# Dedekind Sums

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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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# Abstract

Dedekind sums arose out of the study of elliptic functions and modular forms. They were initially discovered by Dedekind but have since been studied for their many arithmetic properties. Much work has been done on Dedekind sums and in 1972 Rademacher and Grosswald released a book that summarised much of what was known, as well as providing a history of Dedekind sums. This encouraged greater interest in this topic and provided groundwork for further research. In our essay we seek to update Rademacher and Grosswald's book by providing an overview of some of the research that has been done since its publication. We will also extend some results of Myerson and Phillips, regarding fixed points of Dedekind sums. It is our intention that our findings will provide a foundation for future research in the study of Dedekind sums.

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# 1
# Introduction

We define the Dedekind sum for integers $h$ and $k$, where $k$ is positive, by

$$s(h,k) = \sum_{\mu=1}^{k} \left(\left(\frac{h\mu}{k}\right)\right)\left(\left(\frac{\mu}{k}\right)\right). \tag{1.1}$$

The symbol $((x))$ is the sawtooth function (see Figure 1.1), which is defined by

$$((x)) = \begin{cases} x - [x] - 1/2 & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer,} \end{cases} \tag{1.2}$$

where $[x]$ is the integer part of $x$.

Dedekind sums arose out of the study of the Dedekind-eta function,

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{m=1}^{\infty} \left(1 - e^{2\pi i m \tau}\right), \quad \mathrm{Im}(\tau) > 0, \tag{1.3}$$

and of modular forms. They were observed by taking the modular transformation

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \eta(\tau) + \frac{1}{2}\log\frac{c\tau + d}{i\mathrm{sign}(c)} + \pi i \frac{a+d}{12c} + \pi i s(d, |c|) \tag{1.4}$$

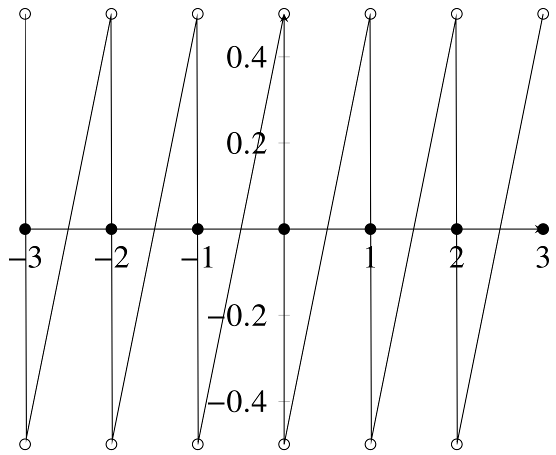


Figure 1.1: The Sawtooth Function

where $a, b, c$ and $d$ are integers and $ad - bc = 1$. Within this study Dedekind proved the now well-known reciprocity formula for Dedekind sums. However, Dedekind sums have been studied independently of modular forms due to their arithmetic properties. Dedekind sums have found applications in number theory, as well as topology, geometry and computer science. We will discuss some of these applications in this essay.

The main resource on Dedekind sums is a book by Rademacher and Grosswald [29]. This book stated many of the known facts about Dedekind sums and laid a foundation for further research. It also provided some open questions which have since been answered. We note that there has been no attempt to synthesise all the work that has been done on Dedekind sums in the 40 years since the publication of [29]. The purpose of our research is to present many of the new results over this period of time and add to the literature on Dedekind sums. Due to the sheer volume of papers that have been published it would be impossible to provide a complete account of all that has been researched on Dedekind sums. However, we have provided a small snapshot of some of the significant advances in this area and we hope that by presenting this information it will encourage more research and further our collective knowledge of Dedekind sums. In the final chapter we also present a number of original results that we have been able to prove, many of which extend previous results of Myerson and Phillips [21]. In our thesis we also lay the groundwork for our own further research on this topic.

# 2

# Elementary Properties

## 2.1 Basic Properties

In this chapter we will discuss some of the elementary properties of the Dedekind sum. Although these results appear in [29], they will be instrumental in progressing further in our discussion.

**Theorem 2.1.1.** *Suppose* gcd$(h,k)$=*1. Then*

$$\begin{aligned} s(-h,k) &= -s(h,k), \\ s(h,-k) &= s(h,k), \\ s(h,k) &= s(h',k) \ \ \textit{where } h \equiv h' \pmod{k}, \\ s(h,k) &= s(h',k) \ \ \textit{where } hh' \equiv 1 \pmod{k}, \\ s(ah,ak) &= s(h,k). \end{aligned} \tag{2.1}$$

Before we prove these we provide the following lemma which is given in [29, Ch. 2].

**Lemma 2.1.1.** *Suppose $x$ is a real number and $k$ is a positive integer, then*

$$\sum_{\lambda \bmod k)} \left(\left(\frac{\lambda + x}{k}\right)\right) = ((x)).$$

*Proof.* Consider $D(x) = \sum_{\lambda \bmod k} \left(\left(\frac{\lambda+x}{k}\right)\right) - ((x))$. We can see that this function is periodic in $x$ with period 1. So we can consider $0 \le x < 1$. Now,

$$D(0) = \sum_{\lambda=1}^{k-1} \left(\frac{\lambda}{k} - \frac{1}{2}\right) = \frac{k-1}{2} - \frac{k-1}{2} = 0.$$

Similarly, for $x \neq 0$,

$$D(x) = \sum_{\lambda=0}^{k-1} \left(\frac{\lambda}{k} + \frac{x}{k} - \frac{1}{2}\right) - \left(x - \frac{1}{2}\right) = \frac{k-1}{2} + x - \frac{k}{2} - x + \frac{1}{2} = 0.$$

□

We now prove our theorem.

*Proof.* The first two of these equations are proved trivially due to the sawtooth function being an odd function. The third result is due to the periodicity of the sawtooth function. That is, we can suppose $h' = nk + h$ for some integer $n$. Then,

$$\begin{aligned} s(h', k) &= \sum_{\mu=1}^{k} \left(\left(\frac{(h+nk)\mu}{k}\right)\right)\left(\left(\frac{\mu}{k}\right)\right) \\ &= \sum_{\mu=1}^{k} \left(\left(\frac{h\mu}{k} + n\mu\right)\right)\left(\left(\frac{\mu}{k}\right)\right) \\ &= \sum_{\mu=1}^{k} \left(\left(\frac{h\mu}{k}\right)\right)\left(\left(\frac{\mu}{k}\right)\right) \\ &= s(h, k). \end{aligned}$$

The fourth equation we can prove by observing that $\mu$ runs through a complete residue system modulo $k$. Hence, $h'\mu$ also runs through a complete residue system modulo $k$ since the $\gcd(h, k) = 1$. Therefore

$$\begin{aligned} s(h, k) &= \sum_{\mu=1}^{k} \left(\left(\frac{h\mu}{k}\right)\right)\left(\left(\frac{\mu}{k}\right)\right) \\ &= \sum_{\mu=1}^{k} \left(\left(\frac{h(h'\mu)}{k}\right)\right)\left(\left(\frac{(h'\mu)}{k}\right)\right) \\ &= \sum_{\mu=1}^{k} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h'\mu}{k}\right)\right) \\ &= s(h', k). \end{aligned}$$

For this final equation we first need to show that $s(h,k) = \sum_{\mu=1}^{k} \frac{u}{k}\left(\left(\frac{h\mu}{k}\right)\right)$. We observe by Lemma 2.1.1 that $\sum_{\mu=1}^{k}\left(\left(\frac{h\mu}{k}\right)\right) = 0$. Now, by our definition of the sawtooth function, we obtain

$$\begin{aligned} s(h,k) &= \sum_{\mu=1}^{k}\left(\frac{\mu}{k} - \left[\frac{\mu}{k}\right] - \frac{1}{2}\right)\left(\left(\frac{h\mu}{k}\right)\right) \\ &= \sum_{\mu=1}^{k}\left(\frac{\mu}{k} - \frac{1}{2}\right)\left(\left(\frac{h\mu}{k}\right)\right) \\ &= \sum_{\mu=1}^{k}\frac{\mu}{k}\left(\left(\frac{h\mu}{k}\right)\right). \end{aligned}$$

Therefore we have,

$$\begin{aligned} s(ah,ak) &= \sum_{\mu=1}^{ak}\frac{\mu}{ak}\left(\left(\frac{ah\mu}{ak}\right)\right) \\ &= \frac{1}{ak}\sum_{\mu=1}^{ak}\mu\left(\left(\frac{h\mu}{k}\right)\right) \\ &= \frac{1}{ak}\left(\sum_{\mu=1}^{k}\mu\left(\left(\frac{h\mu}{k}\right)\right) + \sum_{\mu=1}^{k}(\mu+k)\left(\left(\frac{h\mu}{k}\right)\right) + \cdots + \sum_{\mu=1}^{k}(\mu+(a-1)k)\left(\left(\frac{h\mu}{k}\right)\right)\right) \\ &= \frac{a}{ak}\sum_{\mu=1}^{k}\mu\left(\left(\frac{h\mu}{k}\right)\right) \\ &= s(h,k). \end{aligned}$$

□

Due to the final equation of Theorem 2.1.1 we only need to consider Dedekind sums for coprime variables $h, k$. For the remainder of our discussion we will assume $h$ and $k$ are coprime. We will also assume that $0 < h < k$ unless stated otherwise.

In general, there is no closed form for evaluating Dedekind sums. However, there are some values for which it is possible to give an explicit evaluation. Some of these are listed in

[2, Ch. 3], and we will show some new ones in a later section. Here we will evaluate $s(1,k)$:

$$\begin{aligned} s(1,k) &= \sum_{\mu=1}^{k} \left(\left(\frac{\mu}{k}\right)\right)^2 \\ &= \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k}\left(\frac{\mu}{k}-\frac{1}{2}\right)\right) \\ &= \sum_{\mu=1}^{k-1} \left(\left(\frac{\mu}{k}\right)^2-\frac{\mu}{2k}\right) \\ &= \frac{(k-1)(2k-1)}{6k} - \left(\frac{k}{4}-\frac{1}{4}\right) \\ &= \frac{k^2-6k+2}{12k}+\frac{1}{4} \\ &= -\frac{1}{4}+\frac{1}{6k}+\frac{k}{12}. \end{aligned} \tag{2.2}$$

**Theorem 2.1.2.** $2k\theta s(h,k)$ *is an integer, where* $\theta = \gcd(k,3)$.

*Proof.* We observe,

$$\begin{aligned} s(h,k) &= \sum_{\mu=1}^{k-1} \frac{\mu}{k}\left(\frac{h\mu}{k} - \left[\frac{h\mu}{k}\right] - \frac{1}{2}\right) \\ &= \frac{h}{k^2}\sum_{\mu=1}^{k-1}\mu^2 + \frac{A}{2k} \\ &= \frac{h(k-1)(2k-1)}{6k} + \frac{A}{2k}, \end{aligned}$$

where $A$ is some integer. We now notice that if 3 doesn't divide $k$ then 3 divides $(k-1)$ or $(2k-1)$ , so our denominator divides $2k$. Now if 3 divides $k$ it is easy to see that our denominator divides $6k$, hence $2k\theta s(h,k)$ is an integer. □

Salié showed [33] that $6ks(h,k)$ always satisfies one of the congruences

$$6ks(h,k) \equiv 0, \pm 1, \pm 3 \pmod 9. \tag{2.3}$$

## 2.2 The Reciprocity Formula

The most remarkable result concerning the Dedekind sum is the well-known reciprocity formula.

**Theorem 2.2.1** (The Reciprocity Formula)**.** *Suppose* $h$ *and* $k$ *are positive integers and* $\gcd(h,k) = 1$. *Then*

$$s(h,k) + s(k,h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} + \frac{1}{hk}\right). \tag{2.4}$$

The reciprocity formula was proved by Dedekind in his study of the Dedekind eta-function and of modular forms. However, it exists as a purely arithmetic result and many strictly arithmetic proofs of the reciprocity formula exist. Indeed, Rademacher and Grosswald provide several examples [29, Ch. 2]. This result allows for significantly faster computation of the Dedekind sum and helps to provide us with more properties which will be mentioned throughout our discussion. We will provide one proof of the reciprocity law but we encourage the reader to research some of the other proofs.

