Applications of Asymptotic methods
in
Quantitative Finance and Insurance

by
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DECLARATION

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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I dedicate this thesis to my family.

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SYNOPSIS

DESHPANDE, AMOGH. Applications of Asymptotic methods in Quantitative Finance and Insurance. (Under the direction of Tak Kuen Siu.)

This thesis [1] deals with three essays related to studying asymptotic behavior of a portfolio tail loss, asymptotic behavior of options price and asymptotic stability of a class of jump diffusion process.

In the first article that constitutes our first essay , we study an enhancement to the CreditRisk$^+$ model termed as the 2-stage CreditRisk$^+$. We determine under what conditions on the portfolio does the 2-stage CreditRisk$^+$ credit risk model gives higher Value at Risk than the CreditRisk$^+$. This entails studying rare event probability of large portfolio loss event. For the same we use technique from the theory of large deviations.

In the second article, we consider an asymptotic options pricing problem in a Markov modulated regime switching market. In such market, the key model parameters are modulated by a continuous-time, finite-state, Markov chain. Such a market is incomplete and hence there exist a range of options price. For an asymptotic analysis , we consider two variations of the chain, namely, a slow chain and a fast chain. It has been observed that there exists an asymptotic option price for the slow chain case while it is been argued that such price may not exist for the fast chain case. In this article, we attempt to show why this is so by determining the range of options price for the slow chain and the fast chain.

In the third and the last article, we consider a jump-diffusion process whose drift, diffusion and the jump kernel is modulated by a semi-Markov process. The semi-Markov process is a generalization over the Markov chain case since its sojourn time need not be exponentially distributed. We study the issue of asymptotic stability of this process with regards to almost sure and moment exponential sense. We study this issue here because of its motivational connection to the ruin theory in insurance. We also study the issue of stabilization and de-stabilization of a non-linear system of differential equation perturbed by a semi-Markov modulated jump diffusion process. We thereby comment on the interesting behaviour that we observe with regards to (de)-stabilization of the system of differential equation in one and in higher dimension.

[1]The thesis is in the format of thesis by publication. The guidelines for the same can be found on the webpage `http://www.businessandeconomics.mq.edu.au/current_students/higher_degree_research_students/guidelines/thesis_by_publication_guidelines`.

Chapter 1

Introduction

In this thesis[1] we focus on understanding and quantifying asymptotic phenomena observed in problems motivated from finance and insurance. The tools that we use to study them are derived from the large deviations theory , asymptotic perturbation analysis and asymptotic stability. We apply these to a problem in credit risk management, options pricing and insurance respectively. These three problems are motivated from the published works in Deshpande and Iyer [24], Basu and Ghosh[9] and Yin and Xi [84] respectively. The work on large deviations application to credit risk modeling has appeared in Deshpande [20]. The work on asymptotic stability that constitutes chapter 4 of this thesis has been published as Deshpande [21]. The central theme of this thesis is to understand and quantify the asymptotic behavior of the random phenomena observed in these three problems.

The thesis is organized as follows. Chapter 2 compares the Value at Risk performance of two competing credit risk models utilizing the tool of large deviations. Chapter 3 is based on studying existence of asymptotic options price . Chapter 4 is on understanding the asymptotic stability of semi-Markov modulated jump diffusion.

In this chapter, we first provide technical introduction to the large deviations theory followed by a brief literature survey of its applications to finance. We next detail the theoretical underpinnings related to the Markov modulated regime switching models which we shall use in Chapter 3 for options pricing and a brief literature survey of its application to finance. We conclude this chapter by presenting in brief, the theoretical background behind the

[1]The references occurring in this chapter and in the main text of Ch.2, Ch.3 and Ch.4 are detailed in a collective reference list towards the end of the thesis (refer p.82). References occurring inside the articles are mentioned in a reference list at the end of that respective article.

semi-Markov modulated jump diffusion and detail the previous study done on understanding asymptotic stability of random processes.

1.1 Concepts of Large deviations theory

Large deviations theory is a topic in probability that deals with the description of events in which a sum of random variables deviates from its mean by more than a "normal" amount, i.e. beyond what is described by the central limit theorem. Large deviations theory finds applications in probability theory, statistics, operations research, financial mathematics and many other areas. The estimation of the probability of rare events, amounting to an asymptotic expansion of the tails of the probability density functions of a given random variable or stochastic process, is the subject of this theory.

Consider generically the empirical means $\hat{T}_N = \frac{1}{N}\sum_{j=1}^{N} X_j$ for $\mathbb{R}$-valued random variables $X_1, ..., X_N, ...$ $\hat{T}_N$ is distributed according to the probability law $P_N \in M_1(\mathbb{R})$ i.e $\hat{T}_N =^d P_N$, where $M_1(\mathbb{R})$ denotes the space of all probability measures on $\mathbb{R}$. The large deviations principle characterizes the limiting behavior as $N \to \infty$ of $(P_N)_{N=1}^{\infty}$ on the space $(\mathbb{R}, B(\mathbb{R}))$ in terms of a (rate) function. This characterization is via the asymptotic upper or lower exponential bounds on the values that P_N assigns to measurable subsets of $\mathbb{R}$. We formally define it as follows.

Definition (Large deviations principle) $(P_N)_{N=1}^{\infty}$ *satisfies the Large Deviations Principle (LDP) with a rate function* Λ^* *if, for all* $\Gamma \subset \mathbb{R}$,

$$- \inf_{x \in \Gamma^0} \Lambda^*(x) \leq \liminf_{N \to \infty} \frac{1}{N} \log P_N(\Gamma) \leq \limsup_{N \to \infty} \frac{1}{N} \log P_N(\Gamma) \leq - \inf_{x \in \bar{\Gamma}} \Lambda^*(x),$$

where Γ^0 *is the interior of* Γ *and* $\bar{\Gamma}$ *is the closure of* Γ*. A rate function is a lower semicontinuous mapping* $\Lambda^* : \mathbb{R} \to [0, \infty]$*. It is a "good" rate function if the level sets* $\{x \in \mathbb{R} : \Lambda^*(x) \leq M\}$ *are compact for all* $M < \infty$*.*

An important result in Large deviations theory called Cramer's theorem considers an LDP setup for X_j for $j \in \{1, ..., N\}$ being i.i.d. $\mathbb{R}$-valued random variables with (speed) $\frac{1}{N}$. Another fundamental result, the Gärtner–Ellis theorem covers the case when X_j for $j \in \{1, ..., N\}$ is non-i.i.d. Refer to (Dembo and Zeitouni,[18]) for an accessible introduction to large deviations theory.

We state here the Gärtner–Ellis theorem. We utilize it in our first article while we try understanding rare large portfolio loss values.

Assumption 1
1. We represent the moment-generating function for the sequence (X_n) as $\phi_n(t) = \mathbb{E}e^{\langle t,X_n \rangle}$ $t \in \mathbb{R}^d, n \in \mathbb{N}$ and let $\Lambda \triangleq \lim_{n\to\infty} \frac{1}{n} \log \phi_n(t)$ exists.
2. $0 \in int(\mathcal{D}_\Lambda)$, with $\mathcal{D}_\Lambda = \{t \in \mathbb{R}^d : \Lambda(t) < \infty\}$.
Theorem (Gärtner–Ellis theorem) *Let Assumption 1 be valid. Let $P_n(\cdot) = \mathbb{P}(X_n \in \cdot)$. $(P_n)_{n=1}^\infty$ satisfies the LDP with rate function Λ^*. In other words,*
(a) For every closed set $C \subset \mathbb{R}^d$,

$$\limsup_{n\to\infty} \frac{1}{n} \log P_n(C) \le -\Lambda^*(C);$$

(b) and for every open set $O \subset \mathbb{R}^d$,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_n(O) \ge -\Lambda^*(O \cap E).$$

where $E = E(\Lambda, \Lambda^*)$ is the set of exposed points of Λ^* whose exposing hyperplane belongs to $int(\mathcal{D}_\Lambda)$
(c) Suppose, in addition, that Λ satisfies:
(1) Λ is lower semi-continuous on $\mathbb{R}^d$
(2) Λ is differentiable on $int(\mathcal{D}_\Lambda)$
(3) Either $(\mathcal{D}_\Lambda) = \mathbb{R}^d$or Λ is steep at $\partial\mathcal{D}_\Lambda$, i.e., $\lim_{t\to\partial\mathcal{D}_\Lambda : t\in\mathcal{D}_\Lambda} | \bigtriangledown |\Lambda(t) = \infty$
Then $O \cap E$ may be replaced by O in the RHS of (b). Consequently $(P_n)_{n=1}^\infty$ satisfies LDP on $\mathbb{R}^d$ with rate n and with rate function Λ^*.
Proof We sketch the proof here. Refer to den Hollander [64] for details.
Upper bound We begin with compact sets. Pick an arbitrary $\delta > 0$. For $x \in \mathbb{R}^d$ define

$$\Lambda^*_\delta(x) \triangleq \min\{\Lambda^*(x) - \delta, \frac{1}{\delta}\}.$$

For every $x \in \mathbb{R}^d$ there exists $t_x \in \mathbb{R}^d$ such that

$$\langle x, t_x \rangle - \Lambda(t_x) \ge \Lambda^*_\delta(x).$$

Moreover, for every $x \in \mathbb{R}^d$, there exists a neighbourhood A_x of x such that

$$\inf_{y\in A_x} \langle y - x, t_x \rangle \ge -\delta.$$

By the exponential Chebyshev's inequality we therefore have

$$\begin{aligned}
P_n(A_x) &= \mathbb{P}(X_n \in A_x) \\
&\leq \mathbb{P}(\langle X_n - x, t_x \rangle \geq -\delta) \\
&\leq e^{\delta n}\mathbb{E}[e^{n\langle X_n - x, t_x\rangle}] \\
&= e^{\delta n}\phi_n(nt_x)e^{-n\langle x, t_x\rangle}.
\end{aligned}$$

Let $K \subset \mathbb{R}^d$ be compact. Then the covering of $\cap_{x\in K} A_x$ of K has a finite subcovering $\cap_{i=1,\ldots,N} A_{x_i}$ and by the property that the largest exponent wins we have

$$\begin{aligned}
\frac{1}{n}\log P_n(K) &\leq \frac{1}{n}\log[N \max_{i=1,\ldots,N} P_n(A_{x_i})] \\
&\leq \frac{1}{n}\log N + \delta - \min_{i=1,\ldots,N}[\langle x_i, t_{x_i}\rangle - \frac{1}{n}\log\phi_n(nt_{x_i})].
\end{aligned}$$

As $n \to \infty$ and $\delta \downarrow 0$,

$$\limsup_{n\to\infty} \frac{1}{n}\log P_n(K) \leq -\Lambda^*(K).$$

The extension from compact sets to closed sets amounts to showing the exponential tightness of $(P_n)_{n=1}^{\infty}$, which then concludes the proof for the upper bound.

Lower bound Let $\ni$ be the symbol for "such that". Let $B_\epsilon(x)$ denote the open ball of radius ϵ around x. It suffices to prove the following.

$$\lim_{\epsilon\downarrow 0}\liminf_{n\to\infty} \frac{1}{n}\log P_n(B_\epsilon(x)) \geq -\Lambda^*(x) \quad x \in E.$$

Indeed, for any open set O we have,

$$P_n(O) \geq P_n(B_\epsilon(x)) \quad \forall \;\; x \ni (O \cap E) \; \forall \; \epsilon \leq \epsilon_0(x),$$

and so the claim follows after letting $n \to \infty$ and $\epsilon \downarrow 0$, and optimising over $x \in O \cap E$. Fix $x \in E$ and let $\tau \in int(\mathcal{D}_\Lambda)$ where $\mathcal{D}_\Lambda = \{t \in \mathbb{R}^d : \Lambda(t) < \infty\}$ be an exposing hyperplane for x. Then $\phi_n(n\tau) < \infty$ for n large enough and so we can define a tilted probability measure $\hat{P}_n$ by writing

$$\frac{d\hat{P}_n}{dP_n}(y) = \frac{e^{n\langle y, \tau\rangle}}{\phi_n(n\tau)} \quad y \in \mathbb{R}^d.$$

Thus we have

$$\begin{aligned}\frac{1}{n}\log P_n(B_\epsilon(x)) &= \frac{1}{n}\log\int_{B_\epsilon(x)} P_n(dy) \\ &= \frac{1}{n}\log\phi_n(n\tau)+\frac{1}{n}\log\int_{B_\epsilon(x)} e^{-n\langle y,\tau\rangle}\hat{P}_n(dy) \\ &\geq \frac{1}{n}\log\phi_n(n\tau)-\langle x,\tau\rangle-\epsilon|\tau|+\frac{1}{n}\log\hat{P}_n(B_\epsilon(x)),\end{aligned}$$

where the last inequality uses the fact that $|y-x|\leq\epsilon$ for $y\in B_\epsilon(x)$. Hence

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\liminf_{n\to\infty}\frac{1}{n}\log P_n(B_\epsilon(x)) &\geq [\Lambda(\tau)-\langle x,\tau\rangle]+\lim_{\epsilon\downarrow 0}\liminf_{n\to\infty}\frac{1}{n}\log\hat{P}_n(B_\epsilon(x)) \\ &= -\Lambda^*(x)+\lim_{\epsilon\downarrow 0}\liminf_{n\to\infty}\frac{1}{n}\log\hat{P}_n(B_\epsilon(x)).\end{aligned}$$

To end the proof of the lower bound, we show that

$$\lim_{\epsilon\downarrow 0}\liminf_{n\to\infty}\frac{1}{n}\log\hat{P}_n(B_\epsilon(x))=0,$$

followed by proving that $\Lambda^*(O\cap E)=\Lambda^*(O)$. $\square$

1.1.1 Literature overview of the applications of large deviations theory in finance.

Large deviations theory as described above is an asymptotic approach to determine the probability of rare events. It has been applied to important problems in finance where occurrence of rare events play a crucial role. Its significance can be gauged from the current global financial crisis, which was triggered by the collapse of Lehman Brothers in 2009. The shockwave generated by this collapse affected many financial companies and banks. The failure of such big corporations, which were perceived to be in quite robust financial health, is therefore a rare event scenario. If hedged for such worst-case scenarios, such drastic situations may not have arisen. This elucidates the need for exhaustive application of large deviations theory to problems that involve computation of the probabilities of very rare events causing large portfolio losses. The existing body of work in this work is small since applications of large deviations to problems in finance are fairly recent.

Monte Carlo simulation is commonly used as a tool to price options [10], in which the importance sampling technique critically enhances the simulation's performance. The basic idea of the importance sampling technique is to reduce the variance of the estimator that denotes the options price by changing the probability measure from which price paths are generated. More specifically, the idea is to change the distributions of similar price processes by taking the specifications of the payoff function into account and deriving the process from the region of high contribution to the required expectation. An interesting approach to optimally change the probability measure is by a large deviations approximation of the required expectation . Refer [39] for details.

The application of large deviations is seen while determining the first-passage probability observed in barrier options pricing. Baldi et al. [7] provides a sharp large deviations estimate for the first-passage probability. Large deviations provide a powerful tool for describing the limiting behavior of implied volatilities. Various asymptotics (small time, large time, fast mean-reverting, extreme strike) for stochastic volatility have been studied in Feng et al. [32], Forde and Jacquier[33] and Tehranchi [76].

Large deviations have also found application in the optimal long-term investment problem, where investors are interested in maximizing the probability that their wealth exceeds a predetermined index. Stutzer [57] considers an asymptotic version of this out performance criterion when the time horizon goes to infinity, which leads to a large deviations portfolio criterion.

The application of large deviations to credit risk management is also relatively new, although it has a natural appeal in helping us compute the rare event probability of a large loss to a portfolio. Dembo et al.[19] first suggested that generically, rare events are exponentially rare in the dimension of the portfolio. In other words, the tail of the loss distribution of the portfolio decays at the rate of $e^{-\lambda N}$ where N is the number of assets in the portfolio and λ is some positive constant. The same conclusion was obtained in a recent work by Glasserman [40]. A fairly new application of large deviations is in the study of rare event analysis for Collateralized Debt Obligations (CDOs). These financial instruments provide ways of aggregating risk from a large number of sources like bonds and reselling it in a number of parts, each part having different risk–reward characteristics. The Financial Crisis Inquiry Commission (FCIC), the US congressional panel mandated to scrutinize the 2009 financial crisis deliberated the role CDOs played in the crisis. Interesting reasonings that became pertinent during the hearing were the surge in transactions of CDOs, the timing of these transactions, inaccuracies defining key CDO parameters (e.g. maturity, degree of diversification, average rating), erroneous risk models and perhaps inadequate monetary-policy expectations that created faulty scenarios. Analysis that could have connected such faulty scenarios with rare large event scenarios could perhaps help provide robust strategies that financial companies dealing with CDO's could employ in the future. An article by Sowers [55] analyzed rare events related to losses in senior tranches of CDOs. In chapter 2 we use large deviations to compare the Value at Risk performance of the two competing credit risk models, the CreditRisk$^+$ and its recent enhancement called the 2-stage CreditRisk$^+$.

We next detail what do we mean by Markov modulated diffusion process and follow it up with its application in finance. As said earlier, utilize this process in our second article that constitutes chapter 3 wherein we determine the existence of an asymptotic options price of a European option in a Markov-modulated regime-switching economy.

1.2 Markov-modulated regime-switching models

We consider a simplified continuous-time financial market consisting of two primitive securities, namely, a (locally) risk-free bond and a risky share. These securities can be traded continuously over time in a finite time horizon $\mathcal{T} := [0, T]$, where $T < \infty$. As usual, we suppose that there are no transaction costs and taxes, that any fractional units of the securities can be traded, and that the borrowing and lending rates are the same. To describe uncertainty, we consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability measure. Let $\boldsymbol{\theta} := \{\boldsymbol{\theta}(t) | t \in \mathcal{T}\}$ be a continuous-time, finite-state, Markov chain on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with state space $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_M\} \subset \Re^M$, where the j^{th}-component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij} for each $i, j = 1, 2, \cdots, M$. The state space $\mathcal{E}$ is called a canonical state space for the chain θ. We suppose that the Markov chain is homogeneous and irreducible. To specify the probability law of the chain, we define a rate matrix, or an intensity matrix, $\mathbf{\Lambda} := [\lambda_{ij}]_{i,j=1,2,\cdots,M}$, where λ_{ij} is the constant transition intensity of the chain θ from state $\mathbf{e}_i$ to state $\mathbf{e}_j$. Note that for each $i, j = 1, 2, \cdots, M$ with $i \neq j$, $\sum_{j=1}^{M} \lambda_{ij} = 0$ and $\lambda_{ij} \geq 0$, so $\lambda_{ii} \leq 0$. Let $\mathbb{F}^{\boldsymbol{\theta}} := \{\mathcal{F}^{\boldsymbol{\theta}}(t) | t \in \mathcal{T}\}$ be the right-continuous, $\mathcal{P}$-completed, filtration generated by the values of the chain $\boldsymbol{\theta}$. We now provide the proof for the semi-martingale dynamics of the Markov chain provided in Elliott et al. [25].

Theorem $\mathbf{V}(t)$ *defined by the following equation is an $\mathbb{R}^M$-valued, square-integrable $(\mathbb{F}^{\boldsymbol{\theta}}, \mathcal{P})$-martingale.*

$$\mathbf{V}(t) := \boldsymbol{\theta}(t) - \boldsymbol{\theta}(0) - \int_0^t \mathbf{\Lambda}\boldsymbol{\theta}(u-)du \ , \quad t \in \mathcal{T} \ .$$

Hence the semi-martingale dynamics of the Markov chain are given by

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + \int_0^t \mathbf{\Lambda}\boldsymbol{\theta}(u)du + \mathbf{V}(t) \ , \quad t \in \mathcal{T}.$$

Proof Let the state space for the Markov chain $\{\boldsymbol{\theta}(t), t \geq 0\}$ be denoted by the set S=$\{e_1, ..., e_M\}$. Write $p_t^i = \mathcal{P}(\boldsymbol{\theta}(t) = e_i), 0 \leq i \leq M$. We shall suppose that for some family of matrices $\mathbf{\Lambda}$, $p_t = (p_t^1, ..., p_t^N)'$ satisfies the forward Kolmogorov equation

$$\frac{dp_t}{dt} = \mathbf{\Lambda} p_t.$$

$\{\mathbf{\Lambda}\}$, is, therefore, the family of so-called Q matrices of the process. The fundamental transition matrix associated with $\boldsymbol{\theta}$ will be denoted by $\Phi(t,s)$, so with I the $M \times M$ identity matrix we have

$$\frac{d\Phi(t,s)}{dt} = \mathbf{\Lambda}\Phi(t,s), \quad \Phi(s,s) = I \tag{1}$$

$$\frac{d\Phi(t,s)}{dt} = -\Phi(t,s)\mathbf{\Lambda} \quad \Phi(t,t) = I. \tag{2}$$

Consider the process in state, say, $i \in \mathcal{X}$ at time s and write $\boldsymbol{\theta}_{(s,t)}(e_i)$ for its state at the later time $t \geq s$. Then $E[\boldsymbol{\theta}_{(s,t)}(i)] = E_{s,i}[\boldsymbol{\theta}(t)] = \Phi(t,s)e_i$. Defining $\mathbf{V}(t) := \boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + \int_0^t \mathbf{\Lambda}\boldsymbol{\theta}(u-)du\ , \quad t \in \mathcal{T}$, we now need to show that it is an $\mathcal{F}_t = \sigma\{\boldsymbol{\theta}(s), 0 \leq s \leq t\}$ martingale. We do so in the following way.

$$\begin{aligned}
E[V_t - V_s|\mathcal{F}_s] &= E[\boldsymbol{\theta}(t) - \boldsymbol{\theta}(s) - \int_s^t \mathbf{\Lambda}\boldsymbol{\theta}(u-)du|\mathcal{F}_s] \\
&= E[\boldsymbol{\theta}(t) - \boldsymbol{\theta}(s) - \int_s^t \mathbf{\Lambda}\boldsymbol{\theta}(u-)du|\boldsymbol{\theta}(s)] \\
&= E[\boldsymbol{\theta}(t) - \boldsymbol{\theta}(s) - \int_s^t \mathbf{\Lambda}\boldsymbol{\theta}(u)du|\boldsymbol{\theta}(s)].
\end{aligned}$$

Because $\boldsymbol{\theta}(u) = \boldsymbol{\theta}(u-) = \lim_{\epsilon>0,\epsilon\to 0} \boldsymbol{\theta}(u-\epsilon)$ for each $\omega \in \Omega$, except for countably many u, this is

$$\begin{aligned}
E[V_t - V_s|\mathcal{F}_s] &= E[\boldsymbol{\theta}(t)|\boldsymbol{\theta}(s)] - \boldsymbol{\theta}(s) - \int_s^t \mathbf{\Lambda} E[\boldsymbol{\theta}(u)|\boldsymbol{\theta}(s)]du \\
&= \Phi(t,s)\boldsymbol{\theta}(s) - \boldsymbol{\theta}(s) - \int_s^t \mathbf{\Lambda}\Phi(u,s)du = 0
\end{aligned}$$

from above. This gives

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + \int_0^t \mathbf{\Lambda}\boldsymbol{\theta}(u)du + \mathbf{V}(t)\ , \quad t \in \mathcal{T}.$$

□

A European option price governed by a Markov modulated stock price dynamics is well known to satisfy a system of partial differential equations with weak coupling. Refer Deshpande and Ghosh [23] for details. We will now formulate this partial differential equation, that is satisfied by a European call option in a Markov-modulated regime-switching economy. However before we do so, for the sake of completeness, we describe first a purely probabilistic approach to pricing European-style options payoff in a Markov-modulated regime-switching economy. This discussion is based on the work of Elliott et al. (2005)[26]. In their work they provide option valuations based on the joint characteristic function of the occupation times of the Markov chain. In this approach they consider that the market interest rate, the appreciation rate and the volatility of the underlying risky asset, depend on unobservable states of the economy which are modeled by a continuous-time Hidden Markov process X_t. Mathematically, $r_t := \langle r, X_t \rangle, \mu_t := \mu(t, X_t) = \langle \mu, X_t \rangle, \sigma_t := \sigma(t, X_t) = \langle \sigma, X_t \rangle$. States are as usual denoted by $i \in \{1, ..., M\}$. Define $Z_t = (\mu_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t$. The filtration generated by Z_t is denoted by $\mathcal{F}_t^Z$. Let $J_i(t, T)$ denote the occupation time of $\{X_t\}_{t \in (t,T]}$ in state i over the duration $[t, T]$. Then,

$$P_{t,T} = \int_t^T \langle r, X_s \rangle \, ds = \sum_{i=1}^{M} r_i J_i(t, T)$$

$$U_{t,T} = \int_t^T \langle \sigma, X_s \rangle^2 \, ds = \sum_{i=1}^{M} \sigma_i^2 J_i(t, T).$$

At any time $t \leq T$, the price of a European-style option written on a Markov-modulated stock price dynamics S with payoff $V(S_T)$ at maturity T for the underlying filtration $\mathcal{G}_t$ generated by $\{X_t, S_t, t \leq T\}$ is given by

$$V(t, T, S_t, P_{t,T}, U_{t,T}) = E^{\mathcal{Q}}[\exp(-\int_t^T r_s ds)V(S_T)|\mathcal{G}_t],$$

where $E^{\mathcal{Q}}$ is the expectation computed under the martingale probability measure $\mathbb{P}^{\mathcal{Q}}$. Since $P_{t,T}$ and $U_{t,T}$ are unknown in practice, the price $V(t, T, S_t, P_{t,T}, U_{t,T})$ is also unknown. As in Buffington and Elliott [12], one can take a second expectation of $V(t, T, S_t, P_{t,T}, U_{t,T})$ with respect to the probability distributions of $P_{t,T}$ and $U_{t,T}$, which can be interpreted as a statistical estimation of the unobservable price $V(t, T, S_t, P_{t,T}, U_{t,T})$, given observable market information. Indeed to determine the distribution of $P_{t,T}$ and $U_{t,T}$ one needs to determine the joint distribution of the occupation times $J(t, T) := (J_1(t, T), J_2(t, T), ..., J_M(t, T))$. Let

