Covering Systems

by

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The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

Thesis directed by Senior Lecturer Dr. Gerry Myerson and Prof. Paul Smith.

Statement of Candidate

I certify that the work in this thesis entitled "**Covering Systems**" has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree to any other university or institution other than Macquarie University.

I also certify that the thesis is an original piece of research and it has been written by me. Any help and assistance that I have received in my research work and the preparation of the thesis itself have been appropriately acknowledged.

In addition, I certify that all information sources and literature used are indicated in the thesis.

Paul Emanuel (40091686)

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Summary

Covering systems were introduced by Paul Erdős [8] in 1950. A covering system is a collection of congruences of the form $x \equiv a_i (\text{mod } m_i)$ whose union is the integers. These can then be specialised to being incongruent (that is, having distinct moduli), or disjoint, in which each integer satisfies exactly one congruence.

This thesis studies incongruent restricted disjoint covering systems (IRDCS), collections of congruence classes which cover a finite interval of the integers disjointly, subject to an additional technical condition. There exist IRDCS of length 11 and all lengths greater than or equal to 17. These IRDCS are used to study questions analogous to those of interest in covering systems. We focus on the following questions.

(1) Can the smallest modulus of some IRDCS be arbitrarily large?

(2) Do there exist IRDCS with all moduli odd?

(3) What is the appropriate two-dimensional generalisation?

This thesis addresses these questions and makes significant headway towards their resolution.

Chapter 5 studies IRDCS with large minimum modulus. We present, amongst other examples, one IRDCS with minimum modulus 50.

In Chapter 6 it is shown that there are IRDCS with only odd moduli. The smallest example is one of length 83. This chapter will present information on all of the known examples of what will be referred to as odd IRDCS.

Finally, in Chapter 7, we extend the definition of IRDCS to two dimensions, determining conditions on the relevant parameters for the existence of such structures. In this chapter we also study some of the structural properties, analogous to those of one-dimensional IRDCS, for these new constructions.

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Chapter 1

Introduction

1.1 Covering Systems

Covering systems were introduced by Paul Erdős [8] in 1950. Erdős used covering systems to show that there exists an arithmetic progression of odd numbers containing no terms of the form $2^k + p, p$ a prime. Numbers of this form have positive density in the integers, as shown by Romanoff [28]. We begin with some elementary definitions.

Definition 1.1. A covering system is a collection of congruences of the form $x \equiv a_i (\text{mod } m_i)$ whose union is the integers.

Definition 1.2. An incongruent covering system is a covering system with congruences whose moduli are all distinct.

Definition 1.3. A disjoint covering system is a covering system in which each integer satisfies exactly one congruence.

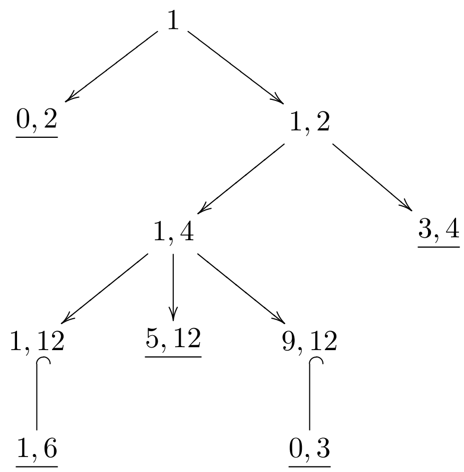
For the course of this thesis, unless otherwise stated, all covering systems discussed are assumed to be incongruent, and the modulus 1 is disallowed to avoid triviality.

These covering systems provide the classical setting for the problems of interest in this thesis. Our approach is to study the related construction introduced in [23], **incongruent restricted disjoint covering systems**. Many questions about this construction are analogous to those asked for covering systems, and only require minor modifications to the original statement.

While it is not immediately obvious that any incongruent covering systems exist, Erdős provides the simplest covering system:

$$0 \pmod 2, 0 \pmod 3, 3 \pmod 4, 1 \pmod 6, \text{and } 5 \pmod{12}.$$

This covering system can be represented geometrically as:



where a, m represents the congruence $a \pmod m$. A collection of p, a prime, downward arrows leaving a congruence class modulo m implies that the congruence class modulo m is being split disjointly into the p congruence classes modulo pm which cover it. The inclusion arrow, as in above the arrow joining $1, 12$ and $1, 6$, shows that we are using the lower congruence class to cover the one from which it branches. All underlined congruences are used in the final cover.

For technical reasons, this simplest covering system did not solve the problem Erdős was studying, so he also produced the covering system

$$0 \pmod 2, 0 \pmod 3, 1 \pmod 4, 3 \pmod 8, 7 \pmod{12}, 23 \pmod{24}.$$

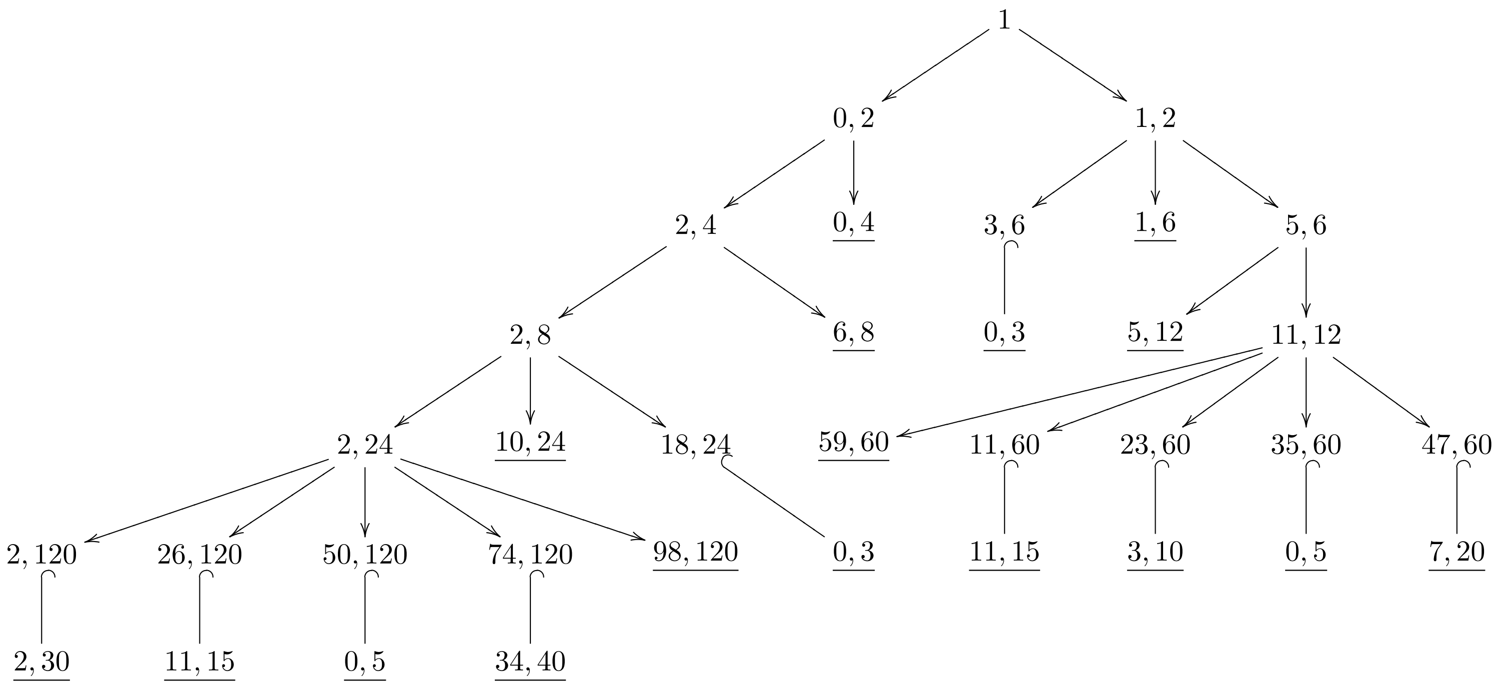
This can be represented geometrically as in the previous example, where we instead keep 1 (mod 4) and cover 3 (mod 4) as

$$
\begin{tikzcd}
 & 3,4 \arrow[dl] \arrow[d] & \\
\underline{3,8} & 7,8 \arrow[dl] \arrow[d] \arrow[dr] & \\
7,24 \arrow[d, dash] & 15,24 \arrow[d, dash] & \underline{23,24} \\
\underline{7,12} & \underline{0,3} &
\end{tikzcd}
$$

Erdős also gave the covering system made up of the following congruences with smallest modulus 3:

$$
\begin{aligned}
&0 \pmod 3, 0 \pmod 4, 0 \pmod 5, 1 \pmod 6, 6 \pmod 8, 3 \pmod{10},\\
&5 \pmod{12}, 11 \pmod{15}, 7 \pmod{20}, 10 \pmod{24}, 2 \pmod{30},\\
&34 \pmod{40}, 59 \pmod{60}, 98 \pmod{120},
\end{aligned}
$$

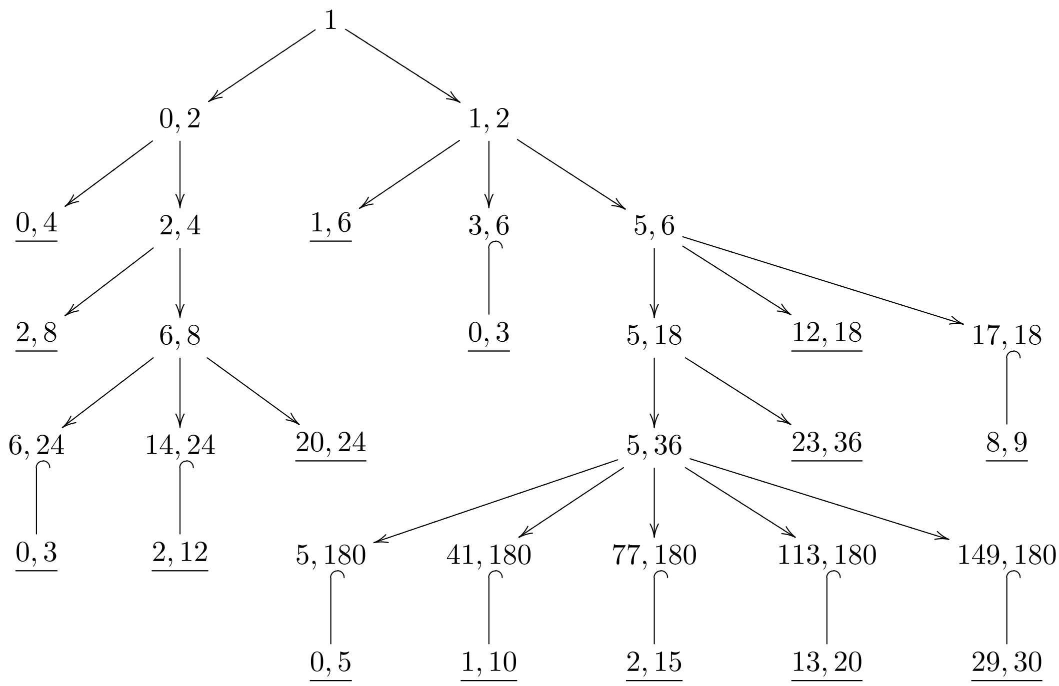
which can be represented as:



Note that while the congruences 0 (mod 3), 0 (mod 5) and 11 (mod 15) appear twice in this geometric representation. This is allowed given that their initial branch is on the prime 2 which is coprime to 3, 5 and 15. The components on the left side of the tree are used to cover the even elements of these congruences while the right side covers the odd elements.

The moduli in this cover have least common multiple 120. This is the smallest such least common multiple in any known cover with minimum modulus 3. Furthermore, it is strongly conjectured by Churchhouse [7] that this is the smallest possible least common multiple of the moduli for a cover with least modulus 3. It is also known that the only possible smaller least common multiples are 60 and 90.

A cover with least modulus 3 must have largest modulus at least 36 (see Theorems 2.6 and 2.7 in [20]). An example of such a covering system is:



The moduli in this covering system have least common multiple 360.

1.2 Arbitrarily Large Minimum Modulus

Questions raised in Erdős' original paper are still unsolved today. After noting the original covering system which used the modulus 2, along with these covers with minimum modulus 3, Erdős conjectured that "It seems likely that for every c there exists such a system (a covering system) all the moduli of which are $> c$" [8].

Conjecture 1.1 (Erdős [8])**.** There exists a covering system with minimum modulus c for any $c \in \mathbb{N}$.

This question remains open to this day, despite being an area of continual research. Erdős also provided additional motivation to solving this as if the minimum modulus conjecture holds, then for all c there exists an arithmetic progression containing no term of the form $2^k + u$, where the number of distinct prime factors of u is less than c [8]. Erdős referred to the least modulus conjecture as his favourite problem in [10]. An analogous question to this will be further explored in Chapter 5.

For any covering system with moduli m_i it is well known, and easily shown with density considerations, that

$$\sum_i \frac{1}{m_i} \geq 1,$$

and that this sum is equal to 1 only in the case of a disjoint cover. Erdős and John Selfridge conjectured that for any number B, there is some N_B such that an incongruent covering system with least modulus N_B has sum of reciprocals of its moduli greater than B [11]. This conjecture was proved when Filaseta, Ford, Konyagin, Pomerance and Yu showed that for a covering system with minimum modulus m_1, as m_1 goes to infinity, so to must $\sum \frac{1}{m_i}$ [12]. These results have been generalized to number fields by Sun Kim [18].

A number of authors have contributed to continued improvement in the record for the largest minimum modulus in a covering system. Churchhouse [7] in 1968 found covering systems with minimum modulus $m_1 = 2, 3, \ldots, 9$ using early computational techniques. Following this, Krukenberg [20] in 1971 gave examples with minimum modulus $m_1 = 2, 3, \ldots, 18$. Krukenberg states in reference to his example with minimum modulus 18 that "The structure at this stage is, to say the least, quite complicated and not easy to visualize".

Soon after, Choi [5] found a covering system with minimum modulus 20, and proposed that further elaboration of the methods using his theoretical result may be capable of finding a larger minimum modulus, but would be limited by a prohibitively large amount of computations. Krukenberg and Choi do not appear to have used computers to aid them in their searches.

More recently there was a claim of a covering system with minimum modulus 24 due to Morikawa [22] which was followed by Gibson [14] in 2008, who found a covering system with minimum modulus 25, and Nielsen [25] in 2009, who found one with minimum modulus 40. Gibson's cover was found using a combination of a greedy algorithm similar to that used by Churchhouse and some theoretical results that turn 'near covers' into covers, while Nielsen's cover uses more than 10^{50} congruences and does not rely on computers.

1.3 Covering Systems With Only Odd Moduli

The survey of Porubsky [26] serves as an excellent initial resource for many of the other questions in the area of covering systems. Other valuable surveys can be found in [27] and [32]. Another difficult and well known problem discussed in these papers, first considered by Erdős and Selfridge [9], is the result of the following conjecture.

Conjecture 1.2 (Erdős [9])**.** There exists a covering system with only odd moduli.

This conjecture provides the motivation for Chapter 6 and was proposed in the negative by Selfridge [9] who offered \$2000 for an explicit odd covering, according to [13]. Erdős and Graham also ask whether a covering system exists with all moduli divisible by a given integer d. Both of these questions remain open.

In a paper studying the reducibility of polynomials Schnizel made the following conjecture which is intrinsically related to the previous conjecture.

Conjecture 1.3 (Schinzel [29])**.** In every covering system at least one modulus divides another.

Selfridge's conjecture (that there is no odd cover) has been shown to imply Schinzel's conjecture due to work of Schinzel [29], in which he gives credit to Selfridge for previously proving the implication. Other than this, little progress has been made on Conjecture 1.2 or its negation. Churchhouse [7] shows that there is no incongruent covering system with all moduli of the form $3^a5^b7^c11^d$, and that it is highly unlikely that a covering set based on the divisors of $3^a5^b7^c11^d13^e$ exists. Berger, Felzenbaum and Fraenkel [3] show that a necessary condition for an incongruent covering system with only odd moduli is

$$\prod_{i=1}^{n} \frac{p_i - 1}{p_i - 2} - \sum_{i=1}^{n} \frac{1}{p_i - 2} > 2,$$

where the p_i are the distinct prime divisors of the moduli of the system, using finite geometry. It has also been shown by Guo and Sun [15] that if there exists an odd covering system with square-free moduli then the least common multiple of the moduli must have at least 22 distinct prime divisors. This has improved the previous results of 13 [3] and 18 [31] distinct prime divisors.

Work has also been done on analogous questions to Conjecture 1.2 in number fields [17]. However this work does not shed light on the question for the integers.

Nielsen asks whether the methods he used to find a covering system with large minimum modulus might be used to find an odd covering system. While attempts here fail, perhaps more can be done by developing the ideas in [25] further.

1.4 Incongruent and Disjoint Covering Systems

It is well known that there can be no finite incongruent and disjoint covering system, a fact proved independently by Mirsky, Newman, Davenport and Rado [26]. The first proof of this uses complex numbers, a tool not used elsewhere in the literature of covering systems. We will reproduce the proof here.

Take a disjoint covering system using the congruences $x \equiv a_i \pmod{m_i}, i = 1, 2, \ldots, t$ where $0 \leq a_i < m_i$ and $1 < n_1 < n_2 < \cdots < n_t$. Let z be a complex number with $|z| < 1$, then

$$\frac{1}{1-z} = \sum_{i=1}^{t} \frac{z^{a_i}}{1-z^{n_i}}.$$

Let $z \to \zeta$, where ζ is a primitive n_t^{th} root of unity. The fact that the moduli increase yields a contradiction with the left side approaching a finite limit, as do all but the last term in the sum on the right side. This proof shows that every disjoint covering system contains at least two residue classes with modulus n_t. For some time no proof of this result was known which does not use complex numbers. Berger, Felzenbaum and Fraenkel [2] have since given an elementary proof of this result using geometry.

Work was done soon after this result by Stein, Znam and Porubsky [26] to completely categorise disjoint covering systems with a single modulus repeated up to 5 times, allowing only the largest modulus in the covering system to be repeated. Znam and Newman [26] also showed that for a disjoint covering system with greatest modulus n_t having least prime divisor p then the covering system contains at least p congruence classes modulo n_t.

There are a number of other results on how many times the largest modulus must be repeated in a disjoint, but not incongruent, covering system, based on the prime divisors of the moduli in the covering system. These results are due to Burshtein [4], Simpson [30] and Berger, Felzenbaum and Fraenkel [1].

While there is no such cover for $\mathbb{Z}$, it is known that there exists a disjoint and incongruent cover for $\mathbb{Z}^3$ using only a finite number of pairs of congruences, namely,

$$\begin{aligned}\mathbb{Z}^3 = \{x \equiv y \equiv 0 \pmod 2\} &\cup \{\{x \equiv z \equiv 0 \pmod 2\} + (1,0,0)\} \\ &\cup \{\{y \equiv z \equiv 0 \pmod 2\} + (0,1,1)\} \cup \{\{x \equiv y \equiv z \pmod 2\} + (0,1,0)\}.\end{aligned}$$

This is equivalently presented as

$$\frac{1}{(1-x)(1-y)(1-z)} = \frac{1}{(1-x^2)(1-y^2)(1-z)} + \frac{x}{(1-x^2)(1-y)(1-z^2)} + \frac{yz}{(1-x)(1-y^2)(1-z^2)} + \frac{y+xz}{(1-x^2)(1-y^2)(1-z^2)}.$$

This cover can be extended to cover $\mathbb{Z}^n$ for $n \geq 3$ by adjoining $n-1$ new variables with no restrictions to each of the above subgroup cosets. The only remaining question is whether $\mathbb{Z}^2$ can be covered with a finite union of disjoint cosets of distinct subgroups. This question is still open, and provides some of the motivation for Chapter 7.

The main focus of this thesis will be to discuss and further the idea of an incongruent restricted disjoint covering system (IRDCS). The definition of an IRDCS is as follows.

Definition 1.4. An incongruent restricted disjoint covering system of length n is a collection of congruence classes which covers the integers in the interval $[1, n]$, where no modulus is repeated, each integer is contained in exactly one congruence class and each congruence class contains at least two numbers in the interval.

Note: we take the plural of IRDCS to be IRDCS. It will always be clear from context whether we are referring to the singular or the plural.

We will use these IRDCS to study the analogous questions to those of interest in covering systems. This thesis discusses incongruent restricted disjoint covering systems and some related constructions in the coming chapters. We focus on the following previously proposed questions from [23].

(1) Can the smallest modulus of some IRDCS be arbitrarily large?

(2) Do there exist IRDCS with all moduli odd?

(3) Do there exist IRDCS where all moduli are used at least k times for some $k > 2$?

This thesis addresses these questions and makes significant headway towards their resolution.

Chapter 5 studies IRDCS with large minimum modulus. We present, amongst other examples, one IRDCS with minimum modulus 50.

In Chapter 6 it is shown that there are IRDCS with only odd moduli. The smallest example is one of length 83. This chapter will present information on all of the known examples of what will be referred to as odd IRDCS.

Finally, in Chapter 7, we extend the definition of IRDCS to two dimensions, determining conditions on the relevant parameters for the existence of such structures. In this chapter we also study some of the structural properties, analogous to those of one-dimensional IRDCS, for these new constructions. The next natural question would be to extend to three and higher dimensions, though two dimensions will be the extent of explorations for this thesis.

Chapter 2

IRDCS Introduction

This chapter describes the rudiments of incongruent restricted disjoint covering systems, giving the grounding from which to build. The vast majority of results and data in this Chapter initially appear in [23] and will be cited as such. From this point onwards our focus will be almost exclusively on incongruent restricted disjoint covering systems rather than the classical setting of covering systems. We begin by restating the definition of IRDCS from Chapter 1.

Definition 2.1. An incongruent restricted disjoint covering system (hereafter IRDCS) of length n is a collection of congruence classes which covers the integers in the interval $[1, n]$, where no modulus is repeated, each integer is contained in exactly one congruence class and each congruence class contains at least two numbers in the interval.

Notation. $S(m, a)$ will denote the congruence class $\{x : x \equiv a \pmod{m}\}$.

It is easy to construct simple covering systems such as the collection

$$S(2, 0), S(2, 1).$$

While this covering system is disjoint, it is not particularly interesting. Including the additional condition that the covering system must be incongruent, then as in [8] the simplest such covering is the collection, as seen in Chapter 1,

$$S(2, 0), S(3, 0), S(4, 3), S(6, 1), S(12, 5)$$

(see Chapter 1 for a geometric representation of this covering system).

IRDCS require covers of the first n integers (or more generally any n consecutive integers) that are simultaneously incongruent and disjoint. This is not possible in the classical setting ([26]). Moreover, to avoid trivialities the definition requires that each congruence class contains at least two points in the interval. These conditions are precisely what makes our situation interesting. It may be of interest to consider related constructions where a small number of congruence classes are allowed to cover only one position in the interval. This is equivalent to having IRDCS with a small number of uncovered positions, since we may then choose any suitably large modulus to cover these remaining positions. These constructions will not be studied here.

It is not immediately obvious that IRDCS exist. An exhaustive search, by an algorithm which will be presented in Chapter 3, shows that the first example is the collection of congruences

$$S(6,1), S(9,2), S(3,0), S(4,0), S(5,0), \tag{2.1}$$

being an IRDCS of length 11. This notation does not provide a clear picture of the given IRDCS, and the following notation is used.

Notation. Rather than expressing an IRDCS of length n as a collection of congruences, write a sequence of n integers where the i^{th} member of the sequence represents the modulus of the unique congruence class to which i belongs. This is the **alternate notation** for an IRDCS.

For example, the first IRDCS (2.1) expressed in alternate notation is

$$6, 9, 3, 4, 5, 3, 6, 4, 3, 5, 9.$$

Thus it would be equivalent to define an IRDCS of length n as a sequence of integers $s_1, s_2, \ldots, s_n$ with the property that $s_i = m$ for some m if and only if $s_{i+km} = m$ for all $k \in \mathbb{Z}$ such that $i + km \in [1, n]$, $s_j \neq m$ for all j which cannot be expressed as

$i + km$ for such k, and all integers in the sequence appear at least twice. Myerson, Poon and Simpson [23] note that this is reminiscent of Langford Sequences [21]. A Langford Sequence of order n is defined as a sequence $l_1, l_2, \ldots, l_{2n}$ of $2n$ integers in which each integer from 1 to n appears exactly twice, and such that if $l_i = l_j$ then $l_i = |i - j| - 1$, for example

$$4, 1, 3, 1, 2, 4, 3, 2.$$

Nothing more on Langford Sequences will be studied in this thesis, but may provide an interesting avenue for related work.

Rather large IRDCS will be required to discuss some of the questions considered in later chapters, having lengths over 100. In these instances even the alternate notation will become unwieldy.

Notation. The **compact notation** of an IRDCS is achieved by listing the moduli of the congruences which cover the n integers in the order in which they first appear in the IRDCS.

The compact notation is clearly unique for a given length, since if two IRDCS of the same length use the same moduli in the same order then they must be equal. The compact notation for the IRDCS (2.1) is $6, 9, 3, 4, 5$.

A few more definitions are required to further discuss the structure of IRDCS. Given an IRDCS covering all of the elements in $[1, n]$ as $\{S(m_1, a_1), S(m_2, a_2), \ldots, S(m_t, a_t)\}$ then:

Definition 2.2. The number of integers n covered by the IRDCS is its **length**.

Definition 2.3. The **order** t of an IRDCS is the number of congruences used.

Definition 2.4. The **heft** of an IRDCS with moduli $m_1, m_2, \ldots, m_t$ is $\sum_{i=1}^{t} 1/m_i$.

So the initial example (2.1) is an IRDCS of length 11 with **order** 5 and **heft** $191/180 = 1.0611\ldots$.

There are instances where the same compact notation can provide IRDCS of different lengths, particularly in some examples to be presented in Chapter 5. These IRDCS are clearly very similar, since they use the same moduli, and have the same heft and order. The process of generating two IRDCS of different lengths with the same compact notation can be viewed as extending the length of the shorter IRDCS to the right, for so long as there is no clash and no position remains uncovered. One such example is the length 17 IRDCS with compact notation $9, 11, 4, 5, 12, 6, 8,$ where the modulus 6 alone also covers the 18^{th} position. Attempting to increase it to a length 19 IRDCS sees the moduli 4 and 5 clash. One can also manufacture a different compact notation (which will just be a reordering of the original) in some circumstances by extending the length of the original IRDCS to the left, rather than the right. Again there is a length 17 IRDCS which provides an example, namely the length 17 IRDCS with compact notation $12, 8, 4, 5, 11, 6, 9$ producing the length 18 IRDCS with compact notation $6, 12, 8, 4, 5, 11, 9,$ since we could cover what was position 0 by the modulus 6. These two length 17 (and then 18) IRDCS are related in that they are **reversals** which will be defined shortly. Neither of these constructions will significantly alter any of our analysis or methods. This construction will be prevalent in examples that occur in Chapter 5, but does not otherwise warrant any further consideration.

2.1 Computational and Structural Results

There exists an algorithm to find all IRDCS of given length and in particular with given conditions. These conditions include restricting the moduli in the IRDCS to be all larger than a given minimum, or to be all odd. This along with structural properties of IRDCS that will be developed will be exploited to say all that we can about IRDCS. The following lemma will be used to show the existence of IRDCS for all sufficiently large lengths.

Lemma 2.1. *[23] For any IRDCS* $\mathcal{A} = \{S(m_i, a_i) : i = 1, 2, \ldots, t\}$ *of length* n.

$$\mathcal{A}' = \{S(2m_i, 2a_i) : i = 1, 2, \ldots, t\} \cup S(2, 1)$$

produces IRDCS of lengths $2n-1, 2n$ *and* $2n+1$. *Moreover if* $\mathcal{A}$ *has order* t *and heft* h *then* $\mathcal{A}'$ *has order* $t+1$ *and heft* $\frac{1}{2}(1+h)$. *We call this* ***doubling***.

Proof. If the alternate notation of the initial IRDCS is $m_1, m_2, \ldots, m_n$ then the doubling process, which doubles both the moduli and the initial position of all of the congruences, before adding the congruence modulo 2 will produce a partial IRDCS which looks like

$$\_, 2m_1, \_, 2m_2, \_, \ldots, \_, 2m_n, \_,$$

to which adding the congruence modulo 2 gives

$$2, 2m_1, 2, 2m_2, 2, \ldots, 2, 2m_n, 2,$$

which is an IRDCS of length $2n+1$. It can be contracted to an IRDCS of length $2n$ (respectively, $2n-1$) by removing one of (respectively, both of) the 2s at the ends of the IRDCS of length $2n+1$ (and shifting everything down one, if necessary).

Also the order of $\mathcal{A}'$ is clearly $t+1$, and the heft of $\mathcal{A}'$ is

$$\frac{1}{2} + \sum_{i=1}^{t} \frac{1}{2m_i} = \frac{1}{2}(1 + \sum_{i=1}^{t} \frac{1}{m_i}) = \frac{1}{2}(1+h),$$

as required □

This doubling process can be iterated to produce arbitrarily long IRDCS with heft approaching 1 and order $O(\log n)$. A consequence of this observation is the following theorem.

Theorem 2.1. *[23] There exist IRDCS for all lengths greater than 16.*

A little more discovery is required before we can present a proof for this theorem. Firstly, all IRDCS for smaller lengths need to be manually found. This is done via the exhaustive algorithm to be presented in Chapter 3. An exhaustive search for all IRDCS of lengths up to 32 gives the number of IRDCS and structural properties presented in the table below, first appearing in [23].

Length	Number of IRDCS	Orders	Heft Range
11	2	5	1.06111
17	4	7	1.0123 - 1.02702
18	6	7, 8	1.0123 - 1.02702
19	18	7, 8	1.00394 - 1.04488
20	14	7, 8	1.00394 - 1.03238
21	26	6 - 9	0.9968 - 1.03056
22	84	6, 8 - 10	0.9968 - 1.05156
23	88	6, 8 - 10	0.9968 - 1.04225
24	46	8 - 10	0.991306 - 1.03013
25	176	8 - 10	0.996205 - 1.02775
26	380	8 - 12	0.996205 - 1.0506
27	812	8 - 12	0.996205 - 1.05051
28	844	8 - 12	0.989552 - 1.04808
29	1770	9 - 13	0.989552 - 1.04947
30	2164	9 - 13	0.989406 - 1.04288
31	5554	9 - 14	0.991297 - 1.05823
32	9244	9 - 14	0.992184 - 1.0578

Note that there are no IRDCS with $n < 11$ or $12 \leq n \leq 16$. With these results we are now in position to prove the theorem.

Proof. As shown in Lemma 2.1, for any IRDCS of length n IRDCS of lengths $2n-1, 2n$ and $2n+1$ can be constructed. So, beginning with the above table, IRDCS of lengths 33 through to 65 are constructed by doubling those that exist of length 17 through 32 (for example, the length 33 comes from doubling the length 17 and removing the 2's at either end, while the length 65 comes from doubling the length 32 example). There are now IRDCS for a block of consecutive lengths twice as long as previous. All of these new IRDCS may now be doubled, removing the 2's when required, to produce IRDCS of all required lengths up to and including length 131. Iterating this process gives us IRDCS of all lengths above 16. □

Given this information concerning the existence of IRDCS, we begin to study their structure. If the collection

$$\mathcal{A} = \{S(m_i, a_i) : i = 1, \ldots, t\}$$

is an IRDCS of length n on $[1, n]$, then so too is

$$\mathcal{A}' = \{S(m_i, n+1-a_i) : i = 1, \ldots, t\}.$$

This modified IRDCS is called the **reversal** of $\mathcal{A}$, since it is generated by writing the alternate notation of the original IRDCS in reverse order.

While the proof of the following lemma is given in [23], we present it in detail here as we will need similar arguments in a later chapter.

Lemma 2.2. *[23] No IRDCS equals its reversal.*

Proof. Assume that such an IRDCS exists and call it **palindromic**.

If the length of this palindromic IRDCS is even, then the two central positions must be covered by the same congruence, which is not possible since the modulus 1 is not allowed.

If the length is odd, $n = 2r+1$, then the two positions around the centre must be covered by the modulus 2. If an IRDCS contains the modulus 2, then it must contain only even moduli. Say the IRDCS has moduli $2, m_2, \ldots, m_t$. Then it can be replaced by one of length r or $r+1$, whichever is odd, with moduli $m_2/2, m_3/2, \ldots, m_t/2$. This new construction is an IRDCS since removing the 2s halves the distance between any pair of points, which means that all moduli must also be halved. We can think of this new IRDCS as the 'un-doubled' version of the original IRDCS.

If the original odd length IRDCS is palindromic, then this un-doubled IRDCS must also be palindromic. The un-doubled IRDCS still has a central element, so its length can't be even. The process of un-doubling the original IRDCS may be repeated until either there is no modulus 2, or the length of the IRDCS is under 11, both of which lead to a contradiction.

Hence, no palindromic IRDCS exists. □

By the above lemma, the number of IRDCS of any given length is even. For example there are two IRDCS of length 11. These length 11 IRDCS are effectively equivalent, since one is the reversal of the other.

2.2 Bounds on Order

For the remainder of the chapter write $n(\mathcal{A}), t(\mathcal{A})$ and $h(\mathcal{A})$ for the length, order and heft of an IRDCS $\mathcal{A}$ respectively. These will revert to n, t and h when it will cause no confusion. Lemma 2.1 shows that doubling produces IRDCS with $t(\mathcal{A}) = O(\log(n(\mathcal{A})))$, but how large can the order be in terms of $n(\mathcal{A})$?

Details of the proof of the following Theorem appear in [23], but they are presented here as we will be using similar arguments in Section 2.2.1.

Theorem 2.2. *[23] For an IRDCS $\mathcal{A}$,*

$$t(\mathcal{A}) \leq \frac{n(\mathcal{A}) - 1}{2} \tag{2.1}$$

with equality if and only if $n(\mathcal{A}) = 11$.

Proof. For $n = 11$ we have $t = 5 = \frac{11-1}{2}$, giving equality in (2.1). There are no IRDCS for $n = 12, \ldots, 16$ as seen in the table following Theorem 2.1.

For $n \geq 17$ odd, let $n = 2r - 1$. The middle position, position r, must be used to cover at least 3 or a greater odd number of positions, since its hits are all symmetric about the middle of the IRDCS. If it is used 5 or more times then there are at most $n - 5$ other positions to be covered and thus

$$t \leq 1 + \frac{n-5}{2} < \frac{n-1}{2},$$

as each modulus must be used at least twice. Thus, we may assume that the modulus covering r covers exactly three positions. If any other modulus is used 3 or more times then clearly $t \leq 2 + \frac{n-6}{2} < \frac{n-1}{2}$.

Note that for $n \geq 17$ with middle modulus hitting only 3 positions the middle modulus must be larger than 4. Now for all the other congruences covering exactly two positions, the positions $r \pm 1$ must be covered by moduli $r - 1$ and r. Without loss of generality, let the modulus $r - 1$ cover positions $r - 1$ and $2r - 2 = n - 1$ and modulus r cover positions $r + 1$ and 1. Then the modulus covering $r + 2$ must be $r - 2$, also covering position 4, and the modulus covering $r-2$ must be $r+1$, also covering position $2r - 1 = n$. The modulus covering $r - 3$ must be $r - 3$ also covering $2r - 6 = n - 5$ and then there is no way to cover position $r + 3$, so $t < \frac{n-1}{2}$.

Now suppose that n is even and let $n = 2r$. Either r or $r + 1$ must be covered by a modulus less than r, thus belonging to a class of size at least 3, and as above only one class may be of size 3 or greater else $t < \frac{n-1}{2}$. Suppose that the modulus r is used to cover positions r and $2r = n$ and that the class containing position $r + 1$ is the only one of size greater than 2. By Theorem 2.1 $r > 8$ and so the modulus covering position $r + 1$ must be larger than 4. Then the modulus $r - 1$ must cover positions $r - 1$ and $2r - 2 = n - 2$, and so position $r + 2$ must be covered by the modulus $r + 1$ covering positions 1 and $r + 2$ and then position $r - 2$ must be covered by the modulus $r - 2$ covering positions $r - 2$ and $2r - 4 = n - 4$. This leaves no modulus to cover position $r + 3$. A similar argument holds, by symmetry, assuming the position $r + 1$ is covered by the modulus r and thus $t < (n-1)/2$. □

2.2.1 A new result on the order bound

The fact that this bound is no better than the worst case, since every congruence must hit at least two points, begs the question of how often can the order of an IRDCS be close to this bound.

There do exist some IRDCS with $n = 2t + 2$ and $n = 2t + 3$. Data in the appendix presents summary statistics for all IRDCS up to and including $n = 50$. The following theorem gives the necessary conditions required for there to exist such IRDCS for $n > 50$.

Theorem 2.3. *For there to exist an IRDCS with $n = 2t + 2$ then two moduli in the IRDCS are used 3 times, the others are used twice and either*

- $n = 18, 22, 26,$ *or*
- $n > 50$ *and the moduli used three times must cover the two separate middle positions of the IRDCS.*

Also, for there to exist an IRDCS with $n = 2t+3$ then three moduli in the IRDCS are used 3 times, one of them covering the middle position, all others are used twice and $n = 11, 17, 19, 21, 23, 27, 29, 31, 35, 37, 41$ or $n > 50$. Further, at least one of the positions $t-1, t, t+1, t+3, t+4, t+5$ must be covered by one of the congruences that covers 3 positions.

Proof. A look at the summary table before the proof of Theorem 2.1 shows that this is possible for $t = 5$ and for all $t = 7, 8, \ldots, 14$ excluding $t = 11$. Data presented in Appendix A shows that up to length 49, and thus $t \leq 23$, it is possible only for $t = 16, 17$ and 19.

If $n = 2t + 2$, then either all but one of the moduli are used precisely twice, and one modulus is used four times, or all but two of the moduli are used precisely twice and two of the moduli are used precisely three times.

In the first case, the middle positions of this IRDCS are $t + 1$ and $t + 2$. Let position $t + 1$ be inside a congruence with two hits. Then the modulus covering this position must be $t+1$. Then position $t+2$ cannot be covered by moduli $t+1$ or by t, as it will clash with the already chosen modulus, so that this congruence will have at least three hits and must thus have four hits. As such, it must have one hit to the right, and two to the left of position $t + 2$. The first restriction on the modulus m forces $2m > t$, while the second gives $2m \leq t + 1$, so that we must have t odd and $2m = t + 1$. This congruence covers the positions $1, \frac{t+3}{2}, t + 2, \frac{3t+5}{2}$. Next position t must be covered by the modulus t, and position $t + 3$ cannot be covered. So there must be precisely two congruences covering three positions and all of the remaining congruences covering 2 positions.

The proof of the bound for order shows that the two congruences covering three positions must cover at least one of $t + 1, t + 2$ and if not both of these another from the set $\{t-1, t, t+3, t+4\}$. For two of these points to be covered by the same three hit congruence, then modulus must satisfy $m \leq 4$ (note that hitting $t-1$ and $t+4$ will force the congruence to hit four positions as it is symmetric about the middle). However this congruence can only have three hits for $n \leq 17$, so that four of these positions must be in a two hit congruence. Firstly, lets assume that only one of $t+1, t+2$ are covered by a three hit congruence, and without loss of generality let it be position $t+1$. If congruences with precisely two hits are forced for as long as possible while the IRDCS is filled from the middle out (as in the algorithm in Chapter 3) then the following congruences are forced:

Initial Position	Modulus	Other Positions Covered
$t+2$	$t+1$	1
$t+1$	$\left[\frac{t+1}{2}\right]+1 \leq m \leq t$	-
$t+3$	t	3
t	$t+2$	$2t+2$
$t+4$	$t-1$	5
$t-1$	-	Three hits
$t+5$	$t-2$	7
$t-2$	$t+3$	$2t+1$
$t+6$	$t-3$	9

and then there is no choice available for a modulus to cover position $t-3$, where the congruence has only two hits. This collection of choices will fail so long as $t-3, t+6$ or more central positions cannot be covered by one of the three hit congruences. If it were possible, then the one with largest modulus will be the $t-1, t+6$ pair. This implies $t-8 \geq 1, t-15 \leq 0$ and $t+13 \geq 2t+3$, so that $9 \leq t \leq 10$, and all possibilities with these orders have been manually checked.

If two hit congruences stop being forced one step earlier, we get:

Initial Position	Modulus	Other Positions Covered
$t+2$	$t+1$	1
$t+1$	-	-
$t+3$	t	3
t	$t+2$	$2t+2$
$t+4$	-	Three hits
$t-1$	$t-1$	$2t-2$
$t+5$	$t-2$ or $t+3$	7 or 2
$t-2$	$t+3$ or $t-2$	$2t+1$ or $2t-4$
$t+6$	$t-3$ or $t+4$	9 or 2
$t-3$	$t+4$ or $t-3$	$2t+1$ or $2t-6$,

in either case both positions $2, 2t+1$ must be covered. Then position $t+7$ must be covered by the modulus $t-4$ and there is nothing to cover position $t-4$. So this fails so long as none of these positions is covered by the three hit congruence. For this to be true, the largest available modulus would be $t+6-(t-4) = 10$, for which $t-4-10 \leq 0$, and these cases have already been manually checked.

By the same arguments it is possible to show that positions t or $t+3$ cannot be the second position with a three hit congruence. The only remaining case is where the two middle positions are both covered by the three hit congruences.

If $n = 2t+3$ then since n is odd, the middle position $t+2$ must be covered by a congruence with either three or five hits. If $t+2$ is covered by a modulus with five hits, then all of the remaining moduli must be used precisely twice. The modulus covering the middle position must satisfy the inequalities $2m \leq t+1$ and $3m \geq t+2$. The moduli covering the positions $t+1$ and $t+3$ must be either $t+1$ or $t+2$. With this we also cover either the pair $1, 2t+2$ or $2, 2t+3$. The positions t and $t+4$ must be covered by the moduli t and $t+3$. For these congruences the additional positions covered are either $1, 2t$ or $4, 2t+3$. In either case positions 1 and $2t+3$ are now covered. Then position $t+5$ must be covered by the modulus $t-1$ and there is no modulus available to cover $t-1$. So long as $t-1$ is not in the same congruence as the middle position, then $t+2$ cannot be covered by a modulus which is used 5 times. This would require modulus $m=3$ which for this to have 5 hits implies $n \leq 17$.

Next, consider whether there can be one modulus being used precisely four times, one modulus being used precisely three times, which must be the modulus covering the middle position, and all other congruences being used precisely twice. More calculations of the form used for the $n=2t+2$ case show that this is not possible.

It remains to see whether there can be precisely three congruences which hit precisely three points, one of which must include the middle position, and all other congruences covering precisely two points. The order proof shows that the second of these three hit congruences must occur at least as early, in the sense of filling around the middle, as the position $t+5$. To see this replace $t+2$ with r in the order proof. □

2.3 Bounds on Heft

A density argument alluded to in Chapter 1 shows that for a classical covering system the heft is always at least 1, and equals 1 if and only if the system is disjoint. The following result gives us bounds on heft for IRDCS. These results are an improvement on results on heft appearing in [23].

