

Discretising Quantum Field Theory using Wavelets

By

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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Bryte Hagan

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To my father, who instilled in me never to become complacent.

To my mother, whose endless wisdom inspired me beyond all.

To my uncle, my academic exemplar.

To my partner, of immeasurable kindness I forever seek to remunerate.

To my supervisors, through continual guidance, who have enabled me to learn so much.

To all of the people in my life who have shaped who I have become, and have given me the opportunity to pursue and realise my greatest goals.

Thank you.

Abstract

In this thesis, I develop a new discretisation scheme offered by wavelet analysis. Daubechies wavelets of order $\mathcal{K} \geq 3$ are used to encode the quantum field theory of the 1D transverse Ising model in a discrete approximation. I then analytically determine the energy spectrum, which is used to construct the correlation matrix. This allows for the calculation of bipartite von Neumann entropy, which acts a measure of entanglement entropy for block spin chains. These results are qualitatively and quantitatively compared to the discrete and continuous Ising models as a gauge of the method's success as an alternative discretisation scheme.

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Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions.

Felix Klein

1

Introduction

The subtle interplay between the discrete and continuous has given rise to many rich studies within physics and mathematics. Although the study of continuous parameters in field theory was developed as a continuous extension to a discrete theory, given the experimental successes of quantum field theory, it would seem that the former is a better representative for nature. The role of discrete theories, therefore, is to function as a tool to shape our understanding of the continuous theories. The solubility of discrete models, either through numerical tractability or analytic manipulations of finite quantities, are one their greatest assets.

Unfortunately, the path between the discrete and continuous is not well lit, or even well defined for the majority of discrete models. The process of deriving a continuous field from a discrete theory is generally ad hoc and restricted to specific regimes in the discrete theory. Divergences that emerge in the transition from countable to uncountable are tamed by selectively pairing limiting parameters. An argument in favour of this process is that if hand-waving mathematics can still uncover truths about nature, then the methods utilised are largely irrelevant. While still paying respect to the physics that

our mathematical models are designed to study, the question remains as to whether one can develop mathematically rigorous theories that are capable of utilising the tractability of discrete models, while still being able to produce the powerful results of field theory.

A hint of the direction in which the answer may lie is offered by renormalisation. The theory was built to deal with the divergences encountered in formal calculations of essentially all non-trivial field theories. The insight key to the theory is that one does not measure bare observables; the net electric charge of a wire is not simply the collective sum of individual electrons and protons. It is a *renormalised* charge that is a result of the complex, many-body interactions averaged out to some finite value at our observable length scale. This *physical* observation guided the ensuing mathematics, and lead to a (debatable) unification between the formal theory and experimental results.

Returning to the controversial disjunction between discrete and continuous theories, this thesis proposes study of a discretisation strategy built around renormalisation: the wavelet transform. Wavelets are a mathematical construction developed to rigorously reflect the rescaling process. By applying wavelets (specifically wavelets associated with the discrete wavelet transform) to continuum theories, we seek to obtain a reversible path by which discrete and continuum theories can be linked. In addition, the renormalisation process (being readily obtainable from wavelet theory) is encoded in the model automatically.

The result of this thesis is the development of a discretised version of a continuous model, obtained through the discrete wavelet transform. The goal of this thesis then to validate the result obtained. The model chosen is the Ising model, which has well studied discrete and continuous variants. This thesis utilises both models as benchmarks to validate the within-constructed wavelet-Ising model. The second chapter of this thesis discusses three variants of the discrete Ising model, namely the one and two dimensional classical model, the one dimensional quantum model as well as the continuous Ising field theory. The third chapter discusses wavelet theory to the level of detail necessary for their implementation in this research. The final chapter develops the discrete wavelet-Ising model, and culminates by verifying continuum eigenenergies for the model, as well as qualitatively comparing sub-system entanglement between the discrete quantum models through calculation of von Neumann entropy.

Problems worthy

of attack

prove their worth

by hitting back.

Piet Hein

2

The Ising Model

The Ising model is a powerful model of cooperative phenomena: the physics of systems that evolve in tandem. While the Ising model is intuitively simple to describe, it is associated with a deep well of physics. The model effectively captures the phenomena of phase transitions (for dimension ≥ 2) exhibited by ferromagnetic materials. There exists an analytic mapping from the 2D Ising model (in the extreme anisotropic regime, where orthogonal lattice couplings divaricate) to the 1D quantum Ising model, hence, solutions and qualitative results are shared by both models. The quantum model is analytically solvable by means of Jordan-Wigner and Bogoliubov transformations, the former being discussed in this chapter. As an extension of the quantum model, the corresponding field theory is obtained in the thermodynamic limit of increasing lattice sites constrained to a fixed volume.

The Ising model was first investigated and named after the German physicist, Ernst Ising. While his name is commonly attributed to the model, it was actually devised by Wilhelm Lenz, who proposed the model to Ising for his PhD thesis. During his thesis, Ising studied the one-dimensional model. Due to mathematical circumstances specific to one-dimension, Ising correctly calculated that his model

did not exhibit phase transitions, but erroneously conjectured that the same would be true in higher dimensions.

Exhausted by his doctorate, Ising decided against continuing a career in physics and pursued business instead. However, his efforts were not for naught, as his work was continued by others in the field. The American chemist, Lars Onsager, devoted many years to developing solutions to the two-dimensional case, and in 1944 published a paper [1] which is regarded as one of the most prominent papers in statistical mechanics to date. The fame garnered by Onsager's solution to the 2D case is well deserved, and much of the applicability of the Ising model stems from the ability to obtain exact solutions to the model via Onsager's work. His result proved that the two-dimensional case exhibited a phase transition at finite temperature, contrary to Ising's assertion. The proposal by Onsager is now known as the *transfer matrix* method.

2.1 The Classical Model

The classical Ising model considers an evenly spaced, one-dimensional chain of objects. The objects have the restriction that they can only interact with their immediate neighbour, and they themselves can exist as one of two variations. The simplest example is a sequence of magnets, where each magnet is either aligned 'upwards' or 'downwards'*.

Given these circumstances, the Ising model associates an energy value to the system as a whole by weighing contributions from each pair of neighbouring objects. A positive contribution is given by similar neighbours, and a negative contribution by differing neighbours. These systems are often studied in the presence of an external field, which is applied transversally to the chain.

As the statements above are extremely general, the model has the ability to be applied broadly to many systems. The archetypal example previously mentioned is that of a chain of magnets, but another realisations is chains of molecules. One can consider an evenly spaced sequence of two molecules, where each site hosts exactly one of the two molecules. Another case is a chain of a single type of molecule that is *unevenly* spaced. The staggered distribution can be captured by considering empty sites to be occupied by 'holes', which analogously plays the role of the spin-down magnet.

*An obvious limitation of this model is the lack of acknowledgement for superposition states, which any sensible model of magnetism should incorporate. This gives motivation to consider the quantum model instead.

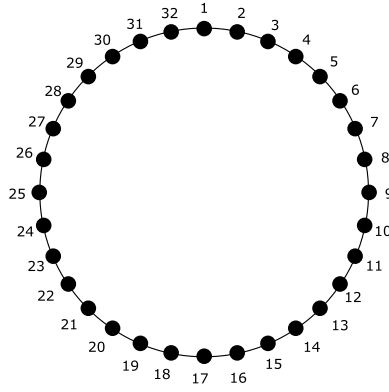


Figure 2.1: An Ising chain with 32 sites

Formally, the Ising model is expressed as the equation

$$H(\{\sigma\}) = \sum_{j=0}^{N-1} [-J_{j,j+1}\sigma_j\sigma_{j+1} - h_j\sigma_j] \quad (2.1)$$

Here, H is the familiar Hamiltonian of the system, containing all relevant information regarding the system's energy. Each object in the chain is denoted by $\sigma_j \in \{-1, 1\}$, and the sequence $\{\sigma\} \equiv (\sigma_0, \sigma_1, \dots, \sigma_{N-1})$ denotes the entire collection of objects. A particular choice for every element in the chain is called a *configuration*. As an example, the configuration for which every second magnet is spin down would be given by $\{\sigma\} = (1, -1, 1, \dots)$. The constants $J_{j,j+1}$ and h_j allow for particular sites to have greater influence over the overall energy. Generally one assumes symmetry over all sites by imposing $J_{j,j+1} = J$ and $h_j = h$. Under these simplifications, we have

$$H = \sum_{j=0}^{N-1} [-J\sigma_j\sigma_{j+1} - h\sigma_j] \quad (2.2)$$

The purpose of the overall minus sign is to ensure that the lowest energy configurations (in zero external field: $h = 0$) of the system are ones where neighbouring pairs are similar: $\sigma_j\sigma_{j+1} = 1$. This corresponds to a ferromagnet, which has neighbouring domains that are magnetically coaligned.

2.1.1 Exact Solubility

The Ising model is often studied under the assumption of periodic boundary conditions, which is implemented by imposing $\sigma_N \equiv \sigma_0$. This is for convenience in later calculations, and the demand is insignificant as physical realisations are ultimately interested in the thermodynamic limit of $N \rightarrow \infty$.

Non-periodic boundary conditions are considered in [2] and are noted to only function as bookkeeping, rather than being fundamental to the model.

For any large ensemble*, keeping track of individual interactions between elements quickly becomes an intractable feat. The insight of statistical mechanics is the ansatz that most of the time, these interactions even out. This is typically a valid assumption under equilibrium conditions. In this case it becomes economical to consider the macroscopically measurable *average* values of the system. In fact, it turns out to be but extremely accurate simplification, with deviations falling off proportionally to \sqrt{N} . Variables of this nature are known as *thermodynamic variables*, and can generally all be expressed in terms of a single function: the partition function. Due to the power of this function, a statistical model in thermodynamic equilibrium is often described as ‘solved’ if the partition function is analytically known.

The partition function takes the form

$$\mathcal{Z} = \sum_{\{\sigma\}} e^{-\beta H(\{\sigma\})} \quad (2.3)$$

where $\sum_{\{\sigma\}}$ enumerates all possible configurations. For a system of N particles occupying one of two states, there are 2^N possible configurations. Without further simplifications, direct computation of \mathcal{Z} quickly succumbs to the curse of dimensionality, and is an unwieldy problem for systems of merely $N \sim 30$. One cannot study a statistical system in detail without first addressing the partition function. The solution offered by Onsager’s method, through various clever matrix identities (for details, see [3]), allows for the partition function for the Ising model to be expressed as the trace of the N th power of the 2×2 *transfer matrix*,

$$\begin{aligned} \mathcal{Z} &= \text{Tr}(\mathbf{T}^N) \\ &= \lambda_1^N + \lambda_2^N \end{aligned} \quad (2.4)$$

In the large N limit, the larger of the two eigenvalues dominates the expression.

2.1.2 Classical to Quantum Mapping

One dimensional models are invaluable for gathering intuition, but are limited in their physical applicability. In particular, the lack of phase transitions in the classical case is one indication of the

*Macroscopic systems are ensembles on the order of Avogadro’s constant: $N_A \approx 10^{24}$.

limitations of the model. A necessary generalisation to push this model towards practicality is to study the multi-dimensional case. For the 2D case, we again consider binary objects with nearest neighbour interactions, but we extend our notion of the chain to a $N \times M$ lattice. The Hamiltonian for the square lattice Ising model is given by

$$H = \sum_{j=0}^{N-1} \sum_{k=0}^{M-1} \left[-J \sigma_j^k \sigma_{j+1}^k - K \sigma_j^k \sigma_j^{k+1} - h \sigma_j^k \right] \quad (2.5)$$

Applying Onsager's transfer matrix method to this model, one obtains a similar expression to eq. (2.4), with an exponentially larger transfer matrix. However, the form of the transfer matrix closely resembles that of an imaginary time path integral under the evolution of the quantum Ising model. The standard link between statistical and quantum theories is established with the aid of a Wick rotation; the only caveat in this case is that one also needs to consider a limiting regime of J and K , performed in a specific way such that an exact mapping can be produced. For a discussion of details, one can consult [4]. The existence of this mapping and the failures of the 1D Ising model to predict phase transitions are motivation for us to consider the quantum model, in addition to the fact that the quantum model is a self-contained and well studied model.