*Proof.* The proof we provide is given by Berndt [6] and makes use of contour integrals. We note, as Berndt did, that other proofs involving contour integration exist; indeed one is given in [29, Ch. 2], but this one differs somewhat in its construction. The proof relies on a result given in [6],

$$s(h,k) = \frac{1}{2\pi} \sum_{\substack{n=1 \\ n \not\equiv 0 \pmod{k}}}^{\infty} \frac{\cot(\pi h n/k)}{n}. \tag{2.5}$$

We will discuss the above result in Section 2.3. Now we consider $C_N$, the positively oriented circle with radius $R_N$, $1 \leq N < \infty$, centred at the origin. We assume that $R_N$ is increasing to infinity and is chosen so the distance from our circle to the points $m/h$ and $n/k$, for integers $m$ and $n$, is greater then some fixed number. Let,

$$\begin{aligned} I_N &= \frac{1}{2\pi i} \int_{C_N} \cot(\pi h z)\cot(\pi k z)\frac{dz}{z} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cot(\pi h R_N e^{i\theta})\cot(\pi k R_N e^{i\theta}) d\theta. \end{aligned} \tag{2.6}$$

We can now evaluate this integral in two different ways. First we consider the 2nd expression and observe

$$\begin{aligned} \cot(Re^{i\theta}) &= \frac{e^{iRe^{i\theta}} + e^{-iRe^{i\theta}}}{-i(e^{iRe^{i\theta}} - e^{-iRe^{i\theta}})} \\ &= i\frac{e^{iR(\cos\theta + i\sin\theta)} + e^{-iR(\cos\theta+i\sin\theta)}}{e^{iR(\cos\theta+i\sin\theta)} - e^{-iR(\cos\theta+i\sin\theta)}} \\ &= i\frac{e^{Ri\cos\theta}e^{-R\sin\theta} + e^{-Ri\cos\theta}e^{R\sin\theta}}{e^{Ri\cos\theta}e^{-R\sin\theta} - e^{-Ri\cos\theta}e^{R\sin\theta}}. \end{aligned} \tag{2.7}$$

We can see that as $R \to \infty$ and $0 < \theta < \pi$ then $\cot(Re^{i\theta})$ tends to $-i$ and similarly on $\pi < \theta < 2\pi$ it tends to $i$. It follows that $I_N \to -1$ as $N \to \infty$.

We now evaluate the integral by contour integration and the residue theorem. We consider the first equation for $I_N$ and notice that there are simple poles at $m/h$ and $n/k$, where $m$ and $n$ are not zero and the respective fractions are not integers. There is a double pole at each non-zero integer $\mu$ and there is a triple pole at 0. The Laurent expansion of the cotangent function enables us to calculate the residues of $I_N$. Thus,

$$\begin{aligned} I_N = \frac{2}{\pi} \sum_{\substack{0<m/h<R_N \\ m\not\equiv 0 \ (\mathrm{mod}\ h)}} \frac{\cot(\pi k m/h)}{m} + \frac{2}{\pi} \sum_{\substack{0<n/k<R_N \\ n\not\equiv 0 \ (\mathrm{mod}\ k)}} \frac{\cot(\pi h n/k)}{n} \\ - \frac{2}{\pi^2 hk} \sum_{0<\mu<R_N} \frac{1}{\mu^2} - \frac{k}{3h} - \frac{h}{3k}. \end{aligned} \tag{2.8}$$

We now let $N \to \infty$, and by our initial evaluation for this integrand and by (2.5) we obtain

$$-1 = \lim_{N\to\infty} I_N = 4s(k,h) + 4s(h,k) - \frac{1}{3hk} - \frac{k}{3h} - \frac{h}{3k}. \tag{2.9}$$

This finishes our proof of the reciprocity formula. □

We can multiply the reciprocity formula by $12hk$ to arrive at the following result:

$$12hks(h,k) + 12hks(k,h) = -3hk + h^2 + k^2 + 1. \tag{2.10}$$

We now recall that $2\theta k s(h,k)$ is an integer, where $\theta = \gcd(k,3)$. We also assume that $h$ and $k$ are coprime. Therefore,

$$12hks(h,k) \equiv h^2 + 1 \ (\mathrm{mod}\ \theta k). \tag{2.11}$$

Rademacher and Grosswald [29, Ch. 3] used this result to show that $s(h,k)$ is an integer if and only if $k$ divides $(h^2+1)$ and the only integer $s(h,k)$ can take is 0. In particular, if $k$ doesn't divide $(h^2+1)$ then $12s(h,k)$ is not an integer. The proof of this is quite simple. First, assume that $h^2 + 1 \equiv 0 \ (\mathrm{mod}\ k)$. Then, $hh' \equiv 1 \ (\mathrm{mod}\ k)$ has the solution $h' = -h$, and by Theorem 2.1.1 $s(h,k) = s(h',k) = s(-h,k) = -s(h,k) = 0$. Conversely, suppose $12s(h,k)$ is an integer. Then, by (2.10), and recalling $6hs(k,h)$ is an integer, we have $h^2 + 1 \equiv 0 \ (\mathrm{mod}\ k)$. This proves our result.

We also note that the reciprocity formula has been generalised to a three term reciprocity formula, given by Rademacher in 1954 [27].

**Theorem 2.2.2.** *For coprime integers $a, b$ and $c$ we have,*

$$s(bc',a) + s(ca',b) + s(ab',c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right), \tag{2.12}$$

*where $aa' \equiv 1 \ (\mathrm{mod}\ bc)$ and similarly for $bb'$ and $cc'$.*

It is clear that the three term reciprocity law can be simplified to the two term case by taking $c = c' = 1$ and recalling $s(a', b) = s(a, b)$. Another reciprocity law is given by Pommersheim [25] as a consequence of relating the Todd class of a toric variety to Dedekind sums, but we will not discuss the details of Pommersheim's proof here.

**Theorem 2.2.3.** *Let $p, q, u, v, u'$ and $v'$ be natural numbers with $(p, q) = 1$ and $(u, v) = 1$ and $u', v'$ chosen such that $uu' + vv' = 1$. Let $x = qv' - pu'$ and $y = pv + qu$. Then*

$$s(p, q) + s(u, v) + s(x, y) = \lambda(q, v, y), \tag{2.13}$$

*where $\lambda(q, v, y) = -\frac{1}{4} + \frac{1}{12}\left(\frac{q}{vy} + \frac{v}{yq} + \frac{y}{qv}\right)$.*

Two independent proofs are also given by Girstmair [13]; the first of which shows the equivalence of Pommersheim's result to the 3 term reciprocity law (2.12) and the second we will describe below.

*Proof.* It can be shown that the residue class of $x$ mod $y$ is independent of the choices of $u'$ and $v'$. We can then define

$$\Delta(p, q, u, v) = s(p, q) + s(u, v) + s(x, y) - \lambda(q, v, y). \tag{2.14}$$

Our proof will then be to show that this is 0. We note that this is true for $v = 1$ and the above equation reduces to the two term reciprocity law since

$$\begin{aligned} x &= qv' - pu' \\ &= q(1 - uu') - pu' \\ &= q - u'(p + qu) \\ &= q - u'y. \end{aligned} \tag{2.15}$$

Hence

$$s(p, q) + s(u, v) + s(x, y) = s(y, q) + 0 + s(q, y) = \lambda(q, 1, y). \tag{2.16}$$

Our proof will therefore be to reduce our equation to one where $v = 1$. Due to periodicity we have, for $u > v$,

$$\Delta(p + q, q, u - v, v) = \Delta(p, q, u, v). \tag{2.17}$$

Also, it can be shown by using the identity $\lambda(q, v, y) + \lambda(p, u, y) = \lambda(p, q, 1) + \lambda(u, v, 1)$,

$$\Delta(q, p, v, u) = -\Delta(p, q, u, v). \tag{2.18}$$

By repeated application of (2.17) and (2.18) we obtain

$$\Delta(p, q, u, v) = \Delta(p*, q*, u*, 1) = 0 \tag{2.19}$$

for some natural numbers $p*$, $q*$ and $u*$. □

## 2.3 Alternative Forms Of The Dedekind Sum

The Dedekind sum can be written in a few different forms, which may be more useful in dealing with certain problems. We note that we have already given one example of a different form and an application in which it is useful, with

$$s(h, k) = \sum_{\mu=1}^{k} \frac{\mu}{k} \left(\left(\frac{h\mu}{k}\right)\right).$$

We can also express the Dedekind sum in terms of $k$th roots of unity. The proof of this and the related sum in terms of cotangents is given in [29, Ch. 2].

$$s(h, k) = \frac{1}{4k} \sum_{m=1}^{k-1} \frac{1 + \eta^m}{1 - \eta^m} \frac{1 + \eta^{-hm}}{1 - \eta^{-hm}} \tag{2.20}$$

Here, $\eta$ is any primitive $k$th root of unity. By specifying $\eta = e^{2\pi i/k}$, we obtain

$$s(h, k) = \frac{1}{4k} \sum_{m=1}^{k-1} \cot \frac{\pi m}{k} \cot \frac{\pi hm}{k}. \tag{2.21}$$

We have another result which we provided as part of the proof of the reciprocity formula:

$$s(h, k) = \frac{1}{2\pi} \sum_{\substack{n=1 \\ n \not\equiv 0 \ (\mathrm{mod}\ k)}}^{\infty} \frac{\cot(\pi hn/k)}{n}.$$

A proof of this is given in [6]. Berndt arrives at this result by expressing the sawtooth function as the Fourier series

$$((x)) = -\sum_{m=1}^{\infty} \frac{\sin(2\pi mx)}{\pi m}. \tag{2.22}$$

After some manipulation and recognising a sum that relates the sine and cotangent functions, he arrives at his result.

Another form of the Dedekind sum is proved as part of a larger result due to Hickerson [17]. We will provide this result and its proof in the following chapter.

# 3

# The Distribution Of Dedekind Sums

## 3.1 Continued Fractions And The Dedekind Sum

From here on we will define

$$[a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}$$

as the finite simple continued fraction. It should not come as a surprise that a relationship exists between the Dedekind sum and the simple continued fractions as the reciprocity law evaluates the Dedekind sum in a way that is equivalent to the Euclidean algorithm. Here we will prove Hickerson's alternative form of the Dedekind sum, as well as providing some other results proved by the use of continued fractions. We first observe that we can set $s(h, k) = s(h/k)$ by Theorem 2.1.1. We can now prove Hickerson's result.

**Theorem 3.1.1.** *Let $h/k = [a_0, a_1, \dots, a_n]$ and $n \geq 0$. Then,*

$$s(h/k) = \frac{-1 + (-1)^n}{8} + \frac{1}{12}([0, a_1, \dots, a_n] + (-1)^{n+1}[0, a_n, \dots, a_1] \\ + a_1 - a_2 + \cdots + (-1)^{n+1}a_n). \quad (3.1)$$

*Proof.* Our proof will follow that of Hickerson [17] and it will rely on induction on $n$. We first note that when $n = 0$ both sides of our equation are equal to 0 since $h/k = a_0/1$, and $s(a_0, 1) = 1$. Now we assume our result is true for $n - 1$. We suppose $0 < h < k$ and $h/k = [0, a_1, \dots, a_n]$ and recall that this means $[a_1, \dots, a_n] = k/h$. Now, by our reciprocity formula,

$$s(h/k) = -\frac{1}{4} + \frac{1}{12}\left([0, a_1, \dots, a_n] + [a_1, \dots, a_n] + \frac{1}{hk}\right) - s([a_1, \dots, a_n]).$$

We can now apply our inductive step on $s([a_1, \dots, a_n])$. Hence,

$$\begin{aligned} s(h/k) = &-\frac{1}{4} + \frac{1}{12}\left([0, a_1, \dots, a_n] + [a_1, \dots, a_n] + \frac{1}{hk}\right) \\ &- \left(\frac{-1 + (-1)^{n-1}}{8} + \frac{1}{12}([0, a_2, \dots, a_n] + (-1)^n[0, a_n, \dots, a_2] + a_2 - \cdots + (-1)^n a_n)\right) \\ &= \frac{-1 + (-1)^n}{8} + \frac{1}{12}([0, a_1, \dots, a_n] + (-1)^{n+1}[0, a_n, \dots, a_2] + \frac{1}{hk} \\ &\qquad + a_1 - a_2 + \cdots + (-1)^{n+1}a_n) \end{aligned}$$

where we notice that $[a_1, \dots, a_n] - [0, a_2, \dots, a_n] = a_1$. To complete our proof we need

$$(-1)^{n+1}[0, a_n, \dots, a_2] + \frac{1}{hk} = (-1)^{n+1}[0, a_n, \dots, a_1].$$

By a suitable substitution, this is equivalent to

$$[0, b_1, \dots, b_n] - [0, b_1, \dots, b_{n-1}] = \frac{(-1)^{n+1}}{hk}.$$

This is a well known result of continued fractions (see [30, Ch. 13]) and completes our proof. □

Rademacher and Grosswald [29, Ch. 3] asked whether the points $(h/k, s(h, k))$ lie everywhere dense in the plane. Hickerson [17] used the previous result on continued fractions to answer Rademacher and Grosswald's question. We present Hickerson's result as a theorem.

**Theorem 3.1.2.** *The set of points $(h/k, s(h, k))$ is dense in the plane.*

A corollary of this is that the values of $s(h,k)$ are dense on the real-axis. Recently, Girstmair [15] followed up the above results by asking what values can $s(h,k)$ take. We know that the Dedekind sum produces rational numbers, and Girstmair showed that the fractional part of $S(h,k) = 12s(h,k)$ takes all possible values for suitable choices of $h$ and $k$. We present his result in the following theorem.

**Theorem 3.1.3.** *Let integers n and q,* $0 \leq q \leq n-1$, $\gcd(q,n) = 1$*, be given. Then there are integers* $m, n', 0 \leq m \leq n'-1, (m,n') = 1$*, such that*

$$S(m,n') \in \frac{q}{n} + \mathbb{Z}. \tag{3.2}$$

Girstmair's proof is constructive in the sense that it allows one to find integers $h, k$ such that $S(h,k) = q/n + \mathbb{Z}$ for any choice of $q/n$. Indeed, he gives a helpful example of applying his proof to show that the fractional part of the Dedekind sum being $7/132$ occurs when we take $S(1319, 134376) \;=\; 120 + 7/132$.