Theorem 2.4. *For any IRDCS $\mathcal{A}$,*

$$1-\frac{(n-1)t-t^2}{n(n-1)} \leq h(\mathcal{A}) \leq 1+\frac{1}{n+1}\left(t-3+\frac{2}{t+1}\right). \tag{2.1}$$

Proof. Consider the IRDCS $\mathcal{A}=\{S(m_i,a_i) : i=1,\ldots,t\}$ and assume, without loss of generality, that $m_1<m_2<\cdots<m_t$, and $1 \leq a_i \leq m_i$ for all i, so that a_i always represents the first position that the given congruence $S(m_i,a_i)$ covers in $[1,n]$. Define the last position that the congruence $S(m_i,a_i)$ covers in $[1,n]$ to be $n+1-b_i$, so that the a_i are positive and distinct and so are the b_i. Given this information, the number

of elements of $[1,n]$ belonging to $S(m_i,a_i)$ is

$$\frac{n+1-b_i-a_i}{m_i}+1.$$

Since each member of $[1,n]$ belongs to exactly one class

$$\sum_{i=1}^{t}\left(\frac{n+1-b_i-a_i}{m_i}+1\right)=n.$$

Thus,

$$n\sum_{i=1}^{t}\frac{1}{m_i}=\sum_{i=1}^{t}\frac{b_i+a_i-1}{m_i}+n-t. \tag{2.2}$$

The right hand side is minimised when

$$a_i=b_i=i$$

for each i, so that

$$n\sum_{i=1}^{t}\frac{1}{m_i}\geq\sum_{i=1}^{t}\frac{2i-1}{m_i}+n-t.$$

Now

$$\begin{aligned}\sum_{i=1}^{t}\frac{2i-1}{m_i}&\geq\frac{1}{m_t}\sum_{i=1}^{t}(2i-1)\\&=\frac{t^2}{m_t},\end{aligned}$$

so that

$$\begin{aligned}nh&\geq\frac{t^2}{m_t}+n-t\\&\geq\frac{t^2}{n-1}+n-t,\\(nh-n)(n-1)&\geq t^2-(n-1)t,\end{aligned}$$

which gives

$$h\geq 1+\frac{t^2-(n-1)t}{n(n-1)}. \tag{2.3}$$

In the other direction note that $a_i\leq m_i$ and $b_i\leq m_i$ for each i. This gives congruences which do not cover the first and last m_1-1 positions in $[1,n]$, so that

$$\sum_{i=1}^{t}\frac{a_i+b_i-1}{m_i}\leq\frac{1}{m_t}+\frac{2}{m_{t-1}}+\cdots+\frac{m_1-1}{m_{t+1-(m_1-1)}}+\sum_{i=1}^{t+1-m_1}\frac{2m_i-1}{m_i}, \tag{2.4}$$

where since $m_1 \geq 2$ at worst then

$$\begin{aligned}\sum_{i=1}^{t} \frac{a_i + b_i - 1}{m_i} &\leq \frac{1}{m_t} + \sum_{i=1}^{t-1} \frac{2m_i - 1}{m_i} \\ &= \frac{2}{m_t} + 2(t-1) - h \\ &\leq \frac{2}{2 + (t-1)} + 2(t-1) - h,\end{aligned}$$

and so

$$\begin{aligned}nh &\leq n - t + \frac{2}{t+1} + 2(t-1) - h \\ h(n+1) &\leq n + t + \frac{2}{t+1} - 2\end{aligned}$$

and then

$$\sum_{i=1}^{t} \frac{1}{m_i} \leq \frac{1}{n+1}\left(n - 2 + t + \frac{2}{t+1}\right). \tag{2.5}$$

Combining (2.3) and (2.5) completes the proof. □

Corollary 2.1. *For any length n IRDCS,*

$$\frac{3}{4} + \frac{1}{4n} \leq h \leq \frac{3n-5}{2(n+1)} + \frac{4}{(n+1)^2}. \tag{2.6}$$

Proof. Substitute $t = (n-1)/2$ into (2.1) on noting in the lower bound that the quadratic in t in (2.3) is minimised for $t = \frac{n-1}{2}$, and that the function in t in (2.5) is increasing in t. □

In the case of the lower bound, more care can be taken to slightly improve the given bound for a given IRDCS. The bound retains the same asymptotic value of $\frac{3}{4}$. Take

$$\sum_{i=1}^{t} \frac{2i-1}{m_i} = \frac{2t}{m_t} + 2\sum_{i=1}^{t-1} \frac{i}{m_i} - h,$$

where

$$\begin{aligned}\sum_{i=1}^{t-1} \frac{i}{m_i} &\geq \frac{1}{m_1} + \frac{1}{m_2} + \sum_{i=3}^{t-1} \frac{i}{m_i} + \frac{1}{m_t} \\ &= h + \sum_{i=3}^{t-1} \frac{i-1}{m_i} \\ &\geq h + \frac{1}{m_{t-1}} \sum_{i=3}^{t-1} (i-1)\end{aligned}$$

which gives

$$nh \geq h + \frac{2t}{m_t} + \frac{t(t-3)}{m_{t-1}} + n - t.$$

Taking $m_t \leq n-1$ and $m_{t-1} \leq n-3$ (if it were $n-2$ then the largest modulus would be $n-1$ and it would have already covered both of the end points, a contradiction) and minimising the resultant polynomial in t gives

$$h \geq \frac{n}{n-1} - \frac{(n^2-3n+6)^2}{4(n-1)^3(n-3)},$$

which can be seen to be a slightly better bound than that in the theorem, but still no better than $\frac{3}{4}$ in the limit. Moreover any improvement given by this formula over that in the theorem is bounded by 0.015 for $n \geq 50$. Data to be presented in the Appendices will give actual heft ranges for all IRDCS with $n \leq 50$.

Attempting to improve the upper bound using the same or similar methods to those used for the lower bound yields no improvement.

It is apparent from the empirical evidence that we should be able to improve the upper bound as we did the lower bound. The bounds effectively only tell us that heft is bounded as $\frac{3}{4} < h < \frac{3}{2}$, the upper bound for $n > 15$, which is clearly fine since the only counter example to this is the length 11 IRDCS. Of all IRDCS up to and including length 50, the heft is actually bounded by $0.987952 \leq h \leq 1.07287$, and it appears that methods such as these presented are not sufficient to significantly improve either of the bounds. An examiner of this thesis has found an improvement of this bound. We present his argument in Appendix C.

Studying IRDCS leads to some natural questions, first asked in [23], motivated in particular by the relationship between covering systems and IRDCS.

(1) Do there exist IRDCS with all moduli odd?

(2) Can the smallest modulus of some IRDCS be arbitrarily large?

(3) Our definition requires that each congruence be used at least twice. Do there exist IRDCS where all moduli are used at least k times for some $k > 2$?

(4) Do there exist IRDCS where no modulus is divisible by any of the first k primes?

These questions provide much of the motivation for the remainder of the thesis. Many of these questions will be studied in later chapters.

2.4 Open Questions

At this stage it is worth highlighting a few areas in which we have some questions of interest where enough progress has not been made to warrant any significant exposition.

2.4.0.1 Moduli Divisibility

In [23] the following question is asked. Do there exist IRDCS with all moduli divisible by some integer d for $d > 2$? Doubling produces examples where all moduli are divisible by 2. For an example with all moduli divisible by three, take two IRDCS of the same length with disjoint moduli sets. Two IRDCS of length 43 are given in [23], but as we will see in one of the following questions, the smallest examples are for length 33 IRDCS with compact notation

$$24, 2, 16, 8, 10, 22, 12, 18, \text{ and } 21, 17, 11, 4, 6, 25, 14, 9, 23, 13, 15.$$

Generally for any two sets of congruences $\{S(m_1, a_1), \ldots, S(m_t, a_t)\}$ and $\{S(n_1, b_1), \ldots, S(n_s, b_s)\}$ giving IRDCS on $[1, n]$ with distinct moduli the collection

$$\{S(3m_i, 3a_i + 1) : i = 1, \ldots, t\} \cup \{S(3n_i, 3b_i + 2) : i = 1, \ldots, s\} \cup \{S(3, 0)\}$$

is an IRDCS on $[1, 3n]$ where every modulus is divisible by 3.

2.4.0.2 Families of IRDCS

The paper [23] introduces the notion of families of IRDCS. Much like doubling, this involves finding conditions for which we may always find or produce IRDCS. The example given in the paper is called good IRDCS.

Definition 2.5. An IRDCS on $[1, n]$ is **good** if,

(1) n is an odd multiple of 3,

(2) if m_1 is the modulus of the class containing 1 then $m_1 > 2n/3$. Along with the first point this implies that

$$3n > 3m_1 \geq 2n + 3,$$

(3) $3m_1 - n - 1$ is not a power of 2,

(4) no modulus in the collection is a power of 2.

These good IRDCS do exist. The presented example is the collection $\{S(19, 1), S(13, 2), S(9, 3), S(5, 4), S(6, 5), S(10, 6), S(11, 7), S(17, 8), S(12, 10), S(14, 13)\}$. There is then an algorithm which creates another good IRDCS from this IRDCS.

Let $\{S(m_i, a_i) : i = 1, \ldots, t\}$ be a good IRDCS on $[1, n]$ where $a_1 = 1$, so that $m_1 > 2n/3$. The following collection of congruence classes form a good IRDCS on $[1, 3n]$. Set $\mathcal{A} = \{S(3, 0)\}$. Set $\mathcal{B} = \{S(3m_i, 3a_i - 2) : i = 1, \ldots, t\}$. Labeling $x_1 = 1$ and $x_2 = 1 + 3m_1$ then $\mathcal{B}$ covers all of $S(3, 1) \cap [1, 3n]$ excluding x_1 and x_2.

Let $\theta = [\log_2(n/3)]$ and $m = 3(2^\theta)$ and set $y_1 = n+2$ and $y_2 = n+2m+2$, then set $\mathcal{C} = \{S(3(2^i), y_1 + 3(2^{i-1})) : i = 1, \ldots, \theta+1\}$. $\mathcal{C}$ covers all of $S(3,2) \cap [1, 3n]$ excluding y_1 and y_2. Lastly set $\mathcal{D} = \{S(y_2 - x_1, x_1), S(x_2 - y_1, y_1)\}$. Then $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ is a good IRDCS on $[1, 3n]$.

The previous two questions were raised in [23]. We now present two new questions.

A possibility for another family of IRDCS is the following based on the use of the moduli 9, 6 and 3 in consecutive positions. Any IRDCS with 963 somewhere in its alternate notation looks like

$$\ldots, 9, 6, 3, -, -, 3, -, 6, 3, 9, -, 3, -, 6, 3, -, -, 3, \ldots,$$

where this pattern repeats indefinitely. So long as $n \geq 18$ the modulus 9 always covers at least two positions, and so a completed interval will be an IRDCS. The question is whether there exists some N such for all $n \geq N$ there exists a length n IRDCS using this 963 construction.

2.4.0.3 IRDCS using only even moduli that aren't doublings

We know that there are IRDCS with only even moduli, namely doublings. It will be shown in Chapter 6 that there also exist IRDCS with only odd moduli. This leads to the question of whether there are any IRDCS with only even moduli that are not doublings, meaning that they do not use the modulus 2. These IRDCS do exist. The first such solution is one with $n = 66$, having heft 1.01857 and order 19, with compact notation 48, 42, 4, 34, 32, 22, 8, 16, 12, 50, 20, 28, 44, 18, 46, 24, 26, 36, 30.

Say that we have such an IRDCS with congruences $S(m_i, a_i)$. Since all moduli are even, if a congruence has odd a_i then all of the positions it covers will be odd. Similarly congruences with even a_i only cover even positions. If we remove all congruences with a_i odd and for the remaining congruences set the moduli m_i to $m_i/2$ and the initial point a_i to $a_i/2$, then it is easy to see that this will be an IRDCS. Similarly removing congruences with a_i even, halving the moduli and sending the respective a_i to $(a_i+1)/2$ will give an IRDCS. Thus any even IRDCS not using the modulus 2 must be generated by two IRDCS with disjoint moduli sets, which may or may not have the same length. If the original IRDCS has length $2n$ then the two smaller IRDCS must both have length n, while if the original IRDCS is of length $2n+1$ then the two smaller IRDCS must have length n and $n+1$.

Chapter 3

Computing IRDCS

Given the existence and structural properties of IRDCS the natural question to ask is how to compute all IRDCS of a given length n? This question naturally extends to how to compute all IRDCS of given length n with given conditions, for example all moduli being odd or with given minimum modulus. The answer comes in the form of the following algorithm, which will first be presented technically, the details being first published in [24], and then the algorithm will be explained in words, supported by an example.

The algorithm is presented in detail, even though details have already been published, in order to familiarize the reader with concepts and techniques needed in Chapter 7, when we tackle the algorithmically more challenging extension of the IRDCS concept to higher dimensions.

3.1 The Algorithm, Technically

3.1.0.4 Backtracking

The algorithm is based on backtracking. Backtracking refers to the process of exhausting all possibilities at a given position and then transitioning back to the most recent decision and updating there. Begin with an array x of length n, initially containing a 1 in each position. This will represent the IRDCS in **alternate notation**, built up one modulus at a time. If a modulus is found to clash with a congruence that has already been filled (i.e. attempting to cover a single position with two separate moduli) then the algorithm attempts to use a larger modulus in the current position, if one is available. Otherwise backtrack to the position most recently filled with a new modulus and attempt to use a larger modulus in that position if one is available, repeating as necessary. This is backtracking.

The algorithm also requires some other arrays to keep track of the process. The first will be the **modusage** array, again of length n. This array will take the value 1 (true) in position k if the modulus k cannot be selected at that point by the algorithm, and a 0 otherwise. This array in particular will be useful for searching for IRDCS with given conditions. For example, to find an IRDCS with all odd moduli, before commencing set the modusage of all even moduli to the value true. To find an IRDCS with a given smallest modulus begin by setting the modusage of all moduli smaller than this value to 1 before beginning the search, with similar considerations for other search possibilities. Next is the array **primary**, once again of length n. This array will keep track of the particular congruences, by storing the first position, in the sense of how the algorithm fills the congruence, where moduli are used. This array is used to determine the point to which the algorithm must backtrack when looking for the next position to attempt to fill with a new congruence. The **primary** array takes the value true if that position in the x array is empty or if it was the first position which used some modulus, and is false otherwise. Given that the method used for filling moduli alternates around the middle of the x array leaving no position unfilled, to be shown more precisely in what follows, backtracking will never move through an empty position. Thus the algorithm may always backtrack until $primary = true$, taking the process to the next position to attempt a new congruence. For more exposition through example see Section 3.2.

To show how the IRDCS is filled some variables are required. The variable **position** stores the current position of the algorithm, where the primary modulus, the first position to be filled, will be inserted. This variable starts at $[n/2] + 1$ and is adjusted by the variables **increment** and **polarity**. The variable **increment** begins at value 1, and is increased by 1 with every change in position. **Polarity** takes the values ± 1, starting with value -1 and changing sign with every change in position. The position changes by $increment * polarity$ at each step in the algorithm. This setup enables the algorithm to fill the IRDCS by alternating around the middle position. For instance, for length 11 the algorithm starts at position 6, then once that is filled it moves to position 5, then position 7, position 4 and so on. The IRDCS is filled by reflecting around the middle, which was found to be faster and more intuitive than filling from one end of the IRDCS when computing examples by hand. This method of filling also seems to work best in proving various properties of IRDCS.

The variable **clash** takes the value true if a clash is generated on trying to fill a given congruence, and takes the value false at other times. The variable *maxmodulus* will be constantly updated to calculate the maximum possible modulus at the current position. It will not be shown how to update this variable in the algorithm, but it should be clear to the reader how to do so in an appropriate fashion. The update depends on the conditions of the IRDCS, such as all congruences having at least 3 hits in the IRDCS. Last of all the variable **finished** will take the value $false$ so long as there remains the possibility of more IRDCS with given conditions, and takes the value *true* once all possibilities are exhausted.

The algorithm is now presented, firstly in technical form.

3.1.0.5 The Algorithm

The algorithm was first published in [24], and our version is not significantly different. Any bold sentences prefaced with a '%' will refer to a comment.

% Initialisation

Input n

Set all entries of vectors x to 1, **primary** to **true**, **modusage**$[2, 3, \ldots, n-1]$ to **false** and **modusage**$[1]$ and **modusage**$[n]$to **true**. Set $position = [n/2]+1, polarity = -1$, $increment = 1, finished = false$ and $clash = false$.

% Begin main loop

while not **finished** **do**

  % Calculate *maxmodulus* **and choose next modulus**

  Set $m =$ next unused modulus after $x[position]$

  Calculate *maxmodulus* based off of current *position* and conditions for the IRDCS

  if $m \leq maxmodulus$ **then**

    % Feasible modulus found - enter while checking for clashes.

    $x[position] := m$

    $i := position \pmod m$

    while $i \leq n$ and not **clash do**

      if $i \neq position$ **then**

        if $x[i] = 1$ **then**

          $x[i] := x[position]$

          $primary[i] := false$

        else

          $clash := true$

          $i := i - m$

        end if

      end if

      $i := i + m$

    end while

    if *clash* **then**

% If clash has occurred clear last modulus except at current position

$modusage[m] := false$

while $i \geq 1$ and $primary[i] = false$ **do**

  if $i \neq position$ **then**

    $x[i] := 1$

    $primary[i] := true$

  end if

  $i := i - m$

end while

$clash := false$

else

% Check to see if finished current system

while $x[position] > 1$ and $1 \leq position \leq n$ **do**

  $position := position + increment * polarity$

  $increment := increment + 1$

  $polarity := polarity * (-1)$

end while

if $position < 1$ or $position > n$ **then**

  % Output system and backtrack

  $modusage[m] := false$

  while $primary[position] = false$ **do**

    $increment := increment - 1$

    $polarity := polarity * (-1)$

    $position := position - increment * polarity$

  end while

  % Remove previously filled congruence

  $i := position \pmod{m}$

```
          while i < n do
            if i ≠ position then
              x[i] := 1
              primary[i] := true
            end if
            i := i + m
          end while
        end if
      end if
    else
      % Backtrack or finish
      if position = [n/2] + 1 then
        finished := true
        Output "No more solutions"
      else
        % Backtrack to previously filled position
        while primary[position] = false do
          increment := increment − 1
          polarity := polarity ∗ (−1)
          position := position − increment ∗ polarity
        end while
        % Remove previously filled congruence
        m := x[position]
        i := position (mod m)
        while i < n do
          if i ≠ position then
            x[i] := 1
```

$primary[i] := true$

end if

$i := i + m$

end while

end if

end if

end while

3.2 The Algorithm In Words

To make some more sense of this algorithm, it will be illustrated with a simple example, keeping track of the variables and vectors as the algorithm progresses. For the sake of simplicity, all length 11 IRDCS will be calculated.

Begin with vectors and variables initialised based on length 11, with no special conditions on the IRDCS. Note that the variables *clash* and *finished* are both *false* and will remain so until otherwise stated.

x	1,1,1,1,1,1,1,1,1,1,1
modusage	1,0,0,0,0,0,0,0,0,0,1
primary	1,1,1,1,1,1,1,1,1,1,1
position / polarity / increment	6 / -1 / 1

Firstly, start filling in the $[n/2] + 1 = 6$ position, using the first available modulus 2. After filling, move to position $6 + (-1) \times 1 = 5$, since **polarity** is (-1) and **increment** is 1. The algorithm is now in the state below.

x	1,2,1,2,1,2,1,2,1,2,1
modusage	1,1,0,0,0,0,0,0,0,0,1
primary	1,0,1,0,1,1,1,0,1,0,1
position / polarity / increment	5 / 1 / 2

Position 5 has primary 1, and so the algorithm attempts to fill it with a new modulus. The next smallest modulus available is 3, however this will clash with the modulus 2 at position 8, as attempting to fill the modulus 3 will create $clash = true$ at $position = 8$ and so the algorithm backtracks the modulus 3 to get back to the same state as above including resetting $clash = false$. The next modulus available is 4, which will not clash, and so fill and move to position $5 + 1 \times 2 = 7$.

x	4,2,1,2,4,2,1,2,4,2,1
modusage	1,1,0,1,0,0,0,0,0,0,1
primary	0,0,1,0,1,1,1,0,0,0,1
position / polarity / increment	7 / -1 / 3

At position 7, the next available modulus is 3, which will clash with modulus 2 at position 4, so the algorithm attempts modulus 5. The modulus 5 will clash with modulus 2 at position 2, modulus 6 will clash with the modulus 4 at position 1, and then any larger modulus will only intersect $[1, 11]$ in position 7, and so is invalid. Thus the algorithm backtracks. As such set **polarity** to $(-1) \times (-1)$, **increment** to $3 - 1 = 2$ and move to position $7 - 1 \times 2 = 5$, which has the modulus 4 and primary value 1, so that this was the first position used for this modulus. As such remove the modulus 4 and attempt to fill this position with the next available modulus. The modulus 3 has already been attempted in this position, so the next available modulus is 5. The modulus 5 clashes with the modulus 2 in position 10, so try and use the modulus 6, which works.

x	1,2,1,2,6,2,1,2,1,2,6
modusage	1,1,0,0,0,1,0,0,0,0,1
primary	1,0,1,0,1,1,1,0,1,0,0
position / polarity / increment	7 / -1 / 3

At position 7, modulus 3 clashes with 2 at position 4, modulus 4 clashes with modulus 6 at position 11, modulus 5 clashes with modulus 2 at position 2, modulus 6 is already used, and modulus 7 will not intersect the interval of the IRDCS twice. Thus the algorithm needs to backtrack. Removing the modulus 6, the next modulus will be 7 which will only intersect $[1, 11]$ once, and so will be invalid. As such, backtrack to position 6 and try the next available modulus at that position, which will be 3.

x	1,1,3,1,1,3,1,1,3,1,1
modusage	1,0,1,0,0,0,0,0,0,0,1
primary	1,1,0,1,1,1,1,1,0,1,1
position / polarity / increment	5 / 1 / 2

At position 5, the moduli 2 and 4 will both clash, so we fill the modulus 5.

x	1,1,3,1,5,3,1,1,3,5,1
modusage	1,0,1,0,1,0,0,0,0,0,1
primary	1,1,0,1,1,1,1,1,0,0,1
position / polarity / increment	7 / -1 / 3

Now at position 7 moduli 2 and 4 will once again clash, but the modulus 6, which is then the next available modulus, will work, and move to position $7 + (-1) \times 3 = 4$.

x	6,1,3,1,5,3,6,1,3,5,1
modusage	1,0,1,0,1,1,0,0,0,0,1
primary	0,1,0,1,1,1,1,1,0,0,1
position / polarity / increment	4 / 1 / 4

Now at position 4, the modulus 2 will clash, but the modulus 4 will work, and then move to position $4 + 1 \times 4 = 8$.

x	6,1,3,4,5,3,6,4,3,5,1
modusage	1,0,1,1,1,1,0,0,0,0,1
primary	0,1,0,1,1,1,1,0,0,0,1
position / polarity / increment	8 / -1 / 5

Position 8 is already filled, so the algorithm moves to position $8 + (-1) \times 5 = 3$, and the polarity changes to 1, increment to 6. The position 3 is already filled, and so we move to position $3 + 1 \times 6 = 9$, which again is full. Thus once more adjust the increment and polarity and move to position $9 + (-1) \times 7 = 2$ which is empty. Next cycle through the available moduli looking for the next valid modulus. Modulus 2 will clash, as will 7 and 8. Modulus 9 will work and take the system to:

x	6,9,3,4,5,3,6,4,3,5,9
modusage	1,0,1,1,1,1,0,0,1,0,1
primary	0,1,0,1,1,1,1,0,0,0,0
position / polarity / increment	2 / 1 / 8

which is a full IRDCS, and the algorithm has found a solution of length 11.

Now to find all of the solutions the algorithm backtracks. The most recently filled modulus is first removed and thus is the modulus 9, but clearly nothing else works here, so iterate the changes to **position, increment** and **polarity** until reaching position 4 where the modulus 4 was used, removing this also.

x	6,1,3,1,5,3,6,1,3,5,1
modusage	1,0,1,1,1,1,0,0,0,0,1
primary	0,1,0,1,1,1,1,1,0,0,1
position / polarity / increment	4 / 1 / 4

The only other valid modulus here would be 7 (to intersect again at position 11),

x	6,1,3,7,5,3,6,1,3,5,7
modusage	1,0,1,0,1,1,1,0,0,0,1
primary	0,1,0,1,1,1,1,1,0,0,0
position / polarity / increment	8 / 1 / 4

but this leaves only positions 2 and 8 free, which would require the already used modulus 6, and so is invalid.

At this point note that the manual method has a nice way to either quickly finish, or show that given the current status of the system show it cannot be finished, an almost full IRDCS by analysing the remaining free positions and seeing what moduli must be used to fill them. This is particularly useful in finding the large minimum modulus examples of Chapter 5. It does not seem that there is an efficient way to implement this way of thinking into the algorithm, but this bears further consideration.

Backtrack further now to position 7 and remove the modulus 6. However, this was the largest possible modulus at this position, any other modulus will only hit once, and so backtrack further still to position 5 and replace the modulus here with the modulus 6.

x	1,1,3,1,6,3,1,1,3,1,6
modusage	1,0,1,0,0,1,0,0,0,0,1
primary	1,1,0,1,1,1,1,1,0,1,0
position / polarity / increment	7 / -1 / 3

At this point notice that the positions of the moduli 3 and 6 are reflected from those in the first IRDCS. Thus, continuing with this system will at least find the **reversal** of the first IRDCS. Based on the previous results of the search it is known that position 7 cannot use moduli 2, 3, or 4, that the only time it can use modulus 5 is in the reversal of the first IRDCS, and any modulus larger than 6 will only hit the IRDCS in the one position, so that the only IRDCS with the 3 and the 6 in these position is the reversal of the first IRDCS. Thus we have found the second solution

x	9,5,3,4,6,3,5,4,3,9,6

and can immediately backtrack to trying the next available modulus to replace the modulus 6 in position 5, for which there are no valid options. Thus backtrack to the beginning and try the modulus 4.

x	1,4,1,1,1,4,1,1,1,4,1
modusage	1,0,0,1,0,0,0,0,0,0,0
primary	1,0,1,1,1,1,1,1,1,0,1
position / polarity / increment	5 / 1 / 2

Shortening the analysis somewhat, next try a 2

$$x = 1, 4, 2, 1, 2, 4, 2, 1, 2, 4, 2,$$

which takes us to position 4, where there are no valid moduli available, so we weren't able to use the modulus 2. The modulus 2 can't be replaced with a 3 or a 5 as they will cause a clash, but this position can be filled with a 6

$$x = 1, 4, 1, 1, 6, 4, 1, 1, 1, 4, 6,$$

taking us to position 7, where 2, 3, 5 and 6 are all unavailable, and 7 is too large, so remove the 6. However, replacing the 6 with anything larger will not hit the interval twice, which means the modulus 4 must be replaced with a 5

$$x = 5, 1, 1, 1, 1, 5, 1, 1, 1, 1, 5.$$

Now in position 5 the moduli 2, 3 and 4 all cause clashes, and 6 is too large, so the 5 was invalid. Now backtracking to the start shows that clearly anything larger than a 5 as our starting modulus is invalid, and so we have found all length 11 IRDCS and the algorithm terminates.

3.3 Knuth's Algorithm X and Dancing Links

Donald Knuth describes Algorithm X in [19]. This algorithm was designed to solve the NP-complete Exact Cover problem. In particular, Knuth suggests using the dancing links implementation of this algorithm to solve these problems. The algorithm has been used to study tilings with polyominoes, the N queens problem [19] and more recently has been used in Sudoku solvers [6]. We will not discuss the dancing links implementation here; details can be found in [19].

To turn the problem of finding an IRDCS into an exact cover problem, it must be represented by a matrix consisting of only 0's and 1's. In this matrix, each row will represent an available congruence. For an IRDCS of length n, the first n columns will represent the positions in the IRDCS, where a 1 in the j^{th} column will mean that the congruence represented by that row contains the number j, and then the remaining $n-2$ columns will represent the modulus of the congruence. The matrix is filled from the top down by congruences $S(2,0), S(2,1), S(3,0), S(3,1), \ldots$. So for a length 6 IRDCS the component of the matrix which represents the congruences is:

$$\left(\begin{array}{cccccc|cccc}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right).$$

After adding all of the rows that represent congruences, $n-2$ additional rows which have all 0's for the positions within the IRDCS and a single 1 in the modulus section are required. These rows are added so that once we have a full IRDCS, the moduli that have not been used for the IRDCS can be filled out to give an appropriate matrix solution. This necessity will become clearer once Algorithm X is discussed more formally.

3.3.0.6 Algorithm X

Start with the matrix A which in our case is as described above. Then Algorithm X is described, as in [19], by the following.

(1) If the matrix A is empty, the problem is solved; terminate successfully.

(2) Otherwise choose a column c deterministically (for example, first column from the left with the lowest number of 1's).

(3) Choose a row r such that $A_{r,c} = 1$ (choose this row randomly, but implement in such a way that you can backtrack when the search for a solution fails or finishes).

(4) Include row r in the partial solution matrix.

(5) For each column j such that $A_{r,j} = 1$, for each row i such that $A_{i,j} = 1$,

- delete row i from matrix A,
- delete column j from matrix A.

(6) Repeat this algorithm recursively on the reduced matrix A.

Note that in the above example the algorithm can terminate unsuccessfully if there is a column containing zero 1's. This corresponds to not being able to cover the equivalent position in the IRDCS given the current system.

The dancing links implementation alters Algorithm X by the presentation of the matrix A. Rather, each row and column in the matrix will consist of a circular doubly linked list of nodes. These links are designed to significantly reduce the time needed during the search for 1's and removal of the relevant rows and columns. In a naive implementation of Algorithm X it is the search for 1's and column and row removal which occupies the majority of the computing time for the algorithm, thus the dancing links implementation.

For our problem A will have $n^2/4 + O(n)$ rows ([24]). The modulus $n-1$ can only be used in one congruence in a length n IRDCS, namely $x \equiv 1(\text{mod } n-1)$, and thus only requires one row in A. The moduli 2 and $n-2$ both have two possible congruences in a length n IRDCS and will thus both generate two rows in the matrix A. Similarly the moduli 3 and $n-3$ will generate three rows, and so on. This pairing of moduli continues until, for n even, the modulus $\frac{n}{2}$ which generates $\frac{n}{2}$ rows or for n odd the moduli $\frac{n-1}{2}$ and $\frac{n+1}{2}$ which both generate $\frac{n-1}{2}$ rows. In either case, counting these gives $n^2/4 + O(n)$ rows. Adding on the final $n-2$ rows, used to fill in the unused moduli once the IRDCS is finished, maintains this bound due to the $O(n)$.

This algorithm can be systematically extended to find all possible IRDCS of a given length, and can also be easily altered to find all IRDCS with a given condition, such as only odd moduli, by removing invalid congruence rows wherever appropriate.

The dancing links implementation of Algorithm X is significantly more efficient than the backtracking algorithm at finding the first solution for a given IRDCS. For instance, in finding the first odd IRDCS of length 83, the backtracking algorithm took approximately 7 minutes while dancing links found a solution in approximately 30 seconds. We used a quad-core 2.67 GHz processor with 4 GB of memory on a 64-bit Windows 7 operating system.

Chapter 4

IRDCS with minimum hits 3 and higher

In the paper [23] by Myerson, Poon and Simpson, the authors ask the following question.

Question. Can we have an IRDCS where every congruence intersects the IRDCS in 3 or more positions?

Definition 4.1. An IRDCS of length n is said to have **minhits** k if the minimum number of times any congruence used in the IRDCS intersects the interval $[1, n]$ is k.

An exhaustive search has thus far shown that there are no IRDCS with minhits ≥ 3 for lengths up to and including 105.

Given a length n IRDCS with minhits 3, then the maximum possible modulus is $\left[\frac{n-1}{2}\right]$, which can cover the positions

$$1, \frac{n}{2}, n-1, \text{or}, 2, \frac{n}{2}+1, n \text{ if } n \text{ is even, or}$$
$$1, \frac{n+1}{2}, n, \text{ if } n \text{ is odd.}$$

Say that, in the sense of the exhaustive backtracking algorithm from Chapter 3, the primary position for a congruence being filled is p, which is to the left of the middle of the IRDCS. Given that the algorithm fills by alternating around the middle, all positions closer to the middle on either side will already be filled by other moduli. Thus the modulus at position p must be at least as large as

$$m = 2\left(\left[\frac{n}{2}\right] + 1 - p\right).$$

On the other hand, for the congruence to hit another position to the left of p, then $m \leq p-1$. If the congruence hits no points to the left of p then $m \geq p$ and positions at least as large as $2p$ and $3p$ are covered. To avoid clashing with the already filled middle then $2p \geq 2\left[\frac{n}{2}\right]+2-p$, so that $3p \geq 2\left[\frac{n}{2}\right]+2 > n$, a contradiction, thus $m \leq p-1$ and the congruence must cover positions both to the left and the right of p. So

$$2\left(\left[\frac{n}{2}\right]+1-p\right) \leq m \leq p-1.$$

Therefore, we must have, for n even

$$\begin{aligned} n+2-2p \leq p-1 &\Rightarrow n+3 \leq 3p \\ &\Rightarrow p \geq \frac{n}{3}+1, \end{aligned}$$

and $p \geq \frac{n+2}{3}$ for n odd.

If the algorithm were attempting to fill a congruence with primary position q to the right of the middle of the IRDCS, then to avoid clashing with those positions in the middle already covered, the modulus at position q must satisfy

$$m \geq 2\left(q-\left[\frac{n}{2}\right]-1\right)+1.$$

A symmetric argument will show that we must hit a position to the right of q, so that $m \leq n-q$, and

$$2q-2\left[\frac{n}{2}\right]-1 \leq m \leq n-q.$$

Hence $2q-2\left[\frac{n}{2}\right]-1 \leq n-q \Rightarrow 3q \leq n+2\left[\frac{n}{2}\right]+1$, which gives $q \leq \frac{2n}{3}$ for n odd and $q \leq \frac{2n+1}{3}$ for n even.

Thus for a length n IRDCS with minhits 3, the primary positions for all congruences must be numbers x satisfying

$$\frac{n+2}{3} \leq x \leq \frac{2n+1}{3}.$$

Alternately all positions with $x < \frac{n+2}{3}$ or $x > \frac{2n+1}{3}$ must be filled by a congruence with primary position closer to the middle.

Recall that we use t for the **order** of our IRDCS, as defined in Definition 2.3. If $t = [\frac{n}{3}]$, then the IRDCS must contain a modulus at least as small as

$$\begin{aligned} A &= \left[\frac{n-1}{2}\right] - \left(\left[\frac{n}{3}\right] - 1\right) \\ &= \begin{cases} \frac{n}{2} - \frac{n}{3} & \text{if } n \equiv 0 \pmod 6, \\ \frac{n-1}{2} - \frac{n-1}{3} + 1 & \text{if } n \equiv 1 \pmod 6, \\ \frac{n}{2} - \frac{n-2}{3} & \text{if } n \equiv 2 \pmod 6, \\ \frac{n-1}{2} - \frac{n}{3} + 1 & \text{if } n \equiv 3 \pmod 6, \\ \frac{n}{2} - \frac{n-1}{3} & \text{if } n \equiv 4 \pmod 6, \\ \frac{n-1}{2} - \frac{n-2}{3} + 1 & \text{if } n \equiv 5 \pmod 6, \end{cases} \\ &\leq \left[\frac{n}{6}\right] + 2. \end{aligned}$$

For any $k \in \mathbb{N}$, all moduli m with $\left[\frac{n}{k+1}\right] < m \leq [\frac{n}{k}]$ will hit the IRDCS at least k times and at most $k+1$ times. If for an IRDCS with minhits 3 only the largest available moduli are used this produces the fewest possible hits in the IRDCS. Define $\#hits$ to be the number of positions covered by the congruences in this IRDCS, thus

$$\begin{aligned} \#hits &\geq 3\left(\left[\frac{n-1}{2}\right] - \left[\frac{n}{3}\right]\right) + 3\left(\left[\frac{n}{3}\right] - \left[\frac{n}{4}\right]\right) + 4\left(\left[\frac{n}{4}\right] - \left[\frac{n}{5}\right]\right) + 5\left(\left[\frac{n}{5}\right] - \left[\frac{n}{6}\right] - 1\right) \\ &= 3\left[\frac{n-1}{2}\right] + \left[\frac{n}{4}\right] + \left[\frac{n}{5}\right] - 5\left[\frac{n}{6}\right] - 5 \\ &\geq 3\left(\frac{n}{2} - 1\right) + \left(\frac{n}{4} - \frac{3}{4}\right) + \left(\frac{n}{5} - \frac{4}{5}\right) - \frac{5n}{6} - 5 \\ &= \frac{67}{60}n - \frac{191}{20}, \end{aligned}$$

so that $\#hits > n$ so long as $n > \frac{191\times 60}{20\times 7} = 81\frac{6}{7}$. These lengths have been checked for minhits 3 exhaustively using the algorithm, finding no results. Thus $t < \left[\frac{n}{3}\right]$.

Repeating this process assuming that $t = [\frac{n}{3}] - k$, where $\left[\frac{n-1}{2}\right] - t \leq [\frac{n}{5}]$ gives

$\#hits \geq \frac{67}{60}n - \frac{191}{20} - 5k$. Comparing this to n

$$\begin{aligned}\#hits \geq \frac{67}{60}n - \frac{191}{20} - 5k > n \\ \frac{7}{60}n - \frac{191}{20} - 5k > 0,\end{aligned}$$

where substituting $k = 1, 2, \ldots$ will give the minimal length possible for such an IRDCS to have order $t = [\frac{n}{3}] - k$. For example, to have order $t = \left[\frac{n}{3}\right] - 1$ the particular solution with minimum hits 3 must have length $n < \frac{291\times 60}{20\times 7} = 124.714\ldots$. Note that a solution in this range does not imply that it must have order $t = \left[\frac{n}{3}\right] - 1$.

Moreover, this method will always provide a contradiction so long as

$$\frac{7n}{60} - 5k > \frac{191}{20} \quad \to \quad k < \frac{7n}{300} - \frac{191}{100}.$$

Thus

$$t \leq \left[\frac{n}{3}\right] - \frac{7n}{300} + \frac{191}{100} \leq \frac{31n}{100} + \frac{191}{100},$$

so that

$$t \leq \left[\frac{31n + 191}{100}\right].$$

This bound is a tighter bound than the trivial bound $\left[\frac{n}{3}\right]$. We may hope to try to extend this bound to $t \leq \frac{31n}{100}$. However while for $5k < \frac{7n}{60}$ a large enough length n can always be found which will give $\#hits > n$, it is not true in general for all such n, and thus the constant term is required in the bound. As such the best possible constant associated to this linear relationship that can be achieved by this method is sought.

For an IRDCS with *minhits* $= 3$ and **order** t a modulus at least as small as

$\left[\frac{n-1}{2}\right]-t+1$ must be used. Let $t=\left[\frac{31n}{100}+\gamma\right]$, then so long as $\left[\frac{n}{6}\right]\leq\left[\frac{n-1}{2}\right]-t+1\leq\left[\frac{n}{5}\right]$,

$$\begin{aligned}
\#hits &\geq 3\left(\left[\frac{n-1}{2}\right]-\left[\frac{n}{4}\right]\right)+4\left(\left[\frac{n}{4}\right]-\left[\frac{n}{5}\right]\right)\\
&\quad+5\left(\left[\frac{n}{5}\right]-\left[\frac{n-1}{2}\right]+\left[\frac{31n}{100}+\gamma\right]\right)\\
&=-2\left[\frac{n-1}{2}\right]+\left[\frac{n}{4}\right]+\left[\frac{n}{5}\right]+5\left[\frac{31n}{100}+\gamma\right]\\
&\geq -n+1+\frac{n}{4}-\frac{3}{4}+\frac{n}{5}-\frac{4}{5}+5\left(\frac{31n}{100}+\gamma-1\right)\\
&=n+5\gamma-\frac{111}{20}.
\end{aligned}$$

Now

$$\begin{aligned}
\left[\frac{n-1}{2}\right]-\left[\frac{31n}{100}+\frac{111}{100}\right]+1 &\geq \frac{n}{2}-1-\frac{31n+111}{100}+1\\
&=\frac{19n}{100}-\frac{111}{100},
\end{aligned}$$

so that

$$\frac{19n}{100}-\frac{111}{100}\geq\left[\frac{n}{6}\right],$$

if

$$\frac{19n}{100}-\frac{n}{6}\geq\frac{111}{100},$$

which is always true for $n\geq 47\frac{4}{7}$, where there do not exist any $minhits=3$ examples up to this length. Similarly

$$\begin{aligned}
\left[\frac{n-1}{2}\right]-\left[\frac{31n}{100}+\frac{111}{100}\right]+1 &\leq \frac{n-1}{2}-\frac{31n+111+99}{100}+1\\
&=\frac{19n-160}{100}\\
&\leq\left[\frac{n}{5}\right].
\end{aligned}$$

Thus, whenever $\gamma>\frac{111}{100}$ we have $\#hits>n$ and so for an IRDCS with $minhits=$

3

$$t\leq\left[\frac{31n+111}{100}\right].$$

To highlight a slightly different approach to get a simpler proof of the first order bound, for an IRDCS with order t the largest possible smallest modulus for the IRDCS is $\left[\frac{n-1}{2}\right] - (t-1)$. Using similar methods to previously, take

$$\left[\frac{n}{\alpha+1}\right] < \left[\frac{n-1}{2}\right] - (t-1) \leq \left[\frac{n}{\alpha}\right],$$

which implies that

$$\left[\frac{n-1}{2}\right] + 1 - \left[\frac{n}{\alpha}\right] \leq t < \left[\frac{n-1}{2}\right] + 1 - \left[\frac{n}{\alpha+1}\right].$$

Following the same argument as previously, if $\alpha = 3$, then

$$\begin{aligned}\#hits &\geq 3\left(\left[\frac{n-1}{2}\right] - \left[\frac{n}{3}\right]\right) + 3\left(\left[\frac{n}{3}\right] - \left(\left[\frac{n-1}{2}\right] - (t-1)\right)\right) \\ &\geq 3\left(\left[\frac{n-1}{2}\right] - \left[\frac{n}{3}\right]\right) \\ &\geq \frac{n}{2} - 3,\end{aligned}$$

which is fine. With the same analysis if $\alpha = 4$ then $\#hits \geq \frac{3n}{4} - 3$, and for $\alpha = 5$, $\#hits \geq \frac{19n}{20} - \frac{3}{4}$, both of which are always fine. Finally, if $\alpha = 6$, then $\#hits \geq \frac{67}{60}n - \frac{91}{20}$ as before. Thus

$$t \leq \left[\frac{n-1}{2}\right] - \left[\frac{n}{6}\right],$$

as with the original method.

Any hope of finding a lower bound for the order of a minimum hits 3 or higher IRDCS is dashed by **doubling**. Any IRDCS with minimum hits 3 can always be doubled to be another IRDCS of minimum hits 3, doubling the length but increasing the number of moduli by just one. So should there exist any IRDCS with **minhits** 3 after repeating the doubling process you can find $t < \epsilon n$ for any $\epsilon > 0$. No upper bound can be much better than $\log n$, as in the standard IRDCS case.

We now extend this to studying IRDCS with $minhits = 4$. The maximum modulus in this case is $\left[\frac{n-1}{3}\right]$ and if the order is $t = \left[\frac{n}{4}\right]$ a modulus at least as small as

$$B = \left[\frac{n-1}{3}\right] - \left[\frac{n}{4}\right] + 1 \leq \left[\frac{n}{12}\right] + 2$$

must be used. As for the previous case assuming that only the largest available moduli are used and each of these moduli only hits the IRDCS in the fewest possible places

$$\begin{aligned}\#hits &\geq 4\left(\left[\frac{n-1}{3}\right] - \left[\frac{n}{4}\right]\right) + 4\left(\left[\frac{n}{4}\right] - \left[\frac{n}{5}\right]\right) + \cdots + 11\left(\left[\frac{n}{11}\right] - \left[\frac{n}{12}\right] - 1\right)\\ &\geq \frac{37511}{27720}n - 21\frac{1759}{27720},\end{aligned}$$

which is larger than n so long as $n > 60$. So the order must be $t < \left[\frac{n}{4}\right]$.

Repeating the process, assuming that $t = \left[\frac{n}{4}\right] - k$, where $\left[\frac{n-1}{3}\right] - t \leq \left[\frac{n}{11}\right]$ then $\#hits \geq \frac{37511}{27720}n - 21\frac{1759}{27720} - 11k$. This will always produce a contradiction so long as

$$k < \frac{9791n - 583879}{11 \times 27720},$$

so that

$$t \leq \left[\frac{n}{4}\right] - \frac{9791n - 583879}{11 \times 27720} \leq \frac{66439n + 583879}{11 \times 27720} \approx 0.218n + 1.915,$$

and the process may in a similar fashion attempt to improve associated constant.

These methods to do not provide any improvement on the bounds for order in the standard IRDCS case. It is the fact that the congruences with large modulus, between $\left[\frac{n}{3}\right]$ and $\left[\frac{n}{2}\right]$, must cover 3 positions, rather than 2, that gives the reductions in the order bound that are found here.

Chapter 5

IRDCS with large minimal modulus

Perhaps the most famous open problem of covering systems is the one posed by Erdős [8], which asks whether there is a covering system with arbitrarily large minimal modulus. There was famously a prize of $1,000 on offer for a solution to this problem [10]. For our purposes, the problem of interest is explicitly stated in [23].

Question. Can the smallest modulus of an IRDCS be arbitrarily large?

The following table shows the smallest length for which an IRDCS with given minimum modulus $M > 2$, can be found, and a comparison of the length to $4M$.

Minimal Modulus	Length	Length Formula
3	11	$4M$ - 1
4	17	$4M$ + 1
5	18	$4M$ - 2
6	22	$4M$ - 2
7	26	$4M$ - 2
8	30	$4M$ - 2
9	34	$4M$ - 2
10	37	$4M$ - 3
11	41	$4M$ - 3
12	45	$4M$ - 3
13	49	$4M$ - 3
14	53	$4M$ - 3
15	57	$4M$ - 3

It would seem that for all M we can find such IRDCS.

Conjecture 5.1. [23] For all sufficiently large M there exists an IRDCS of length $4M - 3$ with minimum modulus M.

In this chapter we establish the conjecture for $M < 20$, and present results on the weaker conjecture that for all M there exists an IRDCS with minimum modulus M. Before considering this further, we have the following result on the lower bound for the length of an IRDCS with minimum modulus M.

