ASSET PRICING AND PORTFOLIO OPTIMIZATION UNDER REGIME SWITCHING MODELS

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Contents

List of Tables

List of Figures

Signed Statement

I, Yang Shen, declare that this thesis, submitted in fulfilment of the requirements for the award of Doctor of Philosophy in Applied Finance and Actuarial Studies at Macquarie University, wholly represents my own work unless otherwise referenced or acknowledged. The document has not been previously included in a thesis, dissertation or report submitted to this university or any other institution for a degree, diploma or other qualifications.

The results in Chapters 2, 3 and 5 have been published on The Journal of Futures Markets (DOI: 10.1002/fut.21613), Operations Research Letters (2013; 41, 180-187), and Insurance: Mathematics and Economics (2013; 53, 757-768), respectively. I finished the works independently with necessary direction from my supervisor, Tak Kuen Siu. I thank Kun Fun for her assistance with computer programming for numerical results in Chapter 2.

Chapter 4 is unpublished working papers. I finished it independently with necessary direction from my supervisor, Tak Kuen Siu.

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Dedication

To my parents, Qianrong Shen and Jilu Li
and my girlfriend, Yueheng Peng

Abstract

Recently, there has been a considerable interest in applications of regime-switching models in various aspects of finance and insurance. One of the main features of these models is that some model parameters are modulated by a finite-state Markov chain. This makes regime-switching models very useful to describe structural changes in macro-economic conditions, periodical fluctuations in business cycles and sudden transitions in market modes.

In this thesis, a continuous-time, finite-state, observable Markov chain is adopted to model the regime switches. Our regime-switching models are a set of diffusion models, jump-diffusion models or Lévy models coupled by the underlying Markov chain. Under this modeling set up, the financial market is incomplete. So asset pricing and portfolio optimization problems are more involved.

Roughly speaking, this thesis can be divided into two parts. The first part is devoted to asset pricing problems under regime-switching models. Due to the market incompleteness, the equivalent martingale measure is not unique. Therefore, we either choose a particular equivalent martingale measure using the Esscher transform or start directly from a risk-neutral measure. We present analytical pricing formulae for European options and variance swaps in Chapters 2 and 3, respectively. Numerical and empirical implementations of these formulae show that the regime-switching effect is

material for asset pricing problems.

In the second part of this thesis, we apply the stochastic optimal control theory to discuss portfolio optimization problems under regime-switching models. In Chapter 4, we use the dynamic programming principle approach to solve a mean-variance portfolio selection problem with uncertain investment horizon. Explicit expressions of the efficient portfolio and the efficient frontier are obtained. In Chapter 5, the stochastic optimal control theory for portfolio optimization problems is borrowed to investigate an fundamental issue in asset pricing problems, i.e. the selection of equivalent martingale measures. We derive and compare equivalent martingale measures selected by three different approaches, that is, the stochastic differential game approach, the Esscher transformation approach and the general equilibrium approach.

Chapter 1

Introduction

Asset pricing, portfolio optimization and risk management are called the three pillars of modern finance. They are not only important research topics in the fields of financial mathematics and actuarial science, but also play vital roles in financial activities of both individual and institutional investors. Indeed, various problems arising in insurance and finance, such as option pricing and hedging, insurance product pricing, asset allocation, asset-liability management and optimal reinsurance, can be categorized into these three topics.

A major revolution in asset pricing came with the path-breaking works of Black and Scholes (1973) and Merton (1973). The so-called Black-Scholes-Merton formula provided an acceptable method of evaluating and pricing option contracts. Soon afterwards, the new formula was applied on the new option exchanges all over the world. In the last several decades, this formula has been extensively generalized in all possible directions, and has evolved as a methodology to determine the value of derivatives. For this, Robert C. Merton and Myron S. Scholes received the 1997 Nobel Prize in Economics. Although Fischer Black became ineligible for the prize because of his death in 1995, he was mentioned as a contributor by the Swedish academy.

The first pioneering contribution in modern portfolio selection theory was made by Markowitz's Nobel winning work. Markowitz (1952) proposed a mathematical elegant mean-variance paradigm, which simplified the multidimensional problem of investing in a large number of assets into the issue of a trade-off between only two dimensions, namely the mean and the variance of the return of the portfolio. Taking into account of consumption opportunity and preference difference, Samuelson (1969) and Merton (1969, 1971) considered investment-consumption problems in a multi-period setting and a continuous-time set up, respectively. In particular, Samuelson (1969) and Merton (1969) introduced the stochastic optimal control theory to study portfolio optimization problems in discrete and continuous time, respectively.

One of the main shortcomings of both the Black-Scholes option pricing model and the Merton asset allocation model is the use of the Geometric Brownian Motion (GBM) for the underlying asset dynamics. It is known that the GBM assumption is not realistic, fails to incorporate many important stylised empirical features of assets' returns, and makes the Black-Scholes option pricing model unable to explain some important empirical behavior of option prices, namely, the implied volatility smile or smirk. Moreover, the investment opportunity set remains constant over time under the GBM assumption. However, more economic insights and implications can be gained if the investment opportunity set is allowed to vary stochastically over time.

To investigate asset pricing and portfolio optimization problems, therefore, the first task for us is to select more suitable models for primitive assets. As said by a famous statistician, George Box, "All models are wrong, but some are useful." Although there is no perfect model, we can choose some useful models as our underlying dynamics. Among them, regime-switching models are one of the most important candidates. The history of regime-switching models may be dated back to the work of Quant (1958) and Goldfeld and Quandt (1973), where regime-switching regression models were introduced to study nonlinearities in economic data. Tong (1978, 1983) and Tong and Lim (1980) introduced the idea of regime switching to parametric nonlinear time series analysis. Hamilton (1989) popularized the applications of regime-switching models in econometrics. Since then, there has been a growing interest in applying regime-switching models in various areas of finance and economics, particularly asset pricing and portfolio optimization.

The basic idea of regime-switching models is to allow model parameters to change over time according to the states of an underlying Markov chain, which represent different market, or economic, modes. This provides flexibility in incorporating the impact of structural changes in macro-economic conditions and business cycles. For example, the states of the market can be roughly divided into two regimes, "bullish" and "bearish", in which the interest rate, appreciation rate and volatility of the asset differ. Indeed, many empirical studies reveal that regime-switching models are consistent with important stylized features of financial time series, such as time-varying conditional volatility, the asymmetry and heavy tailedness of the unconditional distribution of asset returns, volatility smile and skew and regime switches. Furthermore, through the underlying Markov chain, regime-switching models allow us to introduce stochastic interest rate, appreciation rate and volatility into the modeling framework. Then investment opportunity sets under regime-switching models may evolve in a stochastic fashion over time. Since the regime-switching models overcome many shortcomings of the GBM model, they are ideal candidates to be used as the underlying dynamics in asset pricing and portfolio optimization problems.

In recent years, much attention has been paid to studying asset pricing (particularly, option valuation) and asset allocation under regime-switching models. Naik (1993) was an early attempt to investigate option pricing under a two-state regime-switching model. Elliott at el. (2005) developed a regime-switching Esscher transform approach to derive a martingale condition for the risk-neutral measure under a regime-switching GBM model and then obtained an analytical pricing formula for European call options. Interested readers may also refer to Guo (2001), Buffington and Elliott (2002), Boyle and Draviam (2007), Siu (2008) and Yuen and Yang (2010) for option pricing, Elliott and Mamon (2003), Elliott and Siu (2009), Siu (2010) and Elliott et al. (2011) for bond pricing, and Elliott and Swishchuk (2007), Elliott et al. (2007) and Elliott and Lian (2013) for volatility derivatives pricing under various regime-switching models. Zhou and Yin (2003) considered a continuous-time mean-variance portfolio selection problem with regime-switching and obtained mean-variance efficient portfolios and efficient frontiers in closed forms. Sotomayor and Cadenillas (2009) investigated an optimal investment-consumption problem in financial markets with regime-switching and derived explicit optimal portfolio and consumption policies. Elliott and Siu (2010) studied a risk minimizing portfolio problem under a regime-switching Black-Scholes economy and formulated the problem as a stochastic differential game. Other recent works on portfolio optimization under regime-switching models include Cheung and Yang (2007), Chen et al. (2008), Wu and Li (2012) and Zhang et al. (2012), just to name a few.

This thesis includes four self-contained chapters, each of which is concerned with one asset pricing or portfolio optimization problem under regime-switching models. Option pricing has long been a challenging and interesting problem in the history of modern finance. Chapter 2 shall develop a methodology to incorporate macro-economic risks into option prices. Volatility derivatives, such as variance swaps and volatility swaps, are gaining increasing attention in the market. Which factors have impacts on determining the price of variance swaps? We shall answer this question in Chapter 3. If the terminal time is uncertain, could the classical results of the dynamic mean-variance portfolio selection problem still hold? This is the major concern of Chapter 4. Is there any relationship between asset pricing and portfolio optimization? Chapter 5 shall discuss how to find equivalent martingale measures using the stochastic optimal control theory, that is, how we can apply the method in portfolio optimization problems to solve asset pricing problems.

In Chapter 2, we consider an option valuation problem under a double regime-switching model. Roughly speaking, regime switching refers to the feature in which the model dynamics are allowed to change over time according to an underlying Markov chain. In a double regime-switching model, the parameters (such as the appreciation rate, volatility, and interest rate) and the price level are both allowed to switch when a regime switch occurs. This is in contrast to the single regime-switching models in which only the model parameters (and not the price level) are allowed to switch. The jump component of the share price is modeled by a jump martingale related to the modulating Markov chain, which introduces macro-economic risks or regime-switching risk into the modeling framework. We develop a generalized version of the regime-switching Esscher transform to select an equivalent martingale measure, which allows us to endogenously determine the regime-switching risk from the double regime-switching model. Indeed, one of the main issues in option valuation under regime-switching models is how to price the regime-switching risk. In previous literature, this regime-switching risk was either ignored or determined exogenously. Chapter 2 provides a possible solution to incorporate the regime-switching risk into option prices. Next, we use a Fourier transform to derive the integral pricing formula for an European call option. We then adopt the fast Fourier transform method to discretize the integral pricing formula. Finally, we provide numerical analysis and empirical study of option prices under single and double regime-switching models to illustrate the implications of our results. Chapter 2 is based on the paper by Shen et al. (2013).

Chapter 3 develops a hybrid of two simpler models: a stochastic interest rate model of Hull and White (1990) and a stochastic volatility model of Schöbel and Zhu (1999). Both of these basic models use the Ornstein-Uhlenbeck processes to model the behavior of two state variables, the interest rate and the volatility. Embedding them in a Markovian regime-switching model, we combine these two state variables and allow for correlation between them. We consider the valuation of a variance swap under this stochastic interest rate and volatility model with regime switches, namely the Markovian regime-switching Schöbel-Zhu-Hull-White hybrid model. The basic Gaussian structure of the distribution of the state variables allows for semi-analytical solutions of the model. We present integral solutions for the value of the variance swap as well as for the fair strike value of the variance swap. Using a Schöbel-Zhu-Hull-White hybrid model with two regimes as an example, numerical results are presented to show the impacts of stochastic interest rate and regime switches on the fair value of the variance swap. This chapter is based on the paper by Shen and Siu (2013a).

Chapter 4 discusses a continuous-time mean-variance portfolio selection problem with uncertain investment horizon under a regime-switching jump-diffusion model. More specifically, the asset price processes are modulated by a continuous-time, finite-state, observable Markov chain. To make the mean-variance problem with uncertain investment horizon tractable, we first transform it to an equivalent min-max problem with fixed investment horizon, where the (quadratic loss) minimization problem can be readily solved via the dynamic HJB programming approach. After giving a version of the verification theorem, we derive a regime-switching HJB equation related to the quadratic loss minimization problem. We provide both the expectation solution and the closed-form solution of the value function via solving the HJB equation. Using the Lagrangian duality technique, we obtain the efficient portfolio and the efficient frontier of our mean-variance problem. To illustrate our results, we provide several numerical examples of the efficient frontier under different values of parameters. This chapter is based on the paper by Shen and Siu (2013b).

In Chapter 5, we discuss three different approaches to select an equivalent martingale measure for the valuation of contingent claims under a Markovian regime-switching Lévy model, which is associated with an incomplete financial market due to jumps and regime switches. These approaches are the game theoretic approach, the Esscher transformation approach and the general equilibrium approach. Indeed, the stochastic optimal control theory, which is tailored to portfolio optimization problems, is employed in the stochastic differential game and the general equilibrium approaches. In this sense, Chapter 5 investigates the interplay between asset pricing and portfolio optimization under regime-switching models. We apply the dynamic programming principle to solve the control problems arising in the stochastic differential game and the general equilibrium approaches. We compare equivalent martingale measures chosen by the three approaches and identify the conditions under which these measures are identical. Chapter 5 is based on the paper by Shen and Siu (2013c).

Chapter 2

Option valuation under a double regime-switching model

2.1 Introduction

Since the last decade or so, there has been an interest on studying option valuation problems in regime-switching models. Switches may occur in the model parameters (e.g. the appreciation rate and the volatility) and the price level of the risky share whenever transitions in the modulating Markov chain occur. The existing literature on option valuation can be divided into two categories in terms of the regime-switching models. The former includes Guo (2001), Buffington and Elliott (2002), Elliott et al. (2005), Liu et al. (2006), Boyle and Draviam (2007), Siu (2008), Siu and Yang (2009), Yuen and Yang (2010) and others, where the regime-switching models can only describe the switches of the model parameters. The latter includes Naik (1993), Yuen and Yang (2009) and Elliott and Siu (2011), where not only the model parameters but also the price level of the share may switch whenever a regime switch occurs. To differentiate these two kinds of models, we call them the single regime-switching model and the double regime-switching model, respectively. Numerous works focus on option valuation under the single regime-switching models, while relatively little attention has been paid to that under the double regime-switching models. However, the double regime-switching models provide a more flexible way than their single regime-switching counterpart to describe stochastic movements of the risky share due to the fact that a jump in the share price level occurs in the former, but not in the latter, when there is a regime switch. Regime switches caused by transitions in the modulating Markov chain are often interpreted as structural changes in macro-economic conditions and in different stages of business cycles. These changes are inevitable in a long time span. They may cause not only shifts in the mean and volatility levels of the share price, but also sudden jumps in the share price level, (see Naik (1993), Yuen and Yang (2009) and Elliott and Siu (2011)). Except Siu and Yang (2009), almost all literature on option valuation under the single regime-switching models do not consider the regime-switching risk in the selection of a pricing kernel. Most of the existing works focus on capturing regime-dependent risk. So option prices may be underestimated under such regime-switching models. Although Siu and Yang (2009) developed a new version of regime-switching Esscher transform and found a martingale condition incorporating both the diffusion risk and the regime-switching risk, it was noted by the authors themselves that there exist more than one risk-neutral measures.

It appears that Naik (1993) was an early attempt on option pricing under the double regime-switching models, where a martingale method was employed for the pricing of a European option under a two-state, double regime-switching model. Yuen and Yang (2009) extended the model of Naik (1993) to a multi-regime case and adopted the extended model for pricing of a European option, an American option and other exotic options using a trinomial tree method. Elliott and Siu (2011) considered a risk-based approach for pricing an American contingent claim under a multi-state, double regime-switching model. Compared with the single regime-switching models, the double regime-switching models allow us to “naturally” price the regime-switching risk when one changes the real-world measure to an equivalent martingale measure. However, like a single regime-switching model, a financial market described by a double regime-switching model is also incomplete. Consequently, not all contingent claims can be perfectly hedged by continuously trading primitive securities and there is more than one pricing kernel, or equivalent martingale measure. A primal problem is how to select an equivalent martingale measure in such a market set up. In Naik (1993) and Yuen and Yang (2009), equivalent martingale measures were selected by either ignoring the regime-switching risk or taking an exogenous regime-switching risk. Neither of them determines the regime-switching risk endogenously from their double regime-switching models.

In this chapter, we consider option valuation under a double regime-switching model. More specifically, the model parameters, including the risk-free interest rate, the appreciation rate and the volatility rate, are modulated by a continuous-time, finite-state, observable Markov chain. In addition, when a regime switch occurs, there is a jump in the price level of the risky share. Consequently, the dynamics of the share is a discontinuous process. The jump component of the share price is modeled by the jump martingale related to the modulating Markov chain. We first apply a generalized version of the regime-switching Esscher transform to select a unique equivalent martingale measure, which takes into account both the diffusion risk from the Brownian motion and the regime-switching risk from the chain. Furthermore, the (local)-martingale condition and the model dynamics of the share are obtained under this equivalent martingale measure. Then we use the inverse Fourier transform to derive an integral pricing formula of a European call option. The fast Fourier transform (FFT) method is adopted to discretize the integral pricing formula. Using the FFT method, we provide the numerical analysis of option prices under both the double regime-switching model and the single regime-switching model and document the pricing implications of these two models. Numerical examples reveal that ignoring the regime-switching risk under the double regime-switching model would result in a significant underestimation of the price of an out-of-money option over 6%, even though the market prices of the regime-switching risk are relatively low in our configurations of the hypothetical values of model parameters. This illustrates the economic importance of endogenizing the regime-switching risk under the double regime-switching model. Finally, we provide an empirical application of the double regime-switching model using the real data set of the S&P 500 index options. Our empirical results reveal that endogenizing the regime-switching risk under the double regime-switching model improves the fitting and prediction errors between market prices and model prices of the S&P 500 index options. This provides empirical evidence that the regime-switching risk is priced in the market.

This chapter contributes to the existing literature in at least three aspects. Firstly, we obtain an analytical pricing formula for the multi-regime case via the Fourier transform method. The pricing formula looks neater than those available in the existing literture. In Naik (1993), the pricing formula involves the density of the occupation time of the two-state chain, which makes it difficult, if not impossible, to extend to a multi-state chain case. The trinomial tree method adopted by Yuen and Yang (2009) only gave a numerical solution to the option pricing problem under the multi-state, double regime-switching models. Our second contribution is to introduce a generalized version of the regime-switching Esscher transform and to derive its corresponding (local)-martingale condition, which admits a unique solution for determining a pricing kernel in the incomplete market described by the double regime-switching model. The (local)-martingale conditions given by Naik (1993) and Yuen and Yang (2009) had more than one solutions, which means the pricing kernels were not uniquely determined. Indeed, the selection of a pricing kernel under the double regime-switching model is still an open and challenging problem. To articulate this challenging problem, we provide a possible solution by introducing the generalized version of the regime-switching Esscher transform. It is also interesting to note that endogenizeing the regime-switching risk is crucial in ensuring the uniqueness of the pricing kernel selected by the generalized version of the regime switching Esscher transform. This may provide some theoretical insights into understanding the use of the Esscher transform for option valuation in an incomplete market. In the seminal work of Gerber and Shiu (1994), the Esscher transform was first applied to option valuation in an incomplete market. It was shown in Gerber and Shiu (1994) that a pricing kernel can be uniquely determined by the Esscher transform in a Lévy-based asset price model. This uniqueness result may not hold in the case of the double regime-switching model. We show that the generalized version of the Esscher transform in coupled with endogenizeing the regime-switching risk would lead to the uniqueness of the pricing kernel in the double regime-switching model. Thirdly, our approach allows us to calculate the market prices of the regime-switching risk endogenously from the model parameters of the double regime-switching models, which provides a quantification for how large the regime-switching risk is.

The rest of this chapter is organized as follows. The next section presents the model dynamics. In Section 2.3, we select an equivalent martingale measure using the generalized version of the regime-switching Esscher transform. Section 2.4 applies the inverse Fourier transform to derive an analytical option pricing formula. In Section 2.5, we give numerical examples to illustrate the valuation of the European call options via the FFT. In Section 2.6, we provide an empirical application of the double regime-switching model. Section 2.7 concludes the chapter.

2.2 The model dynamics

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, under which all sources of randomness are defined, including a standard Brownian motion and a Markov chain. We equip the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} := \{\mathcal{F}(t)|t \in \mathcal{T}\}$ satisfying the usual conditions of right-continuity and $\mathbb{P}$-completeness. Suppose that $\mathbb{P}$ is the real-world probability measure. Let $\mathcal{T}$ denote the time index set $[0, T]$ of the model, where $T < \infty$. We describe the evolution of the state of an economy over time by a continuous-time, finite-state, observable Markov chain $\mathbf{X} := \{\mathbf{X}(t)|t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a finite-state space $\mathcal{S} := \{\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_N\}$. The states of the chain $\mathbf{X}$ are interpreted as different states of an economy or different stages of a business cycle. Indeed, these states may be regarded as proxies for different levels of some observable macro-economic indicators such as Gross Domestic Product, Consumer Price Index, Sovereign Credit Ratings and others. Without loss of generality, we adopt the canonical state space representation of the chain in Elliott et al. (1994) and identify the states of the chain with a finite set of standard unit vectors $\mathscr{E} := \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N\} \subset \mathbb{R}^N$, where the l^{th} component of $\mathbf{e}_j$ is the Kronecker delta δ_{jl}, for each $j, l = 1, 2, \cdots, N$.

Let $\mathbf{A} := [a_{jl}]_{j,l=1,2,\cdots,N}$ be the rate matrix of the chain $\mathbf{X}$ under $\mathbb{P}$, where a_{jl} is a constant transition intensity of the chain $\mathbf{X}$ from state $\mathbf{e}_l$ to state $\mathbf{e}_j$. Note that $a_{jl} \geq 0$, for $j \neq l$ and $\sum_{l=1}^{N} a_{jl} = 0$, so $a_{jj} \leq 0$, for each $j, l = 1, 2, \cdots, N$. Under the canonical state space representation of $\mathbf{X}$, Elliott et al. (1994) obtained the following semi-martingale dynamics for the chain:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}\mathbf{X}(s)ds + \mathbf{M}(t) \ , \quad t \in \mathcal{T} \ ,$$

where $\{\mathbf{M}(t)|t \in \mathcal{T}\}$ is an $\mathbb{R}^N$-valued, $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$ -martingale. Here $\mathbb{F}^{\mathbf{X}} := \{\mathcal{F}^{\mathbf{X}}(t)|t \in \mathcal{T}\}$ is the right-continuous, $\mathbb{P}$-complete, natural filtration generated by the chain $\mathbf{X}$.

In what follows, we introduce a set of basic martingales associated with the chain $\mathbf{X}$. For each $t \in \mathcal{T}$ and $j, l = 1, 2, \cdots, N$, let $J^{jl}(t)$ be the number of jumps of the chain $\mathbf{X}$ from state $\mathbf{e}_j$ to state $\mathbf{e}_l$ up to time t. That is

$$\begin{aligned}
J^{jl}(t) &:= \sum_{0<s\leq t} \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \langle \mathbf{X}(s), \mathbf{e}_l \rangle \\
&= \sum_{0<s\leq t} \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \langle \mathbf{X}(s) - \mathbf{X}(s-), \mathbf{e}_l \rangle \\
&= \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \langle d\mathbf{X}(s), \mathbf{e}_l \rangle \\
&= \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \langle \mathbf{A}\mathbf{X}(s), \mathbf{e}_l \rangle \, ds + \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \langle d\mathbf{M}(s), \mathbf{e}_l \rangle \\
&= a_{jl} \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \, ds + m_{jl}(t) \ ,
\end{aligned}$$

where $m_{jl}(t) := \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \langle d\mathbf{M}(s), \mathbf{e}_l \rangle$ and, for each $j, l = 1, 2, \cdots, N$, $m_{jl} := \{m_{jl}(t) | t \in \mathcal{T}\}$ is an $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingale.

For each $l = 1, 2, \cdots, N$, $\Phi_l(t)$ counts the number of jumps of the chain $\mathbf{X}$ into state $\mathbf{e}_l$ from other states up to time t, i.e.

$$\begin{aligned}
\Phi_l(t) &:= \sum_{j=1, j\neq l}^{N} J^{jl}(t) \\
&= \sum_{j=1, j\neq l}^{N} a_{jl} \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \, ds + \widetilde{\Phi}_l(t) \ ,
\end{aligned}$$

where $\widetilde{\Phi}_l := \{\widetilde{\Phi}_l(t) | t \in \mathcal{T}\}$, with $\widetilde{\Phi}_l(t) := \sum_{j=1, j\neq l}^{N} m_{jl}(t)$, is an $(\mathbb{F}^{\mathbf{X}}, \mathbb{P})$-martingale.

Denote by, for each $l = 1, 2, \cdots, N$,

$$\phi_l(t) := \sum_{j=1, j\neq l}^{N} a_{jl} \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \, ds \ ,$$

and

$$a_l(t) := \sum_{j=1, j\neq l}^{N} a_{jl} \langle \mathbf{X}(t), \mathbf{e}_j \rangle \ .$$

Then, the martingale $\widetilde{\Phi}_l$ and its differential form can be represented as

$$\widetilde{\Phi}_l(t) = \Phi_l(t) - \phi_l(t) \ ,$$

and

$$d\widetilde{\Phi}_l(t) = d\Phi_l(t) - a_l(t-)dt \ ,$$

for each $l = 1, 2, \cdots, N$. The former representation may be related to a version of the Doob-Meyer decomposition for a counting process relating to the chain.

We consider a continuous-time financial market with two primitive assets, namely, a bank account B and a risky share S. The instantaneous market interest rate is given by

$$r(t) := \langle \mathbf{r}, \mathbf{X}(t) \rangle \ ,$$

where $\mathbf{r} := (r_1, r_2, \cdots, r_N)' \in \mathbb{R}^N$ with $r_j > 0$ for each $j = 1, 2, \cdots, N$; $\mathbf{y}'$ is the transpose of a vector or a matrix $\mathbf{y}$; $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^N$. Then the dynamics of the price process $B := \{B(t) | t \in \mathcal{T}\}$ for the bank account is given by

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1 \ .$$

Similarly, the appreciation rate $\mu(t)$ and the volatility $\sigma(t)$ of the risky share are also modulated by $\mathbf{X}$ as follows:

$$\mu(t) := \langle \boldsymbol{\mu}, \mathbf{X}(t) \rangle \ , \quad \sigma(t) := \langle \boldsymbol{\sigma}, \mathbf{X}(t) \rangle \ , \quad t \in \mathcal{T} \ ,$$

where $\boldsymbol{\mu} := (\mu_1, \mu_2, \cdots, \mu_N)' \in \mathbb{R}^N$ and $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \cdots, \sigma_N)' \in \mathbb{R}^N$ with $\mu_j > r_j$ and $\sigma_j > 0$ for each $j = 1, 2, \cdots, N$.

Let $W := \{W(t) | t \in \mathcal{T}\}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. To simplify our discussion, we suppose that W and $\mathbf{X}$ are stochastically independent under $\mathbb{P}$.

The price process of the risky share $S := \{S(t) | t \in \mathcal{T}\}$ then evolves over time according to the following double regime-switching model:

$$\frac{dS(t)}{S(t-)} = \mu(t-)dt + \sigma(t-)dW(t) + \sum_{l=1}^{N} \left(e^{\beta_l(t-)} - 1\right) d\widetilde{\Phi}_l(t) \ , \quad S(0) = S_0 > 0 \ , \tag{2.2.1}$$

where $\beta_l(t) := \langle \boldsymbol{\beta}_l, \mathbf{X}(t) \rangle$ and $\boldsymbol{\beta}_l := (\beta_{1l}, \beta_{2l}, \cdots, \beta_{Nl})' \in \mathbb{R}^N$. Write $\boldsymbol{\beta} := (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_N) \in \mathbb{R}^{N \times N}$. Here we provide the flexibility that the jump size of the share price depends on the states of the chain before and after a state transition, i.e., $e^{\beta_{jl}} - 1$ is the ratio of a jump in the share price level when the chain transits from state $\mathbf{e}_j$ to state $\mathbf{e}_l$. We further assume that $\beta_{ll} = 0$, which implies that there is no jump in the share price level when the chain $\mathbf{X}$ remains in state $\mathbf{e}_l$, for each $l = 1, 2, \cdots, N$.

The double regime-switching model (2.2.1) is an N-state extension of Naik (1993), which was also considered in Yuen and Yang (2009). The key feature of the double regime-switching model is that a change in the state of an underlying economy not only causes a structural change in the dynamics of the risky share price, but also a sudden jump in the price level of the risky share. The assumption that there is a jump in the price level when the regime changes is not unreasonable if the regime-switching risk (the risk of state transitions) is non-diversifiable. For example, when jumps in price levels of different assets in a market during a state transition would not be cancelled out with each other as a whole, a state transition really imposes a change in the price level of an aggregate portfolio consisting of these risky assets in the market. Here we suppose that the regime-switching risk is non-diversifiable. This assumption is also not unreasonable given the fact that macro-economic risks are usually structural in nature. When the regime-switching risk is non-diversifiable, how to price this source of risk becomes a key question. We shall address this important question in the next section.

For each $t \in \mathcal{T}$, we define $Y(t) = \log(S(t)/S_0)$ as the logarithmic return of the share over time horizon $[0, t]$. Then by Itô's differentiation rule, it is easy to see that

$$\begin{aligned} dY(t) &= \left[\mu(t-) - \frac{1}{2}\sigma^2(t-) - \sum_{l=1}^{N} \left(e^{\beta_l(t-)} - 1 - \beta_l(t-)\right) a_l(t-)\right] dt \\ &\quad + \sigma(t-) dW(t) + \sum_{l=1}^{N} \beta_l(t-) d\widetilde{\Phi}_l(t) \ , \quad t \in \mathcal{T} \ . \end{aligned} \tag{2.2.2}$$

Write $Y := \{Y(t)|t \in \mathcal{T}\}$. Let $\mathbb{F}^S = \{\mathcal{F}^S(t)|t \in \mathcal{T}\}$ and $\mathbb{F}^Y = \{\mathcal{F}^Y(t)|t \in \mathcal{T}\}$ be the right-continuous, $\mathbb{P}$-complete, natural filtrations generated by the processes S and Y, respectively. Since $\mathbb{F}^S$ and $\mathbb{F}^Y$ are equivalent, we can use either one of them as an observed information structure. We define the filtration $\mathbb{G} = \{\mathcal{G}(t)|t \in \mathcal{T}\}$ by setting $\mathcal{G}(t) := \mathcal{F}^Y(t) \vee \mathcal{F}^{\mathbf{X}}(t)$, the minimal σ-field containing $\mathcal{F}^Y(t)$ and $\mathcal{F}^{\mathbf{X}}(t)$.

2.3 Esscher transform and equivalent martingale measure

Esscher transform is a time-honored tool in actuarial science. Gerber and Shiu (1994) pioneered the use of the Esscher transform in option valuation. Indeed, it provides a convenient tool to specify an equivalent martingale measure in an incomplete market. Under the single regime-switching model, Elliott et al. (2005) introduced a regime-switching version of the Esscher transform for option valuation. Siu (2008) justified that the equivalent martingale measure selected by the regime-switching Esscher transform is related to a saddle point of a stochastic differential game for the expected power utility maximization. Siu (2011) further verified that this equivalent martingale measure coincides with the minimal relative entropy measure.

In this section, we first present a generalization of the regime-switching Esscher transform to select an equivalent martingale measure under the double regime-switching model. Then we derive the (local)-martingale condition and obtain the model dynamics under the equivalent martingale measure.

Let $L(Y)$ be the space of all processes $\theta := \{\theta(t)|t \in \mathcal{T}\}$ such that

1. For each $t \in \mathcal{T}$, $\theta(t) := \langle \boldsymbol{\theta}, \mathbf{X}(t) \rangle$, where $\boldsymbol{\theta} := (\theta_1, \theta_2, \cdots, \theta_N)' \in \mathbb{R}^N$;

2. θ is integrable with respect to Y in the sense of stochastic integration.

For each $\theta \in L(Y)$, we write

$$(\theta \cdot Y)(t) := \int_0^t \theta(s) dY(s) \ , \quad t \in \mathcal{T} \ ,$$

for the stochastic integral of θ with respect to Y. We call θ the Esscher transform parameter in the sequel.

For each $\theta \in L(Y)$, define a $\mathbb{G}$-adapted exponential process $D^\theta := \{D^\theta(t)|t \in \mathcal{T}\}$ by

$$D^\theta(t) := \exp((\theta \cdot Y)(t)) \ .$$

Applying Itô's differentiation rule to $D^\theta(t)$ under $\mathbb{P}$, we have

$$D^\theta(t) = 1 + \int_0^t D^\theta(s-) dH^\theta(s) \ ,$$

where $H^\theta := \{H^\theta(t)|t \in \mathcal{T}\}$ is defined as a $\mathbb{G}$-adapted process

$$\begin{aligned} H^\theta(t) \ &:= \ \int_0^t \theta(s) \Big[\mu(s) - \frac{1}{2}\sigma^2(s) - \sum_{l=1}^N \big(e^{\beta_l(s)} - 1 - \beta_l(s) \big) a_l(s) \Big] ds \\ &+ \int_0^t \frac{1}{2} \theta^2(s) \sigma^2(s) ds + \int_0^t \sum_{l=1}^N \big(e^{\theta(s)\beta_l(s)} - 1 - \theta(s)\beta_l(s) \big) a_l(s) ds \\ &+ \int_0^t \theta(s)\sigma(s) dW(s) + \int_0^t \sum_{l=1}^N \big(e^{\theta(s)\beta_l(s)} - 1 \big) d\widetilde{\Phi}_l(s) \ . \end{aligned}$$

Consequently, D^θ is the Doléans-Dade stochastic exponential of H^θ, i.e.

$$D^\theta(t) = \mathcal{E}(H^\theta(t)) \ , \quad t \in \mathcal{T} \ .$$

Since H^θ is a special semi-martingale, its predictable part of finite variation is the Laplace cumulant process [1] of the stochastic integral process $(\theta \cdot Y)$ under $\mathbb{P}$. The Laplace cumulant process $\mathcal{M}^\theta := \{\mathcal{M}^\theta(t) | t \in \mathcal{T}\}$ is given by

$$\begin{aligned}\mathcal{M}^\theta(t) \quad &= \quad \int_0^t \theta(s)\Big[\mu(s) - \frac{1}{2}\sigma^2(s) - \sum_{l=1}^{N}\big(e^{\beta_l(s)} - 1 - \beta_l(s)\big)a_l(s)\Big]ds \qquad (2.3.1)\\ &\quad + \int_0^t \frac{1}{2}\theta^2(s)\sigma^2(s)ds + \int_0^t \sum_{l=1}^{N}\big(e^{\theta(s)\beta_l(s)} - 1 - \theta(s)\beta_l(s)\big)a_l(s)ds \ .\end{aligned}$$

The Doléans-Dade exponential $\mathcal{E}(\mathcal{M}^\theta(t))$ of $\mathcal{M}^\theta(t)$ is (up to indistinguishability unique) the solution of the following equation:

$$\mathcal{E}(\mathcal{M}^\theta(t)) = 1 + \int_0^t \mathcal{E}(\mathcal{M}^\theta(s))d\mathcal{M}^\theta(s) \ , \quad t \in \mathcal{T} \ .$$

Given the fact that $\{\mathcal{M}^\theta(t) | t \in \mathcal{T}\}$ is a finite variation process,

$$\mathcal{E}(\mathcal{M}^\theta(t)) = \exp(\mathcal{M}^\theta(t)) \ .$$

Consequently, the logarithmic transform $\widetilde{\mathcal{M}}^\theta := \{\widetilde{\mathcal{M}}^\theta(t) | t \in \mathcal{T}\}$ of $\mathcal{M}^\theta(t)$, for each $\theta \in L(Y)$, is given by

$$\widetilde{\mathcal{M}}^\theta(t) := \log(\mathcal{E}(\mathcal{M}^\theta(t))) = \mathcal{M}^\theta(t) \ , \quad t \in \mathcal{T} \ . \qquad (2.3.2)$$

Let $\Lambda^\theta := \{\Lambda^\theta(t) | t \in \mathcal{T}\}$ be a $\mathbb{G}$-adapted process associated with $\theta \in L(Y)$ as follows:

$$\Lambda^\theta(t) := \exp\big((\theta \cdot Y)(t) - \widetilde{\mathcal{M}}^\theta(t)\big) \ , \quad t \in \mathcal{T} \ .$$

[1] For more discussion on the Laplace cumulant process as well as the Esscher transform given below, interested readers can refer to Kallsen and Shiryaev (2002) and Elliott and Siu (2013).

Then from (2.2.2) and (2.3.2), we obtain

$$\begin{aligned}\Lambda^{\theta}(t) &= \exp\left\{\int_0^t \theta(s)\sigma(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)\sigma^2(s)ds + \int_0^t \sum_{l=1}^{N} \theta(s)\beta_l(s)d\widetilde{\Phi}_l(s)\right. \\ &\quad \left. - \int_0^t \sum_{l=1}^{N} \left[e^{\theta(s)\beta_l(s)} - 1 - \theta(s)\beta_l(s)\right] a_l(s)dt\right\} . \end{aligned} \quad (2.3.3)$$

Lemma 2.3.1. Λ^{θ} *is a* $(\mathbb{G}, \mathbb{P})$*-(local)-martingale.*

Proof. Applying Itô's differentiation rule to (2.3.3) under $\mathbb{P}$ gives

$$\Lambda^{\theta}(t) = 1 + \int_0^t \Lambda^{\theta}(s-)\theta(s)\sigma(s)dW(s) + \int_0^t \Lambda^{\theta}(s-)\sum_{l=1}^{N}\left(e^{\theta(s)\beta_l(s)} - 1\right)d\widetilde{\Phi}_l(s) \ .$$

Since the processes $\{\theta(t)\sigma(t)|t \in \mathcal{T}\}$ and $\{e^{\theta(t)\beta_l(t)} - 1|t \in \mathcal{T}\}$, $l = 1, 2, \cdots N$, can only take finite different values, they are bounded. Consequently, Λ^{θ} is a $(\mathbb{G}, \mathbb{P})$-(local)-martingale. □

For each $\theta \in L(Y)$, we define a new probability measure $\mathbb{Q}^{\theta}$ equivalent to $\mathbb{P}$ on $\mathcal{G}(T)$ by a generalized version of the regime-switching Esscher transform $\Lambda^{\theta}(T)$ as follows:

$$\left.\frac{d\mathbb{Q}^{\theta}}{d\mathbb{P}}\right|_{\mathcal{G}(T)} := \Lambda^{\theta}(T) \ .$$

According to the fundamental theorem of asset pricing established by Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983), the absence of arbitrage is "essentially" equivalent to the existence of an equivalent martingale measure under which discounted asset prices are (local)-martingales. The following lemma presents a necessary and sufficient condition for the (local)-martingale condition.

Lemma 2.3.2. *Define the discounted price of the risky share as follows:*

$$\widetilde{S}(t) := \exp\left\{-\int_0^t r(s)ds\right\} S(t), \quad t \in \mathcal{T} \ .$$

Then the discounted price process $\widetilde{S} := \{\widetilde{S}(t)|t \in \mathcal{T}\}$ *is a* $(\mathbb{G}, \mathbb{Q}^\theta)$*-(local)-martingale if and only if the Esscher transform parameter* θ *satisfies the following equation:*

$$\mu(t) - r(t) + \theta(t)\sigma^2(t) + \sum_{l=1}^{N} \big(e^{\theta(t)\beta_l(t)} - 1\big)\big(e^{\beta_l(t)} - 1\big)a_l(t) = 0 \ . \tag{2.3.4}$$

Proof. By Lemma 7.2.2 in Elliott and Kopp (2004), $\widetilde{S}$ is a $(\mathbb{G}, \mathbb{Q}^\theta)$-(local)-martingale is equivalent to that $\Lambda^\theta \widetilde{S} := \{\Lambda^\theta(t)\widetilde{S}(t)|t \in \mathcal{T}\}$ is a $(\mathbb{G}, \mathbb{P})$-(local)-martingale. By Itô's differentiation rule and the fact that $\{\Phi_j(t)|t \in \mathcal{T}\}$ and $\{\Phi_l(t)|t \in \mathcal{T}\}$, do not change for a common jump when $j \neq l$, $j, l = 1, 2, \cdots N$, we have

$$\begin{aligned}
&\Lambda^\theta(t)\widetilde{S}(t) - \Lambda^\theta(0)\widetilde{S}(0) \\
&= \int_0^t \Lambda^\theta(s-)d\widetilde{S}(s) + \int_0^t \widetilde{S}(s-)d\Lambda^\theta(s) + \int_0^t d[\widetilde{S}(s), \Lambda^\theta(s)]^c + \sum_{0<s\leq t} \Delta\Lambda^\theta(s)\Delta\widetilde{S}(s) \\
&= \int_0^t \Lambda^\theta(s-)\widetilde{S}(s-)(\mu(s) - r(s))ds + \int_0^t \Lambda^\theta(s-)\widetilde{S}(s-)\sigma(s)dW(s) \\
&\quad + \int_0^t \Lambda^\theta(s-)\widetilde{S}(s-) \sum_{l=1}^{N} \big(e^{\beta_l(s)} - 1\big)d\widetilde{\Phi}_l(s) + \int_0^t \Lambda^\theta(s-)\widetilde{S}(s-)\theta(s)\sigma(s)dW(s) \\
&\quad + \int_0^t \Lambda^\theta(s-)\widetilde{S}(s-) \sum_{l=1}^{N} \big(e^{\theta(s)\beta_l(s)} - 1\big)d\widetilde{\Phi}_l(s) + \int_0^t \Lambda^\theta(s-)\widetilde{S}(s-)\theta(s)\sigma^2(s)ds \\
&\quad + \int_0^t \Lambda^\theta(s-)\widetilde{S}(s-) \sum_{l=1}^{N} \big(e^{\beta_l(s)} - 1\big)\big(e^{\theta(s)\beta_l(s)} - 1\big)d\widetilde{\Phi}_l(s) \\
&\quad + \int_0^t \Lambda^\theta(s-)\widetilde{S}(s-) \sum_{l=1}^{N} \big(e^{\beta_l(s)} - 1\big)\big(e^{\theta(s)\beta_l(s)} - 1\big)a_l(s)ds \ .
\end{aligned} \tag{2.3.5}$$

Then $\Lambda^\theta \widetilde{S}$ is a $(\mathbb{G}, \mathbb{Q}^\theta)$-(local)-martingale if and only if the predictable part of finite variation in (2.3.5) is indistinguishable from the zero process. That is

$$\mu(t) - r(t) + \theta(t)\sigma^2(t) + \sum_{l=1}^{N} \big(e^{\theta(t)\beta_l(t)} - 1\big)\big(e^{\beta_l(t)} - 1\big)a_l(t) = 0 \ .$$

This completes the proof. □

Remark 2.3.1. *When the chain is in state* $\mathbf{e}_j$*, Equation (2.3.4) becomes the following* N *equations:*

$$\mu_j - r_j + \theta_j\sigma_j^2 + \sum_{l=1, l\neq j}^{N} (e^{\theta_j\beta_{jl}} - 1)(e^{\beta_{jl}} - 1)a_{jl} = 0, \quad j = 1, 2, \cdots, N \ . \tag{2.3.6}$$

Once the regime-switching parameter θ_j *for state* $\mathbf{e}_j$ *is determined, the market prices of the regime-switching risk from state* $\mathbf{e}_j$ *to state* $\mathbf{e}_l$ *can be calculated as* $e^{\theta_j\beta_{jl}} - 1$*, for each* $j, l = 1, 2, \cdots, N$ *and* $l \neq j$*. Note that when the regime-switching risk is not priced (see Section 5.1 in Yuen and Yang (2009)), the (local)-martingale condition becomes*

$$\mu(t) - r(t) + \theta(t)\sigma^2(t) = 0 \ , \tag{2.3.7}$$

or

$$\mu_j - r_j + \theta_j\sigma_j^2 = 0, \quad j = 1, 2, \cdots, N \ . \tag{2.3.8}$$

The following lemma follows from Lemma 2.3 in Dufour and Elliott (1999). We present it here without giving the proof.