D denote a diagonal matrix consisting of the elements in the vector $\xi := (\xi_1, \xi_2, ..., \xi_M)$ as its diagonal. Then for any ξ, the characteristic function of $J(t,T)$ is given by

$$E[\exp(i \langle \xi, J(t,T) \rangle) | \mathcal{F}_t^Z] = \langle \exp[(A + iD)(T-t)] X_t, \mathbf{I} \rangle \,, \tag{3}$$

where $i = \sqrt{-1}$ and $\mathbf{I} := (1, 1, ..., 1) \in \mathbb{R}^M$. Let $\phi(J_1, J_2, ..., J_M)$ denote the joint probability distribution for the occupation times $(J_1(t,T), J_2(t,T), ..., J_M(t,T))$.This joint probability distribution can be completely determined by the characteristic function $E[\exp(i \langle \xi, J(t,T) \rangle) | \mathcal{F}_t^Z]$. Let Φ be the cumulative distribution function for the standard normal distribution. Hence the price of a European call option at time t with strike price K and maturity time T in the purely probabilistic format is given by

$$V(t, T, S_t, P_{t,T}, U_{t,T}) = \int_t^T \int_t^T \cdots \int_t^T V(t, T, S_t, P_{t,T}, U_{t,T}) \phi(J_1, J_2, ..., J_M) dJ_1 dJ_2 ... dJ_M, \tag{4}$$

where

$$V(t, T, S_t, P_{t,T}, U_{t,T}) = S_t \Phi(d_{1,t,T}) - K \exp(-P_{t,T}) \Phi(d_{2,t,T}) \tag{5}$$

and

$$d_{1,t,T} = (U_{t,T})^{-1/2} (\ln \frac{S_t}{K} + P_{t,T} + \frac{1}{2} U_{t,T}), \quad d_{2,t,T} = d_{1,t,T} - (U_{t,T})^{1/2}. \tag{6}$$

This concludes our discussion on the pure probabilistic approach. We come back to formulate Black-Scholes PDE for Markov-modulated regime switching diffusion that is satisfied by the European Options price. In this discussion we refer to the chain by $\boldsymbol{\theta}$ and to the stock price by S_t for $t \in \mathcal{T}$. Let $r(t, \boldsymbol{\theta}(t))$ be the (locally) risk-free rate of interest of the bond at time t. We suppose that $r(t, \boldsymbol{\theta}(t))$ is modulated by the chain $\boldsymbol{\theta}$ by $r(t, \boldsymbol{\theta}(t)) = \langle \mathbf{r}(t), \boldsymbol{\theta}(t) \rangle$. Here $\mathbf{r}(t) := (r_1(t), r_2(t), \cdots, r_M(t))' \in \Re^M$ and $r_i(t) > 0$ for each $i = 1, 2, \cdots, M$ and each $t \in \mathcal{T}$; $r_i(t)$ is the interest rate when $\boldsymbol{\theta}(t) = \mathbf{e}_i$; the scalar product $\langle \cdot, \cdot \rangle$ selects the component of the vector $\mathbf{r}(t)$ of interest rates in force according to the state of the Markov chain $\boldsymbol{\theta}(t)$ at the current time t. The price process of the (locally) risk-free bond evolves over time as:

$$B(t) = \exp\left(\int_0^t r(u, \boldsymbol{\theta}(u)) du \right), \quad t \in \mathcal{T}, \quad B(0) = 1 \,.$$

As usual, for each $t \in \mathcal{T}$, let $\mu(t, \boldsymbol{\theta}(t))$ and $\sigma(t, \boldsymbol{\theta}(t))$ be the appreciation rate and the volatility of the risky share price at time t. We suppose that the chain $\boldsymbol{\theta}$ modulates $\mu(t, \boldsymbol{\theta}(t))$ and

$\sigma(t, \boldsymbol{\theta}(t))$ as

$$\mu(t, \boldsymbol{\theta}(t)) = \langle \boldsymbol{\mu}(t), \boldsymbol{\theta}(t) \rangle \ , \quad \sigma(t, \boldsymbol{\theta}(t)) = \langle \boldsymbol{\sigma}(t), \boldsymbol{\theta}(t) \rangle \ .$$

Here $\boldsymbol{\mu}(t) := (\mu_1(t), \mu_2(t), \cdots, \mu_M(t))' \in \mathbb{R}^M$ and $\boldsymbol{\sigma}(t) := (\sigma_1(t), \sigma_2(t), \cdots, \sigma_M(t))' \in \mathbb{R}^M$; for each $i = 1, 2, \cdots, M$, $\mu_i(t)$ and $\sigma_i(t)$ are the appreciation rate and the volatility of the risky share when $\boldsymbol{\theta}(t) = \mathbf{e}_i$; $\mu_i(t) > r_i(t)$ and $\sigma_i(t) > 0$.

Let $W := \{W(t) | t \in \mathcal{T}\}$ be the standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ with respect to the right-continuous, $\mathcal{P}$-completion of its natural filtration $\mathbb{F}^W := \{\mathcal{F}^W(t) | t \in \mathcal{T}\}$. Then we suppose that under $\mathcal{P}$, the evolution of the share price process $S \triangleq \{\boldsymbol{\theta}(t) | t \in \mathcal{T}\}$ over time is governed by the following Markovian, regime-switching, geometric Brownian motion:

$$dS(t) = \mu(t, \boldsymbol{\theta}(t)) S(t) dt + \sigma(t, \boldsymbol{\theta}(t)) S(t) dW(t) \ , \quad S_0 = s_0 > 0 \ .$$

Our market includes the two underlying assets B and S. Write

$$V(t, S, \boldsymbol{\theta}) = E[\exp(-\int_0^T r(u) du)(S(T) - K)^+ | \mathcal{G}_t], \tag{7}$$

where $\mathcal{G}_t$ is defined as the filtration generated by $(S(t), \boldsymbol{\theta}(t))$ i.e. $\mathcal{G}_t = \sigma\{S(u), \boldsymbol{\theta}(u) : u \leq t\}$. Write

$$\mathbf{V}(t, S) = (V(t, S, e_1), ..., V(t, S, e_2)),$$

so that $V(t, S_t, \boldsymbol{\theta}_t) = \langle \mathbf{V}(t, S_t, \boldsymbol{\theta}_t) \rangle$. Applying Ito's rule to V we have

$$\begin{aligned} V(t, S(t), \boldsymbol{\theta}(t)) \quad &= \quad V(0, S_t, \boldsymbol{\theta}(t)) + \int_0^t \frac{\partial V}{\partial u} du + \int_0^t \frac{\partial V}{\partial S} (\mu(u), \boldsymbol{\theta}(u)) S_u du + \sigma(u, \theta u) S_u dB_u) \\ &+ \quad \frac{1}{2} \int_0^t \frac{\partial^2 V}{\partial S^2} \sigma^2(u, \boldsymbol{\theta} u) du + \int_0^t \langle V, \Lambda \boldsymbol{\theta}(u) \rangle du \ . \end{aligned} \tag{8}$$

By definition, as V is a martingale, all the time integral terms in the above equation must sum to zero identically. That is,

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} (\mu(t), \boldsymbol{\theta}(t)) S_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2(t, \boldsymbol{\theta}(t))(S_t^2) + \langle \mathbf{V}, \boldsymbol{\Lambda} \rangle = 0 \ .$$

Let $\mathbf{C}$ be the price of, say, a European Call option having a terminal value of $\mathbf{C}(t, T, S, \boldsymbol{\theta}) = (S - K)^+$. Now $\mathbf{V} = \exp(-\int_0^t r_u du) \mathbf{C}$, so with $\mathbf{C}(t, T, S, \cdot) = (C(t, T, S, e_1), ..., C(t, T, S, e_M))$,

C satisfies the following system of coupled Black–Scholes PDE,

$$-r_i C_i + \frac{\partial C_i}{\partial t} + \mu_i S \frac{\partial C_i}{\partial S} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 C_i}{\partial S^2} + \langle \mathbf{C}, \mathbf{\Lambda} e_i \rangle = 0 \tag{9}$$

with the terminal condition

$$C(T, T, S, e_i) = (S - K)^+.$$

This is the underlying PDE used to determine the asymptotic option price for a European option.
We now describe overview of applications of Markov modulated regime-switching models in finance.

1.2.1 Literature overview of Markov-modulated regime-switching models in finance

The third chapter of this thesis is based on asymptotic evaluation of a European option in an economy wherein the interest rate and the volatility switches according to an underlying Markov chain that describes the state of the economy. Such models originated in econometrics, pioneered by Hamilton's work [61]. Hamilton first proposed that the unobserved regime (or state) follows a first-order Markov process. Dai and Singleton [17] pointed out that mere addition of jumps to diffusion models may not always capture "turbulent" and "quiet" periods of bond markets. This feature exemplified the need to have models that addressed these changes in these periods. This also galvanized much work in finance and economics, for example Hamilton and Susmel [63], Schaller and Norden [74] for stock prices; Engle and Hamilton [31] for time series of exchange rates; Elliott and Mamon [27], Mamon and Rodrigo [51], Kallimpalli and Susmel [46] and Wu and Zeng [81] for short-term interest rates. For American options, Buffington and Elliott [12] discussed a model involving a two-state Markov chain while closed-form solutions for perpetual American put options in a regime switching framework were proposed in a recent study by Guo and Zhang [43]. An interesting approach of using game theory to price options in a Markov-modulated economy was proposed by Siu [53].

Although the Markov regime-switching models reflect the tenets of the efficient market hypothesis that states that the entire history of the underlying price process is no more useful than its current value, the catch is that these models make the market incomplete.

This incompleteness is the result of having two sources of randomness (i.e. the randomness due to the Markov chain and that due to the Brownian motion) in a single equation of asset price dynamics. Therefore there exist many Equivalent martingale measures EMMs on which the derivatives could be priced. To resolve this issue Guo [42] proposed to complete the market via the introduction of new securities. Deshpande and Ghosh [23] proposed selecting the minimal martingale measure (MMM) from the space of EMM's and pricing European option under it . Models for pricing zero coupon risky bond under the MMM was discussed recently by Deshpande [22]. Elliott et al. [26] introduced the Gerber–Shiu–Esscher transform technique for pricing options in a Markov-modulated economy.

A novel approach of incorporating a "feedback" effect to pricing and hedging European options in a "double" Markov-modulated regime-switching economy was recently proposed by Elliott et al. [29]. The price dynamics of the risky asset are governed by the double-Markovian regime-switching model. In their approach they incorporated the feedback effect of the price process of the risky asset on the economic conditions by assuming that the rate transition matrix of the chain was modulated by the price process. Elliott and Siu [28] recently studied pricing and hedging of contingent claims in a Markov-modulated economy by incorporating two sources of risk. The first is the risk associated with fluctuations of market prices, referred to as financial risk. The second is due to fluctuations of underlying Markov chain and is referred to as regime switching risk. They combine these by introducing a general pricing kernel defined by the product of two density processes: one for a measure change for a diffusion process and other one for a measure change for a Markov chain.

Jump-diffusion models of the regime-switching type are also widely applied in options pricing theory. There are many references. We mention here just a few as in this thesis we will be generally concentrating on the Markov modulated diffusion process. Yuen and Yang [83] priced options in a jump-diffusion regime-switching model where they used the trinomial tree method for pricing. Elliott et al. [30] considered pricing options under a generalized Markov-modulated jump-diffusion model wherein the underlying measure process was defined to be a generalized mixture of Poisson random measures that encompassed a generalized gamma process. Siu et al. [54] studied pricing life insurance products under a generalized jump-diffusion model with a Markov-switching compensator.

The body of work referenced above dealt with pricing and hedging of options under Markov-modulated regime switching in which an option price was either determined as an solution to the above described PDE expression or by utilizing Monte Carlo simulation. A very recent work of Basu and Ghosh [9] provides asymptotic analysis of option prices in a Markov-modulated market. This generalizes the work of Fouque et al. [34] who do not consider Markov modulation of the stock price dynamics. They considered two variations of the chain, the slow chain and the fast chain. They obtained asymptotic expansion for the slow-chain case while arguing that such an expansion may not exist for the fast-chain case. In the slow chain the transition intensity of the modulating Markov chain becomes very small while for the fast chain case it becomes very large. As noted earlier Markov modulated regime switching market is incomplete and hence is characterized by many equivalent martingale measures if they exist. Hence there is a range of options price. By simple application of Ito's theorem we show that for any small positive perturbation of the underlying Markov chain, the option price for the slow chain case is bounded above by the underlying stock price at current time t and bounded below by the corresponding Black-Scholes options price evaluated at current time t. For the fast chain case, the options price is nothing but the the Black-Scholes price evaluated at current time t for averaged out interest rate, drift and volatility process. Thus we would then infer that there exist an asymptotic options price for the slow chain case contrary to the fast chain case. These results constitute chapter 3 of this thesis.

We now provide technical introduction to semi-Markov modulated jump diffusion. We explain our motivation in studying it's stability property in this thesis. We follow it up with a brief literature overview of study done on the stability of regime switching random processes.

1.3 semi-Markov modulated jump diffusion, stochastic stability and its connection to ruin theory and options pricing theory

Consider the following semi-Markov Modulated Jump Diffusion (sMMJD) described as follows:

$$\begin{aligned} dX_t &= b(X_t, \theta_t)dt + \sigma(X_t, \theta_t)dW_t + dJ_t \\ dJ_t &= \int_\Gamma g(X_t, \theta_t, \gamma)N(dt, d\gamma) \\ X_0 &= x, \theta_0 = \theta, \end{aligned} \tag{10}$$

where X takes values in $\mathbb{R}^r$ and θ_t is a finite-state semi-Markov process taking values in $\mathcal{X} = \{1, ..., M\}$. Let Γ be a subset of $\mathbb{R}^r - 0$; it is the range space of impulsive jumps. For any set B in Γ, $N(t, B)$ counts the number of jumps on $[0, t]$ with values in B and is independent of the Brownian motion W_t, $b(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{X} \to \mathbb{R}^r$, $\sigma(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{X} \to \mathbb{R}^r \times \mathbb{R}^d$, $g(\cdot, \cdot, \cdot) : \mathbb{R}^r \times \mathcal{X} \times \Gamma \to \mathbb{R}^r$. For future use we define the compensated Poisson measure $\tilde{N}(dt, d\gamma) = N(dt, d\gamma) - \lambda\pi(d\gamma)dt$ where $\pi(\cdot)$ is the jump distribution and $0 < \lambda < \infty$ is the jump rate.

In this chapter, as mentioned earlier, we intend to study the stability of Equation (10) with regards to the following stability criteria, namely, almost surely and moment exponential stability. We define this below.

Definition: Almost-sure exponential stability *The trivial solution of Equation (10) is almost surely exponentially stable if*

$$\limsup_{t\to\infty} \frac{1}{t} \log |X_t| < 0 \quad a.s. \; \forall X_0 \in \mathbb{R}^r.$$

Definition: Moment exponential stability *Let $p > 0$. The trivial solution of (10) is said to be p^{th}-moment exponentially stable if there exists a pair of constants $\lambda > 0$ and $c > 0$, such that for any $X_0 \in \mathbb{R}^r$*

$$E[|X_t|^p] \leq C|X_0|^p \exp(-\lambda t) \quad \forall \; t \leq 0.$$

Before we delve further into studying the stability of the sMMJD, we first describe our main motivation in doing so. The motivation stems from its interesting connection with ruin theory, a result recently attributed to the work of Khasminskii and Milstein [47], which in turn is connected to options pricing theory- a result attributed to Gerber and Shiu [37]. We first sketch the connection stability theory has with ruin theory.

Connection of stability theory to ruin theory

Consider the one dimensional SDE of the following form:

$$dX_t = bX_t dt + \sum_{r=1}^{k} \sigma_r X_t dB_r(t). \tag{11}$$

The p^{th}-moment Lyapunov exponent of a solution to (11) is defined by

$$g(p; x) = \limsup_{t\to\infty} \frac{1}{t} \log E|X_t|^p \quad X_0 = x, \; \forall \; \; p \in \mathbb{R}.$$

It was shown in Appendix B of ([47])that under a certain non degeneracy condition on (11), $g(p; x)$ is independent of x, i.e. $g(p; x) = g(p)$ for all $p \in \mathbb{R}$, $x \neq 0$ and

$$g(p) = \lim_{t\to\infty} \frac{1}{t} \log E|X_t|^p \quad x \neq 0, \tag{12}$$

where, if b and σ_r are some constants, they show that $g(p)$ can be explicitly calculated as

$$g(p) = pb + \frac{1}{2}p(p-1)\sum_{r=1}^{k}\sigma_r^2.$$

If $g(p) > 0$ then $E|X_t|^p \to \infty$ as $t \to \infty$ and if $g(p) < 0$ then $E|X_t|^p \to 0$ as $t \to \infty$. If $g(p) = 0$ then there exist two constants $0 < c < C < \infty$ such that $c \leq E|X_t|^p \leq C$, $t \geq 0$. Let $a^* = g^{'}(0) < 0$. In this case $g(p) < 0$ for sufficiently small positive p. If $g(p) \to \infty$ as $p \to \infty$ (proposition B.4 in [47] gives sufficient conditions for such a behavior of $g(p)$), then the equation $g(p) = 0$ has a unique root $\gamma^* > 0$ (recall that $g(0) = 0$). The uniqueness follows from the convexity of g. We call γ^* the stability index. Under certain conditions via Theorem B.8 in [47], it has been shown that for $a^* = g'(0) < 0$, $\gamma^* > 0$ being a root of $(g(p) = 0)$, for some constant $K \geq 1$ and for any $\delta > 0$ under $|x| < \delta$, the inequalities

$$\frac{1}{K}(|x|/\delta)^{\gamma^*} \leq \mathbb{P}\{\sup_{t\geq 0}|X_t| > \delta\} \leq K(|x|/\delta)^{\gamma^*}$$

are fulfilled. The term $P\{\sup_{t\geq 0}|X_t| > \delta\}; |x| < \delta$ is the celebrated ruin probability that occurs often in insurance. Gerber and Shiu [37] connected the ruin theory to the Options pricing theory as shall be seen below.

Connection of stability theory to options pricing theory

Consider some net wealth/surplus process U_t at time t, $t \geq 0$, where the time to ruin T is denoted by $T := \inf\{t : U_t < 0\}$. The probability of ultimate ruin as a function of the initial surplus $U_0 = u \geq 0$ is defined as $\Psi(u) = P[T < \infty | U_0 = u]$. Let $w(x, y)$ be a nonnegative function of $x > 0$ and $y > 0$. We consider that for $u \geq 0$ the function $\Phi(u)$ is defined as

$$\Phi(u) = E[w(U_{T-}, |U_T|)e^{-\delta T}\mathbb{I}_{(T<\infty)}|U_0 = u], \tag{13}$$

In this equation, δ could be interpreted as a force of interest and w is some kind of penalty at ruin now replaced by a net payoff on the exercise of a perpetual American put option. As a special case, if this net payoff "w" is of constant value 1 and $\delta = 0$, then $\Phi(\cdot)$ corresponds to the probability of ultimate ruin. This probability of ultimate ruin, which as we saw from the earlier result of Khasminskii and Milstein ([47]), is connected to the stability theory. By logical implication, stability theory could hence be connected to the options pricing for perpetual American Put option, and, is of constant value 1 , i.e. with no discounting.
We next provide a succinct literature review of stability of regime switching random processes.

1.3.1 Literature overview of regime-switching models in stability theory

The stability of stochastic differential equations (SDEs) has a long history with some key works being those of Arnold [4], Khasminskii and Milstein [47] and Ladde and Lakshmikantham [69]. SDEs with Markov switching have been applied in such diverse areas as finance (as described earlier) and Biology, refer Hanson [60]. The stability of these processes has also garnered much attention, in particular from Ji and Chizeck [66] and Mariton [71] who both studied the stability of a jump-linear equation of the form $\frac{dx_t}{dt} = A(r_t)x_t$, where r_t is a Markov chain. Basak et al. [8] discussed the stability of a semilinear SDE with Markovian regime switching of the form $\dot{x}_t = A(r_t)x_t dt + \sigma(r_t, x_t)dW_t$. Mao [70] studied the exponential stability of a general nonlinear diffusion with Markovian switching of the form $dx_t = f(t, x_t, r_t)dt + g(t, x_t, r_t)dW_t$. A very recent work of Yin and Xi [84] studied the stability of Markov-modulated jump-diffusion processes. In this thesis, we study the asymptotic stability of general semi-Markov-modulated jump diffusion. Unlike the special Markov-modulated case where the x-dependent diffusion is a partial differential operator, the semi-Markov process involves a general integro-partial differential operator. This work constitutes Chapter 4.

As described earlier, we now start with discussing understanding asymptotic phenomenon in credit risk management (see Ch.2), options pricing (see Ch. 3) and asymptotic stability (see Ch.4).

Chapter 2

Value at Risk performance comparison of the CreditRisk$^+$ and the 2-stage CreditRisk$^+$: A Large Deviations Approach.

2.1 Synopsis

Credit risk models, as the name suggests, are used to quantify the risks involved in dealing with credit. There are two primary types of models in the literature that attempt to describe default processes for debt obligations and other defaultable financial instruments. They are usually referred to as structural and reduced-form (or intensity) models. Structural models use the evolution of firms' structural variables, such as asset and debt values, to determine the time of default. Intensity based models do not consider the relation between default and firm value in an explicit manner. In contrast to structural models, the time of default in intensity models is not determined via the value of the firm, but is the first jump of an exogenously given jump process.

Structural default models provide a link between the credit quality of a firm and the firm's economic and financial conditions. Thus, defaults are endogenously generated within the model instead of exogenously given as in the reduced-form approach. Another difference between the two approaches appears in the treatment of recovery rates: whereas reduced models exogenously specify recovery rates, in structural models the value of the firm's assets and liabilities at default will determine its recovery rate. Refer to Arora et al. [5] for an excellent detailed discussion on the differences between structural and reduced-form models.

A third approach, closer to the reduced-form approach is called the "actuarial" approach. The CreditRisk$^+$ (CR$^+$) model that we will discuss later belongs to this particular approach to quantifying credit risk. Under the actuarial approach, default is an "end-of-game" surprise with a known probability that follows the Poisson distribution. The actuarial method ignores all other factors such as leverage, volatility of asset returns, or even downgrade risk, and considers that defaults "arrive" at a certain rate per unit of time (say years) and considers them distributed according to the Poisson distribution. This approach is quite similar to the reduced-form approach as they both define the risk at default loss only. Neither approach uses transition matrices and they both treat recovery rates as deterministic "loss given default". However, there are also some differences between these two approaches. For example, although they both use conditional probabilities for defaults, in the actuarial approach used in CR$^+$, the conditional probabilities for default are a function of common risk factors, while for the intensity-based approach, they are a function of macrofactors. The risk drivers for the actuarial approach are expected default rates while for the reduced-form approach the driver is the hazard rate. As for the numerical implementation, the actuarial approach is a closed-form analytic approach unlike the reduced-form approach, which involves tree-based simulation. As CreditRisk$^+$ focuses only on defaults and the default process is assumed to follow an exogeneous Poisson process, we consider the CreditRisk$^+$ approach in the framework of actuarial approach. For a systematic treatment of all these three approaches, we refer to Crouhy et al. [15].

The CR$^+$ model developed by Credit Suisse Financial Products computes portfolio loss distribution analytically without relying on Monte Carlo simulation. This is its main strength. The loss distribution in the original CR$^+$ model was computed using a recursion scheme due to Panjer [73]. Gordy [44] showed that this method was numerically unstable for large portfolios. Further, it is difficult to extend this method to more complex models. Giese [38] provided a breakthrough by suggesting a method for evaluating the loss distribution that used recursive computation of exponential and logarithmic polynomials. This computation allows incorporation of a wider range of risk factor distributions.

The standard CR$^+$ model apportions the risk of each individual obligor to different sectors, which can be thought of as industry sectors. The sector default rates are assumed uncorrelated and have fixed net exposures. Burgisser [14] was the first to introduce correlations amongst the sector default rates. This was done by adjusting the portfolio default rate standard deviation to account for the sector correlations and then carrying out single-sector analysis. Giese showed that the Burgisser model cannot adequately capture concentration risk arising from sectors with large exposures and large factor variance. In Giese [38], correlation is induced among the sectors via a single variable that follows a compound gamma distribution which introduces a uniform level of covariance between the sector default rates. While the compound gamma model of Giese performs better than the Burgisser model, it distorts the concentration risks by enhancing low levels of correlation and suppressing higher levels of correlation. Further, estimation of this uniform covariance from observed data can be inconsistent. The 2-stage CreditRisk$^+$ (2-CR$^+$) model proposed recently by Deshpande and Iyer [24] is a generalisation over Giese's compound Gamma model.

We will briefly describe the CR$^+$ and the 2-CR$^+$ models. Before we do so, we explain two key portfolio measures of risk called the Value at Risk (VaR) and the Expected Shortfall (ES) risk measure. We use VaR in our portfolio risk computations. We will also provide motivational guidance so as to understand the need to use the large deviations theory to compare the VaR performance of CR$^+$ and 2-CR$^+$.

The VaR measure is defined as follows. For some confidence level $\alpha \in (0, 1)$, the VaR of the portfolio is given by the smallest number l such that the probability that the portfolio loss L exceeds l is not larger than $1 - \alpha$. Mathematically, it is represented as

$$VaR_\alpha = \inf\{l \in \mathbb{R} : P(L > l) \leq (1 - \alpha)\}.$$

Quite in contrast to the VaR, ES produces better incentives for traders than VAR . ES is also sometimes referred to as conditional VaR, or tail loss. Where VaR asks the question "how bad can things get?", ES asks "if things do get bad, what is our expected loss?" If the underlying portfolio loss distribution for L is a continuous distribution, then ES is equivalent to the tail conditional expectation (TCE$_\alpha$)defined by

$$TCE_\alpha = E[-L|L \leq -VaR_\alpha(L)].$$

Informally, this equation says "in case of losses so severe that they occur only alpha percent of the time, what is our average loss?"