Myerson and Phillips [21] also used continued fractions in their paper. They sought to answer a question, asked by Neville Robbins, of what are the solutions of the equation $s(h,k) = h/k$, which we can consider as finding fixed points of Dedekind sums. Here we outline the important results of [21]. We first recognise that we can write any rational as a simple continued fraction in exactly two ways: with an even number of terms or with an odd number of terms. Suppose $h/k = [a_0, \ldots, a_t]$ with t even. We define $I(h,k) \;=\; \sum_{j=0}^{t}(-1)^{j+1}a_j$. It is clear that if $h'/k = [0, a_t, \ldots, a_1]$ then $hh' \equiv -1 \pmod{k}$. We can rewrite the Dedekind sum as

$$s(h,k) = \frac{1}{12}\left(\frac{h}{k} - \frac{h'}{k} + I(h,k)\right). \tag{3.3}$$

Using this result we have the following theorem.

**Theorem 3.1.4.** *For rational* $\alpha$ *the following are equivalent:*

1. $s(h,k) = \alpha h/k$,
2. $(12\alpha - 1)h + h' = I(h,k)k$,
3. $-(12\alpha - 1)h/k = [-I(h,k), a_t, \ldots, a_1]$. *Moreover, each of these imply*
4. $k | (12\alpha - 1)h^2 - 1$.

Myerson and Phillips use these results to prove the following:

**Theorem 3.1.5.** *If $\alpha \neq 1/12$ is rational, then there are infinitely many $x$ such that $s(x) = \alpha x$.*

**Theorem 3.1.6.** *$\{x : s(x) = x\}$ is dense in the reals.*

A constructive proof is given in [21] which we can use to find a rational number $h/k$ which satisfies $s(h, k) = h/k$, arbitrarily close to any given number. The proof of the previous theorem is a consequence of a sequence of lemmas and expressing the continued fraction $[a_0, a_1, \ldots, a_t]$ in terms of the matrix expansion

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_0 \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 & 1 \\ 1 & a_t \end{pmatrix} = \begin{pmatrix} u & h \\ h' & k \end{pmatrix}$$

where $u$ is given by $uk - hh' = 1$. It is currently unknown whether the solutions of $s(x) = \alpha x$ for fixed $\alpha \neq 1/12$ are dense in the reals. It is also unknown whether there are solutions to the problem $s(x) = \alpha x + \beta$ for fixed rational $\alpha$ and $\beta$. A solution to this problem would enable us to find the range of $s(h, k)$, by considering the case when $\alpha = 0$, which is still an open question mentioned in [29, Ch. 3].

## 3.2 Exceptional Values Of The Dedekind Sum

We have previously stated that the Dedekind sum satisfies the congruences

$$6ks(h, k) \equiv 0, \pm 1, \pm 3 \pmod 9.$$

The question can then be asked, will the Dedekind sum achieve every integer that satisfies these congruences? We will let $6ks(h, k) = t(h, k)$. Salié stated [33], without proof, that $t(h, k)$ takes all possible values up to 150 except for 12, 17, 44 and 107. We are unsure how Salié proved this; however we follow the work of Asai [3], Nagasaka [23] and Saito [32] in verifying Salié's result and proving a stronger one.

**Definition 3.2.1.** We define an integer $W \equiv 0, \pm 1, \pm 3 \pmod 9$ to be an 'exceptional value' if $t(h, k) \neq W$ for any integers $h, k$.

Asai [3] developed a series of lemmas, involving Dedekind sums and Farey fractions, to arrive at the following theorem regarding exceptional values.

**Theorem 3.2.1.** *If $D$ is the value of $t(h, k)$ then it can be attained by taking $k < 2D$ for some suitably well chosen $h$ and $k$.*

This result is significant in the further study of exceptional values as it allows one to consider a finite number of sums in determining if some integer is an exceptional value of the Dedekind sum. Asai used his results and computation to show that 152, 172, 197 and 530 are the next 4 exceptional values, as well as verifying the results of Salié.

Nagasaka [23] used Asai's result to find all the exceptional values up to 56645. Nagasaka noticed that all of these exceptional values, except for 12 and 172, are congruent to $-1 \pmod 9$. He used his results to make the following conjecture.

**Conjecture 3.2.1.** Suppose $W \in \mathbb{N}$ and $W^2 \equiv 0, 1 \pmod 9$. Then $W$ is exceptional if and only if $W - 1$ is a square not divisible by any prime $\equiv \pm 3 \pmod 8$, or $W = 12, 44, 107, 152$ or 172.

Saito[32] managed to prove the "if" statement of this conjecture, that is, any $W$ as defined above will be an exceptional value. He was not able to prove the "only if" part of the statement, and as far as we are aware, this remains an open problem.

## 3.3 Equality Of Dedekind Sums

The question has been asked when are two Dedekind sums, $s(a, c)$ and $s(b, c)$, equal. It has been shown that they can only be equal under certain conditions. Jabuka, Robins and Wang [18] proved the following theorem.

**Theorem 3.3.1.** *If* $s(a, c) = s(b, c)$*, then*

$$c | (1 - ab)(a - b). \tag{3.4}$$

*Proof.* The proof is a consequence of the reciprocity law. We recall,

$$b(12acs(a, c) + 12acs(c, a)) = b(-3ac + a^2 + c^2 + 1)$$

$$a(12bcs(b, c) + 12bcs(c, b)) = a(-3bc + b^2 + c^2 + 1).$$

We now subtract the above equations and use $s(a, c) = s(b, c)$, to obtain

$$12abcs(c, a) - 12abcs(c, b) = a^2b + c^2b + b - (ab^2 + c^2a + a).$$

Now we recall that $6as(c, a)$ is an integer. Therefore,

$$\begin{aligned} 0 &\equiv a^2b + b - (ab^2 + a) \pmod c, \\ 0 &\equiv (1 - ab)(a - b) \pmod c. \end{aligned} \tag{3.5}$$

□

We immediately have the following corollary:

**Corollary 3.3.2.** *Let $p$ be prime. Then $s(a,p) = s(b,p)$ if and only if*

$$ab \equiv 1 \pmod{p}, \quad \text{or}$$

$$a \equiv b \pmod{p}.$$

Girstmair [14] extends the previous theorem with the following result.

**Theorem 3.3.3.** *Let $S(h,k) = 12s(h,k)$ and let $a$ and $b$ be integers relatively prime to $c$. Then $S(a,c) - S(b,c) \in \mathbb{Z}$ if, and only if,* (3.4) *holds.*

*Proof.* By (3.1) and by denoting the convergents to $a/c$ by $s_0/t_0, \ldots, s_n/t_n = a/c$, we have

$$S(a,c) = \sum_{j=1}^{n} (-1)^{j-1} a_j + \begin{cases} \frac{a+t_{n-1}}{c} - 3 & \text{if } n \text{ is odd,} \\ \frac{a-t_{n-1}}{c} & \text{if } n \text{ is even.} \end{cases}$$

We can then use (3.3) to arrive at the result

$$S(a,c) \equiv \frac{a+a'}{c} \pmod{\mathbb{Z}}. \tag{3.6}$$

It is then clear that $S(a,c) - S(b,c) \in \mathbb{Z}$ if, and only if,

$$a + a' \equiv b + b' \pmod{c}. \tag{3.7}$$

Now, by multiplying by $ab$,

$$a^2 b + b \equiv ab^2 + a \pmod{c},$$

$$ab(a-b) \equiv a - b \pmod{c},$$

$$(ab-1)(a-b) \equiv 0 \pmod{c}.$$

To complete the proof in the opposite direction we multiply the final line of the above equation by $a'b'$ to obtain (3.7). □

Girstmair goes further by providing a result which allows us, under certain conditions, to find the number of $b$ such that $S(a,c) - S(b,c) \in \mathbb{Z}$, given $a$ and $c$.

**Theorem 3.3.4.** *Let $c = p_1 \ldots p_t$ be square free, $p_i$ prime. For a given $a$ such that $\gcd(a,c) = 1$, we have*

$$|\{b : 0 \leq b < c, (b,c) = 1, S(a,c) - S(b,c) \in \mathbb{Z}\}| = 2^s$$

*where $s = |\{j : 1 \leq j \leq t, a \not\equiv \pm 1 \pmod{p_j}\}|$.*

## 3.4 Distribution Results

Much attention has been given to the distribution of Dedekind sums. In this section we will provide an overview of some of these results.

The large values of the Dedekind sum are considered [9] by taking $2m$th moments averaged over reduced fractions on the interval [0,1] with denominator $k$.

**Theorem 3.4.1.** *Suppose that $k$ is a large prime number and $m \geq 1$. Then,*

$$\sum_{h=1}^{k-1} s(h,k)^{2m} = 2\frac{\zeta(2m)^2}{\zeta(4m)}\left(\frac{k}{12}\right)^{2m} + O\left(\left(k^{9/5} + k^{2m-1+\frac{1}{m+1}}\right)\log^3 k\right).$$

The proof of this relies on a sequence of lemmas and the Hardy-Littlewood circle method. An estimate is also given for arbitrary $k$.

**Theorem 3.4.2.** *Suppose $k$ is large and $m \geq 1$. Then,*

$$\sum_{\substack{h=1 \\ (h,k)=1}}^{k} s(h,k)^{2m} = f_m(k)\left(\frac{k}{12}\right)^{2m} + O\left(\left(k^{9/5} + k^{2m-1+\frac{1}{m+1}}\right)\log^3 k\right)$$

*where*

$$\sum_{k=1}^{\infty} \frac{f_m(k)}{k^s} = 2\zeta(s)\frac{\zeta(2m)^2\zeta(s+4m-1)}{\zeta(4m)\zeta(s+2m)^2}.$$

It is easy to check that the values of the Dedekind sum satisfy $|s(h,k)| < \frac{k}{12}$ and $s(1,k) \to \frac{k}{12}$ as $k \to \infty$. It has been shown by Vardi [35] that the limiting distribution of $s(h,k)/\log k$ is the Cauchy distribution. That is to say,

$$\lim_{N\to\infty} \frac{\#\{0 < h < k < N, \gcd(h,k) = 1 : s(h,k) < x\log(k)\}}{\#\{0 < h < k < N, \gcd(h,k) = 1\}} = \frac{1}{\pi}\left(\arctan(2\pi x) + \frac{\pi}{2}\right). \tag{3.8}$$

If we study the graph of the Dedekind sum for large, prime, $k$ (see Figure 3.1) we can see that there is a large positive spike near the origin and a corresponding negative spike near $h = k-1$. We can also see other, smaller spikes near rational numbers with small denominators. For example, there is also a large negative spike to the left of $1/2$. This phenomenon is easily verified by considering our continued fraction expansion of the Dedekind sum.

Girstmair and Schoissengeier also study [16] the arithmetic mean of the Dedekind sum. They again define $S(h,k) = 12s(h,k)$. We let $x(N) = \min\{\sqrt{N}/\log N, \sqrt{N}/\tau(N)\}$, where $\tau(N)$ is the number of divisors of $N$. Then let $c, d$ be integers such that $0 \leq c \leq d \leq x$ and $(c,d) = 1$. Then define

$$I_{c/d} = [0,N] \cap \{z \in \mathbb{R} : |z - Nc/d| \leq x/d\}.$$

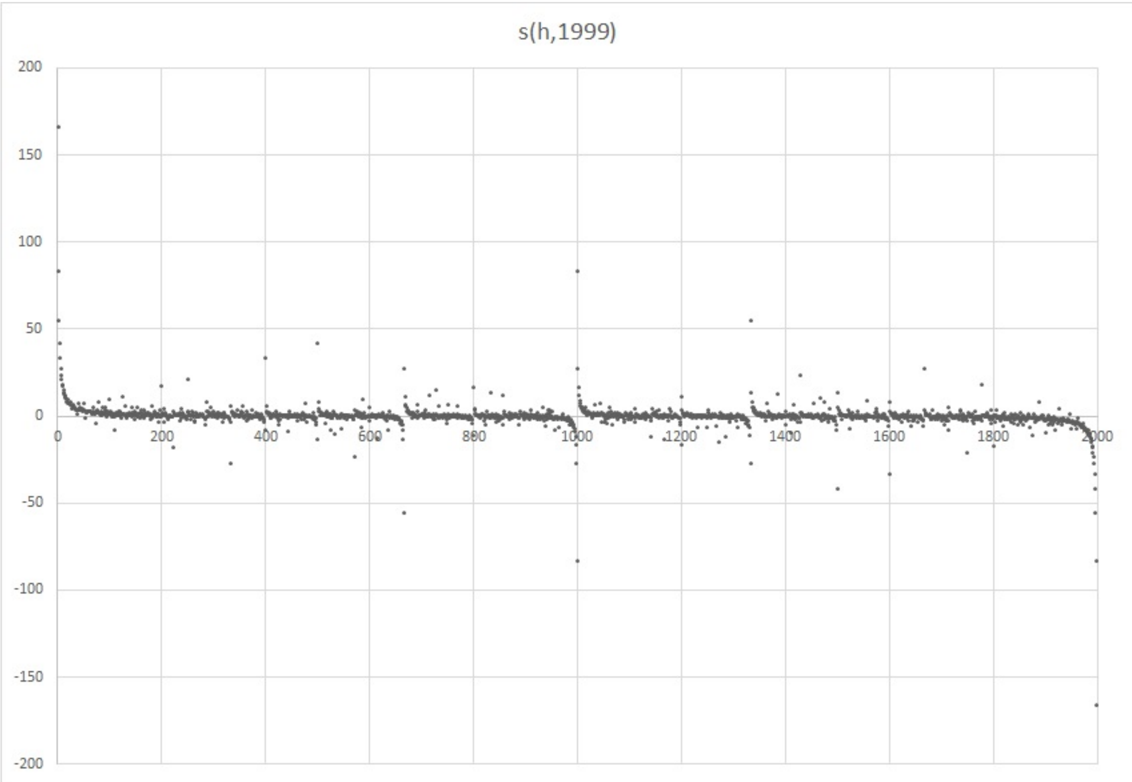


Figure 3.1: Values of $s(h,k)$ for $k = 1999$

Finally, we define

$$\mathcal{F} = \bigcup_{1\le d\le x} \bigcup_{\substack{0\le c\le d\\(c,d)=1}} I_{c/d}.$$

We now present the main result of [16].

