{\mu_2n^2e^{r(T-t)}}\right]V_y^R(t,y) + \frac{\lambda\mu_1^2V_{yy}^R(t,y)}{\mu_2n^2e^{2r(T-t)}} - \frac{m\beta\lambda\mu_1^2V^R(t,y)}{n^2\mu_2} \\ &= V^R(t,y)\left[\frac{2\lambda\mu_1^2m}{\mu_2n} + \frac{\lambda\mu_1^2m(\beta+m)}{\mu_2n^2}\right]. \end{aligned}$$

   Since $V^R(t,y) < 0$, it is obvious that $h''(\eta) < 0$ for any reinsurance premium.

2. When VaR constraint is active for the insurer, inserting η^* and q^* in (4.1.10), and l^* in (4.1.11) into (4.2.9), we obtain

$$h''(\eta^*) = \frac{V^R(t,y)}{(C_1-\lambda\mu_1\eta^*)^4}(C_2 - C_1 + \lambda\mu_1\theta)^2(\beta+m)m\lambda^3\mu_1^2\mu_2e^{2r(T-t)} < 0.$$

These results indicate that the second-order partial derivatives at the optimal reinsurance premium obtained by the first-order optimality conditions are negative. In other words, the obtained optimal reinsurance premium in Theorem 4.1.2 and Theorem 4.2.1 indeed solves the maximization problems and gives the reinsurer an optimal reinsurance premium strategy. ■

Chapter 5

Special cases

This chapter provides some special cases of our model: the reinsurer completely trusts the approximating distribution of the claim process, which implies that the reinsure's ambiguity aversion coefficient β equals 0. In other words, it is supposed that the reinsurer is ambiguity-neutral. Then, the reinsurer's wealth process under probability measure $\mathbb{P}$ is expressed by (3.0.4), and the robust optimization problem degenerates into the traditional optimization problem (3.3.1) without the consideration of ambiguity aversion. Denote the reinsurer's value function under this circumstance by

$$\widetilde{V}^R(t,y) = \sup_{\tilde{\eta}} \mathbb{E}\left[-\frac{1}{m}\exp\left(-mY^{\eta}(T)\right)\right].$$

The corresponding HJB equation is

$$\sup_{\tilde{\eta}}\left\{\widetilde{V}_t^R + \left[ry + \lambda\mu_1\tilde{\eta}(1-q^*)\right]\widetilde{V}_y^R + \frac{1}{2}\lambda\mu_2(1-q^*)^2\widetilde{V}_{yy}^R\right\} = 0,$$

where $\widetilde{V}^R$ is a short notation for $\widetilde{V}^R(t,y)$ with the terminal condition $\widetilde{V}^R(t,y) = U_R(y)$.

Corollary 5.0.1 *(1) When the reinsurer is ambiguity-neutral, and the VaR constraint is binding for the insurer, the optimal reinsurance contract is given by*

$$\begin{cases}\tilde{q}^*(t) = 1 - \frac{C_1}{\lambda\mu_2 m e^{r(T-t)}},\\ \tilde{\eta}^*(t) = \frac{C_1^2 - \lambda\mu_2 m e^{r(T-t)}(C_1 - C_2 - \lambda\mu_1\theta)}{C_1\lambda\mu_1}.\end{cases}$$

Correspondingly, the reinsurer's value function is given by

$$\widetilde{V}^R(t,y) = -\frac{1}{m}\exp\left\{-mye^{r(T-t)} + \tilde{g}_1(t)\right\},$$

where

$$\tilde{g}_1(t) = \frac{m(C_1 - C_2 - \lambda\mu_1\theta)}{r}\left(e^{r(T-t)} - 1\right) - \frac{mC_1^2(T-t)}{2\lambda\mu_2 m},$$

and the insurer's optimal value function is

$$\widetilde{V}^I(t,x) = -\frac{1}{n}\exp\left\{-nxe^{r(T-t)} + \tilde{f}_1(t)\right\},$$

where

$$\tilde{f}_1(t) = \frac{\lambda\mu_2 n^2\left(e^{2r(T-t)} - 1\right)}{4r} - n\left(C_1 - C_2 + \frac{C_1 n}{m}\right)\frac{e^{r(T-t)} - 1}{r} + \frac{C_1^2 n(2m+n)}{2\lambda\mu_2 m^2}.$$

(2) When the reinsurer is ambiguity-neutral, and the VaR constraint is inactive for the insurer, the optimal reinsurance contract is derived as

$$\begin{cases} \tilde{q}^*(t) = \frac{m+n}{m+2n}, \\ \tilde{\eta}^*(t) = \frac{\mu_2 n(m+n)}{\mu_1(m+2n)} e^{r(T-t)}. \end{cases}$$

The reinsurer's value function is given by

$$\widetilde{V}^R(t,y) = -\frac{1}{m}\exp\left\{-mye^{r(T-t)} + \tilde{g}_2(t)\right\},$$

where

$$\tilde{g}_2(t) = \frac{\lambda m n^2\mu_2}{4r(m+2n)}\left(1 - e^{2r(T-t)}\right),$$

and the insurer's value function is

$$\widetilde{V}^I(t,x) = -\frac{1}{n}\exp\left\{-nxe^{r(T-t)} + \tilde{f}_2(t)\right\},$$

where

$$\tilde{f}_2(t) = \frac{\lambda\mu_1\theta n}{r}\left(1 - e^{r(T-t)}\right) + \frac{\lambda\mu_2 n^2(m+n)(m+3n)\left(e^{2r(T-t)} - 1\right)}{4r(m+2n)^2}.$$

Proof. We can see that the worst-case density generator $l^*(t)$ equals 0 if we set $\beta = 0$ in (4.1.11) and (4.2.8). Therefore, Setting $\beta = 0$ in Theorem 4.1.2 and Theorem 4.2.1 immediately gives us the results of this corollary. This completes the proof of this corollary. ■

Letting $\beta = 0$ in Theorem 4.1.1 and inequality (4.2.7), we easily obtain the conditions showing the effects of risk constraints for the two different cases, respectively. Accordingly, we can obtain the following corollary.