In general a measure of risk is said to be coherent if it satisfies the following properties.
- **Monotonicity:** If a portfolio has lower returns than another portfolio for every state of the world, its risk measure should be greater.
- **Translation invariance:** If we add an amount of cash K to a portfolio, its risk measure if coherent should go down by K.
- **Homogeneity:** Changing the size of a portfolio by a factor (say λ) while keeping the rel-

ative amounts of different items in the portfolio the same should result in the risk measure being multiplied by (λ).

- Subadditivity: The risk measure for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged.

The first three conditions are straightforward while the fourth condition states that diversification helps reduce risks. Refer to Artzner et al. [6] for an in-depth treatment on coherent measures of risk. VaR satisfies all the above properties except for being subadditive. ES satisfies all the properties for the risk measures to be coherent. Hence VaR is not a coherent measure of risk. The issue of VaR not satisfying the subadditive property is not just a theoretical issue. Risk managers sometimes find that, when portfolio A is combined with portfolio B to form a single portfolio, for risk management purposes, the total VAR goes up rather than down. Thus this defeats the sole purpose of diversification. Although ES is coherent, it has its own share of problems. The one most discussed is that it fails to eliminate the tail risk. For example suppose that the magnitude of an extreme loss is much higher for portfolio B than for portfolio A. Then, if a risk measure is free of tail risk, the risk measure should choose portfolio A, since its extreme loss is smaller than portfolio B's. Instead, ES may choose portfolio B. Yet another criticism of ES is that its choice of portfolio is inconsistent with that from the utility theory. For details on these two criticisms, refer to Yamai and Yoshiba [82]. Notwithstanding the criticisms of VaR, worldwide adoption of the Basel II Accord gave further impetus to its use. From the definition of VaR we know that we need to determine the large loss event probability $P(L > l)$. To compute the probability of this event we utilise the tools from the theory of large deviations. We now describe here the CR$^+$ and the 2-CR$^+$ models.

Let a suitable base unit of currency ΔL be chosen. An obligor of a portfolio is denoted by a number and there are a finite number of obligors O. Obligors are thus numbered from $1, ..., O$. In CR$^+$, the adjusted exposure E_i of obligors $i \in \{1, ..., O\}$ is replaced by $\nu_i = \lfloor E_i \Delta L \rfloor$, where $\lfloor . \rfloor$ denote the integer part of a real number. Let p_i be the average default probability for obligor i for the time horizon considered (typically one year) . Let N_i denote the default of obligor $i \in \{1, ..., O\}$. Therefore the portfolio loss is represented by the integer random variable

$$L = \sum_{i=1}^{O} \nu_i N_i. \tag{1}$$

γ is a $(K \times 1)$ vector consisting of gamma-distributed random variables that signify sectoral default rates where K is the number of sectors. Conditional on γ, the default variables N_i are assumed independent and Poisson distributed with intensity $X_i(\gamma)$ i.e. $N_i(\gamma) =^d \ Poi(X_i(\gamma))$. Technically, it is reasonable to assume nonexistence of multiple defaults because sequential defaults seldom occur in a short time. For details on this assumption refer to the (CreditRisk$^+$ document, 1997). Hence we assume that $N_i \in \{0, 1\}$ for each i.

2.1.1 CreditRisk$^+$

The standard CR$^+$ model is a popular credit risk management model because of its ability to analytically compute the portfolio loss distribution. This is in contrast to other credit risk models such as Creditmetrics or KMV which utilise computer-intensive Monte Carlo-based simulations. Apart from being able to compute tail losses analytically, CR$^+$ is considered more flexible as it can compute the loss distribution for a portfolio of assets which do not share the common (Bernoulli) distribution. The losses are not independent of each other because the obligors from the same industry or country share similar developments. Moreover it can accommodate obligors characterised by stochastic probability of default with different expected values and a rich structure of default correlations amongst them that are valid for different levels of net exposures. The price one pays for these various benefits is that the Poisson approximation embedded in its analysis requires the expected default probability to be small. Also, only positive default correlations need be considered. CR$^+$ also tends to assume that the recovery rates and defaults are independent of each other. A cautious approach to cope with this problem in practice and to acknowledge the fact of negative correlation is to use conservatively estimated recovery rates. Refer to Grunlach and Lehrbass [41] for details. In the standard CR$^+$ model, the default rate of each sector is represented by a nonnegative gamma random variable $\gamma_k =^d \Gamma(\frac{1}{\sigma_k}, \sigma_k)$ that satisfies $E[\gamma_k] = 1 \quad \forall \ k = 1, ..., K$ and has covariance matrix $Cov(\gamma_k, \gamma_l) = 0, \forall k, \ l = 1, ..., K$.

For each obligor "i" $\in \{1, ..., O\}$ the CR$^+$ framework assumes that the default rates of the obligors depend on the sector default rates via the linear relationship

$$X_i(\gamma) = p_i \sum_{k=1}^{K} g_k^i \gamma_k, \tag{2}$$

where $X_i(\gamma)$ is the default rate of obligor i conditional on the sector default rates $\gamma = (\gamma_1, ..., \gamma_K)$. The risk of each obligor is apportioned among a set of K sectors (industries) by choosing g_k^i such that $\sum_{k=1}^{K} g_k^i = 1$ for each $i \in \{1, ..., O\}$.

For the sake of completeness we provide here the expression for the probability-generating function (p.g.f.) of the loss distribution computed by CR$^+$, denoted by $(G^{CR^+}(\cdot))$.

$$G^{CR^+}(z) = \exp(-\sum_{k=1}^{K} \frac{1}{\sigma_k} log(1 - \sigma_k \sum_{i=1}^{O} g_k^i p_i (z^{\nu_i} - 1)).$$

We now introduce the terms portfolio default rate and mean portfolio default rate for CR$^+$ as follows. The portfolio default rate for CR$^+$ is defined as

$$\begin{aligned} T_K(\gamma) := \sum_{i=1}^{O} X_i(\gamma) &= \sum_{i=1}^{O} p_i \sum_{k=1}^{K} g_k^i \gamma_k \\ &= \sum_{k=1}^{K} \tilde{g}_k \gamma_k, \end{aligned} \tag{3}$$

where $\tilde{g}_k = \sum_{i=1}^{O} g_k^i p_i \quad \forall \quad k = 1, ..., K$, while the mean portfolio default rate for CR$^+$ is $\hat{T}_K(\gamma) := \frac{T_K(\gamma)}{K}$.

2.1.2 The 2-stage CreditRisk$^+$:

The 2-stage CR$^+$ (2- CR$^+$) model proposed recently by (Deshpande and Iyer, 2009) explains default risk at two levels. In the first level, as in CR$^+$, the risk of each obligor is apportioned to a common set of industry sectors using the parameter g_k^i for each $i \in \{1, ..., O\}$. In the second level, the default rates of the industry sectors γ_k $k = 1, ..., K$ are assumed to depend linearly on a set of common independent risk factors $Y_1, ..., Y_M$ i.e.

$$\gamma_k = \sum_{i=1}^{M} a_{ki} Y_i, \tag{4}$$

where Y_i are independent gamma-distributed random variables with mean 1 and variance $\tilde{\sigma_i}$,$\sum_{i=1}^{M} a_{ki} = 1 \quad \forall \quad k = 1, ..., K$ and $a_{ki} < 1$. Since $K \geq M$, we can better approximate γ_k (i.e. with less residual error) if we regress γ_k with $M = K$ macroeconomic random variables Y_i. The variables γ_k, now a sum of independent gamma-distributed random variables, will thus have a general univariate distribution for each k. Also, as they are regressed by a common set of Y_i, they then will be correlated through this common set, for example if γ_1 is regressed by (Y_1, Y_2, Y_3), and γ_2 is regressed by (Y_1, Y_3, Y_4) then $Cov(\gamma_1, \gamma_2) = Cov(Y_1, Y_3) \neq 0$ where $Cov(\cdot)$ is a covariance function. This is in sharp contrast to the γ_k $k \in \{1, ..., K\}$ of the CR$^+$, which are independent nonidentically distributed random variables. The p.g.f. of the loss distribution computed by the 2-Stage CR$^+$ is a closed form expression given as

$$G^{2-CR^+}(z) = exp(-\sum_{i=1}^{M} -\frac{1}{\tilde{\sigma}_i} log(1 - \tilde{\sigma}_i \sum_{k=1}^{K} a_{ki} \sum_{i=1}^{O} g_k^i p_i (z^{\nu_i} - 1))).$$

See Deshpande and Iyer [24] for details. As they observed, the 2-CR$^+$ model is therefore an extended CR$^+$ model that incorporates correlation amongst the sectoral default rates while maintaining the analytical tractability of its primitive, CR$^+$. This feature makes 2-CR$^+$ attractive for the risk managers to implement. The variables Y_i can be obtained from a principal component analysis of macroeconomic variables that influence the sectoral default rates. A factor analysis based on the observed default rate correlation matrix would suffice to obtain an estimate of a_{ki}.

Similarly to CR$^+$, the portfolio default rate and its mean for the 2-CR$^+$ model are defined as follows. The portfolio default rate for 2-CR$^+$ is

$$\begin{aligned} T_M(\gamma) := \sum_{k=1}^{K} \tilde{g}_k \gamma_k &= \sum_{k=1}^{K} \tilde{g}_k \sum_{i=1}^{M} a_{ki} Y_i \\ &= \sum_{i=1}^{M} g^i Y_i, \end{aligned} \tag{5}$$

where $g^i = \sum_{k=1}^{K} \tilde{g}_k a_{ki} \;\; \forall \;\; i = 1, ..., M$, while the mean portfolio default rate for the 2-CR$^+$ is $\hat{T}_M(\gamma) := \frac{T_M(\gamma)}{M}$.

In this chapter, we infer the tail loss probability by understanding the large deviations behaviour of $\hat{T}_K(\gamma)$ and $\hat{T}_M(\gamma)$ (refer Lemma 1) by assuming $\sum_{i=1}^{O} g_k^i p_i := Gp^* \; \forall k \in \{1, ...K\}$ and $\sum_{k=1}^{K} a_{ki} = a_i = a \; \forall \; k \in \{1, ...K\}$ and $\forall \; i \in \{1, ..., M\}$. We utilise a key theorem in large deviations theory called the Gärtner–Ellis theorem . We then obtain explicit representation of the rate functions associated with the rare event asymptotes of $\hat{T}_K(\gamma)$ and $\hat{T}_M(\gamma)$. They are $\frac{(q-Gp^*)^2}{2G^2 p^{*2} \sum_{k=1}^{K} \sigma_k}$, $q \in (Gp^*, \infty)$ for CR$^+$ and $\frac{(q-(G)p^*)^2}{2(G)^2 p^{*2} \sum_{i=1}^{M} \tilde{\sigma}_i}$ for $q \in (Gp^*, \infty)$ for 2-CR$^+$. We conclude this chapter by comparing these two tail decay rates and comment that the VaR produced by 2-CR$^+$ is definitively higher than the one produced by CR$^+$ for a particular type of credit portfolio. We will also support this risk analysis through numerical examples.

2.2 Article "Comparing the Value at Risk performance of the CreditRisk$^+$ and its enhancement: a large deviations approach" – A.Deshpande

2.2.1 Abstract

The standard CreditRisk$^+$ (CR$^+$)is a well-known default-mode credit risk model. An extension to the CR$^+$ that introduces correlation through a two-stage hierarchy of randomness has been discussed by Deshpande and Iyer, [5] and more recently by (Sowers, [9]). It is termed the 2-stage CreditRisk$^+$ (2-CR$^+$) in the former. Unlike the standard CR$^+$, the 2-CR$^+$ model is formulated to allow correlation between sectoral default rates through dependence on a common set of macroeconomic variables. Furthermore the default rates for 2-CR$^+$ are distributed according to a general univariate distribution, which is in stark contrast to the uniformly gamma-distributed sectoral default rates in the CR$^+$. We would then like to understand the behavior of these two models with regards to their computed Value at Risk (VaR) as the number of sectors and macroeconomic variables approaches infinity. The former asymptote refers to portfolio diversification while the later refers to the phenomena of incorporating many independent macro-economic variables thereby making it able to predict the sectoral default rates with less residual regression error (higher precision) as the following article would reveal. In particular we would like to ask whether 2-CR$^+$ produces higher VaR than CR$^+$ and if so, then for which types of credit portfolio. Utilizing the theory of large deviations, we provide a methodology for comparing the VaR performance of these two competing models by computing associated rare event probabilities. In particular we show that this is definitively true for a particular class of credit portfolio, which we call a "balanced" credit portfolio. We support this risk analysis through numerical examples.
Keywords: Value at Risk, CreditRisk$^+$, 2-stage CreditRisk$^+$ model, rare event, large deviations principle, Gärtner–Ellis Theorem.

2.2.2 Introduction

Value at Risk or VaR is an important measure of portfolio risk and is popular amongst portfolio risk managers. Worldwide adoption of the Basel II Accord in 1999 gave further impetus to its use. A large credit portfolio handled by risk managers includes exposures to many obligors whose default probabilities are very small. Occurrence of these rare but large loss events puts emphasis on obtaining the small probabilities of large losses that are relevant in computing VaR. A credit risk model producing high VaR is preferable as then the model puts more emphasis on the occurrence of large loss events. VaR is mathematically expressed as the smallest number l such that the probability that the loss exceeds l is not larger than 1-α for $\alpha \in (0,1)$. This is formally defined as $VaR_\alpha := \inf\{l \in \mathbb{R} : Pr(L > l) \leq (1-\alpha)\}$. Thus, the larger the tail losses, the higher the VaR. Large deviations theory has recently gained importance in computing the probability of rare large loss events. See (Pham, [8]) for an excellent introduction. Let us denote $\mathrm{VaR}_\alpha^{CR^+}$ and $\mathrm{VaR}_\alpha^{2-CR^+}$ as VaR computed by the CR$^+$ and the 2-CR$^+$ model respectively under the same underlying credit portfolio for particular values of α. In this article we seek to discover whether $\mathrm{VaR}_\alpha^{CR^+} \leq \mathrm{VaR}_\alpha^{2-CR^+}$. We employ tools from the theory of large deviations to answer this question. The strength of this theory is that it allows us to compare VaR performance between these two credit risk models without making any distributional assumption on the portfolio loss. This is in stark contrast to the standard mean–variance theory of Markovitz which inherently assumes the normality of the underlying loss distribution.

For the sake of completeness we briefly introduce here the Large Deviations Principle (LDP). Consider generically the empirical means $\hat{T}_N = \frac{1}{N}\sum_{j=1}^{N} X_j$ for $\mathbb{R}$-valued random variables $X_1, ..., X_N, ...$ $\hat{T}_N$, distributed according to the probability law $P_N \in M_1(\mathbb{R})$ i.e. $\hat{T}_N =^d P_N$, where $M_1(\mathbb{R})$ denotes the space of all probability measures on $\mathbb{R}$. The large deviations principle characterizes the limiting behaviour as $N \to \infty$ of $(P_N)_{N=1}^{\infty}$ on the space $(\mathbb{R}, B(\mathbb{R}))$ in terms of a (rate) function. This characterization is via the asymptotic upper or lower exponential bounds on the values that P_N assigns to measurable subsets of $\mathbb{R}$.

Definition 1 (Large deviations principle) *$(P_N)_{N=1}^{\infty}$ satisfies the large deviations principle with a rate function Λ^* if, for all $\Gamma \subset \mathbb{R}$,*

$$-\inf_{x\in\Gamma^0} \Lambda^*(x) \leq \liminf_{N\to\infty} \frac{1}{N} \log P_N(\Gamma) \leq \limsup_{N\to\infty} \frac{1}{N} \log P_N(\Gamma) \leq -\inf_{x\in\bar{\Gamma}} \Lambda^*(x),$$

where Γ^0 is the interior of Γ and $\bar{\Gamma}$ is the closure of Γ. A rate function is a lower semicontinuous mapping $\Lambda^* : \mathbb{R} \to [0,\infty]$. It is a "good" rate function if the level sets $\{x \in \mathbb{R} : \Lambda^*(x) \leq M\}$ are compact for all $M < \infty$.

An important result in large deviations theory called Cramer's theorem considers LDP setup for X_j for $j \in \{1, ..., N\}$ being i.i.d. $\mathbb{R}$-valued random variables with (speed) $\frac{1}{N}$. Another fundamental result, the Gärtner–Ellis theorem that we utilise here covers the case when X_j for $j \in \{1, ..., N\}$ is non-i.i.d. Refer to Dembo and Zeitouni, ([4]) for an accessible introduction to large deviations theory.

Application of the large deviations theory to credit risk analysis is rather new and dates back to the work of (Dembo et al., [3]) who suggested that generically, rare events are exponentially rare in the dimension of the portfolio. In other words, the tail of the loss distribution of the portfolio decays at the rate of $e^{-\lambda N}$, where N is the number of assets in the portfolio and λ is some positive constant. The same conclusion was obtained in a recent work of (Glasserman et al., [7]). Let us denote by L^{CR^+} and L^{2-CR^+} the credit portfolio loss computed by the CR$^+$ and the 2-Stage CR$^+$ respectively. From the definition of VaR it is clear that showing $\mathrm{VaR}_\alpha^{CR^+} \leq \mathrm{VaR}_\alpha^{2-CR^+}$ is equivalent to proving that $\Pr(L^{CR^+} > l) \leq \Pr(L^{2-CR^+} > l)$ for a large positive value l. We will show further in (Section 2.2.3, Lemma 1) that proving $\Pr(L^{CR^+} > l) \leq \Pr(L^{2-CR^+} > l)$ is equivalent to showing $\Pr(\hat{T}_{CR^+}(\gamma) > q) \leq \Pr(\hat{T}_{2-CR^+}(\gamma) > q)$ for some specific value of q (which will be made precise later), where $\hat{T}_{CR^+}(\gamma)$ and $\hat{T}_{2-CR^+}(\gamma)$ are the mean portfolio default rates (empirical means) for the CR$^+$ and the 2-Stage CR$^+$ respectively.

Thus, to prove $\mathrm{VaR}_\alpha^{CR^+} \leq \mathrm{VaR}_\alpha^{2-CR^+}$, we need only show that $\Pr(\hat{T}_{CR^+}(\gamma) > q) \leq \Pr(\hat{T}_{2-CR^+}(\gamma) > q)$. We observe that for a meaningful value of q, this condition is sufficient to prove that VaR computed by CR$^+$ is lower than that for 2-CR$^+$. This *meaningful* value of q is characterized for a class of credit portfolio which we term a "balanced" credit portfolio. In fact, the original portfolios considered both in ([6]) and ([5]) are balanced. Hence, such portfolios can be realistic. Moreover as an exercise in performing a stress test analysis on any other type of credit portfolio, a credit risk manager can perturb it to this framework and analyze accordingly.

We note that $\hat{T}_{CR^+}(\gamma)$ is P_K distributed while $\hat{T}_{2-CR^+}(\gamma)$ is P_M distributed. Via the Gärtner–Ellis theorem (detailed in Section 2.2.4, Theorem 4), we are interested in determining the limiting behaviour as $K \to \infty$ of $(P_K)_{K=1}^{\infty}$ on space $(\mathbb{R}, B(\mathbb{R}))$ in terms of a (rate) function $\Lambda^*_{CR^+}(\cdot)$ for the CR$^+$. Similarly we will determine the limiting behaviour as $M \to \infty$ of $(P_M)_{M=1}^{\infty}$ on space $(\mathbb{R}, B(\mathbb{R}))$ in terms of a (rate) function $\Lambda^*_{2-CR^+}(\cdot)$ for 2-CR$^+$. Knowledge of $\Lambda^*_{CR^+}(\cdot)$ and $\Lambda^*_{2-CR^+}(\cdot)$ will help us compute and compare the probabilities $\Pr(\hat{T}_{CR^+}(\gamma) > q)$ and $\Pr(\hat{T}_{2-CR^+}(\gamma) > q)$.

Thus, in this emerging paradigm of applying large deviations to credit risk analysis, we compare the VaR numbers computed by CR$^+$ and 2-CR$^+$ credit risk models by computing and comparing the rate of the tail decay of $\hat{T}_{CR^+}(\gamma)$ and $\hat{T}_{2-CR^+}(\gamma)$. This paves the way for VaR comparison without making assumptions on the portfolio loss distribution. To the best of our knowledge, no such comparison has been done before between any known competing portfolio credit risk models; our approach is unique because it uses large deviations understanding of the mean portfolio default rate.

The paper is organized as follows. In the second section i.e Section 2.2.3, we briefly describe CR$^+$ and 2-CR$^+$ and formally introduce the terms $\hat{T}_{CR^+}(\gamma)$ and $\hat{T}_{2-CR^+}(\gamma)$. We then connect the portfolio's large deviations behaviour with large losses that will help initiate our analysis. In the third section i.e. 2.2.4, via the Gärtner–Ellis theorem, we quantify and compare the tail decay rate of $\hat{T}_{CR^+}(\gamma)$ and $\hat{T}_{2-CR^+}(\gamma)$. This is followed by a VaR performance comparison of these credit risk models. The paper as usual ends with concluding remarks.

2.2.3 CreditRisk$^+$ and 2-stage CreditRisk$^+$

In this section, we very briefly describe the methodology by which the CR$^+$ and the 2-Stage CR$^+$ compute the p.g.f. of the loss distribution. For details we refer the reader to the (CreditRisk$^+$ technical document, 1997), ([6]) and ([5]). We first briefly describe the structure of the credit portfolio on which the CR$^+$ is based. The same structure is valid for the 2-CR$^+$ model.

The original credit portfolio:

A suitable base unit of currency ΔL is chosen. An obligor of a portfolio is denoted by a number and there are a finite number of obligors O. Obligors are thus denoted from $1, ..., O$. In CR$^+$, the adjusted exposure E_i of obligors $i \in \{1, ..., O\}$ is replaced by $\nu_i = \lfloor E_i \Delta L \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part of a real number, p_i is the average default probability for obligor i for the time horizon considered (typically 1 year) and N_i denotes default of obligor $i \in \{1, ..., O\}$. Henceforth ν_i is the referred to as the exposure of obligor "i". Therefore the portfolio loss is represented by the integer random variable

$$L = \sum_{i=1}^{O} \nu_i N_i. \tag{6}$$

γ is a $(K \times 1)$ vector consisting of gamma-distributed random variables that signify sectoral default rates where K is the number of sectors. Conditional on γ, the default variables N_i are assumed to be independent and Poisson-distributed with intensity $X_i(\gamma)$, i.e. $N_i(\gamma) =^d \;\; Poi(X_i(\gamma))$. Technically, it is reasonable to assume the nonexistence of multiple defaults because sequential defaults seldom occur in a short time. Hence without loss of generality, $N_i, \in \{0, 1\}$ $i \in \{1, ..., O\}$ For details on this assumption refer to the (CreditRisk$^+$ document, 1997).

CreditRisk$^+$:

The standard CR$^+$ model is a popular credit risk management model because of its ability to analytically compute the portfolio loss distribution. This is in contrast to the other credit risk models like Creditmetrics or KMV, which otherwise utilize computer-intensive Monte Carlo-based simulations. In the standard CR$^+$ model, the default rate of each sector is represented by a nonnegative gamma-distributed random variable $\gamma_k =^d \Gamma(\frac{1}{\sigma_k}, \sigma_k)$ that satisfies $E[\gamma_k] = 1 \ \ \forall\ k = 1, ..., K$ and covariance matrix $Cov(\gamma_k, \gamma_l) = 0, \forall k,\ l = 1, ..., K$.

For each obligor $i \in \{1, ..., O\}$, the CR$^+$ framework assumes that the default rates of the obligors depend on the sector default rates via the linear relationship

$$X_i(\gamma) = p_i \sum_{k=1}^{K} g_k^i \gamma_k, \tag{7}$$

where $X_i(\gamma)$ is the default rate of obligor i conditional on the sector default rates $\gamma = (\gamma_1, ..., \gamma_K)$. The risk of each obligor is apportioned among a set of K sectors (industries) by choosing $g_k^i \geq 0 \ni \sum_{k=1}^{K} g_k^i = 1$ for each $i \in \{1, ..., O\}$.

For the sake of completeness we provide here the expression for the p.g.f. of the loss distribution computed by the CR$^+$ denoted by $(G^{CR^+}(\cdot))$.

$$G^{CR^+}(z) = exp(-\sum_{k=1}^{K} \frac{1}{\sigma_k} log(1 - \sigma_k \sum_{i=1}^{O} g_k^i p_i (z^{\nu_i} - 1)).$$

We now introduce the terms “portfolio default rate” and “mean portfolio default rate” for CR$^+$. The portfolio default rate for CR$^+$ is defined as

$$\begin{aligned} T_{CR^+}(\gamma) := \sum_{i=1}^{O} X_i(\gamma) &= \sum_{i=1}^{O} p_i \sum_{k=1}^{K} g_k^i \gamma_k \\ &= \sum_{k=1}^{K} \tilde{g}_k \gamma_k, \end{aligned} \tag{8}$$

where $\tilde{g}_k = \sum_{i=1}^{O} g_k^i p_i \quad \forall \ \ k = 1, ..., K$, while the mean portfolio default rate for CR$^+$ is $\hat{T}_{CR^+}(\gamma) := \frac{T_{CR^+}(\gamma)}{K}$.