Lemma 5.1. *An IRDCS with minimum modulus M has length $n \geq 2M + 6$.*

Proof. The proof follows the ideas of the proof of the upper bound for order in Theorem 2.2. For $n = 2r - 1$ odd, the moduli covering position r and at least one of the positions $r \pm 1, r \pm 2$ and $r - 3$ must cover at least 3 points each in the IRDCS. As the proof considers these positions close to the centre of the IRDCS that are forced to have congruences covering at least 3 positions, as in the proof of Theorem 2.2, the length of the IRDCS is minimised when the positions with three hits are covered by the smallest possible moduli.

If position r is covered by the modulus M then as all moduli are at least large as M, the other congruence which covers 3 positions must be using a larger modulus. The smallest case for this, avoiding clashes, is either of the positions $r \pm 1$ being covered by $M + 2$ or one of $r \pm 2$ being covered by $M + 1$, with this congruence having three hits. So $r - 1 - (M + 2) \geq 1$, or equivalently $r - 2 - (M + 1) \geq 1$, so that $r \geq M + 4$ and thus $n \geq 2M + 7$, where all of these cases provide the same bound by symmetry.

If position r is covered by $M + 1$, then to avoid clashes we have either of $r \pm 1$ covered by a modulus at least as large as $M + 4$. To minimise the length, take both of these positions as being covered by congruences with only two hits, and let one of $r \pm 2$ be covered by the modulus M where this congruence has three hits in the IRDCS. Then this case gives $r + 1 - M - 4 \geq 1$ and thus $r \geq M + 4$ giving $n \geq 2M + 7$.

If position r is covered by $M + 2$, then since this congruence must cover at least 3 positions, namely $r, r - M - 2$ and $r + M + 2$, we have $r - (M + 2) \geq 1$ which gives $n \geq 2M + 5$. If one of $r \pm 1$ are covered by a congruence with three hits, then without loss of generality let $r - 1$ be covered by modulus M. Covering it with $M + 1$ will cause a clash with $r + M + 2$. Then $r + 1$ must be covered by $M + 4$ since using $M + 1$ will clash with $r + M + 2$ and using $M + 3$ will clash with $r - M - 2$, so that $r + 1 - M - 4 \geq 1$ giving $n \geq 2M + 7$. It is easily seen with similar arguments that letting both $r \pm 1$ be covered by congruences with two hits and one of $r \pm 2$ and $r - 3$ having three hits gives the same bound.

For $n = 2r$ even, at least two of the positions $r, r \pm 1$ must be covered by congruences with at least 3 hits in the interval $[1, n]$. If $r + 1$ is one of these and is covered by the modulus M, then either r must be covered by a modulus at least as large as $M + 2$ or $r - 1$ must be covered by a modulus at least as large as $M + 1$ with this position being in a congruence with three positions covered. In either case $r - (M + 2) \geq 1$ so that $n \geq 2M + 6$.

If $r + 1$ is again one of the congruences with 3 hits and is covered by the modulus $M + 1$, then position r must be covered by a modulus at least as large as $M + 3$ or position $r - 1$ must be covered by a modulus at least as large as M and be the congruence with three hits. In the first case, we have $r - M - 3 \geq 1$ which gives $n \geq 2M + 8$. The second case gives $r - 1 - M \geq 1$ so that $n \geq 2M + 4$. For this case the modulus that covers position r must be at least as large as $M + 3$, so that $r + M + 3 \leq 2r$ giving $n \geq 2M + 6$. Any larger modulus covering $r + 1$ will also give this bound.

Lastly if it is positions r and $r - 1$ that are hit three times, then similar analysis shows that the smallest case is when r is covered by $M + 2$ and $r - 1$ is covered by M. Then position $r + 1$ must be covered by the modulus $M + 4$ giving $n \geq 2M + 6$. $\square$

5.1 Smallest Solution Summary Statistics

This section will outline the summary statistics of IRDCS with a given minimal modulus of smallest length. These are found by using the exhaustive search algorithm and taking all examples for the best possible length. Recall our bounds for order $t \leq \frac{n-1}{2}$ and heft

$$\frac{3}{4} + \frac{1}{4n} \leq h \leq \frac{3n-5}{2(n+1)} + \frac{4}{(n+1)^2},$$

which we will use for comparison purposes.

Summary statistics for minimum modulus below 10 will not be presented, as they do not prove to be particularly enlightening.

Minimum modulus 10

At length 37, there are 224 IRDCS with minimum modulus 10. All of the 224 IRDCS use all of the moduli from 10 through to 26 inclusive, giving them heft 1.02545 and order 17, which compares to the formula from Chapter 2 which gives heft between 0.7568 and 1.3975 and order at most 18.

Minimum modulus 11

At length 41, there are 752 IRDCS with minimum modulus 11. All of the 752 IRDCS use all of the moduli from 11 to 28 inclusive and the modulus 30 giving them heft 1.03154 and order 19, which compares to the formula which gives heft between 0.7561 and 1.4070 and order at most 20.

Minimum modulus 12

At length 45, there are 19,752 IRDCS with minimum modulus 12. All of the IRDCS use all of the moduli from 12 to 31 inclusive giving them heft 1.00737 and order 20, which compares to the formula which gives heft between 0.7556 and 1.4149 and order at most 22.

Minimum modulus 13

At length 49, there are 628,332 IRDCS with minimum modulus 13. All of them use all of the moduli from 13 to 32 inclusive, and the moduli 33, 34, 35 and 36 are used 546,144, 338,426, 256,962 and 115,132 times respectively.

There are 256,238 IRDCS which use modulus 33 and 34, 174,774 which use 33 and 35, 82,118 which use 34 and 35 and 115,132 which use 33 and 36 and no others. The IRDCS have heft between 1.01327 and 1.015 and order 22, which compares to the formula which gives heft between 0.7551 and 1.4216 and order at most 24.

Minimum modulus 14

At length 53, there are 11,482,130 IRDCS with minimum modulus 14. The modulus usage is represented in the following table.

Modulus	Number of IRDCS	Proportion of Total
14	11,482,130	1.000
15	11,482,130	1.000
16	11,482,130	1.000
17	11,482,130	1.000
18	11,482,130	1.000
19	11,482,130	1.000
20	11,482,130	1.000
21	11,482,130	1.000
22	11,482,130	1.000
23	11,482,130	1.000
24	11,482,130	1.000
25	11,482,130	1.000
26	11,482,130	1.000
27	11,482,130	1.000
28	11,482,130	1.000
29	11,482,130	1.000
30	11,482,130	1.000
31	11,482,130	1.000
32	11,482,130	1.000
33	11,482,130	1.000
34	11,187,918	0.974
35	10,590,832	0.922
36	8,629,412	0.752
37	6,045,856	0.526
38	4,397,394	0.383
39	2,907,492	0.253
40	1,289,256	0.112
41	880,360	0.077

The heft of these IRDCS ranges from 1.01836 to 1.02145, and all IRDCS have order 24. This compares to the formula which gives heft between 0.7547 and 1.4273 and order at most 26.

The growth in solution numbers for those IRDCS of a given length is comparable to the growth of the total number of IRDCS of a given length, as seen in Section A.2.

Minimum modulus 15, 16, 17, 18, 19 and 20

We know that the first IRDCS with minimum modulus 15 is of length 57, but the computation time to find all such solutions is prohibitively long. We have confirmed via the dancing links implementation of Algorithm X that there exist IRDCS with minimum modulus 16, 17, 18 and 19 with length $n = 4M - 3$. We have not confirmed that these are the shortest possible lengths for these IRDCS. The IRDCS found, in their compact notation, are:

Minimum modulus 15, length 57, IRDCS:

$$38, 34, 32, 29, 36, 26, 35, 19, 42, 24, 17, 18, 43, 15, 16, 21, 23, 20, 31, 37, 22, 33, 30, 25, 27, 28.$$

This IRDCS has heft 1.0234 and order 26, which compares to the formula which gives heft ranging from 0.7554 to 1.4322 and order at most 28. Note that the modulus 28 covers the 26^{th} position, so that all of the moduli are in a single block.

Minimum modulus 16, length 61, IRDCS:

$$41, 37, 33, 31, 32, 28, 36, 45, 20, 35, 48, 18, 19, 24, 16, 17, 39, 22, 25, 21, 40, 38, 23, 34, 30, 26, 27, 29.$$

This IRDCS has heft 1.028 and order 28, which compares to the formula which gives heft ranging from 0.7541 to 1.4365 and order at most 30. Note that the modulus 29 covers the 28^{th} position, so that all of the moduli are in a single block.

Minimum modulus 17, length 65, IRDCS:

$42, 40, 35, 36, 25, 31, 32, 41, 38, 21, 34, 24, 19, 20, 26, 17, 18, 43, 46, 44, 27, 22, 23, 39, 37, 33,$
$30, 28, 29.$

This IRDCS has heft 1.014 and order 29, which compares to the formula which gives heft ranging from 0.7538 to 1.4403 and order at most 32. Once again, the modulus 29 covers the 29^{th} position, and this IRDCS also has all of its moduli in a single block.

Minimum modulus 18, length 69, IRDCS:

$48, 42, 37, 35, 23, 36, 34, 44, 38, 40, 22, 49, 25, 20, 21, 27, 18, 19, 26, 28, 47, 24, 43, 45, 39, 41,$
$32, 29, 30, 31, 33.$

This IRDCS has heft of 1.018 and order 31, which compares to the formula which gives heft ranging from 0.7536 to 1.4437, and order at most 34. This time the moduli in the IRDCS are not presented as a single block with the modulus 23 hitting positions 4 and 27.

Minimum modulus 19, length 73 (also 74), IRDCS:

$51, 46, 42, 40, 38, 36, 24, 33, 41, 44, 52, 23, 49, 26, 21, 22, 29, 19, 20, 27, 30, 50, 48, 25, 28, 47,$
$43, 37, 39, 31, 32, 34, 35.$

This IRDCS has heft of 1.020 and order 33, which compares to the formula which gives heft ranging from 0.7534 to 1.4467, and order at most 36. This IRDCS is also an IRDCS of length 74 (using the modulus 32 to cover position -1), and again the moduli are not presented as a single block with the modulus 24 hitting positions 6 and 30.

For minimum modulus 20, we have not found a solution of length 77, but we have also not confirmed that there are no solutions for this length. For length 78 the first solution found by the Dancing Links implementation of Algorithm X is

$$51, 49, 45, 26, 52, 41, 46, 36, 27, 39, 34, 38, 30, 28, 23, 21, 24, 22, 20, 40, 25, 55, 44, 31, 29, 42, 48, 50, 47, 43, 32, 33, 35, 37.$$

This IRDCS has heft 1.008 and order 34, comparing to the formula which gives heft ranging from 0.7532 to 1.4500 and order at most 38. The moduli are not in a single block, with modulus 26 covering positions 3 and 29.

The first solution for an IRDCS with minimum modulus 20 of length 79 can be presented in the compact notation as;

$$56, 52, 47, 32, 30, 49, 44, 29, 37, 35, 45, 41, 34, 24, 33, 27, 22, 23, 25, 20, 21, 36, 26, 28, 42, 53, 51, 31, 48, 46, 43, 39, 40, 38.$$

This IRDCS has heft of 1.007 and order 34, which compares to the formula which gives heft ranging from 0.7532 to 1.4506, and order at most 39. The moduli in this IRDCS are presented in a single block. We present this solution only to mention that the computation of this example took approximately 13 minutes, while the solution of length 78 took approximately 10 hours. A solution of length 80 took approximately 2 seconds to compute, again using a quad-core 2.67 GHz processor with 4 GB of memory on a 64-bit Windows 7 operating system.

When the dancing links implementation of Algorithm X is used, only one solution is found and the algorithm terminates. As such, no conclusions should be drawn on IRDCS based upon the single solution that is found. For instance, in all of the presented minimum modulus IRDCS, excluding the length 79 IRDCS with minimum modulus 20, the modulus M is used to cover position $M - 1$. This is almost certainly not a general property for all such IRDCS, though it is possible that one may always exist for each M. The growth in the solution numbers of IRDCS seen for minimum modulus up to and including $M = 14$ lends weight to this possibility.

5.2 Manual Method To Find IRDCS With Large Minimum Modulus

Attempting to find IRDCS with large minimal modulus using the exhaustive algorithm quickly becomes too time-consuming a process to produce any new results in a reasonable timeframe. This is especially true since for Conjecture 5.1 to hold it is required to check 3 lengths in full without discovering an example before the 4^{th} length produces the first example, since our conjectured formula for length is $n = 4M - 3$. As such a method for finding an IRDCS with large minimum modulus without the use of the exhaustive algorithm is presented here.

For minimum modulus M, take $[1, 5M] \cap \mathbb{Z}$ for what will be an IRDCS of length $5M$ or shorter. As the IRDCS is filled the length may be shortened to allow for cases where the method of filling either produces clashes or leaves some positions unfilled close to either end of the interval. The concept of closeness will be clarified in what follows.

To create these IRDCS, input modulus M at position M, modulus $M + 1$ at position $M + 1$ and so on, for as long as the congruences do not produce unwanted clashes. That is, use the congruences $S(M, 0), S(M + 1, 0), \ldots, S(M + k, 0)$, where k is chosen so as to not generate a clash that will force the IRDCS to have length too close to $n = 4M$ thus making an IRDCS of this form harder, and perhaps impossible, to construct. This can be represented by the following table:

+	1	2	3	4	...	M - 1	M
0							M
M	M+1	M+2	M+3	M+4			M
2M		M+1		M+2			M
3M			M+1				M
4M				M+1			M

To further clarify the construction, an example with $M = 10$ will be completed. After inserting the initial block of moduli starting with the modulus 10 at position 10 the IRDCS would look like:

+	1	2	3	4	5	6	7	8	9	10
0										10
10	11	12	13	14						10
20		11		12		13		14		10
30			11			12			13	10
40		14		11				12		10

where the modulus 15 is not used at this stage since it would cause a clash at position 30, and the IRDCS is expected to be at best of length 37.

Now add the next available moduli $M + k + 1, M + k + 2$ and so on. Here $M + k + 1 = 15$. The process now becomes more variable, as a choice must be made as to where to insert these moduli. Attempt to insert them as another block, placing modulus 15 at position α, the modulus 16 at $\alpha + 1$ and so on. Start with the modulus 15 at position 1 and then iterate this process at position 2, 3, 4 and so on, to see which starting position allows the largest block of sequential moduli. This leads to the following situation;

+	1	2	3	4	5	6	7	8	9	10
0		15	16	17	18					10
10	11	12	13	14			15		16	10
20	17	11	18	12		13		14		10
30		15	11		16	12		17	13	10
40	18	14		11			15	12		10

where using the modulus 19 will cause a clash with 11 at position 44, which is too close to length $4M$ for our liking at this early stage, and starting at position 2 rather than 1 gives us a longer block of sequential moduli.

Now attempt to add more moduli, adding as many which cover 3 positions of the IRDCS as possible. For the current example start by adding the modulus 19. Attempt to add these new moduli into our IRDCS in increasing positions, starting at the first free position, for no reason other than in examples it seems to be the most efficient method, working as often as any other methods. Continue doing this with consecutive moduli until moduli which hit 3 times without causing a clash can no longer be found. So for our example, this becomes:

+	1	2	3	4	5	6	7	8	9	10
0		15	16	17	18			19	20	10
10	11	12	13	14			15		16	10
20	17	11	18	12		13	19	14	20	10
30		15	11		16	12		17	13	10
40	18	14		11		19	15	12	20	10

where only the moduli 19 and 20 have been added, as no other modulus can hit 3 times. The last step is to attempt to fill the remaining positions with whatever moduli are available. In our current example, position 25 can only be covered with the modulus 24, covering position 1 and clashing at position 49. This forces position 31 to use modulus 25, as it was either that or 24, which forces position 34 to use modulus 27, and then there are no moduli available to cover 18, thus our process must backtrack. If we remove only the modulus 20 in our backtrack, the IRDCS can be finished as follows:

+	1	2	3	4	5	6	7	8	9	10
0	24	15	16	17	18	23	27	19	22	10
10	11	12	13	14	30	21	15	25	16	10
20	17	11	18	12	24	13	19	14	23	10
30	22	15	11	27	16	12	21	17	13	10
40	18	14	25	11	30	19	15	12	24	10

This produces an IRDCS with minimal modulus 10 of length 44 at the shortest, up to length 51 at its longest. This IRDCS has heft 1.007 and order 17. Note that this is an example of an IRDCS whose compact notation provides IRDCS of different lengths. The compact notation in this case is $24, 15, 16, 17, 18, 23, 17, 19, 22, 10, 11, 12, 13, 14, 30, 31, 25$. This particular compact notation provides IRDCS for lengths 45 to 51. The length 44 IRDCS requires removing the first 5 positions and position 50, length 51 comes from an unseen modulus 16 at position 51 which is clash free.

The test for whether or not a clash is allowed during the stages of filling the IRDCS is subjective. Forcing any clashes to be close to the ends of the interval $[1, 5M]$ allows more possibilities for the choices of the particular congruences for the already filled moduli, by removing positions on either end. More importantly, it maintains a longer length IRDCS. Empirical evidence strongly suggests that, other than for some small length IRDCS, the number of IRDCS with given conditions increases very rapidly with length. Thus the larger the interval, the more likely it should be that we will find an IRDCS with minimum modulus M.

If there were a clash close to position $4M$ then many possibilities for the IRDCS would be ruled out prematurely. This would limit the possible congruences modulo $M, M+1, \ldots, M+k$, for which the residues can be altered by adjusting how many positions are left blank at the start of the IRDCS once it is finished. Given that the conjectured smallest length for an IRDCS with minimum modulus M is $n = 4M - 3$, and given that this method produces a special type of IRDCS, at this early stage it is favourable to have as much flexibility in the length of the IRDCS as possible, maintaining $n \approx 5M$.

When there remains only a small number of blank positions left to fill at the ends of the interval, so long as we don't have clashes closer to the middle than the empty spots, the blank ends can simply be removed from consideration. This adjusts the length of the IRDCS constructed. The choice of $5M$ and the starting position of the congruence modulo M is made so that congruences with any residues for the first block of moduli can be utilized in the IRDCS, by removing positions at the beginning of our interval.

This manual construction method gives an IRDCS with given minimum modulus, and sometimes effectively the same IRDCS with multiple lengths. Present results give an IRDCS of minimal modulus up to and including 40, along with minimum modulus 50. For example, an IRDCS of minimum modulus 40 has the following statistics:

Statistic	Observed	Notes
Length	189 to 194	$4 * 40 - 3 = 157$ is our expected best possible length
Order	69	Bounded above by 94
Heft	1.003	For length 189, bounded between 0.7689 and 1.3475

The IRDCS in question has compact notation 97, 102, 107, 88, 110, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 91, 104, 71, 72, 98, 94, 73, 74, 100, 105, 75, 76, 80, 77, 81, 78, 95, 79, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 86, 82, 87, 89, 118, 93, 109, 99, 101, 83, 103, 85, 96, 114, 108.

There is also the following IRDCS with minimum modulus 50.

Statistic	Observed	Notes
Length	241 to 248	$4 * 50 - 3 = 197$ is our expected best possible length
Order	87	Bounded above by 120
Heft	1.0044	For length 241, bounded between 0.7699 and 1.3472

This IRDCS has compact notation 112, 129, 133, 121, 132, 134, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 113, 115, 87, 88, 124, 114, 89, 90, 144, 125, 91, 92, 128, 118, 93, 94, 127, 95, 119, 96, 102, 97, 103, 98, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 135, 137, 139, 110, 100, 141, 143, 99, 106, 142, 105, 150, 104, 108, 109, 101, 136, 111, 140, 123.

5.2.1 Open Question

It does seem that the construction presented requires some luck as to whether or not the particular block selections, especially the second block, gives an IRDCS. In some instances in calculating the following IRDCS we found that the first block chosen did not lead to an IRDCS, or that it was much quicker to chose a different block of moduli. There are also choices to be made in which individual moduli hitting three positions are used in the construction. Can this construction be improved to fully categorise the block selection, and perhaps to guarantee finding an IRDCS with given minimum modulus?

5.2.2 Summary Statistics - Manual Method

In this section we outline the results of the manual method for finding IRDCS with large minimal modulus. The structural properties of the IRDCS found will be compared to the bounds given in the results of Chapter 2. For all of our comparisons with the heft formula, we will adopt the convention that the IRDCS of smallest length along with its order will be used to compare the results to those predicted by formula (2.1) from Theorem 2.4.

The IRDCS are presented in an addition table where the value of the addition gives the position of the IRDCS. All of these IRDCS are presented in this form, which is similar to the alternate notation. It additionally provides more information to highlight the particulars of the construction. The compact notation of the IRDCS is also presented for each example.

Minimum modulus 20

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	57	55	52	47	28	29	30	31	32	33	34	35	36	51	53
15	37	45	38	40	20	21	22	23	24	25	26	27	46	42	61
30	56	54	28	49	29	59	30	39	31	20	32	21	33	22	34
45	23	35	24	36	25	47	26	37	27	52	38	55	57	40	20
60	28	45	21	29	51	22	30	53	23	31	42	24	32	46	25
75	33	39	26	34	20	27	35	49	21	36	54	56	22	28	37
90	61	23	29	38	59	24	30	47	40						

The compact notation of this IRDCS is:

$$57, 55, 52, 47, 28, 29, 30, 31, 32, 33, 34, 35, 36, 51, 53, 37, 45, 38, 40, 20, 21, 22, 23, 24, 25, 26, 27, 46, 42, 61, 56, 54, 49, 59, 39.$$

This compact notation produces IRDCS with lengths from 95 to 99 which have heft 1.003 and order 35. Adjusting the IRDCS to have lengths between 95 and 99 requires removing the appropriate number of hits from the congruences with moduli 24, 30, 47 and 40 from the right end of the IRDCS. This process of removing individual hits of moduli from either or both ends of the IRDCS will almost always be possible for the following examples, and it should always be clear how to find the IRDCS of the relevant lengths. As such, we will not explicitly state how to adjust the lengths for any of the remaining examples, unless the method is decidedly different or particularly interesting. Moduli that are removed must come from congruences which still have at least two hits in the IRDCS, and there must never be any gaps left.

For the minimum modulus 20 IRDCS, the heft and order compare to heft between 0.7688 and 1.3339 and order at most 47 from the relevant formulae.

Minimum modulus 21

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0		58	59	60	62	50	52	47	28	29	30	31	32	33	34
15	35	36	40	42	37	21	22	23	24	25	26	27	45	41	46
30	48	53	57	65	56	51	28	44	29	55	30	21	31	22	32
45	23	33	24	34	25	35	26	36	27	47	50	37	40	52	58
60	42	59	21	60	28	22	62	29	23	41	30	24	45	31	25
75	46	32	26	48	33	27	44	34	21	53	35	51	22	36	57
90	56	23	28	37	55	24	29	40	65	25	30	47	42	26	

The compact notation of this IRDCS is:

$$58, 59, 60, 62, 50, 52, 47, 28, 29, 30, 31, 32, 33, 34, 35, 36, 40, 42, 37, 21, 22, 23, 24, 25, 26, 27,$$
$$45, 41, 46, 48, 53, 57, 65, 56, 51, 44, 55.$$

This compact notation produces IRDCS with lengths from 98 to 103 which have heft 1.000 and order 37, which compares to heft between 0.7665 and 1.3440 and order at most 48.

Minimum modulus 22

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	50	41	57	65	42	43	47	45	54	56	46	22	23	24	25
15	26	27	28	29	30	31	63	32	48	33	49	34	53	35	58
30	36	61	37	22	38	23	39	24	40	25	55	26	41	27	52
45	28	42	29	43	30	50	31	45	47	32	22	46	33	23	57
60	34	24	54	35	25	56	36	26	65	37	27	48	38	28	49
75	39	29	22	40	30	53	23	31	41	63	24	32	58	42	25
90	33	43	61	26	34	55	52								

The compact notation of this IRDCS is:

$$50, 41, 57, 65, 42, 43, 47, 45, 54, 56, 46, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 63, 32, 48, 33, 49,$$
$$34, 53, 35, 58, 36, 61, 37, 38, 39, 40, 55, 52.$$

This IRDCS has length 97 and has heft 1.006 and order 38, which compares to heft between 0.7633 and 1.3577 and order at most 48.

Minimum modulus 23

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	56	61	32	33	34	35	36	37	38	39	40	41	42	52	50
15	43	44	53	45	54	47	55	23	24	25	26	27	28	29	30
30	31	48	65	49	32	59	33	51	34	46	35	64	36	57	37
45	23	38	24	39	25	40	26	41	27	42	28	56	29	43	30
60	44	31	61	45	50	52	32	47	23	33	53	24	34	54	25
75	35	55	26	36	48	27	37	49	28	38	46	29	39	51	30
90	40	23	31	41	59	24	42	65	32	25	57	43	33	26	44
105	64	34	27	45											

The compact notation of this IRDCS is:

$$56, 61, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 52, 50, 43, 44, 53, 45, 54, 47, 55, 23, 24, 25, 26, 27, 28, 29, 30, 31, 48, 65, 49, 59, 51, 46, 64, 57.$$

This compact notation produces IRDCS with lengths from 106 to 109 which have heft 1.003 and order 39, which compares to heft between 0.7687 and 1.3369 and order at most 52.

Minimum modulus 24

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	60	64	66	63	68	70	32	33	34	35	36	37	38	39	40
15	41	42	46	69	43	44	48	45	24	25	26	27	28	29	30
30	31	47	49	51	56	58	65	59	32	74	33	57	34	61	35
45	71	36	24	37	25	38	26	39	27	40	28	41	29	42	30
60	60	31	43	46	44	64	63	45	66	48	32	24	68	33	25
75	70	34	26	47	35	27	49	36	28	51	37	29	69	38	30
90	56	39	31	58	40	24	59	41	57	25	42	65	32	26	61
105	43	33	27	44	46	34	28	45	74	35	29	71	48	36	

The compact notation of this IRDCS is:

$$60, 64, 66, 63, 68, 70, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 46, 69, 43, 44, 48, 45, 24, 25, 26, 27, 28, 29, 30, 31, 47, 49, 51, 56, 58, 65, 59, 74, 57, 61, 71.$$

This compact notation produces IRDCS with lengths from 117 to 119 which have heft 1.000 and order 42, which compares to heft between 0.7710 and 1.3309 and order at most 58.

Minimum modulus 25

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0			69	78	36	37	38	39	40	41	42	43	44	55	50
15	45	46	58	60	47	53	63	48	49	25	26	27	28	29	30
30	31	32	33	34	35	61	51	56	52	71	36	76	37	66	38
45	57	39	59	40	25	41	26	42	27	43	28	44	29	64	30
60	45	31	46	32	50	33	47	34	55	35	48	69	49	53	25
75	58	36	26	60	37	27	78	38	28	63	39	29	51	40	30
90	52	41	31	56	42	32	61	43	33	25	44	34	57	26	35
105	45	59	27	46	66	71	28	36	47	50	29	37	76	48	30
120	38	49	64												

The compact notation of this IRDCS is:

$$69, 78, 36, 37, 38, 39, 40, 41, 42, 43, 44, 55, 50, 45, 46, 58, 60, 47, 53, 63, 48, 49, 25, 26, 27, 28,$$

$$29, 30, 31, 32, 33, 34, 35, 61, 51, 56, 52, 71, 76, 66, 57, 59.$$

This IRDCS has length 121 and has heft 1.003 and order 43, which compares to heft between 0.7720 and 1.3282 and order at most 60.

Minimum modulus 26

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	43	44	45	46	47	76	58	48	49	57	68	50	51	54	61
15	84	53	55	26	27	28	29	30	31	32	33	34	35	59	36
30	63	37	52	38	56	39	66	40	69	41	70	42	73	43	26
45	44	27	45	28	46	29	47	30	65	31	48	32	49	33	60
60	34	50	35	51	58	36	57	54	37	53	26	38	55	27	39
75	61	28	40	68	29	41	76	30	42	52	31	43	59	32	44
90	56	33	45	63	34	46	26	35	47	84	27	36	66	48	28
105	37	49	69	29	38	70	50	30	39	51	73	31	40	65	60
120	32														

The compact notation of this IRDCS is:

$$43, 44, 45, 46, 47, 76, 58, 48, 49, 57, 68, 50, 51, 54, 61, 84, 53, 55, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 59, 36, 63, 37, 52, 38, 56, 39, 66, 40, 69, 41, 70, 42, 73, 65, 60.$$

This compact notation produces IRDCS with lengths from 115 to 121 which have heft 1.009 and order 45, which compares to heft between 0.7632 and 1.3624 and order at most 57.

Minimum modulus 27

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	36	37	38	39	40	41	42	43	44	45	46	47	59	61	48
15	49	66	60	50	51	65	52	57	53	64	69	27	28	29	30
30	31	32	33	34	35	56	36	63	37	58	38	68	39	71	40
45	72	41	79	42	54	43	55	44	27	45	28	46	29	47	30
60	62	31	48	32	49	33	67	34	50	35	51	59	36	52	61
75	37	53	60	38	57	27	39	66	28	40	65	29	41	64	30
90	42	56	31	43	69	32	44	58	33	45	63	34	46	54	35
105	47	55	27	36	68	48	28	37	49	71	29	38	72	50	30
120	39	51	62	31	40	52	79	32	41	53	59	33	42	67	27

The compact notation of this IRDCS is:

$$36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 59, 61, 48, 49, 66, 60, 50, 51, 65, 52, 57, 53, 64, 69,$$
$$27, 28, 29, 30, 31, 32, 33, 34, 35, 56, 63, 58, 68, 71, 72, 79, 54, 55, 62, 67.$$

This compact notation produces IRDCS with lengths from 121 to 135 which have heft 1.005 and order 46, which compares to heft between 0.7656 and 1.3528 and order at most 60.

Minimum modulus 28

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	70	75	40	41	42	43	44	45	46	47	48	49	50	66	67
15	51	52	73	69	53	54	63	71	55	56	74	76	28	29	30
30	31	32	33	34	35	36	37	38	39	57	65	77	40	78	41
45	79	42	82	43	59	44	86	45	61	46	28	47	29	48	30
60	49	31	50	32	68	33	51	34	52	35	70	36	53	37	54
75	38	75	39	55	66	56	67	40	28	63	41	29	69	42	30
90	73	43	31	71	44	32	57	45	33	74	46	34	76	47	35
105	65	48	36	59	49	37	28	50	38	61	29	39	51	77	30
120	52	78	40	31	79	53	41	32	54	82	42	33	68	55	43
135	34	56	86	44											

The compact notation of this IRDCS is:

$$70, 75, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 66, 67, 51, 52, 73, 69, 53, 54, 63, 71, 55, 56, 74,$$
$$76, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 57, 65, 77, 78, 79, 82, 59, 86, 61, 68.$$

This compact notation produces IRDCS with of lengths 138 and 139 which have heft 1.006 and order 49, which compares to heft between 0.7719 and 1.3312 and order at most 68.

Minimum modulus 29

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	85	82	72	76	40	41	42	43	44	45	46	47	48	49	50
15	51	52	65	62	53	77	73	54	55	67	56	74	61	29	30
30	31	32	33	34	35	36	37	38	39	64	66	68	80	83	40
45	84	41	87	42	89	43	70	44	59	45	63	46	29	47	30
60	48	31	49	32	50	33	51	34	52	35	71	36	53	37	72
75	38	54	39	55	76	62	56	65	82	40	85	29	41	61	30
90	42	67	31	43	73	32	44	77	33	45	74	34	46	64	35
105	47	66	36	48	68	37	49	59	38	50	29	39	51	63	30
120	52	70	80	31	40	53	83	32	41	84	54	33	42	55	87
135	34	43	56	89	35	44	71	62	36						

The compact notation of this IRDCS is:

$$85, 82, 72, 76, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 65, 62, 53, 77, 73, 54, 55, 67, 56,$$

$$74, 61, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 64, 66, 68, 80, 83, 84, 87, 89, 70, 59, 63, 71.$$

This compact notation produces IRDCS with lengths 142, 143 and 144 which have heft 1.004 and order 51, which compares to heft between 0.7708 and 1.3359 and order at most 70.

Minimum modulus 30

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	74	78	81	77	40	41	42	43	44	45	46	47	48	49	50
15	51	52	65	68	53	86	73	54	55	67	56	92	61	57	30
30	31	32	33	34	35	36	37	38	39	58	63	59	70	72	40
45	64	41	75	42	85	43	87	44	76	45	66	46	69	47	30
60	48	31	49	32	50	33	51	34	52	35	71	36	53	37	74
75	38	54	39	55	78	77	56	65	81	40	57	68	41	61	30
90	42	67	31	43	73	32	44	58	33	45	59	34	46	63	35
105	47	86	36	48	64	37	49	70	38	50	72	39	51	92	30
120	52	66	75	31	40	53	69	32	41	76	54	33	42	55	85
135	34	43	56	87	35	44	71	57	36	45					

The compact notation of this IRDCS is:

$$74, 78, 81, 77, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 65, 68, 53, 86, 73, 54, 55, 67, 56,$$
$$92, 61, 57, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 58, 63, 59, 70, 72, 64, 75, 85, 87, 76, 66, 69, 71.$$

This compact notation produces IRDCS with lengths 142, 143, 144 and 145 which have heft 1.004 and order 52, which compares to heft between 0.7689 and 1.3429 and order at most 70.

Minimum modulus 31

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	72	77	44	45	46	47	48	49	50	51	52	53	54	71	73
15	74	90	92	94	55	56	82	78	57	58	69	65	59	60	68
30	31	32	33	34	35	36	37	38	39	40	41	42	43	98	80
45	85	44	79	45	66	46	67	47	96	48	81	49	76	50	62
60	51	31	52	32	53	33	54	34	84	35	83	36	72	37	55
75	38	56	39	77	40	57	41	58	42	71	43	59	73	60	74
90	44	65	31	45	69	32	46	68	33	47	78	34	48	82	35
105	49	90	36	50	92	37	51	94	38	52	66	39	53	67	40
120	54	62	41	31	80	42	79	32	43	55	85	33	56	76	44
135	34	81	57	45	35	58	98	46	36	72	59	47	37	60	96
150	48	38	84	83											

The compact notation of this IRDCS is:

$$72, 77, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 71, 73, 74, 90, 92, 94,$$

$$55, 56, 82, 78, 57, 58, 69, 65, 59, 60, 68, 31, 32, 33, 34, 35, 36, 37,$$

$$38, 39, 40, 41, 42, 43, 98, 80, 85, 79, 66, 67, 96, 81, 76, 62, 84, 83.$$

This compact notation produces IRDCS with lengths 153 and 154 which have heft 1.008 and order 55, which compares to heft between 0.7706 and 1.3379 and order at most 76.

Minimum modulus 32

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	82	87	44	45	46	47	48	49	50	51	52	53	54	55	56
15	57	58	70	73	59	60	85	78	61	62	72	63	67	64	74
30	94	32	33	34	35	36	37	38	39	40	41	42	43	69	65
45	88	44	71	45	66	46	79	47	68	48	81	49	84	50	90
60	51	83	52	32	53	33	54	34	55	35	56	36	57	37	58
75	38	77	39	59	40	60	41	82	42	61	43	62	70	87	63
90	44	73	64	45	67	32	46	72	33	47	78	34	48	74	35
105	49	85	36	50	65	37	51	69	38	52	66	39	53	71	40
120	54	68	41	55	94	42	56	32	43	57	79	33	58	88	44
135	34	81	59	45	35	60	84	46	36	83	61	47	37	62	90
150	48	38	63	77	49	39	64	70	50						

The compact notation of this IRDCS is:

$$82, 87, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 70, 73,$$
$$59, 60, 85, 78, 61, 62, 72, 63, 67, 64, 74, 94, 32, 33, 34, 35, 36, 37, 38,$$
$$39, 40, 41, 42, 43, 69, 65, 88, 71, 66, 79, 68, 81, 84, 90, 83, 77.$$

This compact notation produces IRDCS with lengths from 154 to 159 and have heft 1.004 and order 55, which compares to heft between 0.7712 and 1.3357 and order at most 76.

Minimum modulus 33

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	59	74	76	60	61	79	81	62	63	71	64	70	65	77	82
15	78	83	33	34	35	36	37	38	39	40	41	42	43	80	44
30	72	45	73	46	85	47	103	48	84	49	87	50	104	51	67
45	52	97	53	66	54	33	55	34	56	35	57	36	58	37	59
60	38	69	39	60	40	61	41	68	42	62	43	63	75	44	64
75	74	45	65	76	46	71	70	47	33	79	48	34	81	49	35
90	77	50	36	78	51	37	82	52	38	83	53	39	72	54	40
105	73	55	41	80	56	42	67	57	43	66	58	33	44	59	85
120	34	45	84	60	35	46	61	87	36	47	69	62	37	48	63
135	68	38	49	64	103	39	50	65	97	40	51	104	75	41	

The compact notation of this IRDCS is:

$$59, 74, 76, 60, 61, 79, 81, 62, 63, 71, 64, 70, 65, 77, 82, 78, 83, 33, 34, 35, 36, 37, 38,$$
$$39, 40, 41, 42, 43, 80, 44, 72, 45, 73, 46, 85, 47, 103, 48, 84, 49, 87, 50, 104, 51, 67,$$
$$52, 97, 53, 66, 54, 55, 56, 57, 58, 69, 68, 75.$$

This compact notation produces IRDCS with lengths from 147 to 149 which have heft 1.008 and order 57, which compares to heft between 0.7642 and 1.3626 and order at most 73.

It is worth noting that in manually calculating this IRDCS, the original representation found had a block of congruences as described in the method with the moduli 44 through to 59 at positions 1 through to 16. This representation has a clash at position 15, and as such remove the first 15 positions. The standard method of using two long blocks of moduli was used to find this IRDCS, but we allowed for a fairly late clash to see if we could finish the IRDCS, giving the length $n = 4M + 15 = 4M + \frac{M-3}{2}$, relatively short compared to the other large minimum modulus cases.

Minimum modulus 34

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	84	87	92	93	48	49	50	51	52	53	54	55	56	57	58
15	59	60	80	81	61	62	117	86	63	90	121	64	65	83	66
30	75	67	70	34	35	36	37	38	39	40	41	42	43	44	45
45	46	47	76	110	68	79	69	48	73	49	77	50	85	51	94
60	52	89	53	82	54	97	55	34	56	35	57	36	58	37	59
75	38	60	39	88	40	61	41	62	42	84	43	63	44	87	45
90	64	46	65	47	92	66	93	80	67	81	48	34	70	49	35
105	75	50	36	86	51	37	83	52	38	90	53	39	68	54	40
120	69	55	41	76	56	42	73	57	43	79	58	44	77	59	45
135	34	60	46	117	35	47	61	85	36	62	82	121	37	48	63
150	89	38	49	94	64	39	50	65	110	40	51	66	97	41	52
165	67	88	42												

The compact notation of this IRDCS is:

$$84, 87, 92, 93, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 80, 81, 61, 62, 117, 86, 63, 90,$$
$$121, 64, 65, 83, 66, 75, 67, 70, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 76, 110,$$
$$68, 79, 69, 73, 77, 85, 94, 89, 82, 97, 88.$$

This compact notation produces IRDCS with lengths from 167 to 168 which have heft 1.008 and order 60, which compares to heft between 0.7706 and 1.3395 and order at most 73.

Minimum modulus 35

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	96	141	97	92	99	85	48	49	50	51	52	53	54	55	56
15	57	58	59	60	61	62	115	76	63	64	87	83	65	66	72
30	67	75	68	82	35	36	37	38	39	40	41	42	43	44	45
45	46	47	86	70	81	77	73	69	93	48	90	49	91	50	94
60	51	95	52	98	53	100	54	102	55	35	56	36	57	37	58
75	38	59	39	60	40	61	41	62	42	88	43	63	44	64	45
90	85	46	65	47	66	92	96	67	76	97	68	72	48	99	35
105	49	75	36	50	83	37	51	87	38	52	82	39	53	70	40
120	54	69	41	55	73	42	56	77	43	57	81	44	58	86	45
135	59	115	46	60	35	47	61	141	36	62	90	93	37	91	63
150	48	38	64	94	49	39	95	65	50	40	66	98	51	41	67
165	100	52	42	68	102	53	43	88	72						

The compact notation of this IRDCS is:

$$96, 141, 97, 92, 99, 85, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 115, 76, 63, 64, 87, 83, 65, 66, 72, 67, 75, 68, 82, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 86, 70, 81, 77, 73, 69, 93, 90, 91, 94, 95, 98, 100, 102, 88.$$

This compact notation produces IRDCS with lengths from 174 to 175 which have heft 1.006 and order 62, which compares to heft between 0.7714 and 1.3373 and order at most 86.

Minimum modulus 36

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	90	94	106	93	48	49	50	51	52	53	54	55	56	57	58
15	59	60	61	62	63	118	96	64	65	74	77	66	67	71	68
30	75	80	69	81	70	36	37	38	39	40	41	42	43	44	45
45	46	47	76	72	86	82	78	48	73	49	89	50	84	51	87
60	52	88	53	91	54	92	55	95	56	97	57	36	58	37	59
75	38	60	39	61	40	62	41	63	42	85	43	64	44	65	45
90	90	46	66	47	67	94	93	68	74	71	48	69	77	49	70
105	75	50	36	106	51	37	80	52	38	81	53	39	96	54	40
120	72	55	41	76	56	42	73	57	43	78	58	44	82	59	45
135	86	60	46	118	61	47	84	62	36	89	63	87	37	48	88
150	64	38	49	65	91	39	50	92	66	40	51	67	95	41	52
165	68	97	42	53	85										

The compact notation of this IRDCS is:

$$90, 94, 106, 93, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 118, 96, 64, 65, 74, 77, 66, 67, 71, 68, 75, 80, 69, 81, 70, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 76, 72, 86, 82, 78, 73, 89, 84, 87, 88, 91, 92, 95, 97, 85.$$

This IRDCS has length 170 only and has heft 1.003 and order 62, which compares to heft between 0.7691 and 1.3452 and order at most 84.

Minimum modulus 37

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	90	95	52	53	54	55	56	57	58	59	60	61	62	63	64
15	65	66	112	84	67	68	120	82	69	70	80	88	71	72	94
30	73	77	85	74	86	76	37	38	39	40	41	42	43	44	45
45	46	47	48	49	50	51	75	102	79	52	83	53	78	54	91
60	55	96	56	81	57	100	58	101	59	104	60	89	61	37	62
75	38	63	39	64	40	65	41	66	42	93	43	67	44	68	45
90	90	46	69	47	70	48	95	49	71	50	72	51	84	73	82
105	80	52	74	77	53	37	76	54	38	88	55	39	85	56	40
120	86	57	41	94	58	42	75	59	43	112	60	44	79	61	45
135	78	62	46	83	63	47	120	64	48	81	65	49	37	66	50
150	91	38	51	67	102	39	68	96	52	40	89	69	53	41	70
165	100	54	42	101	71	55	43	72	104	56	44	73	93	57	45
180	90	74	58	46	37										

The compact notation of this IRDCS is:

$$90, 95, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 112, 84, 67, 68, 120, 82, 69, 70,$$
$$80, 88, 71, 72, 94, 73, 77, 85, 74, 86, 76, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50,$$
$$51, 75, 102, 79, 83, 78, 91, 96, 81, 100, 101, 104, 89, 93.$$

This compact notation produces IRDCS with lengths from 178 to 185 which have heft 1.006 and order 64, which compares to heft between 0.7705 and 1.3410 and order at most 88.

Minimum modulus 38

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	94	99	101	124	52	53	54	55	56	57	58	59	60	61	62
15	63	64	65	66	67	68	84	135	69	82	96	70	71	90	95
30	72	73	77	74	81	75	76	38	39	40	41	42	43	44	45
45	46	47	48	49	50	51	119	78	80	85	87	52	79	53	86
60	54	97	55	91	56	83	57	98	58	93	59	102	60	105	61
75	38	62	39	63	40	64	41	65	42	66	43	67	44	68	45
90	92	46	69	47	94	48	70	49	71	50	99	51	72	101	73
105	84	82	74	52	77	75	53	76	38	54	81	39	55	90	40
120	56	96	41	57	95	42	58	124	43	59	78	44	60	80	45
135	61	79	46	62	85	47	63	87	48	64	86	49	65	83	50
150	66	38	51	67	91	39	68	135	97	40	52	69	93	41	53
165	98	70	42	54	71	119	43	55	102	72	44	56	73	105	45
180	57	74	92	46	58	75	77	47							

The compact notation of this IRDCS is:

$$94, 99, 101, 124, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 84, 135,$$
$$69, 82, 96, 70, 71, 90, 95, 72, 73, 77, 74, 81, 75, 76, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47,$$
$$48, 49, 50, 51, 119, 78, 80, 85, 87, 79, 86, 97, 91, 83, 98, 93, 102, 105, 92.$$

This compact notation produces IRDCS with lengths from 183 to 188 which have heft 1.006 and order 66, which compares to heft between 0.7701 and 1.3426 and order at most 91.