Corollary 5.0.2 *(1) When the reinsurer is ambiguity-neutral, and the VaR constraint is binding for the insurer, anyone of the following conditions should be satisfied:*

(a)

$$ne^{r(T-t)}\tilde{A}_2(t) < \tilde{A}_1(t) < \frac{rC_3}{e^{rh}-1},$$

and

$$\lambda\mu_1\theta + C_2 - C_1 > 0.$$

(b)

$$\frac{rC_3}{e^{rh}-1} < \tilde{A}_1(t) < ne^{r(T-t)}\tilde{A}_2(t),$$

and

$$\lambda\mu_1\theta + C_2 - C_1 > 0,$$

where

$$\begin{cases} \tilde{A}_1(t) = C_1^2 - \lambda\mu_2 me^{r(T-t)}(C_1 - C_2 - \lambda\mu_1\theta), \\ \tilde{A}_2(t) = \lambda\mu_2 C_1 - \frac{C_1^2}{me^{r(T-t)}}. \end{cases}$$

(2) When the reinsurer is ambiguity-neutral, and the VaR constraint is inactive for the insurer, the following condition should be satisfied:

$$\begin{aligned} \lambda\Big(\theta\mu_1(m+2n)^2 - \mu_2 n(m+n)(m+2n)e^{r(T-t)} + \mu_2 n(m+n)^2 e^{r(T-t)}\Big) \\ - C_1(m+n)(m+2n) + C_2(m+2n)^2 \geq 0. \end{aligned}$$

Chapter 6

Expected utility loss of a suboptimal reinsurance contract

This chapter discusses the expected utility loss of an ambiguity-averse reinsurer. To this end, we assume that the reinsurer is ambiguous about the claim process, but she doesn't take the optimal reinsurance contracts given in Theorem 4.1.2 or Theorem 4.2.1. Instead, she designs a reinsurance contract as if she is ambiguity-neutral, i.e., she follows the contract given in Corollary 5.0.1. Under this case, we define the reinsurer's suboptimal value function by

$$\widehat{V}^R(t,y) := \inf_l \mathbb{E}_{\mathbb{Q}} \left\{ -\int_t^T \frac{ml^2 \widehat{V}^R(s,y)}{2\beta} \mathrm{d}s + \left[-\frac{1}{m} \exp\left\{ -mY^{\tilde{\eta}^*}(T) \right\} \right] \right\}.$$

It should be noted that the reinsurance contract is now pre-specified, by which the worse-case density generator $l(t)$ is endogenously determined. Following Hu et al. (2018b) and Li et al. (2018), we define the expected utility loss of the reinsurer under the suboptimal reinsurance contract as follows

$$UL(t) := 1 - \frac{V^R(t,y)}{\widehat{V}^R(t,y)}, \tag{6.0.1}$$

where $V^R(t,y)$ given in Theorem 4.1.2 and Theorem 4.2.1 under different cases is the robust optimal value function of the reinsurer.

The suboptimal value function $\widehat{V}^R(t,y)$ associated with the optimal reinsurance contract $(\tilde{q}^*(t), \tilde{\eta}^*(t))$ solves the following infimum problem

$$\begin{aligned} \inf_l \Bigg\{ & \widehat{V}_t^R(t,y) + \left[ry + \lambda\mu_1\tilde{\eta}^*(1-\tilde{q}^*) + \sqrt{\lambda\mu_2}(1-\tilde{q}^*)l \right] \widehat{V}_y^R(t,y) \\ & + \frac{1}{2}\lambda\mu_2(1-\tilde{q}^*)^2 \widehat{V}_{yy}^R(t,y) - \frac{ml^2\widehat{V}^R(t,y)}{2\beta} \Bigg\} = 0. \end{aligned} \tag{6.0.2}$$

Differentiating (6.0.2) with respect to l, we find that $l^*(t)$ is given by

$$l^*(t) = \frac{\beta\sqrt{\lambda\mu_2}(1-\tilde{q}^*(t))\widehat{V}^R_y(t,y)}{m\widehat{V}^R(t,y)}. \tag{6.0.3}$$

When the VaR constraint is active for the insurer, we try to find the solution to (6.0.2) having the following form

$$\widehat{V}^R(t,y) = -\frac{1}{m}\exp\left\{-mye^{r(T-t)} + \hat{g}_1(t)\right\}. \tag{6.0.4}$$

Plugging the relative derivatives of $\widehat{V}^R(t,y)$, (6.0.3) and $(\tilde{q}^*(t), \tilde{\eta}^*(t))$ in the first case of Corollary 5.0.1 into HJB equaion (6.0.2), we obtain

$$\hat{g}_1'(t) + \frac{\beta C_1^2}{2m\lambda\mu_2} + m(C_1 - C_2 - \lambda\mu_1\theta)e^{r(T-t)} - \frac{C_1^2}{2\lambda\mu_2} = 0.$$

Incorporating the boundary condition $\hat{g}_1(T) = 0$, we have

$$\hat{g}_1(t) = \frac{m(C_1 - C_2 - \lambda\mu_1\theta)}{r}\left(e^{r(T-t)} - 1\right) + \frac{(\beta - m)(T-t)C_1^2}{2m\lambda\mu_2}.$$

Substituting (6.0.4) and (4.1.12) into (6.0.1), we get the expected utility loss due to ignoring model ambiguity in an explicit form:

$$UL_1(t) = 1 - e^{g_1(t) - \hat{g}_1(t)},$$

where

$$g_1(t) - \hat{g}_1(t) = -\frac{\beta^2 C_1^2 (T-t)}{2\lambda\mu_2 m(\beta + m)}.$$

Similarly, when the VaR constraint is inactive for the insurer, we assume that the solution to (6.0.2) has the following form

$$\widehat{V}^R(t,y) = -\frac{1}{m}\exp\left\{-mye^{r(T-t)} + \hat{g}_2(t)\right\}.$$

The ODE that $\hat{g}_2(t)$ should satisfy becomes

$$\hat{g}_2'(t) + \frac{mn^2\lambda\mu_2 e^{2r(T-t)}}{(m+2n)^2}\left(\frac{\beta}{2} - \frac{m}{2} - n\right) = 0, \tag{6.0.5}$$

combining with the terminal condition $\hat{g}_2(T) = 0$, we then obtain the solution to (6.0.5) is given by

$$\hat{g}_2(t) = \frac{\lambda\mu_2 mn^2(\beta - m - 2n)\left(e^{2r(T-t)} - 1\right)}{4r(m+2n)^2}.$$

Therefore, the expected utility loss of the suboptimal reinsurance contract is given by:

$$UL_2(t) = 1 - e^{g_2(t) - \hat{g}_2(t)},$$

where

$$g_2(t) - \hat{g}_2(t) = \frac{\lambda m n^2 \mu_2 \left(1 - e^{2r(T-t)}\right)}{4r} \left(\frac{1}{m + \beta + 2n} + \frac{\beta - m - 2n}{(m + 2n)^2} \right).$$

Chapter 7

Numerical examples

In this chapter, we conduct some numerical experiments to provide sensitivity analysis for the robust optimal reinsurance contract and the reinsurer's expected utility loss in two different scenarios. In Case I the VaR constraint is active for the insurer, and the model parameters as our benchmark are shown in Table 7.1. It can be verified that the condition (2) in Theorem 4.1.1 is satisfied in this parameter setting. In Case II the VaR constraint is inactive for the insurer, and the parameters are the same as that of Case I except that m changes to 0.2 and n becomes 0.3. It is also can be verified that the condition in (4.2.7) is satisfied in this parameter setting. In each of the following figures, we vary the value of one parameter and study the sensitivity of robust optimal reinsurance contract and the reinsurer's expected utility loss with respect to the change of that parameter. It has been ensured that condition (2) in Theorem 4.1.1 for Case I and condition (4.2.7) for Case II are correspondingly satisfied when the parameters that we are interested in vary in specific ranges.

