# Modelling short term equilibrium and long term change in a natural way

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**Certificate of originality**

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person, nor material which to a substantial extent has been accepted for the award of any other degree or diploma of a university or other institute of higher learning, except where due acknowledgement has been made in the text.

Signed: Doug McLeod

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**Abstract**

Why would an agent produce and supply something if it got nothing in return? In order to investigate how complex systems, such as biological and economic systems, organize themselves, McLeod (2015) constructs a simple economic model for a biological system. In the context of a dimension model, it was shown that if exchange of resources between creatures is based on relative scarcity, we get a similar outcome to that produced by a market economy, even though such exchanges are not reciprocal. Specifically the 'biological economy' constructed in McLeod (2015) promotes the development of specialization and interdependence, and the number of creatures increases over time. These may be construed as large scale, or system, trends.

The work presented in this thesis extends McLeod (2015). It develops a multi–sector general equilibrium model of an economy in which resource-based processes are modelled, in order to understand evolution from an economic perspective. The model is based on habitual behaviour represented by Markov chains. It applies particularly, but not exclusively, to biological systems and to pre–market human economies. Interestingly, the interplay between producers of scarce resources and consumers of those resources generates various kinds of agent number and system trajectories. These range from expanding to collapsing and oscillating to stable, depending on the 'efficiency' of the agents. Such dynamics occur even though we do not assume any explicit law of motion, objective function, or maximisation principle. The model demonstrates that: $(i)$ mutation/learning will cause a progressive increase in the specialization, interdependence and size of the economy; and $(ii)$ a path dependent outcome is possible. Overall, the work contributes to our economic understanding of systems by grounding the dynamics of those systems in the cut and thrust of evolutionary

competition, rather than in the more aloof view of agent behaviour suggested by abstract optimization economics.

## Reference

McLeod, Douglas J, 2015. “An economic approach to the evolution of an ecology”. In: Sanayei A, Rossler O, Zelinka I Editors, ISCS 2014: Interdisciplinary Symposium on Complex Systems, Springer, Berlin.

# Chapter 1

# Introductory Remarks

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## 1.0 Introduction

This chapter describes the context of the work presented in this thesis and its motivation. That context and motivation can be summarised in the following question: why would an agent produce and supply something if it got nothing in return? And there was a cost to manufacturing it? And if it didn't produce it, it could get the good free of charge anyway? From an economic point of view, these are puzzling questions. Yet this is precisely how biological systems operate, and pre-market human economies as well. Game theory offers some answers based on reciprocal expectations and obligations. But what if there is no personal relationship? Here we examine non-reciprocal trade.

## 1.1 Biological and economic systems

The parallels between economic and biological systems have long intrigued economists (for instance, Marshall 1920). Initially these parallels were regarded as analogies, but the modern attitude is that the same principles are being expressed within different substrates (for a epistemological discussion of this point see Witt 2008). Biological economics looks at how concepts such as production, scarcity, price, competition, investment and systemic equilibrium apply to biological organisms. Much of this work is carried out by biologists rather than economists (Vermeij 2004 offers a summary). Economists have focused on the concept of system equilibrium, something which is emphasized is economics, and produced various 'law of the jungle' type models which typically emphasise the role of predation and demonstrate that equilibrium can be defined and attained in natural setting (see for example Piccione and Rubinstein 2007). Conversely biological concepts such as survival, mutation, adaptation, replication and the fitness landscape have entered economics though the field of evolutionary economics. As these concepts are not standard in economics, there is a patchwork quilt of models with various assumptions and contexts; authors such as Foster (2011) and Markey-Towler (2016) have called for a general theory to order these biological concepts, in the same way as general equilibrium theory fills that need for received economic concepts.

### *1.1.1 The meeting point of economic and biological behaviour*

Neoclassical economic theory assumes the following about behaviour, explicitly and implicitly (Eatwell et al. 1998):

- An individual can formulate a criterion function which allows them to make consistent choices about any set of alternatives they may be offered. Utility is a measure of all the benefits and costs of a choice.
- Agents choose the alterative which maximizes the value of their criterion function.
- The agent has correct knowledge of the choices available, and the full consequences of the choices are either known completely or to the extent of a probability distribution of outcomes.
- There is no cost to the decision-making process itself.

These assumptions form an abstracted description of human behaviour which was originally adopted for its philosophical simplicity and clarity rather than its ability to describe behaviour in more than a stylized way. The abstractness of the assumptions has the advantage that the theory can be extended to the decision making of firms and nations. Three questions which arise are (*i*) how sensitive economic theories are to these particular assumptions; (*ii*) are the consequent theories sufficiently accurate to guide real world policy making; (*iii*) what are the alternatives?

One alternative is to carry out empirical research into economic behaviour, but this raises questions of its own. How different, if at all, is such a study from psychology, and does this matter? How specific will the conclusions be to the particular situation under study? How can such research be used to construct a general model of economic behaviour which can serve as a basis for theoretical elaboration? A more common approach is to relax the neoclassical assumptions in ways which seem plausible *apriori* and analytically tractable. For instance modern macroeconomic theories usually assume bounded rationality, which means that agents confine their attention to the available information rather than using perfect foresight (Simon 1972). Simon also introduced satisficing (1956). Studies in finance often assume ‘myopic’ investors who look only one period ahead.

Another alternative is to assume habitual behaviour. Hodgson (2004) presents a review of thought on habitual behaviour in economics. A habitual behaviour assumption is a broadly descriptive of real-world behaviour in both economic and biological systems, can incorporate adaptation and learning, and is mathematically tractable, and so it has been adopted for the purposes of the current study.

### *1.1.2 The meeting point of economic and biological concepts*

*A comparison of paradigms*: When we look at economics and ecology we find three fundamental concepts in play: the production function/morphologically feasible set, the preference function/natural selection, and learning/mutation. It is noteworthy that in economics the production function and the preference function are central, but a learning concept is something of a bolted on extra confined to specialized treatments. It is unlikely that most undergraduate students of economics would encounter the concept. Most economists do not use the concept in their work, except maybe to validate the disturbance term in a regression or an assumption of structural change. By contrast, in ecology mutation and natural selection are fundamental, but morphological and physiological constraints (the limitations to physical form and functioning) are confined to separate discussions. This may be because biologists tend to think of morphological development and natural selection as processes not states, so these processes are thought of as competing considerations rather than the two blades of a pair of scissors.

*Progress*: While economists tend to take progress for granted, that is not the case for biology. Biologist Stephen Gould stated at the 1987 Spring Systematics Symposium at the Chicago Field Museum that "progress is a noxious, culturally embedded, untestable, nonoperational, intractable idea that must be replaced if we wish to understand the patterns of history" (Gould 1988). Gould's remarks were a reaction to the historical baggage weighing on the term in biology, nonetheless he felt that there was no concept of progress in biology at all. Gould (1997) took the position that there are no trends which last for the entire history of evolution on earth, because things which are advantageous in one context will not be so in another. He took the apparent increase in biological complexity to reflect a simple dispersion of this characteristic from a starting point of zero, which has no broader significance. Later writers in biology have been less willing to abandon the concept of progress entirely although they prefer to speak about 'large scale trends'. McShea (1998) identifies eight trends which might be identified in evolution: *(i)* entropy, *(ii)* energy intensiveness, *(iii)* evolutionary versatility, *(iv)* developmental depth, *(v)* structural depth, *(vi)* adaptedness, *(vii)* size of creature, *(viii)* complexity. What is interesting about McShea's list is the omission of two dimensions which to an economist would seem most obvious: 'efficiency' and 'total amount'.

The model developed in this paper incorporates the three core concepts. The feasible set is represented by the state transitions on hand. The preference function is represented by survival and reproduction, and mutation is modelled by altering the matrix coefficients. The model differs from biological models, and resembles economic models, in that it looks explicitly at resources and their processing. Over time the system represented by the model becomes more specialized and an ecology/economy will develop. Resource utilization becomes progressively more efficient and the population increases. These unidirectional changes represent evolutionary development of the system.

## 1.2 Methodology

We specify an agent-based model using Markov chains to represent agent behaviour, and then use the Perron-Frobenius theorem to establish the existence and uniqueness of a solution. To examine processes within the system we convert the Markov chain formulation to an equivalent set of linear equations and use the techniques of linear algebra. The model has a heuristic flavour, with more assumptions than would be desirable in a final theory.

The method of inquiry is purely theoretical, and abstractly theoretical at that. The model does not include even as much real-world detail as general equilibrium models do – there is no consumption sector, separately identified labour supply, wages or even money. However, it is not as abstract as cellular automata models which have only cells in a grid and a period transition rule. This approach is motivated by the belief that an abstract model may be able to uncover general principles that more specific models take for granted. For instance cellular automata models have revealed that certain cells which act as a repository of system state are critical to driving system evolution. This is a fundamental insight into emergence – why systems seem to be more than the sum of their parts. Detail produces precise conclusions but requires more explicit assumptions, more implicit assumptions and more maintained hypotheses. By backing away from the immediate circumstances, we hope to see the wood not the trees.

### *1.2.1 Structure*

Chapter 2 reviews the literature which underpins this thesis. Chapter 3 builds a Markov matrix model of the economy, shows that a unique, stable distributional equilibrium exists, and shows how the Markov matrix can be converted into a linear production model. Chapter

4 uses the linear production model to determine the dynamic properties of the system and investigate the effect of random mutations. Chapter 5 concludes.

*1.2.2 Some notational conventions*

Throughout this thesis the vector inequality is defined as follows:

$\mathbf{a} \geq \mathbf{b}$: $a_i \geq b_i$ for all $i$ and $\mathbf{a} \neq \mathbf{b}$ (i.e. at least one $a_i > b_i$) (1)

$\mathbf{a} >= \mathbf{b}$: $a_i \geq b_i$ for all $i$ (i.e. $\mathbf{a}$ may equal $\mathbf{b}$) (2)

$\mathbf{a} >\neq \mathbf{b}$: there exists some $b_j < a_j$ (i.e. $\mathbf{a} >= \mathbf{b}$ does not apply) (3)

## 1.3 Conclusion

While economics and biology appear to be parallel sciences in that they examine how systems work, this chapter has described some significant differences in the received concepts of each discipline. Things which are taken for granted in one science are controversial in the other. Economists do not question progress; biologists do not question that agents can viably supply resources for free. Our aim is to develop a modelling framework rich enough to embrace insights from both disciplines.

# Chapter 2

# Literature Review

## 2.0 Introduction

The work reported in this thesis started as an attempt to explain the phenomena observed within a computer simulation of a biological system whereby agents collect resources to survive and reproduce. A brief description of the model is given in the appendix. Although based on simple assumptions, the model exhibited many characteristics found in real world economic and biological systems such as the development of specialization and interdependence, boom-bust cycles, and periods of stasis interrupted by upheaval. In order to explain these phenomena we turn to three broad traditions in economics – linear production theory, Markov chain modelling, and biological modelling.

## 2.1 Linear production theory

The seminal Von Neumann paper (1937) introduced the concept of modelling production functions with vectors of inputs and outputs rather than more general functional forms. Consumption is relegated to a fixed negative component of production which is not even separately identified. "Consumption of goods takes place only through the processes of production which include the necessities of life consumed by workers and employees" (Von Neumann 1937 pg2). These rigid and apparently restrictive assumptions made possible a corresponding increase in the power of the macroeconomic analysis. For the first time the macro economy was modelled with different sectors of production rather than in aggregate. The Von Neumann model uses output from one period as capital input into the next, and the dynamics are driven by an external objective function, which is to maximise the rate of economic growth. The solution rate of expansion turns out to be uniform across sectors and equal to the interest rate. This model was the inspiration for post-war developments in general equilibrium analysis; Leontief developed the sectoral analysis into a practical tool with his input-output analysis (1966). Here we adopt the linear production function but we do not use a capital concept or a maximisation principle, but rather a fixed endowment of resources in every period.

## 2.2 Evolutionary economics.

As stated in the previous chapter, economists have looked at biological evolutionary theory for concepts which might be serviceable in economics, particularly for processes of growth and change. For instance, Alchian (1950) argued that neoclassical assumptions cannot explain firm behaviour under uncertainty; but profit maximisation might evolve through the survival of successful strategies. Alchian's article is notable for its early use of biological concepts such as survival, adaptation, inheritance, and the fitness valley. Reproduction and expiry can be used as metaphors for economic viability of all kinds.

Sandholm (2008) in the context of game theory observes that:

"In economics, the initial phase of research on deterministic evolutionary dynamics … focused on populations of agents who are randomly matched to play normal form games, with evolution described by the replicator dynamic or other closely related dynamics. The motivation behind the dynamics continued to be essentially biological: individual agents are pre-programmed to play specific strategies, and the dynamics themselves are driven by differences in birth and death rates. Since that time the purview of the literature has broadened considerably, allowing more general sorts of large population interactions, and admitting dynamics derived from explicit models of active myopic decision making."

These elements are equally applicable here in the context of resource management:

- agents play preprogrammed strategies
- agents replicate according to the success of these strategies
- there are different agent types

- the dynamics are driven by the difference in birth and death rates of each agent type.

### 2.3 Markov chain modelling

Models based on Markov chains are widely used in economics for what might be termed pragmatic models, models which seek to simulate something without being derived from choice theoretic (i.e. set theoretic) foundations. They are particularly common in financial modelling to represent fluctuations in market volatility (Prasad 1974 is an early example). Markov chains have been used to represent habitual behaviour (for instance Schneider 2013). In biology a slightly modified version of a Markov Chain matrix referred to as a Leslie matrix (Leslie 1945) is used to model population dynamics by using a lifecycle analysis. The properties of Markov chains are well understood and there are analytically powerful techniques for working with them. In particular the Perron-Frobenius theorem which can be used to show a unique and stable solution for a system. The modification of behaviour in the light of experience can be represented by adjustment of the transition probabilities.

The behaviour described by the Markov chain determines the agent's chances of survival and its influence on the system. Although agents in a Markov chain model do not have goals as such, effectively their goal can be taken to be survival not utility. It follows that classical consumer theory is not well captured by a Markov chain model, but such an assumption does capture the motivations of a broad class of other actors, from firms in an economy and traders in a financial market to creatures in an ecosystem.

### 2.4 The dimension model

McLeod (2015) constructs an economic life cycle model using Markov chains with variable coefficients to represent resource procurement, manufacturing, trading and consumption. A standard general equilibrium model consists of the following elements: (*i*) a number of autonomous agents, (*ii*) a division of agents into producers and consumers, (*iii*) consumption decisions which are a function of exogenous preferences, (*iv*) resource endowments, (*v*) production decisions which are a function of profit maximisation, technologies and resource endowments, and (*vi*) a tatonnement mechanism which creates a set of prices which clear both the goods market and the labour market. McLeod (2015) compares as follows:

- Agents are autonomous, but the number of agents is not fixed
- There is no division of agents into producers and consumers. All agents produce and consume resources. This is a general way of looking at the role of producers and consumers, both of whom do in fact produce and consume. For instance standard consumers produce labour and producers consume the overhead costs of production. This is an accurate description of biological systems, pre-industrial economies where production is carried out by individual artisans, and also agent-based computer simulations.
- Consumption consists of a fixed vector of goods. There is no preference function as in a general equilibrium model. Each agent receives an endowment of resources, but unlike a typical general equilibrium model the endowment is the same for every agent. The consumption model is the major respect in which the Markov Chain Model is not as rich than a general equilibrium model.
- Production functions are linear as per the Von Neumann (1937) model. Producers do not aim to maximize their profit but the system selects those who maximise the number of their descendants or in other words their enterprise size. It can be debated which assumption is more behaviourally accurate but behaviour is not very sensitive to the difference.
- There is no explicit price system. A non-reciprocal trading mechanism is defined in place of tatonnement. The system equilibrium exhibits an implicit set of prices (shadow prices in the context of a linear programming problem) which exactly clear

> production in all markets. Although price can be identified from the outside, it is not observable by the agents in the model and does not motivate their behaviour.

A model without reciprocal exchange is a realistic description of a biological system and of a pre-industrial economy, but at first sight it appears quite different to a modern market economy. This difference may be more apparent than real. Game theory research has clarified the economic viability of non-commercial, non-reciprocal trading, referred to as cooperation, within modern society (for instance Gintis et al 2005, Nowak and Highfield 2011). Institutions such as the family, the firm, the school, the local community and the international community all impose non-reciprocal trading obligations on their members through a network of relationships and expectations.

McLeod (2015) uses two agent types, two resources, and five scalar equations to show the development of specialization and interdependence from an initially undifferentiated state. Population increases over time as the economy becomes more adept at using the resources on hand, and there is a progressive increase in the order embodied by the system.

The author’s aim is to redevelop the model for an indefinite number of resources and agent types, so as to create a general tool for exploring questions in complexity theory.

## 2.5 Conclusion

While the specific formulation of the model in this thesis is novel, it is grounded in the well-established general equilibrium, linear production, and Markov chain model types.

# Chapter 3

# The Markov Chain agent model

## 3.0 Introduction

In this chapter we define a Markov chain model of agent behaviour, and specify various processes of resource creation and disposition by which the agents sustain themselves. We then construct a Markov matrix for the system as a whole using Kronecker multiplication and use the Perron-Frobenius theorem to show that the system has a unique distributional equilibrium. We convert the system Markov chain to a set of linear equations, which make it possible to apply the concepts of linear production theory.

## 3.1 Definition of the Markov Chain agent model

We consider a system of atomistic agents which may be either biological or economic in nature. In a biological context the agents are creatures, in an economic context they are consumers and producers. There are various resources which an agent needs to sustain itself and the agent can both produce and consume these resources. The resources are essential for survival and if the agent runs out of any resource then it expires. Some of the resources are endowed as a 'gift of nature': there is a fixed amount of such resources made available in each period and it must be shared amongst all extant agents. Agents manufacture other resources from the endowed resources. The agent's resource holdings at any point in time are sufficient to define the state of the agent. Resource holdings of agent *j* at time *t* are given by:

$$\mathbf{R}_{jt} = \begin{bmatrix} R_{jt}^{1} \\ R_{jt}^{2} \\ R_{jt}^{R} \end{bmatrix} : R \times 1, \text{ resource vector of agent } j \text{ at time } t \qquad (4)$$

i.e. resources are always shown as superscripts, agents and time periods as subscripts.

**Assumption 3.1**: Agent life cycle. Agents start at state *B* with an initial stock of each resource. When an agent has increased resources to a reproduction state *L* it reproduces, creating $S-1$ new agents which commence their lives back at the *B* state. The parent agent continues with whatever resources are left after equipping the offspring. If an agent runs out of any resource, then it expires. These states with zero amount of some resource are *E* states. There is an upper limit to the amount of each resource which an agent can hold. We refer to states with the maximum amount of some resource as *U* states. The lifecycle of an agent is depicted in Figure 3.1:

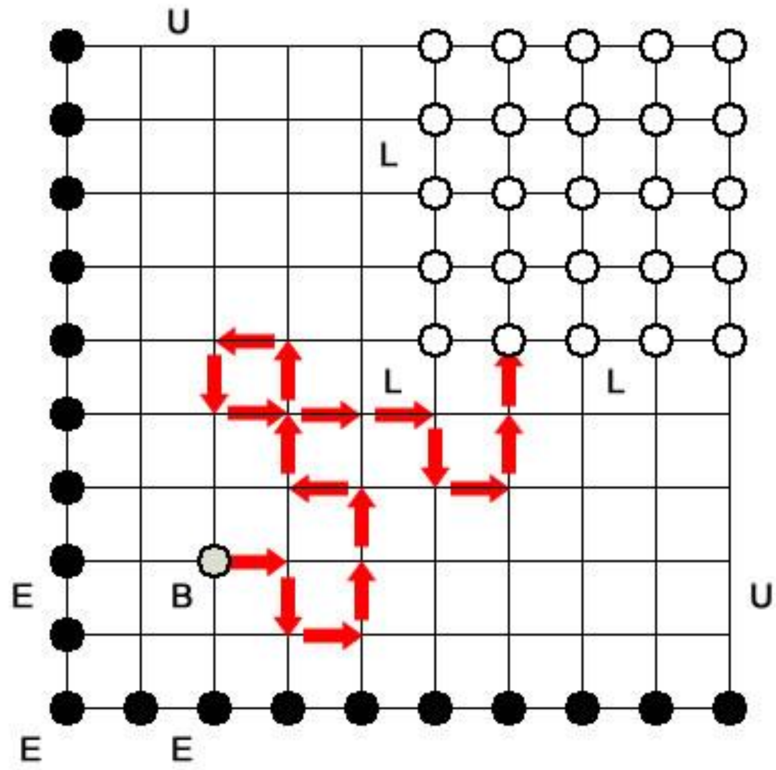


**Figure 3.1**: Agent life cycle in a world with two resources. Horizontal axis shows the agent's holding of resource *A*, vertical axes shows holding of resource *B*. The agent starts (born or instituted) at state *B*. If holdings fall to zero in either resource then the agent expires (dies or wound up). This occurs on the axes and is shown by *E*. There is a maximum amount of resource which the agent can hold and any excess is shed. This occurs at the right hand and top and is shown by *U*. If the agent acquires a threshold amount of both resources then it reproduces. This occurs in the top right hand quadrant and is shown by *L*. The agent gains or

loses resources through various processes which are invoked at random in each period, so the agent follows a random walk through the above landscape. An example is shown in red.

*Processes*. There are eight agent processes which can occur in any given period. These occur with probabilities which may be constant, or may depend in various ways on the parameters of the system. These are (1) production of manufactured resources, (2) production of endowed resources, (3) consumption, (4) trade, (5) resource shedding, (6) resting, (7) reproduction, (8) expiry. The last two are referred to as vital processes.

*1. Production of manufactured resources*.

**Assumption 3.2** Manufactured resources are produced from other resources via a linear production function, i.e. a fixed rate of exchange represented by a vector.

**Remark**: We do not assume a one-to-one relationship between resources and production vectors - a production vector may produce more than one resource and there may more than one production vector which produces the resource.

A resource is *scarce* if agents expire regularly for want of the resource.

A resource is *abundant* if it is not scarce and in general every agent has enough.

*2. Production of endowed resources*.

**Assumption 3.3** There is at least one *endowed* resource *r* which is made available in total amount $L^r$ in each period. Endowed resources can be regarded as harvested from the environment rather than manufactured.

**Assumption 3.4** There is a fixed resource cost to harvesting this resource, so not every agent may wish to produce it.

**Assumption 3.5** The amount available $L^r$ is divided evenly between the agents which choose to harvest that resource. In this model, we take it that all agents so choose.

$$N\ p_j^r = L^r \tag{5}$$

where $p_j^r : scalar$, the probability of agent $j$ producing resource $r$ in the period

$N : scalar$, the total number of agents of all kinds, i.e. the total population

$L^r : scalar$, total number of units of resource r endowed in each period

**Example 3.1** Examples of resources which are endowed in both biological and economic systems are energy, water, air and minerals.

*3. Consumption*.

**Assumption 3.6** In each period every agent must consume a fixed vector of resources in order to survive.

**Remark** Since resource count is discrete, this requirement is interpreted probabilistically. Every resource is given a probability that it will decrease by one unit in a given period, such that the probability generates an expected value equal to consumption.

*4. Trade*.

Agents move around their environment and they meet randomly. Where one agent has a resource and meets another agent, there is some chance that a unit of resource will be transferred to the other agent. If the transfer actually happens, then trade occurs.

**Remark**: Trade is symbiotic, every agent has resources which other agents can use. Symbiotic relationships can be distinguished from parasitic relationships, where the flow of resources can only go one way. Although the resource flow can go either way, there is no reciprocity in a given transaction. Nor is there any expectation of reciprocity between a particular pair of agents in the future, although it may work out that way.

**Example 3.2** In practice most symbiotic relationships in biology occur at the cellular level, where different kinds of bacteria have various chemical roles to play; bacteria are the chemical processing plants of the living world. The chains of predation which we tend to associate with biology are one kind of parasitic relationship.

**Assumption 3.7** The probability of resource transfer between agents is proportional to the resource differential between the agents.

**Assumption 3.8** The total chance of meeting another agent in a time period is constant regardless of the number of agents.

**Remark**: This reflects the tendency of agents to cluster together when there are few of them.

**Remark**: For any particular amount of resource holding $R^r$, the probability $p_j^r$ is the same for every agent.

It follows that the probability of one unit of resource *r* being received by agent *j* in state *s* from agent *k* in state *S* is given by:

$$p_{jk}^r = \frac{k^r \left( R_{kS}^r - R_{js}^r \right)}{N} \tag{6}$$

where $k^r$ is a scalar proportionality constant for the resource. We establish the operational version of the trading definition above.

**Proposition 3.1**. *The expected amount of resource r traded by agent j is given by:*

$$\overset{TRA\,r}{b_j} = k^r \left( \rho^r - \rho_j^r \right) \tag{7}$$

*where*

$$\overset{TRA\,r}{b_j} : scalar\text{, the expected amount of resource r received by j per period} \tag{8}$$

$$k^r > 0, scalar\text{, a trading constant determining the probability of transfer} \tag{9}$$

$$\rho^r : scalar\text{, the average amount of resource r in the population} \tag{10}$$

$$\rho_j^r : scalar\text{, the average stock holdings of agent j} \tag{11}$$

**Proof**: Take expected value of (6) and sum over the population, including agent itself.

$$\overset{TRA\,r}{b_j} = E\left[ p_j^r \right] = E\left[ \sum_i \frac{k^r \left( R_i^r - R_{js}^r \right)}{N} \right] = \sum_i \frac{k^r \left( \rho^r - \rho_j^r \right)}{N} = k^r \left( \rho^r - \rho_j^r \right) \qquad \# \tag{12}$$

**Remark**: For a given stock holding $R_j^r$ the result is

$$\overset{TRA\,r}{b_j} = k^r\left(\rho^r - R_j^r\right) \tag{13}$$

*5. Resource shedding*.

Shedding is the loss of resource by the agent.

**Assumption 3.9** Each agent has an upper bound on the amount of resource which it can store (state *U*). If an agent reaches the upper bound then any additional units of resource which the agent receives are disposed of.

**Assumption 3.10** There is no additional resource cost to shedding, i.e. free disposal.

**Assumption 3.11** A shedding event is taken to be coincident with the production event and cancels all or part of the production out. If production consists of more than one unit, then as many shedding events as are necessary to cancel the excess are deemed to occur.

**Example 3.3** If the agent is one unit below the upper bound in a particular resource, and the manufacturing process produces three units, then we have one production event (+3 units) and two shedding events (-2 units) giving a net production of +1 unit. This raises the agent to the upper bound for that resource.

**Remark**: Where a resource is abundant, i.e. more is produced than the agents need, then these upper bounds will be reached and are an important consideration.

*6. Resting*.

If the agent stays in its current state and nothing changes then it is said to be resting. Resting has the residual probability after other processes have been accounted for.

**Assumption 3.12**: The probability that any particular process will occur in a period is proportional to the length of the period.

**Remark**. We set the duration of the periods sufficiently small that there is a positive chance of resting for every possible agent state. Trading probability in particular is a function of agent state and can vary significantly.

*7. Reproduction*.

Reproduction is the duplication of the agent type.

**Assumption 3.13** When an agent has the resources $\mathbf{R}^{L}$ required for reproduction then it gives rise to *S* agents, *S* is an integer greater than one. $S-1$ new agents are created at the birth state *B*, and the original agent continues with the resources left over.

**Remark**: No resources are created or lost by reproduction, so

$$\mathbf{R}_{0j} = (S-1)\cdot\mathbf{R}^{B} + \mathbf{R}_{1j} \tag{14}$$

*8. Expiry*.

At expiry an agent becomes inactive.

**Assumption 3.14** If consumption or production causes an agent to run out of any of the resources then it expires. In the next period all resources left on hand are forfeited by the agent and transferred to other agents, so that no resources are lost from the system.

**Remark.** We argue that resource recycling is a realistic description for both biological and economic systems, although typically there is a loss of economic value as order is degraded.

**Assumption 3.15** The receipt of resources through the resource recycling is independent of the other processes of the receiving agents, and can occur in the same time period.

*Resource interaction:*

**Assumption 3.16**: AGENT INDEPENDENCE. Processes occur in one agent independently of other agents, except for the reproduction and resource recycling processes.

**Remark**: Several of the processes have the same implications for resource movements. The cell entries in the Markov chain do not represent processes but resource movements. The probabilities for all processes having the same resource outcome are added together to get the probability which is entered in the cell. The question arises as to whether the processes occur independently, in which case two or more can occur in a period, or exclusively, i.e. only one process can occur in a particular period. Pursuant to Assumptions 3.11 and 3.15, shedding and resource recycling processes can be coincident with other processes. We regard the other processes as mutually exclusive, but it is not necessary to make this assumption.

*Agent types:*

**Assumption 3.17**: AGENT TYPES. We introduce different types of agent, indicated by an agent subscript. Agents differ with respect to the resource processing they undertake, its efficiency, and their consumption requirements. An agent can implement as many of the resource production vectors as it chooses. All have the same resource upper bound *U*. We take it that there are *J* agent types, and *N* agents in total.

*The Markov chain transition matrix*:

**Assumption 3.18** In any period the agent may transition from one state to another. This behaviour is described by a Markov chain matrix $\mathbf{M}: D \times D$, which gives the probabilities of the possible transitions.

**Remark**: Each element $m_{ij}$ of **M** gives the probabilities of transition from state *j* in this period to state *i* in the next period. Thus we read down the columns to see what happens to state *j*. The agent may remain in the same state between one period and the next, this is represented by a probability at the diagonal element $m_{jj}$.

Each possible combination of resource amounts corresponds to a state, so the dimension *M* of the matrix **M** is found by calculating the number of distinct resource states, i.e.

$$D = \left(R^{1MAX} + 1\right) \cdot \left(R^{2MAX} + 1\right) \cdot \left(R^{R\,MAX} + 1\right) : scalar \text{, number of states in } \mathbf{M} \qquad (15)$$

The number of states *D* equals the number of dots in Diagram 3.1 above. Certain of these resource states are inaccessible, namely states where the agent has more of all resources than are necessary to reproduce, but this is of no particular consequence. We prove below that an agent will eventually transition to the two absorbing states *E* and *L*. The agent Markov matrices $\mathbf{M}$ do not include any provision for the reproduction or expiry processes as these processes affect more than one agent and must be handled at system level.

$$\begin{array}{cccc} & E & & L \\ \mathbf{M} = & \multicolumn{3}{c}{\begin{bmatrix} 1 & \mathbf{m}^E & 0 \\ \mathbf{0} & \mathbf{M}^M & \mathbf{0} \\ 0 & \mathbf{m}^L & 1 \end{bmatrix}} \end{array} \tag{16}$$

where $\mathbf{m}^E : 1 \times D-2$ Probability of transition to the *E* state in each period

$\mathbf{m}^L : 1 \times D-2$ Probability of transition to the L state in each period

$\mathbf{M}^M : (D-2) \times (D-2)$ central part of **M** for intermediate transitions

As a Markov chain matrix, **M** has a eigenvalue of 1 (each column sums to unity so $\boldsymbol{\iota} \cdot \mathbf{M} = 1 \cdot \boldsymbol{\iota}$ where $\boldsymbol{\iota} : D \times 1$ is a vector of unity elements). The eigenvalue corresponds to two LHS eigenvectors. One of these eigenvectors can be interpreted as giving the probability of reaching expiry state *E* from the current state, and the other eigenvector gives the probability of reaching life state *L*. We set out this standard result for reference.