**Theorem 3.4.3.** *Let $N$ tend to $\infty$. Then*

$$\frac{1}{\phi(N)} \sum_{\substack{m\in\mathcal{F}\\(m,N)=1}} |S(m,N)| = \frac{3}{\pi^2}\log^2 N + O(\log^2 N/\log\log N)$$

*where $\mathcal{F}$ is defined as above and $\phi(N)$ is the Euler function.*

An obvious corollary is

$$\frac{1}{\phi(N)} \sum_{\substack{0\le m<N\\(m,N)=1}} |S(m,N)| \ge \frac{3}{\pi^2}\log^2 N + O(\log^2 N/\log\log N).$$

It is also known [16], by considering the continued fraction expansion of the Dedekind sum, that

$$\frac{1}{\phi(N)} \sum_{\substack{0\le m<N\\(m,N)=1}} |S(m,N)| \le \frac{6}{\pi^2}\log^2 N + O(\log N).$$

Vardi [34] and Myerson [22] related Dedekind sums to Kloosterman sums to observe more distribution results. We define the Kloosterman sum by

$$K(m,n;c) = \sum_{\substack{0\le d\le c\\(d,c)=1\\ad\equiv 1 \pmod c}} e\left(\frac{am+dn}{c}\right) \tag{3.9}$$

where $e(x) = e^{2\pi i x}$.

**Theorem 3.4.4.** *Let $m \in \mathbb{N}$. Then*

$$\sum_{\substack{0\le d\le c\\(d,c)=1}} e(12ms(d,c)) = K(m,m;c). \tag{3.10}$$

*Proof.* To prove this we first need to show that $12s(d,c) - \frac{a+d}{c}$ is always an integer when $ad \equiv 1 \pmod c$. This is true by (3.6). Now we prove the theorem. First,

$$\sum_{\substack{0\le d\le c\\(d,c)=1}} e(12ms(d,c)) = \sum_{\substack{0\le d\le c\\(d,c)=1\\ad\equiv 1 \pmod c}} e\left(\frac{am+dm}{c}\right) = K(m,m;c).$$

The first equality is achieved by considering

$$12ms(d,c) = 12ms(d,c) - \frac{am+dm}{c} + \frac{am+dm}{c} = M + \frac{am+dm}{c}$$

for some integer $M$. □

Vardi also proves in [34]:

**Theorem 3.4.5.** *Let $m \in \mathbb{N}$. Then*

$$\sum_{0<c<x} \sum_{\substack{0\le d\le c\\(d,c)=1}} e(12ms(d,c)) < x^{3/2+\epsilon},\ \forall \epsilon > 0. \tag{3.11}$$

The proof is a consequence of the previous theorem and the Kloosterman sum satisfying Weil's estimate [36]. Vardi used (3.10) and (3.11) to prove the following theorem.

**Theorem 3.4.6.** $\{\langle 12ms(d,c)\rangle\}_{\substack{c>0\\0<d<c\\(d,c)=1}}$ *is uniformly distributed on* $[0,1)$.

A sequence $u_1, u_2, \ldots$ of terms in $[0,1)$ is defined to be uniformly distributed if for all $a$ and $b$ with $0 < a < b < 1$ we have

$$\lim_{n\to\infty} \frac{\#\{k < n : a < u_k < b\}}{n} = b - a. \tag{3.12}$$

The proof of the previous theorem uses Weyl's criterion for uniform distribution [37]. This result is further generalised to give:

**Theorem 3.4.7.** *Let* $r \in \mathbb{R}^{+}$*. Then,* $\{\langle rs(d,c)\rangle\}_{\substack{c>0\\0<d<c\\(d,c)=1}}$ *is uniformly distributed on* $[0,1)$.

Obviously, the case when $r = 1$ shows us that the Dedekind sum is uniformly distributed on $[0,1)$. Myerson [22] follows much of Vardi's working but proves the 2-dimensional result:

**Theorem 3.4.8.** *Let* $r \in \mathbb{R}^{+}$*. Then,* $\{\langle rs(d,c), d/c\rangle\}_{\substack{c>0\\0<d<c\\(d,c)=1}}$ *is uniformly distributed (modulo one).*

# 4

# Related Sums, Applications and Other Properties of Dedekind Sums

## 4.1 Introduction

Dedekind sums, and various related sums, have lent themselves to applications in a wide range of mathematics. In this chapter we will outline areas of mathematics where these sums appear, as well as developing further theory of Dedekind sums.

## 4.2 Generalised Dedekind Sums

The classical Dedekind sum has been generalised in a number of different ways. Here we will mention one of these ways and we will call it the generalised Dedekind sum.

**Definition 4.2.1.** Let $h$ and $k$ be positive integers and $c$ be real. We define the generalised

Dedekind sum to be

$$\sigma(h,k,c) = \sum_{j=0}^{k-1} \left(\left(\frac{hj+c}{k}\right)\right)\left(\left(\frac{j}{k}\right)\right). \tag{4.1}$$

This is often also called the Dedekind-Rademacher sum.

The generalised Dedekind sum has found applications in the study of pseudo-random numbers [20]. In this section we will outline many of the properties of the generalised Dedekind sum and also mention some of these applications.

We observe that the generalised sum reduces to the regular Dedekind sum if we just take $c = 0$. We now want to prove analogous results to the ones we have for ordinary Dedekind sums for the generalised sum. We recall,

$$\sum_{\lambda \bmod k} \left(\left(\frac{\lambda + x}{k}\right)\right) = ((x)) \tag{4.2}$$

and obtain the following lemmas.

**Lemma 4.2.1.** *Let* $(h,k) = 1$ *and* $d$ *be a positive integer. Then*

$$\sigma(dh,dk,dc) = \sigma(h,k,c). \tag{4.3}$$

*Proof.* We have

$$\begin{aligned} \sigma(dh,dk,dc) &= \sum_{j=0}^{dk-1} \left(\left(\frac{hj+c}{k}\right)\right)\left(\left(\frac{j}{dk}\right)\right) \\ &= \sum_{j=0}^{k-1}\sum_{i=0}^{d-1} \left(\left(\frac{hj+c}{k}\right)\right)\left(\left(\frac{ik+j}{dk}\right)\right) \end{aligned} \tag{4.4}$$

.

We now apply (4.2) and obtain

$$\begin{aligned} \sigma(dh,dk,dc) &= \sum_{j=0}^{k-1} \left(\left(\frac{hj+c}{k}\right)\right)\left(\left(\frac{j}{k}\right)\right) \\ &= \sigma(h,k,c). \end{aligned} \tag{4.5}$$

□

We also have the following interesting lemma, which enables us to evaluate the generalised Dedekind sum by only considering integer values.

**Lemma 4.2.2.** *Suppose $h$, $k$ and $c$ are integers, $hh' \equiv 1 \pmod{k}$ and $0 < \theta < 1$. Then,*

$$\sigma(h, k, c + \theta) = \sigma(h, k, c) + \frac{1}{2}\left(\left(\frac{h'c}{k}\right)\right). \tag{4.6}$$

*Proof.* We first define

$$\delta(x) = \begin{cases} 1 & \text{if } x \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Now,

$$\sigma(h, k, c + \theta) = \sum_{j=0}^{k-1} \left(\left(\frac{hj + c + \theta}{k}\right)\right)\left(\left(\frac{j}{k}\right)\right). \tag{4.8}$$

Since $0 < \theta < 1$, we have

$$\begin{aligned} &= \sum_{j=0}^{k-1} \left(\left(\left(\frac{hj + c}{k}\right)\right) + \frac{\theta}{k} - \frac{1}{2}\delta\left(\frac{hj + c}{k}\right)\right)\left(\left(\frac{j}{k}\right)\right) \\ &= \sum_{j=0}^{k-1} \left(\left(\frac{hj + c}{k}\right)\right)\left(\left(\frac{j}{k}\right)\right) + \sum_{j=0}^{k-1} \left(\left(\frac{j}{k}\right)\right)\left(\frac{\theta}{k} - \frac{1}{2}\delta\left(\frac{hj + c}{k}\right)\right) \\ &= \sum_{j=0}^{k-1} \left(\left(\frac{hj + c}{k}\right)\right)\left(\left(\frac{j}{k}\right)\right) + 0 - \frac{1}{2}\sum_{j=0}^{k-1} \left(\left(\frac{j}{k}\right)\right)\delta\left(\frac{hj + c}{k}\right). \end{aligned} \tag{4.9}$$

It is clear that $hj + c \equiv 0 \pmod{k}$ when $j = -h'c$. Hence

$$\begin{aligned} \sigma(h, k, c + \theta) &= \sum_{j=0}^{k-1} \left(\left(\frac{hj + c}{k}\right)\right)\left(\left(\frac{j}{k}\right)\right) - \frac{1}{2}\left(\left(\frac{-h'c}{k}\right)\right) \\ &= \sum_{j=0}^{k-1} \left(\left(\frac{hj + c}{k}\right)\right)\left(\left(\frac{j}{k}\right)\right) + \frac{1}{2}\left(\left(\frac{h'c}{k}\right)\right). \end{aligned} \tag{4.10}$$

□

We can also establish a reciprocity law for the generalised Dedekind sum as follows.

**Theorem 4.2.1.** *Let $h, k$ and $c$ be integers and suppose $0 \leq c < k$. Furthermore, we will suppose*

$$hh' + kk' = 1. \tag{4.11}$$

*Then,*

$$\sigma(h, k, c) + \sigma(k, h, c) = s(h, k) + s(k, h) + \frac{c(c-1)}{2hk} + \frac{1}{2}\left(\left(\left(\frac{h'c}{k}\right)\right) + \left(\left(\frac{k'c}{h}\right)\right) - \left[\frac{c}{h}\right]\right). \tag{4.12}$$

*Proof.* Our proof will follow that given in [20], however we make some changes in our presentation. We first note,

$$\left(\left(\frac{hj+c+1}{k}\right)\right) = \left(\left(\frac{hj+c}{k}\right)\right) + \frac{1}{k} - \frac{1}{2}\delta\left(\frac{hj+c}{k}\right) - \frac{1}{2}\delta\left(\frac{hj+c+1}{k}\right) \tag{4.13}$$

and now, by applying a similar construction to that of the previous lemma, we obtain

$$\sigma(h,k,c+1) = \sigma(h,k,c) + \frac{1}{2}\left(\left(\frac{h'c}{k}\right)\right) + \frac{1}{2}\left(\left(\frac{h'(c+1)}{k}\right)\right). \tag{4.14}$$

It follows that,

$$\sigma(h,k,c) = \sigma(h,k,0) + \sum_{j=1}^{c-1}\left(\left(\frac{h'j}{k}\right)\right) + \frac{1}{2}\left(\left(\frac{h'c}{k}\right)\right). \tag{4.15}$$

We now observe from (4.11) that

$$\begin{aligned}\left(\left(\frac{h'j}{k}\right)\right) &= \left(\left(\frac{j}{hk} - \frac{k'j}{h}\right)\right)\\ &= -\left(\left(\frac{k'j}{h} - \frac{j}{hk}\right)\right)\\ &= -\left(\left(\frac{k'j}{h}\right)\right) + \frac{j}{hk} - \frac{1}{2}\delta\left(\frac{k'j}{h}\right).\end{aligned} \tag{4.16}$$

We now apply (4.15) and (4.16) together and, noticing $\sigma(h,k,0) = s(h,k)$, we obtain

$$\begin{aligned}&\sigma(h,k,c) + \sigma(k,h,c)\\ &= s(h,k) + s(k,h) + \sum_{j=1}^{c-1}\left(\left(\frac{h'j}{k}\right)\right) + \frac{1}{2}\left(\left(\frac{h'c}{k}\right)\right) + \sum_{j=1}^{c-1}\left(\left(\frac{k'j}{h}\right)\right) + \frac{1}{2}\left(\left(\frac{k'c}{h}\right)\right)\\ &= s(h,k) + s(k,h) + \sum_{j=1}^{c-1}\left(\frac{j}{hk} - \frac{1}{2}\delta\left(\frac{k'j}{h}\right)\right) + \frac{1}{2}\left(\left(\frac{h'c}{k}\right)\right) + \frac{1}{2}\left(\left(\frac{k'c}{h}\right)\right)\\ &= s(h,k) + s(k,h) + \frac{c(c-1)}{2hk} + \frac{1}{2}\left(\left(\left(\frac{h'c}{k}\right)\right) + \left(\left(\frac{k'c}{h}\right)\right) - \left[\frac{c}{h}\right]\right).\end{aligned} \tag{4.17}$$

□

We see from above that this expression is not quite symmetric. Lack of symmetry is due to a missing $[c/k]$, but given our restriction on $c$ it is clear that this is 0. We also note that our previous lemma allows us to have a reciprocity law for $c$ being any real. It is also sufficient to consider $0 \leq c < k$ as the sawtooth function reduces any larger $c$ to one in this interval.