Table 7.1: Model parameters

t	T	r	R	h	α	β
0	4	0.05	4.5	0.25	0.05	0.1
θ	m	n	λ	μ_1	μ_2	
0.25	2	5	6	3	7	

Figure 7.1 depicts the effects of the reinsurer's ambiguity aversion coefficient on the optimal reinsurance contract when the VaR constraint is active for the insurer. We note that $q^*(0)$ and $\eta^*(0)$ are both increasing functions of β. The results

in this figure coincide with our intuition in the sense that the reinsurer is prone to increase the reinsurance premium when her level of ambiguity aversion increases in order to offset the adverse effects of model misspecification. In response, the insurer reduces the optimal reinsurance demand due to an increase in the reinsurance premium. And equivalently, the insurer's retained proportion of the claims would increase. Furthermore, for a fixed ambiguity aversion level, the reinsurance premium increases with the growth of the reinsurer's risk aversion parameter m in the exponential utility function. This is because the more risk-averse the reinsurer is, the less the insurance risk she would like to bear, and so she tends to increase the reinsurance premium, which leads to an increase in the insurer's retention level of the claims. In other words, the optimal reinsurance demand of the insurer decreases. It should be noted that the insurer's risk aversion parameter n takes no effects on the robust optimal reinsurance contract when VaR constraint is active. One possible explanation would be that the risk constraint offsets the impact of the insurer's risk aversion parameter. So we do not intend to study the responses of $q^*(0)$ or $\eta^*(0)$ to n in this case.

Figure 7.2 reveals that the insurer's risk aversion parameter n takes effects on the robust optimal reinsurance contract in Case II. The analysis for the effects of β and m on the robust optimal reinsurance contract is the same as that in Figure 7.1. Regarding the insurer's risk aversion parameter, a larger n implies that the insurer is more risk averse, and he tends to decrease his respect retention level of insurance risk and acquire more reinsurance protection.

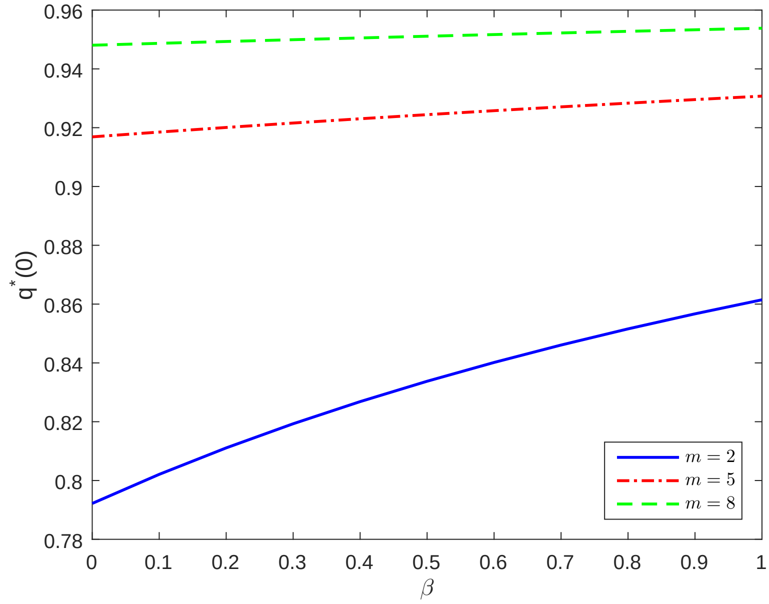
Figure 7.3 displays the influence of α on the insurer's optimal risk retention level when the dynamic VaR constraint is binding. Most of the literature fixes the confidence level α at a relatively small number. In this experiment, we vary the value of α from 0.05 to 0.15. It can be seen that the optimal retained proportional of the claims is significantly affected by α. A larger α corresponds to an insurer who is less conservative about risk, and so he reduces the amount of insurance risk transferred to the reinsurer. Moreover, for a fixed probability level, as the reinsurer's risk aversion parameter grows the reinsurer becomes less willing to undertake the ceded insurance risk and hence increases the reinsurance premium. This then induces the insurer to retain more insurance risk.

In Figure 7.4, we show the effects of the reinsurer's ambiguity aversion parameter β on the expected utility loss $UL_i(0)$, for $i = 1, 2$, in two different cases. We find that $UL_i(0)$ increases significantly as β varies. The expected utility loss reaches up to about 95% in the first case and 85% in the second case. These results indicate that if a reinsurer with less information about the reference measure $\mathbb{P}$ ignores ambiguity aversion, she would suffer greater expected utility loss. Moreover, for a fixed ambiguity aversion level, a higher-risk-averse reinsurer ignoring the impact of model uncertainty would select more conservative premium strategy, and hence she will incur less expected utility loss.

Figure 7.5 illustrates that $UL_i(0)$ has remarkable upward trend as the horizon time T extends. One possible explanation would be that the reinsurer faces a higher level of model uncertainty when the reinsurance horizon T is longer. Therefore, it seems to be essential for the reinsurer to consider ambiguity aversion if she wants to develop a medium and long-term cooperative relationship with the insurer. The explanations for the effects of the reinsurer's risk aversion parameter on the expected utility loss may be similar with those in Figure 7.4 and, thus, we don't repeat them here.

Figures 7.6-7.7 compare the optimal reinsurance contract when the reinsurer is an ambiguity-averse reinsurer (AAR) and an ambiguity-neutral reinsurer (ANR) for Case I and Case II, respectively. We can see that the reinsurer's ambiguity aversion not only influences the reinsurance premium but also affects the optimal reinsurance demand. Specifically, considering that model uncertainty makes the reinsurer more conservative to the risk and so she will reduce her exposure to potential insurance risk by increasing the reinsurance premium, which may in turn induce the insurer to increase his own risk retention by ceding less risk to the reinsurer. The result shown in the first subfigure of Figure 7.7 is in accordance with our intuition that as the insurer becomes more risk-averse, he tends to retain less insurance risk.