2.2 Quantum Ising Model

With motivation to study the quantum model established, we have the starting point for this thesis: the quantum Ising model in one dimension. While the quantum Ising model has been studied independently to its classical counterparts, the existence of the mapping means that results obtained have implications for both systems.

By analogy with the 1D classical case, the Hamiltonian for the quantum Ising model is given by (normalised to unit interaction coupling)

$$H = \sum_{j=0}^{N-1} \left[-Z_j Z_{j+1} - \lambda X_j \right] \quad (2.6)$$

where X_j and Z_j are Pauli operators acting solely on the j th site in the chain. Explicitly, the Pauli operators are (for a single qubit system)

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (2.7)$$

and the single-site operators are formally defined as a tensor product of N trivial identity operations, with the j th operation corresponding to the relevant operation:

$$Z_j \equiv \left(\bigotimes_{k=0}^{j-1} I \right) Z \left(\bigotimes_{k=j+1}^{N-1} I \right) \quad (2.8)$$

The Hilbert space for this model is therefore the tensor product of N qubit spaces*. In the classical ferromagnetic model, we required the most energy favourable configuration to correspond to perfectly aligned neighbours. This requirement is captured by the product of Z operators when evaluated in the computational basis (the eigenbasis of $\prod_{j=0}^{N-1} Z_j$). The lone X operator fulfils the role of our transversally applied field. As previously, we assume periodic boundary conditions in the form of $Z_N \equiv Z_0$. Thus, the relation between the classical and quantum models is twofold: one via analogy with the 1D case, and again by the classical-to-quantum mapping.

While the above form is instructive in construction, for convenience of future calculations we consider an alternate form. We invoke a global rotation given by $H \rightarrow \text{Had} \cdot H \cdot \text{Had}^\dagger$. Under this rotation, the Hamiltonian transforms as

$$H \rightarrow \sum_{j=0}^{N-1} [-X_j X_{j+1} - \lambda Z_j] \quad (2.9)$$

It is also worth noting that the Hamiltonian has a discrete, \mathbb{Z}_2 symmetry. This is shown by defining the global identity and global spin-flip operators,

$$\mathcal{E} \equiv \prod_{j=0}^{N-1} I_j \quad \mathcal{P} \equiv \prod_{j=0}^{N-1} Z_j \quad (2.10)$$

and noting that both operators commute with the Hamiltonian. Under the operation of matrix multiplication, the two operators form a representation of the group $(\mathbb{Z}_2, \times) = \{1, -1\}$, with \mathcal{E} acting as the identity element for the group.

2.2.1 Jordan-Wigner transformation

The typical analysis procedure for the Ising model begins by analysing the algebra associated with the model, inherited from the Pauli matrices. The anomaly of the quantum model in its current statement

*A qubit space is a Hilbert space spanned by two states.

$^\dagger \text{Had} \equiv \prod_{j=0}^{N-1} \text{Had}_j$ denotes a Hadamard rotation applied to every site in the Ising chain. An individual rotation is given by $\text{Had} = \frac{1}{\sqrt{2}}(X + Z)$ and is considered a rotation due to the effect: $\text{Had} \cdot Z \cdot \text{Had} \rightarrow X$ and $\text{Had} \cdot X \cdot \text{Had} \rightarrow Z$.

is that it doesn't obey typical algebras of fermions (or bosons). We would expect that the Ising model, being a model developed to study magnetism, should correspond to fermions. This is rectified by the Jordan-Wigner transformation.

The quantum Hamiltonian operates on the product of spin-1/2 spaces. Within each qubit space, the algebra of the Pauli matrices is given by the anticommutation relations*

$$\begin{aligned}\{X_j, Y_j\} &= \{Y_j, Z_j\} = \{Z_j, X_j\} = 0 \\ \{X_j, X_j\} &= \{Y_j, Y_j\} = \{Z_j, Z_j\} = 2\end{aligned}\quad (2.11)$$

Due to the construction of the single site operators eq. (2.8), the Pauli operators must commute on different sites

$$[X_j, Y_k] = [Y_j, Z_k] = [Z_j, X_k] = 0 \quad (j \neq k) \quad (2.12)$$

In the study of a single mode Fermi oscillator, one introduces ladder and number operators

$$\alpha_j \equiv \frac{1}{2} (X_j - iY_j) \quad \alpha_j^\dagger \equiv \frac{1}{2} (X_j + iY_j) \quad N_j \equiv \alpha_j^\dagger \alpha_j \quad (2.13)$$

However, one cannot meaningfully make this association for our model, as we would find that such induced ladder operators would correspond to neither bosons nor fermions. They obey mixed commutation relations:

$$\begin{aligned}\{\alpha_j, \alpha_j\} &= 0 & \{\alpha_j, \alpha_j^\dagger\} &= 1 & (j = k) \\ [\alpha_j, \alpha_k] &= 0 & [\alpha_j, \alpha_k^\dagger] &= 0 & (j \neq k)\end{aligned}\quad (2.14)$$

We would like to interpret our model as acting on a space of either fermions or bosons, so that we can make use of the vast array of techniques from the study of such particles. The choice naturally falls on fermions, given the fact that the model was initially designed to model magnetism. This choice will eventually allow us to reduce the difficult problem of diagonalising H (a $2^n \times 2^n$ matrix) to the diagonalisation of its quadratic form (a $2n \times 2n$ matrix), similar outcome to the reduction of the classical partition function to the diagonalisation of the transfer matrix.

The solution to the mixed commutation relations is given by invoking the non-local Jordan-Wigner transformation, defined as

$$c_j \equiv \nu_j \alpha_j \quad c_j^\dagger \equiv \nu_j \alpha_j^\dagger \quad (2.15)$$

*We denote $\{a, b\}$ as shorthand for $ab + ba$ and $[a, b] \equiv ab - ba$.

where ν_j is a j -local spin-flip operator encompassing the non-local component of the transformation,

$$\nu_j \equiv \prod_{l=0}^{j-1} Z_l \quad (2.16)$$

The intuition behind defining such an operator is that we find ν_j and α_k commute for $j \leq k$, but anticommute for $j > k$. This modification is the only requirement to obtain a purely fermionic theory. From this transformation, products of ladder operators follow the modification $\alpha_j \alpha_k \rightarrow \pm c_j c_k$, where $+$ is taken for $j \leq k$, and $-$ only for $j > k$. This has the effect of transforming the commutation relations to anticommutation relations at different sites, while preserving the existing same-site anticommutation relations. A quick calculation verifies that $(c_j^\dagger, c_j)_{j=0}^{N-1}$ obey the correct fermionic algebra

$$\{c_j, c_k\} = 0 \quad \{c_j, c_k^\dagger\} = \delta_{jk} \quad (2.17)$$

These operators are referred to as Jordan-Wigner fermions. Noting that $\nu_j^2 = 1$,

$$Z_j = c_j^\dagger c_j - c_j c_j^\dagger \quad X_j X_{j+1} = (c_j - c_j^\dagger) (c_{j+1}^\dagger + c_{j+1}) \quad (2.18)$$

we can now benefit from the earlier Hadamard rotation. These expressions allow us to express our Hamiltonian in a fermionic basis. Doing so yields

$$\begin{aligned} H &= - \sum_{j=0}^{N-1} \left[(c_j - c_j^\dagger) (c_{j+1}^\dagger + c_{j+1}) + \lambda (c_j^\dagger c_j - c_j c_j^\dagger) \right] \\ &= \sum_{j=0}^{N-1} \left[c_j^\dagger c_{j+1}^\dagger + c_j^\dagger c_{j+1} - c_j c_{j+1}^\dagger - c_j c_{j+1} - \lambda c_j^\dagger c_j + \lambda c_j c_j^\dagger \right] \end{aligned} \quad (2.19)$$

Note that the transformation of $X_j X_{j+1}$ relies on the order in which the operators appear (higher indices to the right). The periodic term in the chain, $X_{N-1} X_0$, does not follow this pattern and we obtain a boundary correction term. In the eventual limit of large N , the contribution from this lone term will become insignificant, so we ignore it entirely.

The total number of Jordan-Wigner fermions is not constant in this model, as indicated by non-conserving $c_j c_k$ and $c_j^\dagger c_k^\dagger$ terms. Indeed, by defining the total fermion number and fermion parity operators as

$$\mathcal{N} = \sum_{j=0}^{N-1} c_j^\dagger c_j \quad \mathcal{P} = -e^{i\pi\mathcal{N}} = \prod_{j=0}^{N-1} (2c_j^\dagger c_j - 1) \quad (2.20)$$

we find that $[H, \mathcal{N}] \neq 0$, but $[H, \mathcal{P}] = 0$. This says that total fermion number is not conserved, but *fermion parity* is (that is, the total number of fermions modulo 2). This commutativity is because the

parity operator is nothing more than the global spin flip operator in the Jordan-Wigner basis, which was already shown to commute with the Hamiltonian. The parity conservation property is often associated with Cooper pairs in the BCS theory of superconductivity, as explored in [5]. We will push this analogy further when we apply the Bogoliubov transformation in §4.2.2. The transformation is a diagonalisation technique, initially developed to simplify the BCS Hamiltonian.

The Hamiltonian can also be condensed by introducing the spinors*

$$\mathbf{c}_j \equiv \begin{bmatrix} c_j \\ c_j^\dagger \end{bmatrix} \quad \mathbf{c}_j^\dagger \equiv \begin{bmatrix} c_j^\dagger & c_j \end{bmatrix} \quad (2.21)$$

This tidies up the Hamiltonian[†], resulting in

$$H = \sum_{j=0}^{N-1} \left[\mathbf{c}_j^\dagger (\mathbf{Z} + i\mathbf{Y}) \mathbf{c}_{j+1} - \lambda \mathbf{c}_j^\dagger \mathbf{Z} \mathbf{c}_j \right] \quad (2.22)$$

where \mathbf{Y} and \mathbf{Z} are Pauli matrices acting on the space of spinors. It is worth noting that all terms are quadratic in the fermionic operators, and hence the expression can be assigned to a quadratic form. Dropping the explicit index reference, we define

$$\mathbf{c} \equiv \begin{bmatrix} c_0 \\ c_0^\dagger \\ \vdots \\ c_{N-1} \\ c_{N-1}^\dagger \end{bmatrix} \quad \mathbf{c}^\dagger \equiv \begin{bmatrix} c_0^\dagger & c_0 & \dots & c_{N-1}^\dagger & c_{N-1} \end{bmatrix} \quad (2.23)$$

which gives

$$H = \mathbf{c}^\dagger \mathbf{\Upsilon} \mathbf{c} \quad (2.24)$$

where $\mathbf{\Upsilon}$ is a symmetric matrix defined so that eq. (2.19) and eq. (2.24) agree (symmetry is achieved by use of the anticommutation relations). Quadratic forms are an important representation, as the task of diagonalising H can be reduced to diagonalising $\mathbf{\Upsilon}$. We will exploit this fact during the analysis section of the final model.

*The distinction between spinors and vectors is that the rotations of the former are performed by elements of $SU(n)$, whereas the latter by elements of $SO(n)$. This difference is not important for this research, but vectors of fermions are spinors. Spinors and transformations on them will be signified by boldface font.

[†]Note that in order for $\mathbf{c}_j^\dagger \mathbf{c}_k$ to form an inner product we must conjugate and transpose the elements of \mathbf{c}_j^\dagger .

2.3 Continuum limit

Often with discrete theories, we can learn important details by considering behaviour in limiting cases. For example, intuition for dealing with awkward summations involving vectors can be obtained by considering the corresponding integrals of continuous functions. Although this makes the process seem like nothing more than a mathematical trick for simplifying equations, this process is important enough to have the entire subject of field theory devoted to it. This is because the continuous functions (called fields) act as a far better representation for nature than the discrete analogues.

The field theory for the quantum Ising model is obtained by first introducing a spatial parameter to the Ising chain. We can consider our N sites to be located on a disk (or Möbius strip, if one wishes to consider antiperiodic boundary conditions), as shown in fig. 2.1. As the interaction strength is assumed to be homogeneous, we assume a constant lattice spacing a .