**Proposition 3.2**. *The probability of the Markov process **M** transitioning to either the E state or the L state is unity.*

**Proof**. We obtain expressions for the two LHS eigenvectors, then use the fact that $\boldsymbol{\iota}_{D \times 1}$ is a LHS eigenvector of a Markov chain matrix to obtain an expression into which the LHS eigenvectors can be substituted.

$$\begin{bmatrix} 1 & \boldsymbol{\upsilon}^E & 0 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{m}^E & 0 \\ \mathbf{0} & \mathbf{M}^M & \mathbf{0} \\ 0 & \mathbf{m}^L & 1 \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{\upsilon}^E & 0 \end{bmatrix} \tag{17}$$

$1 \cdot \mathbf{m}^E + \boldsymbol{\upsilon}^E \mathbf{M}^M + 0 \cdot \mathbf{m}^L = \boldsymbol{\upsilon}^E$ Multiplying out middle column (18)

so $\boldsymbol{\upsilon}^E = \mathbf{m}^E \left( \mathbf{I} - \mathbf{M}^M \right)^{-1}$ Rearranging (19)

Similarly $\boldsymbol{\upsilon}^L = \mathbf{m}^L \left( \mathbf{I} - \mathbf{M}^M \right)^{-1}$ (20)

Now the sum of the columns of a Markov chain matrix M is unity.

$$\begin{bmatrix}1 & \boldsymbol{\iota}_{1*M-2} & 1\end{bmatrix}\begin{bmatrix}1 & \mathbf{m}^E & 0\\ \mathbf{0} & \mathbf{M}^M & \mathbf{0}\\ 0 & \mathbf{m}^L & 1\end{bmatrix}=\begin{bmatrix}1 & \boldsymbol{\iota}_{1*M-2} & 1\end{bmatrix} \tag{21}$$

$$\mathbf{m}^E+\boldsymbol{\iota}\cdot\mathbf{M}^M+\mathbf{m}^L=\boldsymbol{\iota} \qquad \text{Multiplying out middle column} \tag{22}$$

$$\boldsymbol{\iota}=\left(\mathbf{m}^E+\mathbf{m}^L\right)\left(\mathbf{I}-\mathbf{M}^M\right)^{-1}=\boldsymbol{\upsilon}^E+\boldsymbol{\upsilon}^L \qquad \text{Rearrange (22), sub in using (19),(20)} \tag{23}$$

i.e. the probabilities of reaching either the *E* state or the *L* state sum to 1 in every state. #

### 3.2 Existence and stability of equilibrium

Here we explore the existence, uniqueness and stability of a solution to the system we have defined. The starting point of the inquiry is to note that the state of a single agent *j* at time *t* is defined by the resources which it currently holds.

$$S_{jt}=\begin{pmatrix}R_{jt}^1 & R_{jt}^2 & R_{jt}^R\end{pmatrix} \tag{24}$$

The state of the whole system is the concatenation of the state of each constituent agent.

$$S_{jt}^{SYS}=\left(\begin{pmatrix}R_{1t}^1 & R_{1t}^2 & R_{1t}^R\end{pmatrix}\ ,\ \begin{pmatrix}R_{jt}^1 & R_{jt}^2 & R_{jt}^R\end{pmatrix}\ ,\ \begin{pmatrix}R_{Nt}^1 & R_{Nt}^2 & R_{Nt}^R\end{pmatrix}\right) \tag{25}$$

We build a matrix to represent the system as a whole in nine steps carried out in order.

1. *Set agent positions*. Firstly, we set an upper limit $N_j^{UPPER}$ for the number of each agent type in the system, well above what resource constraints could conceivably support. The chances of the system reaching this upper bound are effectively zero. The upper bound for the number of agents in the system is given by

$$N^{UPPER}=J\cdot N_j^{UPPER} \tag{26}$$

This upper bound $N_j^{UPPER}$ defines the number of agent slots available to be filled by each agent type. Each agent is assigned a slot. Slots which are not filled by active agents are filled by expired agents which are currently in the inactive (expired) state.

2. *Build the system matrix*. Define:

$$\mathbf{M}^{VOID} = \mathbf{I} : D \times D \text{, identity matrix the same size as } \mathbf{M} \tag{27}$$

We construct a matrix $\mathbf{M}^{SYS\,1}$ through Kronecker multiplication.

$$\mathbf{M}^{SYS\,1} = \mathbf{M}_1^{VOID} \otimes \mathbf{M}_2^{VOID} \otimes \mathbf{M}_{NUPPER}^{VOID} : D^{SYS} \times D^{SYS} \tag{28}$$

where $D^{SYS} = \left(D\right)^{NUPPER}$ (29)

For every cell in this matrix the state of the system is completely defined and this allows us to fill in the probabilities of each event.

3. *Manufactured resources and consumption.* These processes are carried out in every agent state except the *L* and *E* states. They have set probabilities. When a probability is entered into a cell, a corresponding deduction is made to the resting probability for the agent.

$$\mathbf{M}^{SYS\,2} = \mathbf{M}^{SYS\,1} + \mathbf{M}^{SYS\,MANU} + \mathbf{M}^{SYS\,CONS} \tag{30}$$

4. *Production of endowed resources.* The production of endowed resources depends upon the number of active agents *N*. In each cell of the system matrix we can now determine the number of active agents *N* and use it to determine the probability $\frac{L^r}{N}$ of each agent receiving a unit of endowed resource *r*. (If $\frac{L^r}{N} > 1$ then allocate more than one unit of resource.) The entry of the probability values for endowed resources corresponds to the addition of a modifying matrix:

$$\mathbf{M}^{SYS\,3} = \mathbf{M}^{SYS\,2} + \mathbf{M}^{SYS\,ENDOWED} \tag{31}$$

5. *Trading*. Trading will not affect the vital processes, because it averages stock towards the mean. Reproduction and expiry only take place at extreme positions, so if a vital event was not indicated prior to trade, it will not be indicated after trade. Agents already in the *E* and *L* states do not participate in trade.

While trade in principle involves two agents, for simplicity we take it that the trading probabilities in each period are valued independently for each agent according to (7) using the current resource average $\rho^r$. It is therefore possible that the amount which agents gain from trade does not exactly equal the amount which agents supply to trade. Because the deviation of agent stock levels $R_j^r$ around the mean $\rho^r$ sum to zero in every period, and the trading coefficient $k^r$ is constant for all agents, the expected value of transfers in each period is zero. The discrepancy between the amount of stock received and supplied will average out to zero over time and is ignored. Indeed in a real system, stock is held in storage and transit, so such a model may be a more realistic model of real world trading. The trading process therefore depends on a parameter which is determined by the state of the system, namely the mean amount of each resource $\rho^r$.

$$\mathbf{M}^{SYS\ 4} = \mathbf{M}^{SYS\ 3} + \mathbf{M}^{SYS\ TRADE} \tag{32}$$

6. *Shedding*. Shedding is determined after the other processes and the balance of resources on hand is clear.

$$\mathbf{M}^{SYS\ 5} = \mathbf{M}^{SYS\ 4} + \mathbf{M}^{SYS\ SHEDDING} \tag{33}$$

7. *Reproduction and expiry (vital events)*. When the agent transfers to a life state *L* or an expiry state *E* in the previous period, it does not qualify for any of the other processes. Every cell in the Kronecker matrix contains all the information about both the current state of the system and the new state, so the adjustment to be made to account for these vital events is well-defined. The current state corresponds to a particular column in the $\mathbf{M}^{SYS\,1}$ matrix. Modifying matrix $\mathbf{M}^{SYS\ \ VITAL}$ has the effect of transferring each entry from its current location in the column to a new location which reflects the system state after vital events are processed. The $\mathbf{M}^{SYS\ \ VITAL}$ matrix does this by subtracting probabilities

from some cell and adding probabilities to others; the sum of elements in each column of $\mathbf{M}^{SYS\ VITAL}$ is zero.

The rules which determine the transition from the *L* and *E* states are as follows:

*Creation*: When an agent is born:

- It is assigned an agent slot immediately after the highest current slot currently occupied. If the upper bound $N_j^{UPPER}$ has been reached then the next slot wraps around to the first position. If this immediately higher position is already occupied by a living agent then the assignment does not go ahead and the resources are forfeited from the system, as a separate system process described below.
- Its birth resources are allocated to that position.

*Expiry and resource recycling*: When an agent expires:

- All active (living) agents in lower positions are shifted up the position list so that the vacant slot is filled, and the active agents are all in one block.
- The resources held by the expired agent are allocated out to the agents in order of position, and each agent's resource holdings are adjusted accordingly. This can have the effect of moving an agent to a life state *L* as may other processes. An agent is not allocated a resource which would lift it above the upper limit for that resource. There is the technical possibility that it is not possible to allocate out all the resources available, either because all the agents are at their resource limits or all agents have gone extinct. This invokes a separate system process described below.

Reproduction and expiry do not consume resources, so the total resource count of the adjusted state is the same as that of the original state.

$$\mathbf{M}^{SYS\,6} = \mathbf{M}^{SYS\,5} + \mathbf{M}^{SYS\ VITAL} \tag{34}$$

8. *System processes*. In addition to the eight agent processes, there are three system processes of a technical nature which are necessary to complete the model. These are:

*System extinction*: The LHS column in the system matrix $\mathbf{M}^{VOID}$ represents the expiry of all agents in the system. It has a positive albeit small probability of being reached from other states. An extinction state is an absorbing state, and the distribution eigenvector becomes

$$\boldsymbol{\mu}^{SYS} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{35}$$

i.e. eventual extinction of every agent is certain, notwithstanding the infinitesimally small transition probabilities. To avoid this situation, we assume that the system will eventually be regenerated. While the probability of regeneration in any particular period is also infinitesimally small, the long period of time prior to regeneration can be ignored. We place a 1 in the cell which represents one each of the original agent types, at the birth position.

$$\begin{array}{ccc} E & B & O \end{array} \qquad\qquad \begin{array}{ccc} E & B & O \end{array}$$
$$\begin{bmatrix} 1 & p_E & p_E \\ . & M_{BB} & M_{BO} \\ . & M_{OB} & M_{OO} \end{bmatrix} \Rightarrow \begin{bmatrix} . & p_E & p_E \\ 1 & M_{BB} & M_{BO} \\ . & M_{OB} & M_{OO} \end{bmatrix} \quad E\text{=expiry, } B\text{=birth, } O\text{=other} \tag{36}$$

*Species reproduction limit*: If the agent limit $N_j^{UPPER}$ is reached, birth is not possible and the resources set aside for that agent are forfeited and disappear from the system. This process is purely technical; the probability of it occurring can be made arbitrarily low by increasing the agent limit $N_j^{UPPER}$ as required.

*Recycling limit:* If all agents go extinct then the last agent will have no successors which can inherit its resources. It is also possible that if too many agents go extinct that all the resources cannot be distributed without breaching the resource limits of the remaining agents. In these cases the excess resources are forfeit from the system.

**Remark**: The effect of these three assignments is that the corresponding probabilities in the system matrix $\mathbf{M}^{SYS\,3}$ are subtracted from one cell and added onto another cell in the same column which represents the adjusted final state. Unlike agent reproduction and expiry, these system processes do affect the total resource count of the system, however they are considered processes of such low probability that they need not be considered in the resource accounting.

$$\mathbf{M}^{SYS} = \mathbf{M}^{SYS\,6} + \mathbf{M}^{SYS\ PROCESS} \tag{37}$$

**Assumption 3.19** The system processes use negligible resources, taken to be zero resources.

9. *Evaluation of resting probabilities*. In each step, resting probability is adjusted by the opposite of the probability alterations required by the other processes, so that probabilities sum to zero in each column of each adjustment matrix. Because of the values taken by the trading parameters $k^r$ and others, it is possible that some of the resting probabilities are negative. In this case, reduce the size of each time period, which will reduce the process probabilities correspondingly. A sufficient reduction in the length of the time periods will render all the resting probabilities positive.

As system matrix $\mathbf{M}^{SYS}$ is now determined and all variables have been evaluated, we can solve the system uniquely. To do this we make use of the following standard result.

**Theorem 3.3**: PERRON-FROBENIUS THEOREM.

*Let* $\mathbf{A}$ *be an irreducible non-negative square matrix of period 1. Then the following statements hold.*

- *There is a unique maximal eigenvalue r of* ***A***
- *r is a positive real number*
- *The eigenvectors corresponding to eigenvalue r are strictly positive.*

**Proof**: See for instance Meyer (2001).

The following result establishes that the system specified here is capable of solution.

**Theorem 3.4**: EXISTENCE OF UNIQUE SOLUTION. *System matrix* $\mathbf{M}^{SYS}$ *has a unique, positive RHS eigenvector* $\boldsymbol{\mu}^{SYS}$ *corresponding to a maximal eigenvalue of unity.*

$$\boldsymbol{\mu}^{SYS} > \mathbf{0}_{DSYS\times 1} \tag{38}$$

**Proof**: We verify that the conditions for application of Perron-Frobenius theorem are present.

| | | | |
|---|---|---|---|
| | $\mathbf{M}^{SYS} \geq 0$ | by construction | (39) |
| | $\mathbf{M}^{SYS}$ is irreducible | all states are communicating by construction | |
| | $\mathbf{M}^{SYS}$ is aperiodic | contains resting states, so $\gcd\left(n \geq 1, p_{ii}^{n} > 0\right) = 1$ | (40) |
| so | $\exists!$ maximal eigenvalue $\lambda^{MAX}$ | by Perron-Frobenius theorem above | |
| Now | $\boldsymbol{\iota}\mathbf{M}\boldsymbol{\mu}^{SYS} = \boldsymbol{\iota}\left(\mathbf{M}\boldsymbol{\mu}^{SYS}\right) = \lambda^{MAX}\boldsymbol{\iota}\,\boldsymbol{\mu}^{SYS}$ | multiplying by RHS eigenvector | (41) |
| | $= \left(\boldsymbol{\iota}\mathbf{M}\right)\boldsymbol{\mu}^{SYS} = 1 \cdot \boldsymbol{\iota}\,\boldsymbol{\mu}^{SYS}$ | multiplying by LHS eigenvector | (42) |
| So | $\lambda^{MAX} = 1$ | # | (43) |

**Remark**: As the eigenvalue is maximal, any initial RHS vector will converge to the RHS eigenvector and the solution is stable.

**Remark**: The RHS eigenvector can be interpreted as a probability distribution over the states of the system. Since it is positive, every state has a positive chance of occurring, albeit small for most of them.

**Remark**: Given that the system does not occupy any one state in equilibrium but all of them, the concept of an equilibrium solution needs to be examined. An equilibrium solution would normally be interpreted as a particular state which satisfies all the constraints, and which the system will not leave even if it is an unstable solution. Here we cannot speak of such a solution: a state may be stable, but the system will nonetheless leave it. This is a distributional equilibrium, not a situational equilibrium. We can interpret the situational equilibrium solution as a stable point: a point to which the system tends to return, and the pressure to leave is not biased in any particular direction.

### 3.3 Resource accounting

In order to explore the consequences of the processes and resource constraints, we introduce the following variables:

$$\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : D \times 1 \quad \text{current state of the agent, as distinct from distribution vector } \boldsymbol{\mu} \tag{44}$$

$\mathbf{X} : R \times D$ stock matrix: amount of each of $R$ resources in each of $D$ states of $\mathbf{M}$

$$\mathbf{R} = \mathbf{X}\boldsymbol{\mu} : R \times 1 \quad \text{expected amount of resource held by an agent} \tag{45}$$

$$\Delta\mathbf{R} = \mathbf{R}_1 - \mathbf{R}_0 : R \times 1 \quad \text{change in resource of the agent in a period} \tag{46}$$

$$\begin{aligned} \mathbf{X}_1 \oplus \mathbf{X}_2 &= \begin{bmatrix} \mathbf{x}_{1-1} & \mathbf{x}_{1-2} \end{bmatrix} \oplus \begin{bmatrix} \mathbf{x}_{2-1} & \mathbf{x}_{2-2} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{x}_{1-1} + \mathbf{x}_{2-1} & \mathbf{x}_{1-1} + \mathbf{x}_{2-2} & \mathbf{x}_{1-2} + \mathbf{x}_{2-1} & \mathbf{x}_{1-2} + \mathbf{x}_{2-2} \end{bmatrix} : R \times 2D \end{aligned} \tag{47}$$

where $\mathbf{x}_{1-1}$ is $R \times 1$ first column of $\mathbf{X}_1$ etc. Only two columns shown for simplicity.

$$\mathbf{X}^{SYS} = \mathbf{X}_1 \oplus \mathbf{X}_2 \oplus \mathbf{X}_J : R \times D^{SYS} \text{ stock matrix for the system as a whole} \tag{48}$$

*Decomposition of the transition matrix by process*. We now decompose Markov matrix $\mathbf{M}$ into its component processes, and calculate the resource impact of each process according to

$$\boldsymbol{\Delta}\mathbf{R} = \mathbf{R}_1 - \mathbf{R}_0 = \mathbf{X}\mathbf{M}\boldsymbol{\mu} - \mathbf{X}\boldsymbol{\mu} = \mathbf{X}\left(\mathbf{M} - \mathbf{I}\right)\boldsymbol{\mu} \tag{49}$$

The eight processes can be divided into three types. In the first case, the process produces zero net resources. Reproduction, expiry and resting are this kind of process. In the second case, the process is fixed in terms of its probability $p_j$ and resource demands **e** for any active state of the agent. These processes are production of manufactured resources, and consumption. The probability of these processes is zero in the $L$ state but this can be disregarded because the $L$ state is infrequent. For every agent $j$, each process has a process vector $\mathbf{a}_j$, which is the expected value in each period of the process across all states in which the agent is active.

$$\mathbf{a}_j = \frac{\sum_{s\in A} \overset{s}{\mathbf{e}}_j \cdot \overset{s}{p}_j \cdot \overset{s}{\mu}}{\overset{A}{\mu}} \tag{50}$$

$$= \mathbf{e}_j \cdot p_j \cdot \frac{\sum_{s\in A} \overset{s}{\mu}_j}{\overset{A}{\mu}_j} \quad \text{noting } \overset{s}{\mathbf{e}}_j = \mathbf{e}_j,\ \overset{s}{p}_j = p_j \text{ as process is not a function of state} \tag{51}$$

$$= \mathbf{e}_j \cdot p_j \tag{52}$$

where $s \in A$, agent *j* is active (not expired) in state *s* (53)

$\overset{s}{\mu} \in \boldsymbol{\mu} : scalar$, the probability of state *s* in distribution vector $\boldsymbol{\mu}$ in a given period (54)

$\overset{A}{\mu}_j = \sum_{s\in A} \overset{s}{\mu}$ $: scalar$, probability across all system states that *j* is active (55)

$\overset{E}{\mu}_j = 1 - \overset{A}{\mu}_j : scalar$, probability across all system state that *j* is expired (56)

$\overset{s}{p}_j, scalar$, probability of the process per period for agent *j* in state *s* (57)

$p_j, scalar$, probability of the process per period for agent *j* in any active state $A$ (58)

$\mathbf{e}_j = \begin{bmatrix} e^r_{j1} \\ e^r_{j2} \\ e^r_{jR} \end{bmatrix} : R \times 1$, integral amounts of resource used in the process by agent *j* (59)

$\mathbf{a}_j : R \times 1$, expected value per period of the process for agent *j* in active states (60)

In the third case, the process is not fixed. It may vary either because (i) the production vector **e** varies throughout transition matrix $\mathbf{M}$. Production will then vary according to the state of the agent **v**. This process is resource shedding; (ii) The probability of the process *p* varies according to the state of the system $\mathbf{v}^{SYS}$. This process is manufacture of endowed resources;

(iii) the probability of the process depends on both the state of the agent and the state of the system. This process is trade. We denote a variable resource vectors by $\mathbf{b}_j$.

$$\mathbf{b}_j = \frac{\sum_{s\in A} \overset{s}{\mathbf{e}} \cdot \overset{s}{p}_j \cdot \overset{s}{\mu}}{\overset{A}{\mu}_j} \text{ : cannot be simplified} \tag{61}$$

where $\mathbf{b}_j : R\times 1$, expected value per period of the process for agent *j* in active states (62)

As we can see, the difference between **a** and **b** is that in the latter case, the probability $\overset{s}{p}_j$ and the process $\overset{s}{\mathbf{e}}_j$ are functions of the system state *s*.

*Interpretation of distribution vector* $\boldsymbol{\mu}$: The reproduction state *L* operates automatically: there is only one transition which is certain to be selected. No resources are produced in this state, unlike the other states. If we regard the time periods as being short then agents will spend little time in this automatic state compared to others. We suppose:

$$\overset{L}{\mu}_j \approx 0 : scalar \text{, probability that agent } j \text{ is in reproduction state } L \tag{63}$$

so the reproduction state need not enter into our calculations of resource yield below.

*Thematic representation of the resource processes*. We calculate the resource flows for each process for one agent. Each result is a vector representing the flow when the agent is active, and is multiplied by $\overset{A}{\mu}_j$, the percentage of time the agent is active to give the expected value of the process across all states active and expired. It is impossible (or at least pointlessly tedious) to represent the full generality of the Markov matrix $\mathbf{M}$ and stock matrix **X** with all the possible combinations of resources and functions. The following matrix representations are thematic only.

*1.Production of manufactured resources:*

$$\overset{PRO}{\Delta\mathbf{R}} = \mathbf{X}\overset{PRO}{\mathbf{M}}\boldsymbol{\mu} \tag{64}$$

$$= \mathbf{X}\begin{array}{c} \begin{array}{cccc} E & & & L \end{array} \\ \begin{bmatrix} . & . & . & . \\ . & -p^r & . & . \\ . & p^r & -p^r & . \\ . & . & p^r & . \end{bmatrix} \end{array} \boldsymbol{\mu} \tag{65}$$

$$= \begin{bmatrix} . & p^r \cdot \overset{PRO\,r}{\mathbf{e}} & p^r \cdot \overset{PRO\,r}{\mathbf{e}} & . \end{bmatrix}_{R\times M} \begin{bmatrix} \overset{E}{\mu} \\ \overset{S1}{\mu} \\ \overset{S2}{\mu} \\ \overset{L}{\mu} \end{bmatrix} \tag{66}$$

$$= \overset{PRO\,r}{\mathbf{a}} \sum_{s\in A} \overset{s}{\mu} = \frac{\overset{PRO\,r}{\mathbf{a}} \sum_{s\in A} \overset{s}{\mu}}{\overset{A}{\mu}_j} = \overset{PRO\,r}{\mathbf{a}} \cdot \overset{A}{\mu}_j \quad \text{using (52), (55)} \tag{67}$$

*2. Production of endowed resources*: Here the probability of the production varies according to the number of agents as per (5). Other resources may be required for the production. We take it that the other resources are consumed whether production is successful or not, with a fixed probability.

$$\overset{END\,r}{\Delta\mathbf{R}} = \mathbf{X}\overset{END\,r}{\mathbf{M}}\boldsymbol{\mu} + \mathbf{X}\overset{END\,r}{\mathbf{M}}\boldsymbol{\mu} \tag{68}$$

$$= \mathbf{X}\begin{array}{c} \begin{array}{cccc} E & & & L \end{array} \\ \begin{bmatrix} . & . & . & . \\ . & -L^r/N & . & . \\ . & L^r/N & -L^r/N & . \\ . & . & L^r/N & . \end{bmatrix} \end{array} \boldsymbol{\mu} + \mathbf{X}\begin{array}{c} \begin{array}{cccc} E & & & L \end{array} \\ \begin{bmatrix} . & p^r & . & . \\ . & -p^r & p^r & . \\ . & . & -p^r & . \\ . & . & . & . \end{bmatrix} \end{array} \boldsymbol{\mu} \tag{69}$$

$$= \begin{bmatrix} . & \frac{L^r}{\overset{s}{N}} \cdot \overset{END\,r}{\mathbf{e}} & \frac{L^r}{\overset{s}{N}} \cdot \overset{END\,r}{\mathbf{e}} & . \end{bmatrix}_{R\times D} \begin{bmatrix} \overset{E}{\mu} \\ \overset{S1}{\mu} \\ \overset{S2}{\mu} \\ \overset{L}{\mu} \end{bmatrix} + \begin{bmatrix} . & p^r \cdot \overset{END\,r}{\mathbf{e}} & p^r \cdot \overset{END\,r}{\mathbf{e}} & . \end{bmatrix}_{R\times D} \begin{bmatrix} \overset{E}{\mu} \\ \overset{S1}{\mu} \\ \overset{S2}{\mu} \\ \overset{L}{\mu} \end{bmatrix} \tag{70}$$

$$= \overset{END\,r}{\mathbf{b}} \cdot \sum_{s\in A} \overset{s}{\mu}_j + \overset{END\,r}{\mathbf{a}} \cdot \sum_{s\in A} \overset{s}{\mu}_j = \overset{END\,r}{\mathbf{b}} \cdot \overset{A}{\mu}_j + \overset{END\,r}{\mathbf{a}} \cdot \overset{A}{\mu}_j \qquad \text{using (50), (61)} \tag{71}$$

*3.Consumption*: Consumption reduces resources, possibly to the expiry state *E*.

$$\overset{CON}{\Delta\mathbf{R}} = \mathbf{X}\overset{CON}{\mathbf{M}}\boldsymbol{\mu} \tag{72}$$

$$= \mathbf{X}\begin{array}{c} \begin{array}{cccc} E & \phantom{-p^r} & \phantom{-p^r} & L \end{array} \\ \begin{bmatrix} \cdot & p^r & \cdot & \cdot \\ \cdot & -p^r & p^r & \cdot \\ \cdot & \cdot & -p^r & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \end{array}\boldsymbol{\mu} \tag{73}$$

$$= \overset{CON\,r}{\mathbf{a}} \cdot \overset{A}{\mu}_j \qquad \text{as for manufactured resources} \tag{74}$$

*4. Trade*

$$\overset{TRA\,r}{\Delta\mathbf{R}} = \mathbf{X}\overset{TRA\,r}{\mathbf{M}}\boldsymbol{\mu} \tag{75}$$

$$= \mathbf{X}\begin{array}{c} \begin{array}{cccc} E & \phantom{-p^r} & \phantom{-p^r} & L \end{array} \\ \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & -p^r & p^r & \cdot \\ \cdot & p^r & -p^r & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \end{array}\boldsymbol{\mu} \tag{76}$$

$$= \begin{bmatrix} \cdot & \sum_r k^r \left( \overset{S1\,r}{\rho} - \overset{S1\,r}{R} \right) & \sum_r k^r \left( \overset{S2\,r}{\rho} - \overset{S2\,r}{R} \right) & \cdot \end{bmatrix} \begin{bmatrix} \overset{E}{\mu} \\ \overset{S1}{\mu} \\ \overset{S2}{\mu} \\ \overset{L}{\mu} \end{bmatrix} \qquad \text{as per (13)} \tag{77}$$

$$= k^r \left( \rho^r - \rho^r_j \right) \cdot \overset{A}{\mu}_j \tag{78}$$

$$= \overset{TRA\,r}{\mathbf{b}} \cdot \overset{A}{\mu}_j \tag{79}$$

*5. Resource shedding*. Similar to consumption except that it occurs only in the upper bound state *U* and possibly those states immediately below it. It is coincident with production. If the manufacturing vector produces more than one unit, then more than one shedding event can occur in one period so as to cancel out the excess units.

$$\overset{SHED\,r}{\Delta\mathbf{R}} = \mathbf{X}\overset{SHED\,r}{\mathbf{M}}\boldsymbol{\mu} \tag{80}$$

$$= \mathbf{X}\begin{array}{c} \begin{array}{cccc} E & \phantom{.} & U & L \end{array} \\ \begin{bmatrix} . & . & . & . \\ . & . & p^r & . \\ . & . & -p^r & . \\ . & . & . & . \end{bmatrix} \end{array} \begin{bmatrix} \overset{E}{\mu} \\ \overset{S1}{\mu} \\ \overset{U}{\mu} \\ \overset{L}{\mu} \end{bmatrix} \tag{81}$$

$$= \frac{\sum_{s\in A} \overset{s}{\mathbf{e}} \cdot \overset{s}{p}_j \cdot \overset{s}{\mu}}{\overset{A}{\mu}_j} \cdot \overset{A}{\mu}_j \quad \text{using (61)} \tag{82}$$

$$= \overset{SHED\,r}{\mathbf{b}} \cdot \overset{A}{\mu}_j <= \mathbf{0}_{R\times 1} \tag{83}$$

*6. Resting.*

$$\overset{REST}{\Delta\mathbf{R}} = \mathbf{0}_{R\times 1} \tag{84}$$

*7. Reproduction*: Reproduction is represented on the system level as described above not on the agent level.

*8. Expiry*. Expiry is represented on the system level not on the agent level. The amount gained by active agents is variable.

We now amalgamate the results across all resources:

$$\overset{FN}{\Delta\mathbf{R}} = \sum_r \mathbf{X}\overset{FN\,r}{\mathbf{M}}\boldsymbol{\mu} = \sum_r \overset{FN\,r}{\mathbf{a}} \cdot \overset{A}{\mu}_j = \overset{FN}{\mathbf{a}} \cdot \overset{A}{\mu}_j \tag{85}$$

$$\overset{FN}{\Delta\mathbf{R}} = \sum_r \mathbf{X}\overset{FN\,r}{\mathbf{M}}\boldsymbol{\mu} = \sum_r \overset{FN\,r}{\mathbf{b}} \cdot \overset{A}{\mu}_j = \overset{FN}{\mathbf{b}} \cdot \overset{A}{\mu}_j \tag{86}$$

$$\overset{FN}{\mathbf{a}} = \sum_r \overset{FN\,r}{\mathbf{a}} : R\times 1, \text{ sum over all resources. FN=PRO, END, CON} \tag{87}$$

$$\overset{FN}{\mathbf{b}} = \sum_r \overset{FN\,r}{\mathbf{b}} : R\times 1, \text{ sum over all resources. FN=END, TRA, SHED} \tag{88}$$

To establish the relationship of the model to linear production theory we start with a functional decomposition of the Markov matrix.