Figure 7.8 demonstrates the effects of risk constraint and ambiguity aversion on the optimal reinsurance contracts. As analyzed in Figure 7.7, in Case II an AAR charges a higher reinsurance premium than an ANR, which in turn leads the insurer to purchase less reinsurance protection and retain a higher level of insurance risk. On the other hand, in the optimization problems studied from the insurer's perspective, keeping other factors unchanged the dynamic VaR constraint often makes the insurer more conservative to the risk and retain less insurance risk, see, for example, Liu and Yiu (2013). However, from Figure 7.8, it can be seen that the insurer would retain more risk when the VaR constraint is active than that of when the VaR constraint is inactive. This result may indicate that the VaR constraint seems to have the same effect as the ambiguity aversion of the reinsurer. Recalling the expression of the optimal reinsurance premium in (4.1.10) of Theorem 4.1.2, we know that the risk constraint imposed on the insurer also influences the reinsurer's decision making through the parameters C_1 and C_2. Specifically, from the second subfigure in Figure 7.8, it can be seen that the VaR constraint induces the reinsurer to offer a higher reinsurance premium since she has a greater risk-aversion parameter in Case I than that of Case II, and this is how risk constraint magnifies the effect of ambiguity aversion of the reinsurer. In other words, different from the optimization problems that only consider the insurer's objective, under our principal-agent modelling framework, the dynamic VaR constraint makes the insurer more risk-seeking due to the strategic interaction between the insurer and the reinsurer.



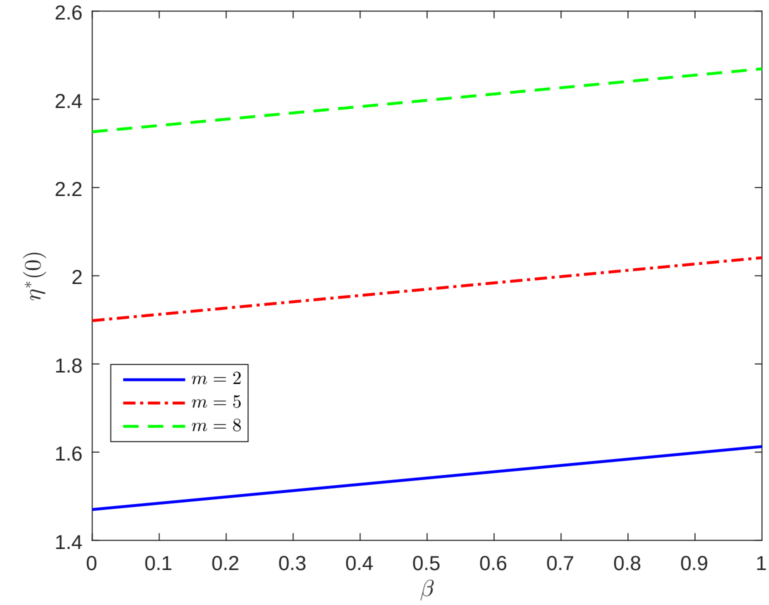


Figure 7.1: Effects of the reinsurer's ambiguity aversion parameter β on the robust optimal reinsurance contract in Case I.

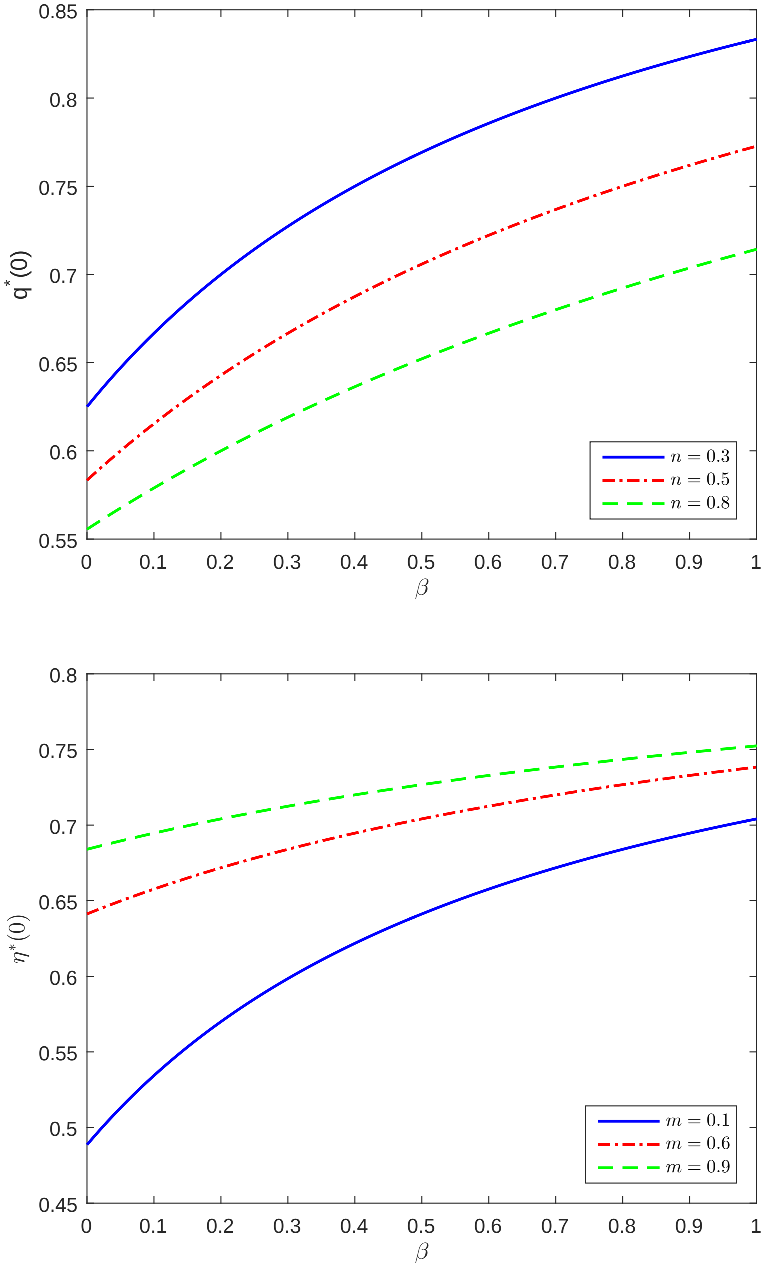


Figure 7.2: Effects of the reinsurer's ambiguity aversion parameter β on the robust optimal reinsurance contract in Case II.