Now, define the continuous operator spinor

$$\mathbf{c}(x) \equiv \begin{bmatrix} c(x) \\ c^\dagger(x) \end{bmatrix} \quad (2.25)$$

The domain of $\mathbf{c}(x)$ is the interval $x \in [0, V]$, where $V \equiv Na$ represents the total length of the Ising chain. We have appropriately named this continuous operator, as its form will be defined precisely by our discrete spinors $(\mathbf{c}_j)_{j=0}^{N-1}$.

We endow our function with the continuous version of the fermionic anticommutation relations,

$$\begin{aligned} \{c(x), c(y)\} &= 0 \\ \{c(x), c^\dagger(y)\} &= \delta(x - y) \end{aligned} \quad (2.26)$$

The discrete operators can be expressed in terms of the new continuous operator. For each site j , assign $x_j = ja$ and we have

$$\begin{aligned} \mathbf{c}_j &= \mathbf{c}(x_j) \\ \mathbf{c}_{j+1} &= \mathbf{c}(x_j + a) \\ &= \mathbf{c}(x_j) + a\partial_x \mathbf{c}(x_j) + O(a^2) \end{aligned} \quad (2.27)$$

To second order in a , the discrete Ising Hamiltonian becomes

$$\begin{aligned}
 H &= \sum_j \left[\mathbf{c}_j^\dagger (\mathbf{Z} + i\mathbf{Y}) \mathbf{c}_{j+1} - \lambda \mathbf{c}_j^\dagger \mathbf{Z} \mathbf{c}_j \right] \\
 &\approx \sum_j \left[\mathbf{c}^\dagger(x_j) (\mathbf{Z} + i\mathbf{Y}) \mathbf{c}(x_j) + a \mathbf{c}^\dagger(x_j) (\mathbf{Z} + i\mathbf{Y}) \partial_x \mathbf{c}(x_j) - \lambda \mathbf{c}^\dagger(x_j) \mathbf{Z} \mathbf{c}(x_j) \right] \\
 &= \sum_j \left[\frac{1-\lambda}{a} \mathbf{c}^\dagger(x_j) \mathbf{Z} \mathbf{c}(x_j) + \mathbf{c}^\dagger(x_j) (\mathbf{Z} + i\mathbf{Y}) \partial_x \mathbf{c}(x_j) \right] a
 \end{aligned} \tag{2.28}$$

which can readily be taken to the continuum. Note that $\mathbf{c}^\dagger(x_j) \mathbf{Y} \mathbf{c}(x_j) = 0$ by the anticommutation relations. Now consider the limit of our Hamiltonian as we take $N \rightarrow \infty$ and $a \rightarrow 0$ such that V remains constant. We replace our discrete sums with integrals,

$$\lim_{\substack{N \rightarrow \infty \\ a \rightarrow 0}} \sum_{j=0}^{N-1} f(x_j) a \equiv \int_0^V f(x) dx \tag{2.29}$$

which gives the field theory for the quantum Ising model in terms of Jordan-Wigner fermions

$$H = \int_0^V \left[i \mathbf{c}^\dagger(x) \mathbf{Y} \partial_x \mathbf{c}(x) + m \mathbf{c}^\dagger(x) \mathbf{Z} \mathbf{c}(x) \right] dx \tag{2.30}$$

Two items of note are the introduction of the constant mass* term, defined as

$$m \equiv \lim_{\substack{a \rightarrow 0 \\ \lambda \rightarrow 1}} \frac{1-\lambda}{a} \tag{2.31}$$

The limit of $a \rightarrow 0$ gives us our field theory, but we are forced to consider the critical (with respect to λ) behaviour in order to maintain a sensible theory. The missing derivative term has also been eliminated as a surface integral

$$\int_0^V \mathbf{c}^\dagger(x) \mathbf{Z} \partial_x \mathbf{c}(x) dx = \int_0^V \left[c^\dagger(x) \partial_x c(x) - c(x) \partial_x c^\dagger(x) \right] dx \tag{2.32}$$

Note that the derivative operator, being linear, can be freely moved outside of the anticommutation relations

$$\begin{aligned}
 \{c(x), \partial_x c^\dagger(x)\} &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\{c(x), c^\dagger(x+h)\} - \{c(x), c^\dagger(x)\} \right] \\
 &= 0
 \end{aligned} \tag{2.33}$$

*The interpretation of this term as mass comes from computing the equations of motion under this Hamiltonian. In the Majorana representation, this Hamiltonian generates the equations of motion for a massive Majorana spinor in (1+1)D [6].

and as a result, we obtain

$$\begin{aligned}
 \int_0^V [c^\dagger(x)\partial_x c(x) + \partial_x c^\dagger(x)c(x)] dx &= \int_0^V \partial_x (c^\dagger(x)c(x)) dx \\
 &= [c^\dagger(x)c(x)]_0^V \\
 &= c^\dagger(V)c(V) - c^\dagger(0)c(0)
 \end{aligned} \tag{2.34}$$

where the periodic (or antiperiodic) boundary conditions force this term to be zero.

We have now obtained a continuous field theory for the 1D quantum Ising model. The remaining chapters are in service of developing discrete analogues of the Hamiltonian in eq. (2.30). The discretised versions will be characterised with renormalisation in mind, and will be constructed in such a way that the mapping between the continuous and discrete versions is as direct as possible. This is achieved through wavelet analysis, which is a mathematical structure that incorporates renormalisation into the very definition.

*Nature uses only the longest
threads to weave her patterns,
so that each small piece of her
fabric reveals the organization
of the entire tapestry.*

Richard Feynman

3

Wavelets

This chapter builds the background, motivation and technical details necessary for discussing wavelets. We start by discussing the failures of some methods preceding wavelets, namely the windowed Fourier transform. Wavelets are developed to directly combat the issues discussed, and accomplish this by establishing the concept of a multiresolution analysis. Daubechies wavelets, the wavelet family utilised in this thesis, are obtained when one imposes the condition of compact support on the scale and wavelet functions. By analysing the implications of such a restriction, one uncovers a finite set associated with each wavelet realisation. This means that Daubechies wavelets can be used to generate discrete analogues of continuous theories.

3.1 Background & History

Wavelets are a tool that were developed to deal with shortcomings of Fourier analysis. The issue with Fourier analysis is that ‘nice’ functions (the canonical example being square integrable functions:

$f \in L^2(\mathbb{R})$) with non-zero, compact support* cannot have a compactly supported Fourier transform [7]. This is essentially a more technical statement of the Heisenberg uncertainty principle, and is a well-known issue regarding studying continuous functions to arbitrary precision in both time and frequency space (or position and momentum space) simultaneously.

Compact support is a natural restriction in the realm of signal analysis, which is where wavelets were first developed. In this field, one typically analyses finite data over finite time intervals, which means that signals are effectively modelled by compactly supported functions. Fourier analysis is an useful technique for studying such signals, but one encounters the previously discussed complications through blind application. As an extension to Fourier analysis, wavelets were able to satisfy the ambitious goal of recovering both roughly localised time and roughly localised frequency information. Put less formally, wavelets are able to obtain information regarding *which* frequency appeared *when* in a signal, albeit to within a fundamentally limited degree of accuracy.

Learning from the so far discussed and other issues within harmonic analysis, wavelets are a carefully constructed orthonormal basis for $L^2(\mathbb{R})$. Daubechies wavelets, in particular, are an orthonormal wavelet basis that is also compactly supported. The set of wavelets generates the *wavelet transform*, which maps a function to a representative set of expansion coefficients. This transform respects the Heisenberg uncertainty principle by dynamically sacrificing frequency precision when temporal information is more important, and vice versa. Fundamental to such a technique is the framework of a *multiresolution analysis*. A multiresolution analysis is a decomposition of a signal into a hierarchy of individual decompositions, each to be interpreted as a ‘scaled’ representation of the original signal. Historically, the motivation for such a procedure has roots in renormalisation theory, and, tautologically, the relevance of wavelets to this research are their renormalisation properties.

3.2 Multiresolution Analysis

The concept of a multiresolution analysis revolves around building successive approximations for a function that are consistent with one another. We associate each approximation with an index which will be referred to as the *scale* or *resolution*.

Discussion of multiresolution analysis starts by considering a collection of indexed vector spaces, \mathcal{V}_j

*Support is the region in the domain of a function that maps to non-zero outputs. A compact interval is closed and bounded.

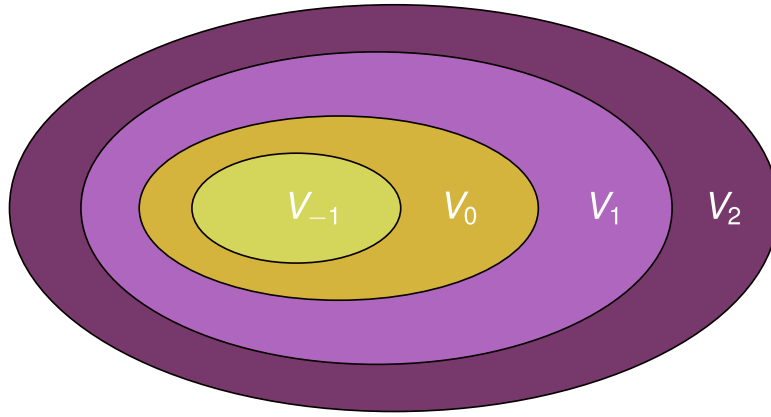


Figure 3.1: A collection of nested vector spaces. Each space is fully contained within the others in the hierarchy.

(as depicted in fig. 3.1), that satisfy the following 6 properties [8]:

1. $\dots \subset \mathcal{V}_{-2} \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \dots$; this condition implies that there is an ordering to the set of spaces. This condition (in combination with condition 4) means that higher indexed spaces capture the intricacies of lower indexed spaces, and more.
2. $\bigcap_{j \in \mathbb{Z}} \mathcal{V}_j = \{0\}$; this condition removes unnecessary redundancy, and leads to the linear independence of the wavelet basis generated.
3. $\overline{\bigcup_{j \in \mathbb{Z}} \mathcal{V}_j} = L^2(\mathbb{R})$; which means that $\mathbf{P}_\infty f = f$, where \mathbf{P}_n is the projector onto \mathcal{V}_n and $f \in L^2(\mathbb{R})$. This implies that a basis for \mathcal{V}_∞ will also be a suitable basis for $L^2(\mathbb{R})$, or equivalently, for arbitrarily small Euclidean error ϵ , there exists some \mathcal{V}_n such that $|\mathbf{P}_n f - f| < \epsilon$.
4. $f(x) \in \mathcal{V}_0 \iff f(2^j x) \in \mathcal{V}_j$; this condition captures the ‘multiresolution’ aspect, and is the property that relates all of the spaces together. All spaces in the chain are rescaled versions of the base space, \mathcal{V}_0 . A rescaling is a transformation of the domain of f by $x \rightarrow 2x$ (i.e. shrinking support in the case of compactly supported functions).
5. $f(x) \in \mathcal{V}_0 \implies f(x - n) \in \mathcal{V}_0$ for all $j \in \mathbb{Z}$; here we assume that the spaces are invariant under integer translations. An integer translation is a transformations of the domain of f by $x \rightarrow x - j$.
6. There exists $s \in \mathcal{V}_0$ such that $\{s_j^0(x) \equiv s(x - j)\}_{j \in \mathbb{Z}}$ is an orthonormal basis for \mathcal{V}_0 ; the final condition builds upon the previous one and assumes that there is some function in the space for which its integer translates form an orthonormal basis for the space. Note that this condition, together with condition 4, implies that if $\{s_j^0\}_{j \in \mathbb{Z}}$ is an orthonormal basis for \mathcal{V}_0 , then $\{s_j^n\}_{j \in \mathbb{Z}}$ is an orthonormal basis for \mathcal{V}_n (where $s_j^n \equiv \sqrt{2}^n s(2^n x - j)$).