**Lemma 3.5**. *In the defined model the agent transition matrix may be decomposed as:*

$$\mathbf{M} = \overset{\substack{PRO\\ENDOW}}{\mathbf{M}} + \overset{\substack{PRO\\MANU}}{\mathbf{M}} + \overset{CON}{\mathbf{M}} + \overset{TRA}{\mathbf{M}} + \overset{SHED}{\mathbf{M}} + \mathbf{I} \tag{89}$$

**Proof**: We evaluate RHS using the expressions derived above for each component. The matrices below are representational only. It is only meaningful to add the diagonal elements representing the resting state, the other cells do not correspond to each other. The expiry state *E* and life state *L* are shown explicitly.

$$\overset{\substack{PRO\\ENDOW}}{\mathbf{M}} + \overset{\substack{PRO\\MANU}}{\mathbf{M}} + \overset{CON}{\mathbf{M}} + \overset{TRA}{\mathbf{M}} + \overset{SHED}{\mathbf{M}} = \begin{array}{c} \begin{array}{cccc} E & & & L \end{array} \\ \begin{bmatrix} . & p & p & . \\ . & -\sum_{FN} \overset{FN}{p} & p & . \\ . & p & -\sum_{FN} \overset{FN}{p} & . \\ . & p & p & . \end{bmatrix} \end{array} \quad FN\text{: five processes at left} \tag{90}$$

$$= \begin{array}{c} \begin{array}{cccc} E & & & L \end{array} \\ \begin{bmatrix} . & p & p & . \\ . & \overset{REST}{p} - 1 & p & . \\ . & p & \overset{REST}{p} - 1 & . \\ . & p & p & . \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} E & & & L \end{array} \\ \begin{bmatrix} 1 & p & p & . \\ . & \overset{REST}{p} & p & . \\ . & p & \overset{REST}{p} & . \\ . & p & p & 1 \end{bmatrix} \end{array} - \mathbf{I} \quad \text{noting} \quad \overset{REST}{p} + \sum_{FN} \overset{FN}{p} = 1 \tag{91}$$

$$= \mathbf{M} - \mathbf{I} \text{ recalling } \mathbf{M} \text{ does not include the vital processes} \qquad \# \tag{92}$$

The following is necessary to allow manipulation of the system Markov matrix $\mathbf{M}^{SYS}$.

**Lemma 3.6**. *Suppose that* $\mathbf{X}_1, \mathbf{X}_2$ *are* $R \times D$ *matrices and* $\mathbf{M}_1, \mathbf{M}_2$ *are* $D \times D$ *matrices with columns adding to unity. Then*

$$\begin{gathered} \left(\mathbf{X}_1 \oplus \mathbf{X}_2\right)^r Eval\left(\mathbf{M}_1 \otimes \mathbf{M}_2\right) = \\ \left[\begin{pmatrix} x_{1-r1} \overset{AA}{m}_{1-11} + \\ x_{1-r2} \overset{AA}{m}_{1-21} \end{pmatrix} \begin{pmatrix} x_{1-r1} \overset{AB}{m}_{1-11} + \\ x_{1-r2} \overset{AB}{m}_{1-21} \end{pmatrix} \begin{pmatrix} x_{1-r1} \overset{BA}{m}_{1-12} + \\ x_{1-r2} \overset{BA}{m}_{1-22} \end{pmatrix} \begin{pmatrix} x_{1-r1} \overset{BB}{m}_{1-12} + \\ x_{1-r2} \overset{BB}{m}_{1-22} \end{pmatrix}\right] + \\ \left[\begin{pmatrix} x_{2-r1} \overset{AA}{m}_{2-11} + \\ x_{2-r2} \overset{AA}{m}_{2-21} \end{pmatrix} \begin{pmatrix} x_{2-r1} \overset{AB}{m}_{2-12} + \\ x_{2-r2} \overset{AB}{m}_{2-22} \end{pmatrix} \begin{pmatrix} x_{2-r1} \overset{BA}{m}_{2-11} + \\ x_{2-r2} \overset{BA}{m}_{2-21} \end{pmatrix} \begin{pmatrix} x_{2-r1} \overset{BB}{m}_{2-12} + \\ x_{2-r2} \overset{BB}{m}_{2-22} \end{pmatrix}\right] \end{gathered} \tag{93}$$

*where* $Eval\left(\mathbf{M}_1 \otimes \mathbf{M}_2\right)$ *refers to a Markov chain matrix Kronecker product* $\mathbf{M}_1 \otimes \mathbf{M}_2$ *evaluated at a system state (in the way described for the system matrix* $\mathbf{M}^{SYS}$ *above). The system state is the concatenation of the state of each constituent matrix.*

*AA is the system state which represents the first state of* $\mathbf{M}_1$ *and the first state of* $\mathbf{M}_2$

*AB is the system state which represents the first state of* $\mathbf{M}_1$ *and the second state of* $\mathbf{M}_2$ *etc.*

*The column in the constituent matrices represent their current state, so column 1 in matrix* $\mathbf{M}_1$ *corresponds to state A for matrix* $\mathbf{M}_1$*. The column number must match the corresponding state letter, e.g.* $\overset{BA}{m_{2-21}}$ *indicates column 1 of the second agent, and this matches the system state designation of A for the second agent.*

*The* $\oplus$ *operation is defined above.*

$x_{1-r1}$ *refers to matrix* $\mathbf{X}_1$*, row r column 1 etc.*

**Proof**: We evaluate the premise. We show the result for the $r$ th row of $\mathbf{X}_1 \oplus \mathbf{X}_2$ and the $i$ th column of $\mathbf{M}_1 \otimes \mathbf{M}_2$, it applies the same way to the remainder. For brevity the proof is shown for just two elements. The $i$ th column of $\mathbf{M}_1 \otimes \mathbf{M}_2$ results from the $j$ th column of $\mathbf{M}_1$ and the $k$ th column of $\mathbf{M}_2$.

$$i = j * D + k \tag{94}$$

$$\begin{aligned}&\left[\left(\mathbf{X}_1 \oplus \mathbf{X}_2\right)\left(\mathbf{M}_1 \otimes \mathbf{M}_2\right)\right]^r \\ &= \left[x_{1-r1} + x_{2-r1} \quad x_{1-r1} + x_{2-r2} \quad x_{1-r2} + x_{2-r1} \quad x_{1-r2} + x_{2-r2}\right] Eval\left(\begin{bmatrix} m_{1-1j} \\ m_{1-2j} \end{bmatrix} \otimes \begin{bmatrix} m_{2-1k} \\ m_{2-2k} \end{bmatrix}\right)\end{aligned} \tag{95}$$

$$= \left( \begin{bmatrix} x_{1-r1} & x_{1-r1} & x_{1-r2} & x_{1-r2} \end{bmatrix} + \begin{bmatrix} x_{2-r1} & x_{2-r2} & x_{2-r1} & x_{2-r2} \end{bmatrix} \right) \begin{bmatrix} \overset{JK}{m}_{1-1j} \begin{bmatrix} \overset{JK}{m}_{2-1k} \\ \overset{JK}{m}_{2-2k} \end{bmatrix} \\ \overset{JK}{m}_{1-2j} \begin{bmatrix} \overset{JK}{m}_{2-1k} \\ \overset{JK}{m}_{2-2k} \end{bmatrix} \end{bmatrix} \tag{96}$$

$$= \sum_{1 \text{ to } D\times D} \begin{bmatrix} x_{1-r1} \overset{JK}{m}_{1-1j} \begin{bmatrix} \overset{JK}{m}_{2-1k} \\ \overset{JK}{m}_{2-2k} \end{bmatrix} \\ x_{1-r2} \overset{JK}{m}_{1-2j} \begin{bmatrix} \overset{JK}{m}_{2-1k} \\ \overset{JK}{m}_{2-2k} \end{bmatrix} \end{bmatrix} + \sum_{1 \text{ to } D\times D} \begin{bmatrix} \overset{JK}{m}_{1-1j} \begin{bmatrix} x_{2-r1} \overset{JK}{m}_{2-1k} \\ x_{2-r2} \overset{JK}{m}_{2-2k} \end{bmatrix} \\ \overset{JK}{m}_{1-2j} \begin{bmatrix} x_{2-r1} \overset{JK}{m}_{2-1k} \\ x_{2-r2} \overset{JK}{m}_{2-2k} \end{bmatrix} \end{bmatrix} \tag{97}$$

where $x_1$ is applied to the outside factor, $x_2$ applied to inside factor of $\mathbf{M}_1 \otimes \mathbf{M}_2$

and $\sum_{1 \text{ to } D\times D}$ shows that the terms are added together.

$$= \left( x_{1-r1} \overset{JK}{m}_{1-1j} + x_{1-r2} \overset{JK}{m}_{1-2j} \right) + \left( x_{2-r1} \overset{JK}{m}_{2-1k} + x_{2-r2} \overset{JK}{m}_{2-2k} \right) \text{ given } m_1, m_2 \text{ cols sum to 1} \tag{98}$$

Similarly for each column of $\mathbf{M}_1 \otimes \mathbf{M}_2$ and row of $\mathbf{X}_1 \oplus \mathbf{X}_2$. #

**Remark**: The result can be extended to progressive multiplications:

$$\left( \left( \mathbf{X}_1 \oplus \mathbf{X}_2 \right) \oplus \mathbf{X}_3 \right)^r \, Eval\left( \left( \mathbf{M}_1 \otimes \mathbf{M}_2 \right) \otimes \mathbf{M}_3 \right) \text{ etc} \tag{99}$$

The following uses the previous result to obtain an expression in terms of the system matrices for the agent's resource holdings.

**Lemma 3.7**. *Expected resources in period 1 of an agent are given by:*

$$\mathbf{R}_{j1} = \mathbf{X} \cdot \mathbf{M}_j^{EVAL} \cdot \boldsymbol{\mu}_j \tag{100}$$

*where* $\boldsymbol{\mu}_j = \begin{bmatrix} \overset{A}{\mu} \\ \overset{B}{\mu} \end{bmatrix} : D \times 1$, *system distribution vector* $\boldsymbol{\mu}^{SYS}$ *aggregated for agent j's states* (101)

$$\overset{A}{\mu} = \sum_{States} \overset{AA}{\mu} + \overset{AB}{\mu}, \text{ aggregation of all system states where agent } j \text{ is in state } A \tag{102}$$

$$\overset{B}{\mu} = \sum_{States} \overset{BA}{\mu} + \overset{BB}{\mu} \qquad \textit{i.e. cumulated over all states of other agents} \tag{103}$$

*where superscript AA denotes the first state of agent 1 and the first state of agent 2, AB denotes the first state of agent 1 and the second state of agent 2, etc. Superscript A denotes the first state of agent 1 cumulated over all states of agent 2, superscript B denotes the second state of agent 1 cumulated over all the states of agent 2, etc. Only two agents shown for simplicity, but in general one letter is used for each state of each agent, so if there are five agents we might have system state JCJKA for instance.*

$$\mathbf{M}_j^{EVAL} = \begin{bmatrix} \overset{A}{m}_{11} & \overset{B}{m}_{12} \\ \overset{A}{m}_{21} & \overset{B}{m}_{22} \end{bmatrix} \tag{104}$$

*where* $$\overset{A}{m}_{11} = \overset{AA}{m}_{11} \frac{\overset{AA}{\mu}}{\overset{AA}{\mu} + \overset{AB}{\mu}} + \overset{AB}{m}_{11} \frac{\mu^{AB}}{\overset{AA}{\mu} + \overset{AB}{\mu}} \qquad \textit{weighted average over states of other agents} \tag{105}$$

*and* $$\overset{A}{m}_{21} = \overset{AA}{m}_{21} \frac{\overset{AA}{\mu}}{\overset{AA}{\mu} + \overset{AB}{\mu}} + \overset{AB}{m}_{21} \frac{\overset{AB}{\mu}}{\overset{AA}{\mu} + \overset{AB}{\mu}} \quad \textit{etc.} \tag{106}$$

**Proof**: We evaluate $\mathbf{R}_1$ using its definition and the previous result. Without loss of generality the proof is shown for only two resources, states and agents.

$$\mathbf{R}_1 = \mathbf{X}^{SYS}\mathbf{M}^{SYS}\boldsymbol{\mu} \qquad \mathbf{R}_1 \text{ denotes resource holdings in period 1} \tag{107}$$

$$= \mathbf{X}^{SYS}\left(\mathbf{M}^{SYS\,2} + \mathbf{M}^{SYS\ VITAL} + \mathbf{M}^{SYS\ PROCESS}\right)\boldsymbol{\mu} \tag{108}$$

$$= \mathbf{X}^{SYS}\mathbf{M}^{SYS\,2}\boldsymbol{\mu} \tag{109}$$

$$\text{given} \qquad \mathbf{X}^{SYS}\mathbf{M}^{SYS\ VITAL} = \mathbf{0}_{DSYS\times 1} \qquad \text{by construction} \tag{110}$$

$$\mathbf{X}^{SYS}\mathbf{M}^{SYS\ PROCESS} = \mathbf{0}_{DSYS\times 1} \quad \text{by Assumption 3.19} \tag{111}$$

$$\mathbf{R}_1 = \left(\mathbf{X} \oplus \mathbf{X}\right) Eval\left(\mathbf{M}_1 \otimes \mathbf{M}_2\right)\boldsymbol{\mu}^{SYS} \tag{112}$$

$$=\left(\mathbf{X}\oplus\mathbf{X}\right)Eval\left(\mathbf{M}_1\otimes\mathbf{M}_2\right)\begin{bmatrix}\overset{AA}{\mu}\\ \overset{AB}{\mu}\\ \overset{BA}{\mu}\\ \overset{BB}{\mu}\end{bmatrix} \tag{113}$$

$$=\left(\begin{matrix}\left[\left(\mathbf{x}_1\overset{AA}{m}_{1-11}+\mathbf{x}_2\overset{AA}{m}_{1-21}\right)\;\left(\mathbf{x}_1\overset{AB}{m}_{1-11}+\mathbf{x}_2\overset{AB}{m}_{1-21}\right)\;\left(\mathbf{x}_1\overset{BA}{m}_{1-12}+\mathbf{x}_2\overset{BA}{m}_{1-22}\right)\;\left(\mathbf{x}_1\overset{BB}{m}_{1-12}+\mathbf{x}_2\overset{BB}{m}_{1-22}\right)\right]+\\ \left[\left(\mathbf{x}_1\overset{AA}{m}_{2-11}+\mathbf{x}_2\overset{AA}{m}_{2-21}\right)\;\left(\mathbf{x}_1\overset{AB}{m}_{2-12}+\mathbf{x}_2\overset{AB}{m}_{2-22}\right)\;\left(\mathbf{x}_1\overset{BA}{m}_{2-11}+\mathbf{x}_2\overset{BA}{m}_{2-21}\right)\;\left(\mathbf{x}_1\overset{BB}{m}_{2-12}+\mathbf{x}_2\overset{BB}{m}_{2-22}\right)\right]\end{matrix}\right)\begin{bmatrix}\overset{AA}{\mu}\\ \overset{AB}{\mu}\\ \overset{BA}{\mu}\\ \overset{BB}{\mu}\end{bmatrix}$$

using (93) to break up $\mathbf{X}^{SYS}$ into separate stock matrices $\mathbf{X}$ for each agent and cols $\mathbf{x}$ (114)

We can therefore deal with resources agent by agent, as would be expected:

$$\mathbf{R}_{j1}=\left[\left(\mathbf{x}_1\overset{AA}{m}_{1-11}+\mathbf{x}_2\overset{AA}{m}_{1-21}\right)\;\left(\mathbf{x}_1\overset{AB}{m}_{1-11}+\mathbf{x}_2\overset{AB}{m}_{1-21}\right)\;\left(\mathbf{x}_1\overset{BA}{m}_{1-12}+\mathbf{x}_2\overset{BA}{m}_{1-22}\right)\;\left(\mathbf{x}_1\overset{BB}{m}_{1-12}+\mathbf{x}_2\overset{BB}{m}_{1-22}\right)\right]\begin{bmatrix}\overset{AA}{\mu}\\ \overset{AB}{\mu}\\ \overset{BA}{\mu}\\ \overset{BB}{\mu}\end{bmatrix}$$

where $\mathbf{x}_1$ refers to column 1 of $\mathbf{X}$ etc. (115)

(Note that $\mathbf{x}_2$ is being applied to state *AA* at $\overset{AA}{m}_{1-21}$ because this represents a transition from State 1 in period 0 to State 2 in period 1. $\mathbf{R}_1$ measures the resources in the next period 1, not the current period 0.)

$$=\left[\mathbf{x}_1\quad\mathbf{x}_2\right]\begin{bmatrix}\overset{AA}{m}_{1-11}\overset{AA}{\mu}+\overset{AB}{m}_{1-11}\overset{AB}{\mu}+\overset{BA}{m}_{1-12}\overset{BA}{\mu}+\overset{BB}{m}_{1-12}\overset{BB}{\mu}\\ \overset{AA}{m}_{1-21}\overset{AA}{\mu}+\overset{AB}{m}_{1-21}\overset{AB}{\mu}+\overset{BA}{m}_{1-22}\overset{BA}{\mu}+\overset{BB}{m}_{1-22}\overset{BB}{\mu}\end{bmatrix} \tag{116}$$

$$=\left[\mathbf{x}_1\quad\mathbf{x}_2\right]\begin{bmatrix}\overset{AA}{m}_{1-11}\dfrac{\overset{AA}{\mu}}{\overset{AA}{\mu}+\overset{AB}{\mu}}+\overset{AB}{m}_{1-11}\dfrac{\mu^{AB}}{\overset{AA}{\mu}+\overset{AB}{\mu}} & \overset{BA}{m}_{1-12}\dfrac{\overset{BA}{\mu}}{\overset{BA}{\mu}+\overset{BB}{\mu}}+\overset{BB}{m}_{1-12}\dfrac{\overset{BB}{\mu}}{\overset{BA}{\mu}+\overset{BB}{\mu}}\\ \overset{AA}{m}_{1-21}\dfrac{\overset{AA}{\mu}}{\overset{AA}{\mu}+\overset{AB}{\mu}}+\overset{AB}{m}_{1-21}\dfrac{\mu^{AB}}{\overset{AA}{\mu}+\overset{AB}{\mu}} & \overset{BA}{m}_{1-22}\dfrac{\overset{BA}{\mu}}{\overset{BA}{\mu}+\overset{BB}{\mu}}+\overset{BB}{m}_{1-22}\dfrac{\overset{BB}{\mu}}{\overset{BA}{\mu}+\overset{BB}{\mu}}\end{bmatrix}\begin{bmatrix}\overset{AA}{\mu}+\overset{AB}{\mu}\\ \overset{BA}{\mu}+\overset{BB}{\mu}\end{bmatrix} \tag{117}$$

$$=\begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2\end{bmatrix}\begin{bmatrix}\overset{A}{m}_{1-11} & \overset{B}{m}_{1-12} \\ \overset{A}{m}_{1-21} & \overset{B}{m}_{1-22}\end{bmatrix}\begin{bmatrix}\overset{A}{\mu} \\ \overset{B}{\mu}\end{bmatrix} \text{ noting probabilities of each case sum to unity \#} \quad (118)$$

**Remark**: Similarly this result extends to progressive multiplications:

$$\mathbf{R}_1 = \left(\left(\mathbf{X}_1 \oplus \mathbf{X}_2\right) \oplus \mathbf{X}_3\right)^r \mathit{Eval}\left(\left(\mathbf{M}_1 \otimes \mathbf{M}_2\right) \otimes \mathbf{M}_3\right) \boldsymbol{\mu}^{SYS} \quad (119)$$

We use the three previous results to convert the Markov chain formulation of the system to a linear production model of the system.

**Lemma 3.8**. *Total resource change in the system is given by:*

$$\Delta\mathbf{R} = \mathbf{L} + \overset{PRO}{\mathbf{A}}\,\mathbf{m} + \overset{END}{\mathbf{A}}\,\mathbf{m} + \overset{CON}{\mathbf{A}}\,\mathbf{m} + \overset{SHED}{\mathbf{B}}\,\mathbf{m} \quad (120)$$

$$= \overset{PRO}{\mathbf{A}}\,\mathbf{m} + \overset{END}{\mathbf{A}}\,\mathbf{m} + \overset{CON}{\mathbf{A}}\,\mathbf{m} + \overset{END}{\mathbf{B}}\,\mathbf{m} + \overset{SHED}{\mathbf{B}}\,\mathbf{m} \quad (121)$$

*where* $\mathbf{L}: R \times 1$ *vector of resource endowments, for each resource which is endowed. For manufactured resources,* $L^r = 0$.

$$\overset{PRO}{\mathbf{A}} = \begin{bmatrix}\overset{PRO}{\mathbf{a}}_1 & \overset{PRO}{\mathbf{a}}_2 & \overset{PRO}{\mathbf{a}}_J\end{bmatrix} : R \times J \quad \text{matrix of production vectors for each agent type} \quad (122)$$

$$\overset{END}{\mathbf{B}} = \begin{bmatrix}\overset{END}{\mathbf{b}}_1 & \overset{END}{\mathbf{b}}_2 & \overset{END}{\mathbf{b}}_J\end{bmatrix} : R \times J \quad \text{matrix of endowed resource production vectors} \quad (123)$$

$\overset{CON}{\mathbf{A}} : R \times J$ *matrix of consumption vectors for each type*

$\overset{SHED}{\mathbf{B}} \le \mathbf{0} : R \times J$ *matrix resource shedding vectors, averaged over states*

$\mathbf{m} : J \times 1$ *expected number of agents of each type* or *'position' in agent space*

**Proof**: We use the previous three lemmas and the process definition to evaluate $\mathbf{R}_1$.

and $$\mathbf{M}_j^{EVAL} = \overset{END\,EVAL}{\mathbf{M}_j} + \overset{PRO\,EVAL}{\mathbf{M}_j} + \overset{CON\,EVAL}{\mathbf{M}_j} + \overset{TRA\,EVAL}{\mathbf{M}_j} + \overset{SHED\,EVAL}{\mathbf{M}_j} + \mathbf{I} \text{ applying (89)} \quad (124)$$

so $$\mathbf{R}_{j1} = \mathbf{X}\left(\overset{END\,EVAL}{\mathbf{M}_j} + \overset{PRO\,EVAL}{\mathbf{M}_j} + \overset{CON\,EVAL}{\mathbf{M}_j} + \overset{TRA\,EVAL}{\mathbf{M}_j} + \overset{SHED\,EVAL}{\mathbf{M}_j} + \mathbf{I}\right)\boldsymbol{\mu}_j \text{ apply (124) to (100)} \quad (125)$$

$$= \left(\overset{PRO}{\mathbf{a}}_j + \overset{END}{\mathbf{a}}_j + \overset{CON}{\mathbf{a}}_j + \overset{END}{\mathbf{b}}_{j0} + \overset{TRA}{\mathbf{b}}_{j0} + \overset{SHED}{\mathbf{b}}_{j0}\right)\overset{A}{\mu}_j + \mathbf{R}_{j0} \quad \text{(67),(71),(74),(79),(82),(127)} \quad (126)$$

noting $\mathbf{X}\cdot\mathbf{I}\cdot\boldsymbol{\mu}_j = \mathbf{X}\boldsymbol{\mu}_j = \mathbf{R}_{j0}$ recall $\boldsymbol{\mu}_j : D\times 1$ comprises aggregated states for agent $j$ (127)

so $$\mathbf{R}_1 = \sum_j \left( \overset{PRO}{\mathbf{a}}_j + \overset{END}{\mathbf{a}}_j + \overset{CON}{\mathbf{a}}_j + \overset{END}{\mathbf{b}}_{j0} + \overset{TRA}{\mathbf{b}}_{j0} + \overset{SHED}{\mathbf{b}}_{j0} \right)\overset{A}{\mu}_j + \mathbf{R}_0$$ aggregating over all agents (128)

We now simplify the expressions for trade and endowed production, $\overset{TRA\,r}{\mathbf{b}}_j$ , $\overset{END\,r}{\mathbf{b}}_j$ .

$$\sum_j \overset{TRA}{\mathbf{b}}_j \cdot \overset{A}{\mu}_j = \sum_j \sum_{\substack{STATES\\ s}} \overset{TRA\,s}{\mathbf{b}}_j \overset{s}{\mu} = \sum_{\substack{STATES\\ s}} \overset{s}{\mu} \sum_j \overset{TRA\,r}{\mathbf{b}}_j = \sum_{\substack{STATES\\ s}} \overset{s}{\mu}\cdot \mathbf{0}_{R\times 1} = \mathbf{0}_{R\times 1} \tag{129}$$

and $$\sum_j \overset{END\,r}{\mathbf{b}}_j \cdot \overset{A}{\mu}_j = \sum_j \sum_{\substack{STATES\\ s}} \overset{END\,sr}{\mathbf{b}}_j \overset{s}{\mu}$$ breaking probability $\overset{A}{\mu}_j$ up into single states $\overset{s}{\mu}$ (130)

$$= \sum_{\substack{STATES\\ s}} \sum_j \frac{\begin{bmatrix} L^{r1} \\ L^{r2} \\ 0 \end{bmatrix}}{\overset{s}{N}} \cdot \overset{s}{\mu} = \sum_{\substack{STATES\\ s}} \overset{s}{N} \cdot \frac{\begin{bmatrix} L^{r1} \\ L^{r2} \\ 0 \end{bmatrix}}{\overset{s}{N}} \cdot \overset{s}{\mu} = \begin{bmatrix} L^{r1} \\ L^{r2} \\ 0 \end{bmatrix} = \mathbf{L} \quad \text{noting} \sum_{\substack{STATES\\ s}} \overset{s}{\mu} = 1 \tag{131}$$

where $\overset{s}{N}$ is no. of active agents in state $s$

to get $$\mathbf{R}_1 = \mathbf{L} + \sum_j \left( \overset{PRO}{\mathbf{a}}_j + \overset{END}{\mathbf{a}}_j + \overset{CON}{\mathbf{a}}_j + \overset{SHED}{\mathbf{b}}_j \right)\overset{A}{\mu}_j + \mathbf{R}_0$$ by (129), (131) (132)

We can break this up by agent type to express it in terms of position **m**:

$$\Delta\mathbf{R} = \mathbf{L} + \sum_{k\in TYPES} \sum_{j\in TYPE\,k} \left( \overset{PRO}{\mathbf{a}}_j + \overset{END}{\mathbf{a}}_j + \overset{CON}{\mathbf{a}}_j + \overset{SHED}{\mathbf{b}}_j \right)\overset{A}{\mu}_j \tag{133}$$

$$= \mathbf{L} + \sum_{k\in TYPES} \left( \overset{PRO}{\mathbf{a}}_j + \overset{END}{\mathbf{a}}_j + \overset{CON}{\mathbf{a}}_j + \overset{SHED}{\mathbf{b}}_j \right) m_k \tag{134}$$

using $$\sum_{j\in TYPE\,k} \overset{A}{\mu}_j = N_k^{UPPER} \cdot \overset{A}{\mu}_j = m_k \tag{135}$$

so we can organize (132) by agent type as:

$$\Delta\mathbf{R} = \mathbf{L} + \begin{bmatrix} \overset{PRO}{\mathbf{a}}_1 & \overset{PRO}{\mathbf{a}}_2 & \overset{PRO}{\mathbf{a}}_J \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_J \end{bmatrix} + \begin{bmatrix} \overset{CON}{\mathbf{a}}_1 & \overset{CON}{\mathbf{a}}_2 & \overset{CON}{\mathbf{a}}_J \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_J \end{bmatrix} + \begin{bmatrix} \overset{SHED}{\mathbf{b}}_1 & \overset{SHED}{\mathbf{b}}_2 & \overset{SHED}{\mathbf{b}}_J \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_J \end{bmatrix} \tag{136}$$

i.e. $$\Delta\mathbf{R} = \mathbf{L} + \overset{PRO}{\mathbf{A}}\,\mathbf{m} + \overset{END}{\mathbf{A}}\,\mathbf{m} + \overset{CON}{\mathbf{A}}\,\mathbf{m} + \overset{SHED}{\mathbf{B}}\,\mathbf{m} \qquad (137)$$

Similarly $\overset{END}{\mathbf{B}}\,\mathbf{m} = \mathbf{L}$ from (131), sub in (137) for (121). # (138)

We apply the properties of a Markov chain to complete the formulation of the linear production model.

**Corollary 3.9**. *Expected system resource use in equilibrium is zero.*

$$\Delta\mathbf{R} = \mathbf{0} \qquad (139)$$

$$\mathbf{L} + \overset{PRO}{\mathbf{A}}\,\mathbf{m} + \overset{END}{\mathbf{A}}\,\mathbf{m} + \overset{CON}{\mathbf{A}}\,\mathbf{m} + \overset{SHED}{\mathbf{B}}\,\mathbf{m} = \mathbf{0}_{R\times 1} \qquad (140)$$

$$\overset{PRO}{\mathbf{A}}\,\mathbf{m} + \overset{END}{\mathbf{A}}\,\mathbf{m} + \overset{CON}{\mathbf{A}}\,\mathbf{m} + \overset{END}{\mathbf{B}}\,\mathbf{m} + \overset{SHED}{\mathbf{B}}\,\mathbf{m} = \mathbf{0}_{R\times 1} \qquad (141)$$

**Proof**: We use the eigenvalue properties of a Markov matrix.

$$\Delta\mathbf{R} = \mathbf{R}_1 - \mathbf{R}_0 = \mathbf{X}^{SYS}\mathbf{M}^{SYS}\boldsymbol{\mu}^{SYS} - \mathbf{X}^{SYS}\boldsymbol{\mu}^{SYS} = \left(\lambda^{SYS} - 1\right)\mathbf{X}^{SYS}\boldsymbol{\mu}^{SYS} \qquad (142)$$

$= \mathbf{0}_{MSYS\times 1}$ as $\mathbf{M}^{SYS}$ has unity eigenvalue. Sub in (120), (121). # (143)

### 3.4 Properties of system equilibrium

The nature of system equilibrium is that every state in the system matrix $\mathbf{M}^{SYS}$ is instantiated at some time or other but most are unlikely. The question arises as to how situational equilibrium can be defined in this context. We take it that the system spends only fleeting time in unstable states and is pushed into stable states, and a situational equilibrium solution is a system state which does not exert systematic pressure to leave.