Minimum modulus 39

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	106	104	101	97	52	53	54	55	56	57	58	59	60	61	62
15	63	64	65	66	67	68	136	151	69	100	84	70	71	99	83
30	72	73	89	74	81	75	82	76	39	40	41	42	43	44	45
45	46	47	48	49	50	51	103	90	77	85	78	52	79	53	86
60	54	87	55	88	56	121	57	91	58	93	59	94	60	105	61
75	95	62	39	63	40	64	41	65	42	66	43	67	44	68	45
90	92	46	69	47	96	48	70	49	71	50	97	51	72	101	73
105	104	106	74	52	84	75	53	83	76	54	81	39	55	82	40
120	56	89	41	57	100	42	58	99	43	59	77	44	60	78	45
135	61	79	46	62	85	47	63	90	48	64	86	49	65	87	50
150	66	88	51	67	103	39	68	136	91	40	52	69	93	41	53
165	94	70	42	54	71	95	43	55	151	72	44	56	73	105	45
180	57	74	92	46	58	75	121	47	59	76	96	48	60	84	39

The compact notation of this IRDCS is:

$$106, 104, 101, 97, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 136, 151,$$
$$69, 100, 84, 70, 71, 99, 83, 72, 73, 89, 74, 81, 75, 82, 76, 39, 40, 41, 42, 43, 44, 45, 46, 47,$$
$$48, 49, 50, 51, 103, 90, 77, 85, 78, 79, 86, 87, 88, 121, 91, 93, 94, 105, 95, 92, 96.$$

This compact notation produces IRDCS with lengths from 191 to 195 which have heft 1.007 and order 68, which compares to heft between 0.7714 and 1.3387 and order at most 95.

Minimum modulus 40

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0		97	102	107	88	110	56	57	58	59	60	61	62	63	64
15	65	66	67	68	69	70	91	104	71	72	98	94	73	74	100
30	105	75	76	80	77	81	78	95	79	40	41	42	43	44	45
45	46	47	48	49	50	51	52	53	54	55	86	82	87	89	118
60	93	109	56	99	57	101	58	83	59	103	60	85	61	96	62
75	114	63	108	64	40	65	41	66	42	67	43	68	44	69	45
90	70	46	88	47	71	48	72	49	97	50	73	51	74	52	102
105	53	75	54	76	55	107	77	91	80	78	110	81	79	56	40
120	94	57	41	98	58	42	104	59	43	100	60	44	95	61	45
135	105	62	46	82	63	47	86	64	48	87	65	49	89	66	50
150	83	67	51	93	68	52	85	69	53	40	70	54	99	41	55
165	71	101	42	72	96	109	43	103	73	56	44	74	118	57	45
180	88	75	58	46	76	108	59	47	77	114	60	48	78	80	61

The compact notation of this IRDCS is:

$$97, 102, 107, 88, 110, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 91, 104, 71, 72,$$
$$98, 94, 73, 74, 100, 105, 75, 76, 80, 77, 81, 78, 95, 79, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49,$$
$$50, 51, 52, 53, 54, 55, 86, 82, 87, 89, 118, 93, 109, 99, 101, 83, 103, 85, 96, 114, 108.$$

This compact notation produces IRDCS with lengths from 189 to 194 which have heft 1.003 and order 69, which compares to heft between 0.7689 and 1.3475 and order at most 94.

Minimum modulus 50

+	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	112	129	133	121	132	134	68	69	70	71	72	73	74	75
14	76	77	78	79	80	81	82	83	84	85	86	113	115	87
28	88	124	114	89	90	144	125	91	92	128	118	93	94	127
42	95	119	96	102	97	103	98	50	51	52	53	54	55	56
56	57	58	59	60	61	62	63	64	65	66	67	135	137	139
70	110	100	141	143	68	99	69	106	70	142	71	105	72	150
84	73	104	74	108	75	109	76	101	77	136	78	111	79	140
98	80	50	81	51	82	52	83	53	84	54	85	55	86	56
112	112	57	87	58	88	59	123	60	89	61	90	62	121	63
126	91	64	92	65	129	66	93	67	94	133	132	95	113	134
140	96	115	68	97	114	69	98	102	70	50	103	71	51	124
154	72	52	118	73	53	125	74	54	119	75	55	128	76	56
168	127	77	57	100	78	58	99	79	59	144	80	60	110	81
182	61	106	82	62	105	83	63	104	84	64	101	85	65	108
196	86	66	109	50	67	87	135	51	88	137	111	52	139	89
210	68	53	90	141	69	54	143	91	70	55	92	142	71	56
224	112	93	72	57	94	136	73	58	95	150	74	59	96	140
238	75	60	97	123	76	61	98	121	77	62				

The compact notation of this IRDCS is:

$$112, 129, 133, 121, 132, 134, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 113, 115, 87, 88, 124, 114, 89, 90, 144, 125, 91, 92, 128, 118, 93, 94, 127, 95, 119, 96, 102, 97, 103, 98, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 135, 137, 139, 110, 100, 141, 143, 99, 106, 142, 105, 150, 104, 108, 109, 101, 136, 111, 140, 123.$$

This compact notation produces IRDCS with lengths from 241 to 248 which have heft 1.004 and order 87, which compares to heft between 0.7699 and 1.3472 and order at most 120.

Chapter 6

IRDCS with only odd moduli

In the classical setting of covering systems Erdős and Selfridge ask in Conjecture 1.2 whether there exists a covering system with only odd moduli. This question has proven famously difficult to decide. Myerson, Poon and Simpson [23] provide an analogous problem in our new setting, namely, do there exist IRDCS with only odd moduli?

Using the exhaustive algorithm presented in Chapter 3 with the additional condition that $modusage[2i] = true$ for all i such that $2 \leq 2i \leq n$, we may search for IRDCS with only odd moduli for given length n.

The first odd IRDCS is one of length 83. This IRDCS and its reversal are the only odd IRDCS of length 83. One of these IRDCS, written in the alternate notation, is:

$$61, 41, 21, 65, 9, 43, 53, 59, 11, 15, 37, 17, 13, 9, 23, 19, 27, 33, 55, 11, 35, 25, 9, 21, 15, 13, 31, 29,$$
$$17, 51, 11, 9, 49, 45, 19, 47, 39, 23, 13, 15, 9, 11, 41, 27, 21, 17, 25, 37, 43, 9, 33, 13, 11, 19, 15, 35,$$
$$29, 31, 9, 53, 23, 61, 17, 11, 13, 21, 59, 9, 65, 15, 27, 25, 19, 55, 11, 39, 9, 13, 45, 17, 51, 49, 47,$$

the other being the reversal of this. The compact notation for the above IRDCS is

$$61, 41, 21, 65, 9, 43, 53, 59, 11, 15, 37, 17, 13, 23, 19, 27, 33, 55, 35, 25, 31, 29, 51, 49, 45, 47, 39.$$

These IRDCS have heft $= 1.02042$, order 27 and use all of the available odd moduli from 9 to 65 excluding 57 and 63.

Theorem 6.1. *There exist odd IRDCS for lengths 83 to 101 inclusive.*

We have computed exhaustive data for lengths 83 to 90 inclusive, and will present the results in what follows. In discussing comparisons to formulae for heft and order, we refer to the formulae from Theorems 2.4 and 2.2 respectively, and we compare heft based on the orders of the particular IRDCS, not for all IRDCS of a given length, thus giving a tighter bound.

Once we reach length 90, the computation time required to exhaustively find all odd IRDCS for a given length becomes unfeasible. As such, for lengths 91 and higher we have used Knuth's Dancing Links implementation of Algorithm X [19] to find a single solution and thus prove the existence of at least 2 odd IRDCS for the given length. We have computed odd IRDCS of lengths up to and including 101 using this algorithm.

Length 84 Odd IRDCS

There are 14 odd IRDCS of length 84. Below are the 7 unique IRDCS up to reversals, along with some summary statistics.

The IRDCS in alternate notation are:

57, 13, 59, 47, 17, 21, 9, 29, 11, 39, 49, 35, 37, 15, 13, 9, 23, 27, 19, 11, 25, 17, 45, 31, 9, 55, 21, 13, 15, 53, 11, 33, 51, 9, 43, 41, 29, 19, 17, 23, 13, 11, 9, 15, 27, 25, 35, 21, 39, 37, 47, 9, 11, 13, 31, 17, 19, 57, 15, 49, 9, 59, 23, 11, 33, 29, 13, 45, 21, 9, 25, 27, 17, 15, 11, 19, 41, 43, 9, 13, 55, 35, 53, 51;

49, 13, 59, 47, 17, 21, 9, 29, 11, 39, 57, 35, 45, 15, 13, 9, 23, 27, 19, 11, 25, 17, 37, 31, 9, 55, 21, 13, 15, 53, 11, 33, 51, 9, 43, 41, 29, 19, 17, 23, 13, 11, 9, 15, 27, 25, 35, 21, 39, 49, 47, 9, 11, 13, 31, 17, 19, 45, 15, 37, 9, 59, 23, 11, 33, 29, 13, 57, 21, 9, 25, 27, 17, 15, 11, 19, 41, 43, 9, 13, 55, 35, 53, 51;

59, 13, 55, 51, 17, 21, 9, 29, 11, 41, 19, 35, 37, 15, 13, 9, 23, 27, 43, 11, 25, 17, 53, 57, 9, 31, 21, 13, 15, 19, 11, 33, 45, 9, 49, 47, 29, 39, 17, 23, 13, 11, 9, 15, 27, 25, 35, 21, 19, 37, 41, 9, 11, 13, 51, 17, 31, 55, 15, 59, 9, 43, 23, 11, 33, 29, 13, 19, 21, 9, 25, 27, 17, 15, 11, 53, 39, 45, 9, 13, 57, 35, 47, 49;

49, 13, 59, 53, 17, 21, 9, 29, 11, 41, 19, 35, 45, 15, 13, 9, 23, 27, 57, 11, 25, 17, 37, 31, 9, 55, 21, 13, 15, 19, 11, 33, 51, 9, 43, 47, 29, 39, 17, 23, 13, 11, 9, 15, 27, 25, 35, 21, 19, 49, 41, 9, 11, 13, 31, 17, 53, 45, 15, 37, 9, 59, 23, 11, 33, 29, 13, 19, 21, 9, 25, 27, 17, 15, 11, 57, 39, 43, 9, 13, 55, 35, 47, 51;

59, 13, 55, 53, 17, 21, 9, 29, 11, 41, 19, 35, 37, 15, 13, 9, 23, 27, 57, 11, 25, 17, 39, 31, 9, 51, 21, 13, 15, 19, 11, 33, 45, 9, 49, 47, 29, 43, 17, 23, 13, 11, 9, 15, 27, 25, 35, 21, 19, 37, 41, 9, 11, 13, 31, 17, 53, 55, 15, 59, 9, 39, 23, 11, 33, 29, 13, 19, 21, 9, 25, 27, 17, 15, 11, 57, 51, 45, 9, 13, 43, 35, 47, 49;

59, 13, 55, 51, 17, 21, 9, 29, 11, 41, 19, 35, 37, 15, 13, 9, 23, 27, 57, 11, 25, 17, 39, 53, 9, 31, 21, 13, 15, 19, 11, 33, 45, 9, 49, 47, 29, 43, 17, 23, 13, 11, 9, 15, 27, 25, 35, 21, 19, 37, 41, 9, 11, 13, 51, 17, 31, 55, 15, 59, 9, 39, 23, 11, 33, 29, 13, 19, 21, 9, 25, 27, 17, 15, 11, 57, 53, 45, 9, 13, 43, 35, 47, 49.

All of these IRDCS have heft 1.00619 and order 26. They all use every available odd modulus from 9 to 59 inclusive. Our formula from Chapter 2 predict that for the 12 new IRDCS the heft is between 0.7874 and 1.2715, and that the order is less than 41.

The final two length 84 odd IRDCS are the length 83 odd IRDCS extended in length by, in this instance, covering the position before the start of the previously presented length 83 IRDCS with the modulus 13, and the other is its reversal. The IRDCS in alternate notation is

$$13, 61, 41, 21, 65, 9, 43, 53, 59, 11, 15, 37, 17, 13, 9, 23, 19, 27, 33, 55, 11, 35, 25, 9, 21, 15, 13, 31,$$
$$29, 17, 51, 11, 9, 49, 45, 19, 47, 39, 23, 13, 15, 9, 11, 41, 27, 21, 17, 25, 37, 43, 9, 33, 13, 11, 19, 15,$$
$$35, 29, 31, 9, 53, 23, 61, 17, 11, 13, 21, 59, 9, 65, 15, 27, 25, 19, 55, 11, 39, 9, 13, 45, 17, 51, 49, 47,$$

and again has heft 1.02042 and order 27.

Length 85 Odd IRDCS

There are 80 odd IRDCS of length 85, all with minimum modulus 9. The minimum heft is 1.00504, while the maximum is 1.02091. There are 22 IRDCS of order 26, and 58 IRDCS of order 27, which compares to the predicted range for heft of between 0.7845 and 1.2799 and order at most 42. The number of times each modulus is used in these IRDCS is presented in the following table.

Modulus	Times Used	Modulus	Times Used
9 to 49	80	63	34
51	78	65	16
53	66	67	6
55	70	69	12
57	56	71	14
59	50	73	6
61	48	75	2

Length 86 Odd IRDCS

There are 382 odd IRDCS of length 86, all with minimum modulus 9. The heft ranges from 1.00373 to 1.02091, with 14 IRDCS of order 26 and 368 of order 27. This compares to the formulae which predict heft between 0.7858 and 1.2767 and order at most 42. The number of times each modulus is used in these IRDCS is summarised in the table below:

Modulus	Times Used	Modulus	Times Used
9 to 41	382	67	36
43 and 45	378	69	54
47 and 49	370	71	64
51	360	73	86
53	300	75	52
55	306	77	28
57	282	79	6
59	248	81	8
61	210	83	12
63	146	85	8
65	104		

Length 87 Odd IRDCS

There are 474 odd IRDCS of length 87, all with minimum modulus 9 and with heft ranging from 1.00563 to 1.0196. Among these odd IRDCS there are 4 with order 26, 462 with order 27 and 8 with order 28. This compares to the heft range of from 0.7829 to 1.2846, and order at most 43. The number of times each modulus is used in these IRDCS is presented in the table below.

Modulus	Times Used	Modulus	Times Used
9 to 27 and 31 to 37	474	59	276
29	466	61	214
39	468	63	252
41	470	65	184
43	464	67	114
45	468	69	100
47	454	71	58
49	442	73	50
51	408	75	40
53	394	77	44
55	396	79	28
57	356	81 / 83	12 / 8

Length 88 Odd IRDCS

There are 152 odd IRDCS of length 88, all with minimum modulus 9. The heft ranges from 1.00988 to 1.01884, and all solutions have order 27. This compares to the predicted heft range of from 0.7884 to 1.2705 and order at most 43. The number of times the various moduli are used in these IRDCS is presented in the table below:

Modulus	Times Used	Modulus	Times Used
9 - 33, 39, 41, 45 and 47	152	63	62
35	146	65	44
37	142	67	12
43	150	69	4
49	144	71	14
51	128	73	10
53	106	75	18
55	122	77	32
57	140	79	24
59	126	81	20
61	76		

Length 89 Odd IRDCS

There are at 80 odd IRDCS of length 89. These IRDCS have heft ranging from 1.00619 to 1.01739 and 78 of these IRDCS have order 26 with the other 2 having order 25. These figures compare to the formulae which give heft ranging from 0.7942 to 1.2564, and order at most 44. The number of times the various moduli are used in these IRDCS is presented in the following table:

Modulus	Times Used	Modulus	Times Used
9 - 33 and 39 - 47	80	63	40
35	76	65	20
37	72	67	2
49	74	69	6
51	76	71	2
53	56	73	6
55	54	75	10
57	74	77	18
59	56	79	16
61	48	81	12

Length 90 Odd IRDCS

There are only 4 odd IRDCS of length 90, made up of two IRDCS and their reversals. The two unique IRDCS up to reversals are presented below in their compact notations.

$$55, 15, 23, 51, 31, 17, 9, 19, 11, 49, 13, 61, 35, 27, 39, 21, 25, 37, 29, 59, 53, 41, 33, 43, 47, 45.$$

$$55, 15, 23, 51, 31, 17, 9, 19, 11, 63, 13, 47, 35, 27, 39, 21, 25, 37, 29, 57, 53, 41, 33, 43, 49, 45.$$

All four IRDCS have order 26, with heft either 1.00504 or 1.00511 and minimum modulus 9. These IRDCS use all of the moduli from 9 to 55 inclusive, with the first presented example above and its reversal additionally using the moduli 59 and 61, and the second example and its reversal using additional moduli 57 and 63. The formulae predicted heft between 0.7955 and 1.2536, and order at most 44.

Larger Lengths Using Dancing Links

For length 91 and higher, Knuth's dancing links implementation of Algorithm X [19] is used to compute a single example of an IRDCS for the particular lengths. For length 91, the IRDCS has heft 1.010 and order 27. The IRDCS in compact notation is:

$$63, 69, 45, 49, 15, 51, 17, 13, 9, 29, 11, 31, 19, 35, 41, 43, 25, 21, 23, 27, 57, 59, 33, 55, 47, 39, 53.$$

For length 92, the IRDCS has heft 1.003 and order 26. The IRDCS in compact notation is:

$$51, 17, 9, 19, 11, 57, 13, 47, 35, 59, 43, 15, 21, 25, 33, 29, 23, 27, 39, 41, 63, 31, 49, 55, 37, 45.$$

For length 93, the IRDCS has heft 1.005 and order 26. It is also an IRDCS of length 94, by moving the modulus 25 to the front of the compact notation. The compact notation of the length 93 IRDCS is:

$$39, 15, 17, 45, 13, 9, 49, 11, 57, 19, 35, 53, 21, 29, 43, 63, 23, 41, 25, 27, 31, 33, 47, 37, 55, 51.$$

For length 94, the IRDCS has heft 1.016 and order 30. The IRDCS in compact notation is:

$$55, 47, 63, 57, 65, 23, 13, 9, 29, 11, 71, 19, 17, 27, 21,$$
$$61, 33, 39, 41, 25, 31, 35, 67, 51, 59, 37, 53, 43, 49, 45.$$

For length 95, the IRDCS has heft 1.014 and order 30. The IRDCS in compact notation is:

$$57, 77, 47, 63, 59, 37, 23, 13, 9, 61, 11, 45, 19, 17, 27,$$
$$43, 29, 33, 21, 71, 25, 31, 35, 67, 39, 49, 41, 55, 53, 51.$$

For length 96, the IRDCS has heft 1.0108 and order 28. The IRDCS in compact notation is:

$$49, 57, 15, 17, 61, 13, 9, 33, 11, 47, 19, 35, 63, 21,$$

$$81, 45, 29, 23, 41, 59, 27, 39, 31, 37, 53, 55, 43, 51.$$

For length 97, the IRDCS has heft 1.0184 and order 31. The IRDCS in compact notation is:

$$69, 59, 53, 15, 63, 23, 65, 9, 19, 11, 29, 43, 83, 49, 31, 17,$$

$$21, 39, 35, 25, 27, 33, 61, 67, 55, 47, 37, 57, 41, 51, 45.$$

For lengths 98 and 99, the computed IRDCS for length 98 can be extended to one of length 99. The length 98 IRDCS has heft 1.0194 and order 32. The length 99 example is found by moving the modulus 47 to the first position in the compact notation of the length 98 example, which is:

$$9, 25, 65, 59, 55, 77, 23, 61, 13, 15, 67, 19, 51, 43, 17, 21,$$

$$27, 29, 71, 31, 33, 37, 69, 35, 63, 39, 57, 41, 53, 45, 49, 47.$$

Below is an independent IRDCS of length 99 which has heft 1.018 and order 32. In compact notation this IRDCS is:

$$63, 59, 49, 53, 69, 9, 65, 47, 19, 13, 15, 17, 81, 45, 21, 33,$$

$$75, 29, 25, 23, 31, 67, 27, 35, 37, 57, 39, 41, 61, 43, 51, 55.$$

In searching for this length 99 odd IRDCS I assumed that the modulus covering the middle position was 9, in order to produce input for our implementation of that algorithm that was of an acceptable size. Excluding length 90, all of the lengths for which exhaustive data has been computed had at least one solution with middle modulus 9.

For lengths 100 and 101 assume once more that the middle modulus is 9. For length 100, the IRDCS has heft 1.0187 and order 32. The IRDCS of length 100 can also be modified to be an IRDCS of length 101 by letting the modulus 41 cover position -1. The IRDCS in compact notation is:

$$63, 55, 69, 57, 17, 9, 47, 19, 31, 13, 75, 23, 21, 15, 67, 77, 45,$$
$$33, 25, 29, 43, 27, 65, 37, 35, 59, 61, 39, 41, 49, 53, 51.$$

For length 101 our independent IRDCS has heft 1.0181 and order 32. The IRDCS in compact notation is:

$$13, 59, 61, 53, 47, 9, 49, 57, 25, 67, 75, 17, 15, 33, 19, 29,$$
$$71, 21, 27, 23, 31, 73, 69, 41, 39, 35, 37, 63, 43, 55, 45, 51.$$

6.1 Open Questions

All of the odd IRDCS that we have found so far have minimum modulus 9. Is it true that all odd IRDCS must use moduli at least as large as 9? Moreover all of the odd IRDCS found, excluding those of length 90, use the modulus 9 to cover the middle position. Is there a good reason for this, or is it just chance?

The number of odd IRDCS for a given length decays from length 87 to length 90, and perhaps does not grow after this given that we have not calculated all IRDCS for larger lengths. Is there a reason for this? This may be similar to the standard IRDCS case where there is a solution of length 11 and then no solutions until length 17. The numbers in the standard case also don't grow all the time, for example from length 23 to 24 and length 27 to 28.

Chapter 7

2 Dimensional IRDCS Properties

7.1 Introduction

The paper by Myerson, Poon and Simpson [23] gives motivation for finding an IRDCS of more than one dimension. Begin by taking the box $[0, X) \times [0, Y) \cap \mathbb{Z}^2$, with $X, Y \geq 2$ for what we will call an X by Y IRDCS, or an $X \times Y$ IRDCS. A two-dimensional IRDCS will be constructed by filling $[0, X) \times [0, Y) \cap \mathbb{Z}^2$ with congruences of the form $ax + by \equiv c \pmod{m}$.

As in the one-dimensional case, notation to assist in visualising these two-dimensional IRDCS, to be fully defined shortly, is required. We call this the **alternate notation**. Putting aside for a moment the precise definition of an $X \times Y$ IRDCS, rather than presenting an $X \times Y$ IRDCS as the collection of congruences that make it up; for example the 9×3 IRDCS

$$x + y \equiv 2 \pmod 3,\ x + y \equiv 4 \pmod 5,\ x + y \equiv 0 \pmod 6,$$

$$x + 5y \equiv 7 \pmod 8,\ x + 5y \equiv 1 \pmod{10},\ x + 3y \equiv 3 \pmod{11},$$

present it as an $X \times Y$ array of integers where, again as in the one-dimensional case, the number at position (x, y) is the modulus of the congruence in the IRDCS containing this position, taking in this instance the lower left corner to be $(0, 0)$. This previous IRDCS is thus presented in its alternate notation as

3	10	5	3	6	8	3	5	11
11	3	8	5	3	6	10	3	5
6	10	3	11	5	3	6	8	3

.

We now look to precisely define the conditions on a two-dimensional IRDCS. The emphasis for the construction of the IRDCS in two-dimensions is on the congruences, as was the case in the one-dimensional case. As a result the conditions that define a two-dimensional IRDCS will be related to these congruences. In the one-dimensional case, having constructions that avoided trivialities meant requiring that each congruence intersect the interval at least twice and that each modulus be used at most once. The moduli here will again be required to be distinct.

The first of the constructions considered trivial that we wish to disallow is one where the finished $X \times Y$ IRDCS has **alternate notation**

k	$k+1$	$k+2$	$\dots$	$k+X-1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
k	$k+1$	$k+2$	$\dots$	$k+X-1$
k	$k+1$	$k+2$	$\dots$	$k+X-1$

.

As a collection of congruences this is

$$x \equiv 0 \pmod{k},\ x \equiv 1 \pmod{k+1}, \dots,\ x \equiv X-1 \pmod{k+X-1}.$$

This construction works for any $k \geq X$, and is not particularly interesting. Extending this further, if congruences are able to cover positions directly above one another, then taking a length X IRDCS with alternate notation $a_1, a_2, \dots, a_X$ this $X \times Y$ IRDCS can always be constructed;

a_1	a_2	a_3	$\dots$	a_X
$\vdots$	$\vdots$	$\vdots$		$\vdots$
a_1	a_2	a_3	$\dots$	a_X
a_1	a_2	a_3	$\dots$	a_X

.

As such, congruences of the form $\{(x, y) \in \mathbb{Z}^2 : x \equiv a \pmod{m}\}$ or $\{(x, y) \in \mathbb{Z}^2 : y \equiv b \pmod{m}\}$ are removed from consideration. Moreover, two-dimensional IRDCS will only use cosets of subgroups of $\mathbb{Z}^2$ which are the solution set to a single congruence $ax + by \equiv c \pmod{m}$ where $1 < a, b < m$. This is motivated by the work in [16]. The following definition is required to further discuss these congruence solution sets in work that follows.

Definition 7.1. A subgroup of $\mathbb{Z}^2$ is said to have **corank** 2 if and only if it is the solution set of a system of 2 homogenous congruences, and not of any smaller system.

For example, the set $\{(2a, 4b) : a, b \in \mathbb{Z}\}$ is of corank 2, as it is the solution set of $x \equiv 0 \pmod{2}$ and $y \equiv 0 \pmod{4}$ and not the solution of a single congruence.

The definition of two-dimensional IRDCS should also remove any constructions that are too easily finished once a certain point is reached in the construction. This is equivalent to not allowing any congruences in the one-dimensional case to cover only a single point in the interval. In two dimensions any two suitably distant positions may be covered by a congruence with a large modulus that will not intersect $[0, X) \times [0, Y)$ in any other positions. So to avoid trivialities all congruences must cover at least 3 positions in the box. If these 3 positions are all allowed to be collinear, then any $n \times 3$ box with $n \geq 5$ could be filled as:

$n-1$	$n+1$	$n+2$	$n+3$	$\dots$	n	$n-1$
$n-2$	$n-1$	$n+1$	$n+2$	$\dots$	$n-2$	n
n	$n-2$	$n-1$	$n+1$	$\dots$	$2n-4$	$n-2$

where the congruences used are $x + y \equiv 0 \pmod{n}, x + y \equiv 1 \pmod{n-2}, x + y \equiv 2 \pmod{n-1}$ and $x + y \equiv i \pmod{n+i-2}$ for $i = 3, \dots n-2$. As in the first example, the moduli $n+1$ through to $2n-4$ inclusive may be replaced with any larger moduli not already being used. Thus each congruence is required to cover at least 3 positions in the box, not all of these positions being collinear. This is then enough to remove any trivial cases.

We are now in a position to define the two-dimensional IRDCS, recalling the standard definition of $gcd(a, b)$ being the greatest common divisor of a and b.

Definition 7.2. An $X \times Y$ two-dimensional incongruent restricted disjoint covering system (henceforth two-dimensional IRDCS) is a collection of congruences in two variables such that

- each modulus is used at most once,
- every element in $[0, X) \times [0, Y) \cap \mathbb{Z}^2$ satisfies precisely one congruence,
- all congruences are satisfied by at least three elements of $[0, X) \times [0, Y) \cap \mathbb{Z}^2$ that are not all collinear, and
- all congruences are of the form $ax + by \equiv c \pmod{m}$ with $1 < a, b < m$ and $gcd(a, b) = 1$.

Note that since $gcd(a, b, m) = 1$, if for a congruence $gcd(a, b) = k > 1$ then take

$$k\left(\frac{a}{k}x + \frac{b}{k}y\right) \equiv c \pmod{m},$$

and since $gcd(k, m) = 1$ there must exist some k^{-1} such that $kk^{-1} \equiv 1 \pmod{m}$ so that the congruence can be taken to be

$$\frac{a}{k}x + \frac{b}{k}y \equiv ck^{-1} \pmod{m},$$

which is the equivalent congruence but with smaller coefficients, and will thus be used in its place.

In order to clarify this construction we provide some examples of two-dimensional IRDCS in their alternate notation. Firstly, the smallest X for which there exists an $X \times 2$ IRDCS is $X = 10$, as discovered by an exhaustive search. Two such IRDCS are

9	3	4	5	3	6	4	3	5	9
6	9	3	4	5	3	6	4	3	5

, and

9	5	6	4	8	7	5	4	6	9
7	8	4	5	9	6	4	7	5	8

.

The first may look familiar as it is based on the length 11 one-dimensional IRDCS, a construction we will study further in Lemma 7.5. It uses the congruences $x + y \equiv 2 \pmod 3, x + y \equiv 3 \pmod 4, x + y \equiv 4 \pmod 5, x + y \equiv 0 \pmod 6$ and $x + y \equiv 1 \pmod 9$. The second example uses the congruences $x + 3y \equiv 2 \pmod 4, x + 2y \equiv 3 \pmod 5, x + 3y \equiv 5 \pmod 6, x + 2y \equiv 0 \pmod 7, x + 5y \equiv 1 \pmod 8$ and $x + 4y \equiv 4 \pmod 9$.

The smallest case with $Y = 3$ is the 9×3 case. Among these, there is again a construction related to the length 11 one-dimensional IRDCS, but there are also other IRDCS including the following three

3	10	5	3	6	8	3	5	11
11	3	8	5	3	6	10	3	5
6	10	3	11	5	3	6	8	3

,

5	8	10	11	6	5	7	13	9
13	6	7	9	5	8	10	6	11
10	8	11	5	6	7	13	9	5

,

and

9	5	7	13	6	11	5	8	10
11	6	8	10	5	9	7	6	13
13	9	5	7	6	8	11	5	10

.

The first of these uses the congruences $x + y \equiv 2 \pmod 3, x + y \equiv 4 \pmod 5, x + y \equiv 0 \pmod 6, x + 5y \equiv 7 \pmod 8, x + 5y \equiv 1 \pmod{10}$, and $x + 3y \equiv 3 \pmod{11}$. The second uses the congruences $x + 4y \equiv 3 \pmod 5, x + 3y \equiv 4 \pmod 6, x + 3y \equiv 5 \pmod 7, x + 4y \equiv 1 \pmod 8, x + 4y \equiv 7 \pmod 9, x + 4y \equiv 0 \pmod{10}, x + 5y \equiv 2 \pmod{11}$ and $x + 6y \equiv 6 \pmod{13}$. The third of these IRDCS uses the congruences $x + 3y \equiv 2 \pmod 5, x + 3y \equiv 4 \pmod 6, x + 4y \equiv 3 \pmod 7, x + 3y \equiv 5 \pmod 8,$

$x+5y \equiv 1 \pmod 9, x+5y \equiv 8 \pmod{10}, x+6y \equiv 6 \pmod{11}$ and $x+5y \equiv 0 \pmod{13}$. Interestingly the last two examples use the same moduli set, but do not appear to be otherwise related.

The next case is $Y = 4$ for which the smallest example is a 6×4 IRDCS. There is effectively only one example for these dimensions, the rest being reflections of this one.

8	7	6	5	9	6
4	5	4	7	4	8
6	9	8	6	5	7
7	4	5	4	9	4

.

This IRDCS uses the congruences $2x + y \equiv 2 \pmod 4, x + 3y \equiv 2 \pmod 5, 2x + y \equiv 1 \pmod 6, x + 2y \equiv 0 \pmod 7, x + 5y \equiv 7 \pmod 8$, and $x + 3y \equiv 4 \pmod 9$.

Some other examples include this 7×7 IRDCS

3	13	20	3	15	9	3
9	3	10	21	3	14	18
12	14	3	12	9	3	12
3	15	18	3	10	13	3
20	3	13	9	3	20	21
21	10	3	14	15	3	10
3	12	9	3	12	18	3

,

which uses the congruences $x + y \equiv 0 \pmod 3, x + 4y \equiv 2 \pmod 9, 2x + 7y \equiv 9 \pmod{10}, 4x + y \equiv 4 \pmod{12}, x + 10y \equiv 9 \pmod{13}, x + 10y \equiv 13 \pmod{14}, x + 9y \equiv 13 \pmod{15}, x + 7y \equiv 5 \pmod{18}, 4x + 3y \equiv 6 \pmod{20}, x + 15y \equiv 15 \pmod{21}$, and the 10×8 IRDCS

2	16	2	6	2	30	2	38	2	6
18	2	32	2	6	2	20	2	24	2
2	24	2	16	2	6	2	18	2	32
6	2	38	2	22	2	6	2	20	2
2	6	2	18	2	16	2	6	2	22
30	2	6	2	24	2	30	2	6	2
2	20	2	6	2	32	2	16	2	6
16	2	22	2	6	2	18	2	38	2

,

which uses the congruences $x+y \equiv 1 \pmod 2, x+y \equiv 4 \pmod 6, x+9y \equiv 0 \pmod{16}$, $x+7y \equiv 6 \pmod{18}, x+11y \equiv 12 \pmod{20}, x+5y \equiv 2 \pmod{22}$, $x+17y \equiv 14 \pmod{24}, 5x+y \equiv 2 \pmod{30}, x+7y \equiv 12 \pmod{32}, x+11y \equiv 8 \pmod{38}$. This uses only even moduli, but the modulus 2 cannot be removed to generate a smaller IRDCS. Removing the 2's causes the collections of positions covered by congruences in the original to shift and no longer be solution sets of a congruence. The details of why this fails will be explained in detail in Section 7.4.

The primary method for describing the structure of IRDCS is to discuss the way that the particular congruences fill the IRDCS. The terms **generators** and **lattice** will be used to describe these congruences, the standard use of these terms being adjusted to suit our purposes as follows.

Definition 7.3. Define a set of **generators** of a homogenous congruence to be a pair of vectors $\mathbf{u}, \mathbf{v}$ such that $\mathbf{w} = (x, y)$ satisfies the congruence if and only if there exists $a, b \in \mathbb{Z}$ such that $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$. These generators are not unique.

Definition 7.4. We call u, v a set of **generators** of a non-homogeneous congruence if they are generators of the associated homogeneous congruence

Definition 7.5. Define the **lattice** of a congruence to be the collection of all points in $\mathbb{Z}^2$ which satisfy the congruence.

As in the one-dimensional case, the quantities **heft** and **order** tell something of the structure of the IRDCS.

Definition 7.6. An $X \times Y$ two-dimensional IRDCS

$$\bigcup_{i=1}^{t}\{(x,y) \in \mathbb{Z}^2 : a_i x + b_i y \equiv c_i \pmod{m_i}\},$$

has, analogous to the one-dimensional case, **order** t and **heft** h where

$$h = \sum_{i=1}^{t} \frac{1}{m_i}.$$

The following definition will be important in our analysis.

Definition 7.7. Call the **first point** of a congruence the point satisfying the congruence in $[0, X) \times [0, Y)$ with the smallest possible y coordinate, and smallest x coordinate for this particular y.

All these concepts will be illustrated with examples in the next few pages

It is generally easiest to analyze a congruence lattice by using its generators. This will be used in discussing many of the structural properties of two-dimensional IRDCS and proving various results. Take a congruence $ax + by \equiv c \pmod{m}$ in which $gcd(a, b) = 1$ so that it is in lowest form. This congruence forms a lattice with generators of the form (x_0, d) and $(m/d, 0)$, where $d = gcd(a, m)$. To see this, if $gcd(a, m) = 1$ then $ax \equiv c \pmod{m}$ has a solution for all c, and thus for all y there exists an x such that $ax+by \equiv c \pmod{m}$, hence giving the generator $(x_0, 1)$. Meanwhile, since the modulus of the congruence is m the congruence must repeat every m/k points on any given row, for some natural number k dividing m. This makes our generators $(x_0, 1)$ and $(m/k, 0)$. Putting them in a 2×2 matrix and setting the absolute value of the determinant to the modulus m gives $k = 1$ as required.

On the other hand, if $gcd(a, m) = d > 1$ then

$$ax \equiv \gamma \pmod{m} \to d(a/d)x \equiv \gamma \pmod{d(m/d)}$$

and this has a solution if and only if $d|\gamma$. Therefore, since $gcd(a, b) = 1$, if there exists some point (x_1, y_1) such that $ax_1 + by_1 \equiv c \pmod{m}$ then $ax + by \equiv c \pmod{m} \leftrightarrow d(a/d)+by \equiv c \pmod{d(m/d)}$ has solutions if and only if $y \equiv y_1 \pmod{d}$. Using similar arguments to above gives us the generators (x_0, d) and $(m/d, 0)$, where $d = gcd(a, m)$.

It is not difficult to see that any congruence with these generators can also be seen to have generators (e, y_0) and $(0, m/e)$, for some e with $gcd(d, e) = 1$.

Lemma 7.1. *A subgroup H of $\mathbb{Z}^2$ has corank 2 if and only if there is an integer $e > 1$ such that for every $(x, y) \in H$ we have $e|x$ and $e|y$.*

Proof. For a proof of this lemma see Theorem 9 in [16]. □

Corollary 7.1. *Congruences with an integer $e > 1$ dividing all of the components of the generator pair (x_0, d) and $(m/d, 0)$ have solution set the coset of a subgroup of corank 2.*

Proof. All lattice points that are the solution to a homogenous congruence with this generator pair can be written in the form

$$(x, y) = s(x_0, d) + t(m/d, 0),$$

for $s, t \in \mathbb{Z}$. So if $e|\{x_0, d, m/d\}$ then $e|x$ and $e|y$ for all points (x, y) in the homogenous congruence. The lattice points of a non-homogenous congruence are a coset of the subgroup associated to the homogenous congruence. □

Definition 7.8. A **rectangular congruence** has generators $(a, 0)$ and $(0, b)$.

Corollary 7.2. *The only valid rectangular congruences have generator pairs $(m/d, 0)$ and $(0, d)$ with $d > 1$ and $gcd(m/d, d) = 1$.*

Proof. The definition of IRDCS does not allow congruences with $d = 1$ and $x_0 = 0$. For $d > 1$ the condition $gcd(m/d, d) = 1$ forces the congruence to have solution set which is a coset of a subgroup not of corank 2, and thus be a single valid congruence. □

Lemma 7.2. *For two congruences $ax + by \equiv 0 \pmod{m}$ and $dx + ey \equiv 0 \pmod{m}$ to have the same solution set, either:*

(1) If $gcd(r, m) = 1$ for some $r \in \{a, b, d, e\}$, then

$$bd \equiv ae \pmod{m}, \textit{ or}$$

(2) If $gcd(r, m) > 1$ for all $r \in \{a, b, d, e\}$, then if $gcd(a, m) = s$ then

$$bd \equiv ae \left(\bmod \frac{m}{s}\right).$$

Proof. In the first case, assume without loss of generality that $gcd(a, m) = 1$, then there exists an $r \in \{0, 1, \ldots, m-1\}$ such that $ar \equiv d \pmod{m}$. Thus

$$dx + bry \equiv 0 \pmod{m},$$

so that for the congruences to be equivalent $br \equiv e \pmod{m} \rightarrow abr \equiv ae \pmod{m} \rightarrow bd \equiv ae \pmod{m}$.

In the second case $gcd(a, m) > 1$, and similarly for b, d and e. Take $gcd(a, m) = s$, then there exists an $r \in \{0, 1, \ldots, m-1\}$ such that $\frac{a}{s}r \equiv d \pmod{\frac{m}{s}}$. Then $ar \equiv ds \pmod{m}$, $dsx + bry \equiv 0 \pmod{m}$, so that if $br \equiv es \pmod{m}$, then the lattices are equivalent. Thus $abr \equiv aes \pmod{m} \rightarrow dsb \equiv aes \pmod{m} \rightarrow bd \equiv ae \pmod{\frac{m}{s}}$, where $gcd(a, m) = s$. □

This lemma is important for the computation of two-dimensional IRDCS, being used to avoid attempting to fill a position with a congruence which will certainly fail. Note that whatever **first point** (x_1, y_1) this congruence will cover, there is a unique c such that $ax_1 + by_1 \equiv c \pmod{m}$, so that there is only one such congruence to attempt.

Forcing the requirement $gcd(a, b) = 1$ is important. It will be shown in Section 7.6.1 that the algorithm selects congruences by cycling through possible values of c, b and a in that order. If $gcd(a, b) = 1$ were not forced then the algorithm would consider the congruence $2x + 5y \equiv 0 \pmod 6$ equivalent to $2x + 2y \equiv c \pmod 6$ and thus it would not be attempted. However the second congruence is actually a congruence class modulo 3, so long as c is even, and thus invalid, and has no solutions for c odd. So the algorithm would never attempt to use the congruence $2x + 5y \equiv 0 \pmod 6$ in the IRDCS, even though it is valid.

We next prove a simple result which will help to simplify our analysis of two-dimensional IRDCS.

Lemma 7.3. *Any $X \times Y$ IRDCS can be reflected along either diagonal to a $Y \times X$ IRDCS.*

Proof. Take any $X \times Y$ IRDCS with congruences with generator pairs (x_0, d) and $(m/d, 0)$. Transforming all of these generators to (d, x_0) and $(0, m/d)$ we get a $Y \times X$ IRDCS. The translation from one IRDCS to the other takes the points in the corner of the original IRDCS $(0,0), (X-1,0), (0,Y-1)$ and $(X-1,Y-1)$ to $(0,0), (0,Y-1), (X-1,0)$ and $(X-1,Y-1)$ respectively. This can be seen as in the figures below.

$m_{(0,Y-1)}$	$m_{(1,Y-1)}$	$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$	$m_{(X-1,Y-1)}$
$\vdots$ $\vdots$			$\vdots$ $\vdots$
$m_{(0,0)}$	$m_{(1,0)}$	$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$	$m_{(X-1,0)}$

$\downarrow$

$m_{(X-1,0)}$	$m_{(X-1,1)}$	$\cdots \quad \cdots \quad \cdots$	$m_{(X-1,Y-1)}$
$\vdots$ $\vdots$ $\vdots$ $\vdots$			$\vdots$ $\vdots$ $\vdots$ $\vdots$
$m_{(0,0)}$	$m_{(0,1)}$	$\cdots \quad \cdots \quad \cdots$	$m_{(0,Y-1)}$

For this translation clearly the modulus m of a congruence remains the same, while the congruence $ax + by \equiv c \pmod{m}$ becomes $bx + ay \equiv c \pmod{m}$ as the x and y coordinates of the original congruence are being switched.

In the other case the generators may instead be replaced by $(0, -m/d)$ and $(-d, -x_0)$, swapping the corner positions $(0, 0)$ and $(X - 1, Y - 1)$. This produces a two-dimensional IRDCS which behaves as the matrix transpose. For this construction the congruence $ax + by \equiv c \pmod{m}$ becomes $a^*x + b^*y \equiv c^* \pmod{m}$, where $ax_0 + bd \equiv 0 \pmod{m}$ and $a^*(-d) + b^*(-x_0) \equiv 0 \pmod{m}$, so that $a^* = b$ and $b^* = a$. Here c^* may or may not equal c. $\square$

An example of this is the 6×4 IRDCS presented previously

8	7	6	5	9	6
4	5	4	7	4	8
6	9	8	6	5	7
7	4	5	4	9	4

,

which can be reflected to

4	7	8	6
9	5	4	9
4	6	7	5
5	8	4	6
4	9	5	7
7	6	4	8

.

An immediate consequence of this lemma is that any $X \times Y$ IRDCS with $Y > X$ can be reflected to a $Y \times X$ IRDCS. As such when studying two-dimensional IRDCS we may consider them as being of dimension $X \times Y$ with $X \geq Y$ and if required reflecting after our analysis.

7.2 Some two-dimensional IRDCS that always work

The following lemma will be useful in what follows.

Lemma 7.4. *For all lengths $n = 11$ and $n \geq 17$ there exists an IRDCS which does not use the modulus $n - 1$.*

Proof. Firstly, a search with the exhaustive algorithm shows that the statement is true for all lengths up to and including $n = 32$. Now using the doubling technique from the proof of Theorem 2.1, generate IRDCS of all possible lengths. Since our first collection of IRDCS do not use the modulus $n - 1$, and given that an IRDCS with alternate notation $a_1, a_2, \ldots, a_n$ doubles to one of the following

$$2a_1, 2, 2a_2, 2, \ldots, 2, 2a_n,$$

$$2, 2a_1, 2, 2a_2, 2, \ldots, 2, 2a_n, \text{ or}$$

$$2, 2a_1, 2, 2a_2, 2, \ldots, 2, 2a_n, 2,$$

the largest modulus these IRDCS can contain is $2(n-2) < (2n-1) - 1$, where $2n - 1$ is the length of the shortest doubled IRDCS. And so the last two doubled IRDCS fit the statement of the lemma. This argument is finished as in the proof of Theorem 2.1. $\square$

There are length n IRDCS that use the modulus $n-1$. The smallest such example is the length 22 IRDCS

$$21, 12, 10, 4, 6, 13, 8, 4, 9, 11, 6, 4, 10, 12, 8, 4, 6, 9, 13, 4, 11, 21,$$

which has heft 1.05156 and order 9. The fact that IRDCS which do not use this modulus exist will be used to establish conditions for which two-dimensional IRDCS can always be found.