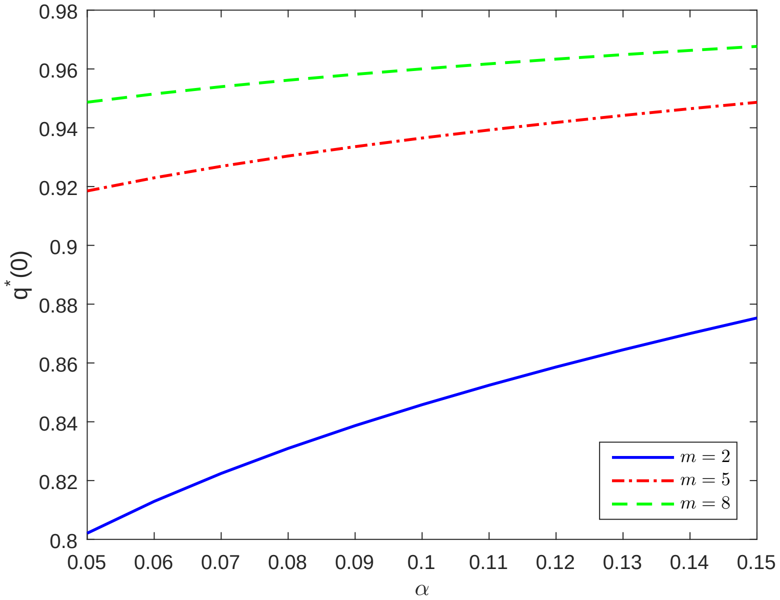


Figure 7.3: Effects of the insurer's confidence level α on the optimal risk retention level in Case I.

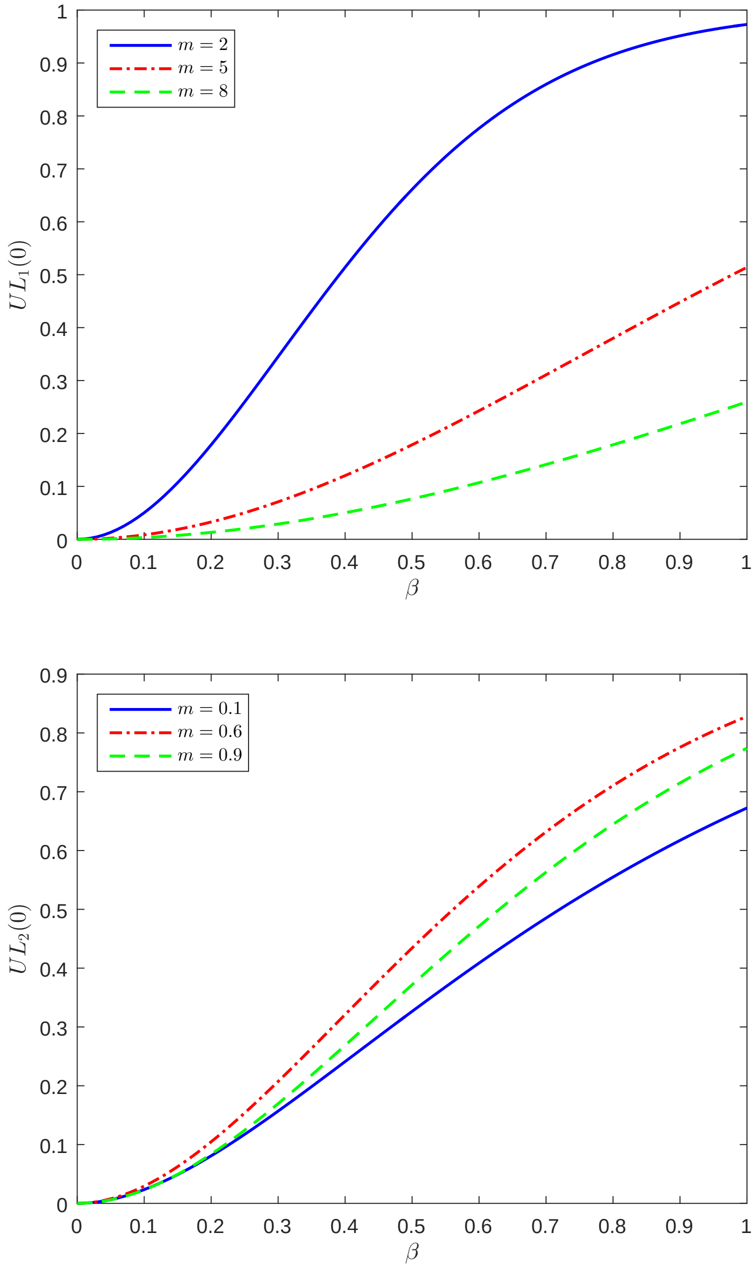
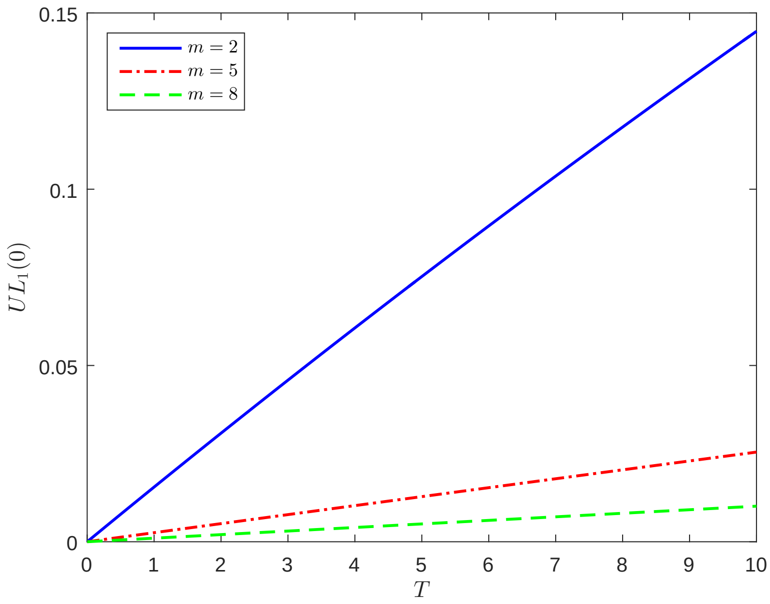


Figure 7.4: Effects of the reinsurer's ambiguity aversion parameter β on the reinsurer's expected utility losses in two cases.