Lemma 2.3.3. *For each* $t \in \mathcal{T}$*, let*

$$W^\theta(t) := W(t) - \int_0^t \theta(s)\sigma(s)ds \ ,$$

and

$$\begin{aligned} \widetilde{\Phi}_l^\theta(t) &:= \Phi_l(t) - \phi_l^\theta(t) \\ &= \Phi_l(t) - \int_0^t a_l^\theta(s-)ds \ , \end{aligned}$$

where

$$\phi_l^\theta(t) := e^{\theta(t)\beta_l(t)}\phi_l(t)$$

$$= \sum_{j=1, j\neq l}^{N} e^{\theta_j \beta_{jl}} a_{jl} \int_0^t \langle \mathbf{X}(s-), \mathbf{e}_j \rangle \, ds \ ,$$

and

$$\begin{aligned} a_l^{\theta}(t) &:= e^{\theta(t)\beta_l(t)} a_l(t) \\ &= \sum_{j=1, j\neq l}^{N} e^{\theta_j \beta_{jl}} a_{jl} \langle \mathbf{X}(t), \mathbf{e}_j \rangle \ . \end{aligned}$$

Then $W^{\theta} := \{W^{\theta}(t) | t \in \mathcal{T}\}$ *is a standard Brownian motion under* $\mathbb{Q}^{\theta}$*, and* $\widetilde{\Phi}_l^{\theta} := \{\widetilde{\Phi}_l^{\theta}(t) | t \in \mathcal{T}\}$ *is an* $(\mathbb{F}^{\mathbf{X}}, \mathbb{Q}^{\theta})$*-martingale, for each* $l = 1, 2, \cdots, N$*.*

Furthermore, suppose $\mathbf{A}^{\theta}$ *is an* $(N \times N)$*-matrix with the following entries:*

$$a_{jl}^{\theta} := \begin{cases} e^{\theta_j \beta_{jl}} a_{jl} \ , & j \neq l \ , \\ -\displaystyle\sum_{l=1, l\neq j}^{N} e^{\theta_j \beta_{jl}} a_{jl} \ , & j = l \ . \end{cases}$$

Then the chain $\mathbf{X}$ *has the following semimartingale decomposition under* $\mathbb{Q}^{\theta}$

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}^{\theta} \mathbf{X}(s) ds + \mathbf{M}^{\theta}(t) \ ,$$

where $\mathbf{M}^{\theta} := \{\mathbf{M}^{\theta}(t) | t \in \mathcal{T}\}$ *is an* $\mathbb{R}^N$*-valued,* $(\mathbb{F}^{\mathbf{X}}, \mathbb{Q}^{\theta})$*-martingale.*

Lemma 2.3.4. *Under* $\mathbb{Q}^{\theta}$*, the dynamics of the return process are given by:*

$$\begin{aligned} dY(t) &= \left[r(t-) - \frac{1}{2}\sigma^2(t-) - \sum_{l=1}^{N} e^{\theta(t-)\beta_l(t-)} (e^{\beta_l(t-)} - 1 - \beta_l(t-)) a_l(t-) \right] dt \\ &\quad + \sigma(t-) dW^{\theta}(t) + \sum_{l=1}^{N} \beta_l(t-) d\widetilde{\Phi}_l^{\theta}(t) \ , \quad t \in \mathcal{T} \ . \qquad (2.3.9) \end{aligned}$$

Proof. From Lemmas 2.3.2 and 2.3.3, we immediately have

$dY(t)$

$$= \left[\mu(t-) - \frac{1}{2}\sigma^2(t-) - \sum_{l=1}^{N}\big(e^{\beta_l(t-)} - 1 - \beta_l(t-)\big)a_l(t-)\right]dt$$

$$+ \sigma(t-)\big(dW^\theta(t) + \theta(t-)\sigma(t-)dt\big) + \sum_{l=1}^{N}\beta_l(t-)\big(d\widetilde{\Phi}_l^\theta(t) + (a_l^\theta(t-) - a_l(t-))dt\big)$$

$$= \left[r(t-) - \frac{1}{2}\sigma^2(t-) - \sum_{l=1}^{N} e^{\theta(t-)\beta_l(t-)}(e^{\beta_l(t-)} - 1 - \beta_l(t-))a_l(t-)\right]dt$$

$$+ \sigma(t-)dW^\theta(t) + \sum_{l=1}^{N}\beta_l(t-)d\widetilde{\Phi}_l^\theta(t) \ .$$

This completes the proof. □

The generalized version of the regime-switching Esscher transform allows us to find an equivalent martingale measure incorporating not only the diffusion risk described by the Brownian motion but also the regime-switching risk modeled by the Markov chain. Furthermore, the regime-switching risk is endogenously determined in our modeling framework. This advances over the works of Naik (1993) and Yuen and Yang (2009), where the regime-switching risk is either ignored or taken exogenously. More specifically, our method provides us with the flexibility to evaluate the market prices of the regime-switching risk once the model parameters are known. For an N-state Markov chain, the double regime-switching model reduces to the single one when $\boldsymbol{\beta}$ is an $(N \times N)$-zero matrix. In this sense, the double regime-switching model incorporates the single one. The regime-switching Esscher transform adopted by Elliott et al. (2005) and Siu (2008, 2011) is a particular case of our generalized version of the regime-switching Esscher transform.

2.4 Option valuation using the fast Fourier transform

In this section, we apply the inverse Fourier transform to derive an analytical option pricing formula under the double regime-switching model. For ease of computation, we then use the FFT method to discretize the pricing formula.

Consider a European call option written on the share S with strike K and maturity $T > 0$. Under the risk-neutral probability measure $\mathbb{Q}^\theta$, the option price $C(0, T, K)$ at time zero is given by

$$C(0,T,K) = \mathrm{E}^\theta\left[\exp\left(-\int_0^T r(t)dt\right)(S_0 e^{Y(T)} - K)_+\right] ,$$

where $\mathrm{E}^\theta[\cdot]$ denotes an expectation under $\mathbb{Q}^\theta$. Denote by $k = \log(K/S_0)$ the modified strike price, then the above equation can be written as

$$C(0,T,k) = S_0\mathrm{E}^\theta\left[\exp\left(-\int_0^T r(t)dt\right)(e^{Y(T)} - e^k)_+\right] .$$

As in Carr and Madan (1999) and Liu et al. (2006), we define the dampened call option price by

$$c(0,T,k) := e^{\alpha k}\frac{C(0,T,k)}{S_0} ,$$

where α is called the dampening coefficient and is assumed to be positive such that $c(0, T, k)$ is square integrable with respect to k over the entire real line. We consider the Fourier transform of the dampened call price $c(0, T, k)$:

$$\psi(0,T,u) = \int_{\mathbb{R}} e^{iuk}c(0,T,k)dk , \quad i = \sqrt{-1} . \tag{2.4.1}$$

For each $t \in \mathcal{T}$ and $u \in \mathbb{R}$, let

$$\varphi_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u) := \mathrm{E}^\theta\left[e^{iuY(t)}|\mathcal{F}^{\mathbf{X}}(t)\right] ,$$

and

$$\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u) := \exp\bigg(- \int_0^t r(s)ds \bigg) \varphi_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u) \ ,$$

be the conditional characteristic function and its discounted version of $Y(t)$ given $\mathcal{F}^{\mathbf{X}}(t)$ under $\mathbb{Q}^\theta$. So the unconditional, discounted characteristic function of $Y(t)$ under $\mathbb{Q}^\theta$ is given by

$$\widetilde{\varphi}_{Y(t)}(0,t,u) = \mathrm{E}^\theta[\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)] \ .$$

Before giving the option pricing formula, we present two useful lemmas.

Lemma 2.4.1. *Under $\mathbb{Q}^\theta$, the Fourier transform $\psi(0,T,u)$ of the the dampened call price and the unconditional, discounted characteristic function $\widetilde{\varphi}_{Y(T)}(0,T,u)$ of $Y(T)$ has the following relationship*

$$\psi(0,T,u) = \frac{\widetilde{\varphi}_{Y(T)}(0,T,u-i(\alpha+1))}{\alpha^2+\alpha-u^2+(2\alpha+1)iu} \ . \tag{2.4.2}$$

Proof. For notational simplicity, write $R_T := \int_0^T r(t)dt$. Let $F_{Y(T)|\mathcal{F}^{\mathbf{X}}(T)}(y)$ be the conditional distribution function of $Y(T)$ given $\mathcal{F}^{\mathbf{X}}(T)$ under $\mathbb{Q}^\theta$. Then

$$\begin{aligned}
\psi(0,T,u) &= \int_{\mathbb{R}} e^{iuk} c(0,T,k)dk \\
&= \int_{\mathbb{R}} e^{iuk} e^{\alpha k} \mathrm{E}^\theta[e^{-R_T}(e^{Y(T)} - e^k)_+]dk \\
&= \mathrm{E}^\theta\bigg[\int_{\mathbb{R}} e^{iuk} e^{\alpha k} \mathrm{E}^\theta[e^{-R_T}(e^{Y(T)} - e^k)_+ | \mathcal{F}^{\mathbf{X}}(T)]dk \bigg] \\
&= \mathrm{E}^\theta\bigg[\int_{\mathbb{R}} e^{-R_T} e^{iuk} e^{\alpha k} \int_k^\infty (e^y - e^k) F_{Y(T)|\mathcal{F}^{\mathbf{X}}(T)}(dy)dk \bigg] \\
&= \mathrm{E}^\theta\bigg[\int_{\mathbb{R}} e^{-R_T} \int_{-\infty}^y (e^y e^{(\alpha+iu)k} - e^{(1+\alpha+iu)k})dk F_{Y(T)|\mathcal{F}^{\mathbf{X}}(T)}(dy) \bigg] \\
&= \mathrm{E}^\theta\bigg[\int_{\mathbb{R}} e^{-R_T} \bigg(\frac{e^{(1+\alpha+iu)y}}{\alpha+iu} - \frac{e^{(1+\alpha+iu)y}}{1+\alpha+iu} \bigg) F_{Y(T)|\mathcal{F}^{\mathbf{X}}(T)}(dy) \bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathrm{E}^\theta[e^{-R_T}\varphi_{Y(T)|\mathcal{F}^{\mathbf{X}}(T)}(u-i(\alpha+1))]}{\alpha^2+\alpha-u^2+(2\alpha+1)iu} \\
&= \frac{\widetilde{\varphi}_{Y(T)}(0,T,u-i(\alpha+1))}{\alpha^2+\alpha-u^2+(2\alpha+1)iu}\ . \qquad (2.4.3)
\end{aligned}
$$

This completes the proof. □

Lemma 2.4.2. *The unconditional, discounted characteristic function of $Y(T)$ under $\mathbb{Q}^\theta$ is given by:*

$$
\widetilde{\varphi}_{Y(T)}(0,T,u) = \Big\langle \mathbf{X}(0)\exp\Big[(diag(\mathbf{g}(u)) + \mathbf{B}^\theta)T\Big], \mathbf{1}\Big\rangle\ ,
$$

where $\mathbf{g}(u) := (g_1(u), g_2(u), \cdots, g_N(u))'$ *and* $\mathbf{B}^\theta = [b^\theta_{jl}]_{j,l=1,2,\cdots,N}$ *with*

$$
\begin{aligned}
g_j(u) \ :=\ & -r_j + iu\big(r_j - \frac{1}{2}\sigma_j^2\big) - \frac{1}{2}u^2\sigma_j^2 \\
& + \sum_{l=1,l\neq j}^{N} e^{\theta_j\beta_{jl}}\Big(\big(e^{iu\beta_{jl}}-1\big) - iu\big(e^{\beta_{jl}}-1\big)\Big)a_{jl}\ ,
\end{aligned}
$$

and

$$
b^\theta_{jl} = \begin{cases} e^{iu\beta_{jl}}a^\theta_{jl}\ , & j\neq l\ , \\ -\displaystyle\sum_{l=1,l\neq j}^{N} e^{iu\beta_{jl}}a^\theta_{jl}\ , & j = l\ . \end{cases}
$$

Proof. Applying Itô's differentiation rule to $e^{iuY(t)}$, we have

$$
\begin{aligned}
de^{iuY(t)} &= e^{iuY(t)}\Big\{iu\Big[r(t) - \frac{1}{2}\sigma^2(t) - \sum_{l=1}^{N} e^{\theta(t)\beta_l(t)}\big(e^{\beta_l(t)} - 1 - \beta_l(t)\big)a_l(t)\Big]dt \\
& -\frac{1}{2}u^2\sigma^2(t)dt + \sum_{l=1}^{N}\big(e^{iu\beta_l(t)} - 1 - iu\beta_l(t)\big)a^\theta_l(t)dt + iu\sigma(t)dW^\theta(t) \\
& + \sum_{l=1}^{N}\big(e^{iu\beta_l(t)}-1\big)d\widetilde{\Phi}^\theta_l(t)\Big\} \\
&= e^{iuY(t)}\Big\{\Big[iu\big(r(t) - \frac{1}{2}\sigma^2(t)\big) - \frac{1}{2}u^2\sigma^2(t) + \sum_{l=1}^{N} e^{\theta(t)\beta_l(t)}\Big(\big(e^{iu\beta_l(t)} - 1\big)
\end{aligned}
$$

$$-iu\big(e^{\beta_l(t)}-1\big)\Big)a_l(t)\Big]dt+iu\sigma(t)dW^\theta(t)+\sum_{l=1}^N\big(e^{iu\beta_l(t)}-1\big)d\widetilde{\Phi}_l^\theta(t)\bigg\}\ .$$

Since $\widetilde{\Phi}_l^\theta$ is an $(\mathbb{F}^{\mathbf{X}},\mathbb{Q}^\theta)$ martingale, $\widetilde{\Phi}_l^\theta$ is adapted to $\mathbb{F}^{\mathbf{X}}$, i.e. $\widetilde{\Phi}^\theta(t)_l$ is an $\mathcal{F}^{\mathbf{X}}(t)$-measurable process, for each $t\in\mathcal{T}$ and $l=1,2,\cdots,N$. Then conditioning both sides on $\mathcal{F}^{\mathbf{X}}(t)$ under $\mathbb{Q}^\theta$, we have

$$\begin{aligned}
&d\varphi_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\\
&=\varphi_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\bigg\{\Big[iu\big(r(t)-\frac{1}{2}\sigma^2(t)\big)-\frac{1}{2}u^2\sigma^2(t)+\sum_{l=1}^N e^{\theta(t)\beta_l(t)}\\
&\quad\Big(\big(e^{iu\beta_l(t)}-1\big)-iu\big(e^{\beta_l(t)}-1\big)\Big)a_l(t)\Big]dt+\sum_{l=1}^N\big(e^{iu\beta_l(t)}-1\big)d\widetilde{\Phi}_l^\theta(t)\bigg\}\ .
\end{aligned}$$

Then

$$\begin{aligned}
&d\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\\
&=\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\bigg\{\Big[-r(t)+iu\big(r(t)-\frac{1}{2}\sigma^2(t)\big)-\frac{1}{2}u^2\sigma^2(t)+\sum_{l=1}^N e^{\theta(t)\beta_l(t)}\\
&\quad\Big(\big(e^{iu\beta_l(t)}-1\big)-iu\big(e^{\beta_l(t)}-1\big)\Big)a_l(t)\Big]dt+\sum_{l=1}^N\big(e^{iu\beta_l(t)}-1\big)d\widetilde{\Phi}_l^\theta(t)\bigg\}\ .
\end{aligned}$$

Note that

$$\sum_{l=1}^N\big(e^{iu\beta_l(t)}-1\big)d\widetilde{\Phi}_l^\theta(t)=\big(\mathbf{D}_0\mathbf{X}(t)-\mathbf{1}+\mathbf{X}(t)\big)'d\widetilde{\mathbf{\Phi}}^\theta(t)\ , \tag{2.4.4}$$

where $\mathbf{1}:=(1,1,\cdots,1)'\in\mathbb{R}^N$, $\widetilde{\mathbf{\Phi}}^\theta:=(\widetilde{\Phi}_1^\theta,\widetilde{\Phi}_2^\theta,\cdots,\widetilde{\Phi}_N^\theta)'\in\mathbb{R}^N$ and

$$\mathbf{D}_0:=[d_{jl}]_{j,l=1,2,\cdots,N}-\mathrm{diag}[(d_{11},d_{22},\cdots,d_{NN})']\ ,$$

with

$$d_{jl}=\begin{cases} e^{iu\beta_{jl}}\ , & j\neq l\ ,\\ \dfrac{\sum_{l=1,l\neq j}^N e^{iu\beta_{jl}}a_{jl}^\theta}{\sum_{l=1,l\neq j}^N a_{jl}^\theta}\ , & j=l\ .\end{cases}$$

Here $\mathrm{diag}(\mathbf{y})$ is a diagonal matrix with the diagonal elements given by the vector $\mathbf{y}$. It is easy to see that $d_{jl} = b^{\theta}_{jl}/a^{\theta}_{jl}$, for each $j, l = 1, 2, \cdots, N$.

Denote by

$$\mathbf{h}(t,u) := \mathbf{X}(t)\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u) \ , \quad t \in \mathcal{T} \ .$$

Using the stochastic integration by parts, we have

$$\begin{aligned} d\mathbf{h}(t,u) &= \big(\mathrm{diag}(\mathbf{g}(u)) + \mathbf{A}^{\theta}\big)\mathbf{h}(t,u)dt + \mathbf{h}(t,u)\sum_{l=1}^{N}\big(e^{iu\beta_l(t)} - 1\big)d\widetilde{\Phi}^{\theta}_l(t) \\ &\quad + \widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)d\mathbf{M}^{\theta}(t) + \Delta\mathbf{X}(t)\Delta\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u) \ . \end{aligned} \tag{2.4.5}$$

From Lemma 2.2 in Dufour and Elliott (1999), the chain $\mathbf{X}$ has the following representation under $\mathbb{Q}^{\theta}$:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \big(\mathbf{I} - \mathbf{X}(s-)\mathbf{1}'\big)d\mathbf{\Phi}^{\theta}(s) \ , \tag{2.4.6}$$

where $\mathbf{\Phi}^{\theta} := (\Phi^{\theta}_1, \Phi^{\theta}_2, \cdots, \Phi^{\theta}_N)' \in \mathbb{R}^N$ and $\mathbf{I}$ is an $(N \times N)$-identity matrix.

Denote by

$$\mathbf{A}^{\theta}_0 := \mathbf{A}^{\theta} - \mathrm{diag}\big[(a^{\theta}_{11}, a^{\theta}_{22}, \cdots, a^{\theta}_{NN})'\big] \ ,$$

and

$$\mathbf{B}^{\theta}_0 := \mathbf{B}^{\theta} - \mathrm{diag}\big[(b^{\theta}_{11}, b^{\theta}_{22}, \cdots, b^{\theta}_{NN})'\big] \ .$$

It is easy to check that

$$\big(\mathbf{I} - \mathbf{X}(t)\mathbf{1}'\big)\mathrm{diag}(\mathbf{A}^{\theta}_0\mathbf{X}(t))\mathbf{X}(t) \equiv \mathbf{0} \ ,$$

where $\mathbf{0} := (0, 0, \cdots, 0)' \in \mathbb{R}^N$.

Combining (2.4.4) and (2.4.6), we follow the proof of Lemma 2.3 in Dufour and Elliott (1999) to derive that

$$
\begin{aligned}
&\Delta\mathbf{X}(t)\Delta\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\\
&= (\mathbf{I}-\mathbf{X}(t)\mathbf{1}')\Delta\mathbf{\Phi}^{\theta}(t)\big(\mathbf{D}_0\mathbf{X}(t)-\mathbf{1}+\mathbf{X}(t)\big)'\Delta\mathbf{\Phi}^{\theta}(t)\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\\
&= \big(\mathbf{I}-\mathbf{X}(t)\mathbf{1}'\big)\mathrm{diag}(d\widetilde{\mathbf{\Phi}}^{\theta}(t))\big(\mathbf{D}_0\mathbf{X}(t)-\mathbf{1}+\mathbf{X}(t)\big)\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\\
&\quad +\big(\mathbf{I}-\mathbf{X}(t)\mathbf{1}'\big)\mathrm{diag}(\mathbf{A}_0^{\theta}\mathbf{X}(t))\big(\mathbf{D}_0\mathbf{X}(t)-\mathbf{1}+\mathbf{X}(t)\big)\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)dt\\
&= \big(\mathbf{I}-\mathbf{X}(t)\mathbf{1}'\big)\mathrm{diag}(d\widetilde{\mathbf{\Phi}}^{\theta}(t))\big(\mathbf{D}_0\mathbf{X}(t)-\mathbf{1}+\mathbf{X}(t)\big)\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\\
&\quad +\big(\mathbf{I}-\mathbf{X}(t)\mathbf{1}'\big)\big(\mathbf{B}_0^{\theta}\mathbf{X}(t)-\mathbf{A}_0^{\theta}\mathbf{X}(t)\big)\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)dt\\
&= \big(\mathbf{I}-\mathbf{X}(t)\mathbf{1}'\big)\mathrm{diag}(d\widetilde{\mathbf{\Phi}}^{\theta}(t))\big(\mathbf{D}_0\mathbf{X}(t)-\mathbf{1}+\mathbf{X}(t)\big)\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\\
&\quad +\big(\mathbf{B}^{\theta}\mathbf{X}(t)-\mathbf{A}^{\theta}\mathbf{X}(t)\big)\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)dt\\
&= \big(\mathbf{I}-\mathbf{X}(t)\mathbf{1}'\big)\mathrm{diag}(d\widetilde{\mathbf{\Phi}}^{\theta}(t))\big(\mathbf{D}_0\mathbf{X}(t)-\mathbf{1}\big)\widetilde{\varphi}_{Y(t)|\mathcal{F}^{\mathbf{X}}(t)}(0,t,u)\\
&\quad +\big(\mathbf{B}^{\theta}-\mathbf{A}^{\theta}\big)\mathbf{h}(t,u)dt\ .
\end{aligned}
$$

Then taking expectation on both sides of (2.4.5) under $\mathbb{Q}^{\theta}$, we have

$$
d\mathrm{E}^{\theta}[\mathbf{h}(t,u)] = \big(\mathrm{diag}(\mathbf{g}(u))+\mathbf{B}^{\theta}\big)\mathrm{E}^{\theta}[\mathbf{h}(t,u)]dt\ .
$$

Solving gives

$$
\mathrm{E}^{\theta}[\mathbf{h}(T,u)] = \mathbf{X}(0)\exp\big[(\mathrm{diag}(\mathbf{g}(u))+\mathbf{B}^{\theta})T\big]\ .
$$

Consequently,

$$
\begin{aligned}
\widetilde{\varphi}_{Y(T)}(0,T,u) &= \big\langle \mathrm{E}^{\theta}[\mathbf{h}(T,u)],\mathbf{1}\big\rangle\\
&= \big\langle \mathbf{X}(0)\exp\big[(\mathrm{diag}(\mathbf{g}(u))+\mathbf{B}^{\theta})T\big],\mathbf{1}\big\rangle\ .
\end{aligned}
$$

This completes the proof. $\square$

Based on Lemmas 2.4.1 and 2.4.2, we obtain the main result of our paper, which gives an integral representation of the call option price.

Theorem 2.4.1. *Under $\mathbb{Q}^\theta$, the call option price under the double regime-switching model (2.2.1) can be represented in the following integral form:*

$$C(0,T,k) = \frac{S_0 e^{-\alpha k}}{\pi} \int_0^\infty e^{-iuk} \psi(0,T,u) du \ , \tag{2.4.7}$$

where

$$\psi(0,T,u) = \frac{\left\langle \mathbf{X}(0) \exp\left[(diag(\mathbf{g}(u - i(\alpha+1))) + \mathbf{B}^\theta) T \right], \mathbf{1} \right\rangle}{\alpha^2 + \alpha - u^2 + (2\alpha+1)iu} \ . \tag{2.4.8}$$

Proof. Applying the inverse Fourier transform to (2.4.1), we have

$$\begin{aligned} C(0,T,k) &= S_0 e^{-\alpha k} c(0,T,k) \\ &= \frac{S_0 e^{-\alpha k}}{2\pi} \int_{\mathbb{R}} e^{-iuk} \psi(0,T,u) du \\ &= \frac{S_0 e^{-\alpha k}}{\pi} \int_0^\infty e^{-iuk} \psi(0,T,u) du \ . \end{aligned}$$

Furthermore, combining Lemmas 2.4.1 and 2.4.2 gives (2.4.8). □

In the rest of this section, we briefly introduce the FFT method for numerical computation. Compared with other approaches such as finite difference method and Monte Carlo simulation, the FFT method is much faster. Indeed if we used finite difference method or Monte Carlo simulation to compute option prices under the regime-switching models, we would solve a system of second-order, coupled PDEs or simulate sample paths of both Brownian motion and Markov chain. This would cost much more computation effort. Furthermore, to apply the FFT method, we first need to derive an analytical option pricing formula, which is very useful in calibration of model parameters in empirical studies.

The FFT method is an efficient method for computing the sum of the following form:

$$w(n) = \sum_{m=1}^{M} e^{-i\frac{2\pi}{M}(m-1)(n-1)} x(m) \ , \quad \text{for } n = 1, 2, \cdots, M \ . \tag{2.4.9}$$

To apply the FFT method, we first need to write the integration (2.4.7) as the summation (2.4.9).

As in Carr and Madan (1999), we can approximate the option price (2.4.7) by the following summation

$$C(0,T,k) \approx \frac{S_0 e^{-\alpha k}}{\pi} \sum_{m=1}^{M} e^{-iu_m k} \psi(0,T,u_m)\eta \ , \tag{2.4.10}$$

where $u_m = (m-1)\eta$, $m = 1, 2, \cdots, M$. Here η represents the grid size in u.
The effective upper limit (UL) for the integration is:

$$UL = M\eta \ .$$

The FFT returns M values of k and the values for k are defined as follows:

$$k_n = -b + \lambda(n-1), \quad \text{for} \quad n = 1, \cdots, M \ , \tag{2.4.11}$$

where $b = \frac{M\lambda}{2}$ and λ is the grid size in k.
Then substituting (2.4.11) into (2.4.10) gives

$$C(0,T,k_n) \approx \frac{S_0 e^{-\alpha k_n}}{\pi} \sum_{m=1}^{M} e^{-iu_m(-b+\lambda(n-1))} \psi(0,T,u_m)\eta, \quad \text{for} \quad n = 1, \cdots, M \ .$$

Noting that $u_m = (m-1)\eta$, we write

$$C(0,T,k_n) \approx \frac{S_0 e^{-\alpha k_n}}{\pi} \sum_{m=1}^{M} e^{-i\lambda\eta(m-1)(n-1)} e^{ibu_m} \psi(0,T,u_m)\eta \ . \tag{2.4.12}$$

To apply the fast Fourier transform, the following restriction need to be imposed:

$$\lambda\eta = \frac{2\pi}{M} \ .$$

Then (2.4.12) becomes

$$C(0,T,k_n) \approx \frac{S_0 e^{-\alpha k_n}}{\pi} \sum_{m=1}^{M} e^{-i\frac{2\pi}{M}(m-1)(n-1)} e^{ibu_m} \psi(0,T,u_m)\eta \ .$$

2.5 Numerical examples

In this section, we perform a numerical analysis for option valuation under the double regime-switching model. For ease of comparison, we also provide the numerical results for option prices under the single regime-switching model. To simplify our computation, we consider a two-state Markov chain $\mathbf{X}$, where State 1 and State 2 of the chain represent a 'Bad' economy and a 'Good' economy, respectively. We write $\mathbf{X}(t) = (1,0)'$ and $\mathbf{X}(t) = (0,1)'$ for State 1 and State 2.

In what follows, we give configurations of the parameter values. The rate matrix of the chain $\mathbf{X}$ under $\mathbb{P}$ is given by

$$\mathbf{A} = \begin{pmatrix} -a & a \\ a & -a \end{pmatrix} ,$$

where a takes discrete values from $\{0, 0.1, 0.2, \cdots, 1\}$. The larger a is, the more volatile the economy is. That is, the probability of the transition of the economy from one state to another increases with a. Note that when $a = 0$, the regime-switching effect is degenerate. Generally speaking, the main features of the financial market in a 'Bad' ('Good') economy are low (high) appreciation rate, low (high) interest rate and high (low) volatility. So we consider the following vectors for the appreciation rate, risk-free interest rate and volatility, respectively:

$$\boldsymbol{\mu} = (0.04, 0.08)' \ , \quad \mathbf{r} = (0.02, 0.04)' \ , \quad \boldsymbol{\sigma} = (0.4, 0.2)' \ .$$

1. **The double regime-switching (DRS) model**

   The jump ratio of the double regime-switching model is described by the matrix

   $$\boldsymbol{\beta} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \ .$$

   To see the effect of the jump ratio on option valuation, we float β from 0 to 1.

2. **The single regime-switching (SRS) model**

   When the jump ratio remains zero during a state transition of the chain, the jump component of the double regime-switching model is absent. That is

   $$\boldsymbol{\beta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ .$$

   In other words, the double regime-switching model reduces to the single one.

Table 2.5.1 presents the prices of the European call options with different strike levels under the DRS model and the SRS model, where we assume $\beta = 0.1$ for the DRS model and $S_0 = 100$, $T = 1$, $a = 0.5$ for both models. Under the above configurations of the hypothetical values of model parameters, the regime-switching Esscher transform parameters are $\theta_1 = -0.1210$ and $\theta_2 = -0.8894$. Then, the market prices of the regime-switching risk are calculated as $e^{\theta_1 \beta} - 1 = -0.012$ in State 1 and $e^{-\theta_2 \beta} - 1 = 0.093$ in State 2, respectively. In Table 2.5.1, the DRS model I and II represent the DRS model where the regime-switching risk is endogenously determined and ignored, respectively. In other words, the regime-switching risk is priced under the DRS model I and not

Table 2.5.1: Option prices calculated via the FFT

	DRS model I		*DRS model II*		*SRS model*	
Strikes	State 1	State 2	State 1	State 2	State 1	State 2
70	34.3847	33.4744	33.9481	33.4357	34.0904	33.1151
	(0.00%)	(0.00%)	(1.27%)	(0.12%)	(0.86%)	(1.07%)
80	27.1651	25.1079	26.6486	25.0263	26.7779	24.5557
	(0.00%)	(0.00%)	(1.90%)	(0.32%)	(1.43%)	(2.20%)
90	21.0267	17.7843	20.4409	17.6632	20.6144	17.1617
	(0.00%)	(0.00%)	(2.79%)	(0.68%)	(1.96%)	(3.50%)
100	15.9804	11.8606	15.3623	11.7194	15.6171	11.3358
	(0.00%)	(0.00%)	(3.88%)	(1.20%)	(2.27%)	(4.42%)
110	11.9595	7.4749	11.3576	7.3378	11.6953	7.1553
	(0.00%)	(0.00%)	(5.03%)	(1.83%)	(2.21%)	(4.28%)
120	8.8423	4.4927	8.2969	4.3757	8.6931	4.3873
	(0.00%)	(0.00%)	(6.17%)	(2.60%)	(1.68%)	(2.35%)

priced under the DRS model II. Indeed, both the DRS model I and II are the two-state double regime-switching models given by Eq. (2.2.1). Their (local)-martingale conditions are given by (2.3.4) and (2.3.7), respectively. This chapter is concerned with the option prices under the DRS model I. For the purpose of comparison, we also present the option prices under the DRS model II, which was considered in Section 5.1 in Yuen and Yang (2009). In the sequel, the DRS model always denotes the DRS model I unless otherwise stated. In each state, the numbers in parentheses under option prices denote the percentages of underestimation of the option prices under other models compared with those under the DRS model I. As shown in Table 2.5.1, for the same strike level, the option prices in State 1 are systematically higher than those in State 2 under all three models. This makes intuitive sense. State 1 ('Bad' economy) has a lower interest rate and higher volatility compared with State 2 ('Good' economy). Consequently, it is reasonable that the option prices in State 1 are higher than the corresponding prices in State 2 due to the additional amount of risk premium required to compensate for a 'Bad' economic condition. The option prices under the DRS model II are lower than those under the DRS model I. This also makes intuitive sense since additional risk premiums are required when the regime-switching risk is priced under the DRS model I. It is worth mentioning that although the market prices of the regime-switching risk are relatively small, the underestimation of option prices is not negligible. If the regime-switching risk is not priced, the percentage of underestimation reaches as high as 6.17% for an out-of-money option with $K = 120$ in a 'Bad' economy (State 1). So it is of economic significance to price the regime-switching risk under the DRS model. Since the risk-free interest rate and the volatility of the asset price, as well as the generator of the Markov chain are assumed to be the same under the DRS model and the SRS model, the DRS model apparently gives higher option prices, due to additional jump risk induced by state transitions. If the DRS model is the more suitable or "true" model, which is indeed evidenced by the empirical studies in Section 2.6, the option prices are underestimated under the SRS model. Note that the option prices converge very quickly. In our illustration, we always adopt the number of discretization $M = 4096$ [2]. Indeed, varying M from $512, 1024, 2048$ to 4096, only slight changes occur in the four decimal places of the option prices. There is almost no difference between the option prices using $M = 2048$ and $M = 4096$. So adopting $M = 4096$, we can achieve accuracy to four decimal places in the option prices.

Furthermore, we also assume $\beta = 0.1$, $T = 1$ and $a = 0.5$ to illustrate option prices under the DRS model with different levels of the initial share price S_0 and modified

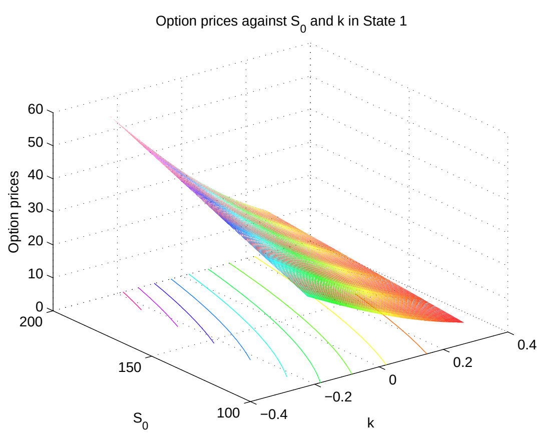


Figure 2.5.1: Option prices corresponding to different S_0 and k in State 1

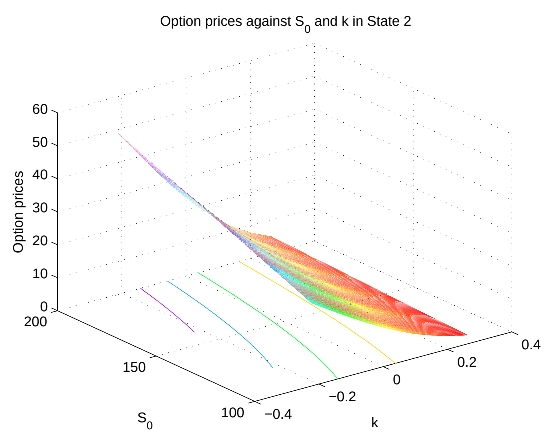


Figure 2.5.2: Option prices corresponding to different S_0 and k in State 2

strike prices $k = \log(K/S_0)$ in both State 1 and State 2. Figs. 2.5.1 and 2.5.2 illustrate that option prices increase with S_0 for fixed k, while decreases with k for fixed S_0 in both State 1 and State 2. This feature is similar to that under the classical Black-Scholes

[2]To control the approximation errors, Carr and Madan (1999) discussed the selection of the upper limit of the integral. Lee (2004) studied the discretization errors of approximation. Liu et al. (2006) showed that the truncation errors are considerably small.

model, even though the double regime-switching effect is present in our model.

We report the implied volatilities of the DRS model when $\beta = 0.1$, $T = 1$ and $a = 0.5$. Figs. 2.5.3 and 2.5.4 show the implied volatility surface with different

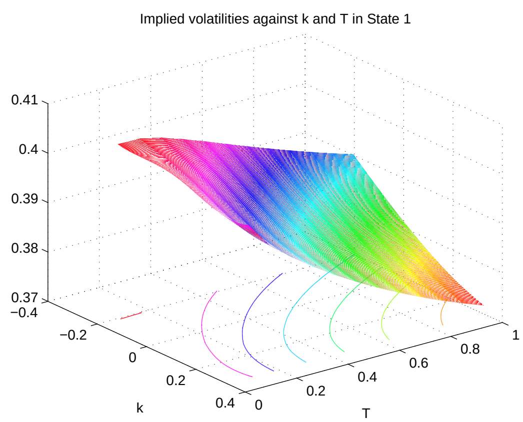


Figure 2.5.3: Implied volatilities corresponding to different k and T in State 1

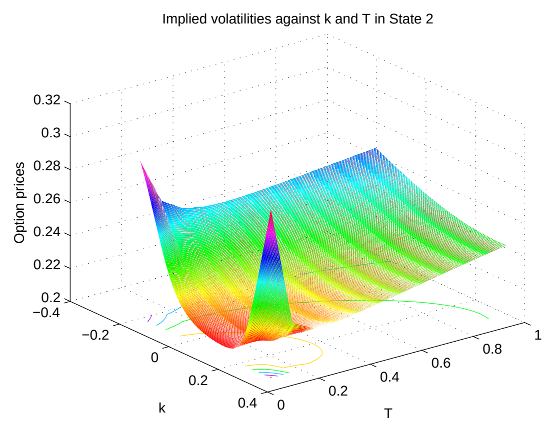


Figure 2.5.4: Implied volatilities corresponding to different k and T in State 2

modified strike prices k and maturities T in State 1 and State 2. The volatilities in a 'Bad' economy (State 1) are higher than the volatilities in a 'Good' economy (State 2). The implied volatilities show the volatility skewness effect in State 1 and the volatility smile effect in State 2, respectively. This may attribute to the different model dynamics in two states. It can be seen that the implied volatilities are relatively low for at-the-money options in State 2 while they are higher for in-the-money and out-of-money options. The volatility skewness and smile effects are more remarkable for options with shorter maturities in both states.

Under the DRS model, we assume $\beta = 0.1$, $S_0 = 100$ and $T = 1$. We provide the sensitivity analysis for the option prices with respect to the rate of transition a. From Figs. 2.5.5 and 2.5.6, we notice that the option prices decrease with a in State 1 while increase with a in State 2. When a increases, the probability of the chain $\mathbf{X}$ transiting

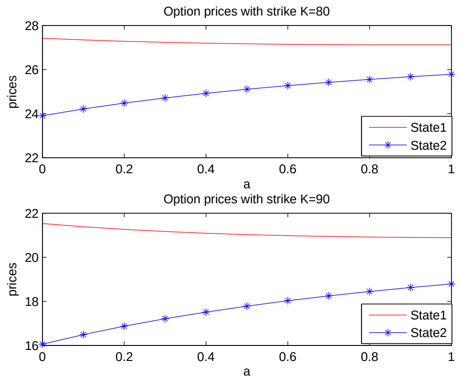


Figure 2.5.5: Option prices corresponding to different a with $K = 80, 90$

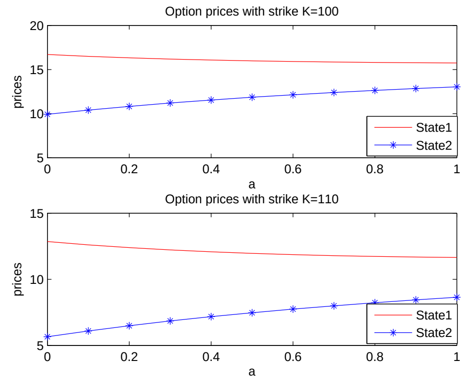


Figure 2.5.6: Option prices corresponding to different a with $K = 100, 110$

between State 1 and State 2 will increase. As explained earlier, the European options are more expensive in State 1 and cheaper in State 2. Thus, the option prices in State 1 decrease with the probability of the chain transiting from State 1 to State 2. On the contrary, the option prices in State 2 increase with the probability of the chain changing from State 2 to State 1. This is the reason why the European options are cheaper when a increases in State 1, while they are more expensive when a increases in State 2. Note that the probability of the chain transiting between State 1 and State 2 is zero when $a = 0$. Under this degenerate case, the regime-switching effect does not exist. Therefore, the option prices are the maximal in State 1 and the minimal in State 2 when $a = 0$.