The following is an important property of the unique system equilibrium.

**Theorem 3.10**: AGENT THEOREM. *At equilibrium the number of each agent type is stable.*

$$\lambda_j = 1 \text{ for all agents } j. \qquad (144)$$

**Proof**: Our strategy is to duplicate the method for resources which leads to the same result at (139). To this end we introduce matrix $\underline{\mathbf{X}}$, same as stock matrix $\mathbf{X}$ except that $\underline{\mathbf{X}}$ gives the number of agents for each system state rather than resources:

$\underline{\mathbf{X}} = \begin{bmatrix} \overset{S1}{X}_j & \overset{S2}{X}_j \\ \overset{S1}{X}_k & \overset{S2}{X}_k \end{bmatrix} : J \times D^{SYS}$ agent matrix: amount of each of J agent types in each state of $\mathbf{M}^{SYS}$

$$\mathbf{m} = \underline{\mathbf{X}}\boldsymbol{\mu}^{SYS} \tag{145}$$

$$\Delta\mathbf{m} = \mathbf{m}_1 - \mathbf{m}_0 = \underline{\mathbf{X}}\,\mathbf{M}^{SYS}\boldsymbol{\mu}^{SYS} - \underline{\mathbf{X}}\,\boldsymbol{\mu}^{SYS} = \left(\lambda^{SYS} - 1\right)\underline{\mathbf{X}}\,\boldsymbol{\mu}^{SYS} = \mathbf{0}_{J\times 1} \tag{146}$$

$$\lambda_j = \frac{m_{j0} + \Delta m_j}{m_{j0}} = \frac{m_{j0} + 0}{m_{j0}} = 1 \qquad \# \tag{147}$$

### 3.5 Resource shedding

In real life, agents have finite storage space and constraints on the amount which they can store. This puts an upper limit on the amount of resource which can be transferred by trading. Without this limit, all the scarce resources which are produced can be transferred to agents who need it and no resource will be wasted. With the upper limits, some resource will be wasted through the resource shedding process. It is important to model resource shedding to understand how non-market (biological or pre-industrial) systems differ from market systems.

We point out that abundant resources do not necessarily need to be stored by individual agents. The free oxygen on earth has been produced biologically, and it is stored in the atmosphere not by any individual agent. In this case the upper bound analysis does not apply.

**Assumption 3.20** The upper limits of the endowed resources are sufficiently high that they are not reached.

**Remark**: Analysis of endowed resources involves us in non-linearities so it is convenient to ignore that here.

Consider the agent type *jMIN* which maintains the lowest average stock level $\rho^r_{jMIN}$. There is some minimum level of *jMIN*'s average stock, $\rho^r_{MIN}$, below which the agent cannot survive even if all the other resources are abundant. This amount does not depend on whether the agent is has a positive growth rate or not, it is purely dependent on the stochastic process for that resource. The following result establishes that the average stock level in the system, $\rho^r_{AVG}$, has a minimum value $\rho^r_{\underset{MIN}{AVG}}$ which is determined by $\rho^r_{MIN}$ and is independent of the number of agents **m**. We then extend this result to show that there is a minimum level of shedding which is determined by the production functions **a** and trading coefficient $k^r$, and is likewise independent of position **m**.

**Lemma 3.11**: *At equilibrium, the minimum level of stock $\rho^r_{MIN}$ necessary for the survival of the low producer jMIN implies a minimum value for the average amount of stock $\rho^r_{AVG}$, given by:*

$$\rho^r_{\underset{MIN}{AVG}} = \rho^r_{MIN} - \frac{\overset{PRO\,r}{a}_{jMIN} + \overset{END\,r}{a}_{jMIN} + \overset{CON\,r}{a}_{jMIN}}{k^r} \tag{148}$$

**Proof**: We evaluate the resource equation using the trade equation and rearrange.

Now $\quad 0 = \overset{TRA\,r}{b}_{jMIN} + \overset{PRO\,r}{a}_{jMIN} + \overset{END\,r}{a}_{jMIN} + \overset{CON\,r}{a}_{jMIN} \quad$ resource constraint for the agent (149)

Note that there is no shedding and by Assumption 3.20, no production of endowed resources.

so $\quad 0 = k^r\left(\rho^r_{\underset{MIN}{AVG}} - \rho^r_{MIN}\right) + \overset{PRO\,r}{a}_{jMIN} + \overset{END\,r}{a}_{jMIN} + \overset{CON\,r}{a}_{jMIN} \quad$ use (7) and rearrange for result. # (150)

**Lemma 3.12**: *At equilibrium and for a given minimum producer jMIN, an agent j which produces resource r has a minimum level of shedding given by:*

$$\overset{SHED\,MIN}{b_j} = \begin{cases} k^r\left(U^r - \rho^r_{jUPPER}\right) & \rho^r_{jUPPER} > U^r \\ 0 & \text{otherwise} \end{cases} \tag{151}$$

*where* $$\rho^r_{jUPPER} = \rho^r_{\underset{MIN}{AVG}} + \frac{\overset{PRO\,r}{a_j} + \overset{END\,r}{a_j} + \overset{CON\,r}{a_j}}{k^r} \tag{152}$$

*is the stock holding which would exhaust production in the absence of upper bound $U^r$.*

**Proof**: We find the value of $\rho^r_{jUPPER}$ which would exhaust production without shedding. We can then substitute this expression for the non-shedding terms in the resource equation. Consider an agent which is a net producer of the resource. The resource equation is:

$$0 = \overset{PRO\,r}{a}_j + \overset{END\,r}{a}_j + \overset{CON\,r}{a}_j + \overset{TRA\,r}{b}_j + \overset{END\,r}{b}_j + \overset{SHED\,r}{b}_j \tag{153}$$

from (141), stated for one agent and one resource. Trade is included. Stock $\rho^r_{jUPPER}$ applies in the absence of upper bound $U^r$ and shedding, so we remove shedding term $\overset{SHED\,r}{b}_j$. We also exclude endowed resource production $\overset{END\,r}{b}_j$, as shedding does not apply to endowed resources by Assumption 3.20. This yields:

$$0 = \overset{PRO\,r}{a}_j + \overset{END\,r}{a}_j + \overset{CON\,r}{a}_j + \overset{TRA\,r}{b}_j \tag{154}$$

$$0 = k^r\left(\rho^r_{\substack{AVG\\MIN}} - \rho^r_{jUPPER}\right) + \overset{PRO\,r}{a}_j + \overset{END\,r}{a}_j + \overset{CON\,r}{a}_j \quad \text{sub using (7). Rearrange for (152).} \tag{155}$$

If $\rho^r_{jUPPER} \le U^r$ then shedding is zero. If $\rho^r_{jUPPER} > U^r$ then the actual level of trade is:

$$\overset{TRA\,r}{b}_j = k^r\left(\rho^r_{\substack{AVG\\MIN}} - U^r\right) \qquad \text{now constrained by the maximum stock } U^r \tag{156}$$

Consider now resource equation (154) with shedding $\overset{SHED\,r}{b}_j$ reintroduced:

$$0 = \overset{SHED\,r}{b}_j + \overset{TRA\,r}{b}_j + \overset{PRO\,r}{a}_j + \overset{END\,r}{a}_j + \overset{CON\,r}{a}_j \tag{157}$$

$$0 = \overset{SHED}{b}_j + k^r\left(\rho^r_{\substack{AVG\\MIN}} - U^r\right) - k^r\left(\rho^r_{\substack{AVG\\MIN}} - \rho^r_{jUPPER}\right) \quad \text{sub (155), (156) in (157).} \qquad \# \tag{158}$$

**Theorem 3.13** *At equilibrium and for a given minimum producer jMIN, an agent j which produces resource r has a minimum level of shedding given by:*

$$\overset{SHED\,MIN}{b}_j = \begin{cases} \left(\overset{PRO\,r}{a}_{jMIN} + \overset{END\,r}{a}_{jMIN} + \overset{CON\,r}{a}_{jMIN}\right) - \left(\overset{PRO\,r}{a}_j + \overset{END\,r}{a}_j + \overset{CON\,r}{a}_j\right) + k^r\left(U^r - \rho^r_{MIN}\right) & \text{if negative} \\ 0 & \text{otherwise} \end{cases} \tag{159}$$

**Proof**: Substitute (148) into (152) and (152) into (151). #

**Remark**: $\left(\overset{PRO\,r}{a}_{jMIN}+\overset{END\,r}{a}_{jMIN}+\overset{CON\,r}{a}_{jMIN}\right)-\left(\overset{PRO\,r}{a}_{j}+\overset{END\,r}{a}_{j}+\overset{CON\,r}{a}_{j}\right)$ is the difference in net production of the producer *j* and the low producer *jMIN*.

$k^r\left(U^r-\rho^r_{MIN}\right)$ is the amount which can be cleared by trade.

**Remark**: Shedding will differ for different agent sets because the agent with the minimum production will be different, and this changes the mean according to (148). This means that a different resource equation applies to every combination of non-zero agents.

**Remark**: The greater the trading coefficient $k^r$, the lower the level of market clearing stock $\rho^r_{jUPPER}$ and the less scarce resource is lost through shedding. Looking at changes in the system as the trading coefficient increases gives us a market system as a special, efficient case of a biological system.

We note that the expression for shedding (159) is constant with respect to position **m**, so shedding is linear in position. Partition total shedding into the minimal level defined by (159) and the excess over this level which occurs if a resource is abundant or the minimum producer maintains higher stock levels than $\rho^r_{MIN}$ :

$$\overset{SHED}{\mathbf{B}}=\overset{SHED\,MIN}{\mathbf{A}}+\overset{EXCESS}{\mathbf{B}} \tag{160}$$

where $$\overset{SHED\,MIN}{\mathbf{A}}:\left(a^r_j=\begin{cases}k^r\left(U^r-\rho^r_{jUPPER}\right) & \rho^r_{jUPPER}>U^r\\ 0 & \text{otherwise}\end{cases}\right):R\times J \tag{161}$$

$\overset{EXCESS}{\mathbf{B}}<=\mathbf{0}:R\times J$ matrix of the excess, if any, of actual shedding over minimal.

**Theorem 3.14**: LINEAR PROGRAMMING EQUIVALENCE. *At equilibrium, the resource constraints which apply to the system can be expressed as the constraints of a linear programming problem:*

$$\mathbf{L}+\overset{NET}{\mathbf{A}}\,\mathbf{m}>=\mathbf{0}_{R\times 1} \qquad i.e.\ -\overset{NET}{\mathbf{A}}\,\mathbf{m}<=\mathbf{L} \tag{162}$$

*When the system is not at equilibrium,*

$$\Delta\mathbf{R} = \mathbf{L} + \overset{NET}{\mathbf{A}}\,\mathbf{m} + \overset{EXCESS}{\mathbf{B}}\,\mathbf{m} \tag{163}$$

*or*

$$\Delta\mathbf{R} = \overset{NET}{\mathbf{A}}\,\mathbf{m} + \overset{END}{\mathbf{B}}\,\mathbf{m} + \overset{EXCESS}{\mathbf{B}}\,\mathbf{m} \tag{164}$$

*where*

$$\overset{NET}{\mathbf{A}} = \overset{PRO}{\mathbf{A}} + \overset{END}{\mathbf{A}} + \overset{CON}{\mathbf{A}} + \overset{SHED\,MIN}{\mathbf{A}} \quad : R \times J \text{ , net production after unavoidable shedding} \tag{165}$$

$$= \begin{bmatrix} \overset{NET}{\mathbf{a}}_1 & \overset{NET}{\mathbf{a}}_2 & \overset{NET}{\mathbf{a}}_J \end{bmatrix} : R \times J \text{ matrix of vectors for each of } J \text{ agents} \tag{166}$$

*where*

$$\overset{NET}{\mathbf{a}}_j = \overset{PRO}{\mathbf{a}}_j + \overset{END}{\mathbf{a}}_j + \overset{CON}{\mathbf{a}}_j + \overset{SHED\,MIN}{\mathbf{a}}_j \quad : R \times 1, \text{ net production for one agent} \tag{167}$$

**Proof**: Apply the breakup of shedding established in this section.

$$\mathbf{L} + \overset{PRO}{\mathbf{A}}\,\mathbf{m} + \overset{END}{\mathbf{A}}\,\mathbf{m} + \overset{CON}{\mathbf{A}}\,\mathbf{m} + \overset{SHED\,MIN}{\mathbf{A}}\,\mathbf{m} + \overset{EXCESS}{\mathbf{B}}\,\mathbf{m} = \mathbf{0} \quad \text{applying (160) to (140)} \tag{168}$$

$$\mathbf{L} + \overset{NET}{\mathbf{A}}\,\mathbf{m} + \overset{EXCESS}{\mathbf{B}}\,\mathbf{m} = \mathbf{0} \quad \text{using (165)} \tag{169}$$

Now

$$\overset{EXCESS}{\mathbf{B}}\,\mathbf{m} <= \mathbf{0} \quad \text{hence result (162).} \tag{170}$$

Apply (165),(160) to (120) for (163). Apply (138) to (163) for (164). #

**Remark**: The minimal amount of shedding is determined by the discrepancy between the lowest producer and others, so if the lowest producer goes extinct then the minimal level can change. The linear programming problem is therefore a function of a particular set of agent types.

**Remark**: The minimal level of shedding $\overset{SHED\,MIN}{\mathbf{A}}$ is not affected by the resource growth term $\Delta\mathbf{R}$, it is purely a function of the Markov process for that resource.

**Remark**: (162) is an upper bound on constraints. In fact the minimal producer may need a higher stock level than $\rho^r_{MIN}$ to survive if other resources are scarce. This implies a higher stock average, more shedding and tighter constraints.

### 3.6 Input-output considerations

*Productivity*: Within standard input-output theory we assume an input-output matrix $\mathbf{A}^{USUAL}$ such that for all non-negative consumption vectors **c**, there exists a non-negative input vector **x** such that

$$\left(\mathbf{I}-\mathbf{A}^{USUAL}\right)\mathbf{x}=\mathbf{c} \tag{171}$$

That property is not necessarily true here. For instance every agent may produce a positive amount of some good, so it is not possible to produce none of that good except at $\mathbf{m}=\mathbf{0}$, and the trivial solution cannot be used if consumption **c** in (171) contains any positive element.

*Bounded output*: It is intuitive that with finite resources the population is bounded, but to prove this we need the standard technical assumption No Land of Cockaigne.

**Assumption 3.21** NO LAND OF COCKAIGNE (NLOC). It is impossible for net output to be non-negative except when there is no input.

For all $\mathbf{m}\geq\mathbf{0}$, $\mathbf{Am}>\neq\mathbf{0}$ (i.e. at least one element is negative) (172)

**Remark**: This model allows endowed resources to be manufactured by agents as well as received by endowment. If manufacturing were sufficiently productive then one might finish with more of every good than one started with. NLOC rules out this possibility.

**Remark**: This statement is a combination of three standard axioms of linear production theory (see for instance Takayama 1974) which, expressed in the form of this model, are:

There is no $\mathbf{m}\geq\mathbf{0}$ such that $\mathbf{Am}\geq\mathbf{0}$ (No Land of Cockaigne) (173)

There is no $\mathbf{m}\geq\mathbf{0}$ such that $\mathbf{Am}=\mathbf{0}$ (Irreversibility) (174)

If $\mathbf{m}=\mathbf{0}$ then $\mathbf{Am}=\mathbf{0}$ (Possibility of Inaction) (175)

NLOC is the economic version of the first law of thermodynamics and Irreversibility is the economic version of the second law of thermodynamics. The Possibility of Inaction could be seen as an analogue of the third law of thermodynamics, except that the third law says that energy measuring zero degrees is impossible whereas here we permit zero action.

We assumed above that the system matrix had a finite number of slots available for agents. We now establish that the system has a finite size.

**Theorem 3.15** UPPER BOUND THEOREM. *Population N in equilibrium is upper bounded.*

**Proof**: We show that every ray drawn from the origin eventually meets the transverse constraints, by applying the NLOC property to a ray segment and magnifying it.

$$\mathbf{L} + \overset{NET}{\mathbf{A}}\,\mathbf{m} >= \mathbf{0}_{R\times 1} \text{ by (162)} \tag{176}$$

$$-\overset{NET}{\mathbf{A}}\,\mathbf{m} <= \mathbf{L} \tag{177}$$

This defines a convex set with linear bounds, *S*. If $\mathbf{m} \in S$ and $0 <= p <= 1$ then

$$-\overset{NET}{\mathbf{A}}\left(p\mathbf{m}\right) = p\left(-\overset{NET}{\mathbf{A}}\right)\mathbf{m} <= \mathbf{L} \tag{178}$$

so the set radiates out from the origin. We need to show the set is closed. Consider $2\mathbf{m} \in S$ on ray *r*.

$$\mathbf{L} + \overset{NET}{\mathbf{A}}\,2\mathbf{m} = \mathbf{b}_1 >= \mathbf{0}_{R\times 1} \tag{179}$$

$$\mathbf{L} + \overset{NET}{\mathbf{A}}\,\mathbf{m} = \mathbf{b}_2 >= \mathbf{0}_{R\times 1} \tag{180}$$

$$\overset{NET}{\mathbf{A}}\,\mathbf{m} = \mathbf{b}_1 - \mathbf{b}_2 = \mathbf{b}_3 \text{ say. (179) - (180)} \tag{181}$$

$\mathbf{b}_3$ must contain a negative, $b_{3k}$ say, or (172) NLOC is violated.

$$\left(\overset{NET}{\mathbf{A}}\,\mathbf{m}\right)_k = b_{3k} < 0 \tag{182}$$

Consider $\mathbf{m}* = \dfrac{-L_k\mathbf{m}}{b_{3k}}$ (183)

$$\left(\mathbf{L} + \overset{NET}{\mathbf{A}}\,\mathbf{m}*\right)_k = L_k + \left(\overset{NET}{\mathbf{A}}\,\mathbf{m}\right)_k \cdot \frac{-L_k}{b_{3k}} = L_k + b_{3k} \cdot \frac{-L_k}{b_{3k}} = 0 \tag{184}$$

So $\mathbf{m}^*$ is the maximum value which a multiple of $\mathbf{m}$ can take, i.e. ray *r* is bounded. As this applies to all rays *r* which pass through *S*, *S* is a closed convex set and $N = \boldsymbol{\iota}'\mathbf{m}$ has a maximum value. #

### 3.7 Conclusion

We have shown that a Markov Chain model of agent behaviour has a unique and stable equilibrium. This equilibrium is distributional rather than situational in nature, so in order to more fully characterize the system we looked at the system in terms of linear production theory. We were able to show that all the system processes were either linear in nature or cancelled each other out, and converted the system to a set of linear constraints. Standard concepts from linear production theory demonstrate that the amount which a finite endowment system can produce is bounded.

The final result is a model with two wings to fly by, on one hand agent processes represented by Markov chains, and on the other hand system constraints represented by input-output equations.

# Chapter 4

# The dynamic model

## 4.0 Introduction

Markov analysis and input-output analysis are both well understood techniques within the economic disciplinary framework. Here we bring them together to show how the characteristics of the agents determine the system trajectory. In the first three sections we examine how the Markov processes and the flow of resources determine agent growth rates and establish a dynamic equation. In the fourth section we use the dynamic equation to show how the system develops from a non-equilibrium starting point to the eventual equilibrium, and establish certain regularities. We represent long-term change in the model by perturbations of the coefficients in the Markov matrices. In the fifth section we apply the dynamic results to the perturbed coefficients in order to examine evolution within the context of the model.

## 4.1 The agent Leslie matrix

In the previous chapter we specified an agent model using Markov matrices. Markov matrices have maximal eigenvalues of unity. A theory of dynamics requires that we introduce a variant form of the Markov matrix whereby the vital events are handled not on the system level but the agent level. The eigenvalue is free to take values other than unity and we can examine growth and decline at agent level.

Consider the agent Markov matrix:

$$\mathbf{M} = \begin{bmatrix} 1 & \mathbf{m}_E & 0 \\ \mathbf{0} & \mathbf{M}_M & \mathbf{0} \\ 0 & \mathbf{m}_L & 1 \end{bmatrix} : D \times D \tag{185}$$

Let us replace the unity element $m_{LL} = 1$ by $m_{BL} = S$ .

$$S > 0 : scalar \text{, number of descendants which the agent produces in the } L \text{ state} \tag{186}$$

If we add an additional partition to the **M** matrix to separate out the birth state *B* which is placed second, then **M** becomes:

$$\begin{bmatrix} 1 & m_{EB} & \mathbf{m}_E & 0 \\ 0 & m_{BB} & \mathbf{m}_{BM} & S \\ 0 & \mathbf{m}_{MB} & \mathbf{M}_M & 0 \\ 0 & m_{LB} & \mathbf{m}_L & 0 \end{bmatrix} \qquad (187)$$

Now delete the expiry state *E* row and column to get:

$$\underline{\mathbf{M}} = \begin{bmatrix} m_{BB} & \mathbf{m}_{BM} & S \\ \mathbf{m}_{MB} & \mathbf{M}_M & 0 \\ m_{LB} & \mathbf{m}_L & 0 \end{bmatrix} : D-1 \times D-1, \text{ Leslie matrix} \qquad (188)$$

This transition matrix $\underline{\mathbf{M}}$ is not a Markov chain matrix, as the final column sums to *S* which in general is not unity. Nor does it have absorbing states. Rather the matrix represents the evolution of the agent and its descendants through time. Within biology such matrices are referred to as Leslie matrices (Leslie 1945). In the biological literature the states in Leslie matrices typically represent different ages or developmental stages, here they represent resource states.

As it is possible to move from any state in $\underline{\mathbf{M}}$ to adjacent resource states (i.e. neighbouring states on the resource grid) and subsequently to any other state, $\underline{\mathbf{M}}$ is irreducible and the Perron Frobenius theorem applies: there is a real, positive, unique maximal eigenvalue $\lambda$, and corresponding eigenvectors are strictly positive. So

$$\underline{\mathbf{M}}\boldsymbol{\mu} = \lambda\,\boldsymbol{\mu} \qquad (189)$$

where $\boldsymbol{\mu} > \mathbf{0} : D-1 \times 1$, steady state distribution of states of the agent

$\lambda : scalar$, growth factor (not growth rate) per period of the agent type.

For the LHS eigenvector:

$$\boldsymbol{\upsilon}\underline{\mathbf{M}} = \lambda\boldsymbol{\upsilon} \qquad (190)$$

The LHS eigenvector of a Markov matrix **M** consists of unity elements. In the case of a Leslie matrix, the LHS eigenvector can be interpreted as the expected number of descendants of a state measured relative to other states. We scale eigenvector $\boldsymbol{\upsilon}$ so that

$$\boldsymbol{\upsilon\mu} = 1 \tag{191}$$

We remind the reader of the standard result, couched in the context of this model, that we can differentiate the eigenvalue $\lambda$ without the need to consider changes in the eigenvectors.

**Lemma 4.1**. *Given matrix* $\underline{\mathbf{M}}$ *with eigenvalue* $\lambda$ *and eigenvectors* $\boldsymbol{\upsilon}, \boldsymbol{\mu}$ *where* $\boldsymbol{\upsilon\mu} = 1$*:*

$$d\lambda = \boldsymbol{\upsilon} \cdot \mathbf{d}\underline{\mathbf{M}} \cdot \boldsymbol{\mu} \tag{192}$$

**Proof**: We differentiate $\lambda\boldsymbol{\mu} = \underline{\mathbf{M}}\boldsymbol{\mu}$ and simplify the result using identities.

$$d\lambda\,\boldsymbol{\mu} + \lambda\,\mathbf{d}\boldsymbol{\mu} = \mathbf{d}\underline{\mathbf{M}}\,\boldsymbol{\mu} + \underline{\mathbf{M}} \cdot \mathbf{d}\boldsymbol{\mu} \tag{193}$$

$$d\lambda\,\boldsymbol{\upsilon\mu} + \lambda\,\boldsymbol{\upsilon} \cdot \mathbf{d}\boldsymbol{\mu} = \boldsymbol{\upsilon}\,\mathbf{d}\underline{\mathbf{M}}\,\boldsymbol{\mu} + \boldsymbol{\upsilon}\underline{\mathbf{M}} \cdot \mathbf{d}\boldsymbol{\mu} \qquad \times\boldsymbol{\upsilon} \text{ on LHS} \tag{194}$$

i.e. $$d\lambda + \lambda\,\boldsymbol{\upsilon} \cdot \mathbf{d}\boldsymbol{\mu} = \boldsymbol{\upsilon}\,\mathbf{d}\underline{\mathbf{M}}\,\boldsymbol{\mu} + \lambda\,\boldsymbol{\upsilon} \cdot \mathbf{d}\boldsymbol{\mu} \qquad \text{by (190), (191). Result (192) follows.\#} \tag{195}$$

**Remark**: $$d\lambda = \lambda_1 - \lambda_0 = \boldsymbol{\upsilon}_1\underline{\mathbf{M}}\,\boldsymbol{\mu}_1 - \boldsymbol{\upsilon}_0\underline{\mathbf{M}}\,\boldsymbol{\mu}_0 \tag{196}$$

so $$\boldsymbol{\upsilon}_1\underline{\mathbf{M}}\,\boldsymbol{\mu}_1 - \boldsymbol{\upsilon}_0\underline{\mathbf{M}}\,\boldsymbol{\mu}_0 = \boldsymbol{\upsilon}_0\mathbf{d}\underline{\mathbf{M}}\,\boldsymbol{\mu}_0 \tag{197}$$

The following result establishes the resource requirements of growth processes.

**Lemma 4.2**. *Average growth in resource holding of an agent type j is proportional to the net eigenvalue* $\lambda_j - 1$.

$$\mathbf{\Delta R}_j = \left(\lambda_j - 1\right)\mathbf{R}_{j0} \tag{198}$$

$$= \Delta\lambda_j \cdot \mathbf{R}_j \tag{199}$$

*where* $$\Delta\lambda_j = \lambda_j - 1 : \textit{scalar},\ \textit{growth rate} \tag{200}$$

**Proof**: $$\mathbf{\Delta R} = \mathbf{R}_1 - \mathbf{R}_0 = \mathbf{XM}\boldsymbol{\mu} - \mathbf{X}\boldsymbol{\mu} = \left(\lambda - 1\right)\mathbf{X}\boldsymbol{\mu} = \left(\lambda - 1\right)\mathbf{R}_0 \qquad \# \tag{201}$$

### 4.2 The dynamic model

In this section we develop the concepts which we need for a dynamic analysis and establish the dynamic model. This initial formulation of the model cannot be solved, so in the following section 4.3 an operational form of the model is developed. The system lifecycle analysis is then carried out in section 4.4.

*The dynamic assumption:*

**Assumption 4.1** Adjustments to the proportion of agent types in the population caused by resource scarcity take place quickly relative to changes in the proportion of agent types brought about by differential agent growth rates $\lambda$.

**Remark**: The rationale for this assumption is that growth takes place over a number of reproduction cycles whereas the demands of resource procurement are felt immediately.

**Remark**: The effect of this assumption is that when determining the growth rates at a position **m**, we take it that the agent types adjust immediately to the proportions which are consistent with that growth.

*Surplus production*: We define surplus production as total net production plus the endowed resources **L**.

$$\mathbf{U} = \mathbf{L} + \overset{NET}{\mathbf{A}}\,\mathbf{m} = \overset{NET}{\mathbf{A}}\,\mathbf{m} + \overset{END}{\mathbf{B}}\,\mathbf{m} : R \times 1, \text{ surplus production vector} \tag{202}$$

$$\mathbf{u} = \frac{\mathbf{U}}{N} : R \times 1, \text{ per capita amount of surplus.} \tag{203}$$

so $$\mathbf{\Delta R} = \mathbf{U} + \overset{SHED\,EXCESS}{\mathbf{B}}\,\mathbf{m} \text{ substituting into (163)} \tag{204}$$

Thus the surplus is divided between the increase in stock, and excess shedding. If the system is contracting then stock change $\mathbf{\Delta R}$ will be negative by (198), and it is possible that the surplus **U** is negative.

*Value matrix*: We introduce a formula which relates the growth factor of an agent $\lambda_j$ to the probability $p^r$ of procuring resources *r* which the agent does not produce for itself. This ratio is regarded as the *value* of the resource to the agent. We need an operational variable to represent the probability $p^r$. If a resource is not produced then it is derived from trading or resource recycling with trading presumed to be the major source. Trading is proportional to average stock level $\rho^r$, but this variable is a stock not a flow and not convenient to use because it does not form part of the resource flow model. Consideration of the underlying Markov chain stochastic process suggests that we can relate stock levels to the period surplus **u** to get an operational relationship between growth rate and probability.

**Assumption 4.2** The marginal probability of procuring a resource is proportion to the marginal amount of surplus production per capita $du^r$ of the resource.

$$dp_j^r = K_j^r \cdot du^r \tag{205}$$

where $K_j^r > 0 : scalar$, rate at which agent *j* procures resource *r* (206)

$dp_j^r : scalar$, change in the marginal probability of procuring the resource by trade and recycling (207)

$du^r : scalar$, change in excess production per capita of resource *r* (208)

The following result relates the growth rate to resource availability.

