Structures in two dimensional category theory and applications to polynomial functors

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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Charles Robert Walker

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Abstract

This thesis seeks to further develop two-dimensional category theory, with a focus on Yoneda structures, (lax-idempotent) pseudomonads, pseudo-distributive laws, and familial representability, in order to gain new insights and tools in the study of polynomial functors.

The first contribution of this thesis concerns Yoneda structures, which give a formalization of the presheaf construction. Our main result shows that any fully faithful lax-idempotent pseudomonad almost gives rise to a Yoneda structure, with all of the axioms holding except for one condition.

The second contribution of this thesis concerns pseudo-distributive laws of a pseudomonad and a lax-idempotent pseudomonad. We show that such distributive laws have a simple algebraic description which only requires three out the usual eight coherence conditions, and another simple description in terms of the data of the near-Yoneda structure recovered from the lax-idempotent pseudomonad.

Our third contribution is to introduce a class of bicategories, which we term *generic bicategories*. These are the bicategories for which horizontal composition admits generic factorisations, and have the interesting property that oplax functors out of them have a reduced description, similar to the axioms of a comonad.

The fourth contribution of this thesis is to establish the universal properties of the bicategory of polynomials, with general and cartesian 2-cells, using the properties of generic bicategories to avoid the majority of the coherence conditions. In addition, we give a new proof of the universal properties of the bicategory of spans and establish the universal properties of the bicategory of spans with invertible 2-cells.

The fifth contribution of this thesis is to give an appropriate notion of familial representability for pseudofunctors $L\colon \mathscr{A} \to \mathscr{B}$ of bicategories, and to describe an equivalence with an analogue of generic factorisations. This improves on work of Weber, who did not provide such an equivalence, and required $\mathscr{A}$ to have a terminal object.

Contents

1

Introduction

1.1 Overview

One of the fundamental constructions in category theory is the so called span construction, which takes a category $\mathcal{E}$ with pullbacks to the bicategory $\mathbf{Span}(\mathcal{E})$ with objects those of $\mathcal{E}$, morphisms $I \nrightarrow J$ given by diagrams of the form

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[dr, "t"] & \\ I & & J \end{tikzcd}$$

called spans, and composition given by forming the pullback. As is commonplace in category theory, to gain an understanding of a construction, we should establish its universal property. In the case of spans, this was done by Hermida [21, Theorem A.2] who showed that composing with the canonical embedding $\mathcal{E} \hookrightarrow \mathbf{Span}(\mathcal{E})$ describes an equivalence

$$\frac{\text{pseudofunctors } \mathbf{Span}(\mathcal{E}) \to \mathscr{C}}{\text{Beck pseudofunctors } \mathcal{E} \to \mathscr{C}} \tag{1.1.1}$$

where a pseudofunctor $F_\Sigma \colon \mathcal{E} \to \mathscr{C}$ is *Beck* if for every morphism f in $\mathcal{E}$ the 1-cell $F_\Sigma f$ has a right adjoint $F_\Delta f$ in $\mathscr{C}$ (such an F_Σ is also known as a sinister pseudofunctor), and if the induced pair of pseudofunctors

$$F_\Sigma \colon \mathcal{E} \to \mathscr{C}, \qquad F_\Delta \colon \mathcal{E}^{\mathrm{op}} \to \mathscr{C}$$

satisfy a Beck-Chevalley condition. A second universal property of the span construction (of which the above is a restriction) was established by Dawson, Paré, and Pronk [9, Theorem

2.15], who showed that composing with the canonical embedding describes an equivalence

$$\frac{\text{gregarious functors } \mathbf{Span}\,(\mathcal{E}) \to \mathscr{C}}{\text{sinister pseudofunctors } \mathcal{E} \to \mathscr{C}} \tag{1.1.2}$$

where gregarious functors are the adjunction-preserving normal[1] oplax functors.

In recent years, interest has appeared in another construction: the so called polynomial construction which takes a locally cartesian closed category $\mathcal{E}$ to the bicategory **Poly** $(\mathcal{E})$ with objects those of $\mathcal{E}$, and morphisms $I \nrightarrow J$ given by diagrams of the form

$$\begin{tikzcd} & E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & \\ I & & & J \end{tikzcd}$$

called polynomials. This construction has appeared in areas ranging from type theory [43] to computer science under the name of containers [1].

This thesis began with the goal of establishing the universal properties of the bicategory of polynomials by giving appropriate analogues of the results in the case of spans, and we indeed achieve this goal in our fourth paper (see Section 1.5). However, this is more complicated than one might initially expect. Indeed, as a consequence of the complexity of polynomial composition, a direct proof of these universal properties would involve very large coherence problems, and would be impractical to verify directly.

Instead of proving these properties directly, we observe that in the case of spans, for a "locally defined functor" $L\colon \mathbf{Span}\,(\mathcal{E}) \to \mathscr{C}$, meaning a family of functors defined on hom-categories (assuming $\mathscr{C}$ has the same objects as **Span** $(\mathcal{E})$ and that $\mathcal{E}$ is small for simplicity), to give an oplax structure on L describing how composition of spans is respected, is equivalent to giving a lax structure on the nerve $R_L\colon \mathscr{C} \to \widehat{\mathbf{Span}}\,(\mathcal{E})$ as below

$$\begin{tikzcd} \mathscr{C} \arrow[r, "R_L"] & \widehat{\mathbf{Span}}\,(\mathcal{E}) \\ & \mathbf{Span}\,(\mathcal{E}) \arrow[ul, "L"] \arrow[u, "Y"'] \end{tikzcd}$$

where $\widehat{\mathbf{Span}}\,(\mathcal{E})$ is the local cocompletion of the bicategory **Span** $(\mathcal{E})$. We will refer to this analogue of Kelly's doctrinal adjunction [27] on diagrams as above (which appear in Yoneda structures [47]) as *doctrinal Yoneda structures*.

The reader will notice that whilst composition in **Span** $(\mathcal{E})$ is given by pullback, composition in $\widehat{\mathbf{Span}}\,(\mathcal{E})$ can be described without pullbacks, instead having a simple description given by taking appropriate sums of presheaves. Indeed, as composition in $\widehat{\mathbf{Span}}\,(\mathcal{E})$ is simpler, we conclude that the problem of exhibiting a lax structure on $R_L\colon \mathscr{C} \to \widehat{\mathbf{Span}}\,(\mathcal{E})$ is simpler than the equivalent problem of exhibiting an oplax structure on $L\colon \mathbf{Span}\,(\mathcal{E}) \to \mathscr{C}$.

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[1] Here "normal" means the unit constraints are invertible.

In this way, one can exhibit an oplax structure on $L\colon \mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ without ever directly using composition of spans.

Fortunately, this approach also works after replacing $\mathbf{Span}(\mathcal{E})$ by the bicategory of polynomials $\mathbf{Poly}_c(\mathcal{E})$ (the "c" here meaning we are restricting to cartesian 2-cells), and thus one can show that a functor $L\colon \mathbf{Poly}_c(\mathcal{E}) \to \mathscr{C}$ is oplax without ever directly using composition of polynomials, and therefore avoiding the majority of the coherence conditions which would arise from polynomial composition.

Given the above idea for proving the universal properties of $\mathbf{Span}(\mathcal{E})$ and $\mathbf{Poly}_c(\mathcal{E})$, it is a natural question to ask what is the special property of these bicategories which makes this method work. It turns out the important point is that both are examples of bicategories $\mathscr{A}$ for which horizontal composition admits generic factorisations (a condition equivalent to familial representability). Thus, before proving the universal properties of polynomials it is worth studying the properties of such bicategories, which we dub *generic bicategories*, and extracting what "doctrinal Yoneda structures" tell us about them. This is done in our third paper (see Section 1.4).

Moreover, it is worth turning "doctrinal Yoneda structures" into a properly-stated theorem in its natural context. But this again is not entirely straightforward. In general an algebraic structure on a category $\mathcal{A}$ (such as a monoidal structure on a category) should only be expected to lift to the free small cocompletion of $\mathcal{A}$ (via Day convolution [11]), but not necessarily to the category of presheaves of $\mathcal{A}$. Thus, one should expect this "natural context" to be the setting where a fully faithful lax-idempotent pseudomonad P lifts to the algebras of another pseudomonad T (equivalent to giving a pseudo-distributive law $\lambda\colon TP \to PT$). This situation is studied in detail in our second paper (see Section 1.3).

This in turn motivates the idea that doctrinal Yoneda structures should apply to fully faithful lax-idempotent pseudomonad, which only makes sense provided these pseudomonads give rise to something close to a Yoneda structure. It turns out that this is indeed the case, as shown in our first paper (see Section 1.3).

In our fifth and final paper (see Section 1.6) we are interested in the special properties of the canonical embeddings $\mathcal{E} \to \mathbf{Span}(\mathcal{E})$ and $\mathcal{E} \to \mathbf{Poly}(\mathcal{E})$; embeddings which are of course central to the universal properties of spans and polynomials. In this paper, we give a description of familiy for pseudofunctors (building on work of Weber [53]), and give a description of familiy in terms of a 2-dimensional analogue of generic factorisations. We then go on to show that $\mathcal{E} \to \mathbf{Span}(\mathcal{E})$ and $\mathcal{E} \to \mathbf{Poly}(\mathcal{E})$ are examples of these familial pseudofunctors, a fact which in future work will be used to explain why pseudomonads on $\mathbf{Fib}(\mathcal{E})$ such as those for fibrations with sums $\Sigma_{\mathcal{E}}$ or fibrations with products $\Pi_{\mathcal{E}}$ have a nicer form than one would generally expect for a pseudomonad on $\mathbf{Fib}(\mathcal{E})$.

We now give a more detailed overview of the material of our five papers.

1.2 Yoneda structures and KZ doctrines

Suppose $L\colon \mathcal{A} \to \mathcal{B}$ is a functor in **CAT** where $\mathcal{A}$ and $\mathcal{B}$ are locally small. We may then form a diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R_L"] & {[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "Y"'] \arrow[ul, phantom, shift right=3ex, "\overset{\Leftarrow}{\varphi_L}"]
\end{tikzcd}$$

by taking R_L to be the nerve $\mathcal{B}(-,L-) : \mathcal{B} \to [\mathcal{A}^{\mathrm{op}},\mathbf{Set}]$, and taking φ_L as the canonical map $\mathcal{A}(-,-) \to \mathcal{B}(L-,L-)$ given by applying L. Such a diagram satisfies two universal properties: namely L is the absolute left lifting of Y through R_L, and R_L is the left extension of Y along L. More generally, diagrams of the form

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"'] \arrow[ul, phantom, shift right=3ex, "\overset{\Leftarrow}{\varphi_L}"]
\end{tikzcd}$$

in a 2-category $\mathscr{C}$ satisfying both of these universal properties (and a couple of additional axioms) form the basis of what is referred to as a "Yoneda structure" [47] on $\mathscr{C}$, allowing for a formal version of the Yoneda lemma as well as an appropriate notion of internal presheaves. In this way Yoneda structures provide a formalization of the presheaf construction.

It is the purpose of this first paper to address the following fundamental question:

Why does the cocompletion construction look like the presheaf construction?

To answer this question, we compare the formalization of the presheaf construction (Yoneda structures) with the formalization of cocompletion operations (lax-idempotent pseudomonads). We show that for any fully faithful lax-idempotent pseudomonad (also called a fully faithful KZ doctrine), one almost gets a Yoneda structure, with every axiom of a Yoneda structure holding except for a right ideal property being replaced by closure under composition.

These KZ-induced "near-Yoneda structures" have the advantage of being quite common (because lax-idempotent pseudomonads are), as well as lifting nicely to 2-categories of algebras. However, they have the disadvantage that in the absence of a right ideal property one cannot easily define a notion of size against such a structure.

We leave as an open question if there is a formal way to recover a right ideal Yoneda structure from a fully faithful KZ doctrine, which would give a correspondence between "cocompletion Yoneda structures" and KZ doctrines. Similar questions are the subject of current research by Di Liberti and Loregian [13], who make use of the more general "relative KZ doctrines " [16].

1.3 Distributive laws via admissibility

In our second paper, we are concerned with the problem of lifting a KZ doctrine P to the 2-category of algebras for a pseudomonad T; which is equivalent to extending P to the Kleisli bicategory of T [8], or giving a pseudo-distributive law $\lambda \colon TP \to PT$ [39]. This is a natural question, which captures for instance the problem of lifting a monoidal structure on a category to the cocompletion of that category (via the Day convolution [11]), or extending a pseudomonad T on locally small categories to the bicategory of profunctors[2] on locally small categories.

The first goal of this paper is to show that such pseudo-distributive laws $\lambda \colon TP \to PT$ have an especially simple form (requiring only three out of the usual eight coherence axioms). Note that it is already known such a pseudo-distributive law has a simple form when T is (co)KZ [39], however this does not capture some of the main cases of interest (such as when T is the pseudomonad for monoidal categories).

The second goal of this paper is to give a description of these pseudo-distributive laws in terms of the data of the near-Yoneda structure arising from the (fully faithful) KZ monad P. It turns out that central to this condition is that the P-admissible maps (morphisms L such that PL has a right adjoint) are preserved upon application of T.

The reason for giving this description of pseudo-distributive laws in terms of the admissible maps is that it is required to properly state "Doctrinal Yoneda structures". Indeed, we show that whenever we have a pseudo-distributive law $\lambda \colon TP \to PT$ over a fully faithful KZ pseudomonad P, we get a bijection between oplax T-morphism structures on $L \colon \mathcal{A} \to \mathcal{B}$ and lax T-morphism structures on $R_L \colon \mathcal{B} \to P\mathcal{A}$ in

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \\
 & \mathcal{A}. \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"'] \arrow[ul, phantom, shift left=3, "\overset{\Leftarrow}{\varphi_L}"]
\end{tikzcd}$$

The bijection between oplax structures on left adjoints and lax structures on right adjoints due to Kelly [27] is a special case of this, given by taking P to be the identity.

1.4 Generic bicategories

In our third paper, we study the class of bicategories $\mathscr{A}$ with the property that each composition functor

$$\circ_{X,Y,Z} \colon \mathscr{A}_{Y,Z} \times \mathscr{A}_{X,Y} \to \mathscr{A}_{X,Z}$$

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[2] By "profunctor" we mean a functor $\mathcal{A} \to P\mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are locally small categories and $P\mathcal{B}$ is the free small cocompletion of $\mathcal{B}$.

admits generic factorisations, meaning that

$$\mathscr{A}_{X,Z}(c,-;-) : \mathscr{A}_{Y,Z} \times \mathscr{A}_{X,Y} \to \mathbf{Set}$$

is a coproduct of representables for every 1-cell $c\colon X \to Z$ in $\mathscr{A}$. We call bicategories with this property *generic*. These are the bicategories $\mathscr{A}$ for which the local cocompletion of $\mathscr{A}$ has an especially nice form, with composition given by a simple coproduct formula, and thus the bicategories for which "Doctrinal Yoneda structures"

$$\begin{tikzcd} \mathscr{C} \arrow[r, "R_L"] & \hat{\mathscr{A}} \\ & \mathscr{A} \arrow[ul, "L"] \arrow[u, "Y"'] \arrow[ul, phantom, shift right=4, "\overset{\Leftarrow}{\varphi_L}"] \end{tikzcd}$$

gives us a non-trivial reduction of the data of an oplax functor $L\colon \mathscr{A} \to \mathscr{C}$ as above. This allows us to give a significantly simpler (but equivalent) description of the data of an oplax functor $L\colon \mathscr{A} \to \mathscr{C}$ which is valid whenever $\mathscr{A}$ is generic. Interestingly, this description turns out to be analogous to the data of comonad, and may be viewed as a generalization of the correspondence between comonads in a 2-category $\mathscr{C}$ and oplax functors $L\colon 1 \to \mathscr{C}$ due to Bénabou [3].

The main advantage of this description of oplax functors $L\colon \mathscr{A} \to \mathscr{C}$ out of a generic $\mathscr{A}$ is that it does not directly involve composition in $\mathscr{A}$. Unsurprisingly, this is especially useful when $\mathscr{A}$ is the bicategory of polynomials with cartesian 2-cells, as we are able to give a description of oplax functors $L\colon \mathbf{Poly}_c(\mathcal{E}) \to \mathscr{C}$ which does not directly involve composition of polynomials.

1.5 Universal properties of bicategories of polynomials

In our fourth paper, we will apply the tools developed in the first three in order to prove the universal properties of polynomials; that is, we give a simple characterization of the data required to construct pseudofunctors (also gregarious functors) $\mathbf{Poly}_c(\mathcal{E}) \to \mathscr{C}$ and $\mathbf{Poly}(\mathcal{E}) \to \mathscr{C}$, giving analogues of (1.1.1) and (1.1.2) for polynomials.

Before doing this however, and in order to demonstrate our method, we start by giving a new proof of the universal properties of the bicategory of spans, exploiting the fact that $\mathbf{Span}(\mathcal{E})$ is a generic bicategory. Note that this new proof addresses the curious observation made in [9] that the bicategory of spans has a universal characterization which does not involve pullbacks (namely (1.1.2)).

We then move on to establish the universal properties of the bicategory of spans with invertible 2-cells $\mathbf{Span}_{\text{iso}}(\mathcal{E})$. Note that $\mathbf{Span}_{\text{iso}}(\mathcal{E})$ is not a generic bicategory, and so the universal property does not have a such a simple proof. Also, its universal property is not as simple to state as the morphisms are no longer generated by simple adjunctions. However, we must give it here as it is required to understand the universal properties of $\mathbf{Poly}_c(\mathcal{E})$.

This gives the required background needed to establish the universal properties of the bicategory of polynomials with and without cartesian 2-cells, which we then address.

In the case of polynomials with general 2-cells, the universal property of $\mathbf{Poly}(\mathcal{E})$ is simple to state (as the morphisms are generated by components of adjoint triples $\Sigma_f \dashv \Delta_f \dashv \Pi_f$) but difficult to prove because $\mathbf{Poly}(\mathcal{E})$ is not a generic bicategory.

Conversely, in the case with cartesian 2-cells, the universal property of $\mathbf{Poly}_c(\mathcal{E})$ is difficult to state (due to a lack of adjunctions), but more straightforward to prove as $\mathbf{Poly}_c(\mathcal{E})$ is a generic bicategory.

Fortunately, as composition in $\mathbf{Poly}(\mathcal{E})$ and $\mathbf{Poly}_c(\mathcal{E})$ is the same, we can use the universal property of $\mathbf{Poly}_c(\mathcal{E})$ to help prove that of $\mathbf{Poly}(\mathcal{E})$, only needing to check an extra coherence condition with respect to the extra 2-cells of $\mathbf{Poly}(\mathcal{E})$ which are not present in the cartesian setting.

As we will see, the universal properties of bicategories of polynomials can be understood in terms of what they are built out of. In particular, pseudofunctors $\mathbf{Poly}(\mathcal{E}) \to \mathscr{C}$ correspond to pairs of pseudofunctors $\mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ and $\mathbf{Span}(\mathcal{E})^{\mathrm{co}} \to \mathscr{C}$ which coincide on spans of the form (s, id) and satisfy a distributivity condition; and pseudofunctors $\mathbf{Poly}_c(\mathcal{E}) \to \mathscr{C}$ correspond to pairs of pseudofunctors $\mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ and $\mathbf{Span}_{\mathrm{iso}}(\mathcal{E}) \to \mathscr{C}$ also coinciding on such spans and satisfying a distributivity condition.

1.6 An elementary view of familial pseudofunctors

Given that the universal properties of spans and polynomials are defined by composing with the canonical embeddings $\mathcal{E} \to \mathbf{Span}(\mathcal{E})$ and $\mathcal{E} \to \mathbf{Poly}(\mathcal{E})$, it is natural to ask if these embeddings have any special properties. This is indeed the case. An obvious (but important) property is that these pseudofunctors are bijective on objects (which turns out to mean that they directly correspond to bi-cocontinuous pseudomonads on fibrations over $\mathcal{E}$). A second important property (which is to be the subject of our fifth paper) is that these pseudofunctors are in fact examples of *familial* pseudofunctors.

Familial pseudofunctors between bicategories are the appropriate two-dimensional analogue of familial functors between categories, and are those pseudofunctors which satisfy the important properties exhibited by the families pseudomonad **Fam** on **CAT**.

The study of these familial pseudofunctors was originally due to Weber [53], who was motivated by parametric right adjoints and their appropriate 2-dimensional analogues. However, as needed for parametric right adjoints, Weber assumes the existence of a terminal object. Also, an equivalence between famility for pseudofunctors and appropriate generic factorisations is not provided.

In this fifth paper, it is our goal to address these concerns. We give a simple description of famility for pseudofunctors $L\colon \mathscr{A} \to \mathscr{B}$ which does not make any assumptions on $\mathscr{A}$ or $\mathscr{B}$, and give a definition of generic factorisations for pseudofunctors L which makes no assumptions on $\mathscr{A}$, $\mathscr{B}$ or L, thus allowing for a theorem describing an equivalence between famility and our appropriate generic factorisations.

Instead of parametric right adjoints, we are motivated by the work of Diers' [15], who considered famility in terms of multiadjoints and spectrums.

2

Yoneda structures and KZ doctrines

Abstract

In this paper we strengthen the relationship between Yoneda structures and KZ doctrines by showing that for any locally fully faithful KZ doctrine, with the notion of admissibility as defined by Bunge and Funk, all of the Yoneda structure axioms apart from the right ideal property are automatic.

Contribution by the author

As the sole author, this paper is entirely my own work. This paper is published in the Journal of Pure and Applied Algebra [51]. Any differences from the journal version are limited to formatting and citation numbering changes.

2.1 Introduction

The majority of this paper concerns Kock-Zöberlein doctrines, which were introduced by Kock [31] and Zöberlein [57]. These KZ doctrines capture the free cocompletion under a suitable class of colimits Φ, with a canonical example being the free small cocompletion KZ doctrine on locally small categories. On the other hand, Yoneda structures as introduced by Street and Walters [47] capture the presheaf construction, with the canonical example being the Yoneda structure on (not necessarily locally small) categories, whose basic data is the Yoneda embedding $\mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ for each locally small category $\mathcal{A}$. When $\mathcal{A}$ is small this coincides with the usual free small cocompletion, but not in general. In this paper we prove a theorem tightening the relationship between these two notions, not just in the context of this example, but in general.

A key feature of a Yoneda structure (which is not present in the definition of a KZ doctrine) is a class of 1-cells called *admissible 1-cells*. In the setting of the usual Yoneda structure on **CAT**, a 1-cell (that is a functor) $L\colon \mathcal{A} \to \mathcal{B}$ is called admissible when the corresponding functor $\mathcal{B}(L-,-) : \mathcal{B} \to [\mathcal{A}^{\text{op}}, \mathbf{SET}]$ factors through the inclusion of $[\mathcal{A}^{\text{op}}, \mathbf{Set}]$ into $[\mathcal{A}^{\text{op}}, \mathbf{SET}]$.

In order to compare Yoneda structures with KZ doctrines, we will also need a notion of admissibility in the setting of a KZ doctrine. Fortunately, such a notion of admissibility has already been introduced by Bunge and Funk [6]. In the case of the free small cocompletion KZ doctrine P on locally small categories, these admissible 1-cells, which we refer to as P-admissible, are those functors $L\colon \mathcal{A} \to \mathcal{B}$ for which the corresponding functor $\mathcal{B}(L-,-) : \mathcal{B} \to [\mathcal{A}^{\text{op}}, \mathbf{Set}]$ factors through the inclusion of $P\mathcal{A}$ into $[\mathcal{A}^{\text{op}}, \mathbf{Set}]$.

The main result of this paper; Theorem 2.4.1, shows that given a locally fully faithful KZ doctrine P on a 2-category $\mathscr{C}$, on defining the admissible maps to be those of Bunge and Funk, one defines all the data and axioms for a Yoneda structure except for the "right ideal property" which asks that the class of admissible 1-cells $\mathbf{I}$ satisfies the property that for each $L \in \mathbf{I}$ we have $L \cdot F \in \mathbf{I}$ for all F such that the composite $L \cdot F$ is defined.

2.2 Background

In this section we will recall the notion of a KZ doctrine P as well as the notions of left extensions and left liftings, as these will be needed to describe Yoneda structures, and to discuss their relationship with KZ doctrines.

Definition 2.2.1. Suppose we are given a 2-cell $\eta\colon I \to R \cdot L$ as in the left diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R"] & \mathcal{C} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "I"'] \arrow[ul, phantom, shift right=4, "\overset{\eta}{\Leftarrow}"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{B} \arrow[r, "R"] \arrow[r, bend left=60, "M"] \arrow[r, phantom, bend left=30, "\Uparrow\sigma"] & \mathcal{C} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "I"'] \arrow[ul, phantom, shift right=4, "\overset{\eta}{\Leftarrow}"]
\end{tikzcd}$$

in a 2-category $\mathscr{C}$. We say that R is exhibited as a *left extension* of I along L by the 2-cell η when pasting 2-cells $\sigma : R \to M$ with the 2-cell $\eta : I \to R \cdot L$ as in the right diagram defines a bijection between 2-cells $R \to M$ and 2-cells $I \to M \cdot L$. Moreover, we say such a left extension is *respected* by a 1-cell $E\colon \mathcal{C} \to \mathcal{D}$ when the whiskering of η by E given by the following pasting diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R"] & \mathcal{C} \arrow[r, "E"] & \mathcal{D} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[ul, phantom, shift right=4, "\overset{\eta}{\Leftarrow}"] \arrow[u, "I"'] \arrow[ur, "E\cdot I"'] \arrow[ur, phantom, shift left=4, "\overset{\mathrm{id}}{\Leftarrow}"] &
\end{tikzcd}$$

exhibits $E \cdot R$ as a left extension of $E \cdot I$ along L.

Dually, we have the notion of a left lifting. We say a 2-cell $\eta\colon I \to R \cdot L$ exhibits L as a *left lifting* of I through R when pasting 2-cells $\delta\colon L \to K$ with the 2-cell $\eta\colon I \to R \cdot L$ defines a bijection between 2-cells $L \to K$ and 2-cells $I \to R \cdot K$. We call such a lifting *absolute* if for any 1-cell $F\colon \mathcal{X} \to \mathcal{A}$ the whiskering of η by F given by the following pasting diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R"] & \mathcal{C} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "I"'] \arrow[ul, phantom, shift right=4, "\overset{\eta}{\Leftarrow}"] \\
& \mathcal{X} \arrow[u, "F"'] \arrow[uul, bend left, "L\cdot F"] \arrow[uul, phantom, "\overset{\mathrm{id}}{\Leftarrow}"]
\end{tikzcd}$$

exhibits $L \cdot F$ as a left lifting of $I \cdot F$ through R.

There are quite a few different characterizations of KZ doctrines, for example those due to Kelly-Lack or Kock [29, 31]. For the purposes of relating KZ doctrines to Yoneda structures, it will be easiest to work with the following characterization given by Marmolejo and Wood [42] in terms of left Kan extensions.

Definition 2.2.2. [42, Definition 3.1] A *KZ doctrine* (P, y) on a 2-category $\mathscr{C}$ consists of

(i) An assignation on objects $P\colon \mathrm{ob}\mathscr{C} \to \mathrm{ob}\mathscr{C}$;

(ii) For every object $\mathcal{A} \in \mathscr{C}$, a 1-cell $y_\mathcal{A}\colon \mathcal{A} \to P\mathcal{A}$;

(iii) For every pair of objects $\mathcal{A}$ and $\mathcal{B}$ and 1-cell $F\colon \mathcal{A} \to P\mathcal{B}$, a left extension

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "\overline{F}"] & P\mathcal{B} \\
\mathcal{A} \arrow[u, "y_\mathcal{A}"] \arrow[ur, "F"'] \arrow[ur, phantom, shift left=4, "\overset{c_F}{\Leftarrow}"] &
\end{tikzcd} \tag{2.2.1}$$

of F along $y_\mathcal{A}$ exhibited by an isomorphism c_F as above.

Moreover, we require that:

(a) For every object $\mathcal{A} \in \mathscr{C}$, the left extension of $y_{\mathcal{A}}$ as in 2.2.1 is given by

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "\mathrm{id}_{P\mathcal{A}}"] & P\mathcal{A} \arrow[dl, phantom, "\overset{\mathrm{id}}{\Leftarrow}" near start] \\
& \mathcal{A} \arrow[ul, "y_{\mathcal{A}}"] \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}$$

Note that this means $c_{y_{\mathcal{A}}}$ is equal to the identity 2-cell on $y_{\mathcal{A}}$.

(b) For any 1-cell $G\colon \mathcal{B} \longrightarrow P\mathcal{C}$, the corresponding left extension $\overline{G}\colon P\mathcal{B} \longrightarrow P\mathcal{C}$ respects the left extension $\overline{F}$ in 2.2.1.

Remark 2.2.3. This definition is equivalent (in the sense that each gives rise to the other) to the well known algebraic definition, which we refer to as a KZ pseudomonad [42, 38]. A *KZ pseudomonad* (P, y, μ) on a 2-category $\mathscr{C}$ is taken to be a pseudomonad (P, y, μ) on $\mathscr{C}$ equipped with a modification $\theta\colon Py \longrightarrow yP$ satisfying two coherence axioms [31].

Just as KZ doctrines may be defined algebraically or in terms of left extensions, one may also define pseudo algebras for these KZ doctrines algebraically or in terms of left extensions.

The following definitions in terms of left extensions are equivalent to the usual notions of pseudo P-algebra and P-homomorphism, in the sense that we have an equivalence between the two resulting 2-categories of pseudo P-algebras arising from the two different definitions [42, Theorems 5.1,5.2].

Definition 2.2.4 ([42])**.** Given a KZ doctrine (P, y) on a 2-category $\mathscr{C}$, we say an object $\mathcal{X} \in \mathscr{C}$ is *P-cocomplete* if for every $G\colon \mathcal{B} \longrightarrow \mathcal{X}$

$$\begin{tikzcd}
P\mathcal{B} \arrow[r, "\overline{G}"] \arrow[dr, phantom, "\overset{c_G}{\Leftarrow}" near start] & \mathcal{X} \\
\mathcal{B} \arrow[u, "y_{\mathcal{B}}"] \arrow[ur, "G"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
P\mathcal{A} \arrow[r, "\overline{F}"] \arrow[dr, phantom, "\overset{c_F}{\Leftarrow}" near start] & P\mathcal{B} \arrow[r, "\overline{G}"] & \mathcal{X} \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[ur, "F"']
\end{tikzcd}$$

there exists a left extension $\overline{G}$ as on the left exhibited by an isomorphism c_G, and moreover this left extension respects the left extensions $\overline{F}$ as in the diagram on the right. We say a 1-cell $E\colon \mathcal{X} \longrightarrow \mathcal{Y}$ between P-cocomplete objects $\mathcal{X}$ and $\mathcal{Y}$ is a *P-homomorphism* when it respects all left extensions along $y_{\mathcal{B}}$ into $\mathcal{X}$ for every object $\mathcal{B}$.

Remark 2.2.5. It is clear that $P\mathcal{A}$ is P-cocomplete for every $\mathcal{A} \in \mathscr{C}$.

The relationship between P-cocompleteness and admitting a pseudo P-algebra structure is as below.

Proposition 2.2.6. *Given a KZ doctrine (P, y) on a 2-category $\mathscr{C}$ and an object $\mathcal{X} \in \mathscr{C}$, the following are equivalent:*

(1) $\mathcal{X}$ is P-cocomplete;

(2) $y_{\mathcal{X}}\colon \mathcal{X} \to P\mathcal{X}$ has a left adjoint with invertible counit;

(3) $\mathcal{X}$ is the underlying object of a pseudo P-algebra.

Proof. For (1) $\iff$ (2) see the proof of [42, Theorem 5.1], and for (2) $\iff$ (3) see [29]. ☐

We now recall the notion of Yoneda structure as introduced by Street and Walters [47].

Definition 2.2.7. A *Yoneda structure* $\mathfrak{Y}$ on a 2-category $\mathscr{C}$ consists of:

(1) A class of 1-cells $\mathbf{I}$ with the property that for any $L \in \mathbf{I}$ we have $L \cdot F \in \mathbf{I}$ for all F such that the composite $L \cdot F$ is defined; we call this the class of admissible 1-cells. We say an object $\mathcal{A} \in \mathscr{C}$ is admissible when $\mathrm{id}_{\mathcal{A}}$ is an admissible 1-cell.

(2) For each admissible object $\mathcal{A} \in \mathscr{C}$, an admissible map $y_{\mathcal{A}}\colon \mathcal{A} \to P\mathcal{A}$.

(3) For each $L\colon \mathcal{A} \to \mathcal{B}$ such that L and $\mathcal{A}$ are both admissible, a 1-cell R_L and 2-cell φ_L as in the diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \\
& \mathcal{A} \arrow[u, "y_{\mathcal{A}}"'] \arrow[ul, "L"] \arrow[ul, phantom, shift right=3ex, "\overset{\varphi_L}{\Leftarrow}"]
\end{tikzcd}$$

Such that:

(a) The diagram above exhibits L as a absolute left lifting and R_L as a left extension via φ_L.

(b) For each admissible $\mathcal{A}$, the diagram

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "\mathrm{id}_{P\mathcal{A}}"] & P\mathcal{A} \\
& \mathcal{A} \arrow[u, "y_{\mathcal{A}}"'] \arrow[ul, "y_{\mathcal{A}}"] \arrow[ul, phantom, shift right=3ex, "\overset{\mathrm{id}}{\Leftarrow}"]
\end{tikzcd}$$

exhibits $\mathrm{id}_{P\mathcal{A}}$ as a left extension.

(c) For admissible $\mathcal{A}, \mathcal{B}$ and L, K as below, the diagram

$$\begin{tikzcd}
P\mathcal{A} & P\mathcal{B} \arrow[l, "R_{y_{\mathcal{B}} \cdot L}"'] & \mathcal{C} \arrow[l, "R_K"'] \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[r, "L"'] \arrow[ur, phantom, "\overset{\varphi_{y_{\mathcal{B}} \cdot L}}{\Rightarrow}"] & \mathcal{B} \arrow[u, "y_{\mathcal{B}}"] \arrow[ur, "K"'] \arrow[ur, phantom, shift left=3ex, "\overset{\varphi_K}{\Rightarrow}"] &
\end{tikzcd}$$

exhibits $R_{y_{\mathcal{B}} \cdot L} \cdot R_K$ as a left extension.

Remark 2.2.8. We note that when the admissible maps form a right ideal, the admissibility of L in condition (c) is redundant. However, in the following sections we will consider a setting in which the admissible maps are closed under composition, but do not necessarily form a right ideal.

Remark 2.2.9. There is an additional axiom (d) discussed in "Yoneda structures" [47] which when satisfied defines a so called *good* Yoneda structure [54]. This axiom asks for every admissible L and every diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "M"] & P\mathcal{A} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"'] \arrow[ul, phantom, shift left=3, "\overset{\phi}{\Leftarrow}"]
\end{tikzcd}$$

that if ϕ exhibits L as an absolute left lifting, then ϕ exhibits M as a left extension. This condition implies axioms (b) and (c) in the presence of (a) [47, Prop. 11].

However, this condition is often too strong. For example one may consider the free **Cat**-cocompletion, and take $\mathbb{N}$ to be the monoid of natural numbers seen as a one object category, yielding the absolute left lifting diagram

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "\text{pick } \mathbb{N}"] & \mathbf{Cat} \\
& \mathbf{1} \arrow[ul, "\mathrm{id}_{\mathbf{1}}"] \arrow[u, "\text{pick } \mathbf{1}"'] \arrow[ul, phantom, shift left=3, "\overset{!}{\Leftarrow}"]
\end{tikzcd}$$

It is then trivial, as we would be extending along an identity, that the left extension property is not satisfied.

2.3 Admissible Maps in KZ Doctrines

Yoneda structures as defined above require us to give a suitable class of admissible maps, and so in order to compare Yoneda structures with KZ doctrines we will need a suitable notion of admissible map in the setting of a KZ doctrine. Bunge and Funk defined a map $L\colon \mathcal{A} \to \mathcal{B}$ in the setting of a KZ pseudomonad P to be P-admissible when PL has a right adjoint, and showed this notion of admissibility may also be described in terms of left extensions [6]. Our definition in terms of left extensions and KZ doctrines is as follows.

Definition 2.3.1. Given a KZ doctrine (P, y) on a 2-category $\mathscr{C}$, we say a 1-cell $L\colon \mathcal{A} \to \mathcal{B}$

is *P-admissible* when

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \arrow[dl, phantom, "\overset{\varphi_L}{\Leftarrow}"] \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \arrow[r, "\overline{H}"] \arrow[dl, phantom, "\overset{\varphi_L}{\Leftarrow}"] & \mathcal{X} \arrow[dl, phantom, "\overset{c_H}{\Leftarrow}"] \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"'] \arrow[ur, "H"']
\end{tikzcd}$$

there exists a left extension (R_L, φ_L) of $y_{\mathcal{A}}$ along L as in the left diagram, and moreover the left extension is respected by any $\overline{H}$ as in the right diagram where $\mathcal{X}$ is P-cocomplete.

Remark 2.3.2. Note that such a $\overline{H}$ is a P-homomorphism, and conversely that a P-homomorphism $\overline{H}\colon P\mathcal{A} \to \mathcal{X}$ is a left extension of $H := \overline{H} \cdot y_{\mathcal{A}}$ along $y_{\mathcal{A}}$ as above. Thus this is saying the left extension R_L is respected by P-homomorphisms.

Lemma 2.3.3. *Suppose we are given a KZ doctrine (P, y) and a P-admissible 1-cell $L\colon \mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ is P-cocomplete, then the 1-cell R_L in*

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \arrow[dl, phantom, "\overset{\varphi_L}{\Leftarrow}"] \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}$$

has a left adjoint $\overline{L}\colon P\mathcal{A} \to \mathcal{B}$.

Proof. Taking $\overline{L}$ to be the left extension

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "\overline{L}"] & \mathcal{B} \arrow[dl, phantom, "\overset{c_L}{\Leftarrow}"] \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[ur, "L"']
\end{tikzcd}$$

we then have $\overline{L} \dashv R_L$ since we may define $n : \mathrm{id}_{P\mathcal{A}} \to R_L \cdot \overline{L}$ and $e : \overline{L} \cdot R_L \to \mathrm{id}_{\mathcal{B}}$ respectively as (since L is P-admissible) the unique solutions to

$$\begin{tikzcd}
& \mathcal{B} \arrow[dr, "R_L"] \arrow[d, phantom, "\Uparrow n"] & \\
P\mathcal{A} \arrow[ur, "\overline{L}"] \arrow[rr, "\mathrm{id}_{P\mathcal{A}}"'] & {} & P\mathcal{A} \arrow[dl, phantom, "\overset{\mathrm{id}}{\Rightarrow}"] \\
& & \mathcal{A} \arrow[ull, "y_{\mathcal{A}}"] \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}
=
\begin{tikzcd}
P\mathcal{A} \arrow[r, "\overline{L}"] & \mathcal{B} \arrow[r, "R_L"] \arrow[dl, phantom, "\overset{c_L}{\Leftarrow}"] & P\mathcal{A} \arrow[dl, phantom, "\overset{\varphi_L}{\Leftarrow}"] \\
& & \mathcal{A} \arrow[ull, bend left, "y_{\mathcal{A}}"] \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R_L"] \arrow[rr, bend left=60, "\mathrm{id}_{\mathcal{B}}"] & P\mathcal{A} \arrow[r, "\overline{L}"] \arrow[u, phantom, "\Uparrow e"] \arrow[dl, phantom, "\overset{\varphi_L}{\Leftarrow}"] & \mathcal{B} \arrow[dl, phantom, "\overset{c_L}{\Leftarrow}"] \\
& & \mathcal{A} \arrow[ull, bend left, "L"] \arrow[ul, "y_{\mathcal{A}}"] \arrow[u, "L"']
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{B} \arrow[r, "\mathrm{id}_{\mathcal{B}}"] & \mathcal{B} \arrow[dl, phantom, "\overset{\mathrm{id}}{\Leftarrow}"] \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "L"']
\end{tikzcd}$$

Verifying the triangle identities is then a simple exercise. □

The following is an easy consequence of this Lemma.

Lemma 2.3.4. *Suppose we are given a KZ doctrine* (P, y) *on a 2-category* $\mathscr{C}$ *and a* P*-admissible 1-cell* $L\colon \mathcal{A} \to \mathcal{B}$*. Then the 1-cell* res_L *defined here as the left extension in the top triangle*

$$\begin{tikzcd}
P\mathcal{A} & P\mathcal{B} \arrow[l, "\mathrm{res}_L"'] \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[r, "L"'] \arrow[ur, phantom, "\overset{\varphi_L}{\Longrightarrow}" near start] & \mathcal{B} \arrow[u, "y_{\mathcal{B}}"'] \arrow[ul, "R_L" description] \arrow[ul, phantom, "\overset{c_{R_L}}{\Longrightarrow}" near end, shift left=3]
\end{tikzcd}$$

has a left adjoint lan_L*, and when* R_L *is* P*-admissible, a right adjoint* ran_L*.*

Proof. First note that it is an easy consequence of the left extension pasting lemma (the dual of [47, Prop. 1]) that $y_{\mathcal{B}} \cdot L$ is P-admissible, which is to say the left extension res_L above is respected by any P-homomorphism $\overline{H}\colon P\mathcal{A} \to \mathcal{X}$. This is since such a $\overline{H}$ will respect the left extension R_L of $y_{\mathcal{A}}$ along L as well as the left extension res_L of R_L along $y_{\mathcal{B}}$. Hence by Lemma 2.3.3 res_L has a left adjoint lan_L given as the left extension as on the left (which is how PL is defined given the data of Definition 2.2.2),

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "\mathrm{lan}_L"] & P\mathcal{B} \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[r, "L"'] \arrow[ur, phantom, "\overset{c_{y_{\mathcal{B}} \cdot L}}{\Longleftarrow}"] & \mathcal{B} \arrow[u, "y_{\mathcal{B}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P\mathcal{A} \arrow[r, "\mathrm{ran}_L"] & P\mathcal{B} \\
 & \mathcal{B} \arrow[u, "y_{\mathcal{B}}"'] \arrow[ul, "R_L"] \arrow[ul, phantom, "\overset{\varphi_{R_L}}{\Longrightarrow}", shift left=3]
\end{tikzcd}$$

and if R_L is P-admissible then we may define $\mathrm{ran}_L := R_{R_L}$ (which is the left extension as on the right) and since $P\mathcal{A}$ is P-cocomplete ran_L has a left adjoint given by $\mathrm{res}_L = \overline{R_L}$ again by Lemma 2.3.3. □

Remark 2.3.5. We have shown that when both L and R_L are P-admissible we have the adjoint triple $PL \dashv \overline{R_L} \dashv R_{R_L}$. Of particular interest is the case where $L = y_{\mathcal{A}}$ for some $\mathcal{A} \in \mathscr{C}$. Clearly in this case both L and R_L are P-admissible and so we may define $\mu_{\mathcal{A}} := \overline{R_{y_{\mathcal{A}}}} = \overline{\mathrm{id}_{P\mathcal{A}}}$ and observe $R_{R_{y_{\mathcal{A}}}} = R_{\mathrm{id}_{P\mathcal{A}}} = y_{P\mathcal{A}}$ to recover the well known sequence of adjunctions $Py_{\mathcal{A}} \dashv \mu_{\mathcal{A}} \dashv y_{P\mathcal{A}}$ as in [38].

The following result is mostly due to Bunge and Funk [6], though we state it in our notation and from the viewpoint of KZ doctrines in terms of left extensions. Also, we will prove the following proposition in full detail in order to clarify some parts of the argument given by Bunge and Funk [6]. For example, in order to check that certain left extensions are respected we will need to know their exhibiting 2-cells. These exhibiting 2-cells will also be needed later to prove our main result.

Proposition 2.3.6. *Given a KZ doctrine (P, y) on a 2-category $\mathscr{C}$ and a 1-cell $L\colon \mathcal{A} \to \mathcal{B}$, the following are equivalent:*

(1) L is P-admissible;

(2) every P-cocomplete object $\mathcal{X} \in \mathscr{C}$ admits, and P-homomorphism respects, left extensions along L. This says that for any given 1-cell $K\colon \mathcal{A} \to \mathcal{X}$, where $\mathcal{X}$ is P-cocomplete, there exists a 1-cell J and 2-cell δ as on the left

$$\begin{tikzcd} \mathcal{B} \arrow[r, "J"] & \mathcal{X} \\ & \mathcal{A} \arrow[u, "K"'] \arrow[ul, "L"] \arrow[ul, phantom, shift right=3ex, "\overset{\delta}{\Leftarrow}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B} \arrow[r, "J"] & \mathcal{X} \arrow[r, "E"] & \mathcal{Y} \\ & \mathcal{A} \arrow[u, "K"'] \arrow[ul, "L"] \arrow[ul, phantom, shift right=3ex, "\overset{\delta}{\Leftarrow}"] & \end{tikzcd}$$

exhibiting J as a left extension, and moreover this left extension is respected by any P-homomorphism $E\colon \mathcal{X} \to \mathcal{Y}$ for P-cocomplete $\mathcal{Y}$ as in the right diagram.

(3) $PL := \mathrm{lan}_L$ given as the left extension

$$\begin{tikzcd} P\mathcal{A} \arrow[r, "PL"] & P\mathcal{B} \\ \mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[r, "L"'] \arrow[ur, phantom, "\overset{c_{y_{\mathcal{B}}\cdot L}}{\Leftarrow}"] & \mathcal{B} \arrow[u, "y_{\mathcal{B}}"'] \end{tikzcd}$$

has a right adjoint. We denote the inverse of the above 2-cell as $y_L := c^{-1}_{y_{\mathcal{B}}\cdot L}$ for every 1-cell L.

Proof. The following implications prove the logical equivalence.

(2) $\implies$ (1) : This is trivial as $P\mathcal{A}$ is P-cocomplete.

(1) $\implies$ (2) : Given a $K\colon \mathcal{A} \to \mathcal{X}$ as in (2). We take the pasting

$$\begin{tikzcd} \mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \arrow[r, "\overline{K}"] & \mathcal{X} \\ & \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"] \arrow[ur, "K"'] \arrow[ul, phantom, shift right=3ex, "\overset{\varphi_L}{\Leftarrow}"] \arrow[ur, phantom, shift left=3ex, "\overset{c_K}{\Leftarrow}"] & \end{tikzcd}$$

as our left extension using that L is P-admissible. This is respected by any P-homomorphism $E\colon \mathcal{X} \to \mathcal{Y}$ where $\mathcal{Y}$ is P-cocomplete as a consequence of the second part of the definition of P-admissibility.

(1) $\implies$ (3) : This was shown in Lemma 2.3.4.

(3) $\implies$ (1) : This implication is where the majority of the work lies in proving this proposition. We suppose that we are given an adjunction $\mathrm{lan}_L \dashv \mathrm{res}_L$ with unit η where lan_L

is defined as in (3). We split the proof into two parts.

Part 1: *The given right adjoint,* res_L*, is a left extension of* $\mathrm{res}_L \cdot y_{\mathcal{B}}$ *along* $y_{\mathcal{B}}$ *as in the diagram*

$$\begin{tikzcd}
P\mathcal{B} \arrow[rr, "\mathrm{res}_L"] & & P\mathcal{A} \\
& P\mathcal{B} \arrow[ur, "\mathrm{res}_L"'] \arrow[ul, phantom, "\overset{\mathrm{id}}{\Leftarrow}"] & \\
\mathcal{B} \arrow[uu, "y_{\mathcal{B}}"] \arrow[ur, "y_{\mathcal{B}}"'] & &
\end{tikzcd}$$

exhibited by the identity 2-cell.[1]

To see this, we consider the isomorphism in the square on the left

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "PL"] & P\mathcal{B} \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[r, "L"'] \arrow[ur, phantom, "\overset{y_L}{\Longrightarrow}"] & \mathcal{B} \arrow[u, "y_{\mathcal{B}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P^2\mathcal{A} \arrow[r, "P^2L"] & P^2\mathcal{B} \\
P\mathcal{A} \arrow[u, "Py_{\mathcal{A}}"] \arrow[r, "PL"'] \arrow[ur, phantom, "\overset{Py_L}{\Longrightarrow}"] & P\mathcal{B} \arrow[u, "Py_{\mathcal{B}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P^2\mathcal{A} \arrow[d, "\mu_{\mathcal{A}}"'] & P^2\mathcal{B} \arrow[l, "P\mathrm{res}_L"'] \arrow[d, "\mu_{\mathcal{B}}"] \arrow[dl, phantom, "\cong"] \\
P\mathcal{A} & P\mathcal{B} \arrow[l, "\mathrm{res}_L"]
\end{tikzcd}$$

and then apply P to get the isomorphism of left adjoints in the middle square (suppressing pseudofunctoriality constraints[2]), which corresponds to an isomorphism of right adjoints in the right square (which we leave unnamed). Now by [42, Theorem 4.2] (and since $\mu_{\mathcal{A}} \cdot P\mathrm{res}_L$ respects the left extension $Py_{\mathcal{B}}$) we have the left extension $\mu_{\mathcal{A}} \cdot P\mathrm{res}_L \cdot Py_{\mathcal{B}}$ of $\mathrm{res}_L \cdot y_{\mathcal{B}}$ along $y_{\mathcal{B}}$ as below

$$\begin{tikzcd}
& & P\mathcal{B} \arrow[dr, "\mathrm{res}_L"] \arrow[d, phantom, "\cong"] & \\
P\mathcal{B} \arrow[r, "Py_{\mathcal{B}}"] \arrow[urr, bend left, "\mathrm{id}_{P\mathcal{B}}"] & P^2\mathcal{B} \arrow[r, "P\mathrm{res}_L"] \arrow[ur, "\mu_{\mathcal{B}}"] \arrow[u, phantom, "\cong"] & P^2\mathcal{A} \arrow[r, "\mu_{\mathcal{A}}"] & P\mathcal{A} \\
\mathcal{B} \arrow[u, "y_{\mathcal{B}}"] \arrow[r, "y_{\mathcal{B}}"'] \arrow[ur, phantom, "\Downarrow y_{y_{\mathcal{B}}}"] & P\mathcal{B} \arrow[u, "y_{P\mathcal{B}}"'] \arrow[r, "\mathrm{res}_L"'] \arrow[ur, phantom, "\Downarrow y_{\mathrm{res}_L}"] & P\mathcal{A} \arrow[u, "y_{P\mathcal{A}}"'] \arrow[ur, "\mathrm{id}_{P\mathcal{A}}"'] \arrow[ur, phantom, "\cong", near start, shift left=2] &
\end{tikzcd}$$

and so pasting with the isomorphism $\mu_{\mathcal{A}} \cdot P\mathrm{res}_L \cdot Py_{\mathcal{B}} \cong \mathrm{res}_L$ constructed as above tells us res_L is also an extension of $\mathrm{res}_L \cdot y_{\mathcal{B}}$ along $y_{\mathcal{B}}$. It follows that res_L respects the left extension

$$\begin{tikzcd}
P\mathcal{B} \arrow[r, "\mathrm{id}_{P\mathcal{B}}"] & P\mathcal{B} \\
\mathcal{B} \arrow[u, "y_{\mathcal{B}}"] \arrow[ur, "y_{\mathcal{B}}"'] \arrow[ur, phantom, "\overset{\mathrm{id}}{\Leftarrow}", shift left=3] &
\end{tikzcd}$$

and this gives the result.

---

[1] This may be seen as an analogue of [6, Prop. 1.3]. However, we emphasize here that considering right adjoints tells us res_L is a P-homomorphism since the adjunctions may be used to construct an isomorphism between res_L and a known P-homomorphism.

[2] These pseudofunctoriality constraints are those arising from the uniqueness of left extensions up to coherent isomorphism.

Part 2: *The following pasting exhibits*

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "y_{\mathcal{B}}"] & P\mathcal{B} \arrow[rr, "\mathrm{res}_L"] & & P\mathcal{A} \arrow[r, "\overline{H}"] & \mathcal{X} \\
& \arrow[l, phantom, "\overset{y_L}{\Leftarrow}"] & P\mathcal{A} \arrow[ul, "\mathrm{lan}_L"] \arrow[ur, "\mathrm{id}_{P\mathcal{A}}"'] \arrow[u, phantom, "\overset{\eta}{\Leftarrow}"] & \arrow[r, phantom, "\overset{c_H}{\Leftarrow}"] & \\
& & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[uull, bend left, "L"] \arrow[uurr, bend right, "H"'] & &
\end{tikzcd}$$

the composite $\overline{H} \cdot \mathrm{res}_L \cdot y_{\mathcal{B}}$ *as a left extension of* H *along* L.

Suppose we are given a 1-cell $K \colon \mathcal{B} \longrightarrow \mathcal{X}$. We then see that our left extension is exhibited by the sequence of natural bijections

$$\begin{array}{rcl}
H & \longrightarrow & K \cdot L \\ \hline
H & \longrightarrow & \overline{K} \cdot y_{\mathcal{B}} \cdot L \\ \hline
H & \longrightarrow & \overline{K} \cdot \mathrm{lan}_L \cdot y_{\mathcal{A}} \\ \hline
\overline{H} & \longrightarrow & \overline{K} \cdot \mathrm{lan}_L \\ \hline
\overline{H} \cdot \mathrm{res}_L & \longrightarrow & \overline{K} \\ \hline
\overline{H} \cdot \mathrm{res}_L \cdot y_{\mathcal{B}} & \longrightarrow & \overline{K} \cdot y_{\mathcal{B}} \\ \hline
\overline{H} \cdot \mathrm{res}_L \cdot y_{\mathcal{B}} & \longrightarrow & K
\end{array}
\qquad
\begin{array}{l}
\overline{K} \cdot y_{\mathcal{B}} \cong K \\
\mathrm{lan}_L \cdot y_{\mathcal{A}} \cong y_{\mathcal{B}} \cdot L \\
c_H \text{ exhibits } \overline{H} \text{ as a left extension} \\
\text{mates correspondence} \\
\text{left extension } \mathrm{res}_L \text{ in Part 1 preserved by } \overline{H} \\
\overline{K} \cdot y_{\mathcal{B}} \cong K
\end{array}$$

It is easily seen this left extension is exhibited by the above 2-cell since when taking $K = \overline{H} \cdot \mathrm{res}_L \cdot y_{\mathcal{B}}$ we may take $\overline{K} = \overline{H} \cdot \mathrm{res}_L$ as a consequence of Part 1 (with the left extension $\overline{K}$ exhibited by the identity 2-cell). Tracing through the bijection to find the exhibiting 2-cell is then trivial. □

Remark 2.3.7. Considering Part 2 in the above proposition with $H = y_{\mathcal{A}}$ and $\overline{H}$ and c_H being an identity 1-cell and 2-cell respectively, we see that for any P-admissible 1-cell $L \colon \mathcal{A} \longrightarrow \mathcal{B}$ and corresponding adjunction $PL \dashv \mathrm{res}_L$ with unit η, we may define our 1-cell R_L and 2-cell φ_L as in Definition 2.3.1 by

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \\
\arrow[ur, phantom, "\overset{\varphi_L}{\Leftarrow}"] & \\
& \mathcal{A} \arrow[uul, "L"] \arrow[uu, "y_{\mathcal{A}}"']
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
\mathcal{B} \arrow[r, "y_{\mathcal{B}}"] & P\mathcal{B} \arrow[r, "\mathrm{res}_L"] & P\mathcal{A} \\
& \arrow[l, phantom, "\overset{y_L}{\Leftarrow}"] \arrow[r, phantom, "\overset{\eta}{\Leftarrow}"] & P\mathcal{A} \arrow[ul, "\mathrm{lan}_L"] \arrow[u, "\mathrm{id}_{P\mathcal{A}}"'] \\
& & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"'] \arrow[uull, bend left, "L"]
\end{tikzcd}$$

We will make regular use of this definition in the next section.

Remark 2.3.8. It is clear from the above proposition that P-admissible 1-cells are closed under composition as noted by Bunge and Funk [6]. We may also note, as in [6], that every left adjoint is P-admissible, as taking $PL := \mathrm{lan}_L$ defines a pseudofunctor [42, Theorem 4.1] and so preserves the adjunction.

2.4 Relating KZ doctrines and Yoneda Structures

We are now ready to prove our main result. In the following statement we call a KZ doctrine locally fully faithful if the unit components are fully faithful; indeed Bunge and Funk [6] noted that a KZ pseudomonad is locally fully faithful precisely when its unit components are fully faithful. Here the admissible maps of Bunge and Funk refer to those maps L for which $PL := \mathrm{lan}_L$ has a right adjoint (which we denote by res_L).

Theorem 2.4.1. *Suppose we are given a locally fully faithful KZ doctrine (P, y) on a 2-category $\mathscr{C}$. Then on defining the class of admissible maps L to be those of Bunge and Funk, with chosen left extensions (R_L, φ_L) those of Remark 2.3.7, we obtain all of the definition and axioms of a Yoneda structure with the exception of the right ideal property (though the admissible maps remain closed under composition).*

Proof. We need only check that:

(1) φ_L exhibits L as an absolute left lifting. Thus, we must exhibit a natural bijection between 2-cells $L \cdot W \to H$ and 2-cells $y_{\mathcal{A}} \cdot W \to R_L \cdot H$ for 1-cells $W \colon \mathcal{D} \to \mathcal{A}$ and $H \colon \mathcal{D} \to \mathcal{B}$ as in the diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "W"] \arrow[dr, bend right, "H"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathcal{A} \arrow[r, "y_{\mathcal{A}}"] \arrow[d, "L"', "\Downarrow\varphi_L"] & P\mathcal{A} \\
& \mathcal{B} \arrow[ur, "R_L"'] \arrow[d, "y_{\mathcal{B}}"', "\Downarrow c_{R_L}"] & \\
& P\mathcal{B} \arrow[uur, bend right, "\mathrm{res}_L"'] &
\end{tikzcd}$$

Such a natural bijection is given by the correspondence

$$\begin{array}{rcl}
L \cdot W & \to & H \\ \hline
y_{\mathcal{B}} \cdot L \cdot W & \to & y_{\mathcal{B}} \cdot H \\ \hline
\mathrm{lan}_L \cdot y_{\mathcal{A}} \cdot W & \to & y_{\mathcal{B}} \cdot H \\ \hline
y_{\mathcal{A}} \cdot W & \to & \mathrm{res}_L \cdot y_{\mathcal{B}} \cdot H \\ \hline
y_{\mathcal{A}} \cdot W & \to & R_L \cdot H
\end{array}
\qquad
\begin{array}{l}
y_{\mathcal{B}} \text{ fully faithful} \\
\mathrm{lan}_L \cdot y_{\mathcal{A}} \cong y_{\mathcal{B}} \cdot L \\
\mathrm{lan}_L \dashv \mathrm{res}_L \\
R_L := \mathrm{res}_L \cdot y_{\mathcal{B}}
\end{array}$$

and the 2-cell exhibiting this absolute left lifting is easily seen to be the 2-cell as given in Remark 2.3.7 by following the above bijection.

(2) $\mathrm{res}_L \cdot R_K$ *is a left extension.* Considering the diagram

$$\begin{tikzcd}
P\mathcal{A} & P\mathcal{B} \arrow[l, "\mathrm{res}_L"'] & \mathcal{C} \arrow[l, "R_K"'] \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[r, "L"'] \arrow[ur, phantom, "\overset{\varphi_L}{\Longrightarrow}"] & \mathcal{B} \arrow[ul, "R_L"] \arrow[u, "y_{\mathcal{B}}"'] \arrow[ur, "K"'] \arrow[ul, phantom, bend right, "\overset{c_{R_L}}{\Longrightarrow}"] \arrow[u, phantom, shift left=4, "\overset{\varphi_K}{\Longrightarrow}"] &
\end{tikzcd}$$

we first note that $\mathrm{res}_L \cdot R_K$ is a left extension of R_L along K since K is P-admissible. We then apply the pasting lemma for left extensions to see the outside diagram also exhibits $\mathrm{res}_L \cdot R_K$ as a left extension. ☐

Remark 2.4.2. We observe that to ask that $\mathrm{res}_L \cdot R_K$ be a left extension in the diagram above for every P-admissible L and K, is to ask by the pasting lemma that the pasting of φ_K and c_{R_L} exhibit $\mathrm{res}_L \cdot R_K$ as a left extension. As c_{R_L} is invertible, this is to say that res_L respects every left extension arising from admissibility. This is equivalent to asking res_L be a P-homomorphism.

Remark 2.4.3. We note here that we do not necessarily have the right ideal property. Indeed given a KZ doctrine on a 2-category every identity arrow is admissible, and so the right ideal property would require all arrows into all objects being admissible (that is all arrows being admissible). This fails for example with the identity KZ doctrine on any 2-category $\mathscr{C}$ which contains an arrow L with no right adjoint.

Remark 2.4.4. Given an object $\mathcal{A} \in \mathscr{C}$ with a P-admissible generalized element $a\colon \mathcal{S} \to \mathcal{A}$ we have a version of the Yoneda lemma in the sense that we have bijections

$$\begin{array}{rcl} y_{\mathcal{A}} \cdot a & \to & K \\ \hline \mathrm{lan}_a \cdot y_{\mathcal{S}} & \to & K \\ \hline y_{\mathcal{S}} & \to & \mathrm{res}_a \cdot K \end{array} \qquad \begin{array}{l} \mathrm{lan}_a \cdot y_{\mathcal{S}} \cong y_{\mathcal{A}} \cdot a \\ \mathrm{lan}_a \dashv \mathrm{res}_a \end{array}$$

for generalized elements $K\colon \mathcal{S} \to P\mathcal{A}$. In the case where P is the usual free small cocompletion KZ doctrine on locally small categories and $\mathcal{S} = \mathbf{1}$ is the terminal category, maps $y_{\mathcal{S}} \to \mathrm{res}_a \cdot K$ are elements of $\mathrm{res}_a \cdot K$ (which may be viewed as K evaluated at a).

The purpose of the following is to give an example in which absolute left liftings (also known as relative adjunctions or partial adjunctions) are preserved[3]. Also, the following proposition does not require locally fully faithfulness, whereas Theorem 2.4.1 does.

Proposition 2.4.5. *Suppose we are given a KZ doctrine (P, y) on a 2-category $\mathscr{C}$. Then for every P-admissible 1-cell $L\colon \mathcal{A} \to \mathcal{B}$ as on the left,*

$$\begin{tikzcd} \mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} & & P\mathcal{B} \arrow[r, "PR_L"] & P^2\mathcal{A} \\ & \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"'] \arrow[ul, phantom, shift left=3, "\overset{\varphi_L}{\Leftarrow}"] & & & P\mathcal{A} \arrow[ul, "PL"] \arrow[u, "Py_{\mathcal{A}}"'] \arrow[ul, phantom, shift left=3, "\overset{P\varphi_L}{\Leftarrow}"] \end{tikzcd}$$

the 2-cell $P\varphi_L$ as on the right (in which we have suppressed the pseudofunctoriality constraints) exhibits PL as an absolute left lifting of $Py_{\mathcal{A}}$ through PR_L.

---

[3] In this case respected by the KZ pseudomonad resulting from the KZ doctrine as in [42].

Proof. Without loss of generality, we define φ_L as in Remark 2.3.7. We then have the sequence of natural bijections

$$\begin{array}{rcl} PL \cdot W & \to & H \\ \hline Py_{\mathcal{B}} \cdot PL \cdot W & \to & Py_{\mathcal{B}} \cdot H \\ \hline P^2L \cdot Py_{\mathcal{A}} \cdot W & \to & Py_{\mathcal{B}} \cdot H \\ \hline Py_{\mathcal{A}} \cdot W & \to & \mathrm{Pres}_L \cdot Py_{\mathcal{B}} \cdot H \\ \hline Py_{\mathcal{A}} \cdot W & \to & PR_L \cdot H \end{array} \qquad \begin{array}{l} Py_{\mathcal{B}} \text{ fully faithful} \\ y_{\mathcal{B}} \cdot L \cong PL \cdot y_{\mathcal{A}} \\ PL \dashv \mathrm{res}_L \\ R_L := \mathrm{res}_L \cdot y_{\mathcal{B}} \end{array}$$

for 1-cells W into $P\mathcal{A}$. Following the bijection we see that the absolute left lifting is exhibited by $P\varphi_L$, suppressing the pseudofunctoriality constraints. □

Some observations made in "Yoneda structures" [47] may be seen more directly in this setting of a KZ doctrine. For example Street and Walters defined an admissible morphism L (in the setting of a Yoneda structure) to be fully faithful when the 2-cell φ_L is invertible (which agrees with a representable notion of fully faithfulness, that is fully faithfulness defined via the absolute left lifting property, when axiom (d) is satisfied). Here we see this in the context of a (locally fully faithful) KZ doctrine.

Proposition 2.4.6. *Suppose we are given a KZ doctrine (P,y) on a 2-category $\mathscr{C}$, and a P-admissible 1-cell $L\colon \mathcal{A} \to \mathcal{B}$*

$$\begin{tikzcd} \mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \arrow[dl, phantom, "\overset{\varphi_L}{\Leftarrow}" near start] \\ & \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"'] \end{tikzcd}$$

with a left extension R_L as in the above diagram. Then the exhibiting 2-cell φ_L is invertible if and only if $PL := \mathrm{lan}_L$ is fully faithful.

Proof. We use the well known fact that the left adjoint of an adjunction is fully faithful precisely when the unit is invertible. Now, given that φ_L is invertible we may define our 2-cell η^* as the unique solution to

$$\begin{tikzcd} P\mathcal{A} & & P\mathcal{A} \arrow[ll, "\mathrm{id}_{P\mathcal{A}}"'] \\ & \mathcal{B} \arrow[ul, "R_L"] \arrow[r, phantom, "\overset{\varphi_L^{-1}}{\Longrightarrow}"] & {} \\ & & \mathcal{A} \arrow[ul, "L"] \arrow[uu, "y_{\mathcal{A}}"'] \end{tikzcd} = \begin{tikzcd} P\mathcal{A} & P\mathcal{B} \arrow[l, "\mathrm{res}_L"'] & P\mathcal{A} \arrow[l, "PL"'] \arrow[ll, bend right=40, "\mathrm{id}_{P\mathcal{A}}"'] \arrow[l, phantom, bend right=20, "\Uparrow\eta^*" near start] \\ & \mathcal{B} \arrow[ul, "R_L"] \arrow[u, "y_{\mathcal{B}}"'] \arrow[ul, phantom, "\Uparrow c_{R_L}" very near end, shift left=4] \arrow[ur, phantom, "\Uparrow c_{y_{\mathcal{B}} \cdot L}"] & \mathcal{A} \arrow[l, "L"] \arrow[u, "y_{\mathcal{A}}"'] \end{tikzcd}$$

That η is the inverse of η^* follows from an easy calculation using Remark 2.3.7. Conversely, if the unit η is invertible then so is φ_L by Remark 2.3.7. □

Remark 2.4.7. If we define a map L to be P-fully faithful when PL is fully faithful, then as a consequence of Proposition 2.3.6 (Part 2) and Proposition 2.4.6 we see that for any P-admissible map L, this L is P-fully faithful if and only if every left extension along L into a P-cocomplete object is exhibited by an invertible 2-cell.

In the following remark we compare PL being fully faithful with L being fully faithful, and point out sufficient conditions for these notions to agree.

Remark 2.4.8. Note that if PL is fully faithful then L is fully faithful assuming P is locally fully faithful, as y is pseudonatural. Conversely if L is fully faithful, then (supposing our corresponding left extension R_L is pointwise) the exhibiting 2-cell is invertible [54, Prop. 2.22], equivalent to PL being fully faithful by the above. This converse may also be seen when the KZ doctrine is locally fully faithful and good (meaning axiom (d) is satisfied for P-admissible maps) as we can use the argument of [47, Prop. 9]. However, as we now see, this converse need not hold in general.

An example in which L is fully faithful but PL is not is given as follows. Take $\mathcal{A}$ to be the 2-category containing the two objects $0, 1$ and two non-trivial 1-cells $x, y \colon 0 \to 1$, and take $\mathcal{B}$ to be the same but with an additional 2-cell $\alpha \colon x \to y$. Define L as the inclusion of $\mathcal{A}$ into $\mathcal{B}$. Then for the free **Cat**-cocompletion of $\mathcal{A}$ given by $y_{\mathcal{A}} \colon \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]$ we note that $y_{\mathcal{A}}$ and $R_L \cdot L$ are not isomorphic, and so the 2-cell φ_L is not invertible meaning PL is not fully faithful (despite L being fully faithful).

2.5 Future Work

We have seen that the notions of pseudo algebras and admissibility for a given KZ doctrine, and KZ doctrines themselves, may be expressed in terms of left extensions. In a soon forthcoming paper we show that pseudodistributive laws over a KZ doctrine may be simply expressed entirely in terms of left extensions and admissibility, allowing us to generalize some results of Marmolejo and Wood [42].

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3

Distributive laws via admissibility

Abstract

This paper concerns the problem of lifting a KZ doctrine P to the 2-category of pseudo T-algebras for some pseudomonad T. Here we show that this problem is equivalent to giving a pseudo-distributive law (meaning that the lifted pseudomonad is automatically KZ), and that such distributive laws may be simply described algebraically and are essentially unique (as known to be the case in the (co)KZ over KZ setting).

Moreover, we give a simple description of these distributive laws using Bunge and Funk's notion of admissible morphisms for a KZ doctrine (the principal goal of this paper). We then go on to show that the 2-category of KZ doctrines on a 2-category is biequivalent to a poset.

We will also discuss here the problem of lifting a locally fully faithful KZ doctrine, which we noted earlier enjoys most of the axioms of a Yoneda structure, and show that a bijection between oplax and lax structures is exhibited on the lifted "Yoneda structure" similar to Kelly's doctrinal adjunction. We also briefly discuss how this bijection may be viewed as a coherence result for oplax functors out of the bicategories of spans and polynomials, but leave the details for a future paper.

Contribution by the author

As the sole author, this paper is entirely my own work. This paper was submitted for publication on June 27th 2017 and was provisionally accepted pending revisions on Jan 31st 2018.

3.1 Introduction

It is well known that to give a lifting of a monad to the algebras of another monad is to give a distributive law [2]. More generally, to give a lifting of a pseudomonad to the pseudoalgebras of another pseudomonad is to give a pseudo-distributive law [39, 8]. However, in this paper we are interested in the problem of lifting a Kock-Zöberlein pseudomonad P (also known as a lax idempotent pseudomonad), as introduced by Kock [31] and Zöberlein [57], to the pseudoalgebras of some pseudomonad T. These KZ pseudomonads are a particular type of pseudomonad for which algebra structures are adjoint to units; an important example being the free cocompletion under a class of colimits Φ.

But what does it mean to give a lifting of a KZ doctrine to the setting of pseudoalgebras such that the lifted pseudomonad is also KZ? One objective of this paper is to show that this problem is equivalent to giving a pseudo-distributive law (meaning a lifting of this pseudomonad automatically inherits the KZ structure), and consequently that such pseudo-distributive laws have a couple of simple descriptions. One simple description being purely algebraic (a generalization and simplification of a description given in [39, Section 11]), and another being a novel description purely in terms of left Kan extensions and Bunge and Funk's admissible maps of a KZ doctrine [6]. In fact, Bunge and Funk's admissible maps are a central tool in the proof of these results. We also see that these distributive laws are essentially unique, a generalization capturing [42, Theorem 7.4] and strengthening parts of [40, Prop. 4.1].

These two descriptions of a pseudo-distributive law correspond to two different descriptions of a KZ pseudomonad. The first, which from now on we call a KZ pseudomonad, is a well known algebraic description similar to Kock's [31]; the second, which we call a KZ doctrine, is to be the description in terms of left Kan extensions due to Marmolejo and Wood [42, Definition 3.1].

Bunge and Funk showed that admissibility in the setting of a KZ pseudomonad also has both an algebraic definition and a definition in terms of left Kan extensions. Indeed, Bunge and Funk defined a morphism f to be admissible in the context of a KZ doctrine P when Pf has a right adjoint [6, Definition 1.1], and showed that this notion of admissibility also has a description in terms of left Kan extensions [6, Prop. 1.5]. We refer to this as P-admissibility.

The central idea here is that instead of thinking about the problem of lifting a KZ doctrine algebraically, we think about the problem in terms of algebraic left Kan extensions. Moreover, this notion of admissibility is crucial here as it allows us to show that certain left extensions exist and are preserved.

A well known and motivating example the reader may keep in mind is the KZ doctrine for the free small cocompletion on locally small categories, with its lifting to the setting of monoidal categories described by Im and Kelly [22] via the Day convolution [11].

In Section 3.2 we give the necessary background for this paper, and recall the basic definitions of pseudomonads, pseudo algebras and morphisms between pseudo algebras. In particular, we recall the notion of a KZ pseudomonad and KZ doctrine and some results concerning them. In addition, we recall some results concerning algebraic left extensions. These notions will be used regularly throughout the paper.

In Section 3.3, which is the bulk of this paper, we use Bunge and Funk's notion of admissibility to generalize some results of Marmolejo and Wood concerning pseudo-distributive laws of (co)KZ doctrines over KZ doctrines, such as the simple form of such distributive laws [39, Section 11] or essential uniqueness of them [42, Theorem 7.4]. Our first improvement here is to show that an axiom concerning the (co)KZ doctrine may be dropped, allowing us to generalize these results to pseudo-distributive laws of *any pseudomonad* over a KZ doctrine. For example, this level of generality allows us to capture the case studied by Im and Kelly [22]; showing that the lifting of the small cocompletion from categories to monoidal categories is essentially unique.

In addition, we use this simplification to give a simple algebraic description of a pseudo-distributive law of a pseudomonad over a KZ pseudomonad, consisting only of a pseudonatural transformation and three invertible modifications subject to three coherence axioms, and prove this definition is equivalent to the usual notion of pseudo-distributive law. However, the main new result of this section is a simple description of pseudo-distributive laws over a KZ doctrine purely in terms of left Kan extensions and admissibility.

Furthermore, through these calculations we find that in the presence of a such a distributive law, the lifting of a KZ doctrine P to pseudo-T-algebras (for a pseudomonad T) is automatically a KZ doctrine. The proof of these results is highly technical, relying on T preserving P-admissible maps; however, the main result of this section is simply stated in Theorem 3.3.8.

In Section 3.4 we study some properties of the lifted KZ doctrine $\widetilde{P}$, such as classifying the $\widetilde{P}$-cocomplete T-algebras as those for which the underlying object is P-cocomplete and the algebra map separately cocontinuous, thus justifying the usual definition of algebraic cocompleteness. We also compare our results to that of Im-Kelly [22], but seen from the KZ doctrine viewpoint.

After checking that the 2-category of KZ doctrines on a 2-category is biequivalent to a poset, we go on to give some examples in which we apply our results. Our first example concerns the case of the small cocompletion and monoidal categories, and our second example concerns multi-adjoints as studied by Diers [14].

In Section 3.5 we consider the problem of lifting a locally fully faithful KZ doctrine. These locally fully faithful KZ doctrines are of interest as they almost give rise to Yoneda structures (see Chapter 2). In particular, it is the goal of this section to describe a bijection between oplax and lax structures on the lifted "Yoneda structure" when we have such a distributive law; that is a bijection between cells α exhibiting L as an oplax T-morphism

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R_L"] \arrow[dr, phantom, "\overset{\varphi_L}{\Leftarrow}"] & P\mathcal{A} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\left(\mathcal{B}, T\mathcal{B} \xrightarrow{y} \mathcal{B}\right) \arrow[r, "(R_L{,}\beta)"] \arrow[dr, phantom, "\overset{\varphi_L}{\Leftarrow}"] & \left(P\mathcal{A}, TP\mathcal{A} \xrightarrow{z_x} P\mathcal{A}\right) \\
& \left(\mathcal{A}, T\mathcal{A} \xrightarrow{x} \mathcal{A}\right) \arrow[ul, "(L{,}\alpha)"] \arrow[u, "(y_{\mathcal{A}}{,}\xi_x)"']
\end{tikzcd}$$

and cells β exhibiting R_L as a lax T-morphism for diagrams as on the right above, underlain by a "Yoneda structure" diagram such as that on the left above. As an instance of this result we recover Kelly's bijection between oplax structures on left adjoints and lax structures on right adjoints [27]. An interesting application of this bijection is as a coherence result for the bicategories of spans and polynomials (and in particular the oplax functors out of these bicategories). We briefly discuss the applications here, but leave this to be explored in more detail in a forthcoming paper.

3.2 Background

It is the purpose of this section to give the background knowledge necessary for this paper. We start off by recalling the basic definitions of pseudomonads, pseudo algebras, and morphisms between pseudo algebras, as these notions will be used regularly throughout the paper. We then recall the notion of a left extension in a 2-category, and consider when these left extensions lift to the setting of pseudo-algebras and morphisms between them (in a sense which will be applicable in later sections). Finally, we go on to recall the notion of a KZ pseudomonad, a special type of pseudomonad for which the algebra structure maps are adjoint to units, and give their basic properties and some examples.

3.2.1 Pseudomonads and their Algebras

In order to define pseudomonads, we first need the notions of pseudonatural transformations and modifications. The notion of pseudonatural transformation is the (weak) 2-categorical version of natural transformation. There are weaker notions also of lax and oplax natural transformations, however those will not be used here. Modifications, defined below, take the place of morphisms between pseudonatural transformations.

Definition 3.2.1. A *pseudonatural transformation* between pseudofunctors $t\colon F \to G\colon \mathscr{A} \to \mathscr{B}$ where $\mathscr{A}$ and $\mathscr{B}$ are bicategories provides for each 1-cell $f\colon \mathcal{A} \to \mathcal{B}$ in $\mathscr{A}$, 1-cells $t_{\mathcal{A}}$ and

$t_{\mathcal{B}}$ and an invertible 2-cell t_f in $\mathscr{B}$ as below

$$\begin{tikzcd}
F\mathcal{A} \arrow[r, "Ff"] \arrow[d, "t_{\mathcal{A}}"'] \arrow[dr, phantom, "\overset{t_f}{\Longrightarrow}"] & F\mathcal{B} \arrow[d, "t_{\mathcal{B}}"] \\
G\mathcal{A} \arrow[r, "Gf"'] & G\mathcal{B}
\end{tikzcd}$$

satisfying coherence conditions outlined in [28, Definition 2.2]. Given two pseudonatural transformations $t, s\colon F \longrightarrow G\colon \mathscr{A} \longrightarrow \mathscr{B}$ as above, a *modification* $\alpha\colon s \longrightarrow t$ consists of, for every object $\mathcal{A} \in \mathscr{A}$, a 2-cell $\alpha_{\mathcal{A}}\colon t_{\mathcal{A}} \longrightarrow s_{\mathcal{A}}$ such that for each 1-cell $f\colon \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathscr{A}$ we have the equality $\alpha_{\mathcal{B}} \cdot Ff \circ t_f = s_f \circ Gf \cdot \alpha_{\mathcal{A}}$.

The following defines the (weak) 2-categorical version of monad to be used throughout this paper. For brevity, we will suppress pseudofunctoriality constraints in this definition and those following.

Definition 3.2.2. A *pseudomonad* on a 2-category $\mathscr{C}$ consists of a pseudofunctor equipped with pseudonatural transformations as below

$$T\colon \mathscr{C} \longrightarrow \mathscr{C}, \qquad u\colon 1_{\mathscr{C}} \longrightarrow T, \qquad m\colon T^2 \longrightarrow T$$

along with three invertible modifications

$$\begin{tikzcd}
T \arrow[r, "uT"] \arrow[dr, "\mathrm{id}"'] & T^2 \arrow[d, "m"] \arrow[dl, phantom, "\overset{\alpha}{\Longleftarrow}" near start] \arrow[dr, phantom, "\overset{\beta}{\Longleftarrow}" near start] & T \arrow[l, "Tu"'] \arrow[dl, "\mathrm{id}"] \\
 & T &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3 \arrow[r, "Tm"] \arrow[d, "mT"'] \arrow[dr, phantom, "\overset{\gamma}{\Longleftarrow}"] & T^2 \arrow[d, "m"] \\
T^2 \arrow[r, "m"'] & T
\end{tikzcd}$$

subject to the two coherence axioms

$$\begin{tikzcd}
T^4 \arrow[r, "T^2m"] \arrow[d, "mT^2"'] \arrow[dr, "TmT"] & T^3 \arrow[dr, "Tm"] \arrow[d, phantom, "\overset{T\gamma}{\Longleftarrow}" near start, shift left=6] & \\
T^3 \arrow[r, phantom, "\overset{\gamma T}{\Longleftarrow}"] \arrow[dr, "mT"'] & T^3 \arrow[r, "Tm"] \arrow[d, "mT"'] \arrow[dr, phantom, "\overset{\gamma}{\Longleftarrow}"] & T^2 \arrow[d, "m"] \\
 & T^2 \arrow[r, "m"'] & T
\end{tikzcd}
=
\begin{tikzcd}
T^4 \arrow[r, "T^2m"] \arrow[d, "mT^2"'] \arrow[dr, phantom, "\overset{m_m^{-1}}{\Longleftarrow}"] & T^3 \arrow[d, "mT"] \arrow[dr, "Tm"] & \\
T^3 \arrow[r, "Tm"'] \arrow[dr, "mT"'] & T^2 \arrow[r, phantom, "\overset{\gamma}{\Longleftarrow}"] \arrow[dr, "m"] \arrow[d, phantom, "\overset{\gamma}{\Longleftarrow}"] & T^2 \arrow[d, "m"] \\
 & T^2 \arrow[r, "m"'] & T
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & T^2 \arrow[dr, "m"] & \\
T^2 \arrow[r, "TuT"] & T^3 \arrow[ur, "Tm"] \arrow[dr, "mT"'] & \Downarrow\gamma & T \\
 & & T^2 \arrow[ur, "m"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & T^3 \arrow[ddr, "Tm"] & & \\
 & \Downarrow T\alpha & & \\
T^2 \arrow[uur, "TuT"] \arrow[rr, "\mathrm{id}"] \arrow[ddr, "TuT"'] & & T^2 \arrow[r, "m"] & T \\
 & \Downarrow\beta T & & \\
 & T^3 \arrow[uur, "mT"'] & &
\end{tikzcd}$$

Remark 3.2.3. One should note here that there are three useful consequences of these pseudomonad axioms [38, Proposition 8.1] originally due to Kelly [25]. Of these, we will only need the consequence that

$$\begin{tikzcd}
& & T^2 \arrow[dr, "m"] \arrow[d, phantom, "\Downarrow\alpha"] & \\
1_{\mathscr{C}} \arrow[r, "u"] & T \arrow[ur, "uT"] \arrow[dr, "Tu"'] \arrow[rr, "\mathrm{id}"] & {} \arrow[d, phantom, "\Downarrow\beta"] & T \\
& & T^2 \arrow[ur, "m"'] &
\end{tikzcd}
=
\begin{tikzcd}
& T \arrow[dr, "uT"] & & \\
1_{\mathscr{C}} \arrow[ur, "u"] \arrow[dr, "u"'] & \Downarrow u_u^{-1} & T^2 \arrow[r, "m"] & T \\
& T \arrow[ur, "Tu"'] & &
\end{tikzcd} \tag{3.2.1}$$

Given a pseudomonad (T, u, m) on a 2-category $\mathscr{C}$ one may consider its strict T-algebras and strict T-morphisms, or the weaker counterparts where conditions only hold up coherent 2-cells. These weaker notions are what will be used throughout this paper, though usually with the coherent 2-cells in question being invertible. For convenience, we will leave the modifications α, β and γ in the above definition as unnamed isomorphisms throughout the rest of the paper.

Definition 3.2.4. Given a pseudomonad (T, u, m) on a 2-category $\mathscr{C}$, a *lax T-algebra* consists of an object $\mathcal{A} \in \mathscr{C}$, a 1-cell $x \colon T\mathcal{A} \to \mathcal{A}$ and 2-cells

$$\begin{tikzcd}
T^2\mathcal{A} \arrow[r, "Tx"] \arrow[d, "m_{\mathcal{A}}"'] \arrow[dr, phantom, "\Downarrow\mu"] & T\mathcal{A} \arrow[d, "x"] \\
T\mathcal{A} \arrow[r, "x"'] & \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \arrow[rr, "\mathrm{id}"] \arrow[dr, "u_{\mathcal{A}}"'] & \Downarrow\nu & \mathcal{A} \\
& T\mathcal{A} \arrow[ur, "x"'] &
\end{tikzcd}$$

such that both

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "u_{\mathcal{A}}"] \arrow[rr, bend left=40, "\mathrm{id}"] \arrow[rr, phantom, bend left=20, "\Downarrow\nu"] & T\mathcal{A} \arrow[r, "x"] & \mathcal{A} \\
& \Downarrow u_x^{-1} \qquad\quad \Downarrow\mu & \\
T\mathcal{A} \arrow[uu, "x"] \arrow[r, "u_{T\mathcal{A}}"'] \arrow[rr, bend right=40, "\mathrm{id}"'] \arrow[rr, phantom, bend right=20, "\cong"] & T^2\mathcal{A} \arrow[uu, "Tx"'] \arrow[r, "m_{\mathcal{A}}"'] & T\mathcal{A} \arrow[uu, "x"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& & T\mathcal{A} \arrow[dr, "x"] & \\
T\mathcal{A} \arrow[urr, bend left, "\mathrm{id}"] \arrow[ur, phantom, "\Downarrow T\nu"] \arrow[r, "Tu_{\mathcal{A}}"'] \arrow[drr, bend right, "\mathrm{id}"'] \arrow[dr, phantom, "\cong"] & T^2\mathcal{A} \arrow[ur, "Tx"] \arrow[dr, "m_{\mathcal{A}}"] & \Downarrow\mu & \mathcal{A} \\
& & T\mathcal{A} \arrow[ur, "x"'] &
\end{tikzcd}$$

paste to the identity 2-cell at x, known as the left and right unit axioms respectively. Moreover, the associativity axiom asks that we have the equality

$$\begin{tikzcd}
 & T^2\mathcal{A} \arrow[r, "Tx"] \arrow[dr, "m_{\mathcal{A}}"'] & T\mathcal{A} \arrow[dr, "x"] \arrow[d, phantom, "\Downarrow\mu"] & & & & T^2\mathcal{A} \arrow[r, "Tx"] \arrow[d, phantom, "\Downarrow T\mu"] & T\mathcal{A} \arrow[dr, "x"] & \\
T^3\mathcal{A} \arrow[ur, "T^2x"] \arrow[dr, "m_{T\mathcal{A}}"'] & \Downarrow m_x^{-1} & T\mathcal{A} \arrow[r, "x"] & \mathcal{A} & = & T^3\mathcal{A} \arrow[ur, "T^2x"] \arrow[r, "Tm_{\mathcal{A}}"] \arrow[dr, "m_{T\mathcal{A}}"'] & T^2\mathcal{A} \arrow[ur, "Tx"] \arrow[dr, "m_{\mathcal{A}}"] \arrow[d, phantom, "\cong"] & \Downarrow\mu & \mathcal{A} \\
 & T^2\mathcal{A} \arrow[r, "m_{\mathcal{A}}"'] \arrow[ur, "Tx"'] & T\mathcal{A} \arrow[ur, "x"'] \arrow[u, phantom, "\Downarrow\mu"] & & & & T^2\mathcal{A} \arrow[r, "m_{\mathcal{A}}"'] & T\mathcal{A} \arrow[ur, "x"'] &
\end{tikzcd}$$

If the above 2-cells ν and μ are isomorphisms, we call this a *pseudo T-algebra*. If ν and μ are identity 2-cells, we call this a *strict T-algebra*.

These T-algebras may be regarded as the objects of a category, with morphisms of (pseudo) T-algebras defined as follows.

Definition 3.2.5. Given a pseudomonad (T,u,m) on a 2-category $\mathscr{C}$, an *oplax T-morphism* of pseudo T-algebras

$$(L,\alpha):\ \left(\mathcal{A}, T\mathcal{A} \xrightarrow{x} \mathcal{A}\right) \to \left(\mathcal{B}, T\mathcal{B} \xrightarrow{y} \mathcal{B}\right)$$

consists of a 1-cell $L\colon \mathcal{A} \to \mathcal{B}$ and a 2-cell

$$\begin{tikzcd}
T\mathcal{B} \arrow[r, "y"] & \mathcal{B} \\
T\mathcal{A} \arrow[u, "TL"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow\alpha"] & \mathcal{A} \arrow[u, "L"']
\end{tikzcd}$$

such that (leaving the pseudo T-algebra coherence cells as unnamed isomorphisms)

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "u_{\mathcal{B}}"] \arrow[rr, bend left=60, "\mathrm{id}"] \arrow[rr, phantom, bend left=40, "\cong"] & T\mathcal{B} \arrow[r, "y"] & \mathcal{B} \\
\mathcal{A} \arrow[u, "L"] \arrow[r, "u_{\mathcal{A}}"'] \arrow[rr, bend right=60, "\mathrm{id}"'] \arrow[rr, phantom, bend right=40, "\cong"'] \arrow[ur, phantom, "\Uparrow u_L"] & T\mathcal{A} \arrow[u, "TL"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow\alpha"] & \mathcal{A} \arrow[u, "L"']
\end{tikzcd}$$

is the identity 2-cell on L, and for which

$$\begin{tikzcd}
& T\mathcal{B} \arrow[dr, "y"] & & & & & \\
T^2\mathcal{B} \arrow[ur, "m_{\mathcal{B}}"] \arrow[r, "Ty"] & T\mathcal{B} \arrow[r, "y"] \arrow[u, phantom, "\cong"] & \mathcal{B} & & T^2\mathcal{B} \arrow[r, "m_{\mathcal{B}}"] & T\mathcal{B} \arrow[r, "y"] & \mathcal{B} \\
T^2\mathcal{A} \arrow[u, "T^2L"] \arrow[r, "Tx"'] \arrow[dr, "m_{\mathcal{A}}"'] \arrow[ur, phantom, "\Uparrow T\alpha"] & T\mathcal{A} \arrow[u, "TL"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow\alpha"] & \mathcal{A} \arrow[u, "L"'] & = & T^2\mathcal{A} \arrow[u, "T^2L"] \arrow[r, "m_{\mathcal{A}}"'] \arrow[ur, phantom, "\Uparrow m_L"] & T\mathcal{A} \arrow[u, "TL"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow\alpha"] & \mathcal{A} \arrow[u, "L"'] \\
& T\mathcal{A} \arrow[ur, "x"'] \arrow[u, phantom, "\cong"] & & & & &
\end{tikzcd}$$

If the 2-cell α goes in the opposite direction, this is the definition of a *lax T-morphism,* and if α is invertible this is then the definition of a *pseudo T-morphism.*

The usual definition of T-transformation between oplax or lax T-morphisms is not general enough for our purposes as we will be considering situations in which we have both oplax and lax T-morphisms, and so we define T-transformations as based on the double category viewpoint [19]. Such transformations are sometimes referred to as generalized T-transformations.

Definition 3.2.6. Suppose we are given a square of morphisms of pseudo T-algebras

$$\begin{tikzcd}
(\mathcal{B}, y) \arrow[r, "{(R,\beta)}"] & (\mathcal{C}, z) \\
(\mathcal{D}, w) \arrow[u, "{(N,\varphi)}"] \arrow[r, "{(M,\varepsilon)}"'] \arrow[ur, phantom, "\overset{\zeta}{\Longleftarrow}"] & (\mathcal{A}, x) \arrow[u, "{(I,\xi)}"']
\end{tikzcd}$$

where the vertical maps are oplax T-morphisms and the horizontal maps are lax T-morphisms. A *T-transformation ζ* as in the above square is a 2-cell $\zeta : I \cdot M \longrightarrow R \cdot N$ for which we have the equality of the two sides of the cube

$$\begin{tikzcd}
& T\mathcal{B} \arrow[r, "y"] & \mathcal{B} \arrow[dr, "R"] & & & & & T\mathcal{B} \arrow[r, "y"] \arrow[dr, "TR"] & \mathcal{B} \arrow[dr, "R"] & \\
T\mathcal{D} \arrow[ur, "TN"] \arrow[r, "w"] \arrow[dr, "TM"'] & \mathcal{D} \arrow[ur, "N"] \arrow[dr, "M"] \arrow[u, phantom, "\Uparrow\varphi"] \arrow[d, phantom, "\Uparrow\varepsilon"] & \Uparrow\zeta & \mathcal{C} & = & & T\mathcal{D} \arrow[ur, "TN"] \arrow[dr, "TM"'] & \Uparrow T\zeta & T\mathcal{C} \arrow[r, "z"] \arrow[u, phantom, "\Uparrow\beta"] \arrow[d, phantom, "\Uparrow\xi"] & \mathcal{C} \\
& T\mathcal{A} \arrow[r, "x"'] & \mathcal{A} \arrow[ur, "I"'] & & & & & T\mathcal{A} \arrow[ur, "TI"] \arrow[r, "x"'] & \mathcal{A} \arrow[ur, "I"'] &
\end{tikzcd}$$

We will call the 2-category of pseudo T-algebras, pseudo T-morphisms, and T-transformations ps-T-alg (we may consider squares where both horizontal maps are identities or both vertical maps are identities to recover the usual notions of transformation between lax/oplax/pseudo T-morphisms).

Remark 3.2.7. Note that in this language it makes sense to talk about the unit and counit of an adjunction where the left adjoint is oplax and the right adjoint lax. Indeed the oplax-lax bijective correspondence in Kelly's doctrinal adjunction [27] is unique such the counit ε (and unit η) of the adjunction is a T-transformation[1]. Note also that in this setting of a doctrinal adjunction $L \dashv R$ (with an oplax structure α on L corresponding a lax structure β on R via the mates correspondence) it makes sense to view the unit and counit as T-transformations as we have squares

$$\begin{tikzcd}
(\mathcal{B},y) \arrow[r, "(\mathrm{id},\mathrm{id})"] & (\mathcal{B},y) & & (\mathcal{B},y) \arrow[r, "(R,\beta)"] & (\mathcal{A},x) \\
(\mathcal{B},y) \arrow[r, "(R,\beta)"'] \arrow[u, "(\mathrm{id},\mathrm{id})"] \arrow[ur, phantom, "\overset{\varepsilon}{\Leftarrow}"] & (\mathcal{A},x) \arrow[u, "(L,\alpha)"'] & & (\mathcal{A},x) \arrow[r, "(\mathrm{id},\mathrm{id})"'] \arrow[u, "(L,\alpha)"] \arrow[ur, phantom, "\overset{\eta}{\Leftarrow}"] & (\mathcal{A},x) \arrow[u, "(\mathrm{id},\mathrm{id})"']
\end{tikzcd}$$

As a convention, will will usually omit these identity T-morphisms. The reader may just remember that it makes sense to consider T-transformations from a lax followed by an oplax T-morphism, into an oplax followed by a lax T-morphism, and that any such transformation may be uniquely expressed as a square in the form of the above definition by inserting the appropriate identity T-morphisms; which is what we have done in the case of the unit and counit above.

Example 3.2.8. *Let* **Cat** *be the category of locally small categories. One may define the category of* **Cat***-enriched graphs, denoted* **CatGrph***, with objects given as families of hom-categories*

$$(\mathscr{C}(X,Y) : X,Y \in \mathrm{ob}\mathscr{C})$$

and morphisms consisting of locally defined functors

$$\left(F_{X,Y} : \mathscr{C}(X,Y) \to \mathscr{D}(FX,FY) : X,Y \in \mathscr{C}\right)$$

which have not been endowed with the structure of a bicategory or a lax/oplax functor respectively [33]. This gives rise to, via a suitable 2-monad T on **CatGrph***, the 2-category of bicategories, oplax functors and icons [34]. We may of course replace oplax here with "lax" or "pseudo". Note that inside this 2-category lives the one object bicategories (isomorphic to monoidal categories), giving the 2-category of monoidal categories, lax/oplax/strong monoidal functors and monoidal transformations (which may also be constructed directly*

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[1] This is shown in more generality in Proposition 3.5.6.

via a suitable 2-monad [34]).

3.2.2 Left Extensions and Algebraic Left Extensions

In this section we will consider how pseudomonads interact with left extensions. In particular, we start off by recalling the notion of a left extension in a 2-category, and go on to give conditions under which such a left extension lifts to a suitable notion of left extension in the setting of pseudo T-algebras, T-morphisms and T-transformations. The results of this section are mostly due to Koudenburg, shown in a more general double category setting [32].

Definition 3.2.9. Suppose we are given a 2-cell $\eta\colon I \longrightarrow R \cdot L$ as in the left diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r, "R"] & \mathcal{C} \\ & \mathcal{A} \arrow[ul, "L"] \arrow[u, "I"'] \arrow[from=1-2, to=2-1, phantom, "\overset{\eta}{\Leftarrow}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B} \arrow[r, "R" description] \arrow[r, bend left=60, "M"] \arrow[r, phantom, bend left=30, "\Uparrow\sigma"] & \mathcal{C} \\ & \mathcal{A} \arrow[ul, "L"] \arrow[u, "I"'] \arrow[from=1-2, to=2-1, phantom, "\overset{\eta}{\Leftarrow}"] \end{tikzcd}$$

in a 2-category $\mathscr{C}$. We say that R is exhibited as a *left extension* of I along L by the 2-cell η when pasting 2-cells $\sigma : R \longrightarrow M$ with the 2-cell $\eta : I \longrightarrow R \cdot L$ as in the right diagram defines a bijection between 2-cells $R \longrightarrow M$ and 2-cells $I \longrightarrow M \cdot L$. Moreover, we say such a left extension (R, η) is *respected* (also called *preserved*) by a 1-cell $E\colon \mathcal{C} \longrightarrow \mathcal{D}$ when the whiskering of η by E, as given by the pasting diagram below

$$\begin{tikzcd} \mathcal{B} \arrow[r, "R"] & \mathcal{C} \arrow[r, "E"] & \mathcal{D} \\ & \mathcal{A} \arrow[ul, "L"] \arrow[u, "I"'] \arrow[ur, "E \cdot I"'] \arrow[from=1-2, to=2-1, phantom, "\overset{\eta}{\Leftarrow}"] \arrow[from=1-3, to=2-2, phantom, "\overset{\mathrm{id}}{\Leftarrow}"] & \end{tikzcd}$$

exhibits $E \cdot R$ as a left extension of $E \cdot I$ along L.

We now give a suitable description of when a lax T-morphism may be regarded as a left extension in the setting of pseudo T-algebras.