Furthermore, we assume $S_0 = 100$, $T = 1$ and $a = 0.5$. We provide sensitivity analysis for the option prices with different β in both State 1 and State 2. Figs. 2.5.7 and 2.5.8 illustrate that the option prices increase with β in both State 1 and State 2. The explanation to this finding is that the larger β is, the larger the jump risk is. Therefore, a higher jump-risk premium leads to a higher option price. It is worth

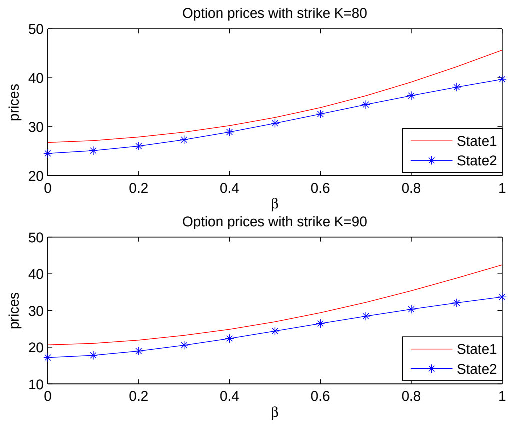


Figure 2.5.7: Option prices corresponding to different β with $K = 80, 90$

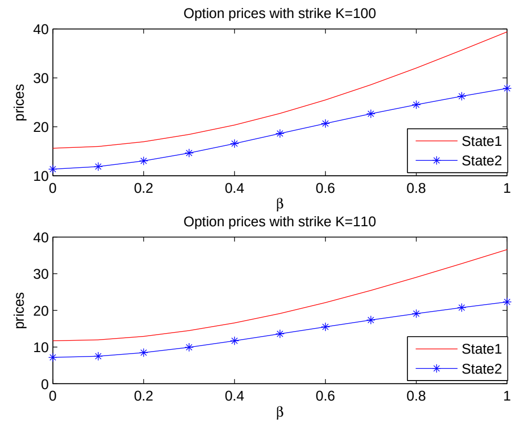


Figure 2.5.8: Option prices corresponding to different β with $K = 100, 110$

mentioning that the European call price increases rapidly as β does. In Figs. 2.5.7 and 2.5.8, the percentage price increases at different strike levels are approximately $70\% - 210\%$ in State 1 and $60\% - 210\%$ in State 2 as β goes from 0 to 1. This indicates that the jump risk in the double regime-switching model has a material effect on the option price. A comparison of Figs. 2.5.5 and 2.5.6 with Figs. 2.5.7 and 2.5.8 implies that the option price is more sensitive to β than a. Therefore, during a structural change in the underlying economy, the sudden jump in the share price level, rather than the change of the model parameters in the dynamics of the share price, may have a greater impact on the option price. This may provide some evidence for justifying the use of the double regime-switching model.

2.6 Empirical studies

In this section, we provide an empirical application of the double regime-switching model to illustrate the practical usefulness of the model, how well the DRS model might fit the observed data and how the DRS model might improve on prior models. More specifically, we calibrate the model parameters to the market prices of the European call options and compare the in-sample fitting errors and out-of-sample prediction errors of different models, including the BS model, the SRS model and the DRS model. As in numerical examples, we only consider a two-regime case for illustration.

For the sake of liquidity and availability of option prices data, we choose European call option prices written on the S&P 500 for five consecutive trading days from 1 October 2012 to 5 October 2012 as our data set. These prices are close prices of corresponding options obtained from the Datastream Database of Reuters. Accordingly, the close prices of the S&P 500 are collected as the initial prices of the underlying asset. For each trading day, the data set consists of 39 call option prices, with 13 strikes ranging from 1300 to 1600 (i.e., the moneyness ratios are approximately 90% to 110%), and 3 maturities: 17 November 2012, 16 March 2013 and 18 January 2014. Consequently, our data set consists of 195 call option prices in total, where we take the first 156 option prices from 1 October 2012 to 4 October 2012 as in-sample data, and the rest 39 option prices from 5 October 2012 as out-of-sample data.

To focus on the stochastic movements of the risky share, we set the risk-free interest rate to be $\mathbf{r} = (0.02, 0.04)'$. Unlike the assumptions imposed in numerical examples, the rate matrix $\mathbf{A} = [a_{jl}]_{j,l=1,2}$ and the jump ratio matrix $\boldsymbol{\beta} = [\beta_{jl}]_{j,l=1,2}$ are not necessarily symmetric matrices in practice. Note that $a_{11} = -a_{12}$, $a_{22} = -a_{21}$ and $\beta_{11} = \beta_{22} = 0$. As in Chen and Hung (2010), we employ the method of nonlinear least squares for calibration using the in-sample data set. Particularly, we calibrate the model parameters $\boldsymbol{\Theta} := (\mu_1, \mu_2, \sigma_1, \sigma_2, a_{12}, a_{21}, \beta_{12}, \beta_{21}, p)$ by minimizing the sum of squared errors between the market prices and model prices over the in-sample reference period, where the model prices are weighted average of the option prices in State 1 and State 2 calculated from Eq. (2.4.7) with weights p and $1 - p$ $(0 \leq p \leq 1)$. Using the language of the Bayesian statistics, the weights p and $1-p$ can be thought of as priori probabilities of the chain $\mathbf{X}$ in State 1 and State 2, respectively. Consequently, the calibrated or implied parameter p may provide information about the market belief on the economic condition, which may be used to represent prior information about the economic condition in the Bayesian context. The parameter estimates of the DRS model based on the in-sample data are as follows:

$$\mathbf{\Theta} = (0.0936, 0.0974, 0.1099, 0.0700, 0.3573, 0.4694, 0.0484, -0.2302, 0.9999)\ .$$

From the (local)-martingale condition (2.3.4), the market prices of the regime-switching risk are $e^{\theta_1 \beta_{12}} - 1 = -0.2428$ in State 1 and $e^{\theta_2 \beta_{21}} - 1 = 0.5042$ in State 2, respectively. This means that the market compensates a regime switch from State 1 to State 2 and penalizes that from State 2 to State 1. This makes intuitive sense if State 1 represents a 'Bad' economy while State 2 represents a 'Good' economy. Indeed, a regime switch from State 1 to State 2 induces an upward jump in the price level of the risky share. An investor could profit from a regime switch from State 1 to State 2. Consequently the market price of the regime-switching risk in State 1 is negative. Similar explanations apply to the positive market price of the regime-switching risk in State 2. From the calibration results of the DRS model, the weight $p = 0.9999$ implies that the chain $\mathbf{X}$ is almost surely in State 1 over the in-sample reference period. In other words, the market may believe that the economy over the in-sample reference period is bad with a 99.99% confidence level.

To show how well the DRS model might fit the observed data and how the DRS model might improve the performances of some existing models, we compare the fitting and prediction errors of the DRS model with those of the BS model and the SRS model, whose model parameters are calibrated on the same in-sample data. The out-of-sample prediction errors of each model are calculated using the implied parameters from the in-sample data. We adopt the root mean square error (RMSE) in percentage of the initial share price as a proxy for the fitting and prediction errors. Table 2.6.1 reports the RMSE of the in-sample and out-of-sample data for each model. As in Table 2.5.1,

Table 2.6.1: In-sample fitting errors and Out-of-sample prediction errors

Errors	*DRS model I*	*DRS model II*	*SRS model*	*BS model*
In-sample	0.1844%	0.2065%	0.2268%	0.5198%
Out-of-sample	0.3019%	0.3240%	0.5668%	0.9534%

the DRS model I and II represent the DRS model where the regime-switching risk is priced and not priced, respectively. It is shown that the DRS model I performs the best, being the one with the lowest RMSE both in fitting the in-sample data and predicting the out-of-sample data. Although the in-sample fitting errors of the DRS model and the SRS model are close to each other, the out-of-sample prediction error of the SRS model is almost twice that of the DRS model I. The in-sample fitting and out-of-sample errors of the BS model are about three times those of the DRS model I. Consequently, the DRS model provides a significant empirical improvement on the existing models, including the SRS model and the BS model.

2.7 Conclusions

We investigate option valuation under the double regime-switching model with an emphasis on how the regime-switching risk is priced. A key feature of the double regime-switching model is that a regime switch causes both a structural change in the underlying share price dynamics and a sudden jump in the share price level. A generalized version of the regime-switching Esscher transform is used to select a pricing kernel. Using the FFT method, we obtain an integral pricing formula and numerically and empirically implemented the European call option pricing. Numerical examples illustrate the regime-switching effects, especially jumps in the price level caused by transitions of the states, have a material effect on option prices. Our empirical results based on real option prices data reveal that the double regime-switching model outperforms the single regime-switching model and the Black-Scholes model in terms of fitting and predicting option prices data.

Chapter 3

Pricing variance swaps under a stochastic interest rate and volatility model with regime switching

3.1 Introduction

Volatility derivatives, such as variance swaps, volatility swaps and VIX options, have been playing an increasingly prominent role in the banking and finance industry. These products can provide information for the volatility level of an underlying risky asset and act as potential tools to manage the market volatility risk. Just as options began trading shortly after the introduction of the Black-Scholes-Merton formula in the 1970s, variance swaps was first launched in the 1990s partly stimulated by the seminal works of Neuberger (1990, 1994) and Dupire (1992, 1993) on variance swaps pricing. Ever since then the rapid development of trading volatility derivatives has attracted considerable attention from both academics and industrial practitioners. In fact, volatility derivatives can be regarded as derivatives contracts written on a specified measure of volatility, which represents risk and uncertainty of the underlying asset prices. There are different measures of volatility, including realized volatility, implied volatility and model-based volatility. Realized volatility (or historical volatility) refers to the standard deviation of financial returns over a fixed period in the past; implied volatility is the volatility implied by the market price data of the option based on an assumed option pricing formula, say, the Black-Scholes-Merton formula; model-based volatility uses various versions of stochastic models to describe the evolution of volatility, such as Heston's model and ARCH/GARCH models. Interested readers may refer to Becker et al. (2007) for predictability in future volatility of implied volatility approach and model-based volatility approach.

Variance swaps have been actively traded in over-the-counter markets since the collapse of the LTCM in late 1998. They allow one to speculate on or hedge risks associated with volatility of some underlying assets, such as exchange rate, interest rate, stock index and so on. A variance swap is a forward contract where the short party pays a floating leg equal to the realized annual variance over the swap's life at maturity in exchange of receiving a fixed leg at maturity. Typically, the fixed leg of a variance swap contract is the predetermined strike price such that the value of the variance swap is zero at transaction. This is called a fair strike value of the variance swap. Like other forward contracts, the dynamics of the interest rate (or the discount factor) and the realized annual variance (i.e. the underlying asset) are two crucial factors for pricing variance swaps. In what follows, various models on these two factors are recalled dating back to the well-known Black-Scholes-Merton formula.

Black and Scholes (1973) and Merton (1973) established the theory of option pricing in their path-breaking works. Despite of the popularity of the Black-Scholes-Merton option pricing formula, it is well-documented that the assumption for the price dynamics of the Black-Scholes-Merton model is not realistic. In the Black-Scholes-Merton model, the interest rate is assumed to be constant, which is untenable in recent years due to the more fluctuating feature of the market interest rate. Thus modeling the stochastic behavior of the interest rate is a topic of crucial importance in finance. In the past three decades or so, numerous short rate models for the term structure of interest rates have been proposed. Some examples include those introduced by Vasicek (1977), Cox et al. (1985), Hull and White (1990), and others. These models and their variants treat the short rate dynamics as continuous-time diffusion processes or jump-diffusion processes. An important feature of these short rate models is that the short rate processes will eventually revert to a long term value. This is called the mean-reverting property of short rates. This property is widely accepted in the theory and practice of interest rate modeling.

Furthermore, the classical Black-Scholes-Merton model cannot explain the "volatility smile" phenomenon, which indicates that implied volatility varies across the strike and the expiry. Stochastic volatility models are an approach to overcome this shortcoming of the Black-Scholes-Merton model. By assuming that the volatility of the underlying price is a stochastic process rather than a constant, the pricing and hedging of derivative securities become more accurate. Indeed, attempts have been made to develop different stochastic volatility models, including the Heston (1993) model and the Schöbel-Zhu (1999) model, which is a generalized version of the Stein-Stein (1991) model.

Stochastic interest rate and volatility models mentioned above are all effective in a short term. However, in the long run, there is strong evidence that structural changes in macro-economic conditions indispensably lead to dramatic transitions in market fundamentals. Consequently, it is of practical relevance to develop stochastic interest rate and volatility models incorporating the impact of regime shifts. Regime-switching models provide a natural and convenient way to describe the effect of changes in macroeconomic conditions on price series and economic series. In particular, regime-switching models do play an important role in interest rate and volatility modeling. The Markovian regime-switching Hull-White model was considered in Elliott and Mamon (2003) and Elliott and Wilson (2007). Adopting the concept of stochastic flows, Elliott and Siu (2009) and Elliott et al. (2011) developed regime-switching term-structure models and exponential-affine forms of bond prices. So et al. (1998) proposed a regime-switching stochastic volatility model and conducted an empirical analysis for the S&P 500's data. Variance swaps for stochastic volatility driven by Markov process were also studied in Elliott and Swishchuk (2007). Elliott et al. (2007) developed a Markov-modulated version of the Heston model for pricing volatility derivatives. Elliott and Lian (2013) presented a set of closed-form exact solutions of pricing discretely sampled variance swaps and volatility swaps under a regime-switching Heston model. Some other applications of regime-switching models include Elliott and van der Hoek (1997) for asset allocation, Elliott et al. (2005) and Mamon and Rodrigo (2005) for option valuation, and Zhang et al. (2012) for mean-variance portfolio selection.

In this chapter, we consider the pricing of variance swaps under a stochastic interest rate and volatility model with regime switches, which is flexible enough to incorporate interest rate risk, volatility risk and economic risk. More specifically, we adopt the Markovian regime-switching Schöbel-Zhu-Hull-White hybrid model, where the regime-switching Hull-White model drives the dynamics of interest rate and the regime-switching Schöbel-Zhu model describes the stochastic movements of volatility. In both models, some parameters, including the mean-reversion levels and the volatility rates of both interest rate and volatility, are assumed to be modulated by a continuous-time, finite-state, observable Markov chain. The states of the chain represent different states of the macro-economic conditions or different stages of a business cycle, which are usually regarded as proxies for different levels of macro-economic indicators such as Gross Domestic Product, Consumer Price Index, Sovereign Credit Ratings and others. We decompose the valuation of variance swaps under our hybrid model into two steps. In the first step, we adopt a PDE approach to derive an exponential form of the bond price under the regime-switching Hull-White model and provide the related forward measure via Girsanov's theorem for the Brownian motion and the Markov chain. In the second step, we obtain an integral representation for the price of variance swaps by virtue of measure changes. In a two-state Markov chain case, we provide a numerical analysis for the fair strike value of variance swaps against different values of the rate matrices of the chain, which illustrates that the effect of both stochastic interest rate and regime-switching is significant in the pricing of variance swaps.

The rest of this chapter is structured as follows. Section 3.2 describes the model dynamics. In Section 3.3, we derive the bond pricing formula and the related forward measure under the Markovian regime-switching Hull-White model. Section 3.4 considers the pricing of variance swaps under the Markovian regime-switching Schöbel-Zhu-Hull-White hybrid model. In Section 3.5, we conduct a numerical analysis for the prices of variance swaps. The final section makes the concluding remarks.

3.2 The model dynamics

We consider a continuous-time financial market with a finite time horizon $\mathscr{T} := [0, T]$, where $T < \infty$. Uncertainty over time is modeled by a complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$, which is assumed to be rich enough to describe all sources of uncertainties in the model. Suppose that $\mathscr{P}$ is a risk-neutral probability measure. Here, like most of the literature about pricing interest-rate related derivatives, we start by the risk-neutral probability measure $\mathscr{P}$ directly. We model the evolution of the state of an economy over time by a continuous-time, finite-state, observable Markov chain $\{\mathbf{X}(t)|t \in \mathscr{T}\}$ on $(\Omega, \mathscr{F}, \mathscr{P})$ taking values in a finite state space $\mathscr{S} := \{\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_N\}$. The states of the chain $\mathbf{X}$ is interpreted as different states of an economy or different stages of a business cycle. Without loss of generality, we adopt the formalism in Elliott et al. (1994) and identify the states of the chain with a set of standard unit vectors $\mathscr{E} := \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N\} \subset \Re^N$, where the j^{th} component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij} for each $i, j = 1, 2, \cdots, N$. This is called the canonical state space representation of the chain. Let $\mathbf{Q} := [q_{ij}]_{i,j=1,2,\cdots,N}$ be the rate matrix of the chain $\mathbf{X}$ under $\mathscr{P}$, where q_{ij} is a constant transition intensity of the chain $\mathbf{X}$ from state $\mathbf{e}_j$ to state $\mathbf{e}_i$. Let $\mathbb{F}^{\mathbf{X}} := \{\mathscr{F}^{\mathbf{X}}(t)|t \in \mathscr{T}\}$ be the right-continuous, $\mathscr{P}$-complete, natural filtration generated by the chain $\mathbf{X}$. With the canonical state space representation of $\mathbf{X}$, Elliott et al. (1994) obtained the following semimartingale dynamics for the chain:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{Q}\mathbf{X}(s)ds + \mathbf{M}(t) \ , \quad t \in \mathscr{T} \ . \tag{3.2.1}$$

Here $\{\mathbf{M}(t)|t \in \mathscr{T}\}$ is an $\Re^N$-valued, $(\mathbb{F}^{\mathbf{X}}, \mathscr{P})$-martingale. It is worth noting that the bounded variation term in the semimartingale dynamics, which is not a (local)-martingale, is predictable. This decomposition is called a canonical decomposition. Consequently, the Markov chain $\mathbf{X}$ is a special semimartingale and the decomposition (3.2.1) is unique.

In what follows, we specify the Markovian regime-switching Schöbel-Zhu-Hull-White hybrid model for the short rate and the volatility. For modeling stochastic interest rate, the Markovian regime-switching Hull-White model was used in Elliott and Mamon (2003) and Elliott and Wilson (2007). For modeling stochastic volatility, the Markovian regime-switching Heston stochastic volatility model was adopted in

Elliott et al. (2007) and Elliott and Lian (2013). The Markovian regime-switching Schöbel-Zhu model proposed in this paper is a new model for stochastic volatility.

Let $\mathbf{y}'$ be the transpose of a vector or a matrix $\mathbf{y}$. $\langle \cdot, \cdot \rangle$ is the scalar product in $\Re^N$. Define $\{\alpha(t) | t \in \mathscr{T}\}$ and $\{a(t) | t \in \mathscr{T}\}$, respectively, as the mean-reversion levels of the short rate process and the volatility process:

$$\alpha(t) := \langle \boldsymbol{\alpha}, \mathbf{X}(t) \rangle \ ,$$

and

$$a(t) := \langle \mathbf{a}, \mathbf{X}(t) \rangle \ ,$$

where $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \cdots, \alpha_N)' \in \Re^N$ with $\alpha_i > 0$, and $\mathbf{a} := (a_1, a_2, \cdots, a_N)' \in \Re^N$ with $a_i > 0$, for each $i = 1, 2, \cdots, N$. They are interpreted as the long-run interest rates and volatilities corresponding to the different possible states of an economy.

Let $\gamma(t)$ and $c(t)$ be the volatility rates of the short rate process and the stochastic volatility process at time t, respectively. Then, we suppose that

$$\gamma(t) := \langle \boldsymbol{\gamma}, \mathbf{X}(t) \rangle \ ,$$

and

$$c(t) := \langle \mathbf{c}, \mathbf{X}(t) \rangle \ ,$$

where $\boldsymbol{\gamma} := (\gamma_1, \gamma_2, \cdots, \gamma_N)' \in \Re^N$ with $\gamma_i > 0$, and $\mathbf{c} := (c_1, c_2, \cdots, c_N)' \in \Re^N$ with $c_i > 0$, for each $i = 1, 2, \cdots, N$.

Let $\beta(t)$ and $b(t)$ be two parameters controlling the speed of mean reversion for the short rate process and the volatility process, both of which are called the mean-reversion coefficients, where $\beta(t)$ and $b(t)$ are assumed to be deterministic, bounded functions of time t and $\beta(t), b(t) > 0$, $\forall t \in \mathscr{T}$.

Then we assume that under $\mathscr{P}$, the evolution of the short rate process $\mathbf{r} := \{r(t)|t \in \mathscr{T}\}$ and the volatility process $\boldsymbol{\sigma} := \{\sigma(t)|t \in \mathscr{T}\}$ over time is governed by the following Markovian regime-switching Schöbel-Zhu-Hull-White hybrid model:

$$\begin{cases} dr(t) = \beta(t)(\alpha(t) - r(t))dt + \gamma(t)dW_r(t) \ , \\ d\sigma(t) = b(t)(a(t) - \sigma(t))dt + c(t)dW_\sigma(t) \ , \end{cases} \tag{3.2.2}$$

where $W_r := \{W_r(t)|t \in \mathscr{T}\}$ and $W_\sigma := \{W_\sigma(t)|t \in \mathscr{T}\}$ are two standard Brownian motions with respect to their right-continuous, $\mathscr{P}$-complete, natural filtrations under $\mathscr{P}$. Suppose that the two Brownian motions W_r and W_σ are correlated and the instantaneous covariance is given by

$$\mathrm{Cov}(dW_r(t), dW_\sigma(t)) = \rho(t)dt \ ,$$

where $\rho(t) := \rho(t, \mathbf{X}(t)) = \langle \boldsymbol{\rho}, \mathbf{X}(t)\rangle$ and $\boldsymbol{\rho} := (\rho_1, \rho_2, \cdots, \rho_N)' \in \Re^N$ with $-1 < \rho_i < 1$, for each $i = 1, 2, \cdots, N$.

Note that in the Schöbel-Zhu-Hull-White hybrid model, there is a positive probability that the interest rate or the volatility will take a negative value. However, the Schöbel-Zhu-Hull-White model leads to some analytically tractable results for pricing volatility derivatives, which will facilitate their uses in practical situations, one may adjust the parameters in the Schöbel-Zhu-Hull-White model so that the probability of getting a negative value of the interest rate or the volatility is small.

We finish this section by introducing the information structure of our model. Let $\mathbb{F}^{\mathbf{r}} := \{\mathscr{F}^{\mathbf{r}}(t)|t \in \mathscr{T}\}$ and $\mathbb{F}^{\boldsymbol{\sigma}} := \{\mathscr{F}^{\boldsymbol{\sigma}}(t)|t \in \mathscr{T}\}$ denote the right-continuous, $\mathscr{P}$-complete filtrations generated by the short rate process $\mathbf{r}$ and the volatility process $\boldsymbol{\sigma}$. We define two enlarged filtrations $\mathbb{G} := \{\mathscr{G}(t)|t \in \mathscr{T}\}$ and $\mathbb{H} := \{\mathscr{H}(t)|t \in \mathscr{T}\}$, where

$$\mathscr{G}(t) := \mathscr{F}^{\mathbf{r}}(t) \vee \mathscr{F}^{\mathbf{X}}(t) \ ,$$

and

$$\mathscr{H}(t) := \mathscr{F}^{\mathbf{r}}(t) \vee \mathscr{F}^{\boldsymbol{\sigma}}(t) \vee \mathscr{F}^{\mathbf{X}}(t) \ .$$

Here $\mathscr{G}(t)$ is the minimal σ-field containing $\mathscr{F}^{\mathbf{r}}(t)$ and $\mathscr{F}^{\mathbf{X}}(t)$, and $\mathscr{H}(t)$ is the minimal σ-field containing $\mathscr{F}^{\mathbf{r}}(t)$, $\mathscr{F}^{\boldsymbol{\sigma}}(t)$ and $\mathscr{F}^{\mathbf{X}}(t)$.

3.3 Bond pricing and the forward measure

In this section, we employ the PDE approach to derive an exponential affine formula for the price of a zero-coupon bond. This method is different from the concept of stochastic flows adopted by Elliott and Siu (2009), Siu (2010) and Shen and Siu (2012) for pricing bonds under the regime-switching Hull-White or Vasicek model. Furthermore, we give the forward measure when taking the zero-coupon bond as the numéraire.

Since $\mathscr{P}$ is a risk-neutral probability measure, the price at time $t \in \mathscr{T}$ of a zero-coupon bond with a unit payoff at maturity time T is:

$$P(t,T) = \mathrm{E}\bigg[\exp\bigg(-\int_t^T r(s)ds\bigg)\bigg|\mathscr{G}(t)\bigg] \ . \tag{3.3.1}$$

Here E is an expectation under $\mathscr{P}$. Note that $(\mathbf{r}, \mathbf{X})$ is a joint Markov process with respect to the enlarged filtration $\mathbb{G}$. Then

$$P(t,T) = \mathrm{E}\bigg[\exp\bigg(-\int_t^T r(s)ds\bigg)\bigg|r(t), \mathbf{X}(t)\bigg] := F(t,T,r(t),\mathbf{X}(t)) \ . \tag{3.3.2}$$

Denote by

$$\begin{aligned} \widetilde{P}(t,T) &:= e^{-\int_0^t r(s)ds}P(t,T) = e^{-\int_0^t r(s)ds}F(t,T,r(t),\mathbf{X}(t)) \\ &= \mathrm{E}\bigg[\exp\bigg(-\int_0^T r(s)ds\bigg)\bigg|\mathscr{G}(t)\bigg] \ , \quad t \in \mathscr{T} \ , \end{aligned}$$

the discounted bond price process.

In what follows, we suppose that for each $\mathbf{x} \in \mathscr{E}$, $(t, r) \to F(t, T, r, \mathbf{x})$ is continuously differentiable with respect to t and twice continuously differentiable with respect to r, where the corresponding partial derivatives are denoted by $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial r}$ and $\frac{\partial^2 F}{\partial r^2}$.

Lemma 3.3.1. *Under the following regularity conditions,*

1. $E[|\exp(-\int_0^T r(t)dt)|] < \infty$;
2. $E[\int_0^T |e^{-\int_0^t r(u)du} \frac{\partial F}{\partial r} \gamma(t)|^2 dt] < \infty$;

the bond price process has the following regime-switching exponential-affine representation:

$$P(t, T) = \exp[A(t, T, \mathbf{X}(t)) - B(t, T) r(t)] \ , \quad t \in \mathscr{T} \ ,$$

where $A(t, T, \mathbf{X}(t))$ and $B(t, T)$ are some "smooth" functions satisfying

$$\begin{aligned} A(t, T, \mathbf{X}(t)) &= \log \left\{ E\left[\exp\left\{ \int_t^T \left(\alpha(s)\beta(s)B(s, T) - \frac{1}{2}\gamma^2(s)B^2(s, T) \right) ds \right\} \middle| \mathbf{X}(t) \right] \right\} \ , \\ B(t, T) &= \int_t^T e^{-\int_t^s \beta(u)du} ds \ , \ t \in \mathscr{T} \ . \end{aligned}$$

Proof. Firstly, we note that the discounted bond price process $\widetilde{P}(t, T)$, $t \in \mathscr{T}$, is a $(\mathbb{G}, \mathscr{P})$-martingale. Applying Itô's differentiation rule to $\widetilde{P}(t, T) = e^{-\int_0^t r(s)ds} F(t, T, r(t), \mathbf{X}(t))$ gives:

$$\begin{aligned} d\widetilde{P}(t, T) &= e^{-\int_0^t r(s)ds} \left\{ \frac{\partial F}{\partial r} \gamma(t) dW_r(t) + \langle \mathbf{F}(t, T, r(t)), d\mathbf{M}(t) \rangle \right. \\ &\quad + \left[-r(t)F(t, T, r(t), \mathbf{X}(t)) + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial r} \beta(t)(\alpha(t) - r(t)) \right. \\ &\quad \left. \left. + \frac{1}{2} \frac{\partial^2 F}{\partial r^2} \gamma^2(t) + \langle \mathbf{F}(t, T, r(t)), \mathbf{Q}\mathbf{X}(t) \rangle \right] dt \right\} \ , \qquad (3.3.3) \end{aligned}$$

where $\mathbf{F}(t, T, r(t)) := (F(t, T, r(t), \mathbf{e}_1), F(t, T, r(t), \mathbf{e}_2), \cdots, F(t, T, r(t), \mathbf{e}_N))' \in \Re^N$.

Since $\widetilde{P}(t,T)$ is a $(\mathbb{G},\mathscr{P})$-martingale, it must be a special semimartingale, which admits a unique decomposition when the process of locally integrable variation is predictable. Consequently, the bounded variation terms, which are not martingales, in the above stochastic integral representation must sum to zero. Therefore, the function $(t,r,\mathbf{x}) \to F(t,T,r,\mathbf{x})$ solves the following regime-switching partial differential equation (PDE):

$$
\begin{aligned}
-rF(t,T,r,\mathbf{x}) + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\beta(t)(\alpha(t)-r) \\
+\frac{1}{2}\frac{\partial^2 F}{\partial r^2}\gamma^2(t) + \langle \mathbf{F}(t,T,r), \mathbf{Q}\mathbf{x}\rangle = 0 \ ,
\end{aligned}
\tag{3.3.4}
$$

with terminal condition:

$$
F(T,T,r,\mathbf{x}) = 1 \ .
$$

We try the following regime-switching exponential-affine form solution:

$$
F(t,T,r,\mathbf{x}) = \exp(A(t,T,\mathbf{x}) - B(t,T)r) \ , \tag{3.3.5}
$$

with the terminal conditions $A(T,T,\mathbf{x}) = B(T,T) = 0$, for each $\mathbf{x} \in \mathscr{E}$. Note that

$$
\frac{\partial F}{\partial t} = \left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)F \ , \quad \frac{\partial F}{\partial r} = -BF \ , \quad \frac{\partial^2 F}{\partial r^2} = B^2 F \ .
$$

Substituting Eq. (3.3.5) into Eq. (3.3.4) gives

$$
\begin{aligned}
&-\left[1 + \frac{\partial B}{\partial t} - \beta(t)B\right]r \\
&+\left[\frac{\partial A}{\partial t} - \alpha(t)\beta(t)B + \frac{1}{2}\gamma^2(t)B^2 + \widetilde{A}^{-1}\left\langle \widetilde{\mathbf{A}}, \mathbf{Q}\mathbf{x}\right\rangle\right] = 0 \ ,
\end{aligned}
\tag{3.3.6}
$$

where $\widetilde{A} := \widetilde{A}(t,T,\mathbf{x}) = \exp(A(t,T,\mathbf{x}))$ and $\widetilde{\mathbf{A}} := (\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_N)' \in \Re^N$ with $\widetilde{A}_i := \widetilde{A}(t,T,\mathbf{e}_i)$, for each $i = 1,2,\cdots,N$.

Since Eq. (3.3.6) must hold for all r, the coefficients of r in this equation must be zeros. Then it implies that

$$B(t,T)=\int_t^T e^{-\int_t^s \beta(u)du}ds\ ,$$

and $\widetilde{A}(t,T,\mathbf{x})$ solves the following regime-switching ordinary differential equation (ODE):

$$\frac{\partial \widetilde{A}}{\partial t}-\widetilde{A}\Big[\alpha(t)\beta(t)B-\frac{1}{2}\gamma^2(t)B^2\Big]+\Big\langle \widetilde{\mathbf{A}},\mathbf{Q}\mathbf{x}\Big\rangle=0\ .$$

Using a version of the Feynman-Kac formula (see Theorem 6 in Elliott and Swishchuk (2007)), we obtain the following expectation representation for $\widetilde{A}$:

$$\widetilde{A}(t,T,\mathbf{x})=\mathrm{E}\Big[\exp\Big(\int_t^T\Big(\alpha(s)\beta(s)B(s,T)-\frac{1}{2}\gamma^2(s)B^2(s,T)\Big)ds\Big|\mathbf{X}(t)=\mathbf{x}\Big]\ ,$$

which obviously leads to the desired result. $\square$

Lemma 3.3.2. *Define the forward measure $\mathscr{P}^T$ equivalent to $\mathscr{P}$ on $\mathscr{G}(T)$ by putting:*

$$\frac{d\mathscr{P}^T}{d\mathscr{P}}\bigg|_{\mathscr{G}(T)}=\Lambda(T):=\frac{\exp(-\int_0^T r(t)dt)}{E[\exp(-\int_0^T r(t)dt)]}\ . \tag{3.3.7}$$

Assume that

$$E\Big[\exp\Big\{\frac{1}{2}\int_0^T\gamma^2(t)B^2(t,T)dt\Big\}\Big]<\infty\ ,$$

and $\widetilde{A}(t,T,\mathbf{x})$ is a suitable function in the sense that

$$\frac{\widetilde{A}(t,T,\mathbf{X}(t))}{\widetilde{A}(0,T,\mathbf{X}(0))}\exp\Big\{-\int_0^t\frac{\frac{\partial\widetilde{A}}{\partial s}+\mathbf{Q}\widetilde{A}(s,T,\mathbf{X}(s))}{\widetilde{A}(s,T,\mathbf{X}(s))}ds\Big\}\ ,\ t\in\mathscr{T}\ ,$$

is a $(\mathbb{G},\mathscr{P})$-martingale.

Then, under the forward measure $\mathscr{P}^T$, *1. the processes*

$$W_r^T(t) = W_r(t) + \int_0^t \gamma(s)B(s,T)ds \ , \ t \in \mathscr{T} \ ,$$

and

$$W_\sigma^T(t) = W_\sigma(t) + \int_0^t \rho(s)\gamma(s)B(s,T)ds \ , \ t \in \mathscr{T} \ ,$$

are standard Brownian motions with respect to $\mathbb{G}$;

2. the rate matrix of the chain $\mathbf{X}$ is $\mathbf{Q}^T(t) := [q_{ij}^T(t)]_{i,j=1,2,\cdots,N}$:

$$q_{ij}^T(t) = \begin{cases} q_{ij}\dfrac{\widetilde{A}(t,T,\mathbf{e}_j)}{\widetilde{A}(t,T,\mathbf{e}_i)} \ , & i \neq j \ , \\ -\displaystyle\sum_{k\neq i} q_{ik}\dfrac{\widetilde{A}(t,T,\mathbf{e}_k)}{\widetilde{A}(t,T,\mathbf{e}_i)} \ , & i = j \ , \end{cases} \tag{3.3.8}$$

and the semimartingale decomposition of the chain is given by:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{Q}^T(s)\mathbf{X}(s)ds + \mathbf{M}^T(t) \ , \quad t \in \mathscr{T} \ , \tag{3.3.9}$$

where $\{\mathbf{M}^T(t)|t \in \mathscr{T}\}$ is an $\Re^N$-valued, $(\mathbb{F}^{\mathbf{X}}, \mathscr{P}^T)$-martingale.

Proof. Recall Eq. (3.3.3) and the exponential-affine form of the bond price process, we have

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - \gamma(t)B(t,T)dW_r(t) + \widetilde{A}(t,T,\mathbf{X}(t))^{-1}\left\langle \widetilde{\mathbf{A}}(t,T), d\mathbf{M}(t)\right\rangle . \tag{3.3.10}$$

Denote by

$$\Lambda(t) := \mathrm{E}[\Lambda(T)|\mathcal{F}(t)] = \frac{\widetilde{P}(t,T)}{P(0,T)} = e^{-\int_0^t r(s)ds}\frac{P(t,T)}{P(0,T)} \ .$$

Then, from Eq. (3.3.10)

$$\frac{d\Lambda(t)}{\Lambda(t)} \quad = \quad \frac{dP(t,T)}{P(t,T)} - r(t)dt$$

$$= \quad -\gamma(t)B(t,T)dW_r(t) + \widetilde{A}(t,T,\mathbf{X}(t))^{-1}\left\langle \widetilde{\mathbf{A}}(t,T), d\mathbf{M}(t)\right\rangle \ . \quad (3.3.11)$$

Since W_r and $\mathbf{M}$ are independent under $\mathscr{P}$, it is easy to see that $\Lambda(t) = \Lambda_1(t)\cdot\Lambda_2(t)$, where

$$\frac{d\Lambda_1(t)}{\Lambda_1(t)} = -\gamma(t)B(t,T)dW_r(t) \ ,$$

and

$$\frac{d\Lambda_2(t)}{\Lambda_2(t)} = \widetilde{A}(t,T,\mathbf{X}(t))^{-1}\left\langle \widetilde{\mathbf{A}}(t,T), d\mathbf{M}(t)\right\rangle \ .$$

Hence

$$\Lambda_1(t) = \exp\left\{ -\int_0^t \gamma(s)B(s,T)dW_r(s) - \frac{1}{2}\int_0^t \gamma^2(s)B^2(s,T)ds \right\} \ ,$$

and

$$\Lambda_2(t) = \frac{\widetilde{A}(t,T,\mathbf{X}(t))}{\widetilde{A}(0,T,\mathbf{X}(0))}\exp\left\{ -\int_0^t \frac{\frac{\partial \widetilde{A}}{\partial s} + \mathbf{Q}\widetilde{A}(s,T,\mathbf{X}(s))}{\widetilde{A}(s,T,\mathbf{X}(s))} ds \right\} \ .$$

From the assumption in this lemma, the $(\mathbb{G},\mathscr{P})$-(local)-martingales $\{\Lambda(t)|t \in \mathscr{T}\}$, $\{\Lambda_1(t)|\ t \in \mathscr{T}\}$ and $\{\Lambda_2(t)|t \in \mathscr{T}\}$ are also $(\mathbb{G},\mathscr{P})$-martingales. Therefore, the forward measure $\mathscr{P}^T$ defined by Eq. (3.3.7) is indeed a probability measure. The desired result follows from Girsanov's theorem for the Brownian motion and Lemma 12.3.3 in Rolski et al. (1999) or Proposition 5.1 in Palmowski and Rolski (2002). □

3.4 Pricing variance swaps

In this section, we derive the price of a variance swap under the regime-switching Schöbel-Zhu-Hull-White hybrid model. By changing the risk-neutral measure to the forward one, we take out the stochastic discount factor from the expectation and separate the interest rate risk and the volatility risk in a variance swap. Then the price of a variance swap and the fair strike value are represented in integral forms.

A variance swap is a forward contract written on realized annual variance. Under the continuous sampling scheme, the price of a T-maturity variance swap at time 0 is

$$C(0,T) = \mathrm{E}\left[\exp\left(- \int_0^T r(t)dt \right) \left(\sigma_R^2 - K_{var} \right) \right] , \tag{3.4.1}$$

where the realized annual variance is given by

$$\sigma_R^2 := \frac{1}{T} \int_0^T v(t)dt , \tag{3.4.2}$$

and the (instantaneous) variance process $\mathbf{v} := \{v(t) | t \in \mathscr{T}\}$ has the following relationship with the (instantaneous) volatility process $\boldsymbol{\sigma}$:

$$v(t) := \sigma^2(t) , \quad t \in \mathscr{T} .$$

In practice, the realized annual variance is evaluated based on predetermined discrete sampling scheme, which is clearly specified for a variance swap. The price of a variance swap defined by Eqs. (3.4.1) and (3.4.2) is only a continuous approximation to that of the actual contract. Since our main concern is the joint effect of interest rate, volatility and regime-switching on the price of a variance swap, we only consider the continuous sampling approximation in this paper. This approximation was also adopted by Elliott and Swishchuk (2007), Elliott et al. (2007), and others.

To simplify our notation, we write in the following that

$$\kappa(t,T) := a(t)b(t) - c(t)\rho(t)\gamma(t)B(t,T) .$$

Then,

$$\kappa(t,T) = \kappa(t,T,\mathbf{X}(t)) = \langle \boldsymbol{\kappa}(t,T), \mathbf{X}(t) \rangle ,$$

where $\boldsymbol{\kappa}(t,T) := (\kappa_1(t,T), \kappa_2(t,T), \cdots, \kappa_N(t,T))' \in \Re^N$ and for each $i = 1, 2, \cdots, N$,

$$\kappa_i(t,T) = a_i b(t) - c_i \rho_i \gamma_i B(t,T) \ . \tag{3.4.3}$$

With a little abuse of notation, we denote $\boldsymbol{\kappa}^2(t,T) := (\kappa_1^2(t,T), \kappa_2^2(t,T), \cdots, \kappa_N^2(t,T))' \in \Re^N$ and $\boldsymbol{\varphi}^2 := (\varphi_1^2, \varphi_2^2, \cdots, \varphi_N^2)' \in \Re^N$, $\varphi = a, c$. Therefore,

$$\kappa^2(t,T) = \left\langle \boldsymbol{\kappa}^2(t,T), \mathbf{X}(t) \right\rangle \ ,$$

and

$$\varphi^2(t) = \left\langle \boldsymbol{\varphi}^2, \mathbf{X}(t) \right\rangle \ .$$

Before deriving the price of a variance swap, we present the following useful lemma about the expectation of the chain $\mathbf{X}$.