**Theorem 4.3**: VALUE THEOREM.

$$d\lambda_j = \bar{\mathbf{V}}_j \cdot \mathbf{dp} \tag{209}$$

$$d\lambda_j = \mathbf{V}_j \cdot \mathbf{du} \tag{210}$$

*where* $d\lambda_j : scalar$, *change in the average growth factor of agent resource groups*

$$\bar{\mathbf{V}}_j = \left.\frac{\partial \lambda_j}{\partial \mathbf{p}}\right|_{\mathbf{m}} : 1 \times R, \text{ vector of marginal value coefficients wrt probability } \mathbf{p} \tag{211}$$

$$\mathbf{V}_j = \left.\frac{\partial \lambda_j}{\partial \mathbf{u}}\right|_{\mathbf{m}} : 1 \times R, \text{ vector of marginal value coefficients wrt surplus } \mathbf{u} \tag{212}$$

*where components are given by:*

$$\bar{v}_j^r = \frac{\partial \lambda_j}{\partial p_j^r} = \boldsymbol{\upsilon}_{j0} \cdot \begin{bmatrix} -1 & . & . \\ 1 & -1 & . \\ . & 1 & . \end{bmatrix}_j^r \cdot \boldsymbol{\mu}_{j0} > 0 \quad \textit{derivative of } \lambda_j \textit{ wrt probability } p_j^r \tag{213}$$

$$v_j^r = \frac{\partial \lambda_j}{\partial u^r} = \boldsymbol{\upsilon}_{j0} \cdot \begin{bmatrix} -1 & . & . \\ 1 & -1 & . \\ . & 1 & . \end{bmatrix}_j^r \cdot \boldsymbol{\mu}_{j0} \cdot K_j^r > 0 \quad \textit{derivative of } \lambda_j \textit{ wrt surplus } u^r \tag{214}$$

$\mathbf{dp}: R \times 1$, *change in the probability of acquiring additional resource in the period*

$\mathbf{du}: R \times 1$, *change in the amount of surplus production per capita of resources*

**Proof**: We apply the matrix calculus result (192) to the Leslie matrix.

First $$\frac{d\lambda_j}{dp_j^r} = \boldsymbol{\upsilon}_{j0} \cdot \frac{\mathbf{d\underline{\underline{M}}}}{dp_j^r} \cdot \boldsymbol{\mu}_{j0} \qquad \text{by (192)} \tag{215}$$

$$= \boldsymbol{\upsilon}_{j0} \left( \frac{d}{dp_j^r} \begin{bmatrix} -p_j^r & . & . \\ p_j^r & -p_j^r & . \\ . & p_j^r & . \end{bmatrix} \right) \boldsymbol{\mu}_{j0} \qquad \text{schematic depiction only} \tag{216}$$

$$= \boldsymbol{\upsilon}_{j0} \cdot \begin{bmatrix} -1 & . & . \\ 1 & -1 & . \\ . & 1 & . \end{bmatrix} \cdot \boldsymbol{\mu}_{j0} \qquad \text{differentiating. This establishes Result (209)} \tag{217}$$

$$= \left[ \Delta\upsilon_1 \quad \Delta\upsilon_2 \quad \Delta\upsilon_D \right] \cdot \boldsymbol{\mu}_{j0} \qquad \text{multiplying out } \boldsymbol{\upsilon}_{j0} \cdot \begin{bmatrix} -1 & . & . \\ 1 & -1 & . \\ . & 1 & . \end{bmatrix} \tag{218}$$

$$> 0 \quad \text{given it can be shown } \Delta\upsilon > 0 \text{ when resource gained (see Remark below)} \tag{219}$$

Second $$\frac{dp^r}{du^r} = K_j^r \qquad \text{from (205)} \tag{220}$$

so $$v_j^r = \frac{d\lambda_j}{dp_j^r} \cdot \frac{dp_j^r}{du^r} \qquad \text{chain rule} \tag{221}$$

$$= \boldsymbol{\upsilon}_{j0} \cdot \begin{bmatrix} -1 & . & . \\ 1 & -1 & . \\ . & 1 & . \end{bmatrix}_j^r \cdot \boldsymbol{\mu}_{j0} \cdot K_j^r \qquad \text{by (217), (220). Establishes Result (210)} \tag{222}$$

$> 0$ by (206), (219) This establishes Results (213), (214) # (223)

**Remark**: It can be shown that the LHS eigenvalue $\boldsymbol{\upsilon}$ is strictly increasing. The proof looks at the outcome of every possible path through resource space to the *E* and *L* states, to establish the dominance of a starting point having one more unit of resource. It is straightforward but rather long and has not been included here. Available from the author on request.

**Definition 4.1** VALUE MATRICES. We compile these row vectors $\mathbf{V}_j$ into matrix $\mathbf{V}^{SYS}$ which contains a row for every agent in the system.

$$\mathbf{V}^{SYS} = \left.\frac{\partial \boldsymbol{\lambda}}{\partial \mathbf{s}}\right|_{\mathbf{m}} : J \times R \text{, system value matrix of marginal value coefficients} \qquad (224)$$

so $\quad \mathbf{d}\boldsymbol{\lambda}^{SYS} = \mathbf{V}^{SYS} \cdot \mathbf{du} : J \times 1$, from (210), change in growth factor for each agent (225)

Value matrix $\mathbf{V}^{SYS}(\mathbf{u})$ is a function of per capita surplus. It is positive as shown, non-linear and decreasing in **u**, because resource shortages become increasingly critical for an agent as supply diminishes. Where an agent produces a resource, the corresponding growth coefficient is approximately zero because the agent has an abundance of that resource. Such entries can be taken as zero for the purposes of the dynamic analysis. Further, the $R - S$ columns for abundant resources are approximately zero for all agents, and are taken as zero.

*Input-output matrix*: Fluctuations in the amount of net production per capita **u** are in turn caused by fluctuations in the relative proportions of the agents. We account for endowed resources as part of this by adding endowed resource production to $\overset{NET}{\mathbf{A}}$.

Define $\quad \mathbf{A}^{SYS}(N) = \overset{NET}{\mathbf{A}} + \overset{END}{\mathbf{B}} : R \times J$, input-output matrix including endowed resources (226)

where $\quad \mathbf{A}^{SYS}(N)$ denotes that $\mathbf{A}^{SYS}$ is a function of population *N*, and write (202) as:

$$\mathbf{U} = \mathbf{A}^{SYS}\mathbf{m} \qquad (227)$$

so $\quad \mathbf{u} = \mathbf{A}^{SYS}\boldsymbol{\omega} \quad$ dividing (227) by *N* (228) where $\boldsymbol{\omega} = \frac{\mathbf{m}}{N} : J \times 1$ proportion of agents of each type (229)

and $\mathbf{du} = \mathbf{A}^{SYS} \cdot \mathbf{d\omega}^{SYS}$ differentiating (228) (230)

where $\mathbf{du} : R \times 1$ surplus fluctuations (231)

$\mathbf{d\omega}^{SYS} : J \times 1$ vector of fluctuations in agent proportions (232)

Note $\boldsymbol{\iota}'_{1\times J} \cdot \mathbf{d\omega}^{SYS} = 0$ sum of fluctuations is zero (233)

The input-output matrix $\mathbf{A}^{SYS}\left(N\right)$ is linear in the fluctuations in proportion $\mathbf{d\omega}^{SYS}$.

*Growth matrix*: Combining the results above allows us to relate growth rates to agent proportions:

$$\mathbf{d\lambda}^{SYS} = \mathbf{V}^{SYS}\left(\mathbf{A}^{SYS} \cdot \mathbf{d\omega}^{SYS}\right) = \mathbf{K}^{SYS} \cdot \mathbf{d\omega}^{SYS} \text{ by (225), (230)} \tag{234}$$

where $\mathbf{K}^{SYS} = \mathbf{V}^{SYS}\mathbf{A}^{SYS} = \frac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{\omega}} : J \times J$, growth matrix (235)

Growth matrix $\mathbf{K}^{SYS}\left(\mathbf{u}, N\right)$ inherits the non-linearity of the value matrix $\mathbf{V}^{SYS}\left(\mathbf{u}\right)$, with its value varying according to which resources are scarce at point **m**.

*Dynamic model:*

The model at (234) implies the following underlying model:

$$\boldsymbol{\lambda} = \mathbf{W}\left(\mathbf{u}\right) : J \times 1, \text{ growth function. } \mathbf{W} \text{ is concave downwards in } \mathbf{u} \tag{236}$$

$$\mathbf{u} = \mathbf{A}^{SYS}\left(N\right) \cdot \boldsymbol{\omega}^{SYS} : R \times 1 = R \times J \cdot J \times 1, \text{ resource surplus} \tag{237}$$

*where* $\frac{\mathbf{dW}\left(\mathbf{u}\right)}{\mathbf{du}} = \mathbf{V}^{SYS} : J \times 1 / R \times 1 = J \times R$ (238)

We derive the dynamic model in terms of agent types.

**Lemma 4.4**.

$$\mathbf{d\lambda}^{SYS} = \mathbf{V}^{SYS}\mathbf{A}^{SYS}\,\mathbf{d\omega}^{SYS} - \mathbf{V}^{SYS}\mathbf{L} \cdot \frac{dN}{N^2} \tag{239}$$

**Proof**: We evaluate the derivative of model (236), (237).

$$\frac{\mathbf{d}\boldsymbol{\lambda}^{SYS}}{\mathbf{dm}} = \frac{\mathbf{d}\boldsymbol{\lambda}^{SYS}}{\mathbf{du}} \cdot \frac{\mathbf{du}}{\mathbf{dm}} \qquad \text{chain rule} \qquad (240)$$

$$= \mathbf{V}^{SYS}\left(\frac{\mathbf{du}}{\mathbf{d}\boldsymbol{\omega}^{SYS}} \cdot \frac{\mathbf{d}\boldsymbol{\omega}^{SYS}}{\mathbf{dm}} + \frac{\mathbf{du}}{dN} \cdot \frac{dN}{\mathbf{dm}}\right) \qquad \text{by (238), (237), product rule} \qquad (241)$$

Now $\mathbf{A}^{SYS} = \overset{NET}{\mathbf{A}} + \overset{END}{\mathbf{B}} = \overset{NET}{\mathbf{A}} + \dfrac{\mathbf{L} \cdot \boldsymbol{\iota}_{1\times J}}{N}$ resource allocated equally to agents (242)

so $\dfrac{\mathbf{du}}{dN} = \dfrac{d}{dN}\left[\mathbf{A}^{SYS}\boldsymbol{\omega}^{SYS}\right] = \dfrac{d}{dN}\left[\left(\overset{NET}{\mathbf{A}} + \dfrac{\mathbf{L} \cdot \boldsymbol{\iota}'_{1\times J}}{N}\right)\boldsymbol{\omega}^{SYS}\right]$ by (230), (242) (243)

$$= \frac{d}{dN}\left[\frac{\mathbf{L}}{N}\right] \qquad \text{by (233), } \overset{NET}{\mathbf{A}} \text{ constant with respect to } N \qquad (244)$$

$$= -\frac{\mathbf{L}}{N^2} \qquad (245)$$

so $\dfrac{\mathbf{d}\boldsymbol{\lambda}^{SYS}}{\mathbf{dm}} = \mathbf{V}^{SYS}\left(\mathbf{A}^{SYS}\dfrac{\mathbf{d}\boldsymbol{\omega}^{SYS}}{\mathbf{dm}} - \dfrac{\mathbf{L}}{N^2} \cdot \dfrac{dN}{\mathbf{dm}}\right)$ (246)

substituting into (241) using (230) for $\dfrac{\mathbf{du}}{\mathbf{d}\boldsymbol{\omega}^{SYS}}$ and using (245) for $\dfrac{\mathbf{du}}{dN}$. #

### 4.3 Resource group form

The result (239) is not useful as it stands because the product $\mathbf{V}^{SYS}\mathbf{A}^{SYS}$ has dimensions

$$J \times R \cdot R \times J = J \times J \qquad (247)$$

and in the first instance we suppose that there are more agent types than resource types, $J > R$, so the product is singular and cannot be inverted for solution purposes. We combine the agents into resource groups according to the scarce resource which they produce (if any), and carry out three steps. Define

$S \le R : scalar$, number of resource groups which are scarce at position **m** (248)

agent *j* is a *producer* of resource *r* if $a_j^{SYS\,r} > 0$ (249)

and introduce the following designations to refer to the resource groups.

*B*: (bound) denotes resource groups which are scarce at position **m**.

*N*: (newly bound) denotes producers of the most recently constrained (scarce) resource in the trajectory. This group is included in *B*. (*N* will not be used until the following section 4.4.)

*F*: (free), producers of resources which are not scarce at position **m**, or do not produce a resource at all.

We carry out three steps to convert agent based equation (239) to a version based on resource groups at (277) below.

*First step*: We assume that proportion fluctuations $\mathbf{d\omega}^{SYS}$ are generated according to resource groups. In doing this we take into account that agents can produce more than one resource.

**Assumption 4.3**. Proportion fluctuations vector $\mathbf{d\omega}^{SYS}$ is generated from fluctuations in the resource groups $\mathbf{d\omega}^{FULL}$, which applies the same fluctuation to all members of a resource group. Where an agent type produces more than one scarce resource, its population $m_j$ is apportioned between all its resource groups, so that it receives an average fluctuation. Agents which produce no scarce group are allocated to the free agent group *F*.

$$\mathbf{d\omega}^{SYS} = \mathbf{S}^{FULL}\,\mathbf{d\omega}^{FULL} \qquad \text{e.g.} \begin{bmatrix} \frac{dm_1}{N} \\ \frac{dm_2}{N} \\ \frac{dm_3}{N} \\ \frac{dm_4}{N} \\ \frac{dm_5}{N} \end{bmatrix} = \begin{bmatrix} \frac{m_1}{m^{r1}} & . & . \\ \frac{p_2^{r1}\cdot m_2}{m^{r1}} & \frac{p_2^{r2}\cdot m_2}{m^{r2}} & . \\ . & \frac{m_3}{m^{r2}} & . \\ . & . & \frac{m_4}{m^F} \\ . & . & \frac{m_5}{m^F} \end{bmatrix} \begin{bmatrix} \frac{dm^{r1}}{N} \\ \frac{dm^{r2}}{N} \\ \frac{dm^F}{N} \end{bmatrix} \tag{250}$$

where $p_j^r : scalar$, proportion of the number $m_j$ of agent type *j* allocated to resource *r* (251)

If agent is allocated to only one resource group, $p_j^r = 1$ (252)

$\sum_r p_j^r = 1$, sum of the portions of $m_j$ allocated to resource groups is unity (253)

$$m^r = \sum_{j \in r} p_j^r m_j : scalar \text{, notional population of agents which produce resource } r \quad (254)$$

$$m^F : scalar \text{, size of free agent group } F \text{ for agents producing no scarce resources} \quad (255)$$

$$\mathbf{d\omega}^{FULL} : S+1 \times 1 \text{, resource-grouped vector of fluctuations in agent proportions} \quad (256)$$

$$\mathbf{S}^{FULL} : J \times S+1 \text{, weighting matrix for } S \text{ scarce resource categories plus one} \quad (257)$$

Note $\boldsymbol{\iota}'_{1\times S+1} \cdot \mathbf{d\omega}^{FULL} = 0$ : sum of fluctuations is zero: multiply (250) on LHS by $\boldsymbol{\iota}'_{1\times J}$ . (258)

Similarly $\boldsymbol{\omega}^{SYS} = \mathbf{S}^{FULL}\, \boldsymbol{\omega}^{FULL}$ as (250) only not differentiated. (259)

**Remark**: In the above example, agent 2 produces both scarce resources and is apportioned to both resource groups according to some rule, relative importance to the group say.

*Second step*: A scarce resource version of the input-output matrix $\mathbf{A}^{SYS}$ is created by grouping the vectors into resource production groups using the same matrix $\mathbf{S}^{FULL}$ .

so $$\mathbf{A}^{FULL} = \mathbf{A}^{SYS} \cdot \mathbf{S}^{FULL} = \mathbf{A}^{SYS} \cdot \begin{bmatrix} \frac{m_1}{m^{r1}} & . & . \\ \frac{p_2^{r1} \cdot m_2}{m^{r1}} & \frac{p_2^{r2} \cdot m_2}{m^{r2}} & . \\ . & \frac{m_3}{m^{r2}} & . \\ . & . & \frac{m_4}{m^F} \\ . & . & \frac{m_5}{m^F} \end{bmatrix}$$ : using same example (260)

where $\mathbf{A}^{FULL} : R \times S+1$, resource group version of matrix $\mathbf{A}^{SYS}$ . (261)

Application of (260) produces summed resource production vectors:

$$\mathbf{a}^{SYS\,r} = \frac{\sum_{j \in r} \mathbf{a}_j^{SYS}\; p_j^r m_j}{\sum_{j \in r} p_j^r m_j} : R \times 1 \text{ average production of agents allocated to resource } r \quad (262)$$

We see that where an agent type produces more than one scarce good, the net production vector for that agent type is apportioned between the different resource vectors.

*Unique producer of more than one output*: If an agent type produces more than one good, and it is the only agent type producing those goods, then we apportion the net production vector between the different resources as usual. Typically, one good would be scarce and the other good would be produced in excess throughout agent space **m**. However, if this is not the case and agents place a similar marginal value on the different resources, it is possible that the final growth matrix **K** has two columns which are similar or the same. This implies that matrix **K** is singular or poorly conditioned and the system cannot be solved. Such a situation can arise if the goods are used together and one is necessary for the other. In this case we can treat the combination as one resource, and remove one of the resources from the system.

**Remark**: The definition here differs from the normal definition of an input-output matrix $\mathbf{A}^{USUAL}_{R\times R}$ in that:

- A standard input-output matrix $\mathbf{A}^{USUAL}$ shows the inputs for one unit of output as positives. Here the net production of each resource is shown. Inputs are negative, net output is positive and is not in general equal to one.

  $$\mathbf{A}^{FULL} = \mathbf{I} - \mathbf{A}^{USUAL} \tag{263}$$

- $\mathbf{A}^{FULL}$ is not in general a square matrix, and each column represents different producers grouped together.
- An agent can produce more than one scarce resource, or none. There is a separate column for agents which do not produce a scarce resource.
- Unlike the net production matrix $\overset{NET}{\mathbf{A}}$, input-output matrix $\mathbf{A}^{FULL}$ incorporates the endowed resource input of $\dfrac{\mathbf{L}}{N}$ per agent valued at position **m**: $N = \boldsymbol{\iota}'\mathbf{m}$.

*Third step*: We define a grouping matrix $\mathbf{S}^{SCARCE}$ for the agent growth differentials $d\lambda_j$ and the value matrix $\mathbf{V}^{SYS}$. Before we do this, we establish how group growth rates relate to the growth rates of the constituents.

**Assumption 4.4** The agents producing resource *r* have different growth rates and processes, but these attributes are such that average production does not vary appreciably from one period to the next and can be taken as constant.

Define $$a^r = \frac{\sum_{j\in r} a_j^r m_{j0}}{\sum_{j\in r} m_{j0}} : scalar \text{ : average production in period 0} \tag{264}$$

then $$a^r = \frac{\sum_{j\in r} a_j^r m_{j1}}{\sum_{j\in r} m_{j1}} = \frac{\sum_r a_j^r \cdot \lambda_j m_{j0}}{\sum_r \lambda_j m_{j0}} \text{ : average production in period 1} \tag{265}$$

**Remark**: This assumption will not be literally true but the necessary adjustments in each period are assumed to be second order.

We use this assumption to demonstrate that the average growth rate of a resource group can be taken as a proxy for the growth in resource production of that group, notwithstanding the averaging involved:

**Lemma 4.5**. *The growth rate of resource production is proportional to the average growth rate of the agents in its resource group.*

$$\sum_r a_j^r \cdot \lambda_j m_j = \lambda^r \cdot a^r \cdot m^r \tag{266}$$

*where* $$\lambda^r = \frac{\sum_r \lambda_j m_j}{\sum_r m_j} : scalar \text{ , } \textit{average rate of growth of producers of resource r} \tag{267}$$

$$m^r = \sum_{j \; for \; r} m_j : scalar \text{ , } \textit{total number of agents producing resource r} \tag{268}$$

**Proof**: We rearrange the assumption above.

$$\sum_r a_j^r \lambda_j m_j = \sum_r \lambda_j m_j \cdot \frac{\sum_r a_j^r m_j}{\sum_r m_j} \qquad \text{by (264),(265)} \qquad (269)$$

$$= \frac{\sum_r \lambda_j m_j}{\sum_r m_j} \cdot \frac{\sum_r a_j^r m_j}{\sum_r m_j} \cdot \sum_r m_j \qquad \text{result follows by (267),(264),(268)} \quad \# \quad (270)$$

We construct weighting matrix $\mathbf{S}^{SCARCE}$ with this result in mind. $\mathbf{S}^{SCARCE}$ differs to $\mathbf{S}^{FULL}$ in that position $m_j$ of an agent which produces more than one scarce resource is not proportioned between the different resources, but is allocated in full to all of them. It is also transposed.

$$\mathbf{S}^{SCARCE} = \begin{bmatrix} \frac{m_1}{m^{r1}} & \frac{m_2}{m^{r1}} & . & . & . \\ . & \frac{m_2}{m^{r2}} & \frac{m_3}{m^{r2}} & . & . \\ . & . & . & \frac{m_4}{m^F} & \frac{m_5}{m^F} \end{bmatrix} : S+1 \times J \text{ , using same example} \qquad (271)$$

Application of $\mathbf{S}^{SCARCE}$ to agent growth differentials $\mathbf{d\lambda}^{SYS}$ produces resource growth differentials $\mathbf{d\lambda}^{FULL}$:

$$d\lambda^r = \frac{\sum_{j \in r} d\lambda_j \cdot m_j}{\sum_{j \in r} m_j} \qquad \text{differentiating (267)} \qquad (272)$$

so $\quad \mathbf{d\lambda}^{FULL} = \mathbf{S}^{SCARCE} \mathbf{d\lambda}^{SYS} : S+1 \times 1 \qquad$ resource-grouped growth rate vector $\qquad$ (273)

$\qquad = \mathbf{S}^{SCARCE} \cdot \mathbf{V}^{SYS} \cdot \mathbf{du} \qquad$ substituting (225) $\qquad$ (274)

$\qquad = \mathbf{V}^{FULL} \cdot \mathbf{du} \qquad$ using (276) below $\qquad$ (275)

where $\quad \mathbf{V}^{FULL} = \mathbf{S}^{SCARCE} \mathbf{V}^{SYS} : S+1 \times R \qquad$ resource-grouped value matrix $\qquad$ (276)

Having carried out the three steps, we now express the dynamic equation (239) in terms of resource groups to get an operational version of the model.

**Theorem 4.6**: DYNAMIC MODEL THEOREM.

$$\mathbf{d\lambda}^{FULL} = \mathbf{V}^{FULL}\mathbf{A}^{FULL}\,\mathbf{d\omega}^{FULL} - \ \mathbf{V}^{FULL}\mathbf{L}\cdot\frac{dN}{N^2} \tag{277}$$

**Proof**: We transform the previous result (239) by breaking $\mathbf{S}^{FULL}$ out of $\mathbf{d\omega}^{SYS}$, and then combining it with $\mathbf{A}^{SYS}$ to get $\mathbf{A}^{FULL}$. Further we multiply by $\mathbf{S}^{SCARCE}$ on the LHS.

$$\mathbf{S}^{SCARCE}\mathbf{d\lambda}^{SYS} = \mathbf{S}^{SCARCE}\mathbf{V}^{SYS}\mathbf{A}^{SYS}\left(\mathbf{S}^{FULL}\mathbf{d\omega}^{FULL}\right) - \ \mathbf{S}^{SCARCE}\mathbf{V}^{SYS}\mathbf{L}\cdot\frac{dN}{N^2} \tag{278}$$

by (239), (250). Apply (273), (276), (260) for result. #

Having derived resource grouped matrices $\mathbf{V}^{FULL}, \mathbf{A}^{FULL}$, we now consider their properties.

*Value matrix*: Partition $\mathbf{V}^{FULL}$:

$$\mathbf{V}^{FULL} = \begin{bmatrix} \mathbf{V}_B & \mathbf{0}_{S\times R-S} \\ \mathbf{v}_F & \mathbf{0}_{1\times R-S} \end{bmatrix} : \begin{bmatrix} S\times S & S\times R-S \\ 1\times S & 1\times R-S \end{bmatrix} : S+1\times R \text{, bound } B\text{, free } F \text{ agents} \tag{279}$$

$$\text{e.g. } \mathbf{V}^{FULL} = \begin{array}{c} \\ B \\ B \\ B \\ F \end{array} \begin{array}{c} \begin{array}{ccccc} S1 & S2 & S3 & R4 & R5 \end{array} \\ \left[\begin{array}{ccc|cc} 0 & + & + & 0 & 0 \\ + & 0 & + & 0 & 0 \\ + & + & 0 & 0 & 0 \\ \hline + & + & + & 0 & 0 \end{array}\right] \end{array} \quad \text{typically.} \tag{280}$$

The partition $\mathbf{V}_B$ applies to the scarce resources. If a price system were in operation then the values would equal the prices, since any incremental resource could be bought or sold at the price. In this case we would have:

$$\mathbf{V}_B = \begin{bmatrix} 0 & P_2 & P_S \\ P_1 & 0 & P_S \\ P_1 & P_2 & 0 \end{bmatrix} \tag{281}$$

and $$\mathbf{V}_B^{-1} = \begin{bmatrix} -\frac{S-2}{P_1(S-1)} & \frac{1}{P_1(S-1)} & \frac{1}{P_1(S-1)} \\ \frac{1}{P_2(S-1)} & -\frac{S-2}{P_2(S-1)} & \frac{1}{P_2(S-1)} \\ \frac{1}{P_S(S-1)} & \frac{1}{P_S(S-1)} & -\frac{S-2}{P_S(S-1)} \end{bmatrix} \tag{282}$$

so $$\mathbf{V}_B^{-1}\boldsymbol{\iota}_{S\times 1} = \begin{bmatrix} \dfrac{1}{P_1(S-1)} \\ \dfrac{1}{P_2(S-1)} \\ \dfrac{1}{P_S(S-1)} \end{bmatrix} > \mathbf{0}_{S\times 1} \text{ i.e. the sum of each row is positive.} \tag{283}$$

On this basis we assume that the value matrix has the same general properties as a price matrix:

**Assumption 4.5** PRICELIKE MATRICES. The scarce resources are of sufficiently similar value to the agents that the value matrix $\mathbf{V}_B(\mathbf{u})$ satisfies the properties

(i) $\mathbf{V}_B$ is invertible

(ii) $\mathbf{V}_B^{-1}\boldsymbol{\iota}_{S\times 1} > 0$ (284)

*Input-output matrix*: We partition $\mathbf{A}^{FULL}$ horizontally between $S$ scarce resources and $R-S$ abundant resources, and vertically between the $S$ bound agent groups producing the scarce resources and the remaining free agent group which does not produce a constrained resource.

$$\mathbf{A}^{FULL} = \begin{bmatrix} \mathbf{A}_{BB}^{FULL} & \mathbf{A}_{BF}^{FULL} \\ \mathbf{A}_{FB}^{FULL} & \mathbf{A}_{FF}^{FULL} \end{bmatrix} : \begin{bmatrix} S\times S & S\times 1 \\ R-S\times S & R-S\times 1 \end{bmatrix} : R\times S+1 \text{, input-output matrix} \tag{285}$$

$$\text{e.g. } \mathbf{A}^{FULL} = \begin{array}{c} \\ S1 \\ S2 \\ S3 \\ R4 \\ R5 \end{array} \begin{array}{c} \begin{array}{cccc} B & B & B & F \end{array} \\ \left[\begin{array}{ccc|c} + & - & - & - \\ - & + & - & - \\ - & - & + & - \\ \hline - & - & - & + \\ - & - & - & + \end{array}\right] \end{array} \text{typically.} \tag{286}$$

*Growth matrix*: We form the growth matrix $\mathbf{K}^{FULL}$ as the product of the value and input-output matrices as per (235).

$$\mathbf{K}^{FULL} = \frac{\partial \boldsymbol{\lambda}^{FULL}}{\partial \boldsymbol{\omega}^{FULL}} = \mathbf{V}^{FULL}\mathbf{A}^{FULL} : S+1\times S+1 \text{, resource grouped growth matrix} \tag{287}$$

We partition $\mathbf{K}^{FULL}$:

$$\mathbf{K}^{FULL}=\begin{bmatrix}\mathbf{K}_{BB}^{FULL} & \mathbf{K}_{BF}^{FULL}\\ \mathbf{k}_{FB}^{FULL} & \mathbf{k}_{FF}^{FULL}\end{bmatrix}:\begin{bmatrix}S\times S & S\times 1\\ 1\times S & 1\times 1\end{bmatrix}:S+1\times S+1 \tag{288}$$

$$=\begin{bmatrix}\mathbf{V}_B & \mathbf{0}_{S\times R-S}\\ \mathbf{v}_F & \mathbf{0}_{1\times R-S}\end{bmatrix}\begin{bmatrix}\mathbf{A}_{BB}^{FULL} & \mathbf{A}_{BF}^{FULL}\\ \mathbf{A}_{FB}^{FULL} & \mathbf{A}_{FF}^{FULL}\end{bmatrix}=\begin{bmatrix}\mathbf{V}_B\mathbf{A}_{BB}^{FULL} & \mathbf{V}_B\mathbf{A}_{BF}^{FULL}\\ \mathbf{v}_F\mathbf{A}_{FB}^{FULL} & \mathbf{v}_F\mathbf{A}_{FF}^{FULL}\end{bmatrix} \tag{289}$$

$$\text{e.g. }\mathbf{K}^{FULL}\quad=\begin{array}{c}\begin{array}{cccc}B & B & B & F\end{array}\\ \left[\begin{array}{ccc|c}- & + & + & -\\ + & - & + & -\\ + & + & - & -\\ \hline + & + & + & -\end{array}\right]\end{array}\begin{array}{l}B\\ B\text{ typically.}\\ B\\ F\end{array} \tag{290}$$

The fluctuations vector $\mathbf{d\omega}^{FULL}$ is linearly dependent since it sums to zero by (258). To solve the system we isolate an independent component $\mathbf{d\omega}_B^{FULL}$

$$\mathbf{d\omega}^{FULL}=\begin{bmatrix}\mathbf{d\omega}_B^{FULL}\\ d\omega_F^{FULL}\end{bmatrix}:\begin{bmatrix}S\times 1\\ 1\times 1\end{bmatrix}:S+1\times 1 \tag{291}$$

where $d\omega_F^{FULL}=-\boldsymbol{\iota}_{1\times S}'\,\mathbf{d\omega}_B^{FULL}:\mathit{scalar}$ by (258) (292)

and re-express $\mathbf{K}^{FULL}\mathbf{d\omega}^{FULL}$ in terms of independent fluctuation $\mathbf{d\omega}_B^{FULL}$ only.