2-stage CreditRisk$^+$:

The 2-stage CR$^+$ (2- CR$^+$) model proposed recently by (Deshpande and Iyer, [5]) explains default risk at two levels. In the first level, as in CR$^+$, the risk of each obligor is apportioned to a common set of industry sectors using the parameters g_k^i for each $i \in \{1, ..., O\}$. In the second level the default rates of the industry sectors γ_k $k = 1, ..., K$ are assumed to depend linearly on a set of common independent risk factors $Y_1, ..., Y_M$, i.e.

$$\gamma_k = \sum_{i=1}^{M} a_{ki} Y_i, \tag{9}$$

where Y_i are independent gamma-distributed r.v.s with mean 1 and variance $\tilde{\sigma}_i$,$\sum_{i=1}^{M} a_{ki} = 1 \ \forall \ k = 1, ..., K$ and $0 \leq a_{ki} < 1$. Since $K \geq M$, one can definitely better approximate γ_k (i.e. with less residual error) if we regress γ_k with $M = K$ macroeconomic random variables Y_i. The variable γ_k is now any general univariate distribution other than the gamma distribution for each k and for the sake of comparison with CR$^+$ is assumed to have variance σ_k. Also, as γ_k for $k \in \{1, , K\}$are regressed by a common set of Y_i, they also will then be correlated through this common set. For example, if γ_1 is regressed by (Y_1, Y_2, Y_3) and γ_2 is regressed by (Y_1, Y_3, Y_4), then $Cov(\gamma_1, \gamma_2) \neq 0$ where $Cov(\cdot)$ is a covariance function. This is in sharp contrast to the γ_ks, $k \in \{1, ..., K\}$ of CR$^+$, which are independent non-identically distributed random variables. The p.g.f. of the loss distribution computed by the 2-Stage CR$^+$ is a closed-form expression given as

$$G^{2-CR^+}(z) = \exp(-\sum_{i=1}^{M} -\frac{1}{\tilde{\sigma}_i} log(1 - \tilde{\sigma}_i \sum_{k=1}^{K} a_{ki} \sum_{i=1}^{O} g_k^i p_i(z^{\nu_i} - 1))).$$

See ([5]) for details. As they observed, the 2-Stage CR$^+$ model is therefore an extended CR$^+$ model that incorporates correlation amongst the sectoral default rates while still maintaining the analytical tractability of its primitive, CR$^+$. This feature makes 2-CR$^+$ attractive for risk managers to implement. The variables Y_i can be obtained from a principal component analysis of macro-economic variables that influence the sectoral default rates. A factor analysis based on the observed default rate correlation matrix would suffice to obtain an estimate of a_{ki}.

Similarly to CR$^+$, the portfolio default rate and its mean for the 2-CR$^+$ model are defined as follows. The portfolio default rate for 2-CR$^+$ is

$$\begin{aligned} T_{2-CR^+}(\gamma) := \sum_{k=1}^{K} \tilde{g}_k \gamma_k &= \sum_{k=1}^{K} \tilde{g}_k \sum_{i=1}^{M} a_{ki} Y_i \\ &= \sum_{i=1}^{M} g^i Y_i, \end{aligned} \tag{10}$$

where $g^i = \sum_{k=1}^{K} \tilde{g}_k a_{ki} \ \ \forall \ \ i = 1, ..., M$, while the mean portfolio default rate for 2-CR$^+$ is $\hat{T}_{2-CR^+}(\gamma) := \frac{T_{2-CR^+}(\gamma)}{M}$.

Remark *We make a choice of gamma distribution for γ and Y since it mainly allows us to obtain an explicit expression for the probability generating function for the loss distribution.*

Definition 2 (Notional credit portfolio) *This is a counterpart of the original credit portfolio except that the obligors have unit exposure, i.e. $\nu_i = 1$ for each $i \in \{1, ..., O\}$.*

We first show in the following lemma why the rare large behaviour of $\hat{T}_{CR^+}(\gamma)$ and $\hat{T}_{2-CR^+}(\gamma)$ is instrumental in our understanding of a large loss in a notional credit portfolio.

Lemma 1 *Consider a notional credit portfolio. If $K = M$ and $Pr(\hat{T}_{CR^+}(\gamma) > q) \leq Pr(\hat{T}_{2-CR^+}(\gamma) > q)$ for q large enough (to be determined later), then $VaR_\alpha^{CR^+} \leq VaR_\alpha^{2-CR^+}$ for every $\alpha \in (0,1)$.*

Proof. From the definition of a notional credit portfolio, the portfolio loss is

$$L = \sum_{i=1}^{O} \nu_i N_i = \sum_{i=1}^{O} N_i = \tilde{N},$$

where $\tilde{N} = \sum_{i=1}^{O} N_i$ is thus a sum of independent Poisson-distributed random variables. As $(N_i(\gamma) =^d Poi(X_i(\gamma)).$, hence from (8) and (2), by the Poisson law of small numbers, for CR$^+$ $(\tilde{N}(\gamma) =^d Poi(T_{CR^+}(\gamma))$ and $(\tilde{N}(\gamma) =^d Poi(T_{2-CR^+}(\gamma))$ for 2-CR$^+$. The probability that the portfolio is subject to more than k defaults in CR$^+$ is given by the simple formula

$$\begin{aligned} Pr^{CR^+}(\tilde{N}(\gamma) > k) &= 1 - \sum_{j=0}^{k} \frac{e^{-T_{CR^+}(\gamma)}(T_{CR^+}(\gamma))^j}{j!} \\ &= 1 - e^{-K\hat{T}_{CR^+}(\gamma)} \sum_{j=0}^{k} \frac{\left(K\hat{T}_{CR^+}(\gamma)\right)^j}{j!} \end{aligned}$$

Similarly for 2-CR$^+$ and assuming $K = M$ we have

$$\begin{aligned} Pr^{2-CR^+}(\tilde{N}(\gamma) > k) &= 1 - e^{-M\hat{T}_{2-CR^+}(\gamma)} \sum_{j=0}^{k} \frac{\left(M\hat{T}_{2-CR^+}(\gamma)\right)^j}{j!} \\ &= 1 - e^{-K\hat{T}_{2-CR^+}(\gamma)} \sum_{j=0}^{k} \frac{\left(K\hat{T}_{2-CR^+}(\gamma)\right)^j}{j!}. \end{aligned}$$

We use $\leq_D$ to signify first-order stochastic dominance. By L'Hôpital's rule, for some $r > 0$ and $x \in \mathbb{R}^+$, e^{rx} grows faster than the power function. Hence for some fixed number of defaults k, $\hat{T}_{CR^+}(\gamma) \leq_D \hat{T}_{2-CR^+}(\gamma)$ iff $Pr^{CR^+}(\tilde{N}(\gamma) > k) \leq Pr^{2-CR^+}(\tilde{N}(\gamma) > k)$ is true. Thus, as the loss variable L is directly related to the number of defaults $\tilde{N}$, we have $\Pr(\mathrm{L}^{CR^+} > l) \leq \Pr(\mathrm{L}^{2-CR^+} > l)$ for large enough l, which by the definition of VaR then implies $\mathrm{VaR}_\alpha^{CR^+} \leq \mathrm{VaR}_\alpha^{2-CR^+}$. $\square$

To utilize Lemma 1, we need to determine a meaningful value of $q \in \mathbb{R}^+$. We also need to consider certain assumptions on the structure of the credit portfolio. The following Lemma will highlight this structure and explains its necessity.

Lemma 2 *For a credit portfolio, if $\sum_{i=1}^{O} g_k^i p_i = Gp^* \ \forall k \in \{1, ...K\}$, then $\hat{T}_{CR^+}(\gamma)$ converges* *to Gp^* in probability as $K \to \infty$, i.e. for each $\delta > 0$, we have*

$$\lim_{K\to\infty} P\left\{\left|\hat{T}_{CR^+}(\gamma) - Gp^*\right| \geq \delta\right\} = 0.$$

Additionally, if $K = M$, and $\sum_{k=1}^{K} a_{ki} = a_i = a$ for each $i \in \{1,...,M\}$, then $\hat{T}_{2-CR^+}(\gamma)$ converges to Gp^* in probability as $M \to \infty$, i.e. for each $\delta > 0$, we have

$$\lim_{M\to\infty} P\left\{\left|\hat{T}_{2-CR^+}(\gamma) - Gp^*\right| \geq \delta\right\} = 0.$$

Proof. As the γ_k in CR$^+$ are independent, gamma-distributed random variables with mean 1, we have $E\left[\frac{T_{CR^+}(\gamma)}{K}\right] = E\left[\frac{\sum_{k=1}^{K}\sum_{i=1}^{O} g_k^i p_i \gamma_k}{K}\right]$. Thus, to meaningfully quantify the convergence of $\hat{T}_{CR^+}(\gamma)$ we select $\sum_{i=1}^{O} g_k^i p_i := Gp^*\ \forall k \in \{1,...K\}$ leading to $E\left[\frac{T_{CR^+}(\gamma)}{K}\right] = Gp^*$ for some positive constants G and p^*. Therefore

$$\begin{aligned} E\left[\left(\hat{T}_{CR^+}(\gamma) - Gp^*\right)^2\right] &= \frac{G^2 p^{*2}}{K^2} E\sum_{k=1}^{K}[(\gamma_k - 1)^2] \\ &= \frac{G^2 p^{*2}}{K^2}\sum_{k=1}^{K}\sigma_k. \end{aligned}$$

It is thus easy to see that $\hat{T}_{CR^+}(\gamma)$ converges to Gp^* in probability for each $\delta > 0$. Similarly, when $K = M$ and $\sum_{k=1}^{K} a_{ki} = a$ we have $a = 1$. This is because as $\sum_{i=1}^{M} a_{ki} = 1$, switching summations we have $\sum_{i=1}^{M}\sum_{k=1}^{K} a_{ki} = K$. Hence $\sum_{i=1}^{M} a = K$, therefore $aM = K$ implies $a = 1$. Following similar lines of proof as above we can easily conclude that $\hat{T}_{2-CR^+}(\gamma) \xrightarrow{p} Gp^*$ as $M \to \infty$ for each $\delta > 0$ $\square$

From Lemma 2 above we have $\Pr(\hat{T}_{CR^+}(\gamma) > q)$=$\Pr(\frac{T_{CR^+}(\gamma)}{K} > q)$ for $q \in (Gp^*, \infty)$, which constitutes the rare event probability for CR$^+$ while $\Pr(\hat{T}_{2-CR^+}(\gamma) > q)$=$\Pr(\frac{T_{2-CR^+}(\gamma)}{M} > q)$ for $q \in (Gp^*, \infty)$ constitutes the rare event probability for 2-CR$^+$.

To summarize, from Lemma 1 and Lemma 2 we conclude that we can make meaningful and definitive comparison of the two credit risk models using the theory of large deviations, for a credit risk portfolio with the appropriate structure. Such a credit portfolio is termed a "balanced" credit portfolio and is formally defined as follows.

Definition 3 (Balanced credit portfolio) *A balanced credit portfolio (either notional or nonnotional) is one in which $K = M$,$\sum_{i=1}^{O} g_k^i p_i = Gp^*$ and $\sum_{k=1}^{K} a_{ki} = a_i = a\ \forall\ k \in \{1,...K\}$ and $\forall\ i \in \{1,...,M\}$.*

We mention here why we have used the word *balanced*. The condition $K = M$ obviously implies that the number of risk sectors is equal to the number of macroeconomic risk drivers. The financial interpretation of the condition $\sum_{i=1}^{O} g_k^i p_i = Gp^*$ is that one provides equal weights Gp^* to each sector $k, k \in \{1, ..., K\}$ in the portfolio. Likewise, the condition $\sum_{k=1}^{K} a_{ki} = a_i = a$ implies that each macroeconomic variable Y_i $\forall i \in \{1, ..., M\}$ is given equal significance in the portfolio risk computations. Note that from equation (9) and switching summations we have

$$\sum_{k=1}^{K} \sigma_k = \sum_{k=1}^{K}\sum_{i=1}^{M} {a_{ki}}^2 \tilde{\sigma}_i \leq \sum_{i=1}^{M}\sum_{k=1}^{K} a_{ki}\tilde{\sigma}_i = \sum_{i=1}^{M} a\tilde{\sigma}_i \leq \sum_{i=1}^{M} \tilde{\sigma}_i. \tag{11}$$

Remark If we assume that the loss L has a Gaussian distribution, as is the case in the Markovitz portfolio theory, then equation (11) to show that $\text{VaR}_{\alpha}^{CR^+} \leq \text{VaR}_{\alpha}^{2-CR^+}$ for every $\alpha \in$(0,1). However, as we do not know the portfolio loss distribution, we hence need to compute the rare event probability of the portfolio default rate exceeding the value q.
We summarise below the steps required to compare VaR numbers computed using CR$^+$ and 2-CR$^+$.

Methodology for comparing VaRs computed by the CR$^+$ and the 2-Stage CR$^+$:
1. We start with the original/nonnotional balanced credit portfolio.
2. We convert the original credit portfolio to a notional credit portfolio by reassigning the obligor exposure values to 1, i.e. $\nu_i = 1 \forall i \in \{1, ..., O\}$.
3. We then compare the VaR numbers computed by CR$^+$ and 2-CR$^+$ for this notional credit portfolio by computing the rare event probabilities $P(\hat{T}_{CR^+}(\gamma) > q)$ and $P(\hat{T}_{2-CR^+}(\gamma) > q)$ for $q \in (Gp^*, \infty)$, utilising the Gärtner–Ellis theorem in large deviations theory.
4. We then extrapolate the inference made in step 3 back to the original credit portfolio.

Remark All the subsequent calculations shown in Section 2.2.4 are also similar for the 2-stage CR$^+$ model, except that σ_k is replaced by $\tilde{\sigma}_i$, $T_{CR^+}(\gamma)$ is replaced by $T_{2-CR^+}(\gamma)$ and $\hat{T}_{CR^+}(\gamma)$ is replaced by $\hat{T}_{2-CR^+}(\gamma)$.

2.2.4 Analysis of the problem

We note from Lemma 2 that $\hat{T}_{CR^+}(\gamma) \xrightarrow{p} Gp^*$ as $K \to \infty$. We thus are interested in the large deviation asymptotics of $P(\hat{T}_{CR^+}(\gamma) > q)$ for $q \in (Gp^*, \infty)$. As usual, we wish to do this by identifying the rate of growth of the logarithmic moment-generating function of $\hat{T}_{CR^+}(\gamma)$. We want to find a sequence $\{A_K\}_{K\in\mathbb{N}}$ such that $A_K \nearrow \infty$ as $K \to \infty$ and such that for each

$\lambda \in \mathbb{R}$,

$$\Lambda_{CR^+}(\lambda) := \lim_{K\to\infty} \frac{1}{A_K} \log E\left[\exp\left\{\lambda A_K \hat{T}_{CR^+}(\gamma)\right\}\right] \tag{12}$$

is an appropriately nontrivial function of λ. We then should compute the Legendre–Fenchel transform of $\Lambda_{CR^+}(\lambda)$, i.e.

$$\Lambda^*_{CR^+}(q)(\lambda) = \sup_{\lambda\in\mathbb{R}}\{\lambda q - \Lambda_{CR^+}(\lambda)\}. \tag{13}$$

In this spirit we first arrive at an expression for the logarithmic moment-generating function. Subsequently we generate an expression for the rate function by proving the applicability of the Gärtner–Ellis theorem.

Lemma 3 *Assume that A_K grows at the rate of K^2 as $K \to \infty$ i.e. $\lim_{K\to\infty} \frac{A_K}{K^2} = 1$ and $\sum_{i=1}^{\infty} \tilde{\sigma}_i < \infty$. The logarithmic moment-generating function $\Lambda_{CR^+}(\cdot)$ for any $\lambda \in \mathbb{R}$ is given by*

$$\Lambda_{CR^+}(\lambda) = G\lambda p^* + \frac{G^2}{2}(\lambda p^*)^2 \sum_{k=1}^{\infty} \sigma_k.$$

Proof. We now proceed with calculating $\Lambda_{CR^+}(\lambda)$.

$$\begin{aligned}
\Lambda_{CR^+}(\lambda) &= \lim_{K\to\infty} \frac{1}{A_K} \log E\left[\exp\left(\lambda A_K \frac{T_{CR^+}(\gamma)}{K}\right)\right] \\
&= \lim_{K\to\infty} \frac{1}{A_K} \log E\left[\exp\left\{\left(\frac{\lambda G p^* A_K}{K}\right)\sum_{k=1}^{K}\gamma_k\right\}\right] \\
&= \lim_{K\to\infty} \frac{1}{A_K} \log \prod_{k=1}^{K} E\left[\exp\left\{\left(\frac{\lambda G p^* A_K \gamma_k}{K}\right)\right\}\right] \\
&= \lim_{K\to\infty} \frac{1}{A_K} \sum_{k=1}^{K} \frac{-1}{\sigma_k} \log\left(1 - \left(\frac{\lambda G p^* A_K}{K}\right)\sigma_k\right) \\
&= \lim_{K\to\infty} \sum_{k=1}^{K} \frac{-1}{\sigma_k}\left[-\left(\frac{\lambda G p^*}{K}\right)\sigma_k - \frac{1}{2}\left(\frac{\lambda G p^*}{K}\right)^2 A_K \sigma_k^2 - \ldots\right]
\end{aligned}$$

By the assumption made on the growth rate of A_K, and the finiteness condition $\sum_{i=1}^{\infty} \tilde{\sigma}_i < \infty$ coupled with (11),

$$\Lambda_{CR^+}(\lambda) = \lambda G p^* + \frac{G^2(\lambda p^*)^2}{2} \sum_{k=1}^{\infty} \sigma_k.$$

□

We now describe the Gärtner–Ellis theorem and then show that it gives a genuine limit, and in our case, not just lower and upper bounds for the closed and open sets.

Theorem 4 (Gärtner–Ellis theorem) *$(P_K)_{K=1}^{\infty}$ satisfies the large deviations principle with rate function $\Lambda^*_{CR^+}$ given by (13) with Λ_{CR^+} described in (12). In other words:*

- *For every $s \geq 0$ and for $q \in (Gp^*, \infty)$,*

$$\{q \in \mathbb{R} : \Lambda^*_{CR^+}(q) \leq s\} \subset\subset \mathbb{R};$$

- *For every closed set $F \subset \mathbb{R}$,*

$$\limsup_{K \to \infty} \frac{1}{A_K} \log P_K(F) \leq -\inf_{q \in F} \Lambda^*_{CR^+}(q);$$

- *and for every open set $O \subset \mathbb{R}$,*

$$\liminf_{K \to \infty} \frac{1}{A_K} \log P_K(O) \geq -\inf_{q \in O} \Lambda^*_{CR^+}(q).$$

Proof. Based on Theorem 2.3.6 in Dembo and Zeitouni ([4]), we need to check whether $\Lambda_{CR^+}(\lambda)$ is convex in $C^1(\mathbb{R})$ and $\Lambda^{'}_{CR^+}(0) = p^*G$. As $\Lambda_{CR^+}(\lambda)$ is quadratic in λ, the existence of the first and second derivative of $\Lambda(\cdot)$, viz. ($\Lambda^{'}_{CR^+}$ and $\Lambda^{''}_{CR^+}$), can easily be shown. Moreover as $\Lambda^{'}_{CR^+}(\lambda)$ is linear in λ, we have $\Lambda^{'}_{CR^+}(\lambda) \in C^1(\mathbb{R})$. As $\Lambda^{''}_{CR^+} = {p^*}^2 G^2 \sum_{k=1}^{\infty} \sigma_k$, which is strictly positive, $\Lambda_{CR^+}(\lambda)$ is convex. From a simple calculation of $\Lambda^{'}_{CR^+}$ it follows that $\Lambda^{'}_{CR^+}(0) = p^*G$. □

The rate function can now be explicitly calculated. For CR$^+$ we have $\Lambda^*_{CR^+}(q) = \sup_{\lambda \in \mathbb{R}}[\lambda q - \Lambda_{CR^+}(\lambda)]$ where, $\Lambda_{CR^+}(\lambda) = \lambda p^* G + \frac{(\lambda p^*)^2}{2} G^2 \sum_{k=1}^{\infty} \sigma_k$. The function inside the supremum is concave and has a unique optimum λ^* given by simple calculus. It is $\lambda^* = \frac{q - p^* G}{p^* G^2 \sum_{k=1}^{\infty} \sigma_k}$. Substituting this in the objective function we find $\Lambda^*_{CR^+}(q) = \frac{(q - Gp^*)^2}{2(Gp^*)^2 \sum_{k=1}^{\infty} \sigma_k}$ for $q \in (Gp^*, \infty)$. Similarly, since Y_i is Gamma distributed, i.e. $\Gamma(\frac{1}{\tilde{\sigma_i}}, \tilde{\sigma_i})$ and from (2.4) we will obtain the rate function as $\Lambda^*_{2-CR^+}(q) = \frac{(q - Gp^*)^2}{2G^2 {p^*}^2 \sum_{i=1}^{\infty} \tilde{\sigma_i}}$ for $q \in (Gp^*, \infty)$. One can easily observe that $\Lambda^*_{CR^+}(Gp^*) = 0$ and $\Lambda^*_{CR^+}(q) > 0$ for $q \neq Gp^*$. Secondly, $\Lambda^*_{CR^+}(q)$ is increasing on $q \in (Gp^*, \infty)$ and decreasing on $(-\infty, Gp^*)$. Analogous observations can be made for $\Lambda^*_{2-CR^+}(q)$. We next have the following corollary.

Corollary 5 Fix q such that $q \in (Gp^*, \infty)$. Then

$$\lim_{K \to \infty} \frac{1}{A_K} \log \left\{ P\left(\hat{T}_{CR^+}(\gamma)\right) > q \right\} = -\Lambda^*_{CR^+}(q).$$

Proof. This is a straightforward consequence of the fact that $\Lambda^*_{CR^+}$ is increasing on (Gp^*, ∞) and is continuous on $\mathbb{R}$. $\square$

We now present an important result in our analysis that compares the VaR numbers computed by the CR$^+$ and the 2-Stage CR$^+$ for both the notional as well as its original *balanced* class of a credit portfolio.

Lemma 6 *For the notional balanced credit portfolio, the Value at Risk computed by CR$^+$ is less than or equal to the one computed by 2-CR$^+$, i.e. $VaR^{CR^+}_\alpha \leq VaR^{2-CR^+}_\alpha$. The same conclusion holds true for the original balanced credit portfolio.*

Proof.

• Consider the notional balanced credit portfolio. From the definition, $P_K(\cdot)$ tends to zero exponentially rapidly in K^2, i.e. $Pr\left(\hat{T}_{CR^+}(\gamma) > q\right) \asymp e^{-K^2\Lambda^*_{CR^+}(q)}$, while $P_M(\cdot)$ tends to zero exponentially rapidly in M^2, i.e. $Pr\left(\hat{T}_{2-CR^+}(\gamma) > q\right) \asymp e^{-M^2\Lambda^*_{2-CR^+}(q)}$ for $q \in (Gp^*, \infty)$. As $\sum_{k=1}^{\infty} \sigma_k \leq \sum_{i=1}^{\infty} \tilde{\sigma}_i$, from the formulas for $\Lambda^*_{CR^+}(q)$ and $\Lambda^*_{2-CR^+}(q)$, $\Lambda^*_{CR^+}(q) \geq \Lambda^*_{2-CR^+}(q)$. This implies that $Pr\left(\hat{T}_{CR^+}(\gamma) > q\right) \leq Pr\left(\hat{T}_{2-CR^+}(\gamma) > q\right)$. Hence from Lemma 1 $\mathrm{VaR}^{CR^+}_\alpha$ $\leq \mathrm{VaR}^{2-CR^+}_\alpha$.

• Original or non-notional balanced credit portfolio. The VaR computed by CR$^+$ and 2-CR$^+$ depends on the values of $p_i, \nu_i, {g_k}^i, \sigma_k$ and $\tilde{\sigma}_j$ for each $i \in \{1, ..., O\}$, $k \in \{1, ..., K\}$ and $j \in \{1, ..., M\}$, of which p_i, g^i_k and ν_i are obligor specific. Since both CR$^+$ and 2-CR$^+$ operate on the same set of obligors, the variables that really affect the VaR are σ_k and $\tilde{\sigma}_j$. In brief, both credit risk models operate on the same set of obligors $i \in \{1, ..., O\}$ whose exposure values ν_i just scale the portfolio loss through expression (6). Thus as $\mathrm{VaR}^{CR^+}_\alpha \leq \mathrm{VaR}^{2-CR^+}_\alpha$ is valid for the notional balanced credit portfolio, the same conclusion also holds true for the non-notional balanced credit portfolio with only the VaR values being *rescaled* in the latter while the *order* is left unchanged, i.e. $\mathrm{VaR}^{CR^+}_\alpha \leq \mathrm{VaR}^{2-CR^+}_\alpha$. $\square$

Remark *We have made a reference earlier of utilizing large deviations technique to understand stress-testing phenomenon. With the credit risk models discussed here, one could design a stress-tested scenario by assigning uniform maximum exposure and uniform maximum default probability to all the obligors. Portfolio II given below provides insights into how stress-testing of Portfolio I (original portfolio) can be carried out. In that portfolio, based on our need to assign uniform obligor exposure we had assigned an exposure of 1 unit to each obligor. In similar fashion, for stress-testing purpose, we could otherwise uniformly assign the highest value of exposure to each obligor and as well uniformly assign highest default probability to them. This would result in one example of the stress tested scenario for our portfolio. In the values used while describing portfolio (***PII***) here, we notice that for the same values of $\sum_{k=1}^{12} \sigma_k$ and $\sum_{i=1}^{12} \tilde{\sigma}_i$ as in portfolio (***PI***), our portfolio (***PII***) produces higher value of VaR. This is an indication of how uniformity of values either for the portfolio exposure or default* *probabilities or both can result in thicker tail loss distributions and higher values of VaR- a scenario typical of stress-testing.*

In support of Lemma 6, we now provide a numerical investigation as to why $\mathrm{VaR}_\alpha^{CR^+} \leq \mathrm{VaR}_\alpha^{2-CR^+}$ for α= 0.1, 0.05, 0.01.

2.2.5 Numerical results

We now consider the class of portfolio described above and compute the VaR numbers using the CR^+ and the 2-CR^+ models. In this example $Gp^* = 6.5$, $a = 1$ and we compute VaR for $\alpha = 0.1$, $\alpha = 0.05$ and $\alpha = 0.01$. We assume that $\lim_{K\to\infty} \sum_{k=1}^{K} \sigma_k$ converges to $\sum_{k=1}^{12} \sigma_k$ and $\lim_{M\to\infty} \sum_{m=1}^{M} \tilde{\sigma}_m$ converges to $\sum_{m=1}^{12} \tilde{\sigma}_m$.