Definition 3.2.10. Suppose we are given an oplax T-morphism (L, α) and lax T-morphisms (R, β) and (I, σ) between pseudo T-algebras equipped with a T-transformation $\eta\colon I \longrightarrow R \cdot L$ as in the diagram

$$\begin{tikzcd} \left(\mathcal{B}, T\mathcal{B} \xrightarrow{y} \mathcal{B}\right) \arrow[r, "(R{,}\beta)"] & \left(\mathcal{C}, T\mathcal{C} \xrightarrow{z} \mathcal{C}\right) \\ & \left(\mathcal{A}, T\mathcal{A} \xrightarrow{x} \mathcal{A}\right) \arrow[ul, "(L{,}\alpha)"] \arrow[u, "(I{,}\sigma)"'] \arrow[from=1-2, to=2-1, phantom, "\overset{\eta}{\Leftarrow}"] \end{tikzcd}$$

We call such a diagram a *T-left extension* if for any given pseudo T-algebra $(\mathcal{D}, w)$, lax T-morphism (M, ε) and oplax T-morphism (N, φ) as below

$$\begin{tikzcd}
& \left(\mathcal{D},T\mathcal{D}\xrightarrow{w}\mathcal{D}\right) \arrow[dr, "{(M,\varepsilon)}"] & \\
\left(\mathcal{B},T\mathcal{B}\xrightarrow{y}\mathcal{B}\right) \arrow[ur, "{(N,\varphi)}"] \arrow[rr, "{(R,\beta)}"'] & \Uparrow\overline{\zeta} & \left(\mathcal{C},T\mathcal{C}\xrightarrow{z}\mathcal{C}\right) \\
& \overset{\eta}{\Leftarrow} & \left(\mathcal{A},T\mathcal{A}\xrightarrow{x}\mathcal{A}\right) \arrow[u, "{(I,\sigma)}"'] \arrow[ull, "{(L,\alpha)}"]
\end{tikzcd}$$

pasting T-transformations of the form $\overline{\zeta}$ above with the T-transformation η defines the bijection of T-transformations:

$$\begin{tikzcd}
(\mathcal{D},w) \arrow[r, "{(M,\varepsilon)}"] & (\mathcal{C},z) & & (\mathcal{D},w) \arrow[r, "{(M,\varepsilon)}"] & (\mathcal{C},z) \\
 & & & (\mathcal{B},y) \arrow[u, "{(N,\varphi)}"] & \\
(\mathcal{B},y) \arrow[uu, "{(N,\varphi)}"] \arrow[r, "{(R,\beta)}"'] \arrow[ur, phantom, "\overset{\overline{\zeta}}{\Leftarrow}"] & (\mathcal{C},z) \arrow[uu, "{(\mathrm{id},\mathrm{id})}"'] & \sim & (\mathcal{A},x) \arrow[u, "{(L,\alpha)}"] \arrow[r, "{(I,\sigma)}"'] \arrow[ur, phantom, "\overset{\zeta}{\Leftarrow}"] & (\mathcal{C},z) \arrow[uu, "{(\mathrm{id},\mathrm{id})}"']
\end{tikzcd}$$

Remark 3.2.11. Note that if $\overline{\zeta}$ and η are both T-transformations then so is the composite $\overline{\zeta}L \cdot \eta$; this is a simple calculation which we omit.

In order to lift left extensions to T-left extensions as above we will require the following algebraic cocompleteness property.

Definition 3.2.12. Given a pseudomonad (T, u, m) on a 2-category $\mathscr{C}$, we say a left extension (H, φ) in $\mathscr{C}$ as on the left below is *T-preserved* by a 1-cell $z \colon T\mathcal{C} \to \mathcal{D}$ when

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "H"] & \mathcal{C} & & T\mathcal{B} \arrow[r, "TH"] & T\mathcal{C} \arrow[r, "z"] & \mathcal{D} \\
\overset{\varphi}{\Leftarrow} & \mathcal{X} \arrow[ul, "G"] \arrow[u, "F"'] & & \overset{T\varphi}{\Leftarrow} & T\mathcal{X} \arrow[ul, "TG"] \arrow[u, "TF"] \arrow[ur, "z\cdot TF"'] & \overset{\mathrm{id}}{\Leftarrow}
\end{tikzcd}$$

the pasting diagram on the right exhibits $(z \cdot TH, z \cdot T\varphi)$ as a left extension.

Remark 3.2.13. Given a pseudo T-algebra $\left(\mathcal{C}, T\mathcal{C} \xrightarrow{z} \mathcal{C}\right)$ if we ask that the underlying object $\mathcal{C}$ is cocomplete in the sense that all left extensions (along a chosen class of maps) into $\mathcal{C}$ exist, and moreover that the algebra structure map z T-preserves these left extensions, then this is (essentially) the notion of algebraic cocompleteness as given by Weber [56, Definition 2.3.1] (except that we are not using pointwise left extensions here). In the setting monoidal categories, this condition of z (when z is an algebra structure map) T-preserving the left extensions is the analogue of asking the tensor product be separately cocontinuous; see [56, Prop. 2.3.2].

We now recall a result for algebraic left extensions mostly due to Koudenburg [32] (though we avoid working in a double categorical setting). We will include some details of the proof as we will need them later.

Proposition 3.2.14. *Suppose we are given a diagram*

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R"] & \mathcal{C} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "I"'] \arrow[ul, phantom, shift right=4, "\overset{\eta}{\Leftarrow}"]
\end{tikzcd}$$

which exhibits R as a left extension in a 2-category $\mathscr{C}$ equipped with a pseudomonad (T,u,m). Suppose further that

$$\left(\mathcal{A}, T\mathcal{A} \xrightarrow{x} \mathcal{A}\right), \quad \left(\mathcal{B}, T\mathcal{B} \xrightarrow{y} \mathcal{B}\right), \quad \left(\mathcal{C}, T\mathcal{C} \xrightarrow{z} \mathcal{C}\right)$$

are pseudo T-algebras. *Suppose even further that the left extension (R,η) is T-preserved by z, and the resulting left extension $(z \cdot TR, z \cdot T\eta)$ is itself T-preserved by z. Then given a lax T*-morphism *structure σ on I and an oplax T*-morphism *structure α on L, there exists a unique lax T*-morphism *structure β on R for which η is a T-transformation. Moreover, this left extension is then lifted to the T-left extension*

$$\begin{tikzcd}
\left(\mathcal{B}, T\mathcal{B} \xrightarrow{y} \mathcal{B}\right) \arrow[r, "(R,\beta)"] & \left(\mathcal{C}, T\mathcal{C} \xrightarrow{z} \mathcal{C}\right) \\
& \left(\mathcal{A}, T\mathcal{A} \xrightarrow{x} \mathcal{A}\right) \arrow[ul, "{(L,\alpha)}"] \arrow[u, "{(I,\sigma)}"'] \arrow[ul, phantom, shift right=4, "\overset{\eta}{\Leftarrow}"]
\end{tikzcd}$$

Proof. Given our structure cells σ and α as below

$$\begin{tikzcd}
T\mathcal{A} \arrow[r, "x"] \arrow[d, "TI"'] \arrow[dr, phantom, "\Uparrow\sigma"] & \mathcal{A} \arrow[d, "I"] \\
T\mathcal{C} \arrow[r, "z"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
T\mathcal{A} \arrow[r, "x"] \arrow[d, "TL"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathcal{A} \arrow[d, "L"] \\
T\mathcal{B} \arrow[r, "y"'] & \mathcal{B}
\end{tikzcd}$$

our lax constraint cell for R is given as the unique β such that η is a T-transformation, that is the unique 2-cell such that

$$\begin{tikzcd}
& T\mathcal{B} \arrow[r, "y"] & \mathcal{B} \arrow[dd, "R"] & & & T\mathcal{B} \arrow[r, "y"] \arrow[dd, "TR"] & \mathcal{B} \arrow[dd, "R"] \\
T\mathcal{A} \arrow[ur, "TL"] \arrow[r, "x"] \arrow[dr, "TI"'] & \mathcal{A} \arrow[ur, "L"'] \arrow[dr, "I"] & {} & = & T\mathcal{A} \arrow[ur, "TL"] \arrow[dr, "TI"'] & {} & {} \\
& T\mathcal{C} \arrow[r, "z"'] & \mathcal{C} & & & T\mathcal{C} \arrow[r, "z"'] & \mathcal{C}
\arrow[from=1-2, to=2-2, phantom, "\Uparrow\alpha"]
\arrow[from=2-2, to=3-2, phantom, "\Uparrow\sigma"]
\arrow[from=2-2, to=2-3, phantom, "\Uparrow\eta"]
\arrow[from=2-5, to=2-6, phantom, "\Uparrow T\eta"]
\arrow[from=2-6, to=2-7, phantom, "\Uparrow\beta"]
\end{tikzcd}$$

as $z \cdot T\eta$ exhibits $z \cdot TR$ as a left extension. From here, the proof of the coherence axioms for β being a lax T-morphism structure on R is the same as in [56, Theorem 2.4.4][2]. Checking that the lax T-morphism (R,β) is then a T-left extension is a straightforward exercise, of which we omit the details. □

3.2.3 KZ Pseudomonads and KZ Doctrines

A KZ pseudomonad is a special type of pseudomonad for which the algebra structure maps are adjoint to units; with typical examples including the cocompletion of a category under some class of colimits Φ. For this paper, we will use two different (but equivalent) characterizations of KZ pseudomonads. The first characterization we will use is a well known algebraic description of a KZ pseudomonad, described via conditions on a "KZ structure cell" (similar to [31]), the second characterization is in terms of left extensions, and will be referred to as a KZ doctrine.

Remark 3.2.15. Note that there are other (still equivalent) characterizations which may be referred to as KZ pseudomonads or KZ doctrines. For example the characterization through adjoint strings [38], or the characterization as lax idempotent pseudomonads [29].

Definition 3.2.16. A *KZ pseudomonad* (P,y,μ) on a 2-category $\mathscr{C}$ consists of a pseudomonad (P,y,μ) on $\mathscr{C}$ along with a modification $\theta\colon Py \longrightarrow yP$ for which

$$\begin{tikzcd}
& & & & & P \arrow[dr, "yP"] & \\
1_{\mathscr{C}} \arrow[r, "y"] & P \arrow[r, bend left, "yP"] \arrow[r, bend right, "Py"'] \arrow[r, phantom, "\Uparrow\theta"] & P^2 & = & 1_{\mathscr{C}} \arrow[ur, "y"] \arrow[dr, "y"'] \arrow[rr, phantom, "\Uparrow y_y"] & & P^2 \\
& & & & & P \arrow[ur, "Py"'] &
\end{tikzcd} \tag{3.2.2}$$

---

[2] The assumptions of [56, Theorem 2.4.4] concerning comma objects are not required for the proof of the coherence axioms.

and

$$\begin{tikzcd}
P \arrow[r, bend left, "yP"] \arrow[r, bend right, "Py"'] \arrow[r, phantom, "\Uparrow\theta"] \arrow[rr, bend left=70, "\mathrm{id}_P"] \arrow[rr, bend right=70, "\mathrm{id}_P"'] & P^2 \arrow[r, "\mu"] \arrow[r, phantom, shift left=20, "\Uparrow\alpha"] \arrow[r, phantom, shift right=20, "\Uparrow\beta"] & P
\end{tikzcd} \quad = \quad \mathrm{id}_{\mathrm{id}_P} \tag{3.2.3}$$

Remark 3.2.17. It is shown in [38, Prop. 3.1, Lemma 3.2] that given the adjoint string characterization we recover the definition given above, and conversely given the above definition it is not hard to recover the adjoint string definition, especially since it suffices to give only one adjunction [38, Theorem 11.1].

The above is an algebraic description of a KZ pseudomonad; however there is another description in terms of left Kan extensions given by Marmolejo and Wood [42] which we refer to as a KZ doctrine.

Definition 3.2.18. [42, Definition 3.1] A *KZ doctrine* (P, y) on a 2-category $\mathscr{C}$ consists of

(i) An assignation on objects $P\colon \mathrm{ob}\mathscr{C} \to \mathrm{ob}\mathscr{C}$;

(ii) For every object $\mathcal{A} \in \mathscr{C}$, a 1-cell $y_{\mathcal{A}}\colon \mathcal{A} \to P\mathcal{A}$;

(iii) For every pair of objects $\mathcal{A}$ and $\mathcal{B}$ and 1-cell $F\colon \mathcal{A} \to P\mathcal{B}$, a left extension

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "\overline{F}"] & P\mathcal{B} \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[ur, "F"'] \arrow[ur, phantom, shift left=8, "\overset{c_F}{\Leftarrow}"]
\end{tikzcd} \tag{3.2.4}$$

of F along $y_{\mathcal{A}}$ exhibited by an isomorphism c_F as above.

Moreover, we require that:

(a) For every object $\mathcal{A} \in \mathscr{C}$, the left extension of $y_{\mathcal{A}}$ as in 3.2.4 is given by

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "\mathrm{id}_{P\mathcal{A}}"] & P\mathcal{A} \\
& \mathcal{A} \arrow[ul, "y_{\mathcal{A}}"] \arrow[u, "y_{\mathcal{A}}"'] \arrow[ul, phantom, shift right=8, "\overset{\mathrm{id}}{\Leftarrow}"]
\end{tikzcd}$$

Note that this means $c_{y_{\mathcal{A}}}$ is equal to the identity 2-cell on $y_{\mathcal{A}}$.

(b) For any 1-cell $G\colon \mathcal{B} \to P\mathcal{C}$, the corresponding left extension $\overline{G}\colon P\mathcal{B} \to P\mathcal{C}$ preserves the left extension $\overline{F}$ in 3.2.4.

Remark 3.2.19. These two descriptions are equivalent in the sense that each gives rise to the other [42, 38]. In Section 3.4 we will express this relationship as a biequivalence between the 2-category of KZ pseudomonads and the preorder of KZ doctrines.

The following definitions in terms of left extensions are equivalent to the preceding notions of pseudo P-algebra and P-homomorphism, in the sense that we have an equivalence between the two resulting 2-categories of pseudo P-algebras arising from the two different definitions [42, Theorems 5.1,5.2].

Definition 3.2.20 ([42])**.** Given a KZ doctrine (P, y) on a 2-category $\mathscr{C}$, we say an object $\mathcal{X} \in \mathscr{C}$ is *P-cocomplete* if for every $G\colon \mathcal{B} \to \mathcal{X}$

$$\begin{tikzcd} P\mathcal{B} \arrow[r, "\overline{G}"] & \mathcal{X} \\ \mathcal{B} \arrow[u, "y_{\mathcal{B}}"] \arrow[ur, "G"'] \arrow[ur, phantom, shift left=3, "\overset{c_G}{\Leftarrow}"] & \end{tikzcd} \qquad \begin{tikzcd} P\mathcal{A} \arrow[r, "\overline{F}"] & P\mathcal{B} \arrow[r, "\overline{G}"] & \mathcal{X} \\ \mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[ur, "F"'] \arrow[ur, phantom, shift left=3, "\overset{c_F}{\Leftarrow}"] & & \end{tikzcd}$$

there exists a left extension $\overline{G}$ as on the left exhibited by an isomorphism c_G, and moreover this left extension respects the left extensions $\overline{F}$ as in the diagram on the right. We say a 1-cell $E\colon \mathcal{X} \to \mathcal{Y}$ between P-cocomplete objects $\mathcal{X}$ and $\mathcal{Y}$ is a *P-homomorphism* (also called *P-cocontinuous*) when it preserves all left extensions along $y_{\mathcal{B}}$ into $\mathcal{X}$ for every object $\mathcal{B}$.

Remark 3.2.21. It is clear that $P\mathcal{A}$ is P-cocomplete for every $\mathcal{A} \in \mathscr{C}$.

We now recall the notion of P-admissibility in the setting of a KZ doctrine P. This notion of admissibility is useful for showing that certain left extensions exist, and moreover are preserved. Note that this notion will be used regularly throughout the paper.

Definition 3.2.22. Given a KZ doctrine (P, y) on a 2-category $\mathscr{C}$, we say a 1-cell $L\colon \mathcal{A} \to \mathcal{B}$ is *P-admissible* if any of the following equivalent conditions are met:

1. In the left diagram below

$$\begin{tikzcd} \mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \\ & \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"'] \arrow[ul, phantom, shift right=3, "\overset{\varphi_L}{\Leftarrow}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \arrow[r, "\overline{H}"] & \mathcal{X} \\ & \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"'] \arrow[ur, "H"'] \arrow[ul, phantom, shift right=3, "\overset{\varphi_L}{\Leftarrow}"] \arrow[ur, phantom, shift left=3, "\overset{c_H}{\Leftarrow}"] & \end{tikzcd}$$

   there exists a left extension (R_L, φ_L) of $y_{\mathcal{A}}$ along L, and moreover the left extension is preserved by any $\overline{H}$ as in the right diagram where $\mathcal{X}$ is P-cocomplete;

2. Every P-cocomplete object $\mathcal{X} \in \mathscr{C}$ admits, and P-homomorphism preserves, left extensions along L. This says that for any given 1-cell $K : \mathcal{A} \to \mathcal{X}$, where $\mathcal{X}$ is P-cocomplete, there exists a 1-cell J and 2-cell δ as in the left diagram below

$$\begin{tikzcd} \mathcal{B} \arrow[r, "J"] & \mathcal{X} \\ & \mathcal{A} \arrow[ul, "L"] \arrow[u, "K"'] \arrow[ul, phantom, shift right=3, "\overset{\delta}{\Leftarrow}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B} \arrow[r, "J"] & \mathcal{X} \arrow[r, "E"] & \mathcal{Y} \\ & \mathcal{A} \arrow[ul, "L"] \arrow[u, "K"'] \arrow[ul, phantom, shift right=3, "\overset{\delta}{\Leftarrow}"] & \end{tikzcd}$$

exhibiting J as a left extension, and moreover this left extension is preserved by any P-homomorphism $E\colon \mathcal{X} \to \mathcal{Y}$ for P-cocomplete $\mathcal{Y}$ as in the right diagram;

3. $PL := \mathrm{lan}_L$ given as the left extension

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "PL"] & P\mathcal{B} \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[r, "L"'] \arrow[ur, phantom, "\overset{c y_{\mathcal{B}} \cdot L}{\Leftarrow}"] & \mathcal{B} \arrow[u, "y_{\mathcal{B}}"']
\end{tikzcd}$$

has a right adjoint.

Remark 3.2.23. For a proof that the descriptions (1), (2) and (3) above are equivalent, we refer the reader to [6] or Chapter 2.

It is well known that pointwise left extensions along fully faithful maps are exhibited by invertible 2-cells; in the following definition we give an analogue of this fact for KZ doctrines.

Definition 3.2.24. Given a KZ doctrine (P, y) on a 2-category $\mathscr{C}$, we say a 1-cell $L\colon \mathcal{A} \to \mathcal{B}$ is *P-fully faithful* if PL is fully faithful.

Remark 3.2.25. The importance of the *P-fully faithful* maps stems from the fact that for a P-admissible map $L\colon \mathcal{A} \to \mathcal{B}$, this L is P-fully faithful if and only if every left extension along L into a P-cocomplete object is exhibited by an isomorphism (see Remark 2.4.7). Clearly each $y_{\mathcal{A}}$ is both P-admissible and P-fully faithful.

For any given KZ doctrine P on a 2-category $\mathscr{C}$ a natural question to ask is: what are the P-cocomplete objects; P-homomorphisms; P-admissible maps and P-fully faithful maps? Let us consider a couple of examples.

Example 3.2.26. *A well known example of a KZ doctrine is the free small cocompletion operation on locally small categories, which sends a locally small category $\mathcal{A}$ to its category of small presheaves. In particular, when $\mathcal{A}$ is small the free small cocompletion is $P\mathcal{A} =$* $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$. *In this example, the P-cocomplete objects are those locally small categories which are small cocomplete and the P-homomorphisms are those functors between such categories preserving small colimits. The P-admissible maps are those functors $L\colon \mathcal{A} \to \mathcal{B}$ for which* $\mathcal{B}(L-,-) : \mathcal{B} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ *factors through $P\mathcal{A}$. Of these P-admissible maps, the P-fully faithful maps are precisely the fully faithful functors.*

Another example is the free *large* cocompletion KZ doctrine on locally small categories. The reader should keep in mind a theorem of Freyd showing that any locally small category which admits all large colimits is a preorder. Consequently, a locally small category is large cocomplete precisely when it is a preorder with all large joins. This KZ doctrine has some unusual properties. For example it is a cocompletion KZ doctrine (in the simple sense that its algebras are described as categories admitting a certain class of colimits) with unit components not always fully faithful. Moreover, every functor is admissible against the large cocompletion. We define this KZ doctrine $P\colon \mathbf{Cat} \to \mathbf{Cat}$ by the assignment

$$P\colon \mathrm{ob}\,\mathbf{Cat} \to \mathrm{ob}\,\mathbf{Cat}\colon \mathcal{A} \mapsto [\mathcal{A}^{\mathrm{op}}, \mathbb{2}]$$

with unit maps for each $\mathcal{A} \in \mathbf{Cat}$ given by

$$y_{\mathcal{A}}\colon \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbb{2}] : X \mapsto \mathcal{A}\langle -, X\rangle$$

with each $\mathcal{A}\langle -, X\rangle$ is defined as

$$\mathcal{A}\langle -, X\rangle : \mathcal{A}^{\mathrm{op}} \to \mathbb{2}\colon S \mapsto \begin{cases} 1, & \exists\, S \xrightarrow{f} X \text{ in } \mathcal{A} \\ 0, & \text{otherwise.} \end{cases}$$

For any functor $F\colon \mathcal{A} \to \mathcal{D}$ where $\mathcal{D}$ is a preordered category with all large joins (such as $P\mathcal{B}$ for any $\mathcal{B}$) we may define a left extension $\overline{F}\colon [\mathcal{A}^{\mathrm{op}}, \mathbb{2}] \to \mathcal{D}$ as in the left diagram

$$\begin{tikzcd}
{[\mathcal{A}^{\mathrm{op}}, \mathbb{2}]} \arrow[r, "\overline{F}"] & \mathcal{D} \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[ur, "F"'] \arrow[ur, phantom, "\overset{\mathrm{id}}{\Leftarrow}", pos=0.3, shift left=3ex] &
\end{tikzcd}
\qquad\qquad
\overline{F}(H) = \sup_{X \in \mathcal{A}\colon HX = 1} FX$$

by the assignment on the right. Hence for this KZ doctrine, the P-cocomplete objects are the large cocomplete categories, and the P-homomorphisms are the order and join preserving maps between such categories. Every map is P-admissible, and it is easily checked that a map $L\colon \mathcal{A} \to \mathcal{B}$ is P-fully faithful precisely when there exists a map $X \to Y$ in $\mathcal{A}$ if and only if there exists a map $LX \to LY$ in $\mathcal{B}$.

Remark 3.2.27. For a set X seen as a discrete category, the large cocompletion of X is $(\mathscr{P}X, \supseteq)$; and dually, the large completion is $(\mathscr{P}X, \subseteq)$, where $\mathscr{P}X$ is the powerset of X.

3.3 Pseudo-Distributive Laws over KZ Doctrines

It was shown by Marmolejo that pseudo-distributive laws of a (co)KZ doctrine over a KZ doctrine have a particularly simple form [39, Definition 11.4]. Here we show that one can give a description which is both simpler (in that less coherence axioms are required) and more general (in that the assumption of the former pseudomonad being (co)KZ may be dropped).

Hence the problem of lifting a cocompletion operation to the 2-category of pseudo algebras may be more easily understood.

Part of the motivation of our method comes from the observation that if a KZ doctrine lifts to a pseudomonad on the 2-category of pseudo algebras, then this pseudomonad is a KZ doctrine automatically[3]. Indeed, this fact means we may consider the problem of lifting a KZ pseudomonad in terms of algebraic left extensions.

In the proof we will make regular use of the admissibility perspective; in fact, the preservation of admissible maps is crucial here, and it is the main goal of this paper to describe such pseudo-distributive laws in terms of this admissibility property.

The proof of these results is quite technical, though the results are summarized in Theorem 3.3.8.

3.3.1 Notions of Pseudo-Distributive Laws

Beck [2] defined a distributive law of a monad (T,u,m) over another monad (P,y,μ) on a category $\mathcal{C}$ to be a natural transformation $\lambda\colon TP \to PT$ rendering commutative the four diagrams

$$\begin{tikzcd}
TP \arrow[r, "\lambda"] & PT \\
& P \arrow[ul, "uP"] \arrow[u, "Pu"'] \arrow[ul, phantom, shift left=3, "="]
\end{tikzcd}
\qquad
\begin{tikzcd}
TP \arrow[r, "\lambda"] & PT \\
& T \arrow[ul, "Ty"] \arrow[u, "yT"'] \arrow[ul, phantom, shift left=3, "="]
\end{tikzcd}$$

$$\begin{tikzcd}
TTP \arrow[r, "T\lambda"] \arrow[d, "mP"'] & TPT \arrow[r, "\lambda T"] \arrow[d, phantom, "="] & PTT \arrow[d, "Pm"] \\
TP \arrow[rr, "\lambda"'] & & PT
\end{tikzcd}
\qquad
\begin{tikzcd}
TPP \arrow[r, "\lambda P"] \arrow[d, "T\mu"'] & PTP \arrow[r, "P\lambda"] \arrow[d, phantom, "="] & PPT \arrow[d, "\mu T"] \\
TP \arrow[rr, "\lambda"'] & & PT
\end{tikzcd}$$

A well known example on **Set** is the canonical distributive law of the monad for monoids over the monad for abelian groups (whose composite is the monad for rings).

More generally, one may talk about a pseudo-distributive law of a pseudomonad over another pseudomonad on a 2-category [39, 26, 48, 8]. In this generalization the four conditions above are replaced by four pieces of data (four invertible modifications) which are then required to satisfy multiple coherence axioms, which we will omit here.

Definition 3.3.1. A *pseudo-distributive law* of a pseudomonad (T,u,m) over a pseudomonad (P,y,μ) on a 2-category $\mathscr{C}$ consists of a pseudonatural transformation $\lambda\colon TP \to PT$, along with four invertible modifications $\omega_1, \omega_2, \omega_3$ and ω_4 in place of the four equalities above. These four modifications are subject to eight coherence axioms; see [41, 39].

---

[3] A fact perhaps most easily seen from the adjoint string definition [38], in view of doctrinal adjunction [27].

As a convention, we choose the direction of these four modifications to be from right to left in the above four diagrams.

In this section, as in the background, we differentiate between “KZ doctrine” defined in terms of left extensions, and “KZ pseudomonad” defined algebraically.

We now define a pseudo-distributive law over such a KZ pseudomonad, though showing this data and these coherence conditions suffice will take some work.

Definition 3.3.2. Suppose we are given a 2-category $\mathscr{C}$ equipped with a pseudomonad (T, u, m) and a KZ pseudomonad (P, y, μ). Then a *pseudo-distributive law over a KZ pseudomonad* $\lambda \colon TP \longrightarrow PT$ consists of a pseudonatural transformation $\lambda \colon TP \longrightarrow PT$ along with three invertible modifications[4]

$$\begin{tikzcd}
TP \arrow[r, "\lambda"] & PT \arrow[d, phantom, "\overset{\omega_1}{\Longleftarrow}" description, shift right=6ex] \\
 & P \arrow[ul, "uP"] \arrow[u, "Pu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
TP \arrow[r, "\lambda"] & PT \arrow[d, phantom, "\overset{\omega_2}{\Longleftarrow}" description, shift right=6ex] \\
 & T \arrow[ul, "Ty"] \arrow[u, "yT"']
\end{tikzcd}
\qquad
\begin{tikzcd}
TTP \arrow[r, "T\lambda"] \arrow[d, "mP"'] & TPT \arrow[r, "\lambda T"] \arrow[d, phantom, "\overset{\omega_3}{\Longleftarrow}" description] & PTT \arrow[d, "Pm"] \\
TP \arrow[rr, "\lambda"'] & & PT
\end{tikzcd}$$

subject to the three coherence axioms:

$$\begin{tikzcd}
TP \arrow[r, "\lambda"] \arrow[d, bend right=40, "TyP"'] \arrow[d, bend left=40, "TPy"] \arrow[d, phantom, "\overset{T\theta}{\Longleftarrow}" description] & PT \arrow[d, "PTy"] \arrow[dr, "PyT"] \arrow[dl, phantom, "\overset{\lambda_y}{\Longleftarrow}" description] \arrow[d, phantom, "\overset{P\omega_2}{\Longleftarrow}" description, shift left=6ex] & & \\
TPP \arrow[r, "\lambda P"'] & PTP \arrow[r, "P\lambda"'] & PPT \arrow[r, "\mu T"'] & PT
\end{tikzcd}
\overset{\text{coh 1}}{=}
\begin{tikzcd}
 & TP \arrow[r, "\lambda"] \arrow[dl, "TyP"'] \arrow[d, "yTP"'] \arrow[d, phantom, "\overset{\omega_2 P}{\Longleftarrow}" description, shift right=6ex] \arrow[dr, phantom, "\overset{y_\lambda^{-1}}{\Longleftarrow}" description] & PT \arrow[d, bend right=40, "yPT"'] \arrow[d, bend left=40, "PyT"] \arrow[d, phantom, "\overset{\theta T}{\Longleftarrow}" description] & \\
TPP \arrow[r, "\lambda P"'] & PTP \arrow[r, "P\lambda"'] & PPT \arrow[r, "\mu T"'] & PT
\end{tikzcd}$$

$$\begin{tikzcd}
P \arrow[r, "uP"] & TP \arrow[r, "\lambda"] & PT \\
1 \arrow[u, "y"] \arrow[r, "u"'] \arrow[ur, phantom, "\overset{u_y}{\Longleftarrow}" description] & T \arrow[u, "Ty"'] \arrow[ur, "yT"'] \arrow[u, phantom, "\overset{\omega_2}{\Longleftarrow}" description, shift left=6ex] &
\end{tikzcd}
\overset{\text{coh 2}}{=}
\begin{tikzcd}
 & TP \arrow[dr, "\lambda"] \arrow[d, phantom, "\Uparrow\omega_1" description] & \\
P \arrow[ur, "uP"] \arrow[rr, "Pu"'] & & PT \\
1 \arrow[u, "y"] \arrow[rr, "u"'] & \overset{y_u^{-1}}{\Longleftarrow} & T \arrow[u, "yT"']
\end{tikzcd}$$

---

[4] Note the direction of the modifications are different in [39]. We use here the direction in which they will naturally arise from left extension and admissiblilty properties. Our direction agrees with that of [49, Section 4].

$$\begin{tikzcd}
TTP \arrow[r, "mP"] & TP \arrow[r, "\lambda"] & PT \\
TT \arrow[u, "T^2y"] \arrow[r, "m"'] \arrow[ur, phantom, "\overset{my}{\Leftarrow}"] & T \arrow[u, "Ty"] \arrow[ur, "yT"'] \arrow[u, phantom, shift left=4ex, "\overset{\omega_2}{\Leftarrow}"] &
\end{tikzcd}
\overset{\text{coh } 3}{=}
\begin{tikzcd}
& & & TP \arrow[drrr, "\lambda"] \arrow[d, phantom, "\Uparrow\omega_3"] & & & \\
TTP \arrow[urrr, "mP"] \arrow[rr, "T\lambda"] & & TPT \arrow[rr, "\lambda T"] & & PTT \arrow[rr, "Pm"] & & PT \\
& & TT \arrow[ull, "T^2y"] \arrow[u, "TyT"'] \arrow[urr, "yT^2"'] \arrow[rr, "m"'] \arrow[ul, phantom, "\overset{T\omega_2}{\Leftarrow}"] \arrow[ur, phantom, "\overset{\omega_2 T}{\Leftarrow}"] & & T \arrow[urr, "yT"'] \arrow[ur, phantom, "\overset{y_m^{-1}}{\Leftarrow}"] & &
\end{tikzcd}$$

Remark 3.3.3. (1) We will see later that ω_1 and ω_3 are uniquely determined by ω_2, due to the last two axioms and left extension properties. (2) Actually, even the naturality cells of λ may be determined given ω_2 and the first coherence axiom. (3) With the 2-cells ω_1 and ω_3 and the last two coherence axioms omitted, we still have sufficient data to lift P to lax T-algebras. (4) These last two axioms may be seen as invertibility conditions on ω_1 and ω_3, analogous to those in [39, Definition 11.4]. (5) During the proof, we will see that each component $\omega_2^{\mathcal{A}}$ necessarily exhibits each component $\lambda_{\mathcal{A}}$ as a left extension. As ω_2 uniquely determines the rest of the data, this will show that such pseudo-distributive laws are essentially unique. (6) In fact, the first coherence axiom above is equivalent to preservation of admissible maps, in the presence of such a pseudonatural transformation λ and invertible modification ω_2.

We will need a notion of separately cocontinuous in the context of KZ doctrines, and so we define the following.

Definition 3.3.4. Suppose we are given a 2-category $\mathscr{C}$ equipped with a pseudomonad (T,u,m) and a KZ doctrine (P,y). We define a 1-cell $z\colon T\mathcal{X} \to \mathcal{C}$ where $\mathcal{X}$ and $\mathcal{C}$ are P-cocomplete objects to be:

1. *T_P-cocontinuous* when every left extension along a unit component $y_{\mathcal{A}}\colon \mathcal{A} \to P\mathcal{A}$ into $\mathcal{X}$ is T-preserved by z;
2. *T_P-adm-cocontinuous* when every left extension along a P-admissible map $L\colon \mathcal{A} \to \mathcal{B}$ into $\mathcal{X}$ is T-preserved by z;

Remark 3.3.5. We will see later in Proposition 3.3.20 that these two notions are equivalent in the presence of a pseudo-distributive law of T over P.

We are now ready to give the definition of a pseudo-distributive law over a KZ doctrine in terms of admissibility and left extensions.

Definition 3.3.6. Suppose we are given a 2-category $\mathscr{C}$ equipped with a pseudomonad (T,u,m) and a KZ doctrine (P,y). Then a *pseudo-distributive law over a KZ doctrine* $\lambda\colon TP \to PT$ consists of the following assertions:

1. T preserves P-admissible maps;

and for every $\mathcal{A} \in \mathscr{C}$,

2. the exhibiting 2-cell $\omega_2^{\mathcal{A}}$ of the left extension $\lambda_{\mathcal{A}}$[5] in

$$\begin{tikzcd}
TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\
& T\mathcal{A} \arrow[ul, "Ty_{\mathcal{A}}"] \arrow[u, "y_{T\mathcal{A}}"'] \arrow[ul, phantom, shift right=3ex, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}"]
\end{tikzcd}$$

is invertible[6];

3. the 1-cell $\lambda_{\mathcal{A}}$ above is T_P-cocontinuous[7];

4. the respective diagrams

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "u_{P\mathcal{A}}"] & TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[r, "u_{\mathcal{A}}"'] \arrow[ur, phantom, "\overset{{}^{u}y_{\mathcal{A}}}{\Leftarrow}"] & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[ur, "y_{T\mathcal{A}}"'] \arrow[ur, phantom, shift left=3ex, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2P\mathcal{A} \arrow[r, "m_{P\mathcal{A}}"] & TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\
T^2\mathcal{A} \arrow[u, "T^2y_{\mathcal{A}}"] \arrow[r, "m_{\mathcal{A}}"'] \arrow[ur, phantom, "\overset{{}^{m}y_{\mathcal{A}}}{\Leftarrow}"] & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[ur, "y_{T\mathcal{A}}"'] \arrow[ur, phantom, shift left=3ex, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}"] &
\end{tikzcd}$$

exhibit both $\lambda_{\mathcal{A}} \cdot u_{P\mathcal{A}}$ and $\lambda_{\mathcal{A}} \cdot m_{P\mathcal{A}}$ as left extensions.

Remark 3.3.7. Note that a pseudo-distributive law as defined above is unique, as it contains only assertions, and these assertions are invariant under the choice of left left extension (unique up to coherent isomorphism).

3.3.2 The Main Theorem

We are now ready to state the main result of this section (and this paper), justifying our definitions above.

Theorem 3.3.8. *Suppose we are given a 2-category $\mathscr{C}$ equipped with a pseudomonad (T, u, m) and a KZ pseudomonad (P, y, μ). Then the following are equivalent:*

(a) P lifts to a KZ doctrine $\widetilde{P}$ on ps-T-alg*;*

(b) P lifts to a KZ pseudomonad $\widetilde{P}$ on ps-T-alg*;*

(c) P lifts to a pseudomonad $\widetilde{P}$ on ps-T-alg*;*

(d) There exists a pseudo-distributive law over a KZ doctrine $\lambda \colon TP \to PT$;

---

[5] The left extension is unique up to coherent isomorphism, and exists since $Ty_{\mathcal{A}}$ is P-admissible.

[6] Equivalently one could ask that each $Ty_{\mathcal{A}}$ is P-fully faithful (see Proposition 2.4.6).

[7] Equivalently one could ask that each $\lambda_{\mathcal{A}}$ is T_P-adm-cocontinuous.

(e) There exists a pseudo-distributive law over a KZ pseudomonad $\lambda\colon TP \to PT$*;*

(f) There exists a pseudo-distributive law $\lambda\colon TP \to PT$.

The proof of this theorem is lengthy, and so we will leave the more difficult aspects of the proof for subsequent subsections. Before moving on to these subsections, we give the remainder of the proof.

Proof of Theorem 3.3.8. In order to prove this theorem, we will complete the cycle of implications

$$\begin{tikzcd}
 & (a) \arrow[r, Rightarrow] & (b) \arrow[dr, Rightarrow] & \\
(d) \arrow[ur, Rightarrow, dotted] & & & (c) \arrow[dl, Rightarrow] \\
 & (e) \arrow[ul, Rightarrow, dotted] & (f) \arrow[l, Rightarrow] &
\end{tikzcd}$$

where the more difficult implications left to later sections are dotted above.

$(a) \implies (b)$: A KZ doctrine gives rise to a pseudomonad whose structure forms a fully faithful adjoint string by [42, Theorem 4.1], and this in turn gives rise to a KZ pseudomonad by [38, Prop. 3.1, Lemma 3.2].

$(b) \implies (c)$: This implication is trivial.

$(c) \implies (f)$: For the correspondence between pseudo-distributive laws and liftings to pseudo T-algebras see [8, Theorem 5.4].

$(f) \implies (e)$: Given a pseudo-distributive law $\lambda\colon TP \to PT$ where P is a KZ pseudomonad, to check that we then have a pseudo-distributive law over a KZ pseudomonad in the sense of Definition 3.3.2 we need only check the first axiom. But this axiom follows from coherences 7 and 8 as given in [39, Section 4] along with the KZ pseudomonad coherence axiom 3.2.3.

$(e) \implies (d)$: This is shown later in Theorem 3.3.17.

$(d) \implies (a)$: This is shown later in Theorem 3.3.21. □

3.3.3 Distributive Laws over KZ Monads to those over KZ Doctrines

We will devote this entire subsection to showing that a pseudo-distributive law over a KZ pseudomonad, as in Definition 3.3.2, gives rise to a pseudo-distributive law over a KZ doctrine, as in Definition 3.3.6. This is $(e) \implies (d)$ of Theorem 3.3.8. As this is the most difficult implication to show, we will break the proof up into a number of propositions and lemmata, starting with the following proposition.

Note for reader. During this subsection and the next, the reader will keep the three equivalent characterizations of P-admissible maps (given in Definition 3.2.22) in mind.

Indeed, all three characterizations are to be used repeatedly throughout these two subsections.

Remark 3.3.9. Most of our diagrams are constructed from the following 2-cells, where P is a KZ doctrine and T a pseudomonad on a bicategory $\mathscr{C}$:

1. As noted in Definition 3.2.22, for any P-admissible 1-cell $L\colon \mathcal{A} \to \mathcal{B}$ we have a left extension (R_L, φ_L) of $y_{\mathcal{A}}$ along L. In particular if $L = Ty_{\mathcal{A}}$ is P-admissible, we will denote this left extension by $\left(\lambda_{\mathcal{A}}, \omega_2^{\mathcal{A}}\right)$. Moreover, by Remark 2.3.7, if we are given a chosen right adjoint res_L to PL, then the canonical way to define (R_L, φ_L) is by

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"'] \arrow[ul, phantom, shift right=3, "\overset{\varphi_L}{\Leftarrow}"]
\end{tikzcd}
:=
\begin{tikzcd}
\mathcal{B} \arrow[r, "y_{\mathcal{B}}"] & P\mathcal{B} \arrow[r, "\mathrm{res}_L"] & P\mathcal{A} \\
& \overset{y_L}{\Leftarrow} & P\mathcal{A} \arrow[ul, bend left, "\mathrm{lan}_L"] \arrow[u, "\mathrm{id}_{P\mathcal{A}}"'] \arrow[ul, phantom, shift left=3, "\overset{\eta}{\Leftarrow}"] \\
& & \mathcal{A} \arrow[uull, bend left, "L"] \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}$$

2. As noted in Definition 3.2.18, for any 1-cell $F\colon \mathcal{A} \to P\mathcal{B}$ we have a left extension $\left(\overline{F}, c_F\right)$ of F along $y_{\mathcal{A}}$ with c_F invertible. If $F = R_L$ for a P-admissible L, we will denote this left extension by $\left(\mathrm{res}_L, c_{R_L}\right)$, and note that res_L defined this way is right adjoint to PL (see Lemma 2.3.4).

Proposition 3.3.10. *Suppose we are given a 2-category $\mathscr{C}$ equipped with a pseudomonad (T, u, m) and a KZ doctrine (P, y). Further suppose that for each object $\mathcal{A} \in \mathscr{C}$, $Ty_{\mathcal{A}}$ is P-admissible, and the left extension*[8] *which we denote $\lambda_{\mathcal{A}}$ in*

$$\begin{tikzcd}
TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\
& T\mathcal{A} \arrow[ul, "Ty_{\mathcal{A}}"] \arrow[u, "y_{T\mathcal{A}}"'] \arrow[ul, phantom, shift right=3, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}"]
\end{tikzcd}$$

is exhibited by an isomorphism denoted $\omega_2^{\mathcal{A}}$. Then for every P-admissible 1-cell $L\colon \mathcal{A} \to \mathcal{B}$ such that $TL\colon T\mathcal{A} \to T\mathcal{B}$ is also P-admissible, the respective pastings

$$\begin{tikzcd}
PT\mathcal{A} & TP\mathcal{A} \arrow[l, "\lambda_{\mathcal{A}}"'] & TP\mathcal{B} \arrow[l, "T\mathrm{res}_L"'] \\
& T\mathcal{A} \arrow[ul, "y_{T\mathcal{A}}"] \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "TL"'] \arrow[ul, phantom, shift left=3, "\overset{\omega_2^{\mathcal{A}}}{\Rightarrow}"] \arrow[r, phantom, shift left=4, "\overset{T\varphi_L}{\Rightarrow}"] & T\mathcal{B} \arrow[ul, "TR_L"] \arrow[u, "Ty_{\mathcal{B}}"'] \arrow[ul, phantom, shift right=3, "\overset{Tc_{R_L}}{\Rightarrow}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
PT\mathcal{A} & PT\mathcal{B} \arrow[l, "\mathrm{res}_{TL}"'] & TP\mathcal{B} \arrow[l, "\lambda_{\mathcal{B}}"'] \\
& T\mathcal{A} \arrow[ul, "y_{T\mathcal{A}}"] \arrow[r, "TL"'] \arrow[r, phantom, shift left=4, "\overset{\varphi_{TL}}{\Rightarrow}"] & T\mathcal{B} \arrow[ull, "R_{TL}"] \arrow[ul, "y_{T\mathcal{B}}"] \arrow[u, "Ty_{\mathcal{B}}"'] \arrow[ul, phantom, shift right=3, "\overset{\omega_2^{\mathcal{B}}}{\Rightarrow}"] \arrow[ull, phantom, shift left=3, "\overset{c_{R_{TL}}}{\Rightarrow}"]
\end{tikzcd}
\tag{3.3.1}$$

exhibit $\lambda_{\mathcal{A}} \cdot T\mathrm{res}_L$ and $\mathrm{res}_{TL} \cdot \lambda_{\mathcal{B}}$ as left extensions of $y_{T\mathcal{A}}$ along $Ty_{\mathcal{B}} \cdot TL$; yielding an

---

[8] This left extension exists since $Ty_{\mathcal{A}}$ is P-admissible.

isomorphism of left extensions:

$$\begin{tikzcd}
TP\mathcal{B} \arrow[r, "\lambda_{\mathcal{B}}"] \arrow[d, "T\mathrm{res}_L"'] \arrow[dr, phantom, "\Uparrow \gamma_L"] & PT\mathcal{B} \arrow[d, "\mathrm{res}_{TL}"] \\
TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"'] & PT\mathcal{A}
\end{tikzcd}$$

Moreover, if the left diagram below exhibits R_L as a left extension

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"'] \arrow[ul, phantom, shift left=3, "\overset{\varphi_L}{\Leftarrow}"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T\mathcal{B} \arrow[r, "TR_L"] & TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\
& T\mathcal{A} \arrow[ul, "TL"] \arrow[u, "Ty_{\mathcal{A}}"] \arrow[ur, "y_{T\mathcal{A}}"'] \arrow[ul, phantom, shift left=3, "\overset{T\varphi_L}{\Leftarrow}"] \arrow[ur, phantom, shift right=3, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}"]
\end{tikzcd}$$

then the right diagram exhibits $\lambda_{\mathcal{A}} \cdot TR_L$ as a left extension.

Proof. Firstly, we consider the diagram

$$\begin{tikzcd}
PT\mathcal{A} & TP\mathcal{A} \arrow[l, "\lambda_{\mathcal{A}}"'] & TP\mathcal{B} \arrow[l, "T\mathrm{res}_L"'] \\
& T\mathcal{A} \arrow[ul, "y_{T\mathcal{A}}"] \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "TL"'] \arrow[ul, phantom, shift right=3, "\overset{\omega_2^{\mathcal{A}}}{\Rightarrow}"] \arrow[r, phantom, shift left=5, "\overset{T\varphi_L}{\Rightarrow}"] & T\mathcal{B} \arrow[ul, "TR_L"] \arrow[u, "Ty_{\mathcal{B}}"'] \arrow[u, phantom, shift left=5, "\overset{Tc_{R_L}}{\Rightarrow}"]
\end{tikzcd}$$

and note that $\lambda_{\mathcal{A}} \cdot T\mathrm{res}_L$ is a left extension since for any 1-cell $H \colon TP\mathcal{B} \to PT\mathcal{A}$ we have the natural bijections

$$\begin{array}{rcl}
\lambda_{\mathcal{A}} \cdot T\mathrm{res}_L & \longrightarrow & H \\
\hline
\lambda_{\mathcal{A}} & \longrightarrow & H \cdot T\mathrm{lan}_L \\
\hline
y_{T\mathcal{A}} & \longrightarrow & H \cdot T\mathrm{lan}_L \cdot Ty_{\mathcal{A}} \\
\hline
y_{T\mathcal{A}} & \longrightarrow & H \cdot Ty_{\mathcal{B}} \cdot TL
\end{array}$$

mates correspondence
since $\lambda_{\mathcal{A}}$ is a left extension
$PL \cdot y_{\mathcal{A}} \cong y_{\mathcal{B}} \cdot L$

and one may check this is the correct exhibiting 2-cell using Remark 2.3.7. We may also consider the diagram

$$\begin{tikzcd}
PT\mathcal{A} & PT\mathcal{B} \arrow[l, "\mathrm{res}_{TL}"'] & TP\mathcal{B} \arrow[l, "\lambda_{\mathcal{B}}"'] \\
& T\mathcal{A} \arrow[ul, "y_{T\mathcal{A}}"] \arrow[r, "TL"'] \arrow[r, phantom, shift left=5, "\overset{\varphi_{TL}}{\Rightarrow}"] & T\mathcal{B} \arrow[ull, "R_{TL}" description] \arrow[ul, "y_{T\mathcal{B}}" description] \arrow[u, "Ty_{\mathcal{B}}"'] \arrow[ull, phantom, shift right=4, "\overset{c_{R_{TL}}}{\Rightarrow}"] \arrow[u, phantom, shift left=5, "\overset{\omega_2^{\mathcal{B}}}{\Rightarrow}"]
\end{tikzcd}$$

and note that since $Ty_{\mathcal{B}}$ is P-admissible the left extension $\lambda_{\mathcal{B}}$ is preserved by res_{TL}. Noting $c_{R_{TL}}$ is invertible, we then apply the pasting lemma for left extensions (the dual of [47, Prop. 1]) to see the outside diagram exhibits $\mathrm{res}_{TL} \cdot \lambda_{\mathcal{B}}$ as a left extension. By uniqueness of left extensions, we derive our desired isomorphism $\gamma_L \colon \lambda_{\mathcal{A}} \cdot T\mathrm{res}_L \cong \mathrm{res}_{TL} \cdot \lambda_{\mathcal{B}}$. Now, to show that

$$\begin{tikzcd}
T\mathcal{B} \arrow[r, "TR_L"] & TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\
& T\mathcal{A} \arrow[ul, "TL"] \arrow[u, "Ty_{\mathcal{A}}"] \arrow[ur, "y_{T\mathcal{A}}"'] \arrow[ul, phantom, shift left=4, "\overset{T\varphi_L}{\Leftarrow}"] \arrow[ur, phantom, shift right=4, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}"] &
\end{tikzcd} \tag{3.3.2}$$

exhibits $\lambda_{\mathcal{A}} \cdot TR_L$ as a left extension, it suffices to show that we have an isomorphism $\lambda_{\mathcal{A}} \cdot TR_L \cong R_{TL}$ and that pasting the left extension (R_{TL}, φ_{TL}) with this isomorphism yields the above. This is the case since all regions in the following diagram commute up to isomorphism

$$\begin{tikzcd}
& & TP\mathcal{A} \arrow[drr, "\lambda_{\mathcal{A}}"] \arrow[dr, phantom, "\cong\gamma_L"] & & \\
T\mathcal{B} \arrow[urr, bend left, "TR_L"] \arrow[r, "Ty_{\mathcal{B}}"] \arrow[rrr, bend right, "y_{T\mathcal{B}}"'] \arrow[rrrr, bend right=40, "R_{TL}"'] \arrow[ur, phantom, "\cong Tc_{R_L}"] \arrow[rr, phantom, shift right=5, "\cong\omega_2^{\mathcal{B}}"] \arrow[rrrr, phantom, shift right=8, "\cong c_{R_{TL}}"] & TP\mathcal{B} \arrow[ur, "T\mathrm{res}_L"] \arrow[rr, "\lambda_{\mathcal{B}}"] & & PT\mathcal{B} \arrow[r, "\mathrm{res}_{TL}"] & PT\mathcal{A}
\end{tikzcd}$$

and it is easy to check φ_{TL} pasted with this isomorphism yields the pasting 3.3.2 if one uses the definition of γ_L. ☐

Remark 3.3.11. Note that the above proposition tells us something about the components of λ being separately cocontinuous, without any assumptions on pseudonaturality of λ. This may seem unusual in view of the following lemma, in which we show pseudonaturality of λ is precisely equivalent to the T_P-cocontinuity of its components.

Lemma 3.3.12. *Suppose we are given a 2-category $\mathscr{C}$ equipped with a pseudomonad (T, u, m) and a KZ doctrine (P, y). Further suppose that for each object $\mathcal{A} \in \mathscr{C}$, $Ty_{\mathcal{A}}$ is P-admissible and the left extension which we call $\lambda_{\mathcal{A}}$ as on the left below*

$$\begin{tikzcd}
TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\
& T\mathcal{A} \arrow[ul, "Ty_{\mathcal{A}}"] \arrow[u, "y_{T\mathcal{A}}"'] \arrow[ul, phantom, shift right=3, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
TP\mathcal{B} \arrow[r, "\lambda_{\mathcal{B}}"] & PT\mathcal{B} \\
TP\mathcal{A} \arrow[u, "TPL"] \arrow[r, "\lambda_{\mathcal{A}}"'] \arrow[ur, phantom, "\Uparrow\lambda_L"] & PT\mathcal{A} \arrow[u, "PTL"']
\end{tikzcd}$$

is exhibited by an isomorphism $\omega_2^{\mathcal{A}}$. Then for all $L \colon \mathcal{A} \to \mathcal{B}$ the naturality squares for λ as *on the right above commute up to a coherent isomorphism λ_L, with coherent meaning*

$$\begin{tikzcd}
& TP\mathcal{B} \arrow[dr, "\lambda_{\mathcal{B}}"] & & & & & \\
TP\mathcal{A} \arrow[ur, "TPL"] \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \arrow[r, "PTL"] \arrow[u, phantom, "\Uparrow\lambda_L"] & PT\mathcal{B} & = & TP\mathcal{A} \arrow[r, "TPL"] & TP\mathcal{B} \arrow[r, "\lambda_{\mathcal{B}}"] & PT\mathcal{B} \\
& T\mathcal{A} \arrow[ul, "Ty_{\mathcal{A}}"] \arrow[u, "y_{T\mathcal{A}}"'] \arrow[r, "TL"'] \arrow[ul, phantom, shift right=4, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}"] \arrow[ur, phantom, "\overset{y_{TL}^{-1}}{\Leftarrow}"] & T\mathcal{B} \arrow[u, "y_{T\mathcal{B}}"'] & & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "TL"'] \arrow[ur, phantom, "\overset{Ty_L^{-1}}{\Leftarrow}"] & T\mathcal{B} \arrow[u, "Ty_{\mathcal{B}}"'] \arrow[ur, "y_{T\mathcal{B}}"'] \arrow[ur, phantom, shift left=4, "\overset{\omega_2^{\mathcal{B}}}{\Leftarrow}"] &
\end{tikzcd}$$

(the condition for ω_2 to be a modification), if and only if each $\lambda_{\mathcal{A}}$ is T_P-cocontinuous.

Proof. The following implications prove the logical equivalence.

$(\Longrightarrow)$: Suppose that for each $L\colon \mathcal{A} \to \mathcal{B}$ the naturality square of λ commutes up to a coherent isomorphism λ_L. Then noting that $\mathrm{id}_{P\mathcal{B}} = R_{y_{\mathcal{B}}}$, we see that for any left extension as on the left (which is isomorphic to $\left(\overline{F}, c_F\right)$ by uniqueness)

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "PF"] & P^2\mathcal{B} \arrow[r, "\mathrm{res}_{y_{\mathcal{B}}}"] & P\mathcal{B} & & TP\mathcal{A} \arrow[r, "TPF"] & TP^2\mathcal{B} \arrow[r, "T\mathrm{res}_{y_{\mathcal{B}}}"] & TP\mathcal{B} \arrow[r, "\lambda_{\mathcal{B}}"] & PT\mathcal{B} \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[r, "F"'] \arrow[ur, phantom, "\overset{y_F^{-1}}{\Leftarrow}"] & P\mathcal{B} \arrow[u, "y_{P\mathcal{B}}"'] \arrow[ur, "\mathrm{id}_{P\mathcal{B}}"'] \arrow[ur, phantom, shift left=4, "\overset{c_{\mathrm{id}_{P\mathcal{B}}}}{\Leftarrow}"] & & & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "TF"'] \arrow[ur, phantom, "\overset{Ty_F^{-1}}{\Leftarrow}"] & TP\mathcal{B} \arrow[u, "Ty_{P\mathcal{B}}"'] \arrow[ur, "T\mathrm{id}_{P\mathcal{B}}"'] \arrow[ur, phantom, shift left=4, "\overset{Tc_{\mathrm{id}_{P\mathcal{B}}}}{\Leftarrow}"] & &
\end{tikzcd}$$

it suffices to check that the right diagram above exhibits $\lambda_{\mathcal{B}} \cdot T\mathrm{res}_{y_{\mathcal{B}}} \cdot TPF$ as a left extension. To see this we note that the pasting

$$\begin{tikzcd}
& PT\mathcal{A} \arrow[r, "PTF"] & PTP\mathcal{B} \arrow[dr, "\mathrm{res}_{Ty_{\mathcal{B}}}"] & \\
TP\mathcal{A} \arrow[ur, "\lambda_{\mathcal{A}}"] \arrow[r, "TPF"] & TP^2\mathcal{B} \arrow[ur, "\lambda_{P\mathcal{B}}"] \arrow[r, "T\mathrm{res}_{y_{\mathcal{B}}}"] \arrow[u, phantom, "\Uparrow\lambda_F^{-1}"] & TP\mathcal{B} \arrow[r, "\lambda_{\mathcal{B}}"] \arrow[u, phantom, "\Uparrow\gamma_{y_{\mathcal{B}}}"] & PT\mathcal{B} \\
T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "TF"'] \arrow[ur, phantom, "\overset{Ty_F^{-1}}{\Leftarrow}"] & TP\mathcal{B} \arrow[u, "Ty_{P\mathcal{B}}"'] \arrow[ur, "\mathrm{id}_{TP\mathcal{B}}"'] \arrow[ur, phantom, shift left=4, "\overset{Tc_{\mathrm{id}_{P\mathcal{B}}}}{\Leftarrow}"] & &
\end{tikzcd}$$

is equal to the pasting (using $\lambda_{\mathcal{B}} = R_{Ty_{\mathcal{B}}}$)

$$\begin{tikzcd}
TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \arrow[r, "PTF"] & PTP\mathcal{B} \arrow[r, "\mathrm{res}_{Ty_{\mathcal{B}}}"] & PT\mathcal{B} \\
& T\mathcal{A} \arrow[ul, "Ty_{\mathcal{A}}"] \arrow[u, "y_{T\mathcal{A}}"'] \arrow[r, "TF"'] \arrow[ul, phantom, shift right=4, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}"] \arrow[ur, phantom, "\overset{y_{TF}^{-1}}{\Leftarrow}"] & TP\mathcal{B} \arrow[u, "y_{TP\mathcal{B}}"'] \arrow[ur, "\lambda_{\mathcal{B}}"'] \arrow[ur, phantom, shift left=4, "\overset{c_{\lambda_{\mathcal{B}}}}{\Leftarrow}"] &
\end{tikzcd}$$

This is shown by first using the coherence condition on λ_F^{-1}, and then using the definition of $\gamma_{y_{\mathcal{B}}}$ from Proposition 3.3.10. Note also this last diagram exhibits $\mathrm{res}_{Ty_{\mathcal{B}}} \cdot PTF \cdot \lambda_{\mathcal{A}}$ as a left extension since $Ty_{\mathcal{A}}$ is P-admissible (using preservation of the left extension $\lambda_{\mathcal{A}}$ by

P-homomorphisms).

$(\Longleftarrow)$: For any $L\colon \mathcal{A} \to \mathcal{B}$, we know $PTL \cdot \lambda_{\mathcal{A}}$ is a left extension of $y_{T\mathcal{B}} \cdot TL$ along $Ty_{\mathcal{A}}$ since $Ty_{\mathcal{A}}$ is P-admissible. Also $\lambda_{\mathcal{B}} \cdot TPL$ is such a left extension as $\lambda_{\mathcal{B}}$ is T_P-cocontinuous, giving us an isomorphism of left extensions λ_F coherent as in the statement of this lemma. □

Remark 3.3.13. A Beck condition is satisfied here. Indeed, the 2-cell γ_L as in Proposition 3.3.10 is the mate of λ_L as in the above lemma. This may be seen by pasting the left diagram of 3.3.1 with the mate of λ_L and then recovering the right diagram making use of Remark 2.3.7 and the coherence condition on λ_L).

In the following lemma we see that for a pseudo-distributive law over a KZ pseudomonad as in Definition 3.3.2, the modification components $\omega_2^{\mathcal{A}}$ necessarily exhibit each $\lambda_{\mathcal{A}}$ as a left extension, and from this we deduce the existence of invertible components $\omega_4^{\mathcal{A}}$.

Lemma 3.3.14. *Suppose we are given a 2-category $\mathscr{C}$ equipped with a pseudomonad (T, u, m) and a KZ pseudomonad (P, y, μ). Suppose further that we are given a pseudo-distributive law over a KZ pseudomonad $\lambda\colon TP \to PT$. Then for each $\mathcal{A} \in \mathscr{C}$, $Ty_{\mathcal{A}}$ is P-admissible, exhibited by an adjunction*

$$PTy_{\mathcal{A}} \dashv \mu_{T\mathcal{A}} \cdot P\lambda_{\mathcal{A}}$$

Moreover, the diagrams as on the left exhibit each $\lambda_{\mathcal{A}}$ as a left extension,

$$\begin{tikzcd} TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\ & T\mathcal{A} \arrow[u, "y_{T\mathcal{A}}"'] \arrow[ul, "Ty_{\mathcal{A}}"] \arrow[ul, phantom, shift left=4, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}"] \end{tikzcd} \qquad \begin{tikzcd} TPP\mathcal{A} \arrow[r, "\lambda_{P\mathcal{A}}"] \arrow[d, "T\mu_{\mathcal{A}}"'] & PTP\mathcal{A} \arrow[r, "P\lambda_{\mathcal{A}}"] \arrow[d, phantom, "\overset{\omega_4^{\mathcal{A}}}{\Leftarrow}"] & PPT\mathcal{A} \arrow[d, "\mu_{T\mathcal{A}}"] \\ TP\mathcal{A} \arrow[rr, "\lambda_{\mathcal{A}}"'] & & PT\mathcal{A} \end{tikzcd}$$

and there exists canonical isomorphisms as on the right for each $\mathcal{A}$.

Proof. We now prove the three assertions of the above lemma.

EACH $Ty_{\mathcal{A}}$ IS P-ADMISSIBLE. Firstly, we note that the below diagram exhibits $\mu_{T\mathcal{A}} \cdot P\lambda_{\mathcal{A}}$ as a left extension

$$\begin{tikzcd} PTP\mathcal{A} \arrow[r, "P\lambda_{\mathcal{A}}"] & P^2T\mathcal{A} \arrow[r, "\mu_{T\mathcal{A}}"] & PT\mathcal{A} \\ TP\mathcal{A} \arrow[u, "y_{TP\mathcal{A}}"] \arrow[r, "\lambda_{\mathcal{A}}"'] \arrow[ur, phantom, "\overset{y^{-1}_{\lambda_{\mathcal{A}}}}{\Leftarrow}"] & PT\mathcal{A} \arrow[u, "y_{PT\mathcal{A}}"] \arrow[ur, "\mathrm{id}_{PT\mathcal{A}}"'] \arrow[ur, phantom, shift left=3, "\cong"] & \end{tikzcd} \tag{3.3.3}$$

Indeed, the construction of a KZ doctrine from a pseudomonad whose structure forms a fully faithful adjoint string is outlined in [42], and the above is an instance of this construction. Noting $\overline{\lambda_{\mathcal{A}}} = \mu_{T\mathcal{A}} \cdot P\lambda_{\mathcal{A}}$, we define our unit η as the unique solution to the left extension problem

$$\begin{tikzcd}
& PTP\mathcal{A} \arrow[dr, "\overline{\lambda_{\mathcal{A}}}"] \arrow[d, phantom, "\Uparrow\eta"] & \\
PT\mathcal{A} \arrow[ur, "PTy_{\mathcal{A}}"] \arrow[rr, "\mathrm{id}"'] & {} & PT\mathcal{A} \\
T\mathcal{A} \arrow[u, "y_{T\mathcal{A}}"] \arrow[urr, bend right, "y_{T\mathcal{A}}"'] & \Uparrow\mathrm{id} &
\end{tikzcd}
=
\begin{tikzcd}
PT\mathcal{A} \arrow[r, "PTy_{\mathcal{A}}"] & PTP\mathcal{A} \arrow[r, "P\lambda_{\mathcal{A}}"] & P^2T\mathcal{A} \arrow[r, "\mu_{T\mathcal{A}}"] & PT\mathcal{A} \\
T\mathcal{A} \arrow[u, "y_{T\mathcal{A}}"] \arrow[r, "Ty_{\mathcal{A}}"'] \arrow[urr, phantom, "\Uparrow y^{-1}_{Ty_{\mathcal{A}}}" near start] \arrow[rr, bend right, "y_{T\mathcal{A}}"', "\Uparrow\omega_2^{\mathcal{A}}"] & TP\mathcal{A} \arrow[u, "y_{TP\mathcal{A}}"] \arrow[r, "\lambda_{\mathcal{A}}"'] \arrow[ur, phantom, "\Uparrow y^{-1}_{\lambda_{\mathcal{A}}}"] & PT\mathcal{A} \arrow[u, "y_{PT\mathcal{A}}"] \arrow[ur, "\mathrm{id}"'] \arrow[ur, phantom, "\cong" near start] &
\end{tikzcd}$$

Note that the unit η must then be given by

$$\begin{tikzcd}
PT\mathcal{A} \arrow[r, "PTy_{\mathcal{A}}"] \arrow[rr, bend left, "Py_{T\mathcal{A}}", "\Downarrow P\omega_2^{\mathcal{A}}"'] \arrow[rrr, bend left=50, "\mathrm{id}_{PT\mathcal{A}}", "\cong"'] & PTP\mathcal{A} \arrow[r, "P\lambda_{\mathcal{A}}"] & P^2T\mathcal{A} \arrow[r, "\mu_{T\mathcal{A}}"] & PT\mathcal{A}
\end{tikzcd}$$

as $y\colon 1 \longrightarrow P$ is a pseudonatural transformation. We define our counit ε as the unique solution to the left extension problem

$$\begin{tikzcd}
PTP\mathcal{A} \arrow[r, "\overline{\lambda_{\mathcal{A}}}"] \arrow[rr, bend left, "\mathrm{id}_{PTP\mathcal{A}}", "\Uparrow\varepsilon"'] & PT\mathcal{A} \arrow[r, "PTy_{\mathcal{A}}"] & PTP\mathcal{A} \\
TPA \arrow[u, "y_{TP\mathcal{A}}"] \arrow[ur, bend right, "\lambda_{\mathcal{A}}"', "\cong"] & &
\end{tikzcd}
=
\begin{tikzcd}
TP\mathcal{A} \arrow[r, bend left, "Ty_{P\mathcal{A}}"] \arrow[r, bend right, "TPy_{\mathcal{A}}"'] \arrow[r, phantom, "\Uparrow T\theta"] \arrow[rr, bend left=60, "y_{TP\mathcal{A}}", "\Uparrow(\omega_2^{P\mathcal{A}})^{-1}"'] \arrow[dr, bend right, "\lambda_{\mathcal{A}}"'] & TP^2\mathcal{A} \arrow[r, "\lambda_{P\mathcal{A}}"] \arrow[d, phantom, "\Uparrow\lambda_{y_{\mathcal{A}}}"] & PTP\mathcal{A} \\
& PTA \arrow[ur, bend right, "PTy_{\mathcal{A}}"'] &
\end{tikzcd}$$

where the unnamed isomorphism above is 3.3.3. One could also define ε directly in terms of θ, but that would result in a more complicated proof. Of the triangle identities:

$$\begin{tikzcd}
PTy_{\mathcal{A}} \arrow[r, "PTy_{\mathcal{A}}\cdot\eta"] \arrow[dr, "\mathrm{id}_{PTy_{\mathcal{A}}}"'] & PTy_{\mathcal{A}}\cdot\overline{\lambda_{\mathcal{A}}}\cdot PTy_{\mathcal{A}} \arrow[d, "\varepsilon\cdot PTy_{\mathcal{A}}"] \\
& PTy_{\mathcal{A}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\lambda_{\mathcal{A}}} \arrow[r, "\eta\cdot\overline{\lambda_{\mathcal{A}}}"] \arrow[dr, "\mathrm{id}_{\overline{\lambda_{\mathcal{A}}}}"'] & \overline{\lambda_{\mathcal{A}}}\cdot PTy_{\mathcal{A}}\cdot\overline{\lambda_{\mathcal{A}}} \arrow[d, "\overline{\lambda_{\mathcal{A}}}\cdot\varepsilon"] \\
& \overline{\lambda_{\mathcal{A}}}
\end{tikzcd}$$

the left identity, which is equivalent to asking for equality when whiskered by $y_{T\mathcal{A}}$, can be proven using that ω_2 is a modification. The right triangle identity, which is equivalent to asking for equality when whiskered by $y_{TP\mathcal{A}}$, amounts to asking that the pasting

$$\begin{tikzcd}
& & & PT\mathcal{A} \arrow[dr, "y_{PT\mathcal{A}}"] \arrow[drr, bend left, "\mathrm{id}_{PT\mathcal{A}}"] \arrow[dl, phantom, "\Uparrow y_{\lambda_{\mathcal{A}}}"] \arrow[drr, phantom, "\cong"] & & \\
TP\mathcal{A} \arrow[urrr, bend left, "\lambda_{\mathcal{A}}"] \arrow[rr, bend left=60, "y_{TP\mathcal{A}}"] \arrow[r, bend left, "Ty_{P\mathcal{A}}"] \arrow[r, bend right, "TPy_{\mathcal{A}}"'] \arrow[r, phantom, "\Uparrow T\theta^{\mathcal{A}}"] \arrow[dr, bend right, "\lambda_{\mathcal{A}}"'] \arrow[rr, phantom, bend left=40, "\Uparrow(\omega_2^{P\mathcal{A}})^{-1}"] & TP^2\mathcal{A} \arrow[r, "\lambda_{P\mathcal{A}}"] \arrow[d, phantom, "\Uparrow\lambda_{y_{\mathcal{A}}}"] & PTP\mathcal{A} \arrow[r, "P\lambda_{\mathcal{A}}"] \arrow[dr, phantom, "\Uparrow P\omega_2^{\mathcal{A}}"] & P^2T\mathcal{A} \arrow[r, "\mu_{T\mathcal{A}}"] \arrow[dr, phantom, "\cong"] & PT\mathcal{A} \\
& PT\mathcal{A} \arrow[ur, bend right, "PTy_{\mathcal{A}}"'] \arrow[urr, bend right, "Py_{T\mathcal{A}}"] \arrow[urrr, bend right=40, "\mathrm{id}_{PT\mathcal{A}}"'] & & &
\end{tikzcd}$$

is the identity. This is where the first axiom for a pseudo-distributive law over a KZ pseudomonad is used, in addition to the second coherence axiom 3.2.3 of a KZ pseudomonad.

Each $\omega_2^{\mathcal{A}}$ exhibits $\lambda_{\mathcal{A}}$ as a left extension. As $Ty_{\mathcal{A}}$ is P-admissible, we know by Remark 2.3.7 that the pasting

$$\begin{tikzcd}
TP\mathcal{A} \arrow[r, "y_{TP\mathcal{A}}"] & PTP\mathcal{A} \arrow[r, "\mathrm{res}_{Ty_{\mathcal{A}}}"] \arrow[dr, phantom, "\overset{\eta\cdot y_{T\mathcal{A}}}{\Leftarrow}"] & PT\mathcal{A} \\
\arrow[r, phantom, "\overset{y_{Ty_{\mathcal{A}}}}{\Leftarrow}"] & PT\mathcal{A} \arrow[u, "PTy_{\mathcal{A}}"] & \\
& & T\mathcal{A} \arrow[ul, "y_{T\mathcal{A}}"] \arrow[uu, "y_{T\mathcal{A}}"'] \arrow[uull, bend left, "Ty_{\mathcal{A}}"]
\end{tikzcd}$$

exhibits $\mathrm{res}_{Ty_{\mathcal{A}}} \cdot y_{TP\mathcal{A}}$ as a left extension, where $\mathrm{res}_{Ty_{\mathcal{A}}} = \overline{\lambda_{\mathcal{A}}} = \mu_{T\mathcal{A}} \cdot P\lambda_{\mathcal{A}}$, and η is the unit of $PTy_{\mathcal{A}} \dashv \mathrm{res}_{Ty_{\mathcal{A}}}$ as just defined. From a substitution of the definition of η (and pasting with a couple of isomorphisms) we see that the pasting

$$\begin{tikzcd}
& PT\mathcal{A} \arrow[rr, "y_{PT\mathcal{A}}"] \arrow[drrr, bend left=50, "\mathrm{id}_{PT\mathcal{A}}"] \arrow[d, phantom, "\Uparrow y_{\lambda_{\mathcal{A}}}"] \arrow[rr, phantom, bend left=30, "\cong"] & & P^2T\mathcal{A} \arrow[dr, "\mu_{T\mathcal{A}}"] & \\
TP\mathcal{A} \arrow[ur, "\lambda_{\mathcal{A}}"] \arrow[rr, "y_{TP\mathcal{A}}"] & \arrow[d, phantom, "\overset{y_{Ty_{\mathcal{A}}}}{\Leftarrow}"] & PTP\mathcal{A} \arrow[ur, "P\lambda_{\mathcal{A}}"] \arrow[r, phantom, "\overset{P\omega_2^{\mathcal{A}}}{\Leftarrow}"] & \arrow[r, phantom, "\cong"] & PT\mathcal{A} \\
& & & PT\mathcal{A} \arrow[ul, "PTy_{\mathcal{A}}"] \arrow[uu, "Py_{T\mathcal{A}}"] \arrow[ur, "\mathrm{id}"'] \arrow[r, phantom, "="] & \\
& & & & T\mathcal{A} \arrow[ul, "y_{T\mathcal{A}}"] \arrow[uu, "y_{T\mathcal{A}}"'] \arrow[uullll, bend left, "Ty_{\mathcal{A}}"]
\end{tikzcd}$$

exhibits $\lambda_{\mathcal{A}}$ as a left extension. Note that this pasting is equal to ω_2 as a consequence of ω_2 being a modification as well as the coherence condition 3.2.1 satisfied by P.

There exists canonical isomorphisms $\omega_4^{\mathcal{A}}$. We have the left extension

$$\begin{tikzcd}
TP^2\mathcal{A} \arrow[r, "\lambda_{P\mathcal{A}}"] & PTP\mathcal{A} \arrow[r, "P\lambda_{\mathcal{A}}"] \arrow[d, phantom, "\overset{\omega_2^{P\mathcal{A}}}{\Leftarrow}"] & PPT\mathcal{A} \arrow[r, "\mu_{T\mathcal{A}}"] \arrow[d, phantom, "\cong"] & PT\mathcal{A} \\
& TP\mathcal{A} \arrow[lu, "Ty_{P\mathcal{A}}"] \arrow[u, "y_{TP\mathcal{A}}"'] \arrow[r, "\lambda_{\mathcal{A}}"'] \arrow[ru, phantom, "\overset{y^{-1}_{\lambda_{\mathcal{A}}}}{\Leftarrow}"] & PT\mathcal{A} \arrow[u, "y_{PT\mathcal{A}}"] \arrow[ru, "\mathrm{id}_{PT\mathcal{A}}"'] &
\end{tikzcd}$$

since $Ty_{P\mathcal{A}}$ is P-admissible, and also the left extension

$$\begin{tikzcd}
TP^2\mathcal{A} \arrow[r, "T\mu_{\mathcal{A}}"] & TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\
TP\mathcal{A} \arrow[u, "Ty_{P\mathcal{A}}"] \arrow[ru, "T\mathrm{id}"'] \arrow[ru, phantom, bend left, "\Uparrow Tc_{\mathrm{id}}"] & &
\end{tikzcd}$$

since $\lambda_{\mathcal{A}}$ is T_P-cocontinuous by Lemma 3.3.12, giving us our isomorphism of left extensions $\omega_4^{\mathcal{A}}$. Note that this means ω_4 satisfies coherence axiom 7 of [39]. □

In the following proposition we show that the admissible maps are preserved. Note that the proof relies on the existence of isomorphisms $\omega_4^{\mathcal{A}}$ as above, which in turn relies on the the admissibility of $y_{\mathcal{A}}$ being preserved (also shown above).

Proposition 3.3.15. *Suppose we are given a 2-category $\mathscr{C}$ equipped with a pseudomonad (T,u,m) and a KZ pseudomonad (P,y,μ). Suppose further that we are given a pseudo-distributive law over a KZ pseudomonad $\lambda\colon TP \longrightarrow PT$. Then T preserves P-admissible maps.*

Proof. Suppose we are given a 1-cell $L\colon \mathcal{A} \longrightarrow \mathcal{B}$ which is P-admissible, meaning that we have an adjunction $PL \dashv \mathrm{res}_L$ with unit and counit denoted η and ε respectively. We show that $TL\colon T\mathcal{A} \longrightarrow T\mathcal{B}$ must then be P-admissible, with the admissibility exhibited by an adjunction

$$PTL \dashv \mu_{T\mathcal{A}} \cdot P\lambda_{\mathcal{A}} \cdot PT\mathrm{res}_L \cdot PTy_{\mathcal{B}}$$

Firstly, we note that this right adjoint is exhibited as the left extension

$$\begin{tikzcd}
PT\mathcal{B} \arrow[r, "PTy_{\mathcal{B}}"] \arrow[d, phantom, "\overset{y^{-1}_{Ty_{\mathcal{B}}}}{\Leftarrow}"] & PTP\mathcal{B} \arrow[r, "PT\mathrm{res}_L"] \arrow[d, phantom, "\overset{y^{-1}_{T\mathrm{res}_L}}{\Leftarrow}"] & PTP\mathcal{A} \arrow[r, "P\lambda_{\mathcal{A}}"] \arrow[d, phantom, "\overset{y^{-1}_{\lambda_{\mathcal{A}}}}{\Leftarrow}"] & P^2T\mathcal{A} \arrow[r, "\mu_{T\mathcal{A}}"] \arrow[d, phantom, "\cong"] & PT\mathcal{A} \\
T\mathcal{B} \arrow[u, "y_{T\mathcal{B}}"] \arrow[r, "Ty_{\mathcal{B}}"'] & TP\mathcal{B} \arrow[u, "y_{TP\mathcal{B}}"] \arrow[r, "T\mathrm{res}_L"'] & TP\mathcal{A} \arrow[u, "y_{TP\mathcal{A}}"] \arrow[r, "\lambda_{\mathcal{A}}"'] & PT\mathcal{A} \arrow[u, "y_{PT\mathcal{A}}"] \arrow[ru, "\mathrm{id}_{PT\mathcal{A}}"'] &
\end{tikzcd} \tag{3.3.4}$$

and denote it $\mathbf{R}_L$ for convenience. We then define our unit n as the unique 2-cell rendering

$$\begin{tikzcd}
& & PT\mathcal{B} \arrow[dr, "\mathbf{R}_L"] \arrow[d, phantom, "\Uparrow n"] & \\
T\mathcal{A} \arrow[r, "y_{T\mathcal{A}}"'] & PT\mathcal{A} \arrow[ur, "PTL"] \arrow[rr, "\mathrm{id}_{PT\mathcal{A}}"'] & & PT\mathcal{A}
\end{tikzcd}$$

equal to

$$\begin{tikzcd}
PT\mathcal{A} \arrow[r, "PTL"] & PT\mathcal{B} \arrow[r, "PTy_{\mathcal{B}}"] & PTP\mathcal{B} \arrow[r, "PT\mathrm{res}_L"] & PTP\mathcal{A} \arrow[r, "P\lambda_{\mathcal{A}}"] & P^2T\mathcal{A} \arrow[r, "\mu_{T\mathcal{A}}"] & PT\mathcal{A} \\
T\mathcal{A} \arrow[u, "y_{T\mathcal{A}}"] \arrow[r, "TL"'] \arrow[ur, phantom, "\overset{y_{TL}^{-1}}{\Longleftarrow}"] \arrow[drr, bend right=15, "Ty_{\mathcal{A}}"] \arrow[rrrr, bend right=50, "\Uparrow\omega_2^{\mathcal{A}}", "y_{T\mathcal{A}}"'] & T\mathcal{B} \arrow[u, "y_{T\mathcal{B}}"] \arrow[r, "Ty_{\mathcal{B}}"'] \arrow[ur, phantom, "\overset{y_{Ty_{\mathcal{B}}}^{-1}}{\Longleftarrow}"] \arrow[d, phantom, "\Uparrow Ty_L"] & TP\mathcal{B} \arrow[u, "y_{TP\mathcal{B}}"] \arrow[r, "T\mathrm{res}_L"'] \arrow[ur, phantom, "\overset{y_{T\mathrm{res}_L}^{-1}}{\Longleftarrow}"] & TP\mathcal{A} \arrow[u, "y_{TP\mathcal{A}}"] \arrow[r, "\lambda_{\mathcal{A}}"'] \arrow[ur, phantom, "\overset{y_{\lambda_{\mathcal{A}}}^{-1}}{\Longleftarrow}"] \arrow[d, phantom, "\Uparrow T\eta"] & PT\mathcal{A} \arrow[u, "y_{PT\mathcal{A}}"] \arrow[ur, "\cong", "\mathrm{id}_{PT\mathcal{A}}"'] & \\
& & TP\mathcal{A} \arrow[u, "TPL"] \arrow[ur, bend right=15, "T\mathrm{id}"'] & & &
\end{tikzcd}$$

and note that the unit n is then given by

$$\begin{tikzcd}
PT\mathcal{A} \arrow[r, "PTL"] \arrow[drr, bend right=15, "PTy_{\mathcal{A}}"] \arrow[rrrr, bend right=40, "\Uparrow P\omega_2^{\mathcal{A}}", "Py_{T\mathcal{A}}"'] \arrow[rrrrr, bend right=60, "\cong", "\mathrm{id}_{PT\mathcal{A}}"'] & PT\mathcal{B} \arrow[r, "PTy_{\mathcal{B}}"] \arrow[d, phantom, "\Uparrow PTy_L"] & PTP\mathcal{B} \arrow[r, "PT\mathrm{res}_L"] & PTP\mathcal{A} \arrow[r, "P\lambda_{\mathcal{A}}"] \arrow[d, phantom, "\Uparrow PT\eta"] & P^2T\mathcal{A} \arrow[r, "\mu_{T\mathcal{A}}"] & PT\mathcal{A} \\
& & PTP\mathcal{A} \arrow[u, "PTPL"] \arrow[ur, bend right=15, "PT\mathrm{id}"'] & & &
\end{tikzcd} \tag{3.3.5}$$

as $y\colon 1 \to P$ is a pseudonatural transformation. We define our counit e as the unique solution to the left extension problem

$$\begin{tikzcd}
T\mathcal{B} \arrow[r, "y_{T\mathcal{B}}"] \arrow[rr, bend right=40, "\cong", "\lambda_{\mathcal{A}}\cdot T\mathrm{res}_L\cdot Ty_{\mathcal{B}}"'] & PT\mathcal{B} \arrow[r, "\mathbf{R}_L"] \arrow[rr, bend left=40, "\mathrm{id}_{PT\mathcal{B}}", "\Uparrow e"'] & PT\mathcal{A} \arrow[r, "PTL"] & PT\mathcal{B}
\end{tikzcd}
=
\begin{tikzcd}
T\mathcal{B} \arrow[r, "Ty_{\mathcal{B}}"] \arrow[d, "Ty_{\mathcal{B}}"'] \arrow[rr, bend left=40, "y_{T\mathcal{B}}", "\Uparrow(\omega_2^{\mathcal{B}})^{-1}"'] \arrow[dr, phantom, "="] & TP\mathcal{B} \arrow[r, "\lambda_{\mathcal{B}}"] \arrow[dr, phantom, "\Uparrow\lambda_L"] & PT\mathcal{B} \\
TP\mathcal{B} \arrow[r, "T\mathrm{res}_L"'] \arrow[ur, "T\mathrm{id}"] \arrow[r, phantom, shift left=4, "\Uparrow T\varepsilon"] & TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"'] \arrow[u, "TPL"] & PT\mathcal{A} \arrow[u, "PTL"']
\end{tikzcd}$$

where the unlabeled isomorphism is 3.3.4. Of the triangle identities:

$$\begin{tikzcd}
PTL \arrow[r, "PTL\cdot n"] \arrow[dr, "\mathrm{id}_{PTL}"'] & PTL\cdot\mathbf{R}_L\cdot PTL \arrow[d, "e\cdot PTL"] \\
& PTL
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{R}_L \arrow[r, "n\cdot\mathbf{R}_L"] \arrow[dr, "\mathrm{id}_{\mathbf{R}_L}"'] & \mathbf{R}_L\cdot PTL\cdot\mathbf{R}_L \arrow[d, "\mathbf{R}_L\cdot e"] \\
& \mathbf{R}_L
\end{tikzcd}$$

the left identity (or equivalently the left triangle identity whiskered by $y_{T\mathcal{A}}$) easily follows from the whiskered definitions of n and e as well as the corresponding triangle identity for $PL \dashv \mathrm{res}_L$, and ω_2 being a modification. The right triangle identity (or that whiskered by

$y_{T\mathcal{B}}$) is more complicated. This identity amounts to checking that

$$\begin{tikzcd}
T\mathcal{B} \arrow[r, "Ty_{\mathcal{B}}"] \arrow[d, "Ty_{\mathcal{B}}"'] \arrow[rr, bend left, "y_{T\mathcal{B}}"] \arrow[rd, phantom, "="] & TP\mathcal{B} \arrow[r, "\lambda_{\mathcal{B}}"] \arrow[r, phantom, shift left=12, "\Uparrow(\omega_2^{\mathcal{B}})^{-1}"] & PT\mathcal{B} \arrow[r, "PTy_{\mathcal{B}}"] & PTP\mathcal{B} \arrow[r, "PT\mathrm{res}_L"] & PTP\mathcal{A} \arrow[r, "P\lambda_{\mathcal{A}}"] & P^2T\mathcal{A} \arrow[r, "\mu_{T\mathcal{A}}"] & PT\mathcal{A} \\
TP\mathcal{B} \arrow[r, "T\mathrm{res}_L"'] \arrow[ur, "T\mathrm{id}" description] \arrow[r, phantom, shift left=12, "\Uparrow T\varepsilon"] & TP\mathcal{A} \arrow[u, "TPL"'] \arrow[r, "\lambda_{\mathcal{A}}"'] \arrow[ur, phantom, "\Uparrow\lambda_L"] & PT\mathcal{A} \arrow[u, "PTL"'] \arrow[r, "PTy_{\mathcal{A}}"] \arrow[ur, phantom, "\Uparrow PTy_L"] \arrow[urrr, bend right, "Py_{T\mathcal{A}}"'] \arrow[urrrr, bend right=40, "\mathrm{id}_{PT\mathcal{A}}"'] & PTP\mathcal{A} \arrow[u, "PTPL"'] \arrow[ur, "PT\mathrm{id}"'] \arrow[u, phantom, shift left=20, "\Uparrow PT\eta"] \arrow[rr, phantom, "\Uparrow P\omega_2^{\mathcal{A}}"] & & \arrow[r, phantom, "\cong"] &
\end{tikzcd}$$

(3.3.6)

is just the isomorphism 3.3.4. The first step here is to make our diagrams more like the first axiom of a pseudo-distributive law over a KZ doctrine. Upon using that ω_2 is a modification and the coherence axiom 3.2.3, the problem reduces to showing that 3.3.6 with the unnamed isomorphism and cell $\omega_2^{\mathcal{B}}$ removed is equal to

$$\begin{tikzcd}
& PT\mathcal{B} \arrow[rr, "PTy_{\mathcal{B}}"] \arrow[d, phantom, "\Uparrow\lambda^{-1}_{y_{\mathcal{B}}}"] & & PTP\mathcal{B} \arrow[rr, "PT\mathrm{res}_L"] \arrow[d, phantom, "\Uparrow\lambda^{-1}_{\mathrm{res}_L}"] & & PTP\mathcal{A} \arrow[r, "P\lambda_{\mathcal{A}}"] & P^2T\mathcal{A} \arrow[r, "\mu_{T\mathcal{A}}"] & PT\mathcal{A} \\
TP\mathcal{B} \arrow[ur, "\lambda_{\mathcal{B}}"] \arrow[rr, "TPy_{\mathcal{B}}"] & \arrow[d, phantom, "\Uparrow Ty^{-1}_{y_{\mathcal{B}}}"] & TP^2\mathcal{B} \arrow[ur, "\lambda_{P\mathcal{B}}"] \arrow[rr, "TP\mathrm{res}_L"] & \arrow[d, phantom, "\Uparrow Ty^{-1}_{\mathrm{res}_L}"] & TP^2\mathcal{A} \arrow[ur, "\lambda_{P\mathcal{A}}"] \arrow[r, phantom, "\overset{\omega_2^{P\mathcal{A}}}{\Leftarrow}"] & \arrow[r, phantom, "\overset{y^{-1}_{\lambda_{\mathcal{A}}}}{\Leftarrow}"] & \arrow[d, phantom, "\overset{\theta^{T\mathcal{A}}}{\Leftarrow}"] & \\
& T\mathcal{B} \arrow[ul, "Ty_{\mathcal{B}}"] \arrow[rr, "Ty_{\mathcal{B}}"'] & & TP\mathcal{B} \arrow[ul, "Ty_{P\mathcal{B}}"] \arrow[rr, "T\mathrm{res}_L"'] & & TP\mathcal{A} \arrow[ul, "Ty_{P\mathcal{A}}"] \arrow[uu, "y_{TP\mathcal{A}}"] \arrow[r, "\lambda_{\mathcal{A}}"'] & PT\mathcal{A} \arrow[uu, bend left, "y_{PT\mathcal{A}}"] \arrow[uu, bend right, "Py_{T\mathcal{A}}"'] &
\end{tikzcd}$$

(3.3.7)

We then simplify 3.3.7 using the first axiom of a pseudo-distributive law over a KZ doctrine, canceling $P\omega_2^{\mathcal{A}}$, and pasting $\lambda_{y_{\mathcal{B}}}$, $\lambda_{y_{\mathcal{A}}}$ and λ_{res_L} to the other side of the desired equation. By pseudonaturality of λ, the problem may then be reduced to showing that

$$\begin{tikzcd}
T\mathcal{B} \arrow[r, "Ty_{\mathcal{B}}"] \arrow[d, "Ty_{\mathcal{B}}"'] \arrow[rd, phantom, "="] & TP\mathcal{B} \arrow[r, "TPy_{\mathcal{B}}"] & TP^2\mathcal{B} \arrow[r, "TP\mathrm{res}_L"] & TP^2\mathcal{A} \arrow[r, "\lambda_{P\mathcal{A}}"] & PTP\mathcal{A} \arrow[r, "P\lambda_{\mathcal{A}}"] & P^2T\mathcal{A} \arrow[r, "\mu_{T\mathcal{A}}"] & PT\mathcal{A} \\
TP\mathcal{B} \arrow[r, "T\mathrm{res}_L"'] \arrow[ur, "T\mathrm{id}" description] \arrow[r, phantom, shift left=12, "\Uparrow T\varepsilon"] & TP\mathcal{A} \arrow[u, "TPL"'] \arrow[r, "TPy_{\mathcal{A}}"'] \arrow[ur, phantom, "\Uparrow TPy_L"] & TP^2\mathcal{A} \arrow[u, "TP^2L"'] \arrow[ur, "TP\mathrm{id}"'] \arrow[u, phantom, shift left=20, "\Uparrow TP\eta"] & & & &
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
TP\mathcal{B} \arrow[r, "TPy_{\mathcal{B}}"] & TP^2\mathcal{B} \arrow[r, "TP\mathrm{res}_L"] & TP^2\mathcal{A} \arrow[r, "\lambda_{P\mathcal{A}}"] & PTP\mathcal{A} \arrow[r, "P\lambda_{\mathcal{A}}"] & P^2T\mathcal{A} \arrow[r, "\mu_{T\mathcal{A}}"] & PT\mathcal{A} \\
T\mathcal{B} \arrow[u, "Ty_{\mathcal{B}}"] \arrow[r, "Ty_{\mathcal{B}}"'] \arrow[ur, phantom, "\Uparrow Ty^{-1}_{y_{\mathcal{B}}}"] & TP\mathcal{B} \arrow[u, "Ty_{P\mathcal{B}}"] \arrow[r, "T\mathrm{res}_L"'] \arrow[ur, phantom, "\Uparrow Ty^{-1}_{\mathrm{res}_L}"] & TP\mathcal{A} \arrow[u, bend left, "Ty_{P\mathcal{A}}"] \arrow[u, bend right, "TPy_{\mathcal{A}}"'] \arrow[u, phantom, "\overset{T\theta^{\mathcal{A}}}{\Leftarrow}"] & & &
\end{tikzcd}$$

Since we have the isomorphism $\omega_4^{\mathcal{A}}$ as in Lemma 3.3.14, and as $T\theta^{\mathcal{B}} \cdot Ty_{\mathcal{B}}$ is invertible (meaning we can paste with $T\theta^{\mathcal{B}}$ and maintain the logical equivalence), we may reduce the problem to showing that

$$\begin{tikzcd}
T\mathcal{B} \arrow[r, "Ty_{\mathcal{B}}"] \arrow[d, "Ty_{\mathcal{B}}"'] \arrow[dr, phantom, "=" near start] & TP\mathcal{B} \arrow[r, bend left, "Ty_{P\mathcal{B}}"] \arrow[r, bend right, "TPy_{\mathcal{B}}"'] \arrow[r, phantom, "\Uparrow T\theta^{\mathcal{B}}"] & TP^2\mathcal{B} \arrow[r, "TP\mathrm{res}_L"] & TP^2\mathcal{A} \arrow[r, "T\mu_{\mathcal{A}}"] & TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\
TP\mathcal{B} \arrow[ur, "T\mathrm{id}"] \arrow[r, "T\mathrm{res}_L"'] \arrow[ur, phantom, "\Uparrow T\varepsilon" near end, shift left=4] & TP\mathcal{A} \arrow[u, "TPL"] \arrow[r, "TPy_{\mathcal{A}}"'] \arrow[ur, phantom, "\Uparrow TPy_L"] & TP^2\mathcal{A} \arrow[u, "TP^2L"] \arrow[ur, "TP\mathrm{id}_{P\mathcal{A}}"'] \arrow[ur, phantom, "\Uparrow TP\eta" near end, shift left=4] & & &
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
TP\mathcal{B} \arrow[r, bend left, "Ty_{P\mathcal{B}}"] \arrow[r, bend right, "TPy_{\mathcal{B}}"'] \arrow[r, phantom, "\Uparrow T\theta^{\mathcal{B}}"] & TP^2\mathcal{B} \arrow[r, "TP\mathrm{res}_L"] & TP^2\mathcal{A} \arrow[r, "T\mu_{\mathcal{A}}"] & TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\
T\mathcal{B} \arrow[u, "Ty_{\mathcal{B}}"] \arrow[r, "Ty_{\mathcal{B}}"'] \arrow[ur, phantom, "\Uparrow Ty_{y_{\mathcal{B}}}"] & TP\mathcal{B} \arrow[u, "Ty_{P\mathcal{B}}"] \arrow[r, "T\mathrm{res}_L"'] \arrow[ur, phantom, "\Uparrow Ty^{-1}_{\mathrm{res}_L}"] & TP\mathcal{A} \arrow[u, bend left, "Ty_{P\mathcal{A}}"] \arrow[u, bend right, "TPy_{\mathcal{A}}"'] \arrow[u, phantom, "\overset{T\theta^{\mathcal{A}}}{\Leftarrow}"] & &
\end{tikzcd}$$

From here, use that θ is a modification, the axioms 3.2.2 and 3.2.3, and pseudonaturality of y to deduce the triangle identity from that of the adjunction $PL \dashv \mathrm{res}_L$. □

Remark 3.3.16. Note that here, as well as in the preceding lemma, we only used that ω_2 is an invertible modification and the first axiom for a pseudo-distributive law over a KZ doctrine, along with pseudo naturality of λ.

We are now ready to prove the main result of this subsection.

Theorem 3.3.17. *In the statement of Theorem 3.3.8 (e) implies (d).*

Proof. We first note by Proposition 3.3.15 that T preserves P-admissible maps. Also, we know by Lemma 3.3.14 that each $\lambda_{\mathcal{A}}$ is a left extension exhibited by the distributive law data as in

$$\begin{tikzcd}
TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \arrow[dl, phantom, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}" near start] \\
 & T\mathcal{A} \arrow[ul, "Ty_{\mathcal{A}}"] \arrow[u, "y_{T\mathcal{A}}"']
\end{tikzcd}$$

with $\omega_2^{\mathcal{A}}$ invertible by assumption. That each $\lambda_{\mathcal{A}}$ is T_P-cocontinuous is a consequence of Lemma 3.3.12 and ω_2 being a modification. Finally, that the diagrams

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "u_{P\mathcal{A}}"] & TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[r, "u_{\mathcal{A}}"'] \arrow[ur, phantom, "\overset{u_{y_{\mathcal{A}}}}{\Leftarrow}"] & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[ur, "y_{T\mathcal{A}}"'] \arrow[ur, phantom, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}" near start, shift left=3] &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2P\mathcal{A} \arrow[r, "m_{P\mathcal{A}}"] & TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\
T^2\mathcal{A} \arrow[u, "T^2y_{\mathcal{A}}"] \arrow[r, "m_{\mathcal{A}}"'] \arrow[ur, phantom, "\overset{m_{y_{\mathcal{A}}}}{\Leftarrow}"] & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[ur, "y_{T\mathcal{A}}"'] \arrow[ur, phantom, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}" near start, shift left=3] &
\end{tikzcd}$$

exhibit both $\lambda_{\mathcal{A}} \cdot u_{P\mathcal{A}}$ and $\lambda_{\mathcal{A}} \cdot m_{P\mathcal{A}}$ as left extensions is due to the last two axioms for a pseudo-distributive law over a KZ pseudomonad (as pasting a left extension with an isomorphism

ω_1 or ω_3 will preserve the left extension property). Indeed, it is clear the left diagram below exhibits $Pu_{\mathcal{A}}$ as a left extension.

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "Pu_{\mathcal{A}}"] & PT\mathcal{A} \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[r, "u_{\mathcal{A}}"'] \arrow[ur, phantom, "\overset{y_{u_{\mathcal{A}}}^{-1}}{\Leftarrow}"] & T\mathcal{A} \arrow[u, "y_{T\mathcal{A}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2P\mathcal{A} \arrow[r, "T\lambda_{\mathcal{A}}"] & TPT\mathcal{A} \arrow[r, "\lambda_{T\mathcal{A}}"] & PT^2\mathcal{A} \arrow[r, "Pm_{\mathcal{A}}"] & PT\mathcal{A}\ A \\
& T^2\mathcal{A} \arrow[ul, "T^2y_{\mathcal{A}}"] \arrow[u, "Ty_{T\mathcal{A}}"] \arrow[ur, "y_{T^2\mathcal{A}}"'] \arrow[r, "m_{\mathcal{A}}"'] & T\mathcal{A} \arrow[ur, "y_{T\mathcal{A}}"'] \arrow[u, phantom, "\overset{y_{m_{\mathcal{A}}}^{-1}}{\Leftarrow}"]
\end{tikzcd}$$

$$T\omega_2^{\mathcal{A}} \Leftarrow \qquad \omega_2T^{\mathcal{A}} \Leftarrow$$

To see that the composite $Pm_{\mathcal{A}} \cdot \lambda_{T\mathcal{A}} \cdot T\lambda_{\mathcal{A}}$ on the right is a left extension, note that Proposition 3.3.10 shows $\lambda_{T\mathcal{A}} \cdot T\lambda_{\mathcal{A}}$ is a left extension above, and since $T^2y_{\mathcal{A}}$ is P-admissible by Proposition 3.3.15, the left extension property is respected upon whiskering by $Pm_{\mathcal{A}}$. $\square$

3.3.4 Lifting a KZ Doctrine to Algebras via a Distributive Law

In this subsection we show that given a pseudo-distributive law of a pseudomonad T over a KZ doctrine P, we may lift P to a KZ doctrine $\widetilde{P}$ on the 2-category of pseudo T-algebras. This is $(d) \implies (a)$ of Theorem 3.3.8. However, before we show this implication we will first need to verify the following proposition.