Lemma 3.4.1. *Let $\mathbf{X}$ be a Markov chain defined in Section 2. Then*

$$E[\mathbf{X}(t)] = e^{\mathbf{Q}t}\mathbf{X}(0) \ , \tag{3.4.4}$$

and

$$E^T[\mathbf{X}(t)] = \mathbf{\Phi}(t)\mathbf{X}(0) \ , \quad t \in \mathscr{T} \ , \tag{3.4.5}$$

where the matrix-valued function $\mathbf{\Phi}(t) \in \Re^{N\times N}$ is the fundamental solution of the following matrix-valued ODE:

$$\frac{d\mathbf{\Phi}(t)}{dt} = \mathbf{Q}^T(t)\mathbf{\Phi}(t) \ , \quad \mathbf{\Phi}(0) = \mathbf{I} \ . \tag{3.4.6}$$

Here E^T is an expectation under the forward measure $\mathscr{P}^T$.

Proof. Taking expectation on both sides of Eq. (3.2.1), we have

$$\mathrm{E}[\mathbf{X}(t)] = \mathbf{X}(0) + \int_0^t \mathbf{Q}\mathrm{E}[\mathbf{X}(s)]ds \ , \tag{3.4.7}$$

which immediately leads to Eq. (3.4.4). The proof of Eq. (3.4.5) is similar. Hence we do not repeat it here. □

The following theorem is the main result of this section, which gives the price of a variance swap under the regime-switching Schöbel-Zhu-Hull-White hybrid model.

Theorem 3.4.1. *Under the Markovian regime-switching Schöbel-Zhu-Hull-White hybrid model, the price of a variance swap has the following integral representation:*

$$\begin{aligned}
&C(0,T) \\
&= \frac{P(0,T)}{T}\int_0^T \Bigg\{ e^{-2\int_0^t b(s)ds}[c^2(0)+2\kappa(0,T)\sigma(0)] \\
&\quad + 2\int_0^t e^{-2\int_s^t b(u)du} e^{-\int_0^s b(u)du}\kappa(0,T)\sigma(0)ds + \int_0^t e^{-2\int_s^t b(u)du}\Bigg[\left\langle \mathbf{c}^2, \mathbf{\Phi}(s)\mathbf{X}(0)\right\rangle \\
&\quad + 2\int_0^s e^{-\int_z^s b(u)du}\left\langle \boldsymbol{\kappa}^2(z,T), \mathbf{\Phi}(z)\mathbf{X}(0)\right\rangle dz\Bigg] ds\Bigg\} dt - P(0,T)K_{var} \ , \qquad (3.4.8)
\end{aligned}$$

and the fair strike value is given by:

$$\begin{aligned}
K_{var} &= \frac{1}{T}\int_0^T \Bigg\{ e^{-2\int_0^t b(s)ds}[c^2(0)+2\kappa(0,T)\sigma(0)] \\
&\quad +2\int_0^t e^{-2\int_s^t b(u)du}e^{-\int_0^s b(u)du}\kappa(0,T)\sigma(0)ds \\
&\quad +\int_0^t e^{-2\int_s^t b(u)du}\Bigg[\left\langle \mathbf{c}^2, \mathbf{\Phi}(s)\mathbf{X}(0)\right\rangle \\
&\quad +2\int_0^s e^{-\int_z^s b(u)du}\left\langle \boldsymbol{\kappa}^2(z,T), \mathbf{\Phi}(z)\mathbf{X}(0)\right\rangle dz\Bigg] ds\Bigg\} dt \ . \qquad (3.4.9)
\end{aligned}$$

Proof. By change of measures, Eq. (3.4.1) becomes

$$\begin{aligned}
C(0,T) &= P(0,T)\mathrm{E}^T[\sigma_R^2 - K_{var}] \\
&= P(0,T)\left\{\frac{1}{T}\int_0^T \mathrm{E}^T[v(t)]dt - K_{var}\right\} \\
&= P(0,T)\left\{\frac{1}{T}\int_0^T \mathrm{E}^T\left[\mathrm{E}^T[v(t)|\mathscr{F}^{\mathbf{X}}(t)]\right]dt - K_{var}\right\} . \qquad (3.4.10)
\end{aligned}$$

Under $\mathscr{P}^T$, the dynamics of stochastic volatility are governed by:

$$d\sigma(t) = b(t)(a(t)-\sigma(t))dt + c(t)(dW_\sigma^T(t) - \rho(t)\gamma(t)B(t,T))$$

$$= \quad \big[\kappa(t,T) - b(t)\sigma(t)\big]dt + c(t)dW_\sigma^T(t)\ . \tag{3.4.11}$$

Conditioning both sides of Eq. (3.4.11) on $\mathscr{F}^{\mathbf{X}}(t)$ under $\mathscr{P}^T$, we have

$$d\mathrm{E}^T[\sigma(t)|\mathscr{F}^{\mathbf{X}}(t)] = \big[\kappa(t,T) - b(t)\mathrm{E}^T[\sigma(t)|\mathscr{F}^{\mathbf{X}}(t)]\big]dt\ . \tag{3.4.12}$$

Solving Eq. (3.4.12) gives

$$\mathrm{E}^T[\sigma(t)|\mathscr{F}^{\mathbf{X}}(t)] = e^{-\int_0^t b(s)ds}\sigma(0) + \int_0^t e^{-\int_s^t b(u)du}\kappa(s,T)ds\ .$$

In the same vein, we can derive that

$$\mathrm{E}^T[\kappa(t,T)\sigma(t)|\mathscr{F}^{\mathbf{X}}(t)] = e^{-\int_0^t b(s)ds}\kappa(0,T)\sigma(0) + \int_0^t e^{-\int_s^t b(u)du}\kappa^2(s,T)ds\ . \tag{3.4.13}$$

Under $\mathscr{P}^T$, applying Itô's differentiation rule to $v(t) = \sigma^2(t)$ gives

$$d\sigma^2(t) = \big[c^2(t) + 2\kappa(t,T)\sigma(t) - 2b(t)\sigma^2(t)\big]dt + 2c(t)\sigma(t)dW_\sigma^T(t)\ . \tag{3.4.14}$$

Conditioning both sides of Eq. (3.4.14) on $\mathscr{F}^{\mathbf{X}}(t)$ under $\mathscr{P}^T$ gives

$$\begin{aligned} d\mathrm{E}^T[\sigma^2(t)|\mathscr{F}^{\mathbf{X}}(t)] \quad &= \quad \big[c^2(t) + 2\mathrm{E}^T[\kappa(t,T)\sigma(t)|\mathscr{F}^{\mathbf{X}}(t)] \\ &\qquad -2b(t)\mathrm{E}^T[\sigma^2(t)|\mathscr{F}^{\mathbf{X}}(t)]\big]dt\ . \end{aligned} \tag{3.4.15}$$

Substituting Eq. (3.4.13) and solving Eq. (3.4.15), we have

$$\begin{aligned} \mathrm{E}^T[\sigma^2(t)|\mathscr{F}^{\mathbf{X}}(t)] \quad &= \quad e^{-2\int_0^t b(s)ds}[c^2(0) + 2\kappa(0,T)\sigma(0)] \\ &\quad + \int_0^t e^{-2\int_s^t b(u)du}\Big\{c^2(s) + 2e^{-\int_0^s b(u)du}\kappa(0,T)\sigma(0) \\ &\quad +2\int_0^s e^{-\int_z^s b(u)du}\kappa^2(z,T)dz\Big\}ds\ . \end{aligned} \tag{3.4.16}$$

Using Lemma 3.4.1 and taking expectation on both sides, we obtain

$$\mathrm{E}^T[v(t)] \quad = \quad \mathrm{E}^T\big[\mathrm{E}^T[\sigma^2(t)|\mathscr{F}^{\mathbf{X}}(t)]\big]$$

$$
\begin{aligned}
&= e^{-2\int_0^t b(s)ds}[c^2(0) + 2\kappa(0,T)\sigma(0)] + 2\int_0^t e^{-2\int_s^t b(u)du} e^{-\int_0^s b(u)du}\kappa(0,T)\sigma(0)ds \\
&\quad + \int_0^t e^{-2\int_s^t b(u)du}\Bigg[\left\langle \mathbf{c}^2, \mathbf{\Phi}(s)\mathbf{X}(0)\right\rangle \\
&\quad +2\int_0^s e^{-\int_z^s b(u)du}\left\langle \boldsymbol{\kappa}^2(z,T), \mathbf{\Phi}(z)\mathbf{X}(0)\right\rangle dz\Bigg]ds\ . \qquad (3.4.17)
\end{aligned}
$$

Consequently, substituting Eq. (3.4.17) into Eq. (3.4.10) gives (3.4.8), the price of a variance swap under the Schöbel-Zhu-Hull-White hybrid model. Furthermore, it is not difficult to see that the fair strike value of a variance swap is represented by the integral form (3.4.9). □

The following corollary is a special case of Theorem 3.4.1. In Corollary 3.4.1, the effect of stochastic interest rate is degenerate. More specifically, it gives the price of a variance swap under a regime-switching Schöbel-Zhu model with deterministic interest rate.

Corollary 3.4.1. *Under the Markovian regime-switching Schöbel-Zhu model with a deterministic interest rate* $\{r(t)|t \in \mathscr{T}\}$*, the price of a variance swap has the following integral representation:*

$$
\begin{aligned}
C(0,T) &= \frac{e^{-\int_0^T r(t)dt}}{T}\int_0^T \Bigg\{e^{-2\int_0^t b(s)ds}[c(0) + 2a(0)b(0)\sigma(0)] \\
&\quad +2\int_0^t e^{-2\int_s^t b(u)du}e^{-\int_0^s b(u)du}a(0)b(0)\sigma(0)ds + \int_0^t e^{-2\int_s^t b(u)du}\Bigg[\left\langle \mathbf{c}^2, e^{\mathbf{Q}s}\mathbf{X}(0)\right\rangle \\
&\quad +2\int_0^s e^{-\int_z^s b(u)du}b^2(z)\left\langle \mathbf{a}^2, e^{\mathbf{Q}z}\mathbf{X}(0)\right\rangle dz\Bigg]ds\Bigg\}dt - e^{-\int_0^T r(t)dt}K_{var}\ , \qquad (3.4.18)
\end{aligned}
$$

and the fair strike value is given by:

$$
\begin{aligned}
K_{var} &= \frac{1}{T}\int_0^T \Bigg\{e^{-2\int_0^t b(s)ds}[c(0) + 2a(0)b(0)\sigma(0)] \\
&\quad +2\int_0^t e^{-2\int_s^t b(u)du}e^{-\int_0^s b(u)du}a(0)b(0)\sigma(0)ds
\end{aligned}
$$

$$+\int_0^t e^{-2\int_s^t b(u)du}\Bigg[\left\langle \mathbf{c}^2, e^{\mathbf{Q}s}\mathbf{X}(0)\right\rangle$$
$$+2\int_0^s e^{-\int_z^s b(u)du} b^2(z)\left\langle \mathbf{a}^2, e^{\mathbf{Q}z}\mathbf{X}(0)\right\rangle dz\Bigg]ds\Bigg\}dt\ . \qquad (3.4.19)$$

3.5 Numerical Implementation

In this section, we perform a numerical analysis for pricing variance swaps under our proposed model. To simplify our computation, we consider a two-state Markov chain $\mathbf{X}$, where State 1 and State 2 of the chain represent a 'Good' economy and a 'Bad' economy, respectively. We write $\mathbf{X}(t) = (1,0)'$ and $\mathbf{X}(t) = (0,1)'$, $\forall t \in \mathscr{T}$, for State 1 and State 2.

First of all, we give configurations of the parameter values. Assume that the rate matrix of the chain $\mathbf{X}$ under $\mathscr{P}$ is

$$\mathbf{Q} = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix} ,$$

where q takes values in $[0,1]$. For simplicity, we consider two positive constants controlling the speed of mean reversion for the short rate and the volatility, $\beta = 0.2$ and $b = 4$. The values of the other parameters of the Schöbel-Zhu-Hull-White hybrid model are given by:

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2)' = (0.04, 0.02)' \ , \qquad \mathbf{a} = (a_1, a_2)' = (0.2, 0.4)' \ ,$$
$$\boldsymbol{\gamma} = (\gamma_1, \gamma_2)' = (0.02, 0.04)' \ , \qquad \mathbf{c} = (c_1, c_2)' = (0.2, 0.4)' \ .$$

Then the two-state Schöbel-Zhu-Hull-White hybrid model becomes

$$\begin{cases} dr(t) = \beta(\alpha_1 - r(t))dt + \gamma_1 dW_r(t) \\ d\sigma(t) = b(a_1 - \sigma(t))dt + c_1 dW_\sigma(t) \end{cases} , \quad \text{if } \mathbf{X}(t) = (1,0)' \ , \qquad (3.5.1)$$

and

$$\begin{cases} dr(t) = \beta(\alpha_2 - r(t))dt + \gamma_2 dW_r(t) \\ d\sigma(t) = b(a_2 - \sigma(t))dt + c_2 dW_\sigma(t) \end{cases}, \quad \text{if } \mathbf{X}(t) = (0,1)' , \tag{3.5.2}$$

where the instantaneous covariance of W_r and W_σ in State i is assumed to be ρ_i, $i = 1, 2$, and $\boldsymbol{\rho} = (\rho_1, \rho_2) = (-0.25, -0.5)$. The initial values of the interest rate and the volatility are $r(0) = 0.02$ and $\sigma(0) = \langle(0.2, 0.4), \mathbf{X}(0)\rangle$. If the interest rate is assumed to be constant (i.e. $r(t) = 0.02$, $t \in \mathscr{T}$), the two-state Schöbel-Zhu-Hull-White (SZHW) hybrid model becomes the two-state Schöbel-Zhu (SZ) model.

From Lemma 3.3.1, Lemma 3.3.2, Theorem 3.4.1 and Corollary 3.4.1, we calculate the fair strike values of variance swaps with a maturity $T = 1$ with stochastic interest rate and deterministic interest rate. Figs. 3.5.1 and 3.5.2 depict the plots of the fair strike values of variance swaps against different values of the rate matrices. Note that the fair strike value of a variance swap has a positive relationship with the price of the variance swap. Therefore the larger the fair strike value is, the higher the price of the variance swap is. From Figs. 3.5.1 and 3.5.2, we have the following findings:

1. The price of a variance swap under the regime-switching Schöbel-Zhu-Hull-White hybrid model is higher than that under the regime-switching Schöbel-Zhu model.

2. The price of a variance swap in State 1 is lower than that in State 2.

3. The price of a variance swap increases with q in State 1 while decreases with q in State 2.

4. In State 1, the price of a variance swap is minimum when $q = 0$; In State 2, it is maximum when $q = 0$.

Our explanations for the above findings are listed below:

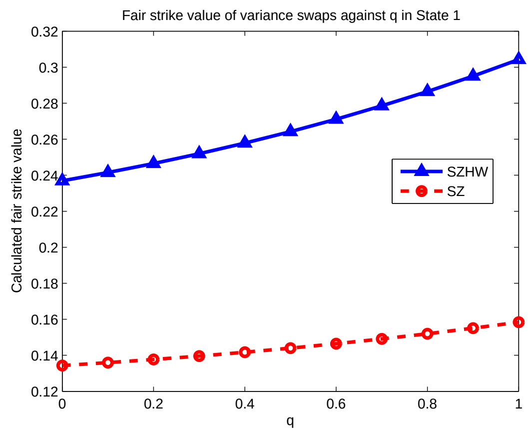


Figure 3.5.1: Fair strike value of variance swaps against q in State 1

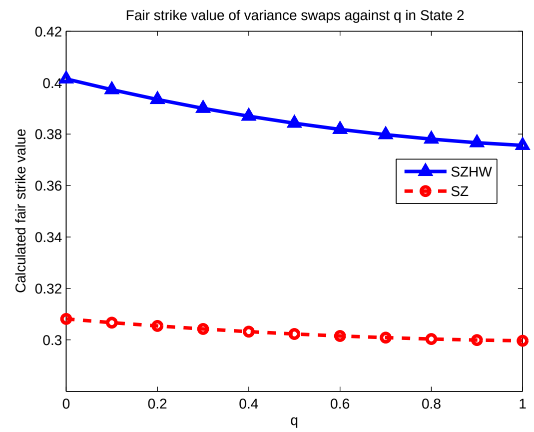


Figure 3.5.2: Fair strike value of variance swaps against q in State 2

1. Compared with the regime-switching Schöbel-Zhu model, the regime-switching Schöbel-Zhu-Hull-White hybrid model incorporates the effect of stochastic interest rate, which leads to a higher price of the variance swap due to fluctuations in the interest rate.

2. Since the rate matrices $\mathbf{Q}$ are symmetric, the probability of the chain transiting from State 1 to State 2 within a fixed period is equal to the probability of the chain transiting from State 2 to State 1. In other words, the probabilities of the chain remaining in State i within the same period are equal, $i = 1, 2$. Variance swaps are cheaper (more expensive) when the interest rate is higher (lower) and the volatility is lower (higher). Hence the price of the variance swap in State 1 (a "Good" economy with a high interest rate and a low volatility) is lower than that in State 2 (a "Bad" economy with a low interest rate and a high volatility).

3. The probability of the chain transiting from State 1 to State 2 increases with q. As explained in Point 2, variance swaps are cheaper (more expensive) when the

interest rate is higher (lower) and the volatility is lower (higher). Furthermore, the interest rate is higher and the volatility is lower in State 1 than in State 2. Thus, the price of variance swaps in State 1 (State 2) increases (decreases) with the probability of the chain transiting from State 1 to State 2.

4. The probability of the chain transiting from State 1 to State 2 or from State 2 to State 1 is zero when $q = 0$. That is, there is no regime-switching effect when $q = 0$. However, if the regime-switching effect is present, the possibilities of the chain transiting from State 1 to State 2 (from State 2 to State 1) would make the price of the variance swap higher (lower).

3.6 Conclusion

As shown in the above theoretical work and numerical analysis, both stochastic interest rate and regime-switching have a considerable impact on the prices of variance swaps. Although variance swaps are volatility derivatives, it is unreasonable to ignore stochastic interest rate and regime-switching in the pricing. Our pricing framework merits extension to other stochastic interest rate and stochastic volatility models, such as the regime-switching Heston-Hull-White hybrid model. However, if stochastic interest rate and volatility are correlated, the square-root term in the Heston model would drastically complicate the derivation of the analytical pricing formula, especially when the regime-switching effect is present. This difficulty will be one of our potential research topics in the future. Other potential research topics include the price dynamics of variance swaps and the valuation of swaptions. The former is important for portfolio optimization problems with variance swaps as investment vehicles. Once the dynamics of variance swaps are obtained, we can consider using variance swaps to hedge volatility risk in portfolio selection problems. The latter is an interesting but challenging problem. The challenge is that the payoff of swaptions depends on the whole path of variance process rather than only the terminal value. The Fourier transform as adopted in Chapter 2 or the Laplace transform of the first passage time of variance process may be useful to solve this challenging problem.

Chapter 4

Mean-variance portfolio selection with uncertain investment horizon under a regime-switching jump-diffusion model

4.1 Introduction

Portfolio selection problem is of great importance in both the theory and practice of insurance, banking and finance. The modern portfolio selection theory can be traced back to the seminal work of Markowitz (1952), where the mean-variance formulation was developed in a single-period setting. Ever since then, there has been a growing interest in extending and generalizing Markowitz's ground-breaking work. Using the embedding techniques, Li and Ng (2000) solved analytically the mean-variance portfolio selection problem in a multi-period setting. Applying the stochastic linear-quadratic theory, Zhou and Li (2000) investigated a continuous-time mean-variance portfolio selection problem.

Although Zhou and Li (2000)'s results on the continuous-time mean-variance problem are mathematically elegant, the use of the Geometric Brownian Motion (GBM) for the asset price dynamics has long been a controversial issue. Please refer to Chapter 1 for the shortcomings of the GBM model. Indeed, more economic insights and implications can be gained if the investment opportunity set is allowed to vary stochastically over time. In recent years, there has been a growing interest in the mean-variance portfolio selection problem under more realistic continuous-time asset price models, which may incorporate aforementioned realistic features of assets' returns and stochastic investment sets. Some examples include stochastic interest rate models (Ferland and Watier (2010)), stochastic volatility models (Dai (2011)), stochastic appreciation rate models (Chiu and Wong (2011)) and so on.

In the new millennium, there arises a growing interest in the mean-variance portfolio selection problems under regime-switching models. Zhou and Yin (2003) considered the mean-variance problem under a continuous-time Markovian regime-switching model. They formulated the problem as a stochastic linear-quadratic (LQ) optimization problem and obtained analytical results for the efficient portfolio and the efficient frontier. Zhang et al. (2012) developed a sufficient maximum principle for a stochastic optimal control problem under Markovian regime-switching jump-diffusion models. They applied the sufficient maximum principle to discuss the mean-variance problem. Donnelly and Heunis (2012) applied a conjugate duality approach to study the mean-variance problem with constraints under a random regime-switching model, where the market parameters are random processes adapted to the joint filtration of the Brownian motion and the Markov chain. They provided explicit optimal portfolios in some special cases. For the mean-variance approach to asset-liability management problems under regime-switching models, interested readers may refer to Chen et al. (2008), Xie (2012) and Wu (2013).

In this chapter, we study a continuous-time mean-variance portfolio selection problem with uncertain investment horizon under a Markovian regime-switching jump-diffusion model. Specifically, we assume that market parameters, including interest rate, appreciation rate, volatility rate, jump ratio, Lévy density of random measure and intensity of uncertain time, are all modulated by a continuous-time, finite-state, observable Markov chain. The Markovian regime-switching jump-diffusion model provides us with flexibility in modeling the asset price dynamics with not only switching regimes but sudden jumps and hence integrates the advantages of both the regime-switching GBM models and the jump-diffusion models. We consider an economic agent allocates his/her wealth into one risky-free bond and multiple risky shares so as to minimize the risk measured by the variance of his/her portfolio for some given expected return. Unlike the traditional literature on the mean-variance problems with fixed investment horizon, the agent does not know with certainty when the portfolio will be liquidated. Interested readers may refer to Li et al. (2008) and Wu and Li (2011) and references therein for the uncertainty of investment horizon in a multi-period setting up. To our best knowledge, the continuous-time mean-variance problem with uncertain investment horizon and regime-switching has not been considered in the literature. This chapter will fill in this gap. We first transform and formulate the original problem as an unconstrained stochastic optimal control problem with fixed investment horizon, which is called a min-max problem. We employ the dynamic programming principle to solve the problem. We provide a verification theorem for a Markovian regime-switching HJB equation related to the control problem. By solving the regime-switching HJB equation, we obtain explicit expressions for the efficient portfolio and the efficient frontier. Even though the investment horizon is uncertain, we prove that the mutual fund theorem still holds in this general setting. To see the impact of uncertain investment horizon, regime switching and sudden jump on the mean-variance problem, we provide several numerical examples in a two-regime economy to illustrate our results.

The rest of this chapter is structured as follows. Section 4.2 introduces the model dynamics and formulates our mean-variance problem. In Section 4.3, we solve the quadratic loss minimization problem related to the mean-variance problem. Section 4.4 derives the efficient portfolio, the efficient frontier and the mutual fund theorem. In Section 4.5, we provide several numerical examples to illustrate our results. Section 4.6 gives concluding remarks.

4.2 Problem formulation

In this section, we first introduce notation to be used throughout this paper. Then we formulate the mean-variance portfolio selection problem with uncertain investment horizon under a regime-switching jump-diffusion model. Finally, we transform the original problem to an unconstrained stochastic optimal control problem with fixed investment horizon, which can be readily solved by the dynamic programming principle.

Throughout this chapter, the following notation will be frequently used:

$C^\top$: the transpose of any vector or matrix C;

C^{-1}: the inverse of a square matrix C;

$\mathrm{tr}(C)$: the trace of a square matrix C;

$\langle C, D\rangle$: the inner product of C and D, that is $\langle C, D\rangle := \mathrm{tr}(C^\top D)$;

$||C||$: the Euclidean norm of C, that is $||C||^2 = \langle C, C\rangle$;

diag(C): the diagonal matrix with the elements of a vector C on the diagonal.

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$ satisfying the usual conditions, where $\mathbb{F}$ is a right-continuous, $\mathcal{P}$-complete filtration generated by a Brownian motion, a Poisson random measure and a Markov chain, which will be defined below. Let $\mathcal{T} := [0, T]$ denote a finite time horizon, where $T < \infty$. We consider a continuous-time, finite-state, homogeneous Markov chain $\mathbf{X} := \{\mathbf{X}(t) | t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with state space $\mathcal{S} := \{\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_N\}$. Without loss of generality, we identify the state space of the chain to be a finite set of unit vectors $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N\} \subset \Re^N$, where the j-th component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij}, for each $i, j = 1, 2, \cdots, N$. Here $\mathcal{E}$ is called the canonical state space of the chain $\mathbf{X}$. To specify the statistical properties of the chain $\mathbf{X}$, we define a constant rate matrix or generator, $\mathbf{A} := [a_{ij}]_{i,j=1,2,\cdots,N}$, of the chain $\mathbf{X}$ under $\mathcal{P}$, where, for $i \neq j$, a_{ij} is the instantaneous intensity of the transition of the chain $\mathbf{X}$ from state $\mathbf{e}_j$ to state $\mathbf{e}_i$. Note that $a_{ij} \geq 0$, for $i \neq j$ and $\sum_{i=1}^{N} a_{ij} = 0$, so $a_{ii} \leq 0$. With the canonical representation of the state space of the chain, Elliott et al. (1994) provided the following semimartingale dynamics of the chain $\mathbf{X}$

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}\mathbf{X}(s)ds + \mathbf{M}(t) \ , \quad t \in \mathcal{T} \ , \tag{4.2.1}$$

where $\{\mathbf{M}(t) | t \in \mathcal{T}\}$ is an $\Re^N$-valued, $(\mathbb{F}, \mathcal{P})$-martingale.

Let $W := \{W(t) | t \in \mathcal{T}\} = \{(W^1(t), W^2(t), \cdots, W^d(t))^\top | t \in \mathcal{T}\}$ be a d-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$. We now introduce a Markov regime-switching Poisson random measure. Let $\Re_0 := \Re \setminus \{0\}$ and $\mathcal{B}(\Re_0)$ be the Borel σ-field generated by open subset O of $\Re_0$, whose closure $\overline{O}$ does not contain the point 0. Suppose that $\gamma^l(dt, dz)$, $l = 1, 2, \cdots, m$, are independent Poisson random measures on the product measurable space $(\mathcal{T} \times \Re_0, \mathcal{B}(\mathcal{T}) \times \mathcal{B}(\Re_0))$ under $\mathcal{P}$. Assume that the Poisson random measure $\gamma^l(dt, dz)$ has the following Markov-modulated compensator

$$\nu^l_{\mathbf{X}(t-)}(dz)dt = \langle \nu^l(dz), \mathbf{X}(t-) \rangle \, dt \ ,$$

where

$$\nu^l(dz) := (\nu^l_{\mathbf{e}_1}(dz), \nu^l_{\mathbf{e}_2}(dz), \cdots, \nu^l_{\mathbf{e}_N}(dz))^\top \in \Re^N .$$

For each $l = 1, 2, \cdots, m$, write

$$\widetilde{\gamma}^l_{\mathbf{X}(t-)}(dt, dz) := \gamma^l(dt, dz) - \nu^l_{\mathbf{X}(t-)}(dz)dt ,$$

for a compensated Poisson random measure, which is a real-valued, $(\mathbb{F}, \mathcal{P})$-martingale. Here we use the subscript $\mathbf{X}(t-)$ in $\nu^l_{\mathbf{X}(t-)}(dz)$, $l = 1, 2, \cdots, m$ to indicate the dependence of the probability law of the Poisson random measure on the Markov chain. Indeed, for each $l = 1, 2, \cdots, m$, $\nu^l_{\mathbf{e}_i}(dz)$ is the Lévy density of jump size of the random measure $\gamma^l(dt, dz)$ if and only if $\mathbf{X}(t-) = \mathbf{e}_i$. To unburden our notation, write

$$\widetilde{\gamma}_{\mathbf{X}(t-)}(dt, dz) := (\gamma^1(dt, dz) - \nu^1_{\mathbf{X}(t-)}(dz)dt, \cdots, \gamma^m(dt, dz) - \nu^m_{\mathbf{X}(t-)}(dz)dt)^\top \in \Re^m ,$$

for an m-dimensional compensated Poisson random measure, which is an $(\mathbb{F}, \mathcal{P})$-martingale.

To simplify our discussion, we assume that the Brownian motion and the Poisson random measure are stochastically independent under $\mathcal{P}$. Let $\mathbb{F}^W := \{\mathcal{F}^W(t) | t \in \mathcal{T}\}$, $\mathbb{F}^{\mathbf{X}} := \{\mathcal{F}^{\mathbf{X}}(t) | t \in \mathcal{T}\}$ be the natural filtrations generated by the Brownian motion W and the Markov chain $\mathbf{X}$, respectively, and $\mathbb{F}^\gamma := \{\mathcal{F}^\gamma(t) | t \in \mathcal{T}\}$ be the natural filtration generated by the Poisson random measure, i.e.

$$\mathcal{F}^\gamma(t) := \sigma\left(\int_0^s \int_E \gamma(dt, dz); 0 \leq s \leq t, E \in \mathcal{B}(\Re_0) \right) .$$

Let $\mathbb{F} := \{\mathcal{F}(t) | t \in \mathcal{T}\}$ be a right-continuous, $\mathcal{P}$-complete, enlarged filtration generated by the Brownian motion, the Poisson random measure and the Markov chain, i.e.

$$\mathcal{F}(t) := \bigcap_{s > t} (\mathcal{F}^W(s) \vee \mathcal{F}^\gamma(s) \vee \mathcal{F}^{\mathbf{X}}(s) \vee \mathcal{N}) ,$$

where $\mathcal{N}$ denotes all $\mathcal{P}$-null sets and $\sigma_1 \vee \sigma_2$ denotes the minimal σ-field generated by both σ_1 and σ_2.

We consider a financial market consisting of $n+1$ primitive assets, namely, one risk-free bond and n risky shares. Let $r(t, \mathbf{X}(t))$ be the instantaneous risk-free rate at time t, which is modulated by the chain $\mathbf{X}$ as follows:

$$r(t, \mathbf{X}(t)) := \langle r(t), \mathbf{X}(t) \rangle \ ,$$

where

$$r(t) := (r(t, e_1), r(t, e_2), \cdots, r(t, e_N))^\top \in \Re^N \ .$$

Here $r(t, e_i)$ is a positive, deterministic, uniformly bounded function in time t, for each $i = 1, 2, \cdots, N$. Then the dynamics of the risk-free bond $S_0 := \{S_0(t) | t \in \mathcal{T}\}$ evolves as

$$dS_0(t) = r(t, \mathbf{X}(t)) S_0(t) dt, \quad S_0(0) = 1 \ . \tag{4.2.2}$$

The other n assets are risky shares whose price processes $S_k := \{S_k(t) | t \in \mathcal{T}\}$, for each $k = 1, 2, \cdots, n$, are governed by the following regime-switching jump-diffusion stochastic differential equations

$$\begin{aligned} \frac{dS_k(t)}{S_k(t-)} &= \mu_k(t, \mathbf{X}(t-))dt + \sum_{j=1}^{d} \sigma_{kj}(t, \mathbf{X}(t-)) dW^j(t) \\ &\quad + \sum_{l=1}^{m} \int_{\Re_0} \eta_{kl}(t, z, \mathbf{X}(t-)) \widetilde{\gamma}^l_{\mathbf{X}(t-)}(dt, dz) \ , \end{aligned} \tag{4.2.3}$$

where $\mu_k(t, \mathbf{X}(t))$ is the appreciation rate of the k-th share at time t; $\sigma_{kj}(t, \mathbf{X}(t))$ is the volatility of the k-th share corresponding to the random shock from the Brownian motion W^j at time t; $\eta_{kl}(t, z, \mathbf{X}(t))$ is the jump ratio in the price level of the k-th share attributed to the l-th random jump with size z at time t.

Furthermore, we assume that $\mu_k(t, \mathbf{X}(t)), \sigma_{kj}(t, \mathbf{X}(t))$ and $\eta_{kl}(t, z, \mathbf{X}(t))$, for each $j = 1, 2, \cdots, d$, $k = 1, 2, \cdots, n$ and $l = 1, 2, \cdots, m$, are also modulated by the chain $\mathbf{X}$ as follows

$$\begin{aligned}
\mu_k(t, \mathbf{X}(t)) &:= \langle \mu_k(t), \mathbf{X}(t) \rangle \ , \\
\sigma_{kj}(t, \mathbf{X}(t)) &:= \langle \sigma_{kj}(t), \mathbf{X}(t) \rangle \ , \\
\eta_{kl}(t, z, \mathbf{X}(t)) &:= \langle \eta_{kl}(t, z), \mathbf{X}(t) \rangle \ ,
\end{aligned}$$

where

$$\begin{aligned}
\mu_k(t) &:= (\mu_k(t, \mathbf{e}_1), \mu_k(t, \mathbf{e}_2), \cdots, \mu_k(t, \mathbf{e}_N))^\top \in \Re^N \ , \\
\sigma_{kj}(t) &:= (\sigma_{kj}(t, \mathbf{e}_1), \sigma_{kj}(t, \mathbf{e}_2), \cdots, \sigma_{kj}(t, \mathbf{e}_N))^\top \in \Re^N \ , \\
\eta_{kl}(t, z) &:= (\eta_{kl}(t, z, \mathbf{e}_1), \eta_{kl}(t, z, \mathbf{e}_2), \cdots, \eta_{kl}(t, z, \mathbf{e}_N))^\top \in \Re^N \ .
\end{aligned}$$

Here $\mu_k(t, \mathbf{e}_i), \sigma_{kl}(t, \mathbf{e}_i)$ and $\eta_{kj}(t, z, \mathbf{e}_i)$ are deterministic, uniformly bounded functions in time t, satisfying $\sigma_{kl}(t, \mathbf{e}_i) > 0$ and $\eta_{kj}(t, z, \mathbf{e}_i) > -1$, for each $t \in \mathcal{T}$ and $i = 1, 2, \cdots, N$. To simplify our notation, we write

$$\sigma(t, \mathbf{e}_i) := [\sigma_{kj}(t, \mathbf{e}_i)]_{n \times d} \in \Re^{n \times d} \ , \quad \eta(t, z, \mathbf{e}_i) := [\eta_{kl}(t, z, \mathbf{e}_i)]_{n \times m} \in \Re^{n \times m} \ ,$$

for the volatility matrix and the jump ratio matrix of the risky shares, for each $i = 1, 2, \cdots, N$, respectively. We assume throughout this paper that the following non-degeneracy condition is satisfied, that is

$$\Theta(t, \mathbf{e}_i) := \sigma(t, \mathbf{e}_i)\sigma(t, \mathbf{e}_i)^\top + \int_{\Re_0} \eta(t, z, \mathbf{e}_i) \mathrm{diag}(\nu_{\mathbf{e}_i}(dz)) \eta(t, z, \mathbf{e}_i)^\top \geq \delta I_{n \times n} \ , \quad (4.2.4)$$

where

$$\nu_{\mathbf{e}_i}(dz) := (\nu^1_{\mathbf{e}_i}(dz), \nu^2_{\mathbf{e}_i}(dz), \cdots, \nu^m_{\mathbf{e}_i}(dz))^\top \in \Re^m \ ,$$

for each $t \in \mathcal{T}$ and $i = 1, 2, \cdots, N$. Here δ is some positive constant and $I_{n\times n}$ is an $(n \times n)$-identity matrix. For notational simplicity, we denote by

$$\rho(t, \mathbf{e}_i) := B(t, \mathbf{e}_i)^\top \Theta(t, \mathbf{e}_i)^{-1} B(t, \mathbf{e}_i) \ ,$$

for each $t \in \mathcal{T}$ and $i = 1, 2, \cdots, N$.

In what follows, we consider the situation where an economic agent invests his wealth into the financial market. Denote by $\pi_k(t)$, the amount of the agent's wealth invested in the k-th risky share at time t. We call $\pi(\cdot) := \{\pi(t)|t \in \mathcal{T}\} = \{(\pi_1(t), \pi_2(t), \cdots, \pi_k(t))^\top|t \in \mathcal{T}\}$ a portfolio strategy of the agent. Denote by $Y(t) := Y^\pi(t)$ the wealth of the agent, i.e. the total wealth of the agent at time t corresponding to the portfolio strategy $\pi(\cdot)$. Note that once $\pi(\cdot)$ is determined, the amount of the agent's wealth invested in the risk-free bond is completely specified and equals $Y(t) - \sum_{k=1}^n \pi_k(t)$ at time t. Suppose that (1) the assets can be traded continuously over time; (2) there are no transaction costs, taxes, and short-selling constraints in trading; (3) the trading strategies are self-financing. Then the wealth process $\{Y(t)|t \in \mathcal{T}\}$ of the agent is governed by the following stochastic differential equation:

$$\begin{aligned} dY(t) \quad = \quad & [r(t, \mathbf{X}(t-))Y(t) + \pi(t)^\top B(t, \mathbf{X}(t-))]dt + \pi(t)^\top \sigma(t, \mathbf{X}(t-))dW(t) \\ & + \int_{\Re_0} \pi(t)^\top \eta(t, z, \mathbf{X}(t-))\widetilde{\gamma}(dt, dz) \ , \quad Y(0) = \ y_0 \ , \end{aligned} \tag{4.2.5}$$

where

$$B(t, \mathbf{e}_i) := (\mu_1(t, \mathbf{e}_i) - r(t, \mathbf{e}_i), \cdots, \mu_n(t, \mathbf{e}_i) - r(t, \mathbf{e}_i))^\top \in \Re^n \ ,$$

for each $t \in \mathcal{T}$ and $i = 1, 2, \cdots, N$.

Unlike the traditional mean-variance portfolio selection problem, we assume that the investment horizon of the agent is $[0, T \wedge \tau]$, i.e. the minimum of the planned terminal time T and a non-negative random variable τ defined on $(\Omega, \mathcal{F}, \mathcal{P})$, which can be interpreted as the lifetime of the agent being alive at time 0 or the liquidation time of the investment fund. In this sense, the uncertainty of investment horizon introduces an additional source of risk to the portfolio, say, the mortality risk or the liquidation risk depending on the situation of interest. Suppose that the intensity of τ is $\lambda(t, \mathbf{X}(t-))$ at time t, which is also modulated by the chain as follows

$$\lambda(t, \mathbf{X}(t-)) = \langle \lambda(t), \mathbf{X}(t-) \rangle \ ,$$

where

$$\lambda(t) := (\lambda(t, \mathbf{e}_1), \lambda(t, \mathbf{e}_2), \cdots, \lambda(t, \mathbf{e}_N))^\top \in \Re^N \ .$$

Here $\lambda(t, \mathbf{e}_i)$ is a positive, deterministic, uniformly bounded function in time t, for each $i = 1, 2, \cdots, N$. Therefore, given that $\mathcal{F}(t)$, we denote by the conditional survival probability $\overline{F}(t)$ at time t:

$$\begin{aligned} \overline{F}(t) &= \mathcal{P}(\tau \geq t | \mathcal{F}(t)) \\ &= \exp\bigg\{ - \int_0^t \lambda(s, \mathbf{X}(s-)) ds \bigg\} \ , \end{aligned} \tag{4.2.6}$$

and by the conditional probability density $f(t)$ at time t:

$$f(t) = \lambda(t, \mathbf{X}(t-)) \exp\bigg\{ - \int_0^t \lambda(s, \mathbf{X}(s-)) ds \bigg\} \ . \tag{4.2.7}$$

Indeed, our assumption on the intensity of the uncertain investment horizon is related to the Markov aging process or the Markov credit rating process. Interested readers may refer to Lin and Liu (2007) and Bielecki et al. (2011) for more details. Note that the states of Markov chain in regime-switching models usually represent different market or economic modes while the Markov process introduced in Lin and Liu (2007) is used to model the health index called physiological age but not economic modes. It seems more reasonable to regard the uncertain time horizon as the liquidation time of the investment fund or the uncertain exit time related to the investment psychology. However, our modeling framework could be also accommodated to describe the regime-switching dependent intensity of the uncertain lifetime of the agent. Indeed, the states of the chain $\mathbf{X}$ can be interpreted as different "combined" or "joint" states of economic and health factors. One may consider two Markov chains $\mathbf{X}_1$ and $\mathbf{X}_2$, one for modeling the evolution of the state of an economy over time and another one for modeling the state of health index over time. When the numbers of states of the two Markov chains are equal, say both of them are equal to N, we can combine the two Markov chains to form an N^2-state Markov chain by $\mathbf{vec}(\mathbf{X}_1 \otimes \mathbf{X}_2)$, where $\otimes$ represents the tensor product in $\Re^N$ and $\mathbf{vec}$ is the vectorization operator. Consequently, to simplify our notation, we could consider here a single Markov chain whose states represent joint economic-health states.

Definition 4.2.1. *A portfolio strategy $\pi(\cdot)$ is said to be admissible if the following conditions hold*

1. *$\pi(\cdot)$ is $\mathbb{F}$-predictable;*

2. *$E[\int_0^T ||\pi(t)||^2 dt] < \infty$;*

3. *the SDE (4.2.5) has a unique strong solution $Y(\cdot)$ corresponding to $\pi(\cdot)$.*

The set of all admissible portfolio strategy is denoted by $\mathcal{A}$.