**Lemma 4.2.3.** *Let c be real and h and k be integers. Then,* $12k\sigma(h,k,c)$ *is an integer.*

*Proof.* From our previous theorem we saw for each integer $c$ that

$$12k\sigma(h,k,c) = 12ks(h,k) + 12k\sum_{j=1}^{c-1}\left(\left(\frac{h'j}{k}\right)\right) + 6k\left(\left(\frac{h'c}{k}\right)\right). \tag{4.18}$$

We know that $12ks(h,k)$ is an integer, and by our definition of the sawtooth function it follows that our remaining terms are also integers. Also, if $c$ were to not be an integer we add the term $6k\left(\left(\frac{h'c}{k}\right)\right)$, which is also an integer. □

Knuth [19] has been able to use the generalised Dedekind sum in his study of random numbers and applying them to the determination of serial correlation coefficients. In [20] Knuth also provides an algorithm for choosing an integer $c$ such that $\sigma(h,k,c)$ is maximised for given $h$ and $k$.

## 4.3 Further Generalisations Of Dedekind Sums

Many other generalisations of Dedekind sums exist. Many of these are mentioned in [29] as Rademacher and Grosswald provide a brief history of Dedekind sums and their variations. In this section we will mention some variations, how they came to be studied and their reciprocity law. We first mention a slightly more generalised version of the Dedekind sum from the previous section. For integers $h$ and $k$ and real numbers $x$ and $y$ we define

$$s(h,k;x,y) = \sum_{j \pmod k}\left(\left(h\frac{j+y}{k}+x\right)\right)\left(\left(\frac{j+y}{k}\right)\right). \tag{4.19}$$

We note that if $x$ and $y$ are both integers, then our sum reduces to the classical Dedekind sum. Dieter and Ahrens [11] also consider these in their study of pseudo-random numbers. They presented a series of lectures which, as far as we are aware, remain unpublished. They show that the exact number of pairs of pseudo-random numbers in a given rectangle can be reduced to the evaluation of these generalised Dedekind sums. Many of their results appear in [10]. We again have a reciprocity formula which is our most useful tool when evaluating these sums.

**Theorem 4.3.1.** *Assume at least one of $x$ and $y$ is not an integer and $h$ and $k$ are integers. Then,*

$$\begin{aligned} &s(h,k;x,y) + s(k,h;y,x) \\ &\qquad = ((x))((y)) + \frac{1}{2}\left(\frac{h}{k}\mathcal{B}_2(y) + \frac{k}{h}\mathcal{B}_2(x) + \frac{1}{hk}\mathcal{B}_2(hy+kx)\right), \end{aligned} \tag{4.20}$$

*where $\mathcal{B}_2(x)$ denotes the second Bernoulli function, which is given by*

$$\mathcal{B}_2(x) = (x - [x])^2 - (x - [x]) + \frac{1}{6}. \tag{4.21}$$

Many proofs of this exist and the first one is credited to Dieter [12], although he only considers $x$ and $y$ being rationals. For examples of later proofs, that consider real $x$ and $y$, see [7] and [26]. As in the case of the classical Dedekind sum, a three-term reciprocity formula also exists for these sums. A statement of the theorem and proof are given in [7].

Another generalisation of the Dedekind sum that has received attention is the Fourier-Dedekind sum. We define it by

$$s_n(a_1, a_2, \ldots, a_d; b) = \frac{1}{b}\sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{(1-\xi_b^{ka_1})(1-\xi_b^{ka_2})\ldots(1-\xi_b^{ka_d})}, \tag{4.22}$$

where $\xi_b$ is a primitive $b$th root of unity and $n$ is a non-negative integer. These sums form the foundations of Erhart quasipolynomials [5], as well as generalising the Dedekind sum.

We first show how this returns the classical Dedekind sum.

**Theorem 4.3.2.** *Let a and b be positive integers. Then,*

$$s_0(a, 1; b) = -s(a, b) + \frac{b-1}{4b}. \tag{4.23}$$

*Proof.* We have

$$\begin{aligned}
s_0(a, 1; b) &= \frac{1}{b}\sum_{k=1}^{b-1} \frac{1}{(1-\xi_b^{ka})(1-\xi_b^{k})} \\
&= \frac{1}{b}\sum_{k=1}^{b-1}\left(\frac{1}{1-\xi_b^{ka}} - \frac{1}{2}\right)\left(\frac{1}{1-\xi_b^{k}} - \frac{1}{2}\right) \\
&+ \frac{1}{2b}\sum_{k=1}^{b-1}\left(\frac{1}{1-\xi_b^{k}} + \frac{1}{1-\xi_b^{ka}}\right) - \frac{1}{4b}\sum_{k=1}^{b-1} 1 \\
&= \frac{1}{4b}\sum_{k=1}^{b-1}\left(\frac{1+\xi_b^{ka}}{1-\xi_b^{ka}}\right)\left(\frac{1+\xi_b^{k}}{1-\xi_b^{k}}\right) + \frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{1-\xi_b^{k}} - \frac{b-1}{4b}.
\end{aligned} \tag{4.24}$$

Our first expression on the right hand side is simply $-s(a, b)$, by recalling the alternative form of the Dedekind sum, (2.20), from Chapter 2. It can also be shown that the middle expression

$$\frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{1-\xi_b^{k}} = \frac{1}{2} - \frac{1}{2b}; \tag{4.25}$$

see [5, Ch. 1] for details. It follows, that

$$s_0(a, 1; b) = -s(a, b) + \frac{b-1}{4b}. \tag{4.26}$$

□

In order to give a reciprocity formula for these sums we need to mention some of the theory of where they came from. A more complete explanation is given in [5, Ch. 1, 8], and we will be using this as our reference point.

We define the restricted partition function to be

$$p_A(n) = \#\{(m_1, \ldots, m_d) \in \mathbb{Z}^d : \text{all } m_j \geq 0,\ m_1 a_1 + \cdots + m_d a_d = n\}, \tag{4.27}$$

where $A = \{a_1, \ldots, a_d\}$ is a set of integers. It also is related to our Fourier-Dedekind sum. We first consider the two dimensional set $A = \{a, b\}$, $\gcd(a, b) = 1$, and consider the function

$$\frac{1}{(1 - z^a)(1 - z^b)} = \sum_{k \geq 0} \sum_{l \geq 0} z^{ak} z^{bl} = \sum_{n \geq 0} p_{(a,b)}(n) z^n. \tag{4.28}$$

The last equality is simple to check from the previous expression and our definition of $p_A(n)$. It then follows immediately that

$$f(z) = \frac{1}{(1 - z^a)(1 - z^b) z^n} = \sum_{k \geq 0} p_{(a,b)}(k) z^{k-n}. \tag{4.29}$$

Hence, $p_{(a,b)}(n)$ is the constant term of this series. To evaluate this we expand $f(z)$ into partial fractions

$$f(z) = \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_n}{z^n} + \frac{B_1}{z - 1} + \frac{B_2}{(z - 1)^2} + \sum_{k=1}^{a-1} \frac{C_k}{z - \xi_a^k} + \sum_{j=1}^{a-1} \frac{D_j}{z - \xi_b^j}, \tag{4.30}$$

where $\xi_l$ is a primitive $l$th root of unity. It can be shown that

$$\begin{aligned} C_k &= -\frac{1}{a(1 - \xi_a^{kb}) \xi_a^{k(n-1)}}, \\ D_j &= -\frac{1}{b(1 - \xi_b^{ja}) \xi_b^{j(n-1)}}, \\ B_2 &= \frac{1}{ab}, \\ B_1 &= \frac{1}{ab} - \frac{1}{2a} - \frac{1}{2b} - \frac{n}{ab}. \end{aligned} \tag{4.31}$$

The first $n$ terms don't contribute to our constant term, so we evaluate the remaining terms when $z = 0$ to obtain

$$p_{(a,b)}(n) = \frac{1}{2a} + \frac{1}{2b} + \frac{n}{ab} + \frac{1}{a} \sum_{k=1}^{a-1} \frac{1}{(1 - \xi_a^{kb}) \xi_a^{kn}} + \frac{1}{b} \sum_{j=1}^{b-1} \frac{1}{(1 - \xi_b^{ja}) \xi_b^{jn}}. \tag{4.32}$$

We can now extend the above results to $A = \{a_1, \ldots, a_d\}$, $\gcd(a_i, a_j) = 1$ for each $i \neq j$, and obtain

$$p_A(n) = \mathrm{const}(f(z)) \tag{4.33}$$

that is the constant term of $f(z)$, where

$$\begin{aligned} f(z) &= \frac{1}{(1-z^{a_1})(1-z^{a_2})\dots(1-z^{a_d})z^n} \\ &= \frac{A_1}{z} + \frac{A_2}{z^2} + \dots + \frac{A_n}{z^n} + \frac{B_1}{z-1} + \dots + \frac{B_d}{(z-1)^d} \\ &+ \sum_{k=1}^{a_1-1} \frac{C_{1k}}{z-\xi_{a_1}^k} + \dots + \sum_{k=1}^{a_d-1} \frac{C_{dk}}{z-\xi_{a_d}^k}. \end{aligned} \tag{4.34}$$

As before, we can show

$$C_{1k} = -\frac{1}{a_1(1-\xi_{a_1}^{ka_2})\dots(1-\xi_{a_1}^{ka_d})\xi_{a_1}^{k(n-1)}} \tag{4.35}$$

and we can obtain a similar expression for each of the other $C_{ik}$. Putting these into each of our sums we arrive at the following theorem.

**Theorem 4.3.3.** *For $A = \{a_1, \dots, a_d\}$ and $\gcd(a_i, a_j) = 1$, for each $i \neq j$, we have*

$$\begin{aligned} p_A(n) = -B_1 + B_2 - \dots + (-1)^d B_d + s_{-n}(a_2, a_3, \dots, a_d; a_1) \\ + s_{-n}(a_1, a_3, a_4, \dots, a_d; a_2) + \dots + s_{-n}(a_1, a_2, \dots, a_{d-1}; a_d) \end{aligned} \tag{4.36}$$

*where the $B_i$'s are given by the partial fraction expression discussed previously.*

We now note that the $B_i$'s are polynomials in $n$, and, as in [5, Ch. 8], we let

$$\mathrm{poly}_A(n) = -B_1 + B_2 - \dots + (-1)^d B_d. \tag{4.37}$$

It is also clear from the definition that for all $a_i$ being positive, $p_A(0) = 1$. We use this to arrive at our reciprocity result.

**Theorem 4.3.4** (Zagier Reciprocity)**.** *For pairwise relatively prime positive integers $a_1, \dots, a_d$,*

$$\begin{aligned} s_0(a_2, a_3, \dots, a_d; a_1) + s_0(a_1, a_3, a_4, \dots, a_d; a_2) + \dots + s_0(a_1, a_2, \dots, a_{d-1}; a_d) \\ = 1 - poly_A(0). \end{aligned} \tag{4.38}$$

The first few expressions of $\mathrm{poly}_A(n)$ are given in [5, Ch. 8], and these can be used to obtain both the two term and three term reciprocity laws for the classical Dedekind sum. We finally mention one more reciprocity law given in [5, Ch. 8], which we will state without proof.

**Theorem 4.3.5** (Rademacher Reciprocity)**.** *For pairwise relatively prime positive integers $a_1, \dots, a_d$ and for $n = 1, 2, \dots, (a_1 + \dots + a_d - 1)$,*

$$\begin{aligned} s_n(a_2, a_3, \dots, a_d; a_1) + s_n(a_1, a_3, a_4, \dots, a_d; a_2) + \dots + s_n(a_1, a_2, \dots, a_{d-1}; a_d) \\ = -poly_A(-n). \end{aligned} \tag{4.39}$$

We mention briefly one of the generalisations of the Dedekind sum involving the Bernoulli function, which was defined for $m = n = 2$ in (4.21). We define the Bernoulli-Dedekind sum by

$$s_{m,n}(a; b, c) = \sum_{k \pmod b} \mathcal{B}_m\left(\frac{kb}{a}\right)\mathcal{B}_n\left(\frac{kc}{a}\right) \tag{4.40}$$

where $\mathcal{B}_m(x) = B_m(x - ((x)))$, $B_m(x)$ is the $m$th Bernoulli polynomial, which is defined in [4]. These were studied by Apostol [1] and Carlitz [8] and are mentioned by Beck in [4]. We can again collapse these down to our classical Dedekind sums by taking $m = n = 1$ and $c = 1$.