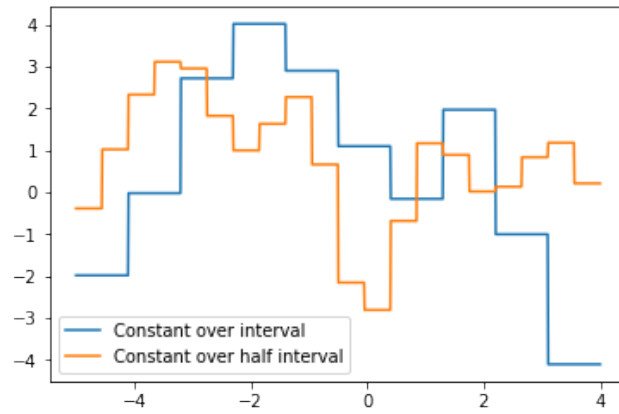


Figure 3.2: Sample functions that are constant over intervals of length 1 and length $\frac{1}{2}$.

3.2.1 Haar Wavelets

As a concrete example of one such wavelet system, consider the set of Haar scale functions, defined as

$$s(x) = \begin{cases} 1 & x \in [0, 1) \\ 0 & \text{otherwise} \end{cases} \quad (3.1)$$

The corresponding wavelet functions (discussed in detail in section §3.2.3) is given by

$$w(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}) \\ -1 & x \in [\frac{1}{2}, 1) \\ 0 & \text{otherwise} \end{cases} \quad (3.2)$$

One can verify that integer translations of the scale function, s , generate a basis for the vector space of functions that are constant over intervals of unit length (and are square integrable). By rescaling the scale function by a factor of 2, one can generate a basis for the vector space of square integrable functions that are constant over intervals of length $\frac{1}{2}$ (fig. 3.2). Further rescaling reveals a nested set of vector spaces (condition 1) that satisfy all of the necessary conditions listed above, and the collection of translated, rescaled wavelet functions form the Haar wavelet basis.

Chronologically, the Haar basis was discovered long before the concept of wavelets was ever formally defined. They were retroactively appointed the post-nominal title of wavelet once the subject was developed. In the discussion of wavelets, the Haar wavelets serve as the most intuitive example of a wavelet system (as they can be easily sketched and visualised, whereas general Daubechies wavelets are universally computer generated), and their existence motivated research into finding other compactly supported scale functions—ideally with increased regularity.

3.2.2 The Scaling Equation

From the definition of a multiresolution analysis, we can immediately generate the most important equation in the discussion of wavelets: the scaling equation.

From condition 1, $s \in \mathcal{V}_0$ means that $s \in \mathcal{V}_1$. Therefore, we should be able to express s as a linear combination of the basis for \mathcal{V}_1 , given by conditions 4 and 6. This leads to the Daubechies *scaling equation*

$$s(x) = \sqrt{2} \sum_{j=0}^{2\mathcal{K}-1} h_j s(2x - j) \quad (3.3)$$

which is a renormalisation group equation. It is not an exaggeration to claim that this equation is the most important one in the discussion of wavelets, as this equation alone defines the exact form of the essential scale function, s . The coefficients $H = \{h_j\}_{j=0}^{2\mathcal{K}-1}$ are called *filter coefficients*, and the set H^* is called a filter. They are obtained through $h_j = \langle s_j^1, s_0^0 \rangle$ with the standard $L^2(\mathbb{R})$ inner product: $\langle f, g \rangle \equiv \int_{-\infty}^{\infty} f^*(x)g(x) dx$. The integer \mathcal{K} is called the *wavelet index*, and it determines the total number of non-zero filter coefficients. Manipulations of the scaling equation that respect the orthogonality of integer translations reveal that the number of filter coefficients must be even. The compact support of s is due to the finite number of filter coefficients.

Since the scaling equation is homogeneous and s is compactly supported (i.e. $\mathcal{V}_0 \subset L^1(\mathbb{R})$), we are free to set normalisation. This is also known as the scale-fixing condition and is expressed as $\int_{-\infty}^{\infty} s(x) dx = 1$. The remarkable property of the scaling equation is that due to the recursive nature of the equation (along with the conditions 2 and 3), simply knowing H is enough to completely determine s , and by extension, \mathcal{V}_0 . Note that each value of \mathcal{K} generates a unique scaling function, and we will soon see that the wavelet index also determines the degree of regularity (smoothness) of s .

3.2.3 The Wavelet Equation

Now that we have a basis for each space determined, we should note that as a corollary, we have constructed frame for $L^2(\mathbb{R})$ in the form of $\{s_j^n\}_{j,n \in \mathbb{Z}}$. Since we seek a basis, we should remove unnecessary redundancy from this set. This is the point at which wavelets enter the picture. We start

*Because of the independent histories of the respective fields, an inevitable clash of notation occurs as H now refers to the filter and to a Hamiltonian. It should be clear from context which interpretation is implied.

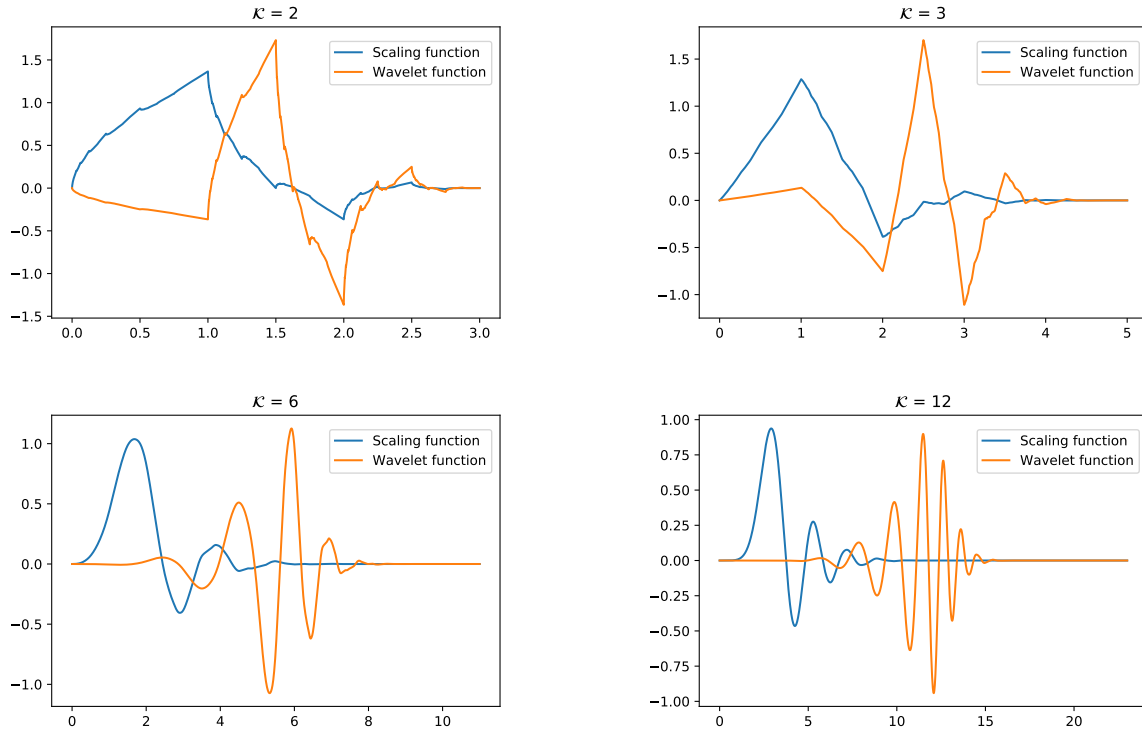


Figure 3.3: Daubechies scale and wavelet functions for indices $K = 2, 3, 6$ and 12 . As K increases, so does the support and smoothness of each function. The effect of higher order vanishing moments manifests in the narrowing of the distributions relative to its support.

by defining the orthocomplement^{*} of \mathcal{V}_0 within \mathcal{V}_1 to be \mathcal{W}_0

$$\mathcal{V}_1 = \mathcal{V}_0 \oplus \mathcal{W}_0 \quad (3.4)$$

The space \mathcal{W}_0 will be the home for our wavelet function w : the partner[†] to the scaling function s . Let $w \in \mathcal{W}_0 \subset \mathcal{V}_1$ be a special function such that integer translates of this function form an orthonormal basis for \mathcal{W}_0 . From the multiresolution axioms, we obtain a similar equation to the scaling equation. We express w as a linear combination of basis elements of \mathcal{V}_1 ,

$$w(x) = \sqrt{2} \sum_{j=0}^{2K-1} g_j s(2x - j) \quad (3.5)$$

^{*}Two subspaces, $V, W \subset U$ are complementary if for any $u \in U$, there exists $v \in V, w \in W$ such that $u = v + w$. We say that V and W are orthogonal if $\langle v, w \rangle = 0$ for all v and w . Orthocomplement is the natural amalgamation of the two terms, and means that every u has a *unique* decomposition in terms of V and W .

[†]These functions are often referred to as mother and father wavelet functions respectively.

which generates the Daubechies *wavelet equation*. If we choose the filter* $G = \{g_j\}_{j=0}^{2K-1}$ such that $g_j = (-1)^j h_{2K-1-j}$ (that is, in the reverse order with alternating signs), then integer translates of w do indeed form a basis for \mathcal{W}_0 , while still satisfying the orthogonality condition $\langle s_j^0, w_{j'}^0 \rangle = 0$. It follows that knowing H is enough to determine G , which uniquely determines both s and w .

We recursively repeat the orthogonal decomposition to achieve a direct sum representation for any space in the chain

$$\mathcal{V}_n = \mathcal{V}_0 \oplus \bigoplus_{k=0}^{n-1} \mathcal{W}_k \quad (3.6)$$

Each space \mathcal{W}_n is spanned by the set $\{w_j^n\}_{j \in \mathbb{Z}}$ which are integer translates of a rescaled version of the wavelet function: $w_j^n(x) \equiv \sqrt{2}^n w(2^n x - j)$. If we continue ad infinitum, we have a set of spaces that satisfy (by condition 3)

$$L^2(\mathbb{R}) = \overline{\bigoplus_{k=-\infty}^{\infty} \mathcal{W}_k} \quad (3.7)$$

and the set $\{w_j^k\}_{j,k \in \mathbb{Z}}$ are an orthonormal basis for $L^2(\mathbb{R})$. We can, however, choose to impose a (physically motivated) scale cutoff n ,

$$L^2(\mathbb{R}) \approx \mathcal{V}_n = \mathcal{V}_0 \oplus \bigoplus_{k=0}^{n-1} \mathcal{W}_k \quad (3.8)$$

Now \mathcal{V}_n represents the minimum scale at which we choose to represent our functions, \mathcal{V}_0 acts as a summary for all large-scale degrees of freedom and \mathcal{W}_k represents a chain of corrections required to obtain finer approximations. In renormalisation language, we associate \mathcal{V}_n with a UV-cutoff momentum, \mathcal{V}_0 with low momentum and \mathcal{W}_k with renormalised states [9].

3.2.4 Vanishing Moments

The first wavelet was developed by Alfréd Haar in 1910s [10], who constructed the Haar basis for $L^2(\mathbb{R})$ long before the theory of wavelets was formalised. The Haar basis is a dyadic set of rescaled and juxtaposed tophat functions which are highly localised (small support), but also non-regular (as they are discontinuous). More than half a century later in the 1980s, Daubechies generalised the Haar

* H and G are sometimes referred to as a low-pass and band-pass filters, particularly in engineering contexts. Elements of \mathcal{V}_n can be thought of approximations for functions up to a certain degree n , meaning \mathcal{V}_n attenuates finer details. Each \mathcal{W}_n yields approximations only within the scale range n to $n + 1$.

basis by developing a family of compactly supported scale functions. The Daubechies wavelets had larger support, but were also smoother (differentiable to higher degrees). The size of the support and degree of regularity of the Daubechies wavelet are both characterised by the wavelet index, \mathcal{K} .

The smoothness of the Daubechies wavelet essentially comes from the fact that they satisfy the vanishing moments condition:

$$\int_{-\infty}^{\infty} x^m w(x) dx = 0 \quad m \in \{0, \dots, \mathcal{K} - 1\} \quad (3.9)$$

This condition allows higher order wavelets act as better approximations to C^∞ class functions, and means that wavelets can exactly represent functions in $L^2 \cap C^{\mathcal{K}-2}$.