**Lemma 4.7**.

$$\mathbf{K}^{FULL}\mathbf{d\omega}^{FULL}=\begin{bmatrix}\mathbf{V}_B\mathbf{A}_B\\ \mathbf{v}_F\mathbf{a}_F\end{bmatrix}\mathbf{d\omega}=\begin{bmatrix}\mathbf{K}_B\\ \mathbf{k}_F\end{bmatrix}\mathbf{d\omega}=\mathbf{K}\,\mathbf{d\omega} \tag{293}$$

$$\textit{where}\quad \mathbf{A}=\begin{bmatrix}\mathbf{A}_B\\ \mathbf{a}_F\end{bmatrix}=\begin{bmatrix}\mathbf{A}_{BB}^{FULL}-\mathbf{A}_{BF}^{FULL}\cdot\boldsymbol{\iota}_{1\times S}'\\ \mathbf{A}_{FB}^{FULL}-\mathbf{A}_{FF}^{FULL}\cdot\boldsymbol{\iota}_{1\times S}'\end{bmatrix}:\begin{bmatrix}S\times S\\ R-S\times S\end{bmatrix}:R\times S \tag{294}$$

*modified input-output matrix, partitioned into bound and free agents*

$$\mathbf{K}=\begin{bmatrix}\mathbf{K}_B\\ \mathbf{k}_F\end{bmatrix}=\begin{bmatrix}\mathbf{K}_{BB}^{FULL}-\mathbf{K}_{BF}^{FULL}\cdot\boldsymbol{\iota}_{1\times S}'\\ \mathbf{k}_{BF}^{FULL}-\mathbf{k}_{FF}^{FULL}\cdot\boldsymbol{\iota}_{1\times S}'\end{bmatrix}:\begin{bmatrix}S\times S\\ 1\times S\end{bmatrix}:S+1\times S \tag{295}$$

*modified grouped growth matrix, partitioned into bound and free agents*

$$\mathbf{d\omega}=\mathbf{d\omega}_B^{FULL}:S\times 1,\ \textit{re-notating fluctuation vector for the bound agents} \tag{296}$$

**Proof**: We transform $\mathbf{K}^{FULL}\mathbf{d\omega}^{FULL}$ by partitioning it, substituting for the fluctuation $d\omega_F^{FULL}$ using (292), and simplifying.

$$\mathbf{K}^{FULL}\mathbf{d\omega}^{FULL} = \begin{bmatrix} \mathbf{V}_B\mathbf{A}_{BB}^{FULL} & \mathbf{V}_B\mathbf{A}_{BF}^{FULL} \\ \mathbf{v}_F\mathbf{A}_{FB}^{FULL} & \mathbf{v}_F\mathbf{A}_{FF}^{FULL} \end{bmatrix} \begin{bmatrix} \mathbf{d\omega}_B^{FULL} \\ d\omega_F^{FULL} \end{bmatrix} \quad \text{by (289), (291)} \tag{297}$$

$$= \begin{bmatrix} \mathbf{V}_B\mathbf{A}_{BB}^{FULL} & \mathbf{V}_B\mathbf{A}_{BF}^{FULL} \\ \mathbf{v}_F\mathbf{A}_{FB}^{FULL} & \mathbf{v}_F\mathbf{A}_{FF}^{FULL} \end{bmatrix} \begin{bmatrix} \mathbf{d\omega}_B^{FULL} \\ -\boldsymbol{\iota}'\mathbf{d\omega}_B^{FULL} \end{bmatrix} \quad \text{by (292)} \tag{298}$$

$$= \begin{bmatrix} \mathbf{V}_B\left(\mathbf{A}_{BB}^{FULL} - \mathbf{A}_{BF}^{FULL}\boldsymbol{\iota}'\right) \\ \mathbf{v}_F\left(\mathbf{A}_{FB}^{FULL} - \mathbf{A}_{FF}^{FULL}\boldsymbol{\iota}'\right) \end{bmatrix} \mathbf{d\omega} \quad \text{making use of (296). Similarly for } \mathbf{K}. \qquad \# \tag{299}$$

Applying result (293) to dynamic equation (277) gives:

$$\mathbf{d\lambda}^{FULL} = \mathbf{K}\,\mathbf{d\omega} - \mathbf{V}^{FULL}\mathbf{L}\cdot\frac{dN}{N^2} \tag{300}$$

## 4.4 Lifecycle analysis

### *4.4.0 Introduction: why produce a surplus?*

We use the tools developed in the preceding section to examine the lifecycle of the system from commencement to maturity. The analysis is heuristic in that it is looking at the forces acting on the system in each situation rather than deriving a system trajectory from first principles.

Suppose that a system of differentiated agents commences from near the origin. The nature of agent surplus in a non-market economy with non-reciprocal trading is that agents will not produce a massive surplus if they are not obliged to. Pursuant to the prisoner’s dilemma, it pays each individual to produce the minimum surplus consistent with its own propagation, and this surplus may be zero. Reasons why non-zero surpluses occur include:

*Group selection*: If agents subsist in isolated groups and resources are not produced for all, then the group will go extinct. Groups which do produce sufficient will thrive, forming new groups.

*Physical impossibility of regulation*: Some chemical processes may be incapable of regulation, or the agent has not evolved a way of doing it, or the resource is co-produced with another. The best example is the creation of oxygen as a by-product during photosynthesis. The plant wants the sugar but does not have an immediate need for the oxygen.

*Coping with lean times*: Although it is not modelled here, variation in natural conditions can require that agents create a buffer of resources to use in poor times such as winter or droughts.

*Parasitism*: Parasitism is not explicitly modelled here, but the free group could be regarded in that light. If agents are subject to parasitism, which may or may not be present in a given situation, then they need to produce an excess against that eventuality.

*Cooperation*: A group of agents can make allowances for each other's needs – in effect, reciprocal trading – for the production of resources. They may even be symbiotic and live in immediate proximity to swap resources. This explanation has been developed extensively within cooperative game theory; it requires an ongoing relationship between particular agents, which we do not assume here.

*Inter-dependent utility*: Agents may take pleasure in the welfare of other agents. This applies to pre-market human systems, family and friend groups in market societies, and possibly also to animal family units.

Whatever the reasons, it is an observable fact that natural and anthropological systems exhibit producers who create more resources than they require for their own immediate needs. Nonetheless, we expect that the feasible set would be relatively narrow because of the tendency for agents to minimise their surplus. Given this and differential growth rates, we expect the trajectory in agent space to hit the manufactured good constraints (which are rays from the origin) as the population expands, rather than move straight to the endowed resource constraints (which are transverse).

### *4.4.1 Starting point*

**Assumption 4.6** STARTING POINT. It is assumed that:

- The system starts from some interior point near the origin in agent space
- Not all agents have the same growth rate.
- For the reasons given above, ray constraints (for manufactured goods) are encountered before transverse constraints (for endowed resources).
- The agent's distribution eigenvector $\boldsymbol{\mu}$ applies, but not the system eigenvector $\boldsymbol{\mu}^{SYS}$.

**Assumption 4.7** When a manufactured resource first becomes constrained then the agent types producing that resource (*N* group) have on average a lower growth rate than the average growth rate for the population.

$$\lambda_{N0} < \Gamma_0 \tag{301}$$

**Remark**: This assumption does not apply to endowed goods.

**Remark**: If the agents producing the manufactured good had the same rate of expansion as the other agents then they would maintain their initial proportion of the population, and the situation whereby the good is abundant would be unlikely to alter. Only random variation early in system history would cause this assumption to be violated.

### *4.4.2 Growth along a ray*

Consider a point on a ray forming the boundary to the feasible set. Here at least one manufactured resource is scarce; endowed resources are scarce along the transverse frontier and abundant here. Unlike the situation at the interior points, the expansion of the agents is now constrained by resource supply.

Recall $\quad \Delta\mathbf{R} = \overset{NET}{\mathbf{A}}\,\mathbf{m} + \overset{END}{\mathbf{B}}\,\mathbf{m} + \overset{EXCESS}{\mathbf{B}}\,\mathbf{m} \qquad$ by (164) above $\qquad$ (302)

Now $\quad \mathbf{\Delta R} = \sum_j \mathbf{\Delta R}_j = \sum_j \lambda_j \cdot \mathbf{R}_j$ by (199) (303)

and $\quad \overset{NET}{\mathbf{A}}\,\mathbf{m} + \overset{END}{\mathbf{B}}\,\mathbf{m} + \overset{EXCESS}{\mathbf{B}}\,\mathbf{m} = \sum_j \left( \overset{NET}{\mathbf{a}}_j + \overset{END}{\mathbf{b}}_j + \overset{EXCESS}{\mathbf{b}}_j \right) m_j$ breaking it by agent (304)

so $\quad \sum_j \Delta\lambda_j \cdot \mathbf{R}_j = \sum_j \left( \overset{NET}{\mathbf{a}}_j + \overset{END}{\mathbf{b}}_j + \overset{EXCESS}{\mathbf{b}}_j \right) m_j = \mathbf{0}$ from (302), (303), (304) (305)

i.e. $\quad \sum_j \left( \overset{NET}{\mathbf{a}}_j + \overset{END}{\mathbf{b}}_j + \overset{EXCESS}{\mathbf{b}}_j - \Delta\lambda_{j0} \cdot \mathbf{R}_{j0} \right) m_j = \mathbf{0}$ (306)

We focus on the $S$ scarce resource groups. The production of endowed resource, $\overset{END}{\mathbf{b}}_j$, can be ignored because endowed goods are not scarce on a ray. Define production vector $\boldsymbol{\alpha}_j$:

$$\boldsymbol{\alpha}_j = \overset{NET}{\mathbf{a}}_j + \overset{EXCESS}{\mathbf{b}}_{j0} - \Delta\lambda_j \cdot \mathbf{R}_j : R \times 1, \text{ use of the resource by agent } j \tag{307}$$

The next assumption and lemma follow the equivalent results for net production above.

**Assumption 4.8** Average production in a resource group does not vary appreciably from one period to the next and can be taken as constant.

Define $\quad \boldsymbol{\alpha}^r = \dfrac{\sum_{j \in r} \boldsymbol{\alpha}_j m_j}{\sum_{j \in r} m_j} : R \times 1$ : vector of average production of resources (308)

then $\quad \boldsymbol{\alpha}^r = \dfrac{\sum_{j \in r} \boldsymbol{\alpha}_j \cdot \lambda_j m_j}{\sum_{j \in r} \lambda_j m_j}$ (309)

**Lemma 4.8**. *The growth rate of resource production in a resource group is proportional to the average growth rate of the agents.*

$$\sum_{j \in r} \boldsymbol{\alpha}_j \lambda_j m_j = \lambda^r \cdot \boldsymbol{\alpha}^r \cdot m^r \tag{310}$$

**Proof**: As for (266).

The following result underpins the dynamic analysis.

**Theorem 4.9**: RAY GROWTH THEOREM. *All resource groups, i.e. the scarce resource groups and the free agent group, have the same growth factor.*

$$\boldsymbol{\lambda} = \Gamma \cdot \boldsymbol{\iota} \tag{311}$$

*where* $\boldsymbol{\lambda} : S+1 \times 1$, *vector of growth factors for each resource group*

$\Gamma :$ *scalar , common growth rate*

$\boldsymbol{\iota} : S+1 \times 1$, *vector of unity elements*

**Proof**: We reorganise the system of production equations into form $\mathbf{Ax} = \mathbf{0}$ and solve.

Consider resource equation (306) divided up by resource producers:

$$\sum_r \sum_{j \in r} \left( \overset{NET}{\mathbf{a}_j} + \overset{EXCESS}{\mathbf{b}_{j0}} - \Delta\lambda_{j0} \cdot \mathbf{R}_{j0} \right) m_j + \sum_{j \in F} \left( \overset{NET}{\mathbf{a}_j} + \overset{EXCESS}{\mathbf{b}_{j0}} - \Delta\lambda_{j0} \cdot \mathbf{R}_{j0} \right) m_j = \mathbf{0} \tag{312}$$

i.e. $$\sum_r \sum_{j \in r} \boldsymbol{\alpha}_j m_j + \sum_{j \in F} \boldsymbol{\alpha}_j m_j = \mathbf{0} \quad \text{by (307)} \tag{313}$$

or $$\sum_r \boldsymbol{\alpha}^r \cdot m^r + \boldsymbol{\alpha}^F \cdot m^F = \mathbf{0} \quad \text{by (308)} \tag{314}$$

In the next period the equation becomes:

$$\sum_r \sum_{j \in r} \boldsymbol{\alpha}_j \lambda_j m_j + \sum_{j \in F} \boldsymbol{\alpha}_j \lambda_j m_j = \mathbf{0} \quad \text{applying agent growth rates} \tag{315}$$

$$\sum_r \lambda^r \cdot \boldsymbol{\alpha}^r \cdot m^r + \lambda^F \cdot \boldsymbol{\alpha}^F \cdot m^F = \mathbf{0} \quad \text{by (310)} \tag{316}$$

Define $\boldsymbol{\beta}^r = \boldsymbol{\alpha}^r \cdot m^r : R \times 1$, net use of resources by group *r*. Similarly $\boldsymbol{\beta}^F$ (317)

to get $$\sum_r \boldsymbol{\beta}^r + \boldsymbol{\beta}^F = \mathbf{0} \quad \text{by (314)} \tag{318}$$

$$\sum_r \lambda^r \cdot \boldsymbol{\beta}^r + \lambda^F \cdot \boldsymbol{\beta}^F = \mathbf{0} \quad \text{by (316)} \tag{319}$$

so $$\sum_r \left( \lambda^r - \lambda^F \right) \cdot \boldsymbol{\beta}^r = \mathbf{0} \quad \text{substituting for } \boldsymbol{\beta}^F \text{ from (318) into (319)} \tag{320}$$

A steady state growth path implies that stock absorption $-\Delta\lambda_j \cdot \mathbf{R}_j$ and shedding rates $\overset{SHED}{\mathbf{b}}_j$ for the scarce resources stay constant. Under these conditions $\boldsymbol{\beta}^r$ values are constant. We replace $\boldsymbol{\beta}^r$ vectors by $\boldsymbol{\beta}^{rs}$ vectors:

$\boldsymbol{\beta}^{rs} : S \times 1$ vector containing the only those rows of $\boldsymbol{\beta}^r$ which pertain to the S scarce resources. Similarly $\boldsymbol{\beta}^{Fs}$.

so $\quad \mathrm{B}\left(\boldsymbol{\lambda}^{rs} - \boldsymbol{\lambda}^{Fs}\right) = \mathbf{0}$ rearranging (320) in matrix form for scarce resources (321)

where $\quad \mathrm{B} = \left[\boldsymbol{\beta}^{r1s} \quad \boldsymbol{\beta}^{r2s} \quad \boldsymbol{\beta}^{Ss}\right] : S \times S$ matrix of use vector coefficients (322)

$$\boldsymbol{\lambda}^r = \begin{bmatrix} \lambda^{r1} \\ \lambda^{r2} \\ \lambda^{S} \end{bmatrix} : S \times 1 \text{ , } \boldsymbol{\lambda}^F = \begin{bmatrix} \lambda^{F} \\ \lambda^{F} \\ \lambda^{F} \end{bmatrix} : S \times 1 \tag{323}$$

we see that trivial solution solves (321), i.e. $\boldsymbol{\lambda}^r = \boldsymbol{\lambda}^F$ # (324)

**Remark**: There will be other solutions if B is singular, but the columns represent resource groups with, typically, large positive diagonal elements so this is unlikely. The case of one agent being the sole producer of two resources is discussed above.

**Remark**: An investigation of possibly oscillating dynamic effects due to variations in the rate of stock accumulation $\Delta\lambda_j \cdot \Delta R_j$ is beyond the scope of this study.

**Remark**: Although the average growth rates of each resource group are equal, in each resource group there can be more than one agent type. These agent types have different growth rates which average out to the resource growth rate. In each period there will be some change in the proportion of agent types because of the different growth rates. The agent types with higher growth rates will become more dominant in the group over time, and this will lift the group growth rate.

**Remark**: It is possible that the number of resource categories is only *S*, because there are no free agents. In this case the dimensions of matrix B become $S \times S - 1$. In this case the system is overdetermined, but nonetheless the trivial solution will apply and the conclusion is unchanged.

### *4.4.3 Encountering a new ray constraint.*

Suppose we have a certain number of scarce resources and a free group, all sharing the common growth rate $\Gamma_0$. It is only the average growth rate of each resource group which is equal, not the growth rates of individual agent types. We expect that some agent types within the free group grow more slowly, and this means that the resource which those agent types produce will eventually become scarce. As shown within section 3.5 Resource Shedding, the stocks of an abundant resource will press up against the upper storage bound *U*, but the stocks of a scarce resource are at minimum levels. As a stock becomes scarce, existing excess stocks will be drawn upon. When this buffer has been consumed, scarcity will hit the system in a fairly abrupt fashion.

When a new manufactured good (i.e. ray) constraint is encountered by the system, the number of scarce goods *S* increases by one to include the newly constrained resource *N*. The initial situation is that

$\boldsymbol{\lambda}_{B0} = \Gamma_0$ the previously bound agents have the common growth rate (325)

$\lambda_N < \Gamma_0$ the newly bound agents have a lower growth rate (326)

$\lambda_{F0} > \Gamma_0$ the remaining free agents have a higher growth rate (proved below). (327)

We now examine how growth rates change to restore a common growth rate as required by the Ray Growth Theorem. The growth equation for resource groups, in the situation where there is no endowed resource constraint, is:

$\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_0 + \mathbf{K}\,\mathbf{d\omega}$ by (300), $\mathbf{V}^{FULL}\mathbf{L} = \mathbf{0}_{S+1\times 1}$ as there are no endowed constraints (328)

where $\boldsymbol{\lambda}_0 : S+1\times 1$ vector of growth coefficients immediately prior to the system trajectory reaching the ray. This vector included the growth factor $\lambda_N$ of the newly constrained resource group $N$, which initially has a lower growth rate as the other bound resource groups in $B$.

$\boldsymbol{\lambda}_1 : S+1\times 1$ vector of growth coefficients after the system trajectory reaches the ray and adjusts as per Assumption 4.1 to equal growth rates (311).

The following result establishes the dynamics of resource constraints. Its implications at first glance are obscure, but we tease out those implications below.

**Theorem 4.10**: *Equilibrium rate of growth* $\Gamma$ *along a ray is given by:*

$$\Gamma_1 = \frac{\mathbf{k}_F\ \mathbf{K}_B^{-1}\ \boldsymbol{\lambda}_{B0} - \lambda_{F0}}{\mathbf{k}_F\ \mathbf{K}_B^{-1}\ \boldsymbol{\iota}_{S\times 1} - 1} \tag{329}$$

$$\mathbf{d}\boldsymbol{\omega} = \frac{\mathbf{K}_B^{-1}\left(\boldsymbol{\lambda}_{B0} - \lambda_{F0}\boldsymbol{\iota}_{S\times 1}\right)}{\mathbf{k}_F\ \mathbf{K}_B^{-1}\ \boldsymbol{\iota}_{S\times 1} - 1} \tag{330}$$

*where* $\Gamma_1$, *scalar is a common growth rate of all the B, F resource groups.*

**Proof**: Solve the system (328) as simultaneous equations.

$$\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_0 + \begin{bmatrix}\mathbf{K}_B\\ \mathbf{k}_F\end{bmatrix}\cdot \mathbf{d}\boldsymbol{\omega} \qquad \text{by (328), (293)} \tag{331}$$

$$\Gamma\begin{bmatrix}\boldsymbol{\iota}\\ 1\end{bmatrix} = \begin{bmatrix}\boldsymbol{\lambda}_{B0}\\ \lambda_{F0}\end{bmatrix} + \begin{bmatrix}\mathbf{K}_B\\ \mathbf{k}_F\end{bmatrix}\cdot \mathbf{d}\boldsymbol{\omega} \qquad \text{where } \begin{bmatrix}\boldsymbol{\lambda}_{B1}\\ \lambda_{F1}\end{bmatrix} = \Gamma_1\begin{bmatrix}\boldsymbol{\iota}_{S\times 1}\\ 1\end{bmatrix} \text{by (311)} \tag{332}$$

$$\mathbf{d}\boldsymbol{\omega} = \mathbf{K}_B^{-1}\left(\Gamma_1\boldsymbol{\iota} - \boldsymbol{\lambda}_{B0}\right) \qquad \text{rearranging top partition} \tag{333}$$

$$\Gamma_1 - \lambda_{F0} = \mathbf{k}_F \cdot \mathbf{K}_B^{-1}\left(\Gamma_1\boldsymbol{\iota} - \boldsymbol{\lambda}_{B0}\right) \quad \text{rearrange lower partition and sub using (333) \#} \tag{334}$$

**Remark**: Compare growth rate expression (329) with the solution to the simple linear system $y = m_1 x + a, \quad y = m_2 x + b$:

$$y = \frac{m_2 a - m_1 b}{m_2 - m_1} = \frac{m_2 m_1^{-1} a - b}{m_2 m_1^{-1}\cdot 1 - 1} \qquad \text{is the one-dimensional case of (329)} \tag{335}$$

$$x = \frac{a-b}{m_2 - m_1} = \frac{m_1^{-1}\left(a-b\right)}{m_2\, m_1^{-1}\cdot 1 - 1} \qquad \text{is the one-dimensional case of (330)} \tag{336}$$

We see that the denominator expression $\mathbf{k}_F\mathbf{K}_B^{-1}\boldsymbol{\iota}-1$ represents the difference in gradients between the free and bound agents, i.e. between $\frac{d\lambda_F}{\mathbf{d\omega}}$ and $\frac{d\Gamma_B}{\mathbf{d\omega}}$.

**Remark**: When the first ray constraint is encountered the above result applies unaltered. The matrices $\mathbf{k}_F, \mathbf{K}_B$ collapse to scalars and the *B* group consists entirely of the newly bound *N* group.

**Remark**: The newly scarce resource will enter for the first time into the value matrix of the bound agents $\mathbf{K}_B$. In the case where some agent type produces more than one scarce resource, some of the production of that resource will be reflected in the coefficients of the resource groups which are already scarce. This does not affect the conclusion.

*Parameter analysis*: To understand the result we look at the parameters.

**Assumption 4.9** BOUND AGENT COEFFICIENTS. Consider full growth matrix $\mathbf{K}^{FULL}$. For the bound agent columns, the average coefficient for the bound agents is less than that of the free agents.

$$\mathbf{k}_{FB}^{FULL} > \frac{\boldsymbol{\iota}'}{R}\cdot\mathbf{K}_{BB}^{FULL} \tag{337}$$

**Remark**: Each bound agent produces one of the constrained resources, so its entry in value matrix **V** for that resource is zero. The consequent element in $\mathbf{K}^{FULL}$ is less positive than it would be otherwise. This consideration does not apply to the free agents.

**Assumption 4.10** FREE AGENT COEFFICIENTS. Consider the columns for free agents within $\mathbf{K}^{FULL}$. The average coefficient for the bound agents is greater than that of the free agents.

$$\mathbf{k}_{FF}^{FULL} < \frac{\boldsymbol{\iota}'}{R}\cdot\mathbf{K}_{BF}^{FULL} \tag{338}$$

**Remark**: The impact of a free agent on the growth coefficients of another free agent is more negative than on the bound agents, because the free agent competes with the other free agents for every resource, whereas each bound agent has a resource which it produces itself and for which it does not face competition.

We are now able to relate the growth matrix coefficients of the free agents to those of the bound agents, and this will allow us to interpret the ray growth theorem.

**Lemma 4.11**.

$$\mathbf{k}_F > \frac{\boldsymbol{\iota}'}{R} \cdot \mathbf{K}_B \tag{339}$$

**Proof**: Rearrange the definition at (295) using assumptions above and inequality rules.

$$\mathbf{k}_F = \mathbf{k}_{FB}^{FULL} - \mathbf{k}_{FF}^{FULL} \cdot \boldsymbol{\iota}'_{1\times S} \qquad \text{by (295)} \tag{340}$$

$$> \frac{\boldsymbol{\iota}'}{R} \cdot \mathbf{K}_{BB}^{FULL} \; - \; \frac{\boldsymbol{\iota}'}{R} \cdot \mathbf{K}_{BF}^{FULL} \cdot \boldsymbol{\iota}'_{1\times S} \qquad \text{by (337), (338)} \tag{341}$$

noting $\boldsymbol{\iota}'_{1\times S}$ is non-negative so inequality is preserved (342)

$$= \frac{\boldsymbol{\iota}'}{R}\left(\mathbf{K}_{BB}^{FULL} - \mathbf{K}_{BF}^{FULL} \cdot \boldsymbol{\iota}'\right) \qquad \text{Result follows by (295)} \qquad \# \tag{343}$$

**Assumption 4.11** INVERTIBLE GROWTH MATRIX. The growth matrix partition $\mathbf{K}_B$ is invertible.

**Remark**: $\mathbf{K}_B = \mathbf{V}_B \mathbf{A}_B = \mathbf{V}_B \left(\mathbf{A}_{BB}^{FULL} - \mathbf{A}_{BF}^{FULL} \boldsymbol{\omega}_F \boldsymbol{\iota}'_{1\times S}\right)$ by (293) (344)

Value matrix $\mathbf{V}_B : S \times S$ is of full rank by Assumption 4.5 Pricelike Matrices (284) above. $\mathbf{A}_B : S \times S$ has large diagonal elements so we expect rank *S*. $\mathbf{K}_B : S \times S$ is negative on the diagonal and roughly zero elsewhere. For these reasons we expect $\mathbf{K}_B$ to be of full rank.

*Non-negative sum*: Consider now reduced input-output matrix $\mathbf{A}_B$. We do not assume that $\mathbf{A}_B^{-1}$ is non-negative (as would normally be the case for input-output matrices) but make the following weaker assumption.

**Assumption 4.12** NON-NEGATIVE SUM.

$$\mathbf{A}_B^{-1} \cdot \mathbf{V}_B^{-1} \boldsymbol{\iota} > \mathbf{0}_{S\times 1} \tag{345}$$

**Remark**: Given that the off-diagonal elements in $\mathbf{A}_{BB}^{FULL}, \mathbf{A}_{BF}^{FULL}$ are expected to be roughly similar, the off-diagonal elements in $\mathbf{A}_B$ are approximately zero. Nonetheless these off-diagonal elements may be positive and this may cause $\mathbf{A}_B^{-1}$ to contain negative elements. The diagonal elements in $\mathbf{A}_{BB}^{FULL}$ are strongly positive and are assumed to dominate the sum of the elements in the row. (We do not assume diagonal dominance however.)

The following result establishes that the denominator of (329), (330) is positive, which allows us to discern the properties of the ray growth expressions.

**Lemma 4.12**. *The difference in gradients between the free and bound agents is positive.*

$$\mathbf{k}_F \mathbf{K}_B^{-1} \boldsymbol{\iota} - 1 > 0 \tag{346}$$

**Proof**: Use the previous result, and assumption above to demonstrate inequality is preserved.

$$\mathbf{k}_F > \frac{\boldsymbol{\iota}'}{R} \mathbf{K}_B \qquad \text{by (339)} \tag{347}$$

Now $\quad \mathbf{K}_B^{-1} \boldsymbol{\iota} = \left( \mathbf{V}_B \mathbf{A}_B \right)^{-1} \boldsymbol{\iota} = \mathbf{A}_B^{-1} \mathbf{V}_B^{-1} \boldsymbol{\iota} > \mathbf{0} \qquad$ by (345) $\qquad$ (348)

so $\quad \mathbf{k}_F \mathbf{K}_B^{-1} \boldsymbol{\iota} > \frac{\boldsymbol{\iota}'}{R} \mathbf{K}_B \cdot \mathbf{K}_B^{-1} \boldsymbol{\iota} = 1$ multiply by $\mathbf{K}_B^{-1} \boldsymbol{\iota}$ which preserves inequality by (348) # (349)

**Remark**: Assumption (345) seems restrictive, but the above result has a statistical character. Even if some elements of vector $\mathbf{A}^{-1} \cdot \mathbf{V}_B^{-1} \boldsymbol{\iota}$ are negative, multiplying (347) by that vector will tend to preserve the inequality.

*Analysis of result*. If more than one ray constraint has been encountered, we can compare the final common growth rate $\Gamma_1$ with the initial growth rate $\Gamma_0$. We employ a methodology of breaking the derivation into stages in order to understand its properties. There is no suggestion that these stages correspond to temporal stages of the transition. We view all these analytical stages of the transition as taking place simultaneously. For the sake of the analysis, we now denote the final common growth rate by $\Gamma_2$, and break down the derivation of the ray growth theorem into sections. We then assign properties to the intermediate results and hence to the whole. The strategy is set out in the following diagram:

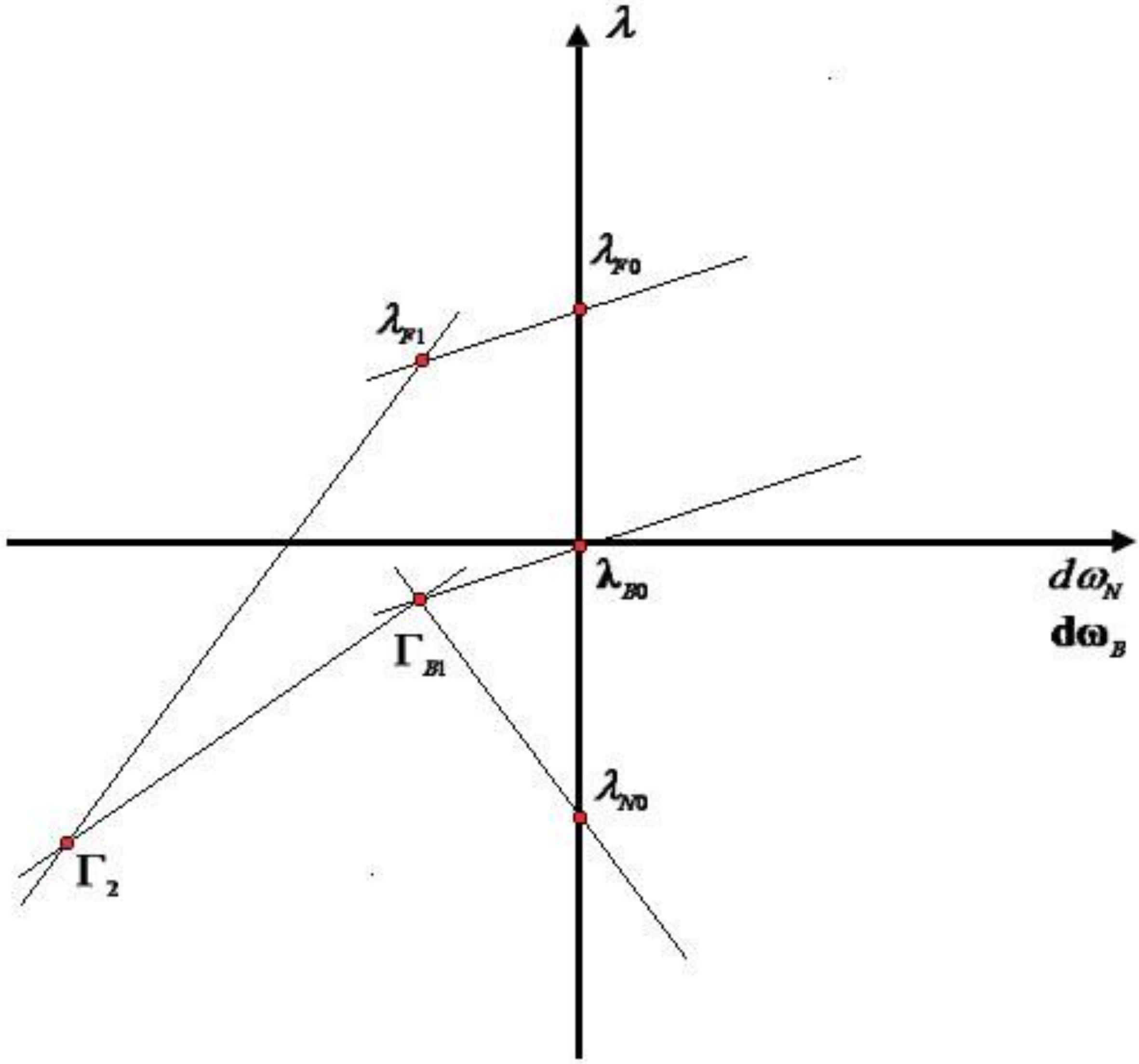


**Diagram 4.1: Impact of a new constraint on agent growth rates.** The vertical axis shows growth rate and the horizontal axis shows changes in the agent resource group proportions $d\omega$. Initially the bound agents grow at rate $\boldsymbol{\lambda}_{B0}$. The newly bound agents must be growing at a lower rate $\lambda_{N0}$ or the resource would not have become scarce, and the remaining free agents must be growing at a greater rate $\lambda_{F0}$. For the first stage of the derivation we consider the decrease in proportion of the newly bound resource group. As the proportion $\omega_N$ of newly

bound agents $N$ decreases, the growth rate of that group increases because of the reduced competition for the particular resources which the group uses. The growth rate of the previously bound resource groups falls as resource $N$ becomes harder to get, but there is a compensating effect through the offsetting increase in the proportion of the other bound resource groups $\boldsymbol{\omega}_B$ so the fall is slight. The $N$ group and $B$ group trajectories meet at $\Gamma_{B1}$, where their growth rates are equalized. The decrease in $\omega_N$ also causes the growth rate of the free group $F$ to decline slightly from $\lambda_{F0}$ to $\lambda_{F1}$.