Proposition 3.3.18. *Suppose we are given statement (d) of Theorem 3.3.8. It then follows that:*

1. *T preserves P-admissible maps;*

and for every pseudo T-algebra $\left(\mathcal{A}, T\mathcal{A} \xrightarrow{x} \mathcal{A}\right)$,

2. *there exists a 1-cell z_x given as the left extension via an isomorphism ξ_x*

$$\begin{tikzcd}
TP\mathcal{A} \arrow[r, "z_x"] & P\mathcal{A} \\
T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow \xi_x"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}$$

   which we call the Day convolution at x;

3. *each z_x is T_P-cocontinuous;*

4. *the respective diagrams*

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "u_{P\mathcal{A}}"] & TP\mathcal{A} \arrow[r, "z_x"] & P\mathcal{A} \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[r, "u_{\mathcal{A}}"'] \arrow[ur, phantom, "\Uparrow u_{y_{\mathcal{A}}}"] & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow \xi_x"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2P\mathcal{A} \arrow[r, "m_{P\mathcal{A}}"] & TP\mathcal{A} \arrow[r, "z_x"] & P\mathcal{A} \\
T^2\mathcal{A} \arrow[u, "T^2y_{\mathcal{A}}"] \arrow[r, "m_{\mathcal{A}}"'] \arrow[ur, phantom, "\Uparrow m_{y_{\mathcal{A}}}"] & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow \xi_x"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}$$

exhibit $z_x \cdot u_{P\mathcal{A}}$ *and* $z_x \cdot m_{P\mathcal{A}}$ *as left extensions.*

Proof. (1) This property is straight from the definition. We include this property here so that this proposition may be taken as one the equivalent conditions of Theorem 3.3.8. We will remark about this later in this subsection. Now, let a pseudo T-algebra $\left(\mathcal{A}, T\mathcal{A} \xrightarrow{x} \mathcal{A}\right)$ be given. (2) The left extension (z_x, ξ_x) is given by the diagram

$$\begin{tikzcd}
TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \arrow[r, "Px"] & P\mathcal{A} \\
& T\mathcal{A} \arrow[ul, "Ty_{\mathcal{A}}"] \arrow[u, "y_{T\mathcal{A}}"'] \arrow[r, "x"'] \arrow[ul, phantom, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}"'] \arrow[ur, phantom, "\overset{y_x^{-1}}{\Leftarrow}"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}$$

where the left extension $\lambda_{\mathcal{A}}$ is preserved by Px as $Ty_{\mathcal{A}}$ is P-admissible. (3) Suppose we are given a left extension as on the left below.

$$\begin{tikzcd}
P\mathcal{D} \arrow[r, "\overline{F}"] & P\mathcal{A} \\
\mathcal{D} \arrow[u, "y_{\mathcal{D}}"] \arrow[ur, "F"'] \arrow[ur, phantom, "\overset{c_F}{\Leftarrow}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
TP\mathcal{D} \arrow[r, "T\overline{F}"] & TP\mathcal{A} \arrow[r, "z_x"] & P\mathcal{A} \\
T\mathcal{D} \arrow[u, "Ty_{\mathcal{D}}"] \arrow[ur, "TF"'] \arrow[ur, phantom, "\overset{Tc_F}{\Leftarrow}"]
\end{tikzcd}$$

As this left extension is T-preserved by $\lambda_{\mathcal{A}}$, which in turn is preserved by Px as $Ty_{\mathcal{D}}$ is P-admissible, the diagram on the right exhibits $z_x \cdot T\overline{F} = Px \cdot \lambda_{\mathcal{A}} \cdot T\overline{F}$ as a left extension. (4) Again noting each $Ty_{\mathcal{A}}$ is P-admissible, we see the left extensions

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "u_{P\mathcal{A}}"] & TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[r, "u_{\mathcal{A}}"'] \arrow[ur, phantom, "\overset{u_y^{\mathcal{A}}}{\Leftarrow}"] & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"'] \arrow[ur, "y_{T\mathcal{A}}"'] \arrow[ur, phantom, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2P\mathcal{A} \arrow[r, "m_{P\mathcal{A}}"] & TP\mathcal{A} \arrow[r, "\lambda_{\mathcal{A}}"] & PT\mathcal{A} \\
T^2\mathcal{A} \arrow[u, "T^2y_{\mathcal{A}}"] \arrow[r, "m_{\mathcal{A}}"'] \arrow[ur, phantom, "\overset{m_y^{\mathcal{A}}}{\Leftarrow}"] & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"'] \arrow[ur, "y_{T\mathcal{A}}"'] \arrow[ur, phantom, "\overset{\omega_2^{\mathcal{A}}}{\Leftarrow}"] &
\end{tikzcd}$$

are preserved upon composing with Px. Trivially, these left extensions are then preserved upon pasting with the isomorphism y_x. □

The following remark is not needed for the proof of Theorem 3.3.8, it merely explains why the consequences in the above proposition are equivalent to the conditions (a) through to (f) of this theorem.

Remark 3.3.19. Note that from this proposition one may recover statement *(d)* of Theorem 3.3.8. This is since given the data of this proposition, one may recover a choice of each $\lambda_{\mathcal{A}}$

and its exhibiting invertible 2-cell $\omega_2^{\mathcal{A}}$ as a left extension, by taking the pasting

$$\begin{tikzcd}
TP\mathcal{A} \arrow[r, "TPu_{\mathcal{A}}"] & TPT\mathcal{A} \arrow[r, "z_{m_{\mathcal{A}}}"] & PT\mathcal{A} \\
T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "Tu_{\mathcal{A}}"'] \arrow[ur, phantom, "\Uparrow Ty^{-1}_{u_{\mathcal{A}}}"] & T^2\mathcal{A} \arrow[u, "Ty_{T\mathcal{A}}"] \arrow[r, "m_{\mathcal{A}}"'] \arrow[ur, phantom, "\Uparrow \xi_{m_{\mathcal{A}}}"] & T\mathcal{A} \arrow[u, "y_{T\mathcal{A}}"']
\end{tikzcd}$$

The condition of each $\lambda_{\mathcal{A}}$ being T_P-cocontinuous is inherited from the corresponding condition on each $z_{m_{\mathcal{A}}}$. Condition (4) of this proposition yields the corresponding conditions on the maps $\lambda_{\mathcal{A}}$. We omit this last calculation, as it is not required for the proof of the main theorem. We just note that this last calculation relies on the pseudo-algebra structure of the maps $z_X \colon TP\mathcal{A} \to P\mathcal{A}$ constructed later on in this subsection. The construction of the algebra structure may be done with all of the axioms for a pseudo-distributive law over a KZ doctrine without the last (which we have recovered from the proposition), in addition to the last condition of the proposition.

The following proposition will be useful in the proof that (d) implies (a).

Proposition 3.3.20. *Suppose we are given a 2-category $\mathscr{C}$ equipped with a pseudomonad (T, u, m) and a KZ doctrine (P, y). Further suppose that we are given a pseudo-distributive law over a KZ doctrine $\lambda \colon TP \to PT$. Then for any two P-cocomplete objects $\mathcal{C}$ and $\mathcal{D}$, a 1-cell $u \colon T\mathcal{C} \to \mathcal{D}$ is T_P-cocontinuous if and only if it is T_P-adm-cocontinuous.*

Proof. Supposing that u is T_P-cocontinuous we check that u is necessarily T_P-adm-cocontinuous. To see this, we first note that we have an induced isomorphism of left extensions as a consequence of having the two left extensions

$$\begin{tikzcd}
TP\mathcal{C} \arrow[r, "\lambda_{\mathcal{C}}"] & PT\mathcal{C} \arrow[r, "Pu"] & P\mathcal{D} \arrow[r, "(y_{\mathcal{D}})_*"] & \mathcal{D} \\
T\mathcal{C} \arrow[u, "Ty_{\mathcal{C}}"] \arrow[ur, "y_{T\mathcal{C}}"'] \arrow[r, "u"'] \arrow[u, phantom, shift left=4, "\overset{\omega_2^{\mathcal{C}}}{\Leftarrow}"] & \mathcal{D} \arrow[ur, "y_{\mathcal{D}}"'] \arrow[urr, bend right, "\mathrm{id}_{\mathcal{D}}"'] \arrow[u, phantom, shift left=4, "\overset{y_u^{-1}}{\Leftarrow}"] & \arrow[u, phantom, "\overset{c_{\mathrm{id}_{\mathcal{D}}}}{\Leftarrow}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
TP\mathcal{C} \arrow[r, "T(y_{\mathcal{C}})_*"] & T\mathcal{C} \arrow[r, "u"] & \mathcal{D} \\
T\mathcal{C} \arrow[u, "Ty_{\mathcal{C}}"] \arrow[ur, "T\mathrm{id}_{\mathcal{C}}"'] \arrow[urr, bend right, "u"'] \arrow[u, phantom, shift left=4, "\overset{Tc_{\mathrm{id}_{\mathcal{C}}}}{\Leftarrow}"] & \arrow[u, phantom, "\cong"] &
\end{tikzcd}$$

We must check that the left extension (where L is P-admissible)

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \arrow[r, "PH"] & P\mathcal{C} \arrow[r, "(y_{\mathcal{C}})_*"] & \mathcal{C} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"'] \arrow[r, "H"'] \arrow[ul, phantom, shift right=3, "\overset{\varphi_L}{\Leftarrow}"] \arrow[ur, phantom, "\overset{y_H^{-1}}{\Leftarrow}"] & \mathcal{C} \arrow[u, "y_{\mathcal{C}}"] \arrow[ur, "\mathrm{id}_{\mathcal{C}}"'] \arrow[ur, phantom, shift left=3, "\overset{c_{\mathrm{id}_{\mathcal{C}}}}{\Leftarrow}"] &
\end{tikzcd}$$

is T-preserved by u. Indeed, on applying T and whiskering by u, and then pasting with this isomorphism of left extensions and a naturality isomorphism of λ (which we have by Lemma

3.3.12), we obtain

$$\begin{tikzcd}
 & P T \mathcal{A} \arrow[r, "PTH"] \arrow[rd, phantom, "\overset{\lambda_H^{-1}}{\Leftarrow}"] & P T \mathcal{C} \arrow[r, "Pu"] \arrow[rd, phantom, "\cong"] & P \mathcal{D} \arrow[rd, "(y_\mathcal{D})_*"] & \\
T \mathcal{B} \arrow[r, "TR_L"] & T P \mathcal{A} \arrow[u, "\lambda_\mathcal{A}"] \arrow[r, "TPH"] \arrow[rd, phantom, "\overset{Ty_H^{-1}}{\Leftarrow}"] & T P \mathcal{C} \arrow[u, "\lambda_\mathcal{C}"'] \arrow[r, "T(y_\mathcal{C})_*"] & T \mathcal{C} \arrow[r, "u"] & \mathcal{D} \\
 & T \mathcal{A} \arrow[lu, "TL"] \arrow[u, "Ty_\mathcal{A}"] \arrow[lu, phantom, bend left, "\overset{T\varphi_L}{\Leftarrow}"] \arrow[r, "TH"'] & T \mathcal{C} \arrow[u, "Ty_\mathcal{C}"] \arrow[ru, "T\mathrm{id}_\mathcal{C}"'] \arrow[ru, phantom, bend left, "\overset{Tc_{\mathrm{id}_\mathcal{C}}}{\Leftarrow}"] & &
\end{tikzcd}$$

Then noting that pasting with invertible 2-cells preserves left extensions and that

$$\begin{tikzcd}
T \mathcal{B} \arrow[r, "TR_L"] & T P \mathcal{A} \arrow[r, "\lambda_\mathcal{A}"] & P T \mathcal{A} \arrow[r, "PTH"] & P T \mathcal{C} \arrow[r, "Pu"] & P \mathcal{D} \arrow[r, "(y_\mathcal{D})_*"] & \mathcal{D} \\
 & T \mathcal{A} \arrow[lu, "TL"] \arrow[u, "Ty_\mathcal{A}"] \arrow[lu, phantom, bend left, "\overset{T\varphi_L}{\Leftarrow}"] \arrow[ru, "y_{T\mathcal{A}}"'] \arrow[ru, phantom, bend right, "\overset{\omega_2}{\Leftarrow}"] & & & &
\end{tikzcd}$$

is a left extension as a consequence of TL being P-admissible (thus the left extension $\lambda_\mathcal{A} \cdot TR_L$ in Proposition 3.3.10 being preserved), we have the result. □

We now have everything required to complete the proof of the main theorem.

Theorem 3.3.21. *In the statement of Theorem 3.3.8 (d) implies (a).*

Proof. Firstly, we observe that each z_x is $T_\mathcal{P}$-adm-cocontinuous as a consequence of Proposition 3.3.20. It follows that we have the left extensions

$$\begin{tikzcd}
T^2 P \mathcal{A} \arrow[r, "Tz_x"] & T P \mathcal{A} \arrow[r, "z_x"] & P \mathcal{A} \\
T^2 \mathcal{A} \arrow[u, "T^2 y_\mathcal{A}"] \arrow[r, "Tx"'] \arrow[ru, phantom, "\Uparrow T\xi_x"] & T \mathcal{A} \arrow[u, "Ty_\mathcal{A}"] \arrow[r, "x"'] \arrow[ru, phantom, "\Uparrow \xi_x"] & \mathcal{A} \arrow[u, "y_\mathcal{A}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3 P \mathcal{A} \arrow[r, "T^2 z_x"] & T^2 P \mathcal{A} \arrow[r, "Tz_x"] & T P \mathcal{A} \arrow[r, "z_x"] & P \mathcal{A} \\
T^3 \mathcal{A} \arrow[u, "T^3 y_\mathcal{A}"] \arrow[r, "T^2 x"'] \arrow[ru, phantom, "\Uparrow T^2\xi_x"] & T^2 \mathcal{A} \arrow[u, "T^2 y_\mathcal{A}"] \arrow[r, "Tx"'] \arrow[ru, phantom, "\Uparrow T\xi_x"] & T \mathcal{A} \arrow[u, "Ty_\mathcal{A}"] \arrow[r, "x"'] \arrow[ru, phantom, "\Uparrow \xi_x"] & \mathcal{A} \arrow[u, "y_\mathcal{A}"']
\end{tikzcd}$$

upon noting that each $T^2 y_\mathcal{A}$ and $T^3 y_\mathcal{A}$ is P-admissible.

Secondly, we check that each $(P\mathcal{A}, z_x)$ is a pseudo T-algebra. We define our algebra structure maps as the unique solutions to the following left extension problems (and note they are invertible as they are isomorphisms of left extensions by Proposition 3.3.18)

$$\begin{tikzcd}
& TP\mathcal{A} \arrow[dr, "z_x"] \arrow[d, phantom, "\Uparrow\sigma_x"] & \\
P\mathcal{A} \arrow[ur, "u_{P\mathcal{A}}"] \arrow[rr, "\mathrm{id}_{P\mathcal{A}}"] & \arrow[d, phantom, "\overset{\mathrm{id}}{\Longleftarrow}"] & P\mathcal{A} \\
& & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"'] \arrow[ull, "y_{\mathcal{A}}"]
\end{tikzcd}
=
\begin{tikzcd}
& TP\mathcal{A} \arrow[r, "z_x"] & P\mathcal{A} \\
P\mathcal{A} \arrow[ur, "u_{P\mathcal{A}}"] \arrow[r, phantom, "\Uparrow u_{y_{\mathcal{A}}}"] & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow\xi_x"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"'] \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[ur, "u_{\mathcal{A}}"'] \arrow[urr, bend right, "\mathrm{id}_{\mathcal{A}}"'] & \cong &
\end{tikzcd}$$

$$\begin{tikzcd}
& TP\mathcal{A} \arrow[dr, "z_x"] \arrow[d, phantom, "\Uparrow\delta_x"] & \\
T^2P\mathcal{A} \arrow[ur, "m_{P\mathcal{A}}"] \arrow[r, "Tz_x"] & TP\mathcal{A} \arrow[r, "z_x"] & P\mathcal{A} \\
T^2\mathcal{A} \arrow[u, "T^2y_{\mathcal{A}}"] \arrow[r, "Tx"'] \arrow[ur, phantom, "\Uparrow T\xi_x"] & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow\xi_x"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}
=
\begin{tikzcd}
& TP\mathcal{A} \arrow[r, "z_x"] & P\mathcal{A} \\
T^2P\mathcal{A} \arrow[ur, "m_{P\mathcal{A}}"] \arrow[r, phantom, "\Uparrow m_{y_{\mathcal{A}}}"] & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow\xi_x"] \arrow[d, phantom, "\cong"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"'] \\
T^2\mathcal{A} \arrow[u, "T^2y_{\mathcal{A}}"] \arrow[ur, "m_{\mathcal{A}}"'] \arrow[r, "Tx"'] & T\mathcal{A} \arrow[ur, "x"'] &
\end{tikzcd}$$

Note that these are the axioms for ξ_x to exhibit $y_{\mathcal{A}}$ as a pseudo T-morphism. To check that the algebra structure coherence axioms are satisfied, we note that the equalities

$$\begin{tikzcd}
& T^2P\mathcal{A} \arrow[r, phantom, "\Uparrow T\sigma_x"'] \arrow[dr, "Tz_x"] \arrow[rr, "m_{P\mathcal{A}}"] & & TP\mathcal{A} \arrow[dr, "z_x"] \arrow[d, phantom, "\Uparrow\delta_x"] & \\
TP\mathcal{A} \arrow[ur, "Tu_{P\mathcal{A}}"] \arrow[rrru, bend left, "\mathrm{id}_{TP\mathcal{A}}", "\cong"'] \arrow[rr, "\mathrm{id}"] & & TP\mathcal{A} \arrow[rr, "z_x"] & & P\mathcal{A} \\
T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[urr, "Ty_{\mathcal{A}}"'] \arrow[u, phantom, shift left=6ex, "="] \arrow[rrrr, "x"'] & & & \Uparrow\xi_x & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}
=
\begin{tikzcd}
TP\mathcal{A} \arrow[r, "z_x"] & P\mathcal{A} \\
T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow\xi_x"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}$$

$$\begin{tikzcd}
& & & TP\mathcal{A} \arrow[ddr, bend left, "z_x"] \arrow[d, phantom, "\Uparrow\delta_x"] & \\
& T^2P\mathcal{A} \arrow[urr, "m_{P\mathcal{A}}"] \arrow[rr, "Tz_x"] \arrow[d, phantom, "\Uparrow u_{z_x}^{-1}"] & & TP\mathcal{A} \arrow[dr, "z_x"] \arrow[d, phantom, "\Uparrow\sigma_x"] & \\
TP\mathcal{A} \arrow[ur, "u_{TP\mathcal{A}}"] \arrow[uurrr, bend left, "\mathrm{id}_{TP\mathcal{A}}", "\cong"'] \arrow[rr, "z_x"'] & & P\mathcal{A} \arrow[ur, "u_{P\mathcal{A}}"] \arrow[rr, "\mathrm{id}_{P\mathcal{A}}"'] & = & P\mathcal{A} \\
T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[rrrr, "x"'] & \Uparrow\xi_x & & & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"'] \arrow[ull, "y_{\mathcal{A}}"]
\end{tikzcd}
=
\begin{tikzcd}
TP\mathcal{A} \arrow[r, "z_x"] & P\mathcal{A} \\
T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow\xi_x"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}$$

and the equality between

$$\begin{tikzcd}
T^2P\mathcal{A} \arrow[r, phantom, "\cong"] \arrow[rr, bend left, "m_{P\mathcal{A}}"] & T^2P\mathcal{A} \arrow[r, "m_{P\mathcal{A}}"] \arrow[dr, "Tz_x"] \arrow[d, phantom, "\Uparrow T\delta_x"] & TP\mathcal{A} \arrow[dr, "z_x"] \arrow[d, phantom, "\Uparrow \delta_x"] & \\
T^3P\mathcal{A} \arrow[u, "m_{TP\mathcal{A}}"] \arrow[ur, "Tm_{P\mathcal{A}}"] \arrow[r, "T^2z_x"] & T^2P\mathcal{A} \arrow[r, "Tz_x"] & TP\mathcal{A} \arrow[r, "z_x"] & P\mathcal{A} \\
T^3\mathcal{A} \arrow[u, "T^3y_{\mathcal{A}}"] \arrow[r, "T^2x"'] \arrow[ur, phantom, "\Uparrow T^2\xi_x"] & T^2\mathcal{A} \arrow[u, "T^2y_{\mathcal{A}}"] \arrow[r, "Tx"'] \arrow[ur, phantom, "\Uparrow T\xi_x"] & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow \xi_x"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & TP\mathcal{A} \arrow[ddr, bend left, "z_x"] \arrow[d, phantom, "\Uparrow \delta_x"] & \\
 & T^2P\mathcal{A} \arrow[ur, "m_{P\mathcal{A}}"] \arrow[r, "Tz_x"] \arrow[d, phantom, "\Uparrow m_{z_x}^{-1}"] & TP\mathcal{A} \arrow[dr, "z_x"] \arrow[d, phantom, "\Uparrow \delta_x"] & \\
T^3P\mathcal{A} \arrow[ur, "m_{TP\mathcal{A}}"] \arrow[r, "T^2z_x"] & T^2P\mathcal{A} \arrow[ur, "m_{P\mathcal{A}}"] \arrow[r, "Tz_x"] & TP\mathcal{A} \arrow[r, "z_x"] & P\mathcal{A} \\
T^3\mathcal{A} \arrow[u, "T^3y_{\mathcal{A}}"] \arrow[r, "T^2x"'] \arrow[ur, phantom, "\Uparrow T^2\xi_x"] & T^2\mathcal{A} \arrow[u, "T^2y_{\mathcal{A}}"] \arrow[r, "Tx"'] \arrow[ur, phantom, "\Uparrow T\xi_x"] & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow \xi_x"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}$$

easily follow from the respective conditions on $(\mathcal{A}, x)$ being a pseudo T-algebra and the definitions of δ_x and σ_x.

We now use the above to define our KZ doctrine

$$\widetilde{P} \colon \text{ps-}T\text{-alg} \to \text{ps-}T\text{-alg}$$

We use the assignment on objects $(\mathcal{A}, x) \mapsto (P\mathcal{A}, z_x)$. We take our units as the pseudo T-morphisms $(y_{\mathcal{A}}, \xi_x) : (\mathcal{A}, x) \to (P\mathcal{A}, z_x)$. Now suppose that we are given a pseudo T-morphism $(F, \phi) : (\mathcal{A}, x) \to (P\mathcal{B}, z_r)$, where $(P\mathcal{B}, z_r) = \widetilde{P}(\mathcal{B}, r)$, as in the diagram

$$\begin{tikzcd}
(P\mathcal{A}, z_x) \arrow[r, "(\overline{F}{,}\overline{\phi})"] \arrow[dr, phantom, "\overset{c_F}{\Leftarrow}"] & (P\mathcal{B}, z_r) \\
(\mathcal{A}, x) \arrow[u, "(y_{\mathcal{A}}{,}\xi_x)"] \arrow[ur, "(F{,}\phi)"'] &
\end{tikzcd}$$

Since z_r is T_P-cocontinuous, we may apply Proposition 3.2.14 to find a lax T-morphism

$\left(\overline{F}, \overline{\phi}\right)$ as above. Indeed the lax structure map $\overline{\phi}$ is given as the unique solution to

$$\begin{tikzcd}
 & TP\mathcal{A} \arrow[r, "z_x"] \arrow[d, phantom, "\Uparrow\xi_x"] & P\mathcal{A} \arrow[dd, "\overline{F}"] \\
T\mathcal{A} \arrow[ur, "Ty_{\mathcal{A}}"] \arrow[r, "x"] \arrow[dr, "TF"'] & \mathcal{A} \arrow[ur, "y_{\mathcal{A}}"'] \arrow[dr, "F"] \arrow[d, phantom, "\Uparrow\phi"] \arrow[r, phantom, "\Uparrow c_F"] & {} \\
 & TP\mathcal{B} \arrow[r, "z_r"'] & P\mathcal{B}
\end{tikzcd}
=
\begin{tikzcd}
 & TP\mathcal{A} \arrow[r, "z_x"] \arrow[dd, "T\overline{F}"'] & P\mathcal{A} \arrow[dd, "\overline{F}"] \\
T\mathcal{A} \arrow[ur, "Ty_{\mathcal{A}}"] \arrow[dr, "TF"'] \arrow[r, phantom, "\Uparrow Tc_F"] & {} \arrow[r, phantom, "\Uparrow\overline{\phi}"] & {} \\
 & TP\mathcal{B} \arrow[r, "z_r"'] & P\mathcal{B}
\end{tikzcd}$$

But we notice that

$$\begin{tikzcd}
TP\mathcal{A} \arrow[r, "T\overline{F}"] & TP\mathcal{B} \arrow[r, "z_r"] & P\mathcal{B} \\
T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[ur, "TF"'] \arrow[ur, phantom, bend left, "\overset{Tc_F}{\Leftarrow}"] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
TP\mathcal{A} \arrow[r, "z_x"] & P\mathcal{A} \arrow[rr, "\overline{F}"] & & P\mathcal{B} \\
T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "x"'] \arrow[drr, bend right, "TF"'] \arrow[ur, phantom, "\Uparrow\xi_x"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"'] \arrow[urr, "F"'] \arrow[urr, phantom, bend left, "\overset{c_F}{\Leftarrow}"] \arrow[dr, phantom, "\cong\phi"] & & \\
 & & TP\mathcal{B} \arrow[uur, "z_r"'] &
\end{tikzcd}$$

are both left extensions since z_r is T_P-cocontinuous and $Ty_{\mathcal{A}}$ is P-admissible respectively. It follows that the lax T-morphism structure map $\overline{\phi}$ is an isomorphism of left extensions, making $\left(\overline{F}, \overline{\phi}\right)$ a pseudo T-morphism. Of course, if we only assume (F, ϕ) to be a lax T-morphism then we can only expect $\overline{F}$ to admit a lax T-morphism structure.

We now check that such left extensions are preserved by other left extensions of this form. Suppose we are given two left extensions of pseudo T-algebras and pseudo T-morphisms

$$\begin{tikzcd}
(P\mathcal{A}, z_x) \arrow[r, "(\overline{F}{,}\overline{\phi})"] & (P\mathcal{B}, z_r) \\
(\mathcal{A}, x) \arrow[u, "(y_{\mathcal{A}}{,}\xi_x)"] \arrow[ur, "(F{,}\phi)"'] \arrow[ur, phantom, bend left, "\overset{c_F}{\Leftarrow}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(P\mathcal{B}, z_r) \arrow[r, "(\overline{G}{,}\overline{\sigma})"] & (P\mathcal{C}, z_h) \\
(\mathcal{B}, r) \arrow[u, "(y_{\mathcal{B}}{,}\xi_r)"] \arrow[ur, "(G{,}\sigma)"'] \arrow[ur, phantom, bend left, "\overset{c_G}{\Leftarrow}"] &
\end{tikzcd}$$

To see that

$$\begin{tikzcd}
(P\mathcal{A}, z_x) \arrow[r, "(\overline{F}{,}\overline{\phi})"] & (P\mathcal{B}, z_r) \arrow[r, "(\overline{G}{,}\overline{\sigma})"] & (P\mathcal{C}, z_h) \\
 & (P\mathcal{B}, z_r) \arrow[ur, "(\overline{G}{,}\overline{\sigma})"'] & \\
(\mathcal{A}, x) \arrow[uu, "(y_{\mathcal{A}}{,}\xi_x)"] \arrow[ur, "(F{,}\phi)"'] \arrow[ur, phantom, bend left, "\overset{(\overline{G}{,}\overline{\sigma})c_F}{\Leftarrow}"] & &
\end{tikzcd}$$

is a left extension we need only observe that the T-morphism structure on $\overline{GF}$ resulting from an application of Proposition 3.2.14 (on the outside diagram) is given by composing $\overline{\phi}$ and $\overline{\sigma}$ as above. This is shown by pasting the defining diagram for $\overline{\phi}$ with $\overline{\sigma}$ which gives

$$\begin{tikzcd}
 & TP\mathcal{A} \arrow[rr, "z_x"] & & P\mathcal{A} \arrow[dd, "\overline{F}"] & & & TP\mathcal{A} \arrow[rr, "z_x"] \arrow[dd, "T\overline{F}"'] & & P\mathcal{A} \arrow[dd, "\overline{F}"] \\
T\mathcal{A} \arrow[ur, "Ty_{\mathcal{A}}"] \arrow[r, "x"] \arrow[dr, "TF"'] & \mathcal{A} \arrow[urr, "y_{\mathcal{A}}"] \arrow[drr, "F"] \arrow[u, phantom, "\Uparrow \xi_x"] \arrow[d, phantom, "\Uparrow \phi"] & \Uparrow c_F & & = & T\mathcal{A} \arrow[ur, "Ty_{\mathcal{A}}"] \arrow[dr, "TF"'] \arrow[r, phantom, "\Uparrow Tc_F"] & & \Uparrow \phi & \\
 & TP\mathcal{B} \arrow[rr, "z_r"'] \arrow[dd, "T\overline{G}"'] & & P\mathcal{B} \arrow[dd, "\overline{G}"] & & & TP\mathcal{B} \arrow[rr, "z_r"'] \arrow[dd, "T\overline{G}"'] & & P\mathcal{B} \arrow[dd, "\overline{G}"] \\
 & & \Uparrow \overline{\sigma} & & & & & \Uparrow \overline{\sigma} & \\
 & TP\mathcal{C} \arrow[rr, "z_h"'] & & P\mathcal{C} & & & TP\mathcal{C} \arrow[rr, "z_h"'] & & P\mathcal{C}
\end{tikzcd} \qquad (3.3.8)$$

which is the defining diagram for the induced lax structure on $\overline{G} \cdot \overline{F}$ from an application of Proposition 3.2.14.

It is an easy consequence of Proposition 3.2.14 that each $(y_{\mathcal{A}}, \xi_x)$ is dense. Indeed since z_x T-preserves the left extension

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "\mathrm{id}_{P\mathcal{A}}"] & P\mathcal{A} \\
\mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[ur, "y_{\mathcal{A}}"'] \arrow[ur, phantom, "=", near end, shift left=3]
\end{tikzcd}$$

(as well the resulting left extension) the density property may be lifted to pseudo-T-algebras applying Proposition 3.2.14. □

3.4 Consequences and Examples

In this section we point out some consequences of Theorem 3.3.8 proven in the previous section, and in particular some properties of the lifted KZ doctrine $\widetilde{P}$ on ps-T-alg. Before considering the properties of $\widetilde{P}$, we mention two easy corollaries.

Corollary 3.4.1. *Pseudo-distributive laws over KZ pseudomonads are essentially unique.*

Proof. As shown in Lemma 3.3.14, the modification components $\omega_2^{\mathcal{A}}$ exhibit $\lambda_{\mathcal{A}}$ as a left extension. The last two coherence axioms of a pseudo-distributive law over a KZ pseudomonad then define the components $\omega_1^{\mathcal{A}}$ and $\omega_3^{\mathcal{A}}$ as unique solutions to a left extension problem. Note that $\omega_4^{\mathcal{A}}$ is also defined as the unique solution to a left extension problem (see the proof of 3.3.14). The essential uniqueness of left extensions then tells us these pseudo-distributive laws are essentially unique. □

Corollary 3.4.2. *When the conditions of Theorem 3.3.8 are met, the lifted pseudomonad* *arising from the pseudo-distributive law is automatically KZ.*

Proof. As a consequence of the essential uniqueness of pseudo-distributive laws over KZ pseudomonads, any lifted pseudomonad must be equivalent to the KZ pseudomonad whose existence is guaranteed by Theorem 3.3.8. ☐

3.4.1 The Lifted KZ Doctrines

We first check that in addition to having a lifting to ps-T-alg, we have a lifting to the 2-category of pseudo-T-algebras, lax (or oplax) T-morphisms, and T-transformations.

Proposition 3.4.3. *Suppose any of the equivalent conditions of Theorem 3.3.8 are satisfied. Then*

(a) P lifts to a KZ doctrine $\widetilde{P}_{\mathrm{oplax}}$ on ps-T-alg$_{\mathrm{oplax}}$*;*

(b) P lifts to a KZ doctrine $\widetilde{P}_{\mathrm{lax}}$ on ps-T-alg$_{\mathrm{lax}}$*;*

Proof. (a) : P lifts to a KZ doctrine $\widetilde{P}_{\mathrm{oplax}}$ on ps-T-alg$_{\mathrm{oplax}}$ since given any oplax structure cell φ on a map $F\colon \mathcal{A} \to P\mathcal{B}$ as below

$$\begin{tikzcd}
(P\mathcal{A}, z_x) \arrow[r, "(\overline{F},\overline{\varphi})"] & (P\mathcal{B}, z_r) \\
(\mathcal{A}, x) \arrow[u, "(y_{\mathcal{A}},\xi_x)"] \arrow[ur, "(F,\varphi)"'] \arrow[ur, phantom, shift left=3ex, "\overset{c_F}{\Longleftarrow}"]
\end{tikzcd}$$

we get an oplax structure cell $\overline{\varphi}$ given as unique the solution to

$$\begin{tikzcd}
TP\mathcal{B} \arrow[r, "z_r"] & P\mathcal{B} \\
TP\mathcal{A} \arrow[u, "T\overline{F}"] \arrow[r, "z_x"] \arrow[ur, phantom, "\Uparrow\overline{\varphi}"] & P\mathcal{A} \arrow[u, "\overline{F}"'] \\
T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow\xi_x"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}
=
\begin{tikzcd}
 & TP\mathcal{B} \arrow[r, "z_r"] & P\mathcal{B} & \\
TP\mathcal{A} \arrow[ur, "T\overline{F}"] \arrow[r, phantom, "\overset{Tc_F^{-1}}{\Longrightarrow}"] & {} \arrow[r, phantom, "\Uparrow\varphi"] & {} \arrow[r, phantom, "\overset{c_F}{\Longrightarrow}"] & P\mathcal{A} \arrow[ul, "\overline{F}"'] \\
 & T\mathcal{A} \arrow[ul, "Ty_{\mathcal{A}}"] \arrow[uu, "TF"'] \arrow[r, "x"'] & \mathcal{A} \arrow[uu, "F"] \arrow[ur, "y_{\mathcal{A}}"'] &
\end{tikzcd}$$

with the coherence conditions for $\overline{\varphi}$ being an oplax T-morphism structure following from Proposition 3.3.18 (Part 4). Note that the induced oplax structure when composed by an oplax T-morphism $\left(\overline{G},\overline{\tau}\right)$ as below

$$\begin{tikzcd}
(P\mathcal{A}, z_x) \arrow[r, "(\overline{F},\overline{\varphi})"] & (P\mathcal{B}, z_r) \arrow[r, "(\overline{G},\overline{\tau})"] & (P\mathcal{C}, z_k) \\
(\mathcal{A}, x) \arrow[u, "(y_{\mathcal{A}},\xi_x)"] \arrow[ur, "(F,\varphi)"'] \arrow[ur, phantom, shift left=3ex, "\overset{c_F}{\Longleftarrow}"] & &
\end{tikzcd}$$

is still $\left(\overline{G}, \overline{\tau}\right) \cdot \left(\overline{F}, \overline{\varphi}\right)$. To see that $\left(\overline{F}, \overline{\varphi}\right)$ is a left extension in the sense of transformations, suppose we are given a transformation $\sigma \colon (F, \varphi) \longrightarrow (H, \psi) \cdot (y_{\mathcal{A}}, \xi_x)$, then the induced cell $\overline{\sigma} \colon \overline{F} \longrightarrow H$ is a transformation since

$$\begin{tikzcd}
TP\mathcal{B} \arrow[r, "z_r"] & P\mathcal{B} & & TP\mathcal{B} \arrow[r, "z_r"] & P\mathcal{B} \\
TP\mathcal{A} \arrow[r, "z_x"] \arrow[u, "T\overline{F}"'] \arrow[u, bend left=60, "TH"] \arrow[u, phantom, shift left=14, "\overset{T\overline{\sigma}}{\Longleftarrow}"] \arrow[ur, phantom, "\Uparrow\overline{\varphi}"] & P\mathcal{A} \arrow[u, "\overline{F}"'] & = & TP\mathcal{A} \arrow[r, "z_x"] \arrow[u, bend left=60, "TH"] \arrow[ur, phantom, "\Uparrow\psi"] & P\mathcal{A} \arrow[u, "\overline{F}"'] \arrow[ul, bend left=30, "H"] \arrow[u, phantom, shift left=14, "\overset{\overline{\sigma}}{\Longleftarrow}"] \\
T\mathcal{A} \arrow[r, "x"'] \arrow[u, "Ty_{\mathcal{A}}"] \arrow[ur, phantom, "\Uparrow\xi_x"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"'] & & T\mathcal{A} \arrow[r, "x"'] \arrow[u, "Ty_{\mathcal{A}}"] \arrow[ur, phantom, "\Uparrow\xi_x"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}$$

as a consequence of σ being a transformation. By Proposition 3.2.14 the density property is still valid in the setting of oplax T-morphisms; this being why we proved the general case of Proposition 3.2.14 in terms of composites of lax and oplax morphisms.

(b) : The proof that P lifts to a KZ doctrine $\widetilde{P}_{\text{lax}}$ on ps-T-alg$_{\text{lax}}$ is essentially given in Theorem 3.3.21. □

We now check that the KZ structure cell $\theta \colon Py \longrightarrow yP$ remains the same upon lifting to algebras.

Proposition 3.4.4. *Suppose any of the equivalent conditions of Theorem 3.3.8 are satisfied. Then the KZ structure cell* $\theta \colon Py \longrightarrow yP$ *for* P *is also the KZ structure cell for* $\widetilde{P}$.

Proof. Recall that the components of θ are recovered as the induced cells out of the left extensions $Py_{\mathcal{A}}$ as in the diagram below

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "Py_{\mathcal{A}}"] & P^2\mathcal{A} \\
\mathcal{A} \arrow[r, "y_{\mathcal{A}}"'] \arrow[u, "y_{\mathcal{A}}"] \arrow[ur, phantom, "\Uparrow y_{y_{\mathcal{A}}}^{-1}"] & P\mathcal{A} \arrow[u, "y_{P\mathcal{A}}"']
\end{tikzcd}$$

such that the composite with this diagram is an identity. Now apply Proposition 3.2.14 to this naturality square noting that each $y_{\mathcal{A}}$ extends to a pseudo T-morphism $(y_{\mathcal{A}}, \xi_x)$ in order to recover the components of the KZ structure cell for $\widetilde{P}$. □

If we are to study the lifted KZ doctrine $\widetilde{P}$, we should consider the $\widetilde{P}$-cocomplete objects and the $\widetilde{P}$-admissible maps. We start with the former.

Algebraic cocompleteness is usually defined by asking that the underlying object be cocomplete, and that the algebra structure map be separately cocontinuous. The following proposition justifies this definition.

Proposition 3.4.5. *Suppose any of the equivalent conditions of Theorem 3.3.8 are satisfied. Then a pseudo T-algebra $(\mathcal{A}, x)$ is*

(a) $\widetilde{P}$-cocomplete iff $\mathcal{A}$ is P-cocomplete and $x \colon T\mathcal{A} \to \mathcal{A}$ is T_P-cocontinuous;

(b) $\widetilde{P}_{\text{lax}}$-cocomplete iff $\mathcal{A}$ is P-cocomplete and $x \colon T\mathcal{A} \to \mathcal{A}$ is T_P-cocontinuous;

(c) $\widetilde{P}_{\text{oplax}}$-cocomplete iff $\mathcal{A}$ is P-cocomplete.

Moreover, the pseudo/lax/oplax T-morphisms (F, ϕ) which are $\widetilde{P}/\widetilde{P}_{\text{lax}}/\widetilde{P}_{\text{oplax}}$-cocontinuous are all classified by those maps for which the underlying F is P-cocontinuous.

Proof. We start off by proving part (a).

$(\Longrightarrow)$: Suppose that $(\mathcal{A}, x)$ is a $\widetilde{P}$-cocomplete pseudo T-algebra. Then, by doctrinal adjunction [27], the pseudo T-morphism $(y_{\mathcal{A}}, \xi_x)$ has a reflection left adjoint $\left((y_{\mathcal{A}})_*, \left(\xi_x^{-1}\right)_*\right)$ for which $\left(\xi_x^{-1}\right)_*$ is defined by the mates correspondence and is invertible. That is, we have isomorphisms

$$\begin{tikzcd}
TP\mathcal{A} \arrow[r, "z_x"] \arrow[rd, phantom, "\Downarrow \xi_x^{-1}"] & P\mathcal{A} \\
T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[r, "x"'] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
TP\mathcal{A} \arrow[r, "z_x"] \arrow[d, "T(y_{\mathcal{A}})_*"'] \arrow[rd, phantom, "\Downarrow \left(\xi_x^{-1}\right)_*"] & P\mathcal{A} \arrow[d, "(y_{\mathcal{A}})_*"] \\
T\mathcal{A} \arrow[r, "x"'] & \mathcal{A}
\end{tikzcd}$$

Now $(y_{\mathcal{A}})_* \dashv y_{\mathcal{A}}$ via a reflection adjoint so $\mathcal{A}$ is P-cocomplete. We thus check that $x \colon T\mathcal{A} \to \mathcal{A}$ is T_P-cocontinuous. Suppose we are given a left extension as on the left

$$\begin{tikzcd}
P\mathcal{D} \arrow[r, "\overline{F}"] & \mathcal{A} \\
\mathcal{D} \arrow[u, "y_{\mathcal{D}}"] \arrow[ru, "F"'] \arrow[u, phantom, shift right=4ex, "\Uparrow c_F"]
\end{tikzcd}
\qquad
\begin{tikzcd}
TP\mathcal{D} \arrow[r, "\overline{F}"] & T\mathcal{A} \arrow[r, "x"] & \mathcal{A} \\
T\mathcal{D} \arrow[u, "Ty_{\mathcal{D}}"] \arrow[ru, "TF"'] \arrow[u, phantom, shift right=4ex, "\Uparrow Tc_F"]
\end{tikzcd}$$

We check that the right diagram is a left extension. We first note this is equivalent to showing that x T-preserves left extensions as on the left below

$$\begin{tikzcd}
P\mathcal{D} \arrow[r, "PF"] & P\mathcal{A} \arrow[r, "(y_{\mathcal{A}})_*"] & \mathcal{A} \\
\mathcal{D} \arrow[u, "y_{\mathcal{D}}"] \arrow[r, "F"'] \arrow[ru, phantom, "\Uparrow c_{y_{\mathcal{A}} \cdot F}"] & \mathcal{A} \arrow[u, "y_{\mathcal{A}}"] \arrow[ru, "\mathrm{id}_{\mathcal{A}}"'] \arrow[u, phantom, shift right=4ex, "\Uparrow c_{\mathrm{id}_{\mathcal{A}}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
TP\mathcal{D} \arrow[r, "TPF"] & TP\mathcal{A} \arrow[r, "T(y_{\mathcal{A}})_*"] & T\mathcal{A} \arrow[r, "x"] & \mathcal{A} \\
T\mathcal{D} \arrow[u, "Ty_{\mathcal{D}}"] \arrow[r, "TF"'] \arrow[ru, phantom, "\Uparrow Tc_{y_{\mathcal{A}} \cdot F}"] & T\mathcal{A} \arrow[u, "Ty_{\mathcal{A}}"] \arrow[ru, "T\mathrm{id}_{\mathcal{A}}"'] \arrow[u, phantom, shift right=4ex, "\Uparrow Tc_{\mathrm{id}_{\mathcal{A}}}"]
\end{tikzcd}$$

and so it suffices to check the right diagram is a left extension. This is seen upon pasting with the isomorphism $\left(\xi_x^{-1}\right)_*$ as z_x is T_P-cocontinuous and $(y_{\mathcal{A}})_*$ is a left adjoint (and hence preserves all left extensions).

$(\Longleftarrow)$: Suppose that $\mathcal{A}$ is P-cocomplete and x is T_P-cocontinuous. Then $(\mathcal{A}, x)$ is $\widetilde{P}$-cocomplete as $(\mathcal{A}, x)$ admits left extensions along $(y_{\mathcal{A}}, \xi_x)$ by Proposition 3.2.14, and showing that such left extensions admit a pseudo T-morphism structure and are preserved is a similar calculation to that in the proof of Theorem 3.3.21.

(b) : The proof of the classification of $\widetilde{P}_{\text{lax}}$-cocomplete pseudo P-algebras is almost the same (as the reflection left adjoint must again be pseudo by doctrinal adjunction [27]), and so we omit the details.

(c) : The $\widetilde{P}_{\text{oplax}}$-cocomplete pseudo P-algebras are those with an underlying P-cocomplete object, as a consequence of doctrinal adjunction [27].

That the T-morphisms (F, ϕ) which are $\widetilde{P}/\widetilde{P}_{\text{lax}}/\widetilde{P}_{\text{oplax}}$-cocontinuous are all classified by those morphisms for which the underlying F is P-cocontinuous is a straightforward calculation. Indeed, given a pseudo T-morphism (F, ϕ) for which F is P-cocontinuous, checking that (F, ϕ) is then $\widetilde{P}$-cocontinuous requires only checking a coherence condition (similar to 3.3.8). Conversely, given that (F, ϕ) is $\widetilde{P}$-cocontinuous, that is, a pseudo $\widetilde{P}$-morphism on ps-T-alg, we know the underlying F must be a pseudo P-morphism on $\mathscr{C}$ (by forgetting that certain morphisms and 2-cells are T-algebraic), so that F is P-cocontinuous. The $\widetilde{P}_{\text{lax}}$ and $\widetilde{P}_{\text{oplax}}$ case may be similarly seen. □

Proposition 3.4.6. *Suppose any of the equivalent conditions of Theorem 3.3.8 are satisfied. Assume* $(L, \alpha) : (\mathcal{A}, x) \longrightarrow (\mathcal{B}, y)$ *is a pseudo* T*-morphism and* $L \colon \mathcal{A} \longrightarrow \mathcal{B}$ *is* P*-admissible. Then* (L, α) *is* $\widetilde{P}$*-admissible if and only if for every* $\widetilde{P}$*-cocomplete pseudo* T*-algebra* $(\mathcal{C}, z)$ *and pseudo* T*-morphism* (I, ξ) *as in the diagram*

$$\begin{tikzcd}
(\mathcal{B}, y) \arrow[r, "{(R,\beta)}"] & (\mathcal{C}, z) \\
& (\mathcal{A}, x) \arrow[ul, "{(L,\alpha)}"] \arrow[u, "{(I,\xi)}"'] \arrow[from=1-1, to=2-2, phantom, shift left=3, "\overset{\delta}{\Leftarrow}"]
\end{tikzcd}$$

the induced lax structure cell β *on the underlying left extension* R *as in Proposition 3.2.14 is invertible. Moreover, for pseudo, lax and oplax* (L, α) *respectively,*

1. (L, α) *is* $\widetilde{P}$*-admissible iff* $\widetilde{P}(L, \alpha)$ *has a pseudo right adjoint;*
2. (L, α) *is* $\widetilde{P}_{\text{lax}}$*-admissible iff* $\widetilde{P}(L, \alpha)$ *is pseudo;*
3. (L, α) *is* $\widetilde{P}_{\text{oplax}}$*-admissible iff* $\widetilde{P}(L, \alpha)$ *has a pseudo right adjoint.*

Proof. The first part of this proposition follows an equivalent characterization of P-admissibility as given by Bunge and Funk (discussed in [6] and Chapter 2), along with Proposition 3.2.14.

The last three properties are a direct consequence of doctrinal adjunction [27]. □

Remark 3.4.7. Note that the conditions of $\widetilde{P}/\widetilde{P}_{\mathrm{oplax}}$-admissibility are analogous to asking a Guitart exactness condition is satisfied [20] (in the presence of some additional structure, and in the context of pointwise left extensions). However, we omit discussion of this as it would take us beyond the scope of this paper.

Remark 3.4.8. Note that if P (and thus $\widetilde{P}$) is locally fully faithful, and (L,α) is a lax T-morphism, then $\widetilde{P}(L,\alpha)$ being pseudo implies (L,α) is. Indeed, the lax structure cell α when whiskered by $y_{\mathcal{A}}$ is invertible (a direct consequence of how the structure cell of $\widetilde{P}(L,\alpha)$ is defined in Proposition 3.2.14). As $y_{\mathcal{A}}$ is fully faithful, this means α is invertible. Hence, in this case, Statement 2 of the above proposition is equivalent to saying (L,α) is pseudo.

Given a KZ doctrine P on a 2-category $\mathscr{C}$ we have an equivalence given by composition with the unit $y_{\mathcal{A}}$, namely $\mathscr{C}_{\mathrm{ccts}}(P\mathcal{A},\mathcal{B}) \simeq \mathscr{C}(\mathcal{A},\mathcal{B})$, with $\mathscr{C}_{\mathrm{ccts}}(P\mathcal{A},\mathcal{B})$ containing left extensions of maps $\mathcal{A} \to \mathcal{B}$ along the unit $y_{\mathcal{A}}$. This is clearly essentially surjective as for an $F\colon \mathcal{A} \to \mathcal{B}$ we may take $\overline{F}\colon P\mathcal{A} \to \mathcal{B}$, and fully faithful as $y_{\mathcal{A}}$ is dense. We can thus recover Im and Kelly's following result.

Corollary 3.4.9 (Im-Kelly [22]). *Suppose we are given a 2-category $\mathscr{C}$ equipped with a pseudomonad (T,u,m) and a KZ doctrine (P,y). Suppose any of the equivalent conditions of Theorem 3.3.8 are met. Then for every pair of pseudo T-algebras $(\mathcal{A},x)$ and $(\mathcal{B},r)$ where $\mathcal{B}$ is P-cocomplete, composition with the unit $(y_{\mathcal{A}},\xi_x)$ defines the equivalence*

$$\mathbf{Oplax}\left[(\mathcal{A},x),(\mathcal{B},r)\right] \simeq \mathbf{Oplax}_{\mathrm{ccts}}\left[(P\mathcal{A},z_x),(\mathcal{B},r)\right]$$

where a morphism of pseudo T-algebras is cocontinuous when the underlying morphism is. Suppose further that r is T_P-cocontinuous. Then composition with the unit $(y_{\mathcal{A}},\xi_x)$ also defines the equivalences

$$\mathbf{Lax}\left[(\mathcal{A},x),(\mathcal{B},r)\right] \simeq \mathbf{Lax}_{\mathrm{ccts}}\left[(P\mathcal{A},z_x),(\mathcal{B},r)\right]$$

$$\mathbf{Pseudo}\left[(\mathcal{A},x),(\mathcal{B},r)\right] \simeq \mathbf{Pseudo}_{\mathrm{ccts}}\left[(P\mathcal{A},z_x),(\mathcal{B},r)\right]$$

Moreover, the above three equivalences restrict to P-admissible underlying morphisms.

Proof. We need only check the restriction. Note that if $\overline{L}\colon P\mathcal{A} \to \mathcal{B}$ is P-admissible then so is the composite $\overline{L} \cdot y_{\mathcal{A}} \cong L$ due to closure under composition. If L is P-admissible, then $\overline{L}$ has a right adjoint by Lemma 2.3.3, and so $P\overline{L}$ also does. □

3.4.2 The Preorder of KZ Doctrines on a 2-Category

In the following discussion of morphisms between KZ pseudomonads and doctrines we will omit most of the details, as this would take us beyond the scope of this paper. Moreover, the calculations are quite similar to those in Section 3.3.

It is the goal of this subsection to show that the 2-category of KZ pseudomonads on a 2-category $\mathscr{C}$ is biequivalent to a preorder. This is a property one might expect given the "property like structure" viewpoint [29]; and the tools of admissible maps give us a method of proving this result.

Definition 3.4.10. Given KZ pseudomonads (P, y, μ) and (P', y', μ') on a 2-category $\mathscr{C}$, a *morphism of KZ pseudomonads* $P \Longrightarrow P'$ (corresponding to a lifting of the identity on $\mathscr{C}$) consists of a pseudonatural transformation $\alpha \colon P \longrightarrow P'$ and an invertible modification

$$\begin{tikzcd}
P \arrow[r, "\alpha"] & P' \\
 & 1_{\mathscr{C}} \arrow[ul, "y"] \arrow[u, "y'"'] \arrow[ul, phantom, shift left=3, "\overset{\psi_y}{\Leftarrow}"]
\end{tikzcd}$$

such that

$$\begin{tikzcd}
P \arrow[r, "\alpha"] \arrow[d, bend right, "yP"'] \arrow[d, bend left, "Py"] \arrow[d, phantom, "\overset{\theta}{\Leftarrow}"] & P' \arrow[d, "P'y"] \arrow[drr, "P'y'"] & & \\
PP \arrow[r, "\alpha P"'] \arrow[ur, phantom, "\overset{\alpha_y}{\Leftarrow}"] & P'P \arrow[r, "P'\alpha"'] \arrow[ur, phantom, "\overset{P'\psi_y}{\Leftarrow}"] & P'P' \arrow[r, "\mu'"'] & P'
\end{tikzcd}
=
\begin{tikzcd}
 & P \arrow[r, "\alpha"] \arrow[dl, "yP"'] \arrow[d, "y'P"'] \arrow[dl, phantom, shift left=3, "\overset{\psi_y P}{\Leftarrow}"] & P' \arrow[d, bend right, "y'P'"'] \arrow[d, bend left, "P'y'"] \arrow[d, phantom, "\overset{\theta'}{\Leftarrow}"] \arrow[dl, phantom, "\overset{(y')_\alpha^{-1}}{\Leftarrow}"] & \\
PP \arrow[r, "\alpha P"'] & P'P \arrow[r, "P'\alpha"'] & P'P' \arrow[r, "\mu'"'] & P'
\end{tikzcd}$$

The reader will notice the following is similar to Lemma 3.3.14, meaning we are justified in omitting most of the details.

Lemma 3.4.11. *Given a morphism of KZ pseudomonads as above, the 2-cell ψ_y exhibits α as a left extension of y' along y.*

Proof. We first observe that $P'y \dashv \mu' \cdot P'\alpha$ (note that this right adjoint is $\overline{\alpha}$, similar to $\overline{\lambda}$ in Lemma 3.3.14) with unit η given by

$$\begin{tikzcd}
 & P'P \arrow[r, "P'\alpha"] & P'P' \arrow[dr, "\mu'"] & \\
P' \arrow[ur, "P'y"] \arrow[urr, bend right, "P'y'"'] \arrow[rrr, bend right=40, "\mathrm{id}_{P'}"'] \arrow[ur, phantom, shift right=6, "\Uparrow P'\psi_y"] & & \cong & P'
\end{tikzcd}$$

We define the counit ε as the unique 2-cell for which

$$\begin{tikzcd}
P \arrow[r, "y'P"] \arrow[dr, bend right, "\alpha"'] & P'P \arrow[r, "P'\alpha"] \arrow[rrr, bend left=30, "\mathrm{id}"] & P'P' \arrow[r, "\mu'"] & P' \arrow[r, "P'y"] & P'P & = & P \arrow[r, bend left, "yP"] \arrow[r, bend right, "Py"'] \arrow[rr, bend left=50, "y'P"] \arrow[dr, bend right, "\alpha"'] \arrow[r, phantom, "\Uparrow\theta"] & PP \arrow[r, "\alpha P"] \arrow[d, phantom, "\Uparrow\alpha_y"] & P'P \\
 & P' \arrow[ur, bend left, "y'P"] \arrow[urr, bend right, "\mathrm{id}_{P'}"'] \arrow[u, phantom, "\Uparrow(y'_\alpha)^{-1}"] & \cong & \Uparrow\varepsilon & & & & P' \arrow[ur, bend right, "P'y"'] & \Uparrow(\psi_y P)^{-1}
\end{tikzcd}$$

We will omit the triangle identities (as this is almost the same calculation as earlier). The result then follows from Remark 2.3.7 and naturality and pseudomonad coherence axioms. ☐

Remark 3.4.12. Given a morphism of KZ pseudomonads, we automatically have an invertible modification

$$\begin{tikzcd}
PP \arrow[r, "\alpha*\alpha"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\cong"] & P'P' \arrow[d, "\mu'"] \\
P \arrow[r, "\alpha"'] & P'
\end{tikzcd}$$

so that multiplication is respected. Indeed $\alpha \cdot \mu$ may be seen as a left extension of y' along $Py \cdot y$ exhibited by the bijections

$$\frac{\dfrac{\alpha \cdot \mu \;\to\; H}{\alpha \;\to\; H \cdot Py}}{y' \;\to\; H \cdot Py \cdot y} \qquad \begin{array}{l}\text{mates correspondence}\\ \text{since } \alpha \text{ is a left extension}\end{array}$$

and $\mu' \cdot \alpha * \alpha$ may be seen as left extension of y' along $yP \cdot y$ by recalling that $R_L = \mathrm{res}_L \cdot y_{\mathcal{B}}$ for admissible $L \colon \mathcal{A} \to \mathcal{B}$ (using Remark 2.3.7) and taking L to be an arbitrary component of $yP \cdot y$ with respect to P'-admissibility. In particular, noting that $P'y \dashv \mu' \cdot P'\alpha$ and $P'yP \dashv \mu' \cdot P'\alpha P$ gives us the necessary data for constructing R_L. Finally, noting that $yP \cdot y \cong Py \cdot y$ gives the result.

Definition 3.4.13. Given KZ doctrines (P, y) and (P', y') on a 2-category $\mathscr{C}$ a *morphism of KZ doctrines $P \Longrightarrow P'$* consists of the assertions that:

1. every P-admissible map is also P'-admissible;

2. for each $\mathcal{A} \in \mathscr{C}$, the resulting 2-cell exhibiting the left extension $\alpha_{\mathcal{A}}$

$$\begin{tikzcd}
P\mathcal{A} \arrow[r, "\alpha_{\mathcal{A}}"] & P'\mathcal{A} \arrow[dl, phantom, "\overset{\psi_y^{\mathcal{A}}}{\Leftarrow}"] \\
 & \mathcal{A} \arrow[ul, "y_{\mathcal{A}}"] \arrow[u, "y'_{\mathcal{A}}"']
\end{tikzcd}$$

is invertible;

3. for each $\mathcal{A}, \mathcal{B} \in \mathscr{C}$, left extensions along $y_{\mathcal{A}}$ into $P\mathcal{B}$ are preserved by $\alpha_{\mathcal{B}}$.[9]

Lemma 3.4.14. *Suppose we are given two KZ doctrines (P, y) and (P', y') on a 2-category $\mathscr{C}$, with corresponding KZ pseudomonads (P, y, μ) and (P', y', μ'). Then morphisms $P \Longrightarrow P'$ of KZ doctrines are in bijection with morphisms $P \Longrightarrow P'$ of KZ pseudomonads (identified via uniqueness of left extensions up to coherent isomorphism).*

Proof. Given that every P-admissible map is also P'-admissible, we know that $P'y$ has a right adjoint (and that we have a left extension α as above, assumed invertible). In particular, this right adjoint may be constructed as in Proposition 3.2.22, and thus we have an adjunction $P'y \dashv \mu' \cdot P'\alpha$ with unit and counit as above. The triangle identities then force the coherence condition. Pseudonaturality of α is equivalent to the preservation condition.

Conversely, given a morphism of KZ pseudomonads (which always gives rise to a usual morphism of pseudomonads) we know that every P'-cocomplete object is also P-cocomplete (as the cocomplete objects may be characterized as algebras), and similarly for homomorphisms. Hence given a P-admissible map $L \colon \mathcal{A} \to \mathcal{B}$ and map $K \colon \mathcal{A} \to \mathcal{X}$ for a P'-cocomplete (and thus also P-cocomplete) object $\mathcal{X}$, there exists a left extension $J \colon \mathcal{B} \to \mathcal{X}$ which is preserved by any P'-homomorphism (as such is necessarily a P-homomorphism also). Consequently, L must be P'-admissible. ☐

Combining this with the results of [42], yields the following proposition.

Proposition 3.4.15. *Given a 2-category $\mathscr{C}$, the assignation of [42, Theorems 4.1,4.2] underlies a biequivalence*

$$\mathbf{KZdoc}\,(\mathscr{C}) \simeq \mathbf{KZps}\,(\mathscr{C})$$

where **KZps** $(\mathscr{C})$ *is the 2-category of KZ pseudomonads, morphisms of KZ pseudomonads and isomorphisms of left extensions, and* **KZdoc** $(\mathscr{C})$ *is the preorder of KZ doctrines and morphisms of KZ doctrines.*

3.4.3 Examples

Consider the 2-monad T on locally small categories for strict monoidal categories, and take P to be the free small cocompletion KZ doctrine on locally small categories. Note that the pseudo-T-algebras are unbiased monoidal categories (equivalent to (strict) monoidal

---

[9] Consequently, components of α are P-homomorphisms.

categories [37]) and so we may write ps-T-alg $\simeq$ $\mathrm{MonCat}_{\mathrm{ps}}$ with the latter being the 2-category of monoidal categories, strong monoidal functors and monoidal transformations.

Given a monoidal category $(\mathcal{A}, \otimes)$ we may define a monoidal structure on $P\mathcal{A}$ by Day's convolution formula

$$F \otimes_{\mathrm{Day}} G := \int^{a,b \in \mathcal{A}} \mathcal{A}(-, a \otimes b) \times Fa \times Gb$$

for small presheaves F and G on $\mathcal{A}$. Note that $F \otimes_{\mathrm{Day}} G$ is then small, see [12, Section 7]. This can be shown to give a monoidal structure by the arguments of Day [11], equivalent to the structure of a pseudo-T-algebra. As the convolution algebra structure map is separately cocontinuous (and hence T_P-cocontinuous [56, Prop. 2.3.2]) we have enough of Proposition 3.3.18 to show condition (a) of Theorem 3.3.8 is met.

We thus know that T preserves P-admissible maps. This says that if we suppose that $L \colon \mathcal{A} \to \mathcal{B}$ is P-admissible, meaning that each $\mathcal{B}(L-, b)$ is a small colimit of representables, then each

$$T\mathcal{B}(TL\mathbf{-}, \mathbf{b}) = T\mathcal{B}\left[(L-, \cdots L-), (b_1, \cdots, b_n)\right] = \prod_{j=1}^{n} \mathcal{B}\left(L-, b_j\right)$$

is also a small colimit of representables.

For simplicity, we will consider the preservation of the admissibility of $L = y_{\mathcal{A}}$ (which is equivalent to preservation for all L). The existence of a pseudo-distributive law of T over P then yields the following example.

Proposition 3.4.16. *Let $X, Y \colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ be two small presheaves on $\mathcal{A}$. Then*

$$X \times Y \colon (\mathcal{A} \times \mathcal{A})^{\mathrm{op}} \to \mathbf{Set}, \qquad (a_1, a_2) \mapsto X(a_1) \times Y(a_2)$$

is a small presheaf on $\mathcal{A} \times \mathcal{A}$.

Proof. Note that $Ty_{\mathcal{A}}$ is P-admissible, and hence

$$TP\mathcal{A}(Ty_{\mathcal{A}}\mathbf{-}, \mathbf{X}) \colon (T\mathcal{A})^{\mathrm{op}} \to \mathbf{Set}$$

is a small presheaf on $T\mathcal{A}$ for each $\mathbf{X} = (X_1, \cdots, X_n)$ in $TP\mathcal{A}$. In particular, if we take

$\mathbf{X} = (X, Y)$ then

$$TP\mathcal{A}\left(y_{\mathcal{A}}-, \mathbf{X}\right) = \begin{cases} TP\mathcal{A}\left[\left(y_{\mathcal{A}}-, y_{\mathcal{A}}-\right), (X, Y)\right], & \mathbf{a} \in (\mathcal{A} \times \mathcal{A})^{\mathrm{op}} \\ \emptyset, & \text{otherwise} \end{cases}$$

$$= \begin{cases} X(-) \times Y(-), & \mathbf{a} \in (\mathcal{A} \times \mathcal{A})^{\mathrm{op}} \\ \emptyset, & \text{otherwise} \end{cases}$$

is a small presheaf on $\sum_{n \in \mathbb{N}} \mathcal{A}^n$ and so $X(-) \times Y(-)$ is a small presheaf on $\mathcal{A} \times \mathcal{A}$. □

Our results also apply to the less general setting of distributing (co)KZ doctrines over KZ doctrines. The following is such an example.

Example 3.4.17. *Consider the KZ doctrine for the free coproduct completion*

$$\mathbf{Fam}_{\Sigma} \colon \mathbf{Cat} \to \mathbf{Cat}.$$

Here a map $L \colon \mathcal{A} \to \mathcal{B}$ is $\mathbf{Fam}_{\Sigma}$-admissible when $\mathbf{Fam}_{\Sigma} L$ is a left adjoint; that is, when L is a left multiadjoint. As noted by Diers [14], this is to say that for any $Z \in \mathcal{B}$ there exists a family of morphisms $(h_i \colon LX_i \to Z)_{i \in \mathcal{I}}$ which is universal in the sense that given any $k \colon LX \to Z$ there exists a unique pair (i, f) with $i \in \mathcal{I}$ and $f \colon X \to X_i$ such that $h_i \cdot Lf = k$.

It is well known the free product completion $\mathbf{Fam}_{\Pi}$ distributes over this doctrine [42, Section 8]. Thus, as a consequence of Theorem 3.3.8, we see that if a functor L is a left multiadjoint, then the functor $\mathbf{Fam}_{\Pi} L$ is a left multiadjoint also.

The following is a simple consequence of the essential uniqueness of distributive laws over KZ doctrines, shown in Corollary 3.4.1.

Example 3.4.18. *Let* **Prof** *be the bicategory of profunctors on small categories, and let* **PROF** *be the Kleisli bicategory of the free small cocompletion KZ doctrine P on locally small categories. Clearly* **Prof** *lies inside* **PROF**. *By Corollary 3.4.1, the extension of a pseudomonad T on locally small categories to the bicategory* **PROF** *is essentially unique.*

3.5 Liftings of Locally Fully Faithful KZ Monads

In this section, we consider the case in which the KZ doctrine P being lifted is locally fully faithful. The reader will recall that a KZ doctrine P is locally fully faithful precisely when each unit map $y_{\mathcal{A}}$ is fully faithful [6].

The main goal of this section is to deduce an analogue of “Doctrinal Adjunction” on the “Yoneda structure” induced by the locally fully faithful KZ doctrine P. We start however with the following basic properties concerning fully faithful and P-fully faithful maps.

Proposition 3.5.1. *Suppose any of the equivalent conditions of Theorem 3.3.8 are satisfied. Then*

(a) if $y_{\mathcal{A}}$ is fully faithful for every $\mathcal{A} \in \mathscr{C}$, then every $Ty_{\mathcal{A}}$ is fully faithful;

(b) T preserves maps which are both P-admissible and P-fully faithful.

Proof. Firstly, note that if each $y_{\mathcal{A}}$ is fully faithful (so that $y_{T\mathcal{A}}$ is fully faithful) then so is $Ty_{\mathcal{A}}$, since we have an isomorphism

$$\begin{tikzcd}
TP \arrow[r, "\lambda_{\mathcal{A}}"] & PT \\
& T\mathcal{A} \arrow[ul, "Ty_{\mathcal{A}}"] \arrow[u, "y_{T\mathcal{A}}"'] \arrow[ul, phantom, shift right=3ex, "\overset{\omega_2}{\Leftarrow}"]
\end{tikzcd}$$

Secondly, note that if L is a P-admissible P-fully faithful map, meaning the unit η of the admissibility adjunction is invertible, then so is the unit n exhibiting the admissibility of TL by Figure 3.3.5. ☐

3.5.1 Doctrinal Partial Adjunctions

In this subsection we study how pseudomonads interact with absolute left liftings (also called partial adjunctions or relative adjunctions), which we now define. In particular, we show that we get an induced oplax structure on a partial left adjoint under suitable conditions, which gives a lifting of the partial adjunction to the setting of pseudo algebras in a suitable sense.

This is in the same spirit as subsection 3.2.2 on algebraic left extensions, but not completely analogous (and therefore not a dual). In particular, here we do not require any algebraic cocompleteness conditions.

Definition 3.5.2. Suppose we are given a diagram of the form

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R"] & \mathcal{C} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "I"'] \arrow[ul, phantom, shift right=3ex, "\overset{\eta}{\Leftarrow}"]
\end{tikzcd} \tag{3.5.1}$$

in a 2-category $\mathscr{C}$ equipped with a 2-cell $\eta \colon I \to R \cdot L$. We call such a diagram a *partial adjunction* and say that L is a *partial left adjoint* to R if given any 1-cells M and N as below, for any 2-cell $\zeta\colon I\cdot M\longrightarrow R\cdot N$ there exists a unique $\overline{\zeta}\colon L\cdot M\longrightarrow N$ such that ζ is equal to the pasting

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R"] \arrow[rd, phantom, "\overset{\overline{\zeta}}{\Leftarrow}" near end, shift right=3] & \mathcal{C} \arrow[ld, phantom, "\overset{\eta}{\Leftarrow}" near start, shift left=3] \\
\mathcal{D} \arrow[u, "N"] \arrow[r, "M"'] & \mathcal{A} \arrow[u, "I"'] \arrow[lu, "L" description]
\end{tikzcd}$$

That is, pasting 2-cells of the form $\overline{\zeta}$ above with η defines a bijection of 2-cells.

Remark 3.5.3. It is an easy and well known exercise to check that we have an adjunction $L\dashv R\colon\mathcal{B}\longrightarrow\mathcal{A}$ with unit η in a 2-category $\mathscr{C}$ if and only if

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R"] & \mathcal{A} \arrow[ld, phantom, "\overset{\eta}{\Leftarrow}" near start, shift left=3] \\
& \mathcal{A} \arrow[u, "\mathrm{id}_{\mathcal{A}}"'] \arrow[lu, "L"]
\end{tikzcd}$$

exhibits L as a partial left adjoint.

We now define a notion of partial adjunction in the context of pseudo T-algebras and T-morphisms.

Definition 3.5.4. Suppose we are given oplax T-morphisms (I,ξ) and (L,α) and a lax T-morphism (R,β) equipped with a T-transformation η (as in Remark 3.2.7 with appropriate identities) as in the diagram

$$\begin{tikzcd}
\left(\mathcal{B},T\mathcal{B}\xrightarrow{y}\mathcal{B}\right) \arrow[r, "{(R,\beta)}"] & \left(\mathcal{C},T\mathcal{C}\xrightarrow{z}\mathcal{C}\right) \arrow[ld, phantom, "\overset{\eta}{\Leftarrow}" near start, shift left=3] \\
& \left(\mathcal{A},T\mathcal{A}\xrightarrow{x}\mathcal{A}\right) \arrow[u, "{(I,\xi)}"'] \arrow[lu, "{(L,\alpha)}"]
\end{tikzcd}$$

We call such a diagram a *T-partial adjunction* if for any given pseudo T-algebra $(\mathcal{D},w)$, lax T-morphism (M,ε), and oplax T-morphism (N,φ) as below

$$\begin{tikzcd}
\left(\mathcal{B},T\mathcal{B}\xrightarrow{y}\mathcal{B}\right) \arrow[r, "{(R,\beta)}"] \arrow[rd, phantom, "\overset{\overline{\zeta}}{\Leftarrow}" near end, shift right=3] & \left(\mathcal{C},T\mathcal{C}\xrightarrow{z}\mathcal{C}\right) \arrow[ld, phantom, "\overset{\eta}{\Leftarrow}" near start, shift left=3] \\
\left(\mathcal{D},T\mathcal{D}\xrightarrow{w}\mathcal{D}\right) \arrow[u, "{(N,\varphi)}"] \arrow[r, "{(M,\varepsilon)}"'] & \left(\mathcal{A},T\mathcal{A}\xrightarrow{x}\mathcal{A}\right) \arrow[u, "{(I,\xi)}"'] \arrow[lu, "{(L,\alpha)}" description]
\end{tikzcd}$$

pasting T-transformations of the form $\overline{\zeta}$ above with the T-transformation η defines the bijection of T-transformations:

$$\begin{tikzcd}
(\mathcal{B},y) \arrow[r, equal] & (\mathcal{B},y) & & (\mathcal{B},y) \arrow[r, "{(R,\beta)}"] & (\mathcal{C},z) \\
(\mathcal{D},w) \arrow[u, "{(N,\varphi)}"] \arrow[r, "{(M,\varepsilon)}"'] \arrow[ur, phantom, "\overset{\overline{\zeta}}{\Leftarrow}"] & (\mathcal{A},x) \arrow[u, "{(L,\alpha)}"'] & \sim & (\mathcal{D},w) \arrow[u, "{(N,\varphi)}"] \arrow[r, "{(M,\varepsilon)}"'] \arrow[ur, phantom, "\overset{\zeta}{\Leftarrow}"] & (\mathcal{A},x) \arrow[u, "{(I,\xi)}"']
\end{tikzcd}$$

This operation of pasting the T-transformation $\overline{\zeta}$ with η is given by pasting the underlying 2-cells. The verification that such a pasting of T-transformations yields a T-transformation is a simple exercise.

Remark 3.5.5. We may be more general here by replacing (M,ε) and (N,φ) by a lax followed by an oplax, and an oplax followed by a lax T-morphism respectively. However, this level of generality will not be necessary for this paper.

We now give the doctrinal properties enjoyed by partial adjunctions.

Proposition 3.5.6. *Suppose we are given a partial adjunction*

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R"] & \mathcal{C} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "I"'] \arrow[u, phantom, shift right=3ex, "\overset{\eta}{\Leftarrow}"]
\end{tikzcd}$$

in a 2-category $\mathscr{C}$ equipped with a pseudomonad (T,u,m). Suppose further that

$$\left(\mathcal{A},T\mathcal{A}\xrightarrow{x}\mathcal{A}\right),\quad \left(\mathcal{B},T\mathcal{B}\xrightarrow{y}\mathcal{B}\right),\quad \left(\mathcal{C},T\mathcal{C}\xrightarrow{z}\mathcal{C}\right)$$

*are pseudo T-*algebras. *Then given an oplax T-*morphism *structure ξ on I and a lax T-*morphism *structure β on R, there exists a unique oplax T-*morphism *structure α on L such that η is a T-transformation. Moreover, this partial adjunction is then lifted to the T-partial adjunction*

$$\begin{tikzcd}
\left(\mathcal{B},T\mathcal{B}\xrightarrow{y}\mathcal{B}\right) \arrow[r, "{(R,\beta)}"] & \left(\mathcal{C},T\mathcal{C}\xrightarrow{z}\mathcal{C}\right) \\
& \left(\mathcal{A},T\mathcal{A}\xrightarrow{x}\mathcal{A}\right) \arrow[ul, "{(L,\alpha)}"] \arrow[u, "{(I,\xi)}"'] \arrow[u, phantom, shift right=5ex, "\overset{\eta}{\Leftarrow}"]
\end{tikzcd}$$

Proof. Given our 2-cells

$$\begin{tikzcd}
T\mathcal{A} \arrow[r, "x"] \arrow[d, "TI"'] \arrow[dr, phantom, "\Downarrow\xi"] & \mathcal{A} \arrow[d, "I"] \\
T\mathcal{C} \arrow[r, "z"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
T\mathcal{B} \arrow[r, "x"] \arrow[d, "TR"'] \arrow[dr, phantom, "\Uparrow\beta"] & \mathcal{B} \arrow[d, "R"] \\
T\mathcal{C} \arrow[r, "z"'] & \mathcal{C}
\end{tikzcd}$$

exhibiting I as an oplax T-morphism and R as a lax T-morphism, we can take our oplax constraint cell of L (which we call α) as the unique solution to

$$\begin{tikzcd}
T\mathcal{B} \arrow[r, "y"] & \mathcal{B} \arrow[dr, "R"] & \\
& & \mathcal{C} \\
T\mathcal{A} \arrow[uu, "TL"] \arrow[r, "x"'] \arrow[uur, phantom, "\Uparrow\alpha"] & \mathcal{A} \arrow[uu, "L"'] \arrow[ur, "I"'] \arrow[u, phantom, "\Uparrow\eta"] &
\end{tikzcd}
=
\begin{tikzcd}
T\mathcal{B} \arrow[rr, "y"] \arrow[dr, "TR"] & & \mathcal{B} \arrow[dr, "R"] \arrow[dl, phantom, "\Uparrow\beta"] & \\
& T\mathcal{C} \arrow[rr, "z"] \arrow[l, phantom, "\Uparrow T\eta"] & & \mathcal{C} \\
T\mathcal{A} \arrow[uu, "TL"] \arrow[ur, "TI"] \arrow[rr, "x"'] & & \mathcal{A} \arrow[ur, "I"'] \arrow[ul, phantom, "\Uparrow\xi"] &
\end{tikzcd}$$

which exists since L is a partial left adjoint. That is, α is the unique oplax structure on L for which $\eta : I \to R \cdot L$ is a T-transformation. The verification that α then satisfies the unitary and multiplicative coherence axioms is a simple exercise which we omit. □

The following example is an easy application of this result which does not involve Yoneda structures.

Proposition 3.5.7. *Suppose $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ are bicategories. Consider a diagram*

$$\begin{tikzcd}
\mathscr{A} \arrow[r, dashed, "F"] \arrow[rr, bend right, "H"'] & \mathscr{B} \arrow[r, "G"] & \mathscr{C}
\end{tikzcd}$$

where G is a lax and locally fully faithful functor, H is an oplax functor, and F is a locally defined functor

$$\left(F_{X,Y} : \mathscr{A}(X,Y) \to \mathscr{B}(FX,FY) : X,Y \in \mathscr{A} \right)$$

where $G \cdot F = H$ locally. It then follows that F extends to an oplax functor.

Proof. To see this, recall that the fully faithfulness of each $G_{M,N}$ (for objects $M,N \in \mathscr{B}$) may be characterized by saying that each

$$\begin{tikzcd}
\mathscr{B}(M,N) \arrow[r, "G_{M,N}"] & \mathscr{C}(HM,HN) \\
& \mathscr{B}(M,N) \arrow[ul, "\mathrm{id}_{\mathscr{B}(M,N)}"] \arrow[u, "G_{M,N}"'] \arrow[u, phantom, shift right=8, "\overset{\mathrm{id}}{\Longleftarrow}"]
\end{tikzcd}$$

is an absolute lifting [54, Example 2.18]. As this absolute left lifting is preserved upon whiskering by

$$F_{X,Y} : \mathscr{A}(X,Y) \to \mathscr{B}(FX,FY)$$

we have the family of partial adjunctions

$$\begin{tikzcd}
\mathscr{B}(FX,FY) \arrow[r, "G_{FX,FY}"] & \mathscr{C}(HX,HY) \\
& \mathscr{A}(X,Y) \arrow[ul, "F_{X,Y}"] \arrow[u, "H_{X,Y}"'] \arrow[ul, phantom, shift right=3ex, "\overset{\text{id}}{\Leftarrow}"']
\end{tikzcd}$$

Endowing with the bicategory structure of $\mathscr{A}$, and full sub-bicategory structures of $\mathscr{B}$ and $\mathscr{C}$ restricted to objects in the images of F and H respectively, we see by Proposition 3.5.6 that F extends to an oplax functor $F \colon \mathscr{A} \to \mathscr{B}$. □

Remark 3.5.8. Clearly, this may be stated more generally in the setting of a pseudo T-algebras. Also, it suffices to only have an isomorphism $GF \cong H$ on the underlying 2-category.

Remark 3.5.9. In Kelly's setting of a doctrinal adjunction [27], if both the left and right adjoint are lax, exhibited by a counit and unit which are T-transformations of lax T-morphisms, then the induced oplax structure on the left adjoint is inverse to the given lax structure. In this partial adjunction case, the best we can say is that if (I,ξ) is pseudo, (L,α^*) lax, and $\eta\colon (I,\xi) \to (R,\beta)\cdot(L,\alpha^*)$ a T-transformation of lax T-morphisms, then the induced oplax structure on L given as α satisfies $\alpha^* \cdot \alpha = \mathrm{id}_{L\cdot x}$. This means the identity 2-cell is a generalized T-transformation from (L,α) to (L,α^*), but not necessarily the other way around.

3.5.2 Doctrinal "Yoneda Structures"

Kelly [27] showed that given an adjunction $L \dashv R$ which lifts to pseudo algebras, oplax structures on the left adjoint are in bijection with lax structures on the right adjoint. The goal of this section is to give a similar result for "Yoneda structure diagrams", that is diagrams of the form

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R"] & P\mathcal{A} \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"'] \arrow[ul, phantom, shift right=3ex, "\overset{\varphi_L}{\Leftarrow}"']
\end{tikzcd}$$

for which L is an absolute left lifting, and R is a left extension exhibited by the same 2-cell φ_L (as appear in Yoneda structures [47], or in the setting of a locally fully faithful KZ doctrine).

We state the following as one of the main results of this paper, due to its applications as a coherence result for oplax functors out of certain bicategories, such as the bicategories of spans or polynomials. This application will be briefly discussed at the end of this section.

Theorem 3.5.10 (Doctrinal Yoneda Structures)**.** *Suppose we are given a 2-category $\mathscr{C}$ equipped with a pseudomonad (T,u,m) and a locally fully faithful KZ doctrine (P,y). Suppose that T pseudo-distributes over P. Suppose we are given pseudo T-algebra structures*

$$\left(\mathcal{A}, T\mathcal{A} \xrightarrow{x} \mathcal{A}\right), \qquad \left(\mathcal{B}, T\mathcal{B} \xrightarrow{y} \mathcal{B}\right)$$

Then for any P-admissible map $L\colon \mathcal{A} \to \mathcal{B}$ we have a Yoneda structure diagram as on the left, underlying a "doctrinal Yoneda structure" diagram as on the right

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "R_L"] & P\mathcal{A} \arrow[dl, phantom, "\overset{\varphi_L}{\Leftarrow}"] \\
& \mathcal{A} \arrow[ul, "L"] \arrow[u, "y_{\mathcal{A}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathcal{B},y) \arrow[r, "(R_L{,}\beta)"] & (P\mathcal{A},z_x) \arrow[dl, phantom, "\overset{\varphi_L}{\Leftarrow}"] \\
& (\mathcal{A},x) \arrow[ul, "(L{,}\alpha)"] \arrow[u, "(y_{\mathcal{A}}{,}\xi)"']
\end{tikzcd}$$

in that 2-cells α as on the left below exhibiting L as an oplax T-morphism

$$\begin{tikzcd}
T\mathcal{B} \arrow[r, "y"] & \mathcal{B} \\
T\mathcal{A} \arrow[u, "TL"] \arrow[r, "x"'] \arrow[ur, phantom, "\Uparrow\alpha"] & \mathcal{A} \arrow[u, "L"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T\mathcal{B} \arrow[r, "y"] \arrow[d, "TR_L"'] \arrow[dr, phantom, "\Uparrow\beta"] & \mathcal{B} \arrow[d, "R_L"] \\
T\mathcal{C} \arrow[r, "z_x"'] & \mathcal{C}
\end{tikzcd}$$

are in bijection with 2-cells β as on the right exhibiting R_L as a lax T-morphism.

Proof. We need only check that the propositions concerning partial adjunctions and left extensions[10] are inverse to each other. But this is just a consequence of the fact that we can go between the defining equalities for these propositions

$$\begin{tikzcd}
T\mathcal{B} \arrow[r, "y"] & \mathcal{B} \arrow[dr, "R_L"] & \\
\arrow[r, phantom, "\Uparrow\alpha"] & \arrow[r, phantom, "\Uparrow\varphi_L"] & P\mathcal{A} \\
T\mathcal{A} \arrow[uu, "TL"] \arrow[r, "x"'] & \mathcal{A} \arrow[uu, "L"] \arrow[ur, "y_{\mathcal{A}}"'] &
\end{tikzcd}
=
\begin{tikzcd}
T\mathcal{B} \arrow[r, "y"] \arrow[dr, "TR_L"] & \mathcal{B} \arrow[dr, "R_L"] \arrow[d, phantom, "\Uparrow\beta"] & \\
\arrow[r, phantom, "\Uparrow T\varphi_L"] & TP\mathcal{A} \arrow[r, "z_x"] \arrow[d, phantom, "\Uparrow\xi_x"] & P\mathcal{A} \\
T\mathcal{A} \arrow[uu, "TL"] \arrow[ur, "Ty_{\mathcal{A}}"] \arrow[r, "x"'] & \mathcal{A} \arrow[ur, "y_{\mathcal{A}}"'] &
\end{tikzcd}$$

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[10] Note that Proposition 3.2.14 applies since each z_x is T_P-cocontinuous by Proposition 3.3.18.

and

$$\begin{tikzcd}
{} & T\mathcal{B} \arrow[r, "y"] \arrow[d, phantom, "\Uparrow\alpha"] & \mathcal{B} \arrow[dd, "R"] \arrow[d, phantom, "\Uparrow\varphi_L"] \\
T\mathcal{A} \arrow[ur, "TL"] \arrow[r, "x"] \arrow[dr, "Ty_{\mathcal{A}}"'] & \mathcal{A} \arrow[ur, "L"'] \arrow[dr, "y_{\mathcal{A}}"] \arrow[d, phantom, "\Uparrow\xi_x^{-1}"] & {} \\
{} & TP\mathcal{A} \arrow[r, "z_x"'] & P\mathcal{A}
\end{tikzcd}
=
\begin{tikzcd}
{} & T\mathcal{B} \arrow[r, "y"] \arrow[dd, "TR_L"] & \mathcal{B} \arrow[dd, "R_L"] \\
T\mathcal{A} \arrow[ur, "TL"] \arrow[dr, "Ty_{\mathcal{A}}"'] \arrow[r, phantom, "\Uparrow T\varphi_L"] & {} \arrow[r, phantom, "\Uparrow\beta"] & {} \\
{} & TP\mathcal{A} \arrow[r, "z_x"'] & P\mathcal{A}
\end{tikzcd}$$

by pasting with ξ_x and ξ_x^{-1}. □

Remark 3.5.11. In the "doctrinal Yoneda structure" of the above, φ_L is a T-transformation exhibiting (R_L, β) as a T-left extension and (L, α) as a T-partial left adjoint, provided α and β correspond via this bijection.

We observe that the bijection between oplax structures on left adjoints and lax structures on right adjoints as in "Doctrinal adjunction" [27] is a special case of this theorem.

Corollary 3.5.12 (Kelly)**.** *Suppose we are given a 2-category $\mathscr{C}$ equipped with a pseudomonad (T,u,m), pseudo T-algebra structures*

$$\left(\mathcal{A}, T\mathcal{A} \xrightarrow{x} \mathcal{A}\right), \qquad \left(\mathcal{B}, T\mathcal{B} \xrightarrow{y} \mathcal{B}\right)$$

and an adjunction $L \dashv R \colon \mathcal{B} \to \mathcal{A}$ in $\mathscr{C}$. Then oplax structures on L are in bijection with lax structures on R.

Proof. Let P be the identity pseudomonad on $\mathscr{C}$, which is clearly a locally fully faithful KZ doctrine. Trivially, any pseudomonad T pseudo-distributes over the identity. Now observe that for the identity pseudomonad, the admissible maps are the left adjoints and the "Yoneda structure diagrams" are the units of adjunctions $\eta \colon \mathrm{id}_{\mathcal{A}} \to R \cdot L$. Applying the above theorem then gives the result. □

3.5.3 Applications and Future Work

The motivating application of this result is not to give an analogous result to doctrinal adjunction, but instead the observation that it may be seen as a coherence result. In particular, consider the following special case of this theorem concerning the bicategory of spans in a category $\mathcal{E}$ with pullbacks, denoted **Span** $(\mathcal{E})$.

For the following corollary, we recall that locally defined functors are the morphisms of **CatGrph**, and **CatGrph** gives rise to bicategories and oplax/lax functors via a suitable 2-monad [34].

Corollary 3.5.13. *Suppose we are given a small*[11] *category with pullbacks $\mathcal{E}$ and a bicategory $\mathscr{C}$ with the same objects, as well as locally defined functors*

$$L_{X,Y} : \mathbf{Span}\,(\mathcal{E})\,(X,Y) \to \mathscr{C}\,(X,Y)$$

with corresponding left extensions $(R_L)_{X,Y}$ as components in the diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "R_L"] & \widehat{\mathbf{Span}}\,(\mathcal{E}) \\
 & \mathbf{Span}\,(\mathcal{E}) \arrow[ul, "L"] \arrow[u, "Y"'] \arrow[from=1-1, to=2-2, phantom, "\overset{\varphi_L}{\Leftarrow}"]
\end{tikzcd}$$

where $\widehat{\mathbf{Span}}\,(\mathcal{E})$ is the local cocompletion[12] *of $\mathbf{Span}\,(\mathcal{E})$. Then oplax structures on L are in bijection with lax structures on R_L.*

To see why this is useful, recall that composition of spans is given by taking the terminal diagram of the form

$$\begin{tikzcd}
 & & \bullet \arrow[dll, dashed, bend right] \arrow[dl, dashed] \arrow[dd, dashed] \arrow[dr, dashed] \arrow[drr, dashed, bend left] & & \\
 & \bullet \arrow[dl, "f"] \arrow[dr, "g"] & & \bullet \arrow[dl, "h"] \arrow[dr, "k"] & \\
\bullet & & \bullet & & \bullet
\end{tikzcd}$$

and so when evaluating the composite of two spans we may recover the two morphisms of spans in the above diagram; that is, there is a relationship between the way 2-cells are defined and how composition of 1-cells is defined.

This relationship between composition and 2-cells is captured in Day's convolution formula [11], and causes the coend defining the Day convolution to collapse to a more workable sum. In particular, composition in $\widehat{\mathbf{Span}}\,(\mathcal{E})$ is given by the convolution formula

$$GF\,(s;t) = \sum_{T \xrightarrow{h} Y} F\,(s;h)\,G\,(h;t)$$

where $s;t$ is an arbitrary span from X to Z through Y, and F and G are presheaves on $\mathbf{Span}\,(\mathcal{E})\,(X,Y)$ and $\mathbf{Span}\,(\mathcal{E})\,(Y,Z)$ respectively. As a result, it is easier to show that a locally defined functor $L\colon \mathbf{Span}\,(\mathcal{E}) \to \mathscr{C}$ is oplax by instead showing that the corresponding $R_L\colon \mathscr{C} \to \widehat{\mathbf{Span}}\,(\mathcal{E})$ is lax. Indeed, the reader should notice here that the problem of showing L is oplax involves pullbacks, whereas the equivalent problem of showing R is lax does not (once this convolution formula has been established).

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[11] Note that one may work in a larger universe to work around this condition.

[12] The monoidal cocompletion as given by the Day convolution structure may be generalized to the setting of bicategories; we call this the local cocompletion.

A more involved application along the same lines deals not with the bicategory of spans, but instead $\mathbf{Poly}_c(\mathcal{E})$, the bicategory of polynomials with cartesian 2-cells as studied by Gambino, Kock and Weber [55, 17]. We see that due to the complicated nature of composition in $\mathbf{Poly}_c(\mathcal{E})$, showing that a locally defined functor $L : \mathbf{Poly}(\mathcal{E}) \rightarrow \mathscr{C}$ is oplax becomes a large calculation (especially for the associativity coherence conditions); however if we instead show that $R_L : \mathscr{C} \rightarrow \hat{\mathbf{Poly}}_c(\mathcal{E})$ is lax our work will be reduced significantly; in fact by this method we can completely avoid coherences involving composition of distributivity pullbacks.

In a soon forthcoming paper we will exploit this fact in more detail to give a complete proof of the universal properties of polynomials which avoids the majority of the coherence conditions.

4

Generic bicategories

Abstract

It is a well known result of Bénabou that monads in a bicategory $\mathscr{C}$ are in bijection with lax functors $L\colon \mathbf{1} \to \mathscr{C}$ where $\mathbf{1}$ is the terminal bicategory. Dually, comonads in $\mathscr{C}$ correspond to oplax functors $L\colon \mathbf{1} \to \mathscr{C}$.

Here we provide a generalization of this dual, exhibiting this correspondence as a special case of a more general result. This is done by replacing the terminal bicategory by any *generic bicategory*, that is a bicategory for which the composition functor admits generic factorisations. We show that for generic bicategories $\mathscr{A}$, the data of an oplax functor $\mathscr{A} \to \mathscr{C}$ has a reduced description which is similar to the data of a comonad; the main advantage of this description being that it does not directly involve composition in $\mathscr{A}$. This in turn allows for a greater understanding of the universal properties of some well known constructions in category theory, particularly those of spans and polynomial functors. Moreover, we will show how this generalization naturally arises from the algebraic properties of Yoneda structures.

Contribution by the author

As the sole author, this paper is entirely my own work. This paper was submitted for publication on May 3rd 2018 and is currently under review.

4.1 Introduction

A beautiful theorem of Bénabou [3] states that lax functors from the terminal bicategory **1** into a bicategory $\mathscr{C}$ correspond to monads in $\mathscr{C}$. Dually, oplax functors $\mathbf{1} \to \mathscr{C}$ correspond to comonads in $\mathscr{C}$.

The purpose of this paper is to provide a generalization of this dual to Bénabou's result, thus providing further insight into questions such as "When is an oplax functor $L\colon \mathscr{A} \to \mathscr{C}$ analogous to a comonad in $\mathscr{C}$?".

This is done by replacing the terminal bicategory with bicategories $\mathscr{A}$ satisfying the following special property: every functor

$$\mathscr{A}_{X,Z}(c, - \circ -) : \mathscr{A}_{Y,Z} \times \mathscr{A}_{X,Y} \to \mathbf{Set}, \qquad X, Y, Z, c \in \mathscr{A}$$

is a coproduct of representables. A more informative and equivalent characterization is as follows: every composition functor

$$\circ\colon \mathscr{A}_{Y,Z} \times \mathscr{A}_{X,Y} \to \mathscr{A}_{X,Z}, \qquad X, Y, Z \in \mathscr{A}$$

admits generic factorisations. We will call bicategories $\mathscr{A}$ satisfying this property *generic*.

This property means that each 2-cell into a binary composite $c \to ba$ in the bicategory $\mathscr{A}$ factors through a generic (also known as "diagonally universal" in the work of Diers [14, 15]) 2-cell $\delta\colon c \to rl$.

A simple example of this is given by taking $\mathscr{A}$ to be a cartesian monoidal category $(\mathcal{E}, \times, \mathbf{1})$ seen as a one-object bicategory. Here the generics are the diagonal maps $\delta_T\colon T \to T \times T$ for each $T \in \mathcal{E}$, and clearly any $\gamma\colon T \to A \times B$ factors as the generic δ_T followed by $\pi_A\gamma \times \pi_B\gamma$ where π_A and π_B are the product projections.

Another example is given by taking $\mathscr{A}$ to be the bicategory of spans $\mathbf{Span}(\mathcal{E})$ in a category $\mathcal{E}$ with pullbacks; here our generic maps are morphisms δ induced into pullbacks as in

$$\begin{tikzcd}
& & T \arrow[d, dashed, "\delta"] \arrow[dddll, bend right, "s"'] \arrow[dddrr, bend left, "t"] & & \\
& & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, phantom, "\text{pb}"] & & \\
& S \arrow[dl, "s"] \arrow[dr, "h"'] & & S \arrow[dl, "h"] \arrow[dr, "t"'] & \\
X & & Y & & Z
\end{tikzcd}$$

such that $\pi_1\delta$ and $\pi_2\delta$ are identities. This can also be done for the bicategory of polynomials $\mathbf{Poly}_c(\mathcal{E})$ with cartesian 2-cells, but becomes more complicated.

Such bicategories also contain "nullary generics" or augmentations; these are the 2-cells into identity 1-cells, and turn out to be unique in such bicategories.

The main result of this paper is that for generic bicategories $\mathscr{A}$, the functors $\mathscr{A} \to \mathscr{C}$ which respect these generic maps are precisely the oplax functors. Here "respecting generics" means that each generic δ and augmentation ε in $\mathscr{A}$ has a corresponding comultiplication map Φ_δ and counit map Λ_ε in $\mathscr{C}$ satisfying coherence conditions much like those for a comonad.

When the domain bicategory $\mathscr{A}$ is generic, this description has an important advantage over the usual definition of an oplax functor: it does not involve composition in the domain bicategory. This reduction being possible since the information concerning composition in $\mathscr{A}$ is encoded into these generic maps. Of course, this property is particularly useful if composition in $\mathscr{A}$ is complicated; the bicategory of polynomials being an archetypal example.

In Section 4.2 we develop the theory of such bicategories $\mathscr{A}$ and their generic maps, and prove the main result of this paper, Theorem 4.2.19, in which we prove the equivalence of oplax functors and functors which respect these generics.

In Section 4.3, we use this result to give a description of oplax functors out of the bicategory of spans which does not involve composition of spans (pullbacks), and then give a description of oplax functors out of the bicategory of polynomials which does not involve composition of polynomials.

These descriptions allow for a simpler proof of the universal properties of spans [9], and a much simpler proof of the universal properties of polynomials. Moreover, these descriptions may be used to explain some curious aspects of these universal properties. For example Dawson, Paré and Pronk made the observation that the span construction has a universal property which does not involve pullbacks [9], a fact which is explained by the results of this paper. Indeed, the primary reason for this paper is to set the stage for future work in which we will use these descriptions to give an efficient proof of the universal properties of the span construction and polynomial functor construction.

In Section 4.4 we discuss how this description of oplax functors naturally arises from the algebraic properties of Yoneda structures, making use of the simpler Day convolution structure on generic bicategories.

4.2 Properties of generic bicategories

In this section we start off by recalling the basic theory of generic morphisms and functors which admit them. We then define generic bicategories and consider the properties of generic morphisms in these generic bicategories. After discussing the coherence properties of these generic morphisms, we go on to give the main result of this paper; showing that the functors which respect these generic morphisms are precisely the oplax functors.

4.2.1 Generic morphisms and factorisations

Generic morphisms (and weaker analogues of them) have historically arisen in the characterization the analytic endofunctors of **Set** [24], as well as the study of qualitative domains [18, 35]. Characterizations of endofunctors which admit them have been studied by Weber [52], and this is known to be related to familial representability as studied by Diers [14].

In this paper we do not consider arbitrary endofunctors which admit generics, but instead composition functors which admit generics, giving us a richer structure to consider.

Definition 4.2.1. Given a functor $T\colon \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$, we say a morphism $\delta\colon B \to TA$ in $\mathcal{B}$ (where $A \in \mathcal{A}$ and $B \in \mathcal{B}$) is *T-generic* if for any commutative square of the form below

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "\delta"'] & TC \arrow[d, "Tg"] \\
TA \arrow[r, "Th"'] \arrow[ur, dotted, "T\overline{f}"] & TD
\end{tikzcd}$$

there exists a unique morphism $\overline{f}$ in $\mathcal{A}$ such that $T\overline{f} \cdot \delta = f$.

Remark 4.2.2. These are precisely the diagonally universal morphisms of Diers [15], who noted that it must follow $g \cdot \overline{f} = h$ since both fillers below

$$\begin{tikzcd}
B \arrow[r, "Tg\cdot f"] \arrow[d, "\delta"'] & TD \arrow[d, "T1_D"] \\
TA \arrow[r, "Th"'] \arrow[ur, dotted, "T\left(g\cdot\overline{f}\right)"] & TD
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B \arrow[r, "Tg\cdot f"] \arrow[d, "\delta"'] & TD \arrow[d, "T1_D"] \\
TA \arrow[r, "Th"'] \arrow[ur, dotted, "Th"] & TD
\end{tikzcd}$$

render commutative the top triangles.