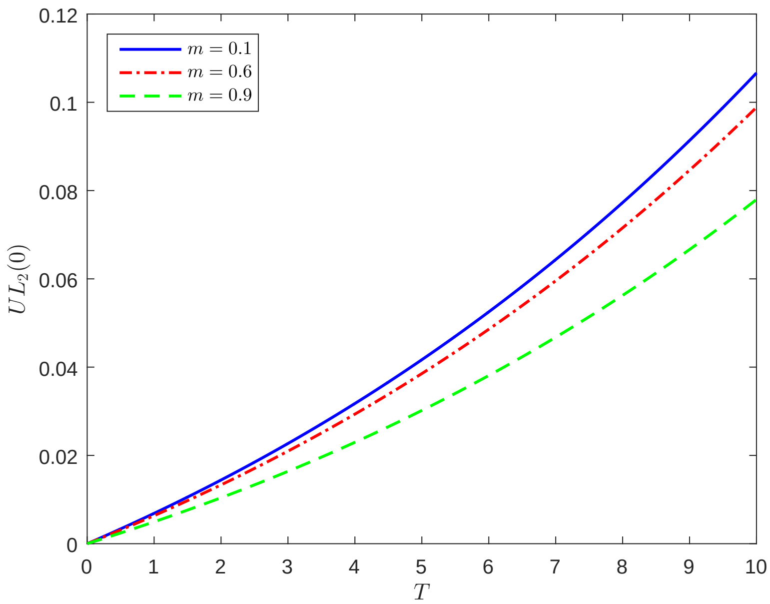
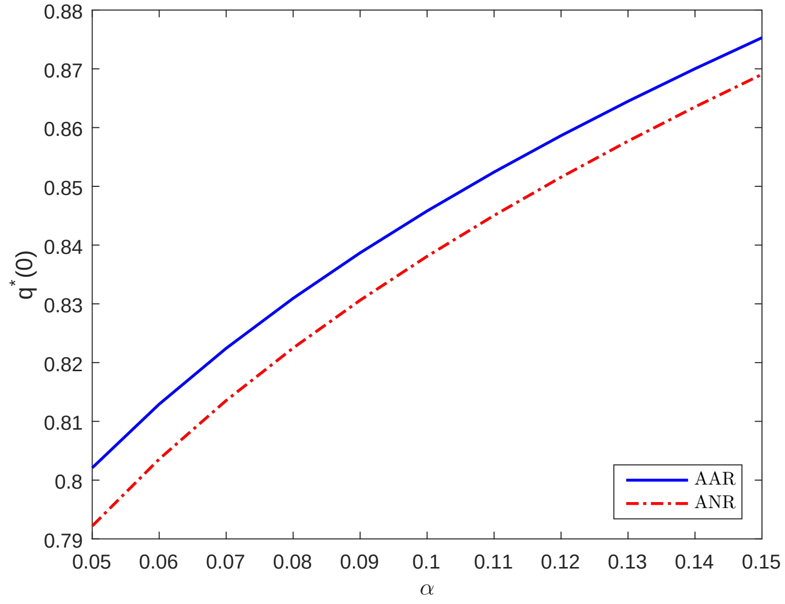


Figure 7.5: Effects of the reinsurance horizon T on the reinsurer's expected utility losses in two cases.



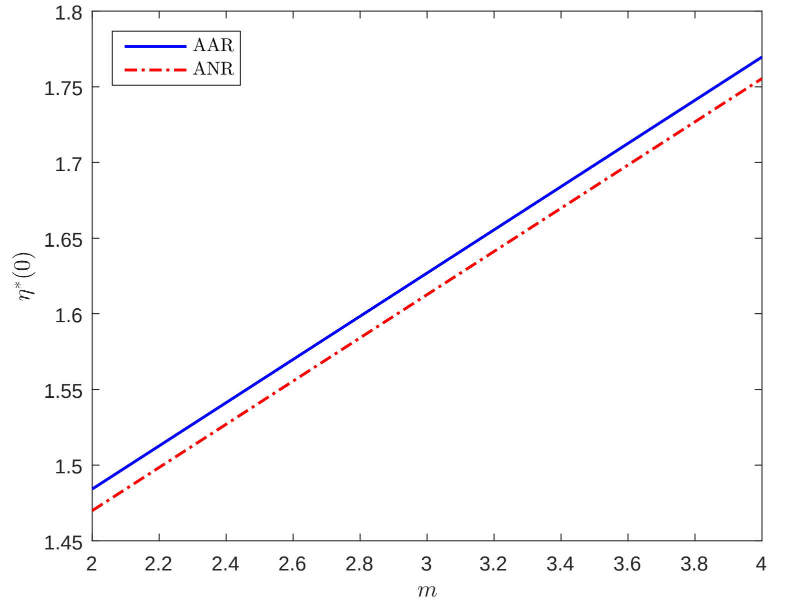


Figure 7.6: Effects of the insurer's confidence level α and the reinsurer's risk aversion parameter m on the robust optimal reinsurance contract in Case I.