Note that the scaling equation has $2\mathcal{K}$ degrees of freedom, given by the set of $2\mathcal{K}$ filter coefficients. Demanding orthogonality of integer translates of the scale function leads to $\mathcal{K} - 1$ linearly independent restrictions on the filter. The compact support of the scale function effectively adds another restriction, square integrability another, and the vanishing moments condition the final $\mathcal{K} - 1$ (the case $m = 0$ is not linearly independent from the rest). This means that the Daubechies wavelets are the *minimal* family of compactly supported wavelets.

3.2.5 Overlap Integrals

In the proceeding chapter, we will be required to evaluate integrals of the following forms

$$\begin{aligned} D_{jj'}^{00} &\equiv \int s_j^0(x) \partial_x s_{j'}^0(x) dx \\ C_{jj'}^{0k'} &\equiv \int s_j^0(x) \partial_x w_{j'}^{k'}(x) dx \\ B_{jj'}^{k0} &\equiv \int w_j^k(x) \partial_x s_{j'}^0(x) dx \\ A_{jj'}^{kk'} &\equiv \int w_j^k(x) \partial_x w_{j'}^{k'}(x) dx \end{aligned} \quad (3.10)$$

These integrals are computed by Beylkin [11] by substituting the scaling equation into the above expressions and resolving the resulting recursion relations. Interestingly, the values are all rational, albeit difficult to obtain analytically for large \mathcal{K} . These values are well defined for $\mathcal{K} \geq 2$.

3.2.6 Summary

As a summary of this quite technical section, we have obtained a set of subspaces for $L^2(\mathbb{R})$ satisfying conditions 1–6. Each space has a basis given by integer translates of rescaled versions of a single scaling function, s . The collection of bases generated by the scale function, however, is overcomplete when considered as a basis for $L^2(\mathbb{R})$. The wavelet function, w , accompanies the scale function and scaled integer translates of this function are a complete basis for $L^2(\mathbb{R})$. The pair satisfy the scaling and wavelet equations eq. (3.3) and eq. (3.5) respectively, as well as the orthogonality relations

$$\begin{aligned} \langle s_j^n, s_{j'}^{n'} \rangle &= \int_{-\infty}^{\infty} s_j^n(x) s_{j'}^{n'}(x) dx = \delta_{jj'} \\ \langle s_j^n, w_{j'}^{n'} \rangle &= \int_{-\infty}^{\infty} s_j^n(x) w_{j'}^{n'}(x) dx = 0 \\ \langle w_j^n, w_{j'}^{n'} \rangle &= \int_{-\infty}^{\infty} w_j^n(x) w_{j'}^{n'}(x) dx = \delta_{jj'} \delta_{nn'} \end{aligned} \quad (n' \geq n) \quad (3.11)$$

The wavelet function satisfies the vanishing moments condition eq. (3.9) for a given wavelet index, \mathcal{K} .

3.3 Discrete Wavelet Transform

We now use the results from the previous sections to construct the discrete wavelet transform. The decomposition of $L^2(\mathbb{R})$ in eq. (3.8) can be used to formulate discrete approximations for any function in $f \in L^2(\mathbb{R})$.

Formally, we have the equality

$$\begin{aligned} f(x) &= \lim_{n \rightarrow \infty} \sum_{j=0}^{2^n V-1} \langle f, s_j^n \rangle s_j^n(x) \\ &= \sum_{j=0}^{V-1} \langle f, s_j^0 \rangle s_j^0(x) + \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \sum_{j=0}^{2^k V-1} \langle f, w_j^k \rangle w_j^k(x) \end{aligned} \quad (3.12)$$

The first expression represents the process of projecting f onto the subspace \mathcal{V}_∞ . The expansion coefficients, $\langle f, s_j^n \rangle$, completely characterise f (that is, if f_1 and f_2 have the same set of expansion coefficients, then $f_1 = f_2$). The second expression describes a process where we project onto a chosen coarsest scale (denoted by s^0) and add an infinite chain of finer details (denoted by w^k for each $k \in [0, \infty)$) until we have a perfect recreation of our starting function. An approximation is given by

simply truncating the limit to a finite value (that is, projecting f onto \mathcal{V}_n for a finite n)

$$\begin{aligned}
 f(x) &\approx \sum_{j=0}^{2^n V-1} \langle f, s_j^n \rangle s_j^n(x) \\
 &= \sum_{j=0}^{V-1} \langle f, s_j^0 \rangle s_j^0(x) + \sum_{k=0}^{n-1} \sum_{j=0}^{2^k V-1} \langle f, w_j^k \rangle w_j^k(x)
 \end{aligned} \tag{3.13}$$

Since f is characterised by its expansion coefficients, we conclude that the collection of expansion coefficients are a discrete analogue for the original function.

*Argument is conclusive, but it
does not remove doubt, so that
the mind may rest in the sure
knowledge of the truth, unless
it finds it by the method of
experiment.*

Roger Bacon

4

Wavelet-Ising Model

Armed with the basics of wavelet theory, we now apply wavelet analysis to our continuous field operators from section 2.2. This will give us an approximate Hamiltonian, but our specific choice of wavelet family (Daubechies wavelets) will also have the side-effect of creating an entirely new set of discrete fermionic ladder operators. The new Hamiltonian is the main result of this thesis, and bares semblance to the original, discrete Ising model as the continuum eigenenergies for both models are equivalent. The link between the models is strengthened by qualitatively considering results for both; namely pairwise ground state correlations as a function of separation length, and ground state subchain entanglement entropy.

Before we dive into the process, we must first note that the previous discussion regarding $L^2(\mathbb{R})$ is overly general for our purposes. Our model only considers periodic, square integrable functions over the interval $\mathcal{I} = [0, V]$. We rectify this by restricting our integrals to \mathcal{I} , and enforce periodicity in the form of $s_V^0 \equiv s_0^0$. This has implications on the minimum size of V , since the scaling equation relates $s_0^0(x)$ to $\{s_0^1, \dots, s_{2\mathcal{K}-1}^1\}$. V must be large enough to contain $2\mathcal{K}$ translates of $s_0^1 \equiv s(2x)$, meaning

$V \geq \mathcal{K}$. If this condition were not met, it would be impossible to satisfy both the scaling equation and all of the vanishing moments conditions, and so a different wavelet family would have to be considered. Since we only investigate Daubechies wavelets in this thesis, we enforce $V \geq \mathcal{K}$. Additionally, in order to avoid discontinuities in the scale field and its derivative, we assume $\mathcal{K} \geq 3$.

4.1 Discretised Hamiltonian

We apply the approximation process in eq. (3.12) to the fermionic field operator from §2.3. For notational convenience, we define $V_n \equiv 2^n V$. The approximations are given by

$$\begin{aligned} \mathbf{c}(x) &\approx \sum_{j=0}^{V_n-1} \mathbf{r}_{j,n} s_j^n(x) \\ \partial_x \mathbf{c}(x) &\approx \sum_{j=0}^{V_n-1} \mathbf{r}_{j,n} \partial_x s_j^n(x) \end{aligned} \quad (4.1)$$

The expansion coefficients are given by

$$\begin{aligned} \mathbf{r}_{j,n} &\equiv \begin{bmatrix} r_{j,n} \\ r_{j,n}^\dagger \end{bmatrix} = \int_0^V \mathbf{c}(x) s_j^n(x) dx \\ &= \begin{bmatrix} \langle c, s_j^n \rangle \\ \langle c^\dagger, s_j^n \rangle \end{bmatrix} \end{aligned} \quad (4.2)$$

Due to the scale orthogonality relations in eq. (3.11), $(r_{j,n}^\dagger, r_{j,n})_{j \in \mathbb{Z}}$ qualifies as a set of fermionic ladder operators

$$\{r_{j,n}, r_{j',n}\} = 0 \quad \{r_{j,n}, r_{j',n}^\dagger\} = \delta_{jj'} \quad (4.3)$$

Applying these expressions to the fermionic Hamiltonian in eq. (2.19) gives

$$\begin{aligned} H &= \int_0^V i \mathbf{c}^\dagger(x) \mathbf{Y} \partial_x \mathbf{c}(x) + m \mathbf{c}^\dagger(x) \mathbf{Z} \mathbf{c}(x) dx \\ &\approx i \sum_{jj'} \mathbf{r}_{j,n}^\dagger \mathbf{Y} \mathbf{r}_{j',n} D_{jj'}^{nn} + m \sum_j \mathbf{r}_{j,n}^\dagger \mathbf{Z} \mathbf{r}_{j,n} \end{aligned} \quad (4.4)$$

where we have used the orthogonality relations for s and the overlap integrals from eq. (3.10). Proposition 1 in [11] states that D only depends on the difference between j and j' , so we can equivalently define $l \equiv j - j'$ and $D_{jj'}^{nn} \rightarrow D_l^{nn}$. Additionally, the proposition asserts that $D_l^{00} = 0$

whenever $|l| > 2(\mathcal{K} - 1) = \mathcal{J}$. Both of these observations lead to the expression

$$H = i \sum_{j=0}^{V_n-1} \sum_{l=-\mathcal{J}}^{\mathcal{J}} \mathbf{r}_{j,n}^\dagger \mathbf{Y} \mathbf{r}_{(j+l),n} D_l^m + m \sum_{j=0}^{V_n-1} \mathbf{r}_{j,n}^\dagger \mathbf{Z} \mathbf{r}_{j,n} \quad (4.5)$$

We have now successfully obtained a discrete, wavelet-based, pseudo-Ising model. The ‘pseudo’ prefix denotes the important distinction between this model and the original, which is that we no longer have nearest-neighbour coupling. Instead, the interaction length is governed entirely by the wavelet index, \mathcal{K} . It might seem that these long-range interactions would draw a complete distinction between the base Ising model and this new model. However, by construction, we know that both models have the same continuum limit, and should (at least qualitatively) behave similarly. In the next section, we will construct a simplified version of this model, which will be used to compare to the discrete Ising model.

4.2 Diagonalisation of the Model

In this section we construct a basis transformation which maps the existing model to a set of uncoupled, Fermi oscillators. These fermions are called Bogoliubov fermions, in honour of the author for the method, Nikolay Bogoliubov. The method was initially developed to simplify the Hamiltonian in the BCS theory of superconductivity[12], but the approach can be broadly applied to Hamiltonians that are quadratic in ladder operators (a related transformation is also defined for bosonic systems). The periodic Ising model is unique in that the exact transformation only requires two relatively easy steps. Being uncoupled, the dynamics of the Bogoliubov fermions are very simply to study. For example, we uncover the ground state of our Hamiltonian simply by identifying the state which is annihilated by every Bogoliubov annihilation operator. We will use the ground state and the diagonalised Hamiltonian to analyse entanglement in the following section.

4.2.1 Fourier Transform

Before directly diagonalising our Hamiltonian, we observe that it is translationally invariant. This means that the quadratic form of the Hamiltonian will be a pseudo-circulant matrix, comprised of blocks that are individually circulant. A property of circulant matrices is that they are diagonalised by the discrete Fourier transform, which implies that the quadratic form for our model will be almost diagonal when expressed in momentum space.