Definition 4.2.3. We say a functor $T\colon \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$ *admits generic factorisations* if for any morphism $f\colon B \to TC$ in $\mathcal{B}$ there exists a T-generic morphism $\delta\colon B \to TA$ in $\mathcal{B}$ and morphism $\overline{f}\colon A \to C$ in $\mathcal{A}$ rendering commutative

$$\begin{tikzcd}
 & TA \arrow[dr, "T\overline{f}"] & \\
B \arrow[ur, "\delta"] \arrow[rr, "f"'] & & TC
\end{tikzcd}$$

We are now ready to define generic bicategories, the structures to be considered in this paper. It will be helpful to write composition in diagrammatic order, denoted by the symbol “;”.

Definition 4.2.4. We say a bicategory $\mathscr{A}$ is *generic* if for every triple of objects $X, Y, Z \in \mathscr{A}$

the composition functor

$$\mathscr{A}_{X,Y} \times \mathscr{A}_{Y,Z} \xrightarrow{;} \mathscr{A}_{X,Z}$$

admits generic factorisations. Moreover, we simply call *generic* those 2-cells $\delta\colon c \to l; r$ which are ;-generic.

Remark 4.2.5. Unpacking the above definition into a more useful form, we see that a 2-cell $\delta\colon c \to l; r$ is generic if and only if every commuting diagram of the form

$$\begin{tikzcd}
c \arrow[r, "\gamma"] \arrow[d, "\delta"'] & f; g \arrow[d, "\phi_1;\phi_2"] \\
l; r \arrow[r, "\theta_1;\theta_2"'] \arrow[ur, dotted, "\gamma_1;\gamma_2"] & m; n
\end{tikzcd}$$

(where $\theta_1, \theta_2, \phi_1, \phi_2$ and γ are arbitrary 2-cells) admits a filler $\gamma_1; \gamma_2$ as displayed, such that the top triangle commutes and the bottom triangle commutes component-wise. Moreover, the pair (γ_1, γ_2) must be unique such that the top triangle commutes, justifying the notation.

Remark 4.2.6. As we will see in Section 4.3, there are a number of well known bicategories and monoidal categories which are generic, such as:

- any cartesian monoidal category;
- finite sets and bijections with the disjoint union monoidal structure;
- the bicategory of spans;
- the bicategory of polynomials with cartesian 2-cells.

Generic bicategories may be alternatively defined in terms of familial representability, a property which is often easier to verify. This is a consequence of the following known relationship[1] between functors which admit generics and the familial representability conditions of Diers [14].

Proposition 4.2.7 (Diers)**.** *Given a functor $T\colon \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$ the following are equivalent:*

1. *the functor T admits generic factorisations;*

---

[1] We include the proof of this relationship due to the difficulty of finding a reference.

2. *for every $B \in \mathcal{B}$ there exists a set $\mathfrak{M}_B$ and function $P_{(-)}\colon \mathfrak{M}_B \longrightarrow \mathcal{A}_{\mathrm{ob}}$ yielding isomorphisms*

$$\mathcal{B}(B,TA) \cong \sum_{\delta\in\mathfrak{M}_B} \mathcal{A}(P_\delta, A)$$

natural in $A \in \mathcal{A}$.

Proof. Suppose that T admits generic factorisations. Call two generic morphisms δ and δ' equivalent if there exists an isomorphism α rendering commutative a diagram as below:

$$\begin{tikzcd}
TM \arrow[rr, "T\alpha"] & & TM' \\
& B \arrow[ul, "\delta'"] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

Now take $\mathfrak{M}_B$ to be the set of equivalence classes of generic morphisms out of B, with each class labeled by a chosen representative. It follows that for any $f\colon B \longrightarrow TA$ we can find a representative generic morphism δ_f and unique morphism $\overline{f}$ rendering commutative

$$\begin{tikzcd}
B \arrow[rr, "f"] \arrow[dr, "\delta_f"'] & & TA \\
& TM \arrow[ur, "T\overline{f}"'] &
\end{tikzcd}$$

We note also that the representative generic δ_f is itself unique (such a generic necessarily lies in the same equivalence class). Therefore the assignment $f \mapsto \left(\delta_f, \overline{f}\right)$ is bijective, where each P_{δ_f} is taken as the M above. Trivially, given a map $x\colon A \longrightarrow A'$ the diagram

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[dr, "\delta_f"'] & TA \arrow[r, "Tx"] & TA' \\
& TM \arrow[u, "T\overline{f}"] \arrow[ur, "T\left(x\overline{f}\right)"'] &
\end{tikzcd}$$

commutes, and by genericity $x\overline{f}$ is the unique such map making the outside commute; thus showing naturality.

Conversely, suppose we are given such a family of isomorphisms[2]

$$\mathcal{B}(B,TA) \cong \sum_{m\in\mathfrak{M}_B} \mathcal{A}(P_m, A)$$

natural in $A \in \mathcal{A}$, where $B \in \mathcal{B}$ is given. We first note that by naturality, the inverse assignment is necessarily defined by

$$m \in \mathfrak{M}_B \quad , \quad P_m \xrightarrow{\alpha} A \quad \mapsto \quad B \xrightarrow{\delta_m} TP_m \xrightarrow{T\alpha} TA$$

---

[2] Here $\mathfrak{M}_B$ is an arbitrary set, so we do not use the suggestive notation δ for its elements.

where δ_m is the morphism corresponding to the identity at P_m. Also, this δ_m is generic since given any commuting diagram as on the outside below

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "\delta_m"'] & TA \arrow[d, "Th"] \\
TP_m \arrow[r, "Tg"'] \arrow[ur, dotted, "T\overline{f}"] & TD
\end{tikzcd}$$

the morphism $Th \cdot f$ must correspond to the pair (δ_m, g) under the bijection. By naturality, f must factor through this same δ_m, and so the pair $\left(\delta_m, \overline{f}\right)$ corresponding to f is unique such that the top triangle commutes. That $g = h \cdot \overline{f}$ is also a consequence of naturality. It is implicit in the above argument that T then admits generic factorisations. □

Taking T to be the composition functor, we have the following.

Corollary 4.2.8. *A bicategory $\mathscr{A}$ is generic if and only if for any triple of objects $X, Y, Z \in \mathscr{A}$ and 1-cell $c\colon X \to Z$ the functor*

$$\mathscr{A}_{X,Z}(c, -; -) : \mathscr{A}_{X,Y} \times \mathscr{A}_{Y,Z} \to \mathbf{Set}$$

is a coproduct of representables, meaning that for any (X, Y, Z, c) there exists a set $\mathfrak{M}_c^{X,Y,Z}$ equipped with projections

$$\left(\mathscr{A}_{X,Y}\right)_{\mathrm{ob}} \xleftarrow{l_{(-)}} \mathfrak{M}_c^{X,Y,Z} \xrightarrow{r_{(-)}} \left(\mathscr{A}_{Y,Z}\right)_{\mathrm{ob}}$$

such that for all $a\colon X \to Y$ and $b\colon Y \to Z$ we have isomorphisms

$$\mathscr{A}_{X,Z}(c, a; b) \cong \sum_{m \in \mathfrak{M}_c^{X,Y,Z}} \mathscr{A}_{X,Y}(l_m, a) \times \mathscr{A}_{Y,Z}(r_m, b)$$

natural in a and b.

We have defined generics as universal maps into a composite of two 1-cells; what one might call “2-generics”. We might ask if there is a corresponding notion for “0-generics” into composites of zero 1-cells, that is, identity 1-cells. However, as for each $n\colon X \to X$ the functor

$$\mathscr{A}_{X,X}(n, 1_X) : \mathbf{1} \to \mathbf{Set}$$

is trivially a coproduct of representables, there is no condition to impose on these 2-cells, and so any 2-cell $\varepsilon\colon n \to 1_X$ may be regarded as a “0-generic”. Regardless, these 2-cells still have an interesting property; they are unique.

Proposition 4.2.9. *Suppose $\mathscr{A}$ is a generic bicategory. Then for each $X \in \mathscr{A}$, the identity 1-cell 1_X is sub-terminal in $\mathscr{A}_{X,X}$.*

Proof. Given a morphism $n\colon X \to X$ and two 2-cells $s, t\colon n \to 1_X$ we have two commuting squares

$$\begin{tikzcd} n \arrow[r, "\delta_1"] \arrow[d, "\delta_2"'] & l;n \arrow[d, "h;s"] \\ n;r \arrow[r, "s;k"'] \arrow[ur, dotted, "\theta;\phi" description] & 1_X;1_X \end{tikzcd} \qquad \begin{tikzcd} n \arrow[r, "\delta_1"] \arrow[d, "\delta_2"'] & l;n \arrow[d, "h;t"] \\ n;r \arrow[r, "t;k"'] \arrow[ur, dotted, "\theta;\phi" description] & 1_X;1_X \end{tikzcd}$$

where δ_1 and $h\colon l \to 1_X$ are given by factorizing the unitor $n \to 1_X; n$ through a generic, and δ_2 and $k\colon r \to 1_X$ are given by factorizing the other unitor $n \to n; 1_X$. Now both of these squares admit a unique filler, and moreover both these fillers must be equal as uniqueness is forced by the top left triangles; we denote this filler $\theta; \phi$. Equating the left components of the bottom right triangles we then find $s = h\theta = t$. ☐

It will be useful to give such 2-cells a name as they still play an important role, despite the lack of a non-trivial universal property.

Definition 4.2.10. We call any 2-cell of the form $\varepsilon\colon n \to 1_X$ in a bicategory $\mathscr{A}$ an *augmentation.*

4.2.2 Coherence of generics

The following two lemmata show that there exists "nice" choices of generics. This will later be useful in regard to stating and checking coherence conditions.

Lemma 4.2.11. *Suppose $\mathscr{A}$ is a generic bicategory. Then for any factorization of a left unitor at a 1-cell $c\colon X \to Y$ through a generic δ as below*

$$\begin{tikzcd} & l;r \arrow[dr, "\theta;\phi"] & \\ c \arrow[ur, "\delta"] \arrow[rr, "\text{unitor}"'] & & 1_X;c \end{tikzcd} \tag{4.2.1}$$

the induced 2-cell ϕ is invertible.

Proof. Define $\phi^*\colon c \to r$ to be the composite

$$c \xrightarrow{\delta} l;r \xrightarrow{\theta;r} 1_X;r \xrightarrow{\text{unitor}} r$$

and note that when this is post-composed by ϕ we recover the identity 2-cell at c, by commutativity of the diagram 4.2.1 and naturality of unitors. We also note that by naturality of unitors the diagram

$$\begin{tikzcd}
c \arrow[r, "\text{unitor}"] \arrow[d, "\delta"'] & 1_X; c \arrow[d, "1_X;\phi^*"] \\
l; r \arrow[r, "\theta;r"'] \arrow[ur, dotted, "\theta;\phi" description] & 1_X; r
\end{tikzcd}$$

commutes and thus admits a filler such that both triangles commute. Moreover, we note that as uniqueness is forced by the top triangle this filler must be $\theta;\phi$. Equating the second components of the bottom right triangle we have established ϕ followed by ϕ^* as being the identity. □

Remark 4.2.12. As ϕ is invertible above, composing the generic δ with ϕ still yields a generic. This shows that there exists "nice" generics $c \longrightarrow l;c$ and augmentations $l \longrightarrow 1_X$ which compose to the unitor. Moreover, it is clear this may be similarly done for right unitors.

Lemma 4.2.13. *Suppose $\mathscr{A}$ is a generic bicategory. Let W,X,Y,Z be objects in $\mathscr{A}$, let T be the functor given by composition*

$$\left(\mathscr{A}_{W,X} \times \mathscr{A}_{X,Y}\right) \times \mathscr{A}_{Y,Z} \longrightarrow \mathscr{A}_{W,Y} \times \mathscr{A}_{Y,Z} \longrightarrow \mathscr{A}_{W,Z}$$

and consider 1-cells

$$d\colon W \longrightarrow Z, \qquad l\colon W \longrightarrow X, \qquad m\colon X \longrightarrow Y, \qquad r\colon Y \longrightarrow Z.$$

Then a 2-cell $d \longrightarrow (l;m);r$ in $\mathscr{A}$ is T-generic if and only if it has the form

$$d \xrightarrow{\delta_1} h;r \xrightarrow{\delta_2;r} (l;m);r$$

for a pair of generics δ_1 and δ_2.

Proof. Suppose we are given generics δ_1 and δ_2 composable as in the diagram on the left below

$$\begin{tikzcd}
d \arrow[r, "\gamma"] \arrow[d, "\delta_1"'] & (a;b);c \arrow[dd, "(\beta_1;\beta_2);\beta_3"] \\
h;r \arrow[d, "\delta_2;r"'] \arrow[ur, dotted, "\gamma_1;\gamma_2" description] & \\
(l;m);r \arrow[r, "(\alpha_1;\alpha_2);\alpha_3"'] & (f;g);h
\end{tikzcd}
\qquad
\begin{tikzcd}
h \arrow[r, "\gamma_1"] \arrow[d, "\delta_2"'] & a;b \arrow[d, "\beta_1;\beta_2"] \\
l;m \arrow[r, "\alpha_1;\alpha_2"'] \arrow[ur, dotted, "\zeta_1;\zeta_2" description] & f;g
\end{tikzcd}$$

where $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3$ and γ are arbitrary 2-cells such that the outside diagram commutes. Then there exists a filler $\gamma_1;\gamma_2$ splitting the diagram into two commuting regions, by genericity of δ_1. Moreover, there exists a filler $\zeta_1;\zeta_2$ for the commuting diagram on the right above as

δ_2 is generic. We thus have a diagonal filler $(\zeta_1;\zeta_2)\,;\gamma_2$ for the diagram on the left above. For uniqueness, suppose we are given another filler $\left(\zeta_1';\zeta_2'\right);\gamma_2'$ and note that since δ_1 is generic, we have $\left[\left(\zeta_1';\zeta_2'\right)\circ\delta_2\right];\gamma_2' = \gamma_1;\gamma_2$ component wise. Hence $\gamma_2' = \gamma_2$ and $\left(\zeta_1';\zeta_2'\right)\circ\delta_2 = \gamma_1$. Since δ_2 is generic it follows that $\zeta_1' = \zeta_1$ and $\zeta_2' = \zeta_2$.

Conversely, suppose we are given a 2-cell $\delta\colon\ d \longrightarrow (l;m)\,;r$ which is T-generic. Now, we know that the T-generic δ can be factored through a generic δ_1 giving the triangle on the left below

$$\begin{tikzcd}
d \arrow[r, "\delta_1"] \arrow[d, "\delta"'] & h';r' \arrow[r, "\delta_2;r'"] \arrow[dl, dotted, "\alpha;\beta"'] & (l';m')\,;r' \arrow[dll, dotted, "(\gamma_1;\gamma_2);\beta"] \\
(l;m)\,;r & &
\end{tikzcd}$$

and the 2-cell α can be factored through a generic δ_2 yielding the right triangle above. In particular, the components of $(\gamma_1;\gamma_2)\,;\beta$ are invertible as this is an induced isomorphism of T-generic morphisms [52, Lemma 5.7]. Hence upon taking δ_1^* to be δ_1 pasted with β, and δ_2^* to be δ_2 pasted with $\gamma_1;\gamma_2$, we see that δ is a pasting of generics δ_1^* and δ_2^*. ☐

Remark 4.2.14. The above lemma is an instance of a more general fact: if $\delta_1\colon C \longrightarrow SB$ is S-generic and $\delta_2\colon B \longrightarrow TA$ is T-generic, then

$$C \xrightarrow{\delta_1} SB \xrightarrow{S\delta_2} STA$$

is ST-generic. Moreover, if both S and T admit generic factorisations then all ST-generics have this form.

Remark 4.2.15. Clearly, we can state and prove an analogue of the above lemma if we replace T by the functor S given as the composite

$$\mathscr{A}_{W,X} \times \left(\mathscr{A}_{X,Y} \times \mathscr{A}_{Y,Z}\right) \longrightarrow \mathscr{A}_{W,Y} \times \mathscr{A}_{Y,Z} \longrightarrow \mathscr{A}_{W,Z}$$

It is also clear that given a composite of generics

$$d \xrightarrow{\delta_1} h;r \xrightarrow{\delta_2;r} (l;m)\,;r$$

which is T-generic, that the composite

$$d \xrightarrow{\delta_1} h;r \xrightarrow{\delta_2;r} (l;m)\,;r \xrightarrow{\text{assoc}} l;(m;r)$$

is S-generic, and hence by the analogue of the above lemma we may write this composite as

$$d \xrightarrow{\delta_3} l;k \xrightarrow{l;\delta_4} l;(m;r)$$

for some pair of generics δ_3 and δ_4.

It is sometimes advantageous to not consider all generics, but instead a smaller class of generics which is still large enough to generate the entire class of generics when completed under isomorphisms. Such a smaller class should satisfy the coherence properties outlined in the following definition.

Definition 4.2.16. Let $\mathscr{A}$ be a generic bicategory. Let Δ_2 and Δ_0 be given collections of generics and augmentations in $\mathscr{A}$ respectively. Denote by Ω_2 the set of domains of the generics in Δ_2. We say the pair (Δ_2, Δ_0) is *coherent* if:

1. (completeness of generics) for every generic $\delta' \colon c' \to l';r'$ in $\mathscr{A}$ there exists a generic $\delta \colon c \to l;r$ in Δ_2 and isomorphisms ζ_1, ζ_2 and ζ rendering commutative

$$\begin{tikzcd} c \arrow[r, "\delta"] \arrow[d, "\zeta"'] & l;r \arrow[d, "\zeta_1;\zeta_2"] \\ c' \arrow[r, "\delta'"'] & l';r' \end{tikzcd}$$

2. (completeness of augmentations) for every augmentation $\varepsilon' \colon n' \to 1_X$ in $\mathscr{A}$ there exists an augmentation $\varepsilon \colon n \to 1_X$ in Δ_0 and isomorphism $\xi \colon n \to n'$ rendering commutative

$$\begin{tikzcd} n \arrow[rr, "\xi"] \arrow[dr, "\varepsilon"'] & & n' \arrow[dl, "\varepsilon'"] \\ & 1_X & \end{tikzcd}$$

3. (associator coherence) for all generics $\delta_1, \delta_2 \in \Delta_2$ composable as below, there exists generics $\delta_3, \delta_4 \in \Delta_2$ rendering commutative

$$\begin{tikzcd} c \arrow[r, equal] \arrow[d, "\delta_3"'] & c \arrow[d, "\delta_1"] \\ l;k \arrow[d, "l;\delta_4"'] & h;r \arrow[d, "\delta_2;r"] \\ l;(m;r) \arrow[r, "\mathrm{assoc}"'] & (l;m);r \end{tikzcd}$$

4. (left unitor coherence) for all $c \colon X \to Y$ in Ω_2 there exists a $\delta \in \Delta_2$ and $\varepsilon \in \Delta_0$

composable as below and rendering commutative

$$\begin{tikzcd} & n;c \arrow[dr, "\varepsilon;c"] & \\ c \arrow[ur, "\delta"] \arrow[rr, "\text{unitor}"'] & & 1_X;c \end{tikzcd}$$

5. (right unitor coherence) for all $c\colon X \to Y$ in Ω_2 there exists a $\delta \in \Delta_2$ and $\varepsilon \in \Delta_0$ composable as below and rendering commutative

$$\begin{tikzcd} & c;n \arrow[dr, "c;\varepsilon"] & \\ c \arrow[ur, "\delta"] \arrow[rr, "\text{unitor}"'] & & c;1_Y \end{tikzcd}$$

Remark 4.2.17. If $\mathscr{A}$ is generic, we may always take (Δ_2, Δ_0) to be the class of all generic 2-cells and augmentations. This is a consequence of the previous two lemmata.

Remark 4.2.18. Informally, the conditions (3) to (5) guarantee that each 1-cell $c \in \Omega_2$ admits the structure of an "$\mathscr{A}$-comonoid"; a simple example of this being that objects in cartesian monoidal categories admit the structure of a comonoid.

4.2.3 Functors which respect generics

It is well known that to give an oplax functor $L\colon \mathbf{1} \to \mathscr{C}$ is to give a comonad in $\mathscr{C}$. The following theorem generalizes this fact, replacing the terminal category by any generic bicategory $\mathscr{A}$.

At the same time, the following theorem may be seen as a coherence result; it provides a reduction in the data of an oplax functor out of such an $\mathscr{A}$, showing that the coherence data of such an oplax functor is completely determined by the data at the diagonals.

The most important property of this result however is that it provides a description of oplax functors $L\colon \mathscr{A} \to \mathscr{C}$ out of generic bicategories $\mathscr{A}$ which does not involve composition in the domain bicategory; by this we mean expressions of the form $L(a;b)$ or $L(1_X)$ do not appear in our description below.

For completeness, we also give a reduced description of oplax natural transformations and icons [34] between such oplax functors.

Theorem 4.2.19. *Let $\mathscr{A}$ and $\mathscr{C}$ be bicategories, and suppose $\mathscr{A}$ is generic. Suppose we are given a coherent class (Δ_2, Δ_0) of generics and augmentations of $\mathscr{A}$. Then given a locally defined functor*

$$L_{X,Y}\colon \mathscr{A}_{X,Y} \to \mathscr{C}_{LX,LY}, \qquad X, Y \in \mathscr{A}$$

the following data are in bijection:

1. *for every pair of composable 1-cells a and b, a constraint 2-cell*

$$\varphi_{a,b} \colon L(a;b) \to L(a);L(b)$$

and for every identity 1-cell 1_X*, a constraint 2-cell*

$$\lambda_X \colon L(1_X) \to 1_{LX}$$

exhibiting L as an oplax functor;

2. *for every generic* $\delta \colon c \to l;r$ *in* Δ_2*, a comultiplication 2-cell*

$$\Phi_\delta \colon L(c) \to L(l);L(r)$$

and for every augmentation $\varepsilon \colon n \to 1_X$ *in* Δ_0*, a counit 2-cell*

$$\Lambda_\varepsilon \colon L(n) \to 1_{LX}$$

satisfying the following coherence axioms:

   (a) *(naturality of comultiplication) for any 2-cell* $\zeta \colon c \to c'$ *and commuting diagram as on the left below with* $\delta_1, \delta_2 \in \Delta_2$

$$\begin{tikzcd} c \arrow[r, "\delta_1"] \arrow[d, "\zeta"'] & l;r \arrow[d, "\zeta_1;\zeta_2"] \\ c' \arrow[r, "\delta_2"'] & l';r' \end{tikzcd} \qquad \begin{tikzcd} Lc \arrow[r, "\Phi_{\delta_1}"] \arrow[d, "L\zeta"'] & Ll;Lr \arrow[d, "L\zeta_1;L\zeta_2"] \\ Lc' \arrow[r, "\Phi_{\delta_2}"'] & Ll';Lr' \end{tikzcd}$$

   the diagram on the right above commutes;

   (b) *(naturality of counits) for any 2-cell* $\xi \colon n \to n'$ *and pair of augmentations* $\varepsilon \colon n \to 1_X$ *and* $\varepsilon' \colon n' \to 1_X$ *in* Δ_0 *giving a commuting diagram as on the left below*

$$\begin{tikzcd} n \arrow[rr, "\xi"] \arrow[dr, "\varepsilon"'] & & n' \arrow[dl, "\varepsilon'"] \\ & 1_X & \end{tikzcd} \qquad \begin{tikzcd} Ln \arrow[rr, "L\xi"] \arrow[dr, "\Lambda_\varepsilon"'] & & Ln' \arrow[dl, "\Lambda_{\varepsilon'}"] \\ & 1_{LX} & \end{tikzcd}$$

   the diagram on the right above commutes;

   (c) *(associativity of comultiplication) for every* $\delta_1, \delta_2, \delta_3, \delta_4 \in \Delta_2$ *yielding an equality* *as on the left below*

$$\begin{tikzcd}
c \arrow[rr, equal] \arrow[d, "\delta_3"'] & & c \arrow[d, "\delta_1"] \\
l; k \arrow[d, "l;\delta_4"'] & & h; r \arrow[d, "\delta_2;r"] \\
l;(m;r) \arrow[rr, "\mathrm{assoc}"'] & & (l;m);r
\end{tikzcd}
\qquad
\begin{tikzcd}
Lc \arrow[rr, equal] \arrow[d, "\Phi_{\delta_3}"'] & & Lc \arrow[d, "\Phi_{\delta_1}"] \\
Ll; Lk \arrow[d, "Ll;\Phi_{\delta_4}"'] & & Lh; Lr \arrow[d, "\Phi_{\delta_2};Lr"] \\
Ll;(Lm;Lr) \arrow[rr, "\mathrm{assoc}"'] & & (Ll;Lm);Lr
\end{tikzcd}$$

the diagram on the right above commutes;

(d) (left counit axiom) for any 1-cell $c\colon X \to Y$, generic $\delta \in \Delta_2$ and augmentation $\varepsilon \in \Delta_0$ yielding an equality as on the left below

$$\begin{tikzcd}
& n; c \arrow[dr, "\varepsilon;c"] & \\
c \arrow[ur, "\delta"] \arrow[rr, "\mathrm{unitor}"'] & & 1_X; c
\end{tikzcd}
\qquad
\begin{tikzcd}
& Ln; Lc \arrow[dr, "\Lambda_\varepsilon;Lc"] & \\
Lc \arrow[ur, "\Phi_\delta"] \arrow[rr, "\mathrm{unitor}"'] & & 1_{LX}; Lc
\end{tikzcd}$$

the diagram on the right above commutes;

(e) (right counit axiom) for any 1-cell $c\colon X \to Y$, generic $\delta \in \Delta_2$ and augmentation $\varepsilon \in \Delta_0$ yielding an equality as on the left below

$$\begin{tikzcd}
& c; n \arrow[dr, "c;\varepsilon"] & \\
c \arrow[ur, "\delta"] \arrow[rr, "\mathrm{unitor}"'] & & c; 1_Y
\end{tikzcd}
\qquad
\begin{tikzcd}
& Lc; Ln \arrow[dr, "Lc;\Lambda_\varepsilon"] & \\
Lc \arrow[ur, "\Phi_\delta"] \arrow[rr, "\mathrm{unitor}"'] & & Lc; 1_{LY}
\end{tikzcd}$$

the diagram on the right above commutes.

Suppose now we are given a locally defined functor L equipped with a collection (φ, λ) as in (1), or equivalently equipped with a collection (Φ, Λ) as in (2). Denote this data by the 5-tuple $(L, \varphi, \Phi, \lambda, \Lambda)$ whilst noting the collections (φ, λ) and (Φ, Λ) uniquely determine each other. Let $(K, \psi, \Psi, \gamma, \Gamma)$ be another such 5-tuple. Then the following data are in bijection:

1. *an oplax natural transformation $\vartheta\colon L \Longrightarrow K$ of oplax functors;*

2. *for every object $X \in \mathscr{A}$, a 1-cell $\vartheta_X\colon LX \to KX$ in $\mathscr{C}$, and for every 1-cell $f\colon X \to Y$ in $\mathscr{A}$, a 2-cell*

$$\begin{tikzcd}
LX \arrow[r, "Lf"] \arrow[d, "\vartheta_X"'] \arrow[dr, phantom, "\Downarrow \vartheta_f"] & LY \arrow[d, "\vartheta_Y"] \\
KX \arrow[r, "Kf"'] & KY
\end{tikzcd}$$

natural in 1-cells $f\colon X \to Y$ and satisfying the following conditions:

(a) for every generic $\delta\colon c \longrightarrow l;r$ in Δ_2,

$$\begin{tikzcd}
LX \arrow[rr, "Lc"] \arrow[d, "\vartheta_X"'] \arrow[drr, phantom, "\Downarrow\vartheta_c"] & & LZ \arrow[d, "\vartheta_Z"] & & LX \arrow[rr, "Lc"] \arrow[d, "\vartheta_X"'] \arrow[dr, bend right, "Ll"'] & \, & LZ \arrow[d, "\vartheta_Z"] \\
KX \arrow[rr, "Kc"] \arrow[dr, bend right, "Kl"'] & \, & KZ & = & KX \arrow[dr, bend right, "Kl"'] & LY \arrow[ur, bend right, "Lr"'] \arrow[d, "\theta_Y"] & KZ \\
& KY \arrow[ur, bend right, "Kr"'] & & & & KY \arrow[ur, bend right, "Kr"'] &
\arrow[from=2-2, to=3-2, phantom, "\Downarrow\Psi_\delta"]
\arrow[from=1-6, to=2-6, phantom, "\Downarrow\Phi_\delta"]
\arrow[from=2-5, to=3-6, phantom, "\Downarrow\vartheta_l"]
\arrow[from=2-7, to=3-6, phantom, "\Downarrow\vartheta_r"]
\end{tikzcd}$$

(b) for every augmentation $\varepsilon\colon n \longrightarrow 1_X$ in Δ_0,

$$\begin{tikzcd}
LX \arrow[rr, "Ln"] \arrow[d, "\vartheta_X"'] \arrow[drr, phantom, "\Downarrow\vartheta_n"] & & LX \arrow[d, "\vartheta_X"] & & LX \arrow[rr, "Ln"] \arrow[rr, bend right=50, "\Downarrow\Lambda_\varepsilon"', "1_{LX}"' near end] \arrow[d, "\vartheta_X"'] & & LX \arrow[d, "\vartheta_X"] \\
KX \arrow[rr, "Kn"] \arrow[rr, bend right=50, "1_{KX}"'] \arrow[rr, phantom, bend right=25, "\Downarrow\Gamma_\varepsilon"'] & & KX & = & KX \arrow[rr, bend right=50, "1_{KX}"'] \arrow[rr, phantom, bend right=25, "\Downarrow\mathrm{id}"'] & & KX
\end{tikzcd}$$

When L and K agree on objects, this restricts to the bijection of the following data:

1. *An icon between oplax functors*

$$\vartheta\colon L \implies K\colon \mathscr{A} \longrightarrow \mathscr{C}$$

2. *A collection of natural transformations*

$$\vartheta_{X,Y}\colon L_{X,Y} \implies K_{X,Y}\colon \mathscr{A}_{X,Y} \longrightarrow \mathscr{C}_{X,Y}, \qquad X,Y \in \mathscr{A}$$

rendering commutative the diagrams

$$\begin{tikzcd}
L(c) \arrow[r, "\Phi_\delta"] \arrow[d, "\vartheta_c"'] & L(l)\,;L(r) \arrow[d, "\vartheta_l;\vartheta_r"] & L(n) \arrow[rr, "\vartheta_n"] \arrow[dr, "\Lambda_n"'] & & K(n) \arrow[dl, "\Gamma_n"] \\
K(c) \arrow[r, "\Psi_\delta"'] & K(l)\,;K(r) & & 1_X &
\end{tikzcd}$$

Proof. We divide the proof into parts, verifying each bijection separately.

Bijection With Oplax Functors. We first show how to pass between the data of (1) and (2), and then verify this defines a bijection.

(1) $\implies$ (2) : Suppose we are given the data (L, φ, λ) of (1). We define Φ_δ for each generic $\delta\colon c \longrightarrow l;r$ by the composite

$$L(c) \xrightarrow{L\delta} L(l;r) \xrightarrow{\varphi_{l,r}} L(l)\,;L(r) \tag{4.2.2}$$

and define Λ_ε for each augmentation $\varepsilon\colon n \longrightarrow 1_X$ by the composite

$$L(n) \xrightarrow{L\varepsilon} L(1_X) \xrightarrow{\lambda_X} 1_{LX} \tag{4.2.3}$$

For naturality of comultiplication, we see that given a diagram as on the left below

$$\begin{tikzcd}
c \arrow[r, "\delta_1"] \arrow[d, "\zeta"'] & l;r \arrow[d, "\zeta_1;\zeta_2"] \\
c' \arrow[r, "\delta_2"'] & l';r'
\end{tikzcd}
\qquad
\begin{tikzcd}
Lc \arrow[r, "L\delta_1"] \arrow[d, "L\zeta"'] & L(l;r) \arrow[r, "\varphi_{l,r}"] \arrow[d, "L(\zeta_1;\zeta_2)"'] & Ll;Lr \arrow[d, "L\zeta_1;L\zeta_2"] \\
Lc' \arrow[r, "L\delta_2"'] & L(l';r') \arrow[r, "\varphi_{l',r'}"'] & Ll';Lr'
\end{tikzcd}$$

the right commutes by naturality of φ and local functoriality of L. For naturality of counits note that given a commuting diagram as on the left below

$$\begin{tikzcd}
n \arrow[dr, "\varepsilon"] \arrow[dd, "\xi"'] & \\
& 1_X \\
n' \arrow[ur, "\varepsilon'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Ln \arrow[dr, "L\varepsilon"] \arrow[dd, "L\xi"'] & & \\
& L1_X \arrow[r, "\lambda_X"] & 1_{LX} \\
Ln' \arrow[ur, "L\varepsilon'"'] & &
\end{tikzcd}$$

the right trivially commutes. For associativity of comultiplication, note that given a commuting diagram

$$\begin{tikzcd}
c \arrow[r, equal] \arrow[d, "\delta_3"'] & c \arrow[d, "\delta_1"] \\
l;k \arrow[d, "l;\delta_4"'] & h;r \arrow[d, "\delta_2;r"] \\
l;(m;r) \arrow[r, "\mathrm{assoc}"'] & (l;m);r
\end{tikzcd}$$

we have the commutativity of the diagram

$$\begin{tikzcd}
Lc \arrow[rrr, equal] \arrow[d, "L\delta_3"'] & & & Lc \arrow[d, "L\delta_1"] \\
L(l;k) \arrow[d, "\varphi_{l,k}"'] \arrow[dr, "L(l;\delta_4)"] & & & L(h;r) \arrow[d, "\varphi_{h,r}"] \arrow[dl, "L(\delta_1;r)"'] \\
Ll;Lk \arrow[d, "Ll;L\delta_4"'] & L(l;(m;r)) \arrow[r, "L(\mathrm{assoc})"'] \arrow[dl, "\varphi_{l,(m;r)}"] & L((l;m);r) \arrow[dr, "\varphi_{(l;m),r}"'] & Lh;Lr \arrow[d, "L\delta_2;Lr"] \\
Ll;L(m;r) \arrow[d, "Ll;\varphi_{m,r}"'] & & & L(l;m);Lr \arrow[d, "\varphi_{l,m};Lr"] \\
Ll;(Lm;Lr) \arrow[rrr, "\mathrm{assoc}"'] & & & (Ll;Lm);Lr
\end{tikzcd}$$

by naturality of φ, associativity of φ and local functoriality of L. For the left counit axiom, suppose we are given a commuting diagram as on the left below

$$\begin{tikzcd}
 & l;c \arrow[dr, "\varepsilon;c"] & \\
c \arrow[ur, "\delta"] \arrow[rr, "\text{unitor}"'] & & 1_X;c
\end{tikzcd}
\qquad
\begin{tikzcd}
Lc \arrow[r, "\delta"] \arrow[drr, bend right, "L(\text{unitor})"'] & L(l;c) \arrow[r, "\varphi_{l,r}"] \arrow[dr, "L(\varepsilon;c)"] & Ll;Lc \arrow[r, "L\varepsilon;Lc"] & L1_X;Lc \arrow[r, "\lambda_X;Lc"] & 1_{LX};Lc \\
 & & L(1_X;c) \arrow[ur, "\varphi_{1_X,c}"] \arrow[r, "L(\text{unitor})"'] & L(c) \arrow[ur, bend right, "\text{unitor}"'] &
\end{tikzcd}$$

and note the composite on the right above is the unitor by local functoriality of L, naturality of φ, and the unit axiom on λ. The right counit axiom is similar.

(2) $\implies$ (1) : Suppose we are given the data (L, Φ, Λ) for a coherent class (Δ_2, Δ_0). Now for any generic $\delta' \colon c' \to l'; r'$ in $\mathscr{A}$ we have a commuting diagram as on the left below with ζ_1, ζ_2, ζ invertible and $\delta \in \Delta_2$

$$\begin{tikzcd}
c \arrow[r, "\delta"] \arrow[d, "\zeta"'] & l;r \arrow[d, "\zeta_1;\zeta_2"] \\
c' \arrow[r, "\delta'"'] & l';r'
\end{tikzcd}
\qquad
\begin{tikzcd}
Lc \arrow[r, "\Phi_\delta"] \arrow[d, "L\zeta"'] & Ll;Lr \arrow[d, "L\zeta_1;L\zeta_2"] \\
Lc' \arrow[r, "\Phi_{\delta'}"'] & Ll';Lr'
\end{tikzcd}$$

and so we may define $\Phi_{\delta'}$ as the unique morphism making the diagram on the right above commute; this being well defined as a consequence of naturality of comultiplication.

Similarly, for any augmentation $\varepsilon' \colon n' \to 1_X$ in $\mathscr{A}$ there exists an augmentation $\varepsilon \colon n \to 1_X$ in Δ_0 and isomorphism $\xi \colon n \to n'$ rendering commutative the left diagram below

$$\begin{tikzcd}
n \arrow[rr, "\xi"] \arrow[dr, "\varepsilon"'] & & n' \arrow[dl, "\varepsilon'"] \\
 & 1_X &
\end{tikzcd}
\qquad
\begin{tikzcd}
Ln \arrow[rr, "L\xi"] \arrow[dr, "\Lambda_\varepsilon"'] & & Ln' \arrow[dl, "\Lambda_{\varepsilon'}"] \\
 & 1_{LX} &
\end{tikzcd}$$

and so we may define $\Lambda_{\varepsilon'}$ as the unique morphism making the right diagram above commute; similarly well defined by naturality of counits.

We have now extended the definition of Φ and Λ to all generic morphisms and augmentations. Moreover, the naturality properties now hold with respect to all generics δ and augmentations ε. Indeed, given any generics δ and δ' in $\mathscr{A}$ and a diagram as on the left below (not assuming ζ, ζ_1 or ζ_2 are invertible)

$$\begin{tikzcd}
c \arrow[r, "\delta"] \arrow[d, "\zeta"'] & l;r \arrow[d, "\zeta_1;\zeta_2"] \\
c' \arrow[r, "\delta'"'] & l';r'
\end{tikzcd}
=
\begin{tikzcd}
c \arrow[r, "\delta"] \arrow[d, "\theta"'] & l;r \arrow[d, "\theta_1;\theta_2"] \\
\widetilde{c} \arrow[r, "\widetilde{\delta}"'] \arrow[d, "\phi"'] & \widetilde{l};\widetilde{r} \arrow[d, "\phi_1;\phi_2"] \\
\widetilde{c'} \arrow[r, "\widetilde{\delta'}"'] \arrow[d, "\gamma"'] & \widetilde{l'};\widetilde{r'} \arrow[d, "\gamma_1;\gamma_2"] \\
c' \arrow[r, "\delta'"'] & l';r'
\end{tikzcd}$$

we can factor as on the right, where $\widetilde{\delta}$ and $\widetilde{\delta'}$ are in Δ_2 and $\theta, \theta_1, \theta_2, \gamma, \gamma_1$ and γ_2 are invertible. Applying the naturality condition to the three squares on the right then gives the naturality condition for the left diagram. A similar calculation may be done concerning augmentations.

To show that one may recover an oplax functor $L\colon \mathscr{A} \to \mathscr{C}$ we note we may define a general oplax constraint cell $\varphi_{a,b}\colon L\,(a;b) \to La;Lb$ by taking a diagram as on the left below with δ generic and then defining the right diagram to commute.

$$\begin{tikzcd}
 & l;r \arrow[dr, "s_1;s_2"] & & & & Ll;Lr \arrow[dr, "Ls_1;Ls_2"] & \\
a;b \arrow[ur, "\delta"] \arrow[rr, "\mathrm{id}"'] & & a;b & & L\,(a;b) \arrow[ur, "\Phi_\delta"] \arrow[rr, "\varphi_{a,b}"'] & & La;Lb
\end{tikzcd} \tag{4.2.4}$$

Note that this is well defined since given two diagrams as on the left above, we have a commuting diagram as on the left below

$$\begin{tikzcd}
a;b \arrow[r, "\delta"] \arrow[d, "\delta'"'] & l;r \arrow[d, "s_1;s_2"] & & La;Lb \arrow[r, "\Phi_\delta"] \arrow[d, "\Phi_{\delta'}"'] & Ll;Lr \arrow[d, "Ls_1;Ls_2"] \\
l';r' \arrow[r, "t_1;t_2"'] \arrow[ur, dotted, "\gamma_1;\gamma_2" description] & a;b & & Ll';Lr' \arrow[r, "Lt_1;Lt_2"'] \arrow[ur, "L\gamma_1;L\gamma_2" description] & La;Lb
\end{tikzcd} \tag{4.2.5}$$

composing to the identity, and this implies the right diagram commutes by naturality of comultiplication (with ζ taken to be the identity). Trivially, we take each unit $\lambda_X\colon L\,(1_X) \to 1_X$ to be the component of Λ at id_{1_X}.

To see that the family φ satisfies naturality of the constraints suppose that we are given a diagram as on the left below with the horizontal paths composing to identities

$$\begin{tikzcd}
a;b \arrow[r, "\delta"] \arrow[d, "\alpha;\beta"'] & l;r \arrow[r, "s_1;s_2"] \arrow[d, dotted, "\gamma_1;\gamma_2"] & a;b \arrow[d, "\alpha;\beta"] & & L\,(a;b) \arrow[r, "\Phi_\delta"] \arrow[d, "L(\alpha;\beta)"'] & Ll;Lr \arrow[r, "Ls_1;Ls_2"] \arrow[d, "L\gamma_1;L\gamma_2"] & La;Lb \arrow[d, "L\alpha;L\beta"] \\
a';b' \arrow[r, "\delta'"'] & l';r' \arrow[r, "s'_1;s'_2"'] & a';b' & & L\,(a';b') \arrow[r, "\Phi_{\delta'}"'] & Ll';Lr' \arrow[r, "Ls'_1;Ls'_2"'] & La';Lb'
\end{tikzcd}$$

and note that the right diagram commutes by naturality of comultiplication.

Before checking associativity we first note that given any generics $\delta'_1, \delta'_2, \delta'_3$ and δ'_4 in $\mathscr{A}$ such that (1) commutes below,

$$\begin{tikzcd}
c \arrow[r, "\zeta^{-1}"] \arrow[d, "\delta_3"'] \arrow[dr, phantom, "(5)"] & c' \arrow[r, equal] \arrow[d, "\delta'_3"'] & c' \arrow[r, "\zeta"] \arrow[d, "\delta'_1"] \arrow[dr, phantom, "(2)"] & c \arrow[d, "\delta_1"] \\
l;k \arrow[r, "\alpha;\beta" description] \arrow[d, "l;\delta_4"'] \arrow[dr, phantom, "(6)"] & l';k' \arrow[r, phantom, "(1)"] \arrow[d, "l';\delta'_4"'] & h';r' \arrow[r, "\zeta_1;\zeta_2" description] \arrow[d, "\delta'_2;r'"] \arrow[dr, phantom, "(3)"] & h;r \arrow[d, "\delta_2;r"] \\
l;(m;r) \arrow[r, "\phi_1^{-1};(\phi_2^{-1};\zeta_2^{-1})"'] & l';(m';r') \arrow[r, "\mathrm{assoc}"'] \arrow[dr, bend right, "\phi_1;(\phi_2;\zeta_2)"'] & (l';m');r' \arrow[r, "(\phi_1;\phi_2);\zeta_2"'] \arrow[d, phantom, "(4)"] & (l;m);r \\
 & & l;(m;r) \arrow[ur, bend right, "\mathrm{assoc}"'] &
\end{tikzcd}$$

we can construct regions (2) and (3) as on the right above, where δ_1 and δ_2 lie in Δ_2. By naturality of the associator (4) commutes. Then since our given class of generics is coherent, we can find a δ_3 and δ_4 in Δ_2 such that the outside diagram commutes above. By genericity of δ_3 we then have induced 2-cells α and β such that (5) and (6) commute (invertible as δ_3' is also generic). Now, by associativity of comultiplication the commutativity of the outside diagram is respected by the transformation $\delta \mapsto \Phi_\delta$, and this is equivalent to the commutativity of (1) being respected as the pasting with (2),(3),(4),(5) and (6) may be undone.

Now, to see that the family φ satisfies associativity of the constraints consider the outside diagram of

$$\begin{tikzcd}
(a;b);c \arrow[r, "\delta_1"] \arrow[dr, "\delta_3"'] \arrow[dddd, "\text{assoc}"'] & h;r \arrow[r, "s_1;s_2"] \arrow[dr, "\delta_2;r"'] & (a;b);c \arrow[r, "\delta_5;c"] \arrow[d, phantom, "(1)"] & (f;g);c \arrow[r, "(t_1;t_2);c"] & (a;b);c \arrow[dddd, "\text{assoc}"] \\
 & l;k \arrow[r, phantom, "(3)"] \arrow[dr, "l;\delta_4"'] \arrow[ddd, dotted, "\gamma_1;\gamma_2"'] & (l;m);r \arrow[d, "\text{assoc}"] \arrow[ur, "(\xi_1;\xi_2);s_2"'] & & \\
 & (5) & l;(m;r) \arrow[d, dotted, "\gamma_1;(\alpha;\beta)"'] \arrow[dr, "\xi_1;(\xi_2;s_2)"] & (4) & \\
 & (6) & \widetilde{l};(\widetilde{m};\widetilde{r}) \arrow[r, phantom, "(7)"] \arrow[dr, "p_1;(\zeta_1;\zeta_2)"] \arrow[d, phantom, "(2)"] & f;(g;c) \arrow[dr, "t_1;(t_2;c)"] & \\
a;(b;c) \arrow[r, "\delta_6"'] & \widetilde{l};\widetilde{k} \arrow[r, "p_1;p_2"'] \arrow[ur, "\widetilde{l};\delta_7"] & a;(b;c) \arrow[r, "a;\delta_8"'] & a;(u;v) \arrow[r, "a;(q_1;q_2)"'] & a;(b;c)
\end{tikzcd}$$

where the appropriate horizontal composites are identity 2-cells. We first factor $\delta_5 s_1$ through a generic δ_2 to recover 2-cells ξ_1 and ξ_2 and the commuting region (1). Similarly, we create the region (2). Now take δ_3 and δ_4 to be generics such that region (3) commutes, which exist by Lemma 4.2.13. We then note that region (4) commutes by naturality of the associator in $\mathscr{A}$. Finally, note that we have an induced $(\gamma_1;\gamma_2)$ by genericity of δ_3, and thus $\delta_7\gamma_2$ yields an induced $(\alpha;\beta)$ through the generic δ_4.

We have now constructed the above diagram and shown each region commutes; all that remains is to notice in the corresponding diagram below

$$\begin{tikzcd}
L\left((a;b);c\right) \arrow[r, "\Phi_{\delta_1}"] \arrow[dddd, "L(\text{assoc})" swap] \arrow[dr, "\Phi_{\delta_3}" swap] & Lh;Lr \arrow[r, "Ls_1;Ls_2"] \arrow[dr, "\Phi_{\delta_2};Lr" swap] & L\left(a;b\right);Lc \arrow[r, "\Phi_{\delta_5};c"] \arrow[d, phantom, "(1)"] & \left(Lf;Lg\right);Lc \arrow[r, "(Lt_1;Lt_2);Lc"] & \left(La;Lb\right);Lc \arrow[dddd, "\text{assoc}"] \\
 & Ll;Lk \arrow[r, phantom, "(3)"] \arrow[dddd, "L\gamma_1;L\gamma_2" swap] \arrow[dr, "Ll;\Phi_{\delta_4}" swap] & \left(Ll;Lm\right);Lr \arrow[d, "\text{assoc}"] \arrow[ur, "(L\xi_1;L\xi_2);Ls_2" swap] & & \\
{} \arrow[r, phantom, "(5)"] & {} & Ll;\left(Lm;Lr\right) \arrow[d, "L\gamma_1;(L\alpha;L\beta)" swap] \arrow[dr, "L\xi_1;(L\xi_2;Ls_2)"] & (4) & \\
 & {} \arrow[r, phantom, "(6)"] & L\widetilde{l};L\left(\widetilde{m};\widetilde{r}\right) \arrow[r, phantom, "(7)"] \arrow[d, phantom, "(2)"] \arrow[dr, "Lp_1;(L\zeta_1;L\zeta_2)"] & Lf;\left(Lg;Lc\right) \arrow[dr, "Lt_1;(Lt_2;Lc)"] & \\
L\left(a;(b;c)\right) \arrow[r, "\Phi_{\delta_6}" swap] & L\widetilde{l};L\widetilde{k} \arrow[r, "Lp_1;Lp_2" swap] \arrow[ur, "L\widetilde{l};\Phi_{\delta_7}"] & La;L\left(b;c\right) \arrow[r, "La;\Phi_{\delta_8}" swap] & La;\left(Lu;Lv\right) \arrow[r, "La;(Lq_1;Lq_2)" swap] & La;\left(Lb;Lc\right)
\end{tikzcd}$$

naturality of comultiplication implies (1), (2), (5) and (6) commute; associativity of comultiplication implies (3) commutes; naturality of the associators in $\mathscr{C}$ implies (4) commutes, and (7) commutes as L is locally a functor.

Before checking the unit axioms on λ we note that given a generic δ' and augmentation ε' composable as in the middle diagram below

$$\begin{tikzcd}
 & & n;c \arrow[d, dotted, "u_1;u_2"] \arrow[ddrr, bend left, "\varepsilon;c"] & & \\
 & & n';c' \arrow[dr, "\varepsilon';c'"] & & \\
c \arrow[r, "\zeta" swap] \arrow[uurr, bend left, "\delta"] & c' \arrow[rr, "\text{unitor}" swap] \arrow[ur, "\delta'"] & & 1_X;c' \arrow[r, "1_X;\zeta^{-1}" swap] & 1_X;c
\end{tikzcd}$$

we have an isomorphism $\zeta\colon c \to c'$ by axiom (1) of a coherent class. By axiom (5) we then have a δ and ε in the coherent class such that the outside diagram commutes. It follows from genericity of δ that we have an induced isomorphism $u_1;u_2$ such that the above diagram commutes. As the commutativity of the outside diagram is respected by assumption, and the commutativity of the left and right regions is respected by naturality of comultiplication and augmentations respectively (and the pasting with these regions can be undone), it follows that the commutativity of the middle diagram is respected.

Now, to see the left unit axiom on λ is satisfied note that given any commuting diagram as on the left below

$$\begin{tikzcd}
1_X;c \arrow[r, "\text{unitor}"] \arrow[d, "\delta"'] & c \arrow[r, "\text{unitor}"] \arrow[d, "\delta'"'] & 1_X;c \\
l;r \arrow[r, "s_1;s_2"'] & l';c \arrow[r, "\varepsilon;c"'] \arrow[ur, "\varepsilon;c"] & 1_X;c \arrow[u, "\text{id}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
L\left(1_X;c\right) \arrow[r, "L(\text{unitor})"] \arrow[d, "\Phi_\delta"'] & Lc \arrow[r, "\text{unitor}"] \arrow[d, "\Phi_{\delta'}"'] & 1_X;Lc \\
Ll;Lr \arrow[r, "Ls_1;Ls_2"'] & Ll';Lc \arrow[r, "L\varepsilon;Lc"'] \arrow[ur, "\Lambda_\varepsilon;Lc"] & L1_X;Lc \arrow[u, "\Lambda_{1_X};Lc"']
\end{tikzcd}$$

we get a commuting diagram as on the right above by naturality of comultiplication, the left counit axiom, and naturality of counits (the bottom composite in this diagram is a φ followed by a λ). The right unit axiom is similar.

Finally, note that the composite assignment

$$(1) \mapsto (2) \mapsto (1)$$

is the identity, since with Φ defined as in (4.2.2), the oplax constraint cells as recovered by (4.2.4), given by the family of constraints

$$\begin{tikzcd}
L\left(a;b\right) \arrow[r, "L\delta"] & L\left(l;r\right) \arrow[r, "\varphi_{l,r}"] & Ll;Lr \arrow[r, "Ls_1;Ls_2"] & La;Lb
\end{tikzcd}$$

are clearly equal to $\varphi_{a,b}$ by naturality. Moreover, the composite assignment

$$(2) \mapsto (1) \mapsto (2)$$

is the identity, since with φ defined as by (4.2.4), the comultiplication cells Φ at an arbitrary generic $\widetilde{\delta} \in \Delta_2$ are given by the composite in the top line on the left below

$$\begin{tikzcd}
Lc \arrow[r, "L\widetilde{\delta}"] \arrow[ddr, "\Phi_{\widetilde{\delta}}"'] & L\left(\widetilde{l};\widetilde{r}\right) \arrow[r, "\Phi_\delta"] \arrow[rr, bend left, "\varphi_{\widetilde{l},\widetilde{r}}", ":="'] & Ll;Lr \arrow[r, "Ls_1;Ls_2"] & L\widetilde{l};L\widetilde{r} \\
& (1) & & \\
& L\widetilde{l};L\widetilde{r} \arrow[uur, "L\widetilde{\delta}_1;L\widetilde{\delta}_2"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& l;r \arrow[dr, "s_1;s_2"] & \\
\widetilde{l};\widetilde{r} \arrow[ur, "\delta"] \arrow[rr, "\text{id}"'] & & \widetilde{l};\widetilde{r}
\end{tikzcd}
\tag{4.2.6}$$

where $\delta \in \Delta_2$ is a generic and the right diagram commutes. Then we note that

$$\begin{tikzcd}
c \arrow[r, "\widetilde{\delta}"] \arrow[d, "\widetilde{\delta}"'] & \widetilde{l};\widetilde{r} \arrow[d, dotted, "\widetilde{\delta}_1;\widetilde{\delta}_2"] \\
\widetilde{l};\widetilde{r} \arrow[r, "\delta"'] & l;r
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, "\widetilde{\delta}"] \arrow[d, "\widetilde{\delta}"'] & \widetilde{l};\widetilde{r} \arrow[d, "\text{id;id}"] \arrow[dl, dotted, "\text{id;id}"'] \\
\widetilde{l};\widetilde{r} \arrow[r, "\text{id;id}"'] & \widetilde{l};\widetilde{r}
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, "\widetilde{\delta}"] \arrow[d, "\widetilde{\delta}"'] & \widetilde{l};\widetilde{r} \arrow[d, "s_1\widetilde{\delta}_1;s_2\widetilde{\delta}_2"] \arrow[dl, dotted, "\text{id;id}"'] \\
\widetilde{l};\widetilde{r} \arrow[r, "\text{id;id}"'] & \widetilde{l};\widetilde{r}
\end{tikzcd}$$

we have an induced $\widetilde{\delta}_1;\widetilde{\delta}_2$ rendering commutative the left diagram above by genericity of $\widetilde{\delta}$, the middle diagram shows that the induced diagonal is necessarily a pair of identities (by component-wise commutativity of the bottom triangle), and whiskering the left diagram with $s_1; s_2$ gives the right diagram, where as we have noted the induced diagonal making the diagram commute is a pair of identities. Consequently, $s_1\widetilde{\delta_1}$ and $s_2\widetilde{\delta_2}$ are identities. We then note that in diagram 4.2.6 the region (1) commutes by naturality of comultiplication, and applying local functoriality of L we then see the given composite is $\Phi_{\tilde{\delta}}$ as required.

The bijection of the nullary data may be similarly proven using the respective naturality properties, and so we omit the details.

Bijection With Oplax Natural Transformations. As the the data of (1) and (2) is the same, we need only check that the coherence conditions correspond.

(1) $\Longrightarrow$ (2) : Suppose we are given an oplax natural transformation $\vartheta\colon L \to K$ in the usual sense. Then by the definition of Φ at a $\delta \in \Delta_2$ we have

$$\begin{tikzcd}
LX \arrow[rr, "Lc"] \arrow[d, "\vartheta_X"'] \arrow[dr, bend right, "Ll"] & \arrow[d, phantom, "\Downarrow\Phi_\delta"] & LZ \arrow[d, "\vartheta_Z"] & & LX \arrow[rr, bend left=40, "L(c)"] \arrow[rr, phantom, bend left=20, "\Downarrow L\delta"] \arrow[rr, "L(l;r)" description] \arrow[d, "\vartheta_X"'] \arrow[dr, bend right, "Ll"] & \arrow[d, phantom, "\Downarrow\varphi_{l,r}"] & LZ \arrow[d, "\vartheta_Z"] \\
KX \arrow[dr, bend right, "Kl"'] \arrow[r, phantom, shift right=6, "\Downarrow\vartheta_l"] & LY \arrow[ur, bend right, "Lr"'] \arrow[d, "\theta_Y"] \arrow[r, phantom, shift right=6, "\Downarrow\vartheta_r"] & KZ & = & KX \arrow[dr, bend right, "Kl"'] \arrow[r, phantom, shift right=6, "\Downarrow\vartheta_l"] & LY \arrow[ur, bend right, "Lr"'] \arrow[d, "\theta_Y"] \arrow[r, phantom, shift right=6, "\Downarrow\vartheta_r"] & KZ \\
& KY \arrow[ur, bend right, "Kr"'] & & & & KY \arrow[ur, bend right, "Kr"'] &
\end{tikzcd}$$

which by compatibility with composition is

$$\begin{tikzcd}
LX \arrow[rr, bend left=40, "L(c)"] \arrow[rr, phantom, bend left=20, "\Downarrow L\delta"] \arrow[rr, "L(l;r)" description] \arrow[d, "\vartheta_X"'] & \arrow[d, phantom, "\Downarrow\vartheta_{l;r}"] & LZ \arrow[d, "\vartheta_Z"] & & LX \arrow[rr, bend left=40, "L(l;r)"] \arrow[rr, phantom, bend left=20, "\Downarrow\vartheta_c"] \arrow[d, "\vartheta_X"'] & & LZ \arrow[d, "\vartheta_Z"] \\
KX \arrow[rr, "K(l;r)" description] \arrow[dr, bend right, "Kl"'] & \arrow[d, phantom, "\Downarrow\psi_{l,r}"] & KZ & = & KX \arrow[rr, bend left=40, "K(c)"] \arrow[rr, phantom, bend left=20, "\Downarrow K\delta"] \arrow[rr, "K(l;r)" description] \arrow[dr, bend right, "Kl"'] & \arrow[d, phantom, "\Downarrow\psi_{l,r}"] & KZ \\
& KY \arrow[ur, bend right, "Kr"'] & & & & KY \arrow[ur, bend right, "Kr"'] &
\end{tikzcd}$$

and by definition of Ψ this gives the required coherence condition. We omit the nullary version.

(2) $\Longrightarrow$ (1) : Suppose we are given the data of (2) subject to the coherence conditions of

(2). Then by the definition of the constraint data φ we have

$$\begin{tikzcd}
LX \arrow[r, "Lf"] \arrow[d, "\vartheta_X"'] \arrow[rr, bend left=50, "L(f;g)"] \arrow[rr, phantom, bend left=25, "\Downarrow\varphi_{f,g}"] & LY \arrow[r, "Lg"] \arrow[d, "\vartheta_Y"] & LZ \arrow[d, "\vartheta_Z"] \\
KX \arrow[r, "Kf"'] \arrow[ru, phantom, "\Downarrow\vartheta_f"] & KY \arrow[r, "Kg"'] \arrow[ru, phantom, "\Downarrow\vartheta_g"] & KZ
\end{tikzcd}
=
\begin{tikzcd}
LX \arrow[r, bend left, "Ll"] \arrow[r, bend right, "Lf"'] \arrow[r, phantom, "\Downarrow Ls_1"] \arrow[d, "\vartheta_X"'] \arrow[rr, bend left=60, "L(f;g)"] \arrow[rr, phantom, bend left=35, "\Downarrow\Phi_\delta"] & LY \arrow[r, bend left, "Lr"] \arrow[r, bend right, "Lg"'] \arrow[r, phantom, "\Downarrow Ls_2"] \arrow[d, "\vartheta_Y"] & LZ \arrow[d, "\vartheta_Z"] \\
KX \arrow[r, "Kf"'] \arrow[ru, phantom, "\Downarrow\vartheta_f"] & KY \arrow[r, "Kg"'] \arrow[ru, phantom, "\Downarrow\vartheta_g"] & KZ
\end{tikzcd}$$

and so applying naturality of ϑ, this is equal to the left below

$$\begin{tikzcd}
LX \arrow[r, "Ll"] \arrow[d, "\vartheta_X"'] \arrow[rr, bend left=50, "L(f;g)"] \arrow[rr, phantom, bend left=25, "\Downarrow\Phi_\delta"] & LY \arrow[r, "Lr"] \arrow[d, "\vartheta_Y"] & LZ \arrow[d, "\vartheta_Z"] \\
KX \arrow[r, bend left, "Kl"] \arrow[r, bend right, "Kf"'] \arrow[r, phantom, "\Downarrow Ks_1"] \arrow[ru, phantom, "\Downarrow\vartheta_l"] & KY \arrow[r, bend left, "Kr"] \arrow[r, bend right, "Kg"'] \arrow[r, phantom, "\Downarrow Ks_2"] \arrow[ru, phantom, "\Downarrow\vartheta_r"] & KZ
\end{tikzcd}
=
\begin{tikzcd}
LX \arrow[rr, bend left=40, "L(f;g)"] \arrow[rr, phantom, bend left=20, "\Downarrow\vartheta_{f;g}"] \arrow[d, "\vartheta_X"'] & & LZ \arrow[d, "\vartheta_Z"] \\
KX \arrow[r, bend left, "Kl"] \arrow[r, bend right, "Kf"'] \arrow[r, phantom, "\Downarrow Ks_1"] \arrow[rr, bend left=60, "K(f;g)"] \arrow[rr, phantom, bend left=35, "\Downarrow\Psi_\delta"] & KY \arrow[r, bend left, "Kr"] \arrow[r, bend right, "Kg"'] \arrow[r, phantom, "\Downarrow Ks_2"] & KZ
\end{tikzcd}$$

which by the assumed coherence axiom is the right above. Applying the definition of ψ, we recover the compatibility of an oplax natural transformation with composition. Again, we will omit the analogous nullary condition.

Bijection With Icons. This trivially follows taking each ϑ_X to be an identity 1-cell in the above bijection. ☐

Remark 4.2.20. The reader will have noticed from the proof of Theorem 4.2.19 that giving binary oplax constraint cells

$$\varphi_{l,r} \colon L\,(l;r) \longrightarrow Ll;Lr$$

for generics $\delta\colon c \longrightarrow l;r$ in Δ_2 completely determines arbitrary oplax constraint cells

$$\varphi_{a,b} \colon L\,(a;b) \longrightarrow La;Lb.$$

This is since these $\varphi_{l,r}$ suffice to construct each Φ_δ. Hence this theorem provides a reduction in the data of an oplax functor when the domain bicategory $\mathscr{A}$ is generic.

Remark 4.2.21. Given a family of hom-categories $\mathscr{A}_{X,Y}$, sets $\mathfrak{M}_c^{X,Y,Z}$, and natural isomorphisms

$$\mathscr{A}_{X,Z}\,(c,a;b) \cong \sum_{m\in\mathfrak{M}_c^{X,Y,Z}} \mathscr{A}_{X,Y}\,(l_m,a) \times \mathscr{A}_{Y,Z}\,(r_m,b)$$

for all X,Y,Z and c, the formal composite $a;b$ is essentially uniquely determined (by essential uniqueness of representing objects).

Given a complete class of generics Δ_2 equipped with their universal properties, one may recover the above by taking $\mathfrak{M}_c^{X,Y,Z}$ to be the set of equivalence classes of generics $\delta\colon c \to l;r$. It follows that composition in the bicategory is essentially uniquely determined by the generics.

We now give another perspective on the above remark, seeing that composition in a generic bicategory is essentially uniquely determined by the generics in that it must obey a universal property similar to that of a product.

Remark 4.2.22. In the setting of a generic bicategory $\mathscr{A}$, one can view composition as a sort of generalized product. Indeed, as each composition functor $\circ_{X,Y,Z}\colon \mathscr{A}_{Y,Z} \times \mathscr{A}_{X,Y} \to \mathscr{A}_{X,Z}$ admits generic factorisations, one may form the spectrum [15] of $\circ_{X,Y,Z}$ given by the presheaf $\mathfrak{M}^{-}_{X,Y,Z}\colon \mathscr{A}^{\mathrm{op}}_{X,Z} \to \mathbf{Set}$. This gives a factorisation

$$\begin{tikzcd}
\mathscr{A}_{Y,Z} \times \mathscr{A}_{X,Y} \arrow[rr, "\circ_{X,Y,Z}"] \arrow[dr] & & \mathscr{A}_{X,Z} \\
& \mathrm{el}\, \mathfrak{M}^{-}_{X,Y,Z} \arrow[ur] &
\end{tikzcd}$$

where the first arrow has a left adjoint (which we may denote by $(-)_r \times (-)_l$) and the second is a discrete fibration [14, 15]. It follows that for all $a \in \mathscr{A}_{X,Y}$ and $b \in \mathscr{A}_{Y,Z}$ there exists a triple

$$\left(a;b \in \mathscr{A}_{X,Z},\ m \in \mathfrak{M}^{a;b}_{X,Y,Z}\right), \qquad \pi_1\colon m_l \to a, \qquad \pi_2\colon m_r \to b$$

which is universal in that given another triple

$$\left(c \in \mathscr{A}_{X,Z},\ m' \in \mathfrak{M}^{c}_{X,Y,Z}\right), \qquad \gamma_1\colon m'_l \to a, \qquad \gamma_2\colon m'_r \to b$$

there exists a unique $\alpha\colon\ (c,m') \to (a;b,m)$ in el $\mathfrak{M}^{-}_{X,Y,Z}$ such that

$$\begin{tikzcd}
m'_l \arrow[rr, "\gamma_1"] \arrow[dr, dotted, "\alpha_l"'] & & a \\
& m_l \arrow[ur, "\pi_1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
m'_r \arrow[rr, "\gamma_2"] \arrow[dr, dotted, "\alpha_r"'] & & b \\
& m_r \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

commute.

4.3 Consequences and examples

In this section we discuss some of the main examples of Theorem 4.2.19. Viewing monoidal categories as one-object bicategories, we first consider the case where $\mathscr{A}$ is a cartesian monoidal category, giving a simple and informative example of this situation. We then go on to consider more complicated examples, namely where $\mathscr{A}$ is the bicategory of spans or the bicategory of polynomials with cartesian 2-cells.

For completeness, we also discuss the case where $\mathscr{A}$ is the category of finite sets and bijections with the disjoint union monoidal structure, but will omit some details as this is a rather trivial example.

4.3.1 Cartesian monoidal categories

Given a category $\mathcal{E}$ with finite products, one may construct the cartesian monoidal category $(\mathcal{E}, \times, \mathbf{1})$ where the tensor product is the cartesian product and the unit is the terminal object. Clearly this monoidal category is generic, as

$$\mathcal{E}(T, - \times -) : \mathcal{E} \times \mathcal{E} \to \mathbf{Set}$$

is representable (no coproducts are necessary). Now, seen as a one object bicategory, the generics are the diagonal morphisms δ_T in $\mathcal{E}$ of the form

$$\begin{tikzcd}
& T \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] \arrow[d, dotted, "\delta_T"] & \\
T & T \times T \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & T
\end{tikzcd}$$

and so we take[3] Δ_2 to be the class of diagonals $\delta_T \colon T \to T \times T$ for each $T \in \mathcal{E}$. Trivially, we take the augmentations as the unique maps into the terminal object from each object $T \in \mathcal{E}$. Applying Theorem 4.2.19 in this case then makes it clear why we may say the data of this theorem is analogous to the data of a comonad; indeed, we have the following.

Corollary 4.3.1. *Let $\mathcal{E}$ be a category with finite products and let $(\mathcal{C}, \otimes, I)$ be a monoidal category. Denote by $(\mathcal{E}, \times, \mathbf{1})$ the category $\mathcal{E}$ equipped with the cartesian monoidal structure. Then to give an oplax monoidal functor*

$$L \colon (\mathcal{E}, \times, \mathbf{1}) \to (\mathcal{C}, \otimes, I)$$

is to give a functor $L \colon \mathcal{E} \to \mathcal{C}$ with comultiplication and counit maps

$$\Phi_T \colon L(T) \to L(T) \otimes L(T), \qquad \Lambda_T \colon L(T) \to I$$

---

[3] Note that even in this simple case there can be different ways to define Δ_2. For example, if we have an isomorphism $T \cong S$ in $\mathcal{E}$ (where $T \neq S$), we could take Δ_2 to include δ_T but not δ_S. However, one should normally use the canonical choice of Δ_2 in order to make the coherence conditions easier to verify.

for every $T \in \mathcal{E}$, such that for every $T \in \mathcal{E}$ the diagrams

$$\begin{tikzcd}
& LT \otimes LT \arrow[dr, "LT\otimes\Lambda_T"] & \\
LT \arrow[ur, "\Phi_T"] \arrow[rr, "\text{unitor}"'] & & LT \otimes I
\end{tikzcd}
\qquad
\begin{tikzcd}
& LT \otimes LT \arrow[dr, "\Lambda_T\otimes LT"] & \\
LT \arrow[ur, "\Phi_T"] \arrow[rr, "\text{unitor}"'] & & I \otimes LT
\end{tikzcd}$$

commute, the diagrams

$$\begin{tikzcd}
LT \arrow[r, equal] \arrow[d, "\Phi_T"'] & LT \arrow[d, "\Phi_T"] \\
LT \otimes LT \arrow[d, "LT\otimes\Phi_T"'] & LT \otimes LT \arrow[d, "\Phi_T\otimes LT"] \\
LT \otimes (LT \otimes LT) \arrow[r, "\text{assoc}"'] & (LT \otimes LT) \otimes LT
\end{tikzcd}$$

commute, and all morphisms $f \colon T \to T'$ in $\mathcal{E}$ render commutative

$$\begin{tikzcd}
L(T) \arrow[rr, "Lf"] \arrow[dr, "\Lambda_T"'] & & L(T') \arrow[dl, "\Lambda_{T'}"] \\
& 1_X &
\end{tikzcd}
\qquad
\begin{tikzcd}
LT \arrow[r, "\Phi_T"] \arrow[d, "Lf"'] & LT \otimes LT \arrow[d, "Lf\otimes Lf"] \\
LT' \arrow[r, "\Phi_{T'}"'] & LT' \otimes LT'
\end{tikzcd}$$

The unitary and associativity conditions above ask that L sends each $T \in \mathcal{E}$ to a comonoid (LT, Φ_T, Λ_T) in $(\mathcal{C}, \otimes, I)$, and the last two conditions ask that morphisms in $\mathcal{E}$ are sent to morphisms of comonoids. Hence this may be simply stated as follows.

Corollary 4.3.2. *Let* **Comon** $(\mathcal{C}, \otimes, I)$ *be the category of comonoids in the monoidal category $(\mathcal{C}, \otimes, I)$. Then oplax monoidal functors $(\mathcal{E}, \times, \mathbf{1}) \to (\mathcal{C}, \otimes, I)$ are in bijection with functors $\mathcal{E} \to$* **Comon** $(\mathcal{C}, \otimes, I)$*.*

4.3.2 Bicategories of spans

Given a category $\mathcal{E}$ with pullbacks, one may form the bicategory of spans in $\mathcal{E}$ denoted **Span** $(\mathcal{E})$ with objects those of $\mathcal{E}$, 1-cells given by spans

$$\begin{tikzcd}
& T \arrow[dl, "s"'] \arrow[dr, "t"] & \\
X & & Z
\end{tikzcd}$$

denoted (s, t), 2-cells given by morphisms f rendering commutative diagrams as on the left below

$$\begin{tikzcd}
& K \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[dd, "f"] & \\
X & & Y \\
& R \arrow[ul, "u"] \arrow[ur, "v"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, phantom, "\text{pb}"] & & \\
& R \arrow[dl, "u"'] \arrow[dr, "v"] & & S \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X & & Y & & Z
\end{tikzcd}$$

and composition of 1-cells given by forming the pullback as on the right above [3].

The reader will then notice that by the universal property of pullback, giving a morphism of spans $(s,t) \longrightarrow (u,v)\,;(p,q)$ as on the left below

$$\begin{tikzcd}
& & T \arrow[d, dotted] \arrow[dddll, bend right, "s"'] \arrow[dddrr, bend left, "t"] & & & & & & T \arrow[ddl, dotted] \arrow[ddr, dotted] \arrow[ddd, "h"] \arrow[dddll, bend right, "s"'] \arrow[dddrr, bend left, "t"] & & \\
& & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & & \sim & & & & & \\
& R \arrow[dl, "u"'] \arrow[dr, "v"] \arrow[rr, phantom, "\mathrm{pb}"] & & S \arrow[dl, "p"'] \arrow[dr, "q"] & & & & R \arrow[dl, "u"'] \arrow[dr, "v"] & & S \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X & & Y & & Z & & X & & Y & & Z
\end{tikzcd}$$

is to give a morphism $h\colon T \longrightarrow Y$ as well as pair of morphisms of spans as on the right above such that each region in the diagram commutes. Therefore

$$\mathbf{Span}\,(\mathcal{E})_{X,Z}\,((s,t)\,,(u,v)\,;(p;q))$$

is isomorphic to

$$\sum_{h\colon H\longrightarrow Y} \mathbf{Span}\,(\mathcal{E})_{X,Y}\,((s,h)\,,(u,v)) \times \mathbf{Span}\,(\mathcal{E})_{Y,Z}\,((h,t)\,,(p,q)) \tag{4.3.1}$$

and so the bicategory of spans is generic. Our class of generics Δ_2 consists of, for each diagram

$$\begin{tikzcd}
& T \arrow[dl, "s"'] \arrow[d, "h"] \arrow[dr, "t"] & \\
X & Y & Z
\end{tikzcd}$$

in $\mathcal{E}$, the morphisms of spans $\delta_{s,h,t}\colon\ (s,t) \longrightarrow (s,h)\,;(h,t)$ corresponding to

$$\begin{tikzcd}
& & T \arrow[ddl, dotted, "\mathrm{id}"] \arrow[ddr, dotted, "\mathrm{id}"'] \arrow[ddd, "h"] \arrow[dddll, bend right, "s"'] \arrow[dddrr, bend left, "t"] & & \\
& & & & \\
& T \arrow[dl, "s"'] \arrow[dr, "h"] & & T \arrow[dl, "h"'] \arrow[dr, "t"] & \\
X & & Y & & Z
\end{tikzcd}$$

under this bijection. Our augmentations are the morphisms of spans as below for each morphism h in $\mathcal{E}$

$$\begin{tikzcd}
& T \arrow[dl, "h"'] \arrow[dr, "h"] \arrow[dd, "h"] & \\
X & & X \\
& X \arrow[ul, "\mathrm{id}"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

and will be denoted by ε_h. Thus, applying Theorem 4.2.19 we have the following.

Corollary 4.3.3. *Let $\mathcal{E}$ be a category with pullbacks and denote by* **Span** $(\mathcal{E})$ *the bicategory* *of spans in $\mathcal{E}$. Let $\mathscr{C}$ be a bicategory. Then to give an oplax functor*

$$L\colon \mathbf{Span}\,(\mathcal{E}) \to \mathscr{C}$$

is to give a locally defined functor

$$L_{X,Y}\colon \mathbf{Span}\,(\mathcal{E})_{X,Y} \to \mathscr{C}_{LX,LY}, \qquad X,Y \in \mathcal{E}$$

with comultiplication and counit maps

$$\Phi_{s,h,t}\colon L\,(s,t) \to L\,(s,h)\,;L\,(h,t), \qquad \Lambda_h\colon L\,(h,h) \to 1_{LX}$$

for every respective diagram in $\mathcal{E}$

$$\begin{tikzcd}
& T \arrow[dl, "s"'] \arrow[d, "h"] \arrow[dr, "t"] & & & T \arrow[d, "h"] \\
X & Y & Z & & X
\end{tikzcd}$$

such that:

1. *for any triple of morphisms of spans as below*

$$\begin{tikzcd}
& R \arrow[dl, "u"'] \arrow[dr, "v"] \arrow[dd, "f"] & & & R \arrow[dl, "u"'] \arrow[dr, "k"] \arrow[dd, "f"] & & & R \arrow[dl, "k"'] \arrow[dr, "v"] \arrow[dd, "f"] & \\
X & & Z & X & & Y & Y & & Z \\
& T \arrow[ul, "s"] \arrow[ur, "t"'] & & & T \arrow[ul, "s"] \arrow[ur, "h"'] & & & T \arrow[ul, "h"] \arrow[ur, "t"'] &
\end{tikzcd}$$

   we have the commuting diagram

$$\begin{tikzcd}
L\,(u,v) \arrow[r, "\Phi_{u,k,v}"] \arrow[d, "Lf"'] & L\,(u,k)\,;L\,(k,v) \arrow[d, "Lf;Lf"] \\
L\,(s,t) \arrow[r, "\Phi_{s,h,t}"'] & L\,(s,h)\,;L\,(h,t)
\end{tikzcd}$$

2. *for any morphism of spans as on the left below*

$$\begin{tikzcd}
& M \arrow[dl, "p"'] \arrow[d, "f"] \arrow[dr, "p"] & & & L\,(p,p) \arrow[rr, "Lf"] \arrow[dr, "\Lambda_p"'] & & L\,(q,q) \arrow[dl, "\Lambda_q"] \\
X & N \arrow[l, "q"] \arrow[r, "q"'] & X & & & 1_{LX} &
\end{tikzcd}$$

   the diagram on the right above commutes;

3. *for all diagrams of the form*

$$\begin{tikzcd}
& & T \arrow[dll, "s"'] \arrow[dl, "h"] \arrow[dr, "k"'] \arrow[drr, "t"] & & \\
W & X & & Y & Z
\end{tikzcd}$$

in $\mathcal{E}$, we have the commuting diagram

$$\begin{tikzcd}
L(s,t) \arrow[rr, equal] \arrow[d, "\Phi_{s,h,t}"'] & & L(s,t) \arrow[d, "\Phi_{s,k,t}"] \\
L(s,h);L(h,t) \arrow[d, "L(s;h);\Phi_{h,k,t}"'] & & L(s,k);L(k,t) \arrow[d, "\Phi_{s,h,k};L(k;t)"] \\
L(s,h);(L(h,k);L(k,t)) \arrow[rr, "\text{assoc}"'] & & (L(s,h);L(h,k));L(k,t)
\end{tikzcd}$$

4. *for all spans (s,t) we have the commuting diagrams*

$$\begin{tikzcd}
& L(s,s);L(s,t) \arrow[dr, "\Lambda_s;L(s,t)"] & \\
L(s,t) \arrow[ur, "\Phi_{s,s,t}"] \arrow[rr, "\text{unitor}"'] & & 1_{LX};L(s,t)
\end{tikzcd}
\qquad
\begin{tikzcd}
& L(s,t);L(t,t) \arrow[dr, "L(s,t);\Lambda_t"] & \\
L(s,t) \arrow[ur, "\Phi_{s,t,t}"] \arrow[rr, "\text{unitor}"'] & & L(s,t);1_{LY}
\end{tikzcd}$$

Remark 4.3.4. Note that this description of an oplax functor out of the bicategory of spans does not involve pullbacks, thus allowing for a simpler for a simpler proof of the universal properties of the span construction [9].

4.3.3 Bicategories of polynomials

Given a locally cartesian closed category $\mathcal{E}$, one may form the bicategory of polynomials in $\mathcal{E}$ with cartesian 2-cells [55, 17]. This bicategory we denote by $\mathbf{Poly}_c(\mathcal{E})$ and has objects those of $\mathcal{E}$, 1-cells given by diagrams

$$\begin{tikzcd}
& E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & \\
X & & & Z
\end{tikzcd}$$

in $\mathcal{E}$ called polynomials and denoted by (s,p,t), and 2-cells given by commuting diagrams as below

$$\begin{tikzcd}
& K \arrow[r, "i"] \arrow[dl, "a"'] \arrow[dd, "f"'] \arrow[ddr, phantom, "\text{pb}"] & I \arrow[dd, "g"] \arrow[dr, "b"] & \\
X & & & Y \\
& R \arrow[ul, "u"] \arrow[r, "j"'] & J \arrow[ur, "v"'] &
\end{tikzcd}$$

where the middle square is a pullback. Composition of 1-cells is more complicated and so will be omitted; especially as it is not necessary to describe oplax functors out of $\mathbf{Poly}_c\,(\mathcal{E})$ once we know the generics.

The reader need only know the following corollary of [55, Prop. 3.1.6], a description of polynomial composition due to Weber.

Corollary 4.3.5. *Consider two polynomials in $\mathcal{E}$ as below:*

$$\begin{tikzcd} & K \arrow[r, "i"] \arrow[dl, "a"'] & I \arrow[dr, "b"] & \\ X & & & Y \end{tikzcd} \qquad \begin{tikzcd} & R \arrow[r, "j"] \arrow[dl, "u"'] & J \arrow[dr, "v"] & \\ Y & & & Z \end{tikzcd}$$

Then to give a cartesian 2-cell $(s,p,t) \longrightarrow (a,i,b)\,;(u,j,v)$ is to give a factorization $p = p_1;p_2$ through an object T, a morphism $h\colon T \longrightarrow Y$, and a pair of cartesian morphisms $(s,p_1,h) \longrightarrow (a,i,b)$ and $(h,p_2,t) \longrightarrow (u,j,v)$

$$\begin{tikzcd}
& & E \arrow[rr, "p_1"] \arrow[rrrr, bend left, "p"] \arrow[dl, dotted, "w"'] \arrow[ddll, bend right, "s"'] \arrow[dr, phantom, "\text{pb}"] & & T \arrow[rr, "p_2"] \arrow[dl, dotted, "x"'] \arrow[dr, dotted, "y"] \arrow[dd, "h"] & & B \arrow[dr, dotted, "z"] \arrow[ddrr, bend left, "t"] \arrow[dl, phantom, "\text{pb}"] & & \\
& K \arrow[rr, "i"] \arrow[dl, "a"'] & & I \arrow[dr, "b"] & & R \arrow[rr, "j"] \arrow[dl, "u"'] & & J \arrow[dr, "v"] & \\
X & & & & Y & & & & Z
\end{tikzcd}$$

such that the above diagram commutes. Here we identify a septuple (p_1,h,p_2,w,x,y,z) as above with another septuple $\left(p'_1,h',p'_2,w',x',y',z'\right)$ if $w = w'$, $z = z'$ and there exists an invertible $\alpha\colon T \longrightarrow T'$ rendering commutative the diagrams[4]

$$\begin{tikzcd} & T \arrow[dr, "p_2"] \arrow[dd, "\alpha"] & \\ E \arrow[ur, "p_1"] \arrow[dr, "p'_1"'] & & B \\ & T' \arrow[ur, "p'_2"'] & \end{tikzcd} \qquad \begin{tikzcd} & T \arrow[dl, "x"'] \arrow[dr, "y"] \arrow[dd, "\alpha"] & \\ I & & R \\ & T' \arrow[ul, "x'"] \arrow[ur, "y'"'] & \end{tikzcd} \qquad \begin{tikzcd} T \arrow[dr, "h"] \arrow[dd, "\alpha"'] & \\ & Y \\ T' \arrow[ur, "h'"'] & \end{tikzcd} \tag{4.3.2}$$

It follows that

$$\mathbf{Poly}_c\,(\mathcal{E})_{X,Z}\,((s,p,t)\,,(a,i,b)\,;(u,j,v))$$

is isomorphic to

$$\overset{\sim}{\sum_{p=p_1;p_2,\ h\colon T\to Y}} \mathbf{Poly}_c\,(\mathcal{E})_{X,Y}\,((s,p_1,h)\,,(a,i,b)) \times \mathbf{Poly}_c\,(\mathcal{E})_{Y,Z}\,((h,p_2,t)\,,(u,j,v)) \tag{4.3.3}$$

where the equivalence relation "∼" indicates the sum is taken over representatives of equivalence classes of triples (p_1,h,p_2) (where two such triples are seen as equivalent if there is

---

[4] It is clear that if the middle diagram commutes then the rightmost diagram also does. Also, such an isomorphism α making the left diagram commute must be unique.

an isomorphism α rendering commutative the left and right diagrams as in Figure 4.3.2). We have thus exhibited the bicategory of polynomials with cartesian 2-cells as a generic bicategory.

Here our class of generics Δ_2 consists of, for each diagram

$$\begin{tikzcd}
 & E \arrow[r, "p_1"] \arrow[dl, "s"'] & T \arrow[r, "p_2"] \arrow[d, "h"] & B \arrow[dr, "t"] & \\
X & & Y & & Z
\end{tikzcd}$$

in $\mathcal{E}$ where $p = p_1 ; p_2$, the cartesian morphisms of polynomials

$$\delta_{s,p_1,h,p_2,t} \colon\ (s,p,t) \to (s,p_1,h)\,;(h,p_2,t)$$

corresponding to

$$\begin{tikzcd}
 & & E \arrow[r, "p_1"] \arrow[rr, bend left, "p"] \arrow[dl, dotted, "\mathrm{id}"] \arrow[ddll, bend right, "s"'] & T \arrow[r, "p_2"] \arrow[dl, dotted, "\mathrm{id}"] \arrow[dr, dotted, "\mathrm{id}"] \arrow[dd, "h"] & B \arrow[dr, dotted, "\mathrm{id}"] \arrow[ddrr, bend left, "t"] & & \\
 & E \arrow[r, "p_1"] \arrow[dl, "s"] & T \arrow[dr, "h"] & & T \arrow[r, "p_2"] \arrow[dl, "h"'] & B \arrow[dr, "t"] & \\
X & & & Y & & & Z
\end{tikzcd}$$

under this bijection. We take as our augmentations the cartesian morphisms

$$\begin{tikzcd}
 & T \arrow[r, "\mathrm{id}"] \arrow[dl, "h"'] \arrow[dd, "h"] & T \arrow[dr, "h"] \arrow[dd, "h"'] & \\
X & & & X \\
 & X \arrow[ul, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & X \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

and denote these by ε_h. There are more general morphisms into identity polynomials where the middle map is invertible; but using those would lead to unnecessary complexity.

Remark 4.3.6. Note that our class of generics Δ_2 does not involve representatives of equivalence classes, unlike the summation formula given.

Now, applying Theorem 4.2.19 we have the following.

Corollary 4.3.7. *Let $\mathcal{E}$ be a locally cartesian closed category and denote by $\mathbf{Poly}_c(\mathcal{E})$ the bicategory of polynomials in $\mathcal{E}$ with cartesian 2-cells. Let $\mathscr{C}$ be a bicategory. Then to give an oplax functor*

$$L \colon \mathbf{Poly}_c(\mathcal{E}) \to \mathscr{C}$$

is to give a locally defined functor

$$L_{X,Y} \colon \mathbf{Poly}_c(\mathcal{E})_{X,Y} \to \mathscr{C}_{LX,LY}, \qquad X,Y \in \mathcal{E}$$

with comultiplication and counit maps

$$\Phi_{s,p_1,h,p_2,t} \colon L\,(s,p,t) \to L\,(s,p_1,h)\,;L\,(h,p_2,t), \qquad \Lambda_h \colon L\,(h,1,h) \to 1_{LX}$$

for every respective diagram in $\mathcal{E}$

$$\begin{tikzcd}
& E \arrow[r, "p_1"] \arrow[dl, "s"'] & T \arrow[r, "p_2"] \arrow[d, "h"] & B \arrow[dr, "t"] & \\
X & & Y & & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
& T \arrow[r, "\mathrm{id}"] \arrow[dl, "h"'] & T \arrow[dr, "h"] & \\
X & & & X
\end{tikzcd}$$

where we assert $p = p_1 ; p_2$ *on the left, such that:*

1. *for any morphisms of polynomials as below*

$$\begin{tikzcd}
& R \arrow[r, "q"] \arrow[dl, "u"'] \arrow[dd, "f"] & I \arrow[dr, "v"] \arrow[dd, "g"] & \\
X & & & Z \\
& E \arrow[ul, "s"] \arrow[r, "p"'] & B \arrow[ur, "t"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
& R \arrow[r, "q_1"] \arrow[dl, "u"'] \arrow[dd, "f"] & S \arrow[dr, "k"] \arrow[dd, "c"] & \\
X & & & Y \\
& E \arrow[ul, "s"] \arrow[r, "p_1"'] & T \arrow[ur, "h"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
& S \arrow[r, "q_2"] \arrow[dl, "k"'] \arrow[dd, "c"] & I \arrow[dr, "v"] \arrow[dd, "g"] & \\
Y & & & Z \\
& T \arrow[ul, "h"] \arrow[r, "p_2"'] & B \arrow[ur, "t"'] &
\end{tikzcd}$$

we have the commuting diagram

$$\begin{tikzcd}
L\,(u,q,v) \arrow[r, "\Phi_{u,q_1,k,q_2,v}"] \arrow[d, "L(f,g)"'] & L\,(u,q_1,k)\,;L\,(k,q_2,v) \arrow[d, "L(f,c);L(c,g)"] \\
L\,(s,p,t) \arrow[r, "\Phi_{s,p_1,h,p_2,t}"'] & L\,(s,p_1,h)\,;L\,(h,p_2,t)
\end{tikzcd}$$

2. *for any morphism of polynomials as on the left below*

$$\begin{tikzcd}
& R \arrow[r, "\mathrm{id}"] \arrow[dl, "s"'] \arrow[dd, "f"] \arrow[ddr, phantom, "\text{pb}"] & R \arrow[dr, "s"] \arrow[dd, "f"'] & \\
X & & & X \\
& T \arrow[ul, "t"] \arrow[r, "\mathrm{id}"'] & T \arrow[ur, "t"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
L\,(s,1,s) \arrow[rr, "L(f,f)"] \arrow[dr, "\Lambda_s"'] & & L\,(t,1,t) \arrow[dl, "\Lambda_t"] \\
& 1_{LX} &
\end{tikzcd}$$

the diagram on the right above commutes;

3. *for all diagrams of the form*

$$\begin{tikzcd}
& F \arrow[r, "a"] \arrow[dl, "s"'] & G \arrow[r, "b"] \arrow[d, "h"'] & H \arrow[r, "c"] \arrow[d, "k"] & K \arrow[dr, "t"] & \\
W & & X & Y & & Z
\end{tikzcd}$$

in $\mathcal{E}$, we have the commuting diagram

$$\begin{tikzcd}
L\left(s,a;b;c,t\right) \arrow[rr, equal] \arrow[d, "\Phi_{s,a,h,b;c,t}"'] & & L\left(s,a;b;c,t\right) \arrow[d, "\Phi_{s,a;b,k,c,t}"] \\
L\left(s,a,h\right);L\left(h,b;c,t\right) \arrow[d, "L(s,a,h);\Phi_{h,b,k,c,t}"'] & & L\left(s,a;b,k\right);L\left(k,c,t\right) \arrow[d, "\Phi_{s,a,h,b,k};L(k,c,t)"] \\
L\left(s,a,h\right);\left(L\left(h,b,k\right);L\left(k,c,t\right)\right) \arrow[rr, "\text{assoc}"'] & & \left(L\left(s,a,h\right);L\left(h,b,k\right)\right);L\left(k,c,t\right)
\end{tikzcd}$$

4. *for all polynomials (s,p,t) the diagrams*

$$\begin{tikzcd}
 & L\left(s,1,s\right);L\left(s,p,t\right) \arrow[dr, "\Lambda_s;L(s,p,t)"] & \\
L\left(s,p,t\right) \arrow[ur, "\Phi_{s,1,s,p,t}"] \arrow[rr, "\text{unitor}"'] & & 1_{LX};L\left(s,p,t\right)
\end{tikzcd}$$

$$\begin{tikzcd}
 & L\left(s,p,t\right);L\left(t,1,t\right) \arrow[dr, "L(s,p,t);\Lambda_t"] & \\
L\left(s,p,t\right) \arrow[ur, "\Phi_{s,p,t,1,t}"] \arrow[rr, "\text{unitor}"'] & & L\left(s,p,t\right);1_{LY}
\end{tikzcd}$$

commute.

Remark 4.3.8. As the above description of oplax functors out of the bicategory of polynomials does not rely on polynomial composition, it may be used for an efficient proof of the universal properties of polynomials. Indeed, this allows us to avoid the large coherence diagrams which would arise in a direct proof. We will discuss this in detail in our next paper.

4.3.4 Finite sets and bijections

We give this example for completeness, but will omit some details as Theorem 4.2.19 becomes rather trivial in this case (due to all generic morphisms being invertible). Here we take $\mathscr{A}$ to be the category of finite sets and bijections with the disjoint union monoidal structure, denoted $(\mathbb{P},\sqcup,\emptyset)$. This monoidal category is generic since we have isomorphisms

$$\mathbb{P}\left(C,A\sqcup B\right)\cong\sum_{C=L\sqcup R}\mathbb{P}\left(L,A\right)\times\mathbb{P}\left(R,B\right)$$

natural in finite sets A and B, where the sum is taken over decompositions of C into the disjoint union of two sets. Here we choose our class of generics Δ_2 to contain the chosen bijection, where $[n]=\{1,\cdots,n\}$,

$$\delta_{n_1,n_2}\colon\ [n_1+n_2]\longrightarrow[n_1]\sqcup[n_2],\qquad n\mapsto\begin{cases}(1,n), & n\leq n_1\\ (2,n), & n>n_1\end{cases}$$

for each pair of non-negative integers n_1 and n_2. Trivially, the only augmentation is the identity map on the empty set. Taking $(\mathcal{C}, \otimes, I)$ to be a monoidal category, it follows from Theorem 4.2.19 that oplax monoidal functors $L\colon (\mathbb{P}, \sqcup, \emptyset) \to (\mathcal{C}, \otimes, I)$ may be specified by giving comultiplication and counit maps

$$\Phi_{n_1,n_2}\colon L\,[n_1 + n_2] \to L\,[n_1] \otimes [n_2], \qquad \Lambda\colon L\,(\emptyset) \to I$$

Of course, this may more easily be seen by simply taking the skeleton.

4.4 Convolution structures and Yoneda structures

By results of Day [11], given a bicategory $\mathscr{A}$ with small hom-categories one may construct the local cocompletion of $\mathscr{A}$, a new bicategory $\hat{\mathscr{A}}$ with objects those of $\mathscr{A}$, hom-categories given by

$$\hat{\mathscr{A}}_{X,Y} := \left[\mathscr{A}_{X,Y}^{\mathrm{op}}, \mathbf{Set}\right], \qquad X, Y \in \mathscr{A}_{\mathrm{ob}}$$

and a composite of two presheaves

$$F\colon \mathscr{A}_{X,Y}^{\mathrm{op}} \to \mathbf{Set}, \qquad G\colon \mathscr{A}_{Y,Z}^{\mathrm{op}} \to \mathbf{Set}$$

given by Day's convolution formula

$$GF\colon \mathscr{A}_{X,Z}^{\mathrm{op}} \to \mathbf{Set}, \qquad GF\,(c) = \int^{a,b} \mathscr{A}_{X,Z}\,(c, a; b) \times Fa \times Gb.$$

With this definition of $\hat{\mathscr{A}}$, the family of Yoneda embeddings

$$y_{\mathscr{A}_{X,Y}}\colon \mathscr{A}_{X,Y} \to \hat{\mathscr{A}}_{X,Y}, \qquad X, Y \in \mathscr{A}_{\mathrm{ob}}$$

underlies a pseudofunctor $y_{\mathscr{A}}\colon \mathscr{A} \to \hat{\mathscr{A}}$. This is of interest since in the case of generic bicategories $\mathscr{A}$, this convolution structure has an especially simple form. Moreover, just as one can gain insight into a category by studying its category of presheaves, one can deduce many of the properties of generic bicategories $\mathscr{A}$ as a consequence of this simple convolution structure on $\hat{\mathscr{A}}$.

Proposition 4.4.1. *Suppose $\mathscr{A}$ is a generic bicategory. Then for any pair of presheaves*

$$F\colon \mathscr{A}_{X,Y}^{\mathrm{op}} \to \mathbf{Set}, \qquad G\colon \mathscr{A}_{Y,Z}^{\mathrm{op}} \to \mathbf{Set}$$

there exists isomorphisms as below

$$\int^{a,b} \mathscr{A}_{X,Z}\,(c, a; b) \times Fa \times Gb \cong \sum_{m \in \mathfrak{M}_c^{X,Y,Z}} Fl_m \times Gr_m$$

thus reducing the Day convolution structure to a simpler formula.

Proof. We have

$$\begin{aligned}
\mathrm{LHS} &= \int^{a,b} \mathscr{A}_{X,Z}(c,a;b) \times Fa \times Gb \\
&\cong \int^{a,b} \left[\sum_{m \in \mathfrak{M}_c^{X,Y,Z}} \mathscr{A}_{X,Y}(l_m,a) \times \mathscr{A}_{Y,Z}(r_m,b) \right] \times Fa \times Gb \\
&\cong \int^{a,b} \sum_{m \in \mathfrak{M}_c^{X,Y,Z}} \mathscr{A}_{X,Y}(l_m,a) \times Fa \times \mathscr{A}_{Y,Z}(r_m,b) \times Gb \\
&\cong \sum_{m \in \mathfrak{M}_c^{X,Y,Z}} \int^{a,b} \mathscr{A}_{X,Y}(l_m,a) \times Fa \times \mathscr{A}_{Y,Z}(r_m,b) \times Gb \\
&\cong \sum_{m \in \mathfrak{M}_c^{X,Y,Z}} \left(\int^{a} \mathscr{A}_{X,Y}(l_m,a) \times Fa \right) \times \left(\int^{b} \mathscr{A}_{Y,Z}(r_m,b) \times Gb \right) \\
&\cong \sum_{m \in \mathfrak{M}_c^{X,Y,Z}} Fl_m \times Gr_m \\
&= \mathrm{RHS}
\end{aligned}$$

as required. □

Remark 4.4.2. Unfortunately, the above formula has some disadvantages. Indeed, as $\mathfrak{M}_c^{X,Y,Z}$ is isomorphic to the set of equivalence classes of generics out of c, it follows that explicitly describing $\mathfrak{M}_c^{X,Y,Z}$ will involve a choice of representatives for each equivalence class. This is problematic since choices of representatives do not nicely behave with respect to composition.

As a consequence of this proposition and the formulas (4.3.1) and (4.3.3) given in the previous section, we have the following.

Corollary 4.4.3. *The Day convolution of two presheaves of spans*

$$F \colon \mathbf{Span}(\mathcal{E})^{\mathrm{op}}_{X,Y} \to \mathbf{Set}, \qquad G \colon \mathbf{Span}(\mathcal{E})^{\mathrm{op}}_{Y,Z} \to \mathbf{Set}$$

is given by

$$GF \colon \mathbf{Span}(\mathcal{E})^{\mathrm{op}}_{X,Z} \to \mathbf{Set}, \quad GF(s,t) \cong \sum_{h \colon T \to Y} F(s,h) \times G(h,t)$$

and the Day convolution of two presheaves of polynomials

$$F \colon \mathbf{Poly}_c(\mathcal{E})^{\mathrm{op}}_{X,Y} \to \mathbf{Set}, \qquad G \colon \mathbf{Poly}_c(\mathcal{E})^{\mathrm{op}}_{Y,Z} \to \mathbf{Set}$$

is given by the formula

$$GF\colon \mathbf{Poly}_c(\mathcal{E})^{\mathrm{op}}_{X,Z} \to \mathbf{Set}, \quad GF(s,p,t) \cong \widetilde{\sum_{p=p_1;p_2,\, h\colon T\to Y}} F(s,p_1,h)\times G(h,p_2,t)$$

The purpose of the following is to describe how Theorem 4.2.19 may be seen as an instance of a more general result, and to see how this theorem follows from the simple convolution structure on $\hat{\mathscr{A}}$. Indeed, as a special case of Theorem 3.5.10 we have the following corollary.