In the second phase, the proportion of all the bound resources $\boldsymbol{\omega}_B$ decreases, and the proportion of the free resource group $\omega_F$ increases. This causes a decline in the growth rate of the bound resource groups, but a greater decline in the growth rate of the free resource group, because that group does not produce any of the scarce resources so is more reliant on importing scarce resources. The greater gradient of the free resource group is captured by Lemma 4.12 above, $\mathbf{k}_F\mathbf{K}_B^{-1}\boldsymbol{\iota}-1>0$. Equality of growth rates is restored at common growth rate $\Gamma_2$, which is necessarily lower than the common growth rate which obtained initially, $\boldsymbol{\lambda}_{B0}$.

The dynamic mechanism of the adjustment (which is distinct from this comparative analysis) is that agents with higher growth rates expand more rapidly, reducing the proportion of the others.

**Lemma 4.13**. *If more than one ray constraint has been encountered, expression 4.5.1 Incremental mutation for the common growth rate can be derived in three stages:*

- *Find the common growth rate* $\Gamma_{B1}$ *for newly bound N and old bound B agents.*
- *Find free agent growth rate* $\lambda_{F1}$ *by applying the consequent adjustment in proportions* $\mathbf{d\omega}$ *to the free agents.*

- *Derive the common growth rate for all agents $\Gamma_2$ from the adjusted growth rates produced by the first two steps.*

**Proof**: We split up the derivation at (329) into three parts. Firstly we allow the proportion of the originally bound agents $\boldsymbol{\omega}_B$ and the newly bound agent $d\omega_N$ to vary, and we apply that fluctuation to the bound agents to obtain an intermediate value of $\boldsymbol{\lambda}_{B0}$ called $\boldsymbol{\lambda}_{B1}$. Secondly we apply the same fluctuation $\mathbf{d\omega}$ to the free agents to obtain an intermediate value of $\lambda_F$ called $\lambda_{F1}$. Finally we apply the original result (329) to $\boldsymbol{\lambda}_{B1}, \lambda_{F1}$ instead of $\boldsymbol{\lambda}_{B0}, \lambda_{F0}$.

$\boldsymbol{\iota} \cdot \mathbf{d\omega} = 0$ where $\boldsymbol{\iota} : 1 \times S+1$, vector of ones (350)

Now $\boldsymbol{\lambda}_{B1} = \boldsymbol{\lambda}_{B0} + \mathbf{K}_B \cdot \mathbf{d\omega}$ by (331) top partition (351)

$\boldsymbol{\iota}\, \mathbf{K}_B^{-1} \boldsymbol{\lambda}_{B1} = \boldsymbol{\iota}\, \mathbf{K}_B^{-1} \boldsymbol{\lambda}_{B0}$ $\times \boldsymbol{\iota}\, \mathbf{K}_B^{-1}$, note $\boldsymbol{\iota}\, \mathbf{K}_B^{-1} \cdot \mathbf{K}_B \cdot \mathbf{d\omega} = \boldsymbol{\iota}\, \mathbf{d\omega} = 0$ by (350) (352)

Now $\boldsymbol{\lambda}_{B1} = \Gamma_{B1} \boldsymbol{\iota}$ common growth rate of bound agents (353)

So $\Gamma_{B1} = \dfrac{\boldsymbol{\iota}\, \mathbf{K}_B^{-1} \boldsymbol{\lambda}_{B0}}{\boldsymbol{\iota}\, \mathbf{K}_B^{-1} \boldsymbol{\iota}}$ applying (353) to (352) (354)

$\mathbf{d\omega} = \mathbf{K}_B^{-1} \left( \boldsymbol{\lambda}_{B1} - \boldsymbol{\lambda}_{B0} \right) = \Gamma_1 \mathbf{K}_B^{-1} \boldsymbol{\iota} - \mathbf{K}_B^{-1} \boldsymbol{\lambda}_{B0}$ from (351) and apply (353) (355)

(ii) $\lambda_{F1} = \lambda_{F0} + \mathbf{k}_F \cdot \mathbf{d\omega}$ by (331) lower partition (356)

$= \lambda_{F0} + \left( \Gamma_1 \mathbf{K}_B^{-1} \boldsymbol{\iota} - \mathbf{K}_B^{-1} \boldsymbol{\lambda}_{B0} \right) = \lambda_{F0} + \Gamma_1 \mathbf{k}_F \mathbf{K}_B^{-1} \boldsymbol{\iota} - \mathbf{k}_F \mathbf{K}_B^{-1} \boldsymbol{\lambda}_{B0}$ by (355) (357)

(iii) $\Gamma_2 = \dfrac{\mathbf{k}_F \mathbf{K}_B^{-1} \boldsymbol{\lambda}_{B1} - \lambda_{F1}}{\mathbf{k}_F \mathbf{K}_B^{-1} \boldsymbol{\iota} - 1} = \dfrac{\Gamma_{B1} \mathbf{k}_F \mathbf{K}_B^{-1} \boldsymbol{\iota} - \lambda_{F1}}{\mathbf{k}_F \mathbf{K}_B^{-1} \boldsymbol{\iota} - 1}$ apply (329) to $\boldsymbol{\lambda}_{B1}, \lambda_{F1}$, use $\boldsymbol{\lambda}_{B1} = \Gamma_{B1} \boldsymbol{\iota}$ (358)

$$= \frac{\Gamma_{B1} \mathbf{k}_F \mathbf{K}_B^{-1} \boldsymbol{\iota} - \left( \lambda_{F0} + \Gamma_{B1} \mathbf{k}_F \mathbf{K}_B^{-1} \boldsymbol{\iota} - \mathbf{k}_F \mathbf{K}_B^{-1} \boldsymbol{\lambda}_{B0} \right)}{\mathbf{k}_F \mathbf{K}_B^{-1} \boldsymbol{\iota} - 1} \qquad (359)$$

$= \dfrac{\mathbf{k}_F \mathbf{K}_B^{-1} \boldsymbol{\lambda}_{B0} - \lambda_{F0}}{\mathbf{k}_F \mathbf{K}_B^{-1} \boldsymbol{\iota} - 1}$ as per original result (329) # (360)

We can now analyse each stage in turn. Recall that an increase in the proportion of a bound agent $B_1$ causes its own growth to decline, i.e. $\frac{\partial \lambda_{B1}}{\partial \omega_{B1}} < 0$, and the growth rate of other bound agents to increase, i.e $\frac{\partial \lambda_{B}}{\partial \omega_{N}} > 0$. To equate the growth rate of the newly bound agent *N* to the higher growth rates of the original bound agents therefore requires a considerable decrease in the proportion of newly bound agents, $d\omega_N < 0$, and small compensating increases in the proportions of the other bound agents. The effect of this on each of the originally bound agents will be a strongly negative effect due to the decrease in $\omega_N$, a weakly negative effect due to the small increase in their own proportion, and a weak positive effect for each of the small increases in the proportion of the other bound agents. On this basis we assume the result after the first stage (i) is given by:

**Assumption 4.13** FIRST STAGE GROWTH RATE FOR BOUND AGENTS.

$$\Gamma_{B1} < \Gamma_0 \tag{361}$$

**Remark**: This is the general tendency but it may be possible to construct examples where this is not the case.

The following allows us to understand the first component of the ray growth rate.

**Lemma 4.14**. *When a manufactured good first becomes constrained, the free agent types which do not produce it (denoted F 0) have a higher growth rate than the common growth rate.*

$$\lambda_{F0} > \Gamma_0 \tag{362}$$

**Proof**: The mean of the free agents growth rate must equal the common growth rate, so we decompose the mean into its parts and apply the assumption above. Initially the growth rate of resource group *N* satisfies:

$$\lambda_{N0} < \Gamma_0 \text{ by (301)} \tag{363}$$

Initially, the average of the growth rate of the free agents is equal to $\Gamma_0$ by (311)

$$\Gamma_0 = \frac{\sum_{j\in F}\lambda_j m_j}{\sum_{j\in F} m_j} = \lambda_{N0}\cdot\frac{m_N}{\sum_{j\in F} m_j} + \lambda_{F0}\cdot\frac{\sum_{j\in F0} m_j}{\sum_{j\in F} m_j} \qquad \# \qquad (364)$$

Turning now to the free agent category *F*, we expect that the fall in the growth rate $\lambda_{F0}$ of the free agents brought about by the reduction in $d\omega_N$ is roughly in line with the fall in the growth rate of the previously bound agents, so the relativity of growth rates is preserved. Again it is possible in anomalous circumstances that this relationship does not hold, for instance the free agents are particularly reliant on the resource produced by the newly bound agent. To upgrade this assumption to a theorem would require other structural assumptions.

**Assumption 4.14** FIRST STAGE GROWTH RATE FOR FREE AGENTS.

$$\lambda_{F1} > \Gamma_{B1} \qquad (365)$$

We are now able to establish the dynamic characteristic of a new ray constraint.

**Theorem 4.15**: *The system growth factor* $\Gamma_2$ *after the new manufactured resource constraint is encountered is lower than the growth factor* $\Gamma_0$ *prior.*

$$\Gamma_2 < \Gamma_0 \qquad (366)$$

**Proof**: By adding and subtracting we show using the assumptions and results above that one component of (358) is negative which establishes the inequality.

$$\Gamma_2 = \frac{\Gamma_{B1}\mathbf{k}_F\mathbf{K}_B^{-1}\boldsymbol{\iota} - \Gamma_{B1} + \left(\Gamma_{B1} - \lambda_{F1}\right)}{\mathbf{k}_F\mathbf{K}_B^{-1}\boldsymbol{\iota} - 1} \quad \text{adding and subtracting } \Gamma_{B1} \text{ to (358)} \qquad (367)$$

$$= \Gamma_{B1} + \frac{\Gamma_{B1} - \lambda_{F1}}{\mathbf{k}_F\mathbf{K}_B^{-1}\boldsymbol{\iota} - 1} \quad \text{simplifying} \qquad (368)$$

$$< \Gamma_{B1} \quad \text{by (365),(346)} \qquad (369)$$

$$< \Gamma_0 \quad \text{by (361)} \qquad \# \qquad (370)$$

We observe from (329) that the final growth factor $\Gamma_2$ varies inversely with the initial growth rate of the free agents $\lambda_{F0}$. This is because a larger reduction in the bound agents is necessary to choke back the growth of the free agents. If the initial growth rate of the free agents $\lambda_{F0}$ is sufficiently high, it is possible that the final growth factor $\Gamma_2$ is less than unity and the system begins to collapse. In this case we define the free agent group as being "superefficient". The following result gives the condition for superefficiency.

**Corollary 4.16**: SUPEREFFICIENT AGENT THEOREM. If

$$\lambda_{F0} > 1 + \mathbf{k}_F\ \mathbf{K}_B^{-1}\left(\boldsymbol{\lambda}_{B0} - \boldsymbol{\iota}_{S\times 1}\right) \tag{371}$$

then the final growth factor $\Gamma_2$ will be contractionary.

**Proof**. Follows immediately by rearrangement of (329). Set $\Gamma_2 < 1$ to get:

$$1 > \frac{\mathbf{k}_F\ \mathbf{K}_B^{-1}\ \boldsymbol{\lambda}_{B0} - \lambda_{F0}}{\mathbf{k}_F\ \mathbf{K}_B^{-1}\ \boldsymbol{\iota}_{S\times 1} - 1} \quad \text{by (329). Solve for result.} \quad \# \tag{372}$$

**Remark**: For example, if the current growth rates of the bound agents $\boldsymbol{\lambda}_{B0} = \boldsymbol{\iota}_{S\times 1}$ then (371) collapses to $\lambda_{F0} > 1$. Given that the free agents have a higher growth rate by (362), criterion (371) will be satisfied and the free agents will force the common growth rate $\Gamma_1$ below unity.

**Remark**: In practical terms, an agent which is too efficient will collapse the system because the proportion of resource providers which is consistent with an equal growth rate in all sectors is so low that not enough resource is produced.

**Remark**: It is also possible to have a 'hyper-efficient' free agent with growth gradient less than the bound agents with respect to the bound resources, contrary to assumptions (337), (338). In this case the decrease in the bound agents causes a widening of the gap between the free and bound agents, and the free agents expand even more aggressively relative to the others. In this case the system will collapse rapidly. Is such a scenario possible? It has been suggested that man's efficiency in the use of resources puts humankind in this position.

*Conclusion*: The expected value of the common growth factor falls with every new ray encountered. The free agents $F$ have a higher growth rate, so the proportion of bound agents, including the newly bound agents, declines. The fall in the proportion of the bound agents continues until the resources which they produce are so scarce that the growth rates of free agents are choked back to match their own. Thereafter, the newly bound agents are not subject to further falls in their numbers relative to the rest, and their place in the system is protected.

*4.4.4 Rebalancing of the growth rate within a resource group*

In general resource categories are composed of more than one type of agent, with different growth rates. Over time the agents with the higher growth rates become relatively more common and the growth rate of the group as a whole is increased. This process runs counter to the declining trend caused by the build-up in the number of bound resource groups.

*4.4.5 Oscillations*

It is possible that the introduction of a new constraint causes the growth factor to fall below unity and subsequent rebalancing causes the growth factor to rise above unity again. This will cause the population to oscillate. At some stage in the process, the growth factor will be exactly unity $\Gamma = 1$. Such a situation is structurally unstable because of the ongoing rebalancing. Nonetheless during the period in which this situation obtains we have:

$$E\left[\Delta\lambda\right] = \Gamma - 1 = 0 \tag{373}$$

In this case the system is non-stationary and population oscillates randomly from period to period – these short-term oscillations are a separate process from the medium-term oscillations due to rebalancing. Such short-term oscillations do not arise when the growth factor is forced to unity at the transverse frontier, because the action of the endowed constraint is a converging process.

*4.4.6 Releasing a ray constraint*

The question arises whether a given constraint can be released when a subsequent ray is met. The solution (360) implies that all previously encountered constraints are still binding. Effectively this conclusion is an artefact of assumptions (337), (338) which compare the growth coefficients of the bound agents with those of the free agents. It is possible, particularly early in the process before many constraints have been encountered, that these assumptions are violated. It can be that a newly encountered constraint is produced by an agent with a low growth factor, and that a previously bound agent is efficient in its use of that resource. Under such conditions the previously bound agent may be able to sustain a growth rate above the common growth rate and rejoin the free agent group. As the trajectory encounters more constraints this scenario becomes less plausible as the escaping agent would have to be efficient in its use of all the bound resources.

### *4.4.7 Encountering an endowed constraint*

If the system does not collapse or oscillate then population will continue to expand to the point where one of the endowed constraints becomes binding. The behaviour of the system when it reaches an endowed constraint requires a separate treatment. When the supplies of the endowed good first become scarce, the population term $-\mathbf{V}^{FULL}\mathbf{L}\cdot\frac{dN}{N^2}$ in the dynamic equation (300) becomes non-zero. It impacts on the producers of the manufactured goods, and they must rebalance in the usual way to offset its impacts and restore a balanced growth path as per the Ray Growth Theorem. However, the common growth rate $\Gamma$ is not forced to equal the growth rate of the exporters of the endowed good, as is the case for manufactured goods. The Ray Growth Theorem does not apply to these producers because the matrix of endowed production $\overset{END}{\mathbf{B}}$ is excluded from that analysis. The growth rate of the endowed good exporters has no particular relationship to the common growth rate – it is not below it as we assumed for ray constraints - and the exporters continue to be included in the free agent category. The scarce good categories are unchanged and the number of scarce goods *S* (in fact, the number of scarce manufactured goods) remains at its previous value.

When the endowed good constraint becoming binding, the initial growth rate changes to reflect the population term:

$$\boldsymbol{\lambda}_0 = \Gamma_0 \boldsymbol{\iota}_{S+1\times 1} - \mathbf{V}^{FULL}\mathbf{L} \cdot \frac{dN}{N^2} \tag{374}$$

Further expansion requires that the production of manufactured goods continues to be rebalanced so growth is equal in all scarce manufactured resource sectors:

$$\boldsymbol{\lambda}_1 = \Gamma_1 \boldsymbol{\iota}_{S+1\times 1} \tag{375}$$

Expansion implies that $dN > 0$ in each period, requiring further rounds of rebalancing. We show below that the system converges to a zero growth rate.

The process of adjustment which occurs when a ray constraint is encountered does not apply here. The growth rate of free agents is not necessarily greater than average, and their proportion does not necessarily increase and reduce the growth rate of the others. It is therefore possible that some agents are less adversely affected by the change than others, and that some previously constrained resource is no longer scarce. In this case the system can move off the ray for that resource onto a less restrictive surface (either a ray with less restrictions or, if there are no manufactured resource constraints at all, the transverse surface) and increase its population further. We discern therefore that a system tends to ‘stick to the sides’ of its feasible set. At the eventual equilibrium, a particular constraint may not be binding, but it may have been binding as the system made its way to the final point.

**Theorem 4.17**: TRANSVERSE GROWTH THEOREM. *For an endowed good constraint the equilibrium growth rate and proportion change are given by:*

$$\Gamma_1 = \Gamma_0 + \frac{\left(\mathbf{v}_F \mathbf{L} - \mathbf{k}_F\ \mathbf{K}_B^{-1} \mathbf{V}_B \mathbf{L}\right) \frac{dN}{N^2}}{\mathbf{k}_F\ \mathbf{K}_B^{-1}\ \boldsymbol{\iota}_{S\times 1} - 1} \tag{376}$$

$$\mathbf{d\omega}=\frac{\mathbf{K}_B^{-1}\left(\boldsymbol{\iota}_{S\times1}\mathbf{v}_F\mathbf{L}-\mathbf{V}_B\mathbf{L}\right)\frac{dN}{N^2}}{\mathbf{k}_F\ \mathbf{K}_B^{-1}\ \boldsymbol{\iota}_{S\times1}-1} \tag{377}$$

**Proof**: We demonstrate this result by putting the premise into form (329).

$$\Gamma_1\boldsymbol{\iota}_{S+1\times1}=\Gamma_0\boldsymbol{\iota}_{S+1\times1}+\begin{bmatrix}\mathbf{K}_B\\ \mathbf{k}_F\end{bmatrix}\mathbf{d\omega}-\begin{bmatrix}\mathbf{V}_B\\ \mathbf{v}_F\end{bmatrix}\mathbf{L}\frac{dN}{N^2} \qquad \text{by (300)} \tag{378}$$

$$=\begin{bmatrix}\Gamma_0\boldsymbol{\iota}_{S\times1}-\mathbf{V}_B\mathbf{L}\frac{dN}{N^2}\\ \Gamma_0-\mathbf{v}_F\mathbf{L}\frac{dN}{N^2}\end{bmatrix}+\begin{bmatrix}\mathbf{K}_B\\ \mathbf{k}_F\end{bmatrix}\mathbf{d\omega} \qquad \text{rearranging into form (332)} \tag{379}$$

$$=\frac{\mathbf{k}_F\ \mathbf{K}_B^{-1}\left(\Gamma_0\boldsymbol{\iota}_{S\times1}-\mathbf{V}_B\mathbf{L}\frac{dN}{N^2}\right)-\left(\Gamma_0-\mathbf{v}_F\mathbf{L}\frac{dN}{N^2}\right)}{\mathbf{k}_F\ \mathbf{K}_B^{-1}\ \boldsymbol{\iota}_{S\times1}-1} \qquad \text{using (329). Rearrange.} \quad \# \tag{380}$$

**Remark**: This result captures the effect on the system of the reduction in per capita resource allocations, but it does not capture the change in proportions caused by the reduction in trading stocks. However unlike the case for manufactured resources, the population of endowed good exporters does not reduce, so this is not as significant.

We know by the Bounded Population Theorem (176) that population *N* cannot increase indefinitely. The outworking of this conclusion here is as follows. As population increases the need for the endowed resource becomes acute and is felt by every agent. We assume:

**Assumption 4.15** EQUAL VALUE. Eventually population increases to the point where the scarce endowed resource has approximately the same value to all agents. In this case:

$$\mathbf{V}_B\mathbf{L}=\boldsymbol{\iota}_{S\times1}\ \mathbf{v}_F\mathbf{L} \tag{381}$$

**Remark**: If this assumption is satisfied then the change in proportions expression (377) becomes zero:

$$\mathbf{d\omega}=\frac{\mathbf{K}_B^{-1}\cdot\mathbf{0}_{S\times1}\cdot\frac{dN}{N^2}}{\mathbf{k}_F\ \mathbf{K}_B^{-1}\ \boldsymbol{\iota}_{S\times1}-1}=\mathbf{0}_{S\times1} \tag{382}$$

**Assumption 4.16** SEDATE PACE. The rate of growth is not so vigorous that the system can overshoot a limiting value.

$$0 < \frac{\mathbf{v}_F \mathbf{L}}{N} < 1 \tag{383}$$

We show below that the population will converge to an upper limit over time.

**Theorem 4.18**: CONVERGENCE THEOREM. *If*

- *the growth factor exceeds unity*
- *population has increased to a point where Assumption 4.15 Equal Value (381) applies*

*then the final growth factor is* $\Gamma = 1$ (384)

**Proof**: We take the previous result (376) and substitute in the assumptions above, then rearrange into a form which makes convergence clear.

$$\text{If } \Gamma_1 > 1 \text{ then } dN > 0 \tag{385}$$

$$\Gamma_1 = \Gamma_0 + \frac{\left(\mathbf{v}_F \mathbf{L} - \mathbf{k}_F\ \mathbf{K}_B^{-1}\,\boldsymbol{\iota}_{S\times 1}\,\mathbf{v}_F \mathbf{L}\right)\frac{dN}{N^2}}{\mathbf{k}_F\ \mathbf{K}_B^{-1}\,\boldsymbol{\iota}_{S\times 1} - 1} \quad \text{substituting (381) into (376)} \tag{386}$$

$$= \Gamma_0 - \mathbf{v}_F \mathbf{L}\,\frac{dN}{N^2} \quad \text{factorize numerator and cancel} \tag{387}$$

$$= \Gamma_0 - \frac{\mathbf{v}_F \mathbf{L}}{N}\left(\Gamma_0 - 1\right) \quad \text{substituting } \frac{dN}{N} = \Gamma_0 - 1 \tag{388}$$

$$\Gamma_1 - 1 = \left(\Gamma_0 - 1\right)\left(1 - \frac{\mathbf{v}_F \mathbf{L}}{N}\right) \quad \text{rearranging} \tag{389}$$

so $$\lim_{t\to\infty} \Gamma_t - 1 = 0 \quad \text{given (383) and } \lim_{t\to\infty} \frac{\mathbf{v}_{Ft}\mathbf{L}}{N_t} \neq 0 \qquad \# \tag{390}$$

**Remark**: If in fact Assumption 4.15 Equal Value does not apply and $\mathbf{V}_B \mathbf{L} < \boldsymbol{\iota}_{S\times 1}\,\mathbf{v}_F \mathbf{L}$ then $\mathbf{d\omega} \geq \mathbf{0}_{S\times 1}$ and the proportion of the free agent will decrease, possibly to zero. The growth rate may increase which reflects the removal of the debilitating effect of the free agents. Eventually all the free agents will be eliminated and this effect will come to an end. If

$\mathbf{V}_B\mathbf{L} > \boldsymbol{\iota}_{S\times 1}\,\mathbf{v}_F\mathbf{L}$ then the decline in the growth rate will be faster than in the proof above with the same result.

*4.4.8 Developments on the transverse frontier*

When the system has reached stability on the transverse frontier, the average growth rate in each resource group is unity. This implies that some of the agent types in each resource group necessarily have growth rates below unity and will be eliminated. As this elimination occurs the system will rebalance itself to restore a unity growth rate in that resource group, and further agent types are pushed below the unity average and elimination. Eventually only one agent type will be left producing each type of resource. The maximum number of agent types at the long run equilibrium is no more than the total number of constrained resources. It may be less than this if one agent type is the most efficient in more than one category. In this model at least, it appears that monopoly, far from being an anomalous deviation from perfect competition, is the normal case.

The bound agents are held to a certain proportion by the need for their resources as shown. By contrast, if a free agent has a lesser growth than the others, and it does not produce a resource which eventually becomes constrained, then its relative proportion of the population will fall until it becomes extinct. Free agents are in a structurally unstable situation.

*4.4.9 Path dependence*

The question arises as to whether the final resting point of the trajectory is unique, or whether a different path would lead to a different final result. The existence of a unique distributional equilibrium does not rule out the possibility of more than one equilibrium modal point although it does suggest that transition from one modal point to another is ultimately possible. In practice this may not be meaningful because if an agent type becomes extinct then the only way it can be regenerated is for the whole system to become extinct and regenerated as per the System Extinction process.

Consider that the first agent type to become bound (i.e. the manufactured resource which it produces becomes constrained) may depend purely on how close the initial point is to the ray boundaries. Then those agent types which depend heavily on that resource will themselves suffer lower growth rates, and face a higher risk of expiry than agents producing the same resources but of a different type. It seems clear that which agents survive the system trajectory to the convex frontier can depend on the initial starting point.

The deeper question is whether the binding constraints at the point where the trajectory comes to rest are themselves functions of the starting point. The model suggests the following mechanism: a change in the starting point will change the trajectory to the frontier of the feasible set. This may (or may not) change the resource constraint encountered, change the growth factors of the agents in each resource group, change the ordering of growth factors in each resource group, alter the agent type selected from each resource group, and ultimately result in a different equilibrium. We suggest that the growth matrix $\mathbf{K}^{SYS}$ implies sets of mutually reliant agents which will be selected in groups according to the starting point.

### 4.5 Modelling system evolution

By representing creatures and their processes with a Markov chain we have a natural entry point for imposing mutation on the production function, namely changing the elements in the transition matrix. We identify four distinct types of mutation in a biological system:

- Incremental shifts, where a transition matrix parameter is perturbed incrementally.
- Deletions, where the creature loses the ability to do something. Deletions are not incremental but involve setting parameters to zero. It is relatively easy for a creature to lose the ability to do something, since doing it requires everything to be functioning correctly. Different groups of Mexican cavefish have repeatedly and independently lost their sight over the past ten thousand to one hundred thousand years (Jeffery Strickler Yamamoto 2003).

- Additions, where two creatures are brought together and merge their productive capabilities. For instance, the mitochondria and the nucleus of the cell possibly evolved independently before combining in the eukaryotic cell around 1.6 billion years ago (Dawkins 2004). In biology this is referred to as horizontal (or lateral) gene transfer, with some suggesting that the process is more important than Darwinian species based evolution (Gogarten 2000). Rutgers University (2017) provides a survey article.
- Saltation, or leaps in phenotype. Within biology this is often referred to as the hopeful monster hypothesis after Goldschmidt (1940). While always controversial, the hypothesis has garnered some empirical support in recent years, for instance Chouard (2010).

Within economic systems analogous situations are easily identified. The mechanism of random chance is augmented both by trial and error learning processes and the calculated application of reason, but the source of the innovation is not important for understanding its impact on the system.

- Incremental shifts are represented by continual improvement in the light of experience. Experience consists of slightly varying practices, and the practices with the most effective outcomes are selected. This is trial and error, or least-squares learning.
- Addition and deletion mutations are represented by organisations bringing functions in-house which were previously outsourced (for instance graphics and printing with computers) or outsourcing things previously done inhouse (staff recruitment, building maintenance). Similarly consumers now do typing for themselves and often out-source cooking. Department stores brought together many different types of retailing, but the shopping mall and credit card removed the convenience advantages of the

department store so recently department stores have evolved back in the direction of separate brand outlets under one roof.

- Saltation is more distinctly recognisable in economic systems than biological as new technology, but it can be argued that the new technology can be broken into small steps in scientific discovery, research and development. The application of reason can provide larger jumps in the development process than trial and error.

### *4.5.1 Incremental mutation*

Mutations may be either positive or negative. We interpret a mutation as the creation of a new agent type with a changed coefficient for one of its activities. This change can affect either the input or the output coefficients (negative or positive) and it can be an improvement (the new coefficient is greater) or a deterioration (the new coefficient is less). Deteriorations imply a lower growth rate and the eventual removal from the population of the new agent type and we do not consider this case further here.

Consider the case of improvements to a bound agent where the transverse frontier has been reached and the system has moved to maturity with only one producer in each bound resource class. Improvements imply, through the dynamic equation (234), an increase in the rate of growth of that agent.

We establish that incremental mutations necessarily increase population over time.