We invoke the discrete Fourier transform, defined as

$$\begin{aligned} r_{j,n} &= \frac{1}{\sqrt{V}} \sum_{k=0}^{V_n-1} p_{k,n} e^{-\frac{2\pi i}{V_n} jk} \\ r_{j,n}^\dagger &= \frac{1}{\sqrt{V}} \sum_{k=0}^{V_n-1} p_{k,n}^\dagger e^{\frac{2\pi i}{V_n} jk} \end{aligned} \quad (4.6)$$

for the second scale term, we have

$$\begin{aligned} r_{(j+l),n} &= \frac{1}{\sqrt{V}} \sum_{k'=0}^{V_n-1} p_{k',n} e^{-\frac{2\pi i}{V_n} jk'} e^{-\frac{2\pi i}{V_n} lk'} \\ r_{(j+l),n}^\dagger &= \frac{1}{\sqrt{V}} \sum_{k'=0}^{V_n-1} p_{k',n}^\dagger e^{\frac{2\pi i}{V_n} jk'} e^{\frac{2\pi i}{V_n} lk'} \end{aligned} \quad (4.7)$$

The unitarity of the Fourier transform guarantees that the anticommutation relations are preserved,

$$\{p_{j,n}, p_{j',n}\} = 0 \quad \{p_{j,n}, p_{j',n}^\dagger\} = \delta_{jj'} \quad (4.8)$$

We will now return to the non-spinor representation for our Hamiltonian. By invoking the Fourier transformation and with judicious use of the Dirichlet kernel identity: $\frac{1}{V_n} \sum_{j=n}^{V_n-1} e^{\frac{2\pi i}{V_n} j(k-k')} = \delta_{kk'}$, the model simplifies to

$$H = \sum_{k=0}^{V_n-1} \sum_{l=-\mathcal{J}}^{\mathcal{J}} D_l^{nn} \left(p_{k,n}^\dagger p_{-k,n}^\dagger e^{-\frac{2\pi i}{V_n} lk} - p_{k,n} p_{-k,n} e^{\frac{2\pi i}{V_n} lk} \right) + m \sum_{k=0}^{V_n-1} \left(p_{k,n}^\dagger p_{k,n} - p_{k,n} p_{k,n}^\dagger \right) \quad (4.9)$$

This can be tidied up by noting $D_l^{nn} = -D_{-l}^{nn}$ is an odd function of l , so we introduce

$$\begin{aligned} Q_k &\equiv \sum_{l=-\mathcal{J}}^{\mathcal{J}} D_l^{nn} e^{-\frac{2\pi i}{V_n} lk} \\ &= -2i \sum_{l=1}^{\mathcal{J}} D_l^{nn} \sin\left(\frac{2\pi}{V_n} lk\right) \end{aligned} \quad (4.10)$$

Under this simplification, the Hamiltonian reads

$$H = \sum_{k=0}^{V_n-1} \left[Q_k p_{k,n}^\dagger p_{-k,n}^\dagger - Q_{-k} p_{k,n} p_{-k,n} + m \left(p_{k,n}^\dagger p_{k,n} - p_{k,n} p_{k,n}^\dagger \right) \right] \quad (4.11)$$

Since all terms are periodic (that is $p_{(k+V),n} \equiv p_{k,n}$ and $Q_{k+V} = Q_k$) we can selectively rearrange the sums as follows

$$\begin{aligned} \sum_{k=0}^{V_n-1} Q_{-k} p_{k,n} p_{-k,n} &= Q_0 p_{0,n} p_{0,n} + Q_{-1} p_{1,n} p_{-1,n} + \cdots + Q_{1-V_n} p_{(V_n-1),n} p_{(1-V_n),n} \\ &= Q_0 p_{0,n} p_{0,n} + Q_{1-V_n} p_{(V_n-1),n} p_{(1-V_n),n} + \cdots + Q_{-1} p_{1,n} p_{-1,n} \\ &= Q_0 p_{0,n} p_{0,n} + Q_1 p_{-1,n} p_{1,n} + \cdots + Q_{V_n-1} p_{(1-V_n),n} p_{(V_n-1),n} \\ &= \sum_{k=0}^{V_n-1} Q_k p_{-k,n} p_{k,n} \end{aligned} \quad (4.12)$$

We apply this procedure only to the second and last terms in eq. (4.11). This gives

$$\begin{aligned}
 H &= \sum_{k=0}^{V_n-1} \left[Q_k p_{k,n}^\dagger p_{-k,n}^\dagger - Q_k p_{-k,n} p_{k,n} + m \left(p_{k,n}^\dagger p_{k,n} - p_{-k,n} p_{-k,n}^\dagger \right) \right] \\
 &= \sum_{k=0}^{V_n-1} \begin{bmatrix} p_{k,n}^\dagger & p_{-k,n} \end{bmatrix} \begin{bmatrix} m & Q_k \\ -Q_k & -m \end{bmatrix} \begin{bmatrix} p_{k,n} \\ p_{-k,n}^\dagger \end{bmatrix}
 \end{aligned} \tag{4.13}$$

which is considerably easier to diagonalise, as the double sum of eq. (4.5) has now been reduced to a single sum, although with more complicated coefficients.

4.2.2 Bogoliubov Transform

With the simplified Hamiltonian, the final step will produce a diagonalised Hamiltonian. That is, we seek to express the Hamiltonian in the form

$$H \propto \sum_{k=0}^{V_n-1} \omega_k \eta_{k,n}^\dagger \eta_{k,n} + \text{const.} \tag{4.14}$$

where $(\eta_{k,n}^\dagger, \eta_{k,n})_{k=0}^{V_n-1}$ are fermionic operators satisfying the correct fermionic algebra and ω_k are the corresponding eigenenergies. To perform such a feat, we utilise the Bogoliubov transformation

$$\begin{aligned}
 \eta_{k,n} &= u_k p_{k,n} + i v_k p_{-k,n}^\dagger & \eta_{-k,n} &= u_{-k} p_{-k,n} + i v_{-k} p_{k,n}^\dagger \\
 \eta_{k,n}^\dagger &= u_k p_{k,n}^\dagger - i v_k p_{-k,n} & \eta_{-k,n}^\dagger &= u_{-k} p_{-k,n}^\dagger - i v_{-k} p_{k,n}
 \end{aligned} \tag{4.15}$$

In order to preserve the anticommutation relations, we have the conditions

$$\begin{aligned}
 \{\eta_{j,n}, \eta_{k,n}^\dagger\} &= \delta_{jk} & \{\eta_{j,n}, \eta_{k,n}\} &= 0 \\
 \Rightarrow u_k^2 + v_k^2 &= 1 & \Rightarrow u_k v_{-k} &= -u_{-k} v_k
 \end{aligned} \tag{4.16}$$

the second of which is satisfied if u_k and v_k are an even and odd function of k respectively. We choose u_k and v_k , subject to the specified constraints, such that H becomes diagonalised. The transformation above can be summarised by the unitary transformations

$$\begin{aligned}
 \begin{bmatrix} \eta_{k,n} \\ \eta_{-k,n}^\dagger \end{bmatrix} &= \begin{bmatrix} u_k & i v_k \\ i v_k & u_k \end{bmatrix} \begin{bmatrix} p_{k,n} \\ p_{-k,n}^\dagger \end{bmatrix} & \begin{bmatrix} \eta_{k,n}^\dagger & \eta_{-k,n} \end{bmatrix} &= \begin{bmatrix} p_{k,n}^\dagger & p_{-k,n} \end{bmatrix} \begin{bmatrix} u_k & -i v_k \\ -i v_k & u_k \end{bmatrix} \\
 &= \mathbf{U}_k \begin{bmatrix} p_{k,n} \\ p_{-k,n}^\dagger \end{bmatrix} & & = \begin{bmatrix} p_{k,n}^\dagger & p_{-k,n} \end{bmatrix} \mathbf{U}_k^\dagger
 \end{aligned} \tag{4.17}$$

where \mathbf{U}_k is clearly unitary. Invoking the inverse transformation, the Hamiltonian reads

$$H = \sum_{k=0}^{V_n-1} \begin{bmatrix} \eta_{k,n}^\dagger & \eta_{-k,n} \end{bmatrix} \begin{bmatrix} u_k & iv_k \\ iv_k & u_k \end{bmatrix} \begin{bmatrix} m & Q_k \\ -Q_k & -m \end{bmatrix} \begin{bmatrix} u_k & -iv_k \\ -iv_k & u_k \end{bmatrix} \begin{bmatrix} \eta_{k,n} \\ \eta_{-k,n}^\dagger \end{bmatrix} \quad (4.18)$$

We wish for the matrix terms to be diagonal, that is $\mathbf{U}_k \mathbf{A}_k \mathbf{U}_k^\dagger = \mathbf{\Omega}_k$. This implies that the columns of \mathbf{U}_k^\dagger are eigenvectors of \mathbf{A}_k ,

$$\begin{bmatrix} m & Q_k \\ -Q_k & -m \end{bmatrix} \begin{bmatrix} u_k \\ -iv_k \end{bmatrix} = \omega_k \begin{bmatrix} u_k \\ -iv_k \end{bmatrix} \quad (4.19)$$

with eigenvalues

$$\omega_k = \pm \sqrt{m^2 - Q_k^2} \quad (4.20)$$

From the eigenvector equation we obtain explicit expressions for u_k and v_k (note that $u_{-k} = u_k$ and $v_{-k} = -v_k$, as required for the anticommutation relations to hold)

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} = \frac{1}{\sqrt{(m + \omega_k)^2 - Q_k^2}} \begin{bmatrix} m + \omega_k \\ iQ_k \end{bmatrix} \quad (4.21)$$

Expanding the Hamiltonian out in this basis gives

$$\begin{aligned} H &= \sum_{k=0}^{V_n-1} \begin{bmatrix} \eta_{k,n}^\dagger & \eta_{-k,n} \end{bmatrix} \begin{bmatrix} \omega_k & 0 \\ 0 & -\omega_k \end{bmatrix} \begin{bmatrix} \eta_{k,n} \\ \eta_{-k,n}^\dagger \end{bmatrix} \\ &= \sum_{k=0}^{V_n-1} \omega_k \eta_{k,n}^\dagger \eta_{k,n} - \omega_k \eta_{-k,n} \eta_{-k,n}^\dagger \end{aligned} \quad (4.22)$$

We now apply the anticommutation relations and repeat the trick from earlier of selectively rearranging the second term and exploiting periodicity. Doing so (and noting $\omega_{-k} = \omega_k$) gives

$$\begin{aligned} H &= 2 \sum_{k=0}^{V_n-1} \omega_k \left(\eta_{k,n}^\dagger \eta_{k,n} - \frac{1}{2} \right) \\ H &= 2 \sum_{k=0}^{V_n-1} \sqrt{m^2 + \left[2 \sum_{l=1}^{2(\mathcal{K}-1)} D_l^{nm} \sin\left(\frac{2\pi}{V_n} kl\right) \right]^2} \eta_{k,n}^\dagger \eta_{k,n} + E_0 \end{aligned} \quad (4.23)$$

where $E_0 \equiv -\sum_k \omega_k$ is the ground state energy of the system.

We can use this expression and the relative ease of renormalisation in the wavelet expression to compare the spectrum of our model with the continuum Ising model. If we restrict the discussion to

low momenta, then in the resolution limit $n \rightarrow \infty$ we make the approximation

$$\begin{aligned} \sin\left(\frac{2\pi}{2^n V} kl\right) &\approx \frac{2\pi k}{2^n V} l \\ &= \frac{\tilde{k}}{2^n} l \end{aligned} \quad (4.24)$$

One can employ change of variables to verify $D_l^{nn} = 2^n D_l^{00}$. In the resolution limit, the energy of such a state becomes

$$\omega_k^2 = m^2 + \left[2 \sum_{l=1}^J D_l^{00} l \tilde{k} \right]^2 \quad (4.25)$$

By utilising proposition 1 in [11] (which is $\sum_{l=1}^J D_l^{00} l = -1/2$), this expression simplifies to

$$\omega_k^2 = m^2 + \tilde{k}^2 \quad (4.26)$$

which is the low momentum dispersion relation for the continuum Ising model.

To summarise the process above, we combine eq. (4.6) with eq. (4.15) to condense the entire mapping as a single, unitary transformation, \mathbf{W}

$$\begin{aligned} \eta_{k,n} &= \frac{u_k + i v_k}{\sqrt{V}} \sum_{j=0}^{V_n-1} \left(r_{j,n} + r_{j,n}^\dagger \right) e^{\frac{2\pi i}{V_n} j k} \\ \boldsymbol{\eta}[n] &= \mathbf{W} \mathbf{r}[n] \end{aligned} \quad (4.27)$$

where the lack of position index implies the spinor representation

$$\boldsymbol{\eta}[n] = \begin{bmatrix} \eta_{0,n} \\ \eta_{0,n}^\dagger \\ \vdots \\ \eta_{(V_n-1),n} \\ \eta_{(V_n-1),n}^\dagger \end{bmatrix} \quad \mathbf{r}[n] = \begin{bmatrix} r_{0,n} \\ r_{0,n}^\dagger \\ \vdots \\ r_{(V_n-1),n} \\ r_{(V_n-1),n}^\dagger \end{bmatrix} \quad (4.28)$$

4.3 Entanglement Entropy

In this section, I compute the block chain entropy of the wavelet-Ising model eq. (4.5) and compare to the traditional Ising model eq. (2.19). A block in the Ising model is a contiguous sequence of fermion sites, which partitions the Ising chain into two sections. Because of the translational invariance of the model, only the length of the block needs to be considered, and not its relative position. One computes

the von Neumann entropy through calculation of the reduced density matrix. As computing the partial trace is a somewhat involved process, a preferred method to obtaining von Neumann entropy is given by considering the correlation matrix, which I define below. The process for computing entropy is detailed in [13], and what follows is largely a reproduction of their process applied to the wavelet-Ising model.