Corollary 4.4.4 (Doctrinal Yoneda Structures)**.** *Let $\mathscr{A}$ and $\mathscr{C}$ be two bicategories with small hom-categories and the same objects. Let $\hat{\mathscr{A}}$ be the free small local cocompletion of $\mathscr{A}$. Then for any locally defined identity on objects functor $L\colon \mathscr{A} \to \mathscr{C}$, with the corresponding locally defined identity on objects functor $R = \mathscr{C}(L-,-)$ as below*

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "R"] \arrow[dr, phantom, "\overset{\varphi}{\Leftarrow}"] & \hat{\mathscr{A}} \\
& \mathscr{A} \arrow[ul, "L"] \arrow[u, "y_{\mathscr{A}}"']
\end{tikzcd}$$

the structure of an oplax functor on L is in bijection with the structure of a lax functor on R.

Supposing that $\mathscr{A}$ is generic, and hence that composition on $\hat{\mathscr{A}}$ has the reduced form given by Proposition 4.4.1, one sees from this corollary that for a given locally defined functor $L\colon \mathscr{A} \to \mathscr{C}$[5], giving L an oplax structure $(L,\varphi,\lambda)\colon \mathscr{A} \to \mathscr{C}$ with constraint cells

$$\varphi_{a,b}\colon L(a;b) \to La;Lb, \qquad \lambda_X\colon L1_X \to 1_X$$

is to give R a lax structure $(R,\phi,\omega)\colon \mathscr{C} \to \hat{\mathscr{A}}$ with constraints

$$\phi_{a,b}\colon Ra;Rb \to R(a;b), \qquad \omega_X\colon 1_X \to R1_X.$$

By the definition of R and composition in $\hat{\mathscr{A}}$, these binary constraints ϕ are functions for each $c\colon X \to Z$

$$\sum_{m\in\mathfrak{M}_c^{X,Y,Z}} \mathscr{C}_{X,Y}(Ll_m,a)\times\mathscr{C}_{Y,Z}(Lr_m,b) \to \mathscr{C}_{X,Z}(Lc,a;b)$$

natural in a,b and c. By naturality, to give such a function is to give an assignment on the identity pair (we may call the result $\Phi_{c,m}$)

$$(\mathrm{id}\colon Ll_m \to Ll_m, \mathrm{id}\colon Lr_m \to Lr_m) \mapsto \Phi_{c,m}\colon Lc \to Ll_m;Lr_m$$

Thus the binary constraints ϕ are determined by giving appropriate Φ. A similar calculation

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[5] Note that here one can use bijective on objects - bijective on 1-cells factorisations to avoid the assumption that L is the identity on objects.

may be done to see the unit constraints ω are determined by augmentations Λ. In this way, one constructs a bijection between the data of an oplax structure on L (containing cells φ and λ) and a lax structure on R (containing cells Φ and Λ).

Remark 4.4.5. It is this observation which was the original motivation for Theorem 4.2.19. However, this approach does not give an efficient proof of this theorem for a number of technical reasons. In particular, we wish to avoid considering equivalence classes of generic morphisms (such as the elements of the set $\mathfrak{M}_c^{X,Y,Z}$) to avoid technicalities involving choices of representatives.

Though this approach is more technical (and thus not the method used in the proof), it is conceptually nicer as it exhibits Theorem 4.2.19 as a natural result concerning the Yoneda structures of Street and Walters [47] and their algebraic properties.

5

Universal properties of bicategories of polynomials

Abstract

We establish the universal properties of the bicategory of polynomials, considering both cartesian and general morphisms between these polynomials. A direct proof of these universal properties would be impractical due to the complicated coherence conditions arising from polynomial composition; however, in this paper we avoid most of these coherence conditions using the properties of generic bicategories.

In addition, we give a new proof of the universal properties of the bicategory of spans, and also establish the universal properties of the bicategory of spans with invertible 2-cells; showing how these properties may be used to describe the universal properties of polynomials.

Contribution by the author

As the sole author, this paper is entirely my own work. This paper was accepted on November 4th 2018 in the Journal of Pure and Applied Algebra, and is currently in press. Any differences from the journal version are limited to formatting and citation numbering changes.

5.1 Introduction

In this paper we are interested in two constructions on suitable categories $\mathcal{E}$: the bicategory of spans $\mathbf{Span}(\mathcal{E})$ as introduced by Bénabou [3], and the bicategory of polynomials $\mathbf{Poly}(\mathcal{E})$ as introduced by Gambino and Kock [17], and further studied by Weber [55] (all to be reviewed in Section 5.2). Here we wish to study the universal properties of these constructions; that is, for an arbitrary bicategory $\mathscr{C}$ we wish to know what it means to give a pseudofunctor $\mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ or $\mathbf{Poly}(\mathcal{E}) \to \mathscr{C}$.

In the case of spans, these results have already been established. In particular, given any category $\mathcal{E}$ with pullbacks, one can form a bicategory denoted $\mathbf{Span}(\mathcal{E})$ whose objects are those of $\mathcal{E}$ and 1-cells are diagrams in $\mathcal{E}$ of the form below

$$\begin{tikzcd}
& \bullet \arrow[dl, "s"'] \arrow[dr, "t"] & \\
\bullet & & \bullet
\end{tikzcd}$$

called spans. The universal property of this construction admits a simple description since for every morphism f in $\mathcal{E}$ we have adjunctions

$$\begin{tikzcd}
& \bullet \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & & & & \bullet \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}"] & \\
\bullet & & \bullet & \dashv & \bullet & & \bullet
\end{tikzcd}$$

in $\mathbf{Span}(\mathcal{E})$, and these adjunctions generate all of $\mathbf{Span}(\mathcal{E})$.

Indeed, it was proven by Hermida [21, Theorem A.2] that composing with the canonical embedding $\mathcal{E} \hookrightarrow \mathbf{Span}(\mathcal{E})$ describes an equivalence

$$\frac{\text{pseudofunctors } \mathbf{Span}(\mathcal{E}) \to \mathscr{C}}{\text{Beck pseudofunctors } \mathcal{E} \to \mathscr{C}}$$

where a pseudofunctor $F_\Sigma \colon \mathcal{E} \to \mathscr{C}$ is Beck if for every morphism f in $\mathcal{E}$ the 1-cell $F_\Sigma f$ has a right adjoint $F_\Delta f$ in $\mathscr{C}$ (such an F_Σ is also known as a sinister pseudofunctor), and if the induced pair of pseudofunctors

$$F_\Sigma \colon \mathcal{E} \to \mathscr{C}, \qquad F_\Delta \colon \mathcal{E}^{\mathrm{op}} \to \mathscr{C}$$

satisfy a Beck-Chevalley condition. A natural question to then ask is what these sinister pseudofunctors correspond to when the Beck–Chevalley condition is dropped. This question was solved by Dawson, Paré, and Pronk [9, Theorem 2.15] who showed that composing with the canonical embedding describes an equivalence

$$\frac{\text{gregarious functors } \mathbf{Span}(\mathcal{E}) \to \mathscr{C}}{\text{sinister pseudofunctors } \mathcal{E} \to \mathscr{C}}$$

where gregarious functors are the adjunction preserving normal[1] oplax functors.

An important special case of this is when $\mathscr{C} = \mathbf{Cat}$, where one may consider sinister pseudofunctors $\mathcal{E} \to \mathbf{Cat}$, or equivalently cosinister[2] pseudofunctors $\mathcal{E}^{\mathrm{op}} \to \mathbf{Cat}$. In this case we recover the equivalence

$$\frac{\text{gregarious functors } \mathbf{Span}\left(\mathcal{E}\right) \to \mathbf{Cat}}{\text{bifibrations over } \mathcal{E}}$$

Note also that on $\mathbf{Cat}/\mathcal{E}$ there is a KZ pseudomonad $\mathit{\Gamma}_{\mathcal{E}}$ for opfibrations and a coKZ pseudomonad $\mathit{\Upsilon}_{\mathcal{E}}$ for fibrations. This yields (via a pseudo-distributive law) the pseudomonad $\mathit{\Gamma}_{\mathcal{E}}\mathit{\Upsilon}_{\mathcal{E}}$ for bifibrations satisfying the Beck–Chevalley condition [50] (also known as fibrations with sums). The above equivalence then restricts to

$$\frac{\text{pseudofunctors } \mathbf{Span}\left(\mathcal{E}\right) \to \mathbf{Cat}}{\text{fibrations with sums over } \mathcal{E}}$$

An archetypal example of this is the codomain fibration over $\mathcal{E}$ corresponding to the canonical pseudofunctor $\mathbf{Span}\left(\mathcal{E}\right) \to \mathbf{Cat}$ defined by

$$\begin{tikzcd} & T \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & & Y \end{tikzcd} \quad \mapsto \quad \mathcal{E}/X \xrightarrow{\Delta_s} \mathcal{E}/T \xrightarrow{\Sigma_t} \mathcal{E}/Y$$

where for every morphism f in $\mathcal{E}$, the functor Σ_f denotes composition with f, and the functor Δ_f denotes pulling back along f.

When considering polynomials it is convenient to assume some extra structure on $\mathcal{E}$. In particular, we will take $\mathcal{E}$ to be a category with finite limits, such that for each morphism f in $\mathcal{E}$ the "pullback along f" functor Δ_f has a right adjoint Π_f. For such a category $\mathcal{E}$ (known as a locally cartesian closed category) one can form a bicategory denoted $\mathbf{Poly}\left(\mathcal{E}\right)$ whose objects are those of $\mathcal{E}$ and 1-cells are diagrams in $\mathcal{E}$ of the form below

$$\begin{tikzcd} & \bullet \arrow[r, "p"] \arrow[dl, "s"'] & \bullet \arrow[dr, "t"] & \\ \bullet & & & \bullet \end{tikzcd}$$

called polynomials[3]. One can also form a bicategory $\mathbf{Poly}_c\left(\mathcal{E}\right)$ with the same objects and 1-cells by being more restrictive on the 2-cells (that is, only taking "cartesian" morphisms of polynomials).

The purpose of this paper is to describe the universal properties of these two bicategories

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[1] Here "normal" means the unit constraints are invertible.

[2] Here "cosinister" means arrows are sent to right adjoint 1-cells instead of left adjoint 1-cells. This is the F_{Δ} of such a pair F_{Σ}-F_{Δ}.

[3] The bicategory of polynomials can be defined on any category $\mathcal{E}$ with pullbacks [55]; however, we will assume local cartesian closure for simplicity.

of polynomials.

Similar to the case of spans, the universal property of **Poly** $(\mathcal{E})$ admits a simple description since for every morphism f in $\mathcal{E}$ we have adjunctions

$$\begin{tikzcd}
& \bullet \arrow[dl, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] & \bullet \arrow[dr, "f"] & & & & \bullet \arrow[dl, "f"'] \arrow[r, "\mathrm{id}"] & \bullet \arrow[dr, "\mathrm{id}"] & & & & \bullet \arrow[dl, "\mathrm{id}"'] \arrow[r, "f"] & \bullet \arrow[dr, "\mathrm{id}"] & \\
\bullet & & & \bullet & \dashv & \bullet & & & \bullet & \dashv & \bullet & & & \bullet
\end{tikzcd} \tag{5.1.1}$$

in **Poly** $(\mathcal{E})$, and these adjunctions generate all of **Poly** $(\mathcal{E})$ (to be shown in Proposition 5.2.25). Using this fact, we show that in the case of polynomials with general 2-cells, composition with the embedding $\mathcal{E} \hookrightarrow \mathbf{Poly}(\mathcal{E})$ describes the equivalence

$$\frac{\text{pseudofunctors } \mathbf{Poly}(\mathcal{E}) \to \mathscr{C}}{\text{DistBeck pseudofunctors } \mathcal{E} \to \mathscr{C}} \tag{5.1.2}$$

where a pseudofunctor $F_\Sigma \colon \mathcal{E} \to \mathscr{C}$ is DistBeck if for every morphism f in $\mathcal{E}$ the 1-cell $F_\Sigma f$ has two successive right adjoints $F_\Delta f$ and $F_\Pi f$ (such an F_Σ is called a 2-sinister pseudofunctor), and if the induced triple of pseudofunctors

$$F_\Sigma \colon \mathcal{E} \to \mathscr{C}, \qquad F_\Delta \colon \mathcal{E}^{\mathrm{op}} \to \mathscr{C}, \qquad F_\Pi \colon \mathcal{E} \to \mathscr{C}$$

satisfies the earlier Beck-Chevalley condition on the pair F_Σ and F_Δ, in addition to a "distributivity condition" on the pair F_Σ and F_Π. Forgetting the distributivity condition yields the notion of a 2-Beck pseudofunctor, so that (5.1.2) may be seen as a restriction of an equivalence

$$\frac{\text{gregarious functors } \mathbf{Poly}(\mathcal{E}) \to \mathscr{C}}{\text{2-Beck pseudofunctors } \mathcal{E} \to \mathscr{C}}$$

Similar to earlier, an important special case of this is when $\mathscr{C} = \mathbf{Cat}$, where one recovers the equivalence

$$\frac{\text{gregarious functors } \mathbf{Poly}(\mathcal{E}) \to \mathbf{Cat}}{\text{fibrations with sums and products over } \mathcal{E}}$$

Note also that on **Fib** $(\mathcal{E})$ there is a KZ pseudomonad $\Sigma_\mathcal{E}$ for fibrations with sums, and a coKZ pseudomonad $\Pi_\mathcal{E}$ for fibrations with products. This yields (via a pseudo-distributive law) a pseudomonad $\Sigma_\mathcal{E}\Pi_\mathcal{E}$ for fibrations with sums and products which satisfy a distributivity condition [50]. Here we recover the equivalence

$$\frac{\text{pseudofunctors } \mathbf{Poly}(\mathcal{E}) \to \mathbf{Cat}}{\text{distributive fibrations with sums and products over } \mathcal{E}}$$

The codomain fibration is again an archetypal example of this, with the corresponding canonical pseudofunctor **Poly** $(\mathcal{E}) \longrightarrow$ **Cat** being defined by

$$\begin{tikzcd}
& E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & & \mapsto & \mathcal{E}/I \arrow[r, "\Delta_s"] & \mathcal{E}/E \arrow[r, "\Pi_p"] & \mathcal{E}/B \arrow[r, "\Sigma_t"] & \mathcal{E}/J \\
I & & & J
\end{tikzcd}$$

which is how one assigns a polynomial to a polynomial functor.

Another example of this situation is given by taking $\mathcal{E}$ to be a regular locally cartesian closed category. In this case we have the 2-Beck pseudofunctor $\mathrm{Sub}\colon \mathcal{E} \longrightarrow$ **Cat** which sends a morphism $f\colon X \longrightarrow Y$ in $\mathcal{E}$ to the existential quantifier $\exists_f\colon \mathrm{Sub}(X) \longrightarrow \mathrm{Sub}(Y)$ mapping subobjects of X to those of Y, which has the two successive right adjoints Δ_f "pullback along f" and $\forall_f$ "universal quantification at f", thus giving a gregarious functor **Poly** $(\mathcal{E}) \longrightarrow$ **Cat** defined by the assignment

$$\begin{tikzcd}
& E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & & \mapsto & \mathrm{Sub}(I) \arrow[r, "\Delta_s"] & \mathrm{Sub}(E) \arrow[r, "\forall_p"] & \mathrm{Sub}(B) \arrow[r, "\exists_t"] & \mathrm{Sub}(J) \\
I & & & J
\end{tikzcd}$$

The distributivity condition here then amounts to asking that $\mathcal{E}$ satisfies the internal axiom of choice.

With only cartesian morphisms we do not have the adjunctions on the right in (5.1.1) since the units and counits of such adjunctions are not cartesian in general, thus making the universal property of $\mathbf{Poly}_c(\mathcal{E})$ more complicated to state. The universal property of this construction is described as an equivalence

$$\frac{\text{pseudofunctors } \mathbf{Poly}_c(\mathcal{E}) \longrightarrow \mathscr{C}}{\text{DistBeck triple } \mathcal{E} \longrightarrow \mathscr{C}}$$

where a DistBeck triple $\mathcal{E} \longrightarrow \mathscr{C}$ is a triple of pseudofunctors

$$F_\Sigma\colon \mathcal{E} \longrightarrow \mathscr{C}, \qquad F_\Delta\colon \mathcal{E}^{\mathrm{op}} \longrightarrow \mathscr{C}, \qquad F_\otimes\colon \mathcal{E} \longrightarrow \mathscr{C}$$

such $F_\Sigma f \dashv F_\Delta f$ for all morphisms f in $\mathcal{E}$, with a Beck–Chevalley condition satisfied for the pair F_Σ and F_Δ, for which F_Δ and $F_\otimes$ are related via invertible Beck–Chevalley coherence data (as we do not have adjunctions $F_\Delta f \dashv F_\otimes f$ this data does not come for free and must be given instead, subject to suitable coherence axioms), such that the pair F_Σ and $F_\otimes$ satisfy a distributivity condition as before[4]. There are also weakened versions of the universal property of $\mathbf{Poly}_c(\mathcal{E})$ which arise from dropping these conditions.

An example of this is given by taking $\mathcal{E}$ to be the category of finite sets **FinSet** and $\mathscr{C}$ to be the 2-category of small categories **Cat**. Taking $(\mathcal{A}, \otimes, I)$ to be a symmetric monoidal category such that $\mathcal{A}$ has finite coproducts, we can assign to any finite set n the category $\mathcal{A}^n$

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[4] The distributivity data need not be given as it may be constructed using the F_Δ-$F_\otimes$ Beck coherence data and the adjunctions $F_\Sigma f \dashv F_\Delta f$.

and to any morphism $f\colon m \to n$ the functors

$$\begin{aligned} \mathrm{lan}_f\colon \mathcal{A}^m \to \mathcal{A}^n, &\quad (a_i\colon i \in m) \mapsto \left(\Sigma_{x \in f^{-1}(j)}\, a_x\colon j \in n\right) \\ (-) \circ f\colon \mathcal{A}^n \to \mathcal{A}^m, &\quad \left(a_j\colon j \in n\right) \mapsto \left(a_{f(i)}\colon i \in m\right) \\ \otimes_f\colon \mathcal{A}^m \to \mathcal{A}^n, &\quad (a_i\colon i \in m) \mapsto \left(\otimes_{x \in f^{-1}(j)}\, a_x\colon j \in n\right) \end{aligned}$$

This gives the data of a Beck triple (that is a DistBeck triple without requiring the distributivity condition). The distributivity condition here holds precisely when the functor $X \otimes (-) : \mathcal{A} \to \mathcal{A}$ preserves finite coproducts for all $X \in \mathcal{A}$.

The reader should note that proving the universal properties concerning the polynomial construction is much more complex than that of the span construction. This is since composition of polynomials is significantly more complicated; this is especially evident in calculations involving associativity of polynomial composition being respected by an oplax or pseudofunctor, or calculations involving horizontal composition of general polynomial morphisms.

Fortunately, we are able to avoid these calculations to some extent. This is done by exploiting the fact that both **Span** $(\mathcal{E})$ and $\mathbf{Poly}_c$ $(\mathcal{E})$ are "generic bicategories" (as detailed in Chapter 4), that is a bicategory $\mathscr{A}$ with the property that each composition functor

$$\circ_{X,Y,Z}\colon \mathscr{A}_{Y,Z} \times \mathscr{A}_{X,Y} \to \mathscr{A}_{X,Z}$$

admits generic factorisations. The main result of Chapter 4 shows that oplax functors out of such bicategories admit a much simpler description; thus allowing for a simple description of oplax functors out **Span** $(\mathcal{E})$ and $\mathbf{Poly}_c$ $(\mathcal{E})$. A problem here is that the bicategory **Poly** $(\mathcal{E})$ does not enjoy this property. However, as $\mathbf{Poly}_c$ $(\mathcal{E})$ embeds into **Poly** $(\mathcal{E})$ and both bicategories have the same composition the universal property of the former will assist in proving the latter.

In Section 5.2 we give the necessary background for this paper. We recall the definitions and basic properties of the bicategories of spans and polynomials, the notions of lax, oplax and gregarious functors, the basic properties of the mates correspondence, and the basic properties of generic bicategories.

In Section 5.3 we give a proof of the universal properties of spans using the properties of generic bicategories. This is to give a complete and detailed proof of these properties demonstrating our method, before applying it the more complicated setting of polynomials later on.

In Section 5.4 we give a proof of the universal properties of spans with invertible 2-cells. This is necessary since the universal properties of polynomials with cartesian 2-cells will be described in terms of this property.

In Section 5.5 we give a proof of the universal properties of polynomials with cartesian

2-cells. It is in this section that our method is of the most use; indeed in our proof we completely avoid coherences involving composition of distributivity pullbacks (the worst coherence conditions which would arise in a direct proof).

In Section 5.6 we give a proof of the universal properties of polynomials with general 2-cells, by using the corresponding properties for polynomials with cartesian 2-cells and checking some additional coherence conditions concerning naturality with respect to these more general 2-cells.

5.2 Background

In this section we give the necessary background knowledge for this paper.

5.2.1 The bicategory of spans

Before studying the bicategory of polynomials we will study the simpler and more well known construction of the bicategory of spans, as introduced by Bénabou [3].

Definition 5.2.1. Suppose we are given a category $\mathcal{E}$ with chosen pullbacks. We may then form *bicategory of spans* in $\mathcal{E}$, denoted $\textbf{Span}\,(\mathcal{E})$, with objects those of $\mathcal{E}$, 1-cells $A \nrightarrow B$ given diagrams in $\mathcal{E}$ of the form

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A & & B \end{tikzcd}$$

called spans, composition of 1-cells given by taking the chosen pullback

$$\begin{tikzcd} & & \bullet \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\ & X \arrow[dl, "p"'] \arrow[dr, "q"] & & Y \arrow[dl, "r"'] \arrow[dr, "s"] & \\ A & & B & & C \end{tikzcd}$$

and 2-cells v given by those morphisms between the vertices of two spans which yield commuting diagrams of the form

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "v"] & \\ A & & B \\ & Y \arrow[ul, "s"] \arrow[ur, "r"'] & \end{tikzcd}$$

The identity 1-cells are given by identity spans $X \xleftarrow{1_X} X \xrightarrow{1_X} X$ and composition extends to 2-cells by the universal property of pullbacks. The essential uniqueness of the limit of a diagram

$$\begin{tikzcd}
 & X \arrow[dl, "p"'] \arrow[dr, "q"] & & Y \arrow[dl, "r"'] \arrow[dr, "s"] & & Z \arrow[dl, "u"'] \arrow[dr, "v"] & \\
A & & B & & C & & D
\end{tikzcd}$$

yields the associators, making **Span** $(\mathcal{E})$ into a bicategory.

We denote by $\mathbf{Span}_{\mathrm{iso}}(\mathcal{E})$ the bicategory as defined above, but only taking the invertible 2-cells.

5.2.2 The bicategory of polynomials

In the earlier defined bicategory of spans the morphisms may be viewed as multivariate linear maps (matrices). In this subsection we recall the bicategory of polynomials, whose morphisms may be viewed as multivariate polynomials, and whose study has applications in areas including theoretical computer science (under the name of containers and indexed containers [1]) and the theory of W-types [43, 44].

Before we can define this bicategory we must recall the notion of distributivity pullback as given by Weber [55].

Definition 5.2.2. Given two composable morphisms $u\colon X \to A$ and $f\colon A \to B$ in a category $\mathcal{E}$ with pullbacks, we say that:

1. a *pullback around* (f, u) is a diagram

$$\begin{tikzcd}
T \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[r, "u"] & A \arrow[d, "f"] \\
Y \arrow[rr, "r"'] & & B
\end{tikzcd}$$

such that the outer rectangle is a pullback, and a *morphism of pullbacks around* (f, u) is a pair of morphisms $s\colon T \to T'$ and $t\colon Y \to Y'$ such that $p's = p$, $q's = tq$ and $r = r't$;

2. a *distributivity pullback* around (f, u) is a terminal object in the category of pullbacks around (f, u).

We also recall the notion of an exponentiable morphism, a condition which ensures the existence of such distributivity pullbacks.

Definition 5.2.3. We say a morphism $f\colon A \to B$ in a category $\mathcal{E}$ with pullbacks is *exponentiable* if the "pullback along f" functor $\Delta_f\colon \mathcal{E}/B \to \mathcal{E}/A$ has a right adjoint. We will denote this right adjoint by Π_f when it exists.

Remark 5.2.4. Note that such an f is exponentiable if and only if for every u there exists a distributivity pullback around (f, u) [55].

The following diagrams are to be the morphisms in the bicategory of polynomials.

Definition 5.2.5. A *polynomial* $P\colon I \nrightarrow J$ in a category $\mathcal{E}$ with pullbacks is a diagram of the form

$$\begin{tikzcd}
 & E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & \\
I & & & J
\end{tikzcd}$$

where p is exponentiable.

We will also need the following universal property of polynomial composition.

Proposition 5.2.6. *[55, Prop. 3.1.6] Suppose we are given two polynomials $P\colon I \nrightarrow J$ and $Q\colon J \nrightarrow K$. Consider a category with objects given by commuting diagrams of the form*

$$\begin{tikzcd}
 & & A_1 \arrow[r] \arrow[dl] & A_2 \arrow[r] \arrow[dl] \arrow[dr] & A_3 \arrow[dr] & & \\
 & E \arrow[r] \arrow[dl] & B \arrow[dr] & & M \arrow[r] \arrow[dl] & N \arrow[dr] & \\
I & & & J & & & K
\end{tikzcd}$$

for which the left and right squares are pullbacks (but not necessarily the middle), and morphisms given by triples $(A_i \longrightarrow B_i\colon i = 1, 2, 3)$ rendering commutative the diagram

$$\begin{tikzcd}
 & & A_1 \arrow[r] \arrow[d] \arrow[ddl] & A_2 \arrow[r] \arrow[d] \arrow[ddl, dashed] \arrow[ddr, dashed] & A_3 \arrow[d] \arrow[ddr] & & \\
 & & B_1 \arrow[r] \arrow[dl] & B_2 \arrow[r] \arrow[dl] \arrow[dr] & B_3 \arrow[dr] & & \\
 & E \arrow[r] \arrow[dl] & B \arrow[dr] & & M \arrow[r] \arrow[dl] & N \arrow[dr] & \\
I & & & J & & & K
\end{tikzcd}$$

Then in this category, the outside composite in the diagram formed below (which is a polynomial $I \nrightarrow K$), where dpb indicates distributivity pullback

$$\begin{tikzcd}
 & \bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[rr] \arrow[d] & & \bullet \arrow[dd] & \\
 & & \text{pb} & \bullet \arrow[dl] \arrow[dr] & \text{dpb} & & \\
 & E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & \text{pb} & M \arrow[r, "q"] \arrow[dl, "u"'] & N \arrow[dr, "v"] & \\
I & & & J & & & K
\end{tikzcd} \tag{5.2.1}$$

is a terminal object.

Definition 5.2.7. Suppose we are given a locally cartesian closed category $\mathcal{E}$ with chosen pullbacks and distributivity pullbacks. We may then form the *bicategory of polynomials with cartesian 2-cells* in $\mathcal{E}$, denoted $\mathbf{Poly}_c(\mathcal{E})$, with objects those of $\mathcal{E}$, 1-cells $A \nrightarrow B$ given by polynomials, composition of 1-cells given by forming the diagram (5.2.1) just above, and cartesian 2-cells given by pairs of morphisms (σ, v) rendering commutative the diagram

$$\begin{tikzcd}
& E \arrow[r, "p"] \arrow[dl, "s"'] \arrow[dd, "\sigma"'] \arrow[ddr, phantom, "\mathrm{pb}"] & B \arrow[dr, "t"] \arrow[dd, "v"] & \\
I & & & J \\
& M \arrow[ul, "u"] \arrow[r, "q"'] & N \arrow[ur, "v"'] &
\end{tikzcd}$$

such that the middle square is a pullback. The identity 1-cells are given by identity polynomials $X \xleftarrow{1_X} X \xrightarrow{1_X} X \xrightarrow{1_X} X$. Composition of 2-cells and the associators may be recovered from Proposition 5.2.6 above.

Definition 5.2.8. Suppose we are given a locally cartesian closed category $\mathcal{E}$ with chosen pullbacks and distributivity pullbacks. We may then form the *bicategory of polynomials with general 2-cells,* denoted $\mathbf{Poly}(\mathcal{E})$, with objects and 1-cells as in $\mathbf{Poly}_c(\mathcal{E})$, and 2-cells given by diagrams as below on the left below

$$\begin{tikzcd}
& E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[d, equal] \arrow[dr, "t"] & & & & E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[d, equal] \arrow[dr, "t"] & \\
X & S_1 \arrow[u, "e_1"'] \arrow[r, "pe_1"] \arrow[d, "f_1"] \arrow[dr, phantom, "\mathrm{pb}"] & B \arrow[d, "g"] & Y & \approx & X & S_2 \arrow[u, "e_2"'] \arrow[r, "pe_2"] \arrow[d, "f_2"] \arrow[dr, phantom, "\mathrm{pb}"] & B \arrow[d, "g"] & Y \\
& M \arrow[ul, "u"] \arrow[r, "q"'] & N \arrow[ur, "v"'] & & & & M \arrow[ul, "u"] \arrow[r, "q"'] & N \arrow[ur, "v"'] &
\end{tikzcd}$$

regarded equivalent to the diagram on the right provided both indicated regions are pullbacks.

For the other operations of this bicategory such as the composition operation on 2-cells we refer the reader to the equivalence $\mathbf{Poly}(\mathcal{E}) \simeq \mathbf{PolyFun}(\mathcal{E})$ [17] where $\mathbf{PolyFun}(\mathcal{E})$ is the bicategory of polynomial functors, described later in Example 5.2.12.

Remark 5.2.9. Note that it suffices to give local equivalences $\mathbf{PolyFun}(\mathcal{E})_{X,Y} \simeq \mathbf{Poly}(\mathcal{E})_{X,Y}$ since from this it follows that the bicategorical structure on $\mathbf{PolyFun}(\mathcal{E})$ endows the family of hom-categories $\mathbf{Poly}(\mathcal{E})_{X,Y}$ with the structure of a bicategory via doctrinal adjunction [27]. This describes the bicategory structure on $\mathbf{Poly}(\mathcal{E})$.

5.2.3 Morphisms of bicategories

There are a few types of morphisms between bicategories we are interested in for this paper. These include oplax functors, lax functors, pseudofunctors, gregarious functors and sinister pseudofunctors. After the following trivial definition we will recall these notions.

Definition 5.2.10. Given two bicategories $\mathscr{A}$ and $\mathscr{B}$, a *locally defined functor* $F\colon \mathscr{A} \to \mathscr{B}$ consists of:

- for each object $X \in \mathscr{A}$ an object $FX \in \mathscr{B}$;
- for each pair of objects $X, Y \in \mathscr{A}$, a functor $F_{X,Y}\colon \mathscr{A}_{X,Y} \to \mathscr{B}_{FX,FY}$;

subject to no additional conditions.

It is one of the main points of this paper that many of the coherence conditions arising from the associativity diagram (5.2.2) for oplax functors out of the bicategories **Span** $(\mathcal{E})$ and $\mathbf{Poly}_c$ $(\mathcal{E})$ may be avoided (for suitable categories $\mathcal{E}$).

Definition 5.2.11. Given two bicategories $\mathscr{A}$ and $\mathscr{B}$, a *lax functor* $F\colon \mathscr{A} \to \mathscr{B}$ is a locally defined functor $F\colon \mathscr{A} \to \mathscr{B}$ equipped with

- for each object $X \in \mathscr{A}$, a 2-cell $\lambda_X\colon 1_{FX} \to F1_X$;
- for each triple of objects $X, Y, Z \in \mathscr{A}$ and pair of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, a 2-cell $\varphi_{g,f}\colon Fg \cdot Ff \to Fgf$ natural in g and f,

such that the constraints render commutative the associativity diagram

$$\begin{tikzcd}
Fh \cdot (Fg \cdot Ff) \arrow[r, "Fh \cdot \varphi_{g,f}"] & Fh \cdot F(gf) \arrow[r, "\varphi_{h,gf}"] & F(h(gf)) \\
(Fh \cdot Fg) \cdot Ff \arrow[u, "a_{Fh,Fg,Ff}"] \arrow[r, "\varphi_{h,g} \cdot Ff"'] & F(hg) \cdot Ff \arrow[r, "\varphi_{hg,f}"'] & F((hg)f) \arrow[u, "F(\tilde{a}_{h,g,f})"']
\end{tikzcd} \tag{5.2.2}$$

for composable morphisms h,g and f. In addition, the nullary constraint cells must render commutative the diagrams

$$\begin{tikzcd}
Ff \cdot 1_{FX} \arrow[r, "Ff \cdot \lambda_X"] \arrow[d, "r_{Ff}"'] & Ff \cdot F(1_X) \arrow[d, "\varphi_{f,1_X}"] \\
Ff & F(f \cdot 1_X) \arrow[l, "F(r_f)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{FY} \cdot Ff \arrow[r, "\lambda_Y \cdot Ff"] \arrow[d, "l_{Ff}"'] & F(1_Y) \cdot Ff \arrow[d, "\varphi_{1_Y,f}"] \\
Ff & F(1_Y \cdot f) \arrow[l, "F(l_f)"]
\end{tikzcd}$$

for all morphisms $f\colon X \to Y$. If the direction of the constraints φ and λ is reversed, this is the definition of an *oplax functor*. If the nullary constraints λ are invertible (in either the lax or oplax case) we then say our functor is *normal*. If both types of constraint cells φ and λ are required invertible, then this is the definition of a *pseudofunctor*.

Example 5.2.12. *It is well known that given a category $\mathcal{E}$ with pullbacks there is a pseudofunctor* $\mathbf{Span}(\mathcal{E}) \to \mathbf{Cat}$ *which assigns an object $X \in \mathcal{E}$ to the slice category $\mathcal{E}/X$ and on spans is defined the assignment*

$$\begin{tikzcd} & B \arrow[dl, "s"'] \arrow[dr, "t"] & & & \\ I & & J & \mapsto & \mathcal{E}/I \arrow[r, "\Delta_s"] & \mathcal{E}/B \arrow[r, "\Sigma_t"] & \mathcal{E}/J \end{tikzcd}$$

where Σ_t is the "composition with t" functor, and Δ_s is the "pullback along s" functor (the right adjoint of Σ_s).

If $\mathcal{E}$ is locally cartesian closed, meaning that for each morphism p the functor Δ_p has a further right adjoint denoted Π_p, then there is also such a canonical functor out of $\mathbf{Poly}(\mathcal{E})$ *[17] and* $\mathbf{Poly}_c(\mathcal{E})$ *[55], which assigns an object $X \in \mathcal{E}$ to the slice category $\mathcal{E}/X$ and on polynomials is defined the assignment*

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[r, "p"] & B \arrow[dr, "t"] & & & \\ I & & & J & \mapsto & \mathcal{E}/I \arrow[r, "\Delta_s"] & \mathcal{E}/E \arrow[r, "\Pi_p"] & \mathcal{E}/B \arrow[r, "\Sigma_t"] & \mathcal{E}/J \end{tikzcd}$$

A functor isomorphic to one as on the right above is known as a polynomial functor. *The objects of $\mathcal{E}$, polynomial functors, and strong natural transformations form a 2-category* $\mathbf{PolyFun}(\mathcal{E})$ *[17].*

Remark 5.2.13. In the subsequent sections we are interested in pseudofunctors mapping into a general bicategory $\mathscr{C}$, not just **Cat**, however we will still use the above example to motivate our notation.

The following is a special type of oplax functor which turns up when studying the universal properties of the span construction [9, 10]. This notion will also be useful for studying the universal properties of the polynomial construction.

Definition 5.2.14. [9, Definition 2.4] We say a normal oplax functor of bicategories $F\colon \mathscr{A} \to \mathscr{B}$ is *gregarious* (also known as *jointed*) if for any pair of 1-cells $f\colon A \to B$ and $g\colon B \to C$ in $\mathscr{A}$ for which g has a right adjoint, the constraint cell $\varphi_{g,f}\colon F(gf) \to Fg \cdot Ff$ is invertible.

There is also an alternative characterization of gregarious functors worth mentioning, which establishes gregarious functors as a natural concept.

Proposition 5.2.15. *[9, Propositions 2.8 and 2.9] A normal oplax functor of bicategories* $F\colon \mathscr{A} \to \mathscr{B}$ *is gregarious if and only if it preserves adjunctions; that is, if for every adjunction* $(f \dashv u\colon A' \to A, \eta, \varepsilon)$ *in* $\mathscr{A}$ *there exists 2-cells* $\overline{\eta} : 1_{FA} \to Fu \cdot Ff$ *and* $\overline{\varepsilon} : Ff \cdot Fu \to 1_{FA'}$ *which exhibit* Ff *as left adjoint to* Fu *and render commutative the squares*

$$\begin{tikzcd}
F(1_A) \arrow[r, "F\eta"] \arrow[d, "\lambda_A"'] & F(uf) \arrow[d, "\varphi_{u,f}"] \\
1_{FA} \arrow[r, "\overline{\eta}"'] & Fu \cdot Ff
\end{tikzcd}
\qquad
\begin{tikzcd}
F(fu) \arrow[r, "F\varepsilon"] \arrow[d, "\varphi_{f,u}"'] & F(1_{A'}) \arrow[d, "\lambda_{A'}"] \\
Ff \cdot Fu \arrow[r, "\overline{\varepsilon}"'] & 1_{FA'}
\end{tikzcd}$$

We also need a notion of morphism between lax, oplax, gregarious or pseudofunctors. It will be convenient here to use Lack's icons [34], defined as follows.

Definition 5.2.16. Given two lax functors $F, G\colon \mathscr{A} \to \mathscr{B}$ which agree on objects, an *icon* $\alpha\colon F \Rightarrow G$ consists of a family of natural transformations

$$\begin{tikzcd}
\mathscr{A}_{X,Y} \arrow[r, bend left, "F_{X,Y}"] \arrow[r, bend right, "G_{X,Y}"'] \arrow[r, phantom, "\Downarrow\alpha_{X,Y}"] & \mathscr{B}_{FX,FY},
\end{tikzcd} \qquad X, Y \in \mathscr{A}$$

with components rendering commutative the diagrams

$$\begin{tikzcd}
Fg \cdot Ff \arrow[r, "\varphi_{g,f}"] \arrow[d, "\alpha_g * \alpha_f"'] & F(gf) \arrow[d, "\alpha_{gf}"] \\
Gg \cdot Gf \arrow[r, "\psi_{g,f}"'] & G(gf)
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{FX} \arrow[d, "\lambda_X"'] \arrow[dr, "\omega_X"] & \\
F1_X \arrow[r, "\alpha_{1_X}"'] & G1_X
\end{tikzcd}$$

for composable morphisms f and g in $\mathscr{A}$. Similarly, one may define icons between oplax functors.

An important point about icons is that there is a 2-category of bicategories, oplax (lax) functors, and icons. For convenience, we make the following definition.

Definition 5.2.17. We denote by **Icon** (resp. **Greg**) the 2-category of bicategories, pseudofunctors (resp. gregarious functors) and icons.

Finally, we recall the notion of a sinister pseudofunctor, as well as the notion of a sinister pseudofunctor which satisfies a certain Beck condition. These notions are to be used regularly through the paper.

Definition 5.2.18. Let $\mathcal{E}$ be a category seen as a locally discrete 2-category, and let $\mathscr{C}$ be a bicategory. We say a pseudofunctor $F\colon \mathcal{E} \to \mathscr{C}$ of bicategories is *sinister* if for every morphism f in $\mathcal{E}$ the 1-cell Ff has a right adjoint in $\mathscr{C}$.

Supposing further that $\mathcal{E}$ has pullbacks, for any pullback square in $\mathcal{E}$ as on the left below, we may apply F and compose with pseudofunctoriality constraints giving an invertible 2-cell as in the middle square below, and then take mates to get a 2-cell as on the right below

$$\begin{tikzcd} \bullet \arrow[r, "f'"] \arrow[d, "g'"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "Ff'"] \arrow[d, "Fg'"'] \arrow[dr, phantom, "\cong"] & \bullet \arrow[d, "Fg"] \\ \bullet \arrow[r, "Ff"'] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "F_\Sigma f'"] \arrow[dr, phantom, "\Downarrow \mathfrak{b}_{f,g}^{f',g'}"] & \bullet \\ \bullet \arrow[r, "F_\Sigma f"'] \arrow[u, "F_\Delta g'"] & \bullet \arrow[u, "F_\Delta g"'] \end{tikzcd}$$

We say the sinister pseudofunctor $F\colon \mathcal{E} \to \mathscr{C}$ satisfies the *Beck condition* if every such $\mathfrak{b}_{f,g}^{f',g'}$ as on the right above is invertible.

We will denote by **Sin** $(\mathcal{E},\mathscr{C})$ the category of sinister pseudofunctors $\mathcal{E} \to \mathscr{C}$ and invertible icons, and **Beck** $(\mathcal{E},\mathscr{C})$ the subcategory of sinister pseudofunctors satisfying the Beck condition.

Remark 5.2.19. Note that $\mathfrak{b}_{f,g}^{f',g'}$ as above may be defined for any commuting square, not just a pullback. We call such a $\mathfrak{b}_{f,g}^{f',g'}$ the *Beck 2-cell* corresponding to the commuting square, but should not expect it to be invertible if the square is not a pullback (even if the Beck condition holds).

5.2.4 Mates under adjunctions

We now recall the basic properties of mates [30]. Given two pairs of adjoint morphisms

$$\eta_1,\varepsilon_1\colon f_1 \dashv u_1\colon B_1 \to A_1, \qquad \eta_2,\varepsilon_2\colon f_2 \dashv u_2\colon B_2 \to A_2$$

in a bicategory $\mathscr{A}$, we say that two 2-cells

$$\begin{tikzcd} A_1 \arrow[r, "g"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\Downarrow\alpha"] & A_2 \arrow[d, "f_2"] \\ B_1 \arrow[r, "h"'] & B_2 \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[r, "g"] \arrow[dr, phantom, "\Downarrow\beta"] & A_2 \\ B_1 \arrow[r, "h"'] \arrow[u, "u_1"] & B_2 \arrow[u, "u_2"'] \end{tikzcd}$$

are mates under the adjunctions $f_1 \dashv u_1$ and $f_2 \dashv u_2$ if β is given by the pasting

$$\begin{tikzcd}
& A_1 \arrow[r, "g"] \arrow[d, "f_1"] \arrow[dl, phantom, "\Downarrow\varepsilon_1"] \arrow[dr, phantom, "\Downarrow\alpha"] & A_2 \arrow[r, "\mathrm{id}"] \arrow[d, "f_2"] \arrow[dr, phantom, "\Downarrow\eta_2"] & A_2 \\
B_1 \arrow[ur, bend left, "u_1"] \arrow[r, "\mathrm{id}"'] & B_1 \arrow[r, "h"'] & B_2 \arrow[ur, bend right, "u_2"'] &
\end{tikzcd}$$

or equivalently, α is given by the pasting

$$\begin{tikzcd}
A_1 \arrow[r, "\mathrm{id}"] \arrow[dr, bend right, "f_1"'] \arrow[dr, phantom, "\Downarrow\eta_1"] & A_1 \arrow[r, "g"] \arrow[dr, phantom, "\Downarrow\beta"] & A_2 \arrow[dr, bend left, "f_2"] \arrow[dr, phantom, "\Downarrow\varepsilon_2"] & \\
& B_1 \arrow[r, "h"'] \arrow[u, "u_1"] & B_2 \arrow[r, "\mathrm{id}"'] \arrow[u, "u_2"] & B_2
\end{tikzcd}$$

It follows from the triangle identities that taking mates in this fashion defines a bijection between 2-cells $f_2g \to hf_1$ and 2-cells $gu_1 \to u_2h$.

Moreover, it is well known that this correspondence is functorial. Given another adjunction $\eta_3, \varepsilon_3 \colon f_3 \dashv u_3 \colon B_3 \to A_3$ and 2-cells as below

$$\begin{tikzcd}
A_1 \arrow[r, "g"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\Downarrow\alpha_l"] & A_2 \arrow[r, "m"] \arrow[d, "f_2"] \arrow[dr, phantom, "\Downarrow\alpha_r"] & A_3 \arrow[d, "f_3"] \\
B_1 \arrow[r, "h"'] & B_2 \arrow[r, "n"'] & B_3
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "g"] \arrow[dr, phantom, "\Downarrow\beta_l"] & A_2 \arrow[r, "m"] \arrow[dr, phantom, "\Downarrow\beta_r"] & A_3 \\
B_1 \arrow[r, "h"'] \arrow[u, "u_1"] & B_2 \arrow[r, "n"'] \arrow[u, "u_2"] & B_3 \arrow[u, "u_2"']
\end{tikzcd}$$

where α_l and α_r respectively correspond to β_l and β_r under the mates correspondence, it follows that the pasting of α_l and α_r corresponds to the pasting of β_l and β_r under the mates correspondence. Moreover, the analogous property holds for pasting vertically. These vertical and horizontal pasting properties[5] are often referred to as *functoriality of mates*.

Remark 5.2.20. Given an adjunction $\eta, \varepsilon \colon f \dashv u \colon B \to A$ the left square below

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & A \arrow[d, "\mathrm{id}"] \\
B \arrow[r, "\mathrm{id}"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, phantom, "\Downarrow\varepsilon"] & A \\
B \arrow[r, "\mathrm{id}"'] \arrow[u, "u"] & B \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

corresponds to the right above via the mates correspondence, allowing one to see the counit of an adjunction as an instance of the mates correspondence. A similar calculation may be done for the units. This will allow us to see calculations involving units and counits as functoriality of mates calculations.

One consequence of the mates correspondence which will be of interest to us is the following lemma; a special case of [9, Lemma 2.13], showing that the component of an icon

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[5] There are also nullary pasting properties which we will omit.

between gregarious functors at a left adjoint 1-cell is invertible.

Lemma 5.2.21. *Suppose $F, G\colon \mathscr{A} \to \mathscr{B}$ are gregarious functors between bicategories which agree on objects. Suppose that $\alpha\colon F \Rightarrow G$ is an icon. Suppose that a given 1-cell $f\colon X \to Y$ has a right adjoint u in $\mathscr{A}$ with unit ε and counit η. Then the 2-cell $\alpha_f\colon Ff \to Gf$ has an inverse given by the mate of $\alpha_u\colon Fu \to Gu$.*

Proof. As $f \dashv u$ we have $Ff \dashv Fu$ via counit

$$Ff \cdot Fu \xrightarrow{\varphi_{f,u}^{-1}} F(fu) \xrightarrow{F\varepsilon} F1_Y \xrightarrow{\lambda_Y} 1_{FY}$$

and unit

$$1_{FX} \xrightarrow{\lambda_X^{-1}} F1_X \xrightarrow{F\eta} F(uf) \xrightarrow{\varphi_{u,f}} Fu \cdot Ff$$

and similarly $Gf \dashv Gu$. That the mate of α_u constructed as the pasting

$$\begin{tikzcd}
FX \arrow[r, "1_{FX}"] \arrow[d, "1_{FX}"'] \arrow[dr, phantom, "\Uparrow \lambda_X^{-1}"] & FX \arrow[r, "1_{FX}"] \arrow[d, "F1_X"] \arrow[dr, phantom, "\Uparrow F\eta"] & FX \arrow[r, "Ff"] \arrow[d, "Fuf"] \arrow[dr, phantom, "\Uparrow \varphi_{u,f}"] & FY \arrow[r, "1_{FY}"] \arrow[d, "Fu"] \arrow[dr, phantom, "\Uparrow \alpha_u"] & FY \arrow[r, "1_{FY}"] \arrow[d, "Gu"] \arrow[dr, phantom, "\Uparrow \psi_{f,u}"] & FY \arrow[r, "1_{FY}"] \arrow[d, "Gfu"] \arrow[dr, phantom, "\Uparrow G\varepsilon"] & FY \arrow[r, "1_{FY}"] \arrow[d, "G1_Y"] \arrow[dr, phantom, "\Uparrow \omega_Y"] & FY \arrow[d, "1_{FY}"] \\
FX \arrow[r, "1_{FX}"'] & FX \arrow[r, "1_{FX}"'] & FX \arrow[r, "1_{FX}"'] & FX \arrow[r, "1_{FX}"'] & FX \arrow[r, "Gf"'] & FY \arrow[r, "1_{FY}"'] & FY \arrow[r, "1_{FY}"'] & FY
\end{tikzcd}$$

is the inverse of α_f is a simple calculation which we will omit (as the details are in [9, Lemma 2.13]). □

Remark 5.2.22. Under the conditions of the above lemma we have corresponding functors $F^{\mathrm{co}}, G^{\mathrm{co}}\colon \mathscr{A}^{\mathrm{co}} \to \mathscr{B}^{\mathrm{co}}$ which are adjunction preserving (gregarious), and an icon $\alpha^{\mathrm{co}}\colon G^{\mathrm{co}} \Rightarrow F^{\mathrm{co}}$. Thus noting $u \dashv f$ in $\mathscr{A}^{\mathrm{co}}$ we see that in $\mathscr{B}^{\mathrm{co}}$, $\alpha_u^{\mathrm{co}}\colon Gf \to Ff$ has an inverse given as the mate of α_f^{co}. It follows that α_u has an inverse given as the mate of α_f in $\mathscr{B}$.

5.2.5 Adjunctions of spans and polynomials

Later on we will need to discuss gregarious functors out of bicategories of spans and bicategories of polynomials, and so an understanding of the adjunctions in these bicategories will be essential.

We first recall the classification of adjunctions in the bicategory of spans. A proof of this classification is given in [7, Proposition 2], but this proof does not readily generalize to the setting of polynomials. We therefore give a simpler proof using the properties of the mates correspondence.

Proposition 5.2.23. *Up to isomorphism, all adjunctions in* **Span** $(\mathcal{E})$ *are of the form*

$$\begin{tikzcd}
& X \arrow[dl, "1_X"'] \arrow[dr, "f"] & & & & X \arrow[dl, "f"'] \arrow[dr, "1_X"] & \\
X & & Y & \dashv & Y & & X
\end{tikzcd} \tag{5.2.3}$$

with unit and counit

$$\begin{tikzcd}
& X \arrow[dl, "1_X"'] \arrow[dr, "1_X"] \arrow[dd, "\delta"] & & & & X \arrow[dl, "f"'] \arrow[dr, "f"] \arrow[dd, "f"] & \\
X & & X & & Y & & Y \\
& X \times_Y X \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] & & & & Y \arrow[ul, "1_Y"] \arrow[ur, "1_Y"'] &
\end{tikzcd}$$

where $(X \times_Y X, \pi_1, \pi_2)$ *is the pullback of* f *with itself.*

Proof. It is simple to check the above defines an adjunction. We now check that all adjunctions have this form, up to isomorphism. To do this, suppose we are given an adjunction of spans

$$\begin{tikzcd}
& \bullet \arrow[dl, "s"'] \arrow[dr, "t"] & & & & \bullet \arrow[dl, "u"'] \arrow[dr, "v"] & \\
\bullet & & \bullet & \dashv & \bullet & & \bullet
\end{tikzcd}$$

and denote the unit of this adjunction (actually a representation of the unit using the universal property of pullback) by

$$\begin{tikzcd}
& & \bullet \arrow[ddll, bend right, "\mathrm{id}"'] \arrow[dl, "\alpha"'] \arrow[dd, "h"] \arrow[dr, "\beta"] \arrow[ddrr, bend left, "\mathrm{id}"] & & \\
& \bullet \arrow[dl, "s"'] \arrow[dr, "t"] & & \bullet \arrow[dl, "u"'] \arrow[dr, "v"] & \\
\bullet & & \bullet & & \bullet
\end{tikzcd} \tag{5.2.4}$$

noting that $v\beta$ is the identity. We then factor this unit as

$$1 \longrightarrow (s,t)\,;(h,1) \xrightarrow{\mathrm{id};\beta} (s,t)\,;(u,v)$$

where the first morphism is represented by

$$\begin{tikzcd}
& & \bullet \arrow[ddll, bend right, "\mathrm{id}"'] \arrow[dl, "\alpha"'] \arrow[dd, "h"] \arrow[dr, "\mathrm{id}"] \arrow[ddrr, bend left, "\mathrm{id}"] & & \\
& \bullet \arrow[dl, "s"'] \arrow[dr, "t"] & & \bullet \arrow[dl, "h"'] \arrow[dr, "\mathrm{id}"] & \\
\bullet & & \bullet & & \bullet
\end{tikzcd}$$

and $\beta\colon (h,1) \to (u,v)$ is pictured on the right in (5.2.4). Under the mates correspondence this yields two morphisms

$$(u,v) \longrightarrow (h,1) \xrightarrow{\beta} (u,v)$$

which must compose to the identity. As the first morphism of spans is necessarily v we have also established βv as the identity, and hence v as an isomorphism. This allows us to construct an isomorphism of right adjoints $(u, v) \to (f, 1)$ for an f as in (5.2.3), corresponding to an isomorphism of left adjoints $(1, f) \to (s, t)$ and hence showing s is invertible also. $\square$

Remark 5.2.24. If we restrict ourselves to the bicategory $\mathbf{Span}_{\mathrm{iso}}(\mathcal{E})$ then we only have adjunctions as above when f is invertible (necessary to construct the counit).

In the case of polynomials there are more adjunctions to consider.

Proposition 5.2.25. *Up to isomorphism, every adjunction in* $\mathbf{Poly}(\mathcal{E})$ *is a composite of adjunctions of the form*

$$\begin{tikzcd}
& X \arrow[r, "1_X"] \arrow[dl, "1_X"'] & X \arrow[dr, "f"] & & & & X \arrow[r, "1_X"] \arrow[dl, "f"'] & X \arrow[dr, "1_X"] & \\
X & & & Y & \dashv & Y & & & X
\end{tikzcd}$$

with unit and counit

$$\begin{tikzcd}
& X \arrow[r, "1_X"] \arrow[dl, "1_X"'] & X \arrow[d, equal] \arrow[dr, "1_X"] & & & & X \arrow[r, "1_X"] \arrow[dl, "f"'] & X \arrow[d, equal] \arrow[dr, "f"] & \\
X & X \arrow[u, "1_X"] \arrow[r, "1_X"] \arrow[d, "\delta"'] & X \arrow[d, "\delta"] & X & & Y & X \arrow[u, "1_X"] \arrow[r, "1_X"] \arrow[d, "f"] & X \arrow[d, "f"] & Y \\
& X \times_Y X \arrow[ul, "\pi_1"] \arrow[r, "1_{X \times_Y X}"'] & X \times_Y X \arrow[ur, "\pi_2"'] & & & & Y \arrow[ul, "1_Y"] \arrow[r, "1_Y"'] & Y \arrow[ur, "1_Y"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& X \arrow[r, "1_X"] \arrow[dl, "f"'] & X \arrow[dr, "1_X"] & & & & X \arrow[r, "f"] \arrow[dl, "1_X"'] & Y \arrow[dr, "1_Y"] & \\
Y & & & X & \dashv & X & & & Y
\end{tikzcd}$$

with unit and counit

$$\begin{tikzcd}
& Y \arrow[r, "1_Y"] \arrow[dl, "1_Y"'] & Y \arrow[d, equal] \arrow[dr, "1_Y"] & & & & X \times_Y X \arrow[r, "\pi_2"] \arrow[dl, "\pi_1"'] & X \arrow[d, equal] \arrow[dr, "1_X"] & \\
Y & X \arrow[u, "f"] \arrow[r, "f"] \arrow[d, "1_X"] & Y \arrow[d, "1_Y"] & Y & & X & X \arrow[u, "\delta"] \arrow[r, "1_X"] \arrow[d, "1_X"] & X \arrow[d, "1_X"] & X \\
& X \arrow[ul, "f"] \arrow[r, "f"'] & Y \arrow[ur, "1_Y"'] & & & & X \arrow[ul, "1_X"] \arrow[r, "1_X"'] & X \arrow[ur, "1_X"'] &
\end{tikzcd}$$

Proof. It is simple to check that the above define adjunctions of polynomials, indeed this is almost the same calculation as in the case of spans. We now check that all adjunctions have this form, up to isomorphism. To do this, suppose we are given an adjunction of polynomials

$$\begin{tikzcd}
& \bullet \arrow[r, "p"] \arrow[dl, "s"'] & \bullet \arrow[dr, "t"] & & & & \bullet \arrow[r, "q"] \arrow[dl, "u"'] & \bullet \arrow[dr, "v"] & \\
\bullet & & & \bullet & \dashv & \bullet & & & \bullet
\end{tikzcd}$$

and denote the unit of this adjunction by[6]

$$\begin{tikzcd}
& & \bullet \arrow[rrd, bend left, "\mathrm{id}"] \arrow[ddd, ll, bend right, "\mathrm{id}"'] & & & & \\
& & \bullet \arrow[u, "e"] \arrow[r, "p'"] \arrow[dl, "\alpha_1"'] & \bullet \arrow[r, "q'"] \arrow[dl, "\alpha_2"'] \arrow[dr, "\beta_1"] \arrow[dd, "h"] & \bullet \arrow[dr, "\beta_2"] \arrow[ddrr, bend left, "\mathrm{id}"] & & \\
& \bullet \arrow[r, "p"] \arrow[dl, "s"'] & \bullet \arrow[dr, "t"] & & \bullet \arrow[r, "q"] \arrow[dl, "u"'] & \bullet \arrow[dr, "v"] & \\
\bullet & & & \bullet & & & \bullet
\end{tikzcd}$$

noting that $v\beta_2$ is the identity. We then factor this unit as, where $(\beta_1,\beta_2):\ (h,q',1)\longrightarrow(u,q,v)$ is the cartesian morphism of polynomials pictured on the right above,

$$1 \longrightarrow (s,p,t)\,;(h,q',1) \xrightarrow{\mathrm{id};(\beta_1,\beta_2)} (s,p,t)\,;(u,q,v)$$

which under the mates correspondence yields two morphisms

$$(u,q,v) \longrightarrow (h,q',1) \xrightarrow{(\beta_1,\beta_2)} (u,q,v)$$

which must compose to the identity; that is, a diagram below

$$\begin{tikzcd}
& \bullet \arrow[r, "q"] \arrow[ddl, bend right, "u"'] & \bullet \arrow[d, equal] \arrow[ddr, bend left, "v"] & \\
& \bullet \arrow[u] \arrow[r] \arrow[d] & \bullet \arrow[d, "v"] & \\
\bullet & \bullet \arrow[l, "h"'] \arrow[r, "q'"] \arrow[d, "\beta_1"] & \bullet \arrow[r, "1"] \arrow[d, "\beta_2"] & \bullet \\
& \bullet \arrow[ul, "u"] \arrow[r, "q"] & \bullet \arrow[ur, "v"'] &
\end{tikzcd}$$

composing to the identity, showing $\beta_2 v$ is the identity, and hence that v is invertible. This allows us to construct an isomorphism of right adjoints $(u,q,v)\longrightarrow(f,g,1)$ for some f and g, corresponding to an isomorphism of left adjoints $(g,1,f)\longrightarrow(s,p,t)$ and hence showing p is invertible also. □

Remark 5.2.26. If we restrict ourselves to the bicategory $\mathbf{Poly}_c(\mathcal{E})$ then to have the second adjunction of Proposition 5.2.25 we require f to be invertible.

---

[6] Here the cartesian part of the morphism of polynomials is represented using Proposition 5.2.6.

5.2.6 Basic properties of generic bicategories

The bicategories of spans **Span** $(\mathcal{E})$ and bicategories of polynomials with cartesian 2-cells $\mathbf{Poly}_c\,(\mathcal{E})$ defined above both satisfy a special property: they are examples of a bicategory $\mathscr{A}$ which contains a special class of 2-cells (which one may informally think of as the "diagonal" 2-cells[7]) such that any 2-cell into a composite of 1-cells $\alpha\colon c \to a; b$ factors uniquely as some diagonal 2-cell $\delta\colon c \to l; r$ pasted with 2-cells $\alpha_1\colon l \to a$ and $\alpha_2\colon r \to b$.

For the reader familiar with generic morphisms [14, 15, 52], this property can be stated concisely by asking that each composition functor

$$\circ_{X,Y,Z}\colon \mathscr{A}_{Y,Z} \times \mathscr{A}_{X,Y} \to \mathscr{A}_{X,Z}$$

admits generic factorisations. A bicategory $\mathscr{A}$ with this property is called *generic*.

As shown in Chapter 4, one of the main properties of generic bicategories $\mathscr{A}$ is that oplax functors out of them admit an alternative description, similar to the description of a comonad. In particular, for a locally defined functor $L\colon \mathscr{A} \to \mathscr{C}$ one may define a bijection between coherent binary and nullary oplax constraint cells

$$\varphi_{a,b}\colon L\,(a; b) \to La; Lb, \qquad \lambda_X\colon L1_X \to 1_X$$

and "coherent" comultiplication and counit maps

$$\Phi_\delta\colon Lc \to Ll; Lr, \qquad \Lambda_\varepsilon\colon Ln \to 1_X$$

indexed over diagonal maps $\delta\colon c \to l; r$ and augmentations (2-cells into identity 1-cells) $\varepsilon\colon n \to 1_X$. Indeed, given the data (φ, λ) the comultiplication maps Φ_δ and counit maps Λ_ε are given by the composites

$$Lc \xrightarrow{L\delta} L\,(l; r) \xrightarrow{\varphi_{l,r}} Ll; Lr \qquad\qquad Ln \xrightarrow{L\varepsilon} L1_X \xrightarrow{\lambda_X} 1_X$$

and conversely given the data (Φ, Λ) the oplax constraints $\varphi_{a,b}$ and λ_X are recovered by factoring the identity 2-cell through a diagonal as on the left below and defining the right diagram to commute.

$$\begin{tikzcd} & l; r \arrow[dr, "s_1; s_2"] & \\ a; b \arrow[ur, "\delta"] \arrow[rr, "\mathrm{id}"'] & & a; b \end{tikzcd} \qquad \begin{tikzcd} & Ll; Lr \arrow[dr, "Ls_1; Ls_2"] & \\ L\,(a; b) \arrow[ur, "\Phi_\delta"] \arrow[rr, "\varphi_{a,b}"'] & & La; Lb \end{tikzcd}$$

Trivially, we recover each unit $\lambda_X\colon L\,(1_X) \to 1_X$ as the component of Λ at id_{1_X}.

For the full statement concerning the bicategories of spans and cartesian polynomials, see Proposition 5.3.4 and Proposition 5.5.9 respectively.

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[7] Formally, these diagonals are defined as the generic morphisms against the composition functor. See Chapter 4 for details.

5.3 Universal properties of spans

In this section, we give a complete proof of the universal properties of spans [9] using the properties of generic bicategories. This is to demonstrate our method in the simpler case of spans before applying it to polynomials in Section 5.5.

5.3.1 Stating the universal property

Before stating the universal property we recall that we have two canonical embeddings into the bicategory of spans given by the pseudofunctors denoted

$$(-)_\Sigma : \mathcal{E} \to \mathbf{Span}\,(\mathcal{E}), \qquad (-)_\Delta : \mathcal{E}^{\mathrm{op}} \to \mathbf{Span}\,(\mathcal{E})\,.$$

These are defined on objects by sending an object of $\mathcal{E}$ to itself, and are defined on each morphism in $\mathcal{E}$ by the assignments

$$\begin{tikzcd}
(-)_\Sigma : & X \arrow[r, "f"] & Y & \mapsto & X & X \arrow[l, "1_X"'] \arrow[r, "f"] & Y \\
(-)_\Delta : & X \arrow[r, "f"] & Y & \mapsto & Y & X \arrow[l, "f"'] \arrow[r, "1_X"] & X
\end{tikzcd}$$

Remark 5.3.1. Note that the embedding $(-)_\Sigma$ is an example of a pseudofunctor which is both sinister and satisfies the Beck condition.

The universal property of spans is then the following result, as given by Hermida [21] and Dawson, Paré, and Pronk [9, Theorem 2.15].

Theorem 5.3.2 (Universal Properties of Spans)**.** *Given a category $\mathcal{E}$ with chosen pullbacks, composition with the canonical embedding $(-)_\Sigma : \mathcal{E} \to \mathbf{Span}\,(\mathcal{E})$ defines the equivalence of categories*

$$\mathbf{Greg}\,(\mathbf{Span}\,(\mathcal{E}),\mathscr{C}) \simeq \mathbf{Sin}\,(\mathcal{E},\mathscr{C})$$

which restricts to the equivalence

$$\mathbf{Icon}\,(\mathbf{Span}\,(\mathcal{E}),\mathscr{C}) \simeq \mathbf{Beck}\,(\mathcal{E},\mathscr{C})$$

for any bicategory $\mathscr{C}$.

5.3.2 Proving the universal property

Before proving Theorem 5.3.2 we will need to show that given a sinister pseudofunctor $\mathcal{E} \to \mathscr{C}$ one may reconstruct an oplax functor $\mathbf{Span}\,(\mathcal{E}) \to \mathscr{C}$. The following lemma and subsequent propositions describe this construction.

Lemma 5.3.3. *Let $\mathcal{E}$ be a category with pullbacks seen as a locally discrete 2-category, and let $\mathscr{C}$ be a bicategory. Suppose $F\colon \mathcal{E} \longrightarrow \mathscr{C}$ is a given sinister pseudofunctor, and for each morphism $f \in \mathcal{E}$ define $F_\Sigma f := Ff$ and take $F_\Delta f$ to be a chosen right adjoint of Ff (choosing F_Δ to strictly preserve identities). We may then define local functors*

$$L_{X,Y} : \mathbf{Span}\,(\mathcal{E})_{X,Y} \longrightarrow \mathscr{C}_{LX,LY}, \qquad X, Y \in \mathcal{E}$$

by the assignment $T \longmapsto FT$ on objects, and

$$\begin{tikzcd}
& T \arrow[ld, "s"'] \arrow[rd, "t"] \arrow[dd, "f"] & & & & & FT \arrow[rrd, "F_\Sigma t"] & & \\
X & & Y & \longmapsto & FX \arrow[rru, "F_\Delta s"] \arrow[rrd, "F_\Delta u"'] & \Downarrow\alpha & & \Downarrow\gamma & FY \\
& S \arrow[lu, "u"] \arrow[ru, "v"'] & & & & & FS \arrow[uu, "F_\Delta f" description] \arrow[rru, "F_\Sigma v"'] & &
\end{tikzcd}$$

on morphisms, where α is the mate of the isomorphism on the left below

$$\begin{tikzcd}
FT \arrow[r, "1_{FT}"] \arrow[d, "F_\Sigma s"'] \arrow[rd, phantom, "\cong"] & FT \arrow[d, "F_\Sigma u \cdot F_\Sigma f"] & & FT \arrow[r, "F_\Sigma t"] \arrow[d, "F_\Sigma f"'] \arrow[rd, phantom, "\cong"] & FY \arrow[d, "1_{FY}"] \\
FX \arrow[r, "1_{FX}"'] & FX & & FS \arrow[r, "F_\Sigma v"'] & FY
\end{tikzcd}$$

under the adjunctions $F_\Sigma s \dashv F_\Delta s$ and $F_\Sigma u \cdot F_\Sigma f \dashv F_\Delta f \cdot F_\Delta u$, and γ is the mate of the isomorphism on the right above under the adjunctions $F_\Sigma f \dashv F_\Delta f$ and $1_{FY} \dashv 1_{FY}$.

Proof. Functoriality is clear from functoriality of mates and the associativity condition and unitary conditions on F. □

To show that these local functors can be endowed with the structure of an oplax functor it will be useful to recall the following reduced description of such an oplax structure, obtained via the theory of Chapter 4.

Proposition 5.3.4. *Let $\mathcal{E}$ be a category with pullbacks and denote by* **Span** $(\mathcal{E})$ *the bicategory of spans in $\mathcal{E}$. Let $\mathscr{C}$ be a bicategory. Then to give an oplax functor*

$$L\colon \mathbf{Span}\,(\mathcal{E}) \to \mathscr{C}$$

is to give a locally defined functor

$$L_{X,Y}\colon \mathbf{Span}\,(\mathcal{E})_{X,Y} \longrightarrow \mathscr{C}_{LX,LY}, \qquad X, Y \in \mathcal{E}$$

with comultiplication and counit maps

$$\Phi_{s,h,t} \colon L(s,t) \to L(s,h) \,; L(h,t), \qquad \Lambda_h \colon L(h,h) \to 1_{LX}$$

for every respective diagram in $\mathcal{E}$

$$\begin{tikzcd} & T \arrow[dl, "s"'] \arrow[d, "h"] \arrow[dr, "t"] & \\ X & Y & Z \end{tikzcd} \qquad \begin{tikzcd} T \arrow[d, "h"] \\ X \end{tikzcd}$$

such that:

1. *for any triple of morphisms of spans as below*

$$\begin{tikzcd} & R \arrow[dl, "u"'] \arrow[dr, "v"] \arrow[dd, "f"] & \\ X & & Z \\ & T \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd} \quad \begin{tikzcd} & R \arrow[dl, "u"'] \arrow[dr, "k"] \arrow[dd, "f"] & \\ X & & Y \\ & T \arrow[ul, "s"] \arrow[ur, "h"'] & \end{tikzcd} \quad \begin{tikzcd} & R \arrow[dl, "k"'] \arrow[dr, "v"] \arrow[dd, "f"] & \\ Y & & Z \\ & T \arrow[ul, "h"] \arrow[ur, "t"'] & \end{tikzcd}$$

we have the commuting diagram

$$\begin{tikzcd} L(u,v) \arrow[r, "\Phi_{u,k,v}"] \arrow[d, "Lf"'] & L(u,k) \,; L(k,v) \arrow[d, "Lf;Lf"] \\ L(s,t) \arrow[r, "\Phi_{s,h,t}"'] & L(s,h) \,; L(h,t) \end{tikzcd}$$

2. *for any morphism of spans as on the left below*

$$\begin{tikzcd} & M \arrow[dl, "p"'] \arrow[d, "f"] \arrow[dr, "p"] & \\ X & N \arrow[l, "q"] \arrow[r, "q"'] & X \end{tikzcd} \qquad \begin{tikzcd} L(p,p) \arrow[rr, "Lf"] \arrow[dr, "\Lambda_p"'] & & L(q,q) \arrow[dl, "\Lambda_q"] \\ & 1_{LX} & \end{tikzcd}$$

the diagram on the right above commutes;

3. *for all diagrams of the form*

$$\begin{tikzcd} & & T \arrow[dll, "s"'] \arrow[dl, "h"] \arrow[dr, "k"'] \arrow[drr, "t"] & & \\ W & X & & Y & Z \end{tikzcd}$$

in $\mathcal{E}$, we have the commuting diagram

$$\begin{tikzcd}
L(s,t) \arrow[rr, equal] \arrow[d, "\Phi_{s,h,t}"'] & & L(s,t) \arrow[d, "\Phi_{s,k,t}"] \\
L(s,h);L(h,t) \arrow[d, "L(s;h);\Phi_{h,k,t}"'] & & L(s,k);L(k,t) \arrow[d, "\Phi_{s,h,k};L(k;t)"] \\
L(s,h);(L(h,k);L(k,t)) \arrow[rr, "\text{assoc}"'] & & (L(s,h);L(h,k));L(k,t)
\end{tikzcd}$$

4. *for all spans (s,t) we have the commuting diagrams*

$$\begin{tikzcd}
& L(s,s);L(s,t) \arrow[dr, "\Lambda_s;L(s,t)"] & \\
L(s,t) \arrow[ur, "\Phi_{s,s,t}"] \arrow[rr, "\text{unitor}"'] & & 1_{LX};L(s,t)
\end{tikzcd}
\qquad
\begin{tikzcd}
& L(s,t);L(t,t) \arrow[dr, "L(s,t);\Lambda_t"] & \\
L(s,t) \arrow[ur, "\Phi_{s,t,t}"] \arrow[rr, "\text{unitor}"'] & & L(s,t);1_{LY}
\end{tikzcd}$$

We now prove that the locally defined functor L above may be endowed with an oplax structure.

Proposition 5.3.5. *Let $\mathcal{E}$ be a category with pullbacks seen as a locally discrete 2-category, and let $\mathscr{C}$ be a bicategory. Suppose $F\colon \mathcal{E} \to \mathscr{C}$ is a given sinister pseudofunctor. Then the locally defined functor*

$$L_{X,Y} : \mathbf{Span}(\mathcal{E})_{X,Y} \to \mathscr{C}_{LX,LY}, \qquad X,Y \in \mathcal{E}$$

as in Lemma 5.3.3 canonically admits the structure of an oplax functor.

Proof. By Proposition 5.3.4, to equip the locally defined functor L with an oplax structure is to give comultiplication maps $\Phi_{s,h,t}\colon L(s,t) \to L(s,h);L(h,t)$ and counit maps $\Lambda_h\colon L(h,h) \to 1_{LX}$ for diagrams of the respective forms

$$\begin{tikzcd}
& T \arrow[dl, "s"'] \arrow[d, "h"] \arrow[dr, "t"] & \\
X & Y & Z
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& T \arrow[dl, "h"'] \arrow[dr, "h"] & \\
X & & X
\end{tikzcd}$$

satisfying naturality, associativity, and unitary conditions. To do this, we take each $\Phi_{s,h,t}$ and Λ_h to be the respective pastings

$$\begin{tikzcd}
& & {} & & \\
FX \arrow[r, "F_\Delta s"] & FT \arrow[r, "F_\Sigma h"] \arrow[rr, bend left=60, "1_{FT}"] & FY \arrow[r, "F_\Delta h"] \arrow[u, phantom, "\Downarrow\eta_{Fh}"] & FT \arrow[r, "F_\Sigma t"] & FZ
\end{tikzcd}
\qquad
\begin{tikzcd}
& FT \arrow[dr, "F_\Sigma h"] \arrow[d, phantom, "\Downarrow\varepsilon_{Fh}"] & \\
FX \arrow[ur, "F_\Delta h"] \arrow[rr, "1_{FX}"] & {} & FX
\end{tikzcd}
\tag{5.3.1}$$

Associativity of comultiplication is trivial; indeed, given a diagram of the form

$$\begin{tikzcd}
& & & T \arrow[dlll, "s"'] \arrow[dl, "h"] \arrow[dr, "k"'] \arrow[drrr, "t"] & & & \\
W & & X & & Y & & Z
\end{tikzcd}$$

the pasting

$$\begin{tikzcd}
FW \arrow[r, "F_\Delta s"] & FT \arrow[r, "F_\Sigma h"] \arrow[rr, bend left=60, "\mathrm{id}_{FT}"] \arrow[rr, phantom, bend left=30, "\Downarrow\eta_{Fh}"] & FX \arrow[r, "F_\Delta h"] & FT \arrow[r, "F_\Sigma k"] \arrow[rr, bend left=60, "\mathrm{id}_{FT}"] \arrow[rr, phantom, bend left=30, "\Downarrow\eta_{Fk}"] & FY \arrow[r, "F_\Delta k"] & FT \arrow[r, "F_\Sigma t"] & FZ
\end{tikzcd}$$

evaluates to the same 2-cell regardless if we paste the bottom path with first η_{Fh} and then η_{Fk}, or vice versa. The unitary axioms are also trivial, an immediate consequence of the triangle identities for an adjunction.

For the naturality condition, suppose we are given a triple of morphisms of spans

$$\begin{tikzcd}
& R \arrow[dl, "u"'] \arrow[dd, "f"] \arrow[dr, "v"] & & & R \arrow[dl, "u"'] \arrow[dd, "f"] \arrow[dr, "k"] & & & R \arrow[dl, "k"'] \arrow[dd, "f"] \arrow[dr, "v"] & \\
X & & Z & X & & Y & Y & & Z \\
& T \arrow[ul, "s"] \arrow[ur, "t"'] & & & T \arrow[ul, "s"] \arrow[ur, "h"'] & & & T \arrow[ul, "h"] \arrow[ur, "t"'] &
\end{tikzcd}$$

and note that we have the commuting diagram

$$\begin{tikzcd}
L(u,v) \arrow[r, "\Phi_{u,k,v}"] \arrow[d, "Lf"'] & L(u,k)\,;L(k,v) \arrow[d, "Lf;Lf"] \\
L(s,t) \arrow[r, "\Phi_{s,h,t}"'] & L(s,h)\,;L(h,t)
\end{tikzcd}$$

since the top composite is

$$\begin{tikzcd}
& FR \arrow[rr, "1_{FR}"] \arrow[dr, "F_\Sigma k"] \arrow[rr, phantom, "\Downarrow\eta_{Fk}"', shift right=4] & & FR \arrow[dr, "F_\Sigma v"] & \\
FX \arrow[ur, "F_\Delta u"] \arrow[dr, "F_\Delta s"'] \arrow[r, phantom, "\Downarrow"] & {} \arrow[r, phantom, "\Downarrow"] & FY \arrow[ur, "F_\Delta k"] \arrow[dr, "F_\Delta h"'] \arrow[r, phantom, "\Downarrow"] & {} \arrow[r, phantom, "\Downarrow"] & FZ \\
& FT \arrow[uu, "F_\Delta f"] \arrow[ur, "F_\Sigma h"'] & & FT \arrow[uu, "F_\Delta f"] \arrow[ur, "F_\Sigma t"'] &
\end{tikzcd}$$

and the bottom composite is

$$\begin{tikzcd}
& FR \arrow[rr, "1_{FR}"] & & FR \arrow[dr, "F_\Sigma v"] & \\
FX \arrow[ur, "F_\Delta u"] \arrow[dr, "F_\Delta s"'] & \Downarrow\ F_\Delta f & = & F_\Delta f\ \Downarrow & FZ \\
& FT \arrow[uu] \arrow[rr, "1_{FT}"] \arrow[dr, "F_\Sigma h"'] & \Downarrow\eta_{Fh} & FT \arrow[uu] \arrow[ur, "F_\Sigma t"'] & \\
& & FY \arrow[ur, "F_\Delta h"'] & &
\end{tikzcd}$$

where the unlabeled 2-cells are as in Lemma 5.3.3. That these pastings agree is a standard functoriality of mates calculation. We omit the naturality of counits calculation, as it is a simpler functoriality of mates calculation. ☐

Remark 5.3.6. It is trivial that each λ_X given by Λ at 1_X is invertible above.

We now check that the structure given above has its oplax constraints given by Beck 2-cells.

Lemma 5.3.7. *Let the oplax functor* $L\colon \mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ *be constructed as in Proposition 5.3.5. Then the binary oplax constraint cell on L, at a composite of spans constructed as below*

$$\begin{tikzcd}
& & M \arrow[dl, "c'"'] \arrow[dr, "b'"] & & \\
& T \arrow[dl, "a"'] \arrow[dr, "b"] & \text{pb} & S \arrow[dl, "c"'] \arrow[dr, "d"] & \\
X & & Y & & Z
\end{tikzcd} \tag{5.3.2}$$

is given by the Beck 2-cell for the pullback appropriately whiskered by $F_\Delta a$ *and* $F_\Sigma d$.

Proof. Given composable spans (a,b) and (c,d) the composite is given by the diagram (5.3.2). We then have an induced diagonal

$$\delta_{ac',h,db'}\colon\ (ac',db') \to (ac',h)\,;(h,db')$$

and morphisms $c'\colon\ (ac',h) \to (a,b)$ and $b'\colon\ (h,db') \to (c,d)$ for which

$$(a,b)\,;(c,d) \xrightarrow{\delta_{ac',h,db'}} (ac',h)\,;(h,db') \xrightarrow{c';b'} (a,b)\,;(c,d)$$

is the identity on $(a,b)\,;(c,d)$. Hence the oplax constraint cell corresponding to the comultiplication maps Φ, namely

$$\varphi_{(a,b),(c,d)}\colon L\left((a,b)\,;(c,d)\right) \to L\,(a,b)\,;L\,(c,d)$$

is given by the pasting

$$\begin{tikzcd}
 & FM \arrow[rr, "1_{FM}"] \arrow[dr, "F_\Sigma h"] & {} \arrow[d, phantom, "\Downarrow\eta_{Fh}"] & FM \arrow[dr, "F_\Sigma db'"] & \\
FX \arrow[ur, "F_\Delta ac'"] \arrow[dr, "F_\Delta a"'] \arrow[r, phantom, "\Downarrow"] & {} \arrow[r, phantom, "\Downarrow"] & FY \arrow[ur, "F_\Delta h"] \arrow[dr, "F_\Delta c"'] \arrow[r, phantom, "\Downarrow"] & {} \arrow[r, phantom, "\Downarrow"] & FZ \\
 & FT \arrow[uu, "F_\Delta c'"] \arrow[ur, "F_\Sigma b"'] & & FS \arrow[uu, "F_\Delta b'"] \arrow[ur, "F_\Sigma d"'] &
\end{tikzcd} \tag{5.3.3}$$

where $h = bc' = cb'$. It is an easy consequence of functoriality of mates that this pasting is the usual Beck 2-cell for the pullback with the appropriate whiskerings. □

Finally, we will need the following lemma, a consequence of Lemma 5.2.21, in order to complete the proof.

Lemma 5.3.8. *Suppose $L, K\colon \mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ are given gregarious functors. Then any icon $\alpha\colon L \Rightarrow K$ is necessarily invertible.*

Proof. We take identities to be pullback stable for simplicity, so that we have $(s,t) = (s,1)\,;(1,t)$. Let us consider the component of such an icon α at a general span s,t. Since α is an icon, the diagram

$$\begin{tikzcd}
L(s,t) \arrow[r, "\varphi"] \arrow[d, "\alpha_{(s,t)}"'] & L(s,1)\,;L(1,t) \arrow[d, "\alpha_{(s,1)};\alpha_{(1,t)}"] \\
K(s,t) \arrow[r, "\psi"'] & K(s,1)\,;K(1,t)
\end{tikzcd} \tag{5.3.4}$$

commutes. By Lemma 5.2.21 we know $\alpha_{(1,t)}$ is invertible, and by its dual we know $\alpha_{(s,1)}$ is invertible. As F and G are gregarious φ and ψ are invertible above. Hence $\alpha_{(s,t)}$ is invertible. □

We now know enough for a complete proof of the universal properties of the span construction as given by Dawson, Paré, Pronk and Hermida.

Proof of Theorem 5.3.2. We consider the assignment of Theorem 5.3.2, i.e. composition with the embedding $(-)_\Sigma : \mathcal{E} \to \mathbf{Span}(\mathcal{E})$ written as the assignment

$$\begin{tikzcd}
\mathbf{Span}(\mathcal{E}) \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{C}
\end{tikzcd} \quad\mapsto\quad \begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "F_\Sigma"] \arrow[r, bend right, "G_\Sigma"'] \arrow[r, phantom, "\Downarrow\alpha_\Sigma"] & \mathscr{C}
\end{tikzcd}$$

We start by proving the first universal property.

Well defined. This is clear by Corollary 5.3.8.

Fully faithful. That the assignment $\alpha \mapsto \alpha_\Sigma$ is bijective follows from the condition $\left(\alpha_{\Sigma_f}^{-1}\right)^* = \alpha_{\Delta_f}$ forced by Lemma 5.2.21, and the commutativity of (5.3.4). One need only check that any collection

$$\alpha_{s,t} \colon F(s,t) \to G(s,t)$$

satisfying these two properties necessarily defines an icon. Indeed, that such an α is locally natural is a simple consequence of functoriality of mates and α_Σ being an icon. To see that such an α then defines an icon, note that each $\Phi_{s,h,t}$ may be decomposed as the commuting diagram

$$\begin{tikzcd}
F(s,t) \arrow[r, "\Phi_{s,h,t}"] \arrow[d, "\Phi_{s,1,t}"'] & F(s,h)\,;F(h,t) \\
F(s,1)\,;F(1,t) \arrow[r, "F(s,1);\Phi_{1,h,1};F(1,t)"'] & F(s,1)\,;F(1,h)\,;F(h,1)\,;F(1,t) \arrow[u, "\Phi^{-1}_{s,1,h};\Phi^{-1}_{h,1,t}"']
\end{tikzcd}$$

and so the commutativity of the diagram[8]

$$\begin{tikzcd}
F(s,t) \arrow[r, "\Phi_{s,h,t}"] \arrow[d, "\alpha_{s,t}"'] & F(s,h)\,;F(h,t) \arrow[d, "\alpha_{s,h};\alpha_{h,t}"] \\
G(s,t) \arrow[r, "\Psi_{s,h,t}"'] & G(s,h)\,;G(h,t)
\end{tikzcd}$$

amounts to asking that the pastings

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "F_\Delta s"] \arrow[r, bend right, "G_\Delta s"'] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left, "F_\Sigma h"] \arrow[r, bend right, "G_\Sigma h"'] \arrow[r, phantom, "\Downarrow"] \arrow[rr, bend left=70, "\mathrm{id}"] \arrow[rr, phantom, bend left=40, "\Downarrow\eta_{Gh}"] & \bullet \arrow[r, bend left, "F_\Delta h"] \arrow[r, bend right, "G_\Delta h"'] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left, "F_\Sigma t"] \arrow[r, bend right, "G_\Sigma t"'] \arrow[r, phantom, "\Downarrow"] & \bullet
\end{tikzcd}$$

and

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "F_\Delta s"] \arrow[r, bend right, "G_\Delta s"'] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, "G_\Sigma h"'] \arrow[rr, bend left=70, "\mathrm{id}"] \arrow[rr, phantom, bend left=40, "\Downarrow\eta_{Gh}"] & \bullet \arrow[r, "G_\Delta h"'] & \bullet \arrow[r, bend left, "F_\Sigma t"] \arrow[r, bend right, "G_\Sigma t"'] \arrow[r, phantom, "\Downarrow"] & \bullet
\end{tikzcd}$$

agree; which is easily seen by expanding α_{Δ_h} in terms of $\alpha_{\Sigma_h}^{-1}$ and using the triangle identities. The nullary icon condition is trivial. This shows that α indeed admits the structure of an icon.

Essentially surjective. Given any sinister pseudofunctor $F \colon \mathcal{E} \to \mathscr{C}$ we take the gregarious functor $L \colon \mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ from Proposition 5.3.5 and note that $L_\Sigma = F$.

---

[8] This diagram is equivalent to the binary coherence condition on such an icon.

We now verify the second universal property.

Restrictions. The second property is a restriction of the first. Indeed, given a pseudofunctor $L\colon \mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ the corresponding pseudofunctor $L_\Sigma\colon \mathcal{E} \to \mathscr{C}$ satisfies the Beck condition, since the embedding $(-)_\Sigma : \mathcal{E} \to \mathbf{Span}(\mathcal{E})$ satisfies the Beck condition. Moreover, given a sinister pseudofunctor $F\colon \mathcal{E} \to \mathscr{C}$ which satisfies the Beck condition, the corresponding map $\mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ is pseudo since the oplax constraint cells of this functor are Beck 2-cells by Lemma 5.3.7. □

5.4 Universal properties of spans with invertible 2-cells

In this section we derive the universal property of the bicategory of spans with invertible 2-cells, denoted $\mathbf{Span}_{\mathrm{iso}}(\mathcal{E})$. Indeed, an understanding of this universal property will be required for stating the universal property of polynomials with cartesian 2-cells $\mathbf{Poly}_c(\mathcal{E})$ described in the next section.

5.4.1 Stating the universal property

The embeddings $(-)_\Sigma$ and $(-)_\Delta$ into $\mathbf{Span}_{\mathrm{iso}}(\mathcal{E})$ are defined the same as in the case of spans with the usual 2-cells. The difference here is that we no longer have adjunctions $f_\Sigma \dashv f_\Delta$ in general, a fact which we will emphasize by replacing the symbol Σ with $\otimes$. Consequently the universal property is more complicated to state, and so we will need some definitions.

Definition 5.4.1. Given a category $\mathcal{E}$ with chosen pullbacks, we may define the category of *lax Beck pairs* on $\mathcal{E}$, denoted $\mathbf{LaxBeckPair}(\mathcal{E}, \mathscr{C})$. This category has objects given by pairs of pseudofunctors

$$F_\otimes\colon \mathcal{E} \to \mathscr{C}, \qquad\qquad F_\Delta\colon \mathcal{E}^{\mathrm{op}} \to \mathscr{C}$$

which agree on objects, equipped with, for each pullback square

$$\begin{tikzcd} \bullet \arrow[r, "f'"] \arrow[d, "g'"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r, "F_\otimes f'"] & \bullet \\ \bullet \arrow[u, "F_\Delta g'"] \arrow[r, "F_\otimes f"'] \arrow[ur, phantom, "\Downarrow \mathfrak{b}^{f',g'}_{f,g}"] & \bullet \arrow[u, "F_\Delta g"'] \end{tikzcd}$$

in $\mathcal{E}$ as on the left, a 2-cell as on the right (which we call a Beck 2-cell). The collection of these Beck 2-cells comprise the "Beck data" denoted ${}^F\mathfrak{b}$ (or just $\mathfrak{b}$), and are required to satisfy the following coherence conditions:

1. (horizontal double pullback condition) for any double pullback

$$\begin{tikzcd}
\bullet \arrow[r, "f_2'"] \arrow[d, "g''"'] & \bullet \arrow[r, "f_1'"] \arrow[d, "g'"] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "f_2"'] & \bullet \arrow[r, "f_1"'] & \bullet
\end{tikzcd} \tag{5.4.1}$$

we have

$$\begin{tikzcd}
\bullet \arrow[r, "F_\otimes f_2'"] \arrow[rr, bend left=40, "F_\otimes\big(f_1'f_2'\big)", "\cong"'] \arrow[dr, phantom, "\Downarrow\mathfrak{b}^{f'',g''}_{f_2,g'}"] & \bullet \arrow[r, "F_\otimes f_1'"] \arrow[dr, phantom, "\Downarrow\mathfrak{b}^{f_1',g'}_{f_1,g}"] & \bullet \\
\bullet \arrow[u, "F_\Delta g''"] \arrow[r, "F_\otimes f_2"'] \arrow[rr, bend right=40, "F_\otimes\big(f_1'f_2'\big)"', "\cong"] & \bullet \arrow[u, "F_\Delta g'"'] \arrow[r, "F_\otimes f_1"'] & \bullet \arrow[u, "F_\Delta g"']
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "F_\otimes\big(f_1'f_2'\big)"] \arrow[dr, phantom, "\Downarrow\mathfrak{b}^{f_1'f_2',g''}_{f_1f_2,g}"] & \bullet \\
\bullet \arrow[u, "F_\Delta g''"] \arrow[r, "F_\otimes(f_1f_2)"'] & \bullet \arrow[u, "F_\Delta g"']
\end{tikzcd}$$

2. (vertical double pullback condition) for any double pullback

$$\begin{tikzcd}
\bullet \arrow[r, "f''"] \arrow[d, "g_2'"'] & \bullet \arrow[d, "g_2"] \\
\bullet \arrow[r, "f'"] \arrow[d, "g_1'"'] & \bullet \arrow[d, "g_1"] \\
\bullet \arrow[r, "f"'] & \bullet
\end{tikzcd} \tag{5.4.2}$$

we have

$$\begin{tikzcd}
\bullet \arrow[r, "F_\otimes f''"] \arrow[dr, phantom, "\Downarrow\mathfrak{b}^{f'',g_2'}_{f,'g_2}"] & \bullet \\
\bullet \arrow[u, "F_\Delta g_2'"] \arrow[r, "F_\otimes f'"] \arrow[dr, phantom, "\Downarrow\mathfrak{b}^{f',g_1'}_{f,g_1}"] & \bullet \arrow[u, "F_\Delta g_2"'] \\
\bullet \arrow[u, "F_\Delta g_1'"] \arrow[r, "F_\otimes f"'] \arrow[uu, bend left=60, "F_\Delta\big(g_1'g_2'\big)", "\cong"'] & \bullet \arrow[u, "F_\Delta g_1"'] \arrow[uu, bend right=60, "F_\Delta(g_1g_2)"', "\cong"]
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "F_\otimes f''"] \arrow[dr, phantom, "\Downarrow\mathfrak{b}^{f'',g_1'g_2'}_{f,g_1g_2}"] & \bullet \\
\bullet \arrow[u, "F_\Delta\big(g_1'g_2'\big)"] \arrow[r, "F_\otimes f"'] & \bullet \arrow[u, "F_\Delta(g_1g_2)"']
\end{tikzcd}$$

3. (horizontal nullary pullback condition) for any nullary pullback as on the left below, the right pasting below is the identity

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & \bullet \arrow[d, "\mathrm{id}"] \\
\bullet \arrow[r, "f"'] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r, "F_\otimes(f)"] \arrow[dr, phantom, "\Downarrow\mathfrak{b}^{f,\mathrm{id}}_{f,\mathrm{id}}"] & \bullet \\
\bullet \arrow[u, "F_\Delta(\mathrm{id})"'] \arrow[u, bend left=60, "\mathrm{id}", "\cong"'] \arrow[r, "F_\otimes(f)"'] & \bullet \arrow[u, "F_\Delta(\mathrm{id})"] \arrow[u, bend right=60, "\mathrm{id}"', "\cong"]
\end{tikzcd}$$

4. (vertical nullary pullback condition) for any nullary pullback as on the left below, the right pasting below is the identity

$$\begin{tikzcd}
\bullet \arrow[r, "\mathrm{id}"] \arrow[d, "g"'] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "\mathrm{id}"'] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r, "F_\otimes(\mathrm{id})" description] \arrow[r, bend left=60, "\mathrm{id}", "\cong"'] \arrow[dr, phantom, "\Downarrow\mathfrak{b}^{\mathrm{id},g}_{\mathrm{id},g}"] & \bullet \\
\bullet \arrow[u, "F_\Delta g"] \arrow[r, "F_\otimes(\mathrm{id})" description] \arrow[r, bend right=60, "\mathrm{id}"', "\cong"] & \bullet \arrow[u, "F_\Delta g"']
\end{tikzcd}$$

We refer to these conditions as the *Beck–Chevalley coherence conditions*. A morphism in this category $\left(F_\otimes, F_\Delta, {}^F\mathfrak{b}\right) \to \left(G_\otimes, G_\Delta, {}^G\mathfrak{b}\right)$ is a pair of icons $\alpha\colon F_\otimes \Rightarrow G_\otimes$ and $\beta\colon F_\Delta \Rightarrow G_\Delta$ such that for each pullback square as on the left below

$$\begin{tikzcd}
\bullet \arrow[r, "f'"] \arrow[d, "g'"'] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "f"'] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F_\otimes f' \cdot F_\Delta g' \arrow[r, "\alpha_{f'} * \beta_{g'}"] \arrow[d, "{}^F\mathfrak{b}^{f',g'}_{f,g}"'] & G_\otimes f' \cdot G_\Delta g' \arrow[d, "{}^G\mathfrak{b}^{f',g'}_{f,g}"] \\
F_\Delta g \cdot F_\otimes f \arrow[r, "\beta_g * \alpha_f"'] & G_\Delta g \cdot G_\otimes f
\end{tikzcd} \tag{5.4.3}$$

the right diagram commutes. The category **BeckPair** $(\mathcal{E}, \mathscr{C})$ is the subcategory of **LaxBeckPair** $(\mathcal{E}, \mathscr{C})$ containing objects $\left(F_\otimes, F_\Delta, {}^F\mathfrak{b}\right)$ such that every Beck 2-cell in ${}^F\mathfrak{b}$ is invertible.

Before we can state the universal property, we will need to describe how lax Beck pairs arise from suitable functors out of $\mathbf{Span}_{\mathrm{iso}}(\mathcal{E})$.

Definition 5.4.2. Let $\mathcal{E}$ be a category with pullbacks (chosen such that identities pullback to identities) and let $\mathscr{C}$ be a bicategory. Then the category

$$\mathbf{Greg}_{\otimes,\Delta}\left(\mathbf{Span}_{\mathrm{iso}}(\mathcal{E}), \mathscr{C}\right)$$

has objects given by those gregarious functors of bicategories $\mathbf{Span}_{\mathrm{iso}}(\mathcal{E}) \to \mathscr{C}$ which restrict to pseudofunctors when composed with the canonical embeddings $(-)_\otimes : \mathcal{E} \to \mathbf{Span}_{\mathrm{iso}}(\mathcal{E})$ and $(-)_\Delta : \mathcal{E} \to \mathbf{Span}_{\mathrm{iso}}(\mathcal{E})$. Moreover, we require that each oplax constraint

$$F\left(\begin{tikzcd}[column sep=small] & \bullet \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \bullet & & \bullet \end{tikzcd}\right) \to F\left(\begin{tikzcd}[column sep=small] & \bullet \arrow[dl, "s"'] \arrow[dr, "\mathrm{id}"] & \\ \bullet & & \bullet \end{tikzcd}\right); F\left(\begin{tikzcd}[column sep=small] & \bullet \arrow[dl, "\mathrm{id}"'] \arrow[dr, "t"] & \\ \bullet & & \bullet \end{tikzcd}\right) \tag{5.4.4}$$

be invertible. The morphisms of this category are icons.