Lemma 7.5. *For any IRDCS of length n with alternate notation $a_1, a_2, a_3, \ldots, a_n$, where $a_1 \neq n-1$ there exists an $(n-1) \times 2$ two-dimensional IRDCS*

a_2	a_3	a_4	$\ldots$	a_{n-1}	a_n
a_1	a_2	a_3	$\ldots$	a_{n-2}	a_{n-1}

.

Proof. It is required to check that all of the properties for two-dimensional IRDCS are satisfied. Since $a_1, a_2, a_3, \ldots, a_n$ is an IRDCS in alternate notation and congruences in this IRDCS with modulus m have the same modulus in the two-dimensional IRDCS it has distinct moduli. Since $a_1 \neq a_n$, every modulus appears at least once on each row and at least twice on at least one row, giving three non-collinear hits. Lastly $a_i \geq 2, \forall i$, so that positions above one another are not covered by the same congruence, making all congruences of valid form. Thus it is indeed a two-dimensional IRDCS. □

Theorem 7.1. *If there exists an $n \times 2$ IRDCS with congruences*

$$x + y \equiv c_i \pmod{m_i}, \; i = 1, 2, \ldots t,$$

then there exists an $(n-k) \times (k+2)$ IRDCS with congruences

$$x + y \equiv c_i \pmod{m_i}, \; i = 1, 2, \ldots t,$$

for all $0 \leq k \leq n-2$.

Proof. Proceed via induction. Any $n \times 2$ IRDCS with congruences $x + y \equiv c_i \pmod{m_i}$ has alternate notation

a_2	a_3	a_4	$\ldots$	a_n	a_{n+1}
a_1	a_2	a_3	$\ldots$	a_{n-1}	a_n

where $a_1, a_2, a_3, \ldots, a_n, a_{n+1}$ is an IRDCS of length $n+1$ written in alternate notation with $a_1 \neq n$. This is as the congruences must have 3 non-collinear hits and so every modulus appears at least once in every row and at least twice in one row, and the only way to hit only once in both rows is to have $a_1 = a_{n+1} = n$.

Then

a_3	a_4	a_5	$\dots$	a_n	a_{n+1}
a_2	a_3	a_4	$\dots$	a_{n-1}	a_n
a_1	a_2	a_3	$\dots$	a_{n-2}	a_{n-1}

is also a two-dimensional IRDCS. In removing the last column exactly one copy of the moduli a_n, a_{n+1} were removed, both being regained by adding in the top row. It remains to check the collinearity condition. When the moduli a_n, a_{n+1} were added into the top row, it was done in the same diagonal as the copies that were removed. As the previous case was an IRDCS, there must be other copies of these moduli elsewhere in the IRDCS not on this diagonal. So this new construction is an $(n-1) \times 3$ IRDCS.

Now assume that we have an $(n-k) \times (k+2)$ IRDCS of the required form, where $0 \leq k \leq n-3$. Such an IRDCS can be represented as

a_{k+2}	a_{k+3}	a_{k+4}	$\dots$	a_n	a_{n+1}
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
a_2	a_3	a_4	$\dots$	a_{n-k}	a_{n-k+1}
a_1	a_2	a_3	$\dots$	a_{n-k-1}	a_{n-k}

Now in removing the last column one copy of the elements $a_{n-k}, a_{n-k+1}, \dots, a_{n+1}$ are lost, and in adding in the top row the elements $a_{k+3}, a_{k+4}, \dots, a_{n+1}$ are gained. The new possible IRDCS is represented below.

a_{k+3}	a_{k+4}	a_{k+5}	$\dots$	a_n	a_{n+1}
a_{k+2}	a_{k+3}	a_{k+4}	$\dots$	a_{n-1}	a_n
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
a_2	a_3	a_4	$\dots$	a_{n-k-1}	a_{n-k}
a_1	a_2	a_3	$\dots$	a_{n-k-2}	a_{n-k-1}

As in the initial case, all points added back into the IRDCS appear on the same diagonal as those that have been removed. Since the original was a two-dimensional IRDCS, all moduli in the original covered positions on at least two different lines. All moduli that have been removed, excluding the top right corner position, are solutions to congruences which have at least one other solution on that particular diagonal line, where the top right corner is always added back to the top right hand corner at any rate. So the collinearity condition will be maintained. For this to be an IRDCS it remains to check that all congruences have at least three hits.

If $n-k \geq k+3$ then all points that were removed are added back in, creating a new two-dimensional IRDCS. In the other case, one count of $a_{n-k}, a_{n-k+1}, \ldots, a_{k+2}$ are lost. Each of these moduli now appear on the diagonal we have removed it from $n-k-1$ times. As we are only concerned with two-dimensional IRDCS $n-k-1 \geq 2$, and as they all have at least one other non-collinear point all congruences must have at least three hits and thus this is also a two-dimensional IRDCS. □

Corollary 7.3. *If there exists an $s \times t$ IRDCS with congruences*

$$x + y \equiv c_i \pmod{m_i},\ i = 1, 2, \ldots t,$$

then there exists a $k \times (t + (s-k))$ IRDCS with congruences

$$x + y \equiv c_i \pmod{m_i},\ i = 1, 2, \ldots t,$$

for all $2 \leq k \leq s + t - 2$.

Proof. The proof of Theorem 7.1 gives the method for converting an $s \times t$ IRDCS to an $(s-1) \times (t+1)$ IRDCS, while an analogous argument will take the IRDCS to an $(s+1) \times (t-1)$ IRDCS. Repeating this process as many times as is needed will give the required IRDCS. □

Corollary 7.4. *There exist two-dimensional IRDCS of dimensions $s \times t$ for $s + t = 12$ and for $s + t \geq 18$.*

Proof. There exists an $n \times 2$ IRDCS for $n = 10$ and $n \geq 16$ by Lemma 7.5. Thus, by the above corollary, there exists a $k \times (n + 2 - k)$ IRDCS for all $2 \leq k \leq n$. Setting $k = s$, then $t = n + 2 - k$ is required. Thus $s + t = n + 2$, as required. □

These results give two-dimensional IRDCS for most dimensions. However they are all constructed from the one-dimensional case.

7.2.0.1 Using reflections

Call a congruence in a length n IRDCS **centrally symmetric** if for some $k \in \mathbb{N}$ the congruence is satisfied by both k and $n + 1 - k$. Call a one-dimensional IRDCS non-centrally symmetric if none of its congruences are centrally symmetric. Only IRDCS of even length can be non-centrally symmetric, as the congruence that covers the middle position in an odd length IRDCS will always be centrally symmetric.

Lemma 7.6. *For any non-centrally symmetric IRDCS with alternate notation $a_1, \ldots, a_n$, taking the IRDCS and writing its reversal above or below it as*

a_n	a_{n-1}	$\ldots$	a_2	a_1
a_1	a_2	$\ldots$	a_{n-1}	a_n

or

a_1	a_2	$\ldots$	a_{n-1}	a_n
a_n	a_{n-1}	$\ldots$	a_2	a_1

produces two $n \times 2$ IRDCS.

Proof. The fact that the IRDCS is non-centrally symmetric means that in the two-dimensional IRDCS no modulus will appear directly above itself and so the congruences will all be of the valid type. Since the initial case is a one-dimensional IRDCS, all congruences cover at least two positions on each row with generator $(m, 0)$ and by taking the change from the **first point** of the congruence to the nearest point to the right on the top row with the same modulus gives the second generator $(x_0, 1)$ where $0 < x_0 < m$.

All congruences will be of the form $x - x_0 y \equiv a_m \pmod{m}$ where a_m is the congruence class for the modulus m in the one dimensional IRDCS reversal. □

These IRDCS have heft and order the same as that of the original IRDCS.

Definition 7.9. A **near IRDCS** of length n is an exact cover of $[1, n]$ where some of the congruences only have one solution in the interval and where there would be either a clash or an empty position, on extending the length in either direction, before all congruences have two solutions.

So a **near IRDCS** can never be turned into a one-dimensional IRDCS by extending the length in either direction, other than perhaps by adding in new congruences in empty positions that may be created.

Theorem 7.2. *All $n \times 2$ IRDCS are constructed by either*

- *two copies of a single one-dimensional IRDCS, the second row of the IRDCS either being shifted or having had the reversal taken, or*
- *two different IRDCS with the same moduli, one on each row, possibly with a shift, or*
- *made up of at least one near IRDCS on one of the rows, where the second row is either an IRDCS or another near IRDCS, in both cases with the same moduli.*

Proof. Clearly the first two cases can form two-dimensional IRDCS so long as all congruences are of co-rank 1. If it is neither of the first two cases, then as we must have each congruence covering at least 3 non-collinear points in the IRDCS each congruence must hit each row at least once and hit one row at least twice. Since the generators are $(m, 0)$ and $(x_0, 1)$ both rows will have all of the properties of a one-dimensional IRDCS excluding all congruences having at least two hits. The rows are then a section of either an IRDCS or a near IRDCS. $\square$

All of these IRDCS can be summarised by saying that each row is a segment of an IRDCS, or a near-IRDCS, the two rows sharing the same moduli. One example which is of the third type in the above lemma is the 10×2 IRDCS

9	5	6	4	8	7	5	4	6	9
7	8	4	5	9	6	4	7	5	8

where the top row does not use the modulus 8 twice before there is an uncovered position, and similarly for the bottom row with the modulus 9. Note that the top row for this IRDCS is not a shift of the bottom. Moreover for $n \times 2$ IRDCS of the third type in the above Lemma this is not possible, otherwise each row would have at least two hits and be an IRDCS. Also note that this IRDCS has order $6 = \left[\frac{10\times 2}{3}\right]$, which will be alluded to later.

7.3 Open Question

Start with the following easy lemma.

Lemma 7.7. *Take a pair of different length n IRDCS I_1, I_2 with the same moduli sets $m_1 < m_2 < \cdots < m_t$ and with residues $a_1, \ldots a_t$, and $b_1, \ldots b_t$ respectively. If $a_i \neq b_i$ for all i then two different $X \times 2$ IRDCS can be constructed by placing either I_1 above I_2 or vice-versa in the alternate notation.*

Proof. If $a_i \neq b_i$ for all i, putting one of the IRDCS on top of the other clearly gives 3 non-collinear hits and all congruences will be associated to subgroups of corank 1. $\square$

Note that the case where $a_i = b_i$ for all i has been previously studied in Lemmas 7.5 and 7.6.

We believe that the following statement is also true:

Given the conditions of the previous lemma, if $a_i = b_i$ for some but not all i then, so long as neither IRDCS uses the modulus $n-1$, there exists a shift by k places such that putting I_1 shifted by k places above I_2 or vice-versa and removing the ends as needed in the alternate notation gives two $X \times 2$ IRDCS, where $X = n - k$.

7.4 Doubling two-dimensional IRDCS

Doubling with the modulus 2

This section will attempt, as in the one-dimensional case (see Lemma 2.1), to construct a two-dimensional IRDCS of dimensions twice as large from an existing IRDCS.

Take a two-dimensional IRDCS on $[0, X) \times [0, Y)$, with order t and heft h, where the congruences are of the form

$$\{a_i x + b_i y \equiv c_i \pmod{m_i} : i = 1, 2, \ldots t; m_{i+1} > m_i\},$$

and the i^{th} congruence has generators $(m_i/d_i, 0)$ and $(x_{0,i}, d_i)$. This IRDCS has alternate notation

$A_{(0,Y-1)}$	$A_{(1,Y-1)}$	$A_{(2,Y-1)}$	$\ldots$	$A_{(X-1,Y-1)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$A_{(0,1)}$	$A_{(1,1)}$	$A_{(2,1)}$	$\ldots$	$A_{(X-1,1)}$
$A_{(0,0)}$	$A_{(1,0)}$	$A_{(2,0)}$	$\ldots$	$A_{(X-1,0)}$

where $A_{(i,j)}$ is the modulus of the congruence covering $(i, j) \in \mathbb{Z}^2$. Assuming that Y is odd and extending the IRDCS horizontally by adding the congruence $x+y \equiv 1 \pmod 2$ to the collection of congruences already used gives the following

$A^*_{(0,Y-1)}$	2	$A^*_{(1,Y-1)}$	2	$\ldots$	$A^*_{(X-1,Y-1)}$	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$A^*_{(0,2)}$	2	$A^*_{(1,2)}$	2	$\ldots$	$A^*_{(X-1,2)}$	2
2	$A^*_{(0,1)}$	2	$A^*_{(1,1)}$	$\ldots$	2	$A^*_{(X-1,1)}$
$A^*_{(0,0)}$	2	$A^*_{(1,0)}$	2	$\ldots$	$A^*_{(X-1,0)}$	2

,

which is of dimensions $2X \times Y$ and not necessarily an IRDCS. The equivalent extension for Y even is obvious.

If it actually generates an IRDCS, we call this construction, along with the extension by $x + y \equiv 0 \pmod 2$, is **horizontal doubling**.

Theorem 7.3. *If an $X \times Y$ two-dimensional IRDCS can be* ***horizontally doubled*** *to a $2X \times Y$ IRDCS then it must have all of its congruences with solutions on precisely two rows of the IRDCS and the congruences must have generator pair $(m/d, 0)$ and (x_0, d) with d odd and $m/d \leq X - 1$.*

Should it exist this new IRDCS will have heft $\frac{1}{2}(1+h)$ and order $t+1$.

Delaying the proof of the theorem for a moment, any $X \times 2$ IRDCS can be doubled, since all congruences hit only two rows, and must hit one row at least twice in order to have 3 hits in the IRDCS. An example is the following 10×2 IRDCS

9	3	4	5	3	6	4	3	5	9
6	9	3	4	5	3	6	4	3	5

$\Longrightarrow$

18	2	6	2	8	2	10	2	6	2	12	2	8	2	6	2	10	2	18	2
2	12	2	18	2	6	2	8	2	10	2	6	2	12	2	8	2	6	2	10

.

Otherwise these conditions on the original two-dimensional IRDCS prove to be quite restrictive. Computations have so far failed to find any other IRDCS which can be doubled. Along with this, excluding the $X \times 2$ case these **doubled** IRDCS cannot themselves be **doubled** as the congruence modulo 2 fails to be of the required form, clearly covering positions on more than two rows and columns. Thus there is no way to use this construction to prove the existence of two-dimensional IRDCS for all dimensions.

Proof. For all of the congruences in the original IRDCS this operation doubles the length of the horizontal generators to become $(2m_i/d_i, 0)$. This is clear for congruences that have two hits on some row. If a congruence does not have two hits on some row, then in the original two-dimensional IRDCS, due to the non-collinearity condition, one solution to the congruence, (x_2, y_2) say, must be reached using the congruence's generators as

$$(x_2, y_2) = (x, y) + \beta(x_0, d_i) - \gamma(m_i/d_i, 0),$$

assuming $x_0 > 0$. Here $\beta, \gamma \in \mathbb{N}, x + \beta x_0 \geq X$, and no rows between row x and row x_2 contain any solutions to the congruence. The equivalent transition after adding the congruence modulo 2 will again require γ copies of the horizontal generator, but this must move twice as far, and so the horizontal generator again becomes $(2m_i/d_i, 0)$. The argument is symmetric for $x_0 < 0$ and if $x_0 = 0$ for the non-collinearity condition every row with one solution must have at least two.

For all of the congruences where the second generator was originally (x_0, d), the addition of the congruence modulo 2 has not adjusted d and so we have doubled all of the moduli to give $A^*_{i,j} = 2A_{i,j}$. This construction is valid so long as the collections of $A^*_{(i,j)}$ which were originally all of the solutions to a single congruence remain the solution set to a single congruence.

If some congruence has d_i is even, then if the doubling process produces a solution set to a single congruence the generator pair of $(x_{0,i}, d_i)$ and $(m_i/d_i, 0)$ in the original must become $(2x_{0,i}, d_i)$ and $(2m_i/d_i, 0)$, but then all elements of the generator pair are divisible by 2, and Corollary 7.1 tells us this is invalid. On the other hand if d_i were odd, and the congruence covers positions on two rows distance d_i apart then the vector connecting these rows must be $(x^*_{0,i}, d_i)$ where,

$$(x^*_{0,i}, d_i) = (2x_{0,i} + 1, d_i) \text{ when the initial } y\text{−coordinate is even,}$$
$$(x^*_{0,i}, d_i) = (2x_{0,i} - 1, d_i) \text{ when the initial } y\text{−coordinate is odd.}$$

If the congruence doesn't ever cover positions on two rows distance d_i apart then there are two possibilities:

(1) if the congruence only covers positions on rows an even distance apart, then if the new collection of points is the solution set to a single congruence the generators for this new congruence can be set to the vectors joining any three non-collinear points that form a parallelogram of area $2m_i$, and will only have even elements, associating the congruence to a subgroup of corank 2, or

(2) if the congruence covers positions on two rows an odd distance apart, then as above the connecting vector depends on the initial y-coordinate.

This vector $(x^*_{0,i}, d_i)$ must always produce the same $x^*_{0,i}$ otherwise the new collection of points covered will not be a coset of a subgroup in $\mathbb{Z}^2$. Since $x^*_{0,i}$ varies depending on whether the vector is leaving an odd or even numbered row and since d_i is odd, the only possible way to have a consistent $x^*_{0,i}$ is to cover positions on only two rows and have them be an odd distance apart. Given that all congruences must cover at least 3 positions $m_i/d_i \leq X-1$, d_i odd and so the congruence will hit two rows distance d_i apart. Thus for a congruence with **first position** (x_1, y_1),

$$\max(y_1, (Y-y_1)/2) < d \leq Y - y_1.$$

Similarly, extending by the congruence $x + y \equiv 0 \pmod 2$ horizontally, to give for odd Y

2	$A^*_{(0,Y-1)}$	2	$A^*_{(1,Y-1)}$	$\ldots$	2	$A^*_{(X-1,Y-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
2	$A^*_{(0,2)}$	2	$A^*_{(1,2)}$	$\ldots$	2	$A^*_{(X-1,2)}$
$A^*_{(0,1)}$	2	$A^*_{(1,1)}$	2	$\ldots$	$A^*_{(X-1,1)}$	2
2	$A^*_{(0,0)}$	2	$A^*_{(1,0)}$	$\ldots$	2	$A^*_{(X-1,0)}$

it is not difficult to see that the same conditions on all of the congruences in the original IRDCS are required. Thus, if possible, there exist two different horizontally doubled two-dimensional IRDCS. $\square$

We may similarly attempt to double the IRDCS vertically using congruences modulo 2. Firstly take the generators of the i^{th} congruence to be $(0, m_i/d_i)$ and $(d_i, y_{0,i})$, then extend the IRDCS by $x + y \equiv 1 \pmod 2$ to get, assuming X is even

2	$A^*_{(1,Y-1)}$	2	$A^*_{(3,Y-1)}$	2	$\dots$	2	$A^*_{(X-1,Y-1)}$
$A^*_{(0,Y-1)}$	2	$A^*_{(2,Y-1)}$	2	$A^*_{(4,Y-1)}$	$\dots$	$A^*_{(X-2,Y-1)}$	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$A^*_{(0,1)}$	2	$A^*_{(2,1)}$	2	$A^*_{(4,1)}$	$\dots$	$A^*_{(X-2,1)}$	2
2	$A^*_{(1,0)}$	2	$A^*_{(3,0)}$	2	$\dots$	2	$A^*_{(X-1,0)}$
$A^*_{(0,0)}$	2	$A^*_{(2,0)}$	2	$A^*_{(4,0)}$	$\dots$	$A^*_{(X-2,0)}$	2

where this time the length of the vertical generator was doubled to $(0, 2m_i/d_i)$. In a very similar fashion to the previous cases, d_i must be odd and the second generator becomes $(d_i, y^*_{0,i})$, where

$$(d_i, y^*_{0,i}) = (d_i, 2y_{0,i} + 1) \text{ when the initial } x\text{−coordinate is even,}$$

$$(d_i, y^*_{0,i}) = (d_i, 2y_{0,i} - 1) \text{ when the initial } x\text{−coordinate is odd.}$$

So analogous conditions are required on the generators to the previous case, and the case for the other congruence modulo 2 is almost identical.

Corollary 7.5. *For a two-dimensional IRDCS to be able to be* ***vertically doubled****, all of its congruences must have solutions on only two columns and have generator pair* $(0, m/d)$ *and* (d, y_0) *for* d *odd and* $m/d \le Y - 1$*.*

Extending with one-dimensional IRDCS Take an $X \times Y$ IRDCS with $X = 11$ or $X \ge 17$ and attempt to turn it into an $X \times 2Y$ and an $X \times (2Y \pm 1)$ IRDCS by inserting a one-dimensional IRDCS as in the following construction. Write the one-dimensional IRDCS in its alternate notation as $a_0, a_1, a_2, \dots, a_{X-1}$, then insert this IRDCS into a two-dimensional IRDCS, using the notation of the previous section:

$A_{(0,Y-1)}$	$A_{(1,Y-1)}$	$A_{(2,Y-1)}$	$\dots$	$A_{(X-1,Y-1)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$A_{(0,1)}$	$A_{(1,1)}$	$A_{(2,1)}$	$\dots$	$A_{(X-1,1)}$
$A_{(0,0)}$	$A_{(1,0)}$	$A_{(2,0)}$	$\dots$	$A_{(X-1,0)}$

$\Longrightarrow$

$A_{(0,Y-1)}$	$A_{(1,Y-1)}$	$A_{(2,Y-1)}$	$\dots$	$A_{(X-1,Y-1)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$2a_0$	$2a_1$	$2a_2$	$\dots$	$2a_{X-1}$
$A_{(0,1)}$	$A_{(1,1)}$	$A_{(2,1)}$	$\dots$	$A_{(X-1,1)}$
$2a_0$	$2a_1$	$2a_2$	$\dots$	$2a_{X-1}$
$A_{(0,0)}$	$A_{(1,0)}$	$A_{(2,0)}$	$\dots$	$A_{(X-1,0)}$

This is possibly an $X \times (2Y-1)$ IRDCS. To produce the $X \times 2Y$ and $X \times (2Y+1)$ versions add in extra copies of the one-dimensional IRDCS on the top and bottom of the above alternate notation.

Theorem 7.4. *For an $X \times Y$ two-dimensional IRDCS to be extended with a one-dimensional IRDCS as in this construction, then $X \geq 83$ and the moduli in the two-dimensional IRDCS must be distinct from the set of moduli for some length X odd IRDCS. Also, the generators for the congruences in the two-dimensional IRDCS $(m/d, 0)$ and (x_0, d) must have at least one of x_0 and m/d odd.*

Proof. Clearly the (x_0, d) generators for the new congruences from the one-dimensional IRDCS are always $(0, 2)$ making their modulus $2a_i$. Since congruences must be associated to subgroups of corank 1, and these congruences have second generator $(m, 0)$, m the moduli in the IRDCS, all of these congruences must have odd moduli. The generator pair of the congruences in the original two-dimensional IRDCS become $(m/d, 0)$ and $(x_0, 2d)$ in all cases. Thus these congruences can't have both m/d and x_0 even. The moduli in the original two-dimensional IRDCS must be distinct from those in the odd one-dimensional IRDCS as in both instances the moduli are doubled. □

A similar construction will give a $2X \times Y$ and $(2X \pm 1) \times Y$ IRDCS by inserting the one-dimensional IRDCS vertically. In this case $Y \geq 83$, and the analogous conditions hold.

A slightly different construction would be to insert a one-dimensional IRDCS on the second row above, and then insert its reversal, known to be distinct by Lemma 2.2, two rows above, continuing to alternate between them every second row. In this case the congruence in the one-dimensional IRDCS $x \equiv a \pmod{m}$ will cover positions $(a, 1)$ and $(X-1-a, 3)$ in the two-dimensional IRDCS, thus its generator pair must be $(m, 0)$ and $(X-1-2a, 2)$. Now the congruence must also cover the position $(a, 5)$ so that

$$\begin{aligned}(X-1-a)+(X-1-2a) &\equiv a \pmod{m} \\ 2X-2 &\equiv 4a \pmod{m}.\end{aligned}$$

Hence $X \equiv 1 \pmod{2}$, and then the second generator $(X-1-2a, 2)$ has both elements even, so that m must be odd, the one-dimensional IRDCS must again be odd and have distinct moduli from the original two-dimensional IRDCS. Thus this construction requires the same conditions as Theorem 7.4.

One example which works is to take a length 84 IRDCS with even moduli by doubling a length 42 IRDCS. Combine this IRDCS with a copy of itself shifted by 1 in either direction to get an 83×2 IRDCS. All congruences in this IRDCS have even moduli and will have generator pair $(m, 0)$ and one of $(1, 1)$ and $(m-1, 1)$ depending on the direction of the shift, so that at least one of x_0 and m are odd. A length 83 odd IRDCS can then be used to create 83×4 and 83×5 IRDCS. The equivalent construction works for all dimensions $X \times 2$ where there exists an odd IRDCS of length X.

7.4.1 Open Questions

As seen in Lemma 7.2 for $Y = 2$ there is only one type of two-dimensional IRDCS which is not effectively a one-dimensional IRDCS, namely the IRDCS based on a **near IRDCS**. Call these Type 3 $X \times 2$ IRDCS. We know that those $X \times 2$ IRDCS related to the one-dimensional case exist for all X and also that they can be horizontally doubled. It remains to see whether there exist Type 3 $X \times 2$ IRDCS for all $X \geq k$ for some k.

Any $X \times 2$ IRDCS

$$\begin{array}{|c|c|c|}\hline a_{(0,1)} & \dots & a_{(X-1,1)} \\ \hline a_{(0,0)} & \dots & a_{(X-1,0)} \\ \hline \end{array},$$

can be doubled to two $2X \times 2$ IRDCS

$$\begin{array}{|c|c|c|c|c|}\hline 2 & 2a_{(0,1)} & \dots & 2 & 2a_{(X-1,1)} \\ \hline 2a_{(0,0)} & 2 & \dots & 2a_{(X-1,0)} & 2 \\ \hline \end{array}, \text{ and}$$

$$\begin{array}{|c|c|c|c|c|}\hline 2a_{(0,1)} & 2 & \dots & 2a_{(X-1,1)} & 2 \\ \hline 2 & 2a_{(0,0)} & \dots & 2 & 2a_{(X-1,0)} \\ \hline \end{array}.$$

But these only create $2X \times Y$ IRDCS, not $(2X-1) \times Y$ IRDCS. To do this would require at least one of $a_{(0,1)}, a_{(X-1,1)}, a_{(0,0)}$ and $a_{(X-1,0)}$ to not come from a 3 hit congruence, so that one of the columns on the end of these IRDCS can be removed.

A similar question begs in the more general doubling case. Even if some two-dimensional IRDCS were able to be doubled, the construction only creates a $2X \times Y$ IRDCS, not necessarily a $(2X-1) \times Y$ IRDCS.

7.5 Two-dimensional IRDCS reversals

It is relatively simple to see that any $X \times Y$ IRDCS may be reflected horizontally. This is done by changing the generators of the congruences from $(m/d, 0)$ and (x_0, d) to $(m/d, 0)$ and $(-x_0, d)$. This changes the congruence $ax + by \equiv c \pmod{m}$ to $ax - by \equiv c^* \pmod{m}$, where c^* is determined by the **first point** of the reflected congruence $ax_1 - by_1 \equiv c^* \pmod{m}$. Motivated by these, we give the following definition.

Definition 7.10. The **horizontal reversal** of a two-dimensional IRDCS Λ is the two-dimensional IRDCS generated when Λ is reflected horizontally. Similarly define **vertical reversal**. The **complete reversal** of a two-dimensional IRDCS is the combination of the horizontal and vertical reversals, resulting in a rotation about the centre of the IRDCS through π.

Clearly these reflections are themselves IRDCS. It remains to study whether or not these reflections provide different two-dimensional IRDCS.

Lemma 7.8. *For any length Y one-dimensional odd IRDCS on* $[0, Y-1]$

$$\{S(m_i, a_i) : i = 1, \ldots, t\}$$

there exists two $X \times Y$ IRDCS, $X \geq 3$ which are constructed by congruences with generator pairs

$$(2,0),(1,1) \text{ and } (2,0),(1,m_i), i = 1, \ldots, t.$$

If the length Y odd IRDCS has alternate notation $(d_0, d_1, d_2, \ldots, d_{Y-1})$ *then the two-dimensional IRDCS have alternate notation, for Y odd,*

2	$2d_{Y-1}$	2	$2d_{Y-1}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$
$2d_1$	2	$2d_1$	2	$\ldots$
2	$2d_0$	2	$2d_0$	$\ldots$

, or

$2d_{Y-1}$	2	$2d_{Y-1}$	2	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$
2	$2d_1$	2	$2d_1$	$\ldots$
$2d_0$	2	$2d_0$	2	$\ldots$

,

and similarly for Y even.

Proof. Clearly all position in the IRDCS are covered, and each congruence contains three non-collinear points as they are inherited from odd one-dimensional IRDCS so that each column has at least one copy of every modulus. The congruences used for the first of these IRDCS are

$$x + y \equiv 0 \pmod 2$$
$$m_i x + y \equiv a_i \pmod{2m_i}, \text{for } a_i \text{ odd, and}$$
$$m_i x + y \equiv a_i + m_i \pmod{2m_i}, \text{for } a_i \text{ even,}$$

and may be similarly found for the other IRDCS. $\square$

Lemma 7.9. *For a collection of α length Y one-dimensional odd IRDCS, which may include repetitions, where the j^{th} IRDCS has congruences $x \equiv a_{i,j} \pmod{m_{i,j}}, i = 1, \ldots t_j$, there exists a two-dimensional IRDCS with dimensions $(2X+1) \times Y$ for $2X+1 \geq 2^\alpha + 1$ in which the congruences used have generator pairs*

$$(2^j, 0) \text{ and } (0, m_{i,j}),$$

for $j = 1, 2, \ldots, \alpha - 1$ and $i = 1, \ldots, t_j$, and

$$(2^\alpha, 0) \text{ and } (2^{\alpha-1}, 1), \text{ and}$$

$$(2^\alpha, 0) \text{ and } (2^{\alpha-1}, m_{i,\alpha}).$$

If the j^{th} one-dimensional IRDCS has alternate notation $(d_{0,j}, d_{1,j}, \ldots, d_{Y-1,j})$ then this two-dimensional IRDCS has alternate notation, for Y even,

$\ldots$	$2^\alpha d_{Y-1,\alpha}$	$\ldots$	$2d_{Y-1,1}$	2^α	$2d_{Y-1,1}$	$\ldots$	$2^\alpha d_{Y-1,\alpha}$	$\ldots$
$\ldots$	2^α	$\ldots$	$2d_{Y-2,1}$	$2^\alpha d_{Y-2,\alpha}$	$2d_{Y-2,1}$	$\ldots$	2^α	$\ldots$
	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	
$\ldots$	$2^\alpha d_{3,\alpha}$	$\ldots$	$2d_{3,1}$	2^α	$2d_{3,1}$	$\ldots$	$2^\alpha d_{3,\alpha}$	$\ldots$
$\ldots$	2^α	$\ldots$	$2d_{2,1}$	$2^\alpha d_{2,\alpha}$	$2d_{2,1}$	$\ldots$	2^α	$\ldots$
$\ldots$	$2^\alpha d_{1,\alpha}$	$\ldots$	$2d_{1,1}$	2^α	$2d_{1,1}$	$\ldots$	$2^\alpha d_{1,\alpha}$	$\ldots$
$\ldots$	2^α	$\ldots$	$2d_{0,1}$	$2^\alpha d_{0,\alpha}$	$2d_{0,1}$	$\ldots$	2^α	$\ldots$
$\ldots$	$X - 2^{\alpha-1}$	$\ldots$	$X - 1$	X	$X + 1$	$\ldots$	$X + 2^{\alpha-1}$	$\ldots$

,

and similarly for Y odd, where the next columns around the centre not presented would be $4d_{i,2}$, then $8d_{i,3}$ and so on.

Proof. Since the IRDCS is generated by one-dimensional IRDCS, $2X + 1 > 2^\alpha$ and the IRDCS is symmetric about the middle, all of the congruences clearly have at least 3 non-collinear hits. All congruences have modulus $2^j m_{i,j}$, where the $m_{i,j}$ is odd and $m_{i,j} \neq m_{k,j}$ for all i, k. As such, all moduli in the new construction are distinct. Lastly, again since the $m_{i,j}$ are odd, all congruences are associated to subgroups of $\mathbb{Z}^2$ of corank 1, and we do indeed have a two-dimensional IRDCS. $\square$

In this construction the x-axis is labeled to show that the particular IRDCS is symmetric about the middle. This symmetry is not necessary to have an IRDCS, rather it is only required that on the bottom row of the above alternate notation, the furthest left position covered by the congruence with modulus $2^\alpha d_{1,\alpha}$ be at position $(x, 0)$ such that $x + 2^\alpha < X$, to give 3 non-collinear hits.

Definition 7.11. Call a $(2X + 1) \times Y$ two-dimensional IRDCS **special symmetric** when all of its congruences have generator pairs

$$(2^\alpha, 0) \text{ and } (2^{\alpha-1}, d), \text{ or}$$

$$(2^\alpha, 0) \text{ and } (0, d),$$

for some collection of natural numbers α, with $d > 1$ and all of the congruences cover positions that are horizontally symmetric.

Definition 7.12. Call a two-dimensional IRDCS **super special symmetric** when it is **special symmetric** and there is some congruence with horizontal generator $(2, 0)$ which covers the middle position on some row.

Lemma 7.10. *If there exists a* ***special symmetric*** *but not* ***super special symmetric*** *IRDCS then all congruences with horizontal generator* $(2, 0)$ *can be removed from the IRDCS, adjusting all other congruence generators pairs to*

$$(2^{\alpha-1}, 0) \textit{ and } (2^{\alpha-2}, d), \textit{ or}$$

$$(2^{\alpha-1}, 0) \textit{ and } (0, d).$$

Call this operation ***halving****.*

Note that being **special symmetric** but not **super special symmetric** implies that all congruences with horizontal generator $(2, 0)$ have second generator $(0, d)$, d odd.

Proof. All **special symmetric** but not **super special symmetric** IRDCS may be repeatedly **halved** until they are a **super special symmetric** IRDCS. To see this, rather than in the **alternate notation** present a single row of the IRDCS by placing the non-zero element of a congruence's generator $(m/d, 0)$ in the position that it covers on that row. Call this notation the **horizontal alternate notation**. So some row with middle position covered by $(2^\alpha, 0)$, where for the sake of illustration we assume that $\alpha \geq 3$, looks like

...	2	8	2	4	2	2^α	2	4	2	8	2	...

.

Now assume that all of the congruences in the IRDCS with one generator $(2, 0)$ have second generator $(0, d)$, d odd, and that these congruences never cover the middle position of a row. The IRDCS can then be viewed in **horizontal alternate notation** as

...	2	4	2	$2^{\alpha_{Y-1}}$	2	4	2	...
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
...	2	4	2	2^{α_1}	2	4	2	...
...	2	4	2	2^{α_0}	2	4	2	...

where $\alpha_i > 1$ for all i. **Halving** this IRDCS in the above notation removes all of the 2's, and changes the 4's to 2's and so on with 2^{α_i} becoming $2^{\alpha_i - 1}$. This operation takes a $(4X + 1) \times Y$ IRDCS and a $(4X + 3) \times Y$ IRDCS to a $(2X + 1) \times Y$ IRDCS, where the $+3$ or $+1$ depends on whether the columns on either end of the original IRDCS are all $(2, 0), (0, d)$ congruences or not respectively. Being able to **halve** an IRDCS implies that $Y \geq 83$, as those congruences that were removed must have at least two hits of odd distance apart on each column by the collinearity condition, making these columns odd one-dimensional IRDCS. □

On the other hand, if there is a congruence with generator $(2, 0)$ which covers the middle on some row, in performing the **halving** operation, this row would be completely removed. The case where all rows have a congruence with generator $(2, 0)$ covering the middle position will be considered in what follows, and are those in Lemma 7.8. It remains to consider the case where there is some row which does not have its middle position covered by such a congruence.

The congruences that remain on this row after removing those with generator $(2, 0)$ will have d which may become either even or inconsistent, causing an invalid congruence.

No headway has been made on completely answering this question. Considering a row which is not completely removed directly below one that was removed may be something good to attempt. We are not yet saying that it is never possible to remove these whole rows, but more that it is not possible to easily construct explicit cases where it is possible.

Theorem 7.5. *For a two-dimensional IRDCS to equal its* ***horizontal reversal*** *then it must be either*

- *a $(2X + 1) \times Y$ two-dimensional IRDCS generated by a collection of length Y odd one-dimensional IRDCS as in Lemma 7.9, or*
- *a $(2X + 1) \times Y$ two-dimensional* ***special symmetric*** *IRDCS.*

The IRDCS in Lemma 7.9 are always equal to their ***horizontal reversal****.*

Proof. View the two-dimensional IRDCS as $\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}, \ldots, \mathbf{c_X}$, where $\mathbf{c_i}$ is the i^{th} column of the IRDCS. If X is even, then the reflection process is

$$\mathbf{c_1}, \ldots, \mathbf{c_{\frac{X}{2}-1}}, \mathbf{c_{\frac{X}{2}}}, \ldots, \mathbf{c_X} \longrightarrow \mathbf{c_X}, \ldots, \mathbf{c_{\frac{X}{2}}}, \mathbf{c_{\frac{X}{2}-1}}, \ldots, \mathbf{c_1},$$

and for these to be equal, we must have $\mathbf{c_{\frac{X}{2}}} = \mathbf{c_{\frac{X}{2}+1}}$, which since all moduli are used only once means that the generator $(m/d, 0)$ for all of the congruences on this column must be $(1, 0)$, giving congruences associated to subgroups not of corank 1, and so X must be odd.

If X is odd, the reflection process is

$$\mathbf{c_1}, \ldots, \mathbf{c}_{[\frac{\mathbf{X}}{\mathbf{2}}]}, \mathbf{c}_{[\frac{\mathbf{X}}{\mathbf{2}}]+\mathbf{1}}, \mathbf{c}_{[\frac{\mathbf{X}}{\mathbf{2}}]+\mathbf{2}}, \ldots, \mathbf{c_X} \longrightarrow \mathbf{c_X}, \ldots, \mathbf{c}_{[\frac{\mathbf{X}}{\mathbf{2}}]+\mathbf{2}}, \mathbf{c}_{[\frac{\mathbf{X}}{\mathbf{2}}]+\mathbf{1}}, \mathbf{c}_{[\frac{\mathbf{X}}{\mathbf{2}}]}, \ldots, \mathbf{c_1},$$

so that $\mathbf{c}_{\frac{\mathbf{X-1}}{\mathbf{2}}} = \mathbf{c}_{\frac{\mathbf{X-1}}{\mathbf{2}}+\mathbf{2}}, \mathbf{c}_{\frac{\mathbf{X-1}}{\mathbf{2}}-\mathbf{1}} = \mathbf{c}_{\frac{\mathbf{X-1}}{\mathbf{2}}+\mathbf{3}}$ and so on. Thus the generators $(m/d, 0)$ for all congruences with a solution for $x = \frac{X-1}{2}$ must be $(2, 0)$ and so all columns $1 \leq \frac{X-1}{2} + 2k \leq X$ for integer k will be covered by congruences of this type. The next free column, excluding the middle column, is $\mathbf{c}_{\frac{\mathbf{X-1}}{\mathbf{2}}-\mathbf{1}}$. The generators $(m/d, 0)$ of all congruences covering an element in this column must be either $(2, 0)$, covering the middle element in the same row also, or $(4, 0)$. If some row has generator $(2, 0)$ here, then all of that particular row will be full. If the generator is $(4, 0)$ then so will be all other columns k with $k \equiv \frac{X-1}{2} - 1 \pmod 4$. Continuing this analysis, the next possible free element can only be as close to the middle as $\mathbf{c}_{\frac{\mathbf{X-1}}{\mathbf{2}}-\mathbf{3}}$. The generators in this column can either be $(2, 0)$, if this generator was used to cover column $\frac{X-1}{2} - 1$, $(4, 0)$, also covering the middle position or $(8, 0)$. The next free column will then use horizontal generator of length 2, 4, 8 or 16 and so on, so that every horizontal generator must be of length a power of 2.

For the IRDCS to be symmetric through a horizontal reflection X must be odd and a given row must have **horizontal alternate notation**

$\ldots$	2	α	2	β	2	α	2	γ	2	α	2	β	2	α	2	$\ldots$

where γ is covering the middle position and where $\alpha \in \{2, 4\}, \beta \in \{2, 4, 8\}$ and the next free position at either end will be $\delta \in \{2, 4, 8, 16\}$.

Take some congruence in a horizontally reversible two-dimensional IRDCS with generators $(2^\omega, 0)$ and (x_0, d) and assume that $0 < x_0 < 2^\omega$. This congruence has **first point** (x_1, y_1). If the middle position on row y_1 is not covered by this congruence then two points of distance 2^ω apart which are symmetric about the middle must be covered on row y_1. There must then be some point on row $y_1 + d$ covered by the congruence which is between these two points horizontally, achieved by adding (x_0, d) to the y_1 row.

If this point is not in the centre, then it will have a reflected point on the other side of the middle of this row, but as $x_0 > 0$, these two points will be closer than distance 2^ω apart, a contradiction, and so $x_0 = 2^{\omega-1}$. If the middle point were covered on row y_1 by this congruence then the middle will not be covered on row y_1+d since $0 < x_0 < 2^\omega$, and so two points symmetric about the middle column of distance 2^ω apart and no closer must be covered and thus $x_0 = 2^{\omega-1}$.

Thus all congruences for a horizontally reversible two-dimensional IRDCS must have generator pairs

$$(2^\omega, 0) \text{ and } (2^{\omega-1}, d) \text{ or,}$$
$$(2^\omega, 0) \text{ and } (0, d).$$

As two-dimensional IRDCS only allow congruences which are associated to subgroups of $\mathbb{Z}^2$ of co-rank 1, Corollary 7.1 tells us that we cannot have any number which divides every element of the generators. As such, the **rectangular congruences** (Definition 7.2) in the second case can only occur for $d \geq 3$ odd, and for the first type of congruences with $\omega > 1$, d must be odd, and d is free for $\omega = 1$. The rectangular congruences may or may not hit the middle column and the remaining congruences will cover the middle position on every second row on which they cover some point.

Since every congruence which is not rectangular covers the middle in every second row in which it has a solution, if there are more than 2 congruences with $d = 1$, then at least $\frac{3}{2}Y$ middle positions are covered, a contradiction.

If the congruence that covers the middle position of a row has generator $(2^\omega, 0)$, then it will cover all of the positions that could possibly be covered by generators $(2^{\omega+k}, 0)$ for $k \geq 1$ on that row. Thus, whatever congruence covers the middle position of a row will have the largest $(m/d, 0)$ on that row.

If there are precisely 2 congruences with $d = 1$, let the generator pairs for these congruences be $(2^\alpha, 0), (2^{\alpha-1}, 1)$ and $(2^\beta, 0), (2^{\beta-1}, 1)$. Firstly, we must have $\alpha \neq \beta$, else they will have the same modulus. Now these two congruences must cover the middle position on all rows, so say that 2^α covers the middle of the bottom row. Then since 2^β must also be on that row, we must have $\alpha > \beta$. On the other hand, 2^β will cover the middle position on the second row, and since 2^α will also be on that row, we must have $\alpha < \beta$.

Thus we can have at most one congruence with $d = 1$. Call the horizontal generator of this congruence $(2^\alpha, 0)$. This congruence will cover the middle position of every second row, and since every congruence has odd d, excluding possibly some with generator pair $(2, 0)$ and $(1, d)$, every congruence other than those excluded will hit some row where the 2^α covers the middle. If we do use some congruence with generator pair $(2, 0)$ and $(1, d)$ for d even then this congruence will have solutions in at least two rows, covering the middle position in at least one row, as well as only off-middle positions in at least one row. Since these rows are evenly spaced, the congruence with $d = 1$ will only cover off-middle positions in these rows, and the row where $(2, 0), (1, d)$ covers the middle position forces $\alpha = 1$, so that the congruences will clash in some row that the congruence $(2, 0), (1, d)$ covers only off-middle positions. Thus $(2^\alpha, 0)$ is the largest horizontal generator for this IRDCS, and there can be no congruences with even d.

Moreover, any congruence with generator pair $(2^a, 0)$ and $(2^{a-1}, d)$ will cover the middle position on some row, and since the congruence $(2^\alpha, 0), (2^{\alpha-1}, 1)$ will cover some off-middle position on this row $a \geq \alpha$, but $(2^\alpha, 0)$ is the largest horizontal generator, so that $a = \alpha$ and all congruences which are not rectangular must have generator pair $(2^\alpha, 0)$ and $(2^{\alpha-1}, d)$.