**Theorem 4.19**: INCREMENTAL MUTATION THEOREM. *Suppose that*

- *a mutation in agent j is such that* $d\lambda_N > 0$ (391)
- *the system is currently at equilibrium with* $\Gamma_0 = 1$ (392)

*then there will be an increase in population:*

$$N_1 > N_0 \tag{393}$$

*where* $N_0$ *is the equilibrium population prior to the mutation and*

*$N_1$ is the new equilibrium population.*

**Proof**: We treat the variant form as a different agent type and apply the results already established for dynamics. The resource group can be treated as a newly constrained resource with a higher growth rate:

$\lambda_{N0} > \Gamma_0$ given (391). (394)

This is the opposite of Assumption (301) and consequently Assumption (361) is reversed:

$\Gamma_{B1} > \Gamma_0$ opposite (361): first stage growth rate is higher (395)

$\lambda_{F0} \approx \Gamma_0$ cf. result (362), free agents have same growth rate (396)

$\lambda_{F1} \approx \Gamma_{B1}$ cf. assumption (365), first stage here is the same (397)

So $\Gamma_2 = \dfrac{\Gamma_{B1}\mathbf{k}_F\mathbf{K}_B^{-1}\boldsymbol{\iota} - \Gamma_{B1} + \left(\Gamma_{B1} - \lambda_{F1}\right)}{\mathbf{k}_F\mathbf{K}_B^{-1}\boldsymbol{\iota} - 1}$ adding and subtracting $\Gamma_{B1}$ to (358) (398)

$= \Gamma_{B1} + \dfrac{\Gamma_{B1} - \lambda_{F1}}{\mathbf{k}_F\mathbf{K}_B^{-1}\boldsymbol{\iota} - 1}$ simplifying (399)

$\approx \Gamma_{B1}$ by (397), (346) (400)

$> \Gamma_0 = 1$ by (395), (392). Hence increase in population. # (401)

We show that the new variant will displace the old.

**Corollary 4.20**: DISPLACEMENT THEOREM. *Suppose that*

- *agent type j is the sole producer within a bound resource group r*
- *a mutation in agent j, denoted agent type ja, is such that* $d\lambda_{ja} > 0$ (402)

*then agent type j will be forced to expiry.*

**Proof**: We know the growth rate for the group as a whole so we express it as a weighted average of its constituent agent growth rates.

$\Gamma_1 = 1$ and $\lambda^r = 1$ at equilibrium by (384), (311) (403)

$$1 = \frac{m_j \lambda_j}{m_j + m_{ja}} + \frac{m_{ja} \lambda_j}{m_j + m_{ja}} = w_j \lambda_j + w_{ja} \lambda_{ja} \text{ where } w_j + w_{ja} = 1 \text{ by (266)} \tag{404}$$

$$\lambda_j < 1 \text{ given } \lambda_{ja} > 1 \qquad \# \tag{405}$$

**Remark**: It does not follow from this that the mutation causes the resource *r* to be produced more efficiently. In fact since agent *j* has this resource in surplus, agent *j* does not benefit much from producing the resource more efficiently and the increase in growth rate according to premise (391), $\lambda_{ja} > \lambda_j$, is small. It is more likely that the agent reduces its use of some other resource. This raises the population of agent type *j*, which improves the supply of the resource as a side effect. This is the disadvantage of a system without reciprocal trade – because benefits to other agents are not passed onto the producer, the producer has somewhat less incentive to produce efficiently.

**Remark**: Conversely if a free agent becomes more efficient it will have the effect of pushing down the population. The Displacement Theorem above will still apply and the efficient mutation will tend to push out other free agents.

### *4.5.2 Deletion mutations*

Consider the case where the agent deletes a manufacturing process which produces an abundant good and thereby uses less of a scarce good.

**Theorem 4.21**: DELETION MUTATION THEOREM. *Consider a process which produces abundant good 'a' using scarce resource 's'. If that process is deleted, then the agent will increase its rate of growth:*

$$d\lambda_j > 0 \tag{406}$$

**Proof**: We apply Value Matrix result (209) to this situation. Observe that the probability of the agent gaining a unit of resource in a period increases as consumption is reduced.

$$dp_j^s > 0 \tag{407}$$

Further, $V_j^s > 0$ for a scarce resource (408)

Now $d\lambda_j = \mathbf{V}_j\mathbf{dp} = V_j^a \cdot dp_j^a + V_j^s \cdot dp_j^s$ by (209) (409)

$= V_j^s \cdot dp_j^s$ given $V_j^a \approx 0$ by (279) (410)

$> 0$ by (407), (408) # (411)

**Remark**: In this case the mutated agent will again increase at the expense of the original agent type, but there is a critical difference to the incremental mutation case. If the abundant resource becomes constrained as the original agent declines, the binding constraint mechanism will be triggered and any further decline in the original agent population will be arrested.

**Remark**: Pursuant to Result (393) there will be an increase in population.

### 4.6 Conclusion

We have developed a dynamic theory by relating growth rates to fluctuations in agent proportions via a 'growth matrix' **K**. This matrix is itself the product of a 'value matrix' **V**, which is derived from the Markov processes of the agents, and an input-output matrix **A**, which is derived from the linear resource constraints which apply to the system as a whole. We have an apriori expectation that producer surpluses will be as small as the producers can make them (but non-zero), and that as a result the system trajectory will encounter manufactured good or 'ray' constraints as it expands. Using the dynamic theory we have shown that with every additional ray constraint encountered, the growth rate falls. It is possible that the growth rate falls below zero and the system contracts. If the system does expand to reach the constraints imposed by the finite amount of endowed resources, then the growth rate will fall to zero as expected.

We develop a simple evolutionary model and find that evolutionary results fall out immediately as special cases of dynamic phenomena.

# Chapter 5

# Discussion of results

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## 5.0 Introduction and overview

We construct a model of a biological ecology or a pre-market human economy using two of the most useful tools in the economics toolkit, Markov matrices and linear production theory. Within this economy there is a variety of agent types which produce different kinds of resource necessary for survival. There is no reciprocal trade, rather agents supply resources to others according to the disparity in resource holdings. The behaviour of the agents is habitual rather than optimising, and representable by a Markov chain matrix. Resource production and consumption are represented by vectors which specify the possible Markov matrix transitions. We show that the feasible set is bounded by linear resource constraints in the positive orthant of agent space – coordinate space with an axis showing the number of each agent type – as for a linear programming problem. If a resource constraint is binding at a given point in that space then the resource is scarce there, and otherwise it is abundant. Quantity-adjusting forces move the economy to a mixture of agent types which can produce all the resources necessary for survival.

Because the model is constructed using economic principles based on resource abundance and scarcity, it offers heightened perspectives on some aspects of survival and natural selection.

- *Single growth rate*: There is a single system growth rate common to every resource group (group of agent types which produce the same scarce resource), and the group of remaining ‘free’ agents as well.
- *Falling growth rate*. With every resource constraint which is encountered by the system trajectory, the system growth rate will fall. There is a maximum population

which the economy can support based on the amount of resources which are endowed to the system in each period, and at that point growth is zero. It is possible that the maximum is not reached.

- *Survival of the weak*. Resource constraints mean that producers of scarce resource can survive even though their growth rates are initially lower than other agents, because all other sectors of the economy are choked back to the same growth rate. This guarantee of survival does not extend to producers of abundant resources or free-riders.
- *The paradox of efficiency*. It is possible for one agent to be overly efficient, so that the rate of growth which equilibrates the growth of the resource classes is negative. It is even possible for an agent to be hyper-efficient, in that its relative superiority over the other agents increases as the overall growth rate declines. In this case the system is forced into a catastrophic and immediate collapse.
- *Dynamic effects*. The rate of growth increases as more efficient agents become predominant in the mix, and declines when new resource constraints are encountered. The interplay of these effects generates expanding, declining, oscillating and static system trajectories. There is a tendency for the system to 'stick to the sides' of the feasible set. These dynamic effects arise without any explicit assumptions which would force this result. We have assumed no law of motion, maximisation process or objective function.

### 5.1 The economics of non-reciprocal trade

We have assumed that agents supply the resources they produce to other agents without receiving compensation, and are supplied with resources they require without incurring obligation. What are the economics of this system?

Firstly we consider why an agent might produce more than it needs for its own requirements, contrary to the iron dictates of the Prisoners' Dilemma. We presented a variety of explanations, namely group selection, non-controllable production, lean times buffer, parasitism, agent cooperation, and inter-dependent utility. All of these explanations lie outside the current model, but we take it as given that agents do produce a surplus in natural and pre-market economic systems.

Secondly, we consider how such a system operates. In a market system, producers are rewarded for what they supply with something of equal value. Here there is no market price and no market supply and demand curves. We develop an alternative mechanism. Growth rate equation (300) relates changes in quantity to changes in agent proportion. The matrix which mediates between the two, 'growth' matrix **K**, is the product of a 'value' matrix **V** and an input-output matrix **A**. The value matrix derives from the stochastic properties of the Markov matrix and is analogous to changes in marginal utility per unit of resource. The input output matrix derives from the linear production form of the model and gives changes in resource. Multiplied together, the two matrices produce a version of a demand curve. We then relate those changes in quantity to the requirement that at equilibrium all sectors grow at the same pace, the Ray Growth Theorem (311). The Ray Growth condition is more like an equilibrium condition which intersects with the demand curve than a supply curve as such.

The practical effect is that if agents cannot get what they require, their growth rate falls, their proportion in the population falls, and the proportion of producers of that scarce resource increase by default until balance between producers and consumers is restored. Alterations in quantity, brought about by differential birth and death rates, are the alternative device which produce equilibrium. Although producers receive no direct reward for producing, and have no direct incentive to expand production, the fact of survival is an alternative form of reward.

Let us now consider trading efficiency. The system generates resource shedding according to (159). The amount shed represents production lost to the system and is a source of inefficiency. We observe from (159) that the higher the value of the trading constant $k^r$, the less resource will be shed. Now there is an implicit concept of price, $\Delta\upsilon^r$, in the LHS eigenvalue of the agent Leslie matrix, which measures the incremental number of descendants which one extra unit of resource *r* will produce. We can relate price $\Delta\upsilon^r$ to stock levels $\rho_j^r$, stock levels to the trading coefficient $k^r$ and the trading coefficient to the wastage.

> At limit $k^r = 0$, there is no trading and any non-autarkical system is unsustainable. The resource has no value to producers as it is in excess, and infinite value to other agents who perish for want of it.
>
> As $k^r$ increases, resource wastage reduces and efficiency increases. The resource is has a certain marginal value, as measured by price $\Delta\upsilon$, to producers, but it is worth more to importers.
>
> As $k^r$ approaches infinity, all agents have equal access to the resource and equal resource stocks $\rho_j^r$. The implicit price $\Delta\upsilon$ is the roughly the same to all. This is the situation in an ideal market economy – the resource has the same opportunity cost to

every agent. There is no stock-shedding unless every agent is at the upper limit, which implies the resource is abundant, so there is no wastage of a scarce resource.

Increasing $k^r$ can be interpreted as transitioning from a biological economy to a market economy. However, if every agent shares similar stock levels then the differential survival mechanism which underpins equilibrium can no longer operate. We conclude that a system can operate using quantity rather than price adjustments but not as efficiently.

### 5.2 Specialization and interdependence

We represent the processes of biological evolution by altering the resource processing coefficients of the agents. We find that an incremental mutation which increases the efficiency of an agents' resource processing will lead to an increase in the overall population if an agent produces a scarce good. Conversely, a free agent (one which does not produce a scarce good) which becomes more efficient will cause a reduction in population.

It is the deletion mutations of an agent – ceasing some resource transforming process – which results in specialization and interdependence in real world economies. The explanation presented here follows the same logic as for incremental mutations. If a creature stops producing a resource which it can easily obtain because the resource is abundant, then that agent effectively becomes more efficient, with the same consequences as an incremental mutation. In each case the variant agent type will expand at the expense of the original type because it has a higher rate of growth. However, if the process which was deleted is for a necessary resource which no other agent type is producing, then eventually the resource will become scarce and further decline in the original agent type will be arrested.

We conclude that an economy originally consisting of one type of autarkical agent will move to a system of differentiated and inter-dependent agent roles, solely through resource pressure. This is as distinct from the view that the genesis of differentiated roles is the greater production efficiency which flows from specialization – a view which goes right back to Adam Smith's pin factory. In terms of this model, specialization starts after the fact of differentiation so cause and effect are reversed. Once creatures have developed specialized roles they are free to optimize those roles without the compromises necessary to perform other functions as well. The Red Queen Hypothesis (Van Valen 1973) in biology represents coevolution as a competitive and costly thing, a 'genetic arms race', but from the perspective of this model it is a positive thing. We can interpret the ecosystem, or economy, as a global genotype of one original creature, and specialization as a device which allows it to expand the boundaries of its feasible set. Effectively, the agent has broken free from the trade-offs of morphological space and can optimize everything at once. But it is wrong to imagine that central coordination motivates individual actors in either a biological or economic setting. The most enduring theme in economics is that it is the system itself which coordinates individual self-interest so that everyone's needs are met, and the model demonstrates that reciprocal trade is not necessary for that coordination. Natural selection operating within a system of scarcity is sufficient.

However, we have also seen that an agent which is too efficient relative to others risks collapsing the system. This is by no means a hypothetical scenario. Examples include cancer in the body, a corrupt regime in an economy, predation in an ecology, some suggest humankind. A full explication of the role of predation awaits further development of the theory.

In the context of the debate in biology regarding the nature of progress, the movement to more efficient resource utilization and greater population are unidirectional trends which can be expected to persist over the life of the system.

### 5.3 Inhouse processing versus outsourcing

Assumption 3.8 for trading requires agents to meet each other at a constant rate; they cannot become too dispersed to trade effectively. If this assumption does not apply and the rate at which agents meet decreases, then the trading coefficient $k^r$ also decreases. As discussed above, this reduces the economic efficiency of the system. A sufficient decline in $k^r$ will produce an evolutionary incentive for entities to join together - the opposite of the deletion mutation. When entities reunite, they may be able to fix their relative proportions at a level whereby the new entity has production in the right proportion of every resource. We see this incentive to regroup in such diverse phenomena as the reunion of the mitochondria and the nucleus within a cell, the organs in the body, and departments within a firm. When the situation or technology changes, the firm may find it more efficient to outsource and the creature to lose some of its functions. There is a continuing tension between these competing tendencies and this is another direction of future research.

### 5.4 Nature of system equilibrium

We constructed a system Markov chain matrix by Kronecker multiplication, and showed using the Perron-Frobenius theorem an equilibrium distribution to such a system must exist. The Perron-Frobenius theorem provides not only existence but uniqueness, positivity and stability which means that the solution is tightly characterized for a minimum of analytical effort. However the nature of the solution is different from general equilibrium analysis. It is distributional - a distribution of states - rather than situational - a point in production-consumption space. The positivity of the solution eigenvector implies that every state in the system, no matter how improbable, will be instantiated at some point. This interpretation parallels results from quantum mechanics which are possibly derived in a similar way. It does not seem like a disadvantage that the solution comes in this form. Real systems whether economic or biological are subject to fluctuations, some of them very large, and a solution which embraces such fluctuations may be more interesting than the traditional one-point solution.

### 5.5 Path dependence

This model implies a theory of path dependence based on historical price structures, which is perhaps a more general version of the modal lock-in theory of path dependence established by Arthur (1989). The competition which agents face is with other producers in the same sector. Agent growth factors are a function of which resources are scarce. Different starting points can result in different scarcities, different resource price structures (as measured by each agent's incremental survival rate $\Delta \upsilon_j^r$ ), and consequently different growth and survival rates. The final equilibrium will depend to some degree, maybe a large degree, on which agent types survive. This conclusion differs from a standard economic model because the standard model takes the agent types as fixed and does not consider survival.

We note that one outstanding issue in biology is the question of what characteristics determine whether a species can enter an existing ecosystem. A general theory has proved elusive (for instance, Alexander et al. 2014). In terms of this model, the answer lies in the adaptation of that species to the implicit price system of the ecosystem, which derives from the resource processing abilities of the existing agents. Alexander et al. present a biological theory which is also based on relative resource processing efficiency. Explication of path dependent phenomena in the model is an area of future research.

### 5.6 The system as a complex adaptive system

Our results demonstrate that a system based on habitual behaviour and survival rather than explicit optimization can nonetheless exhibit optimizing behaviour. The same principles of scarcity underlie the operation and development of both economic and biological systems, and natural selection and quantity adjustment can take the place of price adjustment. There is evolutionary pressure produce more efficiently, and there is evolutionary pressure to differentiate production. Once an agent has stopped producing resources available in excess, it need no longer maintain a form which can do everything, with all the compromises which that implies, but can adapt that form to the specific requirements of what it does produce. The system specialises, differentiates its forms, and increases its size over time; on a more abstract level, it is processing information and decreasing entropy. The oldest and most profound theme in economics, Adam Smith's invisible hand, is that the system as a whole has organising abilities which cannot be found in any of its parts.

### 5.7 Conclusion

We have built a model using two of the most powerful tools in the economic toolkit - Markov matrices, and linear production theory – by multiplying a Markov matrix defined over resource states **M** by a stock matrix **X** to get production vectors **a**. To the best of the author's knowledge and research, this device has not been used previously. In this way we have shown how specialized and interdependent species – whether economic or biological– become concentrated in a substrate over time. Future research has reasonable prospects of extending this conclusion to certain chemical species as well.

**Appendix – Description of an artificial life simulation developed by the author**

The following is a description of an artificial life simulation developed by the author which stimulated his interest in the topic. The simulation is not unlike others, for instance the Echo simulation built by the Santa Fe institute (Hraber et al. 1997), but there are important differences between the Echo simulation and this. The Echo simulation implements reciprocal trade rather than non-reciprocal trade, and uses different learning algorithms. The Echo simulation does not employ a graphical user interface, this simulation can be watched on the screen as it develops. A brief description of the system follows:

- There is a grid, typically measuring $500 \times 500$ cells, and defined terrain with uphill and downhill. Water flows in streams and rivers downhill into lakes.
- There are autonomous agents and six resources (energy, water, carbon, phosphorus, nitrogen and sulphur – these being the resources essential for life). If agents run out of energy they die. When they have one unit of each resource then they reproduce.
- Endowed resources: there is a standard allocation of energy in each period to each agent, a certain amount of rainfall which is fixed, and a fixed amount of phosphorus available in the environment.
- Each agent consists of cells of different kinds spread over a $5 \times 5$ grid (not necessarily touching). Each cell has an energy cost, and there are different kinds of cells which confer different abilities on the agent. There are particular cells which manufacture the three resources which are not endowed. Because there is an energy overhead for each cell, these cells embody a production function whereby energy is exchanged for the resource. The conversion rate is different for each resource.
- Agents move around the grid collecting water and phosphorus. Agents can be superimposed on the same space and when one agent meets another, it must give the

second agent any resource which the second agent lacks. In this way resources are shuffled throughout the system.

- Agents mutate randomly when they reproduce with a fixed low probability, gaining or losing cells with different abilities.
- Agents are given information about the cells immediately surrounding them, and they can learn different behaviours to respond to that information. Initially the learning method was Holland's (1975) genetic algorithm, but the author found that this method did not work effectively in this context. It was replaced with a variant of least squares learning whereby regression coefficients mutate randomly over time and generations to useful values. It was found that least squares learning could support much more detailed behaviour than the genetic algorithms.

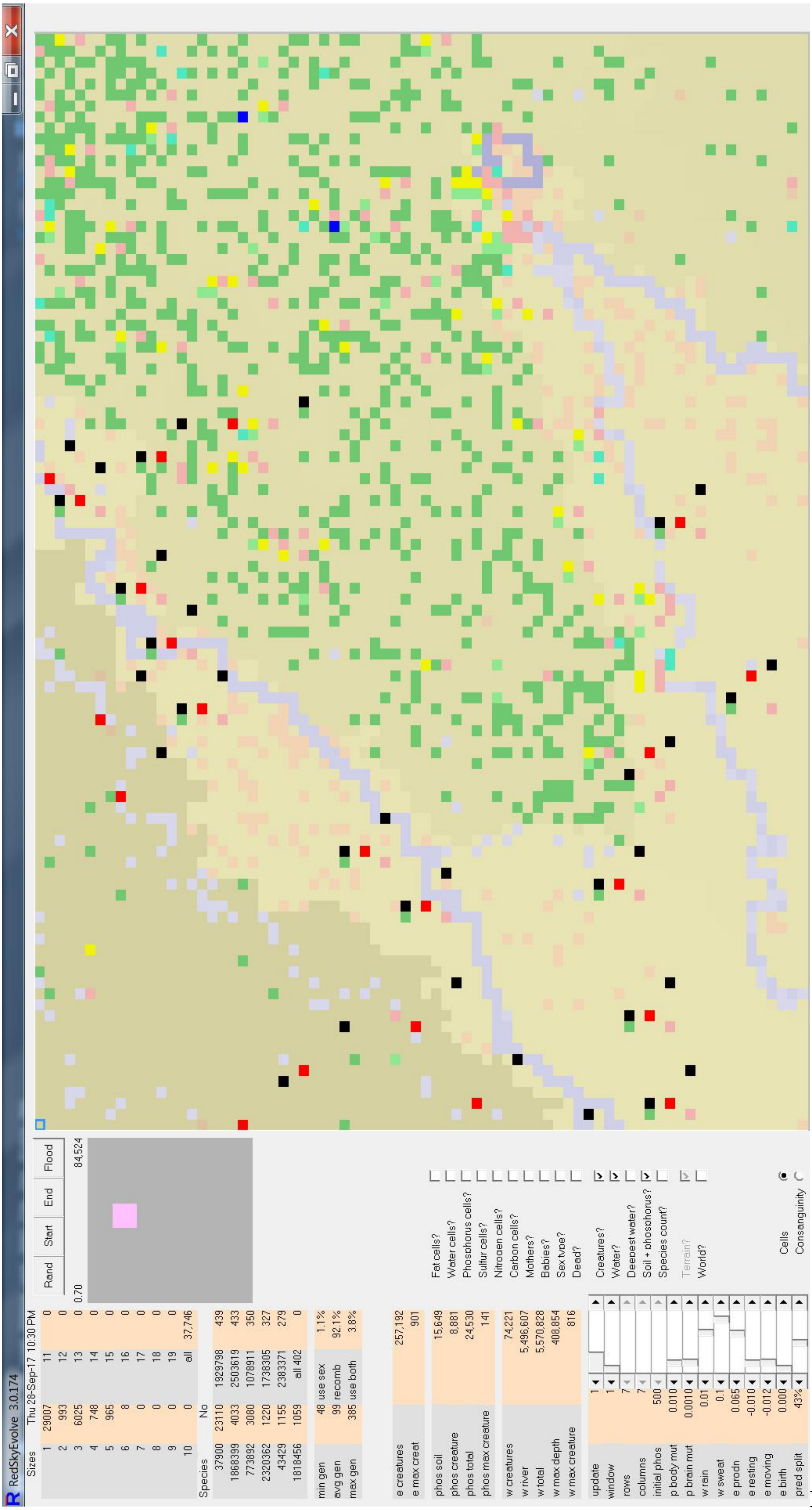
The agents started off as undifferentiated autarkical agents with basic capability. Over time (100,000 periods) a differentiated ecology evolved with a full range of environmental niches being filled. Typically 75% to 80% of the agents are single cell producers, as in real life biologies and economies. One notable tendency was for population to increase over time notwithstanding periodic booms and busts. After 100,000 periods, population settles down to around 7,000 agents. After 1,500,000 iterations, sophisticated behaviour has evolved and the population rises to around 70,000 agents.

The simulation has features not implemented within the mathematical model presented in this thesis, such as differentiated terrain and environmental niches, learning, and predation. It was notable that just as in real world biology, predators lose all cells and functions except the attack cell and a 'stomach' cell, which is required for realism. They do not even maintain the energy cell for the free energy allocation.

These types of artificial life simulation have been criticised (for instance, Horgan 1995) as games, and so they are. But games capture aspects of reality thought worthy of study. In the opinion of the author, this kind of game has scientific value insofar as it inspires and assists mathematical modelling. The mathematical model developed in this thesis studies the development of a specialised ecology from autarkical producers making non-reciprocal trades, and the reasons for the increased population. Future work will study predation and behavioural learning.

**Diagram A.1: Screenshot of the simulation.** The light khaki colour represents lowlands, the dark khaki colour represents highlands, and the grey-blue represents rivers which flow down to the lowest point to form lakes. In this shot, which is early in the simulation before the herbivorous agents have evolved effective defences, the predators have driven the herbivores out of the well-watered areas where they were once dense. They have taken refuge on a poorly watered highland plateau which the predators have not found (agents cannot move up cliffs). Different coloured cells represent different resource producing functions, all of which have an energy overhead. Where agents touch, they exchange resources according to scarcity.

**Diagram A.2: Screenshot of the species listing**. A computer simulation reports the values of variables which can only be inferred in real-life situations. This table shows the consanguinity (degree of interrelatedness) of the ten most populous agent types. It reveals that in the simulation as in real life, the most populous agents are the single cell vegetable types which collect the energy and process the resources – in this case 29,007 out of 37,746 creatures or 76.8%. Carnivores and large herbivores are relatively uncommon.


RedSkyEvolve 3.0.174
Sizes
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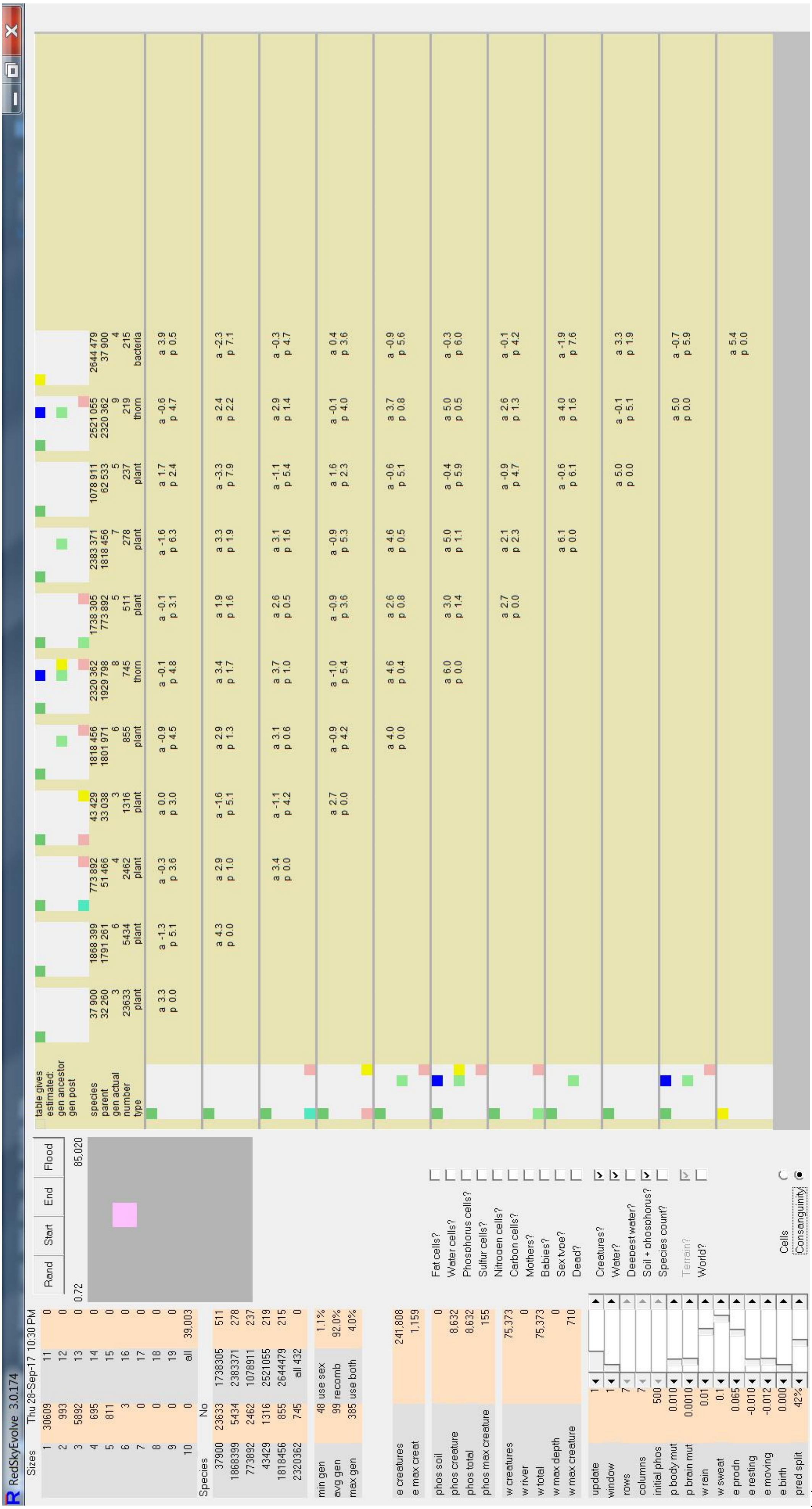
2 993 12 0
3 6025 13 0
4 748 14 0
5 965 15 0
6 8 16 0
7 0 17 0
8 0 18 0
9 0 19 0
10 0 all 37,746
Species No
37900 23110 1929798 439
1868399 4033 2503619 433
773892 3080 1078911 350
2320362 1220 1738305 327
43429 1155 2383371 279
1818456 1059 all 402 0
min gen 48 use sex 1.1%
avg gen 99 recomb 92.1%
max gen 385 use both 3.8%
e creatures 257,192
e max creat 901
phos soil 15,649
phos creature 8,881
phos total 24,530
phos max creature 141
w creatures 74,221
w river 5,496,607
w total 5,570,828
w max depth 408,854
w max creature 816
update 1
window 1
rows 7
columns 7
initial phos 500
p body mut 0.010
p brain mut 0.0010
w rain 0.01
w sweat 0.1
e prodn 0.065
e resting -0.010
e moving -0.012
e birth 0.000
pred split 43%
Rand
Start
End
Flood
0.70
84,524
Fat cells?
Water cells?
Phosphorus cells?
Sulfur cells?
Nitrogen cells?
Carbon cells?
Mothers?
Babies?
Sex type?
Dead?
Creatures?
Water?
Deepest water?
Soil + phosphorus?
Species count?
Terrain?
World?
Cells
Consanguinity

Diag. A.1


RedSkyEvolve 3.0.174
Thu 28-Sep-17 10:30 PM
Rand
Start
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Flood
Fat cells?
Water cells?
Phosphorus cells?
Sulfur cells?
Nitrogen cells?
Carbon cells?
Mothers?
Babies?
Sex type?
Dead?
Creatures?
Water?
Deepest water?
Soil + phosphorus?
Species count?
Terrain?
World?
Cells
Consanguinity

Diag. A.2

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