Proposition 5.4.3. *Let $\mathcal{E}$ be a category with pullbacks (chosen such that identities pullback* *to identities) and let $\mathscr{C}$ be a bicategory. We then have a functor*

$$(-)_{\otimes\Delta} : \mathbf{Greg}_{\otimes,\Delta}\left(\mathbf{Span}_{\mathrm{iso}}(\mathcal{E}),\mathscr{C}\right) \to \mathbf{LaxBeckPair}\,(\mathcal{E},\mathscr{C})$$

defined by the assignment taking such a gregarious functor $F\colon \mathbf{Span}_{\mathrm{iso}}(\mathcal{E}) \to \mathscr{C}$ to the pair of pseudofunctors

$$F_{\otimes}\colon \mathcal{E} \to \mathscr{C}, \qquad F_{\Delta}\colon \mathcal{E}^{\mathrm{op}} \to \mathscr{C}$$

equipped with Beck data ${}^{F}\mathfrak{b}$ given by, for each pullback square as on the left below (with the chosen pullback on the right below)

$$\begin{tikzcd} \bullet \arrow[r, "f'"] \arrow[d, "g'"'] & \bullet \arrow[d, "g"] & & \bullet \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{g}"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "f"'] & \bullet & & \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

the composite of:

1. *the inverse of an oplax constraint cell*

$$F\left(\begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[dl, "g'"'] \arrow[dr, "\mathrm{id}"] & \\ \bullet & & \bullet \end{tikzcd}\right);F\left(\begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f'"] & \\ \bullet & & \bullet \end{tikzcd}\right) \to F\left(\begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[dl, "g'"'] \arrow[dr, "f'"] & \\ \bullet & & \bullet \end{tikzcd}\right)$$

2. *the application of F to the induced isomorphism of pullbacks*

$$F\left(\begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[dl, "g'"'] \arrow[dr, "f'"] & \\ \bullet & & \bullet \end{tikzcd}\right) \to F\left(\begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[dl, "\widetilde{g}"'] \arrow[dr, "\widetilde{f}"] & \\ \bullet & & \bullet \end{tikzcd}\right)$$

3. *the oplax constraint cell*

$$F\left(\begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[dl, "\widetilde{g}"'] \arrow[dr, "\widetilde{f}"] & \\ \bullet & & \bullet \end{tikzcd}\right) \to F\left(\begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\ \bullet & & \bullet \end{tikzcd}\right);F\left(\begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[dl, "g"'] \arrow[dr, "\mathrm{id}"] & \\ \bullet & & \bullet \end{tikzcd}\right)$$

Proof. We must check the Beck 2-cells defined as above satisfy the required coherence conditions. The nullary conditions on the Beck 2-cells are trivially equivalent to the nullary conditions on the constraints of F. To see the "horizontal double pullback condition" holds, we note that since $F\colon \mathbf{Span}_{\mathrm{iso}}(\mathcal{E}) \to \mathscr{C}$ is normal oplax, we have a resulting natural transformation

$$N(F) : N\left(\mathbf{Span}_{\mathrm{iso}}(\mathcal{E})\right) \to N(\mathscr{C})$$

where the functor $N\colon \mathbf{Bicat} \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$ is given by the geometric nerve [46]. In particular (as in [5]), on 2-simplices the assignment

$$\begin{tikzcd}
& \bullet \arrow[dr, "b"] \arrow[d, phantom, "\Uparrow\alpha"] & & & & \bullet \arrow[dr, "Fb"] \arrow[d, phantom, "\Uparrow\overline{F\alpha}"] & \\
\bullet \arrow[ur, "a"] \arrow[rr, "c"'] & {} & \bullet & \mapsto & \bullet \arrow[ur, "Fa"] \arrow[rr, "Fc"'] & {} & \bullet
\end{tikzcd}$$

(where $\overline{F\alpha}$ is $F\alpha$ composed with the appropriate oplax constraint cell) satisfies the condition that

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[d] & & \bullet \arrow[r] \arrow[dr, "\delta\Nearrow", "\Uparrow\gamma"'] & \bullet \arrow[d] \\
\bullet \arrow[u] \arrow[r] \arrow[ur, "\beta\Nwarrow", "\Uparrow\alpha"'] & \bullet & = & \bullet \arrow[u] \arrow[r] & \bullet
\end{tikzcd}$$

implies that

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[d] & & \bullet \arrow[r] \arrow[dr, "\overline{F\delta}\Nearrow", "\Uparrow\overline{F\gamma}"'] & \bullet \arrow[d] \\
\bullet \arrow[u] \arrow[r] \arrow[ur, "\overline{F\beta}\Nwarrow", "\Uparrow\overline{F\alpha}"'] & \bullet & = & \bullet \arrow[u] \arrow[r] & \bullet
\end{tikzcd}$$

Now consider the three spans

$$\begin{tikzcd}
& \bullet \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f_2"] & & & \bullet \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f_1"] & & & \bullet \arrow[dl, "g"'] \arrow[dr, "\mathrm{id}"] & \\
\bullet & & \bullet & \bullet & & \bullet & \bullet & & \bullet
\end{tikzcd}$$

which we denote by shorthand as $(1, f_2)$, $(1, f_1)$ and $(g, 1)$ respectively (where f_1, f_2 and g are as in (5.4.1)). Applying the above implication to the equality below, where each of the four regions contains a canonical isomorphism or equality of spans

$$\begin{tikzcd}
\bullet \arrow[r, "{(1,f_1)}"] & \bullet \arrow[d, "{(g,1)}"] & & \bullet \arrow[r, "{(1,f_1)}"] \arrow[dr, "{(g',f_1')}" description] & \bullet \arrow[d, "{(g,1)}"] \\
\bullet \arrow[u, "{(1,f_2)}"] \arrow[r, "{(g'',f_1'f_2')}"'] \arrow[ur, "{(1,f_1f_2)}" description] & \bullet & = & \bullet \arrow[u, "{(1,f_2)}"] \arrow[r, "{(g'',f_1'f_2')}"'] & \bullet
\end{tikzcd}$$

then gives the horizontal double pullback condition (after composing with the appropriate pseudofunctoriality constraints of F_Σ and constraints of the form (5.4.4)). The proof of the vertical condition is similar. Finally, it is clear the canonical assignation on morphisms is well defined, and the assignment given by composing with the canonical embeddings is trivially functorial. □

We can now state the universal property of $\mathbf{Span}_{\mathrm{iso}}(\mathcal{E})$.

Theorem 5.4.4. *Given a category $\mathcal{E}$ with chosen pullbacks (chosen such that identities pullback to identities), the functor $(-)_{\otimes\Delta}$ of Proposition 5.4.3 defines the equivalence of categories*

$$\mathbf{Greg}_{\otimes,\Delta}\left(\mathbf{Span}_{\mathrm{iso}}\left(\mathcal{E}\right),\mathscr{C}\right) \simeq \mathbf{LaxBeckPair}\left(\mathcal{E},\mathscr{C}\right)$$

which restricts to the equivalence

$$\mathbf{Icon}\left(\mathbf{Span}_{\mathrm{iso}}\left(\mathcal{E}\right),\mathscr{C}\right) \simeq \mathbf{BeckPair}\left(\mathcal{E},\mathscr{C}\right)$$

for any bicategory $\mathscr{C}$.

5.4.2 Proving the universal property

We prove Theorem 5.4.4 directly, as the properties of generic bicategories cannot be used here. Also, for simplicity we assume without loss of generality that $\mathscr{C}$ is a 2-category and that the gregarious functors in question strictly preserve identities. This is justified since every bicategory is equivalent to a 2-category and every normal oplax functor is isomorphic to one which preserves identity 1-cells strictly.

Proof of Theorem 5.4.4. We start by proving the first universal property. We must prove that the functor

$$(-)_{\otimes\Delta} : \mathbf{Greg}_{\otimes,\Delta}\left(\mathbf{Span}_{\mathrm{iso}}\left(\mathcal{E}\right),\mathscr{C}\right) \to \mathbf{LaxBeckPair}\left(\mathcal{E},\mathscr{C}\right)$$

defines an equivalence of categories.

Essentially Surjective. Given such a pair F_Σ and F_Δ with Beck data $\mathfrak{b}$ we may define local functors

$$L_{X,Y} : \mathbf{Span}_{\mathrm{iso}}\left(\mathcal{E}\right)_{X,Y} \to \mathscr{C}_{X,Y}, \qquad X,Y \in \mathcal{E}$$

by the assignment (suppressing pseudofunctoriality of F_Σ and F_Δ)

$$\begin{tikzcd}
& E \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[d, "f" description] & & & & & FE \arrow[dr, "F_\otimes f"] & & \\
X & M \arrow[l, phantom] \arrow[ul, phantom] & Y & \mapsto & FX \arrow[r, "F_\Delta u"] & FM \arrow[ur, "F_\Delta f"] \arrow[dr, "F_\otimes 1"'] & \Downarrow \mathfrak{b}^{f,f}_{1,1} & FM \arrow[r, "F_\otimes v"] & FY \\
& M \arrow[ul, "u"] \arrow[ur, "v"'] & & & & & FM \arrow[ur, "F_\Delta 1"'] & &
\end{tikzcd}$$

which is functorial by the Beck coherence conditions. An oplax constraint cell

$$L\left(\begin{tikzcd}[column sep=small] & \bullet \arrow[dl, "up'"'] \arrow[dr, "qv'"] & \\ \bullet & & \bullet \end{tikzcd}\right) \to L\left(\begin{tikzcd}[column sep=small] & \bullet \arrow[dl, "u"'] \arrow[dr, "v"] & \\ \bullet & & \bullet \end{tikzcd}\right) ; L\left(\begin{tikzcd}[column sep=small] & \bullet \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \bullet & & \bullet \end{tikzcd}\right)$$

is given by (suppressing pseudofunctoriality of $F_\otimes$ and F_Δ)

$$\begin{tikzcd}
 & & \bullet \arrow[dr, "F_\otimes v'"] & & \\
\bullet \arrow[r, "F_\Delta u"] & \bullet \arrow[ur, "F_\Delta p'"] \arrow[dr, "F_\Sigma v"'] & \Downarrow \mathfrak{b}^{v',p'}_{v,p} & \bullet \arrow[r, "F_\otimes q"] & \bullet \\
 & & \bullet \arrow[ur, "F_\Delta p"'] & &
\end{tikzcd}$$

That these constraints satisfy the identity conditions trivially follows from the unit condition on the Beck 2-cells. For the associativity condition, suppose we are given diagrams of chosen pullbacks as below, with p the induced isomorphism of generalized pullbacks, that is the associator for the triple $(a,b),(c,d),(e,f))$,

$$\begin{tikzcd}
 & & & & & & & & & & & X \arrow[d, dotted, "p"] & & & \\
 & & & X \arrow[dl, "i"'] \arrow[ddrr, "j"] & & & & & & & & Y \arrow[ddll, "m"'] \arrow[dr, "n"] & & & \\
 & & \bullet \arrow[dl, "g"'] \arrow[dr, "h"] & & & & & = & & & & & \bullet \arrow[dl, "k"'] \arrow[dr, "\ell"] & & \\
 & \bullet \arrow[dl, "a"'] \arrow[dr, "b"] & & \bullet \arrow[dl, "c"'] \arrow[dr, "d"] & & \bullet \arrow[dl, "e"'] \arrow[dr, "f"] & & & & \bullet \arrow[dl, "a"'] \arrow[dr, "b"] & & \bullet \arrow[dl, "c"'] \arrow[dr, "d"] & & \bullet \arrow[dl, "e"'] \arrow[dr, "f"] & \\
\bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet
\end{tikzcd}$$

Then we must check that

$$\begin{tikzcd}
L(agi, fj) \arrow[r, "\varphi"] \arrow[d, "Lp"'] & L(ag, dh) ; L(e,f) \arrow[r, "\varphi"] & (L(a,b) ; L(c,d)) ; L(e,f) \arrow[d, equal] \\
L(am, f\ell n) \arrow[r, "\varphi"'] & L(a,b) ; L(ck, f\ell) \arrow[r, "\varphi"'] & L(a,b) ; (L(c,d) ; L(e,f))
\end{tikzcd}$$

commutes. The top path is a pasting of Beck 2-cells corresponding[9] to the left diagram in (5.4.5) below, and the bottom path is the pasting of Beck 2-cells corresponding to the right diagram in (5.4.5) below (suppressing pseudofunctoriality constraints of F_Σ and F_Δ) which

---

[9] By "corresponding" we mean that one assigns each pullback square in (5.4.5) to the Beck data for that square, as in Definition 5.4.1.

are equal by the Beck coherence conditions.

$$\begin{tikzcd}[column sep=small]
 & & & \bullet \arrow[dl, "i"'] \arrow[dr, "np"] & & & \\
 & & \bullet \arrow[dl, "g"'] \arrow[dr, "h"] & & \bullet \arrow[dl, "k"'] \arrow[dr, "\ell"] & & \\
 & \bullet \arrow[dl, "a"'] \arrow[dr, "b"] & & \bullet \arrow[dl, "c"'] \arrow[dr, "d"] & & \bullet \arrow[dl, "e"'] \arrow[dr, "f"] & \\
\bullet & & \bullet & & \bullet & & \bullet
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & & & \bullet \arrow[dl, "p"'] \arrow[dr, "p"] & & & \\
 & & \bullet \arrow[dl, "ip^{-1}"'] \arrow[dr, "\mathrm{id}"'] & & \bullet \arrow[dl, "\mathrm{id}"] \arrow[dr, "n"] & & \\
 & \bullet \arrow[dl, "g"'] \arrow[dr, "\mathrm{id}"'] & & \bullet \arrow[dl, "ip^{-1}"'] \arrow[dr, "n"] & & \bullet \arrow[dl, "\mathrm{id}"] \arrow[dr, "\ell"] & \\
\bullet \arrow[dr, "\mathrm{id}"'] & & \bullet \arrow[dl, "g"'] \arrow[dr, "h"] & & \bullet \arrow[dl, "k"'] \arrow[dr, "\ell"] & & \bullet \arrow[dl, "\mathrm{id}"] \\
 & \bullet \arrow[dl, "a"'] \arrow[dr, "b"] & & \bullet \arrow[dl, "c"'] \arrow[dr, "d"] & & \bullet \arrow[dl, "e"'] \arrow[dr, "f"] & \\
\bullet & & \bullet & & \bullet & & \bullet
\end{tikzcd}
\tag{5.4.5}$$

For checking the oplax constraint cells are natural, consider a pair of morphisms of spans

$$\begin{tikzcd}
 & \bullet \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, "f"] & \\
\bullet & & \bullet \\
 & \bullet \arrow[ul, "u"] \arrow[ur, "v"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \bullet \arrow[dl, "m"'] \arrow[dr, "n"] \arrow[dd, "g"] & \\
\bullet & & \bullet \\
 & \bullet \arrow[ul, "p"] \arrow[ur, "q"'] &
\end{tikzcd}$$

We must check the commutativity of

$$\begin{tikzcd}
L\,(sm',nt') \arrow[r, "\varphi"] \arrow[d, "L(f;g)"'] & L\,(s,t)\,;L\,(m,n) \arrow[d, "Lf;Lg"] \\
L\,(up',qv') \arrow[r, "\varphi"'] & L\,(u,v)\,;L\,(p,q)
\end{tikzcd}$$

The top path of this diagram corresponds to a pasting of Beck 2-cells for the left diagram below, and the bottom path corresponds to the pasting of Beck 2-cells for the right diagram below. Hence the commutativity of this diagram amounts to applying the Beck coherence conditions to the diagrams of pullbacks (which compose to the same pullback)

$$\begin{tikzcd}
\bullet \arrow[rr, "t'"] \arrow[dd, "m'"'] & & \bullet \arrow[r, "g"] \arrow[d, "g"'] & \bullet \arrow[d, "1"] \\
 & & \bullet \arrow[r, "1"] \arrow[d, "p"'] & \bullet \arrow[d, "p"] \\
\bullet \arrow[r, "f"] \arrow[d, "f"'] & \bullet \arrow[r, "v"] \arrow[d, "1"'] & \bullet \arrow[r, "1"] \arrow[d, "1"'] & \bullet \arrow[d, "1"] \\
\bullet \arrow[r, "1"'] & \bullet \arrow[r, "v"'] & \bullet \arrow[r, "1"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "h"] \arrow[d, "h"'] & \bullet \arrow[r, "v'"] \arrow[d, "1"] & \bullet \arrow[d, "1"] \\
\bullet \arrow[r, "1"'] \arrow[d, "p'"'] & \bullet \arrow[r, "v'"] \arrow[d, "p'"'] & \bullet \arrow[d, "p"] \\
\bullet \arrow[r, "1"'] & \bullet \arrow[r, "v"'] & \bullet
\end{tikzcd}$$

where h is the morphism of spans arising from horizontally composing the morphisms of spans f and g.

Inherited from Proposition 5.2.23 is the fact that all adjunctions in $\mathbf{Span}_{\text{iso}}(\mathcal{E})$ are of the form $(1,f) \dashv (f,1)$ up to isomorphism, where f must be invertible. To see F is gregarious, meaning that this gives an adjunction $F_\Sigma f \dashv F_\Delta f$ in $\mathscr{C}$, note that we may construct the unit and counit as the Beck 2-cells arising from the pullback squares

$$\begin{tikzcd} \bullet \arrow[r, "1"] \arrow[d, "1"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "f"'] & \bullet \arrow[d, "1"] \\ \bullet \arrow[r, "1"'] & \bullet \end{tikzcd}$$

Fully Faithful. Suppose we are given two gregarious functors $F, G \colon \mathbf{Span}_{\text{iso}}(\mathcal{E}) \to \mathscr{C}$ along with their restrictions $F_\otimes, G_\otimes$ and F_Δ, G_Δ and families of Beck 2-cells ${}^F\mathfrak{b}$ and ${}^G\mathfrak{b}$.

We first check the assignment of icons is surjective. Suppose we are given icons $\alpha \colon F_\otimes \to G_\otimes$ and $\beta \colon F_\Delta \to G_\Delta$ such that (5.4.3) holds. Then we may define an icon $\gamma \colon F \to G$ on each span (s,t) by

$$\begin{tikzcd} F(s,t) \arrow[rr, "\gamma_{s,t}"] \arrow[d, "\varphi"'] & & G(s,t) \arrow[d, "\psi"] \\ F_\otimes t \cdot F_\Delta s \arrow[rr, "\alpha_t * \beta_s"'] & & G_\otimes t \cdot G_\Delta s \end{tikzcd} \tag{5.4.6}$$

where φ and ψ are the appropriate oplax constraint cells (necessarily invertible above). Now (5.4.3) forces γ to be locally natural, as it suffices to check naturality on generating 2-cells, that is diagrams such as

$$\begin{tikzcd} & \bullet \arrow[dl, "f"'] \arrow[dr, "f"] \arrow[dd, "f"] & \\ \bullet & & \bullet \\ & \bullet \arrow[ul, "1"] \arrow[ur, "1"'] & \end{tikzcd}$$

with f invertible (this only needs trivial pullbacks corresponding to $\mathfrak{b}^{f,f}_{1,1}$). For checking γ is an icon, the identity condition on γ is from that of α and β. The composition condition is precisely (5.4.3).

We now check that the assignment of icons is injective. Suppose two given icons σ, δ both restrict to icons α and β. Then since the icons σ and δ respect the composite of the spans

$$\begin{tikzcd} & \bullet \arrow[dl, "s"'] \arrow[dr, "1"] & \\ \bullet & & \bullet \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[dl, "1"'] \arrow[dr, "t"] & \\ \bullet & & \bullet \end{tikzcd}$$

both σ and δ must satisfy (5.4.6) (in place of γ) and so are equal.

Restrictions. It is clear from the above that the oplax constraints are invertible precisely when the Beck data is invertible. □

5.5 Universal properties of polynomials with cartesian 2-cells

In this section we prove the universal property of the bicategory of polynomials with cartesian 2-cells, denoted $\mathbf{Poly}_c\left(\mathcal{E}\right)$. We will keep the proof as analogous to the case of spans as possible, though it still becomes somewhat more complicated.

5.5.1 Stating the universal property

This universal property of $\mathbf{Poly}_c\left(\mathcal{E}\right)$ turns out to be an amalgamation of that of $\mathbf{Span}\left(\mathcal{E}\right)$ and $\mathbf{Span}_{\mathrm{iso}}\left(\mathcal{E}\right)$; in particular to give a pseudofunctor $\mathbf{Poly}_c\left(\mathcal{E}\right) \to \mathscr{C}$ is to give a pair of pseudofunctors

$$\mathbf{Span}\left(\mathcal{E}\right) \to \mathscr{C}, \qquad \mathbf{Span}_{\mathrm{iso}}\left(\mathcal{E}\right) \to \mathscr{C}$$

which "Δ-agree" , that is coincide on objects and on spans of the form

$$Y \xleftarrow{f} X \xrightarrow{1_X} X$$

with an additional condition asking that certain "distributivity morphisms" be invertible. For the purposes of the proof we will give a slightly different but equivalent description, for which we will need the following definitions.

Definition 5.5.1. Given a category $\mathcal{E}$ with chosen pullbacks, we may define the category of *lax Beck triples* from $\mathcal{E}$ to a bicategory $\mathscr{C}$, denoted $\mathbf{LaxBeckTriple}\left(\mathcal{E},\mathscr{C}\right)$. An object consists of a triple of pseudofunctors which agree on objects

$$F_\Sigma \colon \mathcal{E} \to \mathscr{C}, \qquad F_\Delta \colon \mathcal{E}^{\mathrm{op}} \to \mathscr{C}, \qquad F_\otimes \colon \mathcal{E} \to \mathscr{C}$$

such that $F_\Sigma f \dashv F_\Delta f$ for all morphisms f in $\mathcal{E}$, along with "Beck data" denoted by ${}^F\mathfrak{b}$ and consisting of for each pullback square

$$\begin{tikzcd} \bullet \arrow[r, "f'"] \arrow[d, "g'"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r, "F_\otimes f'"] \arrow[d, phantom, shift left=4ex, "\Downarrow \mathfrak{b}^{f',g'}_{f,g}"] & \bullet \\ \bullet \arrow[u, "F_\Delta g'"] \arrow[r, "F_\otimes f"'] & \bullet \arrow[u, "F_\Delta g"'] \end{tikzcd} \tag{5.5.1}$$

in $\mathcal{E}$ as on the left, a 2-cell as on the right subject to the binary and nullary Beck coherence conditions as in Definition 5.4.1.

A morphism $\left(F_\Sigma, F_\Delta, F_\otimes, {}^F\mathfrak{b}\right) \to \left(G_\Sigma, G_\Delta, G_\otimes, {}^G\mathfrak{b}\right)$ in this category consists of an invertible icon $\beta \colon F_\Delta \Rightarrow G_\Delta$ and icon $\gamma \colon F_\otimes \Rightarrow G_\otimes$ such that for each pullback square in $\mathcal{E}$ as above, the diagram

$$\begin{tikzcd}
F_\otimes f' \cdot F_\Delta g' \arrow[r, "\gamma_{f'} * \beta_{g'}"] \arrow[d, "{}^F\mathfrak{b}^{f',g'}_{f,g}"'] & G_\otimes f' \cdot G_\Delta g' \arrow[d, "{}^G\mathfrak{b}^{f',g'}_{f,g}"] \\
F_\Delta g \cdot F_\otimes f \arrow[r, "\beta_g * \gamma_f"'] & G_\Delta g \cdot G_\otimes f
\end{tikzcd} \tag{5.5.2}$$

commutes.

There are a number of conditions which may be imposed on a lax Beck triple; these are defined as follows.

Definition 5.5.2. We say a lax Beck triple $\left(F_\Sigma, F_\Delta, F_\otimes, {}^F\mathfrak{b}\right)$ from $\mathcal{E}$ to a bicategory $\mathscr{C}$ is a *Beck triple* if both:

1. the $\Delta\otimes$ condition holds; meaning each component of the Beck data $\mathfrak{b}^{f',g'}_{f,g}$ is invertible;
2. the $\Sigma\Delta$ condition holds; meaning each component of the F_Σ-F_Δ Beck data is invertible[10];

Furthermore, we say such a Beck triple is a *distributive Beck triple* if in addition:

3. the $\Sigma\otimes$ condition (distributivity condition) holds; meaning that for any distributivity pullback in $\mathcal{E}$ as on the left below

$$\begin{tikzcd}
\bullet \arrow[r, "q"] \arrow[d, "p"'] & \bullet \arrow[dd, "r"] \\
\bullet \arrow[d, "u"'] & \\
\bullet \arrow[r, "f"'] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \bullet \arrow[rr, "F_\otimes q"] & & \bullet \arrow[drr, "F_\Sigma r"] & & \\
\bullet \arrow[ur, "F_\Delta p"] \arrow[r, "\mathrm{id}"] \arrow[dr, "F_\Sigma u"'] \arrow[ur, phantom, "=", near end, shift right=3] & \bullet \arrow[u, "F_\Delta p"] & \Downarrow \mathfrak{b}^{q,up'}_{f,r} & & \Downarrow \varepsilon_{Fr} & \bullet \\
 & \bullet \arrow[u, "F_\Delta u"] \arrow[rr, "F_\otimes f"'] \arrow[ul, phantom, "\Downarrow \eta_{Fu}", near start, shift left=3] & & \bullet \arrow[uu, "F_\Delta r"] \arrow[urr, "\mathrm{id}"'] & &
\end{tikzcd} \tag{5.5.3}$$

the corresponding “distributivity morphism” (defined as the pasting on the right above) is invertible.

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---

[10] This is equivalent to asking the gregarious functor $\mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ resulting from F_Σ be a pseudofunctor.

In particular, we define a *Beck triple* to be a lax Beck triple such that both conditions (1) and (2) hold, and a *DistBeck triple* to be a Beck triple also satisfying (3). We denote the corresponding subcategories of $\mathbf{LaxBeckTriple}(\mathcal{E},\mathscr{C})$ as $\mathbf{BeckTriple}(\mathcal{E},\mathscr{C})$ and $\mathbf{DistBeckTriple}(\mathcal{E},\mathscr{C})$ respectively.

There are a number of canonical embeddings into $\mathbf{Poly}_c(\mathcal{E})$ to mention; the most obvious being the embeddings

$$(-)_\Sigma : \mathcal{E} \to \mathbf{Poly}_c(\mathcal{E}), \quad (-)_\Delta : \mathcal{E}^{\mathrm{op}} \to \mathbf{Poly}_c(\mathcal{E}), \quad (-)_\otimes : \mathcal{E} \to \mathbf{Poly}_c(\mathcal{E})$$

which are defined on objects by sending an object of $\mathcal{E}$ to itself, and are defined on each morphism in $\mathcal{E}$ by the assignments

$$\begin{array}{llll}
(-)_\Sigma : & X \xrightarrow{f} Y & \mapsto & X \xleftarrow{1_X} X \xrightarrow{1_X} X \xrightarrow{f} Y \\
(-)_\Delta : & X \xrightarrow{f} Y & \mapsto & Y \xleftarrow{f} X \xrightarrow{1_X} X \xrightarrow{1_X} X \\
(-)_\otimes : & X \xrightarrow{f} Y & \mapsto & Y \xleftarrow{1_X} X \xrightarrow{f} Y \xrightarrow{1_Y} Y
\end{array}$$

We also have the inclusion $(-)_{\Sigma\Delta} : \mathbf{Span}(\mathcal{E}) \to \mathbf{Poly}_c(\mathcal{E})$ of spans into polynomials given by the assignment

$$\begin{tikzcd}
& \bullet \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, "f"] & & & & \bullet \arrow[r, "\mathrm{id}"] \arrow[dl, "s"'] \arrow[dd, "f"] & \bullet \arrow[dr, "t"] \arrow[dd, "f"] & \\
\bullet & & \bullet & \mapsto & \bullet & & & \bullet \\
& \bullet \arrow[ul, "u"] \arrow[ur, "v"'] & & & & \bullet \arrow[r, "\mathrm{id}"'] \arrow[ul, "u"] & \bullet \arrow[ur, "v"'] &
\end{tikzcd}$$

The less obvious embedding $(-)_{\Delta\otimes} : \mathbf{Span}_{\mathrm{iso}}(\mathcal{E}) \to \mathbf{Poly}_c(\mathcal{E})$ is the canonical embedding of spans with invertible 2-cells into polynomials, given by the assignment

$$\begin{tikzcd}
& \bullet \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, "f"] & & & & \bullet \arrow[r, "t"] \arrow[dl, "s"'] \arrow[dd, "f"] & \bullet \arrow[dr, "\mathrm{id}"] \arrow[dd, "\mathrm{id}"] & \\
\bullet & & \bullet & \mapsto & \bullet & & & \bullet \\
& \bullet \arrow[ul, "u"] \arrow[ur, "v"'] & & & & \bullet \arrow[r, "v"'] \arrow[ul, "u"] & \bullet \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

where one must note the appropriate square is a pullback since f is invertible.

We will need to consider gregarious functors which restrict to pseudofunctors on the embeddings we have just defined, and so we make the following definition.

Definition 5.5.3. Let $\mathcal{E}$ be a locally cartesian closed category, let $\mathscr{C}$ be a bicategory and form the category $\mathbf{Greg}\left(\mathbf{Poly}_c(\mathcal{E}),\mathscr{C}\right)$. We define $\mathbf{Greg}_\otimes\left(\mathbf{Poly}_c(\mathcal{E}),\mathscr{C}\right)$ as the subcategory of gregarious functors $F\colon \mathbf{Poly}_c(\mathcal{E}) \to \mathscr{C}$ such that the restriction $F_{\otimes}\colon \mathcal{E} \to \mathscr{C}$ is pseudo. Define $\mathbf{Greg}_{\Sigma\Delta,\Delta\otimes}\left(\mathbf{Poly}_c(\mathcal{E}),\mathscr{C}\right)$ as the subcategory of gregarious functors for which both restrictions $F_{\Sigma\Delta}\colon \mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ and $F_{\Delta\otimes}\colon \mathbf{Span}_{\mathrm{iso}}(\mathcal{E}) \to \mathscr{C}$ are pseudo.

Remark 5.5.4. Note that a gregarious functor $F\colon \mathbf{Poly}_c(\mathcal{E}) \to \mathscr{C}$ automatically restricts to pseudofunctors F_{Σ} and F_{Δ}. This is why we have omitted these conditions. Also note that oplax constraints of the form

$$F\left(\begin{tikzcd} & \bullet \arrow[dl, "s"'] \arrow[r, "t"] & \bullet \arrow[dr, "\mathrm{id}"] & \\ \bullet & & & \bullet \end{tikzcd}\right) \to F\left(\begin{tikzcd} & \bullet \arrow[dl, "s"'] \arrow[r, "\mathrm{id}"] & \bullet \arrow[dr, "\mathrm{id}"] & \\ \bullet & & & \bullet \end{tikzcd}\right); F\left(\begin{tikzcd} & \bullet \arrow[dl, "\mathrm{id}"'] \arrow[r, "t"] & \bullet \arrow[dr, "\mathrm{id}"] & \\ \bullet & & & \bullet \end{tikzcd}\right)$$

are automatically invertible by gregariousness.

Definition 5.5.5. Given a category $\mathcal{E}$ with pullbacks, we define

$$\mathbf{Greg}\left(\mathbf{Span}(\mathcal{E}),\mathscr{C}\right) \times_{\Delta} \mathbf{Greg}_{\otimes,\Delta}\left(\mathbf{Span}_{\mathrm{iso}}(\mathcal{E}),\mathscr{C}\right)$$

to be the full subcategory of

$$\mathbf{Greg}\left(\mathbf{Span}(\mathcal{E}),\mathscr{C}\right) \times \mathbf{Greg}_{\otimes,\Delta}\left(\mathbf{Span}_{\mathrm{iso}}(\mathcal{E}),\mathscr{C}\right)$$

consisting of pairs $H\colon \mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ and $K\colon \mathbf{Span}_{\mathrm{iso}}(\mathcal{E}) \to \mathscr{C}$ which coincide on objects and on spans of the form

$$\begin{tikzcd} Y & X \arrow[l, "f"'] \arrow[r, "1_X"] & X \end{tikzcd}$$

Noting that this forces $H_{\Sigma}f \dashv H_{\Delta}f = K_{\Delta}f$ for all morphisms f in $\mathcal{E}$, we denote by Ξ the assignment of such a H and K to the lax Beck triple

$$H_{\Sigma}\colon \mathcal{E} \to \mathscr{C}, \qquad K_{\Delta}\colon \mathcal{E}^{\mathrm{op}} \to \mathscr{C}, \qquad K_{\otimes}\colon \mathcal{E} \to \mathscr{C}$$

with K_{Δ}-$K_{\otimes}$ Beck data ${}^{K}\mathfrak{b}$ given by Proposition 5.4.3.

We now have enough to state the universal property of polynomials.

Theorem 5.5.6 (Universal Properties of Polynomials: Cartesian Setting)**.** *Given a locally cartesian closed category $\mathcal{E}$ with chosen pullbacks and distributivity pullbacks, denote by Υ*

the composite operation

$$\begin{tikzcd}
\mathbf{Greg}_{\otimes}\left(\mathbf{Poly}_{c}\left(\mathcal{E}\right),\mathscr{C}\right) \arrow[d] \\
\mathbf{Greg}\left(\mathbf{Span}\left(\mathcal{E}\right),\mathscr{C}\right)\times_{\Delta}\mathbf{Greg}_{\otimes,\Delta}\left(\mathbf{Span}_{\mathrm{iso}}\left(\mathcal{E}\right),\mathscr{C}\right) \arrow[d] \\
\mathbf{LaxBeckTriple}\left(\mathcal{E},\mathscr{C}\right)
\end{tikzcd}$$

where the first operation is composition with the embeddings $(-)_{\Sigma\Delta}$ *and* $(-)_{\Delta\otimes}$*, and the second operation is* Ξ *from Definition 5.5.5. Then* Υ *defines the equivalence of categories*

$$\mathbf{Greg}_{\otimes}\left(\mathbf{Poly}_{c}\left(\mathcal{E}\right),\mathscr{C}\right)\simeq\mathbf{LaxBeckTriple}\left(\mathcal{E},\mathscr{C}\right)$$

which restricts to the equivalence

$$\mathbf{Greg}_{\Sigma\Delta,\Delta\otimes}\left(\mathbf{Poly}_{c}\left(\mathcal{E}\right),\mathscr{C}\right)\simeq\mathbf{BeckTriple}\left(\mathcal{E},\mathscr{C}\right)$$

and further restricts to the equivalence

$$\mathbf{Icon}\left(\mathbf{Poly}_{c}\left(\mathcal{E}\right),\mathscr{C}\right)\simeq\mathbf{DistBeckTriple}\left(\mathcal{E},\mathscr{C}\right)$$

for any bicategory $\mathscr{C}$*.*

Remark 5.5.7. There are five other equivalences of categories since each of the three independent conditions $\Sigma\Delta$, $\Delta\otimes$ and $\Sigma\otimes$ of Definition 5.5.2 may or may not be enforced (giving a total of eight conditions). However, as the three above appear to be the most useful, we will not mention the others.

5.5.2 Proving the universal property

Before proving Theorem 5.5.6 we will need to show that given a lax Beck triple $\mathcal{E}\to\mathscr{C}$ one may reconstruct an oplax functor $\mathbf{Poly}_{c}\left(\mathcal{E}\right)\to\mathscr{C}$. The following lemma and subsequent propositions describe this construction. Also note that we are keeping the proof as similar as possible to the case of spans, starting with the below lemma which is the analogue of Lemma 5.3.3.

Lemma 5.5.8. *Let* $\mathcal{E}$ *be a locally cartesian closed category seen as a locally discrete 2-category, and let* $\mathscr{C}$ *be a bicategory. Suppose we are given a lax Beck triple consisting of pseudofunctors*

$$F_{\Sigma}\colon\mathcal{E}\to\mathscr{C},\qquad F_{\Delta}\colon\mathcal{E}^{\mathrm{op}}\to\mathscr{C},\qquad F_{\otimes}\colon\mathcal{E}\to\mathscr{C}$$

and Beck 2-cells $\mathfrak{b}$*. We may then define local functors*

$$L_{X,Y} : \mathbf{Poly}_c(\mathcal{E})_{X,Y} \to \mathscr{C}_{LX,LY}, \qquad X, Y \in \mathcal{E}$$

by the assignment $T \longrightarrow FT$ *on objects, and*

$$\begin{tikzcd}
& E \arrow[r, "p"] \arrow[dl, "s"'] \arrow[d, "f"] \arrow[dr, phantom, "\mathrm{pb}"] & B \arrow[d, "g"'] \arrow[dr, "t"] & & & & E \arrow[r, "F_\otimes p"] & B \arrow[dr, "F_\Sigma t"] & \\
X & M \arrow[l, "u"] \arrow[r, "q"'] & N \arrow[r, "v"'] & Y & \mapsto & X \arrow[ur, "F_\Delta s"] \arrow[dr, "F_\Delta u"'] \arrow[r, phantom, "\Downarrow\alpha"] & M \arrow[u, "F_\Delta f" description] \arrow[r, "F_\otimes q"'] \arrow[ur, phantom, "\Downarrow\mathfrak{b}"] & N \arrow[u, "F_\Delta g" description] \arrow[r, phantom, "\Downarrow\gamma"] \arrow[ur, "F_\Sigma v"'] & Y
\end{tikzcd}$$

on morphisms, where α *is the mate of the isomorphism on the left below*

$$\begin{tikzcd}
FE \arrow[r, "1_{FE}"] \arrow[d, "F_\Sigma s"'] \arrow[dr, phantom, "\cong"] & FE \arrow[d, "F_\Sigma u \cdot F_\Sigma f"] & & FB \arrow[r, "F_\Sigma t"] \arrow[d, "F_\Sigma g"'] \arrow[dr, phantom, "\cong"] & FY \arrow[d, "1_{FY}"] \\
FX \arrow[r, "1_{FX}"'] & FX & & FN \arrow[r, "F_\Sigma v"'] & FY
\end{tikzcd}$$

under the adjunctions $F_\Sigma s \dashv F_\Delta s$ *and* $F_\Sigma u \cdot F_\Sigma f \dashv F_\Delta f \cdot F_\Delta u$*,* γ *is the mate of the isomorphism on the right above under the adjunctions* $F_\Sigma g \dashv F_\Delta g$ *and* $1_{FY} \dashv 1_{FY}$*, and* $\mathfrak{b}^{p,f}_{q,g}$ *(simply denoted* $\mathfrak{b}$ *for convenience) is the component of the Beck data at the given pullback.*

Proof. The local functor $L_{X,Y}$ sends the components of the composite

$$\begin{tikzcd}
& E \arrow[r, "p"] \arrow[dl, "s"'] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dr, "t"] & \\
X & M \arrow[l, "u"'] \arrow[r, "q"] & N \arrow[r, "v"] & Y \\
& T \arrow[ul, "m"] \arrow[u, "h"] \arrow[r, "r"'] & S \arrow[u, "k"'] \arrow[ur, "n"'] &
\end{tikzcd}$$

to the top and bottom halves of the pasting diagram below:

$$\begin{tikzcd}
& FE \arrow[r, "F_\otimes p"] & FB \arrow[dr, "F_\Sigma t"] & \\
FX \arrow[ur, "F_\Delta s"] \arrow[r, "F_\Delta u" description] \arrow[dr, "F_\Delta m"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha_1"] \arrow[r, phantom, shift right=4, "\Downarrow\alpha_2"] & FM \arrow[u, "F_\Delta f" description] \arrow[r, "F_\otimes q" description] \arrow[ur, phantom, "\Downarrow\mathfrak{b}"] & FN \arrow[u, "F_\Delta g" description] \arrow[r, "F_\Sigma v" description] \arrow[r, phantom, shift left=4, "\Downarrow\gamma_1"] \arrow[r, phantom, shift right=4, "\Downarrow\gamma_2"] & FY \\
& FT \arrow[u, "F_\Delta h" description] \arrow[r, "F_\otimes r"'] \arrow[ur, phantom, "\Downarrow\mathfrak{b}"] & FS \arrow[u, "F_\Delta k" description] \arrow[ur, "F_\Sigma n"'] &
\end{tikzcd}$$

To see this is functorial, we insert an unlabeled constraint of F_Δ and its inverse on both the left and right side of the above diagram, giving the pasting below

$$\begin{tikzcd}
& FE \arrow[r, "\mathrm{id}"] & FE \arrow[r, "\mathrm{id}"] & FE \arrow[r, "F_{\otimes}p"] & FB \arrow[r, "\mathrm{id}"] & FB \arrow[r, "\mathrm{id}"] & FB \arrow[dr, "F_{\Sigma}t"] & \\
FX \arrow[ur, "F_{\Delta}s"] \arrow[r, "F_{\Delta}u"] \arrow[dr, "F_{\Delta}m"'] & FM \arrow[u, "F_{\Delta}f"'] \arrow[ul, phantom, "\Downarrow\alpha_1"] \arrow[dl, phantom, "\Downarrow\alpha_2"] \arrow[r, phantom, "\cong"] & {} \arrow[r, phantom, "\cong"] & FM \arrow[u, "F_{\Delta}f"'] \arrow[r, "F_{\otimes}q"] \arrow[ur, phantom, "\Downarrow\mathfrak{b}"] & FN \arrow[u, "F_{\Delta}g"'] \arrow[r, phantom, "\cong"] & {} \arrow[r, phantom, "\cong"] & FN \arrow[u, "F_{\Delta}g"'] \arrow[r, "F_{\Sigma}v"] \arrow[ur, phantom, "\Downarrow\gamma_1"] \arrow[dr, phantom, "\Downarrow\gamma_2"] & FY \\
& FT \arrow[u, "F_{\Delta}h"'] \arrow[r, "\mathrm{id}"'] & FT \arrow[uu, "F_{\Delta}hf" description] \arrow[r, "\mathrm{id}"'] & FT \arrow[u, "F_{\Delta}h"'] \arrow[r, "F_{\otimes}r"'] \arrow[ur, phantom, "\Downarrow\mathfrak{b}"] & FS \arrow[u, "F_{\Delta}k"'] \arrow[r, "\mathrm{id}"'] & FS \arrow[uu, "F_{\Delta}kg" description] \arrow[r, "\mathrm{id}"'] & FS \arrow[u, "F_{\Delta}k"'] \arrow[ur, "F_{\Sigma}n"'] &
\end{tikzcd}$$

and then apply the vertical double pullback condition on Beck data and use functoriality of mates. This shows that the above diagram is $L_{X,Y}$ applied to the composite. That the identity maps are preserved is similar to the case of spans, but using the vertical nullary pullback condition on Beck 2-cells $\mathfrak{b}$. ☐

As in the case of spans, it will be helpful to recall from Chapter 4 the reduced description of an oplax structure on local functors out of the bicategory of polynomials.

Proposition 5.5.9. *Let $\mathcal{E}$ be a locally cartesian closed category and denote by $\mathbf{Poly}_c(\mathcal{E})$ the bicategory of polynomials in $\mathcal{E}$ with cartesian 2-cells. Let $\mathscr{C}$ be a bicategory. Then to give an oplax functor*

$$L\colon \mathbf{Poly}_c(\mathcal{E}) \to \mathscr{C}$$

is to give a locally defined functor

$$L_{X,Y}\colon \mathbf{Poly}_c(\mathcal{E})_{X,Y} \to \mathscr{C}_{LX,LY}, \qquad X,Y \in \mathcal{E}$$

with comultiplication and counit maps

$$\Phi_{s,p_1,h,p_2,t}\colon L(s,p,t) \to L(s,p_1,h)\,;L(h,p_2,t), \qquad \Lambda_h\colon L(h,1,h) \to 1_{LX}$$

for every respective diagram in $\mathcal{E}$

$$\begin{tikzcd}
& E \arrow[dl, "s"'] \arrow[r, "p_1"] & T \arrow[d, "h"] \arrow[r, "p_2"] & B \arrow[dr, "t"] & \\
X & & Y & & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
& T \arrow[dl, "h"'] \arrow[r, "\mathrm{id}"] & T \arrow[dr, "h"] & \\
X & & & X
\end{tikzcd}$$

where we assert $p = p_1;p_2$ on the left, such that:

1. *for any morphisms of polynomials as below*

$$\begin{tikzcd}
& R \arrow[r, "q"] \arrow[dl, "u"'] \arrow[dd, "f"] & I \arrow[dd, "g"] \arrow[dr, "v"] & \\
X & & & Z \\
& E \arrow[ul, "s"] \arrow[r, "p"'] & B \arrow[ur, "t"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \arrow[r, "q_1"] \arrow[dl, "u"'] \arrow[dd, "f"] & S \arrow[dd, "c"] \arrow[dr, "k"] & \\
X & & & Y \\
& E \arrow[ul, "s"] \arrow[r, "p_1"'] & T \arrow[ur, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& S \arrow[r, "q_2"] \arrow[dl, "k"'] \arrow[dd, "c"] & I \arrow[dd, "g"] \arrow[dr, "v"] & \\
Y & & & Z \\
& T \arrow[ul, "h"] \arrow[r, "p_2"'] & B \arrow[ur, "t"'] &
\end{tikzcd}$$

we have the commuting diagram

$$\begin{tikzcd}
L(u,q,v) \arrow[r, "\Phi_{u,q_1,k,q_2,v}"] \arrow[d, "L(f,g)"'] & L(u,q_1,k)\,;L(k,q_2,v) \arrow[d, "L(f,c);L(c,g)"] \\
L(s,p,t) \arrow[r, "\Phi_{s,p_1,h,p_2,t}"'] & L(s,p_1,h)\,;L(h,p_2,t)
\end{tikzcd}$$

2. *for any morphism of polynomials as on the left below*

$$\begin{tikzcd}
& R \arrow[r, "\mathrm{id}"] \arrow[dl, "s"'] \arrow[dd, "f"] & R \arrow[dd, "f"'] \arrow[dr, "s"] & \\
X & & \mathrm{pb} & Z \\
& T \arrow[ul, "t"] \arrow[r, "\mathrm{id}"'] & T \arrow[ur, "t"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
L(s,1,s) \arrow[rr, "L(f,f)"] \arrow[dr, "\Lambda_s"'] & & L(t,1,t) \arrow[dl, "\Lambda_t"] \\
& 1_{LX} &
\end{tikzcd}$$

the diagram on the right above commutes;

3. *for all diagrams of the form*

$$\begin{tikzcd}
& O \arrow[r, "a"] \arrow[dl, "s"'] & G \arrow[r, "b"] \arrow[d, "h"'] & H \arrow[r, "c"] \arrow[d, "k"] & K \arrow[dr, "t"] & \\
W & & X & Y & & Z
\end{tikzcd}$$

in $\mathcal{E}$, we have the commuting diagram

$$\begin{tikzcd}
L(s,a;b;c,t) \arrow[r, equal] \arrow[d, "\Phi_{s,a,h,b;c,t}"'] & L(s,a;b;c,t) \arrow[d, "\Phi_{s,a;b,k,c,t}"] \\
L(s,a,h)\,;L(h,b;c,t) \arrow[d, "L(s,a,h);\Phi_{h,b,k,c,t}"'] & L(s,a;b,k)\,;L(k,c,t) \arrow[d, "\Phi_{s,a,h,b,k};L(k,c,t)"] \\
L(s,a,h)\,;(L(h,b,k)\,;L(k,c,t)) \arrow[r, "\mathrm{assoc}"'] & (L(s,a,h)\,;L(h,b,k))\,;L(k,c,t)
\end{tikzcd}$$

4. *for all polynomials (s, p, t) the diagrams*

$$\begin{tikzcd}
& L(s,1,s)\,;L(s,p,t) \arrow[dr, "\Lambda_s;L(s,p,t)"] & \\
L(s,p,t) \arrow[ur, "\Phi_{s,1,s,p,t}"] \arrow[rr, "\text{unitor}"'] & & 1_{LX};L(s,p,t)
\end{tikzcd}$$

$$\begin{tikzcd}
& L(s,p,t)\,;L(t,1,t) \arrow[dr, "L(s,p,t);\Lambda_t"] & \\
L(s,p,t) \arrow[ur, "\Phi_{s,p,t,1,t}"] \arrow[rr, "\text{unitor}"'] & & L(s,p,t)\,;1_{LY}
\end{tikzcd}$$

commute.

We now prove that the locally defined functor L above may be endowed with an oplax structure.

Lemma 5.5.10. *Let $\mathcal{E}$ be a locally cartesian closed category seen as a locally discrete 2-category, and let $\mathscr{C}$ be a bicategory. Suppose we are given a lax Beck triple*

$$F_\Sigma\colon \mathcal{E} \to \mathscr{C}, \qquad F_\Delta\colon \mathcal{E}^{\mathrm{op}} \to \mathscr{C}, \qquad F_\otimes\colon \mathcal{E} \to \mathscr{C}$$

with Beck 2-cells $\mathfrak{b}$. Then the locally defined functor

$$L_{X,Y}\colon \mathbf{Poly}_c(\mathcal{E})_{X,Y} \to \mathscr{C}_{LX,LY}, \qquad X, Y \in \mathcal{E}$$

as in Lemma 5.5.8 canonically admits the structure of an oplax functor.

Proof. By Proposition 5.5.9, to equip the locally defined functor L with an oplax structure is to give comultiplication maps $\Phi_{s,p_1,h,p_2,t}\colon L(s,p,t) \to L(s,p_1,h)\,;L(h,p_2,t)$ and counit maps $\Lambda_h\colon L(h,1,h) \to 1_{LX}$ for all diagrams of the respective forms, where $p = p_2 p_1$,

$$\begin{tikzcd}
& E \arrow[r, "p_1"] \arrow[dl, "s"'] & T \arrow[r, "p_2"] \arrow[d, "h"] & B \arrow[dr, "t"] & \\
X & & Y & & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
& T \arrow[r, "\mathrm{id}"] \arrow[dl, "h"'] & T \arrow[dr, "h"] & \\
X & & & X
\end{tikzcd}$$

satisfying naturality, associativity, and unitary conditions. To do this, we take each $\Phi_{s,p_1,h,p_2,t}$ to be the pasting

$$\begin{tikzcd}
FX \arrow[r, "F_\Delta s"] & FE \arrow[r, "F_\otimes p_1"] \arrow[rrrr, bend left=60, "F_\otimes p"] \arrow[rrrr, phantom, bend left=40, "\cong"] & FT \arrow[r, "F_\Sigma h"] \arrow[rr, bend left=50, "1_{FT}"] \arrow[rr, phantom, bend left=25, "\Downarrow\eta_{Fh}"] & FY \arrow[r, "F_\Delta h"] & FT \arrow[r, "F_\otimes p_2"] & FB \arrow[r, "F_\Sigma t"] & FZ
\end{tikzcd}$$

and each Λ_h to be the pasting

$$\begin{tikzcd}
& FT \arrow[r, bend left, "F_\otimes 1_T"] \arrow[r, bend right, "1_{FT}"'] \arrow[r, phantom, "\cong"] & FT \arrow[dr, "F_\Sigma h"] & \\
FX \arrow[ur, "F_\Delta h"] \arrow[rrr, "1_{FX}"'] \arrow[rrr, phantom, "\Downarrow \varepsilon_{Fh}", yshift=4ex] & & & FX
\end{tikzcd}$$

Associativity of comultiplication is almost trivial; indeed, given a diagram of the form

$$\begin{tikzcd}
& O \arrow[r, "a"] \arrow[dl, "s"'] & G \arrow[r, "b"] \arrow[d, "h"'] & H \arrow[r, "c"] \arrow[d, "k"] & K \arrow[dr, "t"] & \\
W & & X & Y & & Z
\end{tikzcd}$$

both paths in the associativity of comultiplication condition compose to

$$\begin{tikzcd}
FW \arrow[r, "F_\Delta s"] & FO \arrow[r, "F_\otimes a"] \arrow[rrrrrrr, bend left=40, "F_\otimes cba"] \arrow[rrrrrrr, phantom, "\cong", yshift=6ex] & FG \arrow[r, "F_\Sigma h"] \arrow[rr, bend left, "1_{FG}"] \arrow[rr, phantom, "\Downarrow \eta_{Fh}", yshift=2ex] & FX \arrow[r, "F_\Delta h"] & FG \arrow[r, "F_\otimes b"] & FH \arrow[r, "F_\Sigma k"] \arrow[rr, bend left, "1_{FH}"] \arrow[rr, phantom, "\Downarrow \eta_{Fk}", yshift=2ex] & FY \arrow[r, "F_\Delta k"] & FH \arrow[r, "F_\otimes c"] & FK \arrow[r, "F_\Sigma t"] & FZ
\end{tikzcd}$$

by associativity of the constraints of $F_\otimes$. The unitary axioms are also almost trivial, a consequence of the triangle identities for an adjunction and the unitary axioms on $F_\otimes$.

For the naturality condition, suppose we are given a triple of cartesian morphisms of polynomials

$$\begin{tikzcd}
& R \arrow[r, "q"] \arrow[dl, "u"'] \arrow[dd, "f"] & I \arrow[dr, "v"] \arrow[dd, "g"] & \\
X & & & Z \\
& E \arrow[ul, "s"] \arrow[r, "p"'] & B \arrow[ur, "t"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \arrow[r, "q_1"] \arrow[dl, "u"'] \arrow[dd, "f"] & S \arrow[dr, "k"] \arrow[dd, "c"] & \\
X & & & Y \\
& E \arrow[ul, "s"] \arrow[r, "p_1"'] & T \arrow[ur, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& S \arrow[r, "q_2"] \arrow[dl, "k"'] \arrow[dd, "c"] & I \arrow[dr, "v"] \arrow[dd, "g"] & \\
Y & & & Z \\
& T \arrow[ul, "h"] \arrow[r, "p_2"'] & B \arrow[ur, "t"'] &
\end{tikzcd}$$

and consider the diagram

$$\begin{tikzcd}
L(u,q,v) \arrow[r, "\Phi_{u,q_1,k,q_2,v}"] \arrow[d, "L(f,g)"'] & L(u,q_1,k)\,;L(k,q_2,v) \arrow[d, "L(f,c);L(c,g)"] \\
L(s,p,t) \arrow[r, "\Phi_{s,p_1,h,p_2,t}"'] & L(s,p_1,h)\,;L(h,p_2,t)
\end{tikzcd}$$

Now the top composite is

$$\begin{tikzcd}
& FR \arrow[r, "F_{\otimes}q_1"] \arrow[rrrr, bend left, "F_{\otimes}q"] \arrow[rrrr, phantom, bend left=20, "\cong"] & FS \arrow[rr, "1_{FR}"] \arrow[dr, "F_{\Sigma}k"] \arrow[drr, phantom, "\Downarrow\eta_{Fk}"] & & FS \arrow[r, "F_{\otimes}q_2"] & FI \arrow[dr, "F_{\Sigma}v"] & \\
FX \arrow[ur, "F_{\Delta}u"] \arrow[dr, "F_{\Delta}s"'] \arrow[r, phantom, "\Downarrow"] & \arrow[r, phantom, "\Downarrow\mathfrak{b}"] & \arrow[r, phantom, "\Downarrow"] & FY \arrow[ur, "F_{\Delta}k"] \arrow[dr, "F_{\Delta}h"] \arrow[r, phantom, "\Downarrow"] & \arrow[r, phantom, "\Downarrow\mathfrak{b}"] & \arrow[r, phantom, "\Downarrow"] & FZ \\
& FE \arrow[r, "F_{\otimes}p_1"] \arrow[uu, "F_{\Delta}f"] & FT \arrow[uu, "F_{\Delta}c"] \arrow[ur, "F_{\Sigma}h"] & & FT \arrow[r, "F_{\otimes}p_2"] \arrow[uu, "F_{\Delta}c"] & FT \arrow[uu, "F_{\Delta}g"] \arrow[ur, "F_{\Sigma}t"'] &
\end{tikzcd}$$

where the unlabeled 2-cells are as in Lemma 5.5.8, and one may rewrite the pasting of the three middle triangles above as an "identity square" and pasting with η_{Fh}. It follows that this is equal to the bottom composite given by the pasting

$$\begin{tikzcd}
& FR \arrow[rrrr, "F_{\otimes}q"] & & & & FI \arrow[dr, "F_{\Sigma}v"] & \\
FX \arrow[ur, "F_{\Delta}u"] \arrow[dr, "F_{\Delta}s"'] \arrow[r, phantom, "\Downarrow"] & & & \Downarrow\mathfrak{b} & & \arrow[r, phantom, "\Downarrow"] & FZ \\
& FE \arrow[uu, "F_{\Delta}f"] \arrow[rrrr, "F_{\otimes}p"] \arrow[dr, "F_{\otimes}p_1"'] & & \arrow[d, phantom, "\cong"] & & FB \arrow[uu, "F_{\Delta}g"] \arrow[ur, "F_{\Sigma}t"'] & \\
& & FT \arrow[rr, "1_{FT}"] \arrow[dr, "F_{\Sigma}h"'] & \arrow[d, phantom, "\Downarrow\eta_{Fh}"] & FT \arrow[ur, "F_{\otimes}p_2"'] & & \\
& & & FY \arrow[ur, "F_{\Delta}h"'] & & &
\end{tikzcd}$$

using the horizontal binary axiom on elements of $\mathfrak{b}$; thus showing naturality of comultiplication. Naturality of counits is similar to the case of spans (except that one must use the horizontal nullary axiom on elements of $\mathfrak{b}$) and so will be omitted. □

It will be useful to have a description of the oplax constraint cells φ corresponding to our comultiplication maps Φ. This is described by the following lemma.

Lemma 5.5.11. *Let the oplax functor $L\colon \mathbf{Poly}_c(\mathcal{E}) \to \mathscr{C}$ be constructed as in Proposition 5.5.10. Then the binary oplax constraint cell on L at a composite of polynomials constructed as below*

$$\begin{tikzcd}
& & H \arrow[r, "p_1"] \arrow[dl, "w"'] \arrow[d, phantom, "\text{pb}"] & M \arrow[r, "p_2"] \arrow[dl, "x"] \arrow[dr, "y"'] & K \arrow[dr, "z"] \arrow[d, phantom, "\text{pb}"] & & \\
& A \arrow[r, "m"] \arrow[dl, "a"'] & B \arrow[dr, "b"] & & C \arrow[r, "n"] \arrow[dl, "c"'] & D \arrow[dr, "d"] & \\
X & & & Y & & & Z
\end{tikzcd} \tag{5.5.4}$$

is given by the pasting

$$\begin{tikzcd}
& & FH \arrow[rr, "F_\otimes p_1"] \arrow[rrrrrrrr, bend left, "F_\otimes p"] \arrow[rrrrrrrr, phantom, bend left=15, "\cong"] & & FM \arrow[rrrr, "1_{FM}"] \arrow[rrrr, phantom, "\Downarrow\eta_{Fh}"'] \arrow[rrd, "F_\Sigma h"] & & & & FM \arrow[rr, "F_\otimes p_2"] & & FK \arrow[rrd, "F_\Sigma dz"] & & \\
FX \arrow[rru, "F_\Delta aw"] \arrow[rrd, "F_\Delta a"'] & \Downarrow & & \Downarrow\mathfrak{b} & & \Downarrow & FY \arrow[rru, "F_\Delta h"] \arrow[rrd, "F_\Delta c"] & \Downarrow & & \Downarrow\mathfrak{b} & & \Downarrow & FZ \\
& & FA \arrow[uu, "F_\Delta w"] \arrow[rr, "F_\otimes m"] & & FB \arrow[uu, "F_\Delta x"] \arrow[rru, "F_\Sigma b"] & & & & FC \arrow[uu, "F_\Delta y"] \arrow[rr, "F_\otimes n"] & & FD \arrow[uu, "F_\Delta z"] \arrow[rru, "F_\Sigma d"] & &
\end{tikzcd} \tag{5.5.5}$$

where $p = p_2 p_1$ *and* $h = bx = cy$.

Proof. Given composable polynomials (a,m,b) and (c,n,d) the composite is given by the terminal diagram as in (5.5.4) . We then have an induced diagonal

$$\delta_{ac',h,db'} \colon \ (aw,p,dz) \longrightarrow (aw,p_1,h)\,;(h,p_2,dz)$$

and morphisms $(w,x) : \ (aw,p_1,h) \longrightarrow (a,m,b)$ and $(y,z) : \ (h,p_2,dz) \longrightarrow (c,n,d)$ for which

$$(a,m,b)\,;(c,n,d) \xrightarrow{\delta_{ac',h,db'}} (aw,p_1,h)\,;(h,p_2,dz) \xrightarrow{(w,x);(y,z)} (a,m,b)\,;(c,n,d)$$

is the identity on $(a,m,b)\,;(c,n,d)$. It follows that the binary oplax constraint cell corresponding to the comultiplication maps Φ, namely

$$\varphi_{(a,m,b),(c,n,d)} \colon L\left((a,m,b)\,;(c,n,d)\right) \longrightarrow L\,(a,m,b)\,; L\,(c,n,d)$$

is given by (5.5.5). □

We now know enough for a complete proof of the universal properties of the polynomials with cartesian 2-cells.

Proof of Theorem 5.5.6. We consider the assignment

$$\Upsilon \colon \mathbf{Greg}_\otimes\left(\mathbf{Poly}_c\,(\mathcal{E})\,,\mathscr{C}\right) \longrightarrow \mathbf{LaxBeckTriple}\,(\mathcal{E},\mathscr{C})$$

of Theorem 5.5.6, i.e. given a gregarious functor $F \colon \mathbf{Poly}_c\,(\mathcal{E}) \longrightarrow \mathscr{C}$ which restricts to a pseudofunctor when composed with $(-)_\otimes$, we extract, via Theorem 5.4.4, the pseudofunctors F_Σ, F_Δ and $F_\otimes$ equipped with the Beck data $\mathfrak{b}$. This data defines a lax Beck triple $\mathcal{E} \longrightarrow \mathscr{C}$.

Well defined. Given an icon $\alpha \colon F \Rightarrow G \colon \mathbf{Poly}_c\,(\mathcal{E}) \longrightarrow \mathscr{C}$ we know $\alpha_\Delta \colon F_\Delta \Rightarrow G_\Delta$ is invertible, as it is a restriction of an icon $\alpha_{\Sigma\Delta} \colon F_{\Sigma\Delta} \Rightarrow G_{\Sigma\Delta} \colon \mathbf{Span}\,(\mathcal{E}) \longrightarrow \mathscr{C}$ which is necessarily invertible by Lemma 5.3.8.

We start by proving the first universal property.

Fully faithful. That the assignment $\alpha \mapsto (\alpha_\Delta, \alpha_\otimes)$ is bijective follows from the fact the assignment $\alpha_{\Sigma\Delta} \mapsto \alpha_\Delta$ is bijective, and the necessary commutativity of

$$\begin{tikzcd}
F(s,p,t) \arrow[r, "\varphi"] \arrow[d, "\alpha_{(s,p,t)}"'] & F(s,1,1) \,;\, F(1,p,1) \,;\, F(1,1,t) \arrow[d, "\alpha_{(s,1,1)};\alpha_{(1,p,1)};\alpha_{(1,1,t)}"] \\
G(s,p,t) \arrow[r, "\psi"'] & G(s,1,1) \,;\, G(1,p,1) \,;\, G(1,1,t)
\end{tikzcd} \tag{5.5.6}$$

where φ and ψ must be invertible constraints since F and G are gregarious.

Again, that $\left(\alpha_{\Sigma_f}^{-1}\right)^* = \alpha_{\Delta_f}$ is forced by Lemma 5.2.21 and one need only check that any collection

$$\alpha_{s,p,t} \colon F(s,p,t) \to G(s,p,t)$$

satisfying this property and (5.5.6) necessarily defines an icon.

We omit the calculation showing α is locally natural. Indeed, this calculation is almost the same as in the proof of Theorem 5.3.2, except we must interchange a Beck 2-cell with the components α_Δ and $\alpha_\otimes$ using the condition (5.5.2).

To see that such an α then defines an icon, note that each $\Phi_{s,p_1,h,p_2,t}$ may be decomposed as the commuting diagram

$$\begin{tikzcd}
F(s,p,t) \arrow[r, "\Phi_{s,p_1,h,p_2,t}"] \arrow[d, "\Phi_{s,p_1,1,p_2,t}"'] & F(s,p_1,h) \,;\, F(h,p_2,t) \\
F(s,p_1,1) \,;\, F(1,p_2,t) \arrow[r, "F(s,p_1,1);\Phi_{1,1,h,1,1};F(1,p_2,t)"'] & F(s,p_1,1) \,;\, F(1,1,h) \,;\, F(h,1,1) \,;\, F(1,p_2,t) \arrow[u, "\Phi^{-1}_{s,p_1,1,1,h};\Phi^{-1}_{h,1,1,p_2,t}"']
\end{tikzcd}$$

and so the commutativity of the diagram

$$\begin{tikzcd}
F(s,p,t) \arrow[r, "\Phi_{s,p_1,h,p_2,t}"] \arrow[d, "\alpha_{s,p,t}"'] & F(s,p_1,h) \,;\, F(h,p_2,t) \arrow[d, "\alpha_{s,p_1,h};\alpha_{h,p_2,t}"] \\
G(s,p,t) \arrow[r, "\Psi_{s,p_1,h,p_2,t}"'] & G(s,p_1,h) \,;\, G(h,p_2,t)
\end{tikzcd}$$

amounts to checking that the pastings

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "F_\Delta s"] \arrow[r, bend right, "G_\Delta s"'] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left, "F_\otimes p_1"] \arrow[r, bend right, "G_\otimes p_1"'] \arrow[r, phantom, "\Downarrow"] \arrow[rrrr, bend left=60, "F_\otimes p"] \arrow[rrrr, phantom, bend left=40, "\Downarrow\varphi"] & \bullet \arrow[r, bend left, "F_\Sigma h"] \arrow[r, bend right, "G_\Sigma h"'] \arrow[r, phantom, "\Downarrow"] \arrow[rr, bend left=70, "\mathrm{id}"] \arrow[rr, phantom, bend left=30, "\Downarrow\eta_{Gh}"] & \bullet \arrow[r, bend left, "F_\Delta h"] \arrow[r, bend right, "G_\Delta h"'] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left, "F_\otimes p_2"] \arrow[r, bend right, "G_\otimes p_2"'] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left, "F_\Sigma t"] \arrow[r, bend right, "G_\Sigma t"'] \arrow[r, phantom, "\Downarrow"] & \bullet
\end{tikzcd}$$

and

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "F_\Delta s"] \arrow[r, bend right, "G_\Delta s"'] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, "G_\otimes p_1"'] \arrow[rrrr, bend left=75, "F_\otimes p"] \arrow[rrrr, phantom, bend left=65, "\Downarrow\alpha_\otimes"] \arrow[rrrr, bend left=55, "G_\otimes p"] \arrow[rrrr, phantom, bend left=45, "\Downarrow\psi"] & \bullet \arrow[r, "G_\Sigma h"'] \arrow[rr, bend left=70, "\mathrm{id}"] \arrow[rr, phantom, bend left=30, "\Downarrow\eta_{Gh}"] & \bullet \arrow[r, "G_\Delta h"'] & \bullet \arrow[r, "G_\otimes p_2"'] & \bullet \arrow[r, bend left, "F_\Sigma t"] \arrow[r, bend right, "G_\Sigma t"'] \arrow[r, phantom, "\Downarrow"] & \bullet
\end{tikzcd}$$

agree. This is almost the same calculation as in spans except here we must use that $\alpha_\otimes$ is an icon.

Essentially surjective. Suppose we are given a lax Beck triple $(F_\Sigma, F_\Delta, F_\otimes, \mathfrak{b})$. Then by Proposition 5.5.10, we get a normal oplax functor $F\colon \mathbf{Poly}_c(\mathcal{E}) \to \mathscr{C}$ (which is gregarious as a consequence Proposition 5.2.25, and clearly restricts to a pseudofunctor on $\otimes$), and this constructed F clearly restricts to the same lax Beck triple when Υ is applied.

We now prove the remaining two universal properties, seen as restrictions of the first.

Restrictions. It is clear that for any $\mathrm{Greg}_\otimes$ functor $F\colon \mathbf{Poly}_c(\mathcal{E}) \to \mathscr{C}$ we may write $F \cong \widetilde{F}$ where $\widetilde{F}$ is given by sending F to its lax Beck triple and recovering a map $\widetilde{F}\colon \mathbf{Poly}_c(\mathcal{E}) \to \mathscr{C}$ under the above equivalence.

Also it is clear that F (or equivalently $\widetilde{F}$) restricts to pseudofunctors on $\Sigma\Delta$ and $\Delta\otimes$ precisely when this lax Beck triple is a Beck triple. This is seen by using the general expression for an oplax constraint cell (5.5.5) on composites of polynomials $(s, 1, t)\,;(u, 1, v)$ and $(s, t, 1)\,;(u, v, 1)$.

Now as each oplax constraint cell may be constructed from "Beck composites" as above and "distributivity composites" of the form $(1, 1, u)\,;(1, f, 1)$ (by the proof of [17, Prop. 1.12]), it follows that asking F be pseudo corresponds to asking that, in addition, the oplax constraint cells for composites $(1, 1, u)\,;(1, f, 1)$ be invertible. But this is precisely the $\Sigma\otimes$ distributivity condition. □

5.6 Universal properties of polynomials with general 2-cells

In this section we prove the universal property of the bicategory of polynomials with general 2-cells, denoted **Poly** $(\mathcal{E})$. As this bicategory is not generic, the methods of the previous section do not directly apply. However, as composition in $\mathbf{Poly}_c(\mathcal{E})$ and **Poly** $(\mathcal{E})$ is the same we can still apply some results of the previous section to help prove this universal property.

5.6.1 Stating the universal property

The universal property of **Poly** $(\mathcal{E})$ ends up being simpler to state than that of $\mathbf{Poly}_c(\mathcal{E})$ due to the existence of more adjunctions. To state this property we will first require a strengthening of the notions of "sinister" and "Beck" pseudofunctors as described in Definition 5.2.18. For the following definition, recall that the categories of such pseudofunctors and invertible icons are denoted **Sin** $(\mathcal{E},\mathscr{C})$ and **Beck** $(\mathcal{E},\mathscr{C})$ respectively.

Definition 5.6.1. Let $\mathcal{E}$ be a category with pullbacks, and let $\mathscr{C}$ be a bicategory. We denote by **2Sin** $(\mathcal{E},\mathscr{C})$ the subcategory of **Sin** $(\mathcal{E},\mathscr{C})$ consisting of pseudofunctors $F\colon \mathcal{E} \to \mathscr{C}$ for which Ff has two successive right adjoints for every morphism $f \in \mathcal{E}$. We denote by **2Beck** $(\mathcal{E},\mathscr{C})$ the subcategory of **2Sin** $(\mathcal{E},\mathscr{C})$ consisting of those pseudofunctors which in addition satisfy the Beck condition.

Remark 5.6.2. The above Beck condition is on the pair F_Σ-F_Δ, but one could also ask a Beck condition on the pair F_Δ-F_Π. The reason for not using the latter is that the Beck 2-cells (arising from adjunctions $F_\Delta f \dashv F_\Pi f$) are not in the direction required for constructing a lax Beck triple, and are invertible if and only if the former Beck 2-cells are invertible.

The following lemma will be needed to describe a distributivity condition which may be imposed on such pseudofunctors.

Lemma 5.6.3. *Let $\mathcal{E}$ be a locally cartesian closed category seen as a locally discrete 2-category, and let $\mathscr{C}$ be a bicategory. Suppose $F\colon \mathcal{E} \to \mathscr{C}$ is a given 2-Beck pseudofunctor, and for each morphism $f \in \mathcal{E}$ define $F_\Sigma f := Ff$, take $F_\Delta f$ to be a chosen right adjoint of Ff (choosing F_Δ to strictly preserve identities), and take $F_\Pi f$ to be a chosen right adjoint of $F_\Delta f$ (again choosing F_Π to strictly preserve identities). We may then define a Beck triple with underlying pseudofunctors*

$$F_\Sigma\colon \mathcal{E} \to \mathscr{C}, \qquad F_\Delta\colon \mathcal{E}^{\mathrm{op}} \to \mathscr{C}, \qquad F_\Pi\colon \mathcal{E} \to \mathscr{C}$$

and for each pullback as on the left below, the mate of the middle isomorphism below whose existence is asserted by the Beck condition

$$\begin{tikzcd}
\bullet \arrow[r, "f'"] \arrow[d, "g'"'] & \bullet \arrow[d, "g"] & & \bullet \arrow[d, "F_\Sigma g'"'] \arrow[dr, phantom, "\cong"] & \bullet \arrow[l, "F_\Delta f'"'] \arrow[d, "F_\Sigma g"] & & \bullet \arrow[r, "F_\Pi f'"] \arrow[dr, phantom, "\Downarrow \mathfrak{b}^{f',g'}_{f,g}"] & \bullet \\
\bullet \arrow[r, "f"'] & \bullet & & \bullet & \bullet \arrow[l, "F_\Delta f"] & & \bullet \arrow[u, "F_\Delta g'"] \arrow[r, "F_\Pi f"'] & \bullet \arrow[u, "F_\Delta g"']
\end{tikzcd} \tag{5.6.1}$$

defining the Beck data as on the right above.

Proof. One needs to check that the defined Beck data satisfies the necessary coherence conditions, but this trivially follows from functoriality of mates. Also, every component of the Beck data $\mathfrak{b}^{f',g'}_{f,g}$ defined as in the above lemma must be invertible. This is since an isomorphism of left adjoints must correspond to an isomorphism of right adjoints under the mates correspondence. □

It will be useful to give the Beck triples arising this way a name, and so we make the following definition.

Definition 5.6.4. We call a Beck triple $\mathcal{E} \to \mathscr{C}$ *cartesian* if for every morphism $f \in \mathcal{E}$ there exists adjunctions $F_\Sigma f \dashv F_\Delta f \dashv F_\Pi f$ and the $\Delta\Pi$ Beck data corresponds to the $\Sigma\Delta$ data via the mates correspondence as in (5.6.1).

We may also ask that a cartesian Beck triple (or the corresponding 2-Beck functor) satisfies a distributivity condition.

Definition 5.6.5. Given the assumptions and data of Lemma 5.6.3, we say a 2-Beck pseudofunctor $F \colon \mathcal{E} \to \mathscr{C}$ satisfies the *distributivity condition* if the cartesian Beck triple recovered from Lemma 5.6.3 satisfies the distributivity condition of Definition 5.5.2 (meaning this cartesian Beck triple is a DistBeck triple).

Similar to the case of $\mathbf{Poly}_c(\mathcal{E})$, we again have embeddings

$$(-)_\Sigma : \mathcal{E} \to \mathbf{Poly}(\mathcal{E}), \quad (-)_\Delta : \mathcal{E}^{\mathrm{op}} \to \mathbf{Poly}(\mathcal{E}), \quad (-)_\Pi : \mathcal{E} \to \mathbf{Poly}(\mathcal{E})$$

The main difference here is that with these embeddings we have triples of adjunctions $f_\Sigma \dashv f_\Delta \dashv f_\Pi$ for every morphism $f \in \mathcal{E}$.

Trivially we have the inclusion $(-)_{\Sigma\Delta} : \mathbf{Span}(\mathcal{E}) \to \mathbf{Poly}(\mathcal{E})$ of spans into polynomials given by the assignment

$$\begin{tikzcd}
 & \bullet \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, "f"] & & & & \bullet \arrow[r, "\mathrm{id}"] \arrow[dl, "s"'] & \bullet \arrow[d, equal] \arrow[dr, "t"] & \\
\bullet & & \bullet & \mapsto & \bullet & \bullet \arrow[u, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] \arrow[d, "f"] & \bullet \arrow[d, "f"'] & \bullet \\
 & \bullet \arrow[ul, "u"] \arrow[ur, "v"'] & & & & \bullet \arrow[ul, "u"] \arrow[r, "\mathrm{id}"'] & \bullet \arrow[ur, "v"'] &
\end{tikzcd}$$

The less obvious embedding $(-)_{\Delta\Pi} : \mathbf{Span}\,(\mathcal{E})^{\mathrm{co}} \to \mathbf{Poly}\,(\mathcal{E})$ is the canonical embedding of spans with reversed 2-cells into polynomials, given by the assignment

$$\begin{tikzcd}
 & \bullet \arrow[dl, "s"'] \arrow[dr, "t"] & & & & \bullet \arrow[r, "t"] \arrow[dl, "s"'] & \bullet \arrow[d, equal] \arrow[dr, "\mathrm{id}"] & \\
\bullet & & \bullet & \mapsto & \bullet & \bullet \arrow[u, "f"] \arrow[r, "v"] \arrow[d, "\mathrm{id}"'] & \bullet \arrow[d, "\mathrm{id}"] & \bullet \\
 & \bullet \arrow[ul, "u"] \arrow[ur, "v"'] \arrow[uu, "f"] & & & & \bullet \arrow[ul, "u"] \arrow[r, "v"'] & \bullet \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

We now have enough to state the universal property of polynomials.

Theorem 5.6.6 (Universal Properties of Polynomials: General Setting)**.** *Given a locally cartesian closed category $\mathcal{E}$ with chosen pullbacks and distributivity pullbacks, composition with the canonical embedding $(-)_\Sigma : \mathcal{E} \to \mathbf{Poly}\,(\mathcal{E})$ defines the equivalence of categories*

$$\mathbf{Greg}\,(\mathbf{Poly}\,(\mathcal{E}), \mathscr{C}) \simeq \mathbf{2Beck}\,(\mathcal{E}, \mathscr{C})$$

which restricts to the equivalence

$$\mathbf{Icon}\,(\mathbf{Poly}\,(\mathcal{E}), \mathscr{C}) \simeq \mathbf{DistBeck}\,(\mathcal{E}, \mathscr{C})$$

for any bicategory $\mathscr{C}$.

Remark 5.6.7. One might ask if there is a universal property without the Beck condition being required. The problem is that if the restrictions to $\mathbf{Span}\,(\mathcal{E})$ and $\mathbf{Span}\,(\mathcal{E})^{\mathrm{co}}$ are only required gregarious, but not pseudo, we do not have a canonical way to construct the necessary $\Delta\Pi$ Beck data $\mathfrak{b}$, and so such a universal property would be unnatural.

5.6.2 Proving the universal property

Before proving Theorem 5.6.6 we will need to show how to reconstruct a gregarious functor $\mathbf{Poly}\,(\mathcal{E}) \to \mathscr{C}$ from a 2-Beck pseudofunctor $\mathcal{E} \to \mathscr{C}$. The following proposition describes this construction.

Proposition 5.6.8. *Let $\mathcal{E}$ be a locally cartesian closed category seen as a locally discrete 2-category, and let $\mathscr{C}$ be a bicategory. Suppose $F\colon \mathcal{E} \longrightarrow \mathscr{C}$ is a given 2-Beck pseudofunctor, and for each morphism $f \in \mathcal{E}$ define $F_\Sigma f := Ff$, take $F_\Delta f$ to be a chosen right adjoint of Ff (choosing F_Δ to strictly preserve identities), and take $F_\Pi f$ to be a chosen right adjoint of $F_\Delta f$ (again choosing F_Π to strictly preserve identities). We may then:*

1. *define a lax Beck triple as in Lemma 5.6.3;*
2. *define a gregarious functor $\overline{L}\colon \mathbf{Poly}_c(\mathcal{E}) \longrightarrow \mathscr{C}$ satisfying the $\Sigma\Delta$ and $\Delta\Pi$ Beck conditions;*
3. *define local functors*

$$L\colon \mathbf{Poly}(\mathcal{E})_{X,Y} \longrightarrow \mathscr{C}_{LX,LY}, \qquad X,Y \in \mathcal{E}$$

assigning each general morphism of polynomials

$$\begin{tikzcd}
 & E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[d, equal] \arrow[dr, "t"] & \\
X & S \arrow[u, "e"] \arrow[r, "pe"] \arrow[d, "f"'] & B \arrow[d, "g"] & Y \\
 & M \arrow[ul, "u"] \arrow[r, "q"'] & N \arrow[ur, "v"'] &
\end{tikzcd}$$

to the pasting

$$\begin{tikzcd}
 & FE \arrow[r, "F_\Pi p"] \arrow[d, "F_\Delta e"] \arrow[dr, phantom, "\Downarrow\mathfrak{m}"] & FB \arrow[d, equal] \arrow[dr, "F_\Sigma t"] & \\
FX \arrow[ur, "F_\Delta s"] \arrow[dr, "F_\Delta u"'] \arrow[r, phantom, "\Downarrow\alpha"] & FS \arrow[r, "F_\Pi pe"] & FB \arrow[r, phantom, "\Downarrow\gamma"] & FY \\
 & FM \arrow[u, "F_\Delta f"] \arrow[r, "F_\Pi q"'] \arrow[ur, phantom, "\Downarrow\mathfrak{b}"] & FN \arrow[u, "F_\Delta g"'] \arrow[ur, "F_\Sigma v"'] &
\end{tikzcd}$$

where of the diagrams

$$\begin{tikzcd}
FS \arrow[r, "1_{FS}"] \arrow[d, "F_\Sigma s\cdot F_\Sigma e"'] \arrow[dr, phantom, "\cong"] & FS \arrow[d, "F_\Sigma u\cdot F_\Sigma f"] \\
FX \arrow[r, "1_{FX}"'] & FX
\end{tikzcd}
\qquad
\begin{tikzcd}
FE \arrow[r, "F_\Pi p"] \arrow[dr, phantom, "\cong"] & FB \\
FS \arrow[u, "F_\Pi e"] \arrow[r, "F_\Pi pe"'] & FB \arrow[u, "1_{FB}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FB \arrow[r, "F_\Sigma t"] \arrow[d, "F_\Sigma g"'] \arrow[dr, phantom, "\cong"] & FY \arrow[d, "1_{FY}"] \\
FN \arrow[r, "F_\Sigma v"'] & FY
\end{tikzcd}$$

   (a) *α is constructed as the mate of the left diagram under the adjunctions $F_\Sigma s \cdot F_\Sigma e \dashv F_\Delta e \cdot F_\Delta s$ and $F_\Sigma u \cdot F_\Sigma f \dashv F_\Delta f \cdot F_\Delta u$;*

(b) $\mathfrak{m}$ *is constructed as the mate of the middle diagram under the adjunctions* $F_{\Delta}e \dashv F_{\Pi}e$ *and* $1_{FB} \dashv 1_{FB}$*;*

(c) $\mathfrak{b}$ *is the component of the Beck data at the given pullback;*

(d) γ *is the mate of the isomorphism on the right above under the adjunctions* $F_{\Sigma}g \dashv F_{\Delta}g$ *and* $1_{FY} \dashv 1_{FY}$*.*

4. *define a gregarious functor* $L\colon \mathbf{Poly}\,(\mathcal{E}) \to \mathscr{C}$*.*

Proof. We prove the different parts of the statement separately.

Part 1. See Lemma 5.6.3.

Part 2. It then follows from Theorem 5.5.6 that this cartesian Beck triple gives rise to a gregarious functor $\overline{L}\colon \mathbf{Poly}_c\,(\mathcal{E}) \to \mathscr{C}$. The $\Sigma\Delta$ invertibility condition translates to an $\Delta\Pi$ invertibility condition via the mates correspondence; an isomorphism of left adjoints must correspond to an isomorphism of right adjoints. Therefore each component of the Beck data $\mathfrak{b}$ is invertible.

Part 3. The goal here is to show that we have local functors

$$L\colon \mathbf{Poly}\,(\mathcal{E})_{X,Y} \to \mathscr{C}_{LX,LY}, \qquad X, Y \in \mathcal{E}$$

We first note, for well definedness, that given two general morphisms of polynomials as below

$$\begin{tikzcd}
& E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] \arrow[d, equal] & \\
X & S_1 \arrow[u, "e_1"] \arrow[r, "pe_1"] \arrow[d, "f_1"] & B \arrow[d, "g"] & Y \\
& M \arrow[r, "q"'] \arrow[ul, "u"] & N \arrow[ur, "v"'] &
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}
& E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] \arrow[d, equal] & \\
X & S_2 \arrow[u, "e_2"] \arrow[r, "pe_2"] \arrow[d, "f_2"] & B \arrow[d, "g"] & Y \\
& M \arrow[r, "q"'] \arrow[ul, "u"] & N \arrow[ur, "v"'] &
\end{tikzcd}$$

equivalent in that there exists an isomorphism $\nu\colon S_1 \to S_2$ such that $f_2\nu = f_1$ and $e_2\nu = e_1$, it follows from a straightforward functoriality of mates calculation that $L_{X,Y}$ assigns both morphisms of polynomials to equal pastings.

As local functoriality with respect to cartesian morphisms was shown in Lemma 5.5.8, local functoriality with respect to “triangle morphisms” is a straightforward functoriality of mates calculation, and the case of a triangle morphism followed by a cartesian morphism is almost by definition, it suffices to consider the case of a cartesian morphism followed by a triangle morphism (the only non trivial case to consider).

Suppose we are given a composite of polynomial morphisms as on the left below

$$\begin{tikzcd}
& E \arrow[r, "p"] \arrow[dl, "s"'] \arrow[d, "f"] \arrow[dr, phantom, "\mathrm{pb}"] & B \arrow[d, "g"] \arrow[dr, "t"] & \\
X & M \arrow[l, "u"'] \arrow[r, "q"] & N \arrow[r, "v"] & Y \\
& J \arrow[ul, "a"] \arrow[u, "e"] \arrow[r, "r"'] & N \arrow[u, equal] \arrow[ur, "b"'] &
\end{tikzcd}
=
\begin{tikzcd}
& E \arrow[rr, "p"] \arrow[dl, "s"'] & & B \arrow[d, equal] \arrow[dr, "t"] & \\
X & P \arrow[u, "e'"] \arrow[r, "e'"] \arrow[d, "f'"] \arrow[dr, phantom, "\mathrm{pb}"] & E \arrow[r, "p"] \arrow[d, "f"] \arrow[dr, phantom, "\mathrm{pb}"] & B \arrow[d, "g"] & Y \\
& J \arrow[ul, "a"] \arrow[r, "e"'] & M \arrow[r, "q"'] & N \arrow[ur, "b"'] &
\end{tikzcd}$$

evaluated as the diagram on the right above. We must check that

$$\begin{tikzcd}
& FE \arrow[r, "F_{\Pi}p"] & FB \\
FX \arrow[ur, bend left, "F_{\Delta}s"] \arrow[dr, bend right, "F_{\Delta}a"'] \arrow[r, "F_{\Delta}u"] & FM \arrow[u, "F_{\Delta}f"] \arrow[d, "F_{\Delta}e"] \arrow[r, "F_{\Pi}q"] \arrow[ul, phantom, "\cong"] \arrow[dl, phantom, "\cong"] \arrow[ur, phantom, "\Downarrow\mathfrak{b}"] \arrow[dr, phantom, "\Downarrow\mathfrak{m}"] & FN \arrow[u, "F_{\Delta}g"'] \arrow[d, equal] \\
& FJ \arrow[r, "F_{\Pi}r"'] & FN
\end{tikzcd}
=
\begin{tikzcd}
& FE \arrow[r, "F_{\Pi}p"] \arrow[d, "F_{\Delta}e'"] \arrow[dr, phantom, "\Downarrow\mathfrak{m}"] & FB \arrow[d, equal] \\
FX \arrow[ur, bend left, "F_{\Delta}s"] \arrow[dr, bend right, "F_{\Delta}a"'] \arrow[r, phantom, "\Downarrow\alpha"] & FP \arrow[r, "F_{\Pi}pe'"] \arrow[dr, phantom, "\Downarrow\mathfrak{b}"] & FB \\
& FJ \arrow[u, "F_{\Delta}f'"] \arrow[r, "F_{\Pi}r"'] & FN \arrow[u, "F_{\Delta}g"']
\end{tikzcd}
\qquad (5.6.2)$$

To see this, we paste both sides with the inverse of the $\mathfrak{b}$ appearing on the right above, and check that have an equality. Starting with the observation that the left side pasted with this inverse is the left diagram below, we see

$$\begin{tikzcd}
& FE \arrow[rr, "F_{\Pi}p"] \arrow[dr, phantom, "\Downarrow\mathfrak{b}"] & & FB \\
& & FN \arrow[ur, "F_{\Delta}g"] \arrow[r, phantom, "\Downarrow\mathfrak{b}^{-1}"] & {} \\
FX \arrow[uur, bend left, "F_{\Delta}s"] \arrow[ddr, bend right, "F_{\Delta}a"'] \arrow[r, "F_{\Delta}u"] & FM \arrow[uu, "F_{\Delta}f"] \arrow[dd, "F_{\Delta}e"] \arrow[ur, "F_{\Pi}q"] \arrow[uul, phantom, "\cong"] \arrow[ddl, phantom, "\cong"] \arrow[dr, phantom, "\Downarrow\mathfrak{m}"] & & FE \arrow[uu, "F_{\Pi}p"'] \\
& & FM \arrow[uu, "F_{\Pi}q"] \arrow[ur, "F_{\Delta}f"] \arrow[r, phantom, "\Downarrow\mathfrak{b}^{-1}"] & {} \\
& FJ \arrow[ur, "F_{\Pi}e"] \arrow[rr, "F_{\Delta}f'"'] & & FP \arrow[uu, "F_{\Pi}e'"']
\end{tikzcd}
=
\begin{tikzcd}
& FE \arrow[rr, "F_{\Pi}p"] \arrow[drr, phantom, "="] & & FB \\
FX \arrow[ur, bend left, "F_{\Delta}s"] \arrow[dr, bend right, "F_{\Delta}a"'] \arrow[r, "F_{\Delta}u"] & FM \arrow[u, "F_{\Delta}f"] \arrow[d, "F_{\Delta}e"] \arrow[r, "\mathrm{id}", "\Downarrow\eta"'] \arrow[ul, phantom, "\cong"] \arrow[dl, phantom, "\cong"] & FM \arrow[r, "F_{\Delta}f"'] \arrow[dr, phantom, "\Downarrow\mathfrak{b}^{-1}"] & FE \arrow[u, "F_{\Pi}p"'] \\
& FJ \arrow[ur, "F_{\Pi}e"] \arrow[rr, "F_{\Delta}f'"'] & & FP \arrow[u, "F_{\Pi}e'"']
\end{tikzcd}$$

upon realizing $\mathfrak{m}$ as a whiskering of a unit and canceling the $\mathfrak{b}$. Transferring the unit along the mates correspondence gives the left diagram below

$$\begin{tikzcd}
& FE \arrow[r, "F_{\Pi}p"] & FB & \\
FX \arrow[ur, bend left, "F_{\Delta}s"] \arrow[dr, bend right, "F_{\Delta}a"'] \arrow[r, "F_{\Delta}u"] & FM \arrow[u, "F_{\Delta}f"] \arrow[d, "F_{\Delta}e"] \arrow[r, "F_{\Delta}f"] \arrow[ul, phantom, "\cong"] \arrow[dl, phantom, "\cong"] \arrow[ur, phantom, "="] \arrow[dr, phantom, "\cong"] & FE \arrow[u, "F_{\Pi}p"] \arrow[d, "F_{\Delta}e'"] \arrow[r, "\mathrm{id}"] \arrow[ur, phantom, "="] \arrow[dr, phantom, "\Downarrow\eta"] & FE \arrow[ul, bend right, "F_{\Pi}p"'] \\
& FJ \arrow[r, "F_{\Delta}f'"'] & FP \arrow[ur, bend right, "F_{\Pi}e'"'] &
\end{tikzcd}
=
\begin{tikzcd}
& FE \arrow[r, "F_{\Pi}p"] & FB \\
FX \arrow[ur, bend left, "F_{\Delta}s"] \arrow[dr, bend right, "F_{\Delta}a"'] \arrow[r, "F_{\Delta}u"] & FM \arrow[u, "F_{\Delta}f"] \arrow[d, "F_{\Delta}e"] \arrow[r, "F_{\Delta}f"] \arrow[ul, phantom, "\cong"] \arrow[dl, phantom, "\cong"] \arrow[ur, phantom, "="] \arrow[dr, phantom, "\cong"] & FE \arrow[u, "F_{\Pi}p"] \arrow[d, "F_{\Delta}e'"] \arrow[r, phantom, "\Downarrow\mathfrak{m}"] & {} \\
& FJ \arrow[r, "F_{\Delta}f'"'] & FP \arrow[u, bend right=80, "F_{\Pi}pe'"', start anchor=east, end anchor=east]
\end{tikzcd}$$

which is seen as the right diagram after writing the whiskering of the unit back in terms of $\mathfrak{m}$.

This is clearly the right side of (5.6.2) with the pasting of the 2-cell $\mathfrak{b}$ having been undone.

Part 4. The goal here is to show that this now defines a gregarious functor

$$L\colon \mathbf{Poly}\,(\mathcal{E}) \to \mathscr{C}$$

Now, as we already have a gregarious functor $\overline{L}\colon \mathbf{Poly}_c\,(\mathcal{E}) \to \mathscr{C}$, given by the restriction of L to the cartesian setting, and composition of 1-cells $\mathbf{Poly}_c\,(\mathcal{E})$ and $\mathbf{Poly}\,(\mathcal{E})$ is defined the same way, it suffices to check that the oplax constraint data φ, λ of $\overline{L}$ defines oplax constraint data on L. Indeed, φ and λ are already known to satisfy the nullary and associativity axioms and so we need only check naturality of the constraint data with respect to our larger class of 2-cells.

Taking $\theta\colon P \to P''$ and $\phi\colon Q \to Q''$ to be general morphisms of polynomials, canonically decomposed into triangle parts θ_t, ϕ_t and cartesian parts θ_c, ϕ_c, we note that to see that the left diagram commutes below

$$\begin{tikzcd}
L\,(P;Q) \arrow[r, "\varphi_{P,Q}"] \arrow[d, "L(\theta;\phi)"'] & LP;LQ \arrow[d, "L\theta;L\phi"] \\
L\,(P'';Q') \arrow[r, "\varphi_{P'',Q''}"'] & LP'';LQ''
\end{tikzcd}
\qquad
\begin{tikzcd}
L\,(P;Q) \arrow[r, "\varphi_{P,Q}"] \arrow[d, "L(\theta_t;\phi_t)"'] & LP;LQ \arrow[d, "L\theta_t;L\phi_t"] \\
L\,(P';Q') \arrow[r, "\varphi_{P',Q'}"'] \arrow[d, "L(\theta_c;\phi_c)"'] & LP';LQ' \arrow[d, "L\theta_c;L\phi_c"] \\
L\,(P'';Q') \arrow[r, "\varphi_{P'',Q''}"'] & LP'';LQ''
\end{tikzcd}$$

it suffices to check that the top square in the right diagram commutes. This is since the bottom square on the right commutes by naturality of the constraint data φ with respect to $\overline{L}$. To prove the commutativity of this square, it will be helpful to decompose further into

$$\begin{tikzcd}
L\,(P;Q) \arrow[r, "\varphi_{P,Q}"] \arrow[d, "L(\theta_t;Q)"'] & LP;LQ \arrow[d, "L\theta_t;LQ"] \\
L\,(P';Q) \arrow[r, "\varphi_{P',Q}"] \arrow[d, "L(P';\phi_t)"'] & LP';LQ \arrow[d, "LP';L\phi_t"] \\
L\,(P';Q') \arrow[r, "\varphi_{P',Q'}"'] & LP';LQ'
\end{tikzcd}$$

and so we need only prove naturality for whiskerings of triangle morphisms.

We first check the naturality condition for right whiskerings of triangle morphisms. The whiskering of such an x is constructed as the induced map x' into the pullback as in the diagram below

$$\begin{tikzcd}
\bullet \arrow[ddd, "\ell'"'] \arrow[dddrrr, bend left, "p'_1"] & & & & & & \\
& \bullet \arrow[ul, dotted, "x'"'] \arrow[rr, "p'_2"] \arrow[ddd, "\ell"'] & & \bullet \arrow[rr] \arrow[d, "e"'] & & \bullet \arrow[ddd] & \\
\bullet \arrow[dd, "s_1"'] \arrow[drr, bend left, "p_1"] & & \text{pb} & \bullet \arrow[dl, "u'"'] \arrow[dr] & \text{dpb} & & \\
& E \arrow[ul, "x"] \arrow[dl, "s_2"'] \arrow[r, "p_2"] & B \arrow[dr, "t"] & \text{pb} & M \arrow[dl, "u"'] \arrow[r, "q"] & N \arrow[dr, "v"] & \\
I & & & J & & & K
\end{tikzcd}$$

The naturality condition then amounts to checking that

$$\begin{tikzcd}
& \bullet \arrow[r, "F_\Pi p'_1"] \arrow[d, phantom, "\Downarrow\mathfrak{b}" near start, shift left=4ex] & \bullet \\
\bullet \arrow[ur, bend left, "F_\Delta s_1\ell'"] \arrow[dr, bend right, "F_\Delta s_2"'] \arrow[r, "F_\Delta s_1"] & \bullet \arrow[u, "F_\Delta\ell'"] \arrow[d, "F_\Delta x"] \arrow[r, "F_\Pi p_1"] \arrow[d, phantom, "\Downarrow\mathfrak{m}", shift left=4ex] & \bullet \arrow[u, "F_\Delta u'e"'] \arrow[d, equal] \\
& \bullet \arrow[r, "F_\Pi p_2"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
& \bullet \arrow[r, "F_\Pi p'_1"] \arrow[d, "F_\Delta x'"] \arrow[d, phantom, "\Downarrow\mathfrak{m}", shift left=4ex] & \bullet \arrow[d, equal] \\
\bullet \arrow[ur, bend left, "F_\Delta s_1\ell'"] \arrow[dr, bend right, "F_\Delta s_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet \arrow[r, "F_\Pi p'_2"] \arrow[d, phantom, "\Downarrow\mathfrak{b}", shift left=4ex] & \bullet \\
& \bullet \arrow[u, "F_\Delta\ell"] \arrow[r, "F_\Pi p_2"'] & \bullet \arrow[u, "F_\Delta u'e"']
\end{tikzcd}$$

which is similar to the calculation in (5.6.2).

We now check left whiskerings of triangle morphisms, which is significantly more complicated than the above situation. To simplify this calculation, we consider only simpler triangle morphisms of the form $x\colon (u_1, 1, 1) \to (u_2, x, 1)$. It will turn out that it suffices to consider only these simpler triangle morphisms.