So a horizontally reversible IRDCS with some congruence with $d = 1$ must have 3 columns of distance $2^{\alpha-1}$ apart, including the middle column, covered as

	$\vdots$		$\vdots$		$\vdots$	
$\ldots$	$2^\alpha, 1$	$\ldots$	$2^\alpha, d_2$	$\ldots$	$2^\alpha, 1$	$\ldots$
$\ldots$	$2^\alpha, d_{k_2}$	$\ldots$	$2^\alpha, 1$	$\ldots$	$2^\alpha, d_{k_2}$	$\ldots$
$\ldots$	$2^\alpha, 1$	$\ldots$	$2^\alpha, d_1$	$\ldots$	$2^\alpha, 1$	$\ldots$
$\ldots$	$2^\alpha, d_{k_1}$	$\ldots$	$2^\alpha, 1$	$\ldots$	$2^\alpha, d_{k_1}$	$\ldots$

where we write $2^\alpha, d$ to represent the congruence with generator pair $(2^\alpha, 0)$ and $(2^{\alpha-1}, d)$. These and any other equidistant columns in either direction are the only solutions to these congruences, so that removing $2^\alpha, 1$ and concatenating the two leftmost columns shows that these d_i's must form an odd IRDCS, as they must be odd, each covering at least two positions and covering Y consecutive integers disjointly, so $Y \geq 83$. If $\alpha = 1$ then this is a horizontally reversible two-dimensional IRDCS. If $\alpha \geq 2$ the two columns on either side of the middle are at this stage empty, and so must be covered by congruences with generators of the form $(2, 0)$ and $(0, d)$ for $d > 1$ odd. These congruences will need to cover positions in at least two rows for non-collinearity and so these d's must again form an odd one-dimensional IRDCS of length Y and provide congruences with modulus $2d$. Continuing this for $\alpha \geq 3$ and so on shows that all of the reversible two-dimensional IRDCS containing a congruence with $d = 1$ are the IRDCS from Lemma 7.9. They can be viewed as generalisations of either a single one-dimensional odd IRDCS, or collections of one-dimensional odd IRDCS if we use different length Y odd IRDCS for different column sets in the two-dimensional IRDCS. This two-dimensional IRDCS has heft and order

$$h = \frac{1}{2^\alpha} + \sum_{j=1}^{\alpha} \frac{h_i}{2^j} \to 1, \qquad t = 1 + \sum_{j=1}^{\alpha} t_j,$$

j as in the notation of Lemma 7.9. $\square$

The only remaining case is that where there are no congruences with $d = 1$. These must all be **special symmetric** IRDCS, and so it remains to consider the case where all congruences have $d > 1$ and there is at least one congruence with horizontal generator $(2, 0)$ which covers the middle of some row.

Corollary 7.6. *For a two-dimensional IRDCS to equal its* ***vertical reversal*** *then it must be either*

- *an* $X \times (2Y+1)$ *IRDCS generated by a collection of length* X *odd one-dimensional IRDCS as in the obvious analogue to Lemma 7.9, or*
- *an* $X \times (2Y+1)$ *IRDCS which is the analogue to a* ***special symmetric*** *IRDCS in the obvious way.*

Lemma 7.11. *For a two-dimensional IRDCS to equal its* ***complete reversal*** *it must have both* X *and* Y *even.*

Proof. For an IRDCS to equal its **complete reversal** then since the **complete reversal** operation switches their places, the pair of positions (i, j) and $(X-1-i, Y-1-j)$ must have the same modulus. If X is odd and Y is even, then considering the middle column by taking $i = \frac{X-1}{2}$ and $j = \frac{Y}{2}$ in the above, positions $\left(\frac{X-1}{2}, \frac{Y}{2}\right)$ and $\left(\frac{X-1}{2}, \frac{Y}{2}-1\right)$ must be covered by the same congruence, implying a generator $(0, 1)$, so that the congruence is not of corank 1 and is invalid. Similarly for Y odd and X even, considering the middle row gives a contradiction.

If X and Y are both odd, then presenting the **horizontal alternate notation** of the middle row gives, after similar consideration as in the horizontal reversal proof,

...	2^β	2	2^α	2	$*$	2	2^α	2	2^β	...

,

where $\alpha \in \{1, 2\}, \beta \in \{1, 2, 3\}$ and so on. So all congruences which cover a position in the middle row other than the very middle have one generator $(2^a, 0)$ for some $a \in \mathbb{N}$. Similar analysis on the middle column gives all congruences which cover a position in the middle column other than the very middle have one generator $(0, 2^b)$ for some $a \in \mathbb{N}$. Now considering the next closest rows to the middle

$\ldots$	d	b	$*$	a	c	$\ldots$
$\ldots$	$*$	$*$	$*$	$*$	$*$	$\ldots$
$\ldots$	c	a	$*$	b	d	$\ldots$

,

clearly the labeled letters must be in the same congruence. So that so long as they don't cover the middle position the congruence that covers a must have generator $(2, 2)$ and the congruence that covers b must have generator $(2, -2)$, c must have generator $(4, 2)$ and d must have generator $(4, -2)$. The elements $*$ here are either the middle position, or have already been considered. Continuing this analysis, all congruences excluding the single congruence that covers the middle position will have generators with only even elements, and thus by Corollary 7.1 be invalid. Since only one congruence covers the middle position and it clearly can't cover every position in the IRDCS, there is at least one congruence which has solution set the coset of a subgroup with corank 2, and so X and Y cannot both be odd. $\square$

If X and Y are both even and an $X \times Y$ IRDCS equals its complete reversal, then the IRDCS must have alternate notation

B	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$A^\sharp$
$\boldsymbol{\alpha_2}$	m_2	m_1	$\bar{\boldsymbol{\alpha_1}}$
$\boldsymbol{\alpha_1}$	m_1	m_2	$\bar{\boldsymbol{\alpha_2}}$
A	$\bar{\boldsymbol{\beta_2}}$	$\bar{\boldsymbol{\beta_1}}$	$B^\sharp$

,

where the α_i are row vectors and the β_i are column vectors and where if

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_j),$$

then

$$\bar{\alpha} = (\alpha_j, \ldots, \alpha_2, \alpha_1),$$

and similarly for the column vectors. The A and B are $(X/2-1) \times (Y/2-1)$ matrices and $A^\sharp_{(i,j)} = A_{(X/2-i, Y/2-j)}$ with the equivalent statement for B.

m_1 must be associated to a congruence with generators $(1, 1)$ and $(m_1, 0)$, and we can do similar analysis on the other elements as we move out from the middle. For instance the lowest element of β_1 must have one generator $(1, -3)$. Other than this we cannot categorize this case any further at this stage.

7.5.1 Open Questions

We suspect that we should be able to double all super special symmetric IRDCS, but at the moment we are unable to categorise this. Can it be done? It is certainly only possible if $Y \geq 83$.

Also, can completely reversible IRDCS be further categorised?

7.6 Computing Two-Dimensional IRDCS

7.6.0.1 Structural Qualities to Assist in Computing IRDCS

Earlier lemmas showed two-dimensional IRDCS which are intrinsically related to the one-dimensional case. It remains to see how many IRDCS there are, particularly IRDCS not of these forms or some other form related to the traditional IRDCS, and to produce an algorithm to find all such IRDCS.

The algorithm needs to be told at what point all possible moduli for a congruence with given **first point** have been attempted. As in the one-dimensional case, the algorithm will increment the moduli, in this case as we exhaust all possible congruences with given moduli at given **first point**.

Lemma 7.12. *For an $X \times Y$ IRDCS the largest modulus possible for any congruence class is $(X - 1)(Y - 1)$.*

This is a special case of the following result.

Lemma 7.13. *For an $X \times Y$ IRDCS the largest modulus possible for a congruence class with **first point** (x_0, y_0) is $(X-1)(Y-y_0-1)$.*

Proof. First, the congruence which hits the points $(0, y_0), (X-1, y_0)$ and $(1, Y-1)$ has modulus $(X-1)(Y-y_0-1)$ since it has generators $(X-1, 0)$ and $(1, Y-y_0-1)$.

For any congruence, take three non-collinear points A, B, C in its lattice that determine a parallelogram with no other points in the lattice in its interior. Name these A, B, C in order of increasing x-coordinate, followed by y-coordinate should they have the same x-coordinate. Firstly take the differences $B-A$ and $C-A$ as the generators of this congruence. In the case where two of these points are on a single row, this produces a generator $(m/d, 0)$ with $m/d \leq X-1$. The other generator must be of the form (x_1, d), where $d \leq Y-y_0-1$, so that the modulus $m \leq (X-1)(Y-y_0-1)$.

Similarly for the case where two of these points are on a single column, there is a vertical generator $(0, m/d)$ and second generator (d, y_1), we have $m/d \leq Y-y_0-1$ and $d \leq X-1$, so that $m \leq (X-1)(Y-y_0-1)$.

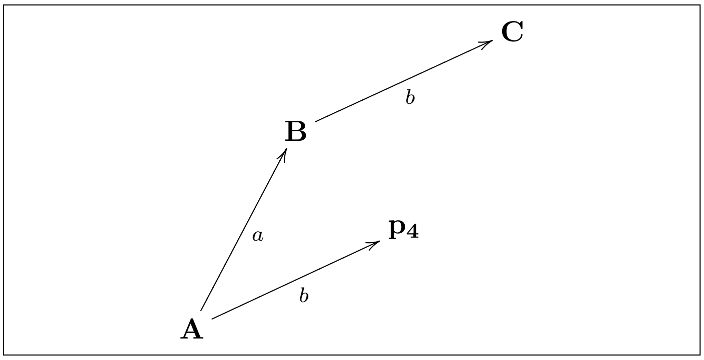
Otherwise this congruence has either B below A and C, or B vertically in between A and C, or B above A and C.

If the B is below A and C, then take the generators as $B-A$ and $C-A$, so that they are (x_1, y_1) and $(-x_2, y_2)$, where $x_1, x_2, y_1, y_2 > 0$. Thus

$$m = \left|\det\begin{pmatrix} x_1 & y_1 \\ -x_2 & y_2 \end{pmatrix}\right| = x_1y_2 + x_2y_1.$$

Since all three points must be inside the box $y_1, y_2 \leq Y-y_0-1$, and $x_1+x_2 \leq X-1$, so that $m \leq (x_1+x_2)(Y-y_0-1) \leq (X-1)(Y-y_0-1)$. An analogous argument works where the B is above A and C, with generators $(-x_1, -y_1)$ and $(x_2, -y_2)$.

If B is vertically in the middle of A and C, then take the generators to be $\mathbf{a} = A-B$ and $\mathbf{b} = B-C$. Adding the generator $\mathbf{b}$ to the A will go to another point still in the IRDCS, since A is below and to the left of B, and generator $\mathbf{b}$ applied to B moves up and to the right and still lands inside the IRDCS. Hence applying the generator **b** to the leftmost point will also land inside the IRDCS, as in the following diagram.



Thus the points A, B, C and p_4 form a parallelogram. The area of this parallelogram is the modulus of the congruence, and the fact that it sits entirely within a box of dimensions $(X-1) \times (Y-y_0-1)$ implies $m \leq (X-1)(Y-y_0-1)$. This argument clearly also holds for the case where the points A, B, C increase in height from right to left, rather than from left to right. □

As the algorithm will be a backtracking algorithm, establishing the position with greatest y coordinate in the two-dimensional IRDCS which may be the **first point** for a congruence will give the point at which the algorithm will always backtrack.

Lemma 7.14. *A congruence in an* $X \times Y$ *IRDCS with first point* (x, y) *must have* $y \leq [Y/2] - 1$.

Proof. Take A, B and C as in the proof of the previous lemma, where $A = (x_a, y_a)$ and so on, and where $y_a, y_b, y_c \geq [Y/2]$. Assume that one of these points is the **first point** of the congruence and thus has the smallest y-coordinate.

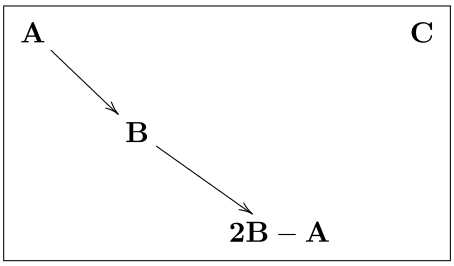
In the first case, if any pair of these three points lie on the same row, then the congruence has generator $(m/d, 0)$ with $m/d \leq X-1$. The second generator can be taken to be (x_0, d) which takes any of these two points to the third point, and where $d \leq Y-1-[Y/2]$. Since $m/d \leq X-1$, this congruence will cover at least one position on every d^{th} row, and so will cover a position on a row at most as low as

$$\left[\frac{Y}{2}\right] - \left(Y - 1 - \left[\frac{Y}{2}\right]\right) = 2\left[\frac{Y}{2}\right] - Y + 1$$
$$\geq 0,$$

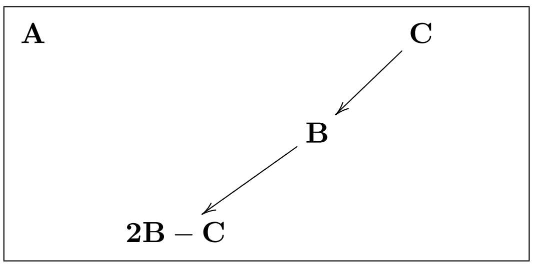
a contradiction, so that the first point of the congruence must be below the $[Y/2]$ row.

Otherwise, three cases remain. Either B is below or above A and C vertically, or B is vertically in between A and C.

First consider the case where B is below both A and C. Either $x_b < [X/2]$ or $x_b \geq [X/2]$. If $x_b < [X/2]$, then heuristically since B is less than halfway across the box and A is further left still, and since the change in height between the two must be less than half the height of the array then B with the addition of the vector $B - A$ will land inside the box. More precisely, since $0 \leq x_a < x_b < [X/2]$ and $[Y/2] \leq y_b < y_a \leq Y - 1$ then $x_b + (x_b - x_a) < X$ and $0 \leq y_b + (y_b - y_a) < y_b$, so that $B + (B - A) \in [0, X) \times [0, Y)$ and is lower than B, a contradiction. This is additionally presented graphically below.

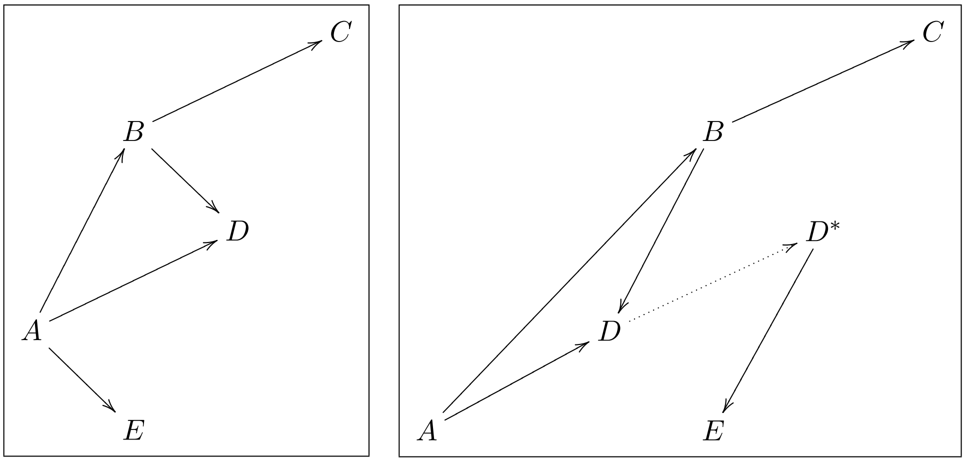


If $x_b \geq [X/2]$, then a similar argument gives $0 \leq y_b + (y_b - y_c) < y_b$, while $x_b < x_c$ gives $0 \leq 2[X/2] - x_c \leq x_b + (x_b - x_c) \leq [X/2] - 1$, so that $B + (B - C) \in [0, X) \times [0, Y)$.



Similarly if B is above both A and C then it is not difficult to show that depending on the x-coordinate of B at least one of $A+(C-B)$ and $C+(A-B)$ will land inside the array and be lower than both A and C.

Lastly if B is in between A and C vertically then without loss of generality assume that A is below B. The slopes of the vectors $B-A$ and $C-B$ must be different. Assume that the slope of $B-A$ is larger than the slope of $C-B$. Apply the vector $C-B$ to A to get the point D in one of the following two figures, depending on whether $x_c - x_b \geq x_b - x_a$ or $x_c - x_b < x_b - x_a$ respectively.



In the first case, the newly created vector $D-B$ is applied to A to generate a point E in the array which is below A. In the second case, first note that the vector $C-B$ may be repeatedly applied to A for so long as the point this produces is to the left of B. Once this produces the first point to the right of B, D^* in the above diagram, apply the vector $D-B$ to this point. This takes the point D^* to the left, but not so far as to fall out of the array on the left, since D^* is to the right of B. It also takes this point down below A, but not out of the array, since D^* is necessarily below B and $y_{d^*} > y_a \geq [Y/2]$. Thus we generate a new point in the array E which is below A. The more precise details for these two cases are very similar to those for the case of B being below both A and C.

The case for the slope of $C-B$ being larger is equivalent, and we are done.

On the other hand there exists a congruence which starts in the row $[Y/2]-1$, namely the congruence lattice covering the points $(0, [Y/2]-1), (X-1, [Y/2]-1)$ and $(1, Y-1)$ so that $y = [Y/2]-1$ is the last row which can contain a congruence's first point. □

If we use the generators $(2, 0)$ and $(1, [Y/2])$ then there is a valid congruence with first point $(x, [Y/2]-1)$ with modulus Y if Y is even, and $Y-1$ if Y is odd. The question remains as to whether any smaller moduli than $2\left[\frac{Y}{2}\right]$ can be used on this row.

If $2 \leq m/d \leq X$, then every row has a position covered where $y \equiv y_0 \pmod{d}$, where the **first point** of the congruence is (x_0, y_0), so that $d \geq \left[\frac{Y}{2}\right]$. Therefore, if $2 \leq m/d \leq X$ then $m \geq 2\left[\frac{Y}{2}\right]$. On the other hand if $m/d > X$, then $m > dX > X > Y$, so that the smallest possible modulus must be $2\left[\frac{Y}{2}\right]$.

Extending this, in attempting to fill a congruence with **first point** (x_0, y_0), if $2 \leq m/d \leq X$ then $d \geq y_0 + 1$, so that $m \geq 2(y_0+1)$. Meanwhile, if $m/d > X$ once again $m > Y \geq 2(y_0+1)$ since $y_0 \leq \left[\frac{Y}{2}\right] - 1$.

This implies that if $Y = 3$, then since $[Y/2]-1 = 0$ all moduli must hit the first row.

7.6.1 An Algorithm for the Two-Dimensional Case

As with the one-dimensional IRDCS problem, an algorithm to find all of the two-dimensional IRDCS for given dimensions and with given conditions will be constructed. The algorithm is again a backtracking algorithm, being based on the one-dimensional IRDCS algorithm, with the adjustments required to take it into the two-dimensional case.

Before introducing the various vectors and variables required and presenting the algorithm in technical detail, the workings of the algorithm will be described.

Take the $X \times Y$ lattice for which all of the possible two-dimensional IRDCS are to be found. Analogous to the one-dimensional case, start by trying to fill the IRDCS from the middle of the bottom row, attempting all congruences with the next unused modulus, and once a valid congruence is found fill it and move to the next free position. Define this next position to be the first empty position the algorithm comes across in firstly cycling around the middle of the current row, as in the one-dimensional case, and once the end is reached, moving up one row, returning to the middle and repeating.

Continue in this fashion until either the lattice is full or a position which cannot be filled by any of the available moduli is reached. In either case, backtrack to the previously filled position, remove the current congruence and try the next available congruence, which may or may not involve choosing a new modulus. Continue with this until the algorithm is at the first position $([X/2], 0)$ and there are no more valid congruences as the maximum modulus has been exceeded. At this point all possible IRDCS have been found and the algorithm terminates.

7.6.1.1 The technical algorithm

Begin by creating an $X \times Y$ vector **lattice** to house the 2D IRDCS and a vector **modusage** of length $(X-1)(Y-1)$ to, as in the one-dimensional case, store whether a modulus is already used. We then need a variable **maxmodulus** to keep track of the largest possible modulus for a congruence with the current first position, **position**, a two-element vector, which stores the current x and y position in the IRDCS, and variables a, b, c, x_0 and d, the coefficients and generator elements for a given congruence as in the congruence $ax + by \equiv c \pmod{m}$ with generators $(m/d, 0)$ and (x_0, d).

The $(X-1)(Y-1) \times 5$ vector **congruences** will store the coefficients a, b, c as well as x_0 and d for the congruence with modulus m in the m^{th} row, with the m^{th} row of this vector empty if the modulus m is not being used. This vector is required to keep track of how to choose the next congruence for the current modulus after backtracking.

The following variables are used to negotiate our way around the IRDCS, both in the filling and the backtracking components of the algorithm. The variables **polarity** and **increment** act in the same way as in the one-dimensional IRDCS algorithm, with **polarity** moving the current position around the lattice to find the next free position by alternating it around the middle of the row, while **increment** makes these shifts larger as the algorithm gets away from the middle of a row, so that it doesn't double over any positions. The algorithm will reset these as it moves up a row while filling the lattice. Associated to these the backtracking portion of the algorithm will use a vector **primary** of dimension $X \times Y$, which will store whether the particular hit of a congruence at that position is the first position of that congruence covered by the algorithm, where this will be the position closest to the middle horizontally, rather than furthest left as in the definition of **first point**.

As in the one-dimensional case the variable **clash** will track whether a congruence will clash with the current lattice and **issolution** will change to **false** when there are no more IRDCS for this dimension.

Lastly the variable **collinear counter** will be used to ensure that each congruence hits the lattice in three non-collinear points, **flag nextmodulus** will track when all possible congruences for the current modulus have been attempted, and **first height** will calculate the y-coordinate of the **first point** of the congruence, and be used to check that it matches the current position in the algorithm.

If a congruence fits, it will be stored in the congruence storage lattice **congruences**, and the algorithm will move on to the next free position to try to fill a new congruence, while adjusting the polarity and increment variables. If, while choosing the next congruence, the modulus exceeds the maximum possible modulus the algorithm backtracks, in a very similar fashion as in the one-dimensional case, using the **congruences** vector to keep track of the next congruence to attempt.

Backtracking will occasionally force the algorithm to move down a row. Whether the last position cycled through on the lower row was on the left or right-hand end will determine the values of *increment* and *polarity* required. Given that the first position filled is $[X/2]$ and *polarity* on any given row starts at (-1), for X odd the algorithm starts on the very middle position and thus finishes the left side of the row first, so when the algorithm moves down a row, it resets to $x = X - 1$ with $polarity = -1$ and $increment = X$. On the other hand for X even, the algorithm started at the right of the two central positions, and so in filling finishes the right side of the row first. Thus it resets to $x = 0$ with $polarity = 1$ and $increment = X$.

The algorithm refers to $position[0]$ as the x-coordinate of *position*, and similarly $position[1]$ for the y-coordinate.

Any bold sentences in the algorithm prefaced with a '%' will refer to a comment. The algorithm is written in a mixture of pseudocode and plain English, in an attempt to highlight its key features.

% Initialisation

Input X and Y, the dimensions in the search for an X by Y IRDCS.

Set all of the entries of vectors **lattice** and **congruences** to 1, **primary** to **true**.

Set **modusage**$[2, 3, \ldots, maxmodulus]$ to **false** and **modusage**$[1]$ to **true**.

Set $maxmodulus = (X - 1)(Y - 1)$, $position[0] = [X/2], position[1] = 0$,

$polarity = -1, increment = 1, finished = false$ and $clash = false$.

Set **collinear counter** to 1 and initialise the constants for the congruence to $a = 1, b = 1, c = 0, x_0 = 1, d = 1$.

Set **first height** $= -1$ and **flag nextmodulus** $= true$.

% Begin main loop

while issolution do

  $maxmodulus := (X - 1)(Y - 1 - position[1])$

  valid congruence $:= false$

  % Choose next modulus and associated congruence

  while valid congruence is **false do**

    if flag nextmodulus is **true then**

```
    Iterate to next available modulus and set m := modulus
    if m > max modulus then
      Break from while loop
    end if
    set flag nextmodulus = false
    reset a = 1, b = 1, c = 0
  end if
  while The congruence does not cover the current position do
    while c < m and the congruence does not cover the current position do
      c := c + 1
    end while
    if c = m then
      c := 0
      while b < m and (gcd(a, b) > 1 OR the congruence is a repeat of a previous
      congruence) do
        b := b + 1
      end while
    end if
    if b = m then
      b := 1
      a := a + 1
      if a = m then
        set flag nextmodulus = true
        Break from while loop
      end if
    end if
  end while
```

if **flag nextmodulus** $= false$ **then**

Find the first position the congruence hits, and set **first height** to the y-coordinate

if $first\ height = position[1]$ **then**

Calculate the generators (x_0, d) and $(m/d, 0)$

valid congruence $:= true$

end if

end if

end while

if the modulus is less than or equal to **max modulus** **then**

% Feasible modulus found - enter while checking for clashes and non-collinear hits

Fill the congruence using the generators, while checking that there are no clashes and at least 3 non-collinear hits using the variable **collinear counter**

Set $primary$ to $false$ for all but the original position of the congruence

if There is a clash, only two hits, or the points are all collinear **then**

flag nextmodulus $= false$

Erase all filled values

else

% valid congruence found

flag nextmodulus $= true$

% Store the congruence coefficients and generators in the congruences vector

$congruences[m][0] := a, congruences[m][1] := b, congruences[m][2] := c,$

$congruences[m][3] := x_0$ and $congruences[m][4] := d$.

while lattice at the current **position** is already filled **do**

% Iterate the position using polarity and increment to find the

next free position

$position[0] := position[0] + polarity \times increment$

$polarity := polarity \times (-1), increment := increment + 1$

if $position[0] < 0$ **OR** $position[0] \geq X$ **then**

$position[1] := position[1] + 1$

$position[0] := \left[\frac{X}{2}\right]$

$polarity = -1, increment = 1$

if $position[1] = Y$ **then**

% We have a two-dimensional IRDCS

Calculate **heft** and **order** and output the IRDCS

Backtrack: remove the most recently filled congruence and clear the *congruences* vector

Backtrack: remove the second most recently filled congruence, setting a, b and c as in the *congruences* vector and then clearing that row of the *congruences* vector

% This is so that we can iterate for the next available congruence at this position

end if

end if

end while

end if

else

% Modulus has exceeded maximum possible value, backtrack

while $primary[position] = false$ **do**

if $position[0] = \left[\frac{X}{2}\right]$ **then**

$position[1] := position[1] - 1$

if X is even **then**

$position[0] = 0, polarity = 1, increment = X$

else

$position[0] = X - 1, polarity = -1, increment = X$

end if

if $position[1] = -1$ **then**

Break from the while loop

end if

else

$polarity := polarity \times (-1), increment := increment - 1$

$position[0] := position[0] - polarity \times increment$

end if

end while

if $position[1] = -1$ **then**

% there are no more two-dimensional IRDCS

Set **issolution** to $false$

else

Remove the congruence at the current position

Set a, b, c to the values from the *congruences* vector for the removed congruence, from which to iterate

Clear the row of the *congruences* vector for this congruence

end if

end if

end while

It is possible to adjust this algorithm to search for two-dimensional IRDCS with specific properties. Most often this can be achieved by adjusting either the allowable moduli, or the specific allowable congruences, by enforcing specific generator pairs, or both. For instance, one might want to search for reversible or double-able two-dimensional

IRDCS, or for examples that are explored in what follows.

There may also be more efficient ways to calculate two-dimensional IRDCS. These may be in the form of the dancing links algorithm, or by adjusting the algorithm presented above. One possibility is to alter the starting point of the algorithm from $(X/2, 0)$ to $(X/2, Y/2)$ and then adjust the way that the algorithm works by introducing a second pair of variables $increment_y$ and $polarity_y$. These variables would function as *increment* and *polarity*, once a particular row is finished move to a new row by taking $y = y + increment_y \times polarity_y$, and then adjust these two variables by increasing $increment_y$ and adjusting the sign of $polarity_y$. We have not implemented any of these approaches.

7.7 A miscellaneous two-dimensional IRDCS question

Question. Can an $X \times Y$ IRDCS be translated to a length XY IRDCS, where the transformation takes the rows of the two-dimensional IRDCS and places them next to one another, the transformation in their respective **alternate notation's** being

$$\boxed{\begin{matrix} \mathbf{r_Y} \\ \vdots \\ \mathbf{r_2} \\ \mathbf{r_1} \end{matrix}} \implies \mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_Y}?$$

If so, what are the necessary and sufficient conditions for this transformation to produce an IRDCS?

Note that if this is possible, the one-dimensional IRDCS will have **minhits** 3 or greater, so that $XY > 105$ as seen in Chapter 4. This was the original motivation for asking this question.

Assume that for all congruences $x_0 \geq 0$, then the following holds.

Theorem 7.6. *An $X \times Y$ IRDCS may be transformed into a length XY IRDCS by the above transformation only if all congruences in the two dimensional IRDCS with modulus m have generators $(x_0, 1)$ and $(m, 0)$, where $X + x_0 = \lambda m, \lambda \in \mathbb{N}$. For $m \geq X, \lambda = 1$ while for $m \leq X - 1, \lambda$ is the maximum number of positions covered in a single row of the two-dimensional IRDCS by that congruence.*

Proof. Call the two-dimensional IRDCS α and for a congruence in α with modulus m call the modulus in the one-dimensional IRDCS m_1. We study α by analysing the possible generators of the congruences.

In the first instance if α has a congruence with generators $(m, 0)$ and $(x_0, 1)$, where $m \geq X$ and where the congruence hits two consecutive rows at some point then assume without loss of generality that $x_0 > 0$, since if $x_0 < 0$ it may be replaced with $x_0 + m$. If at some stage this congruence covers positions in two sequential rows, these positions must appear as either

								✓	
			✓						

or

			✓						
								✓	

.

The first case implies a modulus in the one-dimensional IRDCS of $m_1 = X + x_0$ while in the second case $m_1 < X$. In either case 3 non-collinear hits are required which without loss of generality may be assumed to occur above the section of consecutive rows with hits, the argument otherwise being symmetric. In the first diagram above if there is no row skipped before the third position and this position appears to the left as in:

	✓								
								✓	
					✓				

then the second transition here forces $m_1 < X$, a contradiction. If two or more rows are skipped then $m_1 > 2X$, also a contradiction since here clearly $x_0 < X$. If precisely one row is skipped, then position (x_1, y_1) goes to position $(x_1 + 2x_0 - km, y_1 + 2)$ for some $k \in \mathbb{N}$. This transition implies

$$\begin{aligned} m_1 &= (X - x_1) + X + x_1 + 2x_0 - km \\ &= 2X + 2x_0 - km, \end{aligned}$$

which must be $X + x_0$, and so $m_1 = X + x_0 \equiv 0 \pmod{m}$ is required, where $x_0 < X$ so that $m = X + x_0$.

Similar analysis of the second case shows that this scenario cannot generate any valid congruences, since we have already shown that $m_1 < X$.

If for this type of congruence two consecutive rows are never hit then since hits cannot be collinear and consecutive hits cannot appear one above the other, at some point in the IRDCS there must be a section where 3 hits appear as either

	✓								
								✓	
					✓				

or the horizontal reflection of this scenario, where skipped rows have been omitted in our illustration and where the positions covered needn't appear in this precise arrangement, but merely non-collinear. In this case, say that between the first two hits A rows are skipped and between the second two hits B rows are skipped. The first transition implies the modulus in the one-dimensional IRDCS satisfies $AX < m_1 < (A+1)X$, where the inequalities are strict since the positions covered by the congruence in this arrangement cannot appear directly above one another. Similarly the transition from the middle row to the top row implies $(B-1)X < m_1 < BX$, so that $A + 1 = B$. The same analysis on the reflected case gives $A = B + 1$.

Now the movement from the bottom row to the middle row takes position (x_1, y_1)

to position $(x_1 + (A+1)x_0 - km, y_1 + A + 1)$, which implies

$$\begin{aligned} m_1 &= (x_1 + (A+1)x_0 - km - x_1) + (A+1)X \\ &= (A+1)(X + x_0) - km. \end{aligned}$$

The movement to the top row takes the second position above to $(x_1 + (A+1)x_0 - km + ((B+1)x_0 - lm), y_1 + A + 1 + B + 1)$ which implies

$$\begin{aligned} m_1 &= (B+1)x_0 - lm + (B+1)X \\ &= (A+2)(X + x_0) - lm. \end{aligned}$$

Forcing these to be equal takes $X + x_0 = (l-k)m$ so that $X + x_0 = m$, and thus $m_1 = (A+1-k)m$.

If α has generators $(x_0, 1)$ and $(m, 0)$ with $m \leq X-1$ then the condition $m \leq X-1$ implies that every row in the IRDCS must be hit by this congruence. In the first case, if some row in the two-dimensional IRDCS is hit two or more times by the congruence, then the modulus of the one-dimensional IRDCS must be $m_1 = m$. The transition from one row to the next as in

	✓										
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for the bottom position having x coordinate x_1 must give

$$m_1 = X - x_1 + \{(x_1 + x_0) \pmod{m}\} = X - x_1 + x_1 + x_0 - km,$$

for some $k \in \mathbb{Z}$, which implies that

$$(k+1)m = X + x_0,$$

where $k+1$ is the maximum number of times that the congruence hits any row.

In the other case, if no row is hit twice, then the fact that not all of the hits can be collinear means that at some stage there must be three rows that look either like

	✓										
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						✓					

or the above reflected horizontally. The third hit here needn't be to the left of the first, or in fact to the left of the second hit, but the points must be non-collinear, and a similar argument to what follows will still hold. In the illustrated case the transition from the bottom row to the middle row implies that the one-dimensional IRDCS has modulus larger than X, while the middle row to the top row implies modulus smaller than X, a contradiction.

Last of all if α has generators (x_0, d) and $(m/d, 0)$ with $d > 1$, apply the methods for the case of $d = 1$ with some minor adjustments to see the following.

- If $m/d \geq X$ study a section of the two-dimensional IRDCS with 3 non-collinear hits, following the same arguments as previously, and since $d > 1$ we cannot have any valid moduli.
- If $m/d < X$ where some row is hit twice, then the modulus of the one-dimensional IRDCS must be m/d, but given that $d - 1 \geq 1$ rows are skipped the modulus must also satisfy $m_1 > X$, a contradiction.
- If $m/d < X$ and no row is hit twice, then as in the equivalent $(x_0, 1), (m, 0)$ case we take three non-collinear hits and perform a similar analysis to show that this will not work.

Thus there can be no congruences with $d > 1$. $\square$

7.8 Two-dimensional Heft

This section will be devoted to finding bounds for the heft of a two-dimensional IRDCS. We begin by calculating how many times a congruence satisfying certain conditions will intersect the box $[0, X) \times [0, Y) \cap \mathbb{Z}^2$ of our IRDCS.

First of all, fix X, Y and the **first point** (x_*, y_*), where the first point of a given congruence is defined as Definition 7.7 to be the position of the congruence which is on the lowest row hit and furthest left on that row. We will now study the various possible congruences. For the entirety of the analysis, all of the variables are taken to be non-negative integers, and we will study the congruence $ax + by \equiv c \pmod{m}$.

7.8.1 If $X \equiv 0 \pmod{m}$ or $Y \equiv 0 \pmod{m}$

We take the generators for this congruence to be $(m/d, 0)$ and (x_0, d) in the first case, and $(0, m/d)$ and (d, x_0) in the second. We will focus only on the first case, and the second case follows in a similar fashion.

If $d = 1$, then we have the generators $(m, 0)$ and $(x_0, 1)$. Every row is hit precisely X/m times, as the hits on a given row form a single congruence modulo m and X is a multiple of m. Thus, the number of hits inside the box is $\frac{XY}{m}$.

If $d > 1$, then we have the generators $(m/d, 0)$ and (x_0, d). If $Y \equiv 0 \pmod{d}$, then since the congruence hits every d^{th} row it will hit precisely $\frac{Y}{d}$ rows, and will hit each row $\frac{X}{m/d}$ times, again since X is a multiple of m. Thus the number of hits inside the box is $\frac{Y}{d} \cdot \frac{X}{m/d} = \frac{XY}{m}$.

If $d > 1$ and $Y \equiv n \pmod{d}$, where $n \neq 0$ is the residue of Y modulo d, then let

$Y = ld + n$. This congruence will hit l rows if $y_* \geq n$ and $l+1$ rows if $y_* \leq n-1$. Thus

$$\begin{aligned}
\text{number of hits} &= \begin{cases} \frac{X}{m/d}(l+1) & \text{if } y_* \leq n-1, \\ \frac{X}{m/d}l & \text{if } y_* \geq n. \end{cases} \\
&= \begin{cases} \frac{XY}{m} - \frac{n-d}{m} & \text{if } y_* \leq n-1, \\ \frac{XY}{m} - \frac{n}{m} & \text{if } y_* \geq n. \end{cases} \\
&= \frac{XY}{m} - \frac{n-\alpha}{m},
\end{aligned}$$

where $\alpha = 0$ if $y_* \geq n$ and $\alpha = d$ otherwise.

7.8.2 If $X \not\equiv 0 \pmod{m}$ and $Y \not\equiv 0 \pmod{m}$

7.8.2.1 Case 1: $X > m$, generators $(m, 0)$ and $(x_0, 1)$ where $gcd(m, x_0) = 1$

If we have the generators $(m, 0)$ and $(x_0, 1)$ where $gcd(m, x_0) = 1$, begin by letting $X = mu + v$ and $Y = mp + q$, where $u, v, q > 0$. Since $gcd(m, x_0) = 1$, in each collection of m consecutive rows our starting points on the rows, namely the leftmost points hit, will form a complete set of residues modulo m. As such, in every m consecutive rows we will hit v of those rows $u+1$ times and $m-v$ of them u times. Thus, the number of hits on the first mp rows equals

$$p\big(v(u+1) + (m-v)u\big) = p(mu+v) = pX.$$

On the last q rows, we will hit each row either u or $u+1$ times, depending on whether the first position is smaller than v. So the number of hits on the last q rows equals $qu + \beta$, where $\beta \in \{\max(0, q-(m-v)), \max(0, q-(m-v))+1, \ldots, \min\{v, q\}\}$ counts the number of rows which are hit $u+1$ times. Thus the total number of hits equals

$$\begin{aligned} pX + qu + \beta &= X\frac{Y-q}{m} + qu + \beta \\ &= \frac{XY}{m} + q(u - X/m) + \beta \\ &= \frac{XY}{m} - \frac{vq}{m} + \beta. \end{aligned}$$

7.8.2.2 Case 2: $X > m$, generators $(m, 0)$ and $(x_0, 1)$ where $gcd(m, x_0) = k > 1$

Let $X = mu + kv_1 + v_2$ where $0 \leq kv_1 < m, 0 \leq v_2 < k$ and $kv_1 + v_2 > 0$, and let $Y = \frac{m}{k}p + q$, where $0 \leq q < \frac{m}{k}$ and $Y \not\equiv 0 \pmod{m}$.

The first position hit by the modulus is $(x_*, 0)$ so that all of the furthest left points hit on the rows will be of the form $x_* + kl \in [0, m), l \in \mathbb{Z}$, and we will repeat the set of first hits every $\frac{m}{k}$ rows. This collection will form an incomplete set of residues modulo m.

The number of hits on a given row will be $u+1$ if the first position is less than $kv_1 + v_2$, and will be u otherwise. Thus in any $\frac{m}{k}$ rows there will be $v_1 + 1$ rows with $u+1$ hits if $x^* \equiv x_* \pmod{k} < v_2$, and there will be v_1 such rows otherwise.

Thus, the number of hits on any $\frac{m}{k}$ rows will be

$$(u+1)v_1 + u\bigg(\frac{m}{k} - v_1\bigg) + \gamma_1,$$

where $\gamma_1 \in \{0, 1\}$ and is 1 if and only if $x^* < v_2$. So the number of hits on the first $\frac{m}{k}p$ rows is

$$\begin{aligned} p\left((u+1)v_1 + u\left(\frac{m}{k} - v_1\right) + \gamma_1\right) &= p\left(\frac{m}{k}u + v_1 + \gamma_1\right) \\ &= p\left(\frac{X - v_2}{k} + \gamma_1\right). \end{aligned}$$

Finally the number of hits on the last q rows is $qu + \gamma_2$, where γ_2 counts the number of these last q rows which are hit $u+1$ times. Thus γ_2 runs from $\max\big(0, q - \big(\frac{m}{k} - v_1 - \gamma_1\big)\big)$ to $\min(v_1 + \gamma_1, q)$. Thus in calculating the total number of hits by the congruence inside the box we get

$$\begin{aligned}
\#\text{ hits} &= p\left(\frac{X-v_2}{k}+\gamma_1\right)+uq+\gamma_2 \\
&= \frac{p}{k}(X-v_2)+p\gamma_1+uq+\gamma_2 \\
&= \frac{1}{m}(Y-q)(X-v_2)+p\gamma_1+uq+\gamma_2 \\
&= \frac{XY}{m}-\frac{qX}{m}-\frac{v_2Y}{m}+\frac{qv_2}{m}+p\gamma_1+\frac{q(X-kv_1-v_2)}{m}+\gamma_2 \\
&= \frac{XY}{m}-\frac{v_2Y}{m}-\frac{qkv_1}{m}+\frac{k(Y-q)}{m}\gamma_1+\gamma_2
\end{aligned}$$

where $\gamma_2 \in \{\max\left(0, q-\left(\frac{m}{k}-v_1-\gamma_1\right)\right), \ldots, \min(v_1+\gamma_1, q)\}$ and $\gamma_1 \in \{0,1\}, \gamma_1 = 1$ if and only if $x^* < v_2$.

7.8.2.3 Case 3: $X > m/d$, generators $(m/d, 0)$ and (x_0, d) where $d > 1$ and $gcd(m/d, x_0) = 1$

Let $X = \frac{m}{d}u + v$ and $Y = mp + dq_1 + q_2$, where $0 \leq v < \frac{m}{d}$, $X \not\equiv 0 \pmod{m}$ and $0 \leq dq_1 < m, 0 \leq q_2 < d$ with $dq_1 + q_2 > 0$.

Since $gcd(m/d, x_0) = 1$ in every collection of $\frac{m}{d}$ consecutively hit rows (i.e. every m rows) our starting points will take all of the residues modulo $\frac{m}{d}$. Thus in any m rows, there will be v rows which are hit $u+1$ times, and $\frac{m}{d} - v$ rows hit u times.

Thus, the number of hits in the first mp rows is

$$p\bigg(v(u+1)+\left(\frac{m}{d}-v\right)u\bigg) = p\bigg(\frac{m}{d}u+v\bigg) = pX.$$

On the next dq_1 rows there will be $uq_1+\delta_1$ hits, where δ_1 runs from $\max\left(q_1-(m/d-v), 0\right)$ to $\min(q_1, v)$ and represents the number of rows that are hit $u+1$ times. Lastly, the number of hits on the last q_2 rows is $\delta_2 \in \{0, u, u+1\}$, and $\delta_2 \neq 0$ if and only if $y_* < q_2$.

Thus for the total number of hits we have

$$\begin{aligned}
\#\text{ hits} &= pX + uq_1 + \delta_1 + \delta_2 \\
&= \frac{1}{m}\Big(Y - (dq_1 + q_2)\Big)X + \frac{X - v}{m}dq_1 + \delta_1 + \delta_2 \\
&= \frac{XY}{m} - \frac{X}{m}q_2 - \frac{vdq_1}{m} + \delta_1 + \delta_2.
\end{aligned}$$

7.8.2.4 Case 4: $X > m/d$ generators $(m/d, 0)$ and (x_0, d) where $d > 1$ and $gcd(m/d, x_0) = k > 1$

In this case we let $X = \frac{m}{d}u + kv_1 + v_2$ where $0 \leq kv_1 < \frac{m}{d}, 0 \leq v_2 < k$ and $X \not\equiv 0 \pmod m$, and let $Y = \frac{m}{k}p + dq_1 + q_2$ where $0 \leq dq_1 < \frac{m}{k}$ and $0 \leq q_2 < d$ and $Y \not\equiv 0 \pmod m$. Also, since we are only interested in lattices which correspond to subgroups of corank 1, we must have $gcd(k, d) = 1$.