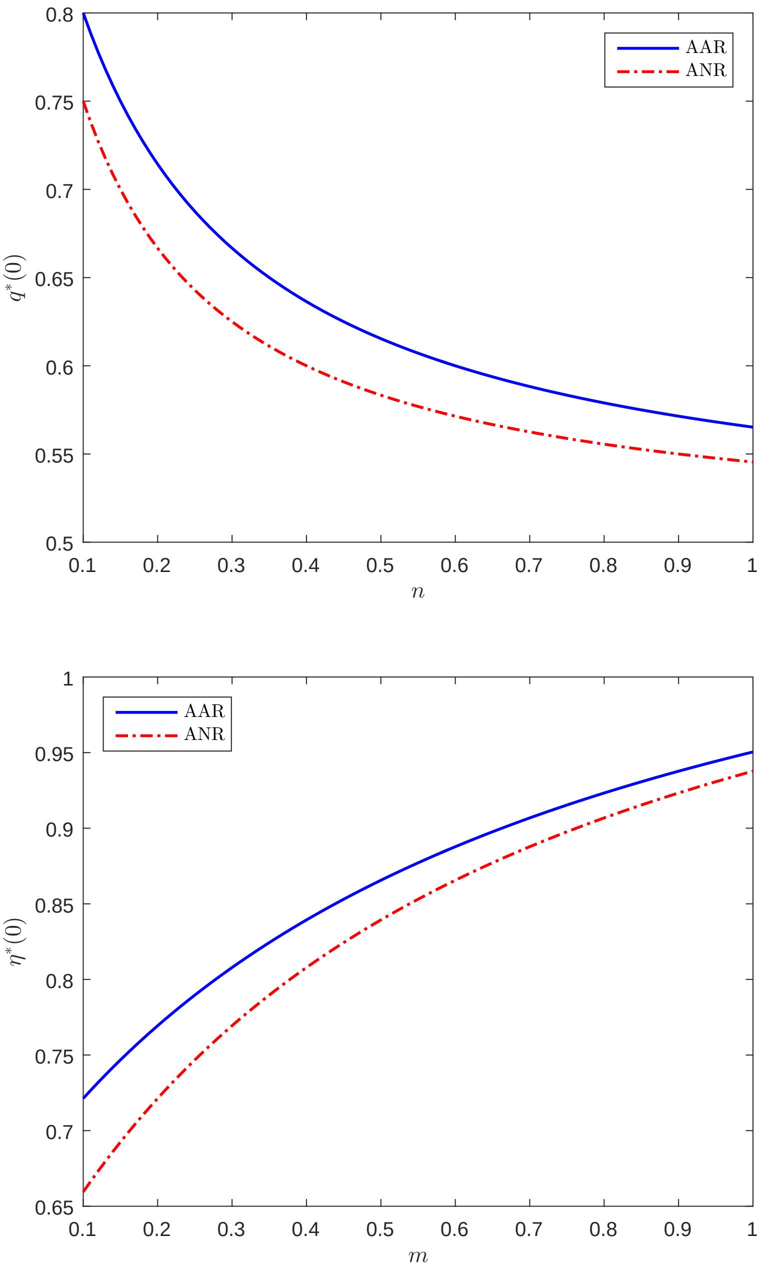


Figure 7.7: Effects of the risk aversion parameters on the robust optimal reinsurance contract in Case II.

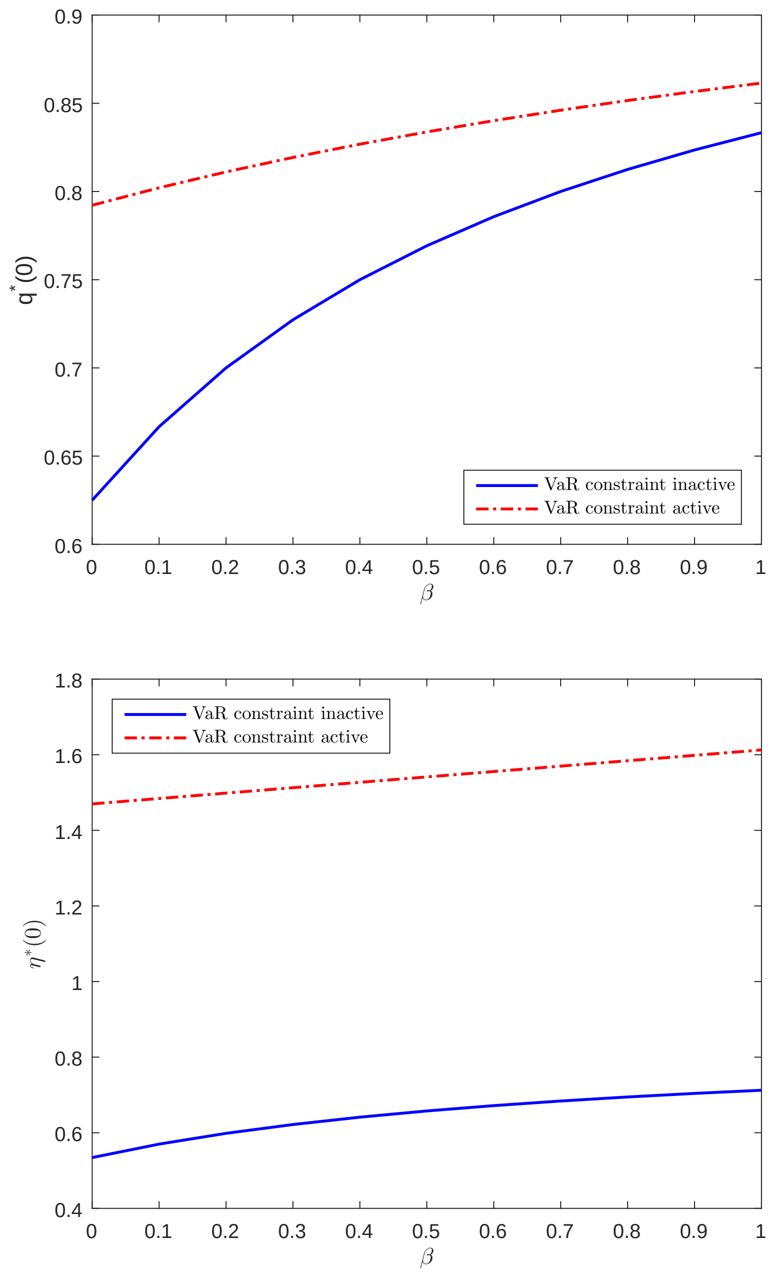


Figure 7.8: Effects of the reinsurer's ambiguity aversion parameter β on the robust optimal reinsurance contracts in two cases.

Chapter 8

Concluding Remarks and Further Research

This thesis investigates an optimal reinsurance contract problem with risk management and model uncertainty consideration from the principal-agent perspective. We apply stochastic optimal control theory to derive the insurer's HJB equation and the reinsurer's HJBI equation. We have discussed two possible scenarios that the VaR constraint is active or inactive for the insurer by adopting KKT conditions. Solving the HJB and HJBI equations, we derive the robust optimal reinsurance contracts under different cases, and we also obtain the corresponding value functions of the insurer and the reinsurer explicitly.

The main results in this thesis are as follows. First, in both of cases wherein VaR constraint is active and inactive for the insurer, the presence of the reinsurer's ambiguity aversion attitudes makes her increase the optimal reinsurance premium, and this in turn decreases the insurer's optimal reinsurance demand. Furthermore, in each case, a more risk-averse reinsurer would offer a higher optimal reinsurance premium and a more risk-averse insurer tends to purchase more reinsurance protection to spread risks, which are in accordance with the results in Hu et al. (2018a,b).

Second, if we don't employ the dynamic VaR to determine the required capitals for the insurer, the model in this thesis can reduce to that in Hu et al. (2018b). Thus, the unconstrained robust reinsurance contract problem is a special case of our model. From the results in Theorem 4.1.2 and Theorem 4.2.1, it can be seen that the parameters representing the characteristics of both the insurer and the reinsurer are required to satisfy specific conditions to make the dynamic VaR constraint for the insurer active or inactive. From the numerical experiments, we can see that the insurer and the reinsurer need to have larger risk aversion parameters when the VaR constraint is active than those when the VaR constraint is inactive for the insurer, thus indicating that the risk constraint makes the insurer and the reinsurer more risk-averse. A quite surprising finding is that the VaR constraint induces the insurer to be more risk-seeking. This is because a reinsurer with a higher level of risk aversion would increase the reinsurance premium, and this is magnified by the effect of the reinsurer's ambiguity aversion attitude, which makes the insurer tend to retain more risk. This result also implies that the effect of the VaR constraint imposed on the insurer offsets the effect of his risk aversion parameter. Moreover, when the VaR constraint is active, a higher confidence level leads the insurer to transfer less risk to the reinsurer.