To construct the left whiskering of the triangle morphism x by a polynomial we first construct the two relevant composites of polynomials as below

$$\begin{tikzcd}
\bullet \arrow[rrr, "p'_1"] \arrow[dddr, "\ell'_1"] & & & \bullet \arrow[rrr, "\text{id}"] \arrow[dddl, "u'_1"] \arrow[ddr, "t'_1"] & & & \bullet \arrow[dddl, "t'_1"] \\
& \bullet \arrow[rr, "p'_2"] \arrow[dd, "\ell'_2"] & & \bullet \arrow[rr, "k"] \arrow[d, "e"] & & \bullet \arrow[dd, "r'_2"] & \\
& & & \bullet \arrow[dl, "u'_2"] \arrow[dr, "t'_2"] & \bullet \arrow[dr, "\text{id}"] \arrow[ddl, "u_1"] & & \\
& E \arrow[r, "p"] \arrow[dl, "s"] & B \arrow[dr, "t"] & & M \arrow[u, "x"] \arrow[r, "x"] \arrow[dl, "u_2"] & V \arrow[dr, "\text{id}"] & \\
I & & & J & & & V
\end{tikzcd}$$

Now since we have a factorization of pullbacks as below

$$\begin{tikzcd}
\bullet \arrow[r, "y"] \arrow[d, "t'_2"'] & \bullet \arrow[r, "u'_1"] \arrow[d, "t'_1"'] & \bullet \arrow[d, "t"'] \\
\bullet \arrow[r, "x"] & \bullet \arrow[r, "u_1"] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "u'_2"] \arrow[d, "t'_2"'] & \bullet \arrow[d, "t"'] \\
\bullet \arrow[r, "u_2"] & \bullet
\end{tikzcd}$$

it follows that we have an induced cartesian morphism of polynomials

$$\begin{tikzcd}
\bullet \arrow[rrr, "h"] \arrow[dr, dotted, "f"] \arrow[dddr, bend right, "j"'] & & & \bullet \arrow[rrr, "y"] \arrow[d, dotted, "m"] \arrow[dddl, "u_2'"'] \arrow[dddr, "t_2'"] & & & \bullet \arrow[dl, dotted, "g"'] \arrow[dddl, bend left, "t_1'"] \\
& \bullet \arrow[rr, "p_2'"] \arrow[dd, "\ell_2'"] & & \bullet \arrow[rr, "k"] \arrow[d, "e"] & & \bullet \arrow[dd, "r_2'"] & \\
& & & \bullet \arrow[dl, "u_2'"'] \arrow[dr, "t_2'"] & & & \\
& E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & & M \arrow[r, "x"] \arrow[dl, "u_2"'] & V \arrow[dr, "\mathrm{id}"] & \\
I & & & J & & & V
\end{tikzcd}$$

where h and j are the pullback of p with u_2'. We then give the factorization of pullbacks

$$\begin{tikzcd}
\bullet \arrow[r, "\alpha"] \arrow[d, "h"] & \bullet \arrow[r, "\ell_1'"] \arrow[d, "p_1'"] & \bullet \arrow[d, "p"] \\
\bullet \arrow[r, "y"] & \bullet \arrow[r, "u_1'"] & \bullet
\end{tikzcd} = \begin{tikzcd}
\bullet \arrow[r, "j"] \arrow[d, "h"] & \bullet \arrow[d, "p"] \\
\bullet \arrow[r, "u_2'"] & \bullet
\end{tikzcd}$$

and see the morphism of polynomials resulting from this whiskering is given by

$$\begin{tikzcd}
& \bullet \arrow[r, "p_1'"] \arrow[dl, "s\ell_1'"'] & \bullet \arrow[d, equal] \arrow[dr, "t_1'"] & \\
\bullet & \bullet \arrow[u, "\alpha"] \arrow[r, "yh"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] & \bullet \\
& \bullet \arrow[ul, "s\ell_2'"] \arrow[r, "kp_2'"'] & \bullet \arrow[ur, "r_2'"'] &
\end{tikzcd}$$

The naturality condition then follows from seeing that, where $z = u_1t_1' = tu_1'$,

$$\begin{tikzcd}
& \bullet \arrow[r, "F_\Pi p_1'"] \arrow[rrrr, bend left, "F_\Pi p_1'"] & \bullet \arrow[rr, "\mathrm{id}"] \arrow[dr, "F_\Sigma z"] & & \bullet \arrow[r, "F_\Pi \mathrm{id}"] & \bullet \arrow[dr, "F_\Sigma t_1'"] & \\
\bullet \arrow[ur, "F_\Delta s\ell_1'"] \arrow[dr, "F_\Delta s"'] & & & \bullet \arrow[ur, "F_\Delta z"] \arrow[dr, "F_\Delta u_1"] \arrow[ddr, bend right, "F_\Delta u_2"'] & & & \bullet \\
& \bullet \arrow[uu, "F_\Delta \ell_1'"] \arrow[r, "F_\Pi p"] & \bullet \arrow[uu, "F_\Delta u_1'"] \arrow[ur, "F_\Sigma t"] & & \bullet \arrow[uu, "F_\Delta t_1'"] \arrow[r, "F_\Pi \mathrm{id}"] \arrow[d, "F_\Delta x"] & \bullet \arrow[uu, "F_\Delta t_1'"] \arrow[ur, "F_\Sigma \mathrm{id}"] \arrow[d, equal] & \\
& & & & \bullet \arrow[r, "F_\Pi x"'] & \bullet \arrow[uur, bend right, "F_\Sigma \mathrm{id}"'] &
\arrow[from=1-2, to=1-6, phantom, bend left=20, "\cong"]
\arrow[from=1-4, to=2-4, phantom, "\Downarrow\eta_{Fz}"]
\arrow[from=2-1, to=2-2, phantom, "\Downarrow"]
\arrow[from=2-2, to=2-3, phantom, "\Downarrow\mathfrak{b}"]
\arrow[from=2-3, to=2-4, phantom, "\Downarrow"]
\arrow[from=2-4, to=2-5, phantom, "\Downarrow"]
\arrow[from=2-5, to=2-6, phantom, "\Downarrow\mathfrak{b}"]
\arrow[from=2-6, to=2-7, phantom, "\Downarrow"]
\arrow[from=3-4, to=3-5, phantom, "\Downarrow"]
\arrow[from=3-6, to=3-7, phantom, "\Downarrow"]
\arrow[from=3-5, to=4-6, phantom, "\Downarrow\mathfrak{m}"]
\end{tikzcd}$$

is equal to the pasting, using an analogue of (5.6.2),

$$\begin{tikzcd}
& \bullet \arrow[r, "F_\Pi p'_1"] \arrow[rrrr, bend left=40, "F_\Pi p'_1"] \arrow[rrrr, phantom, bend left=20, "\cong"] & \bullet \arrow[rr, "\mathrm{id}"] \arrow[dr, "F_\Sigma z"] & {} \arrow[d, phantom, "\Downarrow\eta_{Fz}"] & \bullet \arrow[r, "F_\Pi \mathrm{id}"] \arrow[d, "F_\Delta y"] \arrow[dr, phantom, "\Downarrow\mathfrak{m}"] & \bullet \arrow[d, equal] \arrow[dr, "F_\Sigma t'_1"] & \\
\bullet \arrow[ur, "F_\Delta s\ell'_1"] \arrow[dr, "F_\Delta s"'] \arrow[r, phantom, "\Downarrow"] & {} \arrow[r, phantom, "\Downarrow\mathfrak{b}"] & {} \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[ur, "F_\Delta z"] \arrow[dr, "F_\Delta u_2"] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, "F_\Pi y"] \arrow[dr, phantom, "\Downarrow\mathfrak{b}"] & \bullet \arrow[r, phantom, "\Downarrow"] & \bullet \\
& \bullet \arrow[r, "F_\Pi p"] \arrow[uu, "F_\Delta \ell'_1"] & \bullet \arrow[uu, "F_\Delta u'_1"] \arrow[ur, "F_\Sigma t"] & & \bullet \arrow[r, "F_\Pi x"'] \arrow[u, "F_\Delta t'_2"] & \bullet \arrow[u, "F_\Delta t'_1"] \arrow[ur, "F_\Sigma \mathrm{id}"'] &
\end{tikzcd}$$

which is equal to, where $d = u_2t'_2e = tu'_2e$, noting $dm = zy$ and $u'_2 = u'_1y$,

$$\begin{tikzcd}
& \bullet \arrow[r, "F_\Pi p'_1"] \arrow[rrrr, bend left=40, "F_\Pi p'_1"] \arrow[rrrr, phantom, bend left=20, "\cong"] \arrow[d, "F_\Delta \alpha"] \arrow[dr, phantom, "\Downarrow\mathfrak{b}^{-1}"] & \bullet \arrow[rr, "\mathrm{id}"] \arrow[d, "F_\Delta y"] & {} \arrow[d, phantom, "="] & \bullet \arrow[r, "F_\Pi \mathrm{id}"] \arrow[d, "F_\Delta y"] \arrow[dr, phantom, "\Downarrow\mathfrak{m}"] & \bullet \arrow[d, equal] \arrow[ddr, bend left, "F_\Sigma t'_1"] & \\
{} \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[r, "F_\Pi h"] \arrow[dr, phantom, "\Downarrow\mathfrak{b}"] & \bullet \arrow[rr, "\mathrm{id}"] & {} \arrow[d, phantom, "="] & \bullet \arrow[r, "F_\Pi y"] \arrow[dr, phantom, "\Downarrow\mathfrak{b}"] & \bullet \arrow[dr, phantom, "\Downarrow"] & {} \\
\bullet \arrow[uur, bend left, "F_\Delta s\ell'_1"] \arrow[r, "F_\Delta s\ell'_2"] \arrow[ddr, bend right, "F_\Delta s"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[r, "F_\Pi p'_2"] \arrow[u, "F_\Delta f"] & \bullet \arrow[rr, "\mathrm{id"] \arrow[u, "F_\Delta m"] \arrow[dr, "F_\Sigma d"] & {} \arrow[d, phantom, "\Downarrow\eta_{Fd}"] & \bullet \arrow[r, "F_\Pi k"] \arrow[u, "F_\Delta m"] & \bullet \arrow[r, "F_\Sigma r'_2"] \arrow[u, "F_\Delta g"] \arrow[dr, phantom, "\Downarrow"] & \bullet \\
& {} \arrow[r, phantom, "\Downarrow\mathfrak{b}"] & {} \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[ur, "F_\Delta d"] \arrow[dr, "F_\Delta u_2"] \arrow[r, phantom, "\Downarrow"] & {} \arrow[r, phantom, "\Downarrow\mathfrak{b}"] & {} & \\
& \bullet \arrow[r, "F_\Pi p"] \arrow[uu, "F_\Delta \ell'_2"] & \bullet \arrow[uu, "F_\Delta u'_2e"] \arrow[ur, "F_\Sigma t"] & & \bullet \arrow[r, "F_\Pi x"] \arrow[uu, "F_\Delta t'_2e"] & \bullet \arrow[uu, "F_\Delta r'_2"] \arrow[uur, bend right, "F_\Sigma \mathrm{id}"'] &
\end{tikzcd}$$

finally resulting in

$$\begin{tikzcd}
& \bullet \arrow[rrrr, bend left=30, "F_\Pi p'_1"] \arrow[rrrr, phantom, bend left=15, "\Downarrow\mathfrak{m}"] \arrow[d, "F_\Delta \alpha"] & & & & \bullet \arrow[d, equal] \arrow[ddr, bend left, "F_\Sigma t'_1"] & \\
{} \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[rrrr, bend left=30, "F_\Pi yh"] \arrow[rrrr, phantom, bend left=15, "\Downarrow\mathfrak{b}"] & & & & \bullet \arrow[dr, phantom, "\Downarrow"] & {} \\
\bullet \arrow[uur, bend left, "F_\Delta s\ell'_1"] \arrow[r, "F_\Delta s\ell'_2"] \arrow[ddr, bend right, "F_\Delta s"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[r, "F_\Pi p'_2"] \arrow[u, "F_\Delta f"] \arrow[rrrr, bend left=30, "F_\Pi kp'_2"] \arrow[rrrr, phantom, bend left=15, "\cong"] & \bullet \arrow[rr, "\mathrm{id}"] \arrow[dr, "F_\Sigma d"] & {} \arrow[d, phantom, "\Downarrow\eta_{Fd}"] & \bullet \arrow[r, "F_\Pi k"] & \bullet \arrow[r, "F_\Sigma r'_2"] \arrow[u, "F_\Delta g"] \arrow[dr, phantom, "\Downarrow"] & \bullet \\
& {} \arrow[r, phantom, "\Downarrow\mathfrak{b}"] & {} \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[ur, "F_\Delta d"] \arrow[dr, "F_\Delta u_2"] \arrow[r, phantom, "\Downarrow"] & {} \arrow[r, phantom, "\Downarrow\mathfrak{b}"] & {} & \\
& \bullet \arrow[r, "F_\Pi p"] \arrow[uu, "F_\Delta \ell'_2"] & \bullet \arrow[uu, "F_\Delta u'_2e"] \arrow[ur, "F_\Sigma t"] & & \bullet \arrow[r, "F_\Pi x"] \arrow[uu, "F_\Delta t'_2e"] & \bullet \arrow[uu, "F_\Delta r'_2"] \arrow[uur, bend right, "F_\Sigma \mathrm{id}"'] &
\end{tikzcd}$$

Now, we wish to prove the naturality condition for any left whiskering of a general triangle morphism, written $P; \theta_t$. This can be written as $P; (\theta_x; R)$ for a simpler triangle morphism x as above, and so we are trying to show region (1) commutes below (suppressing associators in $\mathscr{C}$)

$$\begin{tikzcd}
L\left(P;\left(\left(u_1,1,1\right);R\right)\right) \arrow[r, "\varphi"] \arrow[d, "L(P;(\theta_x;R))"'] \arrow[dr, phantom, "(1)"] & LP;L\left(\left(u_1,1,1\right);R\right) \arrow[r, "\varphi"] \arrow[d, "LP;L(\theta_x;R)"] \arrow[dr, phantom, "(2)"] & LP;L\left(u_1,1,1\right);LR \arrow[d, "LP;L\theta_x;LR"] \\
L\left(P;\left(\left(u_2,x,1\right);R\right)\right) \arrow[r, "\varphi"'] & LP;L\left(\left(u_2,x,1\right);R\right) \arrow[r, "\varphi"'] & LP;L\left(u_2,x,1\right);LR
\end{tikzcd}$$

We now note that for the commutativity of (1) it suffices to prove the outside diagram above commutes, as both constraints φ are invertible in region (2) by gregariousness, and region (2) is known to commute as naturality for right whiskerings has been shown.

As associativity of the constraints has been verified, this is the same as showing that the outside of

$$\begin{tikzcd}
L\left(\left(P;\left(u_1,1,1\right)\right);R\right) \arrow[r, "\varphi"] \arrow[d, "L((P;\theta_x);R)"'] & L\left(P;\left(u_1,1,1\right)\right);LR \arrow[r, "\varphi"] \arrow[d, "L(P;\theta_x);LR"] & LP;L\left(u_1,1,1\right);LR \arrow[d, "LP;L\theta_x;LR"] \\
L\left(\left(P;\left(u_2,x,1\right)\right);R\right) \arrow[r, "\varphi"'] & L\left(P;\left(u_2,x,1\right)\right);LR \arrow[r, "\varphi"'] & LP;L\left(u_2,x,1\right);LR
\end{tikzcd}$$

commutes. But the left square commutes as naturality with respect to right whiskerings is known for both triangle and cartesian morphisms, and the right square above commutes by naturality of left whiskerings of θ_x. This gives the result. □

We now have enough to complete the proof of Theorem 5.6.6.

Proof of Theorem 5.6.6. We consider the assignment of Theorem 5.6.6, i.e. composition with the embedding $(-)_\Sigma : \mathcal{E} \to \mathbf{Poly}\left(\mathcal{E}\right)$ written as the assignment

$$\begin{tikzcd}
\mathbf{Poly}\left(\mathcal{E}\right) \arrow[r, bend left, "F", ""{name=U, below}] \arrow[r, bend right, "G"', ""{name=D, above}] \arrow[from=U, to=D, phantom, "\Downarrow \alpha"] & \mathscr{C} & \mapsto & \mathcal{E} \arrow[r, bend left, "F_\Sigma", ""{name=U2, below}] \arrow[r, bend right, "G_\Sigma"', ""{name=D2, above}] \arrow[from=U2, to=D2, phantom, "\Downarrow \alpha_\Sigma"] & \mathscr{C}
\end{tikzcd}$$

We start by proving the first universal property.

Well defined. That each icon is invertible is seen by restricting to spans and applying Corollary 5.3.8.

Fully faithful. That the assignment $\alpha \mapsto \alpha_\Sigma$ is injective follows from the necessary commutativity of

$$\begin{tikzcd}
F\left(s,p,t\right) \arrow[r, "\varphi"] \arrow[d, "\alpha_{(s,p,t)}"'] & F\left(s,1,1\right);F\left(1,p,1\right);F\left(1,1,t\right) \arrow[d, "\alpha_{(s,1,1)};\alpha_{(1,p,1)};\alpha_{(1,1,t)}"] \\
G\left(s,p,t\right) \arrow[r, "\psi"'] & G\left(s,1,1\right);G\left(1,p,1\right);G\left(1,1,t\right)
\end{tikzcd}$$

where φ and ψ are invertible by gregariousness, and the identities $\alpha_{\Delta_f} = \left(\alpha_{\Sigma_f}^{-1}\right)^*$ and $\alpha_{\Pi_f} = \left(\alpha_{\Delta_f}^{-1}\right)^*$ forced by Lemma 5.2.21. For surjectivity, one need only check any collection α consisting of 2-cells

$$\alpha_{s,p,t} \colon F\left(s,p,t\right) \to G\left(s,p,t\right)$$

satisfying these properties defines an icon. As composition is the same in $\mathbf{Poly}_c\left(\mathcal{E}\right)$, the compatibility of the collection α with the oplax constraint cells is the same calculation as in Section 5.5. Thus one need only check local naturality of α. As local naturality with respect to the cartesian morphisms is already known, one need only consider triangle morphisms. But local naturality with respect to triangle morphism is almost the same calculation as in the case of spans; this is expected as the triangle morphisms arise from the canonical embedding $(-)_{\Delta\Pi} : \mathbf{Span}^{\mathrm{co}}\left(\mathcal{E}\right) \to \mathbf{Poly}\left(\mathcal{E}\right)$.

Essentially surjective. Given any 2-Beck pseudofunctor $F \colon \mathcal{E} \to \mathscr{C}$ we take the gregarious functor $L \colon \mathbf{Poly}\left(\mathcal{E}\right) \to \mathscr{C}$ from Proposition 5.6.8 and note that $L_\Sigma = F$.

We now deduce the second universal property.

Restrictions. The second property is a restriction of the first. Indeed, given a pseudofunctor $L \colon \mathbf{Poly}\left(\mathcal{E}\right) \to \mathscr{C}$ the corresponding pseudofunctor $L_\Sigma \colon \mathcal{E} \to \mathscr{C}$ satisfies the distributivity condition since L_Σ is also the restriction of the pseudofunctor $\overline{L} \colon \mathbf{Poly}_c\left(\mathcal{E}\right) \to \mathscr{C}$. Moreover, given a 2-Beck pseudofunctor $F \colon \mathcal{E} \to \mathscr{C}$ which satisfies the distributivity condition, the corresponding map $\mathbf{Poly}\left(\mathcal{E}\right) \to \mathscr{C}$ is pseudo since the map $\mathbf{Poly}_c\left(\mathcal{E}\right) \to \mathscr{C}$ (with the same constraint data) arising from the cartesian Beck triple is pseudo. □

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6

An elementary view of familial pseudofunctors

Abstract

A classical result due to Diers shows that a presheaf $F\colon \mathcal{A} \to \mathbf{Set}$ on a category $\mathcal{A}$ is a coproduct of representables precisely when each connected component of F's category of elements has an initial object. Most often, this condition is imposed on a presheaf of the form $\mathcal{B}(X,T-)$ for a functor $T\colon \mathcal{A} \to \mathcal{B}$, in which case this property says that T admits generic factorisations at X, or equivalently that T has a left multiadjoint at X.

Here we generalize these results to the two dimensional setting, replacing $\mathcal{A}$ with an arbitrary bicategory $\mathscr{A}$, and $\mathbf{Set}$ with $\mathbf{Cat}$. In this two dimensional setting, simply asking that a pseudofunctor $F\colon \mathscr{A} \to \mathbf{Cat}$ be a coproduct of representables is often too strong of a condition. Instead, we will only ask that F be a lax conical colimit of representables. This is turn allows for the weaker notion of lax generic factorisations (and lax multiadjoints) for pseudofunctors of bicategories $T\colon \mathscr{A} \to \mathscr{B}$.

We also compare our lax multiadjoints to Weber's familial 2-functors, finding our description is more general (not requiring a terminal object in $\mathscr{A}$), though essentially equivalent when a terminal object does exist. Moreover, our description of lax generics allows for an equivalence between lax generic factorisations and family.

Finally, we characterize our lax multiadjoints as right lax $\mathbf{F}$-adjoints followed by locally discrete fibrations of bicategories, which in turn yields a more natural definition of parametric right adjoint pseudofunctors.

Contribution by the author

As the sole author, this paper is entirely my own work. This paper was submited to the arXiv preprint server on December 23rd 2018 and will be submitted for publication in the very near future.

6.1 Introduction

Given a category $\mathcal{A}$ and presheaf $F\colon \mathcal{A} \longrightarrow \mathbf{Set}$, it is often a natural question to ask whether this presheaf is a coproduct of representable presheaves; meaning

$$F \cong \sum_{m \in \mathfrak{M}} \mathcal{A}(P_m, -)$$

for some set $\mathfrak{M}$ and function $P_{(-)}\colon \mathfrak{M} \longrightarrow \mathcal{A}$. Such presheaves have a straightforward characterization: a presheaf F is a coproduct of representables precisely when each connected component of $\mathrm{el}\, F$ has an initial object. Said more explicitly, this means that for any (D,w) in $\mathrm{el}\, F$ there exists an (A,x) and morphism $k\colon (A,x) \longrightarrow (D,w)$ where (A,x) satisfies the following property: for any diagram in $\mathrm{el}\, F$ as below

$$\begin{tikzcd}
 & (C,z) \arrow[d, "g"] \\
(A,x) \arrow[ur, dotted, "h"] \arrow[r, "f"'] & (B,y)
\end{tikzcd}$$

there exists a $h\colon (A,x) \longrightarrow (C,z)$ such that the diagram commutes, and moreover h is the unique morphism $(A,x) \longrightarrow (C,z)$.

Of particular interest is the case where F is of the form $\mathcal{B}(X,T-)$ for a functor $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$. Here, asking that each connected component of $\mathrm{el}\, \mathcal{B}(X,T-)$ has an initial object amounts to asking that for any $w\colon X \longrightarrow TD$ there exists an $x\colon X \longrightarrow TA$ and $k\colon A \longrightarrow D$ such that $w = Tk \cdot x$, and x is "generic" meaning that it satisfies the following property: given any commuting square as on the left below

$$\begin{tikzcd}
X \arrow[r, "z"] \arrow[d, "x"'] & TB \arrow[d, "Tg"] \\
TA \arrow[r, "Tf"'] & TC
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "z"] \arrow[d, "x"'] & TB \arrow[d, "Tg"] \\
TA \arrow[ur, "Th"] \arrow[r, "Tf"'] & TC
\end{tikzcd}$$

there exists a unique $h\colon A \longrightarrow B$ such that $Th \cdot x = z$ (note that $g \cdot h = f$ can be shown as a consequence). In this case we say T admits generic factorisations, and call $x\colon X \longrightarrow TA$ a generic morphism.

The reader will notice that the above condition on T makes no mention of terminal objects, and indeed there are natural examples of generic factorisations without terminal objects, such as composition of spans in a category $\mathcal{E}$ with pullbacks

$$\mathbf{Span}\,(\mathcal{E})\,(Y,Z) \times \mathbf{Span}\,(\mathcal{E})\,(X,Y) \to \mathbf{Span}\,(\mathcal{E})\,(X,Z)\,.$$

Thus higher analogues of generic factorisations should also not require the existence of terminal objects.

It is the purpose of this paper to generalize these notions of famility to the two dimensional setting, replacing the category $\mathcal{A}$ with a bicategory $\mathscr{A}$, and replacing **Set** with **Cat**. However, this is not a straightforward generalization, as asking that a pseudofunctor $F\colon \mathscr{A} \to \mathbf{Cat}$ be a coproduct of representables is often too strong of a condition. To see why, consider the case where a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$ is such that each $\mathscr{B}\,(X,T-)$ is a coproduct of representables, meaning we have an equivalence

$$\mathscr{B}\,(X,T-) \simeq \sum_{m\in\mathfrak{M}} \mathscr{A}\,(P_m,-)$$

for some set $\mathfrak{M}$. With such an equivalence, we would then have for each 2-cell α as on the left below

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & TA & \mapsto & m, & P_m \arrow[r, bend left, "\overline{f}"] \arrow[r, bend right, "\overline{g}"'] \arrow[r, phantom, "\Downarrow\overline{\alpha}"] & A \end{tikzcd}$$

assigned to an $\overline{\alpha}\colon \overline{f} \Rightarrow \overline{g}$ as on the right above, that $f \cong T\overline{f} \cdot \delta$ and $g \cong T\overline{g} \cdot \delta$ for the same generic $\delta\colon X \to TP_m$ corresponding to the identity at P_m. This is an unreasonably strong condition: we should not expect two 1-cells to factor through the same generic δ just because there is a comparison map between them. In general, this should only be expected when the comparison map is invertible.

To address this problem, we weaken the condition on $\mathscr{B}\,(X,T-)$, now only asking that it be a *lax conical colimit of representables.* In this paper we will give a characterization of when a pseudofunctor $F\colon \mathscr{A} \to \mathbf{Cat}$ is a lax conical colimit of representables (also giving appropriate notions of generic object and morphism in this setting), and then go on to specialize this characterization to the case where F is of the form $\mathscr{B}\,(X,T-)$. We will see that in this setting, the generics are morphisms $x\colon X \to TA$ for which we have universal factorisations of any 2-cell α as on the left below

$$\begin{tikzcd} X \arrow[r, "z"] \arrow[d, "x"'] \arrow[dr, phantom, "\Uparrow\alpha"] & TB \arrow[d, "Tg"] & & X \arrow[r, "z"] \arrow[d, "x"'] \arrow[dr, phantom, "\Uparrow\gamma", near start] & TB \arrow[d, "Tg"] \\ TA \arrow[r, "Tf"'] & TC & = & TA \arrow[r, "Tf"'] \arrow[ur, "Th" description] \arrow[ur, phantom, "\Uparrow T\nu", near end, shift right=3] & TC \end{tikzcd}$$

into a diagram as on the right above. The factorization being universal means it must satisfy a number of axioms detailed later in Definition 6.5.3.

To see why admitting lax-generic factorisations is a natural condition on a pseudofunctor $T\colon \mathscr{A} \longrightarrow \mathscr{B}$, consider the problem of calculating a left extension as below

$$\begin{tikzcd}
{[\mathscr{A}^{\mathrm{op}}, \mathbf{Cat}]} \arrow[r, dotted, "\mathrm{lan}_T"] & {[\mathscr{B}^{\mathrm{op}}, \mathbf{Cat}]} \\
\mathscr{A} \arrow[u, "y_{\mathscr{A}}"] \arrow[r, "T"'] & \mathscr{B} \arrow[u, "y_{\mathscr{B}}"']
\end{tikzcd}$$

for a given pseudofunctor T (where $\mathscr{A}$ and $\mathscr{B}$ are small). In general this left extension should not be expected to have a nice form. However, if T is a pseudofunctor which admits lax-generic factorisations, so that each $\mathscr{B}(X, T-)$ is a lax conical colimit of representables, then this left extension will have a simple description. An important example of this situation is given by taking T as the canonical inclusion of a small category $\mathcal{E}$ into its bicategory of spans $\mathbf{Span}(\mathcal{E})$

$$\begin{tikzcd}
{[\mathcal{E}^{\mathrm{op}}, \mathbf{Cat}]} \arrow[r, dotted, "\mathrm{lan}_T"] & {\left[\mathbf{Span}(\mathcal{E})^{\mathrm{op}}, \mathbf{Cat}\right]} \\
\mathcal{E} \arrow[u, "y_{\mathcal{E}}"] \arrow[r, "T"'] & \mathbf{Span}(\mathcal{E}) \arrow[u, "y_{\mathbf{Span}(\mathcal{E})}"']
\end{tikzcd}$$

and forming the left extension lan_T as above, with right adjoint res_T given by restricting along T. Now, recognizing $\left[\mathbf{Span}(\mathcal{E})^{\mathrm{op}}, \mathbf{Cat}\right]$ as the 2-category of fibrations with sums (by the universal property of spans) [9], and noting that the extension-restriction adjunction is pseudomonadic (a consequence of T being bijective on objects) [36], the reader will recognize this left extension as the free functor for the pseudomonad $\Sigma_{\mathcal{E}}$ for fibrations over $\mathcal{E}$ with sums. In this way one can derive the pseudomonad for fibrations with sums, and understand why this pseudomonad has a simple description. Note the same can be done for fibrations with products, replacing $\mathbf{Span}(\mathcal{E})$ with $\mathbf{Span}(\mathcal{E})^{\mathrm{co}}$.

6.2 Background

In this section we will recall the necessary background for this paper. We first recall the basic theory of generic factorisations in the one-dimensional case, and then go on to recall the basics of lax conical colimits and the Grothendieck construction, which will replace the category of elements in the two dimensional setting.

6.2.1 Generic factorisations in one dimension.

In the simple one dimensional case, the study of familial representability and generic factorisations stems from the following.

Problem 6.2.1. When is a presheaf $F\colon \mathcal{A} \longrightarrow \mathbf{Set}$ a coproduct of representables, meaning it is equivalent to the colimit of

$$\mathfrak{M}^{\mathrm{op}} \xrightarrow{P_{(-)}^{\mathrm{op}}} \mathcal{A}^{\mathrm{op}} \xrightarrow{y_{\mathcal{A}}} [\mathcal{A}, \mathbf{Set}]$$

for some $\mathfrak{M} \in \mathbf{Set}$ and functor $P_{(-)}\colon \mathfrak{M} \longrightarrow \mathcal{A}$? In particular, when is a functor $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ such that

$$\mathcal{B}(X, T-) : \mathcal{A} \longrightarrow \mathbf{Set}$$

is a coproduct of representables for all $X \in \mathcal{B}$?

The classical answer to these questions is given by Diers [14, 15] (also see [52] for a more recent account), which we will recall after a couple of definitions.

Definition 6.2.2. Given a presheaf $F\colon \mathcal{A} \longrightarrow \mathbf{Set}$, define the category of elements of F as the category with objects given by pairs $(A \in \mathcal{A}, x \in FA)$ and morphisms $(A, x) \nrightarrow (B, y)$ given by maps $f\colon A \longrightarrow B$ such that $Ff(x) = y$. We denote this category el F.

Definition 6.2.3. Given a presheaf $F\colon \mathcal{A} \longrightarrow \mathbf{Set}$, we say an object $(A, x) \in \mathrm{el}\, F$ is *generic* if for any given objects (B, y), (C, z) and morphisms f and g as below

$$\begin{tikzcd}
 & (C, z) \arrow[d, "g"] \\
(A, x) \arrow[ru, dotted, "h"] \arrow[r, "f"'] & (B, y)
\end{tikzcd}$$

there exists a morphism $h\colon (A, x) \longrightarrow (C, z)$ such that the diagram commutes. Moreover, we ask that h is the only morphism $(A, x) \longrightarrow (C, z)$.

Remark 6.2.4. The above may be simply stated by asking (A, x) is initial within its connected component.

Remark 6.2.5. The reader will note that this is stronger than asking for the existence of a unique lifting h. In fact, asking that h be the unique morphism (and not just the unique lifting), is a condition which will turn out to often be too strong in dimension two.

The answer to the first part of Problem 6.2.1 is then the following.

Proposition 6.2.6 (Diers)**.** *Given a presheaf* $F\colon \mathcal{A} \longrightarrow \mathbf{Set}$*, the following are equivalent:*

1. $F\colon \mathcal{A} \longrightarrow \mathbf{Set}$ *is a coproduct of representables;*

2. *each connected component of* el F *has an initial object;*

3. *for any* $(B, y) \in$ el F *there exists a generic object* (A, x) *and morphism* $f\colon (A, x) \to (B, y)$.

Remark 6.2.7. Of course (3) above is simply expanding (2) into more detail. This detailed version will be more analogous to the characterizations we give in the higher dimensional case.

We now consider the second part of Problem 6.2.1 concerning functors $T\colon \mathcal{A} \to \mathcal{B}$, first recalling the notion of "generic morphism" (also known as "diagonally universal morphism" in the work of Diers).

Definition 6.2.8. Given a functor $T\colon \mathcal{A} \to \mathcal{B}$ we say that a morphism $x\colon X \to TA$ for some $X \in \mathcal{B}$ and $A \in \mathcal{A}$ is generic if for any commuting square as on the left below

$$\begin{tikzcd} X \arrow[r, "z"] \arrow[d, "x"'] & TB \arrow[d, "Tg"] \\ TA \arrow[r, "Tf"'] & TC \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "z"] \arrow[d, "x"'] & TB \arrow[d, "Tg"] \\ TA \arrow[r, "Tf"'] \arrow[ur, "Th"] & TC \end{tikzcd}$$

there exists a unique $h\colon A \to B$ such that $Th \cdot x = z$. That $f = g \cdot h$ follows as a consequence of this property.

The following characterization generalizes T having a left adjoint.

Definition 6.2.9. We say a functor $T\colon \mathcal{A} \to \mathcal{B}$ has a *left multiadjoint* if for every $X \in \mathcal{B}$ the presheaf $\mathcal{B}(X, T-)\colon \mathcal{A} \to \mathbf{Set}$ is a coproduct of representables.

Applying Proposition 6.2.6 to presheaves of the form $\mathcal{B}(X, T-)$ for a given functor $T\colon \mathcal{A} \to \mathcal{B}$, we recover the following.

Proposition 6.2.10 (Diers)**.** *Given a functor* $T\colon \mathcal{A} \to \mathcal{B}$ *the following are equivalent:*

1. *the functor* T *has a left multiadjoint;*

2. *for every morphism* $f\colon X \to TW$ *there exists a generic morphism* $\delta\colon X \to TA$ *and morphism* $\overline{f}\colon A \to W$ *such that* $f = T\overline{f} \cdot \delta$.

Remark 6.2.11. Condition (2) is usually stated by saying "T admits generic factorisations".

6.2.2 Lax conical colimits and the Grothendieck construction

Here we give the required background on lax conical colimits and the Grothendieck construction.

Definition 6.2.12 (lax conical colimits)**.** Given a category $\mathcal{A}$, a bicategory $\mathscr{K}$, and pseudofunctor $F\colon \mathcal{A} \to \mathscr{K}$, the *lax colimit* of F consists of an object $T \in \mathscr{K}$, along with for every $A \in \mathcal{A}$ a map $\varphi_A\colon FA \to T$ and for every morphism $f\colon A \to B$ in $\mathcal{A}$ a 2-cell

$$\begin{tikzcd}
& T \arrow[d, phantom, "\overset{\varphi_f}{\Rightarrow}"] & \\
FA \arrow[ur, "\varphi_A"] \arrow[rr, "Ff"'] & {} & FB \arrow[ul, "\varphi_B"']
\end{tikzcd}$$

compatible with the binary and nullary constraints of F. This data, which may be seen as a lax natural transformation $\varphi\colon \Delta\mathbf{1} \Rightarrow \mathscr{K}(F-,T) : \mathcal{A}^{\mathrm{op}} \to \mathscr{K}$, is required to be universal in that

$$\begin{aligned}
&\mathscr{K}(T,S) \to [\mathcal{A}^{\mathrm{op}},\mathbf{Cat}]\,(\Delta\mathbf{1},\mathscr{K}(F-,S)) \\
&\alpha \mapsto \mathscr{K}(F-,\alpha)\cdot\varphi
\end{aligned}$$

defines an equivalence (where $[\mathcal{A}^{\mathrm{op}},\mathbf{Cat}]$ is the 2-category of pseudofunctors, lax natural transformations, and modifications).

Remark 6.2.13. It is worth noting that the above definition can be used when $F\colon \mathcal{A} \to \mathscr{K}$ is only required to be a lax functor. Also, one may note that lax conical colimits can be seen as an instance of weighted bi-colimits (though we will not use this).

When $\mathscr{K} = \mathbf{Cat}$, such a lax colimit can easily be evaluated by the so called Grothendieck construction. We describe this construction below (though we will be more general by replacing the category $\mathcal{A}$ with a bicategory $\mathscr{A}$).

Definition 6.2.14 (Grothendieck construction)**.** Given a bicategory $\mathscr{A}$ and pseudofunctor $F\colon \mathscr{A} \to \mathbf{Cat}$, the category of elements of F, denoted by $\mathrm{el}\,F$ or by

$$\int_{\mathrm{lax}}^{A\in\mathscr{A}} FA$$

is the bicategory with:

Objects An object is a pair of the form $(A \in \mathscr{A}, x \in FA)$;

Morphisms A morphism $(A,x) \nrightarrow (B,y)$ is a morphism $f\colon A \to B$ in $\mathscr{A}$ and a morphism $\alpha\colon Ff(x) \to y$ in FB;

2-cells A 2-cell $(f,\alpha) \Rightarrow (g,\beta) : (A,x) \twoheadrightarrow (B,y)$ is a 2-cell $\nu \colon f \Rightarrow g$ in $\mathscr{A}$ such that

$$Ff(x) \xrightarrow{(F\nu)_x} Fg(x) \xrightarrow{\beta} y$$

is equal to α.

The bicategory $\int_{\mathrm{lax}}^{A\in\mathscr{A}} FA$ with its canonical projection to $\mathscr{A}$ is called the Grothendieck construction of F, especially in the case where $\mathscr{A}$ is a 1-category.

Remark 6.2.15. When $\mathscr{A}$ is a category, the notation $\int_{\mathrm{lax}}^{A\in\mathscr{A}} FA$ is justified as the category of elements can be written as a lax colimit as in Definition 6.2.12. In the case where $\mathscr{A}$ is a bicategory, el F is an appropriate tri-colimit of F, and the notation is still justified (though in a more technical sense that we will not burden this paper with; see [4]).

Taking $[\mathscr{A}, \mathbf{Cat}]$ as the 2-category of pseudofunctors $\mathscr{A} \longrightarrow \mathbf{Cat}$, pseudonatural transformations, and modifications, we are now ready to state the main goal of this paper, which is to answer the following:

Problem 6.2.16. When is a pseudofunctor $F \colon \mathscr{A} \longrightarrow \mathbf{Cat}$ a lax conical colimit of representables, meaning it is equivalent to the lax colimit of

$$\mathfrak{M}^{\mathrm{op}} \xrightarrow{P_{(-)}^{\mathrm{op}}} \mathscr{A}^{\mathrm{op}} \xrightarrow{y_{\mathscr{A}}} [\mathscr{A}, \mathbf{Cat}]$$

for some $\mathfrak{M} \in \mathbf{Cat}$ and pseudofunctor $P_{(-)} \colon \mathfrak{M} \longrightarrow \mathscr{A}$? In particular, when is a pseudofunctor $T \colon \mathscr{A} \longrightarrow \mathscr{B}$ such that

$$\mathscr{B}(X, T-) : \mathscr{A} \longrightarrow \mathbf{Cat}$$

is a lax conical colimit of representables for all $X \in \mathscr{B}$ (such that the construction of these lax colimits is natural in X in an appropriate sense)[1]?

Note that given an F arising as in the first part of this problem, we may write

$$F \simeq \int_{\mathrm{lax}}^{m\in\mathfrak{M}} \mathscr{A}(P_m, -)$$

as the analogue of the usual notation $F \cong \sum_{m\in\mathfrak{M}} \mathcal{A}(P_m, -)$ in one dimension. Moreover, it is easy to see $\int_{\mathrm{lax}}^{m\in\mathfrak{M}} \mathscr{A}(P_m, -)$ is evaluated as the pseudofunctor $\mathscr{A} \longrightarrow \mathbf{Cat}$ sending each $T \in \mathscr{A}$ to the category with objects given by pairs $(m \in \mathfrak{M}, f \colon P_m \longrightarrow T)$ and morphisms

---

[1] An extra condition ensuring naturality in X is not required in the simpler dimension one case.

given by morphisms λ in $\mathfrak{M}$ and 2-cells α in $\mathscr{A}$ as below

$$\begin{tikzcd}
(m \in \mathfrak{M}, f\colon P_m \longrightarrow T) \arrow[dd, squiggly, "(\lambda,\alpha)"] & & m \arrow[dd, "\lambda"'] & P_m \arrow[dd, "P_\lambda"'] \arrow[dr, "f"] & \\
 & : & & \Downarrow\alpha & T\,. \\
(n \in \mathfrak{M}, g\colon P_n \longrightarrow T) & & n & P_n \arrow[ur, "g"'] &
\end{tikzcd}$$

In the next section we will characterize when $F\colon \mathscr{A} \longrightarrow \mathbf{Cat}$ is a lax conical colimit of representables in terms of properties satisfied by el F, using the fact that for such an F we know el F has the form

$$\operatorname{el} F \simeq \int_{\text{lax}}^{A\in\mathscr{A}} \int_{\text{lax}}^{m\in\mathfrak{M}} \mathscr{A}\,(P_m, A) \simeq \int_{\text{lax}}^{m\in\mathfrak{M}} \int_{\text{lax}}^{A\in\mathscr{A}} \mathscr{A}\,(P_m, A)\,.$$

Finally, we recall the notion of a fibration, which characterizes functors $p\colon \mathcal{F} \longrightarrow \mathcal{E}$ (with $\mathcal{E}$ a 1-category) which arise from a pseudofunctor $F\colon \mathcal{E}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ via the Grothendieck construction (here we mean the dual version of Definition 6.2.14 using oplax colimits in place of lax colimits).

Definition 6.2.17. A *fibration* is a functor $p\colon \mathcal{F} \longrightarrow \mathcal{E}$ such that for any morphism $f\colon X \longrightarrow pB$ in $\mathcal{E}$ there exists a morphism $\phi\colon f^*B \longrightarrow B$ in $\mathcal{F}$ such that $p\,(\phi) = f$ and for any $\psi\colon A \longrightarrow B$ and $r\colon pA \longrightarrow X$ rendering commutative the right diagram below

$$\begin{tikzcd}
f^*B \arrow[r, "\phi"] & B & & X \arrow[r, "f"] & pB \\
A \arrow[u, dotted, "\overline{r}"] \arrow[ur, "\psi"'] & & & pA \arrow[u, "r"] \arrow[ur, "p\psi"'] &
\end{tikzcd}$$

there exists a unique $\overline{r}\colon A \longrightarrow f^*B$ such that $p\,(\overline{r}) = r$ and the left diagram commutes. Moreover, we say a morphism $\phi\colon f^*B \longrightarrow B$ in $\mathcal{F}$ is *cartesian* if the above property is satisfied when $f = p\,(\phi)$.

Remark 6.2.18. Dually, we have an equivalence between pseudofunctors $F\colon \mathcal{E} \longrightarrow \mathbf{Cat}$ and *opfibrations* over $\mathcal{E}$, with the equivalence given by Definition 6.2.14. It is worth noting that for such a pseudofunctor $F\colon \mathcal{E} \longrightarrow \mathbf{Cat}$, the morphisms of the form $(f, \alpha)\colon (A, x) \longrightarrow (B, y)$ with α invertible are the opcartesian arrows of el F with respect to the corresponding opfibration el $F \longrightarrow \mathcal{E}$.

6.3 Lax generics in bicategories of elements

Before we can describe lax-generic objects and morphisms in bicategories of elements, we will have to introduce the language needed to describe them. In particular, we define "mixed left liftings" which are similar to left liftings, except that the induced arrow's direction is reversed. Note that basic properties for left liftings, such as the pasting lemma, or the lifting through an identity being itself, do not hold in general for mixed left liftings.

Definition 6.3.1 (mixed left lifting property)**.** Let $\mathscr{C}$ be a bicategory. We say a diagram as on the left below

$$\begin{tikzcd} & \mathcal{C} \arrow[d, "g"] \\ \mathcal{A} \arrow[ur, dotted, "h"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\nu"', shift right=2ex] & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{C} \arrow[d, "g"] \\ \mathcal{A} \arrow[ur, "k"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\psi"', shift right=2ex] & \mathcal{B} \end{tikzcd}$$

exhibits (h, ν) as the *mixed left lifting* of f through g if for any diagram as on the right above, there exists a unique 2-cell $\lambda\colon k \Rightarrow h$ such that

$$\begin{tikzcd} & \mathcal{C} \arrow[d, "g"] \\ \mathcal{A} \arrow[ur, bend left=60, "h"] \arrow[ur, "k"] \arrow[ur, phantom, "\Uparrow\lambda", shift left=2ex] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\psi"', shift right=2ex] & \mathcal{B} \end{tikzcd} = \begin{tikzcd} & \mathcal{C} \arrow[d, "g"] \\ \mathcal{A} \arrow[ur, "h"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\nu"', shift right=2ex] & \mathcal{B}\,. \end{tikzcd}$$

Moreover, we say such a lifting (h, ν) is *strong* if h is sub-terminal in $\mathscr{C}(\mathcal{A}, \mathcal{C})$.

Remark 6.3.2. It is clear that strong mixed liftings are unique up to unique isomorphism. Indeed, it is this stronger notion that will be used though this section.

The following lemma shows that an arrow h which arises as a strong mixed lifting has the property that the strong mixed lifting of h through the identity is itself.

Lemma 6.3.3. *Suppose the left diagram below*

$$\begin{tikzcd} & \mathcal{C} \arrow[d, "g"] \\ \mathcal{A} \arrow[ur, "h"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\nu"', shift right=2ex] & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{C} \arrow[d, "1_{\mathcal{C}}"] \\ \mathcal{A} \arrow[ur, "h"] \arrow[r, "h"'] \arrow[ur, phantom, "\Uparrow\mathrm{id}"', shift right=2ex] & \mathcal{C} \end{tikzcd}$$

exhibits (h, ν) as the strong mixed lifting of f through g. Then the right diagram above exhibits (h, id) as the strong mixed lifting of h through $1_{\mathcal{C}}$.

Proof. Given any $k\colon \mathcal{A} \to \mathcal{C}$ and $\zeta\colon h \Rightarrow k$ we have by universality of (h, ν) an induced $\lambda\colon k \Rightarrow h$ such that

$$\begin{tikzcd}
 & \mathcal{C} \arrow[d, "g"] \\
\mathcal{A} \arrow[ur, bend left=80, "h"] \arrow[ur, bend left=40, "k" description] \arrow[ur, "h" description] \arrow[r, "f"'] \arrow[ur, phantom, bend left=60, "\lambda\,\Nwarrow"] \arrow[ur, phantom, bend left=20, "\zeta\,\Nwarrow"] & \mathcal{B} \arrow[ul, phantom, "\Uparrow\nu"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \mathcal{C} \arrow[d, "g"] \\
\mathcal{A} \arrow[ur, "h"] \arrow[r, "f"'] & \mathcal{B} \arrow[ul, phantom, "\Uparrow\nu"]
\end{tikzcd} ;$$

that is, since h is subterminal, a unique induced $\lambda\colon k \Rightarrow h$ such that $\lambda\zeta$ is the identity. This proves the result. ☐

We now have the required theory to define notions of lax-generic object and lax-generic morphism in bicategories of elements.

Definition 6.3.4 (lax-generic objects)**.** Let $\mathscr{A}$ be a bicategory and $F\colon \mathscr{A} \to \mathbf{Cat}$ be a pseudofunctor. We say that an object (A, x) in el F is *lax-generic* if:

1. for any (B, y), (C, z), (f, α) and (g, β) as below with β invertible

$$\begin{tikzcd}
 & (C, z) \arrow[d, "{(g,\beta)}"] \\
(A, x) \arrow[ur, dotted, "{(h,\gamma)}"] \arrow[r, "{(f,\alpha)}"'] & (B, y) \arrow[ul, phantom, "\Uparrow\nu"]
\end{tikzcd}$$

   there exists a strong mixed left lifting $(h, \gamma) : (A, x) \to (C, z)$ exhibited by a 2-cell $\nu\colon f \Rightarrow gh$;

2. if α is invertible above, then both γ and ν are also invertible.

Remark 6.3.5. If we replace the isomorphism β with an identity above the definition remains equivalent.

Definition 6.3.6 (generic morphisms)**.** Let $\mathscr{A}$ be a bicategory and $F\colon \mathscr{A} \to \mathbf{Cat}$ be a pseudofunctor, and suppose that (A, x) is a lax-generic object in el F. We say that a morphism $(\ell, \phi) : (A, x) \to (D, w)$ out of (A, x) in el F is *generic* if the diagram below

$$\begin{tikzcd}
 & (D, w) \arrow[d, "{(1_D,\mathrm{id})}"] \\
(A, x) \arrow[ur, dotted, "{(\ell,\phi)}"] \arrow[r, "{(\ell,\phi)}"'] & (D, w) \arrow[ul, phantom, "\Uparrow\mathrm{id}"]
\end{tikzcd}$$

exhibits (ℓ, ϕ) as the strong mixed left lifting of (ℓ, ϕ) through $(1_D, \mathrm{id})$.

Remark 6.3.7. It is an easy consequence of the universal property that every 2-cell out of (ℓ, ϕ) is a section (in a unique way); and consequently that any 2-cell between generic 1-cells is invertible. Moreover, as (ℓ, ϕ) is sub-terminal within its hom-category it follows that any isomorphism between generic 1-cells is unique. It follows that if (A, x) and (B, y) are generic objects, then the category of generic morphisms $(A, x) \longrightarrow (B, y)$ is equivalent to a discrete category (a set).

Remark 6.3.8. It is worth noting that for any generic object (A, x) and strong mixed lifting as below

$$\begin{tikzcd}
 & (C,z) \arrow[d, "(g,\beta)"] \\
(A,x) \arrow[ur, dotted, "(h,\gamma)"] \arrow[r, "(f,\alpha)"'] \arrow[ur, phantom, shift right=3, "\Uparrow\nu"] & (B,y)
\end{tikzcd}$$

with β invertible, the induced morphism (h, γ) is a generic morphism as a consequence of Lemma 6.3.3.

The following proposition is a step towards characterizing when an $F\colon \mathscr{A} \longrightarrow \mathbf{Cat}$ is a lax conical colimit of representables.

Proposition 6.3.9. *Let $\mathscr{A}$ be a bicategory and $F\colon \mathscr{A} \longrightarrow \mathbf{Cat}$ be a pseudofunctor. Suppose that generic morphisms between generic objects compose to generic morphisms. Define $\mathscr{A}_g^F$ as the locally full sub-bicategory of* el F *consisting of lax-generic objects and 1-cells. Define $\mathfrak{M}$ as the category consisting of lax-generic objects in* el F *and representatives of isomorphism classes of generic 1-cells between them. Observe $\mathscr{A}_g^F \simeq \mathfrak{M}$. Take $P_{(-)}\colon \mathfrak{M} \longrightarrow \mathscr{A}$ as the assignment taking a generic object (A, x) to A and a representative generic morphism between generic objects $(s, \phi) : (A, x) \longrightarrow (B, y)$ to $s\colon A \longrightarrow B$. Then $P_{(-)}\colon \mathfrak{M} \longrightarrow \mathscr{A}$ defines a pseudofunctor, and for every $T \in \mathscr{A}$ there exists fully faithful functors*

$$\Lambda_T\colon \int_{\mathrm{lax}}^{m \in \mathfrak{M}} \mathscr{A}(P_m, T) \longrightarrow FT$$

pseudo-natural in $T \in \mathscr{A}$.

Proof. Firstly note that $P_{(-)}\colon \mathfrak{M} \longrightarrow \mathscr{A}$ defines a pseudofunctor since it may be written as the composite $\mathfrak{M} \longrightarrow \mathscr{A}_g^F \longrightarrow \mathrm{el}\, F \longrightarrow \mathscr{A}$. We may then define Λ_T on objects by the assignment $(A, x, f) \mapsto Ff(x)$, and on morphisms by the assignment (suppressing the pseudofunctoriality constraints of F)

$$\begin{tikzcd}
(A,x,f\colon A\to T) \arrow[dd, squiggly, "{(h,\gamma,\nu)}"] & A \arrow[dd, "h"'] & Fh\,(x) \arrow[dd, "\gamma"'] & A \arrow[dd, "h"'] \arrow[dr, "f"] & & & Ff\,(x) \arrow[d, "(F\nu)_x"] \\
& & & \Downarrow\nu & T & \mapsto & FgFh\,(x) \arrow[d, "Fg(\gamma)"] \\
(B,y,g\colon B\to T) & B & y & B \arrow[ur, "g"'] & & & Fg\,(y)\ .
\end{tikzcd}\tag{6.3.1}$$

Observe that we have the following conditions satisfied.

Functoriality. Given another

$$\begin{tikzcd}
(B,y,g\colon B\to T) \arrow[dd, squiggly, "{(k,\zeta,\mu)}"] & B \arrow[dd, "k"'] & Fk\,(y) \arrow[dd, "\zeta"'] & B \arrow[dd, "k"'] \arrow[dr, "g"] & & & Fg\,(y) \arrow[d, "(F\mu)_y"] \\
& & & \Downarrow\mu & T & \mapsto & FqFk\,(y) \arrow[d, "Fq(\zeta)"] \\
(C,z,q\colon A\to T) & C & z & C \arrow[ur, "q"'] & & & Fq\,(z)
\end{tikzcd}$$

the commutativity of

$$\begin{tikzcd}
Ff\,(x) \arrow[r, "(F\nu)_x"] & FgFh\,(x) \arrow[r, "Fg(\gamma)"] \arrow[dr, "(F\mu)_{Fh(x)}"'] & Fg\,(y) \arrow[r, "(F\mu)_y"] & FqFk\,(y) \arrow[r, "Fq(\zeta)"] & Fq\,(z) \\
& & FqFkFh\,(x) \arrow[ur, "FqFk(\gamma)"'] & &
\end{tikzcd}$$

by naturality of $F\mu$ exhibits binary functoriality. It is trivial that identities are preserved.

Fullness. Given any $(A,x,f\colon A\to T)$ and $(B,y,g\colon B\to T)$ with a $\phi\colon Ff\,(x)\to Fg\,(y)$, we may construct the universal diagram

$$\begin{tikzcd}
& (B,y) \arrow[d, "{(g,\mathrm{id})}"] \\
(A,x) \arrow[ur, dotted, "{(h,\gamma)}"] \arrow[r, "{(f,\phi)}"'] \arrow[r, phantom, shift left=3, "\Uparrow\nu"] & (B,Fg\,(y))
\end{tikzcd}$$

using lax-genericity of (A,x). Now (h,γ) is generic by Lemma 6.3.3, and without loss of generality we can assume it is a representative generic. Then (h,γ,ν) is assigned to ϕ.

Faithfulness. Given another triple (k,ψ,ω) which also maps to ϕ, we have the diagram

$$\begin{tikzcd}
& (B,y) \arrow[d, "{(g,\mathrm{id})}"] \\
(A,x) \arrow[ur, dotted, "{(k,\psi)}"] \arrow[r, "{(f,\phi)}"'] \arrow[r, phantom, shift left=3, "\Uparrow\omega"] & (B,Fg\,(y))
\end{tikzcd}$$

But as (k, ψ) and (h, γ) are both generics, the induced $(k, \psi) \Rightarrow (h, \gamma)$ arising from universality of (h, γ) must be invertible. Also, as they are both representative, they must be equal. As the identity must then be the induced morphism we conclude $k = h$, $\psi = \gamma$ and $\omega = \nu$.

Pseudo-naturality. Clearly given any 1-cell $\alpha\colon T \longrightarrow S$ in $\mathscr{A}$ the squares

$$\begin{tikzcd}
(A, x, f\colon A \to T) \arrow[r, "\alpha\cdot(-)"] \arrow[d, "\Lambda_T"'] & (A, x, \alpha f\colon A \to S) \arrow[d, "\Lambda_S"] \\
Ff(x) \arrow[r, "F\alpha\cdot(-)"'] & F(\alpha f)(x)
\end{tikzcd}$$

commute up to pseudo-functoriality constraints of F, and the above squares satisfy the required naturality, nullary and binary coherence conditions as a consequence of the corresponding pseudo-functoriality coherence conditions. □

Remark 6.3.10. Given any (h, γ, ν) as in (6.3.1) we also have

$$\begin{tikzcd}
(A, x, f\colon A \to T) \arrow[dd, squiggly, "{(\mathrm{id},\mathrm{id},\nu)}"] & A \arrow[dd, "\mathrm{id}"'] & x \arrow[dd, "\mathrm{id}"'] & A \arrow[dd, "\mathrm{id}"'] \arrow[dr, "f"] & & & Ff(x) \arrow[d, "(F\nu)_x"] \\
& & & \Downarrow\nu & T & \mapsto & Fgh(x) \arrow[d, "Fgh(\mathrm{id})"] \\
(A, x, gh\colon A \to T) & A & x & B \arrow[ur, "gh"'] & & & Fgh(x)
\end{tikzcd}$$

Remark 6.3.11. Each Λ_T is well defined, but not necessarily fully faithful, taking $\mathfrak{M}$ as the category given by el F with no 2-cells (after replacing the bicategory el F with an equivalent 2-category).

We can now characterize precisely when a pseudofunctor $F\colon \mathscr{A} \longrightarrow \mathbf{Cat}$ is a lax conical colimit of representables.

Theorem 6.3.12. *Let $\mathscr{A}$ be a bicategory and $F\colon \mathscr{A} \longrightarrow$ **Cat** be a pseudofunctor. Then the following are equivalent:*

1. *the pseudofunctor $F\colon \mathscr{A} \longrightarrow$ **Cat** is a lax conical colimit of representables;*

2. *the following conditions hold:*

   (a) *for every object (B, y) in* el F *there exists a lax-generic object (A, x) and morphism $(f, \alpha) : (A, x) \twoheadrightarrow (B, y)$ with α invertible;*

   (b) *generic morphisms between lax-generic objects compose to generic morphisms.*

Proof. The direction (2) $\Rightarrow$ (1) is clear from Proposition 6.3.9 as condition (a) means that for any $B \in \mathscr{A}$ and $y \in FB$ we have a lax generic (A, x) and morphism $(f, \alpha) : (A, x) \nrightarrow (B, y)$ in el F with α invertible, so that

$$\Lambda_B\,(A, x, f\colon A \longrightarrow B) = Ff\,(x) \xrightarrow{\alpha} y$$

which witnesses the essential surjectivity of Λ_B at $y \in FB$.

For (1) $\Rightarrow$ (2), suppose we are given a category $\mathfrak{M}$ and pseudofunctor $P_{(-)}\colon \mathfrak{M} \longrightarrow \mathscr{A}$ (assuming without loss of generality that $P_{(-)}$ strictly preserves identities) such that $F \simeq \int_{\text{lax}}^{m\in\mathfrak{M}} \mathscr{A}\,(P_m, -)$, and consequently

$$\text{el}\, F \simeq \int_{\text{lax}}^{m\in\mathfrak{M}} \text{el}\,\mathscr{A}\,(P_m, -)\,.$$

This exhibits el F as the bicategory with:

Objects An object is a triple of the form $(m \in \mathfrak{M}, A \in \mathscr{A}, x\colon P_m \longrightarrow A)$;

Morphisms The morphisms $(m, A, x) \nrightarrow (n, B, y)$ are triples comprising a morphism $u\colon m \longrightarrow n$ in $\mathfrak{M}$, a morphism $f\colon A \longrightarrow B$ in $\mathscr{A}$ and a 2-cell

$$\begin{tikzcd}
P_m \arrow[r, "x"] \arrow[d, "P_u"'] \arrow[dr, phantom, "\Downarrow\theta"] & A \arrow[d, "f"] \\
P_n \arrow[r, "y"'] & B
\end{tikzcd}$$

in $\mathscr{A}$;

2-cell A 2-cell $\lambda\colon (u, f, \theta) \Rightarrow (u, g, \phi) : (m, A, x) \nrightarrow (n, B, y)$ is a 2-cell $\lambda\colon f \Rightarrow g$ in $\mathscr{A}$ such that

$$\begin{tikzcd}
P_m \arrow[r, "x"] \arrow[d, "P_u"'] \arrow[dr, phantom, "\Downarrow\theta"] & A \arrow[d, "f"] \\
P_n \arrow[r, "y"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
P_m \arrow[r, "x"] \arrow[d, "P_u"'] \arrow[dr, phantom, "\Swarrow\phi"] & A \arrow[d, bend right=40, "g"', ""{name=G}] \arrow[d, bend left=40, "f", ""'{name=F}] \\
P_n \arrow[r, "y"'] & B\,.
\arrow[from=F, to=G, Rightarrow, "\lambda"]
\end{tikzcd}$$

Existence of expected lax-generics. We first show that each

$$(m \in \mathfrak{M}, P_m \in \mathscr{A}, \text{id}\colon P_m \longrightarrow P_m)$$

in el F is lax-generic. Consider a diagram

$$\begin{tikzcd}
 & (n,C,z) \arrow[d, "{(\mathrm{id},g,\mathrm{id})}"] \\
(m,P_m,\mathrm{id}) \arrow[ur, dotted, "{(u,h,\gamma)}"] \arrow[r, "{(u,f,\alpha)}"'] \arrow[ur, phantom, "\Uparrow\nu"', shift right=3] & (n,B,y)
\end{tikzcd}$$

where (u,f,α) and $(\mathrm{id},g,\mathrm{id})$ are respectively

$$\begin{tikzcd}
P_m \arrow[r, "\mathrm{id}"] \arrow[d, "P_u"'] \arrow[dr, phantom, "\Downarrow\alpha"] & P_m \arrow[d, "f"] \\
P_n \arrow[r, "y"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
P_n \arrow[r, "z"] \arrow[d, "P_{\mathrm{id}}"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & C \arrow[d, "g"] \\
P_n \arrow[r, "y"'] & B
\end{tikzcd}$$

then we recover a canonical (u,h,γ) as

$$\begin{tikzcd}
P_m \arrow[r, "\mathrm{id}"] \arrow[d, "P_u"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & P_m \arrow[d, "z\cdot P_u"] \\
P_n \arrow[r, "z"'] & C
\end{tikzcd} \tag{6.3.2}$$

with the 2-cell $\nu\colon f \Rightarrow gh = gzP_u = yP_u$ given as α. Now, for universality, suppose we have a (u,k,ϕ) given as

$$\begin{tikzcd}
P_m \arrow[r, "\mathrm{id}"] \arrow[d, "P_u"'] \arrow[dr, phantom, "\Downarrow\phi"] & P_m \arrow[d, "k"] \\
P_n \arrow[r, "z"'] & C
\end{tikzcd}$$

with a 2-cell $\psi\colon f \Rightarrow gk$ such that

$$\begin{tikzcd}
P_m \arrow[r, "\mathrm{id}"] \arrow[dd, "P_u"'] \arrow[ddr, phantom, "\Downarrow\alpha"] & P_m \arrow[dd, "f"] \\
 & \\
P_n \arrow[r, "y"'] & B
\end{tikzcd}
=
\begin{tikzcd}
P_m \arrow[r, "\mathrm{id"] \arrow[d, "P_u"'] \arrow[dr, phantom, "\Downarrow\phi"] & P_m \arrow[d, "k"] \arrow[dd, bend left=60, "f"] \\
P_n \arrow[r, "z"] \arrow[d, "P_{\mathrm{id}}"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & C \arrow[d, "g"] \arrow[r, phantom, "\overset{\Leftarrow}{\psi}"] & {} \\
P_n \arrow[r, "y"'] & B
\end{tikzcd} \tag{6.3.3}$$

Then we can take our induced map $\lambda\colon k \Rightarrow h$ as $\phi\colon k \Rightarrow z \cdot P_u$. It is trivial that

$$\begin{tikzcd}
P_m \arrow[r, "\mathrm{id}"] \arrow[d, "P_u"'] \arrow[dr, phantom, "\Downarrow\phi"] & P_m \arrow[d, "k"] \\
P_n \arrow[r, "z"'] & C
\end{tikzcd}
=
\begin{tikzcd}
P_m \arrow[r, "\mathrm{id}"] \arrow[d, "P_u"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & P_m \arrow[d, "z\cdot P_u"'] \arrow[d, bend left=70, "k"] \arrow[d, phantom, "\overset{\Leftarrow}{\lambda}", shift left=5] \\
P_n \arrow[r, "z"'] & C
\end{tikzcd} \tag{6.3.4}$$

so that λ is a 2-cell $(u,k,\phi) \Rightarrow (u,h,\gamma)$. Also, from (6.3.4) it is clear that $\lambda = \phi$ is the *only* 2-cell $(u,k,\phi) \Rightarrow (u,h,\gamma)$, meaning (u,h,γ) is sub-terminal within its hom-category.

Moreover, (6.3.3) shows ψ pasted with $\lambda = \phi$ is $\alpha = \nu$.

Classification of lax-generics. We now show that an object

$$(m \in \mathfrak{M}, A \in \mathscr{A}, x \colon P_m \to A)$$

in el F is lax-generic if and only if x is an equivalence. It is clear the above argument generalizes if one replaces (m, P_m, id) with (m, A, x) where x is an equivalence. Conversely, if (m, A, x) is a generic object then we may construct the universal diagram

$$\begin{tikzcd}
& (m, P_m, \mathrm{id}) \arrow[d, "{(1,x,\mathrm{id})}"] \\
(m, A, x) \arrow[ur, dotted, "{(1,x^*,\gamma)}"] \arrow[r, "{(1,1,\mathrm{id})}"'] \arrow[r, phantom, shift left=4, "\Uparrow \nu"] & (m, A, x)
\end{tikzcd}$$

noting that ν and γ are both invertible. In fact, this gives an adjoint equivalence. That ν is a 2-cell says

$$\begin{tikzcd}
P_m \arrow[r, "x"] \arrow[dd, "\mathrm{id}"'] \arrow[ddr, phantom, "\Downarrow \mathrm{id}"] & A \arrow[dd, "\mathrm{id}"] \\
& \\
P_m \arrow[r, "x"'] & A
\end{tikzcd}
=
\begin{tikzcd}
P_m \arrow[r, "x"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swarrow \gamma"] & A \arrow[d, "x^*"] \arrow[dd, bend left=60, "\mathrm{id}"] \\
P_m \arrow[r, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swarrow \mathrm{id}"] & P_m \arrow[d, "x"] \arrow[r, phantom, "\Leftarrow \nu"] & {} \\
P_m \arrow[r, "x"'] & A
\end{tikzcd}$$

which gives one triangle identity. For the other identity, note that 2-cells $\xi \colon (1, x^*xx^*, \gamma\gamma) \Rightarrow (1, x^*, \gamma)$, meaning 2-cells ξ such that

$$\begin{tikzcd}
P_m \arrow[r, "x"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swarrow \gamma\gamma"] & A \arrow[d, "x^*xx^*"] \\
P_m \arrow[r, "\mathrm{id}"'] & P_m
\end{tikzcd}
=
\begin{tikzcd}
P_m \arrow[r, "x"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swarrow \gamma"] & A \arrow[d, "x^*"] \arrow[d, bend left=70, "x^*xx^*"] \arrow[d, phantom, shift left=4, "\Leftarrow \xi"] \\
P_m \arrow[r, "\mathrm{id}"'] & A
\end{tikzcd}
\tag{6.3.5}$$

are unique, as $(1, x^*, \gamma)$ is sub-terminal within its hom-category. But we may take ξ to be

$$\gamma x^* \colon (1, x^*xx^*, \gamma\gamma) \Rightarrow (1, x^*, \gamma)$$

or

$$x^*\nu^{-1} \colon (1, x^*xx^*, \gamma\gamma) \Rightarrow (1, x^*, \gamma)$$

which both satisfy (6.3.5). Thus $\gamma x^* = x^*\nu^{-1}$ and so $\gamma x^* \cdot x^*\nu = \mathrm{id}$ giving the other triangle identity.

Existence of lax-generic factorisations . Suppose we are given a $(n, B, y \colon P_n \to B)$

in el F. We have the map $(n, P_n, \mathrm{id}\colon P_n \to P_n) \twoheadrightarrow (n, B, y\colon P_n \to B)$ given as

$$\begin{tikzcd}
P_n \arrow[r, "\mathrm{id}"] \arrow[d, "P_{\mathrm{id}}"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & P_n \arrow[d, "y"] \\
P_n \arrow[r, "y"'] & B
\end{tikzcd}$$

which is of the required form since the 2-cell involved is invertible.

Generic morphisms form a category. Before showing that generic morphisms form a category, we will need a characterization of them. Now, specializing the earlier argument of "existence of expected lax-generics" to the case when g is the identity (though generalizing the identity on P_m to an equivalence $x\colon P_m \to A$) we see that if (m, A, x) is generic (i.e. x is an equivalence)

$$\begin{tikzcd}
 & (n, C, z) \arrow[d, "{(\mathrm{id},\mathrm{id},\mathrm{id})}"] \\
(m, A, x) \arrow[ur, dotted, "{(u,h,\gamma)}"] \arrow[r, "{(u,f,\alpha)}"'] \arrow[ur, phantom, shift right=3, "\Uparrow\nu"] & (n, B, y)
\end{tikzcd}$$

the lifting (u, h, γ) above, constructed as in (6.3.2), has γ invertible. It is also clear that if (u, f, α) is such that α is invertible, then the lifting (u, h, γ) through $(\mathrm{id}, \mathrm{id}, \mathrm{id})$ constructed as in (6.3.2) is given by (u, f, α).

This shows that the generic morphisms between generic objects are diagrams of the form

$$\begin{tikzcd}
P_m \arrow[r, "x"] \arrow[d, "P_u"'] \arrow[dr, phantom, "\Downarrow\alpha"] & A \arrow[d, "f"] \\
P_n \arrow[r, "y"'] & B
\end{tikzcd}$$

with α invertible, and it is clear that these are closed under composition and that identities are such diagrams. ☐

Remark 6.3.13. When $F\colon \mathscr{A} \to \mathbf{Cat}$ is a lax conical colimit of representables, and from a generic object (A, x) we construct the universal diagram

$$\begin{tikzcd}
 & (C, z) \arrow[d, "{(g,\beta)}"] \\
(A, x) \arrow[ur, dotted, "{(h,\gamma)}"] \arrow[r, "{(f,\alpha)}"'] \arrow[ur, phantom, shift right=3, "\Uparrow\nu"] & (B, y)
\end{tikzcd}$$

the 2-cell ν is the *unique* 2-cell $(f, \alpha) \Rightarrow (g, \beta) \cdot (h, \gamma)$. This is since for such an F, generic morphisms compose and any map (g, β) with β invertible is generic. Sub-terminality of

$(g,\beta)\cdot(h,\gamma)$ then gives uniqueness.

Remark 6.3.14. When $F\colon \mathscr{A} \longrightarrow \mathbf{Cat}$ is a lax conical colimit of representables, written $F \simeq \int_{\mathrm{lax}}^{m\in\mathfrak{M}} \mathscr{A}$, then $\mathfrak{M}$ is equivalent to the category of strict[2] lax-generic objects (A,x) and representative generic morphisms in $\mathrm{el}\, F$. This is a consequence of the characterization of lax-generic objects and morphisms given in the above proof of Theorem 6.3.12. Moreover, as Theorem 6.3.12 constructs $\mathfrak{M}$ as the the category of lax-generic objects and morphisms, we conclude this non-strict choice of $\mathfrak{M}$ is also equivalent.

It is a natural question to ask if Theorem 6.3.12 has a variant which does not require generic morphisms to compose; and it turns out that this is the case. Given a pseudofunctor $F\colon \mathscr{A} \longrightarrow \mathbf{Cat}$ one can again define $\mathfrak{M}$ as the category containing generic objects $(A,x)\in \mathrm{el}\, F$ and representative generic morphisms between them, but now defining the composite of two generic morphisms

$$(A,x) \xrightarrow{(h,\gamma)} (B,y) \xrightarrow{(k,\zeta)} (C,z)$$

to be the mixed lifting through the identity as below.

$$\begin{tikzcd}
 & & (C,z) \arrow[d, "{(1,\mathrm{id})}"] \\
(A,x) \arrow[urr, "{(\ell,\phi)}"] \arrow[r, "{(h,\gamma)}"'] & (B,y) \arrow[r, "{(k,\zeta)}"'] \arrow[ur, phantom, "\Uparrow\lambda"] & (C,z)
\end{tikzcd}$$

Now, it is not hard to verify that this situation of generics not directly composing corresponds to the following weaker notion of famility.

Definition 6.3.15. A pseudofunctor $F\colon \mathscr{A} \longrightarrow \mathbf{Cat}$ is a *weak lax conical colimit of representables* if there exists a category $\mathfrak{M}$ and normal[3] lax functor $P_{(-)}\colon \mathfrak{M} \longrightarrow \mathscr{A}$ such that $F \simeq \int_{\mathrm{lax}}^{m\in\mathfrak{M}} \mathscr{A}(P_m,-)$.

Meaning that we find the following variant of Theorem 6.3.12.

Theorem 6.3.16. *Let $\mathscr{A}$ be a bicategory and $F\colon \mathscr{A} \longrightarrow \mathbf{Cat}$ be a pseudofunctor. Then the following are equivalent:*

1. *the pseudofunctor $F\colon \mathscr{A} \longrightarrow \mathbf{Cat}$ is a weak lax conical colimit of representables;*

2. *for every object (B,y) in $\mathrm{el}\, F$ there exists a lax-generic object (A,x) and morphism $(f,\alpha)\colon\ (A,x) \mathrel{+\!\!\!\!\twoheadrightarrow} (B,y)$ with α invertible.*

---

[2] Strict here means if both α and β are identities, then both ν and γ are identities.

[3] By normal we mean the unit constraints are required to be invertible.

Remark 6.3.17. Note that in practice, we will usually want the reindexing $P_{(-)} \colon \mathfrak{M} \to \mathscr{A}$ to be a pseudofunctor. Indeed, $P_{(-)}$ is to be a pseudofunctor in all of the examples of Section 6.7.

The following simple lemmata concern uniqueness of generic factorisations, with a generic factorisation in this abstract setting being an opcartesian map (f, α) out of a lax-generic object (A, x).

Lemma 6.3.18. *A morphism $(h, \gamma) : (A, x) \to (B, y)$ is an equivalence if and only if $h \colon A \to B$ is an equivalence and γ is invertible.*

Proof. Given that (h, γ) has a pseudo-inverse $(k, \psi) : (B, y) \to (A, x)$ it is clear that h has pseudo-inverse k and that $\gamma \colon Fh(x) \to y$ has pseudo-inverse

$$y \xrightarrow{\cong} FhFk(y) \xrightarrow{Fh(\psi)} Fh(x)$$

Conversely, given a $(h, \gamma) : (A, x) \to (B, y)$ such that h has pseudo-inverse k (we may upgrade this equivalence to an adjoint equivalence) and γ is invertible, we have a pseudo-inverse $(k, \psi) : (B, y) \to (A, x)$ where $\psi \colon Fk(y) \to x$ is given by

$$Fk(y) \xrightarrow{Fk(\gamma^{-1})} FkFh(x) \xrightarrow{\cong} x$$

It is then straightforward to verify (h, γ) is pseudo-inverse to (k, ψ). ☐

Whilst generic factorisations are not unique in the sense one may initially expect; they are unique in another sense.

Proposition 6.3.19. *Given two generic factorisations (opcartesian maps out of a generic object) $(f, \alpha) : (A, x) \to (C, z)$ and $(g, \beta) : (B, y) \to (C, z)$ there exists equivalence $(h, \gamma) : (A, x) \to (B, y)$, unique up to unique isomorphism, such that*

$$(A, x) \xrightarrow{(h,\gamma)} (B, y) \xrightarrow{(g,\beta)} (C, z)$$

is isomorphic to (f, α).

Proof. We may form the mixed lifting diagram

$$\begin{tikzcd}
& (B, y) \arrow[d, "{(g,\beta)}"] \\
(A, x) \arrow[ur, "{(h,\gamma)}"] \arrow[r, "{(f,\alpha)}"'] \arrow[r, phantom, shift left=3, "\Uparrow\nu"] & (C, z)
\end{tikzcd}$$

where (h,γ) is necessarily generic and ν and γ invertible. Lifting in the other direction yields the pseudo-inverse (k,ψ). ☐

Lemma 6.3.20. *Every opcartesian map between two generic objects* $(h,\gamma):\ (A,x) \nrightarrow (C,z)$ *is an equivalence.*

Proof. Given such a (h,γ) we may form a (k,ψ) as on the left below

$$\begin{tikzcd}
 & (A,x) \arrow[d, "{(h,\gamma)}"] \\
(C,z) \arrow[ur, "{(k,\psi)}"] \arrow[r, "{(1,\mathrm{id})}"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow\nu"] & (C,z)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (C,z) \arrow[d, "{(k,\psi)}"] \\
(A,x) \arrow[ur, "{(h',\gamma')}"] \arrow[r, "{(1,\mathrm{id})}"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow\mu"] & (A,x)
\end{tikzcd}$$

and one can then form a (h',γ') as on the right above. As ν and μ have inverses

$$(h',\gamma') \cong (h,\gamma)(k,\psi)(h',\gamma') \cong (h,\gamma)$$

so (h,γ) has pseudo-inverse (k,ψ). ☐

6.4 An alternative characterization

In Section 6.3 we gave a characterization of when a pseudofunctor $F\colon \mathscr{A} \longrightarrow \mathbf{Cat}$ is a lax conical colimit of representables in terms of lax-generic objects and morphisms. However, it is natural to ask if we can also give a characterization in terms of what we will call "pseudo-generic" factorisations. Here we address this problem in the case where $\mathscr{A}$ is a 1-category $\mathcal{E}$, giving a simple description of when a pseudofunctor $F\colon \mathcal{E} \longrightarrow \mathbf{Cat}$ is a lax conical colimit of representables.

These pseudo-generics are to be defined in terms of a pseudo-lifting property which we now recall.

Definition 6.4.1 (pseudo-lifting property)**.** Let $\mathscr{C}$ be a bicategory. We say a diagram as on the left below

$$\begin{tikzcd}
 & \mathcal{C} \arrow[d, "g"] \\
\mathcal{A} \arrow[ur, dotted, "h"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow\nu"] & \mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{C} \arrow[d, "g"] \\
\mathcal{A} \arrow[ur, "k"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow\psi"] & \mathcal{B}
\end{tikzcd}$$

with ν invertible exhibits (h,ν) as the *pseudo lifting* of f through g if for any diagram as on the right above with ψ invertible, there exists a unique invertible 2-cell $\lambda\colon k \Rightarrow h$ such that

$$\begin{tikzcd}
& \mathcal{C} \arrow[d, "g"] & & & \mathcal{C} \arrow[d, "g"] \\
\mathcal{A} \arrow[ur, bend left=70, "h"] \arrow[ur, "k" description] \arrow[r, "f"'] \arrow[ur, phantom, bend left=35, "\Uparrow\lambda"] & \mathcal{B} \arrow[ul, phantom, "\Uparrow\psi" near start] & = & \mathcal{A} \arrow[ur, "h"] \arrow[r, "f"'] & \mathcal{B} \arrow[ul, phantom, "\Uparrow\nu" near start]
\end{tikzcd}$$

Moreover, we say such a lifting (h,ν) is *strong* if h is sub-terminal in $\mathscr{C}(\mathcal{A},\mathcal{C})$.

Remark 6.4.2. Note that when $\mathscr{A}$ is a 1-category $\mathcal{E}$, the category of elements $\operatorname{el} F$ is a 1-category, and so the mixed and pseudo lifting properties both become the usual one-dimensional lifting properties.

Definition 6.4.3 (pseudo-generic objects)**.** Let $\mathscr{A}$ be a bicategory and $F\colon \mathscr{A} \longrightarrow \mathbf{Cat}$ be a pseudofunctor. We say that an object (A,x) in $\operatorname{el} F$ is *pseudo-generic* if:

1. for any (B,y), (C,z), (f,α) and (g,β) as below with both α and β invertible

$$\begin{tikzcd}
& (C,z) \arrow[d, "{(g,\beta)}"] \\
(A,x) \arrow[ur, dotted, "{(h,\gamma)}"] \arrow[r, "{(f,\alpha)}"'] & (B,y) \arrow[ul, phantom, "\Uparrow\nu" near start]
\end{tikzcd}$$

   there exists a strong pseudo lifting $(h,\gamma)\colon\ (A,x) \longrightarrow (C,z)$ exhibited by an invertible 2-cell $\nu\colon f \Rightarrow gh$;

2. every pseudo-lifting (h,γ) as above has γ invertible.[4]

We can now give a simple characterization of when a pseudofunctor $F\colon \mathcal{E} \longrightarrow \mathbf{Cat}$ is a lax conical colimit of representables.

Remark 6.4.4. For proving the below theorem, simplified descriptions of pseudo-genericity would suffice as it concerns 1-categories $\mathcal{E}$ (for example every morphism becomes sub-terminal within its hom-category in this case). However, we will leave the descriptions in full generality above in case it is possible to generalize the below theorem to the bicategorical case.

Theorem 6.4.5. *Let $\mathcal{E}$ be a category and $F\colon \mathcal{E} \longrightarrow$ **Cat** be a pseudofunctor. Then the following are equivalent:*

1. *the pseudofunctor $F\colon \mathcal{E} \longrightarrow$ **Cat** is a lax conical colimit of representables;*

---

[4] One could omit this condition and still prove Theorem 6.4.5, however, we give it here as it forces the lax-generic objects and pseudo-generic objects to coincide when $F\colon \mathcal{E} \longrightarrow \mathbf{Cat}$ is a lax conical colimit of representables.

2. *for every object* (B, y) *in* $\mathrm{el}\, F$ *there exists a lax-generic object* (A, x) *and morphism* $(f, \alpha) : (A, x) \nrightarrow (B, y)$ *with* α *invertible;*

3. *the following conditions hold:*

   (a) *for every object* (B, y) *in* $\mathrm{el}\, F$ *there exists a pseudo-generic object* (A, x) *and morphism* $(f, \alpha) : (A, x) \nrightarrow (B, y)$ *with* α *invertible;*

   (b) *for every morphism* $f \colon X \to Y$ *in* $\mathcal{E}$ *the functor* $Ff \colon FX \to FY$ *is a fibration.*

Moreover, if any of the above equivalent conditions hold we then have

$$F \simeq \int_{\mathrm{lax}}^{m \in \mathfrak{M}} \mathcal{E}\,(P_m, -)$$

where $P_{(-)} \colon \mathfrak{M} \to \mathcal{E}$ *is the canonical projection of the category* $\mathfrak{M}$ *with:*

Objects *An object is a pseudo-generic* (A, x) *in* $\mathrm{el}\, F$*;*

Morphisms *A morphism* $(A, x) \nrightarrow (B, y)$ *is a morphism* $f \colon A \to B$ *in* $\mathcal{E}$ *equipped with a morphism* $\alpha \colon Ff\,(x) \to y$ *in* FB*.*

Proof. Firstly note (1) $\Leftrightarrow$ (2) by Theorem 6.3.12. For (1,2) $\Rightarrow$ (3), suppose that F is a lax conical colimit of representables, i.e. that there exists a category $\mathfrak{M}$ and pseudofunctor $P_{(-)} \colon \mathfrak{M} \to \mathcal{E}$ and equivalences

$$FT \simeq \int_{\mathrm{lax}}^{m \in \mathfrak{M}} \mathcal{E}\,(P_m, T)$$

pseudonatural in $T \in \mathcal{E}$. Then as every lax-generic object (A, x) is also pseudo-generic, we have the pseudo-generic factorisations of condition (a). Now consider a morphism $f \colon X \to Y$ in $\mathcal{E}$ and the functor $Ff \colon FX \to FY$. We know that $Ff \colon FX \to FY$ is equivalent to (via an appropriate pseudo-naturality square) the functor

$$f \circ (-) : \int_{\mathrm{lax}}^{m \in \mathfrak{M}} \mathcal{E}\,(P_m, X) \to \int_{\mathrm{lax}}^{m \in \mathfrak{M}} \mathcal{E}\,(P_m, Y)$$

and this functor is a fibration since for any $\lambda \colon (m, u) \Rightarrow f \circ (n, v)$ as on the right below

$$\begin{tikzcd}
m \arrow[dd, dotted, "\lambda"'] & P_m \arrow[dd, dotted, "P_\lambda"'] \arrow[dr, "v \cdot P_\lambda"] & & & m \arrow[dd, "\lambda"'] & P_m \arrow[dd, "P_\lambda"'] \arrow[dr, "u"] & \\
& & X & \mapsto & & & Y \\
n & P_n \arrow[ur, "v"'] & & & n & P_n \arrow[ur, "fv"'] &
\end{tikzcd}$$

we recover the $f\circ(-)$-cartesian lift on the left above. To see this lift is cartesian, and in fact that every morphism in $\int_{\mathrm{lax}}^{m\in\mathfrak{M}}\mathcal{E}(P_m,X)$ is $f\circ(-)$-cartesian, note that given any $\lambda\colon (m,u)\Rightarrow(n,v)$ as on the left below and $\xi\colon (r,fw)\Rightarrow(m,fu)$ as on the top right below

$$\begin{tikzcd}
r \arrow[d, dotted, "\xi"'] & P_r \arrow[d, dotted, "P_\xi"'] \arrow[dr, "w"] & & & r \arrow[d, "\xi"'] & P_r \arrow[d, "P_\xi"'] \arrow[dr, "fw"] & \\
m \arrow[d, "\lambda"'] & P_m \arrow[r, "u"] \arrow[d, "P_\lambda"'] & X & \mapsto & m \arrow[d, "\lambda"'] & P_m \arrow[r, "fu"] \arrow[d, "P_\lambda"'] & Y \\
n & P_n \arrow[ur, "v"'] & & & n & P_n \arrow[ur, "fv"'] &
\end{tikzcd} \tag{6.4.1}$$

for which the right of (6.4.1) can be seen as the result of some assignation

$$\begin{tikzcd}
r \arrow[d, "\phi"'] & P_m \arrow[d, "P_\phi"'] \arrow[dr, "w"] & & & r \arrow[d, "\phi"'] & P_r \arrow[d, "P_\phi"'] \arrow[dr, "fw"] & \\
& & X & \mapsto & & & Y \\
n & P_n \arrow[ur, "v"'] & & & n & P_n \arrow[ur, "fv"'] &
\end{tikzcd}$$

the induced unique lift $\xi\colon (r,w)\Rightarrow(m,u)$ given on the left in (6.4.1) is well defined since $u\cdot P_\xi = v\cdot P_\lambda\cdot P_\xi = v\cdot P_\phi = w$.

(3) $\Rightarrow$ (1) : Define $\mathfrak{M}$ as above, i.e. the full sub-category of el F on the pseudo-generic objects. Now, $\int_{\mathrm{lax}}^{m\in\mathfrak{M}}\mathcal{E}(P_m,T)$ is the category consisting of:

Objects An object is a pair of the form $(A\in\mathcal{E}, x\in FA, f\colon A\to T)$

Morphisms A morphism $(A,x,f\colon A\to T)\nrightarrow(B,y,g\colon B\to T)$ is a morphism $\alpha\colon A\to B$ in $\mathcal{E}$ rendering commutative

$$\begin{tikzcd}
A \arrow[d, "\alpha"'] \arrow[dr, "f"] & \\
& T \\
B \arrow[ur, "g"'] &
\end{tikzcd}$$

equipped with a morphism $\xi\colon F\alpha(x)\to y$ in FB.

It suffices to check that the functor $\int_{\mathrm{lax}}^{m\in\mathfrak{M}}\mathcal{E}(P_m,Y)\to FT$ defined by the assignation $(A,x,f)\mapsto Ff(x)$ on objects, and by

$$\begin{tikzcd}
(A,x,f) \arrow[d, squiggly, "(\alpha{,}\xi)"] & : & A \arrow[d, "\alpha"] & F\alpha(x) \arrow[d, "\xi"] & \mapsto & Ff(x) \arrow[d, "Fg(\xi)"] \\
(B,y,g) & & B & y & & Fg(y)
\end{tikzcd}$$

on morphisms (suppressing pseudo-functoriality constraints) is an equivalence. Functoriality is clear, and so it suffices to check the following.

Essentially Surjective. For any $t \in FT$ we have $(T, t) \in \mathrm{el}\, F$, and thus by (a) a pseudo-generic (A, x) and morphism $(k, \phi) : (A, x) \twoheadrightarrow (Y, t)$ with ϕ invertible. Now note that $(A, x, k) \mapsto Fk\,(x) \cong t$ as required.

Full. Suppose we are given a morphism $\zeta \colon Ff\,(x) \to Fg\,(y)$ in FT. We may then take the Fg-cartesian lift $\overline{\zeta} \colon \zeta^* y \to y$ and construct the universal diagram

$$\begin{tikzcd}
& (B, \zeta^* y) \arrow[d, "{(g,\mathrm{id})}"] \\
(A, x) \arrow[ru, dotted, "{(h,\gamma)}"] \arrow[r, "{(f,\mathrm{id})}"'] \arrow[ru, phantom, shift left=3, "\Uparrow \nu"] & (T, Ff\,(x))
\end{tikzcd}$$

with γ invertible. Note that ν is necessarily an identity and so $Fg\,(\gamma)$ is the identity (suppressing pseudo-functoriality constraints). It then suffices to observe that we have the assignation

$$\begin{tikzcd}
A \arrow[dd, dotted, "h"'] & Fh\,(x) \arrow[d, dotted, "\gamma"] & & Ff\,(x) \arrow[dd, "\zeta"] \\
& \zeta^* y \arrow[d, dotted, "\overline{\zeta}"] & \mapsto & \\
B & y & & Fg\,(y)
\end{tikzcd}$$

Faithful. Now, given another

$$\begin{tikzcd}
A \arrow[d, dotted, "k"'] & Fk\,(x) \arrow[d, dotted, "\phi"] & \mapsto & Ff\,(x) \arrow[d, "\zeta"] \\
B & y & & Fg\,(y)
\end{tikzcd}$$

mapping to ζ, we have $Fg\,(\phi) = \zeta$ and thus a factorization of ϕ through the cartesian lift

$$\begin{tikzcd}
Fk\,(x) \arrow[r, "\lambda"] \arrow[rr, bend right, "\phi"'] & \zeta^* y \arrow[r, "\overline{\zeta}"] & y
\end{tikzcd}$$

with $Fg\,(\lambda)$ the identity. Thus we have a diagram

$$\begin{tikzcd}
& (B, \zeta^* y) \arrow[d, "{(g,\mathrm{id})}"] \\
(A, x) \arrow[ru, dotted, "{(k,\lambda)}"] \arrow[r, "{(f,\mathrm{id})}"'] \arrow[ru, phantom, shift left=3, "\Uparrow \mathrm{id}"] & (T, Ff\,(x))
\end{tikzcd}$$

and so $(k,\lambda) = (h,\gamma)$ by uniqueness. Hence (k,ϕ) is equal to $\left(h,\overline{\zeta}\gamma\right)$ from earlier. □

6.5 Lax generic factorisations and lax multiadjoints

Here we specialize the results of the previous section to the case when $F\colon \mathscr{A} \to \mathbf{Cat}$ is of the form $\mathscr{B}(X,T-)$ for a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$. The following is a generalization of "left multiadjoint" in Definition 6.2.9 to the case of a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$.

Definition 6.5.1. Let $\mathscr{A}$ and $\mathscr{B}$ be bicategories and let $T\colon \mathscr{A} \to \mathscr{B}$ be a pseudofunctor. We say that T has a *left lax multiadjoint* if there exists a pseudofunctor $\mathfrak{M}_{(-)}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ and a pseudofunctor $\mathbf{P}\colon \int_{\mathrm{lax}}^{X\in\mathscr{B}} \mathfrak{M}_X \to \mathscr{A}$ such that

$$\mathscr{B}(X,T-) \simeq \int_{\mathrm{lax}}^{m\in\mathfrak{M}_X} \mathscr{A}\left(P_m^X,-\right)$$

for all $X \in \mathscr{B}$, where each $P_{(-)}^X\colon \mathfrak{M}_X \to \mathscr{A}$ is obtained from $\mathbf{P}$ by including $\mathfrak{M}_X \to \int_{\mathrm{lax}}^{X\in\mathscr{B}} \mathfrak{M}_X$.

Remark 6.5.2. One might wonder why we did not simply define T to have a left lax multiadjoint when every

$$\mathscr{B}(X,T-) : \mathscr{A} \to \mathbf{Cat}$$

is a lax conical colimit of representables. The reason is that this condition would only be sufficient to force $\mathbf{P}$ (which may be constructed from this condition) to be a normal lax functor.

Before applying Theorem 6.3.12 to bi-presheaves of the form $\mathscr{B}(X,T-)$, we will need the appropriate notions of genericity with respect to a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$. The following definitions are recovered by specializing the definitions of genericity in the last section to the case when $F\colon \mathscr{A} \to \mathbf{Cat}$ is of the form $\mathscr{B}(X,T-)$ for a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$.

Definition 6.5.3. Let $\mathscr{A}$ and $\mathscr{B}$ be bicategories and let $T\colon \mathscr{A} \to \mathscr{B}$ be a pseudofunctor. Then a 1-cell $\delta\colon X \to TA$ is lax-generic if for any diagram and 2-cell α as on the left below

$$\begin{tikzcd}
X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Uparrow\alpha"] & TB \arrow[d, "Tg"] & & X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[r, phantom, shift right=4, "\Uparrow\gamma"] & TB \arrow[d, "Tg"] \arrow[d, phantom, shift right=4, "\Uparrow T\nu"] \\
TA \arrow[r, "Tf"'] & TC & = & TA \arrow[r, "Tf"'] \arrow[ur, "Th"] & TC
\end{tikzcd}$$

there exists a diagram and 2-cells ν and γ as on the right above (suppressing the constraint $Tg \cdot Th \cong Tgh$) which is equal to α, such that:

1. the top triangle is "sub-terminal" meaning that given any 2-cells $\omega, \tau \colon k \Rightarrow h$ as below

$$\begin{tikzcd}
X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Uparrow\gamma" near start] & TB \\
TA \arrow[ur, "Th" description] \arrow[ur, bend right=60, "Tk"'] \arrow[ur, phantom, bend right=30, "\Leftarrow T\omega"] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Uparrow\gamma" near start] & TB \\
TA \arrow[ur, "Th" description] \arrow[ur, bend right=60, "Tk"'] \arrow[ur, phantom, bend right=30, "\Leftarrow T\tau"] &
\end{tikzcd}$$

we have $\omega = \tau$;

2. given any other diagram

$$\begin{tikzcd}
X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Uparrow\phi" near start] & TB \arrow[d, "Tg"] \\
TA \arrow[ur, "Tk" description] \arrow[r, "Tf"'] \arrow[ur, phantom, "\Uparrow T\psi" near end, shift right=4] & TC
\end{tikzcd}$$

equal to α, there exists a (necessarily unique) 2-cell $\overline{\psi} \colon k \Rightarrow h$ such that

$$\begin{tikzcd}
X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Uparrow\phi" near start] & TB \\
TA \arrow[ur, "Tk" description] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Uparrow\gamma" near start] & TB \\
TA \arrow[ur, "Th" description] \arrow[ur, bend right=60, "Tk"'] \arrow[ur, phantom, bend right=30, "\Leftarrow T\overline{\psi}"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & B \arrow[d, "g"] \\
A \arrow[ur, bend left=60, "h"] \arrow[ur, "k" description] \arrow[ur, phantom, bend left=30, "\Uparrow\overline{\psi}"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\psi" near end, shift right=4] & C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & B \arrow[d, "g"] \\
A \arrow[ur, "h"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\nu" near end, shift right=4] & C
\end{tikzcd};$$

3. if α is invertible, then both γ and ν are invertible.

We call a factorization

$$\begin{tikzcd}
X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Uparrow\alpha"] & TB \arrow[d, "Tg"] \\
TA \arrow[r, "Tf"'] & TC
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Uparrow\gamma" near start] & TB \arrow[d, "Tg"] \\
TA \arrow[ur, "Th" description] \arrow[r, "Tf"'] \arrow[ur, phantom, "\Uparrow T\nu" near end, shift right=4] & TC
\end{tikzcd}$$

the *universal factorization* of α if both (1) and (2) are satisfied above.

Earlier in Definition 6.3.6 we defined a 1-cell to be generic when it satisfied a certain strong mixed lifting property. Translating this definition into the context of a pseudofunctor $T \colon \mathscr{A} \to \mathscr{B}$ results in the below definition.

Definition 6.5.4. Let $\mathscr{A}$ and $\mathscr{B}$ be bicategories and let $T \colon \mathscr{A} \to \mathscr{B}$ be a pseudofunctor. Let

$\delta\colon X \longrightarrow TA$ be a generic 1-cell. Then a pair (h, γ) of the form

$$\begin{tikzcd} & TA \arrow[dd, "Th"] \\ X \arrow[ur, "\delta"] \arrow[dr, "z"'] \arrow[r, phantom, "\Swarrow_{\gamma}"] & \\ & TB \end{tikzcd}$$

is *generic* if:

1. the diagram is "sub-terminal" meaning that given any 2-cells $\omega, \tau\colon k \Rightarrow h$ as below

$$\begin{tikzcd} & TA \arrow[dd, "Th"'] \arrow[dd, bend left=60, "Tk"] \arrow[dd, phantom, shift left=3, "\overset{\Leftarrow}{T\omega}"] \\ X \arrow[ur, "\delta"] \arrow[dr, "z"'] \arrow[r, phantom, "\Swarrow_{\gamma}"] & \\ & TB \end{tikzcd} = \begin{tikzcd} & TA \arrow[dd, "Th"'] \arrow[dd, bend left=60, "Tk"] \arrow[dd, phantom, shift left=3, "\overset{\Leftarrow}{T\tau}"] \\ X \arrow[ur, "\delta"] \arrow[dr, "z"'] \arrow[r, phantom, "\Swarrow_{\gamma}"] & \\ & TB \end{tikzcd}$$

   we have $\omega = \tau$;

2. given any other diagram

$$\begin{tikzcd} & TA \arrow[dd, "Tk"] \\ X \arrow[ur, "\delta"] \arrow[dr, "z"'] \arrow[r, phantom, "\Swarrow_{\phi}"] & \\ & TB \end{tikzcd}$$

   and $\lambda\colon h \Rightarrow k$ such that

$$\begin{tikzcd} & TA \arrow[dd, "Th"] \\ X \arrow[ur, "\delta"] \arrow[dr, "z"'] \arrow[r, phantom, "\Swarrow_{\gamma}"] & \\ & TB \end{tikzcd} = \begin{tikzcd} & TA \arrow[dd, "Tk"'] \arrow[dd, bend left=60, "Th"] \arrow[dd, phantom, shift left=3, "\overset{\Leftarrow}{T\lambda}"] \\ X \arrow[ur, "\delta"] \arrow[dr, "z"'] \arrow[r, phantom, "\Swarrow_{\phi}"] & \\ & TB \end{tikzcd}$$

   there exists a (necessarily unique) $\lambda^*\colon k \Rightarrow h$ such that

$$\begin{tikzcd} & TA \arrow[dd, "Tk"] \\ X \arrow[ur, "\delta"] \arrow[dr, "z"'] \arrow[r, phantom, "\Swarrow_{\phi}"] & \\ & TB \end{tikzcd} = \begin{tikzcd} & TA \arrow[dd, "Th"'] \arrow[dd, bend left=60, "Tk"] \arrow[dd, phantom, shift left=3, "\overset{\Leftarrow}{T\lambda^*}"] \\ X \arrow[ur, "\delta"] \arrow[dr, "z"'] \arrow[r, phantom, "\Swarrow_{\gamma}"] & \\ & TB \end{tikzcd}$$

   and $\lambda^*\lambda = \mathrm{id}_h$.

From this definition, the following is clear.

Corollary 6.5.5. *For any universal factorization*

$$\begin{tikzcd} X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Uparrow\alpha"] & TB \arrow[d, "Tg"] \\ TA \arrow[r, "Tf"'] & TC \end{tikzcd} = \begin{tikzcd} X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[r, phantom, "\Uparrow\gamma"', near start, yshift=-2ex] & TB \arrow[d, "Tg"] \\ TA \arrow[r, "Tf"'] \arrow[ur, "Th" description] \arrow[r, phantom, "\Uparrow Tv", near end, yshift=2ex] & TC \end{tikzcd}$$

it follows that (h, γ) *is a generic 2-cell.*

Before proving the main theorem of this section, it is worth defining the spectrum of a pseudofunctor. This is to be the two dimensional analogue of Diers' definition of spectrum of a functor [15, Definition 3].

Definition 6.5.6. Let $\mathscr{A}$ and $\mathscr{B}$ be bicategories and let $T\colon \mathscr{A} \to \mathscr{B}$ be a pseudofunctor such that $\mathscr{B}(X, T-)$ is a lax conical colimit of representables for every $X \in \mathscr{B}$. For each $X \in \mathscr{B}$, define $\mathfrak{M}_X$ as the category with objects given by lax-generic morphisms out of X and morphisms given by representative generic cells between them. We define the *spectrum* of T to be the pseudofunctor

$$\mathbf{Spec}_T\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$$

assigning an object $X \in \mathscr{B}^{\mathrm{op}}$ to $\mathfrak{M}_X$ and a morphism $f\colon Y \to X$ in $\mathscr{B}$ to the functor $\mathfrak{M}_f\colon \mathfrak{M}_X \to \mathfrak{M}_Y$ which takes a generic morphism $\delta\colon X \to TA$ to $\delta'\colon Y \to TP$ where $\delta \cdot f \cong Tu \cdot \delta'$ is a chosen generic factorization of $\delta \cdot f$, and takes a generic 2-cell $\gamma\colon Th \cdot \delta \Rightarrow \sigma$ as on the left below to the generic 2-cell $\overline{\gamma}\colon T\overline{h} \cdot \delta' \Rightarrow \sigma'$ as on the right below

$$\begin{tikzcd} & TP \arrow[r, "Tu"] \arrow[d, phantom, "\cong"] & TA \arrow[dd, "Th"] \\ Y \arrow[ur, "\delta'"] \arrow[r, "f"] \arrow[dr, "\sigma'"'] & X \arrow[ur, "\delta"] \arrow[dr, "\sigma"'] \arrow[d, phantom, "\cong"] \arrow[r, phantom, "\Swarrow\gamma"] & {} \\ & TQ \arrow[r, "Tv"'] & TB \end{tikzcd} = \begin{tikzcd} & TP \arrow[r, "Tu"] \arrow[dd, "T\overline{h}"] \arrow[ddr, phantom, "\Swarrow Tv"] & TA \arrow[dd, "Th"] \\ Y \arrow[ur, "\delta'"] \arrow[dr, "\sigma'"'] \arrow[r, phantom, "\Swarrow\overline{\gamma}"] & {} & \\ & TQ \arrow[r, "Tv"'] & TB \end{tikzcd}$$

constructed as the universal factorization of the left pasting above.