The agent's objective is to find an admissible portfolio $\pi(\cdot) \in \mathcal{A}$, such that the expected terminal wealth satisfies $\mathrm{E}_{y_0,\mathbf{e}_i}[Y(T \wedge \tau)] = \xi$ for some given $\xi \in \Re$ while the risk measured by the variance of the terminal wealth

$$\mathrm{Var}_{y_0,\mathbf{e}_i}[Y(T \wedge \tau)] = \mathrm{E}_{y_0,\mathbf{e}_i}[Y(T \wedge \tau) - \mathrm{E}_{y_0,\mathbf{e}_i}[Y(T \wedge \tau)]]^2 = \mathrm{E}_{y_0,\mathbf{e}_i}[Y(T \wedge \tau) - \xi]^2 ,$$

is minimized, where $\mathrm{E}_{y_0,\mathbf{e}_i}[\cdot]$ and $\mathrm{Var}_{y_0,\mathbf{e}_i}[\cdot]$ are the conditional expectation and variance given that $Y(0) = y_0$ and $\mathbf{X}(0) = \mathbf{e}_i$ under $\mathcal{P}$. Finding such a portfolio is referred as the mean-variance portfolio selection problem with uncertain investment horizon. Specifically, we formulate our mean-variance problem as follows:

Definition 4.2.2. *The mean-variance portfolio selection problem with uncertain investment horizon is a constrained stochastic optimization problem. For each given* $\xi \in \Re$*:*

$$\begin{cases} \min\limits_{\pi(\cdot)\in\mathcal{A}} \quad J_{MV}(y_0, \mathbf{e}_i; \pi(\cdot)) = E_{y_0,\mathbf{e}_i}[Y(T\wedge\tau) - \xi]^2 \ , \\ \textit{subject to} \quad \begin{cases} E_{y_0,\mathbf{e}_i}[Y(T\wedge\tau)] = \xi \ , \\ (Y(\cdot), \pi(\cdot)) \ \textit{satisfy} \ (4.2.5) \ . \end{cases} \end{cases} \tag{4.2.8}$$

The mean-variance problem (4.2.8) is called feasible if there is at least one portfolio satisfying all the constraints. An optimal portfolio of the problem is called an efficient portfolio corresponding to ξ. Suppose that there exists an optiaml solution $\pi^*(\cdot)$ to the problem (4.2.8). The wealth process corresponding to $\pi^*(\cdot)$ is denoted by $Y^*(\cdot)$. Then the pair $(\mathrm{Var}_{y_0,\mathbf{e}_i}[Y^*(T\wedge\tau)], \xi)$ is called an efficient point. The set of all efficient points is called the efficient frontier.

To make the original problem tractable, we derive from the definitions of the conditional survival probability (4.2.6) and the conditional probability density (4.2.7) that

$$\begin{aligned} \mathrm{E}_{y_0,\mathbf{e}_i}[Y(T\wedge\tau)] &= \mathrm{E}_{y_0,\mathbf{e}_i}[Y(\tau)\mathbf{1}_{\{\tau<T\}} + Y(T)\mathbf{1}_{\{\tau\geq T\}}] \\ &= \mathrm{E}_{y_0,\mathbf{e}_i}\big[\mathrm{E}[Y(\tau)\mathbf{1}_{\{\tau<T\}} + Y(T)\mathbf{1}_{\{\tau\geq T\}}|\mathcal{F}(T)]\big] \\ &= \mathrm{E}_{y_0,\mathbf{e}_i}\big[\mathrm{E}[Y(\tau)\mathbf{1}_{\{\tau<T\}}]|\mathcal{F}(T)]\big] + \mathrm{E}_{y_0,\mathbf{e}_i}\big[Y(T)\mathrm{E}[\mathbf{1}_{\{\tau\geq T\}}|\mathcal{F}(T)]\big] \\ &= \mathrm{E}_{y_0,\mathbf{e}_i}\left[\int_0^T \lambda(s,\mathbf{X}(s))e^{-\int_0^s \lambda(u,\mathbf{X}(u))du}Y(s)ds + e^{-\int_0^T \lambda(u,\mathbf{X}(u))du}Y(T)\right] . \end{aligned}$$

Similarly, we can derive that

$$\mathrm{E}_{y_0,\mathbf{e}_i}[Y(T\wedge\tau) - \xi]^2$$

$$= \mathrm{E}_{y_0,\mathbf{e}_i}\left[\int_0^T \lambda(s,\mathbf{X}(s))e^{-\int_0^s \lambda(u,\mathbf{X}(u))du}(Y(s)-\xi)^2 ds + e^{-\int_0^T \lambda(u,\mathbf{X}(u))du}(Y(T)-\xi)^2\right] .$$

Therefore, the mean-variance portfolio selection problem with uncertain investment horizon is equivalent to that with fixed investment horizon given below:

Definition 4.2.3. *The equivalent mean-variance portfolio selection problem with fixed investment horizon is a constrained stochastic optimization problem. For each given* $\xi \in \Re$*:*

$$\begin{cases} \min_{\pi(\cdot)\in\mathcal{A}} \quad J_{MV}(y_0,\mathbf{e}_i;\pi(\cdot)) = E_{y_0,\mathbf{e}_i}\Big[\int_0^T \lambda(s,\mathbf{X}(s))e^{-\int_0^s \lambda(u,\mathbf{X}(u))du}(Y(s)-\xi)^2 ds \\ \qquad\qquad + e^{-\int_0^T \lambda(u,\mathbf{X}(u))du}(Y(T)-\xi)^2\Big] , \\ \textit{subject to} \quad \begin{cases} E_{y_0,\mathbf{e}_i}\Big[\int_0^T \lambda(s,\mathbf{X}(s))e^{-\int_0^s \lambda(u,\mathbf{X}(u))du}Y(s)ds \\ \qquad + e^{-\int_0^T \lambda(u,\mathbf{X}(u))du}Y(T)\Big] = \xi , \\ (Y(\cdot),\pi(\cdot)) \textit{ satisfy } (4.2.5) . \end{cases} \end{cases} \tag{4.2.9}$$

Since the problem (4.2.9) involves a constraint, we need to discuss its feasibility. Using Itô's differentiation rule for regime-switching jump-diffusion processes, we could follow Zhou and Yin (2003) to provide two sufficient and necessary conditions under which our mean-variance problem is feasible.

Theorem 4.2.1. *Let* $p(\cdot,\mathbf{e}_i)$*,* $i = 1,2,\cdots,N$*, be the solutions to the following system of linear ordinary differential equations (ODEs):*

$$\begin{cases} p_t(\cdot,\mathbf{e}_i) + [r(t,\mathbf{e}_i)-\lambda(t,\mathbf{e}_i)]p(t,\mathbf{e}_i) + \sum_{j=1}^N a_{ij}p(t,\mathbf{e}_j) + \lambda(t,\mathbf{e}_i) = 0 , \\ p(T,\mathbf{e}_i) = 1 . \end{cases} \tag{4.2.10}$$

Then the following statements are equivalent

1. *The mean-variance problem (4.2.9) is feasible for each given* $\xi \in \Re$*;*

2. $E[\int_0^T e^{-\int_0^t \lambda(u,\mathbf{X}(u))du} p(t,\mathbf{X}(t))^2 B(t,\mathbf{X}(t))^\top B(t,\mathbf{X}(t))dt] > 0$*;*

3. $E[\int_0^T B(t,\mathbf{X}(t))^\top B(t,\mathbf{X}(t))dt] > 0$*.*

Proof. The proof of Theorem 4.2.1 follows the same rationale of Lemmas 3.1-3.2 in Zhou and Yin (2003). So we omit it here. □

Remark 4.2.1. *Although the investment horizon is uncertain in our mean-variance problem, the conditions of feasibility are very mild. For example, Condition 3 implies that the problem is feasible as long as the appreciation rate of at least one share is different from the interest rate in at least market state.*

Having addressed the feasibility of the problem, we apply the Lagrange multiplier technique to deal with the constraint. For each $\beta \in \Re$, we consider an auxiliary performance functional

$$\begin{aligned}
&J_{MVL}(y_0,\mathbf{e}_i;\pi(\cdot),\beta)\\
&:= \mathrm{E}_{y_0,\mathbf{e}_i}\left[\int_0^T \lambda(s,\mathbf{X}(s))e^{-\int_0^s \lambda(u,\mathbf{X}(u))du}(Y(s)-\xi)^2 ds + e^{-\int_0^T \lambda(u,\mathbf{X}(u))du}(Y(T)-\xi)^2\right]\\
&\quad +2\beta\left\{\mathrm{E}_{y_0,\mathbf{e}_i}\left[\int_0^T \lambda(s,\mathbf{X}(s))e^{-\int_0^s \lambda(u,\mathbf{X}(u))du}Y(s)ds + e^{-\int_0^T \lambda(u,\mathbf{X}(u))du}Y(T)\right]-\xi\right\}\\
&= \mathrm{E}_{y_0,\mathbf{e}_i}\left[\int_0^T \lambda(s,\mathbf{X}(s))e^{-\int_0^s \lambda(u,\mathbf{X}(u))du}[Y(s)-(\xi-\beta)]^2 ds\right.\\
&\quad \left. + e^{-\int_0^T \lambda(u,\mathbf{X}(u))du}[Y(T)-(\xi-\beta)]^2\right] - \beta^2 \ .
\end{aligned}$$

Applying the well-known Lagrange duality theorem transforms the original problem (4.2.9) into the following equivalent min-max problem:

$$\begin{cases} \max\limits_{\beta\in\Re}\min\limits_{\pi(\cdot)\in\mathcal{A}} & J_{MVL}(y_0,\mathbf{e}_i;\pi(\cdot),\beta)\ ,\\ \text{subject to} & (Y(\cdot),\pi(\cdot)) \text{ satisfy } (4.2.5)\ . \end{cases} \tag{4.2.11}$$

Denote by $\vartheta := \xi - \beta$. We define a new performance functional

$$\begin{aligned} J_{MVL}(y_0, \mathbf{e}_i; \pi(\cdot)) &= J_{MVL}(y_0, \mathbf{e}_i; \pi(\cdot), \beta) + \beta^2 \\ &= \mathrm{E}_{y_0, \mathbf{e}_i}\Bigg[\int_0^T \lambda(s, \mathbf{X}(s)) e^{-\int_0^s \lambda(u, \mathbf{X}(u)) du} [Y(s) - \vartheta]^2 ds \\ &\quad + e^{-\int_0^T \lambda(u, \mathbf{X}(u)) du} [Y(T) - \vartheta]^2 \Bigg] . \end{aligned}$$

To solve the min-max problem (4.2.11), we first consider the following unconstrained stochastic optimization problem,

$$\begin{cases} \min\limits_{\pi(\cdot) \in \mathcal{A}} \quad J_{MVL}(y_0, \mathbf{e}_i; \pi(\cdot)) \ , \\ \text{subject to} \quad (Y(\cdot), \pi(\cdot)) \text{ satisfy (4.2.5)} \ . \end{cases} \tag{4.2.12}$$

which is called the quadratic-loss minimization problem in the next section.

4.3 Solution to the unconstrained problem

In this section, we employ the dynamic programming principle to solve the unconstrained problem (4.2.12). We first provide a verification theorem for the regime-switching HJB equation related to the problem. Then we derive explicit solutions to the HJB equation.

To pave the way for the dynamic programming principle, we define the dynamic performance functional of the quadratic-loss minimization problem (4.2.12) as follows:

$$\begin{aligned} J_{MVL}(t, y, \mathbf{e}_i; \pi(\cdot)) &= \mathrm{E}_{t, y, \mathbf{e}_i}\Bigg[\int_t^T \lambda(s, \mathbf{X}(s)) e^{-\int_t^s \lambda(u, \mathbf{X}(u)) du} [Y(s) - \vartheta]^2 ds \\ &\quad + e^{-\int_t^T \lambda(u, \mathbf{X}(u)) du} [Y(T) - \vartheta]^2 \Bigg] , \end{aligned}$$

where $\mathrm{E}_{t,y,\mathbf{e}_i}[\cdot]$ is the conditional expectation given that $Y(t) = y$ and $\mathbf{X}(t) = e_i$ under $\mathcal{P}$. The value function of the problem is defined by

$$v(t, y, \mathbf{e}_i) = \inf_{\pi(\cdot) \in \mathcal{A}} J_{MVL}(t, y, \mathbf{e}_i; \pi(\cdot)) \ . \tag{4.3.1}$$

Since the dynamics of the state processes $\{Y(t)|t \in \mathcal{T}\}$ and $\{\mathbf{X}(t)|t \in \mathcal{T}\}$ are joint Markovian, the optimal control processes can be taken to be Markovian (see, for example, Elliott (1982) and Øksendal and Sulem (2007b)). In what follows, we restrict ourselves to consider only Markovian controls for the problem. Let $\mathcal{O} := (0, T) \times (-\infty, +\infty)$ be our solvency region. Suppose that $\mathcal{D}$ denotes the set such that $\pi(t) \in \mathcal{D}$. To restrict ourselves to Markovian controls, we assume that

$$\pi(t) = \overline{\pi}(t, Y(t), \mathbf{X}(t)) \ ,$$

for some functions $\overline{\pi} : \mathcal{O} \times \mathcal{E} \to \mathcal{D}$. In the following, we do not distinguish notationally between π and $\overline{\pi}$ whenever no confusion arises. So, we can simply identify the control processes with a measurable function $\pi(t, y, \mathbf{e}_i)$, for each $(t, y, \mathbf{e}_i) \in \mathcal{O} \times \mathcal{E}$. This is called the feedback control.

Let $V(\cdot, \cdot, \cdot) : \mathcal{O} \times \mathcal{E} \to \Re$ be a function satisfying $V(\cdot, \cdot, \mathbf{e}_i) \in \mathcal{C}^{1,2}(\mathcal{O})$, for each $i = 1, 2, \cdots, N$, where the partial derivatives of V with respect to t and y are denoted by V_t, V_y and V_{yy}. We define the regime-switching generator $\mathcal{L}^{\pi}$ acting on V as

$$\begin{aligned}
&\mathcal{L}^{\pi}[V(t, y, \mathbf{e}_i)] \\
&= -\lambda(t, \mathbf{e}_i)V(t, y, \mathbf{e}_i) + V_t(t, y, \mathbf{e}_i) + [r(t, \mathbf{e}_i)y + \pi^{\top}B(t, \mathbf{e}_i)]V_y(t, y, \mathbf{e}_i) \\
&\quad + \frac{1}{2}\pi^{\top}\sigma(t, \mathbf{e}_i)\sigma(t, \mathbf{e}_i)^{\top}\pi V_{yy}(t, y, \mathbf{e}_i) + \sum_{j=1}^{N} a_{ij}V(t, y, \mathbf{e}_j) \qquad (4.3.2) \\
&\quad + \sum_{l=1}^{m}\int_{\Re_0}\Big[V(t, y + \pi^{\top}\eta_l(t, z, \mathbf{e}_i), \mathbf{e}_i) - V(t, y, \mathbf{e}_i) - V_y(t, y, \mathbf{e}_i)\pi^{\top}\eta_l(t, z, \mathbf{e}_i)\Big]\nu^l_{\mathbf{e}_i}(dz) \ ,
\end{aligned}$$

where

$$\eta_l(t, z, \mathbf{e}_i) := (\eta_{1l}(t, z, \mathbf{e}_i), \eta_{2l}(t, z, \mathbf{e}_i), \cdots, \eta_{nl}(t, z, \mathbf{e}_i))^{\top} \in \Re^n \ ,$$

for each $(t, y, \mathbf{e}_i) \in \mathcal{O} \times \mathcal{E}$ and $l = 1, 2, \cdots, m$.

Proposition 4.3.1. *Let $\overline{\mathcal{O}}$ denote the closure of $\mathcal{O}$. Suppose that there exists a function $V(\cdot,\cdot,\mathbf{e}_i) \in \mathcal{C}^{1,2}(\mathcal{O}) \cap \mathcal{C}(\overline{\mathcal{O}})$, and a Markov control $\pi^*(\cdot) \in \mathcal{A}$ such that:*

1. $\mathcal{L}^{\pi}[V(t,y,\mathbf{e}_i)] + \lambda(t,\mathbf{e}_i)(y-\vartheta)^2 \geq 0$, *for all $\pi(\cdot) \in \mathcal{A}$ and $(t,y,\mathbf{e}_i) \in \mathcal{O} \times \mathcal{E}$;*

2. $\mathcal{L}^{\pi^*}[V(t,y,\mathbf{e}_i)] + \lambda(t,\mathbf{e}_i)(y-\vartheta)^2 = 0$, *for all $(t,y,\mathbf{e}_i) \in \mathcal{O} \times \mathcal{E}$;*

3. *for all $\pi(\cdot) \in \mathcal{A}$,*
$$\lim_{t \to T^-} V(t,Y(t),\mathbf{X}(t)) = (Y(T)-\vartheta)^2 \ ;$$

4. *let $\mathcal{K}$ denote the set of stopping times $\kappa \leq T$. The family $\{V(\kappa,Y(\kappa),\mathbf{X}(\kappa))\}_{\kappa \in \mathcal{K}}$ is uniformly integrable.*

Write, for each $(t,y,\mathbf{e}_i) \in \mathcal{O} \times \mathcal{E}$ and $\pi(\cdot) \in \mathcal{A}$,

$$\begin{aligned} J_{MVL}(t,y,\mathbf{e}_i;\pi(\cdot)) &= E_{t,y,\mathbf{e}_i}\Bigg[\int_t^T \lambda(s,\mathbf{X}(s)) e^{-\int_t^s \lambda(u,\mathbf{X}(u))du}[Y(s)-\vartheta]^2 ds \\ &\quad + e^{-\int_t^T \lambda(u,\mathbf{X}(u))du}[Y(T)-\vartheta]^2\Bigg] \ . \end{aligned}$$

Then,

$$\begin{aligned} V(t,y,\mathbf{e}_i) &= v(t,y,\mathbf{e}_i) \\ &= \inf_{\pi(\cdot)\in\mathcal{A}} J_{MVL}(t,y,\mathbf{e}_i;\pi(\cdot)) \\ &= J_{MVL}(t,y,\mathbf{e}_i;\pi^*(\cdot)) \ , \end{aligned}$$

and $\pi^*(\cdot)$ is an optimal Markovian control.

Proof. The proof can be adapted from the proof of Theorem 3.2 in Mataramvura and Øksendal (2008) to the regime-switching case (see also Elliott and Siu, 2010). We do not repeat it here. □

Rearranging conditions in Proposition 4.3.1, we obtain the regime-switching HJB equation associated with the quadratic-loss minimization problem (4.2.12), i.e.

$$\begin{cases} \inf\limits_{\pi(\cdot)\in\mathcal{A}} \left\{\mathcal{L}^{\pi}[V(t,y,\mathbf{e}_i)] + \lambda(t,\mathbf{e}_i)(y-\vartheta)^2\right\} = 0 \ , \\ V(T,y,\mathbf{e}_i) = (y-\vartheta)^2 \ . \end{cases} \tag{4.3.3}$$

To solve the problem (4.2.12), we only need to find a classical solution $V(\cdot,\cdot,\mathbf{e}_i) \in \mathcal{C}^{1,2}(\mathcal{O})$ to the HJB equation (4.3.3).

Proposition 4.3.2. *Suppose that $\phi(\cdot,\mathbf{e}_i), \varphi(\cdot,\mathbf{e}_i), \psi(\cdot,\mathbf{e}_i) \in \mathcal{C}^1(\mathcal{T})$, for each $i = 1, 2, \cdots, N$. The optimal portfolio strategy and the value function of the quadratic-loss minimization problem (4.2.12) is given by*

$$\pi^*(t,y,\mathbf{e}_i) = -\Theta(t,\mathbf{e}_i)^{-1}B(t,\mathbf{e}_i)\left[y - \vartheta\frac{\varphi(t,\mathbf{e}_i)}{\phi(t,\mathbf{e}_i)}\right] \ , \tag{4.3.4}$$

and

$$V(t,y,\mathbf{e}_i) = y^2\phi(t,\mathbf{e}_i) - 2\vartheta y\varphi(t,\mathbf{e}_i) + \vartheta^2\psi(t,\mathbf{e}_i) \ , \tag{4.3.5}$$

where $\phi(\cdot,\mathbf{e}_i), \varphi(\cdot,\mathbf{e}_i), \psi(\cdot,\mathbf{e}_i)$, for each $i = 1, 2, \cdots, N$, are unique solutions of the following linear systems of ODEs:

$$\begin{cases} \phi_t(t,\mathbf{e}_i) + [2r(t,\mathbf{e}_i) - \rho(t,\mathbf{e}_i) - \lambda(t,\mathbf{e}_i)]\phi(t,\mathbf{e}_i) + \sum\limits_{j=1}^{N} a_{ij}\phi(t,\mathbf{e}_j) + \lambda(t,\mathbf{e}_i) = 0 \ , \\ \phi(T,\mathbf{e}_i) = 1 \ , \end{cases} \tag{4.3.6}$$

$$\begin{cases} \varphi_t(t,\mathbf{e}_i) + [r(t,\mathbf{e}_i) - \rho(t,\mathbf{e}_i) - \lambda(t,\mathbf{e}_i)]\varphi(t,\mathbf{e}_i) + \sum\limits_{j=1}^{N} a_{ij}\varphi(t,\mathbf{e}_j) + \lambda(t,\mathbf{e}_i) = 0 \ , \\ \phi(T,\mathbf{e}_i) = 1 \ , \end{cases} \tag{4.3.7}$$

$$
\begin{cases}
\psi_t(t,\mathbf{e}_i) - \lambda(t,\mathbf{e}_i)\psi(t,\mathbf{e}_i) + \sum_{j=1}^{N} a_{ij}\psi(t,\mathbf{e}_j) + \lambda(t,\mathbf{e}_i) - \rho(t,\mathbf{e}_i)\dfrac{\varphi^2(t,\mathbf{e}_i)}{\phi(t,\mathbf{e}_i)} = 0 \ , \\
\psi(T,e_i) = 1 \ .
\end{cases}
\tag{4.3.8}
$$

Proof. From the terminal condition of the HJB equation (4.3.3), we try the following quadratic form for the value function:

$$
V(t,y,\mathbf{e}_i) = y^2\phi(t,\mathbf{e}_i) - 2\vartheta y\varphi(t,\mathbf{e}_i) + \vartheta^2\psi(t,\mathbf{e}_i) \ , \tag{4.3.9}
$$

where the functions

$$
\phi : \mathcal{T} \times \mathcal{E} \to \Re \ ,
$$

$$
\varphi : \mathcal{T} \times \mathcal{E} \to \Re \ ,
$$

$$
\psi : \mathcal{T} \times \mathcal{E} \to \Re \ ,
$$

are continuously differentiable in t, i.e. $\phi(\cdot,\mathbf{e}_i), \varphi(\cdot,\mathbf{e}_i), \psi(\cdot,\mathbf{e}_i) \in \mathcal{C}^1(\mathcal{T})$, for each $\mathbf{e}_i \in \mathcal{E}$.

For each $(t,y,\mathbf{e}_i) \in \mathcal{O} \times \mathcal{E}$ and $\pi \in \mathcal{A}$, let

$$
\Psi(t,y,\mathbf{e}_i;\pi) := \mathcal{L}^{\pi}[V(t,y,\mathbf{e}_i)] + \lambda(t,\mathbf{e}_i)(y-\vartheta)^2 \ . \tag{4.3.10}
$$

First of all, we assume that $\Psi_{\pi\pi}(t,y,\mathbf{e}_i;\pi) = \Theta(t,\mathbf{e}_i)\phi(t,\mathbf{e}_i)$ is a positive-definite matrix. which is a sufficient condition for a regular interior minimum. Otherwise, the problem (4.2.12) may not have a solution. Since $\Theta(t,\mathbf{e}_i)$ is a positive-definite matrix, the above sufficient condition is equivalent to that

$$
\phi(t,\mathbf{e}_i) > 0, \quad \forall (t,\mathbf{e}_i) \in \mathcal{T} \times \mathcal{E} \ . \tag{4.3.11}
$$

We assume that (4.3.11) is satisfied at this stage. We shall verify it in the later part of this section.

Using the first-order condition to (4.3.10) with respect to π gives that

$$
\Psi_{\pi}(t,y,\mathbf{e}_i;\pi)
$$

$$= B(t,\mathbf{e}_i)V_y(t,y,\mathbf{e}_i) + \sigma(t,\mathbf{e}_i)\sigma(t,\mathbf{e}_i)^\top \pi V_{yy}(t,y,\mathbf{e}_i) \tag{4.3.12}$$
$$+ \sum_{l=1}^{m} \int_{\Re_0} \Big[V_y(t, y + \pi^\top \eta_l(t,z,\mathbf{e}_i), \mathbf{e}_i) - V_x(t,y,\mathbf{e}_i) \Big] \eta_l(t,y,\mathbf{e}_i) \nu^l_{\mathbf{e}_i}(dz) = 0 \ .$$

Substituting (4.3.9) into (4.3.12) leads to

$$B(t,\mathbf{e}_i)[2y\phi(t,\mathbf{e}_i) - 2\vartheta\varphi(t,\mathbf{e}_i)] + 2\Theta(t,\mathbf{e}_i)\pi\phi(t,\mathbf{e}_i) = 0 \ . \tag{4.3.13}$$

Solving (4.3.13) gives

$$\pi^*(t,y,\mathbf{e}_i) = -\Theta(t,\mathbf{e}_i)^{-1} B(t,\mathbf{e}_i) \left[y - \vartheta \frac{\varphi(t,\mathbf{e}_i)}{\phi(t,\mathbf{e}_i)} \right] \ . \tag{4.3.14}$$

Substituting (4.3.9) and (4.3.14) into the regime-switching HJB equation (4.3.3) yields

$$\begin{aligned}
& y^2 \left\{ \phi_t(t,\mathbf{e}_i) + [2r(t,\mathbf{e}_i) - \rho(t,\mathbf{e}_i) - \lambda(t,\mathbf{e}_i)]\phi(t,\mathbf{e}_i) + \sum_{j=1}^{N} a_{ij}\phi(t,\mathbf{e}_j) + \lambda(t,\mathbf{e}_i) \right\} \\
& -2\vartheta y \left\{ \varphi_t(t,\mathbf{e}_i) + [r(t,\mathbf{e}_i) - \rho(t,\mathbf{e}_i) - \lambda(t,\mathbf{e}_i)]\varphi(t,\mathbf{e}_i) + \sum_{j=1}^{N} a_{ij}\varphi(t,\mathbf{e}_j) + \lambda(t,\mathbf{e}_i) \right\} \\
& +\vartheta^2 \left\{ \psi_t(t,\mathbf{e}_i) - \lambda(t,\mathbf{e}_i)\psi(t,\mathbf{e}_i) + \sum_{j=1}^{N} a_{ij}\psi(t,\mathbf{e}_j) + \lambda(t,\mathbf{e}_i) - \rho(t,\mathbf{e}_i)\frac{\varphi^2(t,\mathbf{e}_i)}{\phi(t,\mathbf{e}_i)} \right\} = 0 \ .
\end{aligned} \tag{4.3.15}$$

Setting the coefficients of y^2, y and 1 equal zeros gives the desired results (4.3.6)-(4.3.8).

Next, we prove the uniqueness and existence of solutions to the linear system of ODEs (4.3.6). Denote by

$$\phi(t) := (\phi(t,\mathbf{e}_1), \phi(t,\mathbf{e}_2), \cdots, \phi(t,\mathbf{e}_N))^\top \in \Re^N \ .$$

Then (4.3.6) can be written as the following vector-valued ODE

$$\begin{cases} \phi_t(t) + \big[\mathrm{diag}(2r(t) - \rho(t) - \lambda(t)) + A \big] \phi(t) + \lambda(t) = 0_{N\times 1} \ , \\ \phi(T) = 1_{N\times 1} \ , \end{cases} \tag{4.3.16}$$

where

$$
\begin{aligned}
0_{N\times 1} &:= (0,0,\cdots,0)^\top \in \Re^N \ , \\
1_{N\times 1} &:= (1,1,\cdots,1)^\top \in \Re^N \ .
\end{aligned}
$$

Since each entry in the matrix "diag$(2r(t)-\rho(t)-\lambda(t))+A$" is uniformly bounded in t, the ODE (4.3.16) satisfies the global Lipschitz condition. Therefore, from the theory of the ordinary differential equation, we conclude that there exists a unique solution $\phi(t)$ to (4.3.16), for each $t \in \mathcal{T}$. Equivalently, the linear system of ODEs (4.3.6) admits unique solutions $\phi(t,\mathbf{e}_i)$, for each $(t,\mathbf{e}_i) \in \mathcal{T}\times\mathcal{E}$. The proofs of the uniqueness and existence of solutions to (4.3.7) and (4.3.8) are similar. We do not repeat them here. □

Proposition 4.3.3. *Let $E_{t,\mathbf{e}_i}[\cdot]$ denote the conditional expectation given that $\mathbf{X}(t)=\mathbf{e}_i$. The expectation representations for the solutions $\phi(t,\mathbf{e}_i),\varphi(t,\mathbf{e}_i),\psi(t,\mathbf{e}_i)$ of (4.3.6)-(4.3.8) are given by*

$$
\begin{aligned}
\phi(t,\mathbf{e}_i) &= E_{t,\mathbf{e}_i}\left[e^{\int_t^T[2r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du}\right] \\
&\quad + E_{t,\mathbf{e}_i}\left[\int_t^T \lambda(s,\mathbf{X}(s))e^{\int_t^s[2r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du}ds\right] , \quad (4.3.17)
\end{aligned}
$$

$$
\begin{aligned}
\varphi(t,\mathbf{e}_i) &= E_{t,\mathbf{e}_i}\left[e^{\int_t^T[r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du}\right] \\
&\quad + E_{t,\mathbf{e}_i}\left[\int_t^T \lambda(s,\mathbf{X}(s))e^{\int_t^s[r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du}ds\right] , \quad (4.3.18)
\end{aligned}
$$

and

$$
\psi(t,\mathbf{e}_i) = 1 - E_{t,\mathbf{e}_i}\left[\int_t^T \rho(s,\mathbf{X}(s))\frac{\varphi^2(s,\mathbf{X}(s))}{\phi(s,\mathbf{X}(s))}e^{-\int_t^s \lambda(u,\mathbf{X}(u))du}ds\right] . \quad (4.3.19)
$$

Proof. We only derive (4.3.17) and (4.3.19). The derivation of (4.3.18) is similar. By Itô's differentiation rule,

$$
\begin{aligned}
& d\bigg\{\phi(t,\mathbf{X}(t))e^{\int_0^t [2r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du} \\
& \quad + \int_0^t \lambda(s,\mathbf{X}(s))e^{\int_0^s [2r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du}ds\bigg\} \\
& = e^{\int_0^t [2r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du}\bigg\{\phi_t(t,\mathbf{X}(t)) \\
& \quad +[2r(t,\mathbf{X}(t))-\rho(t,\mathbf{X}(t))-\lambda(t,\mathbf{X}(t))]\phi(t,\mathbf{X}(t)) + \langle \phi(t), A\mathbf{X}(t)\rangle + \lambda(t,\mathbf{X}(t))\bigg\}dt \\
& \quad +e^{\int_0^t [2r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du}\langle \phi(t), d\mathbf{M}(t)\rangle \\
& = e^{\int_0^t [2r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du}\langle \phi(t), d\mathbf{M}(t)\rangle\ ,
\end{aligned}
\tag{4.3.20}
$$

where the second equation is due to (4.3.6) if $\mathbf{X}(t) = \mathbf{e}_i$, for each $i = 1, 2, \cdots, N$. Integrating (4.3.20) from t to T and conditioning on $\mathcal{F}^{\mathbf{X}}(t)$ give

$$
\begin{aligned}
& \phi(t,\mathbf{X}(t))e^{\int_0^t [2r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du} \\
& \quad + \int_0^t \lambda(s,\mathbf{X}(s))e^{\int_0^s [2r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du}ds \\
& = \mathrm{E}\bigg[e^{\int_0^T [2r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du} \\
& \quad + \int_0^T \lambda(s,\mathbf{X}(s))e^{\int_0^s [2r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du}ds\bigg|\mathcal{F}^{\mathbf{X}}(t)\bigg] \\
& = \mathrm{E}\bigg[e^{\int_0^T [2r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du} \\
& \quad + \int_0^T \lambda(s,\mathbf{X}(s))e^{\int_0^s [2r(u,\mathbf{X}(u))-\rho(u,\mathbf{X}(u))-\lambda(u,\mathbf{X}(u))]du}ds\bigg|\mathbf{X}(t)\bigg]\ ,
\end{aligned}
\tag{4.3.21}
$$

where the second equation is due to that $\{\mathbf{X}(t)|t \in \mathcal{T}\}$ is a Markov process with respect to $\mathbb{F}^{\mathbf{X}}$. Therefore, rearranging (4.3.21) and setting $\mathbf{X}(t) = \mathbf{e}_i$ yield the desired the result (4.3.17).

Consider the following transformed function

$$\widetilde{\psi}(t,\mathbf{e}_i)=\psi(t,\mathbf{e}_i)-1\ , \quad \forall (t,\mathbf{e}_i)\in\mathcal{T}\times\mathcal{E}\ . \tag{4.3.22}$$

Substituting (4.3.22) into (4.3.19) gives

$$\begin{cases} \widetilde{\psi}_t(t,e_i)-\lambda(t,e_i)\widetilde{\psi}(t,e_i)+\sum_{j=1}^{N}a_{ij}\widetilde{\psi}(t,e_j)-\rho(t,e_i)\dfrac{\varphi^2(t,e_i)}{\phi(t,e_i)}=0\ , \\ \widetilde{\psi}(T,e_i)=0\ . \end{cases} \tag{4.3.23}$$

Then we can derive as in (4.3.20)-(4.3.21) that

$$\widetilde{\psi}(t,\mathbf{e}_i)=-\mathrm{E}_{t,\mathbf{e}_i}\left[\int_t^T\rho(s,\mathbf{X}(s))\frac{\varphi^2(s,\mathbf{X}(s))}{\phi(s,\mathbf{X}(s))}e^{-\int_t^s\lambda(u,\mathbf{X}(u))du}ds\right]\ . \tag{4.3.24}$$

Therefore, combining (4.3.22) and (4.3.24) gives the desired result (4.3.19). □

Proposition 4.3.4. *For each $(t,\mathbf{e}_i)\in\mathcal{T}\times\mathcal{E}$, the solutions $\phi(t,\mathbf{e}_i),\varphi(t,\mathbf{e}_i),\psi(t,\mathbf{e}_i)$ of (4.3.6)-(4.3.8) satisfy the following relationships:*

(i) $0<\varphi(t,\mathbf{e}_i)\leq\phi(t,\mathbf{e}_i)$;

(ii) $\varphi^2(t,\mathbf{e}_i)\leq\phi(t,\mathbf{e}_i)\psi(t,\mathbf{e}_i)$;

(iii) $0<\psi(t,\mathbf{e}_i)<1$.

Proof. (i) Since $r(t,\mathbf{e}_i),\lambda(t,\mathbf{e}_i)>0$, comparing (4.3.17) with (4.3.18) immediately leads to $0<\varphi(t,\mathbf{e}_i)\leq\phi(t,\mathbf{e}_i)$, for each $(t,\mathbf{e}_i)\in\mathcal{T}\times\mathcal{E}$. Therefore, (4.3.11) is verified.
(ii) We consider two functions $h(\cdot,\cdot),g(\cdot,\cdot):\mathcal{T}\times\mathcal{E}\to\Re$ defined by putting

$$h(t,\mathbf{e}_i):=\frac{\varphi(t,\mathbf{e}_i)}{\phi(t,\mathbf{e}_i)}\ , \tag{4.3.25}$$

and

$$g(t,\mathbf{e}_i):=\psi(t,\mathbf{e}_i)-\frac{\varphi^2(t,\mathbf{e}_i)}{\phi(t,\mathbf{e}_i)}=\psi(t,\mathbf{e}_i)-h^2(t,\mathbf{e}_i)\phi(t,\mathbf{e}_i)\ . \tag{4.3.26}$$

From (4.3.17) and (4.3.18), it is not difficult to show that $h(t, \mathbf{e}_i)$, for each $i = 1, 2, \cdots, N$, satisfy the following system of ODEs

$$\begin{cases} h_t(t, \mathbf{e}_i) - r(t, \mathbf{e}_i)h(t, \mathbf{e}_i) + \dfrac{1}{\phi(t, \mathbf{e}_i)} \displaystyle\sum_{j=1}^{N} a_{ij}\phi(t, \mathbf{e}_j)[h(t, \mathbf{e}_j) - h(t, \mathbf{e}_i)] \\ \qquad\qquad + \dfrac{\lambda(t, \mathbf{e}_i)}{\phi(t, \mathbf{e}_i)}[1 - h(t, \mathbf{e}_i)] = 0 \ , \\ h(T, \mathbf{e}_i) = 1 \ . \end{cases} \tag{4.3.27}$$

Similarly, we can derive from (4.3.17)(4.3.18) and (4.3.27) that $g(t, \mathbf{e}_i)$, for each $i = 1, 2, \cdots, N$, satisfy the following system of ODEs

$$\begin{cases} g_t(t, \mathbf{e}_i) + \displaystyle\sum_{j=1}^{N} a_{ij}\phi(t, \mathbf{e}_j)[h(t, \mathbf{e}_j) - h(t, \mathbf{e}_i)]^2 \\ \qquad\qquad + \lambda(t, \mathbf{e}_i)[1 - h(t, \mathbf{e}_i)]^2 + \displaystyle\sum_{j=1}^{N} a_{ij}g(t, \mathbf{e}_j) = 0 \ , \\ g(T, \mathbf{e}_i) = 0 \ . \end{cases} \tag{4.3.28}$$

As in Proposition 4.3.3, we can derive the following expectation solution for $g(t, \mathbf{e}_i)$:

$$\begin{aligned} g(t, \mathbf{e}_i) \ = \ & \mathrm{E}_{t, \mathbf{e}_i}\Bigg[\int_t^T \Bigg\{ \sum_{j=1}^{N} a_{ij}\phi(s, \mathbf{e}_j)[h(s, \mathbf{e}_j) - h(s, \mathbf{X}(t))]^2 \\ & + \lambda(s, \mathbf{X}(s))[1 - h(s, \mathbf{X}(s))]^2 \Bigg\} ds \Bigg] \ . \end{aligned} \tag{4.3.29}$$

Denote by

$$F(t, \mathbf{e}_i) := \sum_{j=1}^{N} a_{ij}\phi(t, \mathbf{e}_j)[h(t, \mathbf{e}_j) - h(t, \mathbf{e}_i)]^2 + \lambda(t, \mathbf{e}_i)[1 - h(t, \mathbf{e}_i)]^2 \ , \quad \forall (t, \mathbf{e}_i) \in \mathcal{T} \times \mathcal{E} \ ,$$

and

$$F(t) = (F(t, \mathbf{e}_1), F(t, \mathbf{e}_2), \cdots, F(t, \mathbf{e}_N))^\top \in \Re^N \ .$$

Then we further derive that

$$\begin{aligned} g(t,\mathbf{e}_i) &= \mathrm{E}_{t,\mathbf{e}_i}\left[\int_t^T F(s,\mathbf{X}(s))ds\right] \\ &= \mathrm{E}_{t,\mathbf{e}_i}\left[\int_t^T \langle F(s),\mathbf{X}(s)\rangle\, ds\right] \\ &= \int_t^T \langle \exp(A(s-t))\mathbf{e}_i, F(s)\rangle\, ds\ . \end{aligned} \tag{4.3.30}$$

Since $\phi(t,\mathbf{e}_i), \lambda(t,\mathbf{e}_i) > 0$, for each $i = 1, 2, \cdots, N$, and $a_{ij} > 0$, for each $j \neq i$, we have

$$F(t,\mathbf{e}_i) = \sum_{j=1,j\neq i}^{N} a_{ij}\phi(t,\mathbf{e}_j)[h(t,\mathbf{e}_j) - h(t,\mathbf{e}_i)]^2 + \lambda(t,\mathbf{e}_i)[1 - h(t,\mathbf{e}_i)]^2 \geq 0\ .$$

Therefore, we can see from either (4.3.29) or (4.3.30) that $g(t,\mathbf{e}_i) \geq 0$, which immediately results in $\varphi^2(t,\mathbf{e}_i) \leq \phi(t,\mathbf{e}_i)\psi(t,\mathbf{e}_i)$, for each $(t,\mathbf{e}_i) \in \mathcal{T} \times \mathcal{E}$.