4.3.1 Majorana Representation

The Ising model is most naturally described in the language of Majorana spinors, as the dynamics of the Ising field theory correspond to that of free Majorana fermions [6]. Furthermore, we will exploit the compact representation and symplectic structure of the Majorana correlation matrix.

We define the Majorana representation by the transformation

$$g_{j,n}^L = \eta_{j,n}^\dagger + \eta_{j,n} \quad g_{j,n}^R = \frac{\eta_{j,n} - \eta_{j,n}^\dagger}{i} \quad (4.29)$$

The Majorana operators are Hermitian and obey the Majorana anticommutation algebra

$$\{g_{j,n}^\sigma, g_{j,n}^\rho\} = 2\delta_{jk}\delta_{\sigma\rho} \quad (4.30)$$

In the Majorana representation, the diagonalised Hamiltonian has symplectic structure (with constant additive terms omitted)

$$\begin{aligned} H &= \frac{i}{2} \sum_{k=0}^{V_n-1} \omega_k \left(g_{k,n}^L g_{k,n}^R - g_{k,n}^R g_{k,n}^L \right) \\ &= \frac{i}{2} \sum_{k=0}^{V_n-1} \omega_k \begin{bmatrix} g_{k,n}^L & g_{k,n}^R \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} g_{k,n}^L \\ g_{k,n}^R \end{bmatrix} \end{aligned} \quad (4.31)$$

4.3.2 Correlation Matrix

In the fermionic representation, the ground state of H satisfies the property $\eta_{k,n} |G\rangle = 0$ for every choice of k . By making use of the anticommutation relations of the Bogoliubov fermions, we can readily obtain the expectation values for single operators and operator products. Let $\langle M \rangle$ denote the

ground state expectation value of the operator M : $\langle G|M|G\rangle$. We have

$$\begin{aligned}\langle \eta_{j,n} \rangle &= \langle \eta_{j,n}^\dagger \rangle = 0 \\ \langle \eta_{j,n} \eta_{k,n} \rangle &= \langle \eta_{j,n}^\dagger \eta_{k,n}^\dagger \rangle = 0 \\ \langle \eta_{j,n}^\dagger \eta_{k,n} \rangle &= 0 \\ \langle \eta_{j,n} \eta_{k,n}^\dagger \rangle &= \delta_{jk}\end{aligned}\tag{4.32}$$

If required, we can make use of Wick's theorem to reduce expectation values for longer products (for example, $\langle \eta_{j,n} \eta_{k,n}^\dagger \eta_{r,n} \eta_{p,n}^\dagger \rangle$) to combinations of the one and two operator expectation values enumerated in eq. (4.32). Since the ground state is characterised entirely by first and second moments, it is called a *Gaussian state*.

In the Majorana representation, all of the above expectation values can be summarised into a single expression

$$\langle g_{j,n}^\sigma g_{k,n}^\rho \rangle = \delta_{jk} \delta_{\sigma\rho} + i \mathbf{\Gamma}_{\sigma\rho;jk}^G \tag{4.33}$$

where

$$\mathbf{\Gamma}^G = \bigoplus_{j=0}^{V_n-1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{4.34}$$

The latter two indices of $\mathbf{\Gamma}^G$ specify a block location, and the first two indices represent a location within the block (LL corresponds to the upper left, LR corresponds to the upper right, etc). Equation (4.33) is the Majorana *correlation matrix*. The benefit of the Majorana representation is essentially the compactness of this expression, and the subsequent calculation of the von Neumann entropy will be straightforward given this equation. It is simple to directly verify that $\mathbf{\Gamma}^G$ accounts for the expectation values $\langle \eta_{k,n} \eta_{k,n} \rangle$, $\langle \eta_{k,n}^\dagger \eta_{k,n}^\dagger \rangle$, $\langle \eta_{k,n} \eta_{k,n}^\dagger \rangle$ and $\langle \eta_{k,n}^\dagger \eta_{k,n} \rangle$.

In addition to Bogoliubov-Majorana operators, we can construct Majorana operators for the coupled wavelet-Ising model, eq. (4.5). We invoke the same transformation as eq. (4.29)

$$b_{j,n}^L = r_{j,n}^\dagger + r_{j,n} \qquad b_{j,n}^R = \frac{r_{j,n} - r_{j,n}^\dagger}{i} \tag{4.35}$$

which again satisfy the Majorana algebra

$$\{b_j^\sigma, b_k^\rho\} = 2\delta_{jk} \delta_{\sigma\rho} \tag{4.36}$$

In terms of the Majorana operators, the wavelet-Ising model eq. (4.5) takes the form

$$H = -\frac{i}{2} \sum_{j=0}^{V_n-1} \sum_{l=-\mathcal{J}}^{\mathcal{J}} \mathbf{b}_{j,n}^T \mathbf{X} \mathbf{b}_{(j+l),n} D_l^{nn} - \frac{m}{2} \sum_{j=0}^{V_n-1} \mathbf{b}_{j,n}^T \mathbf{Y} \mathbf{b}_{j,n} \quad (4.37)$$

where

$$\mathbf{b}_{j,n} \equiv \begin{bmatrix} b_{j,n}^L \\ b_{j,n}^R \end{bmatrix} \quad \mathbf{b}_{j,n}^\dagger = \mathbf{b}_{j,n}^T \equiv \begin{bmatrix} b_{j,n}^L & b_{j,n}^R \end{bmatrix} \quad (4.38)$$

Since this Hamiltonian is quadratic in Majorana operators, we can assign it a corresponding quadratic form

$$H = \frac{i}{2} \sum_{j=0}^{V_n-1} \sum_{k=0}^{V_n-1} \sum_{\sigma, \rho \in \{L, R\}} \mathbf{S}_{\sigma\rho}^{jk} \mathbf{b}_j^\sigma[n] \mathbf{b}_k^\rho[n] \quad (4.39)$$

Here we have opted to promote the scale-index to discrete function argument for the sake of clarity.

The antisymmetric matrix $\mathbf{S}_{\sigma\rho}^{jk}$ takes the form

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_0 & -\mathbf{S}_1 & -\mathbf{S}_2 & \dots & \mathbf{S}_3 & \mathbf{S}_2 & \mathbf{S}_1 \\ \mathbf{S}_1 & \mathbf{S}_0 & -\mathbf{S}_1 & \dots & \mathbf{S}_4 & \mathbf{S}_3 & \mathbf{S}_2 \\ \mathbf{S}_2 & \mathbf{S}_1 & \mathbf{S}_0 & \dots & \mathbf{S}_5 & \mathbf{S}_4 & \mathbf{S}_3 \\ & \vdots & & \ddots & & \vdots & \\ -\mathbf{S}_1 & -\mathbf{S}_2 & -\mathbf{S}_3 & \dots & \mathbf{S}_2 & \mathbf{S}_1 & \mathbf{S}_0 \end{bmatrix} \quad (4.40)$$

with

$$\mathbf{S}_0 = \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix} \quad \mathbf{S}_j = \begin{bmatrix} 0 & D_j^{nn} \\ D_j^{nn} & 0 \end{bmatrix} \text{ for } j > 0 \quad (4.41)$$

Again, as implied by the structure of eq. (4.40), one should think of $\mathbf{S}_{\sigma\rho}^{jk}$ as block matrix. The upper indices specify the location of the block, and the lower indices specify the position within the block. Next, we determine the form of the correlation matrix for these coupled Majorana operators. It will be determined by the Fourier and Bogoliubov transformations computed above.

4.3.3 Symplectic Diagonalisation

The diagonalisation of the wavelet-Ising model was achieved through use of a Fourier transform, followed by a Bogoliubov transformation. Both operations are summarised as a single unitary transformation, \mathbf{W} . This unitary transforms the quadratic form for eq. (4.5) into a diagonal matrix of

eigenvalues

$$\begin{aligned} H &= \mathbf{r}^\dagger[n] \mathbf{T} r[n] \\ \Rightarrow \mathbf{W}^\dagger \mathbf{T} \mathbf{W} &= \mathbf{\Omega} \end{aligned} \quad (4.42)$$

In the Majorana representation, the free Hamiltonian quadratic form is symplectic, rather than diagonal. The process of mapping an antisymmetric matrix to its symplectic form is entirely contingent on knowing \mathbf{W} explicitly. By symplectic form, we mean a matrix that is of the form of $\mathbf{\Gamma}^G$.

To cast \mathbf{S} into symplectic form, we first note that it is real and antisymmetric (as opposed to \mathbf{T} , the fermionic quadratic form, which was necessarily real and symmetric). As multiplying an antisymmetric matrix by i produces a Hermitian matrix, the eigenvalues of an antisymmetric matrix are always zero or pure imaginary, and come in conjugate pairs. A real matrix cannot map real vectors to complex vectors, and so the imaginary eigenvalues $i\omega_k$ must correspond to complex eigenvectors, $\mathbf{z}_k = \boldsymbol{\beta}_k + i\boldsymbol{\gamma}_k$. The eigenvector corresponding to $-i\omega_k$ is $\bar{\mathbf{z}}_k = \boldsymbol{\beta}_k - i\boldsymbol{\gamma}_k$. We then construct an orthogonal matrix $\mathbf{\Delta}$ whose columns are

$$\mathbf{\Delta} = \begin{bmatrix} \boldsymbol{\beta}_0 & \boldsymbol{\gamma}_0 & \boldsymbol{\beta}_1 & \boldsymbol{\gamma}_1 & \cdots & \boldsymbol{\beta}_{V_n-1} & \boldsymbol{\gamma}_{V_n-1} \end{bmatrix} \quad (4.43)$$

This matrix casts \mathbf{S} into symplectic form:

$$\mathbf{\Delta} \mathbf{S} \mathbf{\Delta}^T = \bigoplus_{j=0}^{V_n-1} \begin{bmatrix} 0 & \omega_j \\ -\omega_j & 0 \end{bmatrix} \quad (4.44)$$

which means that $\mathbf{\Delta}$ is precisely the transformation from coupled to free Majorana operators,

$$\mathbf{b}[n] = \mathbf{\Delta} \mathbf{g}[n] \quad (4.45)$$

The correlation matrix for the coupled Majoranas,

$$\langle b_{j,n}^\sigma b_{k,n}^\rho \rangle = \delta_{jk} \delta_{\sigma\rho} + i \mathbf{\Gamma}_{\sigma\rho;jk}^B \quad (4.46)$$

is also determined by our newly obtained orthogonal matrix via

$$\mathbf{\Gamma}^B = \mathbf{\Delta}^T \mathbf{\Gamma}^G \mathbf{\Delta} \quad (4.47)$$

This matrix has elements

$$\mathbf{\Gamma}^B = \begin{bmatrix} \mathbf{\Pi}_0 & \mathbf{\Pi}_1 & \mathbf{\Pi}_2 & \cdots & \mathbf{\Pi}_{V_n-1} \\ -\mathbf{\Pi}_1 & \mathbf{\Pi}_0 & \mathbf{\Pi}_1 & & \vdots \\ -\mathbf{\Pi}_2 & -\mathbf{\Pi}_1 & \mathbf{\Pi}_0 & & \vdots \\ \vdots & & & \ddots & \vdots \\ -\mathbf{\Pi}_{V_n-1} & \cdots & \cdots & \cdots & \mathbf{\Pi}_0 \end{bmatrix}, \quad \mathbf{\Pi}_l = \begin{bmatrix} 0 & q_l \\ -q_l & 0 \end{bmatrix} \quad (4.48)$$

where q_l are numerically determined and $l \equiv j - k$ specifies block position.