We note that in this chapter we have only mentioned a couple of generalisations of the Dedekind sum. This is by no means an exhaustive list, in fact many different generalisations exist. A long list of references of these is given in [5, Ch. 8]. We also mention [7], which provides examples of other generalisations and even proves reciprocity in some of these cases, as well as [4] which provides many more examples and relates many of these generalisations to a generalised Dedekind sum involving cotangents.

## 4.4 The Farey Series

Although this section seems out of place in this chapter, the relationship between the Dedekind sum and the Farey series is of interest and is worth including in our essay. In [29, Ch. 3] Rademacher and Grosswald provide a quick introduction to Farey fractions. We define the Farey series of order $N$ to be all the reduced rational fractions $h/k$, arranged in ascending order and with $1 \le k \le N$. We recall, if $h_1/k_1 < h_2/k_2$ are two consecutive Farey fractions, then

$$\begin{vmatrix} h_1 & h_2 \\ k_1 & k_2 \end{vmatrix} = -1. \tag{4.41}$$

Hence,

$$\begin{aligned} h_1 k_2 &\equiv -1 \pmod{k_1}, \\ h_2 k_1 &\equiv 1 \pmod{k_2}. \end{aligned} \tag{4.42}$$

This has been used [29, Ch. 3] to relate the Dedekind sum to the Euler phi-function.

Rademacher asked [28] the following problem regarding Farey fractions: If $h_1/k_1 < h_2/k_2$ are adjacent Farey fractions and if $s(h_1, k_1)$ and $s(h_2, k_2)$ are both positive, is $s(h_1+k_1, h_2+k_2)$ non-negative? This question was answered negatively independently by Pinzur [24] and Rosen

[31] who both were able to give an infinite class of examples such that this fails. Asai [3] extends these results by giving a way of constructing these counter examples when given some $h$ and $k$. Asai calls a pair of adjacent Farey fractions $H/K < h/k$ a Rademacher's pair if it satisfies $s(h,k) > 0, s(H,K) > 0$ and $s(h+H, k+K) < 0$. We present his result as a theorem.

**Theorem 4.4.1.** *For each reduced fraction* $0 < h/k < 1$ *with* $s(h,k) > 0$ *there exists a unique reduced fraction* $H/K$ *such that* $H/K < h/k$ *is a Rademacher's pair, unless* $k^2 \equiv -1 \pmod{h}$. *If* $k^2 \equiv -1 \pmod{h}$ *there are no such pairs.*

If $h_1/k_1 < h_2/k_2$ are adjacent Farey fractions, we define the mediant to be $h/k = (h_1+k_1)/(k_1+k_2)$. This gives a new adjacent Farey sequence $h_1/k_1 < h/k < h_2/k_2$. We call $h/k$ the left parent of $h_1/k_1$. Asai used this to present the following theorem in constructing Rademacher's pairs.

**Theorem 4.4.2.** *For all fractions* $h/k$, $k \not\equiv -1 \pmod{h}$, *whose common left parent is a fixed fraction* $0 < h_0/k_0 < 1$, *the Rademacher's pair* $H/K < h/k$ *is given by*

$$H = k - 12hs(k,h), \quad K = 3k - h + 12ks(h,k). \tag{4.43}$$

*Furthermore,*

$$h = 3H - K + 12Hs(K,H), \quad k = H - 12Ks(H,K). \tag{4.44}$$

The proof of these theorems comes as a consequence of a series of lemmas, many of which aren't trivial. We omit these lemmas and the proof of the greater theorems and encourage the reader to work through them.

# 5

# New Results

## 5.1 On A Conjecture Of Myerson And Phillips

In this chapter we present some new results by considering a conjecture presented in [21]. We provide a proof of this conjecture and we present our findings as a theorem.

A weaker version of the following result was left as a conjecture in [21]. However, it was Myerson and Phillips' intention to leave a stronger conjecture which we prove and present below as a theorem.

**Theorem 5.1.1.** *If* $11ab \equiv 1 \pmod{c}$*, then*

$$\delta = s(a, c) + s(b, c) - \frac{a}{c} - \frac{b}{c} \tag{5.1}$$

*is an integer.*

To prove it we will need the following lemmas given in [2, Ch. 3].

**Lemma 5.1.1.** *If k is odd, then*

$$12ks(h,k) \equiv k - 1 + 4 \sum_{r<k/2} \left[\frac{2hr}{k}\right] \pmod{8}. \tag{5.2}$$

A complementary result to this is provided in [29, Ch. 3].

**Lemma 5.1.2.** *Let $k$ be even, $h \geq 1$ and $k = 2^{\lambda} k_1$, where $k_1$ is odd. Then*

$$12hks(h,k) \equiv h^2 + k^2 + 1 + 5k - 4kT_1 \pmod{2^{\lambda+3}}, \tag{5.3}$$

*for some integer $T_1$.*

We can now prove our theorem.

*Proof.* We recall the following result as a consequence of the reciprocity formula,

$$12hks(h,k) \equiv h^2 + 1 \pmod{\theta k} \text{ (where } \theta = \gcd(k,3)). \tag{5.4}$$

Applying the previous equation, we obtain the following,

$$\begin{aligned} 12abc\delta &\equiv a^2b + b + ab^2 + a - 12a^2b - 12ab^2 \pmod{\theta c} \\ &\equiv (a+b)(1-11ab) \pmod{\theta c} \\ &\equiv 0 \pmod{c}. \end{aligned} \tag{5.5}$$

We also know $2\theta ks(h,k)$ is an integer. Therefore, when $\theta = 1$, $12c\delta \equiv 0 \pmod 3$. Putting this together with the above, and observing that $a$ and $b$ don't divide $c$, we obtain $12c\delta \equiv 0 \pmod{3c}$ for $\theta = 1$. For $\theta = 3$ we return to our above congruences to obtain

$$12abc\delta \equiv (a+b)(nc) \equiv 0 \pmod{\theta c} \ (n = 0, 1, 2). \tag{5.6}$$

We obtain the final congruence by observing, when $11ab \equiv 1 \pmod c$, that $\pm a \equiv \mp b \pmod 3$. Since $a, b$ don't divide $3c$, $12c\delta \equiv 0 \pmod{3c}$.

We now split our argument into two cases. First, we suppose that $c$ is odd. By (5.2) we have,

$$\begin{aligned} 12c\delta &\equiv 2c - 2 - 12(a+b) + 4T \pmod 8 \\ &\equiv 2(c-1) - 12(a+b) + 4T \pmod 8 \\ &\equiv 0 \pmod 4, \end{aligned} \tag{5.7}$$

where $T$ is some integer. We observe that $\gcd(4, 3c) = 1$, due to $c$ being odd. Therefore $12c\delta \equiv 0 \pmod{12c}$, hence $\delta$ is an integer.

We now suppose that $c$ is even and $c = 2^\lambda c_1$, where $c_1$ is odd. We observe that $a$ and $b$ are both odd, hence $a + b$ is even. Therefore by (5.3),

$$\begin{aligned} 12abc\delta &\equiv -11ab(a+b) + c^2(a+b) + (a+b) \\ &\quad + 5c(a+b) - 4c(aT_2 + bT_1) \pmod{2^{\lambda+3}} \\ &\equiv (a+b)(-11ab + c^2 + 1 + 5c) \pmod{2^{\lambda+2}}. \end{aligned} \tag{5.8}$$

We recognise that the expression within the right parentheses is a multiple of $c$ since $11ab \equiv 1 \pmod{c}$. Now if $a + b \equiv 0 \pmod 4$ the above equation becomes $0 \pmod{2^{\lambda+2}}$ and our proof is complete. Let's suppose $a + b \equiv 2 \pmod 4$. Then $a \equiv b \equiv \pm 1 \pmod 4$. Therefore, $11ab \equiv -1 \pmod 4$. If we let $11ab = cc' + 1$ for some integer $c'$, then the only possible solutions to both congruences is for $c'$ to be odd and $c$ to be $2c_1$ for some odd $c_1$. We substitute this back into the above congruence to find that

$$\begin{aligned} 12abc\delta &\equiv (a+b)(-(cc'+1) + c^2 + 1 + 5c) \pmod{2^{\lambda+2}} \\ &\equiv c(a+b)(c + 5 - c') \pmod{2^{\lambda+2}} \\ &\equiv 0 \pmod{2^{\lambda+2}}. \end{aligned} \tag{5.9}$$

The last line comes from our definition of $c$ and that the expressions in both pairs of parentheses are now even. Clearly $a$ and $b$ don't divide $2^{\lambda+2}$. Hence, $12c\delta \equiv 0 \pmod{2^{\lambda+2}}$. We now observe $\gcd(2^{\lambda+2}, 3c) = 2^\lambda$. Therefore, $12c\delta \equiv 0 \pmod{12c}$. This completes our proof.

□

We note that we can generalise the previous theorem by the following result.

**Theorem 5.1.2.** *Let $12\alpha - 1 = Q$, where $Q$ is an integer, and let $d = \gcd(12, Q+1)$. If $Qab \equiv 1 \pmod{c}$, then*

$$\delta = \frac{12}{d}(s(a,c) + s(b,c)) - \frac{Q+1}{d}\left(\frac{a}{c} + \frac{b}{c}\right) \tag{5.10}$$

*is an integer.*

*Proof.* The proof of this will use the techniques shown in the previous theorem. We note the case $Q = 11$ is what we proved previously. Now,

$$\begin{aligned} dabc\delta &\equiv a^2b + b + ab^2 + a - (Q+1)a^2b - (Q+1)ab^2 \pmod{\theta c} \\ &\equiv (a+b)(1 - Qab) \pmod{\theta c} \\ &\equiv 0 \pmod{c}. \end{aligned} \tag{5.11}$$

This immediately establishes our theorem for $d = 1$. For $d = 3$ and $d = 6$, we note that $dc\delta \equiv 0 \pmod 3$ when $\theta = 1$, since $3|(Q+1)$ and $3|(s(a,c)+s(b,c))$. Hence, $dc\delta \equiv 0 \pmod{3c}$. For $\theta = 3$,

$$dabc\delta \equiv (a+b)(nc) \equiv 0 \pmod{\theta c}\ (n = 0, 1, 2). \tag{5.12}$$

We obtain the final congruence by observing that if $c \equiv 0 \pmod 3$, then $Q + 1 \equiv 0 \pmod 3$, hence $Q \equiv -1 \pmod 3$. Now, $Qab \equiv 1 \pmod 3$ so $a \equiv -b \pmod 3$, thus $a + b \equiv 0 \pmod 3$. Also, $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$, so $dc\delta \equiv 0 \pmod{3c}$. This completes the proof for $d = 3$.

For $d = 2, 4, 6, 12$ we first suppose $c$ is odd and we apply (5.2) to deduce that

$$\begin{aligned} dc\delta &\equiv 2c - 2 - (Q+1)(a+b) + 4T \pmod 8 \\ &\equiv 2(c-1) - (Q+1)(a+b) + 4T \pmod 8. \end{aligned} \tag{5.13}$$

We notice that the final line is 0 (mod 2) when $d = 2, 6$ and 0 (mod 4) when $d = 4, 12$. Therefore, by applying the conditions (mod 3c) if necessary, $dc\delta \equiv 0 \pmod{dc}$, $d = 2, 4, 6, 12$. We now suppose $c$ is even and we apply (5.3) to obtain

$$\begin{aligned} dabc\delta &\equiv -Qab(a+b) + c^2(a+b) + (a+b) \\ &\quad + 5c(a+b) - 4c(aT_2 + bT_1) \pmod{2^{\lambda+3}} \\ &\equiv (a+b)(-Qab + c^2 + 1 + 5c) \pmod{2^{\lambda+2}}. \end{aligned} \tag{5.14}$$

We now observe that the right side of the previous congruence is a multiple of $c$, and since $c$ is even it follows that $a + b$ is necessarily even. Therefore, the above is $\equiv 0 \pmod{2^{\lambda+1}}$. Combining this with our previous results completes our proof for $d = 2, 6$. For $d = 4, 12$ we note that $Q \equiv -1 \pmod 4$. If $a + b \equiv 0 \pmod 4$ there would be nothing to prove. Suppose $a + b \equiv 2 \pmod 4$. Then $a \equiv b \equiv \pm 1 \pmod 4$ and thus $Qab \equiv -1 \pmod 4$. We let $Qab = cc' + 1$. It follows that $c'$ is odd and $c = 2c_1$ where $c_1$ is also odd. Then,

$$\begin{aligned} 12abc\delta &\equiv (a+b)(-(cc'+1) + c^2 + 1 + 5c) \pmod{2^{\lambda+2}} \\ &\equiv c(a+b)(c + 5 - c') \pmod{2^{\lambda+2}} \\ &\equiv 0 \pmod{2^{\lambda+2}}. \end{aligned} \tag{5.15}$$

Now, $a$ and $b$ don't divide $c$ and so it follows that $dc\delta \equiv 0 \pmod{2^{\lambda+2}}$. Now, since $\gcd(2^{\lambda+2}, c) = 4c$, our proof is established for $d = 4$. Similarly, since $\gcd(2^{\lambda+2}, 3c) = 12c$, our proof is also established for $d = 12$. □

We also provide an alternate, direct proof for the case $d = 2$ and $Q = 1$.