(iii) The non-degeneracy condition (4.2.4) implies that both $\Theta(t,\mathbf{e}_i)$ and its inverse $\Theta^{-1}(t,\mathbf{e}_i)$ are positive-definite matrices. Then

$$\rho(t,\mathbf{e}_i) = B(t,\mathbf{e}_i)^\top \Theta(t,\mathbf{e}_i)^{-1} B(t,\mathbf{e}_i) > 0\ , \quad \forall (t,\mathbf{e}_i) \in \mathcal{T} \times \mathcal{E}\ .$$

So, from the results in (i) and Eq. (4.3.19), we can see that $\psi(t,\mathbf{e}_i) < 1$, for each $(t,\mathbf{e}_i) \in \mathcal{T} \times \mathcal{E}$. On the other hand, through the definition of the function g, we have

$$\psi(t,\mathbf{e}_i) = g(t,\mathbf{e}_i) + h^2(t,\mathbf{e}_i)\phi(t,\mathbf{e}_i) = g(t,\mathbf{e}_i) + \frac{\varphi^2(t,\mathbf{e}_i)}{\phi(t,\mathbf{e}_i)}\ .$$

Therefore, combining the results in (i) and (ii), we obtain that $\psi(t,\mathbf{e}_i) > 0$, for each $(t,\mathbf{e}_i) \in \mathcal{T} \times \mathcal{E}$. This completes the proof. □

Proposition 4.3.5. *Suppose that for each $\mathbf{e}_i \in \mathcal{E}$, the coefficients of the model dynamics are time independent, i.e. $r(t,\mathbf{e}_i) = r(\mathbf{e}_i)$, $\mu(t,\mathbf{e}_i) = \mu(\mathbf{e}_i)$, $\sigma(t,\mathbf{e}_i) = \sigma(\mathbf{e}_i)$, $\lambda(t,\mathbf{e}_i) = \lambda(\mathbf{e}_i)$. Denote by*

$$r := (r(\mathbf{e}_1), r(\mathbf{e}_2), \cdots, r(\mathbf{e}_N))^\top \in \Re^N\ ,$$

$$\rho := (\rho(\mathbf{e}_1), \rho(\mathbf{e}_2), \cdots, \rho(\mathbf{e}_N))^\top \in \Re^N \ ,$$
$$\lambda := (\lambda(\mathbf{e}_1), \lambda(\mathbf{e}_2), \cdots, \lambda(\mathbf{e}_N))^\top \in \Re^N \ .$$

Then the closed-form solutions of (4.3.6)-(4.3.8) are given by

$$\begin{aligned}\phi(t, \mathbf{e}_i) &= \Big\langle \exp\Big\{\big[diag(2r - \rho - \lambda) + A\big](T-t)\Big\}\mathbf{e}_i, 1_N \Big\rangle \\ &\quad + \int_t^T \Big\langle \exp\Big\{\big[diag(2r - \rho - \lambda) + A\big](s-t)\Big\}\mathbf{e}_i, \lambda \Big\rangle ds \ , \quad (4.3.31)\end{aligned}$$

$$\begin{aligned}\varphi(t, \mathbf{e}_i) &= \Big\langle \exp\Big\{\big[diag(r - \rho - \lambda) + A\big](T-t)\Big\}\mathbf{e}_i, 1_N \Big\rangle \\ &\quad + \int_t^T \Big\langle \exp\Big\{\big[diag(r - \rho - \lambda) + A\big](s-t)\Big\}\mathbf{e}_i, \lambda \Big\rangle ds \ , \quad (4.3.32)\end{aligned}$$

and

$$\psi(t, \mathbf{e}_i) = 1 - \int_t^T \Big\langle \exp\Big\{\big[-diag(\lambda) + A\big](s-t)\Big\}\mathbf{e}_i, f(s) \Big\rangle ds \ , \qquad (4.3.33)$$

where

$$f(s) := \Big(\rho(\mathbf{e}_1)\frac{\varphi^2(s, \mathbf{e}_1)}{\phi(s, \mathbf{e}_1)}, \rho(\mathbf{e}_2)\frac{\varphi^2(s, \mathbf{e}_2)}{\phi(s, \mathbf{e}_2)}, \cdots, \rho(\mathbf{e}_N)\frac{\varphi^2(s, \mathbf{e}_N)}{\phi(s, \mathbf{e}_N)}\Big)^\top \in \Re^N \ .$$

Proof. We only derive (4.3.31). The derivations of (4.3.32)-(4.3.33) are similar. Applying Itô's differentiation rule to the process $\{\mathbf{X}(s)e^{\int_t^s[2r(\mathbf{X}(u))-\rho(\mathbf{X}(u))-\lambda(\mathbf{X}(u))]du}|s \in [t, T]\}$ gives

$$\begin{aligned}& d\big\{\mathbf{X}(s)e^{\int_t^s[2r(\mathbf{X}(u))-\rho(\mathbf{X}(u))-\lambda(\mathbf{X}(u))]du}\big\} \qquad (4.3.34) \\ &= e^{\int_t^s[2r(\mathbf{X}(u))-\rho(\mathbf{X}(u))-\lambda(\mathbf{X}(u))]du}\Big\{\big[\mathrm{diag}(2r - \rho - \lambda) + A\big]\mathbf{X}(s)ds + d\mathbf{M}(s)\Big\} \ .\end{aligned}$$

Conditioning on $\mathbf{X}(t) = e_i$ on both sides of (4.3.34) results in

$$d\mathrm{E}_{t,\mathbf{e}_i}\Big[\mathbf{X}(s)e^{\int_t^s[2r(\mathbf{X}(u))-\rho(\mathbf{X}(u))-\lambda(\mathbf{X}(u))]du}\Big] \qquad (4.3.35)$$

$$= \left[\mathrm{diag}(2r-\rho-\lambda)+A\right]\mathrm{E}_{t,\mathbf{e}_i}\left[\mathbf{X}(s)e^{\int_t^s[2r(\mathbf{X}(u))-\rho(\mathbf{X}(u))-\lambda(\mathbf{X}(u))]du}\right]ds \ .$$

Solving (4.3.35) gives

$$\begin{aligned}&\mathrm{E}_{t,\mathbf{e}_i}\left[\mathbf{X}(T)e^{\int_t^T[2r(\mathbf{X}(u))-\rho(\mathbf{X}(u))-\lambda(\mathbf{X}(u))]du}\right]\\ &= \exp\left\{\left[\mathrm{diag}(2r-\rho-\lambda)+A\right](T-t)\right\}\mathbf{e}_i \ .\end{aligned} \tag{4.3.36}$$

Therefore,

$$\begin{aligned}&\mathrm{E}_{t,\mathbf{e}_i}\left[e^{\int_t^T[2r(\mathbf{X}(u))-\rho(\mathbf{X}(u))-\lambda(\mathbf{X}(u))]du}\right]\\ &= \mathrm{E}_{t,\mathbf{e}_i}\left[\langle\mathbf{X}(T),1_N\rangle\, e^{\int_t^T[2r(\mathbf{X}(u))-\rho(\mathbf{X}(u))-\lambda(\mathbf{X}(u))]du}\right]\\ &= \left\langle \mathrm{E}_{t,\mathbf{e}_i}\left[\mathbf{X}(T)e^{\int_t^T[2r(\mathbf{X}(u))-\rho(\mathbf{X}(u))-\lambda(\mathbf{X}(u))]du}\right],1_N\right\rangle\\ &= \left\langle \exp\left\{\left[\mathrm{diag}(2r-\rho-\lambda)+A\right](T-t)\right\}\mathbf{e}_i,1_N\right\rangle \ .\end{aligned} \tag{4.3.37}$$

Since the integrand in the following expectation

$$\mathrm{E}_{t,\mathbf{e}_i}\left[\int_t^T \lambda(\mathbf{X}(s))e^{\int_t^s[2r(\mathbf{X}(u))-\rho(\mathbf{X}(u))-\lambda(\mathbf{X}(u))]du}ds\right] ,$$

are bounded, we can employ Fubini's Theorem to derive that

$$\begin{aligned}&\mathrm{E}_{t,\mathbf{e}_i}\left[\int_t^T \lambda(\mathbf{X}(s))e^{\int_t^s[2r(\mathbf{X}(u))-\rho(\mathbf{X}(u))-\lambda(\mathbf{X}(u))]du}ds\right]\\ &= \int_t^T \mathrm{E}_{t,\mathbf{e}_i}\left[\lambda(\mathbf{X}(s))e^{\int_t^s[2r(\mathbf{X}(u))-\rho(\mathbf{X}(u))-\lambda(\mathbf{X}(u))]du}\right]ds\\ &= \int_t^T \left\langle \exp\left\{\left[\mathrm{diag}(2r-\rho-\lambda)+A\right](s-t)\right\}\mathbf{e}_i,\lambda\right\rangle ds \ .\end{aligned} \tag{4.3.38}$$

Combining (4.3.37) and (4.3.38) gives the desired result (4.3.31). □

4.4 Efficient portfolio and efficient frontier

In this section, we derive the efficient portfolio, the efficient frontier and the mutual fund theorem of the min-max problem (4.2.11), which is equivalent to the original mean-variance problem (4.2.8) or (4.2.9). Surprisingly, although the investment horizon is uncertain, we can still find the so-called efficient portfolio and efficient frontier as in the classical mean-variance problem with fixed investment horizon. Furthermore, the mutual fund theorem also holds in our problem.

Theorem 4.4.1. *The efficient portfolio of the mean-variance portfolio selection problem associated with the expected value $\xi \in \Re$ is an optimal feedback control of $(t, y, \mathbf{e}_i) \in \mathcal{O} \times \mathcal{E}$ as*

$$\pi^*(t, y, \mathbf{e}_i) = -\Theta(t, \mathbf{e}_i)^{-1} B(t, \mathbf{e}_i) \left[y - (\xi - \beta^*) \frac{\varphi(t, \mathbf{e}_i)}{\phi(t, \mathbf{e}_i)} \right] , \tag{4.4.1}$$

where

$$\beta^* = \frac{\xi \psi(0, \mathbf{e}_i) - y_0 \varphi(0, \mathbf{e}_i)}{\psi(0, \mathbf{e}_i) - 1} . \tag{4.4.2}$$

Furthermore, the efficient frontier of the mean-variance problem (4.2.9) is given by

$$\begin{aligned} & Var_{y_0, \mathbf{e}_i}[Y^*(T \wedge \tau)] \\ &= \frac{\psi(0, \mathbf{e}_i)}{1 - \psi(0, \mathbf{e}_i)} \left[\xi - \frac{\varphi(0, \mathbf{e}_i)}{\psi(0, \mathbf{e}_i)} y_0 \right]^2 + \frac{\phi(0, \mathbf{e}_i)\psi(0, \mathbf{e}_i) - \varphi^2(0, \mathbf{e}_i)}{\psi(0, \mathbf{e}_i)} y_0^2 . \end{aligned} \tag{4.4.3}$$

Here $\phi(t, \mathbf{e}_i)$, $\varphi(t, \mathbf{e}_i)$ and $\psi(t, \mathbf{e}_i)$ are given by the expectation expressions (4.3.17)-(4.3.19) in Proposition 4.3.2. Particularly, if for each $\mathbf{e}_i \in \mathcal{E}$, the coefficients of the model dynamics are time independent, then the closed-form expressions of $\phi(t, \mathbf{e}_i)$, $\varphi(t, \mathbf{e}_i)$ and $\psi(t, \mathbf{e}_i)$ are given by (4.3.31)-(4.3.33) in Proposition 4.3.2.

Proof. Denote by

$$V(0, y_0, \mathbf{e}_i; \beta) \;:=\; V(0, y_0, \mathbf{e}_i) - \beta^2 \tag{4.4.4}$$

$$= \quad y_0^2\phi(0,\mathbf{e}_i) - 2(\xi-\beta)y_0\varphi(0,\mathbf{e}_i) + (\xi-\beta)^2\psi(0,\mathbf{e}_i) - \beta^2 \ .$$

The relationship between the performance functionals of the quadratic-loss minimization problem (4.2.12) and the min-max problem (4.2.11) implies that

$$\mathrm{Var}_{y_0,\mathbf{e}_i}[Y^*(T\wedge\tau)] = V(0,y_0,\mathbf{e}_i;\beta^*) \ ,$$

where

$$\beta^* = \arg\max_{\beta\in\Re} V(0,y_0,\mathbf{e}_i;\beta) \ .$$

From (iii) in Proposition 4.3.4, it is clear that

$$\frac{\partial^2 V}{\partial\beta^2} = 2\psi(0,\mathbf{e}_i) - 2 < 0 \ .$$

Then applying the first-order condition to $V(0,y_0,\mathbf{e}_i;\beta)$ with respect to β yields that

$$2y_0\varphi(0,\mathbf{e}_i) + 2(\beta-\xi)\psi(0,\mathbf{e}_i) - 2\beta = 0 \ . \tag{4.4.5}$$

Solving (4.4.5) gives

$$\beta^* = \frac{\xi\psi(0,\mathbf{e}_i) - y_0\varphi(0,\mathbf{e}_i)}{\psi(0,\mathbf{e}_i)-1} \ . \tag{4.4.6}$$

Substituting (4.4.6) into (4.3.4) and (4.4.4) results in

$$\pi^*(t,y,\mathbf{e}_i) = -\Theta(t,\mathbf{e}_i)^{-1}B(t,\mathbf{e}_i)\left[y - (\xi-\beta^*)\frac{\varphi(t,\mathbf{e}_i)}{\phi(t,\mathbf{e}_i)}\right] ,$$

and

$$\begin{aligned}\mathrm{Var}_{y_0,\mathbf{e}_i}[Y^*(T\wedge\tau)] &= V(0,y_0,\mathbf{e}_i;\beta^*) \\ &= \frac{\psi(0,\mathbf{e}_i)}{1-\psi(0,\mathbf{e}_i)}\left[\xi - \frac{\varphi(0,\mathbf{e}_i)}{\psi(0,\mathbf{e}_i)}y_0\right]^2 + \frac{\phi(0,\mathbf{e}_i)\psi(0,\mathbf{e}_i) - \varphi^2(0,\mathbf{e}_i)}{\psi(0,\mathbf{e}_i)}y_0^2 \ .\end{aligned}$$

□

Theorem 4.4.2. *The efficient portfolio that achieves the minimum variance portfolio*

$$Var_{y_0,\mathbf{e}_i}[Y^*_{\min}(T \wedge \tau)] = \frac{\phi(0,\mathbf{e}_i)\psi(0,\mathbf{e}_i) - \varphi^2(0,\mathbf{e}_i)}{\psi(0,\mathbf{e}_i)} y_0^2 \geq 0 \ , \tag{4.4.7}$$

is given by

$$\pi^*_{\min}(t, y, \mathbf{e}_i) = -\Theta(t,\mathbf{e}_i)^{-1} B(t,\mathbf{e}_i) \left[y - \xi_{\min} \frac{\varphi(t,\mathbf{e}_i)}{\phi(t,\mathbf{e}_i)} \right] \ , \tag{4.4.8}$$

with the expected terminal wealth

$$\xi_{\min} = \frac{\varphi(0,\mathbf{e}_i)}{\psi(0,\mathbf{e}_i)} y_0 \ , \tag{4.4.9}$$

and the Lagrange multiplier $\beta^*_{\min} = 0$.

Proof. From (ii) and (iii) in Proposition 4.3.4, we have

$$\frac{\psi(0,\mathbf{e}_i)}{1 - \psi(0,\mathbf{e}_i)} > 0 \ ,$$

and

$$\frac{\phi(0,\mathbf{e}_i)\psi(0,\mathbf{e}_i) - \varphi^2(0,\mathbf{e}_i)}{\psi(0,\mathbf{e}_i)} \geq 0 \ , \quad \forall \mathbf{e}_i \in \mathcal{E} \ .$$

Recalling (4.4.3) in Theorem 4.4.1, the desired results (4.4.7)-(4.4.9) are obvious. In addition, substituting (4.4.9) into (4.4.2) yields that $\beta^*_{\min} = 0$. □

Remark 4.4.1. *Contrary to the case without regime-switching, the efficient frontier is no longer a perfect square and the efficient frontier in the mean-standard deviation diagram is no longer a straight line. One can not achieve an investment resulting in a zero terminal variance. That is, a risk-free investment. This is consistent with the results in Zhou and Yin (2003). Zhou and Yin (2003) provided a sufficient condition, i.e. the interest rate is deterministic, under which the risk-free investment is achievable. However, the deterministic interest rate does not suffice to guarantee the solution of* *the ODEs (4.3.27) is state independent. As a consequence, one is unable to achieve the risk-free investment even if the interest rate is deterministic in the mean-variance problem with uncertain investment horizon under regime-switching models.*

Although the investment horizon is uncertain in our modeling framework, the following mutual fund theorem still holds.

Theorem 4.4.3. *Suppose that an efficient portfolio $\pi_1^*(\cdot)$ is given by (4.4.1) corresponding to $\xi = \xi_1 > \xi_{\min}$. Then a portfolio $\pi^*(\cdot)$ is efficient if and only if there is a $\chi \geq 0$ such that*

$$\pi^*(t) = (1-\chi)\pi_{\min}^*(t) + \chi\pi_1^*(t) \ , \tag{4.4.10}$$

where $\pi_{\min}^*(\cdot)$ is the minimum variance portfolio defined in Theorem 4.4.2.

Proof. The proof is similar to Theorem 5.3 of Zhou and Yin (2003). So we do not repeat it here. □

4.5 Numerical examples

In this section, we provide several numerical examples to illustrate our results. We show the efficient frontier of the mean-variance portfolio selection problem with uncertain investment horizon under a regime-switching jump-diffusion model with a Markov-modulated generalized Gamma distributed jump part.

In our examples, we consider a simple situation where there are only two states of the continuous-time, finite-state Markov chain **X**, i.e. State 1 and State 2, representing a "Good" economy and a "Bad" economy, respectively. The rate matrix of the Markov chain is given by

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} . \tag{4.5.1}$$

The initial wealth and the planned terminal time are assumed to be

$$y_0 = 1 \ , \quad T = 1 \ . \tag{4.5.2}$$

To simplify our computation, we assume that the financial market consists of one risk-free bond and two risky shares. Furthermore, we assume that the standard Brownian motion defined in Section 2 is of two dimensions. That is, $W(t) := (W^1(t), W^2(t))^\top \in \Re^2$, for each $t \in \mathcal{T}$. When $\mathbf{X}(t) = \mathbf{e}_i$, for each $i = 1, 2$, the dynamics of the risk shares are governed by the following regime-switching jump-diffusion models:

$$\begin{cases} \dfrac{dS_1(t)}{S_1(t-)} = \mu_1(\mathbf{e}_i)dt + \sigma_{11}(\mathbf{e}_i)dW^1(t) + \sigma_{12}(\mathbf{e}_i)dW^2(t) \\ \qquad\qquad + \displaystyle\int_{\Re_0} (e^z - 1)[\gamma^1(dt, dz) - \nu^1_{\mathbf{e}_i}(dt, dz)] \ , \\ \dfrac{dS_2(t)}{S_2(t-)} = \mu_2(\mathbf{e}_i)dt + \sigma_{21}(\mathbf{e}_i)dW^1(t) + \sigma_{22}(\mathbf{e}_i)dW^2(t) \\ \qquad\qquad + \displaystyle\int_{\Re_0} (e^z - 1)[\gamma^2(dt, dz) - \nu^2_{\mathbf{e}_i}(dt, dz)] \ , \end{cases} \tag{4.5.3}$$

where

$$\nu^k_{\mathbf{e}_i}(dz) = \frac{1}{\Gamma(1 - \alpha_k(\mathbf{e}_i))} e^{-b_k(\mathbf{e}_i)z} z^{-\alpha_k(\mathbf{e}_i)-1} \ , \quad k = 1, 2 \ .$$

Suppose that the configurations of the parameter values, including the intensity of the uncertain investment horizon and other parameters of the financial market, are given in Table 4.5.1.

In what follows, we use the hypothetical parameter values given in (4.5.1)-(4.5.2) and Table 4.5.1 as our benchmark, where y_0 in (4.5.2) and r, μ_k, σ_k, for each $k = 1, 2$, in Table 4.5.1 are the default choices of unvarying parameters. Varying the values of T, A and λ, we provide several numerical examples for our mean-variance problem. Comparing the efficient frontiers of these examples with that of our benchmark may induce a deep understanding of the problem. Hence with y_0, r, μ_k and σ_k unchanged

Table 4.5.1: Model parameters

	λ	r	μ_1	μ_2	$(\sigma_{11}, \sigma_{12})$	$(\sigma_{21}, \sigma_{22})$	b_1	b_2	α_1	α_2
State 1	0.01	0.06	0.08	0.04	(0.15, 0.20)	(0.20, 0.25)	1.00	1.00	0.00	1.00
State 2	0.02	0.03	0.04	0.02	(0.30, 0.40)	(0.40, 0.50)	1.00	0.50	0.50	0.50

in our benchmark, we also provide the efficient frontier when the jump parameters, the planned terminal time, the rate matrix of the chain and the intensity of the uncertain investment horizon are given by the following alternative values:

(I) when the jump parts are absent, the dynamics of the risk shares follows the regime-switching GBM models:

$$\begin{cases} \dfrac{dS_1(t)}{S_1(t)} = \mu_1(\mathbf{e}_i)dt + \sigma_{11}(\mathbf{e}_i)dW^1(t) + \sigma_{12}(\mathbf{e}_i)dW^2(t) \ , \\ \dfrac{dS_2(t)}{S_2(t)} = \mu_2(\mathbf{e}_i)dt + \sigma_{21}(\mathbf{e}_i)dW^1(t) + \sigma_{22}(\mathbf{e}_i)dW^2(t) \ ; \end{cases} \tag{4.5.4}$$

(II) the terminal time:

$$T = 2 \quad \text{or} \quad 3 \ ; \tag{4.5.5}$$

(III) the rate matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \ ; \tag{4.5.6}$$

(VI) the intensity rate

$$\lambda = (0, 0)^\top \quad \text{or} \quad (0.25, 0.5)^\top \ . \tag{4.5.7}$$

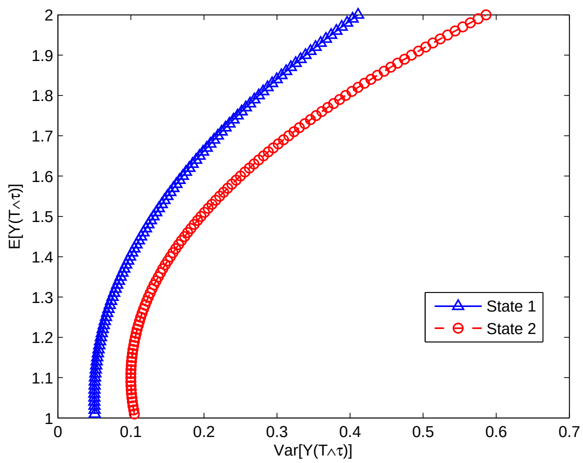


Figure 4.5.1: Efficient frontiers in the benchmark case

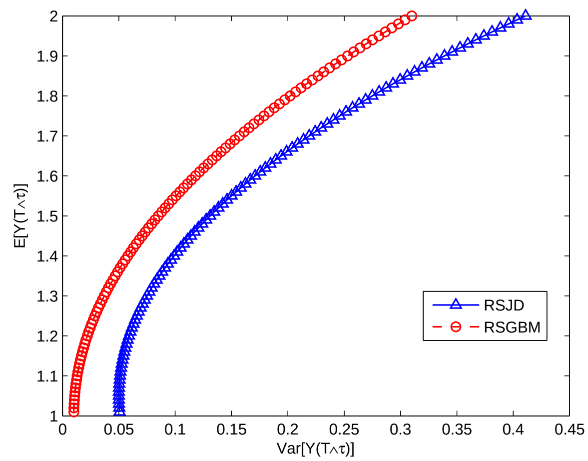
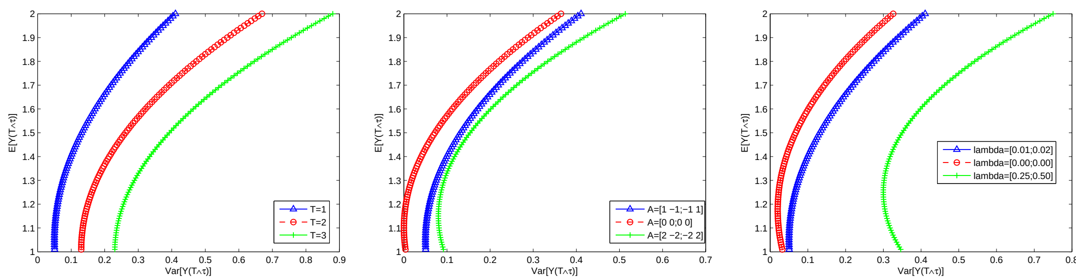


Figure 4.5.2: Efficient frontiers: RSJD vs RS-GBM

Figure 4.5.1 shows the efficient frontiers in States 1 and 2 of the problem under the regime-switching jump-diffusion model (4.5.3) with our benchmark parameters. Since States 1 and 2 represent a "Good" economy and a "Bad" economy, respectively. The risky shares are more volatile in State 2 than in State 1. So the efficient frontier in State 1 lies on the left of that in State 2. It implies that given the same expected terminal wealth, the agent can achieve a smaller variance of the terminal wealth if the current economy is "Good".

In Figure 4.5.2, we compare the efficient frontiers of the problem under the regime-switching jump-diffusion model (RSJD) and the regime-switching GBM model (RS-GBM). We only show the efficient frontiers in State 1 for both models. It can be seen that with the same expected terminal wealth, the variance of the terminal wealth under the regime-switching GBM model is smaller than that under the regime-switching jump-diffusion model. This is intuitive since additional jump risk definitely results in a higher overall risk (i.e. the variance) of the portfolio.

Figure 4.5.3 gives the efficient frontiers under the regime-switching jump-diffusion model (4.5.3). In all subfigures, the blue lines represent the efficient frontiers in State



(a) (b) (c)

Figure 4.5.3: Efficient frontiers with different values of T, A and λ

1 for our benchmark parameters given in (4.5.1)-(4.5.2) and Table 4.5.1, while the red lines and the green lines represent those for the alternative values of the parameters given in (4.5.5)-(4.5.7).

In Subfigure (a), the efficient frontiers with longer planned terminal times lie on the right of those with shorter terminal times. Our explanation to this finding is that the impacts of the regime switching and the uncertain investment horizon on the portfolio increase with the length of the planned terminal time. Indeed, in a longer time span, it is more likely that there exist structural changes in economy and the portfolio may be liquidated prior to the terminal time, which introduces extra economic and liquidation risks to the portfolio and hence increase the variance.

In Subfigure (b), it is shown that the efficient frontiers with rate matrices related to larger transition probabilities fall on the right of those with rate matrices related to smaller transition probabilities. The explanation in Subfigure (a) also applies here. That is, the larger the transition probability of the underlying Markov is, the more economic risk is introduced to the portfolio. Another interesting finding in Subfigure (b) is that the efficient frontier for the zero rate matrix is tangent to the y-axis. In other words, the minimum variance portfolio in this case is a zero variance portfolio, i.e. we can achieve a risk-free investment. When the rate matrix is zero, the regime-switching effect is degenerate in our modeling framework. Therefore, the risk-free investment is certainly expected in the mean-variance problem without regime-switching.

In Subfigure (c), we can see the efficient frontiers with higher intensities lie on the right of those with lower ones. A higher intensity results in a larger probability that the portfolio may be liquidated before the planned terminal time. This introduces extra liquidation risk to the portfolio and increase the variance of the terminal wealth.

4.6 Conclusion

We considered a mean-variance portfolio selection problem with uncertain investment horizon under a regime-switching jump-diffusion model. The dynamic programming principle was applied to solve the problem. Although the investment horizon is uncertain in our problem, it was proved that most results in the classical mean-variance problem with fixed investment horizon still hold. Specifically, we derived the efficient portfolio, the efficient frontier and the mutual fund theorem to the problem. Numerical examples were provided to illustrate our theoretical results.

Chapter 5

Stochastic differential game, Esscher transform and general equilibrium under a Markovian regime-switching Lévy model

5.1 Introduction

The valuation of contingent claims has long been an important topic in economics and finance. It plays a central role in the investment, financing and risk management activities of the finance and insurance markets around the globe. The seminal works of Black and Scholes (1973) and Merton (1973) provided a path-breaking solution to this important problem. Under the assumptions of a Geometric Brownian Motion (GBM) for the price dynamics of the underlying risky asset, a perfect market and the absence of arbitrage opportunities, they derived a preference-free, closed-form option pricing formula for a standard European call option. The pricing formula is widely adopted by market practitioners for pricing, hedging and managing risk of options. Despite its popularity, it is known that the GBM assumption for the price dynamics is not realistic and fails to incorporate many important stylized features of assets returns and option prices. Over the past few decades, various extensions to the Black-Scholes-Merton model have been introduced. These models include jump-diffusion models, GARCH models, stochastic volatility models, pure jump models, Lévy processes, regime-switching models, just to name a few.

One key economic insight behind the Black-Scholes-Merton model is the concept of the risk-neutral valuation, where the price of an option is determined as its discounted expected value under an "artificial" probability measure, namely, a risk-neutral probability measure or an equivalent martingale measure. The market considered by Black-Scholes (1973) and Merton (1973) is complete since there exists only one underlying risky asset with randomness driven by a one-dimensional Brownian motion, and thus any contingent claim can be perfectly replicated by continuously rebalancing the composition of a portfolio consisting of the risk-free asset and the underlying risky asset. However, the financial markets described by other more realistic models are mostly incomplete. As shown by Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983), the market completeness is equivalent to the uniqueness of an equivalent martingale measure. So there exist infinitely many equivalent martingale measures in an incomplete financial market. A natural question is how to choose an equivalent martingale measure among infinitely many equivalent martingale measures. Different approaches have been proposed to address this problem. Recently, Øksendal and Sulem (2007a) introduced a stochastic differential game approach to choose an equivalent martingale measure for option pricing in a jump-diffusion market, where a representative agent chooses a portfolio which maximizes the expected utility of terminal wealth, while the market chooses a probability measure which minimizes this maximal expected utility. It was shown in Øksendal and Sulem (2007a) that choosing an equivalent martingale measure is an optimal strategy for the market (see also Siu (2008) for the stochastic differential game under regime-switching models). The pioneering work by Gerber and Shiu (1994) adopted a time-honored tool in actuarial science, namely the Esscher transform to choose an equivalent martingale measure for option valuation in an incomplete market. The use of the Esscher transform for option valuation can be justified by maximizing the expected power utility of an economic agent. Their works highlighted the interplay between the financial and actuarial pricing in incomplete markets. Applications of the Esscher transform for option pricing under regime-switching models can be found in Elliott et al. (2005), Siu and Yang (2009), Siu (2005, 2008, 2011), Elliott and Siu (2013) and others. Fu and Yang (2012) proposed a general equilibrium approach to choose an equivalent martingale measure for the price dynamics driven by a Lévy process. Many empirical features such as the negative variance risk premium, implied volatility smirk and negative skewness risk premium can be explained based on the derived equivalent martingale measure. Other works with restrictions on the distribution of the jump component on this approach include Pan (2002), Liu and Pan (2003), Liu et al. (2005) and Zhang et al. (2010).

In this chapter, we investigate the game theoretic approach, the Esscher transformation approach and the general equilibrium approach to choose equivalent martingale measures for the valuation of contingent claims under a regime-switching Lévy model. A financial market consisting of a risk-free bond and a risky share is considered. The price dynamics of the risky share are governed by a Markovian regime-switching geometric Lévy process. The market interest rate, the appreciation rate, the volatility and the Lévy measure are assumed to switch over time according to a continuous-time, finite-state, observable Markov chain, whose states may represent some (macro)-

economic factors (e.g. gross domestic product and purchase management index) or credit rating of a region. Firstly, we consider a two-player, zero-sum, stochastic differential game approach to choose an equivalent martingale measure for the valuation of contingent claims. Here the representative agent and the market are the two players in this game. The representative agent has a power/logarithmic utility and chooses his optimal investment-consumption strategy so as to maximize the expected, discounted utility from intertemporal consumption and terminal wealth. Whereas, the market is a fictitious player of the game and selects a real-world probability measure so as to minimize the maximal expected utility of the representative agent. We formulate this min-max problem as a stochastic differential game. We then provide a verification theorem for the Hamilton-Jacobi-Bellman-Issac (HJBI) solution to the game and derive explicit expressions for the optimal strategies of the representative agent, the market and the value function. An equivalent martingale measure is determined by the saddle-point of the game. Secondly, we adopt a generalized version of the Esscher transform using stochastic exponentials and the Laplace cumulant process to choose an equivalent martingale measure. Thirdly, we consider a general equilibrium approach to choose an equivalent martingale measure. We formulate the general equilibrium problem of the representative agent as a stochastic optimal control problem. Then a verification theorem for the Hamilton-Jacobi-Bellman (HJB) solution to the control problem is provided. Under a market clearing condition, we derive explicit expressions for the optimal consumption rate of the representative agent, the value function and the equilibrium equity premium. Finally, we compare equivalent martingale measures chosen by the three approaches and identify the conditions under which these measures are identical. Since we apply the stochastic control theory, which is tailor-made for portfolio optimization problems, to investigate the selection of equivalent martingale measures in asset pricing problems, this chapter may provide a link between asset pricing and portfolio optimization under regime-switching models.

The rest of this chapter is organized as follows. In Section 5.2, we describe the model dynamics and formulate the optimal investment-consumption problem of the representative agent. Section 5.3 presents and compares three different approaches to choose equivalent martingale measures in our modeling framework. The final section gives concluding remarks.

5.2 The Model dynamics

We consider a simplified, continuous-time, financial market with two primitive assets, namely, a risk-free bond and a risky share. These assets are traded continuously over time in a finite horizon $\mathcal{T} := [0, T]$, where $T < \infty$. We fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, which describes randomness in the market. Here $\mathcal{P}$ is a real-world probability measure or a reference probability measure from which a family of real-world probability measures is generated. We further equip $(\Omega, \mathcal{F}, \mathcal{P})$ with a right-continuous, $\mathcal{P}$-complete filtration $\mathbb{F} := \{\mathcal{F}(t) | t \in \mathcal{T}\}$, where $\mathcal{F}(t)$ is the enlarged σ-field generated by information about the values of a Brownian motion, a Poisson random measure and a Markov chain up to time t, which will be defined precisely in the later part of this section.

We model the evolution of the state of an economy over time by a continuous-time, finite-state, observable Markov chain $\mathbf{X} := \{\mathbf{X}(t) | t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$. As in Elliott et al. (1994), we identify the state space of the chain by a set of standard unit vectors $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N\} \subset \Re^N$, where the j^{th} component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij}, for each $i, j = 1, 2, \ldots, N$. This is usually called the canonical state space of the chain $\mathbf{X}$. To describe the statistical laws of the chain $\mathbf{X}$ under $\mathcal{P}$, we consider a constant rate matrix $\mathbf{A} := [a_{ij}]_{i,j=1,2,\cdots,N}$, where a_{ij} is the instantaneous transition rate of the chain $\mathbf{X}$

from state $\mathbf{e}_j$ to state $\mathbf{e}_i$. Let $\mathbb{F}^{\mathbf{X}} := \{\mathcal{F}^{\mathbf{X}}(t)|t \in \mathcal{T}\}$ be the right-continuous, $\mathcal{P}$-complete filtration generated by the chain $\mathbf{X}$. With the canonical state space representation of $\mathbf{X}$, Elliott et al. (1994) obtained the following semimartingale dynamics for $\mathbf{X}$:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}\mathbf{X}(u)du + \mathbf{M}(t) \ , \tag{5.2.1}$$

where $\mathbf{M} := \{\mathbf{M}(t)|t \in \mathcal{T}\}$ is an $\Re^N$-valued, $(\mathbb{F}^{\mathbf{X}}, \mathcal{P})$ -martingale.

Let $J(\cdot,\cdot)$ be a Poisson random measure on the product measurable space $(\mathcal{T} \times \Re_0, \mathcal{B}(\mathcal{T}) \otimes \mathcal{B}(\Re_0))$, where $\mathcal{B}(\mathcal{T})$ and $\mathcal{B}(\Re_0)$ denote the Borel σ-fields generated by open subsets of $\mathcal{T}$ and $\Re_0 := \Re \setminus \{0\}$, respectively. Indeed, the Poisson random measure J can be represented as a counting measure:

$$J(dt, dy) = \sum_{k \geq 1} \delta_{(T_k, \Delta Y(T_k))}(dt, dy) \mathbb{1}_{\{T_k < \infty, \Delta Y(T_k) \neq 0\}} \ .$$

Here, T_k is the random time of the k^{th} jump; $\Delta Y(T_k)$ is the random size of the k^{th} jump at the time epoch T_k; $\delta_{(T_k, \Delta Y(T_k))}(\cdot,\cdot)$ is the random delta function at the point $(T_k, \Delta Y(T_k)) \in \mathcal{T} \times \Re_0$; $\mathbb{1}_E$ is the indicator function of an event E. We assume that the Poisson random measure J has a Markov-modulated Lévy measure under $\mathcal{P}$:

$$\nu_{\mathbf{X}(t)}(dy) := \langle \boldsymbol{\nu}(dy), \mathbf{X}(t) \rangle = \sum_{i=1}^{N} \langle \mathbf{X}(t-), \mathbf{e}_i \rangle \, \nu_i(dy) \ ,$$

where $\boldsymbol{\nu}(dy) := (\nu_1(dy), \nu_2(dy), \cdots, \nu_N(dy))'$. Here $\mathbf{C}'$ is the transpose of a matrix, or a vector, $\mathbf{C}$. The scalar product $\langle \cdot,\cdot \rangle$ selects the component of the vector $\boldsymbol{\nu}(dy)$ of the Lévy measures in force at time t depending on the state of the chain $\mathbf{X}$. In particular, when the chain $\mathbf{X}$ is in the i^{th} state at time t, (i.e. $\mathbf{X}(t) = \mathbf{e}_i$), the Lévy measure of J at time t, say $\nu_{\mathbf{X}(t)}(dy)$, is $\nu_i(dy)$. Write $\widetilde{J}(\cdot,\cdot)$ for a compensated version of the Poisson random measure, i.e.

$$\widetilde{J}(dt, dy) := J(dt, dy) - \nu_{\mathbf{X}(t)}(dy)dt \ .$$

Denote by $\mathbb{F}^J := \{\mathcal{F}^J(t)|t \in \mathcal{T}\}$ the right-continuous, $\mathcal{P}$-complete natural filtration generated by the Poisson random measure J; that is

$$\mathcal{F}_0^J(t) := \sigma\left(\int_0^u \int_E J(dt, dy); u \leq t, E \in \mathcal{B}(\Re_0)\right) ,$$

and

$$\mathcal{F}^J(t) := \bigcap_{u>t}(\mathcal{F}_0^J(u) \vee \mathcal{N}) ,$$

where $\mathcal{N}$ denotes all $\mathcal{P}$-null sets and $\sigma_1 \vee \sigma_2$ denotes the minimal σ-field containing both σ_1 and σ_2. Let $W := \{W(t)|t \in \mathcal{T}\}$ be a one-dimensional standard Brownian motion with respect to its right-continuous, $\mathcal{P}$-complete natural filtration $\mathbb{F}^W := \{\mathcal{F}^W(t)|t \in \mathcal{T}\}$. Based on the above definition, the filtration $\mathbb{F} = \{\mathcal{F}(t)|t \in \mathcal{T}\}$ denotes the enlarged filtration of $\mathbb{F}^W$, $\mathbb{F}^J$ and $\mathbb{F}^{\mathbf{X}}$, i.e. for each $t \in \mathcal{T}$,

$$\mathcal{F}(t) := \mathcal{F}^W(t) \vee \mathcal{F}^J(t) \vee \mathcal{F}^{\mathbf{X}}(t) .$$

In what follows, we describe the model dynamics of the primitive assets. To be more specific, we first extend the Lévy process in Fu and Yang (2012) to its regime-switching variant to describe the dynamics of the share price. Then we formulate investment-consumption problem of a representative agent.

Let $r(t)$ be the instantaneous, continuously compounded risk-free rate at time t, which is modulated by the chain $\mathbf{X}$ as follows:

$$r(t) := \langle \mathbf{r}, \mathbf{X}(t) \rangle ,$$

and

$$\mathbf{r} := (r_1, r_2, \cdots, r_N)' \in \Re^N .$$

We assume that there exists a risk-free bond for instantaneous borrowing and lending at the risk-free rate. Then the dynamics of the risk-free bond $B := \{B(t)|t \in \mathcal{T}\}$ follows

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1 \ . \tag{5.2.2}$$

Let $\mu(t)$ and $\sigma(t)$ be the appreciation rate and the volatility of the share at time t, which are modulated by the chain $\mathbf{X}$ as follows:

$$\mu(t) := \langle \boldsymbol{\mu}, \mathbf{X}(t) \rangle \ , \quad \sigma(t) := \langle \boldsymbol{\sigma}, \mathbf{X}(t) \rangle \ ,$$

and

$$\boldsymbol{\mu} := (\mu_1, \mu_2, \cdots, \mu_N)' \in \Re^N \ , \quad \boldsymbol{\sigma} := (\sigma_1, \sigma_2, \cdots, \sigma_N)' \in \Re^N_{++} \ ,$$

where $\Re_{++}$ is the positive real line. We assume that the share price process $S := \{S(t)|t \in \mathcal{T}\}$ evolves over time as an exponential regime-switching Lévy process:

$$S(t) = S(0) \exp(Y(t)) \ , \quad S(0) = s > 0 \ .$$

The logarithmic return process $Y := \{Y(t)|t \in \mathcal{T}\}$ is a one-dimensional regime-switching Lévy process with the following decomposition:

$$\begin{aligned} Y(t) \ &= \ \int_0^t \left[\mu(u) - \frac{1}{2}\sigma^2(u) - \int_{\Re_0} \left(e^y - 1 - y\mathbb{1}_{|y|<1} \right) \nu_{\mathbf{X}(u)}(dy) \right] du \\ &\quad + \int_0^t \sigma(u) dW(u) + \int_0^t \int_{|y|\geq 1} yJ(du, dy) + \int_0^t \int_{|y|<1} y\widetilde{J}(du, dy) \ , \end{aligned} \tag{5.2.3}$$

where, as in Fu and Yang (2012), we assume that the jump component of Y has finite variation, i.e.

$$\int_{|y|<1} |y| \nu_i(dy) < \infty \ ,$$

for each $i = 1, 2, \cdots, N$. It it clear that (5.2.3) can be simplified as

$$\begin{aligned} Y(t) &= \int_0^t \left[\mu(u) - \frac{1}{2}\sigma^2(u) - \int_{\Re_0} \big(e^y - 1\big)\nu_{\mathbf{X}(u)}(dy)\right] du \\ &\quad + \int_0^t \sigma(u) dW(u) + \int_0^t \int_{\Re_0} y J(du, dy) \ . \end{aligned} \tag{5.2.4}$$

Applying Itô's differentiation rule to $S(t) = S(0)\exp(Y(t))$, we obtain the dynamics of the share price process:

$$dS(t) = S(t-)\left[\big(r(t) + \phi(t)\big)dt + \sigma(t)dW(t) + \int_{\Re_0} \big(e^y - 1\big)\widetilde{J}(dt, dy)\right] , \tag{5.2.5}$$

where

$$\phi(t) = \mu(t) - r(t) \ ,$$

is the equity premium and is also modulated by the chain $\mathbf{X}$ as follows:

$$\phi(t) := \langle \boldsymbol{\phi}, \mathbf{X}(t)\rangle = \langle \boldsymbol{\mu} - \mathbf{r}, \mathbf{X}(t)\rangle \ ,$$

and

$$\boldsymbol{\phi} := (\phi_1, \phi_2, \cdots, \phi_N)' = (\mu_1 - r_1, \mu_2 - r_2, \cdots, \mu_N - r_N)' \in \Re^N \ .$$

Suppose that there exists a representative agent[1] in the market who not only invests his wealth in the risk-free bond and the risky share, but also consumes part of his wealth over time. The objective of the agent is to maximize the expected, discounted utility from intertemporal consumption and terminal wealth. In this chapter, we assume that the agent has the constant relative risk aversion (CRRA) utility:

$$U(v) = \begin{cases} \dfrac{v^{1-\gamma}}{1-\gamma} \ , & \gamma > 0 \ , \gamma \neq 1 \ , \\ \log v \ , & \gamma = 1 \ , \end{cases} \tag{5.2.6}$$

[1]The existence of a representative agent is one of the most fundamental problems in modern economic theory. For more discussions, interested readers can refer to Arrow and Debreu (1954), Rubinstein (1974), Detemple and Gottardi (1998), Cuoco and He (2001) and references therein.

where γ is called the relative risk aversion coefficient.