The first position hit on any given row will be of the form $(x_* + ki, y_* + dj)$, where $i \in \mathbb{Z}, j \in \mathbb{N}_0$ and $x_* + ki \in [0, m/d)$. We will repeat this collection of initial x-coordinates every $\frac{m}{kd}$ rows hit, so every $\frac{m}{k}$ rows. These x-coordinates will not take all residues modulo $\frac{m}{d}$, but $\frac{1}{k}$ of them forming an incomplete set of residues modulo $\frac{m}{d}$. The number of hits on a given row is $u+1$ if $x_* + ki < kv_1 + v_2$ and will be u otherwise. Thus in every $\frac{m}{k}$ rows, there will be either v_1 or $v_1 + 1$ rows with $u+1$ hits, depending as in case 2 on whether $x^* \equiv x_* \pmod k < v_2$. As such the number of hits on the first $\frac{m}{k}p$ rows is

$$\begin{aligned}
\#\text{ hits} &= p\left(v_1(u+1) + \left(\frac{m}{kd} - v_1\right)u + \phi_1\right) \\
&= \frac{p}{k}\left(\frac{m}{d}u + kv_1 + k\phi_1\right) \\
&= \frac{1}{m}\big(Y - (dq_1 + q_2)\big)(X - v_2 + k\phi_1),
\end{aligned}$$

where $\phi_1 \in \{0, 1\}$ and $\phi_1 = 1$ if and only if $x^* < v_2$.

The hits on the next dq_1 rows will be $uq_1 + \phi_2$ where ϕ_2 runs from $\max(q_1 - (\frac{m}{kd} - v_1), 0)$ to $\min(q_1, v_1 + \phi_1)$ and represents the number of these rows that are hit $u + 1$

times. The hits on the final q_2 rows will be $\phi_3 \in \{0, u, u+\phi_1\}$ and $\phi_3 \neq 0$ if and only if $y_* < q_2$. Thus, the total number of hits will be

$$\begin{aligned}
\#\text{ hits} &= \frac{1}{m}\big(Y - (dq_1 + q_2)\big)(X - v_2 + k\phi_1) + uq_1 + \phi_2 + \phi_3 \\
&= \frac{XY}{m} - \left(\frac{X - v_2}{m}\right)(dq_1 + q_2) - \frac{v_2 Y}{m} + \frac{k\phi_1}{m}\big(Y - (dq_1 + q_2)\big) \\
&\quad + dq_1\left(\frac{X - kv_1 - v_2}{m}\right) + \phi_2 + \phi_3 \\
&= \frac{XY}{m} - \frac{X - v_2}{m} q_2 - \frac{v_2 Y}{m} - \frac{kv_1 dq_1}{m} + p\phi_1 + \phi_2 + \phi_3
\end{aligned}$$

7.8.2.5 Case 5: $X < m$, generators $(m, 0), (x_0, 1)$

In this case, not all rows are necessarily hit by the congruence. We can do no better than taking #hits $= \theta$, where θ represents the number of rows hits by the congruence, since if a row is hit, it can be hit only once. Thus $\theta \in \{3, 4, \ldots, \min(X, Y)\}$. However, since we know that any $X \times Y$ IRDCS is equivalent to a $Y \times X$ IRDCS (see Lemma 7.3), we may assume that $X \geq Y$. Thus $\theta \in \{3, 4, \ldots Y\}$. An example satisfying $\theta = Y$ is the congruence $x - 2y \equiv 6 \pmod{10}$ in a 9×3 IRDCS.

7.8.2.6 Case 6: $X < m/d$, generators $(m/d, 0), (x_0, d)$

As in case 5, not all rows are necessarily hit, so that #hits $= \zeta$, where $\zeta \in \left\{3, 4, \ldots, \left[\frac{Y - y_*}{d}\right]\right\}$.

7.8.3 Calculating Heft

Before we can calculate the heft we need some more notation. A complete IRDCS has order t, the following notation will be used to count how many of these congruences in the IRDCS satisfy the particular cases above.

Definition 7.13. Define t_1 to be the number of congruences where $X \equiv 0 \pmod{m}$ and $d = 1$, or where $X \equiv 0 \pmod{m}, d > 1$ and $Y \equiv 0 \pmod{d}$ and define t_2 to be the number of congruences with $X \equiv 0 \pmod{m}, d > 1$ and $Y \not\equiv 0 \pmod{d}$. Also define t_{2+i} to be the number of congruences satisfying, as in the notation above, the conditions for case i, where $i = 1, 2, \ldots, 6$.

By counting the number of hits of the various congruences, we get

$$\begin{aligned} XY = &\sum_{i_1=1}^{t_1} \frac{XY}{m_{i_1}} + \sum_{i_2=1}^{t_2} \frac{XY - (n_{i_2} - \alpha_{i_2})}{m_{i_2}} + \sum_{i_3=1}^{t_3} \left(\frac{XY}{m_{i_3}} - \frac{v_{i_3} q_{i_3}}{m_{i_3}} + \beta_{i_3} \right) \\ &+ \sum_{i_4=1}^{t_4} \left(\frac{XY}{m_{i_4}} - \frac{Y}{m_{i_4}} v_{2,i_4} - \frac{q_{2,i_4} k_{i_4} v_{1,i_4}}{m_{i_4}} + \frac{k_{i_4} \gamma_{1,i_4}}{m_{i_4}} \left(Y - q_{2,i_4} \right) + \gamma_{2,i_4} \right) \\ &+ \sum_{i_5=1}^{t_5} \left(\frac{XY}{m_{i_5}} - \frac{X}{m_{i_5}} q_{2,i_4} - \frac{v_{i_5} d_{i_5} q_{1,i_4}}{m_{i_5}} + \delta_{1,i_4} + \delta_{2,i_4} \right) \\ &+ \sum_{i_6=1}^{t_6} \left(\frac{(X - v_{2,i_6})(Y - q_{3,i_6})}{m_{i_6}} - \frac{k_{i_6} v_{1,i_6} d q_{2,i_6}}{m_{i_6}} + \phi_{1,i_6} k_{i_6} \left(p_{i_6} + \frac{q_{1,i_6}}{k_{i_6}} \right) + \phi_{2,i_6} + \phi_{3,i_6} \right) \\ &+ \sum_{i_7=1}^{t_7} \theta_{i_7} + \sum_{i_8=1}^{t_8} \zeta_{i_8}, \end{aligned} \tag{7.1}$$

where the indexing works as, for example, assigning q_{2,i_4} as the value of q_2 for the congruence with modulus m_{i_4}. So for all terms with a dual index the first index refers to its initial usage, while its second index refers to a particular congruence. We may from time to time compact this notation by writing q_2 rather than q_{2,i_4} where the meaning is clear.

7.8.3.1 The lower bound

We now study the lower bound for heft. To do this we first analyse the more complicated terms in equation (7.1).

7.8.3.2 The third sum

In the third sum, we have

$$-\frac{v_{i_3}q_{i_3}}{m_{i_3}} + \beta_{i_3} \leq -\frac{v_{i_3}q_{i_3}}{m_{i_3}} + \min(v_{i_3}, q_{i_3}).$$

Due to the symmetry of this expression, and the fact that $1 \leq v_{i_3}, q_{i_3} \leq m-1$, we may assume $q_{i_3} \leq v_{i_3}$, so that we are bounding

$$\frac{q_{i_3}}{m_{i_3}}(m_{i_3} - v_{i_3}).$$

If we let $q_{i_3} = v_{i_3} - r$ then

$$\begin{aligned}\frac{q_{i_3}}{m_{i_3}}(m_{i_3} - v_{i_3}) &= \frac{1}{m_{i_3}}(v_{i_3} - r)(m_{i_3} - v_{i_3}) \\ &\leq \left(\frac{m_{i_3} + r}{2} - r\right)\left(m_{i_3} - \frac{m_{i_3} + r}{2}\right)\frac{1}{m_{i_3}} \\ &= \frac{1}{m_{i_3}}\left(\frac{m_{i_3} - r}{2}\right)^2 \leq \frac{m_{i_3}}{4}.\end{aligned}$$

Thus

$$-\frac{v_{i_3}q_{i_3}}{m_{i_3}} + \beta_{i_3} \leq \frac{m_{i_3}}{4}.$$

7.8.3.3 The fourth sum

In the fourth sum, we are trying to find a bound for

$$F_4 = -\frac{Y}{m_{i_4}}v_{2,i_4} - \frac{q_{i_4}k_{i_4}v_{1,i_4}}{m_{i_4}} + \frac{k_{i_4}\gamma_{1,i_4}}{m_{i_4}}\left(Y - q_{i_4}\right) + \gamma_{2,i_4}.$$

There are a number of cases to consider. If $v_1 = 0$, then since $X \not\equiv 0 \pmod{m}$, then we must have $v_2 \neq 0, \gamma_1 \in \{0, 1\}$ and $\gamma_2 \leq \min(\gamma_1, q)$. If $v_2 = 0$ then $v_1 \neq 0, \gamma_1 = 0$ and

$\gamma_2 \leq \min(v_1, q)$. We then have

$$
\begin{aligned}
&-\frac{Yv_{2,i_4}}{m_{i_4}} - \frac{q_{i_4}k_{i_4}v_{1,i_4}}{m_{i_4}} + \frac{k_{i_4}\gamma_{1,i_4}}{m_{i_4}}(Y - q_{i_4}) + \gamma_{2,i_4} \\
&\qquad\qquad \leq \begin{cases} -\frac{Y}{m_{i_4}} + \frac{k_{i_4}}{m_{i_4}}(Y - q_{i_4}) + \gamma_{2,i_4} & \text{if } v_1 = 0 \\ -\frac{q_{i_4}k_{i_4}v_{1,i_4}}{m_{i_4}} + \gamma_{2,i_4} & \text{if } v_2 = 0 \\ -\frac{Y}{m_{i_4}} - \frac{q_{i_4}k_{i_4}v_{1,i_4}}{m_{i_4}} + \frac{k_{i_4}}{m_{i_4}}(Y - q_{i_4}) + \gamma_{2,i_4} & \text{if } v_1, v_2 \neq 0 \end{cases} \\
&\qquad\qquad \leq \begin{cases} -\frac{Y}{m_{i_4}} + \frac{Y - q_{i_4}}{2} + \min(1, q_{i_4}) & \text{if } v_1 = 0 \\ -\frac{2q_{i_4}v_{1,i_4}}{m_{i_4}} + \min(v_{1,i_4}, q_{i_4}) & \text{if } v_2 = 0 \\ -\frac{Y}{m_{i_4}} - \frac{2q_{i_4}v_{1,i_4}}{m_{i_4}} + \frac{Y - q_{i_4}}{2} + \min(v_{1,i_4} + 1, q_{i_4}) & \text{if } v_1, v_2 \neq 0 \end{cases}
\end{aligned}
$$

We now study these terms for the various possibilities of q_{i_4}. If $q_{i_4} \neq 0$, then for $v_{1,i_4} = 0$

$$F_4 \leq -\frac{Y}{m_{i_4}} + \frac{Y+1}{2}.$$

If $v_{2,i_4} = 0$ then by the symmetry of the expression we may take $v_{1,i_4} \leq q_{i_4}$. Then

$$F_4 \leq v_{1,i_4}\left(1 - \frac{2q_{i_4}}{m_{i_4}}\right),$$

where $v_{1,i_4} \leq q_{i_4} < \frac{m_{i_4}}{k_{i_4}} \leq \frac{m_{i_4}}{2}$, so that $\frac{2q_{i_4}}{m_{i_4}} < 1$ and

$$
\begin{aligned}
F_4 &\leq q_{1,i_4}\left(1 - \frac{2q_{i_4}}{m_{i_4}}\right) \\
&\leq \frac{m_{i_4}}{4}\left(1 - \frac{2m_{i_4}}{4m_{i_4}}\right) \\
&= \frac{m_{i_4}}{8},
\end{aligned}
$$

where in the second step we maximised the quadratic in q_{i_4}.

For the final case of $q_{i_4} \neq 0$, if $v_{1,i_4}, v_{2,i_4} \neq 0$ then

$$F_4 \leq \frac{Y}{2} - \frac{Y}{m_{i_4}} - q_{i_4}\left(\frac{1}{2} + \frac{2v_{1,i_4}}{m_{i_4}}\right) + \min(v_{1,i_4} + 1, q_{i_4}).$$

If $q_{i_4} \geq v_{1,i_4} + 1$ then

$$\begin{aligned} -q_{i_4}\left(\frac{1}{2} + \frac{2v_{1,i_4}}{m_{i_4}}\right) + \min(v_{1,i_4}+1, q_{i_4}) &\leq -q_{i_4}\left(\frac{1}{2} + \frac{2v_{1,i_4}}{m_{i_4}}\right) + v_{1,i_4} + 1 \\ &\leq (v_{1,i_4}+1)\left(\frac{1}{2} - \frac{2v_{1,i_4}}{m_{i_4}}\right), \end{aligned}$$

which is a quadratic and can be maximised by taking $v_{1,i_4} = \frac{m_{i_4}}{8} - \frac{1}{2}$ so that

$$F_4 \leq \frac{Y}{2} - \frac{Y}{m_{i_4}} + \left(\frac{m_{i_4}}{8} + \frac{1}{2}\right)\left(\frac{1}{4} + \frac{1}{m_{i_4}}\right).$$

On the other hand if $q_{i_4} < v_{1,i_4} + 1$ then

$$\begin{aligned} -q_{i_4}\left(\frac{1}{2} + \frac{2v_{1,i_4}}{m_{i_4}}\right) + \min(v_{1,i_4}+1, q_{i_4}) &\leq -q_{i_4}\left(\frac{1}{2} + \frac{2v_{1,i_4}}{m_{i_4}}\right) + q_{i_4} \\ &\leq q_{i_4}\left(\frac{1}{2} - \frac{2v_{1,i_4}}{m_{i_4}}\right) \\ &\leq v_{i_4}\left(\frac{1}{2} - \frac{2v_{1,i_4}}{m_{i_4}}\right) \\ &\leq \frac{m_{i_4}}{8}\left(\frac{1}{2} - \frac{1}{4}\right) \\ &= \frac{m}{32}. \end{aligned}$$

Thus for $q_{i_4} \neq 0$

$$F_4 \leq \begin{cases} -\frac{Y}{m_{i_4}} + \frac{Y+1}{2} & \text{if } v_1 = 0 \\ \frac{m_{i_4}}{8} & \text{if } v_2 = 0 \\ \frac{Y}{2} - \frac{Y}{m_{i_4}} + \left(\frac{m_{i_4}}{8} + \frac{1}{2}\right)\left(\frac{1}{4} + \frac{1}{m_{i_4}}\right) & \text{if } v_1, v_2 \neq 0. \end{cases}$$

If $q_{i_4} = 0$ it is easy to see that

$$-\frac{Yv_{2,i_4}}{m_{i_4}} - \frac{q_{i_4}k_{i_4}v_{1,i_4}}{m_{i_4}} + \frac{k_{i_4}\gamma_{1,i_4}}{m_{i_4}}(Y - q_{i_4}) + \gamma_{2,i_4} \leq -\frac{Y}{m_{i_4}} + \frac{Y}{2},$$

which is clearly not as good as our bounds for $q_{i_4} \neq 0$. If $\left(\frac{m_{i_4}}{8} + \frac{1}{2}\right)\left(\frac{1}{4} + \frac{1}{m_{i_4}}\right) \geq \frac{1}{2}$ we use the third bound over the first, and

$$\left(\frac{m_{i_4}}{8} + \frac{1}{2}\right)\left(\frac{1}{4} + \frac{1}{m_{i_4}}\right) - \frac{1}{2} = \frac{m_{i_4}}{32} - \frac{1}{4} + \frac{1}{2m_{i_4}} = \left(\frac{m_{i_4}}{8} - \frac{1}{2}\right)\left(\frac{1}{4} - \frac{1}{m_{i_4}}\right) \geq 0,$$

for $m_{i_4} \geq 4$, which is always true in this case. It remains to compare $\frac{m_{i_4}}{8}$ with $\frac{Y}{2} - \frac{Y}{m_{i_4}} + \frac{m_{i_4}}{32} + \frac{1}{4} + \frac{1}{2m_{i_4}}$. When we compute the heft, we will use $\frac{m_{i_4}}{8} \leq \frac{X-1}{8}$ and $\frac{Y}{2} - \frac{Y}{m_{i_4}} + \frac{m_{i_4}}{32} + \frac{1}{4} + \frac{1}{2m_{i_4}} \leq \frac{Y}{2} - \frac{Y}{X-1} + \frac{X-1}{32} + \frac{1}{4} + \frac{1}{8}$, and so the appropriate bound will depend on the values of X and Y.

7.8.3.4 The fifth sum

Similar to the fourth sum, if $v = 0$ we have $\delta_1 = 0$ and $\delta_2 \in \{0, u\}$. Thus

$$-\frac{X}{m_{i_5}} q_{2,i_4} - \frac{v_{i_5} d_{i_5} q_{1,i_4}}{m_{i_5}} + \delta_{1,i_4} + \delta_{2,i_4} \leq \begin{cases} -\frac{X q_{2,i_4}}{m_{i_5}} + \delta_{2,i_4} & \text{if } v = 0 \\ -\frac{X q_{2,i_4}}{m_{i_5}} - \frac{d_{i_5} q_{1,i_4}}{m_{i_5}} + \delta_{1,i_4} + \delta_{2,i_4} & \text{if } v \neq 0 \end{cases}$$

Now if $q_1 = 0$, then $q_2 \neq 0, \delta_1 = 0$ and so we have

$$-\frac{X}{m_{i_5}} q_{2,i_4} - \frac{v_{i_5} d_{i_5} q_{1,i_4}}{m_{i_5}} + \delta_{1,i_4} + \delta_{2,i_4} \leq \begin{cases} -\frac{X}{m_{i_5}} + u_{i_5} & \text{if } v = 0 \\ -\frac{X}{m_{i_5}} + u_{i_5} + 1 & \text{if } v \neq 0 \end{cases}$$

where in the first case $u_{i_5} = \frac{X}{m_{i_5}/d_{i_5}}$, and in the second $u_{i_5} + 1 = \frac{X - v_{i_5}}{m_{i_5}/d_{i_5}} + 1$, which is larger since $\frac{v_{i_5} d_{i_5}}{m_{i_5}} < 1$. So that here

$$-\frac{X}{m_{i_5}} q_{2,i_4} - \frac{v_{i_5} d_{i_5} q_{1,i_4}}{m_{i_5}} + \delta_{1,i_4} + \delta_{2,i_4} \leq -\frac{X}{m_{i_5}} + \frac{X - v_{i_5}}{m_{i_5}/d_{i_5}} + 1. \tag{7.2}$$

If $q_2 = 0$, then $q_1 \neq 0$ and $\delta_2 = 0$ so that

$$\begin{aligned} -\frac{X}{m_{i_5}} q_{2,i_5} - \frac{v_{i_5} d_{i_5} q_{1,i_5}}{m_{i_5}} + \delta_{1,i_5} + \delta_{2,i_5} &\leq \begin{cases} 0 & \text{if } v = 0 \\ -\frac{d_{i_5} q_{1,i_5}}{m_{i_5}} + \min(q_{1,i_5}, v_{i_5}) & \text{if } v \neq 0 \end{cases} \\ &\leq q_{1,i_5} \left(1 - \frac{d_{i_5}}{m_{i_5}}\right) \\ &\leq \left(\frac{m_{i_5}}{d_{i_5}} - 1\right)\left(1 - \frac{d_{i_5}}{m_{i_5}}\right) \\ &= \frac{m_{i_5}}{d_{i_5}} - 2 + \frac{d_{i_5}}{m_{i_5}}, \end{aligned} \tag{7.3}$$

note that in the second step if we had taken $q_{1,i_5} > v_{i_5}$, this term would become $q_{1,i_5}\left(1 - \frac{d_{i_5}}{m_{i_5}}\right) - r$ for $r > 0$ where $q_{1,i_5} = v_{i_5} + r$, which is clearly smaller.

Finally if both $q_1, q_2 \neq 0$,

$$-\frac{X}{m_{i_5}}q_{2,i_4} - \frac{v_{i_5}d_{i_5}q_{1,i_5}}{m_{i_5}} + \delta_{1,i_5} + \delta_{2,i_5} \leq \begin{cases} -\frac{X}{m_{i_5}} + u_{i_5} & \text{if } v = 0 \\ -\frac{X}{m_{i_5}} + u_{i_5} + 1 + \frac{m_{i_5}}{d_{i_5}} - 2 + \frac{d_{i_5}}{m_{i_5}} & \text{if } v \neq 0 \end{cases}$$
$$\leq \frac{(d_{i_5}-1)X}{m_{i_5}} - \frac{v_{i_5}d_{i_5}}{m_{i_5}} + 1 + \frac{m_{i_5}}{d_{i_5}} - 2 + \frac{d_{i_5}}{m_{i_5}} \tag{7.4}$$

Now $-\frac{X}{m_{i_5}} + \frac{X-v_{i_5}}{m_{i_5}/d_{i_5}} + 1 > 0$ so that (7.4) is a larger bound than (7.3). Similarly, $\left(\frac{m_{i_5}}{d_{i_5}} - 1\right)\left(1 - \frac{d_{i_5}}{m_{i_5}}\right) > 0$ as each term is clearly positive, and so (7.4) provides a larger bound than (7.2).

Thus, we bound this term by

$$\frac{(d_{i_5}-1)X}{m_{i_5}} - \frac{v_{i_5}d_{i_5}}{m_{i_5}} + \frac{m_{i_5}}{d_{i_5}} - 1 + \frac{d_{i_5}}{m_{i_5}} \leq \frac{X}{m_{i_5}} - \frac{d_{i_5}}{m_{i_5}} + (X-1) - 1 + \frac{d_{i_5}}{m_{i_5}}$$
$$\leq \frac{5}{4}X - 2,$$

since $d_{i_5} \geq 2$, $\frac{m_{i_5}}{d_{i_5}} \leq X - 1$ and $m_{i_5} \geq 4$.

7.8.3.5 The sixth sum

We'll leave the details for the sixth sum until after finishing the calculation of the heft lower bound, as the details get quite messy. It will be shown that

$$-\frac{X}{m}q_2 - \frac{v_2Y}{m} + \frac{v_2q_2}{m} - \frac{kv_1dq_1}{m} + p\phi_1 + \phi_2 + \phi_3 \leq \frac{X+Y+3}{2} - \frac{Y}{(X-1)(Y-1)} + \left(\frac{(X-1)(Y-1)}{12} - 1\right)^2.$$

7.8.3.6 Continuing the calculation of the lower bound

We now have

$$\begin{aligned}
1 &\leq \sum_{i=1}^{t_1} \frac{1}{m_i} + \sum_{i_2=1}^{t_2} \left(\frac{1}{m_{i_2}} - \frac{n_{i_2} - d_{i_2}}{m_{i_2} XY} \right) + \sum_{i_3=1}^{t_3} \left(\frac{1}{m_{i_3}} + \frac{m_{i_3}}{4XY} \right) \\
&\quad + \sum_{i_4=1}^{t_4} \left(\frac{1}{m_{i_4}} + \frac{1}{XY} \max \left(\frac{X-1}{8}, \frac{Y}{2} - \frac{Y}{X-1} + \frac{X+11}{32} \right) \right) \\
&\quad + \sum_{i_5=1}^{t_5} \left(\frac{1}{m_{i_5}} + \frac{1}{XY} \left(\frac{5}{4} X - 2 \right) \right) + \frac{t_7}{X} + \frac{1}{XY} \sum_{i_8=1}^{t_8} \left[\frac{Y}{d_{i_8}} \right] \\
&\quad + \sum_{i_6=1}^{t_6} \left(\frac{1}{m_{i_6}} + \frac{1}{XY} \left(\frac{X+Y+3}{2} - \frac{Y}{(X-1)(Y-1)} + \left(\frac{(X-1)(Y-1)}{12} - 1 \right)^2 \right) \right) \\
&\leq h + \sum_{i_2=1}^{t_2} \frac{\frac{m_{i_2}}{2} - 1}{m_{i_2} XY} + \sum_{i_3=1}^{t_3} \frac{m_{i_3}}{4XY} + t_4 \max \left(\frac{X-1}{8XY}, \frac{1}{2X} - \frac{1}{X(X-1)} + \frac{X+11}{32XY} \right) \\
&\quad + \frac{t_5}{XY} \left(\frac{5}{4} X - 2 \right) + \frac{t_7}{X} - \sum_{i_7=1}^{t_7} \frac{1}{m_{i_7}} + \frac{t_8}{2X} - \sum_{i_8=1}^{t_8} \frac{1}{m_{i_8}} \\
&\quad + \frac{t_6}{XY} \left(\frac{X+Y+3}{2} - \frac{Y}{(X-1)(Y-1)} + \left(\frac{(X-1)(Y-1)}{12} - 1 \right)^2 \right) \\
&\leq h + \frac{t_2}{XY} \left(\frac{1}{2} - \frac{1}{X} \right) + \frac{t_3}{4Y} + t_4 \max \left(\frac{X-1}{8XY}, \frac{X+11}{32XY} + \frac{X-3}{2X(X-1)} \right) \\
&\quad + \frac{t_5}{XY} \left(\frac{5}{4} X - 2 \right) + \frac{t_7}{X} + \frac{t_8}{2X} - \frac{t_7 + t_8}{(X-1)(Y-1)} \\
&\quad + \frac{t_6}{XY} \left(\frac{X+Y+3}{2} - \frac{Y}{(X-1)(Y-1)} + \left(\frac{(X-1)(Y-1)}{12} - 1 \right)^2 \right),
\end{aligned}$$

and so finally

$$\begin{aligned}
h &\geq 1 - t_2 \left(\frac{1}{2XY} - \frac{1}{X^2 Y} \right) - t_3 \frac{1}{4Y} - t_4 \max \left(\frac{X-1}{8XY}, \frac{X+11}{32XY} + \frac{1}{4X} \right) \\
&\quad - \frac{t_5}{XY} \left(\frac{5}{4} X - 2 \right) - \frac{t_7}{X} - \frac{t_8}{2X} + \frac{t_7 + t_8}{(X-1)(Y-1)} \\
&\quad - \frac{t_6}{XY} \left(\frac{X+Y+3}{2} - \frac{Y}{(X-1)(Y-1)} + \left(\frac{(X-1)(Y-1)}{12} - 1 \right)^2 \right),
\end{aligned}$$

7.8.3.7 Upper bound

7.8.3.8 The third sum

In the third sum for this bound we take

$$-\frac{v_{i_3}q_{i_3}}{m_{i_3}} + \beta_{i_3} \geq -\frac{v_{i_3}q_{i_3}}{m_{i_3}} + \max(0, v_{i_3} + q_{i_3} - m_{i_3}).$$

If $v_{i_3} + q_{i_3} \leq m_{i_3}$ then we are trying to maximise $\frac{v_{i_3}q_{i_3}}{m_{i_3}}$ subject to this restriction. This occurs when $v_{i_3} = q_{i_3} = \frac{m_{i_3}}{2}$ and

$$-\frac{v_{i_3}q_{i_3}}{m_{i_3}} + \beta_{i_3} \geq -\frac{m_{i_3}}{4}.$$

On the other hand if $v_{i_3} + q_{i_3} > m_{i_3}$, then

$$\begin{aligned}
-\frac{v_{i_3}q_{i_3}}{m_{i_3}} + \beta_{i_3} &= -\frac{v_{i_3}q_{i_3}}{m_{i_3}} + v_{i_3} + q_{i_3} - m_{i_3} \\
&= -\frac{1}{m_{i_3}}(m_{i_3} - v_{i_3})(m_{i_3} - q_{i_3}) \\
&\geq -\frac{1}{m_{i_3}}\left(m_{i_3} - \left(\frac{m_{i_3}}{2} + 1\right)\right)\left(m_{i_3} - \frac{m_{i_3}}{2}\right),
\end{aligned}$$

which is not as small as $-\frac{m_{i_3}}{4}$ so that the number of hits for this sum is bounded below by

$$\frac{XY}{m_{i_3}} - \frac{m_{i_3}}{4}.$$

7.8.3.9 The fourth sum

Here, recalling the facts from the lower bound calculation for this term and that $\gamma_{2,i_4} \geq \max\left(0, q_{i_4} + v_{1,i_4} + \gamma_{1,i_4} - \frac{m_{i_4}}{k_{i_4}}\right)$, we have

$$\begin{aligned}
G_4 &= -\frac{Y v_{2,i_4}}{m_{i_4}} - \frac{q_{i_4} k_{i_4} v_{1,i_4}}{m_{i_4}} + \frac{k_{i_4}}{m_{i_4}}(Y - q_{i_4})\gamma_{1,i_4} + \gamma_{2,i_4} \\
&\geq \begin{cases} -\frac{v_{2,i_4} Y}{m_{i_4}} + \frac{k_{i_4}}{m_{i_4}}(Y - q_{i_4})\gamma_{1,i_4} & \text{if } v_1 = 0 \\ -\frac{q_{i_4} k_{i_4} v_{1,i_4}}{m_{i_4}} + \gamma_{2,i_4} & \text{if } v_2 = 0 \\ -\frac{v_{2,i_4} Y}{m_{i_4}} - \frac{q_{i_4} k_{i_4} v_{1,i_4}}{m_{i_4}} + \frac{k_{i_4}}{m_{i_4}}(Y - q_{i_4})\gamma_{1,i_4} + \gamma_{2,i_4} & \text{if } v_1, v_2 \neq 0. \end{cases} \\
&\geq -\frac{v_{2,i_4} Y}{m_{i_4}} - \frac{q_{i_4} k_{i_4} v_{1,i_4}}{m_{i_4}} + \gamma_{2,i_4} \\
&\geq -\frac{(k_{i_4} - 1) Y}{m_{i_4}} - \frac{q_{i_4} k_{i_4} v_{1,i_4}}{m_{i_4}} + \gamma_{2,i_4},
\end{aligned}$$

taking $\gamma_{1,i_4} = 0$. So it remains to minimise $\max\left(0, q_{i_4} + v_{1,i_4} - \frac{m_{i_4}}{k_{i_4}}\right) - \frac{q_{i_4} k_{i_4} v_{1,i_4}}{m_{i_4}}$. Following almost exactly the same argument as for the third sum above we then have

$$G_4 \geq -\frac{(k_{i_4} - 1) Y}{m_{i_4}} - \frac{m_{i_4}}{4 k_{i_4}},$$

taking $q_{i_4} = v_{1,i_4} = \frac{m_{i_4}}{2 k_{i_4}}$. Since $Y \geq 2$ we take $k_{i_4} = \frac{m_{i_4}}{2}$ and thus

$$\begin{aligned}
G_4 &\geq \left(\frac{1}{m_{i_4}} - \frac{1}{2}\right) Y - \frac{1}{2} \\
&\geq \left(\frac{1}{X} - \frac{1}{2}\right) Y - \frac{1}{2}.
\end{aligned}$$

7.8.3.10 The fifth sum

We analyse this as in the lower bound case to find

$$\begin{aligned}
G_2 &= -\frac{X}{m_{i_5}} q_{2,i_5} - \frac{v_{i_5} d_{i_5} q_{1,i_5}}{m_{i_5}} + \delta_{1,i_5} + \delta_{2,i_5} \\
&\geq \begin{cases} -\frac{X q_{2,i_5}}{m_{i_5}} & \text{if } v = 0 \\ -\frac{X q_{2,i_5}}{m_{i_5}} - \frac{v_{i_5} d_{i_5} q_{1,i_5}}{m_{i_5}} + \max\left(0, q_{i_5} + v_{i_5} - \frac{m_{i_5}}{d_{i_5}}\right) & \text{if } v \neq 0, \end{cases}
\end{aligned}$$

taking $\delta_{2,i_5} = 0$. It remains to study $-\frac{v_{i_5} d_{i_5} q_{1,i_5}}{m_{i_5}} + \max\left(0, q_{i_5} + v_{i_5} - \frac{m_{i_5}}{d_{i_5}}\right)$, and as in the previous case, this is minimised for $q_{i_5} = v_{i_5} = \frac{m_{i_5}}{2d_{i_5}}$. This gives us

$$\begin{aligned} G_2 &\geq -\frac{(d_{i_5} - 1)X}{m_{i_5}} - \frac{m_{i_5}}{4d_{i_5}} \\ &\geq \left(\frac{1}{m_{i_5}} - \frac{1}{2}\right) X - \frac{1}{2} \\ &\geq \left(\frac{1}{(X-1)(Y-1)} - \frac{1}{2}\right) X - \frac{1}{2}. \end{aligned}$$

7.8.3.11 The sixth sum

This will again be left until after the upper bound calculation for heft is completed, as the details are messy. It will be shown that

$$\begin{aligned} -\frac{q_{3,i_6} X}{m_{i_6}} - \frac{v_{2,i_6}(Y - q_{3,i_6})}{m_{i_6}} - \frac{k_{i_6} v_{1,i_6} d q_{2,i_6}}{m_{i_6}} + \phi_{1,i_6} k_{i_6} \left(p_{i_6} + \frac{q_{1,i_6}}{k_{i_6}}\right) + \phi_{2,i_6} + \phi_{3,i_6} \\ \geq -\frac{X}{2} - \frac{Y}{3} + \frac{X+Y+2}{(X-1)(Y-1)} - \frac{1}{4}. \end{aligned}$$

7.8.3.12 Calculating the upper bound

Here we have

$$\begin{aligned} 1 \geq &\sum_{i=1}^{t_1} \frac{1}{m_i} + \sum_{i_2=1}^{t_2} \left(\frac{1}{m_{i_2}} - \frac{d_{i_2} - 1}{m_{i_2} XY}\right) + \sum_{i_3=1}^{t_3} \left(\frac{1}{m_{i_3}} - \frac{m_{i_3}}{4XY}\right) \\ &+ \sum_{i_4=1}^{t_4} \left(\frac{1}{m_{i_4}} + \frac{1}{XY}\left(\left(\frac{1}{X} - \frac{1}{2}\right) Y - \frac{1}{2}\right)\right) \\ &+ \sum_{i_5=1}^{t_5} \left(\frac{1}{m_{i_5}} + \frac{1}{XY}\left(\left(\frac{1}{(X-1)(Y-1)} - \frac{1}{2}\right) X - \frac{1}{2}\right)\right) \\ &+ \sum_{i_6=1}^{t_6} \left(\frac{1}{m_{i_6}} + \frac{1}{XY}\left(-\frac{X}{2} - \frac{Y}{3} + \frac{X+Y+2}{(X-1)(Y-1)} - \frac{1}{4}\right)\right) + \frac{1}{XY}\left(\sum_{i_7=1}^{t_7} \theta_{i_7} + \sum_{i_8=1}^{t_8} \zeta_{i_8}\right). \end{aligned}$$

In the seventh and eighth sum, the number of hits is at least 3, thus

$$
\begin{aligned}
1 > h &- \sum_{i_2=1}^{t_2} \left(\frac{\frac{m_{i_2}}{2} - 1}{m_{i_2} XY} \right) - \sum_{i_3=1}^{t_3} \frac{m_{i_3}}{4XY} + \sum_{i_4=1}^{t_4} \left(\frac{1}{X^2} - \frac{1}{2X} - \frac{1}{2XY} \right) \\
&+ \sum_{i_5=1}^{t_5} \left(\frac{1}{(X-1)(Y-1)Y} - \frac{1}{2Y} - \frac{1}{2XY} \right) \\
&+ \sum_{i_6=1}^{t_6} \left(-\frac{1}{2Y} - \frac{1}{3X} + \frac{X+Y+2}{X(X-1)Y(Y-1)} - \frac{1}{4XY} \right) \\
&+ \frac{3t_7}{XY} - \sum_{i_7=1}^{t_7} \frac{1}{m_{i_7}} + \frac{3t_8}{XY} - \sum_{i_8=1}^{t_8} \frac{1}{m_{i_8}}.
\end{aligned}
$$

Last of all, in the second and third case $X > m$ so that $\frac{1}{m} > X$ and $-m > -X$ respectively, while in the seventh case $X < m \leq (X-1)(Y-1)$ and in the eighth case $2X < dX < m \leq (X-1)(Y-1)$, and so finally

$$
\begin{aligned}
h < 1 &+ \frac{t_2}{XY} \left(\frac{1}{2} - \frac{1}{X} \right) + \frac{t_3}{4Y} + t_4 \left(\frac{1}{2X} + \frac{1}{2XY} - \frac{1}{X^2} \right) \\
&+ t_5 \left(\frac{1}{2Y} + \frac{1}{2XY} - \frac{1}{(X-1)(Y-1)Y} \right) \\
&+ t_6 \left(\frac{1}{2Y} + \frac{1}{3X} + \frac{1}{4XY} - \frac{X+Y+2}{X(X-1)Y(Y-1)} \right) \\
&+ \frac{t_7}{X} \left(1 - \frac{3}{Y} \right) + \frac{t_8}{X} \left(\frac{1}{2} - \frac{3}{Y} \right).
\end{aligned}
$$

This and the lower bound for heft would be much more useful if paired with bounds on the various sub-orders $t_2, t_3, \ldots, t_8$. No insight has been gained on how to bound these, let alone any good bounds on order t as will be seen in Section 7.9.

To compare these bounds to those seen in actual examples, start with dimensions 6×5 and 7×4 for which we have computed all IRDCS. In the first case all IRDCS have heft $0.973785 \leq h \leq 1.03654$ and order $t = 6, 7, 8$. For our bound, in the worst case the IRDCS would have $t = t_6$, producing $-0.04388 < h < 2.1377$. The best case, assuming that heft is in fact approximately 1 as supported by examples is to take $t = t_2$ which gives $0.9333 < h < 1.0667$. Similarly for 7×4 IRDCS, the actual bounds for heft are $0.98195 \leq h \leq 1.0254$ and they have order $t = 6, 7, 8$. The worst case is produced by taking $t = t_6$ giving $-1.0079 < h < 2.246$. The best case here is to again take $t = t_2$ which gives $0.9235 < h < 1.0765$.

These bounds are similar to the one-dimensional case, where the heft is bounded between 0.75 and 1.5, though all examples so far have heft between 0.987289 and 1.07287.

7.8.3.13 Bounding the sixth sum for the lower bound

For this calculation we will remove the i_6 subscript for the sake of clarity. Recall that $\phi_1 \in \{0, 1\}, \phi_3 \in \{0, u, u+\phi_1\}$ and ϕ_2 runs from $\max\left(0, q_1 + v_1 - \frac{m}{kd}\right)$ to $\min(q_1, v_1+\phi_1)$. The function $\phi_1 = 1$ if there are $v_1 + 1$ rows with $u + 1$ hits and so for this we must have $v_2 > 0$, while the function ϕ_3 is the number of hits on the last q_2 rows. Also recall that $X = \frac{m}{d} + kv_1 + v_2$ and $Y = \frac{m}{k}p + dq_1 + q_2$.

For this sum we are bounding

$$F_6 = -\frac{X}{m}q_2 - \frac{v_2 Y}{m} + \frac{v_2 q_2}{m} - \frac{kv_1 dq_1}{m} + p\phi_1 + \phi_2 + \phi_3$$
$$\leq \begin{cases} -\frac{X}{m}q_2 + u & \text{if } v_1 = v_2 = 0 \\ -\frac{X}{m}q_2 + u + \min(v_1, q_1) - \frac{kv_1 dq_1}{m} & \text{if } v_1 \neq 0, v_2 = 0 \\ -\frac{X}{m}q_2 - \frac{Y}{m} + \frac{q_2}{m} + p + u + 1 + \phi_1 & \text{if } v_1 = 0, v_2 \neq 0 \\ -\frac{X}{m}q_2 - \frac{Y}{m} + \frac{q_2}{m} + p + u + 1 + \min\left(q_1, v_1 + \phi_1\right) - \frac{kv_1 dq_1}{m} & \text{if } v_1, v_2 \neq 0, \end{cases} \tag{7.5}$$

since $Y > q_2$ and where $u = \frac{d}{m}(X - kv_1 - v_2)$. We have $kv_1 < \frac{m}{d}$ and so if $q_1 < v_1$ then $\min(q_1, v_1) - \frac{kv_1 dq_1}{m} = q_1\left(1 - \frac{dkv_1}{m}\right) \geq 0$, and similarly if $v_1 \leq q_1$, and so the second bound here beats the first. Along with this, if in the fourth bound we replace $\min\left(q_1, v_1 + \phi_1\right)$ with $\min\left(q_1, v_1\right) + \phi_1$, which is clearly at least as large, then the fourth bound beats the third. Thus we are left with

$$F_6 = -\frac{X}{m}q_2 - \frac{v_2 Y}{m} + \frac{v_2 q_2}{m} - \frac{kv_1 dq_1}{m} + p\phi_1 + \phi_2 + \phi_3$$
$$\leq \begin{cases} -\frac{X}{m}q_2 + u + \min(v_1, q_1) - \frac{kv_1dq_1}{m} & \text{if } v_1 \neq 0, v_2 = 0 \\ -\frac{X}{m}q_2 + u + \min(q_1, v_1) - \frac{kv_1dq_1}{m} + p + 2 - \frac{Y-q_2}{m} & \text{if } v_1, v_2 \neq 0. \end{cases} \tag{7.6}$$

Now

$$\begin{aligned} p + 2 - \frac{Y - q_2}{m} &= \frac{Y - dq_1 - q_2}{m}k - \frac{Y - q_2}{m} + 2 \\ &= \frac{(Y - q_2)(k-1) - dq_1}{m} + 2 \\ &\geq 0, \end{aligned}$$

since $Y \geq dq_1 + q_2$. Thus the second term in (7.6) is larger and

$$F_6 \leq -\frac{X}{m}q_2 + \frac{X - kv_1 - v_2}{m}d + \frac{(Y - q_2)(k-1) - dkq_1}{m} + 2 + \min(q_1, v_1) - \frac{kv_1dq_1}{m}.$$

Take $v_2 = 1$, since it must be non-zero, and taking the terms in q_2

$$-\frac{X}{m}q_2 - \frac{k-1}{m}q_2 = -\frac{(X + k - 1)q_2}{m} \leq 0$$

since $X > k - 1$, so we take $q_2 = 0$ and

$$F_6 \leq \frac{X - kv_1 - 1}{m}d + \frac{(k-1)Y - dkq_1}{m} + 2 + \min(q_1, v_1) - \frac{kv_1dq_1}{m}.$$

It remains to study

$$\beta = -\frac{kv_1d}{m} - \frac{dkq_1}{m} + \min(q_1, v_1) - \frac{kv_1dq_1}{m}.$$

It is clear that increasing q_1 or v_1 decreases all but the $\min(q_1, v_1)$ term, and thus this term is maximised when we take $q_1 = v_1$ and so

$$\begin{aligned} \beta &\leq -2v_1\frac{kd}{m} + v_1 - \frac{kdv_1^2}{m} \\ &= -v_1\frac{kd}{m}\left(v_1 - \left(\frac{m}{kd} - 2\right)\right). \end{aligned}$$

This is maximised at $v_1 = \frac{m}{2kd} - 1$ and so

$$\begin{aligned} \beta &\leq -\left(\frac{m}{2kd} - 1\right)\left(1 - \frac{m}{2kd}\right) \\ &= \left(\frac{m}{2kd} - 1\right)^2 \\ &\leq \left(\frac{m}{12} - 1\right)^2. \end{aligned}$$

So we have

$$F_6 \leq \frac{d(X-1)}{m} + \frac{(k-1)Y}{m} + 2 + \left(\frac{m}{2kd} - 1\right)^2.$$

Now $\frac{dk}{m} \leq 1$, so that

$$F_6 \leq \frac{X-1}{k} + \frac{Y}{d} - \frac{Y}{m} + 2 + \left(\frac{m}{2kd} - 1\right)^2,$$

and so we take $m \leq (X-1)(Y-1)$ and $k, d \geq 2$ with $kd \geq 6$ since $k \neq d$, and so finally

$$F_6 \leq \frac{X+Y+3}{2} - \frac{Y}{(X-1)(Y-1)} + \left(\frac{(X-1)(Y-1)}{12} - 1\right)^2,$$

where if $X - 1 \geq Y$ we could have taken $k = 3$ everywhere, but we cannot be sure of this, and it will not make a large difference in our final bound.