*Proof.* For $Q = 1$, we have $ab \equiv 1 \pmod{c}$. Therefore,

$$\begin{aligned} 6(s(a,c) + s(b,c)) - \frac{a}{c} - \frac{b}{c} &= 12s(a,c) - \frac{a}{c} - \frac{b}{c} \\ &= 12s(b,c) - \frac{a}{c} - \frac{b}{c}. \end{aligned} \tag{5.16}$$

We then multiply both sides by $ac$ and obtain

$$12acs(a,c) \equiv a^2 + 1 \pmod{\theta c}. \tag{5.17}$$

We also know that $6cs(a,c)$ is an integer. Using this we obtain

$$12acs(a,c) - a^2 - ab \equiv 0 \pmod{a} \tag{5.18}$$

together with,

$$\begin{aligned} &12acs(a,c) - a^2 - ab \\ &\equiv a^2 + 1 - a^2 - ab \pmod{c} \\ &\equiv 0 \pmod{c}. \end{aligned} \tag{5.19}$$

Therefore,

$$12acs(a,c) - a^2 - ab \equiv 0 \pmod{ac}. \tag{5.20}$$

□

## 5.2 A Corollary Of The Reciprocity Formula

In our study of Myerson and Phillips' conjecture we discovered the following result.

**Theorem 5.2.1.** *Let $a, b, c$ and $d$ be integers and $(d-1)ab + 1 = cc'$. Then,*

$$\frac{12}{d}(s(a,c) - s(b,c)) - \frac{a}{c} + \frac{b}{c} = \frac{12}{d}(s(a,c') - s(b,c')) - \frac{a}{c'} + \frac{b}{c'}. \tag{5.21}$$

*Proof.* Since $cc' \equiv 1 \pmod{a}$ and $cc' \equiv 1 \pmod{b}$, we know $s(c,a) = s(c',a)$ and similarly $s(c,b) = s(c',b)$. So we can relate $s(a,c)$ to $s(a,c')$ by the reciprocity formula. Indeed,

$$\begin{aligned} s(a,c) &= s(a,c') + \frac{1}{12}\left(\frac{a}{c} + \frac{c}{a} + \frac{1}{ac} - \frac{a}{c'} - \frac{c'}{a} - \frac{1}{ac'}\right) \\ &= s(a,c') + \frac{1}{12}\left(\frac{a(c'-c)}{cc'} - \frac{c'-c}{a} + \frac{c'-c}{acc'}\right) \\ &= s(a,c') + \frac{1}{12}\left(\frac{(c'-c)(a^2 - ((d-1)ab+1) + 1)}{acc'}\right) \\ &= s(a,c') + \left(\frac{c'-c}{12cc'}\right)(a - (d-1)b). \end{aligned}$$

An equivalent relation exists for $s(b,c)$. We obtain,

$$\begin{aligned}
&\frac{12}{d}\left(s(a,c)-s(b,c)\right)-\frac{a}{c}+\frac{b}{c}\\
&=\frac{12}{d}\left(s(a,c')-s(b,c')+\frac{(c'-c)(a-b+(d-1)(a-b))}{12cc'}\right)-\frac{a}{c}+\frac{b}{c}\\
&=\frac{12}{d}\left(s(a,c')-s(b,c')\right)+\left(\frac{c'-c}{cc'}\right)(a-b)-\frac{a}{c}+\frac{b}{c}\\
&=\frac{12}{d}\left(s(a,c')-s(b,c')\right)-\frac{a}{c'}+\frac{b}{c'}.
\end{aligned}$$

□

We can use the above result to evaluate Dedekind sums in special cases. In all these cases we suppose $b=1$ and recall

$$s(1,c)=-\frac{1}{4}+\frac{1}{6c}+\frac{c}{12}=\frac{(c-1)(c-2)}{12c}.$$

a) If $c'=1$, i.e. $(d-1)a+1=c$, or equivalently $c\equiv 1 \pmod{a}$, then,

$$\begin{aligned}
&\frac{12}{d}\left(s(a,c)-s(1,c)\right)-\frac{a}{c}+\frac{1}{c}\\
&=\frac{12}{d}\left(s(a,1)-s(1,1)\right)-a+1\\
&=-a+1.
\end{aligned}$$

Rearranging and observing that $d=(c+a-1)/a$ we obtain an identity given in [2, Ch. 3],

$$\begin{aligned}
s(a,c)&=\frac{(c+a-1)(1-a)(c-1)}{12ac}+\frac{(c-1)(c-2)}{12c}\\
&=\frac{(c-1)((c+a-1)(1-a)+a(c-2))}{12ac}\\
&=\frac{(c-1)(c-a^2-1)}{12ac}.
\end{aligned}\tag{5.22}$$

b) If $c'=2$, or equivalently $2c\equiv 1 \pmod{a}$, then we note that $a$ must be odd and $d$ must be even. Thus,

$$\begin{aligned}
&\frac{12}{d}\left(s(a,c)-s(1,c)\right)-\frac{a}{c}+\frac{1}{c}\\
&=\frac{12}{d}\left(s(a,2)-s(1,2)\right)-\frac{a}{2}+\frac{1}{2}\\
&=\frac{1}{2}-\frac{a}{2}.
\end{aligned}$$

We rearrange the previous equation to obtain an equation equivalent to that in [2, Ch. 3] with different conditions,

$$s(a,c)=\frac{(c-2)(-a^2+2c-1)}{24ac}.\tag{5.23}$$

c) If $(d-1)a+1=cc'$, or equivalently $cc' \equiv 1 \pmod{a}$, and $a \equiv 1 \pmod{c'}$, then

$$\begin{aligned}&\frac{12}{d}\left(s(a,c)-s(1,c)\right)-\frac{a}{c}+\frac{1}{c}\\ &=\frac{12}{d}\left(s(a,c')-s(1,c')\right)-\frac{a}{c'}+\frac{1}{c'}\\ &=\frac{1}{c'}-\frac{a}{c'}.\end{aligned}$$

We now rearrange this equation to deduce that

$$s(a,c)=\frac{(c-1)(c-2)}{12c}+\frac{(cc'+a-1)(1-a)(c-c')}{12acc'}. \tag{5.24}$$

d) If $(d-1)a+1=cc'$, or equivalently $cc' \equiv 1 \pmod{a}$, and $a \equiv -1 \pmod{c'}$, then

$$\begin{aligned}&\frac{12}{d}\left(s(a,c)-s(1,c)\right)-\frac{a}{c}+\frac{1}{c}\\ &=\frac{12}{d}\left(s(a,c')-s(1,c')\right)-\frac{a}{c'}+\frac{1}{c'}\\ &=-2\frac{12}{d}\frac{(c'-1)(c'-2)}{12c'}+\frac{1}{c'}-\frac{a}{c'}.\end{aligned}$$

Therefore,

$$s(a,c)=\frac{(c-1)(c-2)}{12c}-\frac{(c'-1)(c'-2)}{6c'}+\frac{(cc'+a-1)(1-a)(c-c')}{12acc'}. \tag{5.25}$$

e) If $(d-1)a+1=cc'$, or equivalently $cc' \equiv 1 \pmod{a}$, and $a \equiv \pm 2 \pmod{c'}$, then, by recalling $24cs(2,c)=(c-5)(c-1)$,

$$\begin{aligned}&\frac{12}{d}\left(s(a,c)-s(1,c)\right)-\frac{a}{c}+\frac{1}{c}\\ &=\frac{12}{d}\left(s(a,c')-s(1,c')\right)-\frac{a}{c'}+\frac{1}{c'}\\ &=\frac{12}{d}\left(\pm\frac{(c'-5)(c'-1)}{24c'}-\frac{(c'-1)(c'-2)}{12c'}\right)-\frac{a}{c'}+\frac{1}{c'}.\end{aligned}$$

Again we rearrange to obtain

$$\begin{aligned}s(a,c)=&\frac{(c'-1)(\pm(c'-5)-2(c'-2))}{24c'}+\frac{(c-1)(c-2)}{12c}\\ &+\frac{(cc'+a-1)(1-a)(c-c')}{12acc'}.\end{aligned} \tag{5.26}$$

## 5.3 Dense Lines Of Dedekind Sums

We wish to prove that solutions to the problem $s(h,k)=\alpha h/k$, $\alpha \neq 1/12$ are dense in the reals. We will only be considering cases where $12\alpha-1$ is a positive integer and we will follow similar techniques to those shown in [21]. In this section we prove the density of the solutions for $\alpha = 1/6$ and $\alpha = 1/4$.

First we define the following quantities. Let $h/k = [a_0, a_1, \ldots, a_t]$, with $t$ even, and define

$$I(h,k) = \sum_{i=0}^{t} (-1)^{i+1} a_i. \tag{5.27}$$

Similarly, if $h/k = [a_0, a_1, \ldots, a_p]$, with $p$ odd, then we define

$$J(h,k) = \sum_{i=1}^{p} (-1)^{i+1} a_i. \tag{5.28}$$

It is easy to see that $J(h,k) = I(h,k) + [h/k] + 2$.

Let $\gamma = 12\alpha - 1$. Suppose $h, k$ are integers, $k > 0$ and $(\gamma h, k) = 1$. We define $h', r, r''$ and $m$ as follows:

$$\begin{aligned} hh' &\equiv -1 \pmod{k}, \quad 0 < h' < k, \\ \{-\gamma h/k\} &= r/k, \\ rr'' &\equiv 1 \pmod{k}, \quad 0 < r'' < k, \text{ and} \\ m &= (\gamma r'' - h')/k. \end{aligned} \tag{5.29}$$

**Lemma 5.3.1.** *With the above definitions, $m$ is an integer, and $0 \le m \le \gamma - 1$.*

*Proof.* It is clear that $r \equiv -\gamma h \pmod{k}$. If we multiply both sides by $r''h'$ we obtain $h' \equiv 11r'' \pmod{k}$, hence $m$ is an integer. Since $r'' > 0$ and $h' < k$ we have $m > -1$. From $r'' < k, h' > 0$ we get $m < \gamma$ so $0 \le m \le \gamma - 1$. □

**Lemma 5.3.2.** *Given $h, k$ integers, $k > 0$ and $(\gamma h, k) = 1$, let $h/k = [a_0, \ldots, a_t]$ with $t$ even. Let $-\gamma h/k = [-b_0, b_1, \ldots, b_p]$ with $p$ odd. Suppose there are positive integers $c_1, \ldots, c_n$, $n$ odd, satisfying*

$$\sum_{0}^{t} (-1)^i a_i + \sum_{1}^{n} (-1)^i c_i - \sum_{1}^{p} (-1)^i b_i + b_0 = 0, \tag{5.30}$$

*and*

$$ma + \gamma b - c = 0, \tag{5.31}$$

*where $m$ is defined as above and $a, b$ and $c$ are defined by*

$$\begin{pmatrix} 0 & 1 \\ 1 & c_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & c_2 \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 & 1 \\ 1 & c_n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

*Let $x = [a_0, \ldots, a_t, c_1, \ldots, c_n, b_p, \ldots, b_1]$. Then $s(x) = \alpha x$.*

*Proof.* We note that

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_0 \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 & 1 \\ 1 & a_t \end{pmatrix} = \begin{pmatrix} u & h \\ h' & k \end{pmatrix}$$

with $u$ defined by $uk = hh' = 1$, and

$$\begin{pmatrix} 0 & 1 \\ 1 & b_p \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} = \left(\begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 & 1 \\ 1 & b_p \end{pmatrix}\right)^t = \begin{pmatrix} v & r'' \\ r & k \end{pmatrix},$$

with $v$ defined by $vk - rr'' = -1$. If we let $x = H/K, K > 0, (H, K) = 1$, it follows that

$$\begin{pmatrix} u & h \\ h' & k \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v & r'' \\ r & k \end{pmatrix} = \begin{pmatrix} U & H \\ H' & K \end{pmatrix},$$

where $HH' \equiv -1 \pmod K$ and $0 < H' < K$. We need to prove that $\gamma H + H' - KI(H, K) = 0$ as this is equivalent to $s(x) = \alpha x$. By our definitions it is easy to see that $I(H, K) = b_0$. We can multiply out our matrices to see

$$\begin{aligned} H &= ur''a + ukb + hr''c + hkd, \\ H' &= h'va + h'rb + kvc + krd, \\ K &= h'r''a + h'kb + kr''c + k^2 d. \end{aligned} \tag{5.32}$$

Then $\gamma H + H' - KI(H, K) = Aa + Bb + Cc + Dd$. We then solve for $A, B, C$ and $D$ in a way analogous to [21] and find $A = m, B = \gamma, C = -1, D = 0$. We can see that $\gamma H + H' - KI(H, K) = ma + \gamma b - c = 0$. Hence, by Theorem 3.1.4, $s(x) = \alpha x$.