Portfolio (PI). The test portfolio is made up of $K = M = 12$ sectors, each containing 3000 obligors. Obligors in sectors 3–10 belong in equal parts to one of three classes with adjusted exposures $E_1 = 1, E_2 = 2.5$ and $E_3 = 5$ monetary units and their respective default probabilities are $p_1 = 0.55\%, p_2 = 0.08\%, p_3 = 0.02\%$. For the three obligor classes in sectors 1, 2, 11 and 12, we assume the same default rates but the exposures are twice as large ($E_1 = 2, E_2 = 5$ and $E_3 = 10$). We consider the risk factor variances $\tilde{\sigma}_i$ for all $i \in \{1, ..., M\}$ to be sampled from a notional $U(\cdot, \cdot)$ distribution. For the 2-stage model, the correlation between sector default rates would in principle be the outcome of dependence on common set of risk factors, i.e. $\gamma_k = 0.7Y_k + 0.3Y_{k+1}$ for $k \in \{1, ..., 11\}$ while $\gamma_{12} = 0.7Y_{12} + 0.3Y_1$. VaR computations on this non-notional portfolio for the three cases yields the following results.

Table 2.1: $\text{VaR}_{99.0}$ comparison for PI

$\tilde{\sigma}$	$\sum_k^K \sigma_k$	$\sum_i^M \tilde{\sigma}_i$	VaR_{99}-CR^+	VaR_{99}-2 Stage CR^+
$\tilde{\sigma}_i \sim U(0.1, 0.25)$	1.2306	2.1218	0.1427	0.1476
$\tilde{\sigma}_i \sim U(0.26, 0.75)$	3.9033	6.7298	0.1573	0.1710
$\tilde{\sigma}_i \sim U(0.76, 0.95)$	5.988	10.4201	0.1669	0.1839

Table 2.2: $\text{VaR}_{99.5}$ comparison for PI

$\tilde{\sigma}$	$\sum_k^K \sigma_k$	$\sum_i^M \tilde{\sigma}_i$	$\text{VaR}_{99.5}$-CR^+	$\text{VaR}_{99.5}$-2 Stage CR^+
$\tilde{\sigma}_i \sim U(0.1, 0.25)$	1.2306	2.1218	0.1476	0.1532
$\tilde{\sigma}_i \sim U(0.26, 0.75)$	3.9033	6.7298	0.1653	0.1806
$\tilde{\sigma}_i \sim U(0.76, 0.95)$	5.988	10.4201	0.1758	0.1960

Table 2.3: $\text{VaR}_{99.9}$ comparison for PI

$\tilde{\sigma}$	$\sum_k^K \sigma_k$	$\sum_i^M \tilde{\sigma}_i$	$\text{VaR}_{99.9}$-CR^+	$\text{VaR}_{99.9}$-2 Stage CR^+
$\tilde{\sigma}_i \sim U(0.1, 0.25)$	1.2306	2.1218	0.1597	0.1669
$\tilde{\sigma}_i \sim U(0.26, 0.75)$	3.9033	6.7298	0.1823	0.2040
$\tilde{\sigma}_i \sim U(0.76, 0.95)$	5.988	10.4201	0.1960	0.2226

In the next example we construct a notional portfolio counterpart (PII) from the realistic portfolio (PI).

Portfolio (PII). The test portfolio is again made up of $K = M = 12$ sectors, each containing 3000 obligors. Obligors in sectors 1–12 belong in equal parts to one of three classes with adjusted exposures $E_1 = 1, E_2 = 1$ and $E_3 = 1$ monetary units and the respective default probabilities are $p_1 = 0.55\%, p_2 = 0.08\%, p_3 = 0.02\%$. We use the same sector default rate variances as for (PI). For the 2-stage model, as in (PI), the correlation between sector default rates would in principle be the outcome of their dependence on a common set of risk factors, i.e. $\gamma_k = 0.7Y_k + 0.3Y_{k+1}$ for $k \in \{1, ..., 11\}$ while $\gamma_{12} = 0.7Y_{12} + 0.3Y_1$. VaR computations on this notional portfolio for the three cases gave the following results.

Table 2.4: VaR_{99} comparison for PII

$\tilde{\sigma}$	$\sum_k^K \sigma_k$	$\sum_i^M \tilde{\sigma}_i$	VaR_{99}-CR^+	VaR_{99}-2 Stage CR^+
$\tilde{\sigma}_i \sim U(0.1, 0.25)$	1.2306	2.1218	0.2800	0.3083
$\tilde{\sigma}_i \sim U(0.26, 0.75)$	3.9033	6.7298	0.2972	0.3583
$\tilde{\sigma}_i \sim U(0.76, 0.95)$	5.988	10.4201	0.3194	0.3889

Table 2.5: $\mathrm{VaR}_{99.5}$ comparison for PII

$\tilde{\sigma}$	$\sum_k^K \sigma_k$	$\sum_i^M \tilde{\sigma}_i$	$\mathrm{VaR}_{99.5}$-CR^+	$\mathrm{VaR}_{99.5}$-2 Stage CR^+
$\tilde{\sigma}_i \sim U(0.1, 0.25)$	1.2306	2.1218	0.2861	0.3194
$\tilde{\sigma}_i \sim U(0.26, 0.75)$	3.9033	6.7298	0.3056	0.3778
$\tilde{\sigma}_i \sim U(0.76, 0.95)$	5.988	10.4201	0.3306	0.4111

Table 2.6: $\text{VaR}_{99.9}$ comparison for PII

$\tilde{\sigma}$	$\sum_k^K \sigma_k$	$\sum_i^M \tilde{\sigma}_i$	$\text{VaR}_{99.9}$-CR^+	$\text{VaR}_{99.9}$-2 Stage CR^+
$\tilde{\sigma}_i \sim U(0.1, 0.25)$	1.2306	2.1218	0.2901	0.3417
$\tilde{\sigma}_i \sim U(0.26, 0.75)$	3.9033	6.7298	0.3278	0.4194
$\tilde{\sigma}_i \sim U(0.76, 0.95)$	5.988	10.4201	0.3583	0.4639

The following observations are based on the results tabulated above.

Observations

From the above computations, we observe that the VaR numbers in (PI) are scaled down from (PII) with the VaR order preserved, i.e. $\text{VaR}^{CR^+} \leq \text{VaR}^{2-CR^+}$. This confirms the conclusion we drew from Lemma 6. Note from the first observation that in all the above computations, $\sum_{k=1}^K \sigma_k$=1.2306 while $\sum_{i=1}^M \tilde{\sigma}_i = 2.1218$ for $K = M = 12$. As this *difference* between $\sum_{k=1}^K \sigma_k$ and $\sum_{i=1}^M \tilde{\sigma}_i$ is *relatively small*, the VaR results were close enough since the tail decay rates of the portfolio default rate process for both CR^+ and 2-CR^+ were nearly the same. The difference in the VaR numbers computed by the CR^+ and the 2-CR^+ naturally increases as the difference between $\sum_{k=1}^K \sigma_k$ and $\sum_{i=1}^M \tilde{\sigma}_i$ increases. This phenomenon can be observed in the second and third observations of all the above tables.

2.2.6 Conclusion

We have provided a methodology for comparing the VaR values computed by the 2-stage CR^+ model and its primitive, the CR^+ model, using the theory of large deviations. We have shown that for the particular class of credit portfolio called a balanced portfolio, the VaRs produced by the 2-CR^+ model are higher than those computed by the CR^+ model.

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Chapter 3

On the existence of an asymptotic options price in a Markov modulated economy.

3.1 Article " On the existence of an asymptotic options price in a Markov modulated economy"-A. Deshpande and T.K. Siu

3.1.1 Abstract

Asymptotic analysis in option pricing was introduced in Fouque et al. [8]. An asymptotic analysis for option valuation in a Markovian, regime-switching, financial market has been recently proposed by Basu and Ghosh [1]. In such market, the key model parameters are modulated by a continuous-time, finite-state, Markov chain. For an asymptotic analysis, they discussed two variations of the chain, namely, a slow chain and a fast chain. They observed that there exists an asymptotic option price for the slow chain case while argued that such price may not exist for the fast chain case. Since the Markov modulated market is incomplete there are many equivalent martingale measures if they exist. This results in a range of option prices. In this note, we characterize the range of option prices for the slow

and fast chain. More precisely we determine the range of option prices for the slow chain case. Based on the premise that a European call option price in the fast chain case is lower than the corresponding limiting price, we prove that for the fast chain case there exists no range of option prices. We thereby positively conclude that the observation of Basu and Ghosh [1] with regards to the existence of an asymptotic option price is indeed true.
Keywords: Regime Switching Market; Asymptotic Option Pricing.

3.1.2 Synopsis

Markovian regime-switching models have had a long history in economics. Quandt [15] and Goldfeld and Quandt [9] adopted two-state, regime-switching, regression models to model and analyze nonlinear and non-stationary economic data. Tong [17], [18], [20] and Tong and Lim [19] pioneered the idea of regime switching in nonlinear time series analysis and introduced one of the oldest nonlinear time series models, namely, the class of threshold time series models. Hamilton [11] popularized the use of Markovian regime-switching models in economics and econometrics. Since then, much attention has been paid to examining the empirical performance of Markovian regime-switching models in fitting economic and financial time series. Numerous empirical studies support the use of Markovian regime-switching models in economics, finance and actuarial science. Indeed, Markovian regime-switching models provide a natural and convenient way to incorporate structural changes in economic conditions when modeling asset prices movements. They can describe a number of important “stylised” facts of economic and financial time series, such as the heavy-tailedness of assets’ returns, time-varying conditional volatility, volatility clustering, regime switchings, nonlinearity and nonstationarity.

Recently, attention has turned to option valuation under regime-switching models. Some works on this topic include Guo [10], Buffington and Elliott [3], Elliott et al. [7], Siu [16], Deshpande and Ghosh [6]and Basu and Ghosh [1], Deshpande [4], and amongst others. The main difficulty of option valuation in a Markovian regime-switching model is that the market is incomplete, and hence, there is more than one equivalent martingale measure. Guo [10] addressed this problem by completing the market with a set of fictitious assets. Elliott et al. [7] adopted the Esscher transform, a time-honored tool in actuarial science, to pick an equivalent martingale measure for valuation. Deshpande and Ghosh [6] addressed the valuation problem from the perspective of risk minimization. In a recent paper by Basu and Ghosh [1], the concept and existence of asymptotic option prices in Markovian regime-switching diffusion markets were discussed from the perspective of partial differential equations. The authors introduced the asymptotic option prices by fast and slow variations of the chain. They found that an asymptotic option price exists for a slow chain, but observed that it may not exist for a fast chain.

In this paper, we consider an asymptotic analysis for option valuation in a Markovian, regime-switching, financial market by determining the range of option prices in an incomplete market. Here we consider the situation where the modulating Markov chain is observable and interpret the states of the chain as proxies of the levels of some observable economic factors, such as gross domestic product, retail price index and sovereign credit ratings, etc. The introduction of the Markov chain randomness and the Brownian motion randomness in a single equation results in market incompleteness. For the asymptotic analysis of an option price, we consider two variations of the chain, namely, a slow chain and a fast chain. From the deduction of the range of available option prices, we show the existence of an asymptotic option price for a slow chain and the non-existence of such price for a fast chain. This provides a theoretical justification for the key result in the asymptotic analysis for option pricing in a Markovian, regime-switching, financial market obtained in Basu and Ghosh [1].

The paper is structured as follows. The next section presents the model dynamics. In Section 3.1.4, we discuss the concepts of asymptotic option prices in a Markovian regime-switching market. Section 3.1.5 develops the main approach in our current paper and establishes, using the range of option prices, the existence of an asymptotic option price for a slow chain and the non-existence of an asymptotic option price for a fast chain. The final section gives some concluding remarks.

3.1.3 The Model Dynamics

We consider a simplified continuous-time financial market consisting of two primitive securities, namely, a (locally) risk-free bond and a risky share. These securities can be traded continuously over time in a finite-time horizon $\mathcal{T} := [0, T]$, where $T < \infty$. As usual, we suppose that there are no transaction costs and taxes, that any fractional units of the securities can be traded, and that the borrowing and lending rates are the same. To describe uncertainty, we consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability measure.

Let $\boldsymbol{\theta} := \{\boldsymbol{\theta}(t)|t \in \mathcal{T}\}$ be a continuous-time, finite-state, Markov chain on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with state space $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_M\} \subset \Re^M$, where the j^{th}-component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij} for each $i, j = 1, 2, \cdots, M$. The state space $\mathcal{E}$ is called the canonical state space for the chain $\boldsymbol{\theta}$. It has been introduced in Elliott et al. [6] for the purpose of mathematical convenience. We suppose that the Markov chain is homogeneous and irreducible. To specify the probability law of the chain, we define a rate matrix, or an intensity matrix, $\mathbf{\Lambda} := [\lambda_{ij}]_{i,j=1,2,\cdots,M}$, where λ_{ij} is the constant transition intensity of the chain $\boldsymbol{\theta}$ from state $\mathbf{e}_i$ to state $\mathbf{e}_j$. Note that for each $i, j = 1, 2, \cdots, M$ with $i \neq j$, $\sum_{j=1}^{M} \lambda_{ij} = 0$ and $\lambda_{ij} \geq 0$, so $\lambda_{ii} \leq 0$. Let $\mathbb{F}^{\boldsymbol{\theta}} := \{\mathcal{F}^{\boldsymbol{\theta}}(t)|t \in \mathcal{T}\}$ be the right-continuous, $\mathcal{P}$-complete, filtration generated by the values of the chain $\boldsymbol{\theta}$. Then with the canonical state space of the chain, Elliott et al. [6] obtained the following semimartingale dynamics for the chain:

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + \int_0^t \mathbf{\Lambda}\boldsymbol{\theta}(u)du + \mathbf{V}(t) \ , \quad t \in \mathcal{T} \ .$$

Here $\mathbf{V} := \{\mathbf{V}(t)|t \in \mathcal{T}\}$ is an $\Re^M$-valued, square-integrable, $(\mathbb{F}^{\boldsymbol{\theta}}, \mathcal{P})$-martingale.

For each $t \in \mathcal{T}$, let $r(t, \boldsymbol{\theta}(t))$ be the, (locally), risk-free rate of interest of the bond at time t. We suppose that $r(t, \boldsymbol{\theta}(t))$ is modulated by the chain $\boldsymbol{\theta}$ as:

$$r(t, \boldsymbol{\theta}(t)) = \langle \mathbf{r}(t), \boldsymbol{\theta}(t) \rangle \ .$$

Here $\mathbf{r}(t) := (r_1(t), r_2(t), \cdots, r_M(t))' \in \Re^M$ and $r_i(t) > 0$ for each $i = 1, 2, \cdots, M$ and each $t \in \mathcal{T}$; $r_i(t)$ is the interest rate when $\boldsymbol{\theta}(t) = \mathbf{e}_i$; the scalar product $\langle \cdot, \cdot \rangle$ selects the component of the vector $\mathbf{r}(t)$ of interest rates in force according to the state of the Markov chain $\boldsymbol{\theta}(t)$ at the current time t.

Then the price process of the, (locally), risk-free bond evolves over time as:

$$B(t) = \exp\left(\int_0^t r(u, \boldsymbol{\theta}(u))du \right) \ , \quad t \in \mathcal{T} \ , \quad B(0) = 1 \ .$$

For each $t \in \mathcal{T}$, let $\mu(t, \boldsymbol{\theta}(t))$ and $\sigma(t, \boldsymbol{\theta}(t))$ be the appreciation rate and the volatility of the risky share price at time t, respectively. Similarly, we suppose that the chain $\boldsymbol{\theta}$ modulates $\mu(t, \boldsymbol{\theta}(t))$ and $\sigma(t, \boldsymbol{\theta}(t))$ as:

$$\mu(t, \boldsymbol{\theta}(t)) = \langle \boldsymbol{\mu}(t), \boldsymbol{\theta}(t) \rangle \ , \quad \sigma(t, \boldsymbol{\theta}(t)) = \langle \boldsymbol{\sigma}(t), \boldsymbol{\theta}(t) \rangle \ .$$

Here $\boldsymbol{\mu}(t) := (\mu_1(t), \mu_2(t), \cdots, \mu_M(t))' \in \Re^M$ and $\boldsymbol{\sigma}(t) := (\sigma_1(t), \sigma_2(t), \cdots, \sigma_M(t))' \in \Re^M$, respectively; for each $i = 1, 2, \cdots, M$, $\mu_i(t)$ and $\sigma_i(t)$ are the appreciation rate and the volatility of the risky share when $\boldsymbol{\theta}(t) = \mathbf{e}_i$; $\mu_i(t) > r_i(t)$ and $\sigma_i(t) > 0$.

Let $W := \{W(t)|t \in \mathcal{T}\}$ be the standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ with respect to the right-continuous, $\mathcal{P}$-completion of its natural filtration $\mathbb{F}^W := \{\mathcal{F}^W(t)|t \in \mathcal{T}\}$. We assume that $\mu_i(t)$, $r_i(t)$ and $\sigma_i(t)$ are bounded for each $i = 1, 2, \cdots, M$ and $t \in \mathcal{T}$. For each $t \in \mathcal{T}$, let

$$\mathcal{F}(t) := \mathcal{F}^W(t) \vee \mathcal{F}^{\boldsymbol{\theta}}(t) \vee \mathcal{N} \ .$$

Here $\mathcal{A} \vee \mathcal{B}$ is the minimal σ-algebra containing both the σ-algebras $\mathcal{A}$ and $\mathcal{B}$; $\mathcal{N}$ is the collection of all $\mathcal{P}$-null subsets of $\mathcal{F}$. Write $\mathbb{F} := \{\mathcal{F}(t)|t \in \mathcal{T}\}$.

Then we suppose that under $\mathcal{P}$, the evolution of the share price process $X := \{X(t)|t \in \mathcal{T}\}$ over time is governed by the following Markovian, regime-switching, geometric Brownian motion:

$$dX(t) = \mu(t, \boldsymbol{\theta}(t))X(t)dt + \sigma(t, \boldsymbol{\theta}(t))X(t)dW(t) \ , \quad X(0) = x_0 > 0 \ .$$

It is known that the Markovian, regime-switching, market is, in general, incomplete. Consequently, there is more than one equivalent martingale measure for valuation. Here we adopt either the Risk Minimizing options pricing theory as like in Deshpande and Ghosh [6] or an actuarial approach as like the regime-switching Esscher transform proposed in Elliott et al. [7] to pick an equivalent martingale measure. It was shown in Deshpande and Ghosh [6] and Elliott et al. [7] that under an EMM, say $\mathcal{Q}$, the share price process follows the dynamics:

$$dX(t) = r(t, \boldsymbol{\theta}(t))X(t)dt + \sigma(t, \boldsymbol{\theta}(t))X(t)dW^{\mathcal{Q}}(t) \ ,$$

where $W^{\mathcal{Q}} := \{W^{\mathcal{Q}}(t)|t \in \mathcal{T}\}$ defined by:

$$W^{\epsilon}(t) = W^{\mathcal{Q}}(t) := W(t) - \int_0^t \left(\frac{r(u, \boldsymbol{\theta}(u)) - \mu(u, \boldsymbol{\theta}(u))}{\sigma(u, \boldsymbol{\theta}(u))}\right)du \ ,$$

is an $(\mathbb{F}, \mathcal{Q})$-standard Brownian motion. To simplify our discussion, we assume that the Brownian motion $W^{\mathcal{Q}}$ and the chain $\boldsymbol{\theta}$ are independent under $\mathcal{Q}$. It is not our focus to discuss the selection of an equivalent martingale measure here. For a discussion on this issue, interested readers may refer to Deshpande and Ghosh [6] and Elliott et al. [7]. We can represent the Markov chain as a stochastic integral with respect to a Poisson random measure. For

$i,j \in \mathcal{X} = \{1,2,...,M\}$, let $\Delta_{i,j}$ be consecutive (w.r.t. to lexicographic ordering on $\mathcal{X} \times \mathcal{X}$) left closed right open intervals of the real line, each having a length $\lambda_{i,j}$. By embedding $\mathcal{X}$ into $\mathbb{R}^M$, define a function $\bar{h} : \mathcal{X} \times \mathbb{R} \to \mathbb{R}^M$ by,

$$\bar{h}(i,z) = \begin{cases} j-i & \text{if } z \in \Delta_{ij}(y) \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$dX(t) = \int_{\mathbb{R}} \bar{h}(X(t-),z)p(dt,dz)$$

where $p(dt,dz)$ is a Poisson random measure with intensity $dt \times m(dz)$, where $m(dz)$ is the Lebesgue measure on $\mathbb{R}$; $p(\cdot,\cdot)$ and $W(\cdot)$ are independent and where $\tilde{p}(dt,dz)$ is the compensated random measure associated with the chain.

3.1.4 Asymptotic Option Prices

In the case of a Markovian regime-switching model, the main focus of an asymptotic analysis of option pricing is to study the behavior of an option price when the transition intensity of the modulating Markov chain becomes either very small or very large. The mathematical formulation of the analysis is given as follows.

For each $\epsilon > 0$, let $\boldsymbol{\theta}^\epsilon := \{\boldsymbol{\theta}^\epsilon(t)|t \in \mathcal{T}\}$ be a "perturbation" of the original continuous-time Markov chain $\boldsymbol{\theta}$ associated with ϵ so that the transition intensities λ_{ij}^ϵ's of $\boldsymbol{\theta}^\epsilon$ are defined by those of $\boldsymbol{\theta}$ as follows:

$$\lambda_{ij}^\epsilon := \epsilon\lambda_{ij} \ , \quad i,j = 1,2,\cdots,M \ .$$

Then it is easy to see that for each $i,j = 1,2,\cdots,M$ with $i \neq j$, $\sum_{j=1}^M \lambda_{ij}^\epsilon = 0$ and $\lambda_{ij}^\epsilon \geq 0$.

For each $\epsilon \geq 0$, let $\mathbb{F}^\epsilon := \{\mathcal{F}^\epsilon(t)|t \in \mathcal{T}\}$ be the right-continuous, $\mathcal{P}$-complete, natural filtration generated by the perturbed chain $\boldsymbol{\theta}^\epsilon := \{\boldsymbol{\theta}^\epsilon(t)|t \in \mathcal{T}\}$. For each $t \in \mathcal{T}$, let

$$\mathcal{G}^\epsilon(t) := \mathcal{F}^W(t) \vee \mathcal{F}^\epsilon(t) \vee \mathcal{N} \ .$$

Write $\mathbb{G}^\epsilon := \{\mathcal{G}^\epsilon(t)|t \in \mathcal{T}\}$.

Consider, for each $\epsilon \geq 0$, the following $\mathbb{G}^\epsilon$-adapted process $\Lambda^\epsilon := \{\Lambda^\epsilon(t) | t \in \mathcal{T}\}$:

$$\Lambda^\epsilon(t) := \exp\bigg(- \frac{1}{2} \int_0^t \eta^2(u, \boldsymbol{\theta}^\epsilon(u)) du - \int_0^t \eta(u, \boldsymbol{\theta}^\epsilon(u)) dW(u) \bigg) \ ,$$

where

$$\eta(t, \boldsymbol{\theta}^\epsilon(t)) := \frac{\mu(u, \boldsymbol{\theta}^\epsilon(u)) - r(u, \boldsymbol{\theta}^\epsilon(u))}{\sigma(u, \boldsymbol{\theta}^\epsilon(u))} \ .$$

Note that $\{\eta(t, \boldsymbol{\theta}^\epsilon(t)) | t \in \mathcal{T}\}$ is bounded, so the Novikov's condition is satisfied. Consequently, Λ^ϵ is an $(\mathbb{G}^\epsilon, \mathcal{P})$-martingale. Then for each $\epsilon \geq 0$, we define probability measure $\mathcal{Q}^\epsilon \sim \mathcal{P}$ on $\mathcal{G}^\epsilon(T)$ by putting:

$$\frac{d\mathcal{Q}^\epsilon}{d\mathcal{P}}\bigg|_{\mathcal{G}^\epsilon(T)} := \Lambda^\epsilon(T) \ .$$

By a version of Girsanov's theorem, the process

$$W^{\mathcal{Q}^\epsilon}(t) := W(t) + \int_0^t \eta(u, \boldsymbol{\theta}^\epsilon(u)) du \ , \quad t \in \mathcal{T} \ ,$$

is an $(\mathbb{G}^\epsilon, \mathcal{Q}^\epsilon)$-standard Brownian motion.

Let $X^\epsilon := \{X^\epsilon(t) | t \in \mathcal{T}\}$ be the "perturbed" share price process associated with the "perturbed" modulating Markov chain θ^ϵ. Then, under $\mathcal{Q}^\epsilon$, the price process of the share is governed by:

$$dX^\epsilon(t) = r(t, \boldsymbol{\theta}^\epsilon(t)) X^\epsilon(t) dt + \sigma(t, \boldsymbol{\theta}^\epsilon(t)) X^\epsilon(t) dW^{\mathcal{Q}^\epsilon}(t) \ .$$

Hence, for each $\epsilon \geq 0$, $\mathcal{Q}^\epsilon$ is an equivalent martingale measure. Note also that under $\mathcal{Q}^\epsilon$, $W^{\mathcal{Q}^\epsilon}$ and $\boldsymbol{\theta}^\epsilon$ are stochastically independent.

It is known that for each $i = 1, 2, \cdots, M$, the sojourn time of $\boldsymbol{\theta}^\epsilon$ in state $\mathbf{e}_i$ is exponentially distributed with rate $|\lambda^\epsilon_{ii}|$. When ϵ becomes very small, $|\lambda^\epsilon_{ii}|$ is relatively small compared to μ_i for each $i = 1, 2, \cdots, M$. Consequently, the chain $\boldsymbol{\theta}^\epsilon$ moves slowly compared with that of the perturbed share price process X^ϵ. In this case, the chain $\boldsymbol{\theta}^\epsilon$ is said to be a slow chain.
Remark *We note an economic interpretation of slow and fast chain . As ϵ goes to zero, for the slow chain case, the share price movement is more volatile. This is un-usual since correspondingly the economy is not switching rapidly. Likewise for the fast chain case, as ϵ*

gets smaller , the share price is less volatile even though the economy fluctuates rapidly. Consider now a standard European call option with strike price K and maturity at time T. Then for the slow chain case, the payoff of the call option at maturity T is given by:

$$\phi^{\epsilon} := (X^{\epsilon}(T) - K)^{+} \ .$$

Write, for each $i = 1, 2, \cdots, M$, $\phi^{\epsilon}(t, x, i)$ for the value of the call option at time t given that $X^{\epsilon}(t) = x$ and $\boldsymbol{\theta}^{\epsilon}(t) = \mathbf{e}_i$. For each $i = 1, 2, \cdots, M$, we define the following generator, or partial differential operator:

$$L_i := r(t, i)x\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(i, t)x^2\frac{\partial^2}{\partial x^2} \ .$$

For the slow chain it is easy to check that $\phi^{\epsilon}(t, x, i)$, $i = 1, 2, \cdots, M$, satisfy the following system of coupled partial differential equations:

$$\left(\frac{\partial}{\partial t} + L_i\right)[\phi^{\epsilon}(t, x, i)] + \sum_{j=1}^{M} \lambda^{\epsilon}_{ij}\phi^{\epsilon}(t, x, i) = 0 \ , \qquad (1)$$

with terminal condition $\phi^{\epsilon}(T, x, i) = (x - K)^{+}$.