Remark 6.5.7. When $\mathscr{A}$ has a terminal object the spectrum has an especially simple form, namely as the functor $\mathscr{B}(-, T1) : \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$.

Later on we will need to use the following reduced form of the Grothendieck construction of the spectrum.

Lemma 6.5.8. *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be bicategories and let* $T\colon \mathscr{A} \to \mathscr{B}$ *be a pseudofunctor such that* $\mathscr{B}(X, T-)$ *is a lax conical colimit of representables for every* $X \in \mathscr{B}$*. Then the bicategory* *of elements of the spectrum* $\mathbf{Spec}_T \colon \mathscr{B}^{\text{op}} \longrightarrow \mathbf{Cat}$ *is the bicategory*

$$\text{el}\, \mathfrak{M}_{(-)} \cong \int_{\text{oplax}}^{X \in \mathscr{B}} \mathfrak{M}_X$$

consisting of:

Objects *An object is a pair of the form* $(X \in \mathscr{B}, \delta \colon X \longrightarrow TA)$ *where* δ *is a generic out of* X*;*

Morphisms *A morphism* $(X \in \mathscr{B}, \delta \colon X \longrightarrow TA) \nrightarrow (Y \in \mathscr{B}, \sigma \colon Y \longrightarrow TB)$ *is a morphism* $f \colon X \longrightarrow Y$ *in* $\mathscr{B}$ *and a representative generic cell* (h, γ) *as below*

$$\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow_{\gamma}"] & TA \arrow[d, "Th"] \\
Y \arrow[r, "\sigma"'] & TB
\end{tikzcd}$$

2-cells *A 2-cell* $(f, h, \gamma) \Rightarrow (g, k, \phi) \colon (X, \delta) \nrightarrow (Y, \sigma)$ *is a 2-cell* $\nu \colon f \Rightarrow g$ *in* $\mathscr{B}$ *such that*

$$\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, bend right=50, "g"'] \arrow[d, bend left=50, "f"] \arrow[d, phantom, "\overset{\Leftarrow}{\overline{\nu}}"] \arrow[dr, phantom, "\swarrow_{\gamma}"] & TA \arrow[d, "Th"] \\
Y \arrow[r, "\sigma"'] & TB
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, "g"'] \arrow[dr, phantom, "\swarrow^{\phi}"] & TA \arrow[d, bend right=50, "Tk"'] \arrow[d, bend left=50, "Th"] \arrow[d, phantom, "\overset{\Leftarrow}{T\overline{\nu}}"] \\
Y \arrow[r, "\sigma"'] & TB
\end{tikzcd}$$

for some (necessarily unique) $\overline{\nu} \colon h \Rightarrow k$.

Moreover, the cartesian morphisms are precisely those (f, h, γ) *such that* γ *is invertible.*

Proof. We know $\int_{\text{oplax}}^{X \in \mathscr{B}} \mathfrak{M}_{(-)}$ is the bicategory with objects pairs $(X \in \mathscr{B}, m \in \mathfrak{M}_X)$, morphisms $(X \in \mathscr{B}, m \in \mathfrak{M}_X) \nrightarrow (Y \in \mathscr{B}, n \in \mathfrak{M}_Y)$ given by a 1-cell $f \colon X \longrightarrow Y$ and morphism $\alpha \colon m \longrightarrow Ff(n)$ in $\mathfrak{M}_X$, and 2-cells $\nu \colon (f, \alpha) \Rightarrow (g, \beta)$ those 2-cells $\nu \colon f \Rightarrow g$ such that

$$\begin{tikzcd}
m \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta"'] & Ff(n) \arrow[r, "(F\nu)_n"] & Fg(n)
\end{tikzcd}$$

commutes. The objects are clearly as desired. By this formula, a morphism $(X \in \mathscr{B}, \delta \colon X \longrightarrow TA) \nrightarrow (Y \in \mathscr{B}, \sigma \colon Y \longrightarrow TB)$ consists of an $f \colon X \longrightarrow Y$ and an $\alpha \colon \delta \longrightarrow \mathfrak{M}_f(\sigma)$ in $\mathfrak{M}_X$. Hence a morphism is a pair $f, (s, \xi)$ as below

$$\begin{tikzcd}
X \arrow[rr, "\delta"] \arrow[dd, "f"'] \arrow[dr, "\sigma_f"] & \arrow[d, phantom, "\swarrow \xi"] & TA \arrow[dl, "Ts"] \\
 & TT \arrow[dr, "T\overline{f}"] & \\
Y \arrow[rr, "\sigma"'] \arrow[ur, phantom, "\cong"] & & TB
\end{tikzcd}$$

where (s,ξ) is a representative generic cell. Using that generic cells (s,ξ) remain generic when composed with opcartesian cells $\left(\overline{f},\cong\right)$ (because opcartesian cells are themselves generic), the above diagram is itself a generic cell, isomorphic to a unique representative generic cell

$$\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow_{\gamma}"] & TA \arrow[d, "Th"] \\
Y \arrow[r, "\sigma"'] & TB
\end{tikzcd}$$

Conversely, one may form the representative generic factorization of γ

$$\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, "\sigma_f"'] \arrow[d, phantom, shift left=8ex, "\overset{\Leftarrow}{\xi}"] & TA \arrow[d, "Th"] \arrow[dl, "Ts" description] \arrow[d, phantom, shift right=8ex, "\overset{\Leftarrow}{T\zeta}"] \\
TT \arrow[r, "T\overline{f}"'] & TB
\end{tikzcd}$$

to recover (s,ξ) (note that ζ is invertible as genericity of (s,ξ) is preserved by $\left(\overline{f},\mathrm{id}\right)$ and γ is generic). That this is a bijection is a consequence of uniqueness of representative generic factorisations.

It is now worth noting that the opcartesian morphisms, corresponding to the case where (s,ξ) is an equivalence, are those squares where γ is invertible. This is a consequence of Remark 6.5.12, as the case when γ is invertible represents a generic factorization, and to give a choice of generic factorization (h,γ) is to give an equivalence (s,ξ).

By this formula, a 2-cell $\nu\colon (f,s,\xi) \Rightarrow (g,u,\theta)$ consists of a 2-cell $\nu\colon f \Rightarrow g$ such that

$$\begin{tikzcd}
\delta \arrow[r, "{(s,\xi)}"] \arrow[rr, bend right, "{(u,\theta)}"'] & \sigma_f \arrow[r, "(\mathfrak{M}_{\nu})_{\sigma}"] & \sigma_g
\end{tikzcd} \tag{6.5.1}$$

commutes, where $(\mathfrak{M}_{\nu})_{\sigma}$ is given by the representative generic factorization (m,φ) below

$$\begin{tikzcd}
& TT \arrow[dr, bend left, "T\overline{f}"] \arrow[d, phantom, "\cong"] & & & & TT \arrow[dr, bend left, "T\overline{f}"] \arrow[dd, "Tm" description] & \\
X \arrow[ur, bend left, "\sigma_f"] \arrow[dr, bend right, "\sigma_g"'] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\nu"] & Y \arrow[r, "\sigma"] \arrow[d, phantom, "\cong"] & TB & = & X \arrow[ur, bend left, "\sigma_f"] \arrow[dr, bend right, "\sigma_g"'] \arrow[r, phantom, "\Downarrow\varphi"] & {} \arrow[r, phantom, "\Downarrow T\lambda"] & TB \\
& TS \arrow[ur, bend right, "T\overline{g}"'] & & & & TS \arrow[ur, bend right, "T\overline{g}"'] &
\end{tikzcd}$$

Hence given such a ν we have

$$\begin{tikzcd}
X \arrow[rr, "\delta"] \arrow[dd, "\sigma_g"'] \arrow[rrdd, phantom, "\Swarrow\theta" near start] & & TA \arrow[lldd, "Tu" description] \arrow[dd, "Th"] \arrow[lldd, phantom, "\Swarrow T\tau" near end, shift left=6] & & X \arrow[rr, "\delta"] \arrow[dd, "\sigma_g"'] \arrow[dr, "\sigma_f"] \arrow[rr, phantom, "\Swarrow\xi", shift right=4] & & TA \arrow[dd, "Th"] \arrow[dl, "Ts"'] \\
& & & = & \Swarrow\varphi & TT \arrow[dl, "Tm"'] \arrow[dr, "T\overline{f}"] & \Swarrow T\zeta \\
TS \arrow[rr, "T\overline{g}"'] & & TB & & TS \arrow[rr, "T\overline{g}"'] \arrow[rr, phantom, "\Swarrow T\lambda", shift left=4] & & TB
\end{tikzcd}$$

for some (necessarily unique) $\tau\colon h \Rightarrow \overline{g} \cdot u$. Moreover, given a diagram as above we can take the representative generic factorization to recover (6.5.1). $\square$

We can now apply Theorem 6.3.12 to the case where $F\colon \mathscr{A} \to \mathbf{Cat}$ is of the form $\mathscr{B}(X, T-)$ for a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$ to help prove the following theorem.

Theorem 6.5.9. *Let $\mathscr{A}$ and $\mathscr{B}$ be bicategories and let $T\colon \mathscr{A} \to \mathscr{B}$ be a pseudofunctor. Then the following are equivalent:*

1. *the pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$ has a left lax multiadjoint;*
2. *the following conditions hold:*
    - *(a) for every object $X \in \mathscr{A}$ and 1-cell $y\colon X \to TC$ in $\mathscr{B}$, there exists a lax-generic morphism $\delta\colon X \to TA$ and 1-cell $f\colon A \to C$ such that $Tf \cdot \delta \cong y$.*
    - *(b) for any triple of lax-generic morphisms δ, σ and ω, and pair of generic cells (h, θ) and (k, ϕ) as below*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\overset{\theta}{\Nearrow}"] & Y \arrow[r, "g"] \arrow[d, "\sigma" description] \arrow[dr, phantom, "\overset{\phi}{\Nearrow}"] & Z \arrow[d, "\omega"] \\
TA \arrow[r, "Th"'] & TB \arrow[r, "Tk"'] & TC
\end{tikzcd} \tag{6.5.2}$$

the above pasting $(kh, \phi f \cdot \theta)$ is a generic cell[5].

Proof. $(1) \Rightarrow (2)$: Supposing that T has a left lax multiadjoint, it follows that each $\mathscr{B}(X, T-)$ is a lax conical colimit of representables. By Theorem 6.3.12, we have (2)(a), as well as 2(b) when f and g are both the identity at X. To get the full version of (2)(b) we use that

$$\mathbf{P}\colon \int_{\mathrm{lax}}^{X \in \mathscr{B}} \mathfrak{M}_X \to \mathscr{A}$$

is a pseudofunctor, where we have assumed without loss of generality that each $\mathfrak{M}_X$ is the category of generic morphisms out of X and representative cells, using Remark 6.3.14.

---

[5] Suppressing pseudofunctoriality constraints of T.

Indeed, $\int_{\mathrm{lax}}^{X\in\mathscr{B}}\mathfrak{M}_X$ is the bicategory with objects pairs $(X,\delta\colon X\longrightarrow TA)$ and morphisms $(X,\delta\colon X\longrightarrow TA)\nrightarrow(Y,\sigma\colon Y\longrightarrow TB)$ given by triples (f,h,θ) as below

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\delta"'] & Y \arrow[d, "\sigma"] \\
TA \arrow[r, "Th"'] \arrow[ru, Rightarrow, shorten=4mm, "\theta"] & TB
\end{tikzcd}$$

such that (h,θ) is a generic cell. As the lax functoriality constraints of **P** are given by factoring diagrams such as (6.5.2) though a generic, the invertibility of these lax constraints of **P** forces (2)(b).

(2) $\Rightarrow$ (1) : Applying Theorem 6.3.12 to the conditions 2(a) and 2(b) (only needing the case when f and g are identities at X), it follows that we may write

$$\mathscr{B}(X,T-)\simeq\int_{\mathrm{lax}}^{m\in\mathfrak{M}_X}\mathscr{A}\left(P_m^X,-\right)$$

where $\mathfrak{M}_X$ is the category of generic morphisms out of X and representative generic cells between them. From this, we recover the spectrum $\mathbf{Spec}_T\colon\mathscr{B}^{\mathrm{op}}\longrightarrow\mathbf{Cat}$ taking each X to $\mathfrak{M}_X$. Also, we again we have the canonical normal lax functor

$$\mathbf{P}\colon\int_{\mathrm{lax}}^{X\in\mathscr{B}}\mathfrak{M}_X\longrightarrow\mathscr{A}$$

defined as in the reverse implication. The full version of (2)(b) forces this to be a pseudofunctor as required. □

Remark 6.5.10. The reader will notice that condition (2)(b) where f and g are identities at X is what is required to ensure that $P_{(-)}^X\colon\mathfrak{M}_X\longrightarrow\mathscr{A}$ is a pseudofunctor, whilst the full version of 2(b) is what is required to ensure

$$\mathbf{P}\colon\int_{\mathrm{lax}}^{X\in\mathscr{B}}\mathfrak{M}_X\longrightarrow\mathscr{A}$$

is a pseudofunctor.

Under the conditions of this theorem, we also have a notion of generic factorisations on 2-cells, in a sense we now describe.

Remark 6.5.11. Suppose T has a left lax multiadjoint, δ and σ are generic objects, and consider a 2-cell $\alpha\colon Tf \cdot \delta \Rightarrow Tg \cdot \sigma$. Then α has a T-generic factorization

$$\begin{tikzcd}
& TA \arrow[dr, "Tf"] & & & & TA \arrow[dd, "Th"] \arrow[dr, "Tf"] & \\
X \arrow[ur, "\delta"] \arrow[dr, "\sigma"'] & \Downarrow\alpha & TC & = & X \arrow[ur, "\delta"] \arrow[dr, "\sigma"'] & \Downarrow\gamma \qquad \Downarrow T\nu & TC \\
& TB \arrow[ur, "Tg"'] & & & & TB \arrow[ur, "Tg"'] &
\end{tikzcd}$$

Also note that any map $k\colon X \to TC$ can be factored as $T\overline{k} \cdot \xi$ for some generic ξ and morphism $\overline{k}$, and so when T is surjective on objects we have a T-generic factorization of every 1-cell and 2-cell in the bicategory $\mathscr{B}$.

Rephrasing the statements in Section 6.3 concerning uniqueness of generic factorisations in the context of a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$ yields the following.

Remark 6.5.12. Specializing Proposition 6.3.19 to the case where $F\colon \mathscr{A} \to \mathbf{Cat}$ is $\mathscr{B}(X, T-)$ for a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$ and $X \in \mathscr{B}$, says given a 1-cell $f\colon X \to TA$ in $\mathscr{B}$ and two representative generic factorisations

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\delta"'] & \Uparrow\alpha & TA & & X \arrow[rr, "f"] \arrow[dr, "\sigma"'] & \Uparrow\beta & TA \\
& TP \arrow[ur, "T\overline{f}"'] & & & & TQ \arrow[ur, "T\overline{g}"'] &
\end{tikzcd}$$

as above, there exists a unique invertible representative generic cell $(h, \gamma) : \delta \to \sigma$ such that the representative of

$$\begin{tikzcd}
& X \arrow[dl, "\delta"'] \arrow[dr, "\sigma"] \arrow[rr, "f"] \arrow[d, phantom, "\gamma \Nearrow"] & \arrow[d, phantom, "\beta \Nearrow"] & TA \\
TP \arrow[rr, "Th"'] & {} & TQ \arrow[ur, "T\overline{g}"'] &
\end{tikzcd}$$

is equal to $\left(\overline{f}, \alpha\right)$.

6.6 Comparing to Weber's familial 2-functors

The purpose of this section is to compare our definition of a familial 2-functor $T\colon \mathscr{A} \to \mathscr{B}$ between 2-categories (assuming $\mathscr{A}$ has a terminal object) with Weber's definition. It turns out that these two definitions are essentially equivalent. Note also that Weber's definition assumes some "strictness conditions" (such as identity 2-cells factoring into identity 2-cells) which are natural conditions on 2-functors, but arguably less natural in the case of pseudofunctors.

We first recall the notion of generic morphism corresponding to what Weber refers to as the "naive" 2-categorical analogue of parametric right adjoints [53].

Definition 6.6.1. Suppose $\mathscr{A}$ and $\mathscr{B}$ are 2-categories. Given a 2-functor $T\colon \mathscr{A} \to \mathscr{B}$ we say a morphism $x\colon X \longrightarrow TA$ is *naive-generic* if:

1. for any commuting square as on the left below

$$\begin{tikzcd} X \arrow[r, "z"] \arrow[d, "x"'] & TB \arrow[d, "Tg"] \\ TA \arrow[r, "Tf"'] & TC \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "z"] \arrow[d, "x"'] & TB \arrow[d, "Tg"] \\ TA \arrow[r, "Tf"'] \arrow[ur, "Th"] & TC \end{tikzcd}$$

   there exists a unique $h\colon A \longrightarrow B$ such that $Th \cdot x = z$ and $f = gh$;

2. for two commuting diagrams

$$\begin{tikzcd} X \arrow[r, "z_1"] \arrow[d, "x"'] & TB \arrow[d, "Tg"] \\ TA \arrow[r, "Tf"'] \arrow[ur, "Th_1"] & TC \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "z_2"] \arrow[d, "x"'] & TB \arrow[d, "Tg"] \\ TA \arrow[r, "Tf"'] \arrow[ur, "Th_2"] & TC \end{tikzcd}$$

   the 2-cells $\theta\colon z_1 \Rightarrow z_2$ such that $Tg \cdot \theta = \mathrm{id}$ bijectively correspond to 2-cells $\overline{\theta}\colon h_1 \Rightarrow h_2$ such that $T\left(\overline{\theta}\right) \cdot x = \theta$ and $g \cdot \overline{\theta} = \mathrm{id}$.

Definition 6.6.2. Suppose $\mathscr{A}$ and $\mathscr{B}$ are 2-categories, and that $\mathscr{A}$ has a terminal object. We say a 2-functor $T\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a *naive parametric right adjoint* if every $f\colon X \longrightarrow TA$ factors as $T\overline{f} \cdot x$ for a naive-generic morphism x.

Weber's definition of familiity requires certain maps in a 2-category to be fibrations. Thus we will need to recall the definition of fibration in a 2-category $\mathscr{B}$. Note that when $\mathscr{B}$ is finitely complete there are other equivalent characterizations of fibrations [45].

Definition 6.6.3. We say a morphism $p\colon E \longrightarrow B$ in a 2-category $\mathscr{B}$ is a *fibration* if:

1. for every $X \in \mathscr{B}$, the functor $\mathscr{B}(X, p) : \mathscr{B}(X, E) \longrightarrow \mathscr{B}(X, B)$ is a fibration;
2. for every $f\colon X \longrightarrow Y$ in $\mathscr{B}$, the functor $\mathscr{B}(f, E) : \mathscr{B}(Y, E) \longrightarrow \mathscr{B}(X, E)$ preserves cartesian morphisms.

If we have a choice of cartesian lifts which strictly respects composition and identities we say the fibration *splits*.

We now have the required background to define familiity in the sense of Weber.

Definition 6.6.4. Suppose $\mathscr{A}$ and $\mathscr{B}$ are 2-categories and that $\mathscr{A}$ has a terminal object. We say a 2-functor $T\colon \mathscr{A} \longrightarrow \mathscr{B}$ is *Weber-familial* if

1. T is a naive parametric right adjoint;
2. for every $A \in \mathscr{A}$, and unique $t_A \colon A \to 1$ in $\mathscr{A}$, the morphism $Tt_A \colon TA \to T1$ is a split fibration in $\mathscr{B}$.

The following is Weber's analogue of lax-generic morphisms.

Definition 6.6.5. Suppose $\mathscr{A}$ and $\mathscr{B}$ are 2-categories. Given a 2-functor $T \colon \mathscr{A} \to \mathscr{B}$ for which each $Tt_A \colon TA \to T1$ is a split fibration, we say a morphism $x \colon X \to TA$ is *Weber-lax-generic* if for any 2-cell α as on the left below,

$$\begin{tikzcd} X \arrow[r, "z"] \arrow[d, "x"'] \arrow[dr, phantom, "\Uparrow\alpha"] & TB \arrow[d, "Tg"] \\ TA \arrow[r, "Tf"'] & TC \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "z"] \arrow[d, "x"'] \arrow[dr, phantom, "\Uparrow\gamma", near start] & TB \arrow[d, "Tg"] \\ TA \arrow[r, "Tf"'] \arrow[ur, "Th" description] \arrow[ur, phantom, "\Uparrow Tv", near end, shift left=2] & TC \end{tikzcd}$$

there exists a unique factorization (h, γ, v) as above such that (h, γ) is chosen $Tt_B \colon TB \to T1$ cartesian.[6]

The following lemma shows that for Weber-familial 2-functors T, the lax-generics of both our sense and Weber's coincide, and our generic 2-cells can equivalently be characterized as certain cartesian morphisms.

Lemma 6.6.6. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are 2-categories and that $\mathscr{A}$ has a terminal object. Let $T \colon \mathscr{A} \to \mathscr{B}$ be a Weber-familial 2-functor. Define $\mathfrak{M}$ as the category with objects given by chosen naive-generics $\delta \colon X \to TA$ (meaning to be identified with another naive-generic $\sigma \colon X \to TB$ if there exists a pair (h, γ) as below with h invertible and γ an identity), and morphisms given by pairs (h, γ)*

$$\begin{tikzcd} & TA \arrow[dd, "Th"] \\ X \arrow[ur, "\delta"] \arrow[dr, "\sigma"'] \arrow[r, phantom, "\Swarrow\gamma"] & \\ & TB \end{tikzcd}$$

where γ is chosen $Tt_B \colon TB \to T1$ cartesian. Then:

1. *for every $X \in \mathscr{B}$ we have isomorphisms*

$$\mathscr{B}(X, T-) \cong \int_{\mathrm{lax}}^{m \in \mathfrak{M}} \mathscr{A}(P_m, -);$$

---

[6] This definition of lax-generics has the downside that it assumes some family conditions, thus not allowing for a theorem describing an equivalence between family and lax-generic factorisations.

2. *a map $\delta\colon X \to TA$ in $\mathscr{B}$ is naive-generic if and only if it is strict*[7] *lax-generic;*

3. *a 2-cell in $\mathscr{B}$ as below*

$$\begin{tikzcd}
& TA \arrow[dd, "Th"] \\
X \arrow[ur, "\delta"] \arrow[dr, "z"'] \arrow[r, phantom, "\Swarrow\gamma"] & \\
& TB
\end{tikzcd}$$

is generic if and only if it is $Tt_B\colon TB \to T1$ cartesian.

Proof. $(1):$ It suffices to check that the functors

$$\int_{\text{lax}}^{m\in\mathfrak{M}} \mathscr{A}(P_m, W) \to \mathscr{B}(X, TW)$$

are isomorphisms. That this assignment is bijective on objects is a consequence of the well known one-dimensional case (see Proposition 4.2.7). That the assignment on morphisms

$$\begin{tikzcd}
& TP_m \arrow[dd, "Th"] & P_m \arrow[dd, "h"'] \arrow[dr, "f"] & & & & TP_m \arrow[dd, "Th"] \arrow[dr, "Tf"] & \\
X \arrow[ur, "\delta_m"] \arrow[dr, "\delta_{m'}"'] \arrow[r, phantom, "\Downarrow\alpha"] & & \Downarrow\beta & W & \mapsto & X \arrow[ur, "\delta_m"] \arrow[dr, "\delta_{m'}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \Downarrow T\beta & TW \\
& TP_{m'} & P_{m'} \arrow[ur, "g"'] & & & & TP_{m'} \arrow[ur, "Tg"'] &
\end{tikzcd}$$

is bijective follows from the fact each naive-generic is Weber-lax generic [53, Temma 5.8]. Naturality is also an easy consequence of this fact.

$(2):$ If δ is naive-generic, and thus isomorphic to a representative naive-generic, then δ is lax-generic by (1). If δ is strict lax-generic, then from a $\theta\colon z_1 \Rightarrow z_2$ we have a universal factorization

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "x"'] \arrow[dr, phantom, "\Uparrow\theta"] & TA \arrow[d, "Th_2"] & = & X \arrow[r, "x"] \arrow[d, "x"'] \arrow[dr, phantom, "\Uparrow\mathrm{id}", near start] & TA \arrow[d, "Th_2"] \\
TA \arrow[r, "Th_1"'] & TB & & TA \arrow[r, "Th_1"'] \arrow[ur, "T1"] \arrow[ur, phantom, "\Uparrow T\bar{\theta}"', near end] & TB
\end{tikzcd}$$

where we have used that $Tg \cdot \theta$ is an identity to see the top right triangle above can be taken as an identity. In this way, we recover the bijection required of a naive-generic.

$(3):$ Consider a 2-cell

$$\begin{tikzcd}
& TA \arrow[dd, "Th"] \\
X \arrow[ur, "\delta"] \arrow[dr, "z"'] \arrow[r, phantom, "\Swarrow\gamma"] & \\
& TB
\end{tikzcd}$$

---

[7] By strict we mean identity 2-cells universally factor into identity 2-cells.

If this 2-cell is generic, then we have a factorization

$$\begin{tikzcd}
X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Uparrow\gamma"] & TA \arrow[d, "T\mathrm{id}"] & & X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Uparrow\phi", pos=0.25] & TA \arrow[d, "T\mathrm{id}"] \\
TA \arrow[r, "Th"'] & TB \arrow[r, phantom, "="] & & TA \arrow[r, "Th"'] \arrow[ur, "Tk"] \arrow[ur, phantom, "\Uparrow T\lambda"', pos=0.75, shift right=3] & TB
\end{tikzcd} \tag{6.6.1}$$

where ϕ is chosen cartesian. By genericity of γ, we have an $\lambda^*\colon k \Rightarrow h$ such that

$$\begin{tikzcd}
& TA \arrow[dd, "Tk"] & & & TA \arrow[dd, "Th"] \arrow[dd, bend left=60, "Tk"] \arrow[dd, phantom, "\overset{\Leftarrow}{T\lambda^*}", shift left=5] \\
X \arrow[ur, "\delta"] \arrow[dr, "z"'] \arrow[r, phantom, "\Swarrow\phi"] & & = & X \arrow[ur, "\delta"] \arrow[dr, "z"'] \arrow[r, phantom, "\Swarrow\gamma"] & \\
& TB & & & TB
\end{tikzcd} \tag{6.6.2}$$

and $\lambda^*\lambda = \mathrm{id}_h$. Substituting (6.6.1) into (6.6.2) and using that δ is Weber-lax-generic gives $\lambda\lambda^* = \mathrm{id}_k$. Conversely, if this 2-cell is cartesian we then have a factorization

$$\begin{tikzcd}
X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Uparrow\gamma"] & TA \arrow[d, "T\mathrm{id}"] & & X \arrow[r, "z"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Uparrow\phi", pos=0.25] & TA \arrow[d, "T\mathrm{id}"] \\
TA \arrow[r, "Th"'] & TB \arrow[r, phantom, "="] & & TA \arrow[r, "Th"'] \arrow[ur, "Tk"] \arrow[ur, phantom, "\Uparrow T\lambda"', pos=0.75, shift right=3] & TB
\end{tikzcd}$$

where (k, ϕ) is a generic 2-cell (which must also be cartesian by the above argument). Since ϕ and γ are cartesian, and thus isomorphic to chosen cartesian morphisms, it follows that λ is invertible (by uniqueness of chosen cartesian factorisations). □

Finally, we give the main result of this section, showing that for 2-functors $T\colon \mathscr{A} \longrightarrow \mathscr{B}$ our lax-multiadjoint condition is essentially equivalent to Weber's familiarity condition.

Theorem 6.6.7. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are 2-categories and that $\mathscr{A}$ has a terminal object. Then for a 2-functor $T\colon \mathscr{A} \longrightarrow \mathscr{B}$ the following are equivalent:*

1. *T is Weber-familial;*
2. *T has a strict*[8] *left lax multiadjoint.*

Proof. (1) $\Rightarrow$ (2) : Supposing $T\colon \mathscr{A} \longrightarrow \mathscr{B}$ is Weber-familial, we have that each $\mathscr{B}(X, T-)$ is a lax conical colimit of representables by Lemma 6.6.6 part (1). Also, as the generic 2-cells may be identified with the cartesian 2-cells, we know since the fibration $Tt_B\colon TB \longrightarrow T1$ respects precomposition we have the following property: for any generic 2-cell out of an

---

[8] By strict we mean isomorphic to a lax conical colimit of representables in place of equivalent, and that the reindexings $P^X_{(-)}$ are 2-functors instead of pseudofunctors.

$X \in \mathscr{B}$ as on the left below

$$\begin{tikzcd}
 & & TA \arrow[dd, "Th"] \\
X \arrow[urr, "\delta"] \arrow[drr, "z"'] & \Swarrow\gamma & \\
 & & TB
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & & TA \arrow[dd, "Th"] \\
Y \arrow[r, "k"] & X \arrow[urr, "\delta"] \arrow[drr, "z"'] & \Swarrow\gamma & \\
 & & & TB
\end{tikzcd}
\tag{6.6.3}$$

and map $k\colon Y \to X$ in $\mathscr{B}$, the right diagram is a generic 2-cell. It is this property (along with closure of generic cells under composition) which gives (2)(b) of Theorem 6.5.9.

$(2) \Rightarrow (1):$ Suppose $T\colon \mathscr{A} \to \mathscr{B}$ is a strict left lax multiadjoint. Then T is a naive parametric right adjoint since T has strict lax generic factorisations, and lax-generic implies naive generic (shown in the proof of Lemma 6.6.6).

It remains to check that each $Tt_A\colon TA \to T1$ is a split fibration. To see this, note that for each $X \in \mathscr{B}$ the functor $\mathscr{B}(X, TA) \to \mathscr{B}(X, T1)$ may be written as the functor

$$\int_{\text{lax}}^{m\in\mathfrak{M}} \mathscr{A}(P_m, A) \to \int_{\text{lax}}^{m\in\mathfrak{M}} \mathscr{A}(P_m, 1) \cong \mathfrak{M}$$

defined by the assignment

$$\begin{tikzcd}
m \arrow[dd, "\lambda"'] & P_m \arrow[dd, "P_\lambda"'] \arrow[dr, "f"] & & & m \arrow[dd, "\lambda"] \\
 & \Downarrow\beta & A & \longmapsto & \\
m' & P_{m'} \arrow[ur, "g"'] & & & m'\,.
\end{tikzcd}$$

It is straightforward to verify that for each $\left(m', g\colon P'_m \to A\right)$ and $\lambda\colon m \to m'$ we recover a cartesian lift

$$\begin{tikzcd}
m \arrow[dd, "\lambda"'] & P_m \arrow[dd, "P_\lambda"'] \arrow[dr, "g\cdot P_\lambda"] & & & m \arrow[dd, "\lambda"] \\
 & \Downarrow\mathrm{id} & A & \longmapsto & \\
m' & P_{m'} \arrow[ur, "g"'] & & & m'
\end{tikzcd}$$

and it is clear the canonical choice of cartesian lifts given above splits. The cartesian morphisms are diagrams as above (with the identity 2-cell possibly replaced by an isomorphism), and these correspond to generic cells in $\mathscr{B}(X, TA)$. That for each $k\colon Y \to X$ the functor $\mathscr{B}(k, TA) : \mathscr{B}(Y, TA) \to \mathscr{B}(X, TA)$ preserves cartesian morphisms then follows from condition (2)(b) of Theorem 6.5.9. □

6.7 Examples of familial pseudofunctors

We will first consider some simple examples of lax multiadjoints which concern pseudofunctors $T\colon \mathscr{A} \to \mathscr{B}$ where $\mathscr{A}$ is a 1-category. Our first and simplest examples of such pseudofunctors $T\colon \mathscr{A} \to \mathscr{B}$ concern the universal embeddings into bicategories of spans and polynomials.

The reader will also recall that in this setting where $\mathscr{A}$ is a 1-category, $\mathrm{el}\, F = \mathrm{el}\, \mathscr{B}(X, T-)$ is a 1-category for each $X \in \mathscr{B}$, and so the mixed lifting properties become the usual lifting properties. Indeed, it is clear that in such cases every pair (h, γ) out of a generic 1-cell is a generic 2-cell.

Example 6.7.1. *The canonical pseudofunctor* $T\colon \mathcal{E} \to \mathbf{Span}(\mathcal{E})$ *has a left lax multiadjoint. To see this, first observe that a span* $X \nrightarrow TA$ *is generic if it is isomorphic to the form*

$$\begin{tikzcd}
& TA \arrow[dl, "s"'] \arrow[dr, "\mathrm{id}"] & \\
X & & TA
\end{tikzcd}$$

This is since for a general span (s,t) *genericity would imply we can factor the diagram on the left below*

$$\begin{tikzcd}
X \arrow[r, "{(s,1)}"] \arrow[d, "{(s,t)}"'] & TM \arrow[d, "Tt"] \\
TA \arrow[r, "T\mathrm{id}"'] \arrow[ur, phantom, "\Uparrow \mathrm{id}"] & TA
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "{(s,1)}"] \arrow[d, "{(s,t)}"'] \arrow[dr, phantom, "\Uparrow\gamma", pos=0.3] & TM \arrow[d, "Tt"] \\
TA \arrow[r, "T\mathrm{id}"'] \arrow[ur, "Tu" description] & TA \arrow[ul, phantom, "\Uparrow Tv", pos=0.3]
\end{tikzcd}$$

as on the right above, where v *is necessarily an identity and* γ *invertible. Hence* $tu = \mathrm{id}$ *and* ut *is invertible, showing that* t *is invertible. Conversely, to see such a* $(s,1)$ *is generic, note that any diagram as on the left below*

$$\begin{tikzcd}
X \arrow[r, "{(u,v)}"] \arrow[d, "{(s,1)}"'] & TM \arrow[d, "Tq"] \\
TA \arrow[r, "Tp"'] \arrow[ur, phantom, "\Uparrow \alpha"] & TB
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "{(u,v)}"] \arrow[d, "{(s,1)}"'] \arrow[dr, phantom, "\Uparrow\gamma", pos=0.3] & TM \arrow[d, "Tq"] \\
TA \arrow[r, "Tp"'] \arrow[ur, "Tv\theta" description] & TB \arrow[ul, phantom, "\Uparrow T\mathrm{id}", pos=0.3]
\end{tikzcd}$$

universally factors as on the right above, where α *and* γ *are the respective morphisms of spans*

$$\alpha\colon\quad
\begin{tikzcd}
& TA \arrow[dl, "s"'] \arrow[dr, "p"] \arrow[dd, "\theta"] & \\
X & & TB \\
& \bullet \arrow[ul, "u"] \arrow[ur, "qv"'] &
\end{tikzcd}
\qquad
\gamma\colon\quad
\begin{tikzcd}
& TA \arrow[dl, "s"'] \arrow[dr, "v\theta"] \arrow[dd, "\theta"] & \\
X & & TM \\
& \bullet \arrow[ul, "u"] \arrow[ur, "v"'] &
\end{tikzcd}$$

As all cells between generic morphisms are generic, it follows that the category $\mathfrak{M}_X$ *of generics out of* X *is the slice* $\mathcal{E}/X$*, and so for any* $X \in \mathcal{E}$ *we may take* $P_{(-)}$ *as the functor* dom: *$\mathcal{E}/X \longrightarrow \mathcal{E}$, giving*

$$\mathbf{Span}\,(\mathcal{E})\,(X,T-) \cong \int_{\mathrm{lax}}^{m\in\mathcal{E}/X} \mathcal{E}\,(P_m,-)$$

Dual to the above, we see that $T\colon \mathcal{E} \longrightarrow \mathbf{Span}\,(\mathcal{E})^{\mathrm{co}}$ admits oplax-generic factorisations; indeed we may write

$$\mathbf{Span}\,(\mathcal{E})^{\mathrm{co}}\,(X,T-) \cong \int_{\mathrm{oplax}}^{m\in\mathcal{E}/X} \mathcal{E}\,(P_m,-)$$

Moreover, the pseudofunctor $T\colon \mathcal{E} \longrightarrow \mathbf{Span}_{\mathrm{iso}}\,(\mathcal{E})$ admits both lax and oplax generic factorisations, as we may write

$$\mathbf{Span}_{\mathrm{iso}}\,(\mathcal{E})\,(X,T-) \cong \int_{\mathrm{lax}}^{m\in(\mathcal{E}/X)_{\mathrm{iso}}} \mathcal{E}\,(P_m,-) \cong \int_{\mathrm{oplax}}^{m\in(\mathcal{E}/X)_{\mathrm{iso}}} \mathcal{E}\,(P_m,-)$$

where $(\mathcal{E}/X)_{\mathrm{iso}}$ contains the objects of $\mathcal{E}/X$ and only those morphisms which are invertible. The reader will also note that we do not have

$$\mathbf{Span}_{\mathrm{iso}}\,(\mathcal{E})\,(X,T-) \simeq \sum_{\mathrm{ob}\,\mathcal{E}/X} \mathcal{E}\,(P_m,-)$$

As for each $T \in \mathcal{E}$, the right above is a discrete category, but isomorphisms of spans are not unique (and so the canonical assignment is not fully faithful).

The case of spans is also interesting as it gives a simple example in which generic factorisations are not unique in the sense that one might initially expect. That is to say, given two generic factorisations

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\delta"'] & \arrow[d, phantom, "\Uparrow\alpha"] & TA \\ & TP \arrow[ur, "T\overline{f}"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\delta"'] & \arrow[d, phantom, "\Uparrow\beta"] & TA \\ & TP \arrow[ur, "T\overline{g}"'] & \end{tikzcd}$$

(meaning isomorphisms α and β as above), there is not necessarily a coherent comparison 2-cell $\overline{f} \Rightarrow \overline{g}$.

Example 6.7.2. *Consider a span*

$$\begin{tikzcd} & \mathbf{2} \arrow[dl, "!"'] \arrow[dr, "\sigma"] & \\ \mathbf{1} & & \mathbf{2} \end{tikzcd}$$

where σ is the swap map. Here we have the two distinct generic factorisations

$$\begin{tikzcd} \mathbf{1} \arrow[rr, "(!{,}\sigma)"] \arrow[dr, "(!{,}1)"'] & \arrow[d, phantom, "\Uparrow\sigma"] & T\mathbf{2} \\ & T\mathbf{2} \arrow[ur, "T1"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1} \arrow[rr, "(!{,}\sigma)"] \arrow[dr, "(!{,}1)"'] & \arrow[d, phantom, "\Uparrow\mathrm{id}"] & T\mathbf{2} \\ & T\mathbf{2} \arrow[ur, "T\sigma"'] & \end{tikzcd}$$

In the following examples we will omit the verification that the generic morphisms are classified correctly.

Example 6.7.3. *Letting $\mathcal{E}$ be a locally cartesian closed category with chosen pullbacks, the canonical pseudofunctor $T\colon \mathcal{E} \longrightarrow \mathbf{Poly}(\mathcal{E})$ has a left lax multiadjoint. Indeed, a polynomial $X \nrightarrow TA$ is generic precisely when it is isomorphic to the form*

$$\begin{tikzcd}
& TM \arrow[ld, "s"'] \arrow[r, "p"] & TA \arrow[rd, "\mathrm{id}"] & \\
X & & & TA
\end{tikzcd}$$

and one may verify that any cell (general 2-cell of polynomials)

$$\begin{tikzcd}
& TA \arrow[dd, "Tt"] \\
X \arrow[ru, "{(s,p,\mathrm{id})}"] \arrow[rd, "{(u,q,v)}"'] \arrow[r, phantom, "\Downarrow\gamma"] & \\
& TB
\end{tikzcd}$$

is generic. Consequently, we may take $P_{(-)}$ as the functor $\mathrm{pr}\colon \Pi_{\mathcal{E}}(\mathcal{E}/X) \longrightarrow \mathcal{E}$ where $\Pi_{\mathcal{E}}(\mathcal{E}/X)$ is the category with objects given by spans

$$\begin{tikzcd}
X & T \arrow[l, "f"'] \arrow[r, "g"] & U
\end{tikzcd}$$

out of X, and morphisms of spans from $(f,g) \nrightarrow (f',g')$ given by a pair $\alpha\colon W \longrightarrow T$ and $\beta\colon U \longrightarrow U'$ rendering commutative the diagram

$$\begin{tikzcd}
& T \arrow[ld, "f"'] \arrow[r, "g"] & U \arrow[d, dotted, "\beta"] \\
X & W \arrow[u, dotted, "\alpha"] \arrow[ru] \arrow[d] \arrow[rd, phantom, "\mathrm{pb}"] & U' \\
& T' \arrow[lu, "f'"] \arrow[ru, "g'"'] &
\end{tikzcd}$$

such that W is the fixed chosen pullback of β and g'. As a consequence we have

$$\mathbf{Poly}(\mathcal{E})(X,T-) \cong \int_{\mathrm{lax}}^{m\in\Pi_{\mathcal{E}}(\mathcal{E}/X)} \mathcal{E}(P_m,-)$$

for all $X \in \mathbf{Poly}(\mathcal{E})$.

Remark 6.7.4. By the above, the usual inclusion $\mathbf{Span}(\mathcal{E}) \longrightarrow \mathbf{Poly}(\mathcal{E})$ can be seen as coming from the unit components $u_{\mathcal{E}/X}\colon \mathcal{E}/X \longrightarrow \Pi_{\mathcal{E}}(\mathcal{E}/X)$ of the pseudomonad $\Pi_{\mathcal{E}}$ for fibrations with products. Indeed, the family of functors $\mathbf{Span}(\mathcal{E})(X,Y) \longrightarrow \mathbf{Poly}(\mathcal{E})(X,Y)$ may be written as the resulting functors

$$\int_{\mathrm{lax}}^{m\in\mathcal{E}/X} \mathcal{E}\left(P_m,Y\right) \to \int_{\mathrm{lax}}^{m\in\Pi_{\mathcal{E}}(\mathcal{E}/X)} \mathcal{E}\left(P_m,Y\right)$$

We now give a more complicated example, where $\mathscr{A}$ is not a 1-category. In this situation the mixed lifting properties are necessary (unlike the earlier examples where usual liftings would suffice), and so it is no longer the case that every (h,γ) out of a generic morphism is a generic 2-cell.

Example 6.7.5. *The canonical pseudofunctor* $T\colon \mathbf{Span}\left(\mathcal{E}\right)^{\mathrm{co}} \to \mathbf{Poly}\left(\mathcal{E}\right)$ *is such that* T^{op} *has a left lax multiadjoint. Here a polynomial* $TA \nrightarrow X$ *is opgeneric (meaning the opposite polynomial is generic) if it is isomorphic to the form*

$$\begin{tikzcd}
& TA \arrow[r, "\mathrm{id}"] \arrow[dl, "\mathrm{id}"'] & TA \arrow[dr, "f"] & \\
TA & & & X
\end{tikzcd}$$

and a pair $\left((s,t),\gamma\right)$ *out of a opgeneric as below*

$$\begin{tikzcd}
& X \arrow[dd, "T(s,t)=(s,t,1)"] \\
TA \arrow[ur, "{(1,1,f)}"] \arrow[dr, "{(v,u,g)}"'] \arrow[r, phantom, "\Swarrow\gamma"] & \\
& TB
\end{tikzcd}$$

is generic when $\gamma\colon\ (s,t,f) \Rightarrow (v,u,g)$ *is a cartesian morphism of polynomials. We note also that cartesian morphisms of polynomials are closed under vertical composition as well as precomposition by another polynomial.*

Given a general morphism of polynomials $\phi\colon\ (s,t,f) \Rightarrow (v,u,g)$ *as given by the diagram*

$$\begin{tikzcd}
& I \arrow[r, "t"] \arrow[dl, "s"'] & M \arrow[dr, "f"] \arrow[dd, "h"] & \\
TA & P \arrow[u, "e"] \arrow[ur, "u'"] \arrow[d, "h'"'] \arrow[dr, phantom, "\mathrm{pb}"] & & X \\
& J \arrow[ul, "v"] \arrow[r, "u"'] & N \arrow[ur, "g"'] &
\end{tikzcd}$$

the op-generic factorization of ϕ *is given by*

$$\begin{tikzcd}
& & & & M \arrow[dr, "{(1,1,f)}"] & \\
TA \arrow[r, bend left, "{(s,t,f)}"] \arrow[r, bend right, "{(v,u,g)}"'] \arrow[r, phantom, "\Downarrow\phi"] & X & = & TA \arrow[ur, "{T(s,t)}"] \arrow[dr, "{T(v,u)}"'] \arrow[r, phantom, "\Downarrow Tv"] & {} \arrow[r, phantom, "\Downarrow\gamma"] & X \\
& & & & N \arrow[uu, "{T(h,1)}" description] \arrow[ur, "{(1,1,g)}"'] &
\end{tikzcd}$$

where v is the reversed morphism of spans on the left below

$$\begin{tikzcd}
& I \arrow[dl, "s"'] \arrow[dr, "t"] & \\
TA & & M \\
& P \arrow[uu, "e"] \arrow[ul, "vh'"] \arrow[ur, "u'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& M \arrow[r, "1"] \arrow[dl, "h"'] \arrow[dd, "h"] & M \arrow[dr, "f"] \arrow[dd, "h"] & \\
N & & & X \\
& N \arrow[ul, "1"] \arrow[r, "1"'] & N \arrow[ur, "g"'] &
\end{tikzcd}$$

and γ is the cartesian morphism of polynomials on the right above. It follows that for any $X \in \mathcal{E}$ we may take $P_{(-)}$ as the functor

$$\mathcal{E}/X \xrightarrow{\text{dom}} \mathcal{E} \xrightarrow{\iota} \mathbf{Span}\,(\mathcal{E})^{\text{coop}}$$

where ι assigns each morphism $h\colon A \to B$ to $(h, 1_A) \in \mathbf{Span}\,(\mathcal{E})^{\text{coop}}$, and get

$$\mathbf{Poly}\,(\mathcal{E})^{\text{op}}\,(X, T-) \cong \int_{\text{lax}}^{m \in \mathcal{E}/X} \mathbf{Span}\,(\mathcal{E})^{\text{coop}}\,(P_m, -)\,.$$

We now give a natural example which does not come from a pseudofunctor of bicategories $T\colon \mathscr{A} \to \mathscr{B}$. Indeed, the following may be seen as the main motivating example for this paper.

Example 6.7.6. *Consider the pseudofunctor* **Fam**: **CAT** $\to$ **CAT** *sending a category $\mathcal{C}$ to the category* **Fam** $(\mathcal{C})$ *with objects given by families of objects of $\mathcal{C}$ denoted $(A_i \in \mathcal{C} : i \in I)$, and morphisms $(A_i \in \mathcal{C} : i \in I) \nrightarrow \big(B_j \in \mathcal{C} : j \in J\big)$ given by a reindexing $\varphi\colon I \to J$ along with comparison maps $A_i \to B_{\varphi(i)}$ for each $i \in I$.*

Now, the generic objects of el **Fam** *are those elements of the form $(I, (i : i \in I))$ for a set I. And it is clear that for any general element $\big(\mathcal{C}, \big(B_j : j \in J\big)\big)$ of* el **Fam** *that we have the "generic factorization" (that is an opcartesian map from a generic)*

$$(J, (j : j \in J)) \xrightarrow{(B_{(-)}, \text{id})} \big(\mathcal{C}, \big(B_j : j \in J\big)\big)$$

Also, a general morphism out of a generic object

$$(I, (i : i \in I)) \xrightarrow{(H_{(-)}, (\varphi, \gamma))} \big(\mathcal{C}, \big(B_j : j \in J\big)\big)$$

consists of a functor $H_{(-)}\colon I \to \mathcal{C}$, a function $\varphi\colon I \to J$, and morphisms $\gamma_i\colon H_i \to B_{\varphi(i)}$ indexed over $i \in I$. Such a morphism is generic precisely when every γ_i is invertible.

It is then clear that the category of generic objects and generic morphisms between them (note $H_{(-)}$ is uniquely determined by φ in this case) is isomorphic to **Set**. *It follows that the*

Fam *construction is given by*

$$\mathbf{Fam}(\mathcal{C}) = \int_{\mathrm{lax}}^{X \in \mathbf{Set}} \mathcal{C}^X, \qquad \mathcal{C} \in \mathbf{CAT}$$

It is worth noting that restricting to the category of finite sets $\mathbf{Set}_{\mathrm{fin}}$, *yields the finite families construction* $\mathbf{Fam}_f$, *and restricting further the category of finite sets and bijections* $\mathbb{P}$ *yields the free symmetric (strict) monoidal category construction.*

The above shows that **Fam** is familial in the sense that it is a lax conical colimit of representables, however **Fam** is also familial in another sense: it has a left lax multiadjoint.

Example 6.7.7. *The pseudofunctor* $\mathbf{Fam}\colon \mathbf{CAT} \to \mathbf{CAT}$ *has a left lax multiadjoint. Here the generic morphisms are those functors of the form*

$$\delta_F\colon \mathcal{C} \to \mathbf{Fam}(\mathrm{el}\, F) : X \mapsto ((X,x) \in \mathrm{el}\, F\colon x \in FX)$$

for a presheaf $F\colon \mathcal{C} \to \mathbf{Set}$ *(Weber refers to these as "functors endowing* $\mathcal{C}$ *with elements" [53, Definition 5.10]). A cell out of such a generic morphism*

$$\begin{tikzcd}
& \mathbf{Fam}(\mathrm{el}\, F) \arrow[dd, "\mathbf{Fam}(H)"] \\
\mathcal{C} \arrow[ur, "\delta"] \arrow[dr, "z"'] \arrow[r, phantom, "\Swarrow_{\gamma}"] & \\
& \mathbf{Fam}(\mathcal{B})
\end{tikzcd}$$

is generic when the comparison maps (not necessarily the reindexing maps) comprising each γ_X *for* $X \in \mathcal{C}$ *are required invertible. It follows that this lax multiadjoint is exhibited by the formula*

$$\mathbf{CAT}(\mathcal{C}, \mathbf{Fam}(-)) \cong \int_{\mathrm{lax}}^{F\colon \mathcal{C} \to \mathbf{Set}} \mathbf{CAT}(\mathrm{el}\, F, -)$$

for each $\mathcal{C} \in \mathbf{CAT}$.

6.8 The spectrum factorization of a lax multiadjoint

In the simpler dimension one case, Diers [14] showed that familial functors have the following simple characterization:

Theorem 6.8.1 (Diers)**.** *Let* $T\colon \mathcal{A} \to \mathcal{B}$ *be a functor of categories. Then the following are equivalent:*

1. *the functor* T *has a left multiadjoint;*

2. *there exists a factorization*

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "T"] \arrow[dr, "G"'] & & \mathcal{B} \\
& \mathcal{M} \arrow[ur, "V"'] &
\end{tikzcd}$$

such that:

   (a) V is a discrete fibration;

   (b) G has a left adjoint.

When $\mathcal{A}$ has a terminal object, it is not hard to see that $\mathcal{M} \simeq \mathcal{B}/T\mathbf{1}$. This gives the following simple consequence:

Corollary 6.8.2. *Let $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a functor of categories, and assume $\mathcal{A}$ has a terminal object. Then T has a left multiadjoint (is a parametric right adjoint) if and only if the canonical projection*

$$T_1\colon \mathcal{A}/\mathbf{1} \to \mathcal{B}/T\mathbf{1}$$

has a left adjoint.

It is the purpose of this section to find an analogue of these results in the dimension two case. However, as we will see, this is much more complicated than simply asking for a left bi-adjoint. Instead we will require certain types of "lax" adjunctions (or adjunctions up to adjunction).

6.8.1 Lax F-adjunctions

In the setting of an adjunction of functors $F \dashv G\colon \mathcal{A} \longrightarrow \mathcal{M}$ we have natural hom-set isomorphisms $\mathcal{A}(F_m, A) \cong \mathcal{M}(m, GA)$. More generally, one can talk about bi-adjunctions of pseudofunctors $F \dashv G\colon \mathscr{A} \longrightarrow \mathscr{M}$ where we only ask for natural hom-category equivalences $\mathscr{A}(F_m, A) \simeq \mathscr{M}(m, GA)$. However, even this notion is often too strong.

Central to the theory of lax multiadjoints is the theory of lax adjunctions (hence the name), where one only asks that we have adjoint pairs

$$L_{m,A}\colon \mathscr{A}(F_m, A) \longrightarrow \mathscr{M}(m, GA), \qquad R_{m,A}\colon \mathscr{M}(m, GA) \longrightarrow \mathscr{A}(F_m, A)$$

pseudonatural (or even lax natural) in $A \in \mathscr{A}$ and $m \in \mathscr{M}$.

The following type of lax adjunctions, called *lax* **F**-*adjunctions*, appear when studying familial pseudofunctors. These are the lax adjunctions which naturally restrict to biadjunctions on a class of "tight" maps. Before defining lax **F**-adjunctions, we must first define **F**-bicategories and see how they assemble into a tricategory **F-Bicat**.

Definition 6.8.3. The following notions below:

- an **F**-*bicategory* is a bicategory $\mathscr{A}$ equipped with an identity on objects, injective on 1-cells, locally fully faithful functor $\mathscr{A}_{\mathbf{T}} \to \mathscr{A}$. The 1-cells of $\mathscr{A}_{\mathbf{T}}$ are called the *tight* 1-cells of $\mathscr{A}$ and are required to be closed under invertible 2-cells;
- an **F**-*pseudofunctor* $(\mathscr{A}, \mathscr{A}_T) \nvrightarrow (\mathscr{B}, \mathscr{B}_T)$ is a pseudofunctor $F\colon \mathscr{A} \to \mathscr{B}$ which restricts to a pseudofunctor $F_T\colon \mathscr{A}_T \to \mathscr{B}_T$;
- a *lax* **F**-*natural transformation* $\alpha\colon F \Rightarrow G\colon (\mathscr{A}, \mathscr{A}_T) \to (\mathscr{B}, \mathscr{B}_T)$ is a lax natural transformation $\alpha\colon F \Rightarrow G$ such that both:
    1. for all $X \in \mathscr{A}$, $\alpha_X\colon FX \to GX$ is tight;
    2. for all $f\colon X \to Y$ tight, $\alpha_f\colon Gf \cdot \alpha_X \Rightarrow \alpha_Y \cdot Ff$ is invertible.

define the tricategory **F-Bicat** of **F**-bicategories, **F**-pseudofunctors, lax **F**-natural transformations, and modifications.

The above allows for a particularly simple definition of lax **F**-adjunctions.

Definition 6.8.4 (Lax **F**-adjunction)**.** A lax **F**-adjunction of **F**-pseudofunctors

$$\begin{tikzcd}
(\mathscr{A}, \mathscr{A}_T) \arrow[r, shift left=1.5ex, "F"] \arrow[r, phantom, "\perp"] & (\mathscr{B}, \mathscr{B}_T) \arrow[l, shift left=1.5ex, "G"]
\end{tikzcd}$$

is a biadjunction in the tricategory **F-Bicat**.

Remark 6.8.5. It is worth noting that the above immediately tells us that lax **F**-adjunctions enjoy nice properties such as uniqueness of adjoints.

Whilst the above definition is conceptually informative, for our purposes it will be more useful to define these adjunctions in terms of universal arrows. This is due to the connection between the universal arrow definition and notions of genericity.

Remark 6.8.6. From now on we will regard the right adjoint G as a **F**-pseudofunctor $G\colon (\mathscr{A}, \mathscr{A}_T) \to (\mathscr{M}, \mathscr{M}_T)$ to more closely match the notation we will use use later on.

Definition 6.8.7 (Lax **F**-adjunction via universal arrows)**.** Given an **F**-pseudofunctor $G\colon (\mathscr{A}, \mathscr{A}_T) \to (\mathscr{M}, \mathscr{M}_T)$, we say a 1-cell $\eta_m\colon m \to GF_m$ is *universal* if for any 1-cell $f\colon m \to GA$ there exists a $\overline{f}\colon F_m \to A$ and 2-cell

$$\begin{tikzcd}
m \arrow[rr, "f"] \arrow[dr, swap, "\eta_m"] & \Uparrow \gamma_f & GA \\
& GF_m \arrow[ur, swap, "G\overline{f}"] &
\end{tikzcd}$$

such that the pair $\left(\overline{f},\gamma_f\right)$ is *universal*; meaning that for any $\overline{g}\colon F_m \to A$ and 2-cell β as below

$$\begin{tikzcd}
m \arrow[rr, "f"] \arrow[dr, "\eta_m"'] & \arrow[d, phantom, "\Uparrow\beta"] & GA \\
& GF_m \arrow[ur, "G\overline{g}"'] &
\end{tikzcd}
=
\begin{tikzcd}
m \arrow[rr, "f"] \arrow[dr, "\eta_m"'] & \arrow[d, phantom, "\Uparrow\gamma_f"] & GA \\
& GF_m \arrow[ur, "G\overline{f}"] \arrow[ur, bend right, "G\overline{g}"'] \arrow[ur, phantom, bend right=20, "\Uparrow G\widetilde{\beta}"] &
\end{tikzcd}$$

there exists a unique $\widetilde{\beta}\colon \overline{g} \Rightarrow \overline{f}$ such that the above equality holds. If in addition

(i) the 1-cell η_m is tight;

(ii) for every tight 1-cell $f\colon m \to GA$ in $\mathscr{M}$, the 2-cell γ_f is invertible and $\overline{f}\colon F_m \to A$ is tight;

(iii) the diagram

$$\begin{tikzcd}
m \arrow[rr, "\eta_m"] \arrow[dr, "\eta_m"'] & \arrow[d, phantom, "\Uparrow\mathrm{id}"] & GF_m \\
& GF_m \arrow[ur, "G1_{F_m}"'] &
\end{tikzcd}$$

exhibits $\left(1_{F_m},\mathrm{id}\right)$ as a universal pair;

(iv) for any universal pair $\left(\overline{f},\gamma_f\right)$, the G-whiskering by a tight $g\colon A \to B$

$$\begin{tikzcd}
m \arrow[rr, "f"] \arrow[dr, "\eta_m"'] & \arrow[d, phantom, "\Uparrow\gamma_f"] & GA \arrow[r, "Gg"] \arrow[d, phantom, "\cong"] & GB \\
& GF_m \arrow[ur, "G\overline{f}"'] \arrow[urr, bend right, "Gg\overline{f}"'] & &
\end{tikzcd}$$

exhibits $\left(g\overline{f},Gg\cdot\gamma_f\right)$ as a universal pair;

we then say that η_m is $\mathbf{F}$-*universal*[9]. Finally, we say G has a *left lax* $\mathbf{F}$-*adjoint* if:

1. for every object m in $\mathscr{M}$, there exists a $\mathbf{F}$-universal 1-cell $\eta_m\colon m \to GA$;
2. for all 1-cells μ and ν as below, $\overline{\eta_k\nu}\cdot\overline{\eta_n\mu}$ equipped with the 2-cell

$$\begin{tikzcd}
m \arrow[r, "\eta_m"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\swarrow\gamma_{\eta_n\mu}"] & GF_m \arrow[d, "G(\overline{\eta_n\mu})"] \arrow[dd, bend left=60, "G(\overline{\eta_k\nu}\cdot\overline{\eta_n\mu})"] \arrow[dd, phantom, bend left=30, "\cong"] \\
n \arrow[r, "\eta_n"] \arrow[d, "\nu"'] \arrow[dr, phantom, "\swarrow\gamma_{\eta_k\nu}"] & GF_n \arrow[d, "G(\overline{\eta_k\nu})"] \\
k \arrow[r, "\eta_k"'] & GF_k
\end{tikzcd}$$

is universal.

---

[9] The reader will of course notice that such a η_m is unique up to equivalence.

Remark 6.8.8. Note that it comes for free that for all $A \in \mathscr{A}$, the universal pair

$$\begin{tikzcd}
GA \arrow[rr, "1_{GA}"] \arrow[dr, "\eta_{GA}"'] \arrow[rr, phantom, shift right=4ex, "\Uparrow\gamma_{1_{GA}}"] & & GA \\
& GF_{GA} \arrow[ur, "G\overline{\mathrm{id}}"'] &
\end{tikzcd}$$

has the 2-cell component $\gamma_{1_{GA}}$ invertible (as identity 1-cells are necessarily tight). This is one of the triangle identities. The other triangle identity which asks for the composite of F_{η_m} and ε_{F_m} constructed as below

$$\begin{tikzcd}
m \arrow[r, "\eta_m"] \arrow[d, "\eta_m"'] & GF_m \arrow[d, "GF_{\eta_m}"] \arrow[dl, phantom, "\swarrow\gamma_{\eta_{GF_m}\eta_m}"] \\
GF_m \arrow[r, "\eta_{GF_m}"] \arrow[dr, bend right, "1_{GF_m}"'] & GFGF_m \arrow[d, "G\varepsilon_{F_m}"] \arrow[dl, phantom, "\swarrow\gamma_{1_{GF_m}}"] \\
& GF_m
\end{tikzcd}$$

to be isomorphic to the identity, is equivalent to (iii) in the presence of (iv). Pseudofunctoriality of F is clear from (2) and (iii).

The reader will also recognize that $L_{m,A}$ and $R_{m,A}$ are fully pseudonatural in $A \in \mathscr{A}$ and $m \in \mathscr{M}$ respectively; and also fully pseudonatural in $m \in \mathscr{M}_T$ and $A \in \mathscr{A}_T$ respectively. Indeed, $L_{m,A} \colon \mathscr{A}(F_m, A) \longrightarrow \mathscr{M}(m, GA)$ is defined by applying G and composing with η_m, and $R_{m,A} \colon \mathscr{M}(m, GA) \longrightarrow \mathscr{A}(F_m, A)$ is defined by applying F and composing with ε_A. Also, it is not hard to see that η and ε become lax $\mathbf{F}$-natural transformations given the universal arrow viewpoint. Finally, it is worth noting that each γ is invertible if and only if the unit η is fully pseudonatural.

The following property of lax $\mathbf{F}$-adjunctions, that the operations $\widetilde{(-)}$ respect isomorphisms, will be useful later in this section.

Lemma 6.8.9. *Given a pseudofunctor $G\colon \mathscr{A} \longrightarrow \mathscr{M}$ with a left lax $\mathbf{F}$-adjoint (F, η, γ), the operation $\beta \mapsto \widetilde{\beta}$ respects isomorphisms on tight maps.*

Proof. Suppose we have an equality as below where $\overline{g} \colon F_m \longrightarrow A$ is tight

$$\begin{tikzcd}
m \arrow[rr, "f"] \arrow[dr, "\eta_m"'] \arrow[rr, phantom, shift right=4ex, "\Uparrow\beta"] & & GA & & m \arrow[rr, "f"] \arrow[dr, "\eta_m"'] \arrow[rr, phantom, shift right=4ex, "\Uparrow\gamma_f G\overline{f}"] & & GA \\
& GF_m \arrow[ur, "G\overline{g}"'] & & = & & GF_m \arrow[ur] \arrow[ur, bend right=50, "G\overline{g}"'] \arrow[ur, phantom, shift right=2ex, "Gb \Uparrow"] &
\end{tikzcd}$$

and suppose further that β has an inverse, so that we may also form the unique equality

$$\begin{tikzcd}
m \arrow[r, "\eta_m"] \arrow[dr, "\eta_m"'] \arrow[rr, bend right=25, "\overline{f}" description] & GF_m \arrow[r, "G\overline{g}"] \arrow[d, phantom, "\Uparrow\beta^{-1}" near start] & GA \\
& GF_m \arrow[ur, "G\overline{f}"'] \arrow[u, phantom, "\Uparrow\gamma_f" near start] &
\end{tikzcd}
=
\begin{tikzcd}
m \arrow[r, "\eta_m"] \arrow[dr, "\eta_m"'] & GF_m \arrow[r, "G\overline{g}"] \arrow[d, phantom, "\Uparrow\mathrm{id}"] & GA \\
& GF_m \arrow[ur, "G\overline{g}"] \arrow[ur, bend right, "G\overline{f}"'] \arrow[ur, phantom, bend right=15, "Ga\ \Uparrow"] &
\end{tikzcd}$$

where we have used axioms (iii) and (iv) to realize the identity 2-cell as universal. It is then straightforward to verify a is inverse to b. □

Remark 6.8.10. It is not hard to see that in the presence of axiom (iv), the above lemma is equivalent to (iii).

The following theorem, due to Johnstone [23], establishes semi-lax **F**-adjunctions as a fundamental concept. These are the lax **F**-adjunctions such that ε is fully pseudonatural, or equivalently, those for which axiom (iv) holds for all $g\colon A \to B$ (not just on the tight maps).

Theorem 6.8.11 (Johnstone)**.** *A 1-cell $f\colon X \to Y$ in a bicategory $\mathscr{K}$ with pullbacks is a fibration if and only if the functor on the lax slice*

$$\mathscr{K} /\!\!/ X \xrightarrow{\Sigma_f} \mathscr{K} /\!\!/ Y$$

has a right semi-lax **F***-adjoint.*

Remark 6.8.12. Johnstone's choice of "oplax" and "lax" slice is the opposite of ours, and so the above is stated on the oplax slice in [23].

Example 6.8.13. *Let us see the above as an example of a lax* **F***-adjunction via universal arrows*[10]*. Here $G\colon \mathscr{A} \to \mathscr{M}$ is $\Delta_f\colon \mathscr{K} /\!\!/ Y \to \mathscr{K} /\!\!/ X$; which given a $(w,\theta) : a \twoheadrightarrow b$ forms the triangular prism (with commuting faces)*

$$\begin{tikzcd}
N \times_Y X \arrow[rr, "\pi_1"] \arrow[dd, "\pi_2"'] \arrow[dr, dotted, "\widetilde{w}\times_Y 1"] & & N \arrow[dr, "w"] \arrow[dd, "a" near end] \arrow[dl, phantom, "\hat{\varepsilon}\ \Downarrow"] & \\
\arrow[r, phantom, "\hat{\theta}\ \Uparrow"] & N' \times_Y X \arrow[rr, "\pi_1" near start] \arrow[dl, "\pi_2"'] & \arrow[r, phantom, "\theta\ \Uparrow"] & N' \arrow[dl, "b"] \\
X \arrow[rr, "f"'] & & Y &
\end{tikzcd}$$

and $\hat{\varepsilon}$ is invertible (as it is a pseudonaturality square of the counit), resulting in $\left(\widetilde{w} \times_Y 1, \hat{\theta}\right)$.

---

[10] Alternatively, this example may be understood (perhaps more naturally) in terms of the dual notion of co-universal arrows. However, the universal arrow definition will be used for consistency.

Moreover, for a given $p\colon M \longrightarrow X$ *(thought of as an* $m \in \mathscr{M}$*),* $a\colon N \longrightarrow Y$ *(thought of as an* $A \in \mathscr{A}$*), and*

$$\begin{tikzcd}
M \arrow[rr, "\lambda"] \arrow[dr, "p"'] & \arrow[d, phantom, "\overset{\Leftarrow}{\varphi}"] & N \times_Y X \arrow[dl, "\pi_2"] \\
& X &
\end{tikzcd}$$

(thought of as an $f\colon m \longrightarrow GA$*) we have the unit* η_m *given by*

$$\begin{tikzcd}
M \arrow[rr, "{(1,p)}"] \arrow[dr, "p"'] & \arrow[d, phantom, "\overset{\cong}{\Rightarrow}"] & N \times_Y X \arrow[dl, "\pi_2"] \\
& X &
\end{tikzcd}$$

and the induced $\overline{f}\colon F_m \longrightarrow A$ *is given by composing with the bipullback as below*

$$\begin{tikzcd}
M \arrow[rr, "{(1,p)}"] \arrow[dr, "p"'] & \arrow[d, phantom, "\overset{\Leftarrow}{\varphi}"] & N \times_Y X \arrow[rr, "\pi_1"] \arrow[dl, "\pi_2"] & \arrow[d, phantom, "\cong"] & N \arrow[dl, "a"] \\
& X \arrow[rr, "f"'] & & Y &
\end{tikzcd}$$

6.8.2 Factoring through the spectrum

We now have the necessary background on lax adjunctions, and can move towards understanding how a lax multiadjoint factors through the spectrum. This will only require the following simple lemma.

Lemma 6.8.14. *Suppose* $V\colon \mathscr{M} \longrightarrow \mathscr{B}$ *is a locally discrete fibration of bicategories. Then given any 2-cell* $\alpha\colon f \Rightarrow g\colon X \longrightarrow Vm$ *as on the right below*

$$\begin{tikzcd}
f^*m \arrow[drr, "f_c"] \arrow[dd, "\hat{\alpha}"'] & & & & X \arrow[drr, "f"] \arrow[dd, "\mathrm{id}"'] & & \\
& \Downarrow\overline{\alpha} & m & \longmapsto & & \Downarrow\alpha & Vm \\
g^*m \arrow[urr, "g_c"'] & & & & X \arrow[urr, "g"'] & &
\end{tikzcd}$$

with cartesian lifts f_c *and* g_c *of* f *and* g*, there exists a unique pair* $(\hat{\alpha}, \overline{\alpha})$ *as on the left above which is assigned to* α *by* V*. Moreover, if* α *is invertible then both* $\hat{\alpha}$ *and* $\overline{\alpha}$ *are.*

Proof. Suppose without loss of generality that V is the projection $\int_{\mathrm{lax}}^{B \in \mathscr{B}} FB \longrightarrow \mathscr{B}$ for a pseudofunctor $F\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$. Then we may construct a diagram as on the left below

$$\begin{tikzcd}
(X,a) \arrow[rd, "{(f,\cong)}"] \arrow[dd, "{(1,\lambda)}"'] & & & X \arrow[rd, "f"] \arrow[dd, "\mathrm{id}"'] & \\
\Downarrow\alpha & (Y,m) & \mapsto & \Downarrow\alpha & V(Y,m) \\
(X,b) \arrow[ru, "{(g,\cong)}"'] & & & X \arrow[ru, "g"'] &
\end{tikzcd}$$

where λ is the unique map such that

$$a \xrightarrow{\cong} Ff(m) \xrightarrow{(F\alpha)_m} Fg(m) \qquad = \qquad a \xrightarrow{\lambda} b \xrightarrow{\cong} Fg(m)$$

holds. It is clear this is the only choice of such a diagram, and that if α is invertible then so is λ. ☐

Remark 6.8.15. There should be an analogue of the above without assuming V to be locally discrete, so that V is the projection $\int_{\mathrm{lax}}^{B\in\mathscr{B}} FB \to \mathscr{B}$ for a trifunctor $F\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Bicat}$. However, this is beyond the scope of this paper.

We can now prove the main result of this section, which provides a conceptually nice description of lax multiadjoints. This characterization is interesting if one keeps in mind the characterization of fibrations via semi-lax $\mathbf{F}$-adjoints, but is perhaps not entirely unexpected as the connection between the theory of familial 2-functors and the theory of fibrations was already noted by Weber [53].

The reader will also note that if $G\colon \mathscr{A} \to \mathscr{M}$ is such that every 1-cell in $\mathscr{A}$ is tight, then a left lax $\mathbf{F}$-adjoint is equivalently a left semi-lax $\mathbf{F}$-adjoint (as axiom (iv) then holds for all g).

Theorem 6.8.16 (Spectrum factorization)**.** *Let $T\colon \mathscr{A} \to \mathscr{B}$ be a pseudofunctor of bicategories. Then the following are equivalent:*

1. *the pseudofunctor T has a left lax multiadjoint;*
2. *there exists a factorization*

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "T"] \arrow[rd, "G"'] & & \mathscr{B} \\
& \mathscr{M} \arrow[ru, "V"'] &
\end{tikzcd}$$

such that:

   (a) *V is a locally discrete fibration of bicategories;*

   (b) *G has a left lax $\mathbf{F}$-adjoint (where all 1-cells in $\mathscr{A}$ are tight and the V-cartesian 1-cells of $\mathscr{M}$ are tight).*

Proof. (2) $\Rightarrow$ (1) : We first note that for any $f\colon X \longrightarrow TA$ in $\mathscr{B}$, we have a cartesian lift $f_c\colon m \longrightarrow GA$ in $\mathscr{M}$. We thus have an assignment

$$\begin{tikzcd}
m \arrow[rr, "f_c"] \arrow[dr, "\eta_m"'] & {} \arrow[d, phantom, "\Uparrow\gamma_{f_c}"] & GA & \mapsto & X \arrow[rr, "f"] \arrow[dr, "\delta_m"'] & {} \arrow[d, phantom, "\Uparrow V\gamma_{f_c}"] & TA \\
& GF_m \arrow[ur, "G\overline{f_c}"'] & & & & TF_m \arrow[ur, "T\overline{f_c}"'] &
\end{tikzcd}$$

and as γ is invertible on cartesian maps, this is a factorization of f. We thus need only check that each δ_m is lax-generic, and that generic 2-cells compose.

Consider now a 2-cell α as on the right below

$$\begin{tikzcd}
n \arrow[r, "\hat{\alpha}"] \arrow[d, "\eta_n"'] & m \arrow[r, "f_c"] \arrow[d, phantom, "\Uparrow\overline{\alpha}"] & GA \arrow[d, "Gk"] & \mapsto & X \arrow[rr, "f"] \arrow[d, "\delta_n"'] & {} \arrow[d, phantom, "\Uparrow\alpha"] & TA \arrow[d, "Tk"] \\
GF_n \arrow[rr, "Gh"'] & & GC & & TF_n \arrow[rr, "Th"'] & & TC
\end{tikzcd}$$

and its unique preimage as on the left above given by Lemma 6.8.14. This $\overline{\alpha}$ in turn has a factorization as on the left below

$$\begin{tikzcd}
n \arrow[r, "\hat{\alpha}"] \arrow[d, "\eta_n"'] & m \arrow[r, "f_c"] \arrow[d, phantom, "\Uparrow\gamma_{f_c\hat{\alpha}}", pos=0.3] & GA \arrow[d, "Gk"] \arrow[dl, phantom, "\Uparrow G\xi", pos=0.7] & \mapsto & X \arrow[rr, "f"] \arrow[d, "\eta_n"'] & {} \arrow[d, phantom, "\Uparrow V\gamma_{f_c\hat{\alpha}}", pos=0.3] & TA \arrow[d, "Tk"] \arrow[dl, phantom, "\Uparrow T\xi", pos=0.7] \\
GF_n \arrow[rr, "Gh"'] \arrow[urr, "G\overline{f_c\hat{\alpha}}" description] & {} & GC & & TF_n \arrow[rr, "Th"'] \arrow[urr, "T\overline{f_c\hat{\alpha}}" description] & {} & TC
\end{tikzcd}$$

since universality of $\left(f_c\hat{\alpha}, \gamma_{f_c\hat{\alpha}}\right)$ is preserved by Gk, thus giving a factorization of α as on the right above. Note that if α, and hence $\hat{\alpha}$ and $\overline{\alpha}$ are invertible, then $\gamma_{f_c\hat{\alpha}}$ is invertible (as it is on all cartesian 1-cells), and ξ is invertible by Lemma 6.8.9.

Given another factorization as on the right below, we can lift σ by Lemma 6.8.14

$$\begin{tikzcd}
n \arrow[r, "\hat{\sigma}"] \arrow[d, "\eta_n"'] & m \arrow[r, "f_c"] \arrow[d, phantom, "\Uparrow\overline{\sigma}", pos=0.3] & GA \arrow[d, "Gk"] \arrow[dl, phantom, "\Uparrow G\varphi", pos=0.7] & \mapsto & X \arrow[rr, "f"] \arrow[d, "\eta_n"'] & {} \arrow[d, phantom, "\Uparrow\sigma", pos=0.3] & TA \arrow[d, "Tk"] \arrow[dl, phantom, "\Uparrow T\varphi", pos=0.7] \\
GF_n \arrow[rr, "Gh"'] \arrow[urr, "G\overline{g}" description] & {} & GC & & TF_n \arrow[rr, "Th"'] \arrow[urr, "T\overline{g}" description] & {} & TC
\end{tikzcd}$$

giving the left above. Noting that $\hat{\sigma} = \hat{\alpha}$ and that the left pasting above is $\overline{\alpha}$ by uniqueness, we can then factor $\overline{\sigma}$ through $\gamma_{f_c\hat{\alpha}}$ recovering a comparison map $\psi\colon \overline{g} \Rightarrow \overline{f_c\hat{\alpha}}$ satisfying the required conditions. The sub-terminality of each $V\gamma_{f_c\hat{\alpha}}$ stems from the uniqueness of factorisations through $\gamma_{f_c\hat{\alpha}}$.

Finally, to see that generic cells compose, observe that a cell as on the right below

$$\begin{tikzcd}
n \arrow[r, "\eta_n"] \arrow[dd, "\hat{\gamma}"'] \arrow[ddr, phantom, "\Downarrow\overline{\gamma}"] & GF_n \arrow[dd, "Gh"] & & & TF_n \arrow[dd, "Th"] \\
& & \mapsto & X \arrow[ur, "\delta_n"] \arrow[dr, "z"'] \arrow[r, phantom, "\Downarrow\gamma"] & \\
m \arrow[r, "z_c"'] & GC & & & TC
\end{tikzcd}$$

is generic precisely when its lift as on the left above, given by Lemma 6.8.14, exhibits $(h, \overline{\gamma})$ as a universal pair. Also observe that every generic is of the form δ_n, since given any generic δ and cartesian lift δ_c we have an isomorphism

$$\begin{tikzcd}
m \arrow[rr, "\delta_c"] \arrow[dr, "\eta_m"'] & & GA & & X \arrow[rr, "\delta"] \arrow[dr, "\delta_m"'] & & TA \\
& GF_m \arrow[ur, "G\overline{\delta_c}"'] \arrow[u, phantom, "\Uparrow\gamma_{\delta_c}"] & & \mapsto & & TF_m \arrow[ur, "T\overline{\delta_c}"'] \arrow[u, phantom, "\Uparrow V\gamma_{\delta_c}"] &
\end{tikzcd}$$

and we know that $\left(\overline{\delta_c}, V\gamma_{\delta_c}\right)$ is an equivalence by Lemma 6.3.20. It follows that two generic cells as on the right below

$$\begin{tikzcd}
n \arrow[r, "\eta_n"] \arrow[d, "\hat{\gamma}"'] \arrow[ddr, phantom, "\Downarrow\overline{\gamma}"] & GF_n \arrow[dd, "Gh"] & & X \arrow[r, "\delta_n"] \arrow[dd, "f"'] \arrow[ddr, phantom, "\Downarrow\gamma"] & TF_n \arrow[dd, "Th"] \\
\bullet \arrow[d, "f_c"'] & & & & \\
m \arrow[r, "\eta_m"'] \arrow[d, "\hat{\phi}"'] \arrow[ddr, phantom, "\Downarrow\overline{\phi}"] & GF_m \arrow[dd, "Gk"] & \mapsto & Y \arrow[r, "\delta_m"] \arrow[dd, "g"'] \arrow[ddr, phantom, "\Downarrow\phi"] & TF_m \arrow[dd, "Tk"] \\
\bullet \arrow[d, "g_c"'] & & & & \\
w \arrow[r, "\eta_w"'] & GF_w & & Z \arrow[r, "\delta_w"'] & TF_w
\end{tikzcd}$$

compose to a generic, as the composite on the left above is universal.

(1) ⇒ (2) : Supposing that $T\colon \mathscr{A} \to \mathscr{B}$ has a left lax multiadjoint, we may construct the spectrum $\mathfrak{M}_{(-)}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ as in Lemma 6.5.8 and factor T as

$$\mathscr{A} \xrightarrow{G} \int_{\text{oplax}}^{X \in \mathscr{B}} \mathfrak{M}_{(-)} \xrightarrow{\text{pr}} \mathscr{B}$$

where G assigns each $A \in \mathscr{A}$ to $TA \in \mathscr{B}$ with the generic morphism $\delta_A\colon TA \to T\overline{A}$ comprising the generic factorisation

$$TA \xrightarrow{\delta_A} T\overline{A} \xrightarrow{Te_A} TA$$

of the identity. A 1-cell $h\colon A \to B$ in $\mathscr{A}$ is assigned to Th with the pair $\left(\overline{h}, \cong\right)$ comprising the left side

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "Th"'] \arrow[dr, phantom, "\cong"] & T\overline{A} \arrow[r, "Te_A"] \arrow[d, "T\overline{h}"] \arrow[dr, phantom, "\cong"] & TA \arrow[d, "Th"] \\
TB \arrow[r, "\delta_B"'] & T\overline{B} \arrow[r, "Te_B"'] & TB
\end{tikzcd}$$

of the generic factorization above. A given 2-cell $\lambda\colon h \Rightarrow k$ is sent to $T\lambda\colon Th \Rightarrow Tk$, which satisfies

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, bend right=50, "Tk"'] \arrow[d, bend left=50, "Th"] \arrow[d, phantom, "\underset{T\lambda}{\Leftarrow}"] \arrow[dr, phantom, "\cong"] & T\overline{A} \arrow[d, "T\overline{h}"] \\
TB \arrow[r, "\delta_B"'] & T\overline{B}
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "Tk"'] \arrow[dr, phantom, "\cong"] & T\overline{A} \arrow[d, bend right=50, "T\overline{k}"'] \arrow[d, bend left=50, "T\overline{h}"] \arrow[d, phantom, "\underset{T\overline{\lambda}}{\Leftarrow}"] \\
TB \arrow[r, "\delta_B"'] & T\overline{B}
\end{tikzcd}$$

for some (necessarily unique) $\overline{\lambda}\colon \overline{h} \Rightarrow \overline{k}$. To see this, note that the left diagram has a generic factorisation

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, bend right=50, "Tk"'] \arrow[d, bend left=50, "Th"] \arrow[d, phantom, "\underset{T\lambda}{\Leftarrow}"] \arrow[dr, phantom, "\cong"] & T\overline{A} \arrow[d, "T\overline{h}"] \\
TB \arrow[r, "\delta_B"'] & T\overline{B}
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "Tk"'] \arrow[dr, phantom, "\overset{\Swarrow}{\xi}"] & T\overline{A} \arrow[d, bend right=50, "Tm"'] \arrow[d, bend left=50, "T\overline{h}"] \arrow[d, phantom, "\underset{T\overline{\lambda}}{\Leftarrow}"] \\
TB \arrow[r, "\delta_B"'] & T\overline{B}
\end{tikzcd}$$

and thus the left diagram below has the generic factorisation

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, bend right=50, "Tk"'] \arrow[d, bend left=50, "Th"] \arrow[d, phantom, "\underset{T\lambda}{\Leftarrow}"] \arrow[dr, phantom, "\cong"] & T\overline{A} \arrow[r, "Te_A"] \arrow[d, "T\overline{h}"] \arrow[dr, phantom, "T\cong"] & TA \arrow[d, "Th"] \\
TB \arrow[r, "\delta_B"'] & T\overline{B} \arrow[r, "Te_B"'] & TB
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "Tk"'] \arrow[dr, phantom, "\overset{\Swarrow}{\xi}"] & T\overline{A} \arrow[r, "Te_A"] \arrow[d, bend right=50, "Tm"'] \arrow[d, bend left=50, "T\overline{h}"] \arrow[d, phantom, "\underset{T\overline{\lambda}}{\Leftarrow}"] \arrow[dr, phantom, "T\cong"] & TA \arrow[d, "Th"] \\
TB \arrow[r, "\delta_B"'] & T\overline{B} \arrow[r, "Te_B"'] & TB
\end{tikzcd}$$

But this is also the generic factorization of the diagram

$$\begin{tikzcd}
TA \arrow[r, "\delta_A"] \arrow[d, "Tk"'] \arrow[dr, phantom, "\cong"] & T\overline{A} \arrow[r, "Te_A"] \arrow[d, "T\overline{k}"] \arrow[dr, phantom, "T\cong"] & TA \arrow[d, bend right=50, "Tk"'] \arrow[d, bend left=50, "Th"] \arrow[d, phantom, "\underset{T\lambda}{\Leftarrow}"] \\
TB \arrow[r, "\delta_B"'] & T\overline{B} \arrow[r, "Te_B"'] & TB
\end{tikzcd}$$

which has already been factored. By uniqueness of representative generic factorisations we have $(m, \xi) = \left(\overline{k}, \cong\right)$ as required.

Now, we have the pseudofunctor $\mathbf{P}\colon \int_{\mathrm{oplax}}^{X \in \mathscr{B}} \mathfrak{M}_{(-)} \to \mathscr{A}$, and will sketch why $\mathbf{P}$ is a left lax $\mathbf{F}$-adjoint to G. To do this, we take our universal 1-cell $\eta_{(X,\delta)}\colon (X,\delta) \to GF(X,\delta)$ at an object $(X, \delta\colon X \to LA)$ to be the pair (u_A, γ) as below.

$$\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[rd, phantom, "\Swarrow_{\gamma}"] & TA \arrow[d, "Tu_A"] \arrow[rd, "T1_A"] & \\
TA \arrow[r, "\delta_A"'] & T\overline{A} \arrow[r, "Te_A"'] \arrow[ru, phantom, "\Swarrow_{Tv}"] & TA
\end{tikzcd}$$

Moreover, for a given 1-cell $(f, h, \alpha) : (X, \delta) \to GC$ as on the left below, we have

$$\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, "f"'] \arrow[rd, phantom, "\Swarrow_{\alpha}"] & TA \arrow[d, "Th"] \arrow[d, bend left=80, dotted, "T\left(\overline{e_c h u_A}\right)"] \arrow[d, phantom, shift left=4ex, "\overset{\Leftarrow}{T\xi}"] \\
TC \arrow[r, "\delta_c"'] & T\overline{C}
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, "\delta"] \arrow[dd, bend right=60, "f"'] \arrow[rd, phantom, "\Swarrow_{\gamma}"] & TA \arrow[d, "Tu_A"] \\
TA \arrow[r, "\delta_A"'] \arrow[d, "T(e_C h)"] \arrow[l, phantom, "\overset{\Leftarrow}{Te_C\alpha}"] \arrow[rd, phantom, "\cong"] & T\overline{A} \arrow[d, "T\left(\overline{e_C h}\right)"] \\
TC \arrow[r, "\delta_C"'] & T\overline{C}
\end{tikzcd}$$

where ξ is the unique map induced from the fact that the RHS whiskered by Te_C is $Te_C \cdot \alpha$. This defines the universal 2-cell

$$\begin{tikzcd}
(X, \delta) \arrow[rr, "{(f,h,\alpha)}"] \arrow[rd, "\eta_{(X,\delta)}"'] & \Uparrow Te_c \cdot \alpha & GC \\
& GA \arrow[ru, "Ge_c h"'] &
\end{tikzcd}$$

where we have a bijection $\beta \mapsto \widetilde{\beta}$ as below

$$\begin{tikzcd}
(X, \delta) \arrow[rr, "{(f,h,\alpha)}"] \arrow[rd, "\eta_{(X,\delta)}"'] & \Uparrow \beta & GC \\
& GA \arrow[ru, "G\ell"'] &
\end{tikzcd}
=
\begin{tikzcd}
(X, \delta) \arrow[rr, "{(f,h,\alpha)}"] \arrow[rd, "\eta_{(X,\delta)}"'] & \Uparrow Te_C \cdot \alpha & GC \\
& GA \arrow[ru, "Ge_C h"] \arrow[ru, bend right, "G\ell"'] \arrow[ru, phantom, bend right=20, "\Uparrow G\widetilde{\beta}"'] &
\end{tikzcd}$$

or equivalently, a bijection

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\delta"'] \arrow[rd, phantom, "\Uparrow \beta"] & TC \arrow[d, "T\mathrm{id}"] \\
TA \arrow[r, "T\ell"'] & TC
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\delta"'] \arrow[r, phantom, shift right=3ex, "\Uparrow Te_C \cdot \alpha"] & TC \arrow[d, "T\mathrm{id}"] \\
TA \arrow[r, "T\ell"'] \arrow[ru, "Te_c h"] \arrow[r, phantom, shift left=3ex, "\Uparrow T\widetilde{\beta}"] & TC
\end{tikzcd}$$

as genericity of (h, α) is respected by composition with Te_C. The verification that this bijection satisfies the required axioms is left for the reader. □

Finally, the following provides what is perhaps a more natural definition of parametric right adjoint pseudofunctors, obtained by applying the above theorem in the setting where $\mathscr{A}$ has a terminal object.

Corollary 6.8.17 (Parametric right adjoints)**.** *Suppose* $\mathscr{A}$ *is a bicategory with a terminal object. Then a pseudofunctor* $T \colon \mathscr{A} \to \mathscr{B}$ *has a left lax multiadjoint if and only if the canonical projection on the oplax slice*

$$T_1 \colon \mathscr{A} \mathbin{/\!\!/} 1 \to \mathscr{B} \mathbin{/\!\!/} T1$$

has a left lax **F***-adjoint.*

Remark 6.8.18. There are of course four variants of the above, concerning the case when $T/T^{\mathrm{op}}/T^{\mathrm{co}}/T^{\mathrm{coop}}$ is familial.

7

Conclusion and future directions

To conclude, we reflect on some of the highlights of this thesis. We have:

- shown that fully faithful KZ pseudomonads P give rise to near-Yoneda structures;
- given two simple descriptions of pseudo-distributive laws over KZ pseudomonads (one algebraic and one in terms of the near-Yoneda structure arising from the KZ pseudomonad);
- given a generalization of the oplax-lax correspondence in Kelly's doctrinal adjunction [27] to the setting of these KZ-induced near-Yoneda structures;
- introduced a class of bicategories which allows for a generalization of Bénabou's correspondence of (co)monads and (op)lax functors out of the terminal category, also giving a greater understanding of the bicategories of spans and polynomials and maps out of them;
- established the universal properties of the bicategory of polynomials with cartesian and general 2-cells, both for pseudofunctors and the weaker gregarious functors, whilst avoiding the worst of the coherence conditions coming from polynomial composition that would be needed in a direct proof;
- defined a notion of family for pseudofunctors which assumes no completeness conditions, defined an appropriate analogue of generic factorisations for pseudofunctors, and proved that familiality is equivalent to having these generic factorisations (along with a condition on generics ensuring they compose).

However, there remains some unanswered questions to be addressed, which would tie together our work on pseudo-distributive laws, familial pseudofunctors, and the bicategories of spans and polynomials. This is to be the subject of future work.

7.1 Future work

In our fourth paper we showed that pseudofunctors $\mathbf{Poly}(\mathcal{E}) \to \mathscr{C}$ correspond to pairs of pseudofunctors $\mathbf{Span}(\mathcal{E}) \to \mathscr{C}$ and $\mathbf{Span}(\mathcal{E})^{\mathrm{co}} \to \mathscr{C}$ which coincide on spans of the form (s, id) and satisfy a distributivity condition (and also gave a version for cartesian morphisms of polynomials). This however raises the following natural questions:

1. How is the bicategory of polynomials $\mathbf{Poly}(\mathcal{E})$ constructed from the bicategory of spans $\mathbf{Span}(\mathcal{E})$ and bicategory of spans with reversed 2-cells $\mathbf{Span}(\mathcal{E})^{\mathrm{co}}$?
2. How is the bicategory of polynomials $\mathbf{Poly}_c(\mathcal{E})$ constructed from the bicategory of spans $\mathbf{Span}(\mathcal{E})$ and bicategory of spans with invertible 2-cells $\mathbf{Span}_{\mathrm{iso}}(\mathcal{E})$?

For one possible answer, see von Glehn's work on polynomial functors and fibrations with sums and products [50]. We will give a more direct (though closely related) answer, making use of:

Theorem 7.1.1. *Given a bicategory $\mathscr{A}$, we have a correspondence*

$$\frac{\text{Bicategories } \mathscr{B} \text{ equipped with a bijective on objects pseudofunctor } L\colon \mathscr{A} \to \mathscr{B}}{\text{Bicocontinuous pseudomonads on } [\mathscr{A}^{\mathrm{op}}, \mathbf{Cat}]}$$

Remark 7.1.2. Note that there is a question of what the morphisms between such data should be to define an equivalence. This is technical and so will be addressed in the future.

Definition 7.1.3. Under the correspondence of Theorem 7.1.1, we call $\mathscr{B}$ the *representing bicategory* $\mathbf{Rep}(T)$ of the pseudomonad on T on $[\mathscr{A}^{\mathrm{op}}, \mathbf{Cat}]$.

Taking $\mathscr{A}$ to be a 1-category $\mathcal{E}$, so that T is now a pseudomonad on $[\mathcal{E}^{\mathrm{op}}, \mathbf{Cat}] \simeq \mathbf{Fib}(\mathcal{E})$, the correspondence of Theorem 7.1.1 returns the following table

Cocontinuous pseudomonad	Representing bicategory
$T = \Sigma_{\mathcal{E}}$	$\mathbf{Rep}(T) \cong \mathbf{Span}(\mathcal{E})^{\mathrm{op}}$
$T = \Pi_{\mathcal{E}}$	$\mathbf{Rep}(T) \cong \mathbf{Span}(\mathcal{E})^{\mathrm{coop}}$
$T = \otimes_{\mathcal{E}}$	$\mathbf{Rep}(T) \cong \mathbf{Span}_{\mathrm{iso}}(\mathcal{E})^{\mathrm{op}}$
$T = \Sigma_{\mathcal{E}}\Pi_{\mathcal{E}}$	$\mathbf{Rep}(T) \cong \mathbf{Poly}(\mathcal{E})^{\mathrm{op}}$
$T = \Sigma_{\mathcal{E}}\otimes_{\mathcal{E}}$	$\mathbf{Rep}(T) \cong \mathbf{Poly}_c(\mathcal{E})^{\mathrm{op}}$

where $\Sigma_{\mathcal{E}}$ is the pseudomonad for fibrations with sums, $\Pi_{\mathcal{E}}$ is the pseudomonad for fibrations with products, $\otimes_{\mathcal{E}}$ is a pseudomonad which we introduce (to be thought of as "fibrations with tensors"), and the composites $\Sigma_{\mathcal{E}}\Pi_{\mathcal{E}}$ and $\Sigma_{\mathcal{E}}\otimes_{\mathcal{E}}$ are constructed via suitable pseudo-distributive laws.

The reader will also notice that since it makes sense to talk about pseudo-distributive laws between pseudomonads T and S on $\mathbf{Fib}\,(\mathcal{E})$, by Theorem 7.1.1, we must also have a corresponding notion of pseudo-distributive laws between bijective on objects pseudofunctors $L\colon \mathcal{E} \longrightarrow \mathscr{A}$ and $H\colon \mathcal{E} \longrightarrow \mathscr{B}$ out of the same bicategory $\mathscr{A}$. Indeed, we find that such a pseudo-distributive law of bijective on objects pseudofunctors consists of a coherent family of functors

$$\lambda_{X,Y}\colon \int^{I\in\mathcal{E}} \mathscr{A}\,(LX,LI)\times\mathscr{B}\,(HI,HY) \longrightarrow \int^{J\in\mathcal{E}} \mathscr{B}\,(HX,HJ)\times\mathscr{A}\,(LJ,LY)$$

for all X and Y in $\mathcal{E}$, and that such a pseudo-distributive law allows one to construct a new bicategory $\mathscr{C}$ with hom-categories given by

$$\mathscr{C}\,(X,Y) = \int^{J\in\mathcal{E}} \mathscr{B}\,(HX,HJ)\times\mathscr{A}\,(LJ,LY)$$

and composition resulting from the pseudo-distributive law. This gives rise to a pseudofunctor $L * H\colon \mathcal{E} \longrightarrow \mathscr{C}$.

The bicategory of polynomials is an example of this. Indeed, we have a pseudo-distributive law of the pseudofunctors $\mathcal{E} \longrightarrow \mathbf{Span}\,(\mathcal{E})^{\mathrm{op}}$ and $\mathcal{E} \longrightarrow \mathbf{Span}\,(\mathcal{E})^{\mathrm{coop}}$ (corresponding to that of fibrations with sums over fibrations with products) giving rise to the pseudofunctor $\mathcal{E} \longrightarrow \mathbf{Poly}\,(\mathcal{E})^{\mathrm{op}}$. Consequently, we have the formulas for hom-categories of polynomials

$$\mathbf{Poly}\,(\mathcal{E})\,(X,Y) \simeq \int^{A\in\mathcal{E}^{\mathrm{op}}} \mathbf{Span}^{\mathrm{co}}\,(\mathcal{E})\,(X,A)\times\mathbf{Span}\,(\mathcal{E})\,(A,Y)$$

and

$$\mathbf{Poly}_c\,(\mathcal{E})\,(X,Y) \simeq \int^{A\in\mathcal{E}^{\mathrm{op}}} \mathbf{Span}_{\mathrm{iso}}\,(\mathcal{E})\,(X,A)\times\mathbf{Span}\,(\mathcal{E})\,(A,Y)$$

in terms of hom-categories of spans. Also, these formulas are straightforward to evaluate since $\mathcal{E} \longrightarrow \mathbf{Span}\,(\mathcal{E})^{\mathrm{op}}$ has a lax left multiadjoint; so that for example

$$\begin{aligned}
\mathbf{Poly}_c\,(\mathcal{E})\,(X,Y) &\cong \int^{A\in\mathcal{E}^{\mathrm{op}}} \mathbf{Span}_{\mathrm{iso}}\,(\mathcal{E})\,(X,A)\times\mathbf{Span}\,(\mathcal{E})\,(A,Y)\\
&\cong \int^{A\in\mathcal{E}^{\mathrm{op}}} \mathbf{Span}_{\mathrm{iso}}\,(\mathcal{E})\,(X,A)\times\int_{\mathrm{lax}}^{m\in\mathcal{E}/Y} \mathcal{E}\,(P_m,A)\\
&\cong \int_{\mathrm{lax}}^{m\in\mathcal{E}/Y}\int^{A\in\mathcal{E}^{\mathrm{op}}} \mathbf{Span}_{\mathrm{iso}}\,(\mathcal{E})\,(X,A)\times\mathcal{E}\,(P_m,A)\\
&\cong \int_{\mathrm{lax}}^{m\in\mathcal{E}/Y} \mathbf{Span}_{\mathrm{iso}}\,(\mathcal{E})\,(X,P_m)
\end{aligned}$$

where we have used that lax conical colimits (which may be seen as an instance of weighted bi-colimits) commute with bi-colimits.

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