Let $\pi(t)$ and $1-\pi(t)$ be the fractions of the agent's wealth invested in the share S and the bond B at time t, respectively. In addition, the amount of wealth consumed by the agent is $c(t)$ at time t. Suppose that $\pi := \{\pi(t)|t \in \mathcal{T}\}$ and $c := \{c(t)|t \in \mathcal{T}\}$ are an $\mathbb{F}$-predictable, càdlàg process and a nonnegative $\mathbb{F}$-predictable, càdlàg process, which are called the portfolio process and the consumption rate process, respectively. Let $V^{\pi,c} := \{V^{\pi,c}(t)|t \in \mathcal{T}\}$ be the wealth process of the agent associated with the portfolio-consumption pair (π, c). To simplify our notation, we suppress the superscript (π, c) and write $V(t)$ for $V^{\pi,c}(t)$, for each $t \in \mathcal{T}$. Furthermore, we assume that the agent is endowed with an initial wealth $v > 0$. Then under $\mathcal{P}$, the wealth process of the agent evolves over time as:

$$\begin{aligned} dV(t) &= V(t-)\bigg[\big(r(t)+\pi(t)\phi(t)\big)dt + \pi(t)\sigma(t)dW(t) \\ &\quad +\pi(t)\int_{\Re_0}\big(e^y-1\big)\widetilde{J}(dt,dy)\bigg] - c(t)dt \ , \quad V(0)=v>0 \ . \end{aligned} \tag{5.2.7}$$

We say that a portfolio-consumption pair (π, c) is admissible if the following conditions hold:

1. $\int_0^T \pi^2(u)du < \infty$ and $\int_0^T c(u)du < \infty$, $\mathcal{P}$-a.s.;
2. the stochastic differential equation (5.2.7) admits a unique strong solution, such that $V(t) \geq 0$, $\mathcal{P}$-a.s., for each $t \in \mathcal{T}$.

Write $\mathcal{A}$ for the space of all admissible portfolio-consumption pairs (π, c).

5.3 Main results

5.3.1 Stochastic differential game

In this subsection, we consider a two-player, zero-sum stochastic differential game between the representative agent and the market in the Markovian regime-switching Lévy market. Here the goal of the agent is to select an optimal pair of portfolio and consumption rate process so as to maximize the expected, discounted utility from intertemporal consumption and terminal wealth, while the market is interpreted as a fictitious player in the game and acts antagonistically to select a probability measure corresponding to a worst-case scenario that minimizes the maximal expected utility of the agent.

First of all, we generate a family of probability measures $\mathcal{Q}^\xi$ equivalent to $\mathcal{P}$ on $\mathcal{F}(T)$ associated with two $\mathbb{F}$-predictable, càdlàg processes or random fields $\xi_0 := \{\xi_0(t)|t \in \mathcal{T}\}$ and $\xi_1 := \{\xi_1(t,y)|(t,y) \in \mathcal{T} \times \Re_0\}$. Write Ξ for the admissible set of all such processes $\xi := (\xi_0, \xi_1)$ satisfying

1. $\xi_1(t,y) < 1$, for a.a. $(t,y,\omega) \in \mathcal{T} \times \Re_0 \times \Omega$;
2. $\int_0^T \{\xi_0^2(u) + \sum_{i=1}^N [\int_{\Re_0} \xi_1^2(u,y)\nu_i(dy) \langle \mathbf{X}(u), \mathbf{e}_i \rangle]\} du < \infty$, $\mathcal{P}$-a.s.

Define, for each $\xi \in \Xi$, a real-valued, $\mathbb{F}$-adapted process $\Lambda^\xi := \{\Lambda^\xi(t)|t \in \mathcal{T}\}$ as follows:

$$\begin{aligned}\Lambda^\xi(t) \quad := \quad & \exp\Bigg(- \int_0^t \xi_0(u)dW(u) + \int_0^t \int_{\Re_0} \ln(1 - \xi_1(u-,y))\widetilde{J}(du,dy) \\ & -\frac{1}{2}\int_0^t \xi_0^2(u)du + \int_0^t \int_{\Re_0} \big[\ln(1 - \xi_1(u-,y)) + \xi_1(u-,y)\big]\nu_{\mathbf{X}(u)}(dy)du \Bigg) .\end{aligned} \tag{5.3.1}$$

Note that Λ^ξ is the controlled state process of the market.

Applying Itô's differentiation rule to $\Lambda^\xi(t)$ gives

$$d\Lambda^\xi(t) = \Lambda^\xi(t-)\bigg[- \xi_0(t)dW(t) - \int_{\Re_0} \xi_1(t-,y)\widetilde{J}(dt,dy)\bigg] , \quad \Lambda^\xi(0) = 1 . \tag{5.3.2}$$

So Λ^{ξ} is an $(\mathbb{F}, \mathcal{P})$-local-martingale. We suppose that the process ξ is such that Λ^{ξ} is an $(\mathbb{F}, \mathcal{P})$-martingale. Then $\mathrm{E}[\Lambda^{\xi}(T)] = 1$.

For each $\xi \in \Xi$, we define a probability measure $\mathcal{Q}^{\xi}$ equivalent to $\mathcal{P}$ on $\mathcal{F}(T)$ as follows:

$$\frac{d\mathcal{Q}^{\xi}}{d\mathcal{P}}\bigg|_{\mathcal{F}(T)} := \Lambda^{\xi}(T) \ . \tag{5.3.3}$$

So, we can generate a family $\mathcal{Q}(\Xi)$ of real-world probability measures $\mathcal{Q}^{\xi}$ parameterized by $\xi \in \Xi$. Note that the market can choose a real-world probability measure or generalized scenario from $\mathcal{Q}(\Xi)$ through selecting a process $\xi \in \Xi$. Hence, Ξ is the set of admissible controls of the market.

In what follows, we consider the two-person, zero-sum stochastic differential game between the representative agent and the market. For notational simplicity, we define a vector-valued controlled state process $\mathbf{Z} := \{\mathbf{Z}(t) | t \in \mathcal{T}\}$ of the agent and the market:

$$\begin{aligned}
d\mathbf{Z}(t) &= (dZ_0(t), dZ_1(t), dZ_2(t), d\mathbf{Z}_3(t))' \\
&= (dZ_0(t), dZ_1^{\pi,c}(t), dZ_2^{\xi_0,\xi_1}(t), d\mathbf{Z}_3(t))' \ , \\
&= (dt, dV^{\pi,c}(t), d\Lambda^{\xi}(t), d\mathbf{X}(t))' \ , \\
\mathbf{Z}(0) &= \mathbf{z} = (u, z_1, z_2, \mathbf{z}_3) \ .
\end{aligned}$$

Under $\mathcal{P}$, the evolution of the components of the controlled state process $\mathbf{Z}$ over time is governed by:

$$\begin{aligned}
&dZ_0(t) = dt \ , \\
&dZ_1(t) = Z_1(t-)\bigg[\big(r(t) + \pi(t)\phi(t)\big)dt + \pi(t)\sigma(t)dW(t) \\
&\qquad\qquad + \pi(t)\int_{\Re_0} \big(e^y - 1\big)\widetilde{J}(dt, dy)\bigg] - c(t)dt \ , \\
&dZ_2(t) = Z_2(t-)\bigg[- \xi_0(t)dW(t) - \int_{\Re_0} \xi_1(t-, y)\widetilde{J}(dt, dy)\bigg] \ ,
\end{aligned}$$

$$d\mathbf{Z}_3(t) = \mathbf{A}\mathbf{Z}_3(t)dt + d\mathbf{M}(t) \ . \tag{5.3.4}$$

Then the stochastic differential game of the agent and the market can be formulated into finding the value function $\varphi(u, z_1, z_2, \mathbf{z}_3)$, the optimal strategies $\hat{\xi} = (\hat{\xi}_0, \hat{\xi}_1) \in \Xi$ and $(\hat{\pi}, \hat{c}) \in \mathcal{A}$ such that

$$\begin{aligned} \varphi(u, z_1, z_2, \mathbf{z}_3) &= \inf_{(\xi_0,\xi_1)\in\Xi} \sup_{(\pi,c)\in\mathcal{A}} \mathrm{E}^{\xi}_{\mathbf{z}} \left[\int_u^T e^{-\int_u^t \rho(s)ds} U(c(t))dt + e^{-\int_u^T \rho(s)ds} U(Z_1^{\pi,c}(T)) \right] , \\ &= \mathrm{E}^{\hat{\xi}}_{\mathbf{z}} \left[\int_u^T e^{-\int_u^t \rho(s)ds} U(\hat{c}(t))dt + e^{-\int_u^T \rho(s)ds} U(Z_1^{\hat{\pi},\hat{c}}(T)) \right] , \end{aligned} \tag{5.3.5}$$

where $\mathrm{E}^{\xi}_{\mathbf{z}}[\cdot]$ and $\mathrm{E}^{\hat{\xi}}_{\mathbf{z}}[\cdot]$ are the conditional expectations under $\mathcal{Q}^{\xi}$ and $\mathcal{Q}^{\hat{\xi}}$, respectively, given that $\mathbf{Z}(0) = \mathbf{z}$. Here $\rho(t)$ denotes the instantaneous discount rate of the agent at time t. We suppose that $\rho(t)$ is also modulated by the chain as:

$$\rho(t) := \langle \boldsymbol{\rho}, \mathbf{X}(t) \rangle \ ,$$

and

$$\boldsymbol{\rho} := (\rho_1, \rho_2, \cdots, \rho_N)' \in \Re^N \ .$$

By a version of the Bayes' rule, we have

$$\begin{aligned} \varphi(u, z_1, z_2, \mathbf{z}_3) &= \inf_{(\xi_0,\xi_1)\in\Xi} \sup_{(\pi,c)\in\mathcal{A}} \mathrm{E}_{\mathbf{z}} \left[\int_u^T e^{-\int_u^t \rho(s)ds} Z_2(t) U(c(t))dt \right. \\ &\qquad \left. + e^{-\int_u^T \rho(s)ds} Z_2(T) U(Z_1(T)) \right] , \end{aligned} \tag{5.3.6}$$

where $\mathrm{E}_{\mathbf{z}}[\cdot]$ represents an expectation under $\mathcal{P}$ given that $\mathbf{Z}(0) = \mathbf{z}$.

It is noted in Elliott (1982) that if the state processes are Markovian, it may not be unreasonable to consider optimal Markovian controls. Furthermore, under some mild technical conditions, Markovian controls and general adapted controls have essentially the same performance (see, for example, Øksendal and Sulem (2007b)). Since the vector-valued process $\mathbf{Z}$ is Markovian with respect to the enlarged filtration $\mathbb{F}$, it is not unreasonable to assume that the control processes (ξ_0, ξ_1) and (π, c) are Markovian with respect to $\mathbb{F}$. Let $\mathcal{O} := (0, T) \times (0, \infty) \times (0, \infty)$ be our solvency region. Suppose that K_1, K_2, K_3 and K_4 denote the sets such that $\xi_0(t) \in K_1$, $\xi_1(t, y) \in K_2$, $\pi(t) \in K_3$ and $c(t) \in K_4$. To consider Markovian controls, we assume that

$$\xi_0(t) = \overline{\xi}_0(\mathbf{Z}(t)) \ , \quad \xi_1(t, y) = \overline{\xi}_1(\mathbf{Z}(t), y) \ ,$$

and

$$\pi(t) = \overline{\pi}(\mathbf{Z}(t)) \ , \quad c(t) = \overline{c}(\mathbf{Z}(t)) \ ,$$

for some measurable functions $\overline{\xi}_0 : \mathcal{O} \times \mathcal{E} \to K_1$, $\overline{\xi}_1 : \mathcal{O} \times \mathcal{E} \times \Re_0 \to K_2$, $\overline{\pi} : \mathcal{O} \times \mathcal{E} \to K_3$ and $\overline{c} : \mathcal{O} \times \mathcal{E} \to K_4$. These are called feedback controls. To save notation, we do not distinguish $\overline{\xi}_0, \overline{\xi}_1, \overline{\pi}$ and $\overline{c}$ with ξ_0, ξ_1, π and c, respectively.

Suppose that $\mathcal{H}$ denotes the space of functions $h(\cdot, \cdot, \cdot, \cdot) : \mathcal{O} \times \mathcal{E} \to \Re$ such that for each $\mathbf{z}_3 \in \mathcal{E}$, $h(\cdot, \cdot, \cdot, \mathbf{z}_3) \in \mathcal{C}^{1,2,2}(\mathcal{O})$. Write

$$\mathbf{h}(u, z_1, z_2) := \big(h(u, z_1, u_2, \mathbf{e}_1), h(u, z_1, u_2, \mathbf{e}_2), \cdots, h(u, z_1, u_2, \mathbf{e}_N)\big)' \in \Re^N \ .$$

To unburden our notation, write

$$h := h(u, z_1, z_2, \mathbf{z}_3) \ ,$$

and

$$\mathbf{h} := \mathbf{h}(u, z_1, z_2) \ ,$$

whenever no confusion arises. Let h_u, h_{z_1} and h_{z_2} denote the derivatives of h with respect to u, z_1 and z_2, and $h_{z_1 z_1}$ and $h_{z_2 z_2}$ denote the second order derivatives of h with respect to z_1 and z_2.

Define a Markovian regime-switching generator $\mathcal{L}^{\xi_0,\xi_1,\pi,c}$ acting on a function $h \in \mathcal{H}$ for the Markov process $\mathbf{Z}$ as

$$\begin{aligned}
&\mathcal{L}^{\xi_0,\xi_1,\pi,c}[h(u,z_1,z_2,\mathbf{z}_3)] \\
&= -\rho(u)h + h_u + \left[\left(r(u) + \pi(\mathbf{z})\phi(u) - \pi(\mathbf{z})\int_{\Re_0}(e^y - 1)\nu_{\mathbf{z}_3}(dy)\right)z_1 - c(\mathbf{z})\right]h_{z_1} \\
&\quad + \frac{1}{2}\pi^2(\mathbf{z})\sigma^2(u)z_1^2 h_{z_1z_1} + \frac{1}{2}\xi_0^2(\mathbf{z})z_2^2 h_{z_2z_2} - \pi(\mathbf{z})\xi_0(\mathbf{z})\sigma(u)z_1z_2h_{z_1z_2} + \langle \mathbf{h}, \mathbf{A}\mathbf{z}_3\rangle \\
&\quad + \int_{\Re_0}\left[h\big(u, z_1(1+\pi(\mathbf{z})(e^y-1)), z_2(1-\xi_1(\mathbf{z},y)), \mathbf{z}_3\big) - h + \xi_1(\mathbf{z},y)z_2h_{z_2}\right]\nu_{\mathbf{z}_3}(dy)\ .
\end{aligned}$$

The following lemma presents a version of the Dynkin formula for a regime-switching Lévy process and will be used for the development of a verification theorem for the HJBI solution of the stochastic differential game.

Lemma 5.3.1. *Let τ be a stopping time such that $\tau < \infty$, $\mathcal{P}$-a.s.. Assume further that for each $(\xi_0, \xi_1, \pi, c) \in \Xi \times \mathcal{A}$, $h(\mathbf{Z}(t))$ and $\mathcal{L}^{\xi_0,\xi_1,\pi,c}[h(\mathbf{Z}(t))]$ are bounded on $t \in [0, \tau]$. Then,*

$$E_{\mathbf{z}}[e^{-\int_u^\tau \rho(t)dt}h(\mathbf{Z}(\tau))] = h(\mathbf{z}) + E_{\mathbf{z}}\left[\int_u^\tau e^{-\int_u^t \rho(s)ds}\mathcal{L}^{\xi_0,\xi_1,\pi,c}[h(\mathbf{Z}(t))]dt\right]\ .$$

Proof. The result follows immediately by applying Itô's differentiation rule to $e^{-\int_u^t \rho(s)ds}h(\mathbf{Z}(t))$ and conditioning on $\mathbf{Z}(0) = \mathbf{z}$ under $\mathcal{P}$. □

Let $\overline{\mathcal{O}}$ be the closure of $\mathcal{O}$. We now present the HJBI solution to the stochastic differential game between the agent and the market in the following verification theorem. This verification theorem is a saddle-point result.

Theorem 5.3.1. *Suppose that, for each $\mathbf{z}_3 \in \mathcal{E}$, there exists a function $h(\cdot,\cdot,\cdot,\mathbf{z}_3) \in \mathcal{C}^{1,2,2}(\mathcal{O}) \cap \mathcal{C}(\overline{\mathcal{O}})$, and a Markovian control $(\hat{\xi}_0, \hat{\xi}_1, \hat{\pi}, \hat{c}) \in \Xi \times \mathcal{A}$ such that:*

1. $\mathcal{L}^{\xi_0,\xi_1,\hat{\pi},\hat{c}}[h(u,z_1,z_2,\mathbf{z}_3)] + z_2U(\hat{c}(u,z_1,z_2,\mathbf{z}_3)) \geq 0$, *for all* $(\xi_0,\xi_1) \in \Xi$ *and* $(u,z_1,z_2,\mathbf{z}_3) \in \mathcal{O} \times \mathcal{E}$;

2. $\mathcal{L}^{\hat{\xi}_0,\hat{\xi}_1,\pi,c}[h(u,z_1,z_2,\mathbf{z}_3)]+z_2U(c(u,z_1,z_2,\mathbf{z}_3)) \leq 0$, *for all* $(\pi,c) \in \mathcal{A}$ *and* $(u,z_1,z_2,\mathbf{z}_3) \in \mathcal{O} \times \mathcal{E}$*;*

3. $\mathcal{L}^{\hat{\xi}_0,\hat{\xi}_1,\hat{\pi},\hat{c}}[h(u,z_1,z_2,\mathbf{z}_3)] + z_2U(\hat{c}(u,z_1,z_2,\mathbf{z}_3)) = 0$, *for all* $(u,z_1,z_2,\mathbf{z}_3) \in \mathcal{O} \times \mathcal{E}$*;*

4. for all $(\xi_0,\xi_1,\pi,c) \in \Xi \times \mathcal{A}$*,*

$$\lim_{u \to T^-} h(u, Z_1(u), Z_2(u), \mathbf{Z}_3(u)) = Z_2(T)U(Z_1(T)) \ ;$$

5. let $\mathcal{K}$ *denote the set of stopping times* $\tau \leq T$*. The family* $\{h(\mathbf{Z}(\tau))\}_{\tau \in \mathcal{K}}$ *is uniformly integrable.*

Write, for each $\mathbf{z} = (u,z_1,z_2,\mathbf{z}_3) \in \mathcal{O} \times \mathcal{E}$ *and* $(\xi_0,\xi_1,\pi,c) \in \Xi \times \mathcal{A}$*,*

$$J^{\xi_0,\xi_1,\pi,c}(u,z_1,z_2,\mathbf{z}_3) := E_{\mathbf{z}}\left[\int_u^T e^{-\int_u^t \rho(s)ds} Z_2(t)U(c(t))dt + e^{-\int_u^T \rho(s)ds} Z_2(T)U(Z_1(T)) \right] .$$

Then,

$$\begin{aligned} h(u,z_1,z_2,\mathbf{z}_3) &= \varphi(u,z_1,z_2,\mathbf{z}_3) \\ &= \inf_{(\xi_0,\xi_1)\in\Xi} \left(\sup_{(\pi,c)\in\mathcal{A}} J^{\xi_0,\xi_1,\pi,c}(u,z_1,z_2,\mathbf{z}_3) \right) \\ &= \sup_{(\pi,c)\in\mathcal{A}} \left(\inf_{(\xi_0,\xi_1)\in\Xi} J^{\xi_0,\xi_1,\pi,c}(u,z_1,z_2,\mathbf{z}_3) \right) \\ &= \inf_{(\xi_0,\xi_1)\in\Xi} J^{\xi_0,\xi_1,\hat{\pi},\hat{c}}(u,z_1,z_2,\mathbf{z}_3) \\ &= \sup_{(\pi,c)\in\mathcal{A}} J^{\hat{\xi}_0,\hat{\xi}_1,\pi,c}(u,z_1,z_2,\mathbf{z}_3) \\ &= J^{\hat{\xi}_0,\hat{\xi}_1,\hat{\pi},\hat{c}}(u,z_1,z_2,\mathbf{z}_3) \ , \end{aligned}$$

and $(\hat{\xi}_0,\hat{\xi}_1,\hat{\pi},\hat{c})$ *is a saddle point in the space of Markovian strategies of the game.*

Proof. Applying Lemma 5.3.1, the proof of Theorem 5.3.1 resembles that of Theorem 3.2 in Mataramvura and Øksendal (2008). So we do not repeat it here. □

Now we can re-state the conditions of Theorem 5.3.1 as follows:

$$\begin{cases} \inf_{(\xi_0,\xi_1)\in\Xi} \sup_{(\pi,c)\in\mathcal{A}} \{\mathcal{L}^{\xi_0,\xi_1,\pi,c}[\varphi(u,z_1,z_2,\mathbf{z}_3)] + z_2 U(c(u,z_1,z_2,\mathbf{z}_3))\} = 0 \ , \\ \varphi(T,z_1,z_2,\mathbf{z}_3) = z_2 U(z_1) \ . \end{cases} \tag{5.3.7}$$

Theorem 5.3.1 implies that the value function φ is a classical solution of the HJBI equation (5.3.7) and we only need to solve the HJBI equation (5.3.7) for solving the stochastic differential game (5.3.4) and (5.3.6) between the agent and the market. The following theorem gives the value function and the optimal Markovian control $(\hat{\xi}_0, \hat{\xi}_1, \hat{\pi}, \hat{c})$, which is the saddle-point of the game.

Theorem 5.3.2. *The saddle point $(\hat{\xi}_0, \hat{\xi}_1, \hat{\pi}, \hat{c})$ of the game described in (5.3.4) and (5.3.6) between the agent and the market is given by*

$$\phi(u) = \hat{\xi}_0(\mathbf{z})\sigma(u) + \int_{\Re_0} \hat{\xi}_1(\mathbf{z},y)\big(e^y - 1\big)\nu_{\mathbf{z}_3}(dy) \ , \tag{5.3.8}$$

$$\hat{\pi}(\mathbf{z}) = 0 \ , \tag{5.3.9}$$

and

$$\hat{c}(\mathbf{z}) = \begin{cases} \dfrac{z_1}{[P(u,\mathbf{z}_3)]^{\frac{1}{\gamma}}} \ , & \gamma > 0 \ , \gamma \neq 1 \ , \\ \dfrac{z_1}{Q(u,\mathbf{z}_3)} \ , & \gamma = 1 \ . \end{cases} \tag{5.3.10}$$

Furthermore, the value function is

$$\varphi(u,z_1,z_2,\mathbf{z}_3) = \begin{cases} P(u,\mathbf{z}_3) z_2 \dfrac{z_1^{1-\gamma}}{1-\gamma} \ , & \gamma > 0 \ , \gamma \neq 1 \ , \\ Q(u,\mathbf{z}_3) z_2 \log z_1 + R(u,\mathbf{z}_3) z_2 \ , & \gamma = 1 \ . \end{cases} \tag{5.3.11}$$

Here, for each $\mathbf{z}_3 \in \mathcal{E}$, the functions $P(u,\mathbf{z}_3)$, $Q(u,\mathbf{z}_3)$ and $R(u,\mathbf{z}_3)$ are assumed to be continuously differentiable with respect to u. Specifically, $P := P(u,\mathbf{z}_3) > 0$ is the *unique solution of the following Markovian regime-switching nonlinear ODE*

$$\begin{cases} \dfrac{dP}{du} + b(u)P + \gamma P^{1-\frac{1}{\gamma}} + \langle \mathbf{P}, \mathbf{A}\mathbf{z}_3 \rangle = 0 \ , \\ P(T, \mathbf{z}_3) = 1 \ , \end{cases} \tag{5.3.12}$$

where

$$b(u) := (1-\gamma)r(u) - \rho(u) \ ,$$

and

$$\mathbf{P} := \big(P(u, \mathbf{e}_1), P(u, \mathbf{e}_2), \cdots, P(u, \mathbf{e}_N)\big)' \in \Re^N \ .$$

$Q(u, \mathbf{z}_3)$ *and* $R(u, \mathbf{z}_3)$ *are given by the following explicit expressions*

$$\begin{aligned} Q(u, \mathbf{z}_3) \;\; = \;\; & \Big\langle \exp\big[(\mathbf{A} - \boldsymbol{\rho})(T-u)\big]\mathbf{1}_N \\ & + \int_u^T \exp\big[(\mathbf{A} - \boldsymbol{\rho})(t-u)\big]\mathbf{1}_N dt, \mathbf{z}_3 \Big\rangle \ , \end{aligned} \tag{5.3.13}$$

and

$$R(u, \mathbf{z}_3) = \Big\langle \int_u^T \exp\big[(\mathbf{A} - \boldsymbol{\rho})(t-u)\big]\mathbf{f}(t)dt, \mathbf{z}_3 \Big\rangle \ , \tag{5.3.14}$$

where

$$\begin{aligned} \mathbf{1}_N \;\; &:= \;\; (1, 1, \cdots, 1)' \in \Re^N \ , \\ \boldsymbol{\rho} \;\; &:= \;\; diag\big[(\rho_1, \rho_2, \cdots, \rho_N)'\big] \ , \end{aligned}$$

and

$$\begin{aligned} f_i(u) \;\; &:= \;\; r_i Q(u, \mathbf{e}_i) - \log(Q(u, \mathbf{e}_i)) - 1 \ , \\ \mathbf{f}(u) \;\; &:= \;\; (f_1(u), f_2(u), \cdots, f_N(u))' \in \Re^N \ . \end{aligned}$$

Proof. Denote by

$$\Psi(u, z_1, z_2, \mathbf{z}_3; \xi_0, \xi_1, \pi, c) = \mathcal{L}^{\xi_0,\xi_1,\pi,c}[\varphi(u, z_1, z_2, \mathbf{z}_3)] + z_2 U(c(u, z_1, z_2, \mathbf{z}_3)) \ .$$

Using the first order condition that minimizes Ψ with respect to (ξ_0, ξ_1), we have

$$\xi_0(\mathbf{z}) z_2^2 \varphi_{z_2 z_2} - \pi(\mathbf{z})\sigma(u) z_1 z_2 \varphi_{z_1 z_2} = 0 \ , \tag{5.3.15}$$

and

$$\begin{aligned}\int_{\Re_0} \Big[- z_2 \varphi_{z_2}\big(u, z_1(1 + \pi(\mathbf{z})(e^y - 1)), \\ z_2(1 - \xi_1(\mathbf{z}, y)), \mathbf{z}_3\big) + z_2 \varphi_{z_2}\Big] \nu_{\mathbf{z}_3}(dy) = 0 \ .\end{aligned} \tag{5.3.16}$$

Similarly, using the first order condition that maximizes Ψ with respect to (π, c), we have

$$\begin{aligned}&\phi(u) z_1 \varphi_{z_1} + \pi(\mathbf{z})\sigma^2(u) z_1^2 \varphi_{z_1 z_1} - \xi_0(\mathbf{z})\sigma(u) z_1 z_2 \varphi_{z_1 z_2} \\ &+ \int_{\Re_0} (e^y - 1) z_1 \Big[\varphi_{z_1}\big(u, z_1(1 + \pi(\mathbf{z})(e^y - 1)), z_2(1 - \xi_1(\mathbf{z}, y)), \mathbf{z}_3\big) - \varphi_{z_1}\Big] \nu_{\mathbf{z}_3}(dy) = 0 \ ,\end{aligned} \tag{5.3.17}$$

and

$$-\varphi_{z_1} + z_2 U_c(c(\mathbf{z})) = 0 \ . \tag{5.3.18}$$

1. $\gamma > 0$ and $\gamma \neq 1$

From the terminal condition of the HJBI equation (5.3.7), we try a solution of the following form:

$$\varphi(u, z_1, z_2, \mathbf{z}_3) = P(u, \mathbf{z}_3) z_2 \frac{z_1^{1-\gamma}}{1 - \gamma} \ , \tag{5.3.19}$$

where P is assumed to be continuously differentiable with respect to u.

Substituting (5.3.19) into (5.3.15) and (5.3.18) gives the optimal pair of portfolio and consumption rate processes

$$\hat{\pi}(\mathbf{z}) = 0 \ , \tag{5.3.20}$$

and

$$\hat{c}(\mathbf{z}) = \frac{z_1}{[P(u, \mathbf{z}_3)]^{\frac{1}{\gamma}}} \ . \tag{5.3.21}$$

Then substituting (5.3.19) and (5.3.20) into (5.3.17), we obtain the following equity premium equation

$$\phi(u) = \hat{\xi}_0(\mathbf{z})\sigma(u) + \int_{\Re_0} \hat{\xi}_1(\mathbf{z}, y)(e^y - 1)\nu_{\mathbf{z}_3}(dy) \ . \tag{5.3.22}$$

Note that the value function of the form (5.3.19) ensures that (5.3.16) always holds regardless of the value of $\xi_1(\mathbf{z}, y)$. Therefore, substituting (5.3.19)-(5.3.21) into (5.3.7) yields that $P(u, \mathbf{z}_3)$ satisfies the Markovian regime-switching nonlinear ODE (5.3.12). Equivalently, $P_i := P(u, \mathbf{e}_i)$, for each $i = 1, 2, \cdots, N$, satisfy the following system of ODEs

$$\begin{cases} \dfrac{dP_i}{du} + b_i P_i + \gamma P_i^{1-\frac{1}{\gamma}} + \langle \mathbf{P}, \mathbf{A}\mathbf{e}_i \rangle = 0 \ , \\ P(T, \mathbf{e}_i) = 1 \ , \end{cases} \tag{5.3.23}$$

where

$$b_i := (1 - \gamma) r_i - \rho_i \ .$$

Note that Eq. (5.3.23) is a system of N-coupled, nonlinear ODEs. In general, it is difficult, if not possible, to derive the closed-form solution. However, we could modify Lemma 3.2 in Pirvu and Zhang (2011) and prove the existence and uniqueness of a continuously differentiable solution P_i to (5.3.23), for each $i = 1, 2, \cdots, N$. Since the modification is trivial, we omit it here. In the sequel, we verify the positivity of P_i, for each $i = 1, 2, \cdots, N$. Rearranging (5.3.23) gives:

$$\frac{dP_i}{du} + b_i P_i + \gamma P_i^{1-\frac{1}{\gamma}} + a_{ii} P_i + \sum_{j \neq i}^{N} a_{ij} P_j = 0 \ .$$

Then

$$\frac{d[e^{-(b_i+a_{ii})u} P_i]}{du} + e^{-(b_i+a_{ii})u} \left[\gamma P_i^{1-\frac{1}{\gamma}} + \sum_{j \neq i}^{N} a_{ij} P_j \right] = 0 \ ,$$

which immediately leads to:

$$P(u, \mathbf{e}_i) = e^{(b_i+a_{ii})(T-u)} + \int_u^T e^{(b_i+a_{ii})(s-u)} \left[\gamma P^{1-\frac{1}{\gamma}}(s, \mathbf{e}_i) + \sum_{j \neq i}^{N} a_{ij} P(s, \mathbf{e}_j) \right] ds \quad (5.3.24)$$

Define

$$t_0^i := \sup\{t \in [0, T] | P(t, \mathbf{e}_i) \leq 0\} \ , \quad i = 1, 2, \cdots, N \ .$$

We denote by $t_0 := t_0^1 \vee t_0^2 \vee \cdots \vee t_0^N$. Recalling that $\sup \emptyset = -\infty$, we can see that the range of t_0 is $\{-\infty\} \cup [0, T]$. If $t_0 = -\infty$ (i.e. $t_0^1 = t_0^2 = \cdots = t_0^N = -\infty$), the positivity of P_i is satisfied. Otherwise, if $t_0 \in [0, T]$, we can find at least one $k \in \{1, 2, \cdots, N\}$ such that $t_0 = t_0^k \in [0, T]$. From $P(T, \mathbf{e}_i) = 1 > 0$, the continuity of P_i and the definition of t_0, we have that $P(u, \mathbf{e}_i) > 0$, for each $u \in (t_0, T]$ and $i = 1, 2, \cdots, N$. Furthermore, since $a_{kj} > 0$, $j \neq k$ and $\gamma > 0$, setting $u = t_0$ and $i = k$ on both sides of (5.3.24) yields that

$$0 \geq P(t_0, \mathbf{e}_k) = e^{(b_k+a_{kk})(T-t_0)} + \int_{t_0}^T e^{(b_k+a_{kk})(s-t_0)} \left[\gamma P^{1-\frac{1}{\gamma}}(s, \mathbf{e}_k) + \sum_{j \neq k}^{N} a_{kj} P(s, \mathbf{e}_j) \right] ds > 0 \ .$$

This is a contradiction. Therefore, we must have $t_0 = -\infty$ and the positivity of P_i is proved.

2. $\gamma = 1$

From the terminal condition of the HJBI equation (5.3.7), we try the solution of following form

$$\varphi(u, z_1, z_2, \mathbf{z}_3) = Q(u, \mathbf{z}_3) z_2 \log(z_1) + R(u, \mathbf{z}_3) z_2 \ , \tag{5.3.25}$$

where Q and R are assumed to be continuously differentiable with respect to u. Similarly, substituting (5.3.25) into (5.3.15) and (5.3.18) gives the optimal pair of portfolio and consumption rate processes

$$\hat{\pi}(\mathbf{z}) = 0 \ , \quad \hat{c}(\mathbf{z}) = \frac{z_1}{Q(u, \mathbf{z}_3)} \ . \tag{5.3.26}$$

Then substituting the optimal portfolio process $\hat{\pi}(\mathbf{z}) = 0$ in (5.3.26) and (5.3.25) into (5.3.17) leads to the equity premium equation

$$\phi(u) = \hat{\xi}_0(\mathbf{z})\sigma(u) + \int_{\Re_0} \hat{\xi}_1(\mathbf{z}, y)(e^y - 1)\nu_{\mathbf{z}_3}(dy) \ . \tag{5.3.27}$$

Note that the value function of the form (5.3.25) ensures that (5.3.16) always holds regardless of the value of $\xi_1(\mathbf{z}, y)$. To simplify our notation, write $Q := Q(u, \mathbf{z}_3)$, $R := R(u, \mathbf{z}_3)$, $Q_i := Q(u, \mathbf{e}_i)$, $R_i := R(u, \mathbf{e}_i)$, for each $i = 1, 2, \cdots, N$, and

$$\begin{aligned} \mathbf{Q} &:= (Q_1, Q_2, \cdots, Q_N)' \in \Re^N \ , \\ \mathbf{R} &:= (R_1, R_2, \cdots, R_N)' \in \Re^N \ . \end{aligned}$$

Substituting (5.3.25)-(5.3.26) into (5.3.7), we obtain

$$\begin{cases} \left[\dfrac{dQ}{du} - \rho(u)Q + \langle \mathbf{Q}, \mathbf{A}\mathbf{z}_3 \rangle + 1\right] z_2 \log(z_1) \\ + \left[\dfrac{dR}{du} - \rho(u)R + \langle \mathbf{R}, \mathbf{A}\mathbf{z}_3 \rangle + f(u)\right] z_2 = 0 \ , \\ Q(T, \mathbf{z}_3) = 1 \ , \quad R(T, \mathbf{z}_3) = 0 \ , \end{cases} \tag{5.3.28}$$

where

$$f(u) := r(u)Q - \log(Q) - 1 \ .$$

Setting the coefficients of $z_2 \log(z_1)$ and z_2 equal zeros, we obtain the following Markovian regime-switching ODEs:

$$\frac{dQ}{du} - \rho(u)Q + \langle \mathbf{Q}, \mathbf{A}\mathbf{z}_3 \rangle + 1 = 0 \ , \tag{5.3.29}$$

and

$$\frac{dR}{du} - \rho(u)R + \langle \mathbf{R}, \mathbf{A}\mathbf{z}_3 \rangle + f(u) = 0 \ . \tag{5.3.30}$$

Or equivalently,

$$\frac{dQ_i}{du} - \rho_i Q_i + \langle \mathbf{Q}, \mathbf{A}\mathbf{e}_i \rangle + 1 = 0 \ ,$$

and

$$\frac{dR_i}{du} - \rho_i R_i + \langle \mathbf{R}, \mathbf{A}\mathbf{e}_i \rangle + f_i(u) = 0 \ ,$$

for each $i = 1, 2, \cdots, N$.

Then we can rewrite (5.3.29) and (5.3.30) as the following matrix-valued ODEs:

$$\begin{cases} \dfrac{d\mathbf{Q}}{du} + (\mathbf{A} - \boldsymbol{\rho})\mathbf{Q} + \mathbf{1}_N = \mathbf{0}_N \ , \\ \mathbf{Q}(T) = \mathbf{1}_N \ , \end{cases}$$

and

$$\begin{cases} \dfrac{d\mathbf{R}}{du} + (\mathbf{A} - \boldsymbol{\rho})\mathbf{R} + \mathbf{f}(u) = \mathbf{0}_N \ , \\ \mathbf{R}(T) = \mathbf{0}_N \ , \end{cases}$$

where

$$\mathbf{0}_N := (0, 0, \cdots, 0)' \in \Re^N \ .$$

Therefore, the solutions of (5.3.29) and (5.3.30) are given by (5.3.13) and (5.3.14), respectively. $\square$

From Theorem 5.3.2, we can verify that the probability measure $\mathcal{Q}^{\hat{\xi}}$, where $\hat{\xi} = (\hat{\xi}_0, \hat{\xi}_1)$ is determined by Eq. (5.3.8), is an equivalent martingale measure. That is, the discounted share price process $\widetilde{S} := \{\widetilde{S}(t)|t \in \mathcal{T}\}$ is an $(\mathbb{F}, \mathcal{Q}^{\hat{\xi}})$-(local)-martingale, where

$$\widetilde{S}(t) := \exp\left\{-\int_0^t r(u)du\right\} S(t), \quad t \in \mathcal{T} .$$

Applying Itô's differentiation rule to $\Lambda^{\hat{\xi}}(t)\widetilde{S}(t)$, we have

$$\begin{aligned} \Lambda^{\hat{\xi}}(t)\widetilde{S}(t) - \Lambda^{\hat{\xi}}(0)\widetilde{S}(0) &= \int_0^t \big(\sigma(u) - \hat{\xi}_0(u)\big)\Lambda^{\hat{\xi}}(u)\widetilde{S}(u)dW(u) \\ &\quad + \int_0^t \int_{\Re_0} \big(e^y - 1 - e^y\hat{\xi}_1(u,y)\big)\Lambda^{\hat{\xi}}(u)\widetilde{S}(u)\widetilde{J}(du,dy) . \end{aligned}$$

Then the process $\{\Lambda^{\hat{\xi}}(t)\widetilde{S}(t)|t \in \mathcal{T}\}$ is an $(\mathbb{F}, \mathcal{P})$-(local)-martingale. By Lemma 7.2.2 in Elliott and Kopp (2004), we can see that $\widetilde{S}$ is indeed an $(\mathbb{F}, \mathcal{Q}^{\hat{\xi}})$-(local)-martingale.

5.3.2 Esscher transform

In this subsection, we adopt the approach considered in Elliott and Siu (2013) to find an equivalent martingale measure of the regime-switching Lévy model based on stochastic exponential (see also Bühlmann et al., 1997 and Kallsen and Shiryaev, 2002).

Let $L(Y)$ be the space of all processes $\theta := \{\theta(t)|t \in \mathcal{T}\}$ such that

1. For each $t \in \mathcal{T}$, $\theta(t) = \langle \boldsymbol{\theta}, \mathbf{X}(t)\rangle$, where $\boldsymbol{\theta} := (\theta_1, \theta_2, \cdots, \theta_N)' \in \Re^N$;

2. θ is integrable with respect to Y in the sense of stochastic integration.