7.8.3.14 Bounding the sixth sum for the upper bound

We again ignore the i_6 subscripts, and all of the functions satisfy the same conditions as previous. For this sum we are bounding

$$G_6 = -\frac{X}{m}q_2 - \frac{v_2 Y}{m} + \frac{v_2 q_2}{m} - \frac{k v_1 d q_1}{m} + p\phi_1 + \phi_2 + \phi_3.$$

Firstly we consider the various possibilities for the values of v_i to get, on noting that $Y > q_2$,

$$G_6 \geq \begin{cases} -\frac{X}{m}q_2 & \text{if } v_1 = v_2 = 0 \\ -\frac{X}{m}q_2 - \frac{(k-1)Y}{m} + \frac{(k-1)q_2}{m} & \text{if } v_1 = 0, v_2 \neq 0 \\ -\frac{X}{m}q_2 - \frac{kv_1dq_1}{m} + \max\left(0, q_1 + v_1 - \frac{m}{kd}\right) & \text{if } v_1 \neq 0, v_2 = 0 \\ -\frac{X}{m}q_2 - \frac{(k-1)Y}{m} + \frac{(k-1)q_2}{m} - \frac{kv_1dq_1}{m} + \max\left(0, q_1 + v_1 - \frac{m}{kd}\right) & \text{if } v_1, v_2 \neq 0. \end{cases} \tag{7.7}$$

Since $Y > q_2$ we have $-\frac{(Y-q_2)(k-1)}{m} < 0$ so that the second term dominates the first and the fourth term dominates the third. We then have

$$G_6 \geq \begin{cases} -\frac{X}{m}q_2 - \frac{(k-1)Y}{m} + \frac{(k-1)q_2}{m} \\ -\frac{X}{m}q_2 - \frac{(k-1)Y}{m} + \frac{(k-1)q_2}{m} - \frac{kv_1dq_1}{m} + \max\left(0, q_1 + v_1 - \frac{m}{kd}\right), \quad v_1 \neq 0. \end{cases} \tag{7.8}$$

If $q_2 = 0$ then $-\frac{X}{m}q_2 - \frac{(k-1)Y}{m} + \frac{(k-1)q_2}{m} = -\frac{(k-1)Y}{m}$, while if $q_2 \neq 0$ then

$$\begin{aligned} -\frac{X}{m}q_2 - \frac{(k-1)Y}{m} + \frac{(k-1)q_2}{m} &= -\frac{q_2}{m}(X - k + 1) - \frac{k-1}{m}Y \\ &\geq -\frac{d-1}{m}(X - k + 1) - \frac{k-1}{m}Y, \end{aligned}$$

since $X > x_0 > k$ and so we take $q_2 \neq 0$. Next we consider $I = -\frac{kv_1dq_1}{m} + \max\left(0, q_1 + v_1 - \frac{m}{kd}\right)$. Since $kv_1 < \frac{m}{d}$ we have $\frac{kv_1d}{m} < 1$ and thus once $q_1 + v_1 \geq \frac{m}{kd}$ increasing q_1 (or equally, v_1) increases I, thus

$$I \geq -\frac{kv_1dq_1}{m},$$

where $q_1 + v_1 = \frac{m}{kd}$. So we have

$$I \geq -\frac{kd}{m}\left(\frac{m}{2kd}\right)^2 = -\frac{m}{4kd}.$$

This term is negative, so we take the second bound in (7.8) and we have

$$G_6 \geq -\frac{d-1}{m}X - \frac{k-1}{m}Y + \frac{(d-1)(k-1)}{m} - \frac{m}{4kd}.$$

Since $\frac{kd}{m} \leq 1$ we have $\frac{-d}{m} \geq \frac{-1}{k}$ and similarly $\frac{-k}{m} \geq \frac{-1}{d}$ giving us

$$G_6 \geq -\frac{X}{k} - \frac{Y}{d} + \frac{X+Y}{m} + \frac{(d-1)(k-1)}{m} - \frac{m}{4kd},$$

where the first four terms here are best bounded for $k \leq d$ small and m large, while the final term wants the opposite, and so we bound this term as

$$G_6 \geq -\frac{X}{2} - \frac{Y}{3} + \frac{X+Y+2}{(X-1)(Y-1)} - \frac{1}{4}.$$

7.9 Two Dimensional Order

Clearly an $X \times Y$ IRDCS must have order

$$t \leq \left[\frac{XY}{3}\right],$$

as every congruence in the IRDCS must cover at least 3 of the XY total positions within the IRDCS. However, there exist IRDCS with order $t = \left[\frac{XY}{3}\right]$. One such example is the following 10×2 IRDCS, which was presented in Section 7.1.

9	5	6	4	8	7	5	4	6	9
7	8	4	5	9	6	4	7	5	8

This has order $6 = \left[\frac{20}{3}\right]$.

Take an IRDCS with dimensions $(2X+1) \times (2Y+1)$, which recall covers the positions $[0, 2X+1) \times [0, 2Y+1) \cap \mathbb{Z}^2$. Then consider the congruence which covers the position (X, Y). Take the generators of this congruence to be (α_1, β_1) and (α_2, β_2). These vectors when applied to (X, Y) both positively and negatively will land in the box, since (X, Y) is exactly in the middle of the box. Thus the congruence covers at least 5 positions and so

$$t \leq \left[\frac{XY-2}{3}\right].$$

7.9.1 Open Questions

As in the one-dimensional case, we would like to be able to improve this trivial bound. At this stage for anything other than both X and Y odd there is no improvement at all on the trivial bound.

Studying congruences which cover the middle rows or columns in these examples does not force congruences with more than three positions covered. As such, similar analysis to that presented for both X and Y odd will not improve the order bound.

Appendices

Appendix A

One Dimensional IRDCS Additional Data

A.1 All IRDCS of a given length

The first table gives exhaustive data for IRDCS with lengths 33 through to 45 inclusive. We then present data on the proportion of IRDCS for the given length with particular orders. This extends and corrects unpublished work of Jacky Poon.

Length	Number of IRDCS	Orders	Heft Range
33	25,384	8–14	0.992184–1.05867
34	62,092	8–15	0.988973–1.06213
35	68,176	8–16	0.990813–1.06149
36	85,762	8–16	0.990561–1.04313
37	304,892	8–17	0.990192–1.05947
38	855,072	8–17	0.989847–1.06768
39	1,229,050	8–17	0.988657–1.06535
40	1,805,096	8–18	0.988343–1.05875
41	4,433,674	7–19	0.989216–1.05968
42	8,732,554	7–19	0.988944–1.06208
43	19,480,154	7–19	0.987952–1.06461
44	32,765,794	7–20	0.988944–1.06636
45	75,748,582	7–20	0.987289–1.06028

Note that in the following tables, percentages may not sum to 100, due to rounding.

A.1.0.1 Length 33

Order	8–9	10	11	12	13	14
% of total solutions	0	1	3	18	57	20

A.1.0.2 Length 34

Order	8–10	11	12	13	14	15
% of total solutions	0	1	9	43	43	3

A.1.0.3 Length 35

Order	8–10	11	12	13	14	15	16
% of total solutions	0	0.5	5	26	40	28	0.5

A.1.0.4 Length 36

Order	8–11	12	13	14	15	16
% of total solutions	0	3	21	45	30	0.5

A.1.0.5 Length 37

Order	8–12	13	14	15	16	17
% of total solutions	0	10	39	40	11	0

A.1.0.6 Length 38

Order	8–12	13	14	15	16	17
% of total solutions	0	4	22	47	26	1

A.1.0.7 Length 39

Order	8–13	14	15	16	17
% of total solutions	0	13	43	34	8

A.1.0.8 Length 40

Order	8–12	13	14	15	16	17	18
% of total solutions	0	1	6	30	47	16	0

Up to and including length 40, all IRDCS with order 8 are doubled IRDCS of smaller lengths. For length 40 all of the solutions with order 8 and 9 (there are $28 = 2 \times 14$ such IRDCS) are doubled length 20 IRDCS.

A.1.0.9 Length 41

Order	7–13	14	15	16	17	18	19
% of total solutions	0	3	17	44	33	4	0

For length 41, there are IRDCS for small order which are not doublings. The following IRDCS in compact notation have order 10 and heft 0.998007 and 1.0099 respectively: 9, 23, 3, 18, 6, 19, 12, 21, 17, 22, and 18, 23, 3, 9, 6, 27, 12, 16, 14, 22. Given that there are length 21 IRDCS with order 9, there are other IRDCS of length 41 with order 10 that are doublings. The IRDCS presented are the IRDCS with smallest order for length 41 that are not doublings. Interestingly they are both examples of IRDCS with the moduli 9, 6, 3 covering sequential positions in the IRDCS, as in Section 2.4.

A.1.0.10 Length 42

Order	7–13	14	15	16	17	18	19
% of total solutions	0	1	10	31	42	16	1

For length 42, there are IRDCS for small order which are not doublings. The following IRDCS in compact notation have order 10 and heft 1.00299 and 0.998007 respectively: 3, 25, 18, 6, 9, 12, 17, 24, 16, 20, and 3, 6, 18, 22, 9, 21, 12, 17, 19, 23. There are also IRDCS of length 42 with order 10 that are doublings of length 21 IRDCS. The IRDCS presented are the IRDCS with smallest order for length 42 that are not doublings. Once again they are both examples of IRDCS with the moduli 9, 6, 3 covering sequential positions in the IRDCS, as in Section 2.4.

A.1.0.11 Length 43

Order	7–13	14	15	16	17	18	19
% of total solutions	0	1	5	22	45	25	3

A.1.0.12 Length 44

Order	7–14	15	16	17	18	19	20
% of total solutions	0	3	14	38	36	8	0

A.1.0.13 Length 45

Order	7–14	15	16	17	18	19	20
% of total solutions	0	1	7	27	44	20	1

Given the amount of solutions for lengths 46 and above, we present data for them in a different format.

A.1.0.14 Length 46 and 47

	Length 46		**Length 47**	
Order	Solutions	Proportion	Solutions	Proportion
7	4	0.00%	2	0.00%
8	–	0.00%	–	0.00%
9	36	0.00%	26	0.00%
10	96	0.00%	72	0.00%
11	268	0.00%	272	0.00%
12	2,414	0.00%	1,966	0.00%
13	9,704	0.00%	6,494	0.00%
14	79,188	0.03%	54,354	0.01%
15	714,516	0.32%	505,564	0.12%
16	7,165,836	3.16%	5,682,120	1.40%
17	36,716,134	16.20%	38,984,702	9.59%
18	86,298,618	38.09%	123,180,540	30.31%
19	76,842,486	33.91%	169,165,764	41.63%
20	18,404,950	8.12%	64,901,208	15.97%
21	355,336	0.16%	3,910,188	0.96%
Total	226,589,586		406,393,272	
Heft range	min	max	min	max
	0.988485	1.0638	0.988705	1.07287

A.1.0.15 Length 48 and 49

	Length 48		**Length 49**	
Order	Solutions	Proportion	Solutions	Proportion
9	16	0.00%	24	0.00%
10	48	0.00%	82	0.00%
11	222	0.00%	264	0.00%
12	1,530	0.00%	1,170	0.00%
13	2,680	0.00%	7,444	0.00%
14	29,390	0.01%	31,702	0.00%
15	312,200	0.06%	402,254	0.03%
16	3,838,688	0.68%	4,242,342	0.27%
17	30,212,674	5.39%	38,142,260	2.41%
18	120,565,818	21.50%	185,127,866	11.67%
19	243,658,564	43.45%	518,769,882	32.71%
20	145,753,714	25.99%	644,168,204	40.62%
21	16,354,736	2.92%	191,457,490	12.07%
22	-	-	3,470,538	0.22%
Total	560,730,280		1,585,821,522	
Heft range	min	max	min	max
	0.988387	1.0617	0.98832	1.06302

A.1.0.16 Length 50

Length 50

Order	Solutions	Proportion
9	32	0.00%
10	116	0.00%
11	298	0.00%
12	1,010	0.00%
13	11,916	0.00%
14	34,718	0.00%
15	438,654	0.01%
16	5,158,884	0.13%
17	46,155,870	1.19%
18	262,924,406	6.75%
19	892,488,328	22.92%
20	1,633,116,224	41.93%
21	930,723,090	23.90%
22	122,225,234	3.14%
23	1,454,908	0.04%
Total	3,894,733,688	
Heft range	min	max
	0.988021	1.06782

A.2 Minimum modulus data

The conjectured formula for the shortest length of an IRDCS with minimum modulus $M, n = 4M + 3$, holds in a consistent fashion for $M = 10, \ldots, 15$. Below we compare summary statistics for all IRDCS of smallest length with minimum modulus $M = 10, \ldots, 14$.

A.2.0.17 Minimum modulus solutions versus all IRDCS solutions

Minimum modulus	Length	Min. modulus solution counter	Multiplicative increase	All IRDCS solution counter	Multiplicative increase
10	37	224	–	304,892	–
11	41	752	3.357	4,433,674	14.542
12	45	19,752	26.266	75,748,582	17.085
13	49	628,332	31.812	1,585,821,522	20.935
14	53	11,482,130	18.274	–	–

Appendix B

Two Dimensional IRDCS Additional Data

B.1 The existence of two-dimensional IRDCS of given dimensions

Here we present our results for the computation of two-dimensional IRDCS. Given the relative complexity of the algorithm, computation for an $X \times Y$ IRDCS is significantly slower than the computation for a length XY IRDCS. Thus for most lengths only partial computation has been completed.

Two tables of data will be presented. In the first table, if there is no additional notation the solution counter is the total number of solutions for those dimensions. Otherwise for the first table the counter written as n^* means that there are at least n solutions. Moreover this means that there are precisely n solutions for all IRDCS where the modulus covering the position $([X/2], 0)$, the first position covered in the algorithm, is at most 4.

The second table presents the data for dimensions where no standardised end point for our computations exists. This is done for dimensions where calculating up to even the smaller case in the first table takes a prohibitively long time. This table is used to prove the existence rather than frequency of these IRDCS. For example, in the case of 13×10 IRDCS all IRDCS with the modulus 2 covering $([X/2], 0)$ have not been calculated.

Some dimensions will have data in both tables. This will be the case when we have computed some additional data beyond the standard endpoint for those dimensions, but to complete the computation would take many more hours.

Given that the endpoint in the second table is not standardised, no conclusions should be drawn from the relative growth or decay of solutions for different dimensions.

B.1.0.18 Standardised end point data

6				$1,018^*$										
5			0	244	5038^*	410^*								
4		0	0	4	300	48^*	58^*							
3	0	0	0	0	0	0	16	184	62^*					
2	0	0	0	0	0	0	0	20	24	16	300	748	2,352	9,872
Y/X	3	4	5	6	7	8	9	10	11	12	13	14	15	16

B.1.0.19 Non-standardised end point data

13							3
12						6	10
11					4	6	8
10				6	9	21	23
9			2	5	16	13	17
8		108	4	44	4	7	6
7	41	87	237	23	5	13	26
6	532	100	117	146	16	20	5
5	5038	410	370	15	34	16	7
4		666	130	347	107	30	16
3						109	8
Y/X	7	8	9	10	11	12	13

B.2 Data summaries

B.2.0.20 6 by 4 IRDCS

There are 4 IRDCS with dimensions 6×4. All have heft 0.995635 and order 6. The solutions are the horizontal, vertical and complete reversals of one another. The first example found by the exhaustive algorithm from Section 7.6.1 is, in its alternate notation:

$$\begin{array}{cccccc} 8 & 7 & 6 & 5 & 9 & 6 \\ 4 & 5 & 4 & 7 & 4 & 8 \\ 6 & 9 & 8 & 6 & 5 & 7 \\ 7 & 4 & 5 & 4 & 9 & 4 \end{array}$$

B.2.0.21 6 by 5 IRDCS

There are 244 IRDCS with dimensions 6×5. The heft ranges from 0.973785 to 1.03654, and there are 4 solutions with order 6, 36 with order 7 and 204 with order 8. Below we present a few examples in their alternate notation.

$$\begin{array}{cccccc} 20 & 3 & 7 & 9 & 3 & 20 \\ 3 & 8 & 6 & 3 & 10 & 6 \\ 9 & 10 & 3 & 7 & 8 & 3 \\ 7 & 3 & 6 & 9 & 3 & 6 \\ 3 & 20 & 8 & 3 & 7 & 10 \end{array} , \qquad \begin{array}{cccccc} 5 & 3 & 12 & 6 & 3 & 5 \\ 3 & 8 & 10 & 3 & 5 & 6 \\ 12 & 6 & 3 & 5 & 8 & 3 \\ 10 & 3 & 5 & 6 & 3 & 12 \\ 3 & 5 & 8 & 3 & 10 & 6 \end{array}$$

these two IRDCS have, respectively, heft 1.02897 and 1.00833, order 7 and 6 and are the first and second solutions found by the algorithm.

8	12	9	4	6	9		11	20	7	4	10	9
4	7	6	10	4	8		4	8	6	11	4	6
6	4	8	7	12	4	,	10	4	9	7	8	4
9	10	4	9	6	7		7	6	4	10	6	9
7	12	6	4	8	10		20	11	8	4	7	20

these two IRDCS have, respectively, heft 0.978968 and 1.03654 and order 7 and 8.

14	6	5	11	7	9		14	7	6	8	5	13
7	8	10	6	5	8		13	5	11	10	6	7
11	5	9	7	14	6	,	6	8	7	5	14	11
10	6	8	5	11	9		5	10	6	13	8	5
5	14	7	6	10	5		11	14	5	7	6	10

these two IRDCS have, respectively, heft 1.00797 and 0.973785 and both have order 8.

B.2.0.22 6 by 6 IRDCS

We have computed 1138 244 IRDCS with dimensions 6×6. The heft ranges from 0.975372 to 1.06111, and of these solutions there are 4 with order 5, 14 with order 7, 204 with order 8, 880 with order 9 and 36 with order 10. Below we present a few examples in their alternate notation.

8	3	16	6	3	7		15	3	10	7	3	15
12	6	3	7	12	3		12	16	3	11	12	3
3	7	10	3	8	6	,	3	9	7	3	9	16
16	3	8	6	3	10		11	3	15	10	3	7
8	6	3	12	7	3		10	7	3	12	11	3
3	10	7	3	16	6		3	16	9	3	7	9

these two IRDCS have, respectively, heft 1.0137 and 0.9907, order 7 and 8 and are the first and second solutions found by the algorithm.

$$\begin{array}{cccccc} 6 & 9 & 3 & 4 & 5 & 3 \\ 9 & 3 & 4 & 5 & 3 & 6 \\ 3 & 4 & 5 & 3 & 6 & 4 \\ 4 & 5 & 3 & 6 & 4 & 3 \\ 5 & 3 & 6 & 4 & 3 & 5 \\ 3 & 6 & 4 & 3 & 5 & 9 \end{array}, \qquad \begin{array}{cccccc} 15 & 14 & 4 & 11 & 6 & 15 \\ 6 & 17 & 12 & 4 & 9 & 14 \\ 4 & 9 & 6 & 10 & 4 & 17 \\ 10 & 4 & 11 & 15 & 6 & 4 \\ 6 & 12 & 4 & 14 & 9 & 12 \\ 17 & 9 & 6 & 4 & 10 & 11 \end{array}$$

these two IRDCS have, respectively, heft 1.0611 and 0.9989, order 5 and 9 and the first is constructed from the one-dimensional IRDCS of length 11 as in Lemma 7.5 and Theorem 7.1.

$$\begin{array}{cccccc} 13 & 12 & 4 & 18 & 10 & 15 \\ 8 & 16 & 14 & 4 & 8 & 13 \\ 4 & 10 & 15 & 11 & 4 & 16 \\ 18 & 4 & 13 & 8 & 12 & 4 \\ 11 & 12 & 4 & 10 & 14 & 18 \\ 14 & 16 & 8 & 4 & 11 & 15 \end{array}, \qquad \begin{array}{cccccc} 16 & 8 & 10 & 5 & 13 & 8 \\ 14 & 9 & 12 & 11 & 5 & 15 \\ 5 & 15 & 8 & 9 & 16 & 5 \\ 11 & 5 & 13 & 10 & 14 & 9 \\ 10 & 12 & 5 & 8 & 11 & 12 \\ 9 & 14 & 16 & 5 & 15 & 13 \end{array}$$

these two IRDCS have, respectively, heft 0.9823 and 0.9879 and both have order 10. The second of these two solutions is the last solution we have computed with the algorithm.

B.2.0.23 7 by 4 IRDCS

There are 300 IRDCS with dimensions 7×4. The heft ranges from 0.980195 to 1.0254, and of these solutions there are 16 with order 6, 56 with order 7 and 228 with order 8. Below we present a few examples in their alternate notation.

$$\begin{array}{ccccccc} 4 & 10 & 5 & 11 & 4 & 6 & 10 \\ 12 & 4 & 6 & 5 & 8 & 4 & 12 \\ 11 & 8 & 4 & 10 & 5 & 6 & 4 \\ 5 & 12 & 6 & 4 & 11 & 5 & 8 \end{array}, \qquad \begin{array}{ccccccc} 4 & 5 & 9 & 8 & 4 & 7 & 5 \\ 12 & 4 & 5 & 7 & 10 & 4 & 12 \\ 8 & 7 & 4 & 5 & 8 & 9 & 4 \\ 10 & 12 & 9 & 4 & 5 & 10 & 7 \end{array}$$

these two IRDCS have, respectively, heft 1.01591 and 1.0123 both have order 7 and are the first and second solutions found by the algorithm.

$$\begin{array}{ccccccc} 12 & 4 & 6 & 5 & 7 & 4 & 14 \\ 7 & 5 & 9 & 4 & 12 & 6 & 5 \\ 14 & 4 & 6 & 7 & 5 & 4 & 9 \\ 12 & 9 & 5 & 4 & 14 & 6 & 7 \end{array} , \qquad \begin{array}{ccccccc} 12 & 9 & 5 & 7 & 10 & 6 & 12 \\ 7 & 10 & 6 & 8 & 5 & 11 & 9 \\ 11 & 5 & 9 & 12 & 7 & 6 & 5 \\ 8 & 7 & 6 & 5 & 8 & 10 & 11 \end{array}$$

these two IRDCS have, respectively, heft 1.0254 and 1.01988 and order 7 and 8.

$$\begin{array}{ccccccc} 10 & 6 & 5 & 15 & 7 & 9 & 11 \\ 9 & 7 & 11 & 8 & 5 & 6 & 10 \\ 15 & 5 & 10 & 6 & 9 & 7 & 5 \\ 8 & 6 & 7 & 5 & 8 & 11 & 15 \end{array} , \qquad \begin{array}{ccccccc} 8 & 5 & 14 & 10 & 7 & 9 & 5 \\ 9 & 6 & 5 & 12 & 6 & 8 & 7 \\ 10 & 7 & 8 & 5 & 9 & 10 & 14 \\ 12 & 14 & 6 & 7 & 5 & 6 & 12 \end{array}$$

these two IRDCS have, respectively, heft 1.00321 and 1.0004 and order 8.

$$\begin{array}{ccccccc} 6 & 14 & 3 & 6 & 5 & 3 & 6 \\ 10 & 3 & 8 & 5 & 3 & 10 & 8 \\ 3 & 6 & 5 & 3 & 6 & 14 & 3 \\ 14 & 5 & 3 & 8 & 10 & 3 & 5 \end{array} , \qquad \begin{array}{ccccccc} 12 & 11 & 14 & 5 & 10 & 7 & 12 \\ 5 & 6 & 8 & 7 & 6 & 5 & 8 \\ 10 & 7 & 5 & 12 & 11 & 10 & 14 \\ 11 & 14 & 6 & 8 & 5 & 6 & 7 \end{array}$$

these two IRDCS have, respectively, heft 0.996429 and 0.980195 and order 6 and 8.

B.2.0.24 7 by 5 IRDCS

We have calculated 5038 IRDCS with dimensions 7×5. For these, the heft ranges from 0.957875 to 1.07537, and of these solutions there are 4 with order 5, 20 with order 7, 426 with order 8, 2500 with order 9 and 2088 with order 10. Below we present a few examples in their alternate notation.

$$\begin{array}{ccccccc} 9 & 6 & 3 & 11 & 16 & 3 & 12 \\ 3 & 14 & 10 & 3 & 9 & 6 & 3 \\ 11 & 3 & 12 & 6 & 3 & 10 & 14 \\ 16 & 6 & 3 & 9 & 11 & 3 & 12 \\ 3 & 10 & 14 & 3 & 16 & 6 & 3 \end{array} , \qquad \begin{array}{ccccccc} 10 & 3 & 11 & 4 & 3 & 10 & 24 \\ 4 & 9 & 3 & 12 & 4 & 3 & 11 \\ 3 & 4 & 10 & 3 & 9 & 4 & 3 \\ 24 & 3 & 4 & 11 & 3 & 12 & 4 \\ 12 & 9 & 3 & 4 & 10 & 3 & 24 \end{array}$$

these two IRDCS have, respectively, heft 1.01928 and 1.01035 and order 8 and 7. The first of these is the first solution found by the algorithm.

$$\begin{matrix} 8 & 14 & 11 & 4 & 5 & 12 & 9 \\ 4 & 5 & 9 & 8 & 4 & 16 & 5 \\ 16 & 4 & 12 & 5 & 14 & 4 & 8 \\ 5 & 8 & 4 & 9 & 11 & 5 & 4 \\ 14 & 11 & 5 & 4 & 8 & 12 & 16 \end{matrix} \quad , \quad \begin{matrix} 7 & 12 & 6 & 4 & 10 & 13 & 14 \\ 13 & 10 & 4 & 15 & 6 & 7 & 4 \\ 6 & 4 & 14 & 7 & 12 & 4 & 6 \\ 4 & 7 & 6 & 13 & 4 & 10 & 15 \\ 15 & 12 & 10 & 4 & 6 & 14 & 7 \end{matrix}$$

these two IRDCS have, respectively, heft 0.994282 and 0.957875 and both have order 8.

$$\begin{matrix} 15 & 22 & 7 & 4 & 12 & 15 & 8 \\ 4 & 8 & 11 & 10 & 4 & 9 & 7 \\ 12 & 4 & 9 & 7 & 8 & 4 & 22 \\ 7 & 11 & 4 & 15 & 12 & 10 & 4 \\ 22 & 10 & 8 & 4 & 7 & 9 & 11 \end{matrix} \quad , \quad \begin{matrix} 15 & 6 & 18 & 4 & 12 & 14 & 8 \\ 4 & 8 & 13 & 11 & 4 & 6 & 15 \\ 12 & 4 & 14 & 6 & 8 & 4 & 18 \\ 11 & 6 & 4 & 15 & 12 & 13 & 4 \\ 13 & 18 & 8 & 4 & 11 & 6 & 14 \end{matrix}$$

these two IRDCS have, respectively, heft 1.01533 and 0.986483 and both have order 9.

$$\begin{matrix} 14 & 13 & 6 & 4 & 12 & 6 & 11 \\ 8 & 4 & 10 & 9 & 8 & 4 & 12 \\ 13 & 11 & 6 & 4 & 14 & 6 & 10 \\ 10 & 4 & 9 & 8 & 11 & 4 & 13 \\ 12 & 14 & 6 & 4 & 10 & 6 & 9 \end{matrix} \quad , \quad \begin{matrix} 9 & 12 & 16 & 4 & 10 & 18 & 14 \\ 10 & 4 & 11 & 8 & 9 & 4 & 15 \\ 18 & 15 & 14 & 4 & 12 & 11 & 10 \\ 8 & 4 & 10 & 9 & 8 & 4 & 16 \\ 11 & 12 & 16 & 4 & 18 & 14 & 15 \end{matrix}$$

these two IRDCS have, respectively, heft 1.07537 and 1.0165 and both have order 10.

B.2.0.25 7 by 6 IRDCS

We have calculated 532 IRDCS with dimensions 7×6. For these, the heft ranges from 0.978106 to 1.03916, and of these solutions there are 2 with order 6, 52 with order 8, 176 with order 9 and 302 with order 10. Below we present a few examples in their alternate notation.

2	12	2	18	2	6	2		15	3	5	8	3	15	18
18	2	6	2	8	2	10		8	18	3	5	19	3	12
2	8	2	10	2	6	2		3	12	13	3	5	8	3
10	2	6	2	12	2	8	,	5	3	8	15	3	5	13
2	12	2	8	2	6	2		19	5	3	12	18	3	5
8	2	6	2	10	2	18		3	13	5	3	8	19	3

these two IRDCS have, respectively, heft 1.03056 and 0.992444 and order 6 and 8. The first of these is the first solution found by the algorithm, the second is the third. The second solution found is the horizontal reversal of the first solution.

11	3	16	7	3	18	16		9	3	15	13	3	7	21
15	6	3	12	6	3	7		21	12	3	7	19	3	14
3	18	7	3	15	11	3		3	7	14	3	12	9	3
6	3	11	6	3	7	6	,	15	3	13	9	3	15	7
12	7	3	16	12	3	18		19	9	3	21	7	3	13
3	15	6	3	7	6	3		3	12	7	3	14	19	3

these two IRDCS have, respectively, heft 1.00182 and 0.985904 and have order 8 and 9.

13	3	22	6	3	9	16		9	3	20	12	3	13	16
15	6	3	11	13	3	18		19	15	3	10	9	3	15
3	18	9	3	15	6	3		3	16	13	3	14	19	3
11	3	16	6	3	9	22	,	20	3	12	9	3	10	12
22	6	3	13	11	3	16		14	10	3	15	16	3	20
3	15	9	3	18	6	3		3	19	9	3	13	14	3

these two IRDCS have, respectively, heft 1.00912 and 1.00793 with orders 9 and 10.

$$\begin{matrix} 15 & 3 & 17 & 12 & 3 & 15 & 16 \\ 11 & 9 & 3 & 10 & 9 & 3 & 14 \\ 3 & 16 & 14 & 3 & 11 & 17 & 3 \\ 10 & 3 & 12 & 15 & 3 & 10 & 12 \\ 9 & 11 & 3 & 9 & 16 & 3 & 9 \\ 3 & 17 & 10 & 3 & 14 & 11 & 3 \end{matrix} \quad , \quad \begin{matrix} 12 & 17 & 3 & 16 & 12 & 3 & 11 \\ 15 & 3 & 18 & 10 & 3 & 15 & 16 \\ 3 & 11 & 9 & 3 & 13 & 9 & 3 \\ 13 & 10 & 3 & 12 & 11 & 3 & 17 \\ 18 & 3 & 17 & 15 & 3 & 10 & 18 \\ 3 & 9 & 16 & 3 & 9 & 13 & 3 \end{matrix}$$

these two IRDCS have, respectively, heft 0.978106 and 1.03916 with orders 9 and 10.

B.2.0.26 7 by 7 IRDCS

We have calculated 41 IRDCS with dimensions 7×7. For these, the heft ranges from 0.995971 to 1.01502, and of these solutions there are 5 with order 9, 26 with order 10 and 10 with order 11. Below we present a few examples in their alternate notation.

$$\begin{matrix} 3 & 14 & 6 & 3 & 9 & 16 & 3 \\ 6 & 3 & 15 & 13 & 3 & 14 & 6 \\ 19 & 9 & 3 & 18 & 6 & 3 & 15 \\ 3 & 16 & 6 & 3 & 9 & 19 & 3 \\ 6 & 3 & 13 & 14 & 3 & 16 & 6 \\ 18 & 9 & 3 & 15 & 6 & 3 & 13 \\ 3 & 19 & 6 & 3 & 9 & 18 & 3 \end{matrix} \quad , \quad \begin{matrix} 3 & 13 & 6 & 3 & 15 & 6 & 3 \\ 18 & 3 & 16 & 19 & 3 & 14 & 16 \\ 9 & 6 & 3 & 9 & 6 & 3 & 9 \\ 3 & 15 & 14 & 3 & 18 & 13 & 3 \\ 6 & 3 & 13 & 6 & 3 & 15 & 6 \\ 19 & 9 & 3 & 16 & 9 & 3 & 14 \\ 3 & 18 & 6 & 3 & 19 & 6 & 3 \end{matrix}$$

these two IRDCS both have heft 0.996817 and order 9. They are the first two solutions found by the algorithm.

$$\begin{matrix} 3 & 15 & 9 & 3 & 18 & 6 & 3 \\ 13 & 3 & 22 & 6 & 3 & 9 & 16 \\ 12 & 6 & 3 & 12 & 13 & 3 & 12 \\ 3 & 16 & 9 & 3 & 15 & 6 & 3 \\ 15 & 3 & 18 & 6 & 3 & 9 & 22 \\ 22 & 6 & 3 & 13 & 16 & 3 & 18 \\ 3 & 12 & 9 & 3 & 12 & 6 & 3 \end{matrix} \quad , \quad \begin{matrix} 3 & 21 & 9 & 3 & 20 & 19 & 3 \\ 14 & 3 & 12 & 13 & 3 & 9 & 12 \\ 20 & 16 & 3 & 15 & 18 & 3 & 21 \\ 3 & 19 & 9 & 3 & 14 & 16 & 3 \\ 18 & 3 & 13 & 12 & 3 & 9 & 19 \\ 15 & 14 & 3 & 21 & 20 & 3 & 13 \\ 3 & 16 & 9 & 3 & 15 & 18 & 3 \end{matrix}$$

these two IRDCS have, respectively, heft 1.00154 and 1.01111 with orders 9 and 11.

3	15	9	3	24	18	3
18	3	8	13	3	9	8
12	20	3	14	12	3	20
3	8	9	3	15	8	3
15	3	13	18	3	9	24
8	14	3	12	8	3	13
3	24	9	3	20	14	3

,

3	13	20	3	15	9	3
9	3	10	21	3	14	18
12	14	3	12	9	3	12
3	15	18	3	10	13	3
20	3	13	9	3	20	21
21	10	3	14	15	3	10
3	12	9	3	12	18	3

these two IRDCS have, respectively, heft 1.01502 and 0.995971 with order 10.

B.2.0.27 9 by 3 IRDCS

There are 16 IRDCS with dimensions 9×3. For these, the heft ranges from 1.01347 to 1.06111 and there are 4 with order 5 and order 6 and there are 8 with order 8. Of these, there is essentially one solution with orders 5 and 6, and the others are the three possible reversals of this IRDCS. Similarly there are essentially two IRDCS with order 8 and the others are all the possible reversals of these IRDCS. Below we present one of each example in their alternate notation.

3	4	5	3	6	4	3	5	9
9	3	4	5	3	6	4	3	5
6	9	3	4	5	3	6	4	3

this IRDCS is constructed from the one-dimensional IRDCS of length 11 as in Lemma 7.5 and Theorem 7.1. It has heft 1.06111 and order 5.

3	10	5	3	6	8	3	5	11
11	3	8	5	3	6	10	3	5
6	10	3	11	5	3	6	8	3

this IRDCS has heft 1.01591 and order 6.

5	8	10	11	6	5	7	13	9
13	6	7	9	5	8	10	6	11
10	8	11	5	6	7	13	9	5

this IRDCS has heft 1.01347 and order 8.

9	5	7	13	6	11	5	8	10
11	6	8	10	5	9	7	6	13
13	9	5	7	6	8	11	5	10

this IRDCS has heft 1.01347 and order 8.

These last three IRDCS are not at all related to the one dimensional case.

B.2.0.28 10 by 2 IRDCS

There are 20 IRDCS with dimensions 10×2. For these, the heft ranges from 0.995635 to 1.06111 and there are 4 with order 5 and 16 with order 6. All of the order 5 IRDCS are effectively the length 11 one-dimensional IRDCS. For the remaining 16 IRDCS there are 4 different IRDCS and the remainder are all of the possible reversals for these IRDCS. Below we present one of each order 6 example in their alternate notation.

9	5	6	4	8	7	5	4	6	9
7	8	4	5	9	6	4	7	5	8

9	5	6	4	7	8	5	4	6	9
7	8	4	5	9	6	4	7	5	8

9	6	4	5	8	7	4	6	5	9
8	5	7	4	9	6	5	4	8	7

9	6	4	5	7	8	4	6	5	9
8	5	7	4	9	6	5	4	8	7

these all have heft 0.995635.

B.2.0.29 11 by 2 IRDCS

There are 24 IRDCS with dimensions 11×2. All of these have heft 0.995635 and order 6. Two of these (and their reversals) are the first and second examples for the 10×2 case extended by one horizontally to the left in the representatives that we pick. These are:

4	9	5	6	4	8	7	5	4	6	9
6	7	8	4	5	9	6	4	7	5	8

4	9	5	6	4	7	8	5	4	6	9
6	7	8	4	5	9	6	4	7	5	8

The remaining 16 examples are made up by the following 4 IRDCS and their various reversals.

4	9	5	6	4	8	7	5	4	6	9
7	8	4	5	9	6	4	7	5	8	4

4	9	5	6	4	7	8	5	4	6	9
7	8	4	5	9	6	4	7	5	8	4

8	5	7	4	6	9	5	4	8	7	6
4	9	5	6	4	7	8	5	4	6	9

6	7	8	4	5	9	6	4	7	5	8
9	6	4	5	7	8	4	6	5	9	4

These behave quite similarly to the 10×2 case, where in the examples one row has $7, 8$ while the other has $8, 7$ and the IRDCS are otherwise identical.

B.2.0.30 12 by 2 IRDCS

There are 16 IRDCS with dimensions 12×2. All of these have heft 0.995635 and order 6. Two of these (and their various reversals) are the first and last examples above for the 11×2 case extended by one horizontally to the right. These are respectively:

8	5	7	4	6	9	5	4	8	7	6
6	4	5	7	8	4	6	5	9	4	7

7	8	4	5	9	6	4	7	5	8	4
6	4	5	7	8	4	6	5	9	4	7

The other two are quite similar to the 11×2 case.

5	8	6	4	7	5	9	4	6	8	5
6	4	5	7	8	4	6	5	9	4	7

8	6	4	9	5	7	4	6	8	5	4
6	4	5	7	8	4	6	5	9	4	7

B.2.0.31 13 by 2 IRDCS

There are 300 IRDCS of dimension 13×2. These IRDCS have heft ranging from 0.986544 to 1.05397 and 188 of them have order 7 while the remaining 112 having order 8. None of these IRDCS are inherited from the 11×2 or 12×2 cases, and none are inherited from one-dimensional IRDCS since there are no one-dimensional IRDCS of length 14. A few examples of these IRDCS are presented below in their alternate notation.

12	4	9	5	6	4	7	11	5	4	6	9	12
7	11	4	12	5	6	4	7	9	5	4	6	11

this IRDCS has heft 1.04488, order 7 and is the first solution found by the algorithm.

10	4	6	9	7	4	11	8	6	4	10	7	9
11	7	4	6	8	9	4	10	7	6	4	11	8

this IRDCS has heft 0.986544 and order 7.

10	12	6	4	5	7	9	4	6	5	10	4	7
12	4	9	5	6	4	7	10	5	4	6	9	12

this IRDCS has heft 1.05397 and order 7.

12	10	6	7	5	11	8	9	6	5	7	10	12
9	11	8	5	12	6	7	10	5	9	8	6	11

this IRDCS has heft 1.01988 and order 8.

B.2.0.32 14 by 2 IRDCS

There are 748 IRDCS of dimension 14×2. These IRDCS have heft ranging from 0.986544 to 1.03377 and 252 of the IRDCS have order 7 with the remaining 496 having order 8.

B.2.0.33 15 by 2 IRDCS

There are 2352 IRDCS of dimension 15×2. These IRDCS have heft ranging from 0.985689 to 1.06347 and 268 of them have order 7 while the remaining 2084 have order 8.

B.2.0.34 16 by 2 IRDCS

There are 9872 IRDCS of dimension 16×2. These IRDCS have heft ranging from 0.980134 to 1.05797 and 240 of them have order 7, 2144 have order 8 while the remaining 7488 have order 9.

We now present a few of the larger IRDCS calculated. Note that all use the modulus 2 and therefore use only even moduli, but they are not the result of doubling some smaller IRDCS. The first example is an 11×11 IRDCS with heft 0.997505 and order 9.

52	2	4	2	16	2	4	2	20	2	4
2	20	2	4	2	56	2	4	2	16	2
4	2	24	2	4	2	32	2	4	2	24
2	4	2	16	2	4	2	20	2	4	2
20	2	4	2	40	2	4	2	16	2	4
2	56	2	4	2	24	2	4	2	52	2
4	2	16	2	4	2	20	2	4	2	32
2	4	2	32	2	4	2	16	2	4	2
24	2	4	2	52	2	4	2	24	2	4
2	16	2	4	2	20	2	4	2	40	2
4	2	40	2	4	2	16	2	4	2	56

Next is a 12×6 IRDCS with heft 1.01349 and order 8.

2	28	2	4	2	36	2	4	2	20	2	4
4	2	24	2	4	2	40	2	4	2	24	2
2	4	2	12	2	4	2	12	2	4	2	12
36	2	4	2	20	2	4	2	28	2	4	2
2	28	2	4	2	24	2	4	2	20	2	4
4	2	12	2	4	2	12	2	4	2	12	2
2	4	2	40	2	4	2	36	2	4	2	40

Now a 12×12 IRDCS with heft 0.994858 and order 10.

12	2	4	2	36	2	4	2	48	2	4	2
2	48	2	4	2	12	2	4	2	36	2	4
4	2	40	2	4	2	68	2	4	2	12	2
2	4	2	12	2	4	2	52	2	4	2	44
32	2	4	2	44	2	4	2	12	2	4	2
2	12	2	4	2	32	2	4	2	40	2	4
4	2	52	2	4	2	12	2	4	2	32	2
2	4	2	36	2	4	2	48	2	4	2	12
48	2	4	2	12	2	4	2	36	2	4	2
2	68	2	4	2	40	2	4	2	12	2	4
4	2	12	2	4	2	44	2	4	2	68	2
2	4	2	32	2	4	2	12	2	4	2	52

Then we have a 13×8 IRDCS with heft 1.01528 and order 7.

36	2	4	2	16	2	4	2	20	2	4	2	36
2	20	2	4	2	24	2	4	2	16	2	4	2
4	2	12	2	4	2	12	2	4	2	12	2	4
2	4	2	16	2	4	2	20	2	4	2	36	2
20	2	4	2	24	2	4	2	16	2	4	2	24
2	12	2	4	2	12	2	4	2	12	2	4	2
4	2	16	2	4	2	20	2	4	2	36	2	4
2	4	2	24	2	4	2	16	2	4	2	24	2

Lastly the following is one of the 13×13 IRDCS with heft 0.998065 and order 9.

2	4	2	8	2	4	2	48	2	4	2	8	2
8	2	4	2	32	2	4	2	8	2	4	2	64
2	48	2	4	2	8	2	4	2	80	2	4	2
4	2	8	2	4	2	56	2	4	2	8	2	4
2	4	2	40	2	4	2	8	2	4	2	32	2
64	2	4	2	8	2	4	2	40	2	4	2	8
2	8	2	4	2	32	2	4	2	8	2	4	2
4	2	80	2	4	2	8	2	4	2	48	2	4
2	4	2	8	2	4	2	64	2	4	2	8	2
8	2	4	2	48	2	4	2	8	2	4	2	32
2	56	2	4	2	8	2	4	2	56	2	4	2
4	2	8	2	4	2	32	2	4	2	8	2	4
2	4	2	40	2	4	2	8	2	4	2	80	2

Please see the attached DVD for additional data on both the one-dimensional and two-dimensional cases.

Appendix C

One Dimensional Heft

This appendix will present the results and proofs for one-dimensional heft as suggested by one of the examiners.

Theorem C.1. *For any IRDCS $\mathcal{A}$,*

$$h(\mathcal{A}) \geq 1 + \frac{2t^2 - nt}{n(n-t)} > \sqrt{8} - 2 > 0.828.$$

Proof. Starting from equation (2.2), let $u_i = a_i + b_i - 1 \leq 2m_i - 1$ which can be written as $m_i \geq (u_i + 1)/2$. We have $a_i + m_i \leq n + 1 - b_i$, so that $m_i \leq n - u_i$. Combining these gives $(u_i + 1)/2 \leq n - u_i$, which becomes $3u_i \leq 2n - 1$, so that $u_i < (2/3)n$. Thus the u_i are integers in the interval $[1, \frac{2}{3}n]$ with sum at least t^2.

Take the sum

$$\sum_{i=1}^{t} \frac{u_i}{m_i} \geq S = \sum_{i=1}^{t} \frac{u_i}{n - u_i}.$$

Both the u_i and the $n - u_i$ are positive. It can be shown that if $a \leq b$ then $a/(n-a) + b/(n-b) \leq (a-1)/(n-(a-1)) + (b+1)/(n-(b+1))$, so the sum S is minimized when all the u_i are equal, and so $S \geq \frac{t^2}{n-t}$. The stated inequality for h then follows from (2.2) and the numerical lower bound follows from minimising this function of t. $\square$

Theorem C.2. *For any IRDCS $\mathcal{A}$,*

$$h(\mathcal{A}) \leq 1.4055.$$

Proof. Starting from equation (2.2), say there are x_n of the m_i's in $(0, n/3]$, y_n of them in $(n/3, n/2]$ and z_n of them in $(n/2, n]$. For m_i in $(0, n/3]$, we have $\frac{u_i}{m_i} \leq 2$ and for $m_i > n/3$, we have $\frac{u_i}{m_i} \leq \frac{n}{m_i} - 1$. Thus, by (2.2), we have $n(h-1) \leq \sum \frac{n}{m_i} - 1 \leq x_n - 2y_n + n\sum_1 \frac{1}{m_i} - 2z_n + n\sum_2 \frac{1}{m_i}$, where in $\sum_1$ we take the smallest y_n values in $(n/3, n/2]$ and in $\sum_2$ we take the smallest z_n values in $(n/2, n]$. The last two terms, which involve z, together are non-positive, so we drop them and we use the sum of the harmonic series to get that $\sum_1 = \log(1+3y)$ plus an error term that tends to 0 like $O(1/n)$. Hence we get the inequality that $h \leq 1 + x - 2y + log(1+3y)$ with $x \leq 1/3$ and $y \leq 1/6$. Maximising $\log(1+3y) - 2y$ gives the required bound for heft. $\square$

The examiner sketches a proof of $h \leq 1.26425$, but I have been unable to verify the details.

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