The final insight is that an ambiguity-averse reinsurer would greatly suffer expected utility loss if she ignores the effect of model uncertainty on the optimal proportional reinsurance contract. This result demonstrates the importance of incorporating model ambiguity in the optimal reinsurance design problems. Additionally, we find that the expected utility loss of the reinsurer increases with respect to her level of ambiguity aversion and the reinsurance horizon. These results seem to be in line with some of the existing studies, for example, Hu et al. (2018b) and Li et al. (2018).

Several possible extensions of this thesis deserve further investigation. In this thesis, we only allow the insurer and the reinsurer to invest their surpluses in one risk-free asset. Therefore, the first extension is to consider more complicated investment activities of the insurer and the reinsurer. For instance, we can allow them to further invest in stock and defaultable bond to investigate the robust optimal reinsurance and investment problems. It would be interesting to study this kind of problems if the price of the stock is modelled by constant elasticity of variance (CEV) model or Heston's stochastic volatility (SV) model, as the assumption on constant volatility of the risky asset is unrealistic.

Excess-of-loss is a typical type of non-proportional reinsurance treaty, under which the insurer would bear all the claim up to a fixed retention level, while all exceeds that level would be paid by the reinsurance company. It has been shown that under some model settings, the excess-of-loss reinsurance is more profitable than the proportional reinsurance for an insurance company. Thus, another interesting extension is to apply the framework in this thesis and study the interaction between the insurer and the reinsurer when they arrange an excess-of-loss reinsurance treaty .

Finally, in this thesis we assume that the insurer and the reinsurer aim to maximize their expected utilities of their wealth at terminal time. In the future research, other optimization criteria such as mean-variance can be taken into account. Under this circumstance, analytically optimal reinsurance contracts may be unavailable due to complex objective functions. Nevertheless, numerical approximation methods can provide a viable alternative and help us obtain useful economic insights to the contracting parties.

Appendix A

Proof of Proposition 3.1.1

Proof. We have

$$
\begin{aligned}
&\mathbb{P}(X^q(t+h) - e^{rh}X^q(t) \leq L|\mathscr{F}_t) \\
&= \mathbb{P}\left(q(t)\sqrt{\lambda\mu_2}\int_t^{t+h} e^{r(t+h-l)}\mathrm{d}W(l) \leq \ L - \frac{e^{rh}-1}{r}\lambda\mu_1(\theta - \eta(t) + \eta(t)q(t)) \middle| \mathscr{F}_t \right) \\
&= \mathbb{P}\left(\frac{q(t)\sqrt{\lambda\mu_2}\int_t^{t+h} e^{r(t+h-l)}\mathrm{d}W(l)}{q(t)\sqrt{\frac{e^{2rh}-1}{2r}\lambda\mu_2}} \leq \ \frac{L - \frac{e^{rh}-1}{r}\lambda\mu_1(\theta - \eta(t) + \eta(t)q(t))}{q(t)\sqrt{\frac{e^{2rh}-1}{2r}\lambda\mu_2}} \middle| \mathscr{F}_t \right) \\
&= \Phi\left(\frac{L - \frac{e^{rh}-1}{r}\lambda\mu_1(\theta - \eta(t) + \eta(t)q(t))}{q(t)\sqrt{\frac{e^{2rh}-1}{2r}\lambda\mu_2}} \right),
\end{aligned}
$$

where the last equality follows from the fact that the random variable

$$q(t)\sqrt{\lambda\mu_2}\int_t^{t+h} e^{r(t+h-l)}\mathrm{d}W(l),$$

conditionally on the filtration $\mathscr{F}_t$, is normally distributed with zero mean and variance

$$\frac{e^{2rh}-1}{2r}\lambda\mu_2 q^2(t).$$

Thus,

$$
\begin{aligned}
&\mathbb{P}(-\Delta X^q(t) \leq L|\mathscr{F}_t) < \alpha \Leftrightarrow \ \Phi\left(\frac{L - \frac{e^{rh}-1}{r}\lambda\mu_1(\theta - \eta(t) + \eta(t)q(t))}{q(t)\sqrt{\frac{e^{2rh}-1}{2r}\lambda\mu_2}} \right) \leq \alpha \\
&\Leftrightarrow L \leq \frac{e^{rh}-1}{r}\lambda\mu_1(\theta - \eta(t) + \eta(t)q(t)) + \Phi^{-1}(\alpha)q(t)\sqrt{\frac{e^{2rh}-1}{2r}\lambda\mu_2},
\end{aligned}
$$

which implies that

$$Q_t^{\alpha,h} = \frac{e^{rh}-1}{r}\lambda\mu_1(\theta - \eta(t) + \eta(t)q(t)) + \Phi^{-1}(\alpha)q(t)\sqrt{\frac{e^{2rh}-1}{2r}\lambda\mu_2}.$$

Therefore,

$$\mathrm{VaR}_t^{\alpha,h} = (Q_t^{\alpha,h})^- = \left[-\frac{e^{rh}-1}{r}\lambda\mu_1(\theta - \eta(t) + \eta(t)q(t)) - \Phi^{-1}(\alpha)q(t)\sqrt{\frac{e^{2rh}-1}{2r}\lambda\mu_2} \right]^+ .$$

■

Appendix B

Derivation of relative entropy

Given the reference probability measure $\mathbb{P}$ and an alternative measure $\mathbb{Q}$, the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined as the expectation under the alternative probability measure of the log Radon-Nikodym derivative defined in (3.3.2). A lower relative entropy implies that it is harder for the reinsurer to distinguish $\mathbb{P}$ from $\mathbb{Q}$ in statistic sense. Using Itô formula, we obtain

$$\mathrm{d}\ln\nu(t) = l(t)\mathrm{d}W(t) - \frac{1}{2}l^2(t)\mathrm{d}t$$

The relative entropy over the interval from t to $t+\varepsilon$ is given by

$$\begin{aligned} E_{\mathbb{Q}}\left[\ln\frac{\nu(t+\varepsilon)}{\nu(t)}\right] &= E_{\mathbb{Q}}\left[\int_t^{t+\epsilon} l(s)\left(\mathrm{d}W_{\mathbb{Q}}(s) + l(s)\mathrm{d}s\right) - \frac{1}{2}\int_t^{t+\epsilon} l^2(s)\mathrm{d}s\right] \\ &= E_{\mathbb{Q}}\left[\frac{1}{2}\int_t^{t+\epsilon} l^2(s)\mathrm{d}s\right] \end{aligned}$$

Let $\epsilon \to 0$ and we obtain the continuous-time limit of the relative entropy given by $\frac{1}{2}l^2(t)\mathrm{d}t$.

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