In what follows, we call θ the Esscher transform parameter. For each $\theta \in L(Y)$, define an $\mathbb{F}$-adapted process $G^{\theta} := \{G^{\theta}(t)|t \in \mathcal{T}\}$ by

$$G^{\theta}(t) := \exp\biggl(- \int_0^t \theta(u) dY(u) \biggr) .$$

Applying Itô's differentiation rule to $G^{\theta}(t)$ under $\mathcal{P}$, we have

$$G^{\theta}(t) = 1 + \int_0^t G^{\theta}(u-) dH^{\theta}(u) ,$$

where $H^{\theta} := \{H^{\theta}(t)|t \in \mathcal{T}\}$ is an $\mathbb{F}$-adapted process defined by putting

$$\begin{aligned} H^{\theta}(t) \; := \; & \int_0^t \biggl[- \theta(u) \biggl(\mu(u) - \frac{1}{2}\sigma^2(u) - \int_{\Re_0} \bigl(e^y - 1\bigr) \nu_{\mathbf{X}(u)}(dy) \biggr) \\ & + \frac{1}{2}\theta^2(u)\sigma^2(u) + \int_{\Re_0} \bigl(e^{-\theta(u)y} - 1\bigr) \nu_{\mathbf{X}(u)}(dy) \biggr] du \\ & - \int_0^t \theta(u)\sigma(u) dW(u) + \int_0^t \int_{\Re_0} \bigl(e^{-\theta(u)y} - 1\bigr) \widetilde{J}(du, dy) . \end{aligned}$$

Consequently, G^{θ} is the Doléans-Dade stochastic exponential of H^{θ} under $\mathcal{P}$, i.e.

$$G^{\theta}(t) = \mathcal{E}(H^{\theta}(t)) .$$

Since H^{θ} is a special semi-martingale, its predictable part of finite variation is the Laplace cumulant process of $\{- \int_0^t \theta(u) dY(u)|t \in \mathcal{T}\}$ under $\mathcal{P}$. In other words, the Laplace cumulant process, denoted by $\mathcal{M}^{\theta} := \{\mathcal{M}^{\theta}(t)|t \in \mathcal{T}\}$, is given by

$$\begin{aligned} \mathcal{M}^{\theta}(t) \; := \; & \int_0^t \biggl[- \theta(u) \biggl(\mu(u) - \frac{1}{2}\sigma^2(u) - \int_{\Re_0} \bigl(e^y - 1\bigr) \nu_{\mathbf{X}(u)}(dy) \biggr) \\ & + \frac{1}{2}\theta^2(u)\sigma^2(u) + \int_{\Re_0} \bigl(e^{-\theta(u)y} - 1\bigr) \nu_{\mathbf{X}(u)}(dy) \biggr] du . \end{aligned} \tag{5.3.31}$$

Given that $\mathcal{M}^{\theta}$ is a finite variation process, the Doléans-Dade exponential $\mathcal{E}(\mathcal{M}^{\theta}(t))$ of $\mathcal{M}^{\theta}(t)$ is

$$\mathcal{E}(\mathcal{M}^{\theta}(t)) = \exp(\mathcal{M}^{\theta}(t)) .$$

So the logarithm transform $\widetilde{\mathcal{M}}^\theta := \{\widetilde{\mathcal{M}}^\theta(t)|t \in \mathcal{T}\}$ of $\mathcal{M}^\theta(t)$ follows

$$\widetilde{\mathcal{M}}^\theta(t) := \log(\mathcal{E}(\mathcal{M}^\theta(t))) = \mathcal{M}^\theta(t) \ . \tag{5.3.32}$$

Consider an $\mathbb{F}$-adapted process $\Lambda^\theta := \{\Lambda^\theta(t)|t \in \mathcal{T}\}$ associated with $\theta \in L(Y)$ defined by putting

$$\Lambda^\theta(t) := \exp\left(- \int_0^t \theta(u)dY(u) - \widetilde{\mathcal{M}}^\theta(t)\right) \ .$$

This is the generalized version of the Esscher density process, where the Laplace cumulant process is used to replace the moment generating function in the classical Esscher's measure change.

From (5.2.4) and (5.3.31)-(5.3.32), we obtain

$$\begin{aligned}\Lambda^\theta(t) \quad = \quad & \exp\bigg(- \frac{1}{2}\int_0^t \theta^2(u)\sigma^2(u)du - \int_0^t \int_{\Re_0} \big(e^{-\theta(u)y} - 1 + \theta(u)y\big)\nu_{\mathbf{X}(u)}(dy)du \\ & - \int_0^t \theta(u)\sigma(u)dW(u) - \int_0^t \int_{\Re_0} \theta(u)y\widetilde{J}(du, dy)\bigg) \ . \end{aligned} \tag{5.3.33}$$

Applying Itô's differentiation rule to $\Lambda^\theta(t)$ gives

$$\begin{aligned}\Lambda^\theta(t) \quad = \quad & 1 - \int_0^t \Lambda^\theta(u-)\theta(u)\sigma(u)dW(u) \\ & + \int_0^t \int_{\Re_0} \Lambda^\theta(u-)\big(e^{-\theta(u)y} - 1\big)\widetilde{J}(du, dy) \ . \end{aligned} \tag{5.3.34}$$

So Λ^θ is an $(\mathbb{F}, \mathcal{P})$-local-martingale. We suppose that the process θ is such that Λ^θ is an $(\mathbb{F}, \mathcal{P})$-martingale. Then $\mathrm{E}[\Lambda^\theta(T)] = 1$.

For each $\theta \in L(Y)$, we define a new probability measure $\mathcal{Q}^\theta$ equivalent to $\mathcal{P}$ on $\mathcal{F}(T)$ by the Radon-Nikodym derivative:

$$\left.\frac{d\mathcal{Q}^\theta}{d\mathcal{P}}\right|_{\mathcal{F}(T)} := \Lambda^\theta(T) \ .$$

From a version of the second fundamental theorem of asset pricing (see Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983)), the absence of arbitrage is 'essentially' equivalent to the existence of an equivalent martingale measure under which the discounted share price is a martingale. We provide a necessary and sufficient condition for the martingale condition.

Theorem 5.3.3. *The discounted share price process $\widetilde{S} = \{\widetilde{S}(t)|t \in \mathcal{T}\}$ is an $(\mathbb{F}, \mathcal{Q}^{\theta})$-martingale if and only if the Esscher transform parameter θ satisfies the following equation:*

$$\phi(u) = \theta(u)\sigma^2(u) + \int_{\Re_0} \big(1 - e^{-\theta(u)y}\big)\big(e^y - 1\big)\nu_{\mathbf{X}(u)}(dy) \ . \tag{5.3.35}$$

Proof. By Lemma 7.2.2 in Elliott and Kopp (2004), $\widetilde{S}$ is an $(\mathbb{F}, \mathcal{Q}^{\theta})$-martingale if and only if $\Lambda^{\theta}\widetilde{S} := \{\Lambda^{\theta}(t)\widetilde{S}(t)|t \in \mathcal{T}\}$ is an $(\mathbb{F}, \mathcal{P})$-martingale. Using Itô's product rule, we have

$$\begin{aligned}
&\Lambda^{\theta}(t)\widetilde{S}(t) - \Lambda^{\theta}(0)\widetilde{S}(0) \qquad\qquad (5.3.36)\\
&= \int_0^t \Lambda^{\theta}(u-)d\widetilde{S}(u) + \int_0^t \widetilde{S}(u-)d\Lambda^{\theta}(u) + \int_0^t d[\widetilde{S}(u), \Lambda^{\theta}(u)]^c + \sum_{0<u\leq t} \Delta\Lambda^{\theta}(u)\Delta\widetilde{S}(u)\\
&= \int_0^t \left[\phi(u) - \theta(u)\sigma^2(u) + \int_{\Re_0} \big(e^{-\theta(u)y} - 1\big)\big(e^y - 1\big)\nu_{\mathbf{X}(u)}(dy)\right]\Lambda^{\theta}(u)\widetilde{S}(u)du\\
&\quad + \int_0^t (1 - \theta(u))\sigma(u)\Lambda^{\theta}(u)\widetilde{S}(u)dW(u) + \int_0^t \int_{\Re_0} \big(e^{(1-\theta(u))y} - 1\big)\Lambda^{\theta}(u)\widetilde{S}(u)\widetilde{J}(du, dy) \ .
\end{aligned}$$

Then $\Lambda^{\theta}\widetilde{S}$ is an $(\mathbb{F}, \mathcal{Q}^{\theta})$-martingale if and only if the predictable part of finite variation in (5.3.36) is indistinguishable from the zero process. This leads to the desired result immediately. □

For each $\mathbf{e}_i \in \mathcal{E}$, $i = 1, 2, \cdots, N$, the Esscher transform parameter $\theta_i = \langle \boldsymbol{\theta}, \mathbf{e}_i \rangle$ satisfies the following equation

$$\phi_i = \theta_i\sigma_i^2 + \int_{\Re_0} (1 - e^{-\theta_i y})(e^y - 1)\nu_i(dy) \ . \tag{5.3.37}$$

Define a function $g_i : \Re \to \Re$ as follows

$$g_i(\theta_i) = \theta_i \sigma_i^2 + \int_{\Re_0} (1 - e^{-\theta_i y})(e^y - 1)\nu_i(dy) \ .$$

Clearly, g_i is a continuous and strictly increasing function and maps $(-\infty, +\infty)$ into $(-\infty, +\infty)$, since

$$\frac{dg_i}{d\theta_i} = \sigma_i^2 + \int_{\Re_0} e^{-\theta_i y} y(e^y - 1)\nu_i(dy) > 0 \ ,$$

and

$$\lim_{\theta_i \to -\infty} g_i(\theta_i) = -\infty \ , \quad \lim_{\theta_i \to +\infty} g_i(\theta_i) = +\infty \ .$$

Therefore, θ_i is uniquely determined by (5.3.37), for each $i = 1, 2, \cdots, N$.

Note that there is only one equation (5.3.8) or the martingale condition for $(\hat{\xi}_0, \hat{\xi}_1)$ in the stochastic differential game approach. Therefore, $(\hat{\xi}_0, \hat{\xi}_1)$ is not uniquely determined by (5.3.8). As in Siu (2008), if we impose the following additional conditions

$$\hat{\xi}_0(\mathbf{z}) = \lambda(u)\sigma(u) \ , \tag{5.3.38}$$

and

$$\hat{\xi}_1(\mathbf{z}, y) = 1 - e^{-\lambda(u)y} \ , \tag{5.3.39}$$

where $\lambda(u)$ is modulated by the chain as follows:

$$\lambda(u) := \langle \boldsymbol{\lambda}, \mathbf{X}(u) \rangle \ ,$$

and

$$\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \cdots, \lambda_N)' \in \Re^N \ ,$$

then Eq. (5.3.8) becomes

$$\phi(u) = \lambda(u)\sigma^2(u) + \int_{\Re_0} (1 - e^{-\lambda(u)y})(e^y - 1)\nu_{\mathbf{X}(u)}(dy) \ .$$

Equivalently, for each $i = 1, 2, \cdots, N$,

$$\phi_i = \lambda_i\sigma_i^2 + \int_{\Re_0} (1 - e^{-\lambda_i y})(e^y - 1)\nu_i(dy) \ . \tag{5.3.40}$$

Combining (5.3.37) and (5.3.40) gives

$$\phi_i = g_i(\theta_i) = g_i(\lambda_i) \ . \tag{5.3.41}$$

Since g_i is a strictly increasing function, we must have

$$\theta_i = \lambda_i \ , \quad i = 1, 2, \cdots, N \ , \tag{5.3.42}$$

and $\mathcal{Q}^{\theta}$ and $\mathcal{Q}^{\hat{\xi}}$ are identical.

5.3.3 General Equilibrium

In this subsection, we adopt the convention of the literature on equilibrium asset pricing models and consider the general equilibrium of a representative agent under the regime-switching Lévy model. Here we assume the existence of the representative agent and also suppose that the representative agent can trade in both the risk-free bond and the risky share and consume over the time horizon $\mathcal{T}$.

As in Subsection 3.1, we define a vector-valued controlled state process $\overline{\mathbf{Z}} := \{\overline{\mathbf{Z}}(t)|t \in \mathcal{T}\}$ of the agent by putting

$$\begin{aligned} d\overline{\mathbf{Z}}(t) &= (d\overline{Z}_0(t), d\overline{Z}_1(t), d\overline{\mathbf{Z}}_2(t))' \\ &= (d\overline{Z}_0(t), d\overline{Z}_1^{\pi,c}(t), d\overline{\mathbf{Z}}_2(t))' \\ &= (dt, dV^{\pi,c}(t), d\mathbf{X}(t))' \ , \end{aligned}$$

$$\overline{\mathbf{Z}}(0) \;=\; \overline{\mathbf{z}} = (u, \overline{z}_1, \overline{\mathbf{z}}_2)\ ,$$

where, under $\mathcal{P}$, the evolution of the components of the controlled state process $\overline{\mathbf{Z}}$ over time is governed by

$$\begin{aligned}
d\overline{Z}_0(t) &= dt\ ,\\
d\overline{Z}_1(t) &= \overline{Z}_1(t-)\Big[\big(r(t) + \pi(t)\phi(t)\big)dt + \pi(t)\sigma(t)dW(t)\\
&\qquad + \pi(t)\int_{\Re_0}\big(e^y - 1\big)\widetilde{J}(dt, dy)\Big] - c(t)dt\ ,\\
d\overline{\mathbf{Z}}_2(t) &= \mathbf{A}\overline{\mathbf{Z}}_2(t)dt + d\mathbf{M}(t)\ .
\end{aligned} \tag{5.3.43}$$

Let $\mathrm{E}_{\overline{\mathbf{z}}}[\cdot]$ denote an expectation under $\mathcal{P}$ given that $\overline{\mathbf{Z}}(0) = \overline{\mathbf{z}}$. The general equilibrium problem of the agent is to find an optimal portfolio-consumption pair $(\hat{\pi}, \hat{c}) \in \mathcal{A}$ so as to maximize the expected, discounted utility from intertemporal consumption and terminal wealth

$$\mathrm{E}_{\overline{\mathbf{z}}}\bigg[\int_u^T e^{-\int_u^t \rho(s)ds}U(c(t))dt + e^{-\int_u^T \rho(s)ds}U(V(T))\bigg]\ , \tag{5.3.44}$$

subject to the market clearing condition $\hat{\pi}(t) = 1$, for each $t \in \mathcal{T}$. So the value function of the general equilibrium problem is defined as:

$$\overline{\varphi}(u, \overline{z}_1, \overline{\mathbf{z}}_2) = \sup_{(\pi,c)\in\mathcal{A}} \mathrm{E}_{\overline{\mathbf{z}}}\bigg[\int_u^T e^{-\int_u^t \rho(s)ds}U(c(t))dt + e^{-\int_u^T \rho(s)ds}U(V(T))\bigg]\ . \tag{5.3.45}$$

Indeed, the general equilibrium problem of the representative agent is a stochastic control problem with the state processes and performance functional given by (5.3.43) and (5.3.44), respectively.

Let $\mathcal{D} := (0, T) \times (0, \infty)$ be our solvency region. Suppose that L_1 and L_2 denote the sets such that $\pi(t) \in L_1$ and $c(t) \in L_2$. As in Subsection 3.1, we consider Markovian controls and assume that

$$\pi(t) = \overline{\pi}(\overline{\mathbf{Z}}(t))\ ,\quad c(t) = \overline{c}(\overline{\mathbf{Z}}(t))\ ,$$

for some measurable functions $\overline{\pi} : \mathcal{D} \times \mathcal{E} \to L_1$ and $\overline{c} : \mathcal{D} \times \mathcal{E} \to L_2$. Again, to save notation, we do not distinguish $\overline{\pi}$ and $\overline{c}$ with π and c, respectively.

Let $\overline{\mathcal{H}}$ denote the space of functions $\overline{h}(\cdot,\cdot,\cdot) : \mathcal{D} \times \mathcal{E} \to \Re$ such that for each $\overline{\mathbf{z}}_2 \in \mathcal{E}$, $\overline{h}(\cdot,\cdot,\overline{\mathbf{z}}_2) \in \mathcal{C}^{1,2}(\mathcal{D})$. We write $\overline{h}_u$ and $\overline{h}_{\overline{z}_1}$ for the derivatives of $\overline{h}$ with respect to u and $\overline{z}_1$, and $\overline{h}_{\overline{z}_1\overline{z}_1}$ for the second order derivative of $\overline{h}$ with respect to $\overline{z}_1$. Denote by

$$\overline{\mathbf{h}}(u, z_1) := (\overline{h}(u, \overline{z}_1, \mathbf{e}_1), \overline{h}(u, \overline{z}_1, \mathbf{e}_2), \cdots, \overline{h}(u, \overline{z}_1, \mathbf{e}_N))' \in \Re^N .$$

To simplify our notation, we suppress $u, \overline{z}_1$ and $\overline{\mathbf{z}}_2$ and write $\overline{h}$ and $\overline{\mathbf{h}}$ for $\overline{h}(u, \overline{z}_1, \overline{\mathbf{z}}_2)$ and $\overline{\mathbf{h}}(u, \overline{z}_1)$, respectively, whenever no confusion arises. For each $(\pi, c) \in \mathcal{A}$, we define a regime-switching partial differential operator $\mathcal{L}^{\pi,c}$ acting on $\overline{h} \in \overline{\mathcal{H}}$ for the process $\overline{\mathbf{Z}}$ as:

$$\begin{aligned}
&\mathcal{L}^{\pi,c}[\overline{h}(u, \overline{z}_1, \overline{\mathbf{z}}_2)] \\
&= -\rho(u)\overline{h} + \overline{h}_u + \left[\left(r(u) + \pi(\overline{\mathbf{z}})\phi(u) - \pi(\overline{\mathbf{z}}) \int_{\Re_0} \left(e^y - 1\right)\nu_{\overline{\mathbf{z}}_2}(dy)\right) z_1 - c(\overline{\mathbf{z}})\right] \overline{h}_{\overline{z}_1} \\
&+ \frac{1}{2}\pi^2(\overline{\mathbf{z}})\sigma^2(u)\overline{z}_1^2 \overline{h}_{\overline{z}_1\overline{z}_1} + \int_{\Re_0} \left[\overline{h}(u, \overline{z}_1(1 + \pi(\overline{\mathbf{z}})(e^y - 1)), \overline{\mathbf{z}}_2) - \overline{h}\right]\nu_{\overline{\mathbf{z}}_2}(dy) + \left\langle \overline{\mathbf{h}}, \mathbf{A}\overline{\mathbf{z}}_2 \right\rangle .
\end{aligned}$$

As in Subsection 3.1, we need the following version of the Dynkin formula for regime-switching Lévy processes to develop a verification theorem for the HJB solution of the stochastic control problem (i.e. the general equilibrium problem).

Lemma 5.3.2. *Let τ be a stopping time such that $\tau < \infty$, $\mathcal{P}$-a.s.. Assume further that for each $(\pi, c) \in \mathcal{A}$, $\overline{h}(\overline{\mathbf{Z}}(t))$ and $\mathcal{L}^{\pi,c}[\overline{h}(\overline{\mathbf{Z}}(t))]$ are bounded on $t \in [0, \tau]$. Then,*

$$E_{\overline{\mathbf{z}}}[e^{-\int_u^\tau \rho(t)dt}\overline{h}(\overline{\mathbf{Z}}(\tau))] = \overline{h}(\overline{\mathbf{z}}) + E_{\overline{\mathbf{z}}}\left[\int_u^\tau e^{-\int_u^t \rho(s)ds}\mathcal{L}^{\pi,c}[\overline{h}(\overline{\mathbf{Z}}(t))]dt\right] .$$

Proof. Applying Itô's differentiation rule to $e^{-\int_u^t \rho(s)ds}\overline{h}(\overline{\mathbf{Z}}(t))$ and conditioning on $\overline{\mathbf{Z}}(0) = \overline{\mathbf{z}}$ lead to the desired result immediately. □

Let $\overline{\mathcal{D}}$ be the closure of $\mathcal{D}$. We now present the HJB solution to the stochastic control of the agent in the following verification theorem.

Theorem 5.3.4. *Suppose that, for each $\overline{\mathbf{z}}_2 \in \mathcal{E}$, there exists a function $\overline{h}(\cdot, \cdot, \overline{\mathbf{z}}_2) \in \mathcal{C}^{1,2}(\mathcal{D}) \cap \mathcal{C}(\overline{\mathcal{D}})$, and a Markov control $(\hat{\pi}, \hat{c}) \in \mathcal{A}$ such that:*

1. $\mathcal{L}^{\pi,c}[\overline{h}(u, \overline{z}_1, \overline{\mathbf{z}}_2)] + U(c(u, \overline{z}_1, \overline{\mathbf{z}}_2)) \leq 0$, *for all* $(\pi, c) \in \mathcal{A}$ *and* $(u, \overline{z}_1, \overline{\mathbf{z}}_2) \in \mathcal{D} \times \mathcal{E}$*;*

2. $\mathcal{L}^{\hat{\pi},\hat{c}}[\overline{h}(u, \overline{z}_1, \overline{\mathbf{z}}_2)] + U(\hat{c}(u, \overline{z}_1, \overline{\mathbf{z}}_2)) = 0$, *for all* $(u, \overline{z}_1, \overline{\mathbf{z}}_2) \in \mathcal{D} \times \mathcal{E}$*;*

3. *for all* $(\pi, c) \in \mathcal{A}$,
$$\lim_{u \to T^-} \overline{h}(u, \overline{Z}_1(u), \overline{\mathbf{Z}}_2(u)) = U(\overline{Z}_1(T)) \ ;$$

4. *let* $\overline{\mathcal{K}}$ *denote the set of stopping times* $\overline{\tau} \leq T$*. The family* $\{\overline{h}(\overline{\mathbf{Z}}(\overline{\tau}))\}_{\overline{\tau} \in \overline{\mathcal{K}}}$ *is uniformly integrable.*

Write, for each $\overline{\mathbf{z}} = (u, \overline{z}_1, \overline{\mathbf{z}}_2) \in \mathcal{D} \times \mathcal{E}$ *and* $(\pi, c) \in \mathcal{A}$,
$$\overline{J}^{\pi,c}(u, \overline{z}_1, \overline{\mathbf{z}}_2) := E_{\overline{\mathbf{z}}}\left[\int_u^T e^{-\int_u^t \rho(s)ds} U(c(t))dt + e^{-\int_u^T \rho(s)ds} U(\overline{Z}_1(T)) \right] .$$
Then,
$$\begin{aligned} \overline{h}(u, \overline{z}_1, \overline{\mathbf{z}}_2) &= \overline{\varphi}(u, \overline{z}_1, \overline{\mathbf{z}}_2) \\ &= \sup_{(\pi,c) \in \mathcal{A}} \overline{J}^{\pi,c}(u, \overline{z}_1, \overline{\mathbf{z}}_2) \\ &= \overline{J}^{\hat{\pi},\hat{c}}(u, \overline{z}_1, \overline{\mathbf{z}}_2) \ , \end{aligned}$$
and $(\hat{\pi}, \hat{c})$ is an optimal Markovian control.

Proof. Applying Lemma 5.3.2, the proof of Theorem 5.3.4 resembles that of Theorem 3.1 in Øksendal and Sulem (2007b). So we do not repeat it here. □

As in Subsection 3.1, we can re-state the conditions of Theorem 5.3.4 as follows:

$$\begin{cases} \sup_{(\pi,c)\in\mathcal{A}} \{\mathcal{L}^{\pi,c}[\overline{\varphi}(u,\overline{z}_1,\overline{\mathbf{z}}_2)] + U(c(u,\overline{z}_1,\overline{\mathbf{z}}_2))\} = 0 \ , \\ \overline{\varphi}(T,\overline{z}_1,\overline{\mathbf{z}}_2) = U(\overline{z}_1) \ . \end{cases} \tag{5.3.46}$$

Theorem 5.3.5. *In the stochastic control problem (5.3.43) and (5.3.45) of the representative agent, the equilibrium equity premium and the optimal consumption rate process are given by*

$$\phi(u) = \gamma\sigma^2(u) + \int_{\Re_0} \big(1 - e^{-\gamma y}\big)\big(e^y - 1\big)\nu_{\overline{\mathbf{z}}_2}(dy) \ , \tag{5.3.47}$$

and

$$\hat{c}(\overline{\mathbf{z}}) = \begin{cases} \dfrac{\overline{z}_1}{[\overline{P}(u,\overline{\mathbf{z}}_2)]^{\frac{1}{\gamma}}} \ , & \gamma > 0 \ , \gamma \neq 1 \ , \\ \dfrac{\overline{z}_1}{\overline{Q}(u,\overline{\mathbf{z}}_2)} \ , & \gamma = 1 \ . \end{cases} \tag{5.3.48}$$

Furthermore, the value function is

$$\overline{\varphi}(u,\overline{z}_1,\overline{\mathbf{z}}_2) = \begin{cases} \overline{P}(u,\overline{\mathbf{z}}_2)\dfrac{\overline{z}_1^{1-\gamma}}{1-\gamma} \ , & \gamma > 0 \ , \gamma \neq 1 \ , \\ \overline{Q}(u,\overline{\mathbf{z}}_2)\log\overline{z}_1 + \overline{R}(u,\overline{\mathbf{z}}_2) \ , & \gamma = 1 \ . \end{cases} \tag{5.3.49}$$

Here, for each $\mathbf{z}_3 \in \mathcal{E}$, the functions $\overline{P}(u,\overline{\mathbf{z}}_2)$, $\overline{Q}(u,\overline{\mathbf{z}}_2)$ and $\overline{R}(u,\overline{\mathbf{z}}_2)$ are assumed to be continuously differentiable with respect to u. Specifically, $\overline{P} := \overline{P}(u,\overline{\mathbf{z}}_2) > 0$ is the unique solution of the following Markovian regime-switching nonlinear ODE

$$\begin{cases} \dfrac{d\overline{P}}{du} + \overline{b}(u)\overline{P} + \gamma\overline{P}^{1-\frac{1}{\gamma}} + \big\langle \overline{\mathbf{P}}, \mathbf{A}\overline{\mathbf{z}}_2\big\rangle = 0 \ , \\ \overline{P}(T,\overline{\mathbf{z}}_2) = 1 \ , \end{cases} \tag{5.3.50}$$

where

$$\overline{b}(u) = -\rho(u) + (1-\gamma)r(u) + \frac{1}{2}(1-\gamma)\gamma\sigma^2(u) + \int_{\Re_0} \big[(1-\gamma)e^{-\gamma y} + \gamma e^{(1-\gamma)y} - 1\big]\nu_{\overline{\mathbf{z}}_2}(dy) \ ,$$

and

$$\overline{\mathbf{P}} := \left(\overline{P}(u,\mathbf{e}_1), \overline{P}(u,\mathbf{e}_2), \cdots, \overline{P}(u,\mathbf{e}_N)\right)' \in \Re^N .$$

$\overline{Q}(u,\overline{\mathbf{z}}_2)$ *and* $\overline{R}(u,\overline{\mathbf{z}}_2)$ *are given by the following explicit expressions*

$$\begin{aligned}\overline{Q}(u,\overline{\mathbf{z}}_2) \quad &= \quad \Big\langle \exp\left[(\mathbf{A}-\boldsymbol{\rho})(T-u)\right]\mathbf{1}_N \\ &\quad + \int_u^T \exp\left[(\mathbf{A}-\boldsymbol{\rho})(t-u)\right]\mathbf{1}_N dt, \overline{\mathbf{z}}_2 \Big\rangle ,\end{aligned} \tag{5.3.51}$$

and

$$\overline{R}(u,\overline{\mathbf{z}}_2) = \left\langle \int_u^T \exp\left[(\mathbf{A}-\boldsymbol{\rho})(t-u)\right]\overline{\mathbf{f}}(t)dt,\ \overline{\mathbf{z}}_2 \right\rangle , \tag{5.3.52}$$

where

$$\overline{\mathbf{f}}(u) := \left(\overline{f}(u,\mathbf{e}_1), \overline{f}(u,\mathbf{e}_2), \cdots, \overline{f}(u,\mathbf{e}_N)\right)' ,$$

with

$$\overline{f}(u,\mathbf{e}_i) := \left(r_i + \frac{1}{2}\sigma_i^2 - \int_{\Re_0} \left(1-y-e^{-y}\right)\nu_i(dy)\right)\overline{Q}(u,\mathbf{e}_i) - \log(\overline{Q}(u,\mathbf{e}_i)) - 1 ,$$

for each $i = 1, 2, \cdots, N$.

Proof. Since the proof of this theorem is similar to that of Theorem 5.3.2, we only present some key steps of derivations here. Using the first order condition for maximizing $\mathcal{L}^{\pi,c}[\overline{\varphi}(u,\overline{z}_1,\overline{\mathbf{z}}_2)] + U(c(u,\overline{z}_1,\overline{\mathbf{z}}_2))$ with respect to $(\pi,c) \in \mathcal{A}$ gives

$$\begin{aligned}&\left[\phi(u) - \int_{\Re_0} \left(e^y - 1\right)\nu_{\overline{\mathbf{z}}_2}(dy)\right]\overline{z}_1\overline{\varphi}_{\overline{z}_1} + \pi(\overline{\mathbf{z}})\sigma^2(u)\overline{z}_1^2\overline{\varphi}_{\overline{z}_1\overline{z}_1} \\ &+ \int_{\Re_0} \left[(e^y-1)\overline{z}_1\overline{\varphi}_{\overline{z}_1}(u,\overline{z}_1(1+\pi(\overline{\mathbf{z}})(e^y-1)),\overline{\mathbf{z}}_2)\right]\nu_{\overline{\mathbf{z}}_2}(dy) = 0 ,\end{aligned} \tag{5.3.53}$$

and

$$-\overline{\varphi}_{\overline{z}_1} + U_c(c(\overline{\mathbf{z}})) = 0 . \tag{5.3.54}$$

Applying the market clearing condition $\hat{\pi}(\overline{\mathbf{z}}) = 1$ to Eq. (5.3.53) implies that the equilibrium equity premium satisfies

$$\begin{aligned}\phi(t) &= -\frac{1}{\overline{\varphi}_{\overline{z}_1}}\left\{\sigma^2(u)\overline{z}_1\overline{\varphi}_{\overline{z}_1\overline{z}_1} + \int_{\Re_0}\left[(e^y-1)\overline{\varphi}_{\overline{z}_1}(u,\overline{z}_1e^y,\overline{\mathbf{z}}_2))\right]\nu_{\overline{\mathbf{z}}_2}(dy)\right\} \\ &\quad + \int_{\Re_0}\left(e^y-1\right)\nu_{\overline{\mathbf{z}}_2}(dy)\ . \end{aligned} \tag{5.3.55}$$

Then substituting $\hat{\pi}(\overline{\mathbf{z}}) = 1$ and (5.3.55) into (5.3.46) gives

$$\begin{aligned}&-\rho(u)\overline{\varphi} + \overline{\varphi}_u + r(u)\overline{z}_1\overline{\varphi}_{\overline{z}_1} - \frac{1}{2}\sigma^2(u)\overline{z}_1^2\overline{\varphi}_{\overline{z}_1\overline{z}_1} - \int_{\Re_0}\left[(e^y-1)\overline{z}_1\overline{\varphi}_{\overline{z}_1}(u,\overline{z}_1e^y,\overline{\mathbf{z}}_2)\right]\nu_{\overline{\mathbf{z}}_2}(dy) \\ &+\int_{\Re}\left[\overline{\varphi}(u,\overline{z}_1e^y,\overline{\mathbf{z}}_2)) - \overline{\varphi}(u,\overline{z}_1,\overline{\mathbf{z}}_2)\right]\nu_{\overline{\mathbf{z}}_2}(dy) + \langle\overline{\boldsymbol{\varphi}},\mathbf{A}\overline{\mathbf{z}}_2\rangle - \hat{c}(\overline{\mathbf{z}})\overline{\varphi}_{\overline{z}_1} + U(\hat{c}(\overline{\mathbf{z}})) = 0\ .\end{aligned} \tag{5.3.56}$$

Since the agent has a different type of utility functions when γ takes different values, we conjecture that the solution of (5.3.56) has the following parametric form

$$\overline{\varphi}(u,\overline{z}_1,\overline{\mathbf{z}}_2) = \begin{cases}\overline{P}(u,\overline{\mathbf{z}}_2)\dfrac{\overline{z}_1^{1-\gamma}}{1-\gamma}\ , & \gamma > 0\ ,\gamma\neq 1\ , \\ \overline{Q}(u,\overline{\mathbf{z}}_2)\log\overline{z}_1 + \overline{R}(u,\overline{\mathbf{z}}_2)\ , & \gamma = 1\ .\end{cases} \tag{5.3.57}$$

In what follows, we suppress u and $\overline{\mathbf{z}}_2$ and write Φ for $\Phi(u,\overline{\mathbf{z}}_2)$ whenever no confusion arises, where $\Phi := \overline{P}$, $\overline{Q}$ and $\overline{R}$. Furthermore, suppose that Φ is continuously differentiable with respect to u. To simplify our notation, denote by

$$\boldsymbol{\Phi} := \left(\Phi(u,\mathbf{e}_1),\Phi(u,\mathbf{e}_2),\cdots,\Phi(u,\mathbf{e}_N)\right)' \in \Re^N\ ,$$

where $\boldsymbol{\Phi} := \overline{\mathbf{P}}$, $\overline{\mathbf{Q}}$ and $\overline{\mathbf{R}}$.

Substituting (5.3.57) into the first order condition (5.3.54), we obtain the optimal consumption rate

$$\hat{c}(\overline{\mathbf{z}}) = \begin{cases}\dfrac{\overline{z}_1}{[\overline{P}(u,\overline{\mathbf{z}}_2)]^{\frac{1}{\gamma}}}\ , & \gamma > 0\ ,\gamma\neq 1\ , \\ \dfrac{\overline{z}_1}{\overline{Q}(u,\overline{\mathbf{z}}_2)}\ , & \gamma = 1\ .\end{cases} \tag{5.3.58}$$

Substituting (5.3.57)-(5.3.58) into (5.3.56), we obtain the following Markovian regime-switching ODEs:

1. $\gamma > 0$ *and* $\gamma \neq 1$

$$\begin{cases} \dfrac{d\overline{P}}{du} + \overline{b}(u)\overline{P} + \gamma\overline{P}^{1-\frac{1}{\gamma}} + \langle \overline{\mathbf{P}}, \mathbf{A}\overline{\mathbf{z}}_2 \rangle = 0 \ , \\ \overline{P}(T, \overline{\mathbf{z}}_2) = 1 \ . \end{cases} \tag{5.3.59}$$

2. $\gamma = 1$

$$\begin{cases} \left[\dfrac{d\overline{Q}}{du} - \rho(u)\overline{Q} + \langle \overline{\mathbf{Q}}, \mathbf{A}\overline{\mathbf{z}}_2 \rangle + 1\right] \log \overline{z}_1 \\ \quad + \left[\dfrac{d\overline{R}}{du} - \rho(u)\overline{R} + \langle \overline{\mathbf{R}}, \mathbf{A}\overline{\mathbf{z}}_2 \rangle + \overline{f}(u)\right] = 0 \ , \\ \overline{Q}(T, \overline{\mathbf{z}}_2) = 1 \ , \quad \overline{R}(T, \overline{\mathbf{z}}_2) = 0 \ , \end{cases} \tag{5.3.60}$$

where

$$\overline{f}(u) = \left(r(u) + \frac{1}{2}\sigma^2(u) - \int_{\Re_0} \big(1 - y - e^{-y}\big)\nu_{\overline{\mathbf{z}}_2}(dy) \right)\overline{Q} - \log(\overline{Q}) - 1 \ .$$

As in Subsection 3.1, we could follow Lemma 3.2 in Pirvu and Zhang (2011) to prove that (5.3.59) admits a unique continuously differentiable solution. Similarly, we can verify the positivity of this solution as in the proof of Theorem 5.3.2. In addition, the closed-form solutions of (5.3.60) are given by (5.3.51)-(5.3.52).

Consequently, substituting (5.3.57) into (5.3.55) gives the equilibrium equity premium

$$\phi(u) = \gamma\sigma^2(u) + \int_{\Re_0} \big(1 - e^{-\gamma y}\big)\big(e^y - 1\big)\nu_{\overline{\mathbf{z}}_2}(dy) \ . \tag{5.3.61}$$

□

From Theorem 5.3.5, if Eq. (5.3.47) holds, (i.e. the equity premium is in its equilibrium state), we can define a new probability measure $\mathcal{Q}^\gamma$

$$\frac{d\mathcal{Q}^\gamma}{d\mathcal{P}}\bigg|_{\mathcal{F}(T)} = \Lambda^\gamma(T) \ ,$$

where

$$\frac{d\Lambda^\gamma(t)}{\Lambda^\gamma(t-)} = -\gamma\sigma(t)dW(t) + \int_{\Re_0} (e^{-\gamma y} - 1)\widetilde{J}(dt, dy) \ .$$

It is easy to verify that $\mathcal{Q}^\gamma$ is also an equivalent (local)-martingale measure. Indeed, applying Itô's differentiation rule to $\Lambda^\gamma(t)\widetilde{S}(t)$ gives

$$\begin{aligned}
&\Lambda^\gamma(t)\widetilde{S}(t) - \Lambda^\gamma(0)\widetilde{S}(0) \\
&= \int_0^t (1-\gamma)\sigma(u)\Lambda^\gamma(u)\widetilde{S}(u)dW(u) + \int_0^t \int_{\Re_0} \left(e^{(1-\gamma)y} - 1\right)\Lambda^\gamma(u)\widetilde{S}(u)\widetilde{J}(du, dy) \ .
\end{aligned}$$

So $\{\Lambda^\gamma(t)\widetilde{S}(t) | t \in \mathcal{T}\}$ is an $(\mathbb{F}, \mathcal{P})$-(local)-martingale and thus $\{\widetilde{S}(t) | t \in \mathcal{T}\}$ is an $(\mathbb{F}, \mathcal{Q}^\gamma)$-(local)-martingale.

For each $\mathbf{e}_i \in \mathcal{E}$, the equilibrium equity premium for the representative agent is determined by

$$\phi_i = \gamma\sigma_i^2 + \int_{\Re_0} \left(1 - e^{-\gamma y}\right)\left(e^y - 1\right)\nu_i(dy) = g_i(\gamma) \ .$$

Comparing with the martingale condition (5.3.35) of the Esscher transform approach when the equity premium is in its equilibrium state, we obtain

$$\phi_i = g_i(\theta_i) = g_i(\gamma) \ . \tag{5.3.62}$$

Again, since g_i is a strictly increasing function, for each $i = 1, 2, \cdots, N$, we must have

$$\theta_1 = \theta_2 = \cdots = \theta_N = \gamma \ ,$$

and $\mathcal{Q}^\gamma$ and $\mathcal{Q}^\theta$ are identical. Note that if the market is in equilibrium, the Esscher transform parameter is state independent and is equal to the relative risk aversion coefficient in all states.

Remark 5.3.1. *Throughout this section, we have employed three different approaches to choose equivalent martingale measures for the valuation of contingent claims* *in a regime-switching Lévy model. The equivalent martingale measures determined by the stochastic differential game approach and the Esscher transform approach are preference-free. Under conditions (5.3.38)-(5.3.39), the two equivalent martingale measures $\mathcal{Q}^{\hat{\xi}}$ and $\mathcal{Q}^{\theta}$ are identical. So under conditions (5.3.38)-(5.3.39), the prices of any contingent claim under $\mathcal{Q}^{\hat{\xi}}$ and $\mathcal{Q}^{\theta}$ are the same regardless of the preference of the agent. However, the equivalent martingale measure determined by the general equilibrium approach is preference-dependent. The two equivalent martingale measures $\mathcal{Q}^{\theta}$ and $\mathcal{Q}^{\gamma}$ are identical only if the equity premium is in its equilibrium state. So the prices of any contingent claim under $\mathcal{Q}^{\theta}$ and $\mathcal{Q}^{\gamma}$ are the same from the representative agent's perspective.*

5.4 Conclusion

We considered three different approaches, namely, the stochastic differential game, the Esscher transform and the general equilibrium to choose equivalent martingale measures for the valuation of contingent claims under a regime-switching Lévy model. We identified the conditions under which these equivalent martingale measures are identical. Our results are important for pricing various contingent claims in finance and insurance under a general modeling framework, including both Poisson jumps and regime switches. In particular, the pricing framework of our regime-switching models can be used for the valuation of long-dated insurance products, where an economy usually experiences several structural changes in the life spans of these products.

Chapter 6

Conclusion

In this thesis, we have investigated several interesting applications of regime-switching models in finance. Specifically, we have discussed the valuation of European options and variance swaps under different versions of regime-switching models. Then we have considered a mean-variance portfolio selection problem with uncertain investment horizon under a regime-switching jump-diffusion model. Finally, we have applied the stochastic optimal control theory to study the selection of equivalent martingale measures under a regime-switching Lévy model.

Various possible extensions to this work remain. So far, we have only considered the valuation of vanilla options under regime-switching models. For path-dependent and American options, the closed-form pricing formulae for regime-switching models are difficult to derive. In the future research, we plan to derive the closed-form pricing formulae of path-dependent and American options under regime-switching models using the Laplace transform, which are convenient to be implemented via some standard inversion methods. These closed-form pricing formulae are important for the valuation of numerous insurance products with exotic options embedded.

Another potential research topic is portfolio optimization problems under regime-switching models with random coefficients, in the sense that model parameters may depend on not only the current state of but also the full path of the underlying Markov chain. Research on this topic is still at an infant stage. We shall use a BSDE approach to investigate the mean-variance portfolio selection, the investment-consumption and the optimal reinsurance problems under regime-switching models with random coefficients.

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1. Fan, K., **Shen, Y.**, Siu, T.K., Wang, R. (2014). Pricing foreign equity option with regime-switching. Economic Modelling 37, 296-305.

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4. **Shen, Y.**, Siu, T.K. (2013). The maximum principle for a jump-diffusion mean-field model and its application to the mean-variance problem. Nonlinear Analysis: Theory, Methods & Applications 86, 58-73.

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