It can be shown, using the same arguments as in Ladyzhenskaya et al. [12], that the above Cauchy problem (1) has a unique smooth solution given by a family of smooth functions $\{\phi^{\epsilon}(t, x, i)|i = 1, 2, \cdots, M\} \subset \mathcal{C}^{1,2}(\mathcal{T} \times \Re)$ having at most polynomial growth, where $\mathcal{C}^{1,2}(\mathcal{T} \times \Re)$ is the space of functions which are continuously differentiable in $t \in \mathcal{T}$ and twice continuously differentiable $x \in \Re$. It has also been shown in Deshpande and Ghosh [6] that the unique solution $\{\phi^{\epsilon}(t, x, i)|i = 1, 2, \cdots, M\}$ of the Cauchy problem (1) is the set of locally risk minimizing option prices over different states of the economy. Indeed, this set of option prices also coincides with those obtained from the regime-switching Esscher transform. Let for each $i = 1, 2, \cdots, M$,

$$\tilde{r}(t, i) := \frac{1}{T - t}\int_t^T r(u, \mathbf{e}_i)du = \frac{1}{T - t}\int_t^T r_i(u) \left\langle \boldsymbol{\theta}^{\epsilon}(u), \mathbf{e}_i \right\rangle du \ ,$$

and

$$\tilde{\sigma}(t, i) := \sqrt{\frac{1}{T - t}\int_t^T \sigma^2(u, \mathbf{e}_i)du} = \sqrt{\frac{1}{T - t}\int_t^T \sigma_i^2(u) \left\langle \boldsymbol{\theta}^{\epsilon}(u), \mathbf{e}_i \right\rangle du} \ .$$

At each time $t \in [0,T]$, an arbitrage-free price of the contingent claim is given by:

$$\tilde{R}(t,i)\tilde{V}^{\epsilon}(t) \triangleq E^{\epsilon}[\tilde{R}(T,i)\xi^{\epsilon}|\mathcal{G}^{\epsilon}(t)] ,$$

where $E^{\epsilon}[\cdot|\mathcal{G}^{\epsilon}(t)]$ is a conditional expectation given $\mathcal{G}^{\epsilon}(t)$ under $\mathcal{Q}^{\epsilon}$ and

$$\tilde{R}(t,\boldsymbol{\theta}^{\epsilon}(t)) := e^{-\int_0^t \tilde{r}(s,\boldsymbol{\theta}^{\epsilon}(s))ds} .$$

More precisely, $\tilde{R}(t,\boldsymbol{\theta}^{\epsilon}(t))$ should be written as $\tilde{R}(t,\{\boldsymbol{\theta}^{\epsilon}(s); s \in [0,t]\},\boldsymbol{\theta}^{\epsilon}(t))$. For our treatment here, we wish to emphasize that the discounted factor is a function of the terminal regime of the chain, so we adopt the notation $\tilde{R}(t,\boldsymbol{\theta}^{\epsilon}(t))$.
The following result is due to Basu and Ghosh [1], (see Theorem 3.1 (i) therein). We state it here without giving the proof.

Lemma 1. *For a slow moving Markov chain,*

$$\lim_{\epsilon \to 0} \phi^{\epsilon}(t,x,i) = \phi^{0}(t,x,i) = \phi^{BS}(t,x,\tilde{r}(i),\tilde{\sigma}(i),K,T) , \quad \forall \ (t,x,i) ,$$

where ϕ^{BS} is the Black-Scholes-Merton call option price.

Similarly we consider another perturbed Markov chain $\bar{\boldsymbol{\theta}}^{\epsilon} := \{\bar{\boldsymbol{\theta}}^{\epsilon}(t)|t \in \mathcal{T}\}$ with transition intensities:

$$\lambda_{ij} = \bar{\lambda}_{ij} := \frac{\lambda_{ij}}{\epsilon} , \quad i,j = 1,2,\cdots,M .$$

Write $\bar{X}^{\epsilon} := \{\bar{X}^{\epsilon}(t)|t \in \mathcal{T}\}$ for the share price process corresponding to the chain $\bar{\boldsymbol{\theta}}^{\epsilon}$. When ϵ becomes very small, $|\lambda^{\epsilon}_{ii}|$ becomes very large compared to μ_i, for each $i = 1,2,\cdots,M$. Consequently, the perturbed share price process $\bar{X}^{\epsilon}$ moves slowly compared to the chain $\bar{\boldsymbol{\theta}}^{\epsilon}$. In other words, $\bar{\boldsymbol{\theta}}^{\epsilon}$ is a fast chain.

Similarly, for each $i = 1,2,\cdots,M$, let $\bar{\phi}^{\epsilon}(t,x,i)$ be the value of the call option at time t given that $\bar{X}^{\epsilon}(t) = x$ and $\bar{\boldsymbol{\theta}}^{\epsilon}(t) = \mathbf{e}_i$ in the fast chain case. Then it is also easy to see that $\bar{\phi}^{\epsilon}(t,x,i)$, $i = 1,2,\cdots,M$, satisfy the following system of coupled partial differential equations:

$$\left(\frac{\partial}{\partial t} + L_i\right)[\bar{\phi}^{\epsilon}(t,x,i)] + \sum_{j=1}^{M} \bar{\lambda}^{\epsilon}_{ij}\bar{\phi}^{\epsilon}(t,x,i) = 0 , \tag{2}$$

with terminal condition $\bar{\phi}^{\epsilon}(T,x,i) = (x-K)^{+}$.

We now discuss the situation where the underlying Markov chain is a fast chain. Before that, we need to know what typical values the option price takes for the fast chain case. Accordingly we define $\bar{\mu}(t) = \sum_{i=1}^{M} \mu(i,t)\pi_i$, $\bar{r}(t) = \sum_{i=1}^{M} r(i,t)\pi_i$ and $\bar{\sigma}(t) = \sum_{i=1}^{M} \sigma(i,t)\pi_i$ where $\pi = (\pi_1, ..., \pi_M)$ is the unique invariant measure of the Markov chain $\{\boldsymbol{\theta}(t)\}_{t\geq 0}$ generated by its intensity matrix. Let $\hat{r} = \frac{1}{T-t}\int_t^T \bar{r}(u)du$ and $\hat{\sigma} = \sqrt{\frac{1}{T-t}\int_t^T \bar{\sigma}^2(u)du}$.
Lemma 2 *Consider the fast chain dynamics. Then*

$$\lim_{\epsilon\to 0} \phi^{\epsilon}(t, x, r(t, \boldsymbol{\theta}^{\epsilon}(t)), \sigma(t, \boldsymbol{\theta}^{\epsilon}(t)), K, T) = \phi^{BS}(t, x, \hat{r}, \hat{\sigma}, K, T) \ ,$$

where ϕ^{BS} is again the Black-Scholes-Merton price.

Proof. Refer Basu and Ghosh [1], Theorem 3.2. □

In Basu and Ghosh [1], asymptotic expansions for approximating $\phi^{\epsilon}(t, x, i)$ and $\bar{\phi}^{\epsilon}(t, x, i)$ in terms of the Black-Scholes-Merton option pricing formula have been discussed. Loosely speaking, the basic idea of an asymptotic analysis of option prices is to investigate the behavior of option prices when some key model parameters become either very small or very large. We say that for each $(t, x, i) \in \mathcal{T} \times \Re_+ \times \{1, 2, \cdots, M\}$, there exists an asymptotic option price $\phi^{\epsilon}(t, x, i)$ at the point (t, x, i) if there is a sequence of coefficients $\{\phi^{(k)}(t, x, i)|k = 0, 1, \cdots\}$ such that $\phi^{\epsilon}(t, x, i)$ admits the following asymptotic expansion:

$$\phi^{\epsilon}(t, x, i) = \phi^{(0)}(t, x, i) + \epsilon\phi^{(1)}(t, x, i) + \epsilon^2\phi^{(2)}(t, x, i)... \ i \in \{1, ..., M\} \tag{3}$$

Alternatively, we say that there exists an asymptotic option price $\phi^{\epsilon}(t, x, i)$ if we can write it in terms of the Black-Scholes Merton formula i.e $\phi^{\epsilon}(\cdot) = \phi^{BS}(\cdot) + O(\epsilon)$ for any small but positive ϵ. On the other hand, we say that there is no asymptotic option price at $(t, x, i) \in \mathcal{T} \times \Re_+ \times \{1, 2, \cdots, M\}$ if we cannot find a sequence of coefficients $\{\phi^{(k)}(t, x, i)|k = 0, 1, \cdots\}$ such that $\phi^{\epsilon}(t, x, i)$ admits the above asymptotic expansion. In other words, there does not exist an asymptotic option price $\phi^{\epsilon}(t, x, i)$ if we cannot write it in terms of the corresponding Black-Scholes-Merton formula. i.e $\phi^{\epsilon}(\cdot) = \phi^{BS}(\cdot) + O(\epsilon)$ for any small but positive ϵ.

Basu and Ghosh [1] showed that in the slow chain case, there exists an asymptotic option price, (see Theorem 3.1 therein). That is, one can approximate $\phi^{\epsilon}(t, x, i)$ in terms of the sum of the Black-Scholes Merton formula and the correction term. However, in the fast chain case, they mentioned that an asymptotic option price *may* not exist, (see Remark 3.1 therein), i.e. one cannot approximate $\phi^\epsilon(t,x,i)$ in terms of the Black-Scholes Merton formula and the correction term. They attributed it to the possible non-existence of the limiting Markov chain. The objective of this paper is to provide a concrete justification for this claim by determining the range of European options prices. In particular we show that there exists a range of option prices for the slow chain case, while no such range exists for the fast chain case. Our analysis draws an inspiration from Bellamy and Jeanblanc [2].

Remark *We mostly keep the notations in the following text commensurate with the ones used in Basu and Ghosh [1]. This we do so to preserve uniformity in notations thereby providing easy readability between the research article of Basu and Ghosh ([1]) and the following text.*

In the next section we provide a mathematical analysis for this situation.

3.1.5 Analysis

In this section, we present the asymptotic analysis of option pricing considered in Basu and Ghosh [1].

Let ξ^ϵ be a European contingent claim which is a non-negative random variable in $L^2(\Omega, \mathcal{G}^\epsilon(T), \mathcal{P})$. The objective is to characterize the bounds viz. $[\inf_{\epsilon\in\mathcal{E}} \tilde{V}^\epsilon(t), \sup_{\epsilon\in\mathcal{E}} \tilde{V}^\epsilon(t)]$ for the slow and fast chain cases. We start our analysis for the slow chain case.

Slow Chain case

We restrict our attention to the case where ξ^ϵ takes the form $\xi^\epsilon = h(X^\epsilon(T))$ for some convex function h having bounded one sided derivatives. We define the Black-Scholes function $\phi^{BS}(t,x,i)$ by

$$\tilde{R}(t,i)\phi^{BS}(t,x,i) = E[\tilde{R}(T,i)h(X^\epsilon(T))|(X^\epsilon(t)=x, \boldsymbol{\theta}^\epsilon(t)=\mathbf{e}_i)] \quad \phi^{BS}(T,x,i)=h(x)$$
$$\forall\ i\ \in\ \{1,...,M\}$$

where the stock price dynamics of X^ϵ is given by,

$$dX^\epsilon(t) = \tilde{r}(t,\boldsymbol{\theta}^\epsilon(t))X^\epsilon(t)dt + \tilde{\sigma}(t,\boldsymbol{\theta}^\epsilon(t))X^\epsilon(t)dW^\epsilon(t) \quad X^\epsilon(0)=x$$

For a fixed state say i, the drift and the volatility terms of this SDE are constants. Then the Black-Scholes function is known to be given by

$$\phi^{BS}(t,x,i) = \frac{\tilde{R}(T,i)}{\tilde{R}(t,i)} E\left[h\left(\frac{x\tilde{R}(T,i)}{\tilde{R}(t,i)} \exp[\tilde{\Sigma}(t,i)U - \frac{1}{2}\tilde{\Sigma}^2(t,i)] \right) \right]$$

where U is a standard normal variable and $\tilde{\Sigma}(t,i) = \int_t^T \tilde{\sigma}(s,i)ds$. Under the additional assumption on $\tilde{\sigma}$ to be continuous in (t,i), and Hölder continuous in $x \in [0,\infty]$ uniformly in $t \in [0,T]$, coupled with the fact that h is convex, we have that $\phi^{BS}(t,x,i)$ is convex w.r.t. to x and belongs to $\mathcal{C}^{1,2}$ and its Delta is bounded i.e. $|\frac{\partial \phi^{BS}}{\partial x}(t,x,i)| \leq C$ for some positive constant C, Refer [13]. We consider the operators $\mathcal{L}^s$ and Λ^s defined on $\mathcal{C}^{1,2}$ functions by

$$\begin{aligned} \mathcal{L}^s f(t,x,i) &= \frac{\partial f}{\partial t}(t,x) + \tilde{r}(t,i)x\frac{\partial f}{\partial x} + \frac{1}{2}x^2\tilde{\sigma}^2(t,i)\frac{\partial^2 f}{\partial x^2}(t,x) \\ \Lambda^s f(t,x,i) &= \sum_{j=1, j\neq i}^{M} \epsilon\lambda_{ij} f(t,x,j) \ , \end{aligned}$$

where the superscript “s” in the above generators represent the slow chain case.

Here the Black-Scholes function satisfies $\mathcal{L}^s(\tilde{R}\phi^{BS})(t,x,i) = 0$. We have the following important theorem

Theorem 1

Suppose $\tilde{V}^\epsilon(t)$ is an arbitrage-free price process defined by

$$\tilde{R}(t,i)\tilde{V}^\epsilon(t) = E^\epsilon[\tilde{R}(T,i)h(X^\epsilon(T))|\mathcal{G}^\epsilon(t)],$$

for each $i = 1, 2, \cdots, M$.

1. The hedging error caused by jumps in Markov chain is given by $\Lambda^s\phi^{BS}(t,x,i)$. More precisely

$$\tilde{R}(t,i)\tilde{V}^\epsilon(t) = \tilde{R}(t,i)\phi^{BS}(t,x,i) + e^\epsilon(t,i)$$

where $e^\epsilon(t,i) = E^\epsilon[\int_t^T \tilde{R}(s,i)\Lambda^s\phi^{BS}(s,x,i)ds|\mathcal{G}^\epsilon(t)]$.

2. Any arbitrage-free price is bounded below by the Black-Scholes function, evaluated at the underlying asset value i.e.

$$\phi^{BS}(t,x,i) \leq \tilde{V}^\epsilon(t) \quad \forall \ \ \epsilon > 0 \ ,$$

3. If moreover $0 \leq h(x) \leq x, h(0) = 0$ and $h(x)$ is bounded, any viable price is bounded above *by the underlying asset value*

$$\phi^{BS}(t, X^\epsilon(t), i) \le \tilde{V}^\epsilon(t) \le X^\epsilon(t) \quad \forall \quad \epsilon > 0 \; .$$

Proof Let X^ϵ be solution of

$$dX^\epsilon(t) = \tilde{r}(t,i)X^\epsilon(t)dt + \tilde{\sigma}(t,i)X^\epsilon(t)dW^\epsilon(t)$$

As $\phi^{BS} \in \mathcal{C}^{1,2}$. From the Ito's formula for a Markov modulated diffusion process we have,

$$\begin{aligned}
& \tilde{R}(T,i)\phi^{BS}(T, X^\epsilon(T), i) = \tilde{R}(t,i)\phi^{BS}(t, X^\epsilon(t), i) \\
+ & \int_t^T \Big[\mathcal{L}^s(\tilde{R}\phi^{BS})(s, X^\epsilon(s), i) + \tilde{R}(s,i)\Lambda^s\phi^{BS}(s, X^\epsilon(s), i)\Big] ds \\
+ & \int_t^T \tilde{R}(s,i)\frac{\partial \phi^{BS}}{\partial x}(s, X^\epsilon(s), i)X^\epsilon(s)\tilde{\sigma}(s,i)dW^\epsilon(s) \\
+ & \int_{\mathbb{R}} [\phi^{BS}(s, X^\epsilon(s-), i + \bar{h}(i,z)) - \phi^{BS}(s, X^\epsilon(s-), i)]\tilde{p}^\epsilon(du, dz) \; ,
\end{aligned}$$

$\tilde{p}^\epsilon(dt, dz)$ is the compensated random measure associated with the chain under $\mathcal{Q}^\epsilon$.

From the boundedness of the delta and the existence of the moments of the price process, the stochastic integral in above expression is a P^ϵ- martingale. Coupled with the fact that $\mathcal{L}(\tilde{R}\phi^{BS})(s, X^\epsilon(s), i) = 0$ and taking P^ϵ conditional expectation with respect to $\mathcal{G}^\epsilon(t)$ gives,

$$\begin{aligned}
& E^\epsilon(\tilde{R}(T,i)\phi^{BS}(T, X^\epsilon(T), i)|\mathcal{G}^\epsilon(t)) \\
= \; & E^\epsilon(\tilde{R}(T,i)h(X^\epsilon(T), i)|\mathcal{G}^\epsilon(t)) \\
= \; & \tilde{R}(t,i)\phi^{BS}(t, X^\epsilon(t), i) + E^\epsilon \int_t^T \tilde{R}(s,i)\Lambda^s\phi^{BS}(s, X^\epsilon(s), i)ds
\end{aligned}$$

From the convexity of $\phi^{BS}(t,\cdot,\cdot)$ we have $\Lambda^s\phi^{BS}(t,\cdot,\cdot) \ge 0$. Hence we have $\tilde{V}^\epsilon(t) \ge \phi^{BS}(t, X^\epsilon(t), i)$. We also have $\lim_{\epsilon\to 0}\tilde{V}^\epsilon(t) = \phi^{BS}(t,x,i)$. Therefore we have for any small but positive ϵ, $\tilde{V}^\epsilon(t) = \phi^{BS}(t,x,i) + O(\epsilon)$. $\square$

Now let us briefly discuss what will happen for the fast chain case.

Fast Chain case

We define the Black-Scholes function $\phi^{BS}(t,x)$ by

$$\bar{R}(t)\phi^{BS}(t,x) = E[\bar{R}(t)h(\bar{X}^0(T))|(\bar{X}^0(t) = x)] \quad \phi^{BS}(T,x) = h(x)$$

where $\bar{R}(t) = e^{\int_t^T \bar{r}(s)ds}$ and the stock price dynamics of $\bar{X}^0$ is given by,

$$d\bar{X}^0(t) = \hat{r}\bar{X}^0(t)dt + \hat{\sigma}\bar{X}^0(t)dW^0(t) \quad \bar{X}^0(0) = x$$

Note that $\bar{X}^0(t)$ is the limiting stock price dynamics of the fast Markov chain modulated dynamics. In the limit as $\epsilon \to 0$, the drift and the volatility terms of this SDE are constants then, the Black-Scholes function is known to be given by

$$\phi^{BS}(t,x) = \frac{\bar{R}(T)}{\bar{R}(t)} E\Big[h\Big(\frac{x\bar{R}(T)}{\bar{R}(t)} \exp[\bar{\Sigma}(t)(t)U - \frac{1}{2}\bar{\Sigma}^2(t)]\Big)\Big]$$

where U is a standard normal variable and $\bar{R}(t) = e^{-(T-t)\hat{r}}$, $\bar{\Sigma}^2(t) = (T-t)\hat{\sigma}^2$. Under the additional assumption on $\bar{\sigma}$ to be continuous in (t,x), and Hölder continuous in $x \in [0,\infty]$ uniformly in $t \in [0,T]$, coupled with the fact that h is convex, we have that $\phi^{BS}(t,x)$ is convex w.r.t. to x and belongs to $\mathcal{C}^{1,2}$ and its Delta is bounded i.e. $|\frac{\partial \phi^{BS}}{\partial x}(t,x)| \le C$ for some positive constant C [13]. We consider the operators $\mathcal{L}^f$ and Λ^f defined on $\mathcal{C}^{1,2}$ functions by

$$\begin{aligned} \mathcal{L}^f(f)(t,x) &= \frac{\partial f}{\partial t}(t,x) + \bar{r}(t)x\frac{\partial f}{\partial x} + \frac{1}{2}x^2\bar{\sigma}^2(t)\frac{\partial^2 f}{\partial x^2}(t,x) \\ \Lambda^f f(t,x) &= 0 \end{aligned}$$

where the superscript f in the above generators represents the fast chain situation.

By Ito's formula for the Markov-modulated diffusion,

$$\begin{aligned} &\bar{R}(T)\phi^{BS}(T,\bar{X}^0(T)) = \bar{R}(t)\phi^{BS}(t,\bar{X}^0(t)) \\ &+ \int_t^T \Big[\mathcal{L}(\bar{R}\phi^{BS})(s,\bar{X}^0(s))\Big]ds \\ &+ \int_t^T \bar{R}(s)\frac{\partial \phi^{BS}}{\partial x}(s,\bar{X}^0(s))\bar{X}^0(s)\bar{\sigma}(s)\bar{X}^0(s)dW^0(s) \end{aligned} \tag{4}$$

for any $\epsilon > 0$. As $\mathcal{L}(\bar{R}\phi^{BS})(s,\bar{X}^0(s)) = 0$, arguing as in the case of the slow chain case we have , $\bar{V}^0(t) = \phi^{BS}(t,\bar{X}^0(t))$.

Now let us suppose that one can write the European call option price for the fast chain dynamics with $\epsilon > 0$ in an asymptotic fashion i.e. write it around the Black-Scholes option price. Then we have $\bar{V}^\epsilon(t) > \phi^{BS}(t,\bar{X}^0(t))$. From the result in (4), we have $\bar{V}^\epsilon(t) > \bar{V}^0(t)$. This implies that the limiting European call option price is lower than the more riskier European call option price under the fast chain dynamics. This in general is not true and hence our assumption is not true. Hence there exist no such equation like $\bar{V}^\epsilon(t) = \phi^{BS}(t,\bar{X}^0(t)) +$

$O(\epsilon)$. Therefore there exists no asymptotic options price in the fast chain case. $\square$

Remark *For the fast chain case, as ϵ gets smaller, the share price moves slower than the underlying Markov chain that signifies the state of the economy. Hence stock price volatility fluctuates less rapidly than the underlying economy. Therefore for a given expiration, for a European call option, in the limit as ϵ goes to zero, the underlying stock price fluctuates less in comparison to the strike price. Higher value of (implied) volatility of the underlying share price results in higher value of option prices and otherwise. Therefore for the fast chain case, the options prices are lower as $\epsilon \to 0$ and thereby we have lower level of the implied volatility. Hence for the fast chain case, graphing the implied volatility against the strike prices towards expiration would yield a relatively flat curve in comparison to the skewed smile.*

3.1.6 Concluding Remarks

We showed how one can determine the range of options price for proving that there exist an asymptotic expansion for the slow chain case and opposite for the fast chain case.

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Chapter 4

Asymptotic stability of semi-Markov modulated jump diffusions.

4.1 Synopsis

In this chapter, we will study issues of stability for the class of semi-Markov modulated jump diffusions (sMMJDs) whose operator turns out to be an integro-partial differential operator. We find conditions under which the solutions of this class of switching jump-diffusion processes are almost surely exponentially stable and moment-exponentially stable. We also provide conditions that imply almost sure convergence of the trivial solution when the moment-exponential stability of the trivial solution is guaranteed. We further investigate and determine the conditions under which the semi-Markov modulated jump diffusion perturbed nonlinear system of differential equations $\frac{dX_t}{dt} = f(X_t)$ is almost surely exponentially stable. It is observed that for a one-dimensional state space, an unstable system of linear differential equation could be stabilized just by the addition of a jump process and that any addition of a Brownian motion part has no effect on its stability. However, we show that for a state space of dimension 2 or higher, the Brownian motion in fact destabilizes the sMMJD perturbed system of nonlinear differential equations. Our main motivation for studying asymptotic stability in this thesis is its connection to the ruin problem in risk theory as proposed by Khasminskii and Milstein [47]. We have shown in Chapter 1 how in fact the stability of an SDE could be further connected to the options pricing problem in finance. Also, a very recent recent book by Swishchuk and Islam [58] reports discussing connection of stochastic stability to finance. On an explicit application front, jump-diffusion models of the regime-switching type are also widely applied in options pricing theory. There have been many studies of the latter. Yuen and Yang [83] priced options in a jump-diffusion regime-switching model where they used the trinomial tree method for pricing. Elliott et al. [30] considered pricing options under a generalized Markov-modulated jump-diffusion model wherein the underlying measure process was defined to be a generalized mixture of Poisson random measures and encompassed a general class of processes, including a generalized gamma process. Siu et al. [54] studied pricing life insurance products under a generalized jump-diffusion model with a Markov-switching compensator. This, together with the result of Khasminskii and Milstein connecting stability theory to ruin theory encourages us to study the issue of stability with regards to a semi-Markov-modulated jump-diffusion of the following type:

$$\begin{aligned} dX_t &= b(X_t, \theta_t)dt + \sigma(X_t, \theta_t)dW_t + dJ_t \\ dJ_t &= \int_\Gamma g(X_t, \theta_t, \gamma)N(dt, d\gamma) \\ X_0 &= x, \theta_0 = \theta, \end{aligned} \tag{1}$$

where $X(\cdot)$ takes values in $\mathbb{R}^r$ and θ_t is a finite-state semi-Markov process taking values in $\mathcal{X} = \{1, ..., M\}$. Let Γ be a subset of $\mathbb{R}^r - 0$; it is the range space of impulsive jumps. For any set B in Γ, $N(t, B)$ counts the number of jumps on $[0, t]$ with values in B and is independent of the Brownian motion W_t, $b(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{X} \to \mathbb{R}^r$,$\sigma(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{X} \to \mathbb{R}^r \times \mathbb{R}^d$, $g(\cdot, \cdot, \cdot) : \mathbb{R}^r \times \mathcal{X} \times \times\Gamma \to \mathbb{R}^r$. For future use we define the compensated Poisson measure $\tilde{N}(dt, d\gamma) = N(dt, d\gamma) - \lambda\pi(d\gamma)dt$, where $\pi(\cdot)$ is the jump distribution and $0 < \lambda < \infty$ is the jump rate.