4.3.4 Block Chain Entropy

Block chain entropy is a metric to analyse the degree of entanglement within an Ising chain. If $|G\rangle$ is the ground state (a pure state) and ρ is the associated density matrix, then by calculating the reduced density matrix, ρ_M ,

$$\rho_M = \text{Tr}_{V_n-M} |G\rangle\langle G| \quad (4.49)$$

one can quantify entanglement as a function of block length using von Neumann entropy. Formally, von Neumann entropy is given by the equation

$$\mathcal{S}_M = -\text{Tr}\{\rho_M \log_2 \rho_M\} \quad (4.50)$$

where \mathcal{S} denotes the von Neumann entropy, ρ is the density matrix for the ground state and ρ_M is the reduced density matrix for a block of length M . Rather than obtaining ρ_M directly through computation of partial traces, an alternative (and simpler) process is locally diagonalise the Hamiltonian and work with the reduced correlation matrix instead [14]. This method is the process followed by [15], where they analyse the XY model, a generalisation of the Ising model. We obtain the reduced correlation matrix by truncation of the final $V_n - M$ rows and columns of $\mathbf{\Gamma}^B$,

$$\mathbf{\Gamma}_M^B = \begin{bmatrix} \mathbf{\Pi}_0 & \mathbf{\Pi}_1 & \mathbf{\Pi}_2 & \dots & \mathbf{\Pi}_{M-1} \\ -\mathbf{\Pi}_1 & \mathbf{\Pi}_0 & \mathbf{\Pi}_1 & & \vdots \\ -\mathbf{\Pi}_2 & -\mathbf{\Pi}_1 & \mathbf{\Pi}_0 & & \vdots \\ \vdots & & & \ddots & \vdots \\ -\mathbf{\Pi}_{M-1} & \dots & \dots & \dots & \mathbf{\Pi}_0 \end{bmatrix} \quad (4.51)$$

Just as $\mathbf{\Gamma}^B$ is associated with the set of coupled Majorana operators $(b_{j,n}^L, b_{j,n}^R)_{j=0}^{V_n-1}$, the reduced form, $\mathbf{\Gamma}_M^B$, is associated with the truncated set $(b_{j,n}^L, b_{j,n}^R)_{j=0}^{M-1}$. We symplectically diagonalise this matrix,

$$\begin{aligned} \mathbf{\Gamma}_M^F &= \tilde{\mathbf{\Delta}} \mathbf{\Gamma}_M^B \tilde{\mathbf{\Delta}}^T \\ \mathbf{\Gamma}_M^F &= \bigoplus_{j=0}^{M-1} \begin{bmatrix} 0 & \nu_j \\ -\nu_j & 0 \end{bmatrix} \end{aligned} \quad (4.52)$$

to obtain part of the correlation matrix for the uncorrelated Majorana modes, $(f_{j,n}^L, f_{j,n}^R)_{j=0}^{M-1}$. These Majorana operators give rise to a set of uncorrelated fermionic operators,

$$\psi_{j,n} = \frac{f_{j,n}^L + i f_{j,n}^R}{2} \quad \psi_{j,n}^\dagger = \frac{f_{j,n}^L - i f_{j,n}^R}{2} \quad (4.53)$$

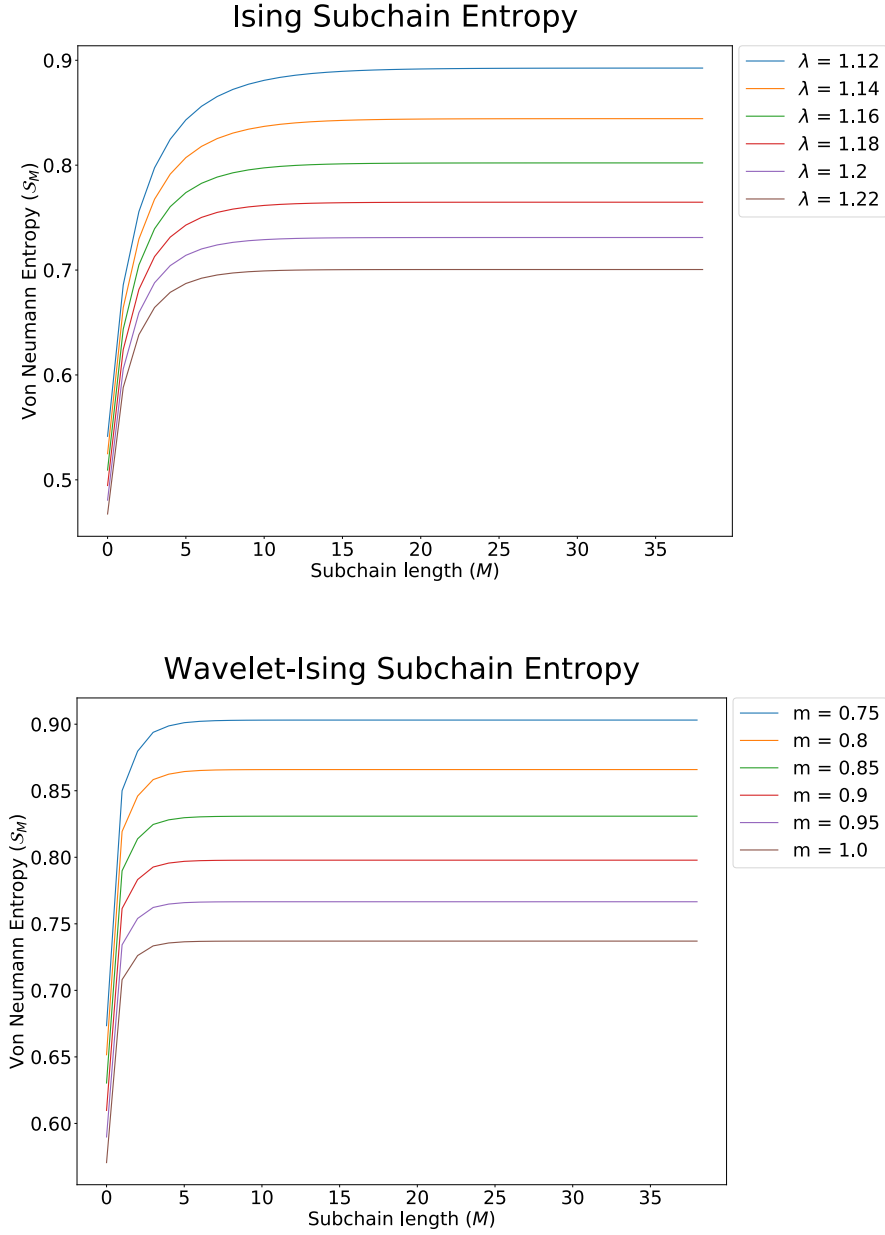


Figure 4.1: Von Neumann entropy of block chains as a function of chain length for both the quantum Ising model (eq. (2.30)) and the wavelet-Ising model (eq. (4.5)) for $\mathcal{K} = 3$. Chain length 700.

which, by direct substitution, have product expectation values

$$\langle \psi_{j,n} \psi_{k,n} \rangle = 0 \quad \langle \psi_{j,n}^\dagger \psi_{k,n} \rangle = \delta_{jk} \frac{1 - \nu_j}{2} \quad \langle \psi_{j,n} \psi_{k,n}^\dagger \rangle = \delta_{jk} \frac{1 + \nu_j}{2} \quad (4.54)$$

and, of course, all operators satisfy the necessary anticommutation relations

$$\{\psi_{j,n}, \psi_{k,n}\} = 0 \quad \{\psi_{j,n}, \psi_{k,n}^\dagger\} = \delta_{jk} \quad \{f_{j,n}^\sigma, f_{k,n}^\rho\} = 2\delta_{jk}\delta_{\sigma\rho} \quad (4.55)$$

Being completely uncorrelated, the density matrix in this basis is a product state,

$$\rho_M = \pi_0 \otimes \pi_1 \otimes \cdots \otimes \pi_{M-1} \quad (4.56)$$

where π_j is the reduced density matrix for the j th site, each of which has eigenvalues $(1 \pm v_j)/2$. For a product state, the total chain entropy is given by the sum of individual entropies on each subspace,

$$\mathcal{S}_M = \sum_{j=0}^{M-1} \mathcal{H}_2\left(\frac{1 + v_j}{2}\right) \quad (4.57)$$

where $\mathcal{H}_2(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$ is the binary entropy function—applicable due to the form of the eigenvalues.

Figure 4.1 shows numerical calculation the entropy for both the traditional Ising model and the wavelet-Ising model. I do not expect the results to exactly correspond to one another, as the parameters m and λ do not have an exactly representable relationship when both are finite. However, it appears that the entropy of the models have similar characteristics, both exhibiting strong saturation for short chain length $M \sim 10$. This implies finite or exponentially decaying correlation length, which is a well known result for the Ising model [16]. Due to the decaying correlation length, the entropy is effectively independent of the total length of the chain.

The vector space equality stated in eq. (3.6) establishes a different perspective of the model. If the model is expressed in the wavelet representation, one can calculate the subchain entropy by following a similar procedure to the one used in this thesis. In this representation, the *scale* of the model appears as an explicit parameter, giving a tunable dial to analyse renormalisation flow with. This is an important feature, and is the ultimate motivation of constructing such a reformulation in terms of wavelets. Unfortunately, analysis of this nature was beyond the scope of this project, and is instead suggested as an immediate extension to this work.

*All truths are easy to understand
once they are discovered—the
point is to discover them.*

Galileo Galilei

5

Conclusion

This thesis explores the construction of a wavelet-based discrete approximation (eq. (4.5)) to the field theory of the 1D quantum Ising model (eq. (2.30)). The discretisation scheme allows for the recovery of the continuous model in the resolution limit $n \rightarrow \infty$. The energy spectrum and ground state of the approximation are analytically determined and the series of transformations undertaken are explicitly detailed. Through numerical simulations, the subchain entanglement entropy of the wavelet-Ising model is compared to the traditional discrete Ising model.

The quantum Ising model is obtained both by analogy with the classical case, and as a result of an exact mapping between statistical mechanics and quantum mechanics. The quantum Ising model maps to a system of spinless fermions via a process described by the non-local Jordan-Wigner transformation. A continuous model is obtained in the limit as the number of sites ($N \rightarrow \infty$), site separation distance ($a \rightarrow 0$) and symmetry breaking parameter ($\lambda \rightarrow 1$) uniformly tend to their respective limits. The discrete wavelet transform is applied to the continuous model, resulting in the wavelet-Ising model. This transformation encodes the continuous fields as discrete analogues. The degree of error in the

approximation is captured by the scale index, n , and the error tends to zero as $n \rightarrow \infty$. There are multiple wavelet encoding schemes; the choice in this thesis was Daubechies wavelets of order $\mathcal{K} \geq 3$. Once the wavelet-Ising model is generated, it is diagonalised through means of a Fourier transform, succeeded by a Bogoliubov transform. The free theory is then used to analyse the coupled theory by quantifying entanglement through calculation of the von Neumann entropy for a truncated density matrix.

The motivation for developing a wavelet-based alternative to quantum field theory ultimately comes from renormalisation. The process of renormalisation is built into the foundations of wavelet theory, and the continuum limit for a discrete wavelet theory is mathematically well defined. The aim of this thesis is to generate results that give validation to a wavelet-based approach to renormalisation. Discretisation strategies can be used to analyse continuous theories that may otherwise require subtle arguments; the benefit of the wavelet-based approach is that one remains within the comfort of a discrete framework that is readily generalised to the continuum. The results obtained in this thesis support the claim that a wavelet-discretised theory can act in place of other, more ad hoc discrete models.

There are many immediate extensions to this work. The most obvious is granted by exploiting the vector space equality of eq. (3.6) to generate the scale-wavelet representation of the model (as opposed to a representation of purely scale fields). In this picture, one obtains a resolution degree of freedom, which allows the renormalisation aspects of the theory to be studied directly. Additionally, the study of bipartite entanglement was only one of many well known quantities regarding the Ising model that are of interest. Additionally, there are schemes for investigating inter-scale correlations, utilising the full scale-wavelet representation of the wavelet theory. One such example is explored in [9], where the authors study the $(1 + 1)\text{D}$ scalar bosonic field theory. As a renormalisable fermionic theory, the wavelet-Ising model may add alternative insights to the results of their methodology.

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