□

Our proof then relies on the existence of $c_i$ as given in the previous lemma. Myerson and Phillips [21] deal with this problem for the case $\alpha = 1$. We will provide a proof of the density in the reals of solutions of $s(x) = \alpha x$ for $\alpha = 1/6$ and $\alpha = 1/4$.

When $\alpha = 1/6$ we note that $m = 0$. Therefore the previous lemma requires that $b = c$ and

$$\sum_{i=0}^{t} (-1)^i a_i + \sum_{i=1}^{n} (-1)^i c_i - \sum_{i=1}^{p} (-1)^i b_i + b_0 = 0.$$

If we consider $(c_1, c_2, c_3)$, it is easy to check that under these conditions $c_1 = c_3$. Therefore,

$$c_2 = \sum_{i=1}^{p} (-1)^i b_i - \sum_{i=0}^{t} (-1)^i a_i - b_0 = -(I(-h, k) - I(h, k) + 2) = -J(b, d). \tag{5.33}$$

We note that we require $c_i$ to be strictly positive and so the above case deals with $J$ being negative. If $J$ were positive, we can consider the construction $(c_1) = (J)$. Finally, by our construction of the $a_i$ and $b_i$ having even and odd length respectively, it will be impossible for $J = 0$, so these cases are sufficient. This gives us the following theorem.

**Theorem 5.3.1.** *The set* $\{x : s(x) = x/6\}$ *is dense in the reals.*

*Proof.* Given $y$ in $\mathbb{R}$ and $\epsilon > 0$, choose integers $h$ and $k$, $k > \sqrt{2/\epsilon}$, such that $\gcd(h, k) = 1$ and $|y - h/k| < \epsilon/2$. Let $h/k = [a_0, \ldots, a_t]$, $t$ even, and let $-h/k = -b_0 + r/k = [-b_0, b_1, \ldots, b_p]$, $p$ odd. Then there exists $c_1, \ldots, c_n$, $n$ odd, such that

$$J(b, d) = I(-h, k) - I(h, k) + 2 \tag{5.34}$$

and

$$b = c \tag{5.35}$$

for $a, b, c$ and $d$ defined previously. Therefore, for $x = [a_0, \ldots, a_t, c_1, \ldots, c_n, b_p, \ldots, b_1]$, $s(x) = x/6$. Furthermore, $|x - h/k| < k^{-2} < \epsilon/2$. Hence, $|y - x| < \epsilon$. □

We will provide an example. Suppose $y = 1/2$ and $\epsilon = 1/10$, therefore $k \geq 5$. So we will take $h/k = 5/11 = [0, 2, 5]$. Therefore, $h' = 2$, $-h/k = -b_0 + r/k = -1 + 6/11 = [-1, 1, 1, 5]$. Hence,

$$J = I(-h, k) - I(h, k) + 2 = 4 - (-3) + 2 = 9.$$

Since $J$ is positive we take the case $(c_1) = (9)$. Therefore, $x = [0, 2, 5, 9, 5, 1, 1] = 516/1133$. And it is easy to check $s(516/1133) = 86/1133$, as required.

We note that our construction will always give us a symmetric continued fraction in the entries beyond $a_0$, since if $a_1 \neq 1$ and $h/k = [a_0, a_1, \ldots, a_n]$, then $-h/k = [-a_0 - 1, 1, a_1 - 1, a_2, \ldots, a_n]$ and if $a_1 = 1$ and $h/k = [a_0, 1, a_2, \ldots, a_n]$ then $-h/k = [-a_0 - 1, a_2 + 1, a_3, \ldots, a_n]$. So our required $x$, when $n$ is even, will be

$$\begin{aligned} x &= [a_0, a_1, \ldots, a_n, c_1, \ldots, c_t, a_n, \ldots, a_2, a_1 - 1, 1] \\ &= [a_0, a_1, \ldots, a_n, c_1, \ldots, c_t, a_n, \ldots, a_2, a_1] \end{aligned}$$

or

$$\begin{aligned} x &= [a_0, 1, a_2, \ldots, a_n, c_1, \ldots, c_t, a_n, \ldots, a_2 + 1] \\ &= [a_0, 1, a_2, \ldots, a_n, c_1, \ldots, c_t, a_n, \ldots, a_2, 1] \end{aligned}$$

respectively. Since the $c_i$'s are symmetric, it follows that $x$ will always be symmetric. We observe from looking at values when $s(x) = x/6$ that every continued fraction expansion of $x$ can be written as a symmetric continued fraction. We leave our findings as a conjecture.

**Conjecture 5.3.1.** If $x$ is a rational number and $s(x) = x/6$, then $x$ can be written as a simple continued fraction of the form

$$[a_0, a_1, \ldots, a_{n-1}, a_n, a_{n-1}, \ldots, a_1].$$

We now consider the case when $\alpha = 1/4$. To prove our result we will need the following lemmas.

**Lemma 5.3.3.** *Let $h$ and $k$ be integers, $k > 0$, with* $\gcd(3h, k) = 1$. *Let $m$ be as in* (5.29) *(we note $m$ can only take 0 or 1). Then $I(-2h, k) - I(h, k) + 2 \equiv m \pmod 3$, where $I(h, k)$ is defined as in* (5.27).

*Proof.* We recall from (3.3) that $12s(h, k) = h/k - h'/k + I(h, k)$. We also note that $-2hr' \equiv -1 \pmod k$, where $r' = k - r''$. Therefore,

$$I(-2h, k) = 12s(-2h, k) + \frac{2h}{k} + \frac{r'}{k} = -12s(r', k) + \frac{2h}{k} + \frac{r'}{k}.$$

Then

$$\begin{aligned} I(-2h, k) - I(h, k) - m & \\ &= 12s(r', k) + \frac{2h}{k} + \frac{r'}{k} - 12s(h, k) + \frac{h}{k} - \frac{h'}{k} - \frac{2r''}{k} + \frac{h'}{k} \\ &= 3\frac{h}{k} + 3\frac{r'}{k} - 12s(h, k) - 12s(r', k) - 2 \\ &= 12\left(\frac{h}{4k} + \frac{r'}{4k} - s(h, k) - s(r', k)\right) - 2. \end{aligned} \tag{5.36}$$

We note that this expression is an integer and since 3 doesn't divide $k$ it follows that

$$\left(\frac{h}{k} + \frac{r'}{k} - 4s(h, k) - 4s(r', k)\right) \tag{5.37}$$

is an integer. Therefore, $I(-2h, k) - I(h, k) + 2 \equiv m \pmod 3$. □

We now show that the $c_1, \ldots, c_n$ exist for this case.

**Lemma 5.3.4.** *Let there be given integers $J$ and $m$, $m$ defined as in* (5.29). *If $J \equiv m \pmod 3$ then there exist positive integers $c_1, \ldots, c_n$, $n$ odd, such that $J(b, d) = J$ and $ma + 2b - c = 0$, where $a, b, c$ and $d$ are defined as before.*

*Proof.* We split our proof into two cases depending on the value of $J$.

1. Assume $J = m + 3q$, $q = 1, 2, \dots$. Let $(c_1, c_2, c_3) = (m + 1 + 2q, 1, q)$. It is easy to check $J(b, d) = J$ and this does indeed satisfy $ma + 2b - c = 0$.

2. Assume $J = m - 3q + 3$, $q = 1, 2, \dots$. Let $(c_1, \dots, c_5) = (m + 2, q, 1, 2q + 1, 1)$. Again, we can check this does satisfy the required conditions. □

**Theorem 5.3.2.** *The set* $\{x : s(x) = x/4\}$ *is dense in the reals.*

*Proof.* Given $y$ in $\mathbb{R}$ and $\epsilon > 0$, choose integers $h$ and $k$, $k > \sqrt{2/\epsilon}$, such that $\gcd(3h, k) = 1$ and $|y - h/k| < \epsilon/2$. Let $h/k = [a_0, \dots, a_t]$, $t$ even, and $-2h/k = -b_0 + r/k = [-b_0, b_1, \dots, b_p]$, $p$ odd. Then, by our previous lemma, there exists $c_1, \dots, c_n$, $n$ odd, such that

$$J(b, d) = I(-2h, k) - I(h, k) + 2 \tag{5.38}$$

and

$$ma + 2b - c = 0 \tag{5.39}$$

for $a, b, c$ and $d$ defined previously. Therefore, for $x = [a_0, \dots, a_t, c_1, \dots, c_n, b_p, \dots, b_1]$, $s(x) = x/4$ . Furthermore, $|x - h/k| < k^{-2} < \epsilon/2$. Hence, $|y - x| < \epsilon$. □

We now provide an application of our theorem. Suppose we take $y = 1/5$ and $\epsilon = 1/10$. Then it is clear we are allowed to take $h/k = 1/5$. Now, $h/k = [0, 4, 1]$, $-2h/k = -b_0 + r/k = -1 + 3/5 = [-1, 2, 1, 1]$, $h' = 4$ and $m = 0$. We calculate,

$$J = I(-2h, k) - I(h, k) + 2 = 1 - 3 + 2 = 0.$$

Therefore, we take $(c_1, \dots, c_5) = (2, 1, 1, 3, 1)$ and $x = [0, 4, 1, 2, 1, 1, 3, 1, 2, 1, 1] = 210/991$. Clearly, $|1/5 - 210/991| < \epsilon$ and we can check that $s(x) = 105/1982$, as required.

We also note that Theorem 5.1.1 allows us to simplify the proof of Theorem 4 of [21]. We state the following lemma as a consequence of our result.

**Lemma 5.3.5.** *Let $h$ and $k$ be integers, $k > 0$, with* $\gcd(\gamma h, k) = 1$, *and* $d = \gcd(12, \gamma + 1)$. *Let $m$ be as in* (5.29). *Then*

$$I(-\gamma h, k) - I(h, k) + 2 \equiv m + 2 - \gamma \pmod{d}.$$

*Proof.* By a similar argument to Lemma 5.3.3 we obtain

$$I(-\delta h, k) - I(h, k) - m = d\left(\frac{\gamma + 1}{d}\left(\frac{h}{k} + \frac{r'}{k}\right) - \frac{12}{d}\left(s(h, k) - s(r', k)\right)\right) - \gamma.$$

By Theorem 5.1.2, it follows that

$$J(b,d) = I(-\gamma h, k) - I(h,k) + 2 \equiv m - \gamma + 2 \pmod{d}.$$

□

We now provide a simpler proof of another lemma in [21].

**Lemma 5.3.6.** *Let there be given integers $J$ and $m$, $m$ defined as before. If $J \equiv m+3 \pmod{12}$ then there exists positive integers $c_1,\ldots,c_n$, $n$ odd, such that $J(b,d) = J$ and $ma+11b-c = 0$, where $a,b,c$ and $d$ are defined as before.*

*Proof.* We use the results of [21], however, by our construction we only need to consider the cases such that $J \equiv m+3 \pmod{12}$. We thus split our proof into 3 cases.

1. Assume $m - J = 12q - 3$, for $q = 1, 2, \ldots$. Let $(c_1,\ldots,c_5) = (m+11, q, 1, 11q+10, 1)$.
2. Assume $J - m = 12q + 3$, for $q = 1, 2, \ldots$. Let $(c_1,\ldots,c_5) = (11q+m+4, 1, 2, 2, q)$.
3. Assume $J - m = 3$. Let $(c_1, c_2, c_3) = (m+12, 10, 1)$.

It is simple to check that these cases do indeed satisfy the required conditions. □

We now provide the statement of the theorem given and provide the simpler proof.

**Theorem 5.3.3.** $\{x : s(x) = x\}$ *is dense in the reals*

*Proof.* Given $y$ in $\mathbb{R}$ and $\epsilon > 0$, choose integers $h$ and $k$, $k > \sqrt{2/\epsilon}$, such that $\gcd(11h, k) = 1$ and $|y - h/k| < \epsilon/2$. Let $h/k = [a_0,\ldots,a_t]$, with $t$ being even. Furthermore, let $-11h/k = -b_0 + r/k = [-b_0, b_1, \ldots, b_p]$, with $p$ being odd. Then, by our previous lemma, there exists $c_1,\ldots,c_n$, $n$ odd, such that

$$\begin{aligned} &J(b,d) = I(-11h,k) - I(h,k) + 2 \text{ and} \\ &ma + 11b - c = 0 \end{aligned} \tag{5.40}$$

for $a,b,c$ and $d$ defined previously. Therefore, for $x = [a_0,\ldots,a_t,c_1,\ldots,c_n,b_p,\ldots,b_1]$, $s(x) = x$ . Furthermore, $|x - h/k| < k^{-2} < \epsilon/2$. Hence, $|y - x| < \epsilon$. □

# 6 Conclusion

Our work on Dedekind sums has provided an overview of some of the results that have been proven in the years following the publication of Rademacher and Grosswald's book. We have provided a foundation for our own further research in this topic, particularly in studying whether solutions of the problem $s(x) = \alpha x$ are dense in the real line, for all $\alpha \neq 1/12$. Our results in the final chapter lead us to believe, at least for $x$ being a rational with denominator dividing 12, that these solutions will be dense in the reals. However, we observe, in its current form, we would require increasingly many cases to give an explicit construction of $x$ when $\alpha$ increases. We hope that our presentation encourages further study of Dedekind sums, to further our collective knowledge of this interesting topic of mathematics.

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