In this article as suggester earlier, we intend to study the stability of Equation 1 with regards to the criterions of almost sure exponential stability and moment exponential stability. We have defined these criterions in chapter 1.

Our first main result, Theorem 3.1, provides conditions under which the trivial solution to (1) is almost surely exponentially stable. We then determine the conditions under which the trivial solution to 1 is moment exponentially stable via Theorem 3.2. Theorem 3.3 determines the conditions that guarantee that the trivial solution to (1) is almost certainly asymptotically stable if it is moment exponentially stable. In section 4 we illustrate these two criterions of stability using some simple examples. We further investigate and determine the conditions under which the sMMJD-perturbed nonlinear system of differential equations $\frac{dX_t}{dt} = f(X_t)$ is almost surely exponentially stable (Theorem 5.1). It is observed that for one-dimensional state spaces, an linear unstable system of differential equation could be stabilized just by the addition of a jump process and that any addition of a Brownian motion part has no effect on its stability. However, we show that for a state space of dimension two or higher, the Brownian motion in fact destabilizes the sMMJD-perturbed system of nonlinear differential equations (Theorem 5.2).

4.2 Article "Asymptotic stability of semi-Markov-modulated jump diffusions"—A. Deshpande

4.2.1 Abstract

We consider the class of semi-Markov modulated jump diffusions (sMMJDs) whose operator turns out to be an integro-partial differential operator. We find conditions under which the solutions of this class of switching jump-diffusion processes are almost surely exponentially stable and moment exponentially stable . We also provide conditions that imply almost sure convergence of the trivial solution when the moment exponential stability of the trivial solution is guaranteed. We further investigate and determine the conditions under which the trivial solution of the sMMJD-perturbed nonlinear system of differential equations $\frac{dX_t}{dt} = f(X_t)$ is almost surely exponentially stable. It is observed that for a one-dimensional state space, a linear unstable system of differential equations when stabilized just by the addition of the jump part of an sMMJD process does not get destabilized by any addition of a Brownian motion. However in a state space of dimension at least two, we show that a corresponding nonlinear system of differential equations stabilized by jumps gets de-stabilized by addition of Brownian motion.

Keywords: semi-Markov modulated jump diffusions, almost sure stability, moment-exponential stability.

4.2.2 Introduction

The stability of stochastic differential equations (SDEs) has a long history with some key works being those of Arnold [4], Hasminskii and Milstein [9], and Ladde and Lakshmikantham [14]. SDEs with switching have been applied in diverse areas such as finance (Deshpande and Ghosh [6]) and biology (Hanson [8]). On the same note, the stability of these processes has been much studied, in particular by Ji and Chizeck [11] and Mariton [17], who both studied the stability of a jump-linear system of the form $\dot{x}_t = A(r_t)x_t$, where r_t is a Markov chain. Basak et al. [9] discussed the stability of a semilinear SDE with Markovian-regime switching of the form $\dot{x}_t = A(r_t)x_t dt + \sigma(r_t, x_t)dW_t$. Mao [16] studied the exponential stability of a general nonlinear diffusion with Markovian switching of the form $dx_t = f(x_t, t, r_t)dt + g(x_t, t, r_t)dW_t$. Yin and Xi [18] studied the stability of Markov-modulated jump-diffusion processes (MMJDs).

Consider the following jump-diffusion equation in which the coefficients are modulated by an underlying semi-Markov process:

$$\begin{aligned} dX_t &= b(X_t, \theta_t)dt + \sigma(X_t, \theta_t)dW_t + dJ_t \\ dJ_t &= \int_\Gamma g(X_t, \theta_t, \gamma)N(dt, d\gamma) \\ X_0 &= x, \theta_0 = i, \end{aligned} \tag{2}$$

where $X(\cdot)$ takes values in $\mathbb{R}^r$ and θ_t is a finite-state semi-Markov process taking values in $\mathcal{X} = \{1, ..., M\}$. Let Γ be a subset of $\mathbb{R}^r - 0$; it is the range space of impulsive jumps. For any set B in Γ, $N(t, B)$ counts the number of jumps on $[0, t]$ with values in B and is independent of the Brownian motion W_t, $b(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{X} \to \mathbb{R}^r$,$\sigma(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{X} \to \mathbb{R}^r \times \mathbb{R}^d$, $g(\cdot, \cdot, \cdot) : \mathbb{R}^r \times \mathcal{X} \times \Gamma \to \mathbb{R}^r$. For future use we define the compensated Poisson measure $\tilde{N}(dt, d\gamma) = N(dt, d\gamma) - \lambda\pi(d\gamma)dt$, where $\pi(\cdot)$ is the jump distribution and $0 < \lambda < \infty$ is the jump rate. Equation (2) can be regarded as the result of the following M equations:

$$\begin{aligned} dX_t &= b(X_t, i)dt + \sigma(X_t, i)dW_t + \int_\Gamma g(X_t, i, \gamma)N(dt, d\gamma) \\ X_0 &= x, \theta_0 = \theta, \end{aligned}$$

that switch from one state to another according to the underlying movement of the semi-Markov process.

Unlike the special Markov-modulated case in which the x-dependent diffusion is a partial differential operator, the semi-Markov case is characterized by an integro-partial differential operator. In this article we study the asymptotic stability of sMMJDs. We also investigate the perturbation of the nonlinear differential equation $\frac{dX_t}{dt} = f(X_t)$ by an sMMJD. We determine the conditions under which the perturbed system is almost surely exponentially stable. We show that for a one-dimensional state space, the deterministic linear unstable system of differential equations that can be stabilized by the addition of a jump component of the process X_t, surprisingly can never be destabilized by an addition of a Brownian motion. An interesting question we may ask here is, can the similar inference hold true for X_t in higher dimension? The answer is surprisingly *no*. We show that for a state space with dimension greater than or equal to 2, a corresponding non-linear system that is stabilized by the jump component of the process X_t can in fact be destabilized by addition of the Brownian motion part. We organize the article as follows.

In Section 4.1.2 we briefly establish a representation of a class of semi-Markov processes as a stochastic integral with respect to a Poisson random measure. We define the concepts of almost sure exponential stability and moment exponential stability. In Section 4.2.4, we present conditions that guarantee almost sure exponential stability and moment exponential stability of the trivial solution of (2). In general there is no connection between these two stability criteria. However, under additional conditions one can say when does the moment exponential stability guarantees or implies almost sure exponential stability. We elaborate on this aspect while concluding this section. In Section 4.2.5 we provide some examples to illustrate these two stability criterion in our context. In Section 4.2.6, we investigate the conditions for which a nonlinear system of differential equation of the type $\frac{dX_t}{dt} = f(X_t)$ is almost surely exponentially stable. We then investigate its behavior in higher-dimensional state space, as mentioned earlier. The article ends with concluding remarks.

4.2.3 Preliminaries

We assume that the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is complete with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets. If v is some vector then $|v|$ is its Euclidean norm and v' is its transpose, while if A is a matrix then its trace norm is denoted as $|A| = \sqrt{tr(A'A)}$. $\mathbb{R}_+$ stands for positive part of the real line while r is a positive integer. Let $\mathcal{C}^{2,1}(\mathbb{R}^r \times \mathcal{X} \times \mathbb{R}_+)$ denote the family of all functions on $\mathbb{R}^r \times \mathcal{X} \times \mathbb{R}_+$ which are twice continuously differentiable in x and continuously differentiable in y. Consider $\{\theta_t\}_{t\geq 0}$ as a semi-Markov process taking values in $\mathcal{X}$ with transition probability $p_{i,j}$ and conditional holding time distribution $F(t|i)$. Thus if $0 \leq t_0 \leq t_1 \leq ...$ are times when jumps occur, then

$$P(\theta_{t_{n+1}} = j, t_{n+1} - t_n \leq t | \theta_{t_n} = i) = p_{ij} F(t|i). \tag{3}$$

Matrix $[p_{ij}]_{\{i,j=1,\ldots,M\}}$ is irreducible and for each i, $F(\cdot|i)$ has continuously differentiable and bounded density $f(\cdot|i)$. Embed $\mathcal{X}$ in $\mathbb{R}^r$ by identifying i with $e_i \in \mathbb{R}^r$. For $y \in [0,\infty)$ $i,j \in \mathcal{X}$, let

$$\lambda_{ij}(y) = p_{ij}\frac{f(y|i)}{1-F(y|i)} \geq 0 \text{ and } \forall \ i \neq j, \lambda_{ii}(y) = -\sum_{j\in\mathcal{X}, j\neq i} \lambda_{ij}(y) \ \forall \ i \ \in \mathcal{X}.$$

Let the stationary distribution of the semi-Markov process be defined as $\nu_i \triangleq \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{I}_{\theta_s=i} ds$ where $\mathbb{I}.$ takes value 1 if $\theta_s = i$ and 0 otherwise for any $i \in \mathcal{X}$.

For $i \neq j \in \mathcal{X}$, $y \in \mathbb{R}_+$ let $\Lambda_{ij}(y)$ be consecutive (with respect to lexicographic ordering on $\mathcal{X}\times\mathcal{X}$) left-closed, right-open intervals of the real line, each having length $\lambda_{ij}(y)$. Define the functions $\bar{h} : \mathcal{X}\times\mathbb{R}_+\times\mathbb{R} \to \mathbb{R}$ and $\bar{g} : \mathcal{X}\times\mathbb{R}_+\times\mathbb{R} \to \mathbb{R}_+$ by

$$\bar{h}(i,y,z) = \begin{cases} j-i & \text{if } z \in \Lambda_{ij}(y) \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{g}(i,y,z) = \begin{cases} y & \text{if } z \in \Lambda_{ij}(y), j \neq i \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{M}(\mathbb{R}_+\times\mathbb{R})$ be the set of all nonnegative integer-valued σ-finite measures on a Borel σ-field of $(\mathbb{R}_+\times\mathbb{R})$. Define the process $\{\theta_t^{'}, Y_t\}$ described by the following stochastic integral equations:

$$\begin{aligned} \theta_t^{'} &= \theta_0^{'} + \int_0^t \int_{\mathbb{R}} \bar{h}(\theta_{u-}, Y_{u-}, z) N_1(du,dz) \\ Y_t &= t - \int_0^t \int_{\mathbb{R}} \bar{g}(\theta_{u-}, Y_{u-}, z) N_1(du,dz), \end{aligned} \tag{4}$$

where $N_1(dt,dz)$ is an $\mathcal{M}(\mathbb{R}_+\times\mathbb{R})$-valued Poisson random measure with intensity $dtm(dz)$ independent of the $\mathcal{X}$-valued random variable $\theta_0^{'}$, where $m(\cdot)$ is a Lebesgue measure on $\mathbb{R}$. We define the corresponding compensated or centered Poisson measure as $\tilde{N}_1(ds,dz) = N_1(ds,dz) - dsm(dz)$. It was shown in Theorem 2.1 of Ghosh et al. [7] that $\theta_t^{'}$ is a semi-Markov process with transition probability matrix $[p_{ij}]_{\{i,j=1,\ldots,M\}}$ with conditional holding time distributions $F(y|i)$. Therefore one can write $\theta_t^{'} = \theta_t$. We assume that $N(\cdot,\cdot), N_1(\cdot,\cdot)$ and θ_0, W_t, S_0 defined on $(\Omega,\mathcal{F},\mathbb{P})$ are independent.

To ensure that zero is the only equilibrium point of (2) we need the following assumption.

Assumption 1: *Assume $g(x,i,\gamma)$ is $\mathcal{B}\Big(\mathbb{R}^r\times\mathcal{X}\times(\mathbb{R}-\{0\})\Big)$-measurable and that constants*

$C > 0$ *exist such that for each* $i \in \mathcal{X}, x_1, x_2$ *being* $\mathbb{R}^r$*-valued and for each* $\gamma \in \Gamma$ *we have*

$$|b(x_1, i) - b(x_2, i)| + |\sigma(x_1, i) - \sigma(x_2, i)| \leq C|x_1 - x_2|,$$

and

$$|g(x_1, i, \gamma) - g(x_2, i, \gamma)| \leq C|x_1 - x_2|.$$

We also need the condition that the generator matrix $Q(\cdot)$ *is bounded and continuous.* $b(0, i) = 0$*,* $\sigma(0, i) = 0$ *and* $g(x, i, 0) = 0$ *and* $g(0, i, \gamma) = 0$ for each $x \in \mathbb{R}^r$, $i \in \mathcal{X}$ and each $\gamma \in \Gamma$.

The process (X_t, θ_t, Y_t)defined on $(\Omega, \mathcal{F}, \mathbb{P})$ in equations (2) and (4) is jointly Markov and has a generator G given as follows. For $f \in \mathcal{C}^{2,1}(\mathbb{R}^r, \mathcal{X}, \mathbb{R}_+)$, we have

$$\begin{aligned} Gf(x, i, y) &= \frac{1}{2}\sum_{k,l=1}^{r} a_{kl}(x, i)\frac{\partial f(x, i, y)}{\partial x_k \partial x_l} + \sum_{k=1}^{r} b_k(x, i)\frac{\partial f(x, i, y)}{\partial x_k} \\ &+ \frac{\partial f(x, i, y)}{\partial y} + \frac{f(y|i)}{1 - F(y|i)} \sum_{j \neq i, j \in \mathcal{X}} p_{ij}[f(x, j, 0) - f(x, i, y)] \\ &+ \lambda \int_{\Gamma} (f(x + g(x, i, \gamma), i, y) - f(x, i, y))\pi(d\gamma), \end{aligned} \tag{5}$$

where $x \in \mathbb{R}^r, a(x, i) = \sigma(x, i)\sigma'(x, i)$ is a $\mathbb{R}^{r \times r}$ matrix and $a_{kl}(x, i)$ is the $(k, l)^{th}$ element of the matrix a while $b_k(x, i)$ is the k^{th} element of the vector $b(x, i)$.

We define the jump times, i.e. time epochs when jumps occur by $\{\tau_n^N\}$, where $\tau_1^N < \tau_2^N < \cdots < \tau_n^N < \cdots$, to be the enumeration of all elements in the domain D_p of the point process $p(t)$ corresponding to the stationary $\mathcal{F}_t$-Poisson point process $N(dt, d\gamma)$. It is easy to see that $\{\tau_n^N\}$ is an $\mathcal{F}_t$-stopping time for each n. Moreover, we have $\lim_{n\to\infty} \tau_n^N = +\infty$ since the characteristic measure $m(\cdot)$ is finite. Next, let us denote the successive switching instants of the second component, which is the semi-Markov process θ_t that switches from one point on the space $\mathcal{X}$ to another and is denoted by $\tau_0^\theta = 0, \tau_n^\theta = \inf\{t : t > \tau_{n-1}^\theta, X_t \neq X_{\tau_{n-1}^\theta}\}$, $n \geq 1$. Since the Poisson random measure $N(\cdot, \cdot)$ is independent of $N_1(\cdot, \cdot)$, one could adapt the proof of Xi ([19]) to show that with probability 1, $\{\tau_n^N : n \geq 1\}$ and $\{\tau_n^\theta : n \geq 1\}$ are mutually disjoint. Hence between two chain-switching epochs the process X_t behaves like an ordinary jump-diffusion process without switching, a fact that we will use below to show the existence and uniqueness of the sMMJD process X_t. Accordingly, we describe next the existence–uniqueness theorem for Equation (2).

Theorem 2.1 *Assume that Assumption 1 holds. Then there exists a unique solution* $(X_t, t \geq 0)$ *with initial data* (X_0, θ_0, Y_0) *to Equation (2).*

Proof We only provide a sketch of the proof here. Consider $[s, t], \tau_1^\theta, ..., \tau_N^\theta \leq t$. Then as described above, on each of the intervals between the chain switching times, i.e. $[s, \tau_1^\theta), ..., (\tau_N^\theta, t]$, the sMMJD process X_t behaves like a jump-diffusion process. We can then use the standard Picard iteration argument in Applebaum [1] to show the existence–uniqueness of solution X_t. □

Before we proceed with our main analysis concerning these two stability issues we introduce a key Lemma.

Lemma 2.2 $\{P(X_t \neq 0, t \neq 0)\} = 1$ *for any* $X_0 = x \neq 0$, *and* $\theta_0 = \theta \in \mathcal{X}$. *Thus almost all sample paths of any solutions of (2) starting from a nonzero state will never reach the origin.*

Proof We show this in a simple way. From the condition on the coefficients, $b(0, i) = 0, \sigma(0, i) = 0$ and $g(0, i, 0) = 0$. So Equation (2) admits a trivial solution $X_t = 0$. From Theorem 2.1 above, due to the uniqueness of the solution of (2) the conclusion now follows. □

We next have the following generalized Ito's formula.

Lemma 2.3 *Utilizing the operator G in (5), the generalized Ito's formula is given by*

$$
\begin{aligned}
f(X_t, \theta_t, Y_t) \quad - \quad & f(x, \theta, y) = \int_0^t Gf(X_s, \theta_s, Y_s)ds + \int_0^t (\nabla f(X_s, \theta_s, Y_s))'\sigma(X_s, \theta_s)dW_s \\
+ \quad & \int_0^t \int_\Gamma [f(X_{s-} + g(X_{s-}, \theta_{s-}, \gamma), \theta_s, Y_{s-}) - f(X_{s-}, \theta_{s-}, Y_{s-})]\tilde{N}(ds, d\gamma) \\
+ \quad & \int_0^t \int_{\mathbb{R}} [f(X_{s-}, \theta_{s-} + \bar{h}(\theta_{s-}, Y_{s-}, z), Y_{s-} - \bar{g}(\theta_{s-}, Y_{s-}, z)) - f(X_{s-}, \theta_{s-}, Y_{s-})]\tilde{N_1}(ds, dz)
\end{aligned}
\tag{6}
$$

where the local martingale terms are explicitly defined as

$dM_1(t) := (\nabla f(X_t, \theta_t, Y_t))'\sigma(X_t, \theta_t)dW_t,$

$dM_2(t) := \int_\Gamma [f(X_{s-} + g(X_{s-}, \theta_{s-}, \gamma), \theta_{s-}, Y_{s-}) - f(X_{s-}, \theta_{s-}, Y_{s-})]\tilde{N}(ds, d\gamma),$

$dM_3(t) := \int_{\mathbb{R}} [f(X_{s-}, \theta_{s-} + \bar{h}(\theta_{s-}, Y_{s-}, z), Y_{s-} - \bar{g}(\theta_{s-}, Y_{s-}, z)) - f(X_{s-}, \theta_{s-}, Y_{s-})]\tilde{N_1}(ds, dz),$

Proof For details refer to Ikeda and Watanabe [10].

We now discuss the two criteria for stochastic stability that we intend to consider.

Definition 2.4: Almost sure exponential stability *The trivial solution of equation (2) is almost surely exponentially stable if*

$$\limsup_{t \to \infty} \frac{1}{t} \log |X_t| < 0 \quad a.s. \; \forall X_0 \in \mathbb{R}^r \quad a.s$$

The quantity on the left hand side of the above equation is termed as the sample Lyapunov exponent.

Definition 2.5: Moment exponential stability *Let $p > 0$. The trivial solution of (2)* *is said to be p^{th} moment exponentially stable if there exists a pair of constants $\lambda > 0$ and $C > 0$, such that for any $X_0 \in \mathbb{R}^r$*

$$E[|X_t|^p] \leq C|X_0|^p \exp(-\lambda t) \quad \forall\ t \leq 0.$$

In the next section we detail the proofs for obtaining the conditions under which the trivial solution of (2) is almost surely exponentially stable and moment exponentially stable.

4.2.4 Almost sure stability and Moment exponential stability

In the sequel we shall always, as standing hypotheses, assume that Assumption 1 holds. From Theorem 2.1 we deduce that there exists a unique solution to Equation (2). By Lemma 2.2 we know that X_t will never reach zero whenever $X_0 \neq 0$. So in what follows we will only need a function $V(x, i, y) \in \mathcal{C}^{2,1}(\mathbb{R}^r \times \mathcal{X} \times \mathbb{R}_+)$ defined on the domain of the deleted neighborhood of zero. Our first main result provides conditions under which the trivial solution to (2) is almost surely exponentially stable.

Theorem 3.1 *Assume that there exist a function $V \in \mathcal{C}^{2,1}(\mathbb{R}^r \times \mathcal{X} \times \mathbb{R}_+)$ in any deleted neighborhood of zero. Moreover assume that there exist positive constants α, β, ρ_1, ρ_2, $\bar{\rho}_1$ and $\bar{\rho}_2$ for each $x \in \mathbb{R}^r, i \in \mathcal{X}$ and for each $\gamma \in \Gamma$ such that*

$$\begin{aligned} G \log V(x, i, y) &\leq -\alpha \\ |(\nabla_x V(x, i, y))' \sigma(x, i)| &\leq \beta V(x, i, y) \\ \rho_1 \leq \left(\frac{V(x + g(x, i, \gamma), i, y)}{V(x, i, y)} \right) &\leq \rho_2 \\ \bar{\rho}_1 \leq \left(\frac{V(x, i + \bar{h}(i, y, z), y - \bar{g}(i, y, z))}{V(x, i, y)} \right) &\leq \bar{\rho}_2, \end{aligned}$$

then the solution to (2) is almost surely exponentially stable.

Proof Note that

$$\log V(X_t, \theta_t, Y_t) = \log V(X_0, \theta_0, Y_0) + \int_0^t G \log V(X_s, \theta_s, Y_s) ds + M_1(t) + M_2(t) + M_3(t). \quad (7)$$

Here the local martingale terms $M_1(t), M_2(t)$ and $M_3(t)$ are respectively

$$M_1(t) = \int_0^t \frac{(\nabla_x V(X_s, \theta_s, Y_s))' \sigma(X_s, \theta_s)}{V(X_s, \theta_s, Y_s)} dW_s,$$

$$M_2(t) = \int_0^t \int_\Gamma \log\left(\frac{V(X_{s-} + g(X_{s-}, \theta_{s-}, \gamma), \theta_{s-}, Y_{s-})}{V(X_{s-}, \theta_{s-}, Y_{s-})}\right)\tilde{N}(ds, d\gamma),$$

$$\begin{aligned} & M_3(t) \\ = \quad & \int_0^t \int_{\mathbb{R}} [\log V(X_{s-}, \theta_{s-} + \bar{h}(\theta_{s-}, Y_{s-}, z), Y_{s-} - \bar{g}(\theta_{s-}, Y_{s-}, z)) - \log V(X_{s-}, \theta_{s-}, Y_{s-})]\tilde{N}_1(ds, dz). \end{aligned}$$

We deal with Equation (7) term by term to derive an upper bound on $\limsup_{t\to\infty} \frac{\log V(X_t, i, Y_t)}{t}$. Consider first the drift term of Equation (7). It is easy to see from the assumptions made that $\int_0^t G \log V(X_s, \theta_s, Y_s)ds$ will be bounded above by $-\alpha t$. Secondly, we now concentrate on the local martingale terms of (7). First consider the quadratic variation of the $M_1(t)$ term. By Ito's isometry we have

$$\begin{aligned} \langle M_1(t), M_1(t)\rangle &= \int_0^t \left|\frac{(\nabla_x V(X_s, \theta_s, Y_s))' \sigma(X_s, \theta_s)}{V(X_s, \theta_s, Y_s)}\right|^2 ds \\ &\leq \int_0^t \beta^2 ds \leq \beta^2 t. \end{aligned}$$

Next consider the quadratic variation of the local martingale term $M_2(t)$. Based on the following result presented in Kunita [13], page 323, and noting that the jump distribution π is a probability measure i.e. $\int_\Gamma \pi(d\gamma) = 1$ we have,

$$\begin{aligned} \langle M_2(t), M_2(t)\rangle &= \int_0^t \int_\Gamma \left(\log\left[\frac{V(X_{s-} + g(X_{s-}, \theta_{s-}, \gamma), \theta_{s-}, Y_{s-})}{V(X_{s-}, \theta_{s-}, Y_{s-})}\right]\right)^2 \pi(d\gamma)ds \\ &\leq \max[(\log \rho_1)^2, (\log \rho_2)^2]t. \end{aligned}$$

On very similar lines one can easily show that the quadratic variation of the local martingale term $M_3(t)$ is given by

$$\langle M_3(t), M_3(t)\rangle \leq \max[(\log \bar{\rho}_1)^2, (\log \bar{\rho}_2)^2]t.$$

Thus by SLLN for local martingales (refer to Lipster and Shiryayaev [15] p. 140–141), we can say that

$$\limsup_{t\to\infty} \frac{M_1}{t} = \limsup_{t\to\infty} \frac{M_2}{t} = \limsup_{t\to\infty} \frac{M_3}{t} = 0.$$

Thus from (7) and the above discussion one can infer that

$$\limsup_{t\to\infty} \frac{\log V(x, i, y)}{t} \leq -\alpha. \tag{8}$$

Thus, since by assumption $\alpha > 0$, from the definition of almost sure exponential stability, the trivial solution to (2) is almost surely exponentially stable.

We now provide conditions under which the trivial solution to (2) is moment exponentially stable.

Theorem 3.2 *Let $p, \alpha, \alpha_1, \alpha_2 > 0$. Assume that there exists a function $V(x, i, y) \in \mathcal{C}^{2,1}(\mathbb{R}^r, \mathcal{X}, \mathbb{R}_+)$ such that*

$$\alpha_1|x|^p \leq V(x, i, y) \leq \alpha_2|x|^p$$

$$\textit{and} \qquad GV(x, i, y) \leq -\alpha|x|^p$$

$$\textit{Then,} \qquad \limsup_{t\to\infty} \frac{1}{t} \log E|X_t|^p \leq \frac{-\alpha}{\alpha_2}|X_0|^p.$$

As a result the trivial solution of (2) is p^{th}-moment exponentially stable under the conditions discussed above and the p^{th} − moment Lyapunov exponent should not be greater than $\frac{-\alpha}{\alpha_2}$.

Proof The proof is omitted as it is a simple extension of the Markov-modulated SDE case discussed in Mao [16].

In the next theorem we provide criteria to connect these two seemingly disparate stabilty criteria. Specifically, we provide conditions under which the p^{th}-moment exponential stability for $p \geq 2$ always implies Almost sure exponential stability for (2).

Theorem 3.3 *Assume that there exists a positive constant C such that for each $i \in \mathcal{X}$*

$$|b(x, i)| \vee |\sigma(x, i)| \vee |g(x, i, \gamma)| \leq C|x|. \tag{9}$$

If $\forall\ X_0 = x_0 \in \mathbb{R}^r$,

$$\limsup_{t\to\infty} \frac{1}{t} \log E(|X_t|^p) \leq -a, \tag{10}$$

then

$$\limsup_{t\to\infty} \frac{1}{t} \log(|X_t|) \leq -\frac{a}{p} \quad a.s.$$

Then p^{th}-moment exponential stability implies almost sure exponential stability.

We need the Burkholder-Davis-Gundy inequality and is detailed in the remark below.

Remark Let us recall that $[X]$ denotes the quadratic variation of a process say X, and $X_t^* \equiv \sup_{s\leq t}|X_s|$ is its maximum process. Then the Burkholder-Davis-Gundy theorem states that for any $1 \leq p < \infty$, there exist positive constants c_p, C_p such that, for all local martingales

X with $X_0 = 0$ and stopping times τ, the following inequality holds,

$$c_p E\left[[X]_\tau^{p/2}\right] \leq E\left[(X_\tau^*)^p\right] \leq C_p E\left[[X]_\tau^{p/2}\right].$$

Furthermore, for continuous local martingales, this statement holds for all $0 < p < \infty$. For its proof refer to Theorem 3.28 pp. 166 in Karatzas and Shreve [12].

Proof of Theorem 3.3 Let $X_0 \in \mathbb{R}^r$. Let ϵ be arbitrarily small positive number . By the definition of p^{th}-moment exponential stability , there exists a constant K such that

$$E|X_t|^p \leq K \exp^{-(a-\epsilon)t}, \quad t \geq 0. \tag{11}$$

Let $\delta > 0$ be sufficiently small such that,

$$5^p C^p (\delta^p + C_p \delta^{\frac{p}{2}}) < \frac{1}{4} \tag{12}$$

From (2) we have

$$\begin{aligned} X_t &= X_0 + \int_0^t b(X_s, \theta_s) ds + \int_0^t \sigma(X_s, \theta_s) dW_s + \int_0^t \int_\Gamma g(X_{s-}, \theta_{s-}, \gamma) \tilde{N}(ds, d\gamma) \\ &+ \lambda \int_0^t \int_\Gamma g(X_{s-}, \theta_{s-}, \gamma) \pi(d\gamma) ds. \end{aligned}$$

Noting that for $a, b, c, d, e \geq 0$

$$\begin{aligned} (a+b+c+d+e)^p \leq [5(a \vee b \vee c \vee d \vee e)]^p &= 5^p (a^p \vee b^p \vee c^p \vee d^p \vee e^p) \\ &\leq 5^p (a^p + b^p + c^p + d^p + e^p) \end{aligned}$$

we have,

$$\begin{aligned} E[\sup_{(k-1)\delta \leq t \leq k\delta} |X_t|^p] &\leq 5^p E[|X_{(k-1)\delta}|^p] + 5^p E(\int_{(k-1)\delta}^{k\delta} |b(X_s, \theta_s)| ds)^p \\ &+ 5^p E(\sup_{(k-1)\delta \leq t \leq k\delta} \int_{(k-1)\delta}^t |\sigma(X_s, \theta_s) dW_s|^p) \\ &+ 5^p E(\sup_{(k-1)\delta \leq t \leq k\delta} \int_{(k-1)\delta}^t \int_\Gamma |g(X_{s-}, \theta_{s-}, \gamma) \tilde{N}(ds, d\gamma)|^p) \\ &+ 5^p \lambda^p E(\int_{(k-1)\delta}^{k\delta} \int_\Gamma |g(X_{s-}, \theta_{s-}, \gamma)| \pi(d\gamma) ds)^p. \end{aligned} \tag{13}$$

Noting that $\int_\Gamma \pi(d\gamma) = 1$ we have

$$\begin{aligned}
E(\sup_{(k-1)\delta\le t\le k\delta}\int_{(k-1)\delta}^{t}\int_\Gamma |g(X_{s-},\theta_{s-},\gamma)\tilde{N}(ds,d\gamma)|^p) &\le C_pE(\int_{(k-1)\delta}^{k\delta}|g(X_{s-},\theta_{s-},\gamma)|^2ds)^{(p/2)} \\
&\le C_pE(\delta\sup_{(k-1)\delta\le s\le k\delta}|g(X_{s-},\theta_{s-},\gamma)|^2)^{(p/2)} \\
&\le C_pC^p\delta^{p/2}E[\sup_{(k-1)\delta\le s\le k\delta}|X_s|^p], \qquad (14)
\end{aligned}$$

Similarly,

$$\begin{aligned}
E(|\int_{(k-1)\delta}^{k\delta}\int_\Gamma |g(X_{s-},\theta_{s-},\gamma)|\pi(d\gamma)ds)^p &\le E[\delta\sup_{(k-1)\delta\le s\le k\delta}|g(X_{s-},\theta_{s-},\gamma)|]^p \\
&\le C^p\delta^pE(\sup_{(k-1)\delta\le s\le k\delta}|X_s|^p). \qquad (15)
\end{aligned}$$

From (11), one can easily show that

$$E[|X_{(k-1)\delta}|^p] \le K\exp^{-(a-\epsilon)(k-1)\delta} \qquad (16)$$

$$E(\int_{(k-1)\delta}^{k\delta}|b(X_s,\theta_s)|ds)^p \le C^p\delta^pE[\sup_{(k-1)\delta\le s\le k\delta}|X_s|^p] \qquad (17)$$

$$E(\sup_{(k-1)\delta\le t\le k\delta}\int_{(k-1)\delta}^{t}|\sigma(X_s,\theta_s)|dW_s)^p \le C_pC^p\delta^{p/2}E[\sup_{(k-1)\delta\le s\le k\delta}|X_s^p|]. \qquad (18)$$

Hence, substituting (14)-(18) in (13) we obtain

$$E[\sup_{(k-1)\delta\le t\le k\delta}|X_t|^p](1-5^p(C^p\delta^p + C_pC^p\delta^{\frac{p}{2}} + C_pC^p\delta^{\frac{p}{2}} + C^p\delta^p)) \le K5^p\exp^{-(a-\epsilon)(k-1)\delta}. \qquad (19)$$

From (12) we obtain that,

$$E[\sup_{(k-1)\delta\le t\le k\delta}|X_t|^p] \le 2\times 5^pK\exp^{-(a-\epsilon)(k-1)\delta},$$

and utilizing the Borel–Cantelli Lemma as in Mao [16] we deduce the desired implication that p^{th}-moment stability implies almost sure exponential stability.

4.2.5 Examples

We now provide some simple examples to illustrate both the almost surely exponential stability and moment exponential stability. We start with an example on almost surely exponential stability. Consider a two state semi-Markov modulated Jump diffusion problem with $X_t \in \mathbb{R}^r$ and $V(X_t, i, Y_t) = |X_t|$ where the generator matrix is given by

$$Q = \begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix}.$$

Let the holding time in each regime be assumed to follow $f(y|i) = \lambda_i e^{-\lambda_i y}, y > 0, i \in \{1, 2\}$. Note that with the choice of the holding time distribution, the sMMJD collapses to the MMJD case in which case the generator G acting on $V(x, i, y)$ is given by,

$$GV(x,i,y) = \frac{1}{2}trace\Bigg[\Bigg(\frac{I}{|x|} - \frac{xx^{'}}{|x|^3}\Bigg)\sigma(x,i)\sigma^{'}(x,i)\Bigg] + \frac{x^{'}}{|x|}b(x,i) + \lambda\int_\Gamma [|x+g(x,i,\gamma)| - |x|]\pi(d\gamma) \tag{20}$$

Now from Assumption 1 as,

$$\begin{aligned} |\sigma(x,i)| &= |\sigma(x,i) - \sigma(0,i)| \leq C|x| \\ |b(x,i)| &= |b(x,i) - b(0,i)| \leq C|x| \end{aligned}$$

and $|g(x, i, \gamma)| \leq C|x|$ we have,

$$\begin{aligned} GV(x,i,y) &\leq C|x| + C|x| + \lambda(2+C)|x| \\ &= (2C + \lambda C + 2\lambda)|x| \end{aligned}$$

if we choose C and λ such that for any $x \in \mathbb{R}^r - \{0\}$, there exists $\alpha := (2 + \lambda)C + 2\lambda \geq 0$ such that $G\log V(x, i, y) \leq -\alpha$. Also $|\frac{\nabla_x V(x,i,y)^{'}\sigma(x,i)}{V(x,i,y)}| \leq C$. Similarly if there exist a positive constant β such that for any $x \in \mathbb{R}^r$, $C \leq \beta$ then, $|\frac{\nabla_x V(x,i,y)^{'}\sigma(x,i)}{V(x,i,y)}| \leq \beta$. If there exists constants ρ_1 and ρ_2 such that $\rho_1 \leq g(x, i, \gamma) \leq \rho_2$ for any $x \in \mathbb{R}^r$, $i \in \mathcal{X}$ and $\gamma \in \Gamma$ then it is easy to see that $(\rho_1) \leq \left(\frac{V(x+g(x,i,\gamma),i,y)}{V(x,i,y)}\right) \leq (\rho_2)$. Thus in brief for certain conditions on the growth of the drift , diffusion and the integrand of the jump component of the process given by (2), we satisfy the conditions of Theorem 3.1 for the solution to (2) to be almost surely exponentially stable.

We next provide a simple example to illustrate Theorem 3.2. Consider that $x \in \mathbb{R}$ and $V(x, i, y) = x^2$. Also assume that the conditional holding time distribution be $f(y|i) = \lambda_i e^{-\lambda_i y}$ for $i \in \{1, 2\}$. Let $g(x, i, \gamma) = x$, $\lambda_i = 1, b(x, i) = a_1 x, \sigma(x, i) = a_2 x$ for $i \in \{1, 2\}$. Then from (4) we have $GV(x, i, y) = (2a_1 + a_2 + 3)x^2$. If $2a_1 + a_2 + 3 < 0$ and $x \neq 0$ then condition (ii) of Theorem 3.2 for $p = 2$ is satisfied. Moreso if we assume that there exist constants α_1 and α_2 such that $\alpha_1|x|^2 \leq V(x, i, y) \leq \alpha_2|x|^2$ is true, then condition (i) of Theorem 3.2 is satisfied. Thus both conditions (i) and (ii) now guarantee that the solution of (2) is moment exponentially stable.

Next we discuss the issue of stochastic stabilization and de-stabilization of non-linear systems.

4.2.6 Stochastic stabilization and destabilization of nonlinear systems

We now investigate the stability of the nonlinear deterministic system of differential equations given by the following dynamics

$$\frac{dX_t}{dt} = f(X_t) \tag{21}$$

on $t \geq 0$ with $X_0 = x_0 \in \mathbb{R}^r$ where $f(x) : \mathbb{R}^r \to \mathbb{R}^r$ is locally Lipschitz continuous and furthermore there exists some constant $K > 0$ such that $|f(x)| \leq K|x| \quad \forall\ x \ \in \mathbb{R}^r$. When perturbed by noise, the non-linear system (21) is either stable if it originally unstable, in the sense that by adding noise we can force the solution of the stochastic differential equation to converge to the trivial solution as time increases indefinitely. This is the aim of stochastic stabilization. Likewise if our original system in stable, then this system is said to destabilize when perturbed by noise if the sample paths of the process escapes to infinity almost surely instead of converging to the trivial solution as time tends to infinity. This is termed as stochastic de-stabilization. Consequently the system then becomes what is known as unstable. Mao [16] and Applebaum and Siakalli [2] have established a general theory of stochastic stabilization/de-stabilization of (21) using a Brownian motion and the general Levy process respectively. However no specific work has been done so far for the case where X_t is a sMMJD. In this article we focus on the first-order nonlinear system of ODEs that is perturbed by a sMMJD. In the following section we show that an unstable linear system counterpart of (21) wherein $\frac{dX_t}{dt} = aX_t$ for $a > 0$ can be stabilized just by the addition of a jump component to the dynamics of the one-dimensional process X_t. We observe that such a jump-stabilized system of DEs cannot be destabilized by further addition of a Brownian motion. On the contrary, we show that such a jump-stabilized nonlinear system of differential equations can surprisingly be destabilized by addition of Brownian motion if the dimension of the state space is at least two. Before we go into the proofs of these statements, we begin by mentioning the key dynamics of the sMMJD process $\{X_t, t \geq 0\}$ that we consider here and some assumptions that follow. Suppose we have an m-dimensional standard $\mathcal{F}_T$-adapted Brownian motion process $B = (B_1(t), ..., B_m(t))$ for each $t \geq 0$. The system (21) is perturbed by the following sMMJD dynamics of X_t given by

$$dX_t = f(X_t)dt + \sum_{k=1}^{m} G_k(\theta_t)X_t dB_k(t) + \lambda \int_\Gamma D(\theta_{t-}, \gamma)X_{t-}N(dt, d\gamma) \;\; \forall \; t \; \geq 0, \tag{22}$$

where $G_k(i)$ is $\mathbb{R}^{r\times r}$ for each $i \in \mathcal{X}$. Likewise $D(i, \gamma)$ is a $\mathbb{R}^{r\times r}$-valued matrix for each $i \in \mathcal{X}$ and $\gamma \in \Gamma \subset \mathbb{R}^r - \{0\}$. We refer to a system (21) perturbed by the dynamics of X_t as in (22) as just a perturbed system. We make the following key assumption that remains valid until the end of this section.

Assumption 2: *Let $||A|| = \sqrt{tr(A'A)}$ be trace norm of matrx A. Assume that for each $i \in \mathcal{X}$ and $\gamma \in \Gamma$ we have*
$\int_\Gamma \bigg(||D(i,\gamma)|| \vee ||D(i,\gamma)||^2\bigg)\pi(d\gamma) < \infty$ and that $D(i,\gamma)$ does not have an eigenvalue equal to -1 π almost surely.

In the following, we will establish the conditions on the coefficients of (22) for the trivial solution of the perturbed system to be almost surely exponentially stable. In particular, this surprisingly demonstrates that the jump process can have a stabilizing effect, as for the Brownian motion part as has been shown by Mao [16]. We state this formally as one of our main theorems.

Theorem 5.1 *Assume that Assumption 2 holds. Suppose that the following conditions are satisfied for $a(i) > 0$, $b(i) \geq 0$:*
(i)$\sum_{k=1}^m |G_k(i)x|^2 \leq a(i)|x|^2$
(ii)$\sum_{k=1}^m |x^{'}G_k(i)x|^2 \geq b(i)|x|^4$ for each $i \in \mathcal{X}$ and $x \in \mathbb{R}^r$.
Then the sample Lyapunov exponent of the solution of (22) exists and satisfies
$\limsup_{t\to\infty} \log|X_t| \;\leq\; K - \sum_{i\in\mathcal{X}}[(b(i) - \frac{a(i)}{2} - \lambda\log(1 + ||D(i,\gamma)||))]\nu_i$ *for any $X_0 \neq 0$. If $-K + \sum_{i\in\mathcal{X}}[b(i) - \frac{a(i)}{2} - \lambda\log(1 + ||D(i,\gamma)||)]\nu_i > 0$, then the trivial solution to the system in (22) is almost surely exponentially stable.*

Proof Step 1: Define $V(x, i, y) = \log|x| \; \forall \; i \; \in \mathcal{X}$. As $V(x, i, y)$ is independent of states i and

y, the following terms in (4) are zero:

$$\frac{f(y|i)}{1-F(y|i)}\sum_{j\neq i, j\in\mathcal{X}} p_{ij}[V(x,j,0)-V(x,i,y)]=0$$
$$\frac{\partial V(x,i,y)}{\partial y}=0.$$

Hence as an application of the generalized Ito's formula we have for $t>0$

$$\begin{aligned}\log|X_t| &= \log|X_0|+\int_0^t \frac{X_s'}{|X_s|^2}f(X_s)ds+\frac{1}{2}\sum_{k=1}^m\int_0^t\left[\frac{|G_k(i)X_s|^2}{|X_s|^2}-\frac{2|X_s'G_k(i)X_s|^2}{|X_s|^4}\right]ds\\ &+ \lambda\int_0^t\int_\Gamma \log\left(\frac{|X_{s-}+D(\theta_{s-}=i,\gamma)X_{s-}|}{|X_{s-}|}\right)\pi(d\gamma)ds+M_1(t)+M_2(t),\end{aligned}$$

where $M_1(t)=\sum_{k=1}^m\int_0^t|\frac{X_s'G_k(i)X_s}{|X_s|^2}|dB_k(s)$ and $M_2(t)=\int_0^t\int_\Gamma\log\left(\frac{(|X_{s-}+D(\theta_{s-},\gamma)X_{s-}|)}{|X_{s-}|}\right)\tilde{N}(ds,d\gamma)$ are the two local martingale terms.

Step 2: Consider now the quadratic variation of the two martingale terms. From Ito's isometry and noting that

$$\begin{aligned}\frac{|X_s'G_k(i)X_s|^2}{|X_s|^4} &= \frac{|X_s'(G_k'(i)+G_k(i))X_s|^2}{4|X_s|^4}\\ &\leq \rho(G_k(i))^2,\end{aligned}$$

where $\rho(G_k(i))$ is the spectral radius of the symmetric $r\times r$ matrix $\frac{G_k(i)+G_k'(i)}{2}$,

$$\langle M_1(t),M_1(t)\rangle\leq\sum_{k=1}^m\int_0^t\frac{|X_s'G_k(i)X_s|^2}{|X_s|^4}ds\leq tm\max_{1\leq k\leq m, i=1,\ldots,M}\rho(G_k(i)).$$

Next, the quadratic variation of the process $M_2(t)$ is given by

$$\begin{aligned}\langle M_2(t),M_2(t)\rangle &= 2\int_\Gamma\int_0^t\log\left[\frac{(|X_{s-}+D(\theta_{s-},\gamma)X_{s-}|)}{|X_{s-}|}\right]ds\pi(d\gamma)\\ &\leq 2t\log(1+\max_{1\leq i\leq M}||D(i,\gamma)||).\end{aligned}$$

Step 3: We work with the rest of the terms in the following way.

$$\limsup_{t\to\infty}|\frac{1}{t}\int_0^t\frac{X_s'f(X_s)}{|X_s|^2}ds|\leq K;$$

also

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\int_0^t\frac{1}{2}\Big[\sum_{k=1}^{m}\Big(\frac{|G_k(i)X_s|^2}{|X_s|^2}-\frac{2|X_sG_k(i)X_s|^2}{|X_s|^4}\Big)\Big]ds\\
\leq\ &\frac{1}{t}\sum_{i\in\mathcal{X}}\int_0^t\left[\frac{a(i)}{2}-b(i)\right]\mathbb{I}_{\theta_s=i}ds\\
\leq\ &\sum_{i\in\mathcal{X}}[\frac{a(i)}{2}-b(i)]\nu_i,
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{\lambda}{t}\int_0^t\int_\Gamma\log\left(\frac{|X_{s-}+D(\theta_{s-},\gamma)X_{s-}|}{|X_{s-}|}\right)\pi(d\gamma)ds\\
\leq\ &\lambda\sum_{i\in\mathcal{X}}\log(1+||D(i,\gamma)||)\nu_i.
\end{aligned}$$

Thus, $\limsup_{t\to\infty}\frac{1}{t}\log|X_t|<0$ if $K+\sum_{i\in\mathcal{X}}[(\frac{a(i)}{2}-b(i)+\lambda\log(1+||D(i,\gamma)||))]\nu_i<0$. $\square$

Remark: Consider a 1-D sMMJD with the dynamics

$$dX_t=aX_tdt+b(i)X_tdB_t+c(i,\gamma)X_td\tilde{N}_t, \tag{23}$$

where $b(x,i)>0$ and $c(i,\gamma)>-1$ for each $x\in\mathbb{R}$, $i\in\{1,...,M\}$ and $\gamma\in\Gamma$. B_t is a 1-D Brownian motion and $\{\tilde{N}_t,t\geq 0\}$ is a compensated Poisson process with $\tilde{N}_t=N_t-\lambda t$, where $\lambda>0$ is the intensity of the Poisson process. Assume that the processes B_t and N_t are independent. Then one can show from the SLLN for a Brownian motion and for a Poisson process (refer to Applebaum [1]) that for each $i\in\{1,2,...,M\}$

$$\limsup_{t\to\infty}\frac{1}{t}\log|X_t|=a+[-\lambda c(i,\gamma)-\frac{1}{2}b^2(i)+\lambda\log(1+c(i,\gamma))]<0\ \ a.s.$$

Note that $b^2(i)\geq 0\ \forall i\in\mathcal{X}$ and has a negative sign attached to it. Hence when the one-dimensional perturbed system $\frac{dX_t}{dt}=aX_t$ for $a>0$ is stabilized by the addition of a jump process infact can never be destabilized by the addition of a Brownian motion. An interesting question we may ask here is: can the same inference hold true in higher dimensions? The answer is surprisingly *no*. In the following theorem we show that for a state space of dimension greater than or equal to two, an unstable nonlinear system of differential equation stabilized by a jump component can still be destabilized by the addition of the Brownian motion . This surprising phenomenon was also observed by Applebaum and Siakalli [3] for the Levy process case.

To prove this assertion let us now consider system of non-linear differential equation (21) stabilized by (22) but with $G_k(i) = 0$ for each $i = 1, ..., M$ and $k = 1, ..., m$. We now show that it gets de-stabilized by further addition of the m-dimensional Brownian motion to (21). This corresponds to $G_k(i) \neq 0$ for each $i = 1, ..., M$ and $k = 1, ..., m$.

Theorem 5.2 *Assume that matrix D is an $r \times r$ symmetric positive definite matrix. Now let*
(i)$\sum_{k=1}^{m} |G_k(i)x|^2 \geq a(i)|x|^2$
(ii)$\sum_{k=1}^{m} |x^{'}G_k(i)x|^2 \leq b(i)|x|^4$,
for $a(i) > 0, b(i) \geq 0$ *for each* $i \in \mathcal{X}$ *,*$x \in \mathbb{R}^r$. *Hence* $\liminf_{t\to\infty} \frac{1}{t} \log |X_t| \geq -K + \sum_{i\in\mathcal{X}} \left[\left(\frac{a(i)}{2} - b(i) + \lambda \log(1 + \min_{1\leq i\leq M} ||D(i,\gamma)||)\right)\right]\nu_i$ *for any* $X_0 \neq 0$. *In particular if* $-K + \sum_{i\in\mathcal{X}}[\frac{a(i)}{2} - b(i) + \lambda\text{log}(1 + \min_{1\leq i\leq M} ||D(i,\gamma)||)]\nu_i > 0$ *,then the trivial solution of (22) tends to infinity almost surely exponentially fast.*

Proof Fix $X_0 \neq 0$. From Lemma 2.2, $X_t \neq 0$ $\forall t \geq 0$. Applying Ito's lemma to $\log |X_t|$, for $t > 0$ and for each $i \in \mathcal{X}$,

$$\begin{aligned}\log |X_t| &= \log |X_0| + \int_0^t \frac{X_s^{'}}{|X_s|^2} f(X_s)ds + \frac{1}{2}\sum_{k=1}^{m}\int_0^t \left[\frac{|G_k(i)X_{s-}|^2}{|X_{s-}|^2} - \frac{2|X_{s-}G_k(i)X_{s-}|^2}{|X_{s-}|^4}\right]ds \\ &+ \lambda\int_0^t\int_\Gamma \log\left(\frac{(|X_{s-} + D(\theta_{s-}= i,\gamma)X_{s-}|)}{|X_{s-}|}\right)\pi(d\gamma)ds + M_1(t) + M_2(t), \qquad (24)\end{aligned}$$

where $M_1(t) = \sum_{k=1}^{m}\int_0^t \frac{|X_{s-}^{'}G_k(i)X_{s-}|}{|X_{s-}|^2}dB_k(s)$ and $M_2(t) = \int_\Gamma\int_0^t \log\left(\frac{(|X_{s-}+D(\theta_{s-}=i,\gamma)X_{s-}|)}{|X_{s-}|}\right)\tilde{N}(ds,d\gamma)$ are the two local martingale terms. Now using methodology similar to Theorem 5.1 we find

$$\liminf_{t\to\infty}\frac{1}{t}\log|X_t| \geq -K + \sum_{i\in\mathcal{X}}\left[(\frac{a(i)}{2} - b(i) + \lambda\log(1 + \min_{1\leq i\leq M}||D(i,\gamma)||))\right]\nu_i$$

for any $X_0 \neq 0$. In particular, if $-K + \sum_{i\in\mathcal{X}}[\frac{a(i)}{2} - b(i) + \lambda\text{log}(1 + \min_{1\leq i\leq M}||D(i,\gamma)||)]\nu_i > 0$, then the trivial solution of the X_t-perturbed system given by (22) tends to infinity almost surely exponentially fast. □

4.2.7 Concluding remarks

We presented conditions under which the solution of an semi Markov Modulated jump diffusion is almost surely exponentially stable and moment exponentially stable . We also provide conditions that connect these two notions of stability. We further determine the conditions under which the trivial solution of the sMMJD-perturbed nonlinear system of differential equation $\frac{dX_t}{dt} = f(X_t)$ is almost surely exponentially stable. We show that an unstable deterministic system can be stabilized by adding jumps. Such jump stabilized system however can get de-stabilized by Brownian motion if the dimension of the state space